❖ **Question # 8:**

$$x\sin\left(\frac{y}{x}\right)dy = \left[y\sin\left(\frac{y}{x}\right) - x\right]dx$$

Solution:

Given equation is

$$x\sin\left(\frac{y}{x}\right)dy = \left[y\sin\left(\frac{y}{x}\right) - x\right]dx$$

$$\Longrightarrow \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) - x}{x\sin\left(\frac{y}{x}\right)}$$

This is a homogenous differential equation in x & y. to solve this, put

$$y = vx$$

$$\Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v + x\frac{dv}{dx} = \frac{vx\sin - x}{x\sin v}$$

$$\Longrightarrow v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$$

$$\Longrightarrow x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} - v$$

$$\Longrightarrow x\frac{dv}{dx} = \frac{v\sin v - 1 - v\sin v}{\sin v}$$

$$\Longrightarrow x\frac{dv}{dx} = -\frac{1}{\sin v}$$

$$\Longrightarrow -\sin v\,dv = \frac{dx}{x}$$

Integrating both sides, we have

$$-\int \sin v\,dv = \int \frac{dx}{x}$$

$$\Longrightarrow \cos v = \ln x + c$$

$$\Longrightarrow \cos\frac{y}{x} = \ln x + c \quad \because v = \frac{y}{x}$$

is required solution.

❖ **Question # 10:**

$$(\sqrt{x+y}+\sqrt{x-y})dx-(\sqrt{x+y}-\sqrt{x-y})dy=0$$

Solution:

Given equation is

$$(\sqrt{x+y}+\sqrt{x-y})dx-(\sqrt{x+y}-\sqrt{x-y})dy=0$$

$$\Rightarrow (\sqrt{x+y}-\sqrt{x-y})dy$$
$$=(\sqrt{x+y}+\sqrt{x-y})dx$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}} ---(a)$$

This is a homogenous differential equation in x & y. to solve this, put

$$y=vx$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v+x\frac{dv}{dx}=\frac{\sqrt{x+vx}+\sqrt{x-vx}}{\sqrt{x+vx}-\sqrt{x-vx}}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}} \times \frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}+\sqrt{1-v}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v+1-v+2\sqrt{1-v^2}}{1+v-1-v}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{2+2\sqrt{1-v^2}}{2v}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{2(1+\sqrt{1-v^2})}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}}{v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}-v^2}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{\sqrt{1-v^2}+(1-v^2)}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{\sqrt{1-v^2}\left(1+\sqrt{1-v^2}\right)}{v}$$

$$\Rightarrow \frac{vdv}{\sqrt{1-v^2}\left(1+\sqrt{1-v^2}\right)} = \frac{dx}{x}$$

Integrating both sides, we have

$$\int \frac{vdv}{\sqrt{1-v^2}\left(1+\sqrt{1-v^2}\right)} = \int \frac{dx}{x}$$

$$Put\ 1+\sqrt{1-v^2} = t$$

$$\Rightarrow \frac{1}{2}(1-v^2)^{\frac{-1}{2}}(-2v)dv = dt$$

$$\Rightarrow \frac{vdv}{\sqrt{1-v^2}} = -dt$$

therefore,

$$\int \frac{-dt}{t} = \int \frac{dx}{x}$$

$$\Longrightarrow -\int \frac{dt}{t} = \int \frac{dx}{x}$$

$$\Longrightarrow -\ln t = lnx + lnc$$

$$\Longrightarrow -\ln\left(1+\sqrt{1-v^2}\right) = lnx + lnc$$

$$\Longrightarrow -\ln\left(1+\sqrt{1-v^2}\right) = lncx$$

$$\Longrightarrow \ln\left(1+\sqrt{1-v^2}\right) = -lncx$$

$$\Longrightarrow \ln\left(1+\sqrt{1-v^2}\right) = ln(cx)^{-1}$$

$$\Longrightarrow \left(1+\sqrt{1-v^2}\right) = \frac{1}{cx}$$

$$\Longrightarrow 1+\sqrt{1-\frac{y^2}{x^2}} = \frac{1}{cx} \quad \because v = \frac{y}{x}$$

$$\Longrightarrow 1+\frac{\sqrt{x^2-y^2}}{x} = \frac{1}{cx}$$

$$\Longrightarrow \frac{x+\sqrt{x^2-y^2}}{x} = \frac{1}{cx}$$

$$\Longrightarrow x+\sqrt{x^2-y^2} = \frac{1}{c} = c \text{ (a constant)}$$

$$\Longrightarrow x+\sqrt{x^2-y^2} = c$$

is required solution.

