Engineering Maths IV

Nov-Dec 2023

(COITAI)

Time (3 hours) **Max Marks: 80**

Note: (1) Question No. 1 is Compulsory

(2) Answer any three questions from Q.2 to Q.6

(3) Figures to the right indicate full marks

1. (a) If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ find the sum and product of Eigen values of A (5)

Solution:

$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}, |A| = 0$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 8-\lambda & -6 & 4 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [8+7+3]\lambda^2 + \left[\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}\right]\lambda - 0 = 0$

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda - 15)(\lambda - 3) = 0$

$\lambda = 0,3,15$

Sum of eigen values $= 18$

Product of eigen values $= 0$

1. (b) Integrate the function $f(z) = z^2$ from $A(0,0)$ to $B(1,1)$ along straight line AB (5)

Solution:

$I = \int z^2 dz$

$I = \int (x+iy)^2(dx+idy)$

Along the line AB,

$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$\frac{y-0}{1-0} = \frac{x-0}{1-0}$

$y = x$

$dy = dx$

The integral becomes,

$I = \int_0^1 (x+ix)^2(dx+idx)$

$I = \int_0^1 x^2(1+i)^2(1+i)dx$

$I = (1+i)^3\left[\frac{x^3}{3}\right]_0^1$

$I = (-2+2i)\left[\frac{1}{3}\right]$

$\boxed{I = \frac{-2+2i}{3}}$

1. (c) Find the Z transform of $f(k) = a^k, k < 0$ (5)

Solution:

We have,

$f(k) = a^k, k < 0$

By definition,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$

$Z\{a^k\} = \sum_{-\infty}^{-1} a^k z^{-k}$

$Z\{a^k\} = \cdots \ldots \ldots \ldots + a^{-3}z^{--3} + a^{-2}z^{--2} + a^{-1}z^{--1}$

$Z\{a^k\} = \frac{z}{a} + \frac{z^2}{a^2} + \frac{z^3}{a^3} + \cdots \ldots \ldots \ldots$

$Z\{a^k\} = \frac{z}{a}\left[1 + \frac{z}{a} + \frac{z^2}{a^2} + \cdots \ldots\right]$

$Z\{a^k\} = \frac{z}{a}\left[1 - \frac{z}{a}\right]^{-1}$

$Z\{a^k\} = \frac{z}{a}\left[\frac{a-z}{a}\right]^{-1}$

$Z\{a^k\} = \frac{z}{a}\left[\frac{a}{a-z}\right]$

$\boxed{Z\{a^k\} = \frac{z}{a-z}}$

1. (d) A transmission channel has a per digit error probability $p = 0.01$. Calculate the probability of more than 1 error in 10 received digit using Poisson distribution. (5)

Solution:

$p = 0.01$

$n = 10$

$m = np = 10 \times 0.01 = 0.1$

By Poisson distribution,

$P(X = r) = \frac{e^{-m}.m^r}{r!} = \frac{e^{-0.1}.(0.1)^r}{r!}$

$$\begin{aligned} P(more\ than\ error) &= P(X > 1) \\ &= 1 - P(X \leq 1) \\ &= 1 - [P(0) + P(1)] \\ &= 1 - [0.9048 + 0.0905] \\ &= 0.0047 \end{aligned}$$

2. (a) Find the eigen values and eigen vectors of the matrix $A=\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$ (6)

Solution:

$A=\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}, |A|=36$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}3-\lambda & -1 & 1\\ -1 & 5-\lambda & -1\\ 1 & -1 & 3-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[3+5+3]\lambda^2+\left[\begin{vmatrix}5 & -1\\ -1 & 3\end{vmatrix}+\begin{vmatrix}3 & 1\\ 1 & 3\end{vmatrix}+\begin{vmatrix}3 & -1\\ -1 & 5\end{vmatrix}\right]\lambda-36=0$

