Non Linear Programming Problem (NLPP)

Weight Distribution of Types

Comp

Type	Name	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
I	Optimization	---	---	---	---	05	---	---	---	06	---
II	Lagrange's Method	---	---	---	---	07	14	14	08	06	08
III	Kuhn Tucker Condition	08	08	08	08	05	08	08	08	08	08
Total Marks		**08**	**08**	**08**	**08**	**17**	**22**	**22**	**16**	**20**	**16**

Type I: Optimization

1. Find maximum or minimum of the function

$z = x_1 + 2x_3 + x_2x_3 - x_1^2 - x_2^2 - x_3^2$

Solution:

Let

$f = x_1 + 2x_3 + x_2x_3 - x_1^2 - x_2^2 - x_3^2$

Consider,

$f_{x_1} = 0$ gives $1 + 0 + 0 - 2x_1 - 0 - 0 = 0$ i.e. $x_1 = \frac{1}{2}$

$f_{x_2} = 0$ gives $0 + 0 + x_3 - 0 - 2x_2 - 0 = 0$ i.e. $2x_2 - x_3 = 0$ .... (1)

$f_{x_3} = 0$ gives $0 + 2 + x_2 - 0 - 0 - 2x_3 = 0$ i..e $x_2 - 2x_3 = -2$ .... (2)

Solving eqn (1) & (2),

$x_2 = \frac{2}{3}, x_3 = \frac{4}{3}$

Now, hessian matrix

$$H = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$\Delta_1 = -2$

$$\Delta_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4$$

$$\Delta_3 = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} = -6$$

Thus, f is a maxima,

$$f_{max} = z_{max} = \frac{1}{2} + 2\left(\frac{4}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{4}{3}\right) - \left(\frac{1}{2}\right)^2 - \left(\frac{2}{3}\right)^2 - \left(\frac{4}{3}\right)^2 = \frac{19}{12}$$

2. Obtain the relative maximum or minimum (if any) of the function
$z = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 10x_2 - 14x_3 + 103$
[N14/MechCivil/6M][M19/Chem/5M][N19/Chem/6M]
Solution:
Let $f = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 10x_2 - 14x_3 + 103$
$f_{x_1} = 0 \Rightarrow 2x_1 - 6 = 0$ ...(1)
$f_{x_2} = 0 \Rightarrow 2x_2 - 10 = 0$ ...(2)
$f_{x_3} = 0 \Rightarrow 2x_3 - 14 = 0$ ...(3)
Solving (1), (2) and (3), we get
$x_1 = 3, x_2 = 5, x_3 = 7$
Now, Hessian matrix,

$$H = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Delta_1 = 2$

$$\Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$$\Delta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$$

Since, all Δs are positive, it is a minima
$\therefore z_{min} = (3)^2 + (5)^2 + (7)^2 - 6(3) - 10(5) - 14(7) + 103$
$\therefore z_{min} = 20$

3. Optimise $z = x_1^2 + x_2^2 + x_3^2 - 4x_1 - 8x_2 - 12x_3 + 100$
[N16/CompIT/6M][M24/CompITAI/6M]
Solution:
Let $f = x_1^2 + x_2^2 + x_3^2 - 4x_1 - 8x_2 - 12x_3 + 100$
$f_{x_1} = 0 \Rightarrow 2x_1 - 4 = 0$ ...(1)
$f_{x_2} = 0 \Rightarrow 2x_2 - 8 = 0$ ...(2)
$f_{x_3} = 0 \Rightarrow 2x_3 - 12 = 0$ ...(3)
Solving (1), (2) and (3), we get
$x_1 = 2, x_2 = 4, x_3 = 6$
Now, Hessian matrix,

$$H = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Delta_1 = 2$

$$\Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$$\Delta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$$

Since, all Δs are positive, it is a minima
$\therefore z_{min} = (2)^2 + (4)^2 + (6)^2 - 4(2) - 8(4) - 12(6) + 100$
$\therefore z_{min} = 44$

4. Optimise $z = x_1^2 + x_2^2 + x_3^2 - 8x_1 - 10x_2 - 12x_3 + 100$

[M22/CompITAI/5M]

Solution:

Let $f = x_1^2 + x_2^2 + x_3^2 - 8x_1 - 10x_2 - 12x_3 + 100$

$f_{x_1} = 0 \Rightarrow 2x_1 - 8 = 0$ ...(1)

$f_{x_2} = 0 \Rightarrow 2x_2 - 10 = 0$ ...(2)

$f_{x_3} = 0 \Rightarrow 2x_3 - 12 = 0$ ...(3)

Solving (1), (2) and (3), we get

$x_1 = 4, x_2 = 5, x_3 = 6$

Now, Hessian matrix,

$$H = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Delta_1 = 2$

$\Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$

$\Delta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$

Since, all Δs are positive, it is a minima

$\therefore z_{min} = (4)^2 + (5)^2 + (6)^2 - 8(4) - 10(5) - 12(6) + 100$

$\therefore z_{min} = 23$

5. Optimise$z = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 8x_2 - 10x_3$

[M14/MechCivil/6M]

Solution:

Let $f = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 8x_2 - 10x_3$

$f_{x_1} = 0 \Rightarrow 2x_1 - 6 = 0$ ...(1)

$f_{x_2} = 0 \Rightarrow 2x_2 - 8 = 0$ ...(2)

$f_{x_3} = 0 \Rightarrow 2x_3 - 10 = 0$ ...(3)

Solving (1), (2) and (3), we get

$x_1 = 3, x_2 = 4, x_3 = 5$

Now, Hessian matrix,

$$H = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Delta_1 = 2$

$$\Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$$\Delta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$$

Since, all Δs are positive, it is a minima

$\therefore z_{min} = (3)^2 + (4)^2 + (5)^2 - 6(3) - 8(4) - 10(5)$

$\therefore z_{min} = -50$

6. Obtain the relative maximum or minimum (if any) of the function
$z = -x_1^2 - x_2^2 - x_3^2 + 9x_1 + x_1x_2 + 6x_3$
[N19/Biot/6M]
Solution:
Let $f = -x_1^2 - x_2^2 - x_3^2 + 9x_1 + x_1x_2 + 6x_3$
$f_{x_1} = 0 \Rightarrow -2x_1 + 9 + x_2 = 0$ ...(1)
$f_{x_2} = 0 \Rightarrow -2x_2 + x_1 = 0$ ...(2)
$f_{x_3} = 0 \Rightarrow -2x_3 + 6 = 0$ ...(3)
Solving (1), (2) and (3), we get
$x_1 = 6, x_2 = 3, x_3 = 3$
Now, Hessian matrix,

$$H = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{bmatrix}$$

$$H = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$\Delta_1 = -2$

$$\Delta_2 = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3$$

$$\Delta_3 = \begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -6$$

Since, Δ_1 is negative, Δ_2 is psoitive and Δ_3 is negative, it is a maxima
$\therefore z_{max} = -(6)^2 - (3)^2 - (3)^2 + 9(6) + (6)(3) + 6(3)$
$\therefore z_{max} = 36$

Type II: Method of Lagrange's Multipliers

1. Using the method of Lagrange's multipliers, solve the following N.L.P.P.

   Optimize $\quad z = 6x_1^2 + 5x_2^2$

   subject to $\quad x_1 + 5x_2 = 7$

   $\quad x_1, x_2 \geq 0$

   [N13/Chem/7M][N15/N16/M17/N17/MechCivil/6M][N18/MTRX/6M]

   Solution:

   Let $f = 6x_1^2 + 5x_2^2$ and $h = x_1 + 5x_2 - 7$

   Consider the Lagrangian function,

   $L = f - \lambda h = (6x_1^2 + 5x_2^2) - \lambda(x_1 + 5x_2 - 7)$

   $L_{x_1} = 0 \Rightarrow 12x_1 - \lambda = 0 \Rightarrow 12x_1 + 0x_2 - \lambda = 0$ ……(1)

   $L_{x_2} = 0 \Rightarrow 10x_2 - 5\lambda = 0 \Rightarrow 0x_1 + 10x_2 - 5\lambda = 0$ …..(2)

   $L_\lambda = 0 \Rightarrow -(x_1 + 5x_2 - 7) = 0 \Rightarrow x_1 + 5x_2 + 0\lambda = 7$ …(3)

   Solving (1), (2) and (3), we get

   $x_1 = \frac{7}{31}, x_2 = \frac{42}{31}, \lambda = \frac{84}{31}$

   Now, hessian matrix,

   $$H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{bmatrix}$$

   $$H = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 12 & 0 \\ 5 & 0 & 10 \end{bmatrix}$$

   $$\Delta_3 = \begin{vmatrix} 0 & 1 & 5 \\ 1 & 12 & 0 \\ 5 & 0 & 10 \end{vmatrix} = -310$$

   Since Δ is negative, it is a minima

   $\therefore z_{min} = 6\left(\frac{7}{31}\right)^2 + 5\left(\frac{42}{31}\right)^2 = \frac{294}{31}$

2. The value of Lagrange's multiplier λ for the following NLPP is

Optimize $z = 6x_1^2 + 5x_2^2$

subject to $x_1 + 5x_2 = 7$

$x_1, x_2 \geq 0$

[M22/CompITAI/2M]

