Example:

$$(x^2+y^2)p + 2xyq = (x+y)z$$

Sol.

A.E:

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{x+y}$$

* $$\frac{dx+dy}{x^2+y^2+2xy} = \frac{dz}{x+y}$$

$$\frac{dx+dy}{(x+y)^2} = \frac{dz}{x+y}$$

$$\frac{dx+dy}{x+y} = dz$$

$$\frac{d(x+y)}{x+y} = dz$$

Integrate

$$\ln(x+y) = z + c_1$$

$$\ln(x+y) - z = c_1$$

* $$\frac{dx+dy}{(x+y)^2} = \frac{dx-dy}{(x-y)^2}$$

$$\int (x+y)^{-2} d(x+y) = \int (x-y)^{-2} d(x-y)$$

$$-(x+y)^{-1} = -(x-y)^{-1} + C_2$$

$$\Rightarrow \quad \frac{1}{x-y} - \frac{1}{x+y} = C_2$$

$$\phi\left(\ln(x+y) - z \,,\; \frac{-1}{x+y} + \frac{1}{x-y}\right) = 0$$

Example..

$$\frac{y-z}{yz}p + \frac{z-x}{zx}q = \frac{x-y}{xy}$$

Sol..

A.E..

$$\frac{dx}{\frac{y-z}{yz}} = \frac{dy}{\frac{z-x}{zx}} = \frac{dz}{\frac{x-y}{xy}}$$

let $1/x$, $1/y$, $1/z$ be the multipliers

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{y-z}{xyz} + \frac{z-x}{xyz} + \frac{x-y}{xyz}}$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\ln x + \ln y + \ln z = \ln C_1$$

$$\ln(xyz) = \ln C_1$$

$$xyz = C_1$$

$$\frac{dx+dy}{\frac{y-z}{yz}+\frac{z-x}{zx}} = \frac{dz}{\frac{x-y}{xy}}$$

$$\frac{dx+dy}{\frac{xy-zx+zy-xy}{xyz}} = \frac{dz}{\frac{x-y}{xy}}$$

$$\frac{dx+dy}{\frac{-z(x-y)}{xyz}} = \frac{dz}{\frac{x-y}{xy}}$$

$$dx+dy = -dz$$

$$x+y+z = c_2$$

$$\phi(xyz,\ x+y+z) = 0$$

Example..

Sol

$$(p-q)z = z^2+(x+y)^2$$

$$zp - zq = z^2+(x+y)^2$$

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$$

$$\frac{dx}{z} = \frac{dy}{-z}$$

$$dx+dy = 0$$

$$x+y = c_1$$

$$\frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$$

$$\frac{dz}{dy} = \frac{z^2+(x+y)^2}{-z}$$

$$\frac{dz}{dy} = \frac{z^2+c_1^2}{-z} \quad \text{—} (1)$$

let $z^2+c_1^2 = t$

$$2z\frac{dz}{dy} = \frac{dt}{dy}$$

$$\Rightarrow z\frac{dz}{dy} = \frac{1}{2}\frac{dt}{dy}$$

(1) $\Rightarrow$ $$-\frac{1}{2}\frac{dt}{dy} = t$$

$$-\frac{1}{2}\frac{dt}{t} = dy$$

$$\ln t = -2y + c_2$$

$$\ln(z^2+c_1^2) + 2y = c_2$$

$$\ln(z^2+(x+y)^2) + 2y = c_2$$

$$\phi(x+y, \ln(z^2+(x+y)^2+2y) = 0$$

Example

$$x^2 p + y^2 q = z(x+y)$$

Sol

A.E.

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z(x+y)}$$

$$\frac{dx - dy}{x^2 - y^2}$$

$$= \frac{d(x-y)}{x^2 - y^2}$$

Compare with 3rd ratio

$$\frac{dz}{z(x+y)} = \frac{d(x-y)}{(x+y)(x-y)}$$

$$\frac{dz}{z} = \frac{d(x-y)}{x-y}$$

Integrate

$$\ln z = \ln(x-y) + \ln c$$

$$\ln\left(\frac{z}{x-y}\right) = \ln c$$

$$\frac{z}{x-y} = c_1$$

and now

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrate

$$\frac{-1}{x} + \frac{1}{y} = c_2$$

Now

G.S

$$\phi\left(\frac{z}{x-y} , \frac{x-y}{xy} \right) .$$