$\lambda^3-11\lambda^2+36\lambda-36=0$

$\lambda=2,3,6$

(i) For $\lambda=2$, $[A-\lambda I]X=0$ gives

$\begin{bmatrix}1 & -1 & 1\\ -1 & 3 & -1\\ 1 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$x_1-x_2+x_3=0$

$-x_1+3x_2-x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}-1 & 1\\ 3 & -1\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}1 & 1\\ -1 & -1\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}1 & -1\\ -1 & 3\end{vmatrix}}$

$\frac{x_1}{-2}=-\frac{x_2}{0}=\frac{x_3}{2}$

$\frac{x_1}{-1}=\frac{x_2}{0}=\frac{x_3}{1}$

Hence, corresponding to $\lambda=2$ the eigen vector is $X_1=[-1,0,1]'$

(ii) For $\lambda=3$, $[A-\lambda I]X=0$ gives

$\begin{bmatrix}0 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$0x_1-x_2+x_3=0$

$-x_1+2x_2-x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}-1 & 1\\ 2 & -1\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}0 & 1\\ -1 & -1\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}0 & -1\\ -1 & 2\end{vmatrix}}$

$\frac{x_1}{-1}=-\frac{x_2}{1}=\frac{x_3}{-1}$

$\frac{x_1}{1}=\frac{x_2}{1}=\frac{x_3}{1}$

Hence, corresponding to $\lambda=3$ the eigen vector is $X_2=[1,1,1]'$

(iii) For $\lambda = 6, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 - x_2 + x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{2} = -\frac{x_2}{4} = \frac{x_3}{2}$$

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 6$ the eigen vector is $X_3 = [1, -2, 1]'$

2. (b) Find $Z\{f(k)\}$ where $f(k) = \cos\left(\frac{k\pi}{4} + a\right)$ where $k \geq 0$ (6)

Solution:

We have,

$$f(k) = \cos\left(\frac{k\pi}{4} + \alpha\right)$$

$$f(k) = \cos\frac{k\pi}{4}\cos\alpha - \sin\frac{k\pi}{4}\sin\alpha$$

$$Z\{f(k)\} = \cos\alpha\ Z\left\{\cos\frac{k\pi}{4}\right\} - \sin\alpha\ Z\left\{\sin\frac{k\pi}{4}\right\}$$

$$Z\{f(k)\} = \cos\alpha\ \left[\frac{z^2 - z\cos\frac{\pi}{4}}{z^2 - 2z\cos\frac{\pi}{4} + 1}\right] - \sin\alpha\ \left[\frac{z\sin\frac{\pi}{4}}{z^2 - 2z\cos\frac{\pi}{4} + 1}\right]$$

$$Z\{f(k)\} = \frac{\cos\alpha\left(z^2 - z\cos\frac{\pi}{4}\right) - \sin\alpha\ z\left(\sin\frac{\pi}{4}\right)}{z^2 - 2z\left(\frac{1}{\sqrt{2}}\right) + 1}$$

$$Z\{f(k)\} = \frac{z^2\cos\alpha - z\cos\alpha\cos\frac{\pi}{4} - z\sin\alpha\sin\frac{\pi}{4}}{z^2 - \sqrt{2}z + 1}$$

$$Z\{f(k)\} = \frac{z^2\cos\alpha - z\left(\cos\frac{\pi}{4}\cos\alpha + \sin\frac{\pi}{4}\sin\alpha\right)}{z^2 - \sqrt{2}z + 1}$$

$$\boxed{Z\{f(k)\} = \frac{z^2\cos\alpha - z\cos\left(\frac{\pi}{4} - \alpha\right)}{z^2 - \sqrt{2}z + 1}}$$

2. (c) Use the dual simplex method to solve the L.P.P. (8)

Minimise $z = 2x_1 + 2x_2 + 4x_3$

subject to $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

Solution:

