ASSIGNMENT 1
THE FINITE ELEMENT METHOD IN COMPUTATIONAL MECHANICS

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Exercise 1

In this assignment we start off by considering the following boundary value problem.

Boundary Value Problem 1. On the two dimensional domain $\Omega \coloneqq (0,1)^2$, consider the problem:

$$-\nabla u = f \text{ in } \Omega, \tag{1}$$

$$u = 0 \text{ for } x = 0 \text{ and } x = 1, \tag{2}$$

$$\frac{\partial u}{\partial n} = 0 \text{ for } y = 0 \text{ and } y = 1. \tag{3}$$

Analytical gobbledygook

We start by assuming $u = \sin(\pi k x)\cos(\pi k y)$ and compute the source term $f = -\Delta u = 2\pi^2 k^2 u$. We wish to compute analytically, the H^p norm. Recall that the H^p norm $\|\cdot\|_p$ is defined by

$$\|u\|_p = \left(\sum_{|\alpha| \leqslant p} \int_\Omega \left(\frac{\partial^{|\alpha|} u}{\partial \mathbf{x}^\alpha}\right)^2 \, d\mathbf{x} \right)^{1/2}$$

where $\alpha \coloneqq (\alpha_1, \ldots, \alpha_d)$ is a multi-index, and $|\alpha| \coloneqq \alpha_1 + \ldots + \alpha_d$. In the case where Ω is a subset of $\mathbb{R}^2$, we have $\alpha = (i, j)$ and $|\alpha| = i + j$. Note that the terms in the sum occur as the L^2 norm squared of the mixed partial derivatives, i.e.,

$$\left\| \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \right\|_{L^2}^2 = (k\pi)^{2(i+j)} \int_0^1 \int_0^1 \cos^2(k\pi x) \sin^2(k\pi y) \, dx dy.$$

Using the fact that both $\sin^2(\pi k y)$ and $\cos^2(\pi k x)$ integrate to $1/2$ over the unit interval, we have that this equals $(k\pi)^{2(i+j)}/4$. For $|\alpha| = n$, we have $n+1$ partial derivatives of order n, hence $\|u\|_p$ can be computed as

$$\|u\|_p = \frac{1}{2} \sum_{|\alpha| \leqslant p} \sum_{r=0}^{|\alpha|} (\pi k)^{i+j}.$$

Numerical error estimates

We solve the system given in Boundary Value Problem 1 in the PYTHON-framework FeniCS. Our mesh is taken to be uniformly spaced with mesh size $h := 1/N$. We examine the error in both the L_2 and the H^1 norms for $k = 1, 10$ and for both first and second order Lagrangian elements, that is

$$\text{error} = \|u - u_h\|_q \qquad \text{for } q = 0, 1.$$

The numerical errors are listed in Table 1.

We now wish to verify the two following error estimates:

$$\|u - u_h\|_1 \leqslant C_\alpha h^\alpha, \tag{4}$$

and

$$\|u - u_h\|_0 \leqslant C_\beta h^\beta. \tag{5}$$

These error estimates can be rewritten as linear equations in h with slopes α, β and constant terms $\log(C_\alpha)$, $\log(C_\beta)$, respectively. That is

$$\log(\|u - u_h\|_1) \leqslant \alpha h + \log(C_\alpha),$$

and similarly for the $\| \cdot \|_0$ error. Sampling the left hand side for several values of h, we can fit a linear function to the data, hence finding the unknown slope and constant terms. This has been done using the function numpy.polyfit(). The results are given in Table 2.

Plotting the errors against the number of elements N in a log-log plot reveals a linear tendency.

Table 1: The L_2 and H^1 errors for varying number of mesh-elements. First order elements to the left and second order elements to the right.

(a) Errors for $k = 1$.

N	L_2	H^1
8	0.62583	3.03866
16	0.64926	3.11820
32	0.65536	3.13876
64	0.65690	3.14394

N	L_2	H^1
8	0.65310	3.11978
16	0.65637	3.13927
32	0.65716	3.14408
64	0.65735	3.14528

(b) Errors for $k = 10$.

N	L_2	H^1
8	0.74298	24.94148
16	0.50061	28.59929
32	0.76293	37.48450
64	0.91167	41.68281

N	L_2	H^1
8	0.65808	26.95569
16	0.71521	32.92179
32	0.90010	40.14219
64	0.95659	42.52210

Table 2: The slopes and coefficients for the error estimates given in Equation (4).

k	C_α	α	C_β	β
1				
10				

Exercise 2

We now consider another boundary value problem:

Boundary Value Problem 2. On the two dimensional domain $\Omega := (0,1)^2$, consider the second order problem:

$$-\mu\Delta u + u_x = 0 \text{ in } \Omega, \tag{6}$$
$$u = 0 \text{ for } x = 0, \tag{7}$$
$$u = 1 \text{ for } x = 1, \tag{8}$$
$$\frac{\partial u}{\partial n} = 0 \text{ for } y = 0 \text{ and } y = 1. \tag{9}$$

Analytical solution

It is possible to derive an analytical solution for the above boundary value problem using seperation of variables. We make the ansatz that we can write $u(x,y) = f(x)g(y)$. Plugging this into Boundary Value Problem 2, and dividing by $-\mu u$ we arrive at the set of equations

$$f''(x) - \frac{1}{\mu}f'(x) - Cf(x) = 0, \tag{10}$$
$$g''(y) + Cg(y) = 0, \tag{11}$$

where C is some unknown constant. Solving for $g(y)$ first, we arrive at the solution

$$g(y) = A\sin(\sqrt{C}y) + B\cos(\sqrt{C}y).$$

Enforcing the Neumann boundary conditions given in Equation (9), we determine g to be constant (with respect to x) equal to

$$g(y) = B\cos(n\pi y),$$

with $n \in \mathbb{N}$. In particular, for $n = 0$, $C = 0$ so we have $g(y) = B$. Furthermore, with this choice of n, Equation (10) reduces to

$$f''(x) - \frac{1}{\mu}f'(x) = 0$$

which has solution $f(x) = De^{\frac{1}{\mu}x} + E$. Enforcing the Dirichlet boundary conditions given in Equations (7) and (8) we determine $E = -1$ and $D = (e^{\frac{1}{\mu}} - 1)^{-1}$, as well as $B = 1$, yielding the final solution

$$u(x,y) = f(x)g(y) = \frac{e^{\frac{1}{\mu}x} - 1}{e^{\frac{1}{\mu}} - 1}.$$