CSC/ITC 301 Engineering Mathematics - III (COMP, IT)
Solutions: End Semester Exam, Nov 2022

Q. 1(a) :Find Laplace of $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$ given that $L[\sin\sqrt{t}] = \dfrac{\sqrt{\pi}}{2s^{3/2}}e^{-(1/4s)}$

Solution: Let $f(t) = \sin\sqrt{t}$

Then $f'(t) = \dfrac{\cos\sqrt{t}}{2\sqrt{t}} \implies \therefore 2f'(t) = \dfrac{\cos\sqrt{t}}{\sqrt{t}}$

Now by the Laplace transform of derivatives

$$\begin{aligned} L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] &= L[2f'(t)] \\ &= 2\{L[f'(t)]\} \\ &= 2\{sL[f(t)] - f(0)\} \\ &= 2\left\{s\frac{\sqrt{\pi}}{2s^{3/2}}e^{-(1/4s)} - 0\right\} \\ &= \frac{\sqrt{\pi}}{2s^{1/2}}e^{-(1/4s)} \end{aligned}$$

Q. 1(b) : Calculate the Spearman's rank correlation coefficient for the following data:

X	32	55	49	60	43	37	43	49	10	20
Y	40	30	70	20	30	50	72	60	45	25

Solution: We have the Spearman's rank correlation coefficient to be :
(Since values (and hence the ranks) are repeated)

$$\rho_{xy} = 1 - 6\frac{\sum d^2 + \sum correction\ factors}{n(n^2-1)}$$

where if the rank k repeats m times, then the correction factor is $\dfrac{m(m^2-1)}{12}$ We get the following table:

X	Y	Rank in X r_x	Rank in Y r_y	$d = r_x - r_y$	d^2
32	40	8	6	2	4
55	30	2	7.5	-5.5	30.25
49	70	3.5	2	1.5	2.25
60	20	1	10	-9	81
43	30	5.5	7.5	-2	4
37	50	7	4	3	9
43	72	5.5	1	4.5	20.25
49	60	3.5	3	0.5	0.25
10	45	10	5	5	25
20	25	9	9	0	0
					$\sum d^2 = 176$

Correction factors

Series	Repeating rank k	No. of times repeated m	correction factor $\frac{m(m^2-1)}{12}$
X	3.5	2	$\frac{1}{2}$
X	5.5	2	$\frac{1}{2}$
Y	7.5	2	$\frac{1}{2}$
			$\sum correction factors = 1.5$

Therefore the Spearman's rank correlation coefficient is

$$\begin{aligned}\rho(=R) &= 1-6\left(\frac{\sum d^2+\sum \ correction \ factors}{n(n^2-1)}\right)\\ &= 1-6\left(\frac{176+1.5}{10\times 99}\right)\\ \Rightarrow \rho &= -0.0758\end{aligned}$$

Q. 1(c) :Find Inverse Laplace Transform of $\dfrac{2s-1}{s^2+8s+29}$

Solution:

$$\begin{aligned}L^{-1}\left(\frac{2s-1}{s^2+8s+29}\right) &= L^{-1}\left(\frac{2s-1}{s^2+8s+16-16+29}\right)\\ &= L^{-1}\left(\frac{2((s+4)-4)-1}{(s+4)^2+13}\right)\\ &= L^{-1}\left(\frac{2(s+4)-9}{(s+4)^2+13}\right)\\ &= e^{-4t}L^{-1}\left(\frac{2s-9}{s^2+13}\right)\\ &= e^{-4t}\left[L^{-1}\left(\frac{2s}{s^2+13}\right)-L^{-1}\left(\frac{9}{s^2+13}\right)\right]\\ &= e^{-4t}\left[2\cos(\sqrt{13}t)-\frac{9}{\sqrt{13}}\sin(\sqrt{13}t)\right]\end{aligned}$$

Q. 1(d) : If $f(z)=qx^2y+2x^2+ry^3-2y^2-i(px^3-4xy-3xy^2)$ is analytic, find p,q,r

Solution: Let $f(z)=u+iv=qx^2y+2x^2+ry^3-2y^2-i(px^3-4xy-3xy^2)$
$\therefore u=qx^2y+2x^2+ry^3-2y^2$ and $v=-px^3+4xy+3xy^2$
$\Rightarrow u_x=2qxy+4x$ and $u_y=qx^2+3ry^2-4y$
And $v_x=-3px^2+4y+3y^2$ and $v_y=4x+6xy$

$f(z)$ is analytic $\Rightarrow u_x = v_y$ and $u_y = -v_x$
$\therefore$

$$\begin{aligned} 2qxy + 4x &= 4x + 6xy \\ \Rightarrow 2q &= 6 \quad \Rightarrow q = 3 \end{aligned}$$

Similarly

$$\begin{aligned} u_y &= -v_x \\ \Rightarrow qx^2 + 3ry^2 - 4y &= -(-3px^2 + 4y + 3y^2) \\ \Rightarrow q = 3p, \ \Rightarrow p = 1; &\qquad 3r = -3 \Rightarrow r = -1 \end{aligned}$$

