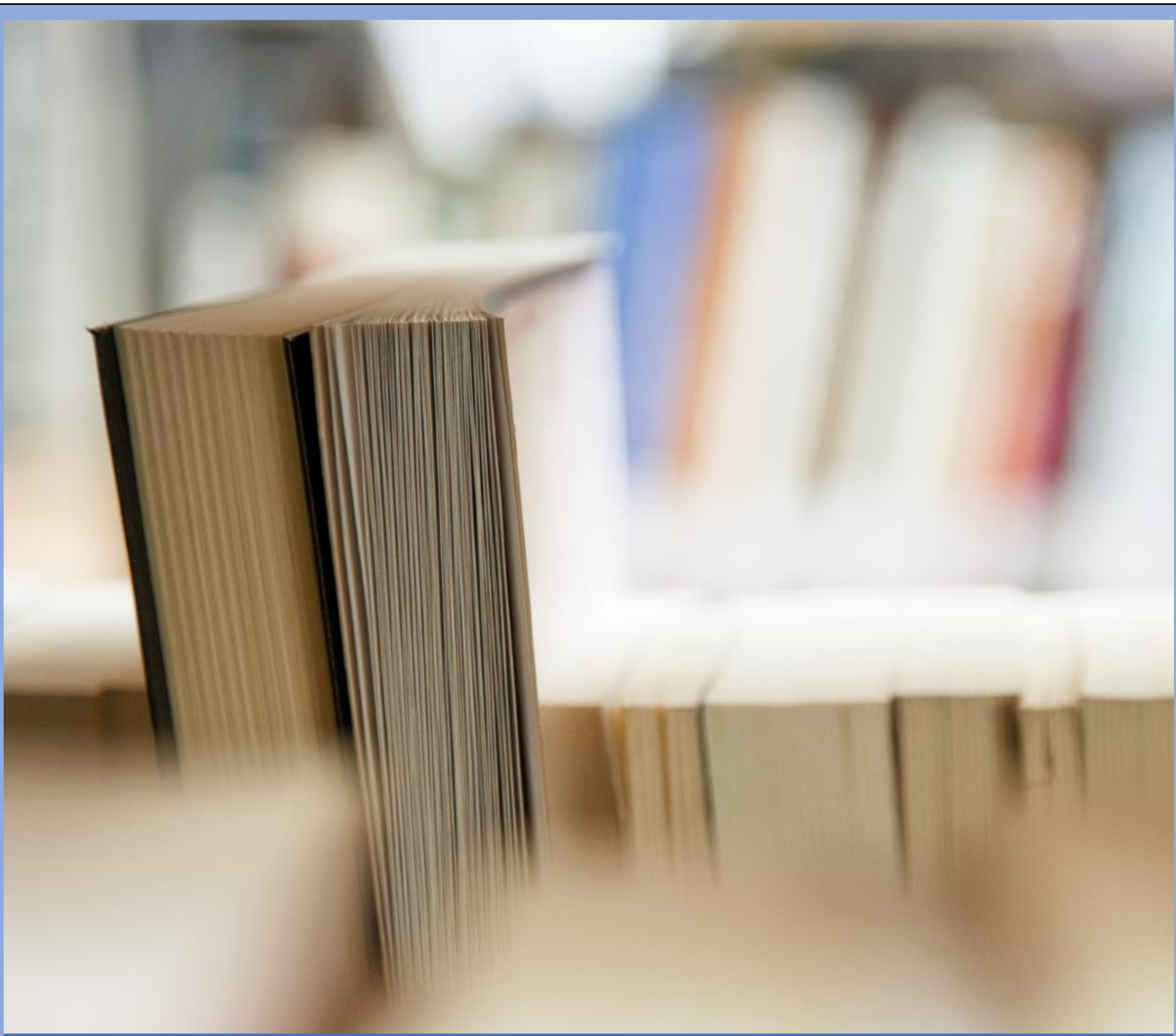

DIFFERENTIAL EQUATIONS

HOMOGENEOUS DIFFERENTIAL EQUATIONS

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HOMOGENEOUS DIFFERENTIAL EQUATIONS

Homogeneous Function:

A function $f(x,y)$ is called homogeneous of degree 'n' if $f(tx,ty) = t^n f(x,y)$

Where 't' is a non-zero real number.

Example:

$$f(x,y) = \sqrt{x^2 + y^2}$$

Put $x = tx, y = ty$

$$f(tx,ty) = \sqrt{t^2x^2 + t^2y^2} = \sqrt{t^2}\sqrt{x^2 + y^2}$$

$$f(tx,ty) = t\sqrt{x^2 + y^2}$$

$$f(tx,ty) = t\, f(x,y)$$

Homogeneous Differential Equations:

A first order Differential equations $\frac{dy}{dx} = f(x,y)$ is called homogeneous if f is homogeneous function of any degree.