The standard form,

Min $z - 2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3 = 0$

s.t. $-2x_1 - 3x_2 - 5x_3 + s_1 + 0s_2 + 0s_3 = -2$

$3x_1 + x_2 + 7x_3 + 0s_1 + s_2 + 0s_3 = 3$

$x_1 + 4x_2 + 6x_3 + 0s_1 + 0s_2 + s_3 = 5$

Simplex table,

Iteration No.	Basic Var	Coefficient of						RHS	Formula
		x_1	x_2	x_3	s_1	s_2	s_3		
0	z	-2	**-2**	-4	0	0	0	0	$X - \frac{2}{3}Y$
s_1 leaves x_2 enters	s_1	**-2**	**-3**	**-5**	**1**	**0**	**0**	**-2**	$\frac{Y}{-3}$
	s_2	3	**1**	7	0	1	0	3	$X + \frac{1}{3}Y$
	s_3	1	**4**	6	0	0	1	5	$X + \frac{4}{3}Y$
Ratio		$\frac{-2}{-2} = 1$	$\frac{-2}{-3} = 0.67$	$\frac{-4}{-5} = 0.8$	-	-	-	-	
1	z	-2/3	0	-2/3	-2/3	0	0	4/3	
	x_2	2/3	1	5/3	-1/3	0	0	2/3	
	s_2	7/3	0	16/3	1/3	1	0	7/3	
	s_3	-5/3	0	-2/3	4/3	0	1	7/3	

The solution is,

$$\boxed{x_1 = 0, x_2 = \frac{2}{3}, x_3 = 0, z_{min} = \frac{4}{3}}$$

3. (a) Evaluate $\int_c \frac{z^2}{(z-1)(z-2)} dz$ where C is the circle $|z-1| = 1$ (6)

Solution:

$I = \int_c \frac{z^2}{(z-1)(z-2)} dz$

Put $(z-1)(z-2) = 0$

$z = 1, z = 2$

C is $|z-1| = 1$

We see that $z = 1$ lies inside C and $z = 2$ lies on C

$I = \int_c \frac{\frac{z^2}{z-2}}{(z-1)} dz$

By CIF,

$I = 2\pi i f(1)$

$I = 2\pi i \left[\frac{1^2}{(1-2)}\right]$

$\boxed{I = -2\pi i}$

3. (b) Verify Cayley-Hamilton theorem and hence find A^{-1} and A^4 where (6)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}, |A| = 40$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [1-1-1]\lambda^2 + \left[\begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}\right]\lambda - 40 = 0$$

$$\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$$

By Cayley Hamilton theorem,

$$A^3 + A^2 - 18A - 40I = 0$$

Consider,

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$

L.H.S. $= A^3 + A^2 - 18A - 40I$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - 18\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.H.S.}$$

Thus, Cayley Hamilton theorem is verified

Now,

$$A^3 + A^2 - 18A - 40I = 0$$

Pre-multiplying by A^{-1}, we get

$$A^2 + A - 18I - 40A^{-1} = 0$$

$$40A^{-1} = A^2 + A - 18I$$

$$40A^{-1} = \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$40A^{-1} = \begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}}$$

Also,

$A^3 + A^2 - 18A - 40I = 0$

Pre-multiplying by$A$, we get

$A^4 + A^3 - 18A^2 - 40A = 0$

$A^4 = -A^3 + 18A^2 + 40A$

$$A^4 = -\begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + 18\begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + 40\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$\boxed{A^4 = \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}}$$

3. (c) Solve the L.P.P. by Big-M method (8)

Maximise $z = 3x_1 - 2x_2$

subject to $2x_1 + x_2 \leq 2$

$x_1 + 3x_2 \geq 3$

$x_1, x_2 \geq 0$

Solution:

The standard form,

Max $z = 3x_1 - 2x_2 + 0s_1 + 0s_2 - MA_2$

Max $z - 3x_1 + 2x_2 + 0s_1 + 0s_2 + MA_2 = 0$ ……(1)

s.t. $2x_1 + x_2 + s_1 = 2$ ……(2)