$\therefore p = 1, q = 3, r = -1$

Q. 2(a) :Find Laplace Transform of $e^{3t}f(t)$ where $f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \\ 0, & \text{otherwise} \end{cases}$

Solution:
By the definition of LT
$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$
Therefore

$$\begin{aligned} L[f(t)] &= \int_0^1 0 \ dt + \int_1^2 e^{-st}(t-1)dt + \int_2^3 e^{-st}(3-t)dt + \int_3^\infty 0 \ dt \\ &= 0 + \left[(t-1)\left[\frac{e^{-st}}{-s}\right] - (1)\left[\frac{e^{-st}}{s^2}\right]\right]_1^2 + \left[(3-t)\left[\frac{e^{-st}}{-s}\right] - (-1)\left[\frac{e^{-st}}{s^2}\right]\right]_2^3 + 0 \\ &= \left\{\left[(1)\left[\frac{e^{-2s}}{-s}\right] - (1)\left[\frac{e^{-2s}}{s^2}\right]\right] - \left[(0)\left[\frac{e^{-s}}{-s}\right] - (1)\left[\frac{e^{-s}}{s^2}\right]\right]\right\} \\ &\quad + \left\{\left[(0)\left[\frac{e^{-3s}}{-s}\right] - (-1)\left[\frac{e^{-3s}}{s^2}\right]\right] - \left[(1)\left[\frac{e^{-2s}}{-s}\right] - (-1)\left[\frac{e^{-2s}}{s^2}\right]\right]\right\} \\ &= \left[\frac{e^{-2s}}{-s}\right] - \left[\frac{e^{-2s}}{s^2}\right] + \left[\frac{e^{-s}}{s^2}\right] + \left[\frac{e^{-3s}}{s^2}\right] - \left[\frac{e^{-2s}}{-s}\right] - \left[\frac{e^{-2s}}{s^2}\right] \\ &= \frac{1}{s^2}[e^{-s} - 2e^{-2s} + e^{-3s}] \end{aligned}$$

Hence by the first shifting theorem
$L[e^{3t}f(t)] = \frac{1}{(s-3)^2}[e^{-(s-3)} - 2e^{-2(s-3)} + e^{-3(s-3)}]$

Q. 2(b) : Two unbiased dice are thrown. If X represents the sum of numbers on the 2 dice,find the probability distribution of X and obtain mean, standard deviation and $P(|X-3| \geq 3)$

Solution: The probability distribution of X is given by:

X	2	3	4	5	6	7	8	9	10	11	12
P(X=x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Now the mean of X is

$$\begin{aligned}
E(X) &= \sum_x xp_x \\
&= 2(\frac{1}{36}) + 3(\frac{2}{36}) + 4(\frac{3}{36}) + 5(\frac{4}{36}) + 6(\frac{5}{36}) + 7(\frac{6}{36}) + 8(\frac{5}{36}) + 9(\frac{4}{36}) + 10(\frac{3}{36}) + 11(\frac{2}{36}) + 12(\frac{1}{36}) \\
&= (\frac{2+6+12+20+30+42+40+36+30+22+12}{36}) \\
&= \frac{252}{36} = 7
\end{aligned}$$

and

$$\begin{aligned}
E(X^2) &= \sum_x x^2 p_x \\
&= 2^2(\frac{1}{36}) + 3^2(\frac{2}{36}) + 4^2(\frac{3}{36}) + 5^2(\frac{4}{36}) + 6^2(\frac{5}{36}) + 7^2(\frac{6}{36}) + 8^2(\frac{5}{36}) \\
&\quad +9^2(\frac{4}{36}) + 10^2(\frac{3}{36}) + 11^2(\frac{2}{36}) + 12^2(\frac{1}{36}) \\
&= (\frac{4+18+48+100+280+294+320+324+300+242+144}{36}) \\
&= \frac{2074}{36} = 56.61
\end{aligned}$$

Therefore the variance

$$\begin{aligned}
Var(x) &= E(X^2) - \{E(X)\}^2 \\
&= 56.11 - 7^2 \\
&= 8.611
\end{aligned}$$

Hence the standard deviation of X is
S.D$(X) = \sqrt{Var(X)} = \sqrt{8.611} = 2.934$
Now

$$\begin{aligned}
P(|X-3| \geq 3) &= P(X-3 \geq 3) + P((X-3) \leq -3) \\
(\because |X| \geq a \Rightarrow & \quad X \geq a \; or \; X \leq -a \text{ if } a \text{ is non-negative}) \\
\Rightarrow P(|X-3| \geq 3) &= P(X \geq 6) + P(X \leq 0) \\
\Rightarrow P(|X-3| \geq 3) &= P(X = 6) + 0 \\
\text{that is } P(|X-3| \geq 3) &= \frac{5}{36}
\end{aligned}$$

