Yield Curve Construction: Exploring Interpolation Methodology

MF 728-D1

Professors Eugene Sorets and Christopher Kelliher

23 February 2022

<u>Group 14</u>

Taiga Schwarz

Dieynaba Awa Ndiaye

Xuyu Cai

Caleb Qi

BOSTON UNIVERSITY

Abstract:

This paper provides an overview behind yield curve construction and the necessity for interpolation methods. The following four interpolation methods have been chosen to be implemented and assessed: raw interpolation, natural cubic spline interpolation, cubic B-spline interpolation, and monotone convex interpolation (Hagan-West method). These specific methods are chosen because they exemplify the primary trade-offs that one would see in yield curve interpolation. For instance, raw interpolation guarantees no-arbitrage curves but has discontinuities in the forward curve at the edges of the intervals. On the other hand, spline methods lead to smooth curves, however they are not necessarily arbitrage free, as we will see later in the paper. Finally, in the last section we apply each method in a hedging strategy to visualize the potential economic impact the interpolation can have.

1 Overview:

In finance, we often require a single, coherent term structure of interest rates– or yield curve for the purpose of valuation or risk management. There are different types of curves corresponding to different types of rates. For instance, a zero curve, i.e. risk-free yield curve, is a function $r(t)$ such that a single payment invested for a length of time t is worth $e^{r(t)t}$ by the end. For this reason, $r(t)$ can also be thought of as the risk-free discount rate. There is also the forward curve, which is made up of the rates at which we can lock in to borrow or lend at some future time, for some predetermined period length.

These rates can be stripped through bootstrapping from various market-traded fixed income instruments such as LIBOR or OIS deposit rates, Eurodollar contracts or FRA's, US T notes and bonds, and annual swaps. However, since the quantity of these instruments is limited, we would need to implement an interpolation method in order to impute the full continuum of rates. One can apply many different interpolation methods to construct a yield curve, and while there is no single correct method, each method has its strengths and weaknesses. A useful paper by Hagan & West (2006) covers many of these interpolation methods and their respective strengths and weaknesses in the context of yield curve construction.

In this paper, we analyze the raw interpolation method, natural cubic spline method, cubic B-spline method, and finally the monotone convex method, which was developed by

Hagan & West in an attempt to combine the strengths of the raw interpolation and splining methods. We will go more into the details of each of the methods in Section 3.

2 Data

Since our goal is to focus on the interpolation methods as opposed to constructing production-level yield curves, we simplify our yield curve construction process by choosing our inputs to be a set of zero rates stripped from par-coupon US Treasury securities, as provided by Bloomberg, at the following maturities: 6M ,1Y, 2Y, 3Y ,4Y, 5Y, 6Y, 7Y, 8Y, 9Y, 10Y, 15Y, 20Y, and 30Y. We also pull data at monthly periods (end-of-month) over the interval 04/2017 to 03/2022. Having this data allows us to visualize how the interpolated curves change over time.

3 Methods Overview:

3.1 Raw Interpolation

This method is linear on the logarithm of discount factors, and as we shall see, corresponds to piecewise constant forward curves. To a good approximation, any forward curve that has the same area between each node would work. This means that if a piecewise linear approximation starts too high, it has to go too low to average to the right value, but then it starts the next interval too low and has to go too high to average to the right value.

$$f(\tau) := \frac{r_{i+1}\tau_{i+1} - r_i\tau_i}{\tau_{i+1} - \tau_i}$$

3.2 Natural Cubic Spline

We resort to the cubic spline method to construct the instantaneous forward curve which is kept continuous and smooth. There are piecewise polynomials of degree 3 between nodes and we also want the process to be continuous with first and second order derivatives:

$$r(\tau) = a_i + b_i(\tau - \tau_i) + c_i(\tau - \tau_i)^2 + d_i(\tau - \tau_i)^3 \quad \tau_i \le \tau \le \tau_{i+1}$$

Then forward curve is calculated by:

$$f(\tau) = \frac{\partial}{\partial \tau} r(\tau)\tau$$

3.3 Cubic B-spline

The B-spline method is thought to be an improvement to the natural cubic spline method. While maintaining the curve continuous and smooth, it also makes the curve more stable than the cubic spline method. A B-spline of degree $d \geq 0$ is a function f(t) of the form:

$$f(t) = \sum_{-\infty<k<\infty} f_k B_k^{(d)}(t),$$

where the basis functions $B_k^{(d)}(t), k = \ldots, -1, 0, 1, 2, \ldots,$ is a family of degree d splines defined as follows. We choose a sequence of knot points:

$$\ldots < t_{-1} < t_0 < t_1 < \ldots < t_k < \ldots,$$

and set:

$$B_k^{(0)}(t) = \begin{cases} 1, & \text{if } t_k \leq t < t_{k+1}. \\ 0, & \text{otherwise.} \end{cases}$$

