Comp KT May-June 2024

Time: 3 hours

Marks: 80

N.B. (1) Question No. 1 is compulsory.

(2) Answer any three questions from Q.2 to Q.6.

(3) Use of Statistical Tables permitted.

(4) Figures to the right indicate full marks

Q1 A If $f(t) = (\sqrt{t} + \frac{1}{\sqrt{t}})^2$, find $L[f(t)]$ and hence find $L\{e^{2t} f(t)\}$ 5

B Find $L^{-1}\{\frac{1}{s(s^2+4)}\}$ 5

C Obtain half-range cosine series for $f(x) = x(2-x)$ in $0<x<2$ 5

D Find moment generating function of the following distribution. Hence find mean and variance. 5

X	1	3	4	5
P(X)	0.4	0.1	0.2	0.3

Q2 A Find the orthogonal trajectories of the family of curves $e^{-x}[x\sin y - y\cos y] = c$ 6

B Find $L\{t(\frac{\cos t}{e^t})^2\}$ 6

C Find the Fourier series expansion for $f(x) = 2, \quad -2 < x < 0$. 8

$= 0, \quad 0 < x < 2$

Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Q3 A Find $L^{-1}\{\log(1 - \frac{1}{s^2})\}$ 6

B Find the analytic function $f(z) = u + iv$ where $u + v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$, using Milne-Thompson's Method. 6

C Fit a parabola $x = a + by + cy^2$ for the following data: 8

X :	1	2	3	4	5
Y :	10	12	15	14	15

Q4 A The first 4 moments of a distribution about origin of the random variable X are -1.5, 17, -30 and 108. Compute Mean, variance, μ_3 and μ_4. 6

B Consider the equations of regression lines 5x-y=22 and 64x-45y=24. Find $\bar{x}$, $\bar{y}$ and correlation coefficient r. 6

C Find $L^{-1}\{\frac{(s+3)^2}{(s^2+6s+13)^2}\}$ 8

Q5 A Find the Laplace transform of $\cos^3 t \cos 5t$. 6

B Find Spearman's rank correlation coefficient for the data below: 6

X :	32	55	49	60	43	37	43	49	10	20
Y :	40	30	70	20	30	50	72	60	45	25

C Obtain Fourier Series for $f(x) = \frac{1}{2}(\pi - x)$ in $(0, 2\pi)$.

Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ 8

Q6 A If f(x) is probability density function of a continuous random variable X, find k, mean and variance. 6

$$f(x) = \begin{cases} kx^2, & 0 \le x \le 1 \\ (2-x)^2, & 1 \le x \le 2 \end{cases}$$

B Check if there exists an analytic function whose real part is $u = \sin x + 3x^2 - y^2 + 5y + 4$. Justify your answer. 6

C Evaluate the following integral by using Laplace transforms 8

$$\int_0^\infty e^{-2t} \left[\int_0^t \left(\frac{e^{3u}\sin^2 2u}{u}\right) du\right] dt$$

The Bombay Salesian Society's
Don Bosco Institute Of Technology
ESE Solution of EM III(KT) - COMP/IT-June2024 (Q.P Code 55380)

Q1 A) If $f(t) = \left(\sqrt{t} + \dfrac{1}{\sqrt{t}}\right)^2$, find $L\{e^{2t} f(t)\}$
Solution:
INCORRECT QUESTION FULL MARKS AWARDED FOR ATTEMPT
Reason:

$$\begin{aligned} L(f(t)) &= L\left\{\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^2\right\} \\ &= L\left\{(\sqrt{t})^2 + 2 \times \sqrt{t} \times \frac{1}{\sqrt{t}} + \left(\frac{1}{\sqrt{t}}\right)^2\right\} \\ &= L\{t\} + L\{2\} + L\left\{\frac{1}{t}\right\} \end{aligned}$$

where $L\left\{\dfrac{1}{t}\right\}$ **is NOT define**

Q1 B) Find $L^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\}$
Solution:
We know that $L^{-1}\left\{\frac{1}{s^2+4}\right\} = \dfrac{1}{2}\sin 2t$

$$\begin{aligned} \therefore L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = L^{-1}\left\{\frac{1}{s}.\frac{1}{s^2+4}\right\} &= \int_0^t \frac{1}{2}\sin 2u \, du \\ &= \frac{1}{2}\left[-\frac{\cos 2u}{2}\right]_0^t \\ &= \frac{-1}{4}[\cos 2t - \cos 0] \\ &= \frac{1}{4}[1 - \cos 2t] \\ \therefore L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} &= \frac{1}{2}\sin^2 t \end{aligned}$$

Q1 C) Obtain half-range cosine series for $f(x) = x(2-x)$ in $0 < x < 2$
Solution: The half range cosine series of $f(x)$ in $(0, 2)$ is given by,
$f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$
where,

$$\begin{aligned} a_0 &= \frac{2}{l}\int_0^l f(x)dx \\ &= \frac{2}{2}\int_0^2 x(2-x)dx \\ &= \int_0^2 2x - x^2 dx \end{aligned}$$

$$\begin{aligned}\therefore a_0 &= \left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_0^2 \\ &= \left[2^2 - \frac{2^3}{3} - 0\right] \\ \therefore a_0 &= \frac{4}{3}\end{aligned}$$

$$\begin{aligned}a_n =& \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ =& \frac{2}{2}\int_0^2 (2x - x^2)\cos\left(\frac{n\pi x}{2}\right)dx \\ =& \left[(2x - x^2)\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right) - (2 - 2x)\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{2^2}}\right) + (-2)\left(\frac{-\sin\left(\frac{n\pi x}{2}\right)}{\frac{n^3\pi^3}{2^3}}\right)\right]_0^2 \\ =& \left[0 - \frac{2\times 2^2}{n^2\pi^2}\cos n\pi + 0 - \left(0 + \frac{2\times 2^2}{n^2\pi^2} + 0\right)\right] ......(As \ \ \sin 0 = 0, \quad \sin n\pi = 0, n \ is \ any \ integer) \\ =& \left[-\frac{8}{n^2\pi^2}(-1)^n - \frac{8}{n^2\pi^2}\right] \\ a_n =& -\frac{8}{n^2\pi^2}(1 + (-1)^n)\end{aligned}$$

