Linear Differential Equation:-

A first order ODE of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is called linear D E because it is linear in 'y' & $\frac{dy}{dx}$.

To solve this D.E find integrating factor

$$I \cdot F = \mu(x) = e^{\int P(x)dx}$$

then Multiply D.E by I.F

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)\, Q(x).$$

$$\frac{d}{dx}\left(\mu(x)\, y\right) = \mu(x)\, Q(x)$$

$$d\left(\mu(x)y\right) = \mu(x)Q(x)dx$$

on applying Integration

$$\boxed{\mu(x)y = \int \mu(x)Q(x)dx + c}$$

Similarly,
A diff eq of the form

$$\frac{dx}{dy} + P(y)\, x = Q(y)$$

is linear in x & $\frac{dx}{dy}$.

Ex # 9.6

Q1:- $$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x} \quad \text{—— (1)}$$

Sol.. It is linear in y

$$I\,F = \exp\left(\int P(x)dx\right) = \exp\left(\int \frac{2x+1}{x}dx\right)$$

$$= \exp\left(\int (2 + 1/x)dx\right) = e^{2x+\ln x} = e^{2x}\cdot e^{\ln x}$$

$$I\cdot F = xe^{2x}.$$

Multiplying eq (1) by $I\cdot F$

$$xe^{2x}\cdot\frac{dy}{dx} + xe^{2x}\left(\frac{2x+1}{x}\right)y = xe^{2x}\cdot e^{-2x}$$

$$\frac{d}{dx}\left(xe^{2x}y\right) = x$$

$$d\left(xe^{2x}y\right) = x\,dx$$

On Integrating

$$\int d\left(xe^{2x}y\right) = \int x\,dx$$

$$xe^{2x}y = x^2/2 + C$$

(10) $(1+x^2)\frac{dy}{dx} + 4xy = \frac{1}{(1+x^2)^2}$

sol...

dividing by $(1+x^2)$

$$\frac{dy}{dx} + \frac{4x}{1+x^2}y = \frac{1}{(1+x^2)^3} \quad \text{--- (1)} \quad (\text{LDE in } y)$$

$$I.F = \exp\left(\int \frac{4x}{1+x^2}dx\right) = \exp\left(2\ln(1+x^2)\right)$$

$$I.F = e^{\ln(1+x^2)^2} = (1+x^2)^2$$

$$(1+x^2)^2\frac{dy}{dx} + 4x(1+x^2)y = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left((1+x^2)^2 y\right) = \frac{1}{1+x^2}$$

$$d\left((1+x^2)y\right) = \frac{1}{1+x^2}dx$$

integrating

$$\int d\left((1+x^2)^2 y\right) = \int \frac{1}{1+x^2}dx$$

$$(1+x^2)^2 y = \tan^{-1}x + C$$

$$y = \frac{1}{(1+x^2)^2}\left[\tan^{-1}x + C\right]$$

is required sol.

BERNOULLI EQUATION.

An equation of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n \quad , n \neq 0, 1$$

is called Bernoulli differential Equation.

It is reducible to linear, dividing by y^n

$$\Rightarrow \quad y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x)$$

put $\quad v = y^{1-n}$

then it reduces to

$$\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)Q(x).$$

Viz linear in 'V'.

(13) $x\frac{dy}{dx} + y = y^2 \ln x$

$\div$ing by x

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\ln x}{x}y^2$$ Bernoulli eq.

dividing by y^2

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = \frac{\ln x}{x} \quad \text{—— (1)}$$

put $\quad y^{-1} = v$

$$-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$$

(1) $\Rightarrow$ $-\frac{dv}{dx} + \frac{1}{x}v = \frac{\ln x}{x}$

$$\frac{dv}{dx} - \frac{1}{x}v = -\frac{\ln x}{x}$$ (LDE in v).

$$I \cdot F = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

Multiply eq ② by I·F

$$\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2} v = -\frac{\ln x}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{x} \cdot v\right) = -\ln x \cdot x^{-2}$$

$$d\left(\frac{1}{x} \cdot v\right) = -\ln x \cdot x^{-2} \, dx$$

Integrating.

$$\frac{1}{x} \cdot v = -\int \ln x \cdot x^{-2} dx$$

$$\frac{v}{x} = -\left(\ln x \cdot (-x^{-1}) - \int \frac{1}{x}(-x^{-1}) dx\right)$$

$$= \frac{1}{x}\ln x - \int x^{-2} dx$$

$$\frac{y^{-1}}{x} = \frac{\ln x}{x} - \frac{x^{-1}}{-1} + c \qquad \because v = y^{-1}$$

$$\frac{1}{y} = \frac{x \ln x}{x} + x x^{-1} + cx$$

$$\frac{1}{y} = \ln x + 1 + cx$$

is required solution.

Q14 :- $\frac{dy}{dx} + y = xy^3$ Bernoulli eq.

dividing by y^3

$$y^{-3}\frac{dy}{dx} + y^{-2} = x \quad \text{——— (1)}$$

put $y^{-2} = V$

$$-2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$$

(1) $\Rightarrow$ $-\frac{1}{2}\frac{dv}{dx} + V = x$

$\Rightarrow$ $\frac{dv}{dx} - 2V = -2x$ ——— (2) LDE in V

$$I.F = e^{\int -2dx} = e^{-2x}$$

Multiplying eq (2) by I.F

$$② \Rightarrow e^{-2x}\frac{dv}{dx} - (e^{-2x})2v = -2xe^{-2x}$$

$$\frac{d}{dx}\left(e^{-2x}v\right) = -2xe^{-2x}$$

$$d\left(e^{-2x}v\right) = -2xe^{-2x}dx$$

Integrating

$$e^{-2x}v = \int -2xe^{-2x}dx$$

let $-2x = t$

$-2dx = dt$

$$e^{-2x}v = \frac{-1}{2}\int te^{t}dt$$

$$e^{-2x}\cdot v = \frac{-1}{2}\left[te^{t} - \int (1)e^{t}dt\right]$$

$$= \frac{1}{2}\left[te^{t} - e^{t}\right] + C$$

$$v e^{-2x} = -\frac{1}{2} e^{t}(t-1) + c$$

$$y^{-2} e^{-2x} = -\frac{1}{2} e^{-2x}(2x-1) + c$$

$$\frac{1}{y^2} = -\frac{1}{2}(2x-1) + c e^{2x}$$