In other words, the degree 0 basis functions are simply indicator functions of the intervals between the knot points. We then define recursively:

$$B_k^{(d)}(t) = \frac{t - t_k}{t_{k+d} - t_k} B_k^{(d-1)}(t) + \frac{t_{k+d+1} - t}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t).$$

We highlight that the knot points are not exactly on the curve but they control the shape of the curve. The choice of these knot points have a significant effect on the shape of the curve. In this case we use the knots with uniform density on the time grid, but undoubtedly, there are several more complicated methods to determine the knots, which are beyond the scope of this project. In order to control the convexity of the curve, we need to add another regularization term on our optimization function, which is called Tikhanov regularization:

$$Q(\lambda, f) = \lambda \int_0^T \frac{\partial^2 f(t)}{\partial t^2} dt$$

where λ is a non-negative (usually, piecewise constant) function. Experience shows that it is a good idea to choose λ smaller in the short end and larger in the back end of the curve. The construction of this optimization is aimed to find the coefficients of basic functions. We use the least square method to make the difference between model rates and market rates smallest.

3.4 Monotone Convex (Hagan-West method)

Very simply, none of the above mentioned methods are aware that they are trying to solve a financial problem. As such, there is no mechanism which ensures that the forward rates generated by the method are positive, and some simple experimentation will uncover a set of inputs to a yield curve which give some negative forward rates under all of the methods mentioned here, as seen in Hagan and West (2006). Thus, in introducing the monotone convex method, we ensure that the continuous forward rates are positive (whenever the discrete forward rates are themselves positive). We follow the algorithms:

(1) Determine the discrete forward rates from the input data.

(2) Define some nodes of instantaneous forward rates.

(3) Add some constraints on the instantaneous forward curve to ensure it is positive.

(4) Construct g(t) with regard to which of the four sectors we are in.

Since the calculations that go into these steps are lengthy, we will direct the reader to the Hagan & West (2006) paper in which it is covered in detail.

4 Yield Curve Methodology Quality Criteria:

We follow the same quality criteria used in Hagan & West (2006) to judge our yield curve constructions and interpolation methods:

1. **Are the forwards positive?**

   We would like our curves to be no-arbitrage. For this to be true, we must have the discount factors implied by the zero curve to be monotonically decreasing in time, which means that we must have the forward rate process to be positive.

2. **Is the forward curve continuous?**

   Continuity in the forward curve is an attractive feature when pricing interest-sensitive instruments, for example a fixed/float swap that depends on 3 month forward rates. A discontinuous forward curve may imply implausible expectations about future spot rates. However, oversmoothing could be an issue, as this might result in higher error in fitting market data.

3. **Is the interpolation method local?**

   If an input to the bootstrap were to be bumped, the interpolation function should only change locally, and not have any significant changes elsewhere in the curve.

4. **Are the curves stable?**

A change in the input data should result in a roughly proportional change in the constructed yield curve. We compare the stability of our interpolation methods by observing the change in the constructed curves over a set interval of time (from end-of-month 11/2021 to 03/2022 at monthly periods). We also quantify the degree of stability by computing the maximum basis point change in the curve given a basis point change in the *i*th input, and taking the average across all *i* inputs. We will refer to this ratio as the max-norm ratio moving forward, as it can be computed as $\left\|\frac{\Delta r(t)}{\Delta r_i}\right\|_\infty$ and $\left\|\frac{\Delta f(t)}{\Delta f_i^d}\right\|_\infty$, where the numerator is the change in the curve and the denominator is the change in the *i*th input, for the zero rates and the instantaneous forwards, respectively.

4.1 Raw Interpolation (Piecewise Constant Forwards)

Forwards positive?	Yes
Curves continuous?	Not continuous
Method local?	Excellent
Curves stable?	Excellent (1st)

An attractive feature of the raw interpolation method is that it guarantees the forwards to be positive as long as the inputs are all positive. This is true by construction, because the instantaneous forwards are set equal to the discrete forwards between each interval.

Furthermore, we also see that the localness of the interpolation is quite excellent. To be more precise, we observe that a bump in the input rate at t_i affects the zero curve primarily between t_i and t_{i+2}, and the forward curve between t_i and t_{i+3}. The change in the forward curve is slightly less local because as explained earlier, the raw interpolation method corresponds to piecewise constant forward curves. This means that if one of the inputs is bumped up, this would cause the piecewise linear approximation to start higher than before, and thus would have to go too low in order to match the next node. This then means the next interval starts too low and therefore has to go too high to average to the correct value. This effect is visible in the figure below, in which the 4Y input is bumped up by 5 basis points.