Solution:

Let $f = 6x_1^2 + 5x_2^2$ and $h = x_1 + 5x_2 - 7$

Consider the Lagrangian function,

$L = f - \lambda h = (6x_1^2 + 5x_2^2) - \lambda(x_1 + 5x_2 - 7)$

$L_{x_1} = 0 \Rightarrow 12x_1 - \lambda = 0 \Rightarrow 12x_1 + 0x_2 - \lambda = 0$ ……(1)

$L_{x_2} = 0 \Rightarrow 10x_2 - 5\lambda = 0 \Rightarrow 0x_1 + 10x_2 - 5\lambda = 0$ …..(2)

$L_\lambda = 0 \Rightarrow -(x_1 + 5x_2 - 7) = 0 \Rightarrow x_1 + 5x_2 + 0\lambda = 7$ …(3)

Solving (1), (2) and (3), we get

$x_1 = \frac{7}{31}, x_2 = \frac{42}{31}, \lambda = \frac{84}{31}$

3. Using the method of Lagrange's multipliers, solve the following N.L.P.P.

Optimise $z = 4x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $x_1 + x_2 = 4$

$x_1, x_2 \geq 0$

[M14/N15/ChemBiot/8M][M15/MechCivil/8M][N16/ChemBiot/6M][M18/Chem/6M][M23/CompIT/6M]

Solution:

Let $f = 4x_1 + 8x_2 - x_1^2 - x_2^2$ and $h = x_1 + x_2 - 4$

Consider the Lagrangian function,

$L = f - \lambda h = (4x_1 + 8x_2 - x_1^2 - x_2^2) - \lambda(x_1 + x_2 - 4)$

$L_{x_1} = 0 \Rightarrow 4 - 2x_1 - \lambda = 0 \Rightarrow 2x_1 + 0x_2 + \lambda = 4$ ……(1)

$L_{x_2} = 0 \Rightarrow 8 - 2x_2 - \lambda = 0 \Rightarrow 0x_1 + 2x_2 + \lambda = 8$ …..(2)

$L_\lambda = 0 \Rightarrow -(x_1 + x_2 - 4) = 0 \Rightarrow x_1 + x_2 + 0\lambda = 4$ …(3)

Solving (1), (2) and (3), we get

$x_1 = 1, x_2 = 3, \lambda = 2$

Now, hessian matrix,

$$H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 4$$

Since Δ is positive, it is a maxima

$\therefore z_{max} = 4(1) + 8(3) - 1^2 - 3^2 = 18$

4. Using the method of Lagrange's multipliers, solve the following N.L.P.P.

Optimise $z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

subject to $x_1 + 2x_2 = 2$

$x_1, x_2 \geq 0$

[M22/CompITAI/5M]

Solution:

Let $f = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$ and $h = x_1 + 2x_2 - 2$

Consider the Lagrangian function,

$L = f - \lambda h = (4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2) - \lambda(x_1 + 2x_2 - 2)$

$L_{x_1} = 0 \Rightarrow 4 - 4x_1 - 2x_2 - \lambda = 0 \Rightarrow 4x_1 + 2x_2 + \lambda = 4$ ……(1)

$L_{x_2} = 0 \Rightarrow 6 - 2x_1 - 4x_2 - 2\lambda = 0 \Rightarrow 2x_1 + 4x_2 + 2\lambda = 6$ …..(2)

$L_\lambda = 0 \Rightarrow -(x_1 + 2x_2 - 2) = 0 \Rightarrow x_1 + 2x_2 + 0\lambda = 2$ …(3)

Solving (1), (2) and (3), we get

$x_1 = \frac{1}{3}, x_2 = \frac{5}{6}, \lambda = 1$

Now, hessian matrix,

$$H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -4 & -2 \\ 2 & -2 & -4 \end{bmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 2 \\ 1 & -4 & -2 \\ 2 & -2 & -4 \end{vmatrix} = 12$$

Since Δ is positive, it is a maxima

$\therefore z_{max} = 4\left(\frac{1}{3}\right) + 6\left(\frac{5}{6}\right) - 2\left(\frac{1}{3}\right)^2 - 2(\frac{1}{3})(\frac{5}{6}) - 2\left(\frac{5}{6}\right)^2 = \frac{25}{6}$

5. Using the method of Lagrange's multipliers, solve the following N.L.P.P.

Optimise $z = 2x_1 + 6x_2 - x_1^2 - x_2^2 + 14$

subject to $x_1 + x_2 = 4$

$x_1, x_2 \geq 0$

[N19/Chem/6M]

Solution:

Let $f = 2x_1 + 6x_2 - x_1^2 - x_2^2 + 14$ and $h = x_1 + x_2 - 4$

Consider the Lagrangian function,

$L = f - \lambda h = (2x_1 + 6x_2 - x_1^2 - x_2^2 + 14) - \lambda(x_1 + x_2 - 4)$

$L_{x_1} = 0 \Rightarrow 2 - 2x_1 - \lambda = 0 \Rightarrow 2x_1 + 0x_2 + \lambda = 2$ ……(1)

$L_{x_2} = 0 \Rightarrow 6 - 2x_2 - \lambda = 0 \Rightarrow 0x_1 + 2x_2 + \lambda = 6$ …..(2)

$L_\lambda = 0 \Rightarrow -(x_1 + x_2 - 4) = 0 \Rightarrow x_1 + x_2 + 0\lambda = 4$ …(3)

Solving (1), (2) and (3), we get

$x_1 = 1, x_2 = 3, \lambda = 0$

Now, hessian matrix,

$$H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 4$$

Since Δ is positive, it is a maxima

$\therefore z_{max} = 2(1) + 6(3) - (1)^2 - (3)^2 + 14 = 24$

6. Using the method of Lagrange's multipliers, solve the following N.L.P.P.

Optimise $z = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

subject to $x_1 + x_2 + x_3 = 7$

$x_1, x_2, x_3 \geq 0$

[M16/ChemBiot/8M][N22/CompITAI/6M]

Solution:

Let $f = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

and $h = x_1 + x_2 + x_3 - 7$

Consider the Lagrangian function,

$L = f - \lambda h$

$L = \left(x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3\right) - \lambda(x_1 + x_2 + x_3 - 7)$

$L_{x_1} = 0 \Rightarrow 2x_1 - 10 - \lambda = 0 \Rightarrow x_1 = \frac{\lambda+10}{2}$

$L_{x_2} = 0 \Rightarrow 2x_2 - 6 - \lambda = 0 \Rightarrow x_2 = \frac{\lambda+6}{2}$

$L_{x_3} = 0 \Rightarrow 2x_3 - 4 - \lambda = 0 \Rightarrow x_3 = \frac{\lambda+4}{2}$

$L_\lambda = 0 \Rightarrow -(x_1 + x_2 + x_3 - 7) = 0$

$x_1 + x_2 + x_3 = 7$

$\frac{\lambda+10}{2} + \frac{\lambda+6}{2} + \frac{\lambda+4}{2} = 7$

$\frac{3\lambda+20}{2} = 7$

$\lambda = -2$

$\therefore x_1 = 4, x_2 = 2, x_3 = 1$

Now, hessian matrix,

$$H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} & h_{x_3} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -4$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$\Delta_4 = -4 - 4 - 4 = -12$

Since both Δs are negative, it is a minima

$\therefore z_{min} = (4)^2 + (2)^2 + (1)^2 - 10(4) - 6(2) - 4(1) = -35$

7. Using the method of Lagrange's multipliers, solve the following N.L.P.P.

Optimise $z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

subject to $x_1 + x_2 + x_3 = 20$

$x_1, x_2, x_3 \geq 0$

[D23/CompITAI/8M]

Solution:

Let $f = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

and $h = x_1 + x_2 + x_3 - 20$

Consider the Lagrangian function,

$L = f - \lambda h$

$L = (2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100) - \lambda(x_1 + x_2 + x_3 - 20)$

$L_{x_1} = 0 \Rightarrow 4x_1 + 10 - \lambda = 0 \Rightarrow x_1 = \frac{\lambda-10}{4}$

$L_{x_2} = 0 \Rightarrow 2x_2 + 8 - \lambda = 0 \Rightarrow x_2 = \frac{\lambda-8}{2}$

$L_{x_3} = 0 \Rightarrow 6x_3 + 6 - \lambda = 0 \Rightarrow x_3 = \frac{\lambda-6}{6}$

$L_\lambda = 0 \Rightarrow -(x_1 + x_2 + x_3 - 20) = 0$

$x_1 + x_2 + x_3 = 20$

$\frac{\lambda-10}{4} + \frac{\lambda-8}{2} + \frac{\lambda-6}{6} = 20$

$\frac{11\lambda-90}{12} = 20$

$\lambda = 30$

$\therefore x_1 = 5, x_2 = 11, x_3 = 4$

Now, hessian matrix,

$$H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} & h_{x_3} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{bmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 6 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 6 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$\Delta_4 = -12 - 24 + 8 = -28$

Since both Δs are negative, it is a minima

$\therefore z_{min} = 2(5)^2 + (11)^2 + 3(4)^2 + 10(5) + 8(11) + 6(4) - 100 = 281$