Q. 2(c) : Find the Fourier series of $f(x) = x \cdot \sin x$ in the interval $(0, 2\pi)$.
Solution: The interval is $[0, 2\pi] \implies l = \pi$
therefore the Fourier series is:
$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 =& \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x dx = \frac{1}{\pi}\left[(x)(-\cos x) - (-\sin x)(1)\right]_0^{2\pi} \\
=& \frac{1}{\pi}\left[-2\pi + 0 - (0+0)\right] = -1
\end{aligned}$$

$$\begin{aligned}
a_n =& \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{2\pi} x\cdot\sin x\cdot\cos nx \ dx \\
=& \frac{1}{\pi}\int_0^{2\pi} x\frac{1}{2}[\sin(n+1)x - \sin(n-1)x]dx .... \because \cos A\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)] \\
=& \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]dx \\
=& \frac{1}{2\pi}\left\{x\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right] - (1)\left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right\}_0^{2\pi} \\
=& \frac{1}{2\pi}\left[2\pi\left(-\frac{\cos 2(n+1)\pi}{(n+1)} + \frac{\cos 2(n-1)\pi}{(n-1)}\right) - 0\right] \\
=& \frac{1}{\pi}\left\{\left[\pi\left(-\frac{(-1)^{(n+1)}}{(n+1)} - \frac{(-1)^{(n-1)}}{(n-1)}\right)\right]\right\} \\
=& -\frac{1}{(n+1)} + \frac{1}{(n-1)} = \frac{2}{(n^2-1)} \quad if n \neq 1
\end{aligned}$$

(If $n = 1$, the value is not defined).
Put $n = 1$ in the formula of a_n

$$\begin{aligned}
a_1 =& \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x dx = \frac{1}{2\pi}\left[(x)(-\frac{\cos 2x}{2}) - (1)(-\frac{\sin 2x}{4})\right]_0^{2\pi} \\
=& \frac{1}{2\pi}\left[2\pi(-\frac{\cos 4\pi}{2}) - 0\right] = -\frac{1}{2}
\end{aligned}$$

Now,

$$\begin{aligned}
b_n =& \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} x\cdot\sin x\cdot\sin nx \ dx \\
=& -\frac{1}{\pi}\int_0^{2\pi} x\frac{1}{2}[\cos(n+1)x - \cos(n-1)x]dx .... \because \cos A\cos B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)] \\
=& -\frac{1}{2\pi}\left\{x\left[-\frac{\sin(n+1)x}{(n+1)} - \frac{\sin(n-1)x}{(n-1)}\right] - (1)\left[-\frac{\cos(n+1)x}{(n+1)^2} + \frac{\cos(n-1)x}{(n-1)^2}\right]\right\}_0^{2\pi} \\
=& -\frac{1}{2\pi}\left\{-(1)\left(-\frac{\cos 2(n+1)\pi}{(n+1)^2} + \frac{\cos 2(n-1)\pi}{(n-1)^2}\right) + (1)\left(-\frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right)\right\} \\
=& -\frac{1}{2\pi}\left[\frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right] = 0 \quad if n \neq 1.
\end{aligned}$$

(If $n = 1$, the value is not defined).
Put $n = 1$ in the formula of b_n

$$\begin{aligned}
b_1 =& \frac{1}{\pi}\int_0^{2\pi} x\sin x \sin x dx = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)dx \\
=& \frac{1}{2\pi}\left[(x)\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi} \\
=& \frac{1}{2\pi}\left[2\pi(2\pi - 0) - \left(\frac{4\pi^2}{2} + \frac{1}{4}\right) - \left(0 - \frac{1}{4}\right)\right] = \frac{1}{2\pi}(2\pi^2) = \pi
\end{aligned}$$

Therefore the Fourier series of the given function is:
$f(x) = -1 - \dfrac{1}{2}\cos x + \displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx + \pi \sin x$

Q. 3(a) : Using Milne-Thompson's method construct an analytic function $f(z) = u + iv$ in terms of z where $u + v = e^x(\cos y + \sin y) + \dfrac{x-y}{x^2+y^2}$
Solution: Given: The analytic function is $f(z) = u + iv$. Then
$if(z) = iu - v$
On adding we get, $f(z) + if(z) = u + iv + iu - v = (u - v) + i(u + v)$
Thus, $(1 + i)f(z) = (u - v) + i(u + v) = U + iV$
$\implies V = u + v$ is the imaginary part of $(1 + i)f(z)$
Now, $V = u + v = e^x(\cos y + \sin y) + \dfrac{x-y}{x^2+y^2}$
Differentiate V partially with respect to x & y, we get

$$\begin{aligned}
V_x = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} &= e^x(\cos y + \sin y) + \frac{x^2+y^2-(x-y)2x}{(x^2+y^2)^2} \\
\Rightarrow V_x &= e^x(\cos y + \sin y) + \frac{-x^2+y^2-2xy}{(x^2+y^2)^2} \\
\Rightarrow V_x(z,0) &= e^z - \frac{1}{z^2}
\end{aligned}$$

Similarly

$$\begin{aligned}
V_y = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} &= e^x(-\sin y + \cos y) + \frac{(-1)x^2+y^2-(x-y)2y}{(x^2+y^2)^2} \\
\Rightarrow V_y &= e^x(-\sin y + \cos y) + \frac{-x^2+y^2-2xy}{(x^2+y^2)^2} \\
\Rightarrow V_y(z,0) &= e^z - \frac{1}{z^2}
\end{aligned}$$