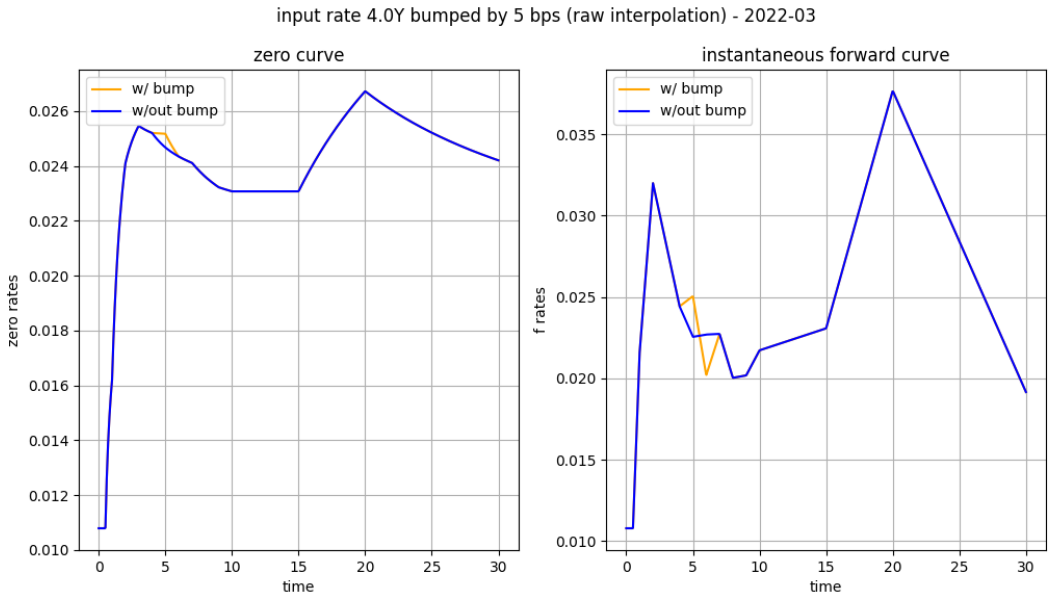
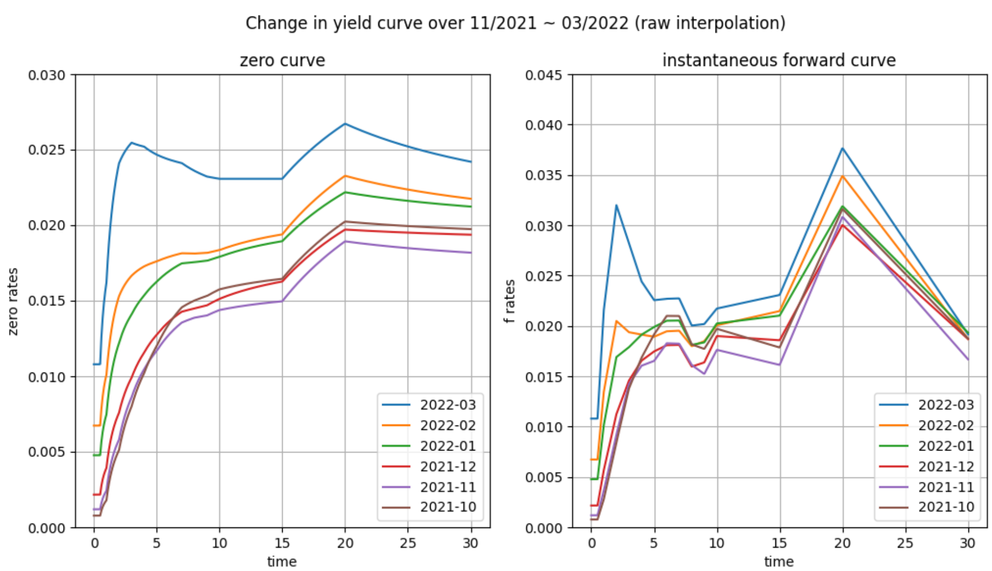


Figure 1. The zero and forward curve before (blue) and after (orange) a 5 basis point bump in the 4Y input rate.

Next, out of the four methods covered in this paper, the raw interpolation method is the most stable. We determine this through inspecting the change in the yield curves over time, specifically from 11/2021 through 03/2022, and also by computing the max-norm ratios. Our results are below:



$$\left\|\frac{\Delta r(t)}{\Delta r_i}\right\|_\infty = 1.00, \qquad \left\|\frac{\Delta f(t)}{\Delta f_i^{\,d}}\right\|_\infty = 1.00$$

We observe that the curves change approximately proportionally to changes in the inputs.

Finally, the raw interpolation method is not continuous, as we can see in the above plots. There are discontinuities at the nodes, as there is nothing ensuring the interpolation is differentiable at the nodes of the piecewise portions. The discontinuities make the raw interpolation method not attractive for practical purposes, for instance pricing instruments that depend on forward rates. However, it is still a useful method for validating more complex methods since it is very simple to implement and does not require subjectivity in the way B-splines do for instance, which depend on the choice of knots.