8. Using the method of Lagrange's multipliers, solve the following N.L.P.P.

Optimise $z = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$

subject to $x_1 + x_2 + x_3 = 10$

$x_1, x_2, x_3 \geq 0$

[M19/Chem/6M][M19/N19/MTRX/6M][M24/CompITAI/6M][D24/CompITAI/8M]

Solution:

Let $f = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$

and $h = x_1 + x_2 + x_3 - 10$

Consider the Lagrangian function,

$L = f - \lambda h$

$L = \left(12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23\right) - \lambda(x_1 + x_2 + x_3 - 10)$

$L_{x_1} = 0 \Rightarrow 12 - 2x_1 - \lambda = 0 \Rightarrow x_1 = \frac{12-\lambda}{2}$

$L_{x_2} = 0 \Rightarrow 8 - 2x_2 - \lambda = 0 \Rightarrow x_2 = \frac{8-\lambda}{2}$

$L_{x_3} = 0 \Rightarrow 6 - 2x_3 - \lambda = 0 \Rightarrow x_3 = \frac{6-\lambda}{2}$

$L_\lambda = 0 \Rightarrow -(x_1 + x_2 + x_3 - 10) = 0$

$x_1 + x_2 + x_3 = 10$

$\frac{12-\lambda}{2} + \frac{8-\lambda}{2} + \frac{6-\lambda}{2} = 10$

$\frac{26-3\lambda}{2} = 10$

$\lambda = 2$

$\therefore x_1 = 5, x_2 = 3, x_3 = 1$

Now, hessian matrix, $H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} & h_{x_3} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}$

$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 4$

$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \\ 1 & 0 & 0 \end{vmatrix}$

$\Delta_4 = -4 - 4 - 4 = -12$

Since both Δ_3 is positive and Δ_4 is negative, it is a maxima

$\therefore z_{max} = 12(5) + 8(3) + 6(1) - (5)^2 - (3)^2 - (1)^2 - 23 = 32$

9. Using the method of Lagrange's multipliers, solve the following N.L.P.P.

Optimise $z = 3x_1^2 + x_2^2 + x_3^2$

subject to $x_1 + x_2 + x_3 = 2$

$x_1, x_2, x_3 \geq 0$

[N13/Biot/8M]

Solution:

Let $f = 3x_1^2 + x_2^2 + x_3^2$ and $h = x_1 + x_2 + x_3 - 2$

Consider the Lagrangian function,

$L = f - \lambda h = (3x_1^2 + x_2^2 + x_3^2) - \lambda(x_1 + x_2 + x_3 - 2)$

$L_{x_1} = 0 \Rightarrow 6x_1 - \lambda = 0 \Rightarrow x_1 = \frac{\lambda}{6}$

$L_{x_2} = 0 \Rightarrow 2x_2 - \lambda = 0 \Rightarrow x_2 = \frac{\lambda}{2}$

$L_{x_3} = 0 \Rightarrow 2x_3 - \lambda = 0 \Rightarrow x_3 = \frac{\lambda}{2}$

$L_\lambda = 0 \Rightarrow -(x_1 + x_2 + x_3 - 2) = 0$

$x_1 + x_2 + x_3 = 2$

$\frac{\lambda}{6} + \frac{\lambda}{2} + \frac{\lambda}{2} = 2$

$\frac{7\lambda}{6} = 2$

$\lambda = \frac{12}{7}$

$\therefore x_1 = \frac{2}{7}, x_2 = \frac{6}{7}, x_3 = \frac{6}{7}$

Now, hessian matrix,

$$H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} & h_{x_3} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 6 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 6 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -8$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 6 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 6 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 6 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$\Delta_4 = -4 - 12 - 12 = -28$

Since both Δs are negative, it is a minima

$\therefore z_{min} = 3\left(\frac{2}{7}\right)^2 + \left(\frac{6}{7}\right)^2 + \left(\frac{6}{7}\right)^2 = \frac{12}{7} = 1.714$

10. Using the method of Lagrange's multipliers, solve the following N.L.P.P.
Optimise $z = 2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 196$
subject to $x_1 + x_2 + x_3 = 11$
$x_1, x_2, x_3 \geq 0$
[M16/MechCivil/6M][M18/Biot/8M]
Solution:
Let $f = 2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 196$
and $h = x_1 + x_2 + x_3 - 11$
Consider the Lagrangian function,
$L = f - \lambda h$
$L = (2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 196) - \lambda(x_1 + x_2 + x_3 - 11)$
$L_{x_1} = 0 \Rightarrow 4x_1 - 24 - \lambda = 0 \Rightarrow x_1 = \frac{\lambda+24}{4}$
$L_{x_2} = 0 \Rightarrow 4x_2 - 8 - \lambda = 0 \Rightarrow x_2 = \frac{\lambda+8}{4}$
$L_{x_3} = 0 \Rightarrow 4x_3 - 12 - \lambda = 0 \Rightarrow x_3 = \frac{\lambda+12}{4}$
$L_\lambda = 0 \Rightarrow -(x_1 + x_2 + x_3 - 11) = 0$
$x_1 + x_2 + x_3 = 11$
$\frac{\lambda+24}{4} + \frac{\lambda+8}{4} + \frac{\lambda+12}{4} = 11$
$\frac{3\lambda+44}{4} = 11$
$\lambda = 0$
$\therefore x_1 = 6, x_2 = 2, x_3 = 3$
Now, hessian matrix,

$$H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} & h_{x_3} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix} = -8$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 4 \\ 1 & 0 & 0 \end{vmatrix}$$

$\Delta_4 = -16 - 16 - 16 = -48$
Since both Δs are negative, it is a minima
$\therefore z_{min} = 2(6)^2 + 2(2)^2 + 2(3)^2 - 24(6) - 8(2) - 12(3) + 196 = 98$

11. Using the method of Lagrange's multipliers, solve the following N.L.P.P.

Maximise $z = 6x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $4x_1 + 3x_2 = 16$

$3x_1 + 5x_2 = 15$

$x_1, x_2 \geq 0$

[N14/ChemBiot/7M][M15/ChemBiot/8M][N18/Chem/6M]

Solution:

Let $f = 6x_1 + 8x_2 - x_1^2 - x_2^2$

Let $h_1 = 4x_1 + 3x_2 - 16$ & $h_2 = 3x_1 + 5x_2 - 15$

Consider the Lagrangian function,

$L = f - \lambda_1 h_1 - \lambda_2 h_2$

$L = 6x_1 + 8x_2 - x_1^2 - x_2^2 - \lambda_1(4x_1 + 3x_2 - 16) - \lambda_2(3x_1 + 5x_2 - 15)$

Now,

$L_{x_1} = 0 \Rightarrow 6 - 2x_1 - 4\lambda_1 - 3\lambda_2 = 0$

$L_{x_2} = 0 \Rightarrow 8 - 2x_2 - 3\lambda_1 - 5\lambda_2 = 0$

$L_{\lambda_1} = 0 \Rightarrow -(4x_1 + 3x_2 - 16) = 0 \Rightarrow 4x_1 + 3x_2 = 16$ …(1)

$L_{\lambda_2} = 0 \Rightarrow -(3x_1 + 5x_2 - 15) = 0 \Rightarrow 3x_1 + 5x_2 = 15$ …(2)

Solving (1) & (2), we get

$x_1 = \frac{35}{11}, x_2 = \frac{12}{11}$

$\therefore z_{max} = 6\left(\frac{35}{11}\right) + 8\left(\frac{12}{11}\right) - \left(\frac{35}{11}\right)^2 - \left(\frac{12}{11}\right)^2 = 16.5041$

12. Using the method of Lagrange's multipliers, solve the following N.L.P.P.

Optimise $z = 6x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $4x_1 + 3x_2 = 16$

$3x_1 + 5x_2 = 15$

$x_1, x_2 \geq 0$

[M23/CompIT/8M]

Solution:

Let $f = 6x_1 + 8x_2 - x_1^2 - x_2^2$

$h_1 = 4x_1 + 3x_2 - 16$

$h_2 = 3x_1 + 5x_2 - 15$

Lagrangian function,

$L = f - \lambda_1 h_1 - \lambda_2 h_2$

$L = (6x_1 + 8x_2 - x_1^2 - x_2^2) - \lambda_1(4x_1 + 3x_2 - 16) - \lambda_2(3x_1 + 5x_2 - 15)$

Consider,

$L_{x_1} = 0$ gives $6 - 2x_1 - 4\lambda_1 - 3\lambda_2 = 0$

$L_{x_2} = 0$ gives $8 - 2x_2 - 3\lambda_1 - 5\lambda_2 = 0$

$L_{\lambda_1} = 0$ gives $-(4x_1 + 3x_2 - 16) = 0$ i.e. $4x_1 + 3x_2 = 16$ .... (1)

$L_{\lambda_2} = 0$ gives $-(3x_1 + 5x_2 - 15) = 0$ i.e. $3x_1 + 5x_2 = 15$ .... (2)

Solving (1) & (2),

$x_1 = \frac{35}{11}, x_2 = \frac{12}{11}$

$$H_B = \begin{bmatrix} O & P \\ P' & Q \end{bmatrix}$$

Where, $P = \begin{bmatrix} h_{1_{x_1}} & h_{1_{x_2}} \\ h_{2_{x_1}} & h_{2_{x_2}} \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix}$