Now,

$$(1+i)f'(z) = V_y + iV_x$$

By Milne-Thompson method

$$\begin{aligned}
(1+i)f'(z) &= V_y(z,0) + iV_x(z,0) \\
(1+i)f'(z) &= e^z - \frac{1}{z^2} + i(e^z - \frac{1}{z^2}) = (1+i)(e^z - \frac{1}{z^2}) \\
\therefore f'(z) &= e^z - \frac{1}{z^2} \\
\text{Integrating w.r.t } z, &\quad \text{we get} \\
f(z) &= \int (e^z - \frac{1}{z^2})dz \\
i.e\ f(z) &= e^z + \frac{1}{z} + c
\end{aligned}$$

Q. 3(b) :Find the Inverse Laplace Transform of $\dfrac{(s+3)^2}{(s^2+6s+5)^2}$ by using convolution Theorem.

Solution:

$$\begin{aligned}
L^{-1}\left(\frac{(s+3)^2}{(s^2+6s+5)^2}\right) &= L^{-1}\left[\left(\frac{(s+3)}{(s^2+6s+5)}\right)^2\right] \\
&= L^{-1}\left[\left(\frac{(s+3)}{(s^2+6s+9-9+5)}\right)^2\right] \\
&= L^{-1}\left[\left(\frac{(s+3)}{(s+3)^2-4}\right)^2\right] \\
L^{-1}\left(\frac{(s+3)^2}{(s^2+6s+5)^2}\right) &= e^{-3t}L^{-1}\left[\left(\frac{s}{s^2-4}\right)^2\right] --------(1)
\end{aligned}$$

Now,we will find $L^{-1}\left[\left(\dfrac{s}{s^2-4}\right)^2\right]$

$L^{-1}\left[\left(\dfrac{s}{s^2-4}\right)^2\right] = L^{-1}\left(\dfrac{s}{s^2-4} \times \dfrac{s}{s^2-4}\right)$

Let $F(s) = G(s) = \dfrac{s}{s^2-4}$

$L^{-1}(F(s)) = L^{-1}(G(s)) = L^{-1}\left(\dfrac{s}{s^2-4}\right) = \cosh 2t = f(t) = g(t)$

By Convolution theorem

$$\begin{aligned}
L^{-1}(F(s)G(s)) &= \int_0^t f(u)g(t-u)du \\
&= \int_0^t \cosh 2u \cosh 2(t-u)du \\
&= \int_0^t \frac{1}{2}[\cosh(4t) + \cosh(4u-2t)]du \\
&= \frac{1}{2}\left[\int_0^t \cosh(4t)du + \int_0^t \cosh(4u-2t)du\right] \\
&= \frac{1}{2}\left[\cosh(4t)u + \frac{\sinh(4u-2t)}{4}\right]_0^t \\
&= \frac{1}{2}\left[\cosh(4t)t + \frac{\sinh(2t)}{4} - \left(0 + \frac{\sinh(-2t)}{4}\right)\right] \\
&= \frac{1}{2}\left[t\cosh(4t) + \frac{\sinh(2t)}{2}\right]
\end{aligned}$$

$\therefore\ \mathrm{L}^{-1}\left[\left(\frac{s}{s^2-4}\right)^2\right] = \frac{1}{2}\left[t\cosh(4t) + \frac{\sinh(2t)}{2}\right]$

From equation (1),

$\therefore\ \mathrm{L}^{-1}\left(\frac{(s+3)^2}{(s^2+6s+5)^2}\right) = \frac{e^{-3t}}{2}\left[t\cosh(4t) + \frac{\sinh(2t)}{2}\right]$

$\therefore\ \mathrm{L}^{-1}\left(\frac{(s+3)^2}{(s^2+6s+5)^2}\right) = \frac{e^{-3t}}{4}\left[2t\cosh(4t) + \sinh(2t)\right]$

Q. 3(c) : Fit a parabola $y = a + bx + cx^2$ to the following data and estimate y when $x = 10$:

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

Solution: We have the equation of least- squares parabola to be

$y = a + bx + cx^2$

The normal equations are given by

$$\begin{aligned}
\sum y &= na + b\sum x + c\sum x^2 \\
\sum xy &= a\sum x + b\sum x^2 + c\sum x^3 \\
\sum x^2y &= a\sum x^2 + b\sum x^3 + c\sum x^4
\end{aligned}$$

We have $n = 9 \sum x = 45, \sum x^2 = 285, \sum y = 74, \sum y^2 = 676, \sum xy = 421,$
$\sum x^3 = 2025, \sum x^2y = 2771, \sum x^4 = 15333,$

Which implies the normal equations are

$$\begin{aligned}
74 &= 9a + 45b + 285c \\
421 &= 45a + 285b + 2025c \\
2771 &= 285a + 2025b + 15333c
\end{aligned}$$

Solving the above system of equations we get,
$a = -0.869;\ b = 3.491;\ c = -0.264$
Hence the best fit parabola is $y = -0.869 + 3.491x - 0.264x^2$

