Non Exact Differential Equation:-

A differential Equation of the form $Mdx + Ndy = 0$ is said to be Non-Exact iff

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

If this equation is multiplied by a suitable function the resulting equation will be Exact diff eq.

The suitable function is called Integrating factor.

Some Rules to find Integrating factors (I.F)

1- $\dfrac{M_y - N_x}{N} = f(x)$ then $I.F = e^{\int f(x)\,dx}$

2- $\dfrac{N_x - M_y}{M} = f(y)$ then $I.F = e^{\int f(y)\,dy}$

3- If $Mdx + Ndy = 0$ is homogenous then $I.F = \dfrac{1}{xM + yN}$

where $xM + yN \neq 0$

4- If diff eq of the form $y\,f(x,y)\,dx + x\,g(x,y)\,dy = 0$

then $I.F = \dfrac{1}{xM - yN}$ where $xM - yN \neq 0$.

Ex # 9.5

Q1:- $(xy^2+y)dx - xdy = 0$

Sol:-

Here

$M = xy^2+y \quad , \quad N = -x$

$M_y = \frac{\partial M}{\partial y} = 2xy + 1 \qquad N_x = \frac{\partial N}{\partial x} = -1$

$\therefore \quad M_y \neq N_x$

It is not an Exact Diff. eq.

Now we find Integrating factor.

$$\frac{M_y - N_x}{N} = \frac{2xy+1+1}{-x} \neq f(x)$$

Now

$$\frac{N_x - M_y}{M} = \frac{-1 - 2xy - 1}{xy^2 + y} = \frac{-2(xy+1)}{y(xy+1)}$$

$$= -\frac{2}{y} = f(y)$$

$$\therefore I \cdot F = e^{\int -2/y \, dy} = e^{-2 \ln y} = e^{\ln y^{-2}} = \frac{1}{y^2}$$

Multiplying both sides of eq ① by $I \cdot F = \frac{1}{y^2}$

$$\frac{1}{y^2}(xy^2 + y)\,dx - \frac{x}{y^2}\,dy = 0$$

$$\left(x + \frac{1}{y}\right) dx - \frac{x}{y^2}\,dy = 0 \quad \text{——— ②}$$

Now $M = x + \frac{1}{y}$ $\qquad N = -\frac{x}{y^2}$

$$M_y = -\frac{1}{y^2} \qquad N_x = -\frac{1}{y^2}$$

Now $\quad M_y = N_x$

therefore eq ② is an Exact diff. eq.

Solution is given by

$$\int M\,dx + \int \bar{N}\,dy = C$$

$$\int \left(x + \tfrac{1}{y}\right) dx + 0 = C$$

$$\frac{x^2}{2} + \frac{x}{y} = C$$

(4) $dy + \left(\frac{y - \sin x}{x}\right) dx = 0$

Sol:-

$M = \frac{y - \sin x}{x}$ $\qquad N = 1$

$M_y = 1/x$ $\qquad N_x = 0$

$$M_y \neq N_x$$

To find I.F

$$\frac{M_y - N_x}{N} = \frac{1/x - 0}{1} = 1/x = f(x)$$

$$I.F = e^{\int 1/x \, dx} = e^{\ln x} = x$$

Multiplying eq (1) by $I.F = x$

$$\textcircled{1} \Rightarrow \quad x\,dy + x\left(\frac{y - \sin x}{x}\right)dx = 0$$

$$x\,dy + (y - \sin x)\,dx = 0 \quad \text{——} \quad \textcircled{2}$$

Now

$M = y - \sin x \qquad N = x$

$M_y = 1 \qquad N_x = 1$

$$M_y = N_x$$

Now eq ② becomes exact diff. eq.

Solution is given by

$$\int M\,dx + \int \bar{N}\,dy = C$$

$$\int (y - \sin x)\,dx + 0 = C$$

$$xy + \cos x = C$$

is required solution.

Q9:- $(3y+4xy^2)dx+(2x+3x^2y)dy=0$

Sol:-

$M = 3y+4xy^2 \qquad N = 2x+3x^2y$

$M_y = 3+8xy \qquad N_x = 2+6xy$

$M_y \neq N_x$

To find I.F :-

$$\frac{M_y - N_x}{N} = \frac{3+8xy-2-6xy}{2x+3x^2y} = \frac{2xy+1}{2x+3x^2y} \neq f(x)$$

$$\frac{N_x - M_y}{M} = \frac{2+6xy-3-8xy}{3y+4xy^2} = \frac{-(2xy+1)}{3y+4xy^2} \neq f(y)$$

$(3y+4xy^2)dx+(2x+3x^2y)dy=0$

as the equation is not homogenous

we can rewrite the given eq.

$y(3+4xy)dx+x(2+3xy)dy=0$

Now Its I.F can be find by Rule 4.

$xM-yN\neq 0$

$x(3y+4xy^2)-y(2x+3x^2y)\neq 0$

$3xy+4x^2y^2-2xy-3x^2y^2\neq 0$

$x^2y^2+xy\neq 0$

So

$$I.F=\frac{1}{xM-yN}=\frac{1}{x^2y^2+xy}$$

Multiplying eq ① by I.F

$$\frac{(3y+4xy^2)}{(x^2y^2+xy)}dx + \frac{2x+3x^2y}{(x^2y^2+xy)}dy = 0$$

$$\underbrace{\frac{3+4xy}{x^2y+x}}_{M}dx + \underbrace{\frac{2+3xy}{xy^2+y}}_{N}dy = 0 \quad \text{—— ②}$$

$$M_y = \frac{(x^2y+x)(4x)-(3+4xy)(x^2)}{(x^2y+x)^2}$$

$$= \frac{4x^3y+4x^2-3x^2-4x^3y}{(x^2y+x)^2}$$

$$= \frac{x^2}{x^2(xy+1)^2}$$

$$M_y = \frac{1}{(xy+1)^2}$$

$$N_x = \frac{(xy^2+y)(3y)-(2+3xy)(y^2)}{(xy^2+y)^2}$$

$$= \frac{3xy^3+3y^2-2y^2-3xy^3}{(xy^2+y)^2}$$

$$= \frac{y^2}{y^2(xy+1)^2}$$

$$N_x = \frac{1}{(xy+1)^2}$$

$\therefore \quad M_y = N_x$

$\Rightarrow$ (2) is an Exact diff. eq.

$$\int M\,dx + \int \bar{N}\,dy = C$$

$$\int \left(\frac{3+4xy}{x+x^2y}\right)dx + \int \frac{2}{y}\,dy = C$$

$$\int \left(\frac{3+3xy+xy}{x(1+xy)}\right)dx + \int \frac{2}{y}\,dy = C$$

$$3\int \frac{(1+xy)}{x(1+xy)}\,dx + \int \frac{xy}{x(1+xy)}\,dx + \int \frac{2}{y}\,dy = C$$

$$3\int \frac{1}{x}\,dx + \int \frac{y}{1+xy}\,dx + \int \frac{2}{y}\,dy = C$$

$$3\ln x + \ln(1+xy) + 2\ln y = \ln C$$

$$\ln x^3 + \ln(1+xy) + \ln y^2 = \ln C$$

$$\ln\left(x^3(1+xy)y^2\right) = \ln C$$

Antilog

$$x^3(1+xy)y^2 = C$$

is required solution.