Where, $P' = \begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix}$

Where, $Q = \begin{bmatrix} L_{x_1x_1} & L_{x_1x_2} \\ L_{x_2x_1} & L_{x_2x_2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

$$H_B = \begin{bmatrix} 0 & 0 & 4 & 3 \\ 0 & 0 & 3 & 5 \\ 4 & 3 & -2 & 0 \\ 3 & 5 & 0 & -2 \end{bmatrix}$$

$$\Delta = 0\begin{vmatrix} 0 & 3 & 5 \\ 3 & -2 & 0 \\ 5 & 0 & -2 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 & 5 \\ 4 & -2 & 0 \\ 3 & 0 & -2 \end{vmatrix} + 4\begin{vmatrix} 0 & 0 & 5 \\ 4 & 3 & 0 \\ 3 & 5 & -2 \end{vmatrix} - 3\begin{vmatrix} 0 & 0 & 3 \\ 4 & 3 & -2 \\ 3 & 5 & 0 \end{vmatrix}$$

$\Delta = 4[55] - 3[33]$

$\Delta = 121$

Since Δ is positive, its a maxima

$$z_{max} = 6\left(\frac{35}{11}\right) + 8\left(\frac{12}{11}\right) - \left(\frac{35}{11}\right)^2 - \left(\frac{12}{11}\right)^2 = 16.5041$$

13. Solve the NLPP using the method of Lagrangian multipliers

Minimize $z = x_1^2 + x_2^2 + x_3^2$

Subject to $x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$

$x_1, x_2, x_3 \geq 0$

[N17/N18/Biot/8M]

Solution:

Let $f = x_1^2 + x_2^2 + x_3^2$

Let $h_1 = x_1 + x_2 + 3x_3 - 2$ & $h_2 = 5x_1 + 2x_2 + x_3 - 5$

Consider the Lagrangian function,

$L = f - \lambda_1 h_1 - \lambda_2 h_2$

$L = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$

Now,

$L_{x_1} = 0 \Rightarrow 2x_1 - \lambda_1 - 5\lambda_2 = 0$ i.e. $x_1 = \frac{\lambda_1 + 5\lambda_2}{2}$

$L_{x_2} = 0 \Rightarrow 2x_2 - \lambda_1 - 2\lambda_2 = 0$ i.e. $x_2 = \frac{\lambda_1 + 2\lambda_2}{2}$

$L_{x_3} = 0 \Longrightarrow 2x_3 - 3\lambda_1 - \lambda_2 = 0$ i.e. $x_3 = \frac{3\lambda_1 + \lambda_2}{2}$

$L_{\lambda_1} = 0 \Rightarrow -(x_1 + x_2 + 3x_3 - 2) = 0 \Rightarrow x_1 + x_2 + 3x_3 = 2$ …(1)

$L_{\lambda_2} = 0 \Rightarrow -(5x_1 + 2x_2 + x_3 - 5) = 0 \Rightarrow 5x_1 + 2x_2 + x_3 = 5$ …(2)

Putting x_1, x_2, x_3 in eqn (1) we get

$\left(\frac{\lambda_1 + 5\lambda_2}{2}\right) + \left(\frac{\lambda_1 + 2\lambda_2}{2}\right) + 3\left(\frac{3\lambda_1 + \lambda_2}{2}\right) = 2$

$11\lambda_1 + 10\lambda_2 = 4$ ….. (3)

Putting x_1, x_2, x_3 in eqn (2) we get

$5\left(\frac{\lambda_1 + 5\lambda_2}{2}\right) + 2\left(\frac{\lambda_1 + 2\lambda_2}{2}\right) + \left(\frac{3\lambda_1 + \lambda_2}{2}\right) = 5$

$10\lambda_1 + 30\lambda_2 = 10$ ….. (4)

Solving (3) & (4), we get

$\lambda_1 = \frac{2}{23}, \lambda_2 = \frac{7}{23}$

Thus, $x_1 = \frac{37}{46}, x_2 = \frac{8}{23}, x_3 = \frac{13}{46}$

$\therefore z_{min} = \left(\frac{37}{46}\right)^2 + \left(\frac{8}{23}\right)^2 + \left(\frac{13}{46}\right)^2 = \frac{39}{46}$

14. Solve the NLPP using the method of Lagrangian multipliers

Optimize $z = x_1^2 + x_2^2 + x_3^2$

subject to $x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$

$x_1, x_2, x_3 \geq 0$

[N22/CompITAI/8M]

Solution:

Let $f = x_1^2 + x_2^2 + x_3^2$

Let $h_1 = x_1 + x_2 + 3x_3 - 2$ & $h_2 = 5x_1 + 2x_2 + x_3 - 5$

Consider the Lagrangian function,

$L = f - \lambda_1 h_1 - \lambda_2 h_2$

$L = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$

Now,

$L_{x_1} = 0 \Rightarrow 2x_1 - \lambda_1 - 5\lambda_2 = 0$ i.e. $x_1 = \frac{\lambda_1 + 5\lambda_2}{2}$

$L_{x_2} = 0 \Rightarrow 2x_2 - \lambda_1 - 2\lambda_2 = 0$ i.e. $x_2 = \frac{\lambda_1 + 2\lambda_2}{2}$

$L_{x_3} = 0 \Longrightarrow 2x_3 - 3\lambda_1 - \lambda_2 = 0$ i.e. $x_3 = \frac{3\lambda_1 + \lambda_2}{2}$

$L_{\lambda_1} = 0 \Rightarrow -(x_1 + x_2 + 3x_3 - 2) = 0 \Rightarrow x_1 + x_2 + 3x_3 = 2$ …(1)

$L_{\lambda_2} = 0 \Rightarrow -(5x_1 + 2x_2 + x_3 - 5) = 0 \Rightarrow 5x_1 + 2x_2 + x_3 = 5$ …(2)

Putting x_1, x_2, x_3 in eqn (1) we get

$\left(\frac{\lambda_1 + 5\lambda_2}{2}\right) + \left(\frac{\lambda_1 + 2\lambda_2}{2}\right) + 3\left(\frac{3\lambda_1 + \lambda_2}{2}\right) = 2$

$11\lambda_1 + 10\lambda_2 = 4$ ….. (3)

Putting x_1, x_2, x_3 in eqn (2) we get

$5\left(\frac{\lambda_1 + 5\lambda_2}{2}\right) + 2\left(\frac{\lambda_1 + 2\lambda_2}{2}\right) + \left(\frac{3\lambda_1 + \lambda_2}{2}\right) = 5$

$10\lambda_1 + 30\lambda_2 = 10$ ….. (4)

Solving (3) & (4), we get

$\lambda_1 = \frac{2}{23}, \lambda_2 = \frac{7}{23}$

Thus, $x_1 = \frac{37}{46}, x_2 = \frac{8}{23}, x_3 = \frac{13}{46}$

Now, hessian matrix

$H^B = \begin{bmatrix} O & P \\ P' & Q \end{bmatrix}$

where, $P = \begin{bmatrix} h_{1_{x_1}} & h_{1_{x_2}} & h_{1_{x_3}} \\ h_{2_{x_1}} & h_{2_{x_2}} & h_{2_{x_3}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 1 \end{bmatrix}$

$$Q=\begin{bmatrix} L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix}=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore H^B=\begin{bmatrix} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

By C_4-C_3, C_5-3C_1

$$H^B=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & -3 & -14 \\ 1 & 5 & 2 & -2 & -6 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

$$\Delta=1\begin{vmatrix} 0 & 0 & -3 & -14 \\ 1 & 5 & -2 & -6 \\ 1 & 2 & 2 & 0 \\ 3 & 1 & 0 & 2 \end{vmatrix}=-3\begin{vmatrix} 1 & 5 & -6 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix}-(-14)\begin{vmatrix} 1 & 5 & -2 \\ 1 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix}$$

$$\Delta=-3(24)+14(38)=460$$

Since, Δ is positive, it is a minima

$$\therefore z_{min}=\left(\frac{37}{46}\right)^2+\left(\frac{8}{23}\right)^2+\left(\frac{13}{46}\right)^2=\frac{39}{46}$$