Therefore Fourier series of the given function is:

$$f(x) = \frac{2}{3} + \sum_{n=1}^{\infty} -\frac{8}{n^2\pi^2}(1 + (-1)^n)\cos\frac{n\pi x}{2}$$

Q1 D) Find moment generating function of the following distribution.

hence find mean and variance.

X	1	3	4	5
P(X)	0.4	0.1	0.2	0.3

Solution: We have the probability distribution of X to be

X	1	3	4	5
P(X)	0.4	0.1	0.2	0.3

Note that X is a discrete r.v.

Hence the moment generating function of X is :

$$\begin{aligned}M_X(t) &= E(e^{tx}) \\ &= \sum_x e^{tx}p_x \\ &= e^{t(1)}0.4 + e^{t(3)}0.1 + e^{t(4)}0.2 + e^{t(5)}0.3 \\ i.e\ M_X(t) &= 0.4e^t + 0.1e^{3t} + 0.2e^{4t} + 0.3e^{5t}\end{aligned}$$

Now, $\quad E(X) = \mu_1', \qquad Var(X) \ = \mu_2' - (\mu_1')^2$

Where, $\mu'_r = \frac{d^r}{dt^r} M_X(t)|_{(t=0)}$, and μ'_r is called the rth raw moment (about the origin)

$$\begin{aligned}
Mean = \mu'_1 &= \frac{d}{dt} M_X(t)|_{(t=0)} \\
&= \frac{d}{dt}(0.4e^t + 0.1e^{3t} + 0.2e^{4t} + 0.3e^{5t})|_{(t=0)} \\
&= (0.4e^t + 0.1 \times 3e^{3t} + 0.2 \times 4e^{4t} + 0.3 \times 5e^{5t})|_{(t=0)} \\
&= (0.4 + 0.3 + 0.8 + 1.5) \\
Mean = \mu'_1 &= 3
\end{aligned}$$

Now

$$\begin{aligned}
\mu'_2 &= \frac{d^2}{dt^2} M_X(t)|_{(t=0)} \\
&= \frac{d^2}{dt^2}(0.4e^t + 0.1e^{3t} + 0.2e^{4t} + 0.3e^{5t})|_{(t=0)} \\
&= \frac{d}{dt}(0.4e^t + 0.3e^{3t} + 0.8e^{4t} + 1.5e^{5t})|_{(t=0)} \\
&= (0.4e^t + 0.3 \times 3e^{3t} + 0.8 \times 4e^{4t} + 1.5 \times 5e^{5t})|_{(t=0)} \\
&= (0.4 + 0.9 + 3.2 + 7.5) \\
\mu'_2 &= 12
\end{aligned}$$

$$\begin{aligned}
\therefore Var(X) &= \mu'_2 - (\mu'_1)^2 \\
\Rightarrow Var(X) &= 12 - (3)^2 \\
i.eVariance &= 3
\end{aligned}$$

Q2 A) Find the orthogonal trajectories of the family of curves $e^{-x}(x \sin y - y \cos y) = c$
Solution:
Let $u(x, y) = e^{-x}(x \sin y - y \cos y)$
Then the family of curves $v(x, y) = c_1$ will be required orthogonal trajectory if $f(z) = u + iv$ is analytic.
Assuming $f(z) = u + iv$ is analytic, we get,
$dv = -u_y dx + u_x dy$.....(by C-R equations)
Above differential equation is Exact..........(u being Harmonic function)
Hence solution is, $\int -u_y dx + \int u_x \ (terms\ free\ from\ x)\ dx = c$......(1)
Now,

$$\begin{aligned}
u_x &= e^{-x}(\sin y) - e^{-x}(x \sin y - y \cos y) \\
\therefore u_x &= e^{-x}(\sin y - x \sin y + y \cos y) \\
and\ u_y &= e^{-x}(x \cos y - (\cos y - y \sin y)) \\
\therefore u_y &= e^{-x}(x \cos y - \cos y + y \sin y)
\end{aligned}$$

From (1), $\int -(e^{-x}(x \cos y - \cos y + y \sin y))dx + \int 0dx = c$
$\therefore \int e^{-x}(\cos y - y \sin y)\ dx - e^{-x}x \cos y\ dx + 0 = c$
$\therefore (\cos y - y \sin y) \int e^{-x} dx - \cos y \int xe^{-x}\ dx = c$

$\therefore (\cos y - y\sin y)\dfrac{e^{-x}}{-1} - \cos y\left[x.\dfrac{e^{-x}}{-1} - 1.e^{-x}\right] = c$
$\therefore (y\sin y - \cos y)e^{-x} + e^{-x}\cos y\,(x+1) = c$
$\therefore e^{-x}(y\sin y - \cos y + xcosy + cosy) = c$
$\therefore e^{-x}(y\sin y + xcosy) = c$
$\therefore v(x,y) = e^{-x}(y\sin y + xcosy)$
Hence, $e^{-x}(y\sin y + xcosy) = c_1$ is the required orthogonal trajectory.

