Mandatory assignment
Finite Element Method in Computational Mechanics
MEK4250

Ivar Stangeby

May 1, 2017

Exercise 7.2: Stokes problem

Problem statement and strong formulation

Let the fluid domain Ω be bounded in $\mathbb{R}^n$ with a smooth boundary. Denote by $u\colon \Omega \to \mathbb{R}^n$ the fluid velocity, and by $p\colon \Omega \to \mathbb{R}$ the fluid pressure. Furthermore, assume that the domain boundary $\partial\Omega$ is partitioned into the Dirichlet boundary, and the Neumann boundary, denoted by $\partial\Omega_D$ and $\partial\Omega_N$, respectively. In its strong form, the Stokes problem reads

$$
\begin{aligned}
-\Delta u + \nabla p &= f, && \text{in} \quad \Omega;\\
\nabla \cdot u &= 0, && \text{in} \quad \Omega;\\
u &= g, && \text{on} \quad \partial\Omega_D;\\
\frac{\partial u}{\partial n} - pn &= h, && \text{on} \quad \partial\Omega_N.
\end{aligned}
$$

Well posedness

We wish to show the well posedness of the weak formulation of Stokes problem. In our case, we deal with the sobolev space $H_0^1(\Omega)$ for the fluid u, and the $L^2(\Omega)$ space for the pressure p. This, involves showing that the following conditions are met:

i)
$$a(u, v) \leq C_1 \|u\|_{H^1} \|v\|_{L^2},$$

ii)
$$b(p, v) \leq C_2 \|p\|_{L^2} \|v\|_{H^1},$$

iii)
$$a(u, u) \geq C_3 \|u\|_{H^1}^2,$$

for all $u, v \in H_0^1(\Omega)$ and all $p \in L^2$. Recall that the bilinear forms in question are given by

$$
\begin{aligned}
a(u, v) &= \int_\Omega \nabla u : \nabla v \,\mathrm{d}x,\\
b(p, v) &= \int_\Omega p \nabla \cdot v \,\mathrm{d}x
\end{aligned}
$$

where $\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j} A_{ij} B_{ij}$ denotes the Frobenius inner product.

i) Applying the Cauchy-Schwartz inequality on a yields the following:

$$
\begin{aligned}
a(u, v) &= \int_\Omega \nabla u : \nabla v \,\mathrm{d}x\\
&= \langle \nabla u \mid \nabla v \rangle \leq |u|_{H^1} |v|_{H^1}.
\end{aligned}
$$

Noting that the seminorm on $H_0^1$is never larger than the full norm on H_0^1, so the condition holds.

ii) Applying Cauchy-Schwartz on b yields

$$b(p, v) \leq \|p\|_{L^2} \|\nabla \cdot v\|_{L^2}.$$

To this end, it suffices to show that $\|\nabla \cdot v\|_{L^2} \leq \|v\|_{H^1}$. For the left hand side, we have that

$$\|\nabla \cdot v\|_{L^2} \leq \left(\int_\Omega \sum_{i=1}^n \left(\frac{\partial v_i}{\partial x_i} \right)^2 \right)^{1/2}.$$

For the right hand side, firstly we have that $(\nabla v)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\partial v_j / \partial x_i\right)^2$. Note that this has a striking resemblance to $\nabla \cdot v$, however, with a lot more positive terms. We can therefore conclude outright that

$$b(p, v) \leq \|p\|_{L^2} \|v\|_{H^1}.$$

iii) Finally, we consider $a(u, u)$. Firstly, we have that $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + |u|_{H_1}^2$. By applying Poincaré, we know that this has to be less than or equal to $(C^2 + 1)|u|_{H^1}^2$. Writing this out, we have that

$$\|u\|_{H^1}^2 \leq (C^2 + 1) \int_\Omega (\nabla u)^2 \, \mathrm{d}x.$$

Note that $(\nabla u)^2 = \nabla u : \nabla u$. Indeed,

$$(\nabla u)^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} \left(\frac{\partial u_i}{\partial x_j} \right)^2 = \nabla u : \nabla u.$$

Consequently, we have that $\|u\|_{H^1}^2 \leq (C^2 + 1) a(u, u)$. Multiplying each side of the equation by $D := 1/(C^2 + 1)$ we get the bound we wanted.

Exercise 7.6: Approximation order in Stokes problem

In this exercise we solve Stokes problem as above, with the known solutions:

$$u = (\sin(\pi y), \cos(\pi x)),$$
$$p = -\sin(2\pi x).$$

It then follows that the source term is given as:

$$f(x) = (\pi^2 \sin(\pi y) - 2\pi \cos(2\pi x), \pi^2 \cos(\pi x)).$$

Ideally, we obtain optimal convergence rate, given by

$$\|u - u_h\|_{H^1} + \|p - p_h\|_{L^2} \leq C h^k \|u\|_{H^{k+1}} + D h^{\ell+1} \|p\|_{H^{\ell+1}},$$

where k and ℓ are the polynomial degree of the velocity and pressure. In order to obtain this optimal convergence, we need to determine the finite element pairs V_h and Q_h for the velocity and pressure, respectively, that satisfy the Babuska–Brezzi condition:

$$\sup_{v_h \in V_{h,g}} \frac{(p_h, \nabla \cdot v_h)}{\|v_h\|_{H^1}} \geq \beta \|p_h\|_{L^2} > 0, \qquad (1)$$

for all $p_h \in Q_h$. Elements satisfying Equation (1) include the following:

Taylor-Hood: Quadratic for velocity, and linear for the pressure.

Crouzeix-Raviart: Linear in velocity, constant in pressure.

Mini element: Linear in both velocity and pressure, and a cubic bubble function is added to the velocity element in order to satisfy Equation (1).

In this exercise, we wish to examine whether the approximation is of the expected order for the $P_4 - P_3$, P_4–P_2, $P_3 - P_2$, and $P_3 - P_1$ elements. We examine these in turn. If the convergence is optimal, we would expect the following:

P_4–P_3: With $k = 4$ and $\ell = 3$, we would expect the error to run as $O(h^4)$.

P_4–P_2: With $k = 4$ and $\ell = 2$, we would expect the error to run as $O(h^3)$.

P_3–P_2: With $k = 3$ and $\ell = 2$, we would expect the error to run as $O(h^3)$.

P_3–P_1: With $k = 3$ and $\ell = 2$, we would expect the error to run as $O(h^2)$.

This is verified by computations, as can be seen in Table 1.

Table 1: The convergence rate of the error is given as α, and the error constant is C in the error estimate

$$\|u - u_h\|_{H^1} + \|p - p_h\|_{L^2} \leq Ch^{\alpha}.$$

These were computed using a linear polynomial fit of the computed errors as h decrease. The convergence rates seem to agree with predictions.

Element	α	C
$P_4 - P_3$	4.04615	0.344373
$P_4 - P_2$	2.92402	1.47968
$P_3 - P_2$	2.91623	1.3616
$P_3 - P_1$	2.02219	2.26125