$x_1 + 3x_2 - s_2 + A_2 = 3$ ……(3)

Multiplying eqn (3) by M and subtracting with eqn (1), we get

$z + (-3 - M)x_1 + (2 - 3M)x_2 + 0s_1 + Ms_2 + 0A_2 = -3M$

Simplex table,

Iteration No.	Basic Var	Coefficient of					RHS	Ratio	Formula
		x_1	x_2	s_1	s_2	A_2			
0	z	-3-M	**2-3M**	0	M	0	-3M	-	$X - \frac{(2-3M)}{3}Y$
A_2 leaves x_2 enters	s_1	2	**1**	1	0	0	2	2	$X - \frac{1}{3}Y$
	A_2	**1**	**3**	**0**	**-1**	**1**	**3**	1	$\frac{Y}{3}$
1	z	**-11/3**	0	0	2/3		-2	-	$X + \frac{11}{5}Y$
s_1 leaves x_1 enters	s_1	**5/3**	**0**	**1**	**1/3**		**1**	0.6	$\frac{3Y}{5}$
	x_2	**1/3**	1	0	-1/3		1	3	$X - \frac{1}{5}Y$
2	z	0	0	11/5	7/5		1/5		
	x_1	1	0	3/5	1/5		3/5		
	x_2	0	1	-1/5	-2/5		4/5		

Thus, the solution is

$\boxed{x_1 = \frac{3}{5}, x_2 = \frac{4}{5}, z_{max} = \frac{1}{5}}$

4. (a) Find inverse Z transform of $F(z) = \frac{1}{(z-1)(z-3)}$ for (i) $|z| < 1$ (ii) $1 < |z| < 3$ (6)

Solution:

We have,

$F(z) = \frac{1}{(z-3)(z-1)}$

Let $\frac{1}{(z-3)(z-1)} = \frac{A}{z-3} + \frac{B}{z-1}$

$1 = A(z-1) + B(z-3)$

Comparing the coefficients, we get

$A + B = 0$

$-A - 3B = 1$

On solving, we get $A = \frac{1}{2}, B = -\frac{1}{2}$

$F(z) = \frac{\frac{1}{2}}{z-3} - \frac{\frac{1}{2}}{z-1}$

(i) $|z| < 1$

$F(z) = \frac{\frac{1}{2}}{-3+z} - \frac{\frac{1}{2}}{-1+z}$

$F(z) = \frac{\frac{1}{2}}{-3\left[1-\frac{z}{3}\right]} - \frac{\frac{1}{2}}{-[1-z]}$

$F(z) = -\frac{1}{2}.\frac{1}{3}\left[1 - \frac{z}{3}\right]^{-1} + \frac{1}{2}[1-z]^{-1}$

$F(z) = -\frac{1}{2}.\frac{1}{3}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots ..\right] + \frac{1}{2}[1 + z + z^2 + z^3 + \cdots ..]$

$F(z) = \frac{1}{2}.\left[-\frac{1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} - \frac{z^3}{3^4} - \cdots \ldots\right] + \frac{1}{2}[1 + z + z^2 + z^3 + \cdots ..]$

$F(z) = \frac{1}{2}[-3^{-1}z^0 - 3^{-2}z^1 - 3^{-3}z^2 - \cdots .] + \frac{1}{2}[z^0 + z^1 + z^2 + \cdots]$

From first series,

Coefficient of $z^k = \frac{1}{2}.\left(-3^{-(k+1)}\right), k \geq 0$

Coefficient of $z^{-k} = -\frac{3^{k-1}}{2}, k \leq 0$

From second series,

Coefficient of $z^k = \frac{1}{2}, k \geq 0$

Coefficient of $z^{-k} = \frac{1}{2}, k \leq 0$

Thus,

$$\boxed{Z^{-1}\left\{\frac{1}{(z-3)(z-1)}\right\} = \frac{1}{2} - \frac{3^{k-1}}{2}, k \leq 0}$$