Q. 4(a) :Find Laplace Transform of $e^{-(1/2)}tf(3t)$ if $L[f(t)] = \dfrac{1}{s\sqrt{s+1}}$

Solution:
Given $L[f(t)] = \frac{1}{s\sqrt{s+1}}$

$$\begin{aligned}
L[f(3t)] &= \frac{1}{3}\frac{1}{\frac{s}{3}\sqrt{\frac{s}{3}+1}} \text{.....by the change of scale property}\\
&= \frac{\sqrt{3}}{s\sqrt{s+3}}\\
L[t\ f(3t)] &= (-1)\frac{d}{ds}\frac{\sqrt{3}}{s\sqrt{s+3}} \text{.....by the multiplication by t property}\\
&= -\sqrt{3}\frac{d}{ds}\left[\frac{1}{s(s+3)^{1/2}}\right]\\
&= -\sqrt{3}\left[\frac{-1}{s^2(s+3)}\left(\frac{s}{2\sqrt{s+3}}+\sqrt{s+3}\right)\right]\\
&= \frac{\sqrt{3}}{2s(s+3)^{3/2}} + \frac{\sqrt{3}}{s^2\sqrt{s+3}}\\
\therefore L[e^{-(1/2)}tf(3t)] &= \frac{\sqrt{3}}{2(s+\frac{1}{2})((s+\frac{1}{2})+3)^{3/2}} + \frac{\sqrt{3}}{(s+\frac{1}{2})^2\sqrt{(s+\frac{1}{2})+3}} .....by the change of scale property\\
&= \frac{\sqrt{3}}{2(s+\frac{1}{2})((s+\frac{7}{2}))^{3/2}} + \frac{\sqrt{3}}{(s+\frac{1}{2})^2(s+\frac{7}{2})^{1/2}}
\end{aligned}$$

Q. 4(b) : Find the half range sine series for $f(x) = x - x^2, \quad 0 < x < 1.$
Hence deduce that $\dfrac{1}{1^3} - \dfrac{1}{3^3} + \dfrac{1}{5^3} - \dfrac{1}{7^3} + \cdots = \dfrac{\pi^3}{32}$
Solution: The half range sine series of $f(x)$ in $(0, l)$ is given by,
$f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$
where
$b_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

Here $(0, 1) = (0, l) \qquad \therefore l = 1$
Therefore, the half range sine series of $f(x)$ is given by,
$f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin n\pi x$

where

$$\begin{aligned}
b_n =& \frac{2}{1}\int_0^1 f(x)\sin n\pi x dx \\
=& 2\int_0^{\pi}(x-x^2)\sin n\pi x dx \\
=& 2\left[(x-x^2)\left(-\frac{1}{n\pi}\cos n\pi x\right)-(1-2x)\left(-\frac{1}{n^2\pi^2}\sin n\pi x\right)+(-2)\left(\frac{1}{n^3\pi^3}\cos n\pi x\right)\right]_0^1 \\
=& 2\left[\left\{0-0-\frac{2}{n^3\pi^3}\cos n\pi\right\}-\left\{0-0-\frac{2}{n^3\pi^3}\right\}\right] \\
=& \frac{4}{n^3\pi^3}(-\cos n\pi+1)
\end{aligned}$$

$$\therefore b_n = \begin{cases} \dfrac{8}{n^3\pi^3}, & \text{if n is odd} \\ 0, & \text{if n is even} \end{cases}$$

$$\therefore f(x) = x - x^2 = \frac{8}{\pi^3}\left[\frac{1}{1^3}\sin\pi x + \frac{1}{3^3}\sin 3\pi x + \frac{1}{5^3}\sin 5\pi x + \cdots\right]$$

For deduction put $x = \frac{1}{2}$

$$\therefore \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{8}{\pi^3}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right]$$

$$\therefore \frac{1}{4} = \frac{8}{\pi^3}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right]$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

Q. 4(c) : Given the regression lines $6y = 5x + 90, \quad 15x = 8y + 130$ and $\sigma_x^2 = 16$, find (i) $\bar{x}, \bar{y}$ (ii) r(iii) σ_y^2

Solution:

Since the point $(\bar{x}, \bar{y})$ lies on both the regression lines,
Solving the two given regression lines, we obtain
$\bar{x} = 30$ and $\bar{y} = 40$
Let us assume that $6y = 5x+90$ is the regression line of y on x and $15x = 8y+130$ is the regression line of x on y
Then

$$\begin{aligned}
6y &= 5x + 90 \\
\Rightarrow y &= \frac{5}{6}x + \frac{90}{6} \\
\Rightarrow \text{the regression coefficient } b_{yx} &= \frac{5}{6}
\end{aligned}$$

And

$$\begin{aligned}
15x &= 8y + 130 \\
\Rightarrow x &= \frac{8}{15}y + \frac{130}{15} \\
\Rightarrow \text{the regression coefficient } b_{xy} &= \frac{8}{15}
\end{aligned}$$