If D.E is written in the form $M(x,y)dx + N(x,y)dy = 0$ is Homogeneous Differential Equation if M and N both are homogeneous function of same degree.

To Solve Homogeneous Differential Equation:

$$\text{Put} \quad y = vx$$

$$\text{Then } \frac{dy}{dx} = v\frac{d\,x}{dx} + x\frac{d\,v}{dx}$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then by method of separable we solve.

EXERCISE # 9.3

Solve:

Q1 : $(x-y)dx+(x+y)dy=0$

Solution:

Hence $M(x,y)=x-y$, $N(x,y)=x+y$

Put $x=tx$ and $y=ty$

Then $M(tx,ty)=tx-ty$, $N(tx,ty)=tx+ty$

$M(tx,ty)=t(\,x-y)$ and $N(tx,ty)=t(x+y)$

$\Rightarrow\ M(tx,ty)=t\,M(x,y)$ and $N(tx,ty)=\ t\,N(x,y)$

We see that both functions are of same degree so it is a homogeneous D.E

Now the question $(x-y)dx+(x+y)dy=0$

$$\Rightarrow (x+y)dy=-(x-y)dx$$

Or

$$\frac{dy}{dx}=\frac{-x+y}{x+y}$$

To solve it

Put $y=vx$

Then $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\Rightarrow\ v+x\frac{dv}{dx}=\frac{-x+vx}{x+vx}$$

$$v+x\frac{dv}{dx}=\frac{x(-1+v)}{x(1+v)}$$

$$x\frac{dv}{dx}=\frac{-1+v}{1+v}-v$$

$$x\frac{dv}{dx}=\frac{-1+v-v-v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{-1 - v^2}{1 + v}$$

$$-\frac{1 + v}{1 + v^2}dv = \frac{1}{x}dx$$

Now it is a separable D.E

Integrating

$$-\int \frac{1 + v}{1 + v^2}dv = \int \frac{1}{x}dx$$

$$-\int \left(\frac{1}{1 + v^2} + \frac{v}{1 + v^2}\right)dv = \ln|x| + \ln c$$

$$-\int \left(\frac{1}{1 + v^2} + \frac{2v}{2(1 + v^2)}\right)dv = \ln\ |cx|$$

$$-\tan^{-1} v - \frac{1}{2}\ln|1 + v^2| = \ln\ |cx|$$

$$-\tan^{-1} v - \ln\left|(1 + v^2)^{\frac{1}{2}}\right| = \ln\ |cx|$$

$$-\tan^{-1} v = \ln\sqrt{(1 + v^2)} + \ln\ |cx|$$

$$-\tan^{-1} v = \ln\left|cx\sqrt{(1 + v^2)}\right|$$

Now by back substitution $v = \frac{y}{x}$

$$-\tan^{-1}\frac{y}{x} = \ln\left|cx\sqrt{\left(1 + \frac{y^2}{x^2}\right)}\right|$$

Q 7 $\frac{dy}{dx} = \frac{4y-3x}{2x-y}$

Solution

Since both function in numerator and denominator are homogeneous hence the equation is homogeneous differential equation

To solve put

Put $y = vx$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

The equation becomes

$$v + x\frac{dv}{dx} = \frac{4vx - 3x}{2x - vx}$$

$$v + x\frac{dv}{dx} = \frac{4v - 3}{2 - v}$$

$$x\frac{dv}{dx} = \frac{4v - 3}{2 - v} - v$$

$$x\frac{dv}{dx} = \frac{4v - 3 - 2v + v^2}{2 - v}$$

$$x\frac{dv}{dx} = \frac{2v - 3 + v^2}{2 - v}$$

Integrating both sides

$$\int \frac{2 - v}{v^2 + 2v - 3} dv = \int \frac{1}{x} dx$$

$$\int \frac{2 - v}{v^2 + 3v - v - 3} dv = \int \frac{1}{x} dx$$

$$\int \frac{2 - v}{(v + 3)(v - 1)} dv = \int \frac{1}{x} dx$$

Now solve it using partial fractions

Suppose

$\frac{2-v}{(v+3)(v-1)} = \frac{A}{(v+3)} + \frac{B}{(v-1)}$ .... (A)

$\Rightarrow 2 - v = (v-1)A + (v+3)B$................ (I)