Q2 B) Find $L\left\{t\left(\dfrac{\cos t}{e^t}\right)^2\right\}$
Solution: We have,

$$\begin{aligned}
L\left\{t\left(\frac{\cos t}{e^t}\right)^2\right\} &= L\left\{\frac{t\cos^2 t}{e^{2t}}\right\}\\
&= L\left\{e^{-2t}t\cos^2 t\right\}\\
&= L\left\{e^{-2t}t\left(\frac{1+\cos 2t}{2}\right)\right\}\\
\therefore L\left\{t\left(\frac{\cos t}{e^t}\right)^2\right\} &= \frac{1}{2}L\left\{e^{-2t}t\,(1+\cos 2t)\right\} -------------(1)
\end{aligned}$$

Now consider,

$$\begin{aligned}
L\{1+\cos 2t\} &= L\{1\} + L\{\cos 2t\}\\
\therefore L\{1+\cos 2t\} &= \frac{1}{s} + \frac{s}{s^2+4} ----------(2)
\end{aligned}$$

By Multiplication by t property

$$\begin{aligned}
L\{t(1+\cos 2t)\} &= -\frac{d}{ds}L\{1+\cos 2t\}\\
&= -\frac{d}{ds}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)\\
&= -\left(\frac{-1}{s^2} + \frac{(1)(s^2+4) - s(2s)}{(s^2+4)^2}\right)\\
&= \left(\frac{1}{s^2} - \frac{-s^2+4}{(s^2+4)^2}\right)\\
\therefore L\{t(1+\cos 2t)\} &= \frac{1}{s^2} + \frac{s^2-4}{(s^2+4)^2}
\end{aligned}$$

By First Shifting property,

$$\begin{aligned}
L\left\{e^{-2t}t\,(1+\cos 2t)\right\} &= L\{t(1+\cos 2t)\}|_{s\to s+2}\\
&= \left(\frac{1}{s^2} + \frac{s^2-4}{(s^2+4)^2}\right)|_{s\to s+2}\\
&= \frac{1}{(s+2)^2} + \frac{(s+2)^2-4}{((s+2)^2+4)^2}\\
\therefore L\left\{e^{-2t}t\,(1+\cos 2t)\right\} &= \frac{1}{(s+2)^2} + \frac{s^2+4s}{(s^2+4s+8)^2}
\end{aligned}$$

$$\therefore L\left\{t\left(\frac{\cos t}{e^t}\right)^2\right\} \quad = \quad \frac{1}{2(s+2)^2}+\frac{s^2+4s}{2(s^2+4s+8)^2}----from(1)$$

Q2 C) Find Fourier series for $f(x)=\begin{cases}2 & -2\le x\le 0\\ 0 & 0\le x\le -2\end{cases}$

Hence deduce that $\dfrac{\pi^2}{8}=\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dots$

Solution: Fourier series of $f(x)$ in the interval $(c, c+2l)$ is given by

$f(x)=\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)$

where

$$a_0=\frac{1}{l}\int_c^{c+2l}f(x)dx$$
$$a_n=\frac{1}{l}\int_c^{c+2l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$
$$b_n=\frac{1}{l}\int_c^{c+2l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

here $(c, c+2l)=(-2,2) \quad \therefore c=-2\& \;\; 2l=4 \quad \Rightarrow l=2$

Therefore the Fourier series for the given function is

$f(x)=\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{2}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{2}\right)$

where

$$\begin{aligned}a_0=&\frac{1}{2}\int_{-2}^{2}f(x)\ dx\\ =&\frac{1}{2}\left(\int_{-2}^{0}2dx+\int_0^2 0\ dx\right)\\ =&\frac{1}{2}[2x]_{-2}^{0}+0\\ \therefore a_0&=2\end{aligned}$$

$$\begin{aligned}a_n=&\frac{1}{2}\int_{-2}^{2}f(x)\cos\left(\frac{n\pi x}{2}\right)\ dx\\ =&\frac{1}{2}\left(\int_{-2}^{0}2\cos\left(\frac{n\pi x}{2}\right)dx+\int_0^2 0\cos\left(\frac{n\pi x}{2}\right)\ dx\right)\\ =&\frac{1}{2}\left\{2\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_{-2}^{0}+0\right\}\\ =&\frac{1}{2}\left\{\frac{4}{n\pi}[\sin 0+\sin n\pi]\right\}\\ \therefore a_n&=0....(\because \sin n\pi=\sin n0=0)\end{aligned}$$

$$\begin{aligned}
b_n =& \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx \\
=& \frac{1}{2}\left(\int_{-2}^{0} 2\sin\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{2} 0\sin\left(\frac{n\pi x}{2}\right)dx\right) \\
=& \frac{1}{2}\left\{2\left[\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_{-2}^{0} + 0\right\} \\
=& \frac{1}{2}\left\{\frac{-4}{n\pi}[\cos 0 - \cos n\pi]\right\} \\
b_n =& \frac{2}{n\pi}[(-1)^n - 1]
\end{aligned}$$

