EXACT DIFFERENTIAL EQUATIONS

A Differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be an exact differential equation if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

To solve this Differential equation:

- Integrate M with respect to x keeping y as constant i.e $I_1 = \int M dx$
- Integrate with respect to y of terms of N free from x. i.e $I_2 = \int \bar{N} dy$
- Add both integrals and equate arbitrary constant.
  The solution is

$$I_1 + I_2 = C$$

EXERCISE 9.4

Solve (problem 1-10)

Question # 1: $(3x^2 + 4xy\ )dx + (2x^2 + 2y)dy = 0$

Solution:-

Given equation is

$$(3x^2 + 4xy\ )dx + (2x^2 + 2y)dy = 0 - - - (1)$$

Here,

$M = 3x^2 + 4xy$	$N = 2x^2 + 2y$
$M_y = \frac{\partial M}{\partial y} = 4x$	$N_x = \frac{\partial N}{\partial x} = 4x$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$\int M dx + \int \bar{N} dy = c$$

$$\int (3x^2 + 4xy)dx + \int 2ydy = c$$

$$x^3 + 2x^2y + y^2 = c$$

Is required solution

Question # 3: $\frac{x+y}{y-1}\, dx - \frac{1}{2}\left(\frac{x+1}{y-1}\right)^2 = 0$

Solution:-

Given equation is

$$\frac{x+y}{y-1}\, dx - \frac{1}{2}\left(\frac{x+1}{y-1}\right)^2 = 0 - - - (1)$$

Here,

$$M = \frac{x+y}{y-1}$$

$$M_y = \frac{\partial}{\partial y}\left(\frac{x+y}{y-1}\right)$$

$$M_y = \frac{(y-1)(1) - (x+y)(1)}{(y-1)^2}$$

$$M_y = \frac{y-1-x-y}{(y-1)^2}$$

$$M_y = -\frac{(x+1)}{(y-1)^2}$$

$$M_y = -\frac{(x+1)}{(y-1)^2}$$

$$N = -\frac{1}{2}\left(\frac{x+1}{y-1}\right)^2$$

$$N_{x=} \frac{\partial}{\partial x}\left[\frac{-1}{2}\left(\frac{x+1}{y-1}\right)^2\right]$$

$$N_{x=} \frac{-1}{2}\frac{\partial}{\partial x}\frac{(x+1)^2}{(y-1)^2}$$

$$N_{x=} \frac{-1}{2(y-1)^2}\frac{\partial}{\partial x}(x+1)^2$$

$$N_{x=} \frac{-1}{2(y-1)^2}\, 2(x+1)$$

$$N_{x=} = -\frac{(x+1)}{(y-1)^2}$$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$I_1 + I_2 = C$$

$$\int M dx + \int \bar{N} dy$$

$$= \int \frac{x+y}{y-1} dx + \int -\frac{1}{2}\frac{1}{(y-1)^2} dy = c$$

$$= \frac{1}{y-1}\int (x+y) dx - \frac{1}{2}\int (y-1)^{-2} dy = c$$

$$= \frac{1}{y-1}\left(\frac{x^2}{2} + xy\right) + \frac{1}{2}\frac{1}{(y-1)} = c$$

$$\frac{1}{y-1}\left(\frac{x^2+2xy+1}{2}\right) = c$$

$$x^2 + 2xy + 1 = c'(y-1)$$

Is required solution

Question # 6: $\frac{ydx+xdy}{1-x^2y^2} + xdx = 0$

Solution:-

Given equation is

$$\frac{ydx + xdy}{1-x^2y^2} + xdx = 0 --- (1)$$

$$\Rightarrow \frac{ydx}{1-x^2y^2} + \frac{xdy}{1-x^2y^2} + xdx = 0$$

$$\Rightarrow \left(\frac{y}{1-x^2y^2} + x\right) dx + \frac{xdy}{1-x^2y^2} = 0$$