Now put $v = -3$ in (I)

$$\Rightarrow 5 = -4A$$

$$\Rightarrow A = -\frac{5}{4}$$

Now put $v = 1$ in (I)

Then $1 = 4B$

$$B = \frac{1}{4}$$

Now eq A becomes $\frac{2-v}{(v+3)(v-1)} = \frac{-5}{4(v+3)} + \frac{1}{4(v-1)}$

Therefore $\int \frac{2-v}{(v+3)(v-1)} dv = \int \frac{-5}{4(v+3)} + \frac{1}{4(v-1)} dv$

Now

$$\int \frac{-5}{4(v+3)} + \frac{1}{4(v-1)} dv = \int \frac{1}{x} dx$$

$$-\frac{5}{4} \ln|v+3| + \frac{1}{4} \ln|v-1| = \ln|x| + \ln c$$

$$-5 \ln|v+3| + \ln|v-1| = 4 \ln|cx|$$

$$\ln|v+3|^{-5} + \ln|v-1| = \ln|cx|^4$$

$$\ln|(v-1)(v+3)^{-5}| = \ln|cx|^4$$

$$\ln \frac{|v-1|}{|v+3|^5} = \ln|cx|^4$$

Now,

$$\frac{v-1}{(v+3)^5} = (cx)^4$$

Now by back substitution

$$\frac{\frac{y}{x}-1}{(\frac{y}{x}+3)^5}=(cx)^4$$

$$\frac{(y-x)x^4}{(y+3x)^5}=(cx)^4$$

$$\boxed{\Rightarrow \frac{(y-x)}{(y+3x)^5}=c'}$$

Where c is arbitrary constant

Q 9 $(x^3+y^2\sqrt{x^2+y^2})dx-xy\sqrt{x^2+y^2}dy=0$

Solution:

$$(x^3+y^2\sqrt{x^2+y^2})dx-xy\sqrt{x^2+y^2}dy=0$$

$$-xy\sqrt{x^2+y^2}dy=-(x^3+y^2\sqrt{x^2+y^2})dx$$

$$\frac{dy}{dx}=\frac{(x^3+y^2\sqrt{x^2+y^2})}{xy\sqrt{x^2+y^2}}$$

Since both function in numerator and denominator are homogeneous hence the equation is homogeneous differential equation

To solve put

Put $y=vx$

Then $\frac{dy}{dx}=v+x\frac{dv}{dx}$

The equation becomes

$$v+x\frac{dv}{dx}=\frac{(x^3+(vx)^2\sqrt{x^2+(vx)^2})}{x(vx)\sqrt{x^2+(vx)^2}}$$

$$v+x\frac{dv}{dx}=\frac{(x^3+v^2x^3\sqrt{1+v^2})}{vx^3\sqrt{1+v^2}}$$

$$v + x\frac{dv}{dx} = \frac{x^3(1 + v^2\sqrt{1 + v^2})}{vx^3\sqrt{1 + v^2}}$$

$$v + x\frac{dv}{dx} = \frac{1 + v^2\sqrt{1 + v^2}}{v\sqrt{1 + v^2}}$$

$$x\frac{dv}{dx} = \frac{1 + v^2\sqrt{1 + v^2}}{v\sqrt{1 + v^2}} - v$$

$$x\frac{dv}{dx} = \frac{1 + v^2\sqrt{1 + v^2} - v^2\sqrt{1 + v^2}}{v\sqrt{1 + v^2}}$$

$$x\frac{dv}{dx} = \frac{1}{v\sqrt{1 + v^2}}$$

$$v\sqrt{1 + v^2}dv = \frac{dx}{x}$$

Integrating both sides we have

$$\int v\sqrt{1 + v^2}dv = \int \frac{dx}{x}$$

$$\frac{1}{2}\int \sqrt{1 + v^2}(2v)dv = \int \frac{dx}{x}$$

$$\frac{1}{2}\frac{(1 + v^2)^{\frac{3}{2}}}{\frac{3}{2}} = \ln x + \ln c$$

$$\frac{1}{3}(1 + v^2)^{\frac{3}{2}} = \ln\,(cx)$$

Now by back substitutions

$$\frac{1}{3}\left(1 + \frac{y^2}{x^2}\right)^{\frac{3}{2}} = \ln\,(cx)$$

$$\left(\frac{x^2 + y^2}{x^2}\right)^{\frac{3}{2}} = 3\ln cx$$

$$(x^2 + y^2)^{\frac{3}{2}} = x^3 \ln(cx)^3$$

Or

$$(x^2 + y^2)^{\frac{3}{2}} = x^3 \ln c' x^3$$

Where c is arbitrary