Type III: Kuhn-Tucker conditions

1. Using the Kuhn-Tucker conditions, solve the following N.L.P.P.
   Maximise $z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$
   subject to $2x_1 + x_2 \leq 5$
   $x_1, x_2 \geq 0$
   [N15/M16/CompIT/6M][N19/MTRX/8M][M22/CompITAI/5M][M23/CompIT/8M]
   Solution:
   Let $f = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$
   Let $h = 2x_1 + x_2 - 5$
   Consider, $L = f - \lambda h$
   $L = 10x_1 + 4x_2 - 2x_1^2 - x_2^2 - \lambda(2x_1 + x_2 - 5)$
   According to Kuhn Tucker conditions,
   $L_{x_1} = 0 \Rightarrow 10 - 4x_1 - 2\lambda = 0$ ……………….(1)
   $L_{x_2} = 0 \Rightarrow 4 - 2x_2 - \lambda = 0$ ……………….(2)
   $\lambda h = 0 \Rightarrow \lambda(2x_1 + x_2 - 5) = 0$ …………………..(3)
   $h \leq 0 \Rightarrow 2x_1 + x_2 - 5 \leq 0$ ……………………………….(4)
   $x_1, x_2, \lambda \geq 0$ ………………(5)
   Case I: If $\lambda = 0$
   From (1), $x_1 = \frac{5}{2}$
   From (2), $x_2 = 2$
   We see that eqn (4) is not satisfied
   Case II: If $\lambda \neq 0$
   From (1), $4x_1 + 0x_2 + 2\lambda = 10$
   From (2), $0x_1 + 2x_2 + \lambda = 4$
   From (3), $2x_1 + x_2 + 0\lambda = 5$
   On solving,
   $x_1 = \frac{11}{6}, x_2 = \frac{4}{3}, \lambda = \frac{4}{3}$
   $z_{max} = 10\left(\frac{11}{6}\right) + 4\left(\frac{4}{3}\right) - 2\left(\frac{11}{6}\right)^2 - \left(\frac{4}{3}\right)^2 = \frac{91}{6}$
   Thus, the optimal solution is $z_{max} = \frac{91}{6}$ at $x_1 = \frac{11}{6}, x_2 = \frac{4}{3}$

2. Using the Kuhn-Tucker conditions, solve the following N.L.P.P.
Maximise $z = 2x_1^2 - 7x_2^2 + 12x_1x_2$
subject to $2x_1 + 5x_2 \leq 98$
$x_1, x_2 \geq 0$
[N13/Chem/7M][M14/CompIT/6M][N14/MechCivil/8M][M18/N19/Comp/8M][M18/N18/Biot/8M][N22/M24/CompITAI/8M][D24/CompITAI/8M]
Solution:
Let $f = 2x_1^2 - 7x_2^2 + 12x_1x_2$
Let $h = 2x_1 + 5x_2 - 98$
Consider, $L = f - \lambda h$
$L = 2x_1^2 - 7x_2^2 + 12x_1x_2 - \lambda(2x_1 + 5x_2 - 98)$
According to Kuhn Tucker conditions,
$L_{x_1} = 0 \Rightarrow 4x_1 + 12x_2 - 2\lambda = 0$ ……………….(1)
$L_{x_2} = 0 \Rightarrow -14x_2 + 12x_1 - 5\lambda = 0$ ……………….(2)
$\lambda h = 0 \Rightarrow \lambda(2x_1 + 5x_2 - 98) = 0$ …………………..(3)
$h \leq 0 \Rightarrow 2x_1 + 5x_2 - 98 \leq 0$ ……………………………….(4)
$x_1, x_2, \lambda \geq 0$ ………………(5)
Case I: If $\lambda = 0$
From (1), $4x_1 + 12x_2 = 0$
From (2), $12x_1 - 14x_2 = 0$
$x_1 = 0, x_2 = 0$
$z_{max} = 2(0)^2 - 7(0)^2 + 12(0)(0) = 0$
Case II: If $\lambda \neq 0$
From (1), $4x_1 + 12x_2 - 2\lambda = 0$
From (2), $12x_1 - 14x_2 - 5\lambda = 0$
From (3), $2x_1 + 5x_2 + 0\lambda = 98$
On solving,
$x_1 = 44, x_2 = 2, \lambda = 100$
$z_{max} = 2(44)^2 - 7(2)^2 + 12(44)(2) = 4900$
Thus, the optimal solution is $z_{max} = 4900$ at $x_1 = 44, x_2 = 2$

3. Using the Kuhn-Tucker conditions, solve the following N.L.P.P.

Maximise $z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

subject to $3x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

[M15/N17/CompIT/8M]

Solution:

Let $f = 8x_1 + 10x_2 - x_1^2 - x_2^2$

Let $h = 3x_1 + 2x_2 - 6$

Consider, $L = f - \lambda h$

$L = 8x_1 + 10x_2 - x_1^2 - x_2^2 - \lambda(3x_1 + 2x_2 - 6)$

According to Kuhn Tucker conditions,

$L_{x_1} = 0 \Rightarrow 8 - 2x_1 - 3\lambda = 0$ ……………….(1)

$L_{x_2} = 0 \Rightarrow 10 - 2x_2 - 2\lambda = 0$ ……………….(2)

$\lambda h = 0 \Rightarrow \lambda(3x_1 + 2x_2 - 6) = 0$ …………………..(3)

$h \leq 0 \Rightarrow 3x_1 + 2x_2 - 6 \leq 0$ ……………………………….(4)

$x_1, x_2, \lambda \geq 0$ ………………(5)

Case I: If $\lambda = 0$

From (1), $x_1 = 4$

From (2), $x_2 = 5$

We see that eqn (4) is not satisfied

Case II: If $\lambda \neq 0$

From (1), $2x_1 + 0x_2 + 3\lambda = 8$

From (2), $0x_1 + 2x_2 + 2\lambda = 10$

From (3), $3x_1 + 2x_2 + 0\lambda = 6$

On solving,

$x_1 = \frac{4}{13}, x_2 = \frac{33}{13}, \lambda = \frac{32}{13}$

$z_{max} = 8\left(\frac{4}{13}\right) + 10\left(\frac{33}{13}\right) - \left(\frac{4}{13}\right)^2 - \left(\frac{33}{13}\right)^2 = \frac{277}{13}$

Thus, the optimal solution is $z_{max} = \frac{277}{13}$ at $x_1 = \frac{4}{13}, x_2 = \frac{33}{13}$

4. Using the Kuhn-Tucker conditions, solve the following N.L.P.P.

Maximise $z = 2x_1 + x_2 - x_1^2$

subject to $2x_1 + 3x_2 \leq 6$

$2x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$

[N14/ChemBiot/7M][N15/ChemBiot/8M][N17/Biot/6M]

Solution:

Let $f = 2x_1 + x_2 - x_1^2$

Let $h_1 = 2x_1 + 3x_2 - 6, h_2 = 2x_1 + x_2 - 4$

Consider, $L = f - \lambda_1 h_1 - \lambda_2 h_2$

$L = 2x_1 + x_2 - x_1^2 - \lambda_1(2x_1 + 3x_2 - 6) - \lambda_2(2x_1 + x_2 - 4)$

According to Kuhn Tucker conditions,

$L_{x_1} = 0 \Rightarrow 2 - 2x_1 - 2\lambda_1 - 2\lambda_2 = 0$ ……………….(1)

$L_{x_2} = 0 \Rightarrow 1 - 3\lambda_1 - \lambda_2 = 0$ ……………….(2)

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(2x_1 + 3x_2 - 6) = 0$ …………………..(3)

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(2x_1 + x_2 - 4) = 0$ ……………….(4)

$h_1 \leq 0 \Rightarrow 2x_1 + 3x_2 - 6 \leq 0$ ……………………………….(5)

$h_2 \leq 0 \Rightarrow 2x_1 + x_2 - 4 \leq 0$ …………………………..(6)

$x_1, x_2, \lambda_1, \lambda_2 \geq 0$ ……………..(7)

Case I: If $\lambda_1 = 0, \lambda_2 = 0$

From (1), $x_1 = 1$

From (2), $1 = 0$ which is absurd

Case II: If $\lambda_1 = 0, \lambda_2 \neq 0$

From (1), $2x_1 + 0x_2 + 2\lambda_2 = 2$

From (2), $\lambda_2 = 1$

From (4), $2x_1 + x_2 + 0\lambda_2 = 4$

On solving,

$x_1 = 0, x_2 = 4, \lambda_2 = 1$

We see that, eqn (5) is not satisfied

Case III: If $\lambda_1 \neq 0, \lambda_2 = 0$

From (1), $2x_1 + 0x_2 + 2\lambda_1 = 2$

From (2), $\lambda_1 = \frac{1}{3}$

From (3), $2x_1 + 3x_2 + 0\lambda_1 = 6$

On solving,

$x_1 = \frac{2}{3}, x_2 = \frac{14}{9}, \lambda_1 = \frac{1}{3}$

$z_{max} = 2\left(\frac{2}{3}\right) + \left(\frac{14}{9}\right) - \left(\frac{2}{3}\right)^2 = \frac{22}{9}$

Case IV: If $\lambda_1 \neq 0, \lambda_2 \neq 0$
From (3), $2x_1 + 3x_2 = 6$
From (4), $2x_1 + x_2 = 4$
On solving,
$x_1 = \frac{3}{2}, x_2 = 1$
From (1), $2\lambda_1 + 2\lambda_2 = -1$
From (2), $3\lambda_1 + \lambda_2 = 1$
On solving,
$\lambda_1 = \frac{3}{4}, \lambda_2 = -\frac{5}{4}$
We see that eqn (7) is not satisfied
Thus, the optimal solution is $z_{max} = \frac{22}{9}$ at $x_1 = \frac{2}{3}, x_2 = \frac{14}{9}$