Therefore the Fourier series is
$f(x) = 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi}[(-1)^n - 1]\sin\left(\frac{n\pi x}{2}\right)$
To deduce, $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$, we use Parseval's identity, $\frac{1}{l}\int_{c}^{c+2l}[f(x)]^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}[a_n^2 + b_n^2]$
L.H.S.,

$$\begin{aligned}
\frac{1}{l}\int_{c}^{c+2l}[f(x)]^2dx &= \frac{1}{2}\int_{-2}^{2}[f(x)]^2dx \\
&= \frac{1}{2}\int_{-2}^{0} 4\ dx \\
&= 2[x]_{-2}^{0} \\
\therefore \frac{1}{l}\int_{c}^{c+2l}[f(x)]^2dx &= 4
\end{aligned}$$

R.H.S.,

$$\begin{aligned}
\frac{a_0^2}{2} + \sum_{n=1}^{\infty}[a_n^2 + b_n^2] &= \frac{2^2}{2} + \sum_{n=1}^{\infty}\left[0^2 + \left(\frac{2}{n\pi}[(-1)^n - 1]\right)^2\right] \\
&= 2 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]^2
\end{aligned}$$

Equating L.H.S & R.H.S,

$$\begin{aligned}
4 &= 2 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]^2 \\
\therefore 4 - 2 &= \frac{4}{\pi^2}\left(\frac{4}{1^2} + \frac{4}{3^2} + \frac{4}{5^2} + ...\right) \\
2.\frac{\pi^2}{4} &= 4\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right) \\
\frac{\pi^2}{8} &= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...
\end{aligned}$$

Q3 A) Find $L^{-1}\left\{\log\left(1-\frac{1}{s^2}\right)\right\}$

Solution: We have ,

$$\begin{aligned}
L^{-1}\left\{\log\left(1-\frac{1}{s^2}\right)\right\} &= L^{-1}\left\{\log\left(\frac{s^2-1}{s^2}\right)\right\} \\
&= L^{-1}\left\{\log\left(s^2-1\right)-\log\left(s^2\right)\right\} \\
\therefore L^{-1}\left\{\log\left(1-\frac{1}{s^2}\right)\right\} &= L^{-1}\left\{\log\left(s^2-1\right)-2\log\left(s\right)\right\} - - - - - (1)
\end{aligned}$$

We know that, $\quad L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{\frac{d}{ds}F(s)\}$

$$\begin{aligned}
\therefore L^{-1}\left\{\log\left(s^2-1\right)-2\log\left(s\right)\right\} &= -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}(\log\left(s^2-1\right)-2\log\left(s\right))\right\} \\
&= -\frac{1}{t}L^{-1}\left\{\frac{1}{s^2-1}\times 2s-\frac{2}{s}\right\} \\
&= -\frac{2}{t}L^{-1}\left\{\frac{s}{s^2-1}-\frac{1}{s}\right\} \\
&= -\frac{2}{t}[\cosh t-1] \\
\therefore L^{-1}\left\{\log\left(s^2-1\right)-2\log\left(s\right)\right\} &= \frac{2}{t}[1-\cosh t] \\
\therefore L^{-1}\left\{\log\left(1-\frac{1}{s^2}\right)\right\} &= \frac{2}{t}[1-\cosh t] - - - - - from(1)
\end{aligned}$$

Q3 B) Find the analytic function $f(z) = u + iv$ where $u + v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$, using Milne-Thompson's Method.

Solution: Let the given analytic function be $f(z) = u + iv$ then $if(z) = iu - v$

On adding we get,

$$\begin{aligned}
f(z) + if(z) &= u + iv + iu - v \\
\therefore f(z)(1+i) &= (u-v) + i(u+v) \\
\therefore F(z) &= U + iV
\end{aligned}$$

$$where, F(z) = f(z)(1+i), \quad U = u - v, \quad and \quad V = u + v$$

$\implies$ $V = u + v$ is a imaginary part of $(1+i)f(z)$ and given $V = u + v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

Step 1: Differentiate u partially with respect to x & y, we get

$V_x = \frac{(\cosh 2y - \cos 2x)2\cos 2x - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$

$= \frac{2\cos 2x\cosh 2y - 2\cos^2 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$

$\therefore V_x = \frac{2\cos 2x\cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$

$V_y = \frac{\sin 2x(-2\sinh 2y)}{(\cosh 2y - \cos 2x)^2} = \frac{-2\sin 2x\sinh 2y}{(\cosh 2y - \cos 2x)^2}$

$\therefore V_y = \frac{-2\sin 2x\sinh 2y}{(\cosh 2y - \cos 2x)^2}$

Step 2: We have $V_x(z,0) = \dfrac{2\cos 2z - 2}{(1-\cos 2z)^2}$ and $V_y(z,0) = 0$

Step 3:We have $F(z) = U + iV \implies F'(z) = U_x + iV_x = V_y + iV_x$
($\because$C-R equations $u_x = v_y$)
By Milne-Thompson method, $F'(z) = V_y(z,0) + iV_x(z,0)$
$\therefore F'(z) = (0) + i\dfrac{2\cos 2z - 2}{(1-\cos 2z)^2}$
$\therefore F'(z) = i\dfrac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$
$\therefore F'(z) = i\dfrac{-2}{1-\cos 2z} = i\dfrac{-2}{2\sin^2 z}$
$\therefore F'(z) = -icosec^2 z$