$\Rightarrow r^2 = b_{yx} \times b_{xy} = \frac{5}{6} \times \frac{8}{15} = \frac{4}{9} < 1,$ Therefore, our assumption is correct.
$\Rightarrow r = \sqrt{\frac{4}{9}} = \frac{2}{3}$ (since $b_{xy} > 0$ *and* $b_{yx} > 0$, we have $r > 0$)
Given:$\sigma_x^2 = 16 \Rightarrow \sigma_x = 4$
Now,

$$\begin{aligned} b_{xy} &= r\frac{\sigma_x}{\sigma_y} \\ i.e\ \frac{8}{15} &= (\frac{2}{3})\frac{4}{\sigma_y} \\ \Rightarrow \sigma_y &= 5 \Rightarrow \sigma_y^2 = 25 \end{aligned}$$

Q. 5(a) : Can the function $u = r + \frac{a^2}{r}\cos\theta$ be considered as real or imaginary part of an analytic function? If yes, find the corresponding analytic function.
Out of syllabus

Q. 5(b) : An unbiased coin is thrown 3 times. If X denotes the absolute difference between the number of heads and the number of tails, find the mgf of X and hence the first moment about the origin and the second moment about the mean.

Solution: The probability distribution of X is given by:

Outcome	{HHH, TTT}	{HHT,HTH,THH,TTH, THT, HTT}
X	0	1
P(X=x)	$\frac{2}{8} = \frac{1}{4}$	$\frac{6}{8} = \frac{3}{4}$

Now the MGF of X is

$$\begin{aligned} M_X(t) &= E(e^{tx}) \\ &= \sum_x e^{tx} p_x \\ &= e^{t(0)}\frac{1}{4} + e^{t(1)}\frac{3}{4} \\ i.e\ M_X(t) &= \frac{1}{4} + \frac{3}{4}e^t \quad \cdots\cdots (*) \end{aligned}$$

Now,the rth raw moment (about the origin) is given by

$\mu'_r = \dfrac{d^r}{dt^r} M_X(t)|_{(t=0)}$

$$\begin{aligned}
(*) \Rightarrow \frac{d}{dt} M_X(t) &= 0 + \frac{3}{4}e^t \\
\Rightarrow \frac{d}{dt} M_X(t)|_{(t=0)} &= \frac{3}{4} \quad \cdots\cdots (1) \\
and \quad \frac{d^2}{dt^2} M_X(t) &= \frac{3}{4}e^t \\
\Rightarrow \frac{d^2}{dt^2} M_X(t)|_{(t=0)} &= \frac{3}{4} \quad \cdots\cdots (2) \\
(1)\&(2) \Rightarrow \mu'_1 &= \frac{3}{4} \quad and \quad \mu'_2 = \frac{3}{4}
\end{aligned}$$

Now

$$\begin{aligned}
\mu_2 &= \mu'_2 - (\mu'_1)^2 \\
\Rightarrow \mu_2 &= \frac{3}{4} - (\frac{3}{4})^2 \\
i.e \quad \mu_2 &= \frac{12-9}{16} = \frac{3}{16}
\end{aligned}$$

Hence the first moment about the origin = Mean $= \mu'_1 = \dfrac{3}{4}$ and

the second moment about the mean = Variance $= \mu_2 = \dfrac{3}{16}$

Q. 5(c) : Evaluate $\displaystyle\int_0^\infty e^{-2t} \cosh t \int_0^t u^2 \sinh u \cosh u \; du \; dt$

Solution:

$$\begin{aligned}
\int_0^\infty e^{-2t} \cosh t \int_0^t u^2 \sinh u \cosh u \; du \; dt &= \int_0^\infty e^{-2t} \left(\frac{e^t + e^{-1}}{2}\right) \int_0^t u^2 \sinh u \cosh u \; du \; dt \\
&= \frac{1}{2}\left[\int_0^\infty e^{-t} \int_0^t u^2 \sinh u \cosh u \; du \; dt + \int_0^\infty e^{-3t} \int_0^t u^2 \sinh u \cosh u \right. \\
&= \frac{1}{2}\left\{L[\int_0^t u^2 \sinh u \cosh u \; du]|_{s=1} + L[\int_0^t u^2 \sinh u \cosh u \; du]|_{s=3}\right\}
\end{aligned}$$

Now

$$
\begin{aligned}
L[\sinh t \cosh t] &= L\left[\frac{\sinh 2t}{2}\right] \\
&= \frac{1}{2}\left(\frac{2}{s^2-4}\right) \\
L[t^2 \sinh t \cosh t] &= \frac{1}{2}(-1)^2 \frac{d^2}{ds^2}\left[\frac{2}{s^2-4}\right] \\
&= \frac{d}{ds}\left[\frac{-1}{(s^2-4)^2}(2s)\right] \\
&= -2\left[\frac{(s^2-4)^2(1)-(s)2(s^2-4)(2s)}{(s^2-4)^4}\right] \\
&= -2\left[\frac{s^2-4-4s^2}{(s^2-4)^3}\right] \\
&= -2\left[\frac{-3s^2-4}{(s^2-4)^3}\right] \\
L[\int_0^t t^2 \sinh t \cosh t] &= \frac{2}{s}\frac{3s^2+4}{(s^2-4)^3}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_0^\infty e^{-2t} \cosh t \int_0^t u^2 \sinh u \cosh u \, du \, dt &= \frac{1}{2}\left\{\left[\frac{2}{s}\frac{3s^2+4}{(s^2-4)^3}\right]_{s=1} + \left[\frac{2}{s}\frac{3s^2+4}{(s^2-4)^3}\right]_{s=3}\right\} \\
&= \left\{\left[\frac{3s^2+4}{s(s^2-4)^3}\right]_{s=1} + \left[\frac{3s^2+4}{s(s^2-4)^3}\right]_{s=3}\right\} \\
&= \left\{\left[\frac{7}{-27}\right] + \left[\frac{31}{375}\right]\right\}
\end{aligned}
$$