Here,

$$M = \frac{y}{1-x^2y^2} + x \qquad N = \frac{xdy}{1-x^2y^2}$$

$$M_y = \frac{\partial}{\partial y}\left(\frac{y}{1-x^2y^2} + x\right) \qquad N_x = \frac{\partial}{\partial x}\left(\frac{x}{1-x^2y^2}\right)$$

$$M_y = \frac{(1-x^2y^2).1-y(-2x^2y)}{(1-x^2y^2)^2} \qquad N_x = \frac{(1-x^2y^2).1-x(-2xy^2)}{(1-x^2y^2)^2}$$

$$M_y = \frac{1-x^2y^2+2x^2y^2}{(1-x^2y^2)^2} \qquad N_x = \frac{1-x^2y^2+2x^2y^2}{(1-x^2y^2)^2}$$

$$M_y = \frac{1+x^2y^2}{(1-x^2y^2)^2} \qquad N_x = \frac{1+x^2y^2}{(1-x^2y^2)^2}$$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$I_1 + I_2 = C$$

$$\int M dx + \int \bar{N} dy$$

$$\int \left(\frac{y}{1-x^2y^2} + x\right)dx + 0 = C$$

$$\int \frac{y}{y^2(\frac{1}{y^2} - x^2)}\,dx + \int x\,dx = C$$

$$\frac{1}{y}\int \frac{dx}{\frac{1}{y^2} - x^2} + \frac{x^2}{2} = C$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$$

$$\frac{1}{y}\left[\frac{1}{2\left(\frac{1}{y}\right)}\ln\left(\frac{\frac{1}{y}+x}{\frac{1}{y}-x}\right)\right] + \frac{x^2}{2} = C$$

$$\ln\left(\frac{1+xy}{1-xy}\right) + x^2 = C' \qquad \therefore C' = 2C$$

Is required solution

Question # 10: $(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)dx + (xe^{xy}\cos 2x - 3)dy = 0$

Solution:-

Given equation is

$$(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)dx + (xe^{xy}\cos 2x - 3)dy = 0 - - - (1)$$

Here,

$M = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x$	$N = xe^{xy}\cos 2x - 3$
$M_y = \frac{\partial}{\partial y}(ye^{xy}\cos 2x - 2e^{xy}\sin 2x\ 2x)$	$N_x = \frac{\partial}{\partial x}(xe^{xy}\cos 2x - 3)$
$M_y = \cos 2x[ye^{xy}(x) + e^{xy}] - 2\sin 2x\,[e^{xy}(x)]$	$N_x = e^{xy}\cos 2x + xye^{xy}\cos 2x - 2xe^{xy}\sin 2x$
$M_y = xy\,e^{xy}\cos 2x + e^{xy}\cos 2x - 2xe^{xy}\sin 2x$	$N_x = e^{xy}(\cos 2x + xy\cos 2x - 2x\sin 2x)$
$M_y = e^{xy}(xy\cos 2x + \cos 2x - 2x\sin 2x)$	$N_x = e^{xy}(xy\cos 2x + \cos 2x - 2x\sin 2x)$

$\because M_y = N_x$. Therefore, given equation is exact.

$$I_1 + I_2 = C$$

$$\Rightarrow \int M dx + \int \overline{N} dy = C$$

$$\Rightarrow \int (ye^{xy} \cos 2x - 2e^{xy} \sin 2x + 2x) dx + \int -3\, dy = C$$

$$\Rightarrow y \int e^{xy} \cos 2x\, dx - 2 \int e^{xy} \sin 2x dx + \frac{2x^2}{2} - 3y = C$$

$$\Rightarrow y \left[cos2x \frac{e^{xy}}{y} - \int -sin2x(2) \frac{e^{xy}}{y} dy \right] - 2 \int e^{xy} sin2xdx + x^2 - 3y = C$$

$$\Rightarrow \cos 2x\, e^{xy} + 2 \int e^{xy} sin2xdx - 2 \int e^{xy} sin2xdx + x^2 - 3y = C$$

$$\Rightarrow \cos 2x\, e^{xy} + x^2 - 3y = C$$

Question # 13:

$$(\mathbf{3x^2y^2 + 2x})\mathbf{dx} + (\mathbf{2x^3y - 2xy^2 + 1}) = \mathbf{0}, \qquad \mathbf{y(-2) = 1}$$

Solution:-

Given equation is

$$(3x^2y^2 + 2x)dx + (2x^3y - 2xy^2 + 1) = 0 - - - (1)$$

Here,

$M = 3x^2y^2 + 2x$

$M_y = \frac{\partial}{\partial y}(3x^2y^2 + 2x)$

$M_y = 3x^2(2y)\text{-}3y^2$

$M_y = 6x^2y - 3y^2$

$N = 2x^3y - 2xy^2 + 1$

$N_x = \frac{\partial}{\partial x}(2x^3y - 2xy^2 + 1)$

$N_x = 2y(3x^2) - 3y^2(1)$

$N_x = 6x^2y - 3y^2$

$\because M_y = N_x$. Therefore, given equation is exact.