Step 4:Integrating w.r.t z, we get
$F(z) = \displaystyle\int -icosec^2 z dz = i\cot z + c$
$\therefore (1+i)f(z) = i\cot z + c$
$\therefore f(z) = \dfrac{i}{(1+i)}\cot z + \dfrac{c}{1+i}$
$\therefore f(z) = \dfrac{i}{(1+i)}\cot z + c'$ ....where $c' = \dfrac{c}{1+i}$
which is the required analytic function

Q3 C) Fit a parabola $x = a + by + cy^2$ for the following data:

X:	1	2	3	4	5
Y:	10	12	15	14	15

Solution:
Here X is the dependent variable.
Let the equation of least- squares second degree parabola be
$x = a + by + cy^2$
The normal equations are given by

$$\begin{aligned}
\sum x &= na + b\sum y + c\sum y^2 \cdots\cdots (1)\\
\sum xy &= a\sum y + b\sum y^2 + c\sum y^3 \cdots\cdots (2)\\
\sum xy^2 &= a\sum y^2 + b\sum y^3 + c\sum y^4 \cdots\cdots (3)
\end{aligned}$$

Now
$n = 5;\quad \sum x = 15;\quad \sum y = 66;\quad \sum y^2 = 890;\quad \sum xy = 210$
$\sum xy^2 = 2972;\quad \sum y^3 = 12222;\quad \sum y^4 = 170402$
Substituting these values in equations (1) (2) and (3) we get

$$\begin{aligned}
15 &= 5a + 66b + 890c\\
210 &= 66a + 890b + 12222c\\
2972 &= 890a + 12222b + 170402c
\end{aligned}$$

Solving the above two equations we get $a = -8.1748;\quad b = 1.0860;\quad c = -0.0177$
Hence the required second degree parabola of best fit is
$y = -8.1748 + 1.0860y - 0.0177y^2$

Q4 A) The first 4 moments of a distribution about origin of the random variable X are -1.5, 17, -30 and 108. Compute mean , variance, μ_3 and $\mu_4 = -1.5$,.
Solution: Given $\mu_1' = -1.5,\ \mu_2' = 17,\ \mu_3' = -30,\ \mu_4' = 108,$
We know that, Mean $= \mu_1' = -1.5$

$$\begin{aligned} And\ Variance &= \mu_2' - (\mu_1')^2 \\ &= 17 - (-1.5)^2 \\ \therefore Variance &= 14.75 \end{aligned}$$

$$As\ we\ know,\quad \mu_r = \mu_r' - {}^rC_1\mu_{r-1}'\mu_1' + {}^rC_2\mu_{r-2}'(\mu_1')^2 - {}^rC_3\mu_{r-3}'(\mu_1')^3 + \cdots\cdots + (-1)^r(\mu_1')^r$$

$$\begin{aligned} Now,\qquad \mu_3 &= \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3 \\ &= -30 - (3 \times -1.5 \times 17) + 2 \times (-1.5)^3 \\ \therefore \mu_3 &= 39.75 \end{aligned}$$

$$\begin{aligned} And\qquad \mu_4 &= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 \\ &= 108 - (4 \times -30 \times -1.5) + 6 \times 17 \times (-1.5)^2 - 3 \times (-1.5)^4 \\ \therefore \mu_3 &= 142.3125 \end{aligned}$$

Q4 B) Consider the equations of regression lines $5x - y = 22$ and $64x - 45y = 24$. Find $\bar{x}, \bar{y}$ and correlation coefficient r.
Solution:
Since the point $(\bar{x}, \bar{y})$ lies on both the regression lines,
Solving the two given regression lines, we obtain
$\bar{x} = 6$ and $\bar{y} = 8$
Let us assume that $5x - y = 22$ is the regression line of x on y and
$64x - 45y = 24$ is the regression line of y on x
Then

$$\begin{aligned} 5x - y &= 22 \\ \Rightarrow x &= \frac{1}{5}y + \frac{22}{5} \\ \Rightarrow \text{the regression coefficient } b_{xy} &= \frac{1}{5} \end{aligned}$$

And

$$\begin{aligned} 64x - 45y &= 24 \\ \Rightarrow y &= \frac{64}{45}x - \frac{24}{45} \\ \Rightarrow \text{the regression coefficient } b_{yx} &= \frac{64}{45} \end{aligned}$$

$$\Rightarrow r^2 = b_{yx} \times b_{xy} = \frac{1}{5} \times \frac{64}{45} = \frac{64}{225} = 0.2844 < 1,$$

$$\Rightarrow r = 0.533\ or = \frac{8}{15}\ (\text{since } b_{xy} > 0\ and\ b_{yx} > 0,\ \text{ we have } r > 0)$$

Q4 C) Find $L^{-1}\left\{\dfrac{(s+3)^2}{(s^2+6s+13)^2}\right\}$

Solution: We have,

$$\begin{aligned}
L^{-1}\left\{\frac{(s+3)^2}{(s^2+6s+13)^2}\right\} &= L^{-1}\left\{\frac{(s+3)^2}{((s^2+6s+9)+4)^2}\right\}\\
&= L^{-1}\left\{\frac{(s+3)^2}{((s+3)^2+4)^2}\right\}\\
\therefore L^{-1}\left\{\frac{(s+3)^2}{(s^2+6s+13)^2}\right\} &= e^{-3t}L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}----(1) \quad (By\ First\ Shifting\ Property\ of\ ILT)
\end{aligned}$$

Now, Let $\dfrac{s^2}{(s^2+4)^2}=\dfrac{s}{s^2+4}.\dfrac{s}{s^2+4}=F(s).G(s)$, where