4.2 Natural Cubic Spline Interpolation

Forwards positive?	No
Curves continuous?	Smooth (class C^3)
Method local?	Poor
Curves stable?	Good (3rd)

The forwards are not guaranteed to be positive for splining methods given. It is worth noting that of course, forward rates could go negative in reality. However, we do not want our interpolation method to result in negative forwards when the input rates are all positive. While splines are a satisfactory interpolation method for dense nodes, any considerable distance between two adjacent nodes may result in excessive convexity of the curve between them. As such, it would be possible to have a set of positive input rates such that the negative forward rates occur. For example, using the continuous yield rates below as input, we can show that applying the cubic spline method results in the forward curve dipping below zero after about 28 years (see Figure 1). A similar example is shown in Hagan & West (2006).

Term	Continuous yield
0.5	8.00%
5	7.00%
10	8.00%
15	7.00%
20	8.00%
30	7.00%

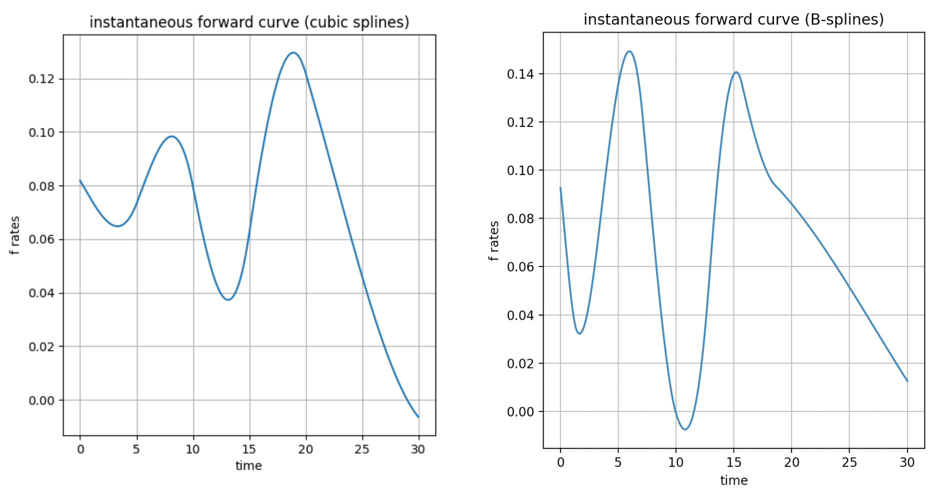


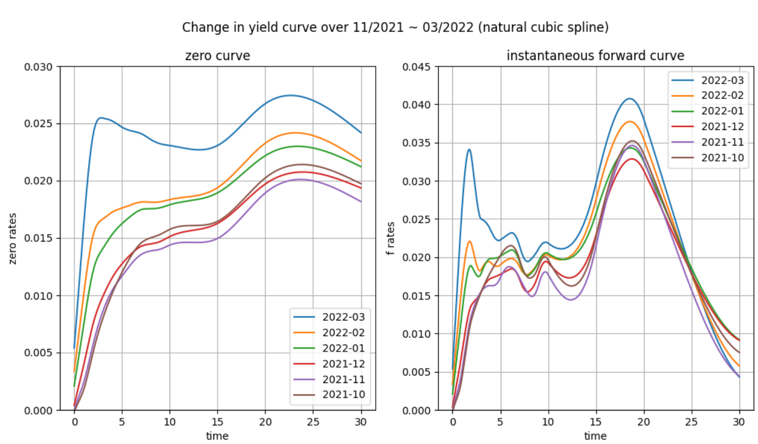
Figure 2. The forward curve becomes negative around 28Y and after for natural cubic splines, and around 11Y for cubic B-splines.

On the other hand, an attractive feature of the cubic spline methods is the level of smoothness achieved in the curves without being overly complex to implement. For a dense set of nodes, the cubic spline method is thus a useful interpolation method to use for yield curve construction.

The cubic spline method is not very local. We see that changes in the inputs, particularly at nodes where the adjacent nodes are far apart, can significantly impact far along the curve. For example, if we bump the 9Y input rate by 5 basis points, we can compare just how much the natural cubic spline interpolated curve changes to how the raw interpolated curve changes (see Figure 3).

Figure 3. The natural cubic spline curve changes all the way through to the 30Y maturity after a 5 bp bump in the 9Y input rate.

Finally, we do our stability analysis of the natural cubic spline method. We conclude that it is third in terms of overall stability. It is slightly less stable than the cubic B-spline method, which makes sense since it does not have the regularization term that the B-spline method has that controls the amount of convexity in the interpolation function. Below are the plot and stability metrics for the natural cubic spline.