Solve the initial value problem

❖ Question # 11:

$$\frac{dy}{dx} = \frac{x+y}{x} \qquad y(1) = 1$$

Solution:

Given equation is

$$\frac{dy}{dx} = \frac{x+y}{x} - - - (a)$$

This is a homogenous differential equation in x & y. to solve this, put

$$y = vx$$

$$\Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v + x\frac{dv}{dx} = \frac{x+vx}{x}$$

$$\Longrightarrow v + x\frac{dv}{dx} = 1 + v$$

$$\Longrightarrow x\frac{dv}{dx} = 1 + v - v$$

$$\Longrightarrow dv = \frac{dx}{x}$$

Integrating both sides, we have

$$\int dv = \int \frac{dx}{x}$$

$$\Longrightarrow v = \ln x + c$$

$$\Longrightarrow \frac{y}{x} = \ln x + c - - - (b) \quad \because v = \frac{y}{x}$$

Applying the condition $y(1) = 1$ on (b), we have

$$1 = 0 + c$$

$$\Longrightarrow c = 1$$

Therefore,

$$\frac{y}{x} = \ln x + 1$$

$$\Longrightarrow y = x \ln x + x$$

is required solution.

❖ **Question # 13:**

$$(2x - 5y)dx + (4x - y)dy = 0 \qquad y(1) = 4$$

Solution:

Given equation is

$$(2x - 5y)dx + (4x - y)dy = 0$$

$$\Longrightarrow (4x - y)dy = -(2x - 5y)dx$$

$$\Longrightarrow (4x - y)dy = (5y - 2x)dx$$

$$\Longrightarrow \frac{dy}{dx} = \frac{5y - 2x}{4x - y} - - - (a)$$

This is a homogenous differential equation in x & y. to solve this, put

$$y = vx$$

$$\Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus equation (a) becomes

$$v+x\frac{dv}{dx}=\frac{5vx-2x}{4x-vx}$$

$$\Longrightarrow v+x\frac{dv}{dx}=\frac{5v-2}{4-v}$$

$$\Longrightarrow x\frac{dv}{dx}=\frac{5v-2}{4-v}-v$$

$$\Longrightarrow x\frac{dv}{dx}=\frac{5v-2-4v+v^2}{4-v}$$

$$\Longrightarrow x\frac{dv}{dx}=\frac{v^2+v-2}{4-v}$$

$$\Longrightarrow \frac{4-v}{v^2+v-2}dv=\frac{dx}{x}$$

$$\Longrightarrow \frac{4-v}{(v+2)(v-1)}dv=\frac{dx}{x}---(b)$$

Consider that

$$\frac{4-v}{(v+2)(v-1)}=\frac{A}{v+2}+\frac{B}{v-1}$$

$$\Longrightarrow 4-v=A(v-1)+B(v+2)---(i)$$

$Put\ v+2=0 \Longrightarrow v=-2\ in\ (i), we\ have$

$$6=A(-3)$$

$$\boxed{\Longrightarrow \boldsymbol{A=-2}}$$

$Put\ v-1=0 \Longrightarrow v=1\ in\ (i), we\ have$

$$3=B(3)$$

$$\boxed{\Longrightarrow \boldsymbol{B=1}}$$

therefore,

$$\frac{4-v}{(v+2)(v-1)}=\frac{-2}{v+2}+\frac{1}{v-1}$$

Thus equation (b) becomes

$$\frac{-2}{v+2}+\frac{1}{v-1}=\frac{dx}{x}$$

Integrating both sides, we have

$$\int \left(\frac{-2}{v+2} + \frac{1}{v-1}\right) dv = \int \frac{dx}{x}$$

$$-2\int \frac{dv}{v+2} + \int \frac{dv}{v-1} = \int \frac{dx}{x}$$

$$\Longrightarrow -2\ln(v+2) + \ln(v-1) = lnx + \ln c$$

$$\Longrightarrow \ln(v+2)^{-2} + \ln(v-1) = ln\, cx$$

$$\Longrightarrow \ln\frac{(v-1)}{(v+2)^2} = ln\, cx$$

$$\Longrightarrow \frac{(v-1)}{(v+2)^2} = cx$$

$$\Longrightarrow \frac{\frac{y}{x} - 1}{\left(\frac{y}{x} + 2\right)^2} = cx \quad \because v = \frac{y}{x}$$

$$\Rightarrow \frac{(y-x)/_x}{(y+2x)^2/_{x^2}} = cx$$

$$\Rightarrow \frac{(y-x)x}{(y+2x)^2} = cx$$

$$\Rightarrow y - x = c(y+2x)^2 - - - (c)$$

Applying the condition $y(1) = 4$ *on* (b)*, we have*

$$4 - 1 = c(4+2)^2$$

$$\boxed{\Rightarrow c = \frac{1}{12}}$$

$$(c) \Rightarrow y - x = \frac{1}{12}(y+2x)^2$$

$$\Rightarrow (y+2x)^2 = 12(y-x)$$

is required solution.