

EXAM SHEET

MEK4250

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June 1, 2017

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Chapter 1

Weak Formulation And Finite Element Error Estimation

Problem. Formulate a finite element method for the Poisson problem with a variable coefficient $\kappa: \Omega \rightarrow \mathbb{R}^{d \times d}$. Show that Lax-Milgram's theorem is satisfied. Consider extensions to e.g. the convection-diffusion and the elasticity equation. Derivate *a priori* error estimates for the finite element method in the energy norm. Describe how to perform an estimation of convergence rates.

1.1 Finite Element Formulation

The Poisson problem is formulated as:

$$-\nabla \cdot (\kappa \nabla u) = f \text{ in } \Omega, \quad (1.1)$$

$$u = u_0 \text{ on } \Gamma_D, \quad (1.2)$$

$$-\kappa \nabla u \cdot n = g \text{ on } \Gamma_N. \quad (1.3)$$

Here u denotes the unknown field. We associate to this strong formulation of the problem, the bilinear operator $a: \hat{V} \times V \rightarrow \mathbb{R}$, and the linear operator

$L: \hat{V} \rightarrow \mathbb{R}$ as follows:

$$a(u, v) := \langle -\nabla \cdot (\kappa \nabla u), v \rangle, \quad (1.4)$$

$$L(v) := \langle f, v \rangle. \quad (1.5)$$

Weak Formulation

We can therefore consider instead the *weak formulation* of the Poisson problem, which is: Find $u \in V$ such that

$$a(u, v) = L(v) \text{ for all } v \in \hat{V}. \quad (1.6)$$

We need to place some requirements to the spaces V and \hat{V} in order to satisfy the boundary conditions. However, one requirement we impose immediately is that $v = 0$ on Γ_D for all $v \in \hat{V}$. What these spaces should be is not immediate from the current formulation. Through the following, $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product on V . Expanding a using among others the Gauss–Green lemma yields:

$$a(u, v) = \langle -\nabla \cdot (\kappa \nabla u), v \rangle = -\langle \kappa \Delta u, v \rangle \quad (1.7)$$

$$= \langle \kappa \nabla u, \nabla v \rangle - \int_{\partial\Omega} -v\kappa \frac{\partial u}{\partial x} \cdot n \, dS. \quad (1.8)$$

The boundary integral above can be rewritten using the partitioning of the boundary:

$$\int_{\partial\Omega} -v\kappa \frac{\partial u}{\partial x} \cdot n \, dS = - \int_{\Gamma_D} v\kappa \frac{\partial u}{\partial x} \cdot n \, dS + \int_{\Gamma_N} -v\kappa \frac{\partial u}{\partial x} \cdot n \, dS. \quad (1.9)$$

Applying the boundary conditions over respective boundaries and condition on \hat{V} we get in total that:

$$a(u, v) = \langle \kappa \nabla u, \nabla v \rangle - \int_{\Gamma_N} gv \, dS. \quad (1.10)$$

Since the boundary term in a is independent of u we decide to transfer this term to L :

$$a(u, v) = \langle \kappa \nabla u, \nabla v \rangle, \quad L(v) = \langle f, v \rangle + \int_{\Gamma_N} gv \, dS. \quad (1.11)$$

We now have what we need to determine the spaces \hat{V} and V . Note that we in the weak form only require u and v to be once differentiable. Furthermore, we require u to reduce to u_0 on Γ_D by the boundary conditions, while we need v to vanish identically at Γ_D . We therefore decide on the test and trial spaces

$$V = H_g^1(\Omega) := \{u \in H^1(\Omega) : u = g \text{ on } \Gamma_D\} \subseteq H^1(\Omega), \quad (1.12)$$

$$\hat{V} = H_0^1(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\} \subseteq H^1(\Omega). \quad (1.13)$$