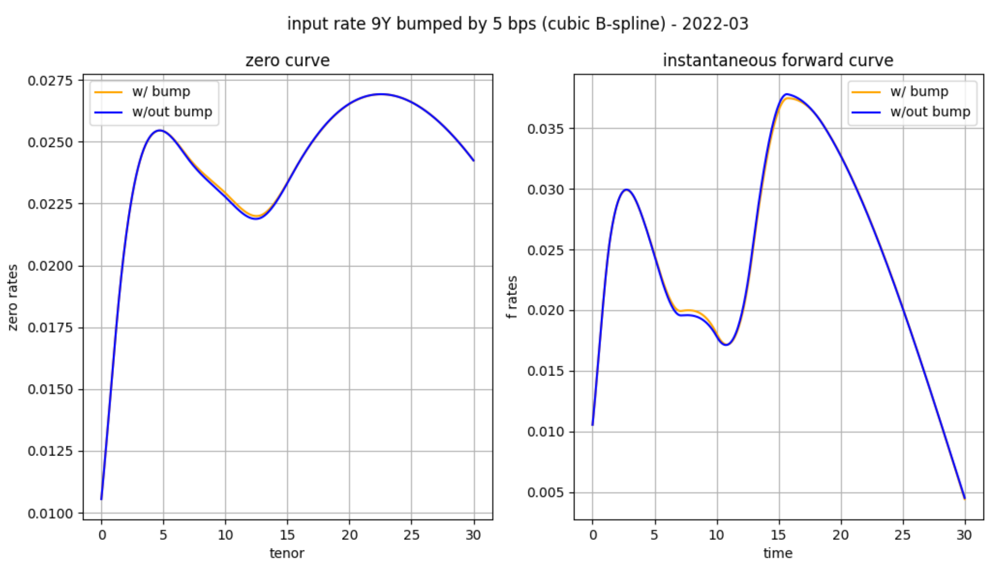
$$\left\|\frac{\Delta r(t)}{\Delta r_i}\right\|_\infty = 1.14, \quad \left\|\frac{\Delta f(t)}{\Delta f_i^d}\right\|_\infty = 2.05.$$

4.3 Cubic B-Spline Interpolation

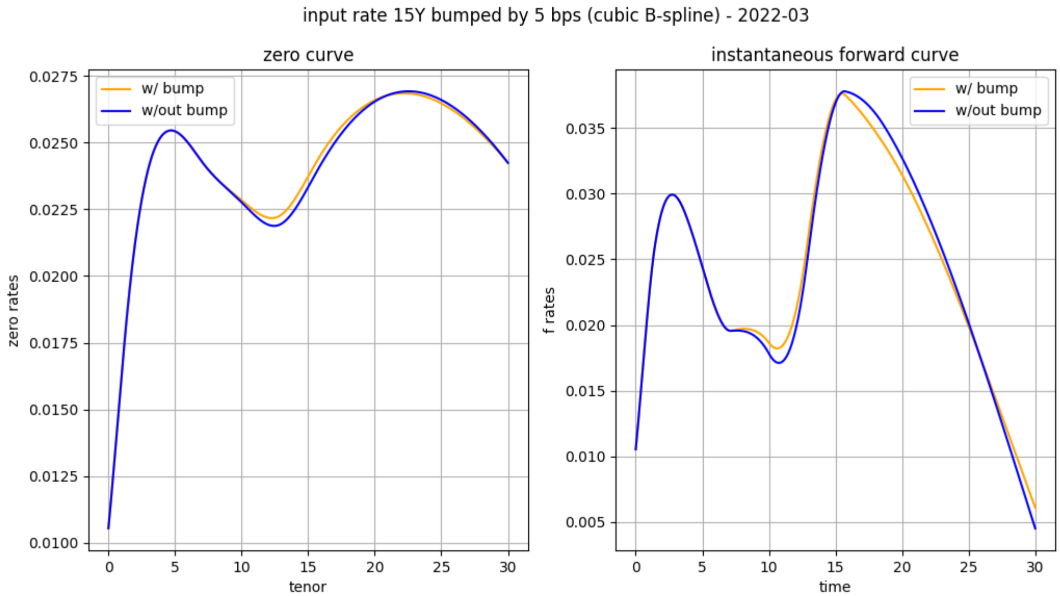
Forwards positive?	No
Curves continuous?	Smooth (class C^3)
Method local?	Good
Curves stable?	Good (2nd)

As detailed before, the cubic B-spline method may not have positive forwards given positive inputs, and also is smooth like the natural cubic spline method. The differences between the cubic B-spline and the natural cubic spline methods appear in the localness of the interpolation and the stability of the interpolation.

