NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS

OR

EQUATION REDUCIBLE TO HOMOGENOUS FORM

Non-Homogeneous Differential Equations of the type

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

Case I

$$\text{If } \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

to solve put $x = X + h, \quad y = Y + k$

$\Rightarrow dx = dX, \quad dy = dY$ so $\frac{dy}{dx} = \frac{dY}{dX}$

$$\frac{dY}{dX} = \frac{a_1(X + h) + b_1(Y + k) + c_1}{a_2(X + h) + b_2(Y + k) + c_2}$$

$$\frac{dY}{dX} = \frac{a_1X + b_1Y + a_1h + b_1k + c_1}{a_2X + b_2Y + a_2h + b_2k + c_2} \ldots \ldots \ldots \ldots \ldots .(a)$$

We find the values of h & k by solving system of equations

$$a_1h + b_1k + c_1 = 0$$

$$a_2h + b_2k + c_2 = 0$$

Then eq (a) reduces to homogenous equation

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

In the variable X & Y

Case II

$$\text{If } \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

$$\text{Put } z = a_1 x + b_1 y$$

$$\Rightarrow \frac{dz}{dx} = a_1 + b_1 \frac{dy}{dx}$$

$$\Rightarrow 1/b_1 \left(\frac{dz}{dx} - a_1\right) = \frac{dy}{dx}$$

The given equation will reduce to separable equation in x & z

❖ ***Question # 15:***

$$\frac{dy}{dx} = \frac{x+3y-5}{x-y-1}$$

Solution:

Given equation is

$$\frac{dy}{dx} = \frac{x+3y-5}{x-y-1} - - - (a)$$

Put

$$x = X + h \qquad \& \qquad y = Y + k$$

$$\Longrightarrow dx = dX \qquad \& \qquad dy = dY$$

Thus equation (a) *becomes*

$$\frac{dY}{dX} = \frac{X + h + 3(Y + k) - 5}{X + h - (Y + k) - 1}$$

$$\Longrightarrow \frac{dY}{dX} = \frac{X + h + 3Y + 3k - 5}{X + h - Y - k - 1}$$

$$\Longrightarrow \frac{dY}{dX} = \frac{X + 3Y + h + 3k - 5}{X - Y + h - k - 1}$$

Put $\quad h + 3k - 5 = 0 - - - (*)$

& $\quad h - k - 1 = 0 - - - (**)$

On solving $(*)$ & $(**)$, *we have*

$h = 2$ & $k = 1$

Now $(i) \Longrightarrow \dfrac{dY}{dX} = \dfrac{X + 3Y}{X - Y} - - - (b)$

This is a homogenous differential equation in X & Y. to solve this, put

$$Y = vX$$

$$\Longrightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$$

Thus equation (b) becomes

$$v + X\frac{dv}{dX} = \frac{X + 3vX}{X - vX}$$

$$\Rightarrow v + X\frac{dv}{dX} = \frac{1 + 3v}{1 - v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1 + 3v}{1 - v} - v$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1 + 3v - v + v^2}{1 - v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{v^2 + 2v + 1}{1 - v}$$

$$\Rightarrow \frac{(1 - v)}{v^2 + 2v + 1}dv = \frac{dX}{X}$$

$$\Rightarrow \frac{(1-v)}{(v+1)^2}dv = \frac{dX}{X}$$

Integrating both sides, we have

$$\int \frac{(1-v)}{(v+1)^2}dv = \int \frac{dX}{X} - - - (i)$$

Suppose that

$$\frac{(1-v)}{(v+1)^2} = \frac{A}{v+1} + \frac{B}{(v+1)^2}$$

$$1 - v = A(v+1) + B - - - (ii)$$

$$1 - v = A(v+1) + B - - - (ii)$$

Put $v + 1 = 0 \Rightarrow v = -1$ in (ii)

