Q12..

Find the general solution of

$$x(x-2)\frac{d^2y}{dx^2} - (x^2-2)\frac{dy}{dx} + 2(x-1)y = 3x^2(x-2)^2e^x.$$

given that $y_1 = x^2$ is a solution.

Sol..

we can write

$$\frac{d^2y}{dx^2} - \frac{(x^2-2)}{x(x-2)}\frac{dy}{dx} + \frac{2(x-1)}{x(x-2)}y = \frac{3x^2(x-2)^2e^x}{x(x-2)}$$

to compare with

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x)$$

For 2nd solution we check

$$P+Qx = -\frac{(x^2-2)}{x(x-2)} + \frac{2(x-1)}{x(x-2)}x \neq 0$$

Now check

$$1+P+Q = 1+\left[-\frac{(x^2-2)}{x(x-2)}\right] + \frac{2(x-1)}{x(x-2)}$$

$$= \frac{x(x-2) - x^2 + 2 + 2x - 2}{x(x-2)}$$

$$= \frac{x^2-2x-x^2+2x}{x(x-2)} = 0$$

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So $y_2 = e^x$ is another solution.

Complementry function

$$y_c = c_1 y_1 + c_2 y_2$$

$$y_c = c_1 x^2 + c_2 e^x$$

Then

$$y_p = u_1 x^2 + u_2 e^x$$

$$W = y_1 y_2' - y_1' y_2$$

$$= x^2 e^x - 2x e^x$$

$$W = x e^x (x-2)$$

$$u_1 = \int \frac{-y_2 F(x)}{W} dx$$

$$= \int \frac{-e^x \, 3x(x-2)e^x}{x e^x (x-2)} dx$$

$$= -3 \int e^x dx$$

$$u_1 = -3e^x$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx$$

$$= \int \frac{x^2 \, 3x(x-2)e^x}{x e^x (x-2)} dx$$

$$= 3 \int x^2 dx$$

$$u_2 = x^3$$

Now

$$y_p = -3e^x x^2 + x e^{3x}$$

$$y = c_1 x^2 + c_2 e^x + x^3 e^x - 3e^x x^2.$$

For third order differential equation we have solution

$$y_c = c_1 y_1 + c_2 y_2 + c_3 y_3$$

&

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

Here $u_1$, $u_2$, $u_3$ will be find using system of equation

i.e

$$u_1' y_1 + u_2' y_2 + u_3' y_3 = 0$$

$$u_1' y_1' + u_2' y_2' + u_3' y_3 = 0$$

$$u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = F(x)$$

We solve this system of equations using cramer's rule & get values of $u_1'$, $u_2'$, $u_3'$ then we integrate & get $u_1$, $u_2$, $u_3$.

Q14.. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = \dfrac{2e^x}{x^2}$

Sol.

$$D^3 - 3D^2 + 3D - 1 = 0$$

$$(D-1)^3 = 0$$

$D = 1, 1, 1$

$y_c = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$

Let $y_p = u_1 e^x + u_2 x e^x + u_3 x^2 e^x$

Here

$y_1 = e^x \quad , \quad y_2 = x e^x \quad , \quad y_3 = x^2 e^x$

$F(x) = \dfrac{2e^x}{x^2}$

Now substituting the values in

* $u_1' y_1 + u_2' y_2 + u_3' y_3 = 0 \Rightarrow u_1' e^x + u_2' x e^x + u_3' x^2 e^x = 0$ —— (1)

* $u_1' y_1' + u_2' y_2' + u_3' y_3 = 0$

$\Rightarrow u_1' e^x + u_2'(e^x + x e^x) + u_3'(2x e^x + x^2 e^x) = 0$ —— (2)

* $u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = F(x)$

$u_1' e^x + u_2'(2e^x + x e^x) + u_3'(2e^x + 4x e^x + x^2 e^x) = \dfrac{2e^x}{x^2}$

—— (3)

Now solving these equations for unknown $u_1', u_2', u_3'$ by Cramer's rule.

