Partial Differential Equations

A differential equation involving partial derivative of dependent variables with respect to more than one independent variable is called a partial Differential Equations.

Ex.:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial y} = 0$$

Degree & order of partial differential Equation is same as we find in ODE.

Formation of Partial Differential Equations:-

Partial Differential Equations can be formed by eliminating the arbitrary constants from equation.

Example :-

$$x^2+y^2+(z-c)^2=a^2 \quad \text{——} \quad (A)$$

Sol.

Diff eq (A) w.r.t x

$$2x+2(z-c)\frac{\partial z}{\partial x}=0$$

$$\Rightarrow x+(z-c)\frac{\partial z}{\partial x}=0 \quad \text{——} \quad (1)$$

Diff. eq (A) w.r.t y

$$y+(z-c)\frac{\partial z}{\partial y}=0 \quad \text{——} \quad (2)$$

Dividing eq (1) by (2)

$$\frac{x}{y}+\frac{\partial z/\partial x}{\partial z/\partial y}=0$$

$$\Rightarrow x\frac{\partial z}{\partial y}-y\frac{\partial z}{\partial x}=0$$

Note:- If $z=f(x,y)$

For simplicity we write

$$\frac{\partial z}{\partial x}=p, \quad \frac{\partial z}{\partial y}=q, \quad \frac{\partial^2 z}{\partial x^2}=r, \quad \frac{\partial^2 z}{\partial x \partial y}=s, \quad \frac{\partial^2 z}{\partial y^2}=t$$

Lagrange Method of Solving linear PDE :-

$$Pp + Qq = R$$

where P, Q, R are functions of x, y, z

Now write down auxiliary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Now solve these equations
we get the solution

$$u(x, y, z) = c_1 \quad , v(x, y, z) = c_2$$

we write general solution as

$$\phi(u, v) = 0$$

Example :-

$$\frac{y^2 z}{x} p + xzq = y^2$$

Sol.

Here

$$P = \frac{y^2 z}{x} \quad , Q = xz \quad , R = y^2$$

Auxiliary eq is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$$

Consider

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz}$$

$$x^2 dx = y^2 dy$$

Integrating

$$\int x^2 dx = \int y^2 dy$$

$$\frac{x^3}{3} = \frac{y^3}{3} + c_1$$

or

$$\frac{x^3}{3} - \frac{y^3}{3} = c_1$$

Consider

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dz}{y^2}$$

$$x\,dx = z\,dz$$

Integrating

$$\int x\,dx = \int z\,dz$$

$$\frac{x^2}{2} - \frac{z^2}{2} = c_2$$

So the General Solution is

$$\phi\left(\frac{x^3}{3} - \frac{y^3}{3}, \frac{x^2}{2} - \frac{z^2}{2}\right) = 0$$

Example :-

$$y^2 p - xyq = x(z-2y)$$

Sol, Auxiliary eq is

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Consider

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$x\,dx = -y\,dy$$

$$\frac{x^2}{2} + \frac{y^2}{2} = c_1$$

Consider

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dy}{-y} = \frac{dz}{z-2y}$$

$$\frac{dy}{dz} = \frac{-y}{z-2y}$$

$$\Rightarrow \frac{dz}{dy} = -\frac{z}{y} + 2$$

$$\frac{dz}{dy} + \frac{1}{y}z = 2 \quad \text{—— (i)}$$

It is linear in z

$$I.F = e^{\int 1/y\,dy} = e^{\ln y} = y$$

Multiply eq (i) by I.F

$$y\frac{dz}{dy} + z = 2y$$

$$\frac{d}{dy}(yz) = 2y$$

Integrating

$$yz = 2\frac{y^2}{2} + C_2$$

$$\Rightarrow yz - y^2 = C_2$$

So the General Solution is.

$$\phi\left(\frac{x^2}{2} + \frac{y^2}{2}, yz - y^2\right) = 0$$

Example:-

$$2p + 3q = 1$$

Sol. A.E $\Rightarrow$ $\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{1}$

$$\frac{dx}{2} = \frac{dy}{3}, \quad \frac{dy}{3} = \frac{dz}{1}, \quad \frac{dx}{2} = \frac{dz}{1}$$

$$3dx = 2dy \qquad\qquad dx = 2dz$$

$$3x - 2y = C_1 \qquad\qquad x - 2z = C_2$$

$$\phi(3x - 2y, x - 2z) = 0$$