(ii) $1 < |z| < 3$

$F(z) = \frac{\frac{1}{2}}{-3+z} - \frac{\frac{1}{2}}{z-1}$

$F(z) = \frac{\frac{1}{2}}{-3\left[1-\frac{z}{3}\right]} - \frac{\frac{1}{2}}{z\left[1-\frac{1}{z}\right]}$

$$F(z) = -\frac{1}{2}.\frac{1}{3}\left[1 - \frac{z}{3}\right]^{-1} - \frac{1}{2}.\frac{1}{z}\left[1 - \frac{1}{z}\right]^{-1}$$
$$F(z) = -\frac{1}{2}.\frac{1}{3}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots..\right] - \frac{1}{2}.\frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots..\right]$$
$$F(z) = \frac{1}{2}\left[-\frac{1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} - \frac{z^3}{3^4} - \cdots ...\right] + \frac{1}{2}\left[-\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \cdots ...\right]$$
$$F(z) = \frac{1}{2}[-3^{-1}z^0 - 3^{-2}z^1 - 3^{-3}z^2 - \cdots.] + \frac{1}{2}[-z^{-1} - z^{-2} - z^{-3} + \cdots]$$

From first series,

Coefficient of $z^k = \frac{1}{2}.\left(-3^{-(k+1)}\right), k \geq 0$

Coefficient of $z^{-k} = -\frac{3^{-(-k+1)}}{2}, k \leq 0$

i.e. Coefficient of $z^{-k} = -\frac{3^{k-1}}{2}, k \leq 0$

From second series,

Coefficient of $z^{-k} = -\frac{1}{2}, k > 0$

Thus,

$$\boxed{Z^{-1}\left\{\frac{1}{(z-3)(z-1)}\right\} = \begin{cases} -\frac{3^{k-1}}{2} & k \leq 0 \\ -\frac{1}{2} & k > 0 \end{cases}}$$

4. (b) The following data represent the marks obtained by 12 students in two tests, one held before the coaching and the other after the coaching

Test I	55	60	65	75	49	25	18	30	35	51	61	72
Test II	63	70	70	81	54	29	21	38	32	50	70	80

Do the data indicate that the coaching was effective in improving performance of the students? (6)

Solution:

x_1	x_2	$x = (x_2 - x_1)$	x^2
55	63	8	64
60	70	10	100
65	70	5	25
75	81	6	36
49	54	5	25
25	29	4	16
18	21	3	9
30	38	8	64
35	32	-3	9
51	50	-1	1
61	70	9	81
72	80	8	64
		Total $= 62$	Total $= 494$

$n = 12$

$\overline{x} = \frac{\sum x}{n} = \frac{62}{12} = 5.1667$

$\sigma = \sqrt{\frac{\sum x^2}{n} - (\overline{x})^2} = \sqrt{\frac{494}{12} - (5.1667)^2} = 3.8042$

(i) Null Hypothesis: $\mu = 0$ (no change)

Alternative Hypothesis: $\mu \neq 0$

(ii) Test statistic:

$$t = \left|\frac{\overline{x}-\mu}{\frac{\sigma}{\sqrt{n-1}}}\right| = \left|\frac{5.1667-0}{\frac{3.8042}{\sqrt{12-1}}}\right| = 4.5045$$

(iii) L.O.S.: $\alpha = 0.05$ (assumed)

(iv) Degree of freedom: $\emptyset = n - 1 = 12 - 1 = 11$

(v) Critical value: $t_\alpha = 2.201$

(vi) Decision: Since, the calculated value of t is more than the critical value, null hypothesis is rejected. Thus, the coaching was effective in improving the performance of students

4. (c) Find all possible Laurent's Series expansions of the function $f(z) = \frac{1}{(z-1)(z+2)}$ about $z = 0$ indicating the region of convergence in each case (8)

