EXERCISE 10.1

Solve :

1. $(9D^2 - 12D + 4)y = 0$
2. $(75D^2 + 50D + 12)y = 0$
3. $(D^3 - 4D^2 + D + 6)y = 0$
4. $(D^3 + D^2 + D + 1)y = 0$
5. $(D^3 - 6D^2 + 12D - 8)y = 0$
6. $(D^3 - 6D^2 + 3D + 10)y = 0$
7. $(D^3 - 27)y = 0$
8. $(4D^4 - 4D^3 - 3D^2 + 4D - 1)y = 0$
9. $(D^4 + 2D^3 - 2D^2 - 6D + 5)y = 0$
10. $(D^4 - 5D^3 + 6D^2 + 4D - 8)y = 0$
11. $(D^4 - 4D^3 - 7D^2 + 22D + 24)y = 0$
12. $(D^4 + 4)y = 0$
13. $(D^4 - D^3 - 3D^2 + D + 2)y = 0$
14. $(16D^6 + 8D^4 + D^2)y = 0$
15. $(D^4 + 6D^3 + 15D^2 + 20D + 12)y = 0$
16. $(D^2 + 8D - 9)y = 0;\ y(1) = 1,\ y'(1) = 0$
17. $(D^2 + 6D + 9)y = 0;\ y(0) = 2,\ y'(0) = -3$
18. $(D^2 + 6D + 13)y = 0;\ y(0) = 3,\ y'(0) = -1$
19. $(D^3 - 6D^2 + 11D - 6)y = 0;\ y(0) = 0 = y'(0),\ y''(0) = 2$
20. $(D^4 - D^3)y = 0;\ y(0) = y'(0) = 1,\ y''(1) = 3e, y'''(1) = e.$

EXERCISE 10.2

Find the general solution of each of the following (**Problems 1 – 15**):

1. $(D^2 + 3D - 4)y = 15e^x$
2. $(D^2 - 3D + 2)y = e^x + e^{-2x}$
3. $(D^2 - 2D - 3)y = 2e^x - 10 \sin x$
4. $(D^4 - 2D^3 + D)y = x^4 + 3x + 1$
5. $(D^3 - D^2 + D - 1)y = 4 \sin x$
6. $(D^3 - 2D^2 - 3D + 10)y = 40 \cos x$
7. $(D^2 + 4)y = 4 \sin^2 x$
8. $(D^3 + D)y = 2x^2 + 4 \sin x$
9. $(D^4 + D^2)y = 3x^2 + 4 \sin x - 2 \cos x$
10. $(D^3 - 2D + 4)y = e^x \cos x$
11. $(D^3 - D^2 + 3D + 5)y = e^x \sin 2x$
12. $(D^3 - 7D - 6)y = e^{2x}(1 + x)$
13. $(D^3 - 7D + 12)y = e^{2x}(x^3 - 5x^2)$
14. $(D^4 + 8D^2 - 9)y = 9x^3 + 5 \cos 2x$
15. $(D^4 + 3D^2 - 4)y = \sinh x - \cos^2 x$

Solve the initial value problems :

16. $y'' - 8y' + 15y = 9x\, e^{2x}$, $y(0) = 5$, $y'(0) = 10$.
17. $y'' - 4y' + 13y = 8 \sin 3x$, $y(0) = 1$, $y'(0) = 2$
18. $y'' - 4y = 2 - 8x$, $y(0) = 0$, $y'(0) = 5$
19. $y'' + y = x \sin x$, $y(0) = 1$, $y'(0) = 2$
20. $y''' + 3y'' + 7y' + 5y = 16e^{-x} \cos 2x$,

    $y(0) = 2$, $y'(0) = -4$, $y''(0) = -2$.

THE METHOD OF UNDETERMINED COEFFICIENTS (U.C.)

(10.12) We have studied some special cases in which particular integral can be evaluated by the inverse operator. Now we consider the method of undetermined coefficients which can prove simpler in finding the particular integral of the equation

$$f(D)y = F(x)$$

when $F(x)$ is

(i) an exponential function: (e^{ax})

(ii) a polynomial : $(b_0 x^n + b_1 x^{n-1} + \dots + b_n)$

(iii) sinusoidal function ($\sin ax$ or $\cos ax$)

and (iv) the more general case in which $F(x)$ is a product of terms of the above types, such as

$$F(x) = e^{ax}(b_o x^n + b_1 x^{n-1} + \dots + b_n)\begin{Bmatrix} \sin ax \\ \cos ax \end{Bmatrix}.$$

A y_p will be constructed according to the following table :

$F(x)$	y_p
1. a	$x^k A$
2. $a x^n$ (n a +ve integer)	$x^k (A_o x^n + A_1 x^{n-1} + \dots + A_{n-1}x + A_n)$
3. $a e^{rx}$	$x^k (A e^{rx})$
4. $a \cos \alpha x$ 5. $a \sin \alpha x$	$x^k (A \cos \alpha x + B \sin \alpha x)$
6. $a x^n e^{rx} \cos \alpha x$ 7. $a x^n e^{rx} \sin \alpha x$	$x^k [(A_o x^n + A_1 x^{n-1} + \dots A_{n-1}x + A_n) e^{rx} \cos \alpha x$ $+(B_o x^n + B_1 x^{n-1} + \dots + B_{n-1}x + B_n) e^{rx} \sin \alpha x)]$

In x^k, k is the smallest non-negative integer which will ensure that no term in y_p is already in the C. F.

If $F(x)$ is sum of several terms, write y_p for each term individually and then add up all of them.