5. Using the Kuhn-Tucker conditions, solve the following N.L.P.P.
Maximise $z = 2x_1 + 3x_2 - x_1^2 - x_2^2$
subject to $x_1 + x_2 \leq 1$
$2x_1 + 3x_2 \leq 6$
$x_1, x_2 \geq 0$
[N13/Biot/8M][M16/MechCivil/8M]
Solution:
Let $f = 2x_1 + 3x_2 - x_1^2 - x_2^2$
Let $h_1 = x_1 + x_2 - 1, h_2 = 2x_1 + 3x_2 - 6$
Consider, $L = f - \lambda_1 h_1 - \lambda_2 h_2$
$L = 2x_1 + 3x_2 - x_1^2 - x_2^2 - \lambda_1(x_1 + x_2 - 1) - \lambda_2(2x_1 + 3x_2 - 6)$
According to Kuhn Tucker conditions,
$L_{x_1} = 0 \Rightarrow 2 - 2x_1 - \lambda_1 - 2\lambda_2 = 0$ ……………….(1)
$L_{x_2} = 0 \Rightarrow 3 - 2x_2 - \lambda_1 - 3\lambda_2 = 0$ ……………….(2)
$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + x_2 - 1) = 0$ …………………..(3)
$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(2x_1 + 3x_2 - 6) = 0$ ……………….(4)
$h_1 \leq 0 \Rightarrow x_1 + x_2 - 1 \leq 0$ ……………………………….(5)
$h_2 \leq 0 \Rightarrow 2x_1 + 3x_2 - 6 \leq 0$ ………………………….(6)
$x_1, x_2, \lambda_1, \lambda_2 \geq 0$ ………………(7)
Case I: If $\lambda_1 = 0, \lambda_2 = 0$
From (1), $x_1 = 1$
From (2), $x_2 = \frac{3}{2}$
We see that eqn (5) is not satisfied
Case II: If $\lambda_1 = 0, \lambda_2 \neq 0$
From (1), $2x_1 + 0x_2 + 2\lambda_2 = 2$
From (2), $0x_1 + 2x_2 + 3\lambda_2 = 3$
From (4), $2x_1 + 3x_2 + 0\lambda_2 = 6$
On solving,
$x_1 = \frac{12}{13}, x_2 = \frac{18}{13}, \lambda_2 = \frac{1}{13}$
We see that, eqn (5) is not satisfied
Case III: If $\lambda_1 \neq 0, \lambda_2 = 0$
From (1), $2x_1 + 0x_2 + \lambda_1 = 2$
From (2), $0x_1 + 2x_2 + \lambda_1 = 3$
From (3), $x_1 + x_2 + 0\lambda_1 = 1$
On solving,
$x_1 = \frac{1}{4}, x_2 = \frac{3}{4}, \lambda_1 = \frac{3}{2}$

$z_{max} = 2\left(\frac{1}{4}\right) + 3\left(\frac{3}{4}\right) - \left(\frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2 = \frac{17}{8}$

Case IV: If $\lambda_1 \neq 0, \lambda_2 \neq 0$

From (3), $x_1 + x_2 = 1$

From (4), $2x_1 + 3x_2 = 6$

On solving,

$x_1 = -3, x_2 = 4$

We see that eqn (7) is not satisfied

Thus, the optimal solution is $z_{max} = \frac{17}{8}$ at $x_1 = \frac{1}{4}, x_2 = \frac{3}{4}$

6. Using the Kuhn-Tucker conditions, solve the following N.L.P.P.

Minimise $z = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$

subject to $x_1 + 3x_2 \leq 6$

$5x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$

[M15/M16/ChemBiot/8M][N15/MechCivil/8M]

Solution:

Let $f = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$

Let $h_1 = x_1 + 3x_2 - 6, h_2 = 5x_1 + 2x_2 - 10$

Consider, $L = f - \lambda_1 h_1 - \lambda_2 h_2$

$L = 2x_1 + 3x_2 - x_1^2 - 2x_2^2 - \lambda_1(x_1 + 3x_2 - 6) - \lambda_2(5x_1 + 2x_2 - 10)$

According to Kuhn Tucker conditions,

$L_{x_1} = 0 \Rightarrow 2 - 2x_1 - \lambda_1 - 5\lambda_2 = 0$ ……………….(1)

$L_{x_2} = 0 \Rightarrow 3 - 4x_2 - 3\lambda_1 - 2\lambda_2 = 0$ ……………….(2)

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + 3x_2 - 6) = 0$ …………………..(3)

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(5x_1 + 2x_2 - 10) = 0$ ……………….(4)

$h_1 \leq 0 \Rightarrow x_1 + 3x_2 - 6 \leq 0$ ……………………………….(5)

$h_2 \leq 0 \Rightarrow 5x_1 + 2x_2 - 10 \leq 0$ ………………………….(6)

$x_1, x_2 \geq 0$ & $\lambda_1, \lambda_2 \leq 0$ ………………(7)

Case I: If $\lambda_1 = 0, \lambda_2 = 0$

From (1), $x_1 = 1$

From (2), $x_2 = \frac{3}{4}$

$z_{min} = 2(1) + 3\left(\frac{3}{4}\right) - (1)^2 - 2\left(\frac{3}{4}\right)^2 = \frac{17}{8} = 2.125$

Case II: If $\lambda_1 = 0, \lambda_2 \neq 0$

From (1), $2x_1 + 0x_2 + 5\lambda_2 = 2$

From (2), $0x_1 + 4x_2 + 2\lambda_2 = 3$

From (4), $5x_1 + 2x_2 + 0\lambda_2 = 10$

On solving,

$x_1 = \frac{89}{54}, x_2 = \frac{95}{108}, \lambda_2 = -\frac{7}{27}$

$z_{min} = 2\left(\frac{89}{54}\right) + 3\left(\frac{95}{108}\right) - \left(\frac{89}{54}\right)^2 - 2\left(\frac{95}{108}\right)^2 = \frac{361}{216} = 1.671$

Case III: If $\lambda_1 \neq 0, \lambda_2 = 0$

From (1), $2x_1 + 0x_2 + \lambda_1 = 2$

From (2), $0x_1 + 4x_2 + 3\lambda_1 = 3$

From (3), $x_1 + 3x_2 + 0\lambda_1 = 6$

On solving,

$x_1 = \frac{3}{2}, x_2 = \frac{3}{2}, \lambda_1 = -1$

$z_{min} = 2\left(\frac{3}{2}\right) + 3\left(\frac{3}{2}\right) - \left(\frac{3}{2}\right)^2 - 2\left(\frac{3}{2}\right)^2 = \frac{3}{4} = 0.75$

Case IV: If $\lambda_1 \neq 0, \lambda_2 \neq 0$

From (3), $x_1 + 3x_2 = 6$

From (4), $5x_1 + 2x_2 = 10$

On solving,

$x_1 = \frac{18}{13}, x_2 = \frac{20}{13}$

From (1), $\lambda_1 + 5\lambda_2 = -\frac{10}{13}$

From (2), $3\lambda_1 + 2\lambda_2 = -\frac{41}{13}$

On solving,

$\lambda_1 = -\frac{185}{169}, \lambda_2 = \frac{11}{169}$

$z_{min} = 2\left(\frac{18}{13}\right) + 3\left(\frac{20}{13}\right) - \left(\frac{18}{13}\right)^2 - 2\left(\frac{20}{13}\right)^2 = 0.7337$

Thus, the optimal solution is $z_{min} = 0.75$ at $x_1 = \frac{3}{2}, x_2 = \frac{3}{2}$

7. Using the Kuhn-Tucker conditions, solve the following N.L.P.P.
Maximise $z = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$
subject to $x_1 + 3x_2 \leq 6$
$5x_1 + 2x_2 \leq 10$
$x_1, x_2 \geq 0$
[M18/Chem/8M]
Solution:
Let $f = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$
Let $h_1 = x_1 + 3x_2 - 6, h_2 = 5x_1 + 2x_2 - 10$
Consider, $L = f - \lambda_1 h_1 - \lambda_2 h_2$
$L = 2x_1 + 3x_2 - x_1^2 - 2x_2^2 - \lambda_1(x_1 + 3x_2 - 6) - \lambda_2(5x_1 + 2x_2 - 10)$
According to Kuhn Tucker conditions,
$L_{x_1} = 0 \Rightarrow 2 - 2x_1 - \lambda_1 - 5\lambda_2 = 0$ ……………….(1)
$L_{x_2} = 0 \Rightarrow 3 - 4x_2 - 3\lambda_1 - 2\lambda_2 = 0$ ……………….(2)
$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + 3x_2 - 6) = 0$ …………………..(3)
$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(5x_1 + 2x_2 - 10) = 0$ ……………….(4)
$h_1 \leq 0 \Rightarrow x_1 + 3x_2 - 6 \leq 0$ ……………………………….(5)
$h_2 \leq 0 \Rightarrow 5x_1 + 2x_2 - 10 \leq 0$ …………………………..(6)
$x_1, x_2 \geq 0$ & $\lambda_1, \lambda_2 \geq 0$ ……………..(7)
Case I: If $\lambda_1 = 0, \lambda_2 = 0$
From (1), $x_1 = 1$
From (2), $x_2 = \frac{3}{4}$
$z_{max} = 2(1) + 3\left(\frac{3}{4}\right) - (1)^2 - 2\left(\frac{3}{4}\right)^2 = \frac{17}{8} = 2.125$
Case II: If $\lambda_1 = 0, \lambda_2 \neq 0$
From (1), $2x_1 + 0x_2 + 5\lambda_2 = 2$
From (2), $0x_1 + 4x_2 + 2\lambda_2 = 3$
From (4), $5x_1 + 2x_2 + 0\lambda_2 = 10$
On solving,
$x_1 = \frac{89}{54}, x_2 = \frac{95}{108}, \lambda_2 = -\frac{7}{27}$
Eqn (7) is not satisfied
Case III: If $\lambda_1 \neq 0, \lambda_2 = 0$
From (1), $2x_1 + 0x_2 + \lambda_1 = 2$
From (2), $0x_1 + 4x_2 + 3\lambda_1 = 3$
From (3), $x_1 + 3x_2 + 0\lambda_1 = 6$
On solving,
$x_1 = \frac{3}{2}, x_2 = \frac{3}{2}, \lambda_1 = -1$