Therefore,

$1 + 1 = B$

$\boxed{\Rightarrow \boldsymbol{B = 2}}$

To find the value of A, we have to solve the (i

Therefore,

$1 - v = Av + A + B$

Comparing the coefficient of v, we have

$\boxed{\boldsymbol{A = -1}}$

$$\int \left[\frac{-1}{v+1} + \frac{2}{v+1^2}\right] dv = \int \frac{dX}{X}$$

$$-\int \frac{dv}{v+1} + 2\int \frac{dv}{(v+1)^2} = \int \frac{dX}{X}$$

$$-\ln(v+1)) + 2\left(\frac{-1}{v+1}\right) = lnX + lnC$$

$$\Rightarrow -\ln\left(\frac{Y}{X} + 1\right) - \frac{2}{\frac{Y}{X} + 1} = lnX + lnC$$

$$\Rightarrow -\ln\left(\frac{X+Y}{X}\right) - \frac{2X}{X+Y} = lnX + lnC$$

Putting the *values* of X and Y, we have

$$-ln\left(\frac{x-2+y-1}{x-2}\right)-\frac{2(x-2)}{x-2+y-1}=ln(x-2)+\ln C$$

$x=X+2$

$y=Y+1$

$$\Rightarrow -ln\left(\frac{x+y-3}{x-2}\right)-\frac{2(x-2)}{x+y-3}=ln(x-2)+\ln C$$

$$\Rightarrow -ln(x+y-3)+ln(x-2)-\frac{2(x-2)}{x+y-3}=ln(x-2)+\ln C$$

$$\Rightarrow -ln(x+y-3)-\frac{2(x-2)}{x+y-3}=\ln C$$

$$\Rightarrow -\frac{2(x-2)}{x+y-3}=\ln C+ln(x+y-3)$$

$$\Rightarrow -\frac{2(x-2)}{x+y-3}=\ln C\,(x+y-3)$$

is required solution.

❖ ***Question # 17:***

$$(3y-7x-3)dx+(7y-3x-7)dy=0$$

Solution:

Given equation is

$$(3y-7x-3)dx+(7y-3x-7)dy=0$$

$$\frac{dy}{dx}=-\frac{(3y-7x-3)}{(7y-3x-7)}---(a)$$

Put

$$x=X+h \qquad \& \qquad y=Y+k$$

$$\Longrightarrow dx=dX \qquad \& \qquad dy=dY$$

Thus equation (a) becomes

$$\frac{dY}{dX}=-\frac{3(Y+k)-7(X+h)-3}{7(Y+k)-3(X+h)-7}$$

$$\Rightarrow \frac{dY}{dX} = -\frac{3Y + 3k - 7X - 7h - 3}{7Y + 7k - 3X - 3h - 7}$$

$$\Rightarrow \frac{dY}{dX} = -\frac{3Y - 7X + 3k - 7h - 3}{7Y - 3X + 7k - 3h - 7}$$

$$\text{Put} \quad 3k - 7h - 3 = 0 - - - (*)$$

$$\& \quad 7k - 3h - 7 = 0 - - - (**)$$

On solving $(*)$ & $(**)$, *we have*

$h = 0 \;\&\; k = 1$

Therefore,

$$\frac{dY}{dX} = \frac{4X + 3Y}{2X + Y} - - - (b)$$

This is a homogenous differential equation in X & Y. to solve this, put

$$Y = vX$$

$$\Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$$

Thus equation (b) becomes

$$v + X\frac{dv}{dX} = \frac{7X - 3vX}{-3X + 7vX}$$

$$\Rightarrow v + X\frac{dv}{dX} = \frac{7 - 3v}{-3 + 7v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{7 - 3v}{-3 + 7v} - v$$

$$\Rightarrow X\frac{dv}{dX} = \frac{7 - 3v + 3v - 7v^2}{-3 + 7v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{7(1 - v^2)}{-3 + 7v}$$

$$\Rightarrow \frac{(-3 + 7v)}{7(1 - v^2)}dv = \frac{dX}{X}$$

$$\Rightarrow \frac{7\left(\frac{-3}{7}+v\right)}{7(1-v^2)}dv = \frac{dX}{X} - - - (c)$$