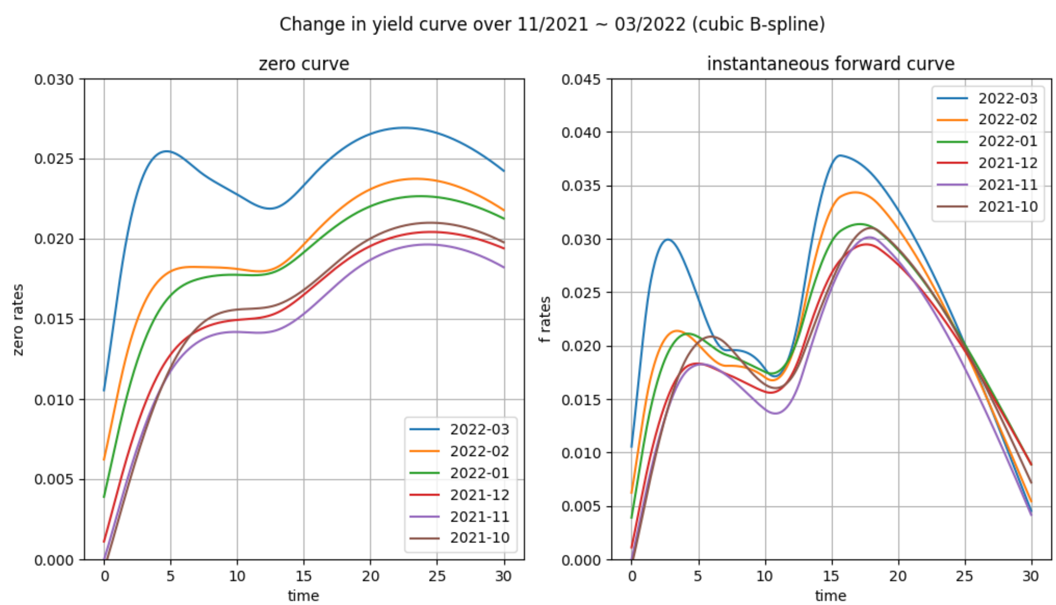
Like the natural cubic spline method, cubic B-spline would technically have the same issue of having changes to one of the inputs leaking into other parts of the curve. However, the changes are much less obvious when the input that changes is part of a dense region of nodes, as the method is considerably more stable. Below is the 9Y input rate bumped up by 5 basis points. The change in either of the curves is considerably less obvious than in the natural cubic spline interpolation which we saw before.



However, if we bump an input rate that is part of a sparse region of nodes, for instance the 15Y input rate, we see the curve change all the way out to the 30Y maturity.



Finally, we do our stability analysis of the cubic B-spline method. It is evident in the plot that the curvature is less steep at some points compared to that of the natural cubic spline. Furthermore, the max-norm metrics confirm that it is more stable than the natural cubic spline method as well.



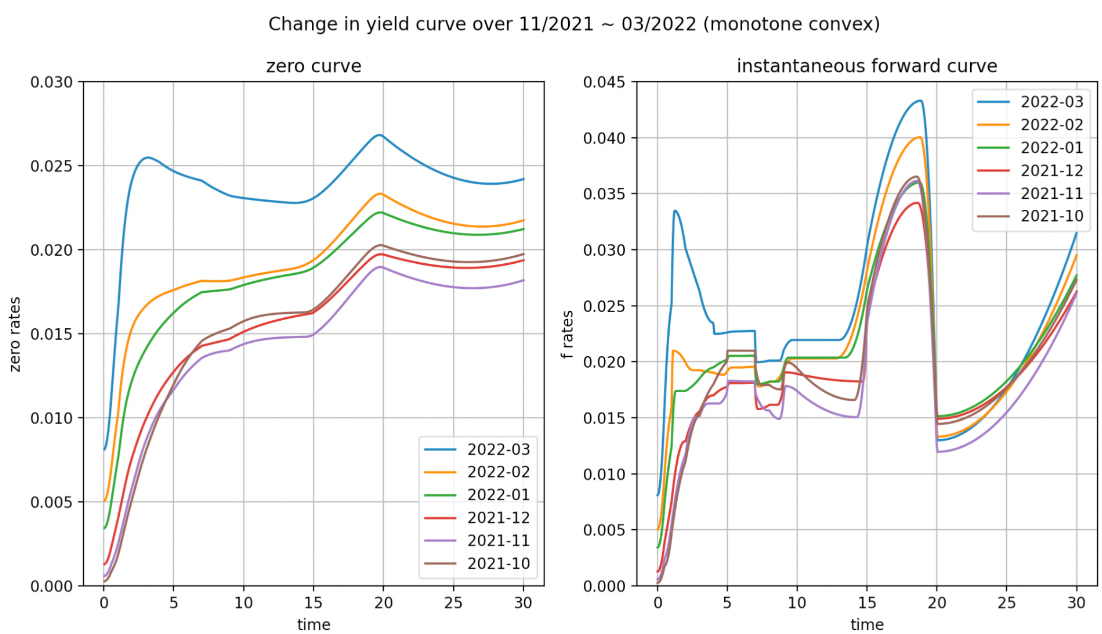
$$\left\|\frac{\Delta r(t)}{\Delta r_i}\right\|_\infty = 1.08, \qquad \left\|\frac{\Delta f(t)}{\Delta f_i^{\,d}}\right\|_\infty = 1.99$$

4.4 Monotone Convex

Forwards positive?	Yes
Curves continuous?	piecewise smooth (class C^2)
Method local?	Very good
Curves stable?	Poor (4^{th})

Similar to the raw interpolation method, the monotone convex method does ensure positivity of forwards (given positive inputs) and also has good localness of interpolation. Both of these are by construction, as the method begins with computing the discrete forwards. The considerable advantage the monotone convex method has over the raw interpolation is that it also provides smooth curves, specifically of class C^2 since it incorporates quadratics to do the interpolation.