$F(s)=\dfrac{s}{s^2+4}$ and $G(s)=\dfrac{s}{s^2+4}$

$\therefore L^{-1}\{F(s)\}=L^{-1}\left\{\dfrac{s}{s^2+4}\right\}=\cos 2t=f(t)$ and

$\therefore L^{-1}\{G(s)\}=L^{-1}\left\{\dfrac{s}{s^2+4}\right\}=\cos 2t=g(t)$

$f(t)=\cos 2t \implies f(u)=\cos 2u$ and $g(t)=\cos 2t \implies g(t-u)=\cos(2(t-u))$

By Convolution theorem, $L^{-1}\{F(s).G(s)\}=\int_0^t f(u)g(t-u)du$

$$\begin{aligned}
\therefore L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} &= \int_0^t \cos 2u.\cos(2(t-u))du\\
&= \int_0^t \frac{1}{2}[\cos(2u+2(t-u))+\cos(2u-2(t-u))]du\\
&\qquad [\because \cos A\cos B=\frac{1}{2}[\cos(A+B)+\cos(A-B)]\\
&= \frac{1}{2}\int_0^t[\cos(2t)+\cos(4u-2t)]du\\
&= \frac{1}{2}\left\{u\cos 2t+\frac{\sin(4u-2t)}{4}\right\}_0^t\\
&= \frac{1}{2}\left\{\left[t\cos 2t+\frac{1}{4}\sin 2t\right]-\left[0+\frac{1}{4}\sin(-2t)\right]\right\}\\
&= \frac{1}{2}\left\{t\cos 2t+\frac{1}{4}\sin 2t+\frac{1}{4}\sin(2t)\right\}\\
\therefore L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} &= \frac{1}{2}\left\{t\cos 2t+\frac{1}{2}\sin 2t\right\}\\
\therefore L^{-1}\left\{\frac{(s+3)^2}{(s^2+6s+13)^2}\right\} &= \frac{e^{-3t}}{2}\left\{t\cos 2t+\frac{1}{2}\sin 2t\right\}-----from(1)
\end{aligned}$$

Q5 A) Find the Laplace transform of $\cos^3 t \cos 5t$
Solution: We have,

$$\begin{aligned}
L\{\cos^3 t \cos 5t\} &= L\left\{\left(\frac{3\cos t + \cos 3t}{4}\right)\cos(5t)\right\} \\
&= \frac{1}{4}L\left\{(3\cos t + \cos 3t)\cos(5t)\right\} \\
&= \frac{1}{4}\left(3L\left\{\cos(5t)\cos t\right\} + L\left\{\cos(5t)\cos 3t\right\}\right) \\
&= \frac{1}{4}\left(\frac{3}{2}L\left\{\cos(6t)+\cos 4t\right\} + \frac{1}{2}L\left\{\cos(8t)+\cos 2t\right\}\right) \\
&\qquad [\because \cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]] \\
&= \frac{1}{8}\left(3\left\{\frac{s}{s^2+36}+\frac{s}{s^2+16}\right\}+\left\{\frac{s}{s^2+64}+\frac{s}{s^2+4}\right\}\right) \\
L\{\cos^3 t \cos 5t\} &= \frac{1}{8}\left(\frac{3s}{s^2+36}+\frac{3s}{s^2+16}+\frac{s}{s^2+64}+\frac{s}{s^2+4}\right)
\end{aligned}$$

Q5 B) Find Spearman's rank correlation coefficient for the data below:

X:	32	55	49	60	43	37	43	49	10	20
Y:	40	30	70	20	30	50	72	60	45	25

Solution: We have the Spearman's rank correlation coefficient to be :
(Since values (and hence the ranks) are repeated)

$$\rho = 1 - \frac{6[\sum d^2 + \sum correction\ factors]}{n(n^2-1)}$$

where if the rank k repeats m times, then the correction factor is $\dfrac{m(m^2-1)}{12}$

We get the following table:

X	Y	Rank in X r_x	Rank in Y r_y	$d = r_x - r_y$	d^2
32	40	3	5	-2	4
55	30	9	3.5	5.5	30.25
49	70	7.5	9	-1.5	2.25
60	20	10	1	9	81
43	30	5.5	3.5	2	4
37	50	4	7	-3	9
43	72	5.5	10	-4.5	20.25
49	60	7.5	8	-0.5	0.25
10	45	1	6	-5	25
20	25	2	2	0	0
					$\sum d^2 = 176$

Correction factors

Series	Repeating rank k	No. of times repeated m	correction factor $\frac{m(m^2-1)}{12}$
X	5.5	2	0.5
X	7.5	2	0.5
Y	3.5	2	0.5
			$\sum correction factors = 1.5$

Therefore the Spearman's rank correlation coefficient is

$$\begin{aligned}\rho(=R) &= 1-\left(\frac{6[\sum d^2+\sum\ correction\ factors]}{n(n^2-1)}\right)\\ &= 1-\frac{6[176+1.5]}{10\times 99}\\ \Rightarrow \rho &= 0.07575\end{aligned}$$

Q5 C) Obtain the Fourier series for $f(x)=\frac{1}{2}(\pi-x)$ in $(0,2\pi)$

Hence, deduce that, $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+......$