EXERCISE 10.3

Solve by the method of U. C. (Problems 1 – 9) :

1. $y'' - 4y' + 4y = e^{2x}$
2. $y'' + 2y' + 5y = 6 \sin 2x + 7 \cos 2x$
3. $2y'' + 3y' + y = x^2 + 3 \sin x$
4. $y'' + 2y' + y = e^{x} \cos x$
5. $y'' + y = 12 \cos^2 x$
6. $y'' - 3y' + 2y = 2x^2 + 2x e^x$
7. $y''' + y' = 2x^2 + 4 \sin x$
8. $y''' + y'' + 3y' - 5y = 5 \sin 2x + 10x^2 + 3x + 7$
9. $y^{(4)} + 8y'' + 16y = \sin x$
10. Write the general form of the P. I. (without evaluating the U. C.) for

    (i) $y'' + 2y' + 2y = 4e^{-x} x^2 \sin x + 3e^{-x} + 2e^{-x} \cos x$

    (ii) $y'' + 3y' + 2y = e^{x}(x^2 + 1) \sin 2x + 4e^{x} + 3e^{-x} \cos x.$

EXERCISE 10.4

Solve :

1. $x^2 \dfrac{d^2y}{dx^2} + 7x \dfrac{dy}{dx} + 5y = x^5$

2. $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 5y = x^2 \sin(\ln x)$

3. $x^2 \dfrac{d^2y}{dx^2} - (2m-1)\, x \dfrac{dy}{dx} + (m^2 + n^2)y = n^2 x^m \ln x$

4. $4x^2 \dfrac{d^2y}{dx^2} - 4x \dfrac{dy}{dx} + 3y = \sin \ln(-x),\ x < 0$

5. $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 2y = 10x + \dfrac{10}{x}$

6. $x^4 \dfrac{d^3y}{dx^3} + 2x^3 \dfrac{d^2y}{dx^2} - x^2 \dfrac{dy}{dx} + xy = 1$

7. $x^3 \dfrac{d^3y}{dx^3} + 4x^2 \dfrac{d^2y}{dx^2} - 5x \dfrac{dy}{dx} - 15y = x^4$

8. $(x+1)^2 \dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} + y = 4\left[\cos \ln(x+1)\right]^2$

9. $(2x+1)^2 \dfrac{d^2y}{dx^2} - 6(2x+1)\dfrac{dy}{dx} + 16y = 8(2x+1)^2$

10. $x^2 y'' + 2xy' - 6y = 10x^2$; $y(1) = 1.\ y'(1) = -6$

11. $x^2 y'' - 2xy' + 2y = x \ln x$; $y(1) = 1.\ y'(1) = 0$

12. $x^3 y''' + 2x^2 y'' + xy - y = 15 \cos(2 \ln x)$;

$$y(1) = 2,\ y'(1) = -3,\ y''(1) = 0.$$

EXERCISE 10.5

Solve :

1. $$\frac{d^2y}{dx^2} + y = \sec^3 x$$

2. $$\frac{d^2y}{dx^2} + 4y = 4\tan 2x$$

3. $$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$ [Legendre's equation of order on.]

4. $$(x - 1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 1$$

5. $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 8x^3$$

6. $$x^2\frac{d^2y}{dx^2} - (x^2 + 2x)\frac{dy}{dx} + (x + 2)y = x^2e^x$$

7. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = \dfrac{1}{(1 + e^x)^2}$

8. $x\dfrac{d^2y}{dx^2} - (2x + 1)\dfrac{dy}{dx} + (x + 1)\,y = (x^2 + x - 1)\,e^x$

9. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = (1 + x + x^2 + \dots + x^{25})\,e^{2x}$, given that

$y = e^{2x}$ is a solution of the associated homogeneous equation

10. $\dfrac{d^2y}{dx^2} - 2\tan x\dfrac{dy}{dx} + 3y = 2\sec x$, given that

$y = \sin x$ is a solution of the associated homogeneous equation.

11. $x\dfrac{d^2y}{dx^2} - 2(x + 1)\dfrac{dy}{dx} + (x + 2)\,y = (x - 2)\,e^{2x}$

12. $(1 - x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x(1 - x^2)^{3/2}$.

EXERCISE 10.6

Find a particular solution of each of the following (**Problems 1 – 9**):

1. $\dfrac{d^2y}{dx^2} + 4y = \sec 2x$

2. $\dfrac{d^2y}{dx^2} + y = \tan x \sec x$

3. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = (1 + e^{-x})^{-1}$

4. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = e^{-2x} \sec x$

5. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = \dfrac{e^{2x}}{1 + x}$

6. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \arcsin x$

7. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = e^x \tan 2x$

8. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^{-x} \ln x$

9. $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 2e^{-x}\tan^2 x$$

10. Find the general solution of

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3 e^x;$$

given that $y_1 = x^2$ is a solution of the associated homogeneous equation.

11. Find the general solution of

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = \frac{1}{1+x};$$

given that $y_1 = \frac{1}{x}$ is a solution of the associated homogeneous equation.

12. Find the general solution of

$$x(x-2)\frac{d^2y}{dx^2} - (x^2-2)\frac{dy}{dx} + 2(x-1)y = 3x^2(x-2)^2 e^x;$$

given that $y_1 = e^x$ is a solution of the associated homogeneous equation.

13. Find the general solution of

$$\sin^2 x\frac{d^2y}{dx^2} - \sin 2x\frac{dy}{dx} + (1+\cos^2 x)y = \sin^3 x;$$

given that $y_1 = \sin x$ and $y_2 = x\sin x$ are linearly independent solutions of the associated homogeneous equation.

14. Use the method of Example 21 to find a particular solution of

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = \frac{2e^x}{x^2}$$

15. By the method of variation of parameters, find a particular solution of

$$\frac{d^3y}{dx^3} - 2\frac{dy}{dx} - 4y = e^{x}\tan x.$$