Finite Element Formulation

In order to compute with these spaces, we need to introduce a basis. However, the spaces might be infinite dimensional, so we approximate by finite subspaces \hat{V}_h and V_h respectively¹. Since we have $\hat{V} \subseteq V$ we have $\hat{V}_h \subseteq V_h$, and we therefore use the same basis vectors for both test and trial functions. We seek a solution $u = \sum_i c_i \varphi_i$ such that

$$\sum_{i=1}^N c_i a(\varphi_i, \varphi_j) = L(\varphi_j) \text{ for } j = 1, \dots, M \quad (1.14)$$

where N, M denotes the dimensions of V_h, \hat{V}_h respectively. This determines a linear system

$$\mathbf{A}\mathbf{c} = \mathbf{b} \quad (1.15)$$

where the matrix entries are determined as follows:

$$A_{i,j} = a(\varphi_i, \varphi_j), \quad b_j = L(\varphi_j). \quad (1.16)$$

1.2 Well Posedness of Weak Formulation

In this section we discuss whether the problem in its weak formulation is infact well posed. Does there exist a solution, and if it does, is this solution unique? The Lax–Milgram theorem provides sufficient conditions for this

¹Much of the finite element theory amounts to determining what the error in this specific approximation is.

problem to be well posed, and hence the solution to exist and be unique. We verify the three properties in turn:

Boundedness of a : Let $u, v \in H^1(\Omega)$. Then we have

$$a(u, v) = \langle \kappa \nabla u, \nabla v \rangle \underset{\text{Using the Cauchy-Schwartz inequality}}{\leq} \overset{\text{subordinate matrix norm}}{\|\kappa\|} \|\nabla u\|_0 \|\nabla v\|_0 = \|\kappa\| \|u\|_1 \|v\|_1 \quad (1.17)$$

This is close to proving boundedness of a , however, I do not know how to deal with the $\|\kappa\|$. However, for $\kappa = 1$ this proves the claim.

Coercivity of a : Let $u \in H^1(\Omega)$ and consider the following:

$$\|u\|_1^2 \stackrel{\text{def}}{=} \|u\|_0^2 + \|\nabla u\|_0^2 \underset{\text{Using the Poincaré inequality on } H_0^1(\Omega)}{\leq} (C^2 + 1) \|\nabla u\|_0^2 \quad (1.18)$$

This shows that $a(u, u) \leq (C^2 + 1)^{-1} \|u\|_1^2$ in the case where $\kappa = 1$ identically.

Boundedness of L :

$$L(v) = \langle f, v \rangle + \int_{\Gamma_N} g v \, dS \quad (1.19)$$

Using Cauchy–Schwartz on both terms yields

$$\leq \|f\|_0 \|v\|_0 + \|g\|_{L^2(\Gamma_N)} \|v\|_{L^2(\Gamma_N)}; \quad (1.20)$$

Using the fact that v is defined on the entirety of Ω , the L^2 norm is certainly larger over the whole domain

$$\leq \|f\|_0 \|v\|_0 + \|g\|_{L^2(\Gamma_N)} \|v\|_0; \quad (1.21)$$

Using that the H^1 norm greater than or equal to the L^2 norm

$$\leq (\|f\|_0 + \|g\|_{L^2(\Gamma_N)}) \|v\|_1, \quad (1.22)$$

$$= D \|v\|_1. \quad (1.23)$$

This proves the boundedness of L .

This means that the weak formulation of the Poisson problem satisfies the Lax–Milgram theorem, hence is well posed.

1.3 Extensions to other problems

Recall that the equation of linear elasticity is given as

$$-2\mu(\nabla \cdot \varepsilon(u)) - \lambda \nabla(\nabla \cdot u) = 0 \quad (1.24)$$

where $\varepsilon(u) := (\nabla u + (\nabla u)^T)/2$ is the strain tensor. Ignoring the second term, first term is supposedly a Poisson-equation². However, in order to check whether we can apply the results found for the abstract Poisson-problem we need to determine what the $\nabla(\nabla \cdot u)$ in the second term is. The Helmholtz decomposition theorem states that any u in $L^2(\Omega)$ can be decomposed into a curl-free part ψ and a divergence-free part φ :

$$u = \psi + \varphi \quad (1.25)$$

where $\nabla \cdot \varphi = 0$ and $\nabla \times \psi = 0$. In the light of this, we can consider the two special cases where either φ or ψ are zero.