Solution: Fourier series of $f(x)$ in the interval $(c,c+2l)$ is given by

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$

where

$$a_0=\frac{1}{l}\int_c^{c+2l}f(x)dx$$

$$a_n=\frac{1}{l}\int_c^{c+2l}f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n=\frac{1}{l}\int_c^{c+2l}f(x)\sin\frac{n\pi x}{l}dx$$

here $(c,c+2l)=(0,2\pi)\therefore c=0\&2l=2\pi$

Therefore the Fourier series for the given function is

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

where

$$\begin{aligned}
a_0 =& \frac{1}{\pi}\int_0^{2\pi} f(x)dx \\
=& \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)dx \\
=& \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)dx \\
=& \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi} \\
=& \frac{1}{2\pi}[(2\pi^2 - 2\pi^2) - (0 - 0)] \\
\therefore \mathbf{a_0} =& \mathbf{0}
\end{aligned}$$

$$\begin{aligned}
a_n =& \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \\
=& \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\cos nx dx \\
=& \frac{1}{2\pi}[(\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)]_0^{2\pi} \\
=& \frac{1}{2\pi}\left[\left(-\pi\frac{\sin 2n\pi}{n} - \frac{\cos 2n\pi}{n^2}\right) - \left(\pi\frac{\sin 0}{n} - \frac{\cos 0}{n^2}\right)\right] \\
=& \frac{1}{2\pi}\left[\left(-\pi\frac{0}{n} - \frac{1}{n^2}\right) - \left(\pi\frac{0}{n} - \frac{1}{n^2}\right)\right] ... \because \sin n\pi = 0 \& \cos 2n\pi = (-1)^{2n} \\
=& \frac{1}{2\pi}\left[-\frac{1}{n^2} + \frac{1}{n^2}\right] \\
\mathbf{a_n} =& \mathbf{0}
\end{aligned}$$

$$\begin{aligned}
b_n =& \frac{1}{\pi}\int_0^{2\pi} f(x) sinnx dx \\
=& \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx dx \\
=& \frac{1}{2\pi}[(\pi - x)\frac{-\cos nx}{n} - (-1)\left(\frac{-\sin nx}{n^2}\right)]_0^{2\pi} \\
=& \frac{1}{2\pi}\left[\left(-\pi\frac{-\cos 2n\pi}{n} - \frac{-\sin 2n\pi}{n^2}\right) - \left(\pi\frac{-\cos 0}{n} - \frac{\sin 0}{n^2}\right)\right] \\
=& \frac{1}{2\pi}\left[\left(\frac{\pi}{n} - \frac{1}{n^2}\right) - \left(\pi\frac{0}{n} + \frac{\pi}{n}\right)\right] ... \because \sin 2n\pi = 0 \& \cos 2n\pi = (-1)^{2n} \\
=& \frac{1}{2\pi}\left[2\frac{\pi}{n}\right] \\
\mathbf{b_n} =& \mathbf{\frac{1}{n}}
\end{aligned}$$

$\therefore$ *Fourier Series is* $f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$

To deduce, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$, substitute $x = \frac{\pi}{2}$ in obtained Fourier Series

$$\therefore f(\frac{\pi}{2}) = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(n\frac{\pi}{2}\right)$$

$$\frac{\left(\pi - \frac{\pi}{2}\right)}{2} = \frac{1}{1}\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\sin\left(2\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(3\frac{\pi}{2}\right) + \frac{1}{4}\sin\left(4\frac{\pi}{2}\right) + \frac{1}{5}\sin\left(5\frac{\pi}{2}\right)$$
$$+ \frac{1}{6}\sin\left(6\frac{\pi}{2}\right) + \frac{1}{7}\sin\left(7\frac{\pi}{2}\right) + \frac{1}{8}\sin\left(8\frac{\pi}{2}\right) + \frac{1}{9}\sin\left(9\frac{\pi}{2}\right) + ...$$

$$\therefore\ \frac{\pi}{4} = \frac{1}{1}(1) + \frac{1}{2}(0) + \frac{1}{3}(-1) + \frac{1}{4}(0) + \frac{1}{5}(1) + \frac{1}{6}(0) + \frac{1}{7}(-1) + \frac{1}{8}(0) + \frac{1}{9}(1) + ...$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

Q6 A) If $f(x)$ is probability density function of a continuous random variable X. find k, mean, variance. $f(x) = \begin{cases} kx^2 & 0 \le x \le 1 \\ (2-x)^2 & 1 \le x \le 2 \end{cases}$

Solution: Since $f(x)$ is a pdf, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\ dx &= 1 \\
\Rightarrow \int_{-\infty}^{0} 0\ dx + \int_{0}^{1} kx^2\ dx + \int_{1}^{2} (2-x)^2\ dx + \int_{2}^{\infty} 0\ dx &= 1 \\
\Rightarrow 0 + k\left(\frac{x^3}{3}\right)_0^1 + \left(\frac{(2-x)^3}{-3}\right)_1^2 + 0 &= 1 \\
\Rightarrow \frac{k}{3} - \frac{1}{3}(0-1) &= 1 \\
\Rightarrow \frac{k}{3} + \frac{1}{3} &= 1
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{k}{3} &= \frac{2}{3} \\
\Rightarrow k &= 2
\end{aligned}$$

$$\Rightarrow\ pdf,\ f(x) = \begin{cases} 2x^2 & 0 \le x \le 1 \\ (2-x)^2 & 1 \le x \le 2 \end{cases}$$