Q. 6(a) :Find the Inverse Laplace Transform of $\dfrac{1}{(s-2)^4(s+3)}$ by using method of partial fractions.

Solution: Let $\dfrac{1}{(s-2)^4(s+3)} = \dfrac{A}{s+3} + \dfrac{B}{s-2} + \dfrac{C}{(s-2)^2} + \dfrac{D}{(s-2)^3} + \dfrac{E}{(s-2)^4}$
$1 = A(s-2)^4 + B(s+3)(s-2)^3 + C(s+3)(s-2)^2 + D(s+3)(s-2) + E(s+3)$....(1)
put s=-3 in equation 1
$\therefore$1=(-5)$^4 A$, $\quad \therefore A = \dfrac{1}{625}$
put s=2 in equation 1,
$\therefore 1 = 5E$, $\quad \therefore$E= $\dfrac{1}{5}$
substituting value of A and E in equation 1, we get,

$$
\begin{aligned}
1 &= \frac{1}{625}(s-2)^4 + B(s+3)(s-2)^3 + C(s+3)(s-2)^2 \\
&\quad + D(s+3)(s-2) + \frac{1}{5}(s+3) \qquad ......(2)
\end{aligned}
$$

Differentiating Equation (2)

$$\begin{aligned} 0 &= \frac{1}{625}4(s-2)^3 + 3B(s+3)(s-2)^2 + B(s-2)^3 + 2C(s+3)(s-2) + \\ & C(s-2)^2 + D(s-2) + D(s+3) + \frac{1}{5} \qquad ......(3) \end{aligned}$$

put s=2, in equation (3) $\therefore \quad 0 = 5D + \frac{1}{5}, \quad \therefore D = \frac{-1}{25}$
substitute value of D in equation (3)

$$\begin{aligned} 0 &= \frac{1}{625}4(s-2)^3 + 3B(s+3)(s-2)^2 + B(s-2)^3 + 2C(s+3)(s-2) + \\ & C(s-2)^2 - \frac{1}{25}(s-2) - \frac{1}{25}(s+3) + \frac{1}{5} \qquad ......(4) \end{aligned}$$

Differentiating Equation (4), we get,

$$\begin{aligned} 0 &= \frac{1}{625}12(s-2)^2 + 6B(s+3)(s-2) + 3B(s-2)^2 + 3B(s-2)^2 + \\ & 2C(s+3) + 2C(s-2) + 2C(s-2) - \frac{1}{25} - \frac{1}{25} \qquad ......(5) \end{aligned}$$

put s=2, in equation (5), $\therefore 0 = 10C - \frac{2}{25}, \quad \therefore C = \frac{2}{250}$
substitute the value of C in equation (5)

$$\begin{aligned} 0 &= \frac{1}{625}12(s-2)^2 + 6B(s+3)(s-2) + 6B(s-2)^2 + \\ & \frac{4}{250}(s+3) + \frac{8}{250}(s-2) - \frac{2}{25} \qquad ......(6) \end{aligned}$$

Differentiating Equation (6)

$$0 = \frac{1}{625}24(s-2) + 6B(s+3) + 6B(s-2) + 12B(s-2) + \frac{12}{250} \qquad ......(7)$$

put s=2, in equation (7), $\therefore 0 = 30B + \frac{12}{250}, \quad \therefore B = -\frac{4}{2500}$
substitute values of A, B, C, D, E in equation (1)

$$\frac{1}{(s-2)^4(s+3)} = \frac{1}{625}\frac{1}{s+3} - \frac{4}{2500}\frac{1}{s-2} + \frac{2}{250}\frac{1}{(s-2)^2} - \frac{1}{25}\frac{1}{(s-2)^3} + \frac{1}{5}\frac{1}{(s-2)^4}$$

$$\begin{aligned} \therefore L^{-1}\left(\frac{1}{(s-2)^4(s+3)}\right) &= \frac{1}{625}L^{-1}\left(\frac{1}{s+3}\right) - \frac{4}{2500}L^{-1}\left(\frac{1}{s-2}\right) + \frac{2}{250}L^{-1}\left(\frac{1}{(s-2)^2}\right) \\ & -\frac{1}{25}L^{-1}\left(\frac{1}{(s-2)^3}\right) + \frac{1}{5}L^{-1}\left(\frac{1}{(s-2)^4}\right) \end{aligned}$$

$$\begin{aligned} L^{-1}\left(\frac{1}{(s-2)^4(s+3)}\right) &= \frac{1}{625}e^{-3t} - \frac{4}{2500}e^{2t} + \frac{2}{250}e^{2t}t \\ & -\frac{e^{2t}}{25}\frac{t^2}{2!} + \frac{e^{2t}}{5}\frac{t^3}{3!} \end{aligned}$$

$L^{-1}\left(\dfrac{1}{(s-2)^4(s+3)}\right) = \dfrac{1}{625}e^{-3t} - \dfrac{4}{2500}e^{2t} + \dfrac{2}{250}e^{2t}t - \dfrac{1}{50}e^{2t}t^2 + \dfrac{1}{30}e^{2t}t^3$

Q. 6(b) : If a continuous random variable X has the following probability density function, $f(x) = \begin{cases} ke^{-x/4}, x > 0 \\ 0, \ elsewhere \end{cases}$

Find k, mean and the variance.