Eqn (7) is not satisfied

Case IV: If $\lambda_1 \neq 0, \lambda_2 \neq 0$

From (3), $x_1 + 3x_2 = 6$

From (4), $5x_1 + 2x_2 = 10$

On solving,

$x_1 = \frac{18}{13}, x_2 = \frac{20}{13}$

From (1), $\lambda_1 + 5\lambda_2 = -\frac{10}{13}$

From (2), $3\lambda_1 + 2\lambda_2 = -\frac{41}{13}$

On solving,

$\lambda_1 = -\frac{185}{169}, \lambda_2 = \frac{11}{169}$

Eqn (7) is not satisfied

Thus, the optimal solution is $z_{max} = 2.125$ at $x_1 = 1, x_2 = \frac{3}{4}$

8. Using the Kuhn-Tucker conditions, solve the following N.L.P.P.
Minimise $z = 7x_1^2 + 5x_2^2 - 6x_1$
subject to $x_1 + 2x_2 \leq 10$
$x_1 + 3x_2 \leq 9$
$x_1, x_2 \geq 0$
[M17/CompIT/8M]
Solution:
Let $f = 7x_1^2 + 5x_2^2 - 6x_1$
Let $h_1 = x_1 + 2x_2 - 10, h_2 = x_1 + 3x_2 - 9$
Consider, $L = f - \lambda_1 h_1 - \lambda_2 h_2$
$L = 7x_1^2 + 5x_2^2 - 6x_1 - \lambda_1(x_1 + 2x_2 - 10) - \lambda_2(x_1 + 3x_2 - 9)$
According to Kuhn Tucker conditions,
$L_{x_1} = 0 \Rightarrow 14x_1 - 6 - \lambda_1 - \lambda_2 = 0$ ……………….(1)
$L_{x_2} = 0 \Rightarrow 10x_2 - 2\lambda_1 - 3\lambda_2 = 0$ ……………….(2)
$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + 2x_2 - 10) = 0$ …………………..(3)
$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(x_1 + 3x_2 - 9) = 0$ ……………….(4)
$h_1 \leq 0 \Rightarrow x_1 + 2x_2 - 10 \leq 0$ ……………………………….(5)
$h_2 \leq 0 \Rightarrow x_1 + 3x_2 - 9 \leq 0$ …………………………..(6)
$x_1, x_2 \geq 0$ & $\lambda_1, \lambda_2 \leq 0$ ……………..(7)
Case I: If $\lambda_1 = 0, \lambda_2 = 0$
From (1), $x_1 = \frac{3}{7}$
From (2), $x_2 = 0$
$z_{min} = 7\left(\frac{3}{7}\right)^2 + 5(0)^2 - 6\left(\frac{3}{7}\right) = -\frac{9}{7} = -1.2857$
Case II: If $\lambda_1 = 0, \lambda_2 \neq 0$
From (1), $14x_1 + 0x_2 - \lambda_2 = 6$
From (2), $0x_1 + 10x_2 - 3\lambda_2 = 0$
From (4), $x_1 + 3x_2 + 0\lambda_2 = 9$
On solving,
$x_1 = \frac{18}{17}, x_2 = \frac{45}{17}, \lambda_2 = \frac{150}{17}$
Eqn (7) is not satisfied
Case III: If $\lambda_1 \neq 0, \lambda_2 = 0$
From (1), $14x_1 + 0x_2 - \lambda_1 = 6$
From (2), $0x_1 + 10x_2 - 2\lambda_1 = 0$
From (3), $x_1 + 2x_2 + 0\lambda_1 = 10$
On solving,
$x_1 = \frac{62}{33}, x_2 = \frac{134}{33}, \lambda_1 = \frac{670}{33}$

Eqn (7) is not satisfied

Case IV: If $\lambda_1 \neq 0, \lambda_2 \neq 0$

From (3), $x_1 + 2x_2 = 10$

From (4), $x_1 + 3x_2 = 9$

On solving,

$x_1 = 12, x_2 = -1$

Eqn (7) is not satisfied

Thus, the optimal solution is $z_{min} = -\frac{9}{7}$ at $x_1 = \frac{3}{7}, x_2 = 0$

9. Using the Kuhn-Tucker conditions, solve the following N.L.P.P.

Maximise $z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

subject to $x_1 + x_2 \leq 2$

$2x_1 + 3x_2 \leq 12$

$x_1, x_2, x_3 \geq 0$

[M14/N16/ChemBiot/8M][M14/N17/MechCivil/8M][N14/CompIT/8M][N18/MTRX/8M][N19/Biot/8M][D23/CompITAI/8M]

Solution:

Let $f = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

Let $h_1 = x_1 + x_2 - 2, h_2 = 2x_1 + 3x_2 - 12$

Consider, $L = f - \lambda_1 h_1 - \lambda_2 h_2$

$L = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2 - \lambda_1(x_1 + x_2 - 2) - \lambda_2(2x_1 + 3x_2 - 12)$

According to Kuhn Tucker conditions,

$L_{x_1} = 0 \Rightarrow -2x_1 + 4 - \lambda_1 - 2\lambda_2 = 0$ ……………….(1)

$L_{x_2} = 0 \Rightarrow -2x_2 + 6 - \lambda_1 - 3\lambda_2 = 0$ ……………….(2)

$L_{x_3} = 0 \Rightarrow -2x_3 = 0 \Rightarrow x_3 = 0$

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + x_2 - 2) = 0$ …………………..(3)

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(2x_1 + 3x_2 - 12) = 0$ ……………….(4)

$h_1 \leq 0 \Rightarrow x_1 + x_2 - 2 \leq 0$ ……………………………….(5)

$h_2 \leq 0 \Rightarrow 2x_1 + 3x_2 - 12 \leq 0$ …………………………..(6)

$x_1, x_2, \lambda_1, \lambda_2 \geq 0$ ………………(7)

Case I: If $\lambda_1 = 0, \lambda_2 = 0$

From (1), $x_1 = 2$

From (2), $x_2 = 3$

We see that eqn (5) is not satisfied

Case II: If $\lambda_1 = 0, \lambda_2 \neq 0$

From (1), $2x_1 + 0x_2 + 2\lambda_2 = 4$

From (2), $0x_1 + 2x_2 + 3\lambda_2 = 6$

From (4), $2x_1 + 3x_2 + 0\lambda_2 = 12$

On solving,

$x_1 = \frac{24}{13}, x_2 = \frac{36}{13}, \lambda_2 = \frac{2}{13}$

We see that, eqn (5) is not satisfied

Case III: If $\lambda_1 \neq 0, \lambda_2 = 0$

From (1), $2x_1 + 0x_2 + \lambda_1 = 4$

From (2), $0x_1 + 2x_2 + \lambda_1 = 6$

From (3), $x_1 + x_2 + 0\lambda_1 = 2$

On solving,

$x_1 = \frac{1}{2}, x_2 = \frac{3}{2}, \lambda_1 = 3$

$z_{max} = -\left(\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - (0)^2 + 4\left(\frac{1}{2}\right) + 6\left(\frac{3}{2}\right) = \frac{17}{2}$

Case IV: If $\lambda_1 \neq 0, \lambda_2 \neq 0$

From (3), $x_1 + x_2 = 2$

From (4), $2x_1 + 3x_2 = 12$

On solving,

$x_1 = -6, x_2 = 8$

We see that eqn (7) is not satisfied

Thus, the optimal solution is $z_{max} = \frac{17}{2}$ at $x_1 = \frac{1}{2}, x_2 = \frac{3}{2}, x_3 = 0$

10. Using the Kuhn-Tucker conditions, solve the following N.L.P.P.

Maximise $z = x_1^2 + x_2^2$

subject to $x_1 + x_2 - 4 \leq 0$

$2x_1 + x_2 - 5 \leq 0$

$x_1, x_2 \geq 0$

[M15/MechCivil/8M][N18/Comp/8M][N19/Chem/8M]

Solution:

Let $f = x_1^2 + x_2^2$

Let $h_1 = x_1 + x_2 - 4, h_2 = 2x_1 + x_2 - 5$

Consider, $L = f - \lambda_1 h_1 - \lambda_2 h_2$

$L = x_1^2 + x_2^2 - \lambda_1(x_1 + x_2 - 4) - \lambda_2(2x_1 + x_2 - 5)$

According to Kuhn Tucker conditions,

$L_{x_1} = 0 \Rightarrow 2x_1 - \lambda_1 - 2\lambda_2 = 0$ ……………….(1)