Now

$$\begin{aligned}
Mean\ = E(X) &= \int_{-\infty}^{\infty} xf(x)\ dx \\
\Rightarrow E(X) &= \int_0^1 2x^3\ dx + \int_1^2 (2-x)^2 x\ dx \\
&= 2\left(\frac{x^4}{4}\right)_0^1 + \int_1^2 (4x - 4x^2 + x^3)\ dx \\
&= \frac{1}{2} + \frac{4}{2}[x^2]_1^2 - \frac{4}{3}[x^3]_1^2 + \frac{1}{4}[x^4]_1^2 \\
&= \frac{1}{2} + 6 - \frac{28}{3} + \frac{15}{4} \\
\Rightarrow E(X) &= \frac{11}{12}
\end{aligned}$$

and

$$\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\ dx \\
\Rightarrow E(X^2) &= \int_0^1 2x^4\ dx + \int_1^2 (2-x)^2 x^2\ dx \\
&= 2\left(\frac{x^5}{5}\right)_0^1 + \int_1^2 (4x^2 - 4x^3 + x^4)\ dx \\
&= \frac{2}{5} + \frac{4}{3}[x^3]_1^2 - \frac{4}{4}[x^4]_1^2 + \frac{1}{5}[x^5]_1^2 \\
&= \frac{2}{5} + \frac{28}{3} - 15 + \frac{31}{5} \\
\Rightarrow E(X^2) &= \frac{14}{15}
\end{aligned}$$

Hence

$$\begin{aligned}
Var(X) &= E(X^2) - (E(X))^2 \\
&= \frac{14}{15} - \left(\frac{11}{12}\right)^2 \\
\Rightarrow Var(X) &= 0.093
\end{aligned}$$

Q6 B) Check if there exists an analytic function whose real part is $u = \sin x + 3x^2 - y^2 + 5y + 4$. Justify your answer.
Solution: Let $u(x,y) = 3x^2 + \sin x - y^2 + 5y + 4$
If u has to be real part of an analytic function, then u should be a harmonic function
i.e. u should satisfy Laplace equation
$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$
Differentiate u partially with respect to x and y, we get
$u_x = 6x + \cos x \implies u_{xx} = 6 - \sin x$
$u_y = -2y + 5 \implies u_{yy} = -2$
$u_{xx} + u_{yy} = 6 - \sin x - 2 = 4 - \sin x \neq 0$
Thus u is not harmonic and cannot be a real part of any analytic function.
$\therefore$ There does not exist an analytic function whose real part is $3x^2 + \sin x - y^2 + 5y + 4$

Q6 C) Evaluate the following integral by using Laplace transforms.
$\displaystyle\int_0^\infty e^{-2t}\left(\int_0^t \left(\frac{e^{3u}\sin^2 2u}{u}\right) du\right) dt$
Solution:
By Definition of Laplace Transform,
$\displaystyle\int_0^\infty e^{-2t}\left(\int_0^t \left(\frac{e^{3u}\sin^2 2u}{u}\right) du\right) dt = L\left\{\int_0^t \left(\frac{e^{3u}\sin^2 2u}{u}\right) du\right\}_{s=2}$ ——-(1)

Consider,

$$\begin{aligned}
L\{\sin^2 2u\} &= L\left\{\frac{1-\cos 4u}{2}\right\} \\
&= \frac{1}{2}L\{1-\cos 4u\} \\
&= \frac{1}{2}\left(L\{1\}-L\{\cos 4u\}\right) \\
\therefore L\{\sin^2 2u\} &= \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+16}\right)
\end{aligned}$$

By Division by t property,

$$\begin{aligned}
L\{\frac{sin^2 2u}{u}\} &= \int_0^\infty L\{\sin^2 2u\}ds \\
&= \int_0^\infty \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+16}\right)ds \\
&= \frac{1}{4}\int_0^\infty \left(\frac{2}{s}-\frac{2s}{s^2+16}\right)ds \\
&= \frac{1}{4}\Big(2\log s-\log\left(s^2+16\right)\Big)_0^\infty \\
&= \frac{1}{4}\log\left(\frac{s^2}{s^2+16}\right)_0^\infty \\
&= \frac{1}{4}\left[\log(1)-\log\left(\frac{s^2}{s^2+16}\right)\right] \\
&= \frac{1}{4}\log\left(\frac{s^2+16}{s^2}\right)
\end{aligned}$$

By First Shifting Property,

$$\begin{aligned}
L\{e^{3u}\frac{sin^2 2u}{u}\} &= \frac{1}{4}\log\left(\frac{s^2+16}{s^2}\right)_{s\to(s-3)} \\
&= \frac{1}{4}\log\left(\frac{(s-3)^2+16}{(s-3)^2}\right)
\end{aligned}$$

By Laplace of Integrals,

$$\begin{aligned}
L\left\{\int_0^t\left(\frac{e^{3u}\sin^2 2u}{u}\right)du\right\} &= \frac{1}{s}L\{e^{3u}\frac{sin^2 2u}{u}\} \\
&= \frac{1}{4s}\log\left(\frac{(s-3)^2+16}{(s-3)^2}\right) \\
\therefore\ from\ (1), & \\
\int_0^\infty e^{-2t}\left(\int_0^t\left(\frac{e^{3u}\sin^2 2u}{u}\right)du\right)dt &= \left[\frac{1}{4s}\log\left(\frac{(s-3)^2+16}{(s-3)^2}\right)\right]_{s=2} \\
&= \left[\frac{1}{8}\log\left(\frac{(2-3)^2+16}{(2-3)^2}\right)\right] \\
&= \frac{1}{8}\log 17
\end{aligned}$$