Solution: Since $f(x)$ is a pdf, we have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\ dx &= 1 \\ \Rightarrow \int_0^{\infty} ke^{-x/4}\ dx &= 1 \\ \Rightarrow k\frac{e^{-x/4}}{-1/4}|_0^{\infty} &= 1 \\ \Rightarrow k(4) &= 1 \\ \Rightarrow k &= \frac{1}{4} \\ \Rightarrow\ pdf,\ f(x) &= \frac{1}{4}e^{-x/4}, x > 0 \end{aligned}$$

Now

$$\begin{aligned} Mean\ = E(X) &= \int_{-\infty}^{\infty} xf(x)\ dx \\ \Rightarrow E(X) &= \int_0^{\infty} x\frac{1}{4}e^{-x/4} \\ &= \frac{1}{4}(x\frac{e^{-x/4}}{-1/4} - (1)\frac{e^{-x/4}}{1/16})\ |_0^{\infty} \\ \Rightarrow E(X) &= 4 \end{aligned}$$

and

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\ dx \\ \Rightarrow E(X^2) &= \int_0^{\infty} x^2\frac{1}{4}e^{-x/4} \\ &= \frac{1}{4}(x^2\frac{e^{-x/4}}{-1/4} - (2x)\frac{e^{-x/4}}{1/16} + (2)\frac{e^{-x/4}}{1/64})\ |_0^{\infty} \\ \Rightarrow E(X^2) &= 32 \end{aligned}$$

Hence

$$\begin{aligned} Var(X) &= E(X^2) - (E(X))^2 \\ &= 32 - 16 \\ \Rightarrow\ Var(X) &= 16 \end{aligned}$$

Q. 6(c) : Find the half range cosine series for $f(x) = x, \quad 0 < x < 2$.

Hence deduce that i) $\dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \dfrac{1}{7^4} + \cdots = \dfrac{\pi^4}{96}$

ii) $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots=\frac{\pi^4}{90}$

Solution: The half range cosine series of $f(x)$ in $(0,l)$ is given by,
$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$
where
$a_0=\frac{2}{l}\int_0^l f(x)dx,\quad a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$

Here $(0,\pi)=(0,2)\qquad \therefore l=2$

Therefore the half range sine series of $f(x)$ is given by,
$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{2}$
where, $a_0=\frac{2}{2}\int_0^2 f(x)dx=\int_0^2 x\ dx=\left[\frac{x^2}{2}\right]_0^2=2$

$$\begin{aligned}
a_n =&\frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}dx\\
=&\int_0^2 x\cdot\cos\frac{n\pi x}{2}dx\\
=&\left[x\cdot\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}-(1)\frac{-\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2\\
=&\left\{\left[2\cdot\frac{\sin n\pi}{\frac{n\pi}{2}}+\frac{\cos n\pi}{\left(\frac{n\pi}{2}\right)^2}\right]-\left[0\cdot\frac{\sin 0}{\frac{n\pi}{2}}+\frac{\cos 0}{\left(\frac{n\pi}{2}\right)^2}\right]\right\}\\
=&\left\{\left[0+\frac{(-1)^n}{\left(\frac{n\pi}{2}\right)^2}\right]-\left[0+\frac{1}{\left(\frac{n\pi}{2}\right)^2}\right]\right\}\\
=&\frac{4}{n^2\pi^2}\left[(-1)^n-1\right]
\end{aligned}$$

$$\therefore a_n=\begin{cases} -\frac{8}{n^2\pi^2}, & \text{if n is odd}\\ 0, & \text{if n is even}\end{cases}$$

Therefore the half range cosine series of the given function is:
$f(x)=x=1-\frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\cdots\right]$

i) By Parseval's identity
$\frac{2}{l}\int_0^l[f(x)]^2dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}a_n^2$
$\frac{2}{l}\int_0^l[f(x)]^2dx=\frac{2}{2}\int_0^2 x^2dx=\left[\frac{x^3}{3}\right]_0^2=\frac{8}{3}$
$\therefore \frac{8}{3}=\frac{2^2}{2}+\frac{64}{\pi^4}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right]$

$\frac{8}{3} - 2 = \frac{64}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$
$\therefore \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots$

ii) Let

$$\begin{aligned} S =& \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \\ =& \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \cdots\right) \\ =& \frac{\pi^4}{96} + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right) = \frac{\pi^4}{96} + \frac{S}{16} \end{aligned}$$

$\therefore S = \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$