Solution:

We have, $f(z) = \frac{1}{(z-1)(z+2)}$

Let $\frac{1}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$

$1 = A(z+2) + B(z-1)$

Comparing the coefficients, we get

$A + B = 0$

$2A - B = 1$

On solving, we get

$A = \frac{1}{3}, B = -\frac{1}{3}$

$f(z) = \frac{\frac{1}{3}}{z-1} - \frac{\frac{1}{3}}{z+2}$

For ROC, put $z - 1 = 0, z + 2 = 0$

$z = 1, z = -2$

$|z| = 1, |z| = 2$

Thus, the ROC are (i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$

(i) $|z| < 1$

$f(z) = \frac{\frac{1}{3}}{-1+z} - \frac{\frac{1}{3}}{2+z}$

$f(z) = \frac{\frac{1}{3}}{-[1-z]} - \frac{\frac{1}{3}}{2\left[1+\frac{z}{2}\right]}$

$f(z) = -\frac{1}{3}[1-z]^{-1} - \frac{1}{6}\left[1 + \frac{z}{2}\right]^{-1}$

$f(z) = -\frac{1}{3}[1 + z + z^2 + z^3 + \cdots.] - \frac{1}{6}\left[1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\right]$

(ii) $1 < |z| < 2$

$f(z) = \frac{\frac{1}{3}}{z-1} - \frac{\frac{1}{3}}{2+z}$

$f(z) = \frac{\frac{1}{3}}{z\left[1-\frac{1}{z}\right]} - \frac{\frac{1}{3}}{2\left[1+\frac{z}{2}\right]}$

$f(z) = \frac{1}{3z}\left[1 - \frac{1}{z}\right]^{-1} - \frac{1}{6}\left[1 + \frac{z}{2}\right]^{-1}$

$f(z) = \frac{1}{3z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots.\right] - \frac{1}{6}\left[1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\right]$

(iii) $|z| > 2$

$f(z) = \frac{\frac{1}{3}}{z-1} - \frac{\frac{1}{3}}{z+2}$

$f(z) = \frac{\frac{1}{3}}{z\left[1-\frac{1}{z}\right]} - \frac{\frac{1}{3}}{z\left[1+\frac{2}{z}\right]}$

$$f(z)=\frac{1}{3z}\left[1-\frac{1}{z}\right]^{-1}-\frac{1}{3z}\left[1+\frac{2}{z}\right]^{-1}$$
$$f(z)=\frac{1}{3z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots.\right]-\frac{1}{3z}\left[1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\cdots\right]$$

5. (a) Determine all basic solutions to the following problem.

Maximise $z = x_1 - 2x_2 + 4x_3$

subject to $x_1 + 2x_2 + 3x_3 = 7$

$3x_1 + 4x_2 + 6x_3 = 15$

$x_1, x_2, x_3 \geq 0$ (6)

Solution:

No	Non-basic var = 0	Basic var	Equations & solutions	Is the solution feasible?	Is the solution degenerate?	Value of z	Is the solution optimal?
1	$x_3 = 0$	x_1, x_2	$x_1 + 2x_2 = 7$ $3x_1 + 4x_2 = 15$ $x_1 = 1, x_2 = 3$	Yes	No	-5	No
2	$x_2 = 0$	x_1, x_3	$x_1 + 3x_3 = 7$ $3x_1 + 6x_3 = 15$ $x_1 = 1, x_3 = 2$	Yes	No	9	Yes
3	$x_1 = 0$	x_2, x_3	$2x_2 + 3x_3 = 7$ $4x_2 + 6x_3 = 15$ Unbounded solution	-	-	-	-

5. (b) Using Normal distribution, find the probability of getting 55 heads in the toss of 100 fair coins. (6)

Solution:

$n = 100, p =$ probability of getting a head $= \frac{1}{2}, q = \frac{1}{2}$

$\mu = np = 100 \times \frac{1}{2} = 50$

$\sigma = \sqrt{npq} = \sqrt{100 \times \frac{1}{2} \times \frac{1}{2}} = 5$

$z = \frac{x-\mu}{\sigma} = \frac{x-50}{5}$

$$\begin{aligned}
P(55\ heads) &= P(x = 55) \\
&= P(54.5 < x < 55.5) \\
&= P\left(\frac{54.5-50}{5} < z < \frac{55.5-50}{5}\right) \\
&= P(0.9 < z < 1.1) \\
&= A(1.1) - A(0.9) \\
&= 0.3643 - 0.3159 \\
&= 0.0484
\end{aligned}$$

5. (c) Solve the N.L.P.P.

Optimise $z = 10x_1 + 8x_2 + 6x_3 + 2x_1^2 + x_2^2 + 3x_3^2 - 100$

subject to $x_1 + x_2 + x_3 = 20$

$x_1, x_2, x_3 \geq 0$ (8)

Solution:

Let $f = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

and $h = x_1 + x_2 + x_3 - 20$

Consider the Lagrangian function,

$L = f - \lambda h$

$L = \left(2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100\right) - \lambda(x_1 + x_2 + x_3 - 20)$

$L_{x_1} = 0 \Rightarrow 4x_1 + 10 - \lambda = 0 \Rightarrow x_1 = \frac{\lambda-10}{4}$

$L_{x_2} = 0 \Rightarrow 2x_2 + 8 - \lambda = 0 \Rightarrow x_2 = \frac{\lambda-8}{2}$

$L_{x_3} = 0 \Rightarrow 6x_3 + 6 - \lambda = 0 \Rightarrow x_3 = \frac{\lambda-6}{6}$

$L_\lambda = 0 \Rightarrow -(x_1 + x_2 + x_3 - 20) = 0$

$x_1 + x_2 + x_3 = 20$

$\frac{\lambda-10}{4} + \frac{\lambda-8}{2} + \frac{\lambda-6}{6} = 20$

$\frac{11\lambda-90}{12} = 20$

$\lambda = 30$

$\therefore x_1 = 5, x_2 = 11, x_3 = 4$

Now, hessian matrix,

$$H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} & h_{x_3} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{bmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 6 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 6 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$\Delta_4 = -12 - 24 + 8 = -28$

Since both Δs are negative, it is a minima

$\therefore z_{min} = 2(5)^2 + (11)^2 + 3(4)^2 + 10(5) + 8(11) + 6(4) - 100$

$\boxed{z_{min} = 281}$

6. (a) Show that the given matrix is diagonalisable and hence find diagonal form and transforming matrix where $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$ (6)

Solution:

$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}, |A| = 4$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [4+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}\right]\lambda - 4 = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$

$\lambda = 1,2,2$

The Algebraic Multiplicity of $\lambda = 1$ is 1 and that of $\lambda = 2$ is 2

(i) For $\lambda = 2$, $[A - \lambda I]X = 0$ gives

$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 - \frac{1}{2}R_1, R_3 + \frac{1}{2}R_1$

$\begin{bmatrix} 2 & 6 & 6 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_3 - R_2$

$\begin{bmatrix} 2 & 6 & 6 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The rank (r) of the matrix is 2 and number of unknowns (n) is 3

Thus, $n - r = 3 - 2 = 1$ vectors to be formed

The Geometric Multiplicity of $\lambda = 2$ is 1

Since, Algebraic Multiplicity $\neq$ Geometric Multiplicity,

matrix A is not diagonalizable.

