Higher Order Differential Equations

A differential equation of the form

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + ---- a_{n-1}\frac{dy}{dx} + a_ny = F(x)$$

is linear Diff. eq with constant coefficients.

If $F(x) = 0$ in above eq then it is called a homogenous Linear Diff. eq.

If $F(x) \neq 0$ then Non homogenous Linear Diff. eq.

Characteristics Equation:-

if we write $\frac{d}{dx} = D$

then

$$a_0D^ny + a_1D^{n-1}y + ---- a_{n-1}Dy + a_ny = 0$$

or

$$(a_0D^n + a_1D^{n-1} + ---- a_{n-1}D + a_n)y = 0$$

$$\Rightarrow a_0D^n + a_1D^{n-1} + --- a_{n-1}D + a_n = 0$$

The last eq is characteristic Equation Auxiliary Equation. LHS is a polynomial in D of degree n.

The polynomial equation has n roots $m_1, m_2 ------ m_n$

Case I

If $m_1, m_2, ... m_n$ all are distinct real roots then General solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + ---- + c_n e^{m_n x}$$

where $c_1, c_2, --- c_n$ are arbitrary constants.

Case II :-

If two roots are Real & equal

e.g $m_1 = m_2$ then general solution is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_2 x} \quad \text{or} \quad y = (c_1 + c_2 x) e^{mx}$$

If k roots out of n roots are real & equal then

$$y = (c_1 + c_2 x + c_3 x^2 + --- c_k x^{k-1}) e^{mx}.$$

Case III :-

If roots are Imaginary & Distinct $a \pm ib$ then G.S is

$$c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx$$

or

$$(c_1 \cos bx + c_2 \sin bx) e^{ax}$$

For Imaginary & repeated roots

If roots are $a \pm ib$ & repeated k times then G.S is

$$\left[(c_1 + c_2 x + c_3 x^2 + \cdots c_k x^{k-1}) \sin bx + (c_{k+1} + c_{k+2} + \cdots c_{2k} x^{k-1}) \cos bx \right] e^{ax}.$$

Ex # 10.1

Q1.. $(9D^2 - 12D + 4)y = 0$

Sol..

Its characteristics eq is

$$9D^2 - 12D + 4 = 0$$

$$9D^2 - 6D - 6D + 4 = 0$$

$$3D(3D-2) - 2(3D-2) = 0$$

$$(3D-2)(3D-2) = 0$$

$$\Rightarrow D = 2/3, 2/3$$

since roots are real & equal

So General solution is

$$y = (c_1 + c_2 x) e^{2/3 x}$$

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Q3.. $(D^3 - 4D^2 + D + 6)y = 0$

Sol.. Its auxiliary eq is

$$D^3 - 4D^2 + D + 6 = 0$$

we see that $D = -1$ is a root of this eq.

by Synthetic Division

-1	1	-4	1	6
		-1	5	-6
	1	-5	6	0

So

$$D^2 - 5D + 6 = 0$$

& we write

$$(D+1)(D^2 - 5D + 6) = 0$$

$$(D+1)(D-2)(D-3) = 0$$

$$D = -1, 2, 3$$

So the G. Sol is (real & Distinct)

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{3x}$$

Q4. $(D^3+D^2+D+1)y=0$

Sol. Its characteristics eq is

$$D^3+D^2+D+1=0$$

$$D^3+D+D^2+1=0$$

$$D(D^2+1)+1(D^2+1)=0$$

$$(D+1)(D^2+1)=0$$

$$D=-1 \quad , \quad D^2=-1$$

$$D=\pm\sqrt{-1}$$

$$D=\pm i$$

$$D=-1,\ 0\pm i$$

(Imaginary roots)

$$y=c_1e^{-x}+\{c_2\cos x+c_3\sin x\}e^{0x}$$

or

$$y=c_1e^{-x}+c_2\cos x+c_3\sin x.$$

Q7.

$(D^3 - 27)y = 0$

Sol.

$$D^3 - 27 = 0$$

$$D^3 - 3^3 = 0$$

$$(D-3)(D^2+3D+9) = 0$$

$D = 3,$ $\quad D^2 + 3D + 9 = 0$

$$D = \frac{-3 \pm \sqrt{9-36}}{2}$$

$$= \frac{-3 \pm \sqrt{-27}}{2}$$

$$= \frac{-3 \pm \sqrt{-1 \times 3^2 \times 3}}{2}$$

$$= \frac{-3 \pm i3\sqrt{3}}{2}$$

$$D = -\frac{3}{2} \pm \frac{3}{2}\sqrt{3}\, i$$

So the G. Sol is

$$y = c_1 e^{3x} + \left\{ c_2 \sin\left(\tfrac{3}{2}\sqrt{3}x\right) + c_3 \cos\left(\tfrac{3}{2}\sqrt{3}x\right) \right\} e^{-\frac{3}{2}x}$$