Now

$$I_1 + I_2 = C$$

$$\int Mdx + \int \bar{N}dy = C$$

$$\int (3x^2y^2 - y^3 + 2x)dx + \int dy = C$$

$$x^3y^2 - xy^3 + x^2 + y = C$$

Applying given condition

$$y(-2) = 1$$

$$(-2)^3(1)^2 - (-2)(1)^3 + (-2)^2 + 1 = C$$

$$-8 + 2 + 4 + 1 = C$$

$$-1 = c$$

Hence

$$x^3y^2 - xy^3 + x^2 + y = -1$$

$$x^3y^2 - xy^3 + x^2 + y + 1 = 0$$

Is required solution

Question # 14:

$$\left(\frac{3-y}{x^2}\right)dx-\left(\frac{y^2-2x}{xy^2}\right)dy=0 \quad ,y(-1)=2$$

Solution:-

Given equation is

$$\left(\frac{3-y}{x^2}\right)dx-\left(\frac{y^2-2x}{xy^2}\right)dy=0 \;---(1)$$

here,

$M=\frac{3-y}{x^2}$	$N=\frac{y^2-2x}{xy^2}$
$M_y=\frac{\partial}{\partial y}\left(\frac{3-y}{x^2}\right)$	$N_x=\frac{\partial}{\partial x}\left(\frac{y^2-2x}{xy^2}\right)$
$M_y=\frac{1}{x^2}(-1)$	$N_x=\frac{-1}{x^2}$
$M_y=\frac{-1}{x^2}$	$N_x=\frac{-1}{x^2}$

$\because M_y=N_x$. Therefore, given equation is exact.

$$I_1+I_2=C$$

$$\int Mdx+\int \bar{N}dy=C$$

$$\int\left(\frac{3-y}{x^2}\right)dx+\int -\frac{2}{y^2}dy=C$$

$$(3-y)\int\frac{1}{x^2}dx-2\int\frac{1}{y^2}dy=C$$

$$(3-y)\left(-\frac{1}{x}\right)-2\left(-\frac{1}{y}\right)=C$$

$$\frac{-y(3-y)+2x}{xy}=C$$

Applying given condition

$$y(-1)=2$$

$$\frac{-2(3-2)+2(-1)}{-2}=C$$

$$\frac{-2-2}{-2}=C$$

$$C=2$$

Now

$$\frac{-y(3-y)+2x}{xy}=2$$

$$-y(3-y)+2x=2xy$$

$$2x-3y+y^2-2xy=0$$

Is required solution

Question # 15:

$\boldsymbol{(4x^3e^{x+y}+x^4e^{x+y}+2x)dx+(x^4e^{x+y}+2y)dy=0, \qquad y(0)=1}$

Solution:-

Given equation is

$$(4x^3e^{x+y}+x^4e^{x+y}+2x)dx+(x^4e^{x+y}+2y)dy=0,---(1)$$

Here,

$\boldsymbol{M=4x^3e^{x+y}+x^4e^{x+y}+2x}$	$\boldsymbol{N=x^4e^{x+y}+2y}$
$M_y=\frac{\partial}{\partial y}(4x^3e^{x+y}+x^4e^{x+y}+2x)$	$N_x=\frac{\partial}{\partial x}(x^4e^{x+y}+2y)$
$M_y=4x^3e^{x+y}+x^4e^{x+y}$	$N_x=e^{x+y}(4x^3)+x^4e^{x+y}$

$\because M_y=N_x$. Therefore, given equation is exact.

Now

$$I_1 + I_2 = C$$

$$\int M dx + \int \bar{N} dy = C$$

$$\int (4x^3 e^{x+y} + x^4 e^{x+y} + 2x) dx + \int 2y dy = C$$

$$\int 4x^3 e^{x+y} dx + \int x^4 e^{x+y} dx + 2\int x dx + 2\int y dy = C$$

$$\int 4x^3 e^{x+y} dx + \left[x^4 e^{x+y} - \int 4x^3 e^{x+y} dx\right] + x^2 + y^2 = C$$

$$x^4 e^{x+y} + x^2 + y^2 = C$$

Applying given condition

$$y(0) = 1$$

$$x^4 e^{x+y} + x^2 + y^2 = C$$

$$0 + 0 + 1 = C$$

$$C = 1$$

Hence

$$x^4 e^{x+y} + x^2 + y^2 = 1$$

Is require solution