6. (b) Of the 64 off-springs of a certain cross between guinea pigs, 34 were red, 10 were black and 20 were white. According to the generic model these numbers should be in the ratio 9:3:4. Use chi-square test to check whether the data are consistent with the model. (6)

Solution:

(i) Null hypothesis: the data is consistent with the generic model

Alternate Hypothesis: the data is inconsistent with the generic model

(ii) Test calculation:

O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
34	$\frac{9}{16}\times 64 = 36$	-2	4	4/36
10	$\frac{3}{16}\times 64 = 12$	-2	4	4/12
20	$\frac{4}{16}\times 64 = 16$	4	16	16/16
Total = 64	64			$\chi^2 = 1.444$

(iii) LOS:0.05

(iv) Degree of freedom: $\phi = n - 1 = 3 - 1 = 2$

(v) Critical value: $\chi^2_\alpha = 5.991$

(vi) Decision: since, the calculated value is less than the critical value, null hypothesis is accepted. The data is consistent with the generic model

6. (c) Maximise $z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$, subject to $x_1 + x_2 \leq 2$ and $2x_1 + 3x_2 \leq 12, x_1, x_2, x_3 \geq 0$ by K.T. condition (8)

Solution:

Let $f = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

Let $h_1 = x_1 + x_2 - 2, h_2 = 2x_1 + 3x_2 - 12$

Consider, $L = f - \lambda_1 h_1 - \lambda_2 h_2$

$L = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2 - \lambda_1(x_1 + x_2 - 2) - \lambda_2(2x_1 + 3x_2 - 12)$

According to Kuhn Tucker conditions,

$L_{x_1} = 0 \Rightarrow -2x_1 + 4 - \lambda_1 - 2\lambda_2 = 0$ ……………….(1)

$L_{x_2} = 0 \Rightarrow -2x_2 + 6 - \lambda_1 - 3\lambda_2 = 0$ ……………….(2)

$L_{x_3} = 0 \Rightarrow -2x_3 = 0 \Rightarrow x_3 = 0$

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + x_2 - 2) = 0$ …………………..(3)

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(2x_1 + 3x_2 - 12) = 0$ ……………….(4)

$h_1 \leq 0 \Rightarrow x_1 + x_2 - 2 \leq 0$ ………………………………(5)

$h_2 \leq 0 \Rightarrow 2x_1 + 3x_2 - 12 \leq 0$ …………………………(6)

$x_1, x_2, \lambda_1, \lambda_2 \geq 0$ ………………(7)

Case I: If $\lambda_1 = 0, \lambda_2 = 0$

From (1), $x_1 = 2$

From (2), $x_2 = 3$

We see that eqn (5) is not satisfied

Case II: If $\lambda_1 = 0, \lambda_2 \neq 0$

From (1), $2x_1 + 0x_2 + 2\lambda_2 = 4$

From (2), $0x_1 + 2x_2 + 3\lambda_2 = 6$

From (4), $2x_1 + 3x_2 + 0\lambda_2 = 12$

On solving,

$x_1 = \frac{24}{13}, x_2 = \frac{36}{13}, \lambda_2 = \frac{2}{13}$

We see that, eqn (5) is not satisfied

Case III: If $\lambda_1 \neq 0, \lambda_2 = 0$

From (1), $2x_1 + 0x_2 + \lambda_1 = 4$

From (2), $0x_1 + 2x_2 + \lambda_1 = 6$

From (3), $x_1 + x_2 + 0\lambda_1 = 2$

On solving,

$x_1 = \frac{1}{2}, x_2 = \frac{3}{2}, \lambda_1 = 3$

$z_{max} = -\left(\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - (0)^2 + 4\left(\frac{1}{2}\right) + 6\left(\frac{3}{2}\right) = \frac{17}{2}$

Case IV: If $\lambda_1 \neq 0, \lambda_2 \neq 0$

From (3), $x_1 + x_2 = 2$

From (4), $2x_1 + 3x_2 = 12$

On solving,

$x_1 = -6, x_2 = 8$

We see that eqn (7) is not satisfied
Thus, the optimal solution is
$\boxed{z_{max} = \frac{17}{2}}$ at $\boxed{x_1 = \frac{1}{2}, x_2 = \frac{3}{2}, x_3 = 0}$