Consider,

$$\Rightarrow \frac{v-\frac{3}{7}}{1-v^2} = \frac{A}{1-v} + \frac{B}{1+v}$$

$$\Rightarrow v-\frac{3}{7} = A(1+v) + B(1-v) - - - (i)$$

Put $1-V=0 \Rightarrow V=1$ in (i), *we have*

$$1-\frac{3}{7} = A(2)$$

$$\Rightarrow \frac{4}{7} = A(2)$$

$$\boxed{\Rightarrow \boldsymbol{A} = \frac{\mathbf{2}}{\mathbf{7}}}$$

Put $1+V=0 \Rightarrow V=-1$ in (i)

$$\Rightarrow -1-\frac{3}{7}=B(2)$$

$$\boxed{\Rightarrow \boldsymbol{B}=\frac{-5}{7}}$$

So,

$$\frac{v-\frac{3}{7}}{1-v^2}=\frac{2}{7(1-v)}+\frac{-5}{7(1+v)}$$

$$Equation\ \ (c)\ will\ be$$

$$\left[\frac{2}{7(1-v)}-\frac{5}{7(1+v)}\right]dv=\frac{dX}{X}$$

$$Integrating\ both\ sides,\ we\ have$$

$$\frac{2}{7}\int\frac{dv}{1-v}-\frac{5}{7}\int\frac{dv}{1+v}=\int\frac{dX}{X}$$

$$\frac{-2}{7}ln(1-v)-\frac{5}{7}\ln(1+v)=lnX+lnC$$

$$(1-v)^{\frac{-2}{7}}(1+v)^{\frac{-5}{7}} = CX$$

$$\Rightarrow \left(1-\frac{Y}{X}\right)^{\frac{-2}{7}}\left(1+\frac{Y}{X}\right)^{\frac{-5}{7}} = CX \quad \because v = \frac{Y}{X}$$

$$\left(\frac{X-Y}{X}\right)^{\frac{-2}{7}}\left(\frac{X+Y}{X}\right)^{\frac{-5}{7}} = CX$$

$$\left(\frac{X-Y}{X}\right)^{-2}\left(\frac{X+Y}{X}\right)^{-5} = (CX)^7$$

$$\frac{X^2}{(X-Y)^2}\frac{X^5}{(X+Y)^5} = C^7X^7$$

$$\frac{X^7}{(X-Y)^2(X+Y)^5} = C^7X^7$$

$$\left(\frac{1}{C}\right)^7 = (X-Y)^2(X+Y)^5$$

Putting values of X and Y

$$x = X + 0$$

$$y = Y + 1$$

$$(x-y+1)^2(x+y-1)^5 = c$$

is required solution.

❖ ***Question # 18:***

$$\frac{dy}{dx}=\frac{3x-4y-2}{3x-4y-3}$$

Solution:

Given equation is

$$\frac{dy}{dx}=\frac{3x-4y-2}{3x-4y-3}----(a)$$

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}$$

Put

$$t=3x-4y$$

$$\Longrightarrow \frac{dt}{dx}=3-4\frac{dy}{dx}$$

$$\Longrightarrow 4\frac{dy}{dx}=3-\frac{dt}{dx}$$

$$\Longrightarrow \frac{dy}{dx}=\frac{3}{4}-\frac{1}{4}\frac{dt}{dx}$$

Thus equation (a) becomes

$$\frac{3}{4}-\frac{1}{4}\frac{dt}{dx}=\frac{t-2}{t-3}$$

$$\Rightarrow \frac{1}{4}\frac{dt}{dx}=\frac{3}{4}-\frac{t-2}{t-3}$$

$$\Rightarrow \frac{1}{4}\frac{dt}{dx}=\frac{3(t-3)-4(t-2)}{4(t-3)}$$

$$\Rightarrow \frac{1}{4}\frac{dt}{dx}=\frac{3t-9-4t+8}{4(t-3)}$$

$$\Rightarrow \frac{dt}{dx}=\frac{-t-1}{t-3}$$

$$\Rightarrow \frac{dt}{dx}=\frac{-(t+1)}{t-3}$$

$$\Rightarrow \frac{t-3}{t+1}dt=-dx$$

Integrating both sides, we have

$$\int \frac{t-3}{t+1}dt=-\int dx$$

$$\Rightarrow \int \frac{t+1-1-3}{t+1}dt=-\int dx$$

$$\Rightarrow \int \frac{t+1}{t+1} dt - 4 \int \frac{dt}{t+1} = -\int dx$$

$$\Rightarrow t - 4\ln(t+1) = -x + c$$

$$\Rightarrow 3x - 4y - 4\ln(3x - 4y + 1) = -x + c$$

is required solution.

$t = 3x - 4y$