However, there is a drawback to using this method, and that is its stability. Although its stability for the zero curve is not bad (neither are the other methods bad), we see that it does not have stable forward curves. This is likely due to just how parameter-heavy the method is, and also our inability to control the convexity of the curves. The instability can be visualized in our plots. We see that the curves change more dramatically compared to the plots of the other methods. There is also the end behavior where the forward curve slopes upwards, which is considerably different from the other methods and could lead to inaccurate forward rates in the long-end. The max-norm ratios confirm the instability that we see in the forward curve.



$$\left\| \frac{\Delta r(t)}{\Delta r_i} \right\|_\infty = 1.11, \qquad \left\| \frac{\Delta f(t)}{\Delta f_i^d} \right\|_\infty = 3.85$$

5 Localness of a Hedge:

One application of a yield curve is in managing the sensitivity of a risky portfolio to changes in interest rates. In the previous sections, we covered the different characteristics of each interpolation method, such as its stability and localness of interpolation. One would expect these characteristics to also have an impact on a hedging strategy, with some methods resulting in less intuitive hedging portfolios than others. Ideally, if given a risky instrument of a given tenor, we would expect to be able to hedge it with a bond of the same maturity.

In this section, we apply each of the interpolation methods in a simple delta-hedge strategy called "bumping". This method, which is described in Hagan & West (2006), finds the portfolio where the first-order partial derivatives with respect to each risk factor are equal to zero. However instead of only considering the key rates as in the case of key rate duration, it treats every input of our bootstrap as a potential risk factor. Of course, in order to do so we must assume that the instruments used to bootstrap the yield curves (in this case zero coupon bonds) are available at each of the maturities for creating hedge portfolios. While this is not very realistic in our case, we will be making simplifying assumptions such as this to aid our task of analyzing the yield curve interpolation methods. We will use a single instrument as our risky portfolio– a fixed/float swap with semi-annual payments at a given tenor.

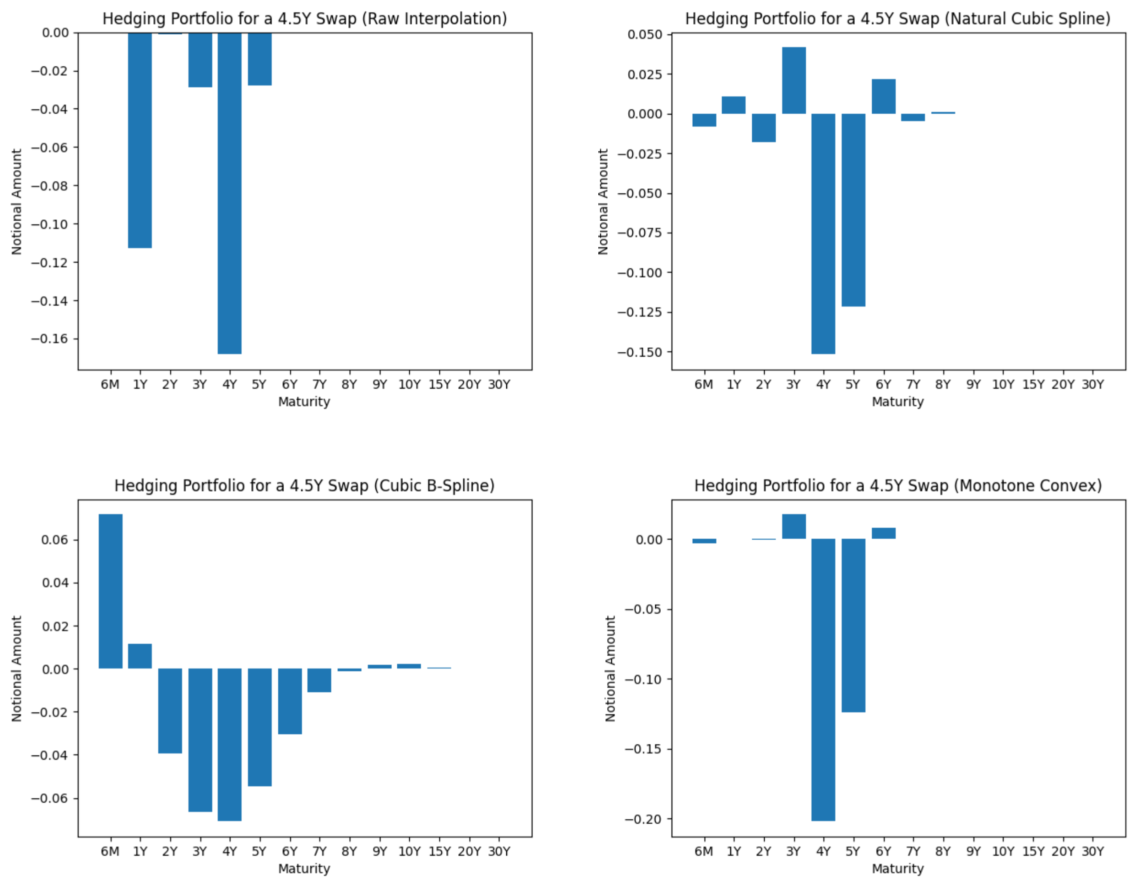
The hedge strategy is outlined below:

1. First, create new curves indexed by i, where $i \in \{1,\ldots,n\}$ and n is the number of bootstrapping instruments. To create the ith curve, we bump up the ith input rate by 1 basis point and bootstrap the curve again. We then reprice the risky portfolio with each new curve, and let the differences between the new and old price be stored in a column vector ΔV.
2. Next, calculate the matrix P, where P_{ij} is the change in price of the jth bootstrapping instrument of the ith curve. In our case, this would be the change in price of each zero-coupon bond for each new curve. This matrix is a square diagonal matrix, as the ith curve is formed by bumping the ith input and the inputs are priced exactly at par.
3. The duration neutral portfolio is then determined by $PQ = \Delta V$, where Q is the hedge vector chosen such that it satisfies the equation. Thus, to solve for Q we can

invert P and compute $Q = P^{-1}\Delta V$. In doing so, we find a portfolio Q that perfectly hedges the risky portfolio under the case where the curve moves from exactly one of the basis points being bumped by 1 basis point, and as such should be an adequate hedge against more general small changes in any collection of the inputs.

5.1 Results:

The risky instrument that we will hedge is a 4.5Y swap. Here we include the resulting hedging portfolios after creating different P matrices for each of the four interpolation methods.



The most intuitive hedge portfolios are given by the monotone convex and natural cubic spline methods. The natural cubic spline method results in stronger correlated interest rates along the curve (as we saw in the localness of the interpolation) and thus this may explain the pattern in the hedge weights outside of the 4Y and the 5Y that we see in the hedge portfolio.

On the other hand, the raw interpolation method and the cubic B-spline method do not result in intuitive hedges. Clearly, this can only be due to the interpolation method, as this is what we are isolating for. However it is difficult to say exactly which characteristic of the interpolation method causes the distribution of the hedge weights that we see for these methods in the plots above. The raw interpolation method does not suffer from poor localness of interpolation. Thus, it is likely due to the fact that it uses piecewise linear interpolation on the forwards. A swap is an instrument that depends on the forwards, thus the discontinuity at the 1Y node may be causing that large 1Y weight that we see in the plot. The cubic B-spline hedge portfolio is peaked at the 4Y maturity, however the rest of the weights are less intuitive. Furthermore, all the weights are relatively low. This implies that there is not a strong connection between the price of the hedging instruments (the ZCB's) and the resulting curves. This would mean that our B-spline interpolation curves are likely not capable of repricing the input instruments given our choice of B-spline knots. Perhaps choosing more optimal knots would improve the performance of the B-spline method.

6 Conclusions:

In our research, we sought out to explore the different interpolation methods that could be applied to yield curve construction. We found that while there really is no "correct" interpolation method, they all have their own unique use cases. For instance, the raw interpolation method is very simple to implement, as it is linear on the log of the discount factors and therefore corresponds to piecewise constant forwards. As such, it can be easily used to validate more complex models. The cubic spline methods are great for a dense set of input instruments, as they too are relatively easy to implement and also ensure smooth curves. An example of when it would be useful is for bootstrapping a swap curve, which has considerably more instruments (e.g. OIS or LIBOR deposit rates, FRA's and Eurodollar contracts, annual swaps) to bootstrap input rates from than that of bootstrapping a bond curve. Finally, the monotone convex method does indeed make up the primary weakness of the raw interpolation (not smooth) and the spline methods (chance for unintended negative forwards), however the trade-off is that it is significantly more complex, and thus it sacrifices some stability and computation time.

Works Cited

Hagan, Patrick, and Graeme West. “Interpolation Methods for Curve Construction.” *Applied Mathematical Finance*, vol. 13, no. 2,89-129, 2006, www.researchgate.net/publication/24071726_Interpolation_Methods_for_Curve_Construction.

Hagan, Patrick, and Graeme West. “Methods for Constructing a Yield Curve.” WILMOTT Magazine, 2006, web.math.ku.dk/~rolf/HaganWest.pdf.