- (i) Assume that $\varphi = 0$. This means that $u = \psi$ consists only of the divergence free part. We then have that $\nabla(\nabla \cdot u) = 0$. Consequently, the second term vanishes in Equation (1.24).
- (ii) Assume that $\psi = 0$, i.e., u consists only of the curl free part. Then using the identity

$$\Delta u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u) \quad (1.26)$$

we see that $u = \nabla(\nabla \cdot u)$. Hence, Equation (1.24) reduces to a Poisson problem.

In these two special cases, we have that the linear elasticity problem reduces to a Poisson problem of the form

$$-(\mu + \lambda) \Delta w = f. \quad (1.27)$$

for some source term f .

²This needs to be verified.

1.4 A priori error estimates

In the following, we work over arbitrary test and trial spaces V , as these estimates are general.

Energy Norm

We now discuss some a priori estimates. Recall that any arbitrary inner product induces a norm by

$$\|x\| := \sqrt{\langle x, x \rangle}. \quad (1.28)$$

It turns out that the bilinear form a may, or may not, constitute an inner product. If it does, then we may talk about the norm induced by this bilinear form. We call this the *energy norm*. We first need to verify that a does indeed constitute an inner product. The only condition that is not trivial is the coercivity, however we already proved this for the Lax–Milgram theorem. We therefore define the energy norm

$$\|u\|_E := \sqrt{a(u, u)}. \quad (1.29)$$

Consider now the error $e := u - u_h$. Let $v \in \hat{V}$ be arbitrary. In the energy norm, we have

$$\|e\|_E^2 = a(e, e) = a(e, u - u_h) = a(e, u - v + v - u_h) \quad (1.30)$$

$$= a(e, u - v) + a(e, \underbrace{u_h - v}_{\in \hat{V}}) = a(e, u - v). \quad (1.31)$$

Using the Cauchy–Schwartz inequality, we have that

$$\|e\|_E^2 \leq \|e\|_E \|u - v\|_E \implies \|e\|_E \leq \|e - v\|_E \quad (1.32)$$

for all $v \in \hat{V}$. This does however not give any sharp bounds, so it is hard to quantify exactly what magnitude the error has. We can however combine this result with an interpolation error estimate:

$$\|e\|_E \leq \|u - \pi_{q,h}u\|_E \leq C(q) \|h^{q+1} D^{q+1}u\|. \quad (1.33)$$

Here, $\pi_{q,h}u$ denotes the q -th order interpolant to u , and h is the maximum mesh size.

Error estimate without coercivity of a

The symmetry of a is quite a strong requirement, and this is not always satisfied. In this section, we do *not* assume that a is symmetric. In the cases where a is bounded however we can consider the error in the vector space norm as follows. Let $v \in \hat{V}$ be arbitrary. Then using the coercivity of a we have:

$$\|e\|_V^2 \leq \frac{1}{\alpha} a(e, e) = \frac{1}{\alpha} a(e, u - v + v - u_h) \quad (1.34)$$

$$= \frac{1}{\alpha} a(e, u - v) \underbrace{\leq}_{\text{Using boundedness of } a} \frac{D}{\alpha} \|e\|_V \|u - v\|_V. \quad (1.35)$$

Dividing by $\|e\|_V$ and combining with an interpolation error estimate, as we did above,

$$\|e\|_V \leq \frac{DC(q)}{\alpha} \|h^{q+1} D^{q+1} u\|. \quad (1.36)$$

1.5 Error Estimation

In the following we assume we solve a constructed problem with known solution. Let N denote the number of basis elements over the domain. Then we can consider the error as a function of N . Using degree p elements over a mesh with maximal mesh size of h . Let u denote the analytic solution, and u_N the computed solution with N basis elements. Denote by e_N the error in the corresponding approximation. We wish to approximate the convergence rate β in the following:

$$\|e\|_V \leq \|h^\beta D^\beta u\|. \quad (1.37)$$

We can rewrite this as a linear equation in h with slope β and constant term $\log(D^\beta u)$. Computing e_N for various N we can find a linear regression line through the data and approximate β .