$L_{x_2} = 0 \Rightarrow 2x_2 - \lambda_1 - \lambda_2 = 0$ ……………….(2)

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + x_2 - 4) = 0$ …………………..(3)

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(2x_1 + x_2 - 5) = 0$ ……………….(4)

$h_1 \leq 0 \Rightarrow x_1 + x_2 - 4 \leq 0$ ……………………………….(5)

$h_2 \leq 0 \Rightarrow 2x_1 + x_2 - 5 \leq 0$ …………………………..(6)

$x_1, x_2, \lambda_1, \lambda_2 \geq 0$ ………………(7)

Case I: If $\lambda_1 = 0, \lambda_2 = 0$

From (1), $x_1 = 0$

From (2), $x_2 = 0$

$z_{max} = 0$

Case II: If $\lambda_1 = 0, \lambda_2 \neq 0$

From (1), $2x_1 + 0x_2 - 2\lambda_2 = 0$

From (2), $0x_1 + 2x_2 - \lambda_2 = 0$

From (4), $2x_1 + x_2 + 0\lambda_2 = 5$

On solving,

$x_1 = 2, x_2 = 1, \lambda_2 = 2$

$z_{max} = 2^2 + 1^2 = 5$

Case III: If $\lambda_1 \neq 0, \lambda_2 = 0$

From (1), $2x_1 + 0x_2 - \lambda_1 = 0$

From (2), $0x_1 + 2x_2 - \lambda_1 = 0$

From (3), $x_1 + x_2 + 0\lambda_1 = 4$

On solving,

$x_1 = 2, x_2 = 2, \lambda_1 = 4$

We see that eqn (6) is not satisfied

Case IV: If $\lambda_1 \neq 0, \lambda_2 \neq 0$
From (3), $x_1 + x_2 = 4$
From (4), $2x_1 + x_2 = 5$
On solving,
$x_1 = 1, x_2 = 3$
From (1), $\lambda_1 + 2\lambda_2 = 2$
From (2), $\lambda_1 + \lambda_2 = 6$
On solving,
$\lambda_1 = 10, \lambda_2 = -4$
We see that eqn (7) is not satisfied
Thus, the optimal solution is $z_{max} = 5$ at $x_1 = 2, x_2 = 1$

11. Using the Kuhn-Tucker conditions, solve the following N.L.P.P.
Maximise $z = 10x_1 + 10x_2 - x_1^2 - x_2^2$
subject to $x_1 + x_2 \leq 8$
$-x_1 + x_2 \leq 5$
$x_1, x_2 \geq 0$
[N16/M17/MechCivil/8M][N18/Chem/8M][M19/MTRX/8M][M19/Comp/8M]
Solution:
Let $f = 10x_1 + 10x_2 - x_1^2 - x_2^2$
Let $h_1 = x_1 + x_2 - 8, h_2 = -x_1 + x_2 - 5$
Consider, $L = f - \lambda_1 h_1 - \lambda_2 h_2$
$L = 10x_1 + 10x_2 - x_1^2 - x_2^2 - \lambda_1(x_1 + x_2 - 8) - \lambda_2(-x_1 + x_2 - 5)$
According to Kuhn Tucker conditions,
$L_{x_1} = 0 \Rightarrow 10 - 2x_1 - \lambda_1 + \lambda_2 = 0$ ……………….(1)
$L_{x_2} = 0 \Rightarrow 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$ ……………….(2)
$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + x_2 - 8) = 0$ …………………..(3)
$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(-x_1 + x_2 - 5) = 0$ ……………….(4)
$h_1 \leq 0 \Rightarrow x_1 + x_2 - 8 \leq 0$ ……………………………….(5)
$h_2 \leq 0 \Rightarrow -x_1 + x_2 - 5 \leq 0$ ………………………….(6)
$x_1, x_2\ \lambda_1, \lambda_2 \geq 0$ ………………(7)
Case I: If $\lambda_1 = 0, \lambda_2 = 0$
From (1), $x_1 = 5$
From (2), $x_2 = 5$
We see that eqn (5) is not satisfied
Case II: If $\lambda_1 = 0, \lambda_2 \neq 0$
From (1), $2x_1 + 0x_2 - \lambda_2 = 10$
From (2), $0x_1 + 2x_2 + \lambda_2 = 10$
From (4), $-x_1 + x_2 + 0\lambda_2 = 5$
On solving,
$x_1 = \frac{5}{2}, x_2 = \frac{15}{2}, \lambda_2 = -5$
We see that, eqn (7) is not satisfied
Case III: If $\lambda_1 \neq 0, \lambda_2 = 0$
From (1), $2x_1 + 0x_2 + \lambda_1 = 10$
From (2), $0x_1 + 2x_2 + \lambda_1 = 10$
From (3), $x_1 + x_2 + 0\lambda_1 = 8$
On solving,
$x_1 = 4, x_2 = 4, \lambda_1 = 2$
$z_{max} = 10(4) + 10(4) - 4^2 - 4^2 = 48$

Case IV: If $\lambda_1 \neq 0, \lambda_2 \neq 0$
From (3), $x_1 + x_2 = 8$
From (4), $-x_1 + x_2 = 5$
On solving,
$x_1 = \frac{3}{2}, x_2 = \frac{13}{2}$
From (1), $\lambda_1 - \lambda_2 = 7$
From (2), $\lambda_1 + \lambda_2 = -3$
On solving,
$\lambda_1 = 2, \lambda_2 = -5$
We see that eqn (7) is not satisfied
Thus, the optimal solution is $z_{max} = 48$ at $x_1 = 4, x_2 = 4$

12. Using the Kuhn-Tucker conditions, solve the following N.L.P.P.

Maximise $z = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

subject to $x_1 + x_2 \leq 10$

$x_2 \leq 8$

$x_1, x_2 \geq 0$

[M19/Chem/8M]

Solution:

Let $f = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

Let $h_1 = x_1 + x_2 - 10, h_2 = 0x_1 + x_2 - 8$

Consider, $L = f - \lambda_1 h_1 - \lambda_2 h_2$

$L = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2 - \lambda_1(x_1 + x_2 - 10) - \lambda_2(0x_1 + x_2 - 8)$

According to Kuhn Tucker conditions,

$L_{x_1} = 0 \Rightarrow 12 + 2x_2 - 4x_1 - \lambda_1 = 0$ ……………….(1)

$L_{x_2} = 0 \Rightarrow 21 + 2x_1 - 4x_2 - \lambda_1 - \lambda_2 = 0$ ……………….(2)

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + x_2 - 10) = 0$ …………………..(3)

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(0x_1 + x_2 - 8) = 0$ ……………….(4)

$h_1 \leq 0 \Rightarrow x_1 + x_2 - 10 \leq 0$ ……………………………….(5)

$h_2 \leq 0 \Rightarrow 0x_1 + x_2 - 8 \leq 0$ …………………………..(6)

$x_1, x_2\ \lambda_1, \lambda_2 \geq 0$ ……………..(7)

Case I: If $\lambda_1 = 0, \lambda_2 = 0$

From (1), $4x_1 - 2x_2 = 12$

From (2), $-2x_1 + 4x_2 = 21$

Solving the equations, we get, $x_1 = \frac{15}{2}, x_2 = 9$

We see that eqn (6) is not satisfied

Case II: If $\lambda_1 = 0, \lambda_2 \neq 0$

From (1), $4x_1 - 2x_2 + 0\lambda_2 = 12$

From (2), $-2x_1 + 4x_2 + \lambda_2 = 21$

From (4), $0x_1 + x_2 + 0\lambda_2 = 8$

On solving,

$x_1 = 7, x_2 = 8, \lambda_2 = 3$

We see that eqn (5) is not satisfied

Case III: If $\lambda_1 \neq 0, \lambda_2 = 0$

From (1), $4x_1 - 2x_2 + \lambda_1 = 12$

From (2), $-2x_1 + 4x_2 + \lambda_1 = 21$

From (3), $x_1 + x_2 + 0\lambda_1 = 10$

On solving,

$x_1 = \frac{17}{4}, x_2 = \frac{23}{4}, \lambda_1 = \frac{13}{2}$

$\therefore z_{max} = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

$z_{max} = 12\left(\frac{17}{4}\right) + 21\left(\frac{23}{4}\right) + 2\left(\frac{17}{4}\right)\left(\frac{23}{4}\right) - 2\left(\frac{17}{4}\right)^2 - 2\left(\frac{23}{4}\right)^2 = \frac{947}{8} = 118.375$

Case IV: If $\lambda_1 \neq 0, \lambda_2 \neq 0$

From (3), $x_1 + x_2 = 10$

From (4), $0x_1 + x_2 = 8$

On solving, we get $x_1 = 2, x_2 = 8$

From (1), $\lambda_1 = 20$

From (2), $\lambda_2 = -27$

We see that eqn (7) is not satisfied

Thus, the optimal solution is $z_{max} = \frac{947}{8}$ at $x_1 = \frac{17}{4}, x_2 = \frac{23}{4}$