$$u_1' = \frac{|A_{u_1'}|}{|A|} = \frac{\begin{vmatrix} 0 & xe^x & x^2e^x \\ 0 & e^x+xe^x & 2xe^x+x^2e^x \\ \frac{2e^x}{x^2} & 2e^x+xe^x & 2e^x+4xe^x+x^2e^x \end{vmatrix}}{\begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x+xe^x & 2xe^x+x^2e^x \\ e^x & 2e^x+xe^x & 2e^x+4xe^x+x^2e^x \end{vmatrix}}$$

$$= \frac{\frac{2e^x}{x^2}\begin{vmatrix} xe^x & x^2e^x \\ e^x+xe^x & 2xe^x+x^2e^x \end{vmatrix}}{\begin{vmatrix} e^x & xe^x & x^2e^x \\ 0 & e^x & 2xe^x \\ 0 & 2e^x & 2e^x+4xe^x \end{vmatrix}} \quad \begin{matrix} \\ R_2-R_1 \\ R_3-R_1 \end{matrix}$$

$$= \frac{\frac{2e^x}{x^2}\left[xe^x(2xe^x+x^2e^x) - x^2e^x(e^x+xe^x)\right]}{e^x\left[e^x(2e^x+4xe^x) - 2e^x(2xe^x)\right]}$$

$$= \frac{\frac{2e^x}{x^2}\left(2x^2e^{2x}+x^3e^{2x}-x^2e^{2x}-x^3e^{2x}\right)}{e^x\left(2e^{2x}+4xe^{2x}-4xe^{2x}\right)}$$

$$u_1' = \frac{2e^{3x}}{2e^{3x}} = 1$$

$$u_2' = \frac{|Au_2'|}{|A|}$$

$$= \frac{\begin{vmatrix} e^x & 0 & x^2e^x \\ e^x & 0 & 2xe^x + x^2e^x \\ e^x & \frac{2e^x}{x^2} & 2e^x + 4xe^x + x^2e^x \end{vmatrix}}{2e^{3x}}$$

$$= \frac{\frac{2e^x}{x^2}\begin{vmatrix} e^x & x^2e^x \\ e^x & 2xe^x + x^2e^x \end{vmatrix}}{2e^{3x}}$$

$$= \frac{\frac{2e^x}{x^2}\left[2xe^{2x} + x^2e^{2x} - x^2e^{2x}\right]}{2e^{3x}} = \frac{4xe^{3x}}{x^2 \cdot 2e^{3x}}$$

$$u_2' = \frac{2}{x}$$

$$u_3' = \frac{|Au_3'|}{|A|} = \frac{\begin{vmatrix} e^x & xe^x & 0 \\ e^x & e^x + xe^x & 0 \\ e^x & 2e^x + xe^x & \frac{2e^x}{x^2} \end{vmatrix}}{2e^{3x}}$$

$$= \frac{\frac{2e^x}{x^2}\left[e^x(e^x + xe^x) - e^x(xe^x)\right]}{2e^{3x}} = \frac{\frac{2e^x}{x^2}\left(e^{2x} + xe^{2x} - xe^{2x}\right)}{2e^{3x}}$$

$$u_3' = \frac{2e^{3x}}{x^2 \cdot 2e^{3x}} = 1/x^2$$

$$u_1 = \int u_1' dx = \int dx = x$$

$$u_2 = \int u_2' dx = \int \frac{2}{x} dx = 2\ln|x|$$

$$u_3 = \int u_3' dx = \int \frac{1}{x^2} dx = -\frac{1}{x}$$

So

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

$$y_p = xe^x + 2\ln|x|(xe^x) + \left(-\frac{1}{x}\right)x^2 e^x$$

$$y_p = xe^x + 2xe^x \ln|x| - xe^x$$

$$y_p = 2xe^x \ln|x|$$

$$y = y_c + y_p$$

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + 2xe^x \ln(x)$$