Matrices

Weight Distribution of Types

MechCivil

Type	Name	Nov 2019	May 2022	Nov 2022	Jan 2023	May SE 2023	May DSE 2023	Nov 2023	June 2024	Nov 2024
I	Eigen Values & Vectors	05	---	06	13	05	11	05	11	05
II	Diagonalising	06	05	08	08	08	08	08	08	08
III	Cayley Hamilton Thm	06	07	05	---	06	---	06	---	06
IV	Functions of matrix	---	05	06	---	---	---	---	---	---
V	Derogatory	06	---	---	---	---	---	---	---	---
Total Marks		**23**	**17**	**25**	**21**	**19**	**19**	**19**	**19**	**19**

Extc

Type	Name	Nov 2019	May 2022	Nov 2022	Jan 2023	May SE 2023	May DSE 2023	Nov 2023	June 2024	Nov 2024
I	Eigen Values & Vectors	05	09	11	17	11	17	11	11	13
II	Diagonalising	08	---	08	08	08	08	08	06	08
III	Cayley Hamilton Thm	06	05	06	---	06	---	06	08	06
IV	Functions of matrix	06	05	---	---	---	---	---	---	---
Total Marks		**25**	**19**	**25**	**25**	**25**	**25**	**25**	**25**	**27**

Elect

Type	Name	Nov 2019	May 2022	Nov 2022	Jan 2023	May SE 2023	May DSE 2023	Nov 2023	June 2024	Nov 2024
I	Eigen Values & Vectors	06	09	13	17	11	17	11	11	13
II	Diagonalising	08	---	08	08	08	08	08	06	08
III	Cayley Hamilton Thm	06	05	06	---	06	---	06	08	06
IV	Functions of matrix	06	05	---	---	---	---	---	---	---
V	Derogatory	06	---	---	---	---	---	---	---	---
VI	Theory	05	---	---	---	---	---	---	---	---
Total Marks		**37**	**19**	**27**	**25**	**25**	**25**	**25**	**25**	**27**

Comp/IT/AI

Type	Name	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
I	Eigen Values & Vectors	11	11	06	12	12	05	11	11	11
II	Diagonalising	---	---	---	05	08	08	06	06	06
III	Cayley Hamilton Thm	06	06	06	---	06	06	06	06	06
IV	Functions of matrix	---	---	05	---	---	---	---	---	---
V	Derogatory	06	06	06	---	---	---	---	---	---
Total Marks		**23**	**23**	**23**	**17**	**26**	**19**	**23**	**23**	**23**

Type I: Eigen Values & Eigen Vectors

1. Find eigen values and eigen vectors of $A = \begin{bmatrix} 3 & 4 \\ 5 & -5 \end{bmatrix}$

Solution:

$A = \begin{bmatrix} 3 & 4 \\ 5 & -5 \end{bmatrix}$

$|A| = (3)(-5) - (5)(4) = -15 - 20 = -35$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 4 \\ 5 & -5-\lambda \end{vmatrix} = 0$

$(3-\lambda)(-5-\lambda) - (5)(4) = 0$

$-15 - 3\lambda + 5\lambda + \lambda^2 - 20 = 0$

$\lambda^2 + 2\lambda - 35 = 0$

$\lambda = 5, -7$ (Eigen Values)

OR

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 4 \\ 5 & -5-\lambda \end{vmatrix} = 0$

$\boldsymbol{\lambda^2 - [sum\ of\ diagonals]\lambda + |A| = 0}$

$\lambda^2 - [3-5]\lambda + (-35) = 0$

$\lambda^2 + 2\lambda - 35 = 0$

$\lambda = 5, -7$

Now,

(i) For $\lambda = 5$

$[A - \lambda I]X = 0$

$\begin{bmatrix} -2 & 4 \\ 5 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2x_1 + 4x_2 = 0$ or $5x_1 - 10x_2 = 0$

$-2x_1 = -4x_2$ or $5x_1 = 10x_2$

$\frac{x_1}{-4} = \frac{x_2}{-2}$ or $\frac{x_1}{10} = \frac{x_2}{5}$

$\frac{x_1}{2} = \frac{x_2}{1}$ or $\frac{x_1}{2} = \frac{x_2}{1}$

Thus, $X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = [2,1]'$

(ii) For $\lambda = -7$

$[A - \lambda I]X = 0$

$\begin{bmatrix} 10 & 4 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$10x_1 + 4x_2 = 0$

$10x_1 = -4x_2$

$\frac{x_1}{-4} = \frac{x_2}{10}$

$\frac{x_1}{-2} = \frac{x_2}{5}$

Thus, $X_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix} = [-2,5]'$

2. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$

[M14/ElexExtcElectBiomInst/6M][N22/MTRX/5M]

Solution:

$A = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}, |A| = 216$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [-2+7-2]\lambda^2 + \left[\begin{vmatrix} 7 & 5 \\ 5 & -2 \end{vmatrix} + \begin{vmatrix} -2 & 4 \\ 4 & -2 \end{vmatrix} + \begin{vmatrix} -2 & 5 \\ 5 & 7 \end{vmatrix}\right]\lambda - 216 = 0$

$\lambda^3 - 3\lambda^2 - 90\lambda - 216 = 0$

$(\lambda - 12)(\lambda + 3)(\lambda + 6) = 0$

$\lambda = 12, -3, -6$

(i) For $\lambda = 12$, $[A - \lambda I]X = 0$ gives

$\begin{bmatrix} -14 & 5 & 4 \\ 5 & -5 & 5 \\ 4 & 5 & -14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-14x_1 + 5x_2 + 4x_3 = 0$

$5x_1 - 5x_2 + 5x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 5 & 4 \\ -5 & 5 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -14 & 4 \\ 5 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -14 & 5 \\ 5 & -5 \end{vmatrix}}$

$\frac{x_1}{45} = -\frac{x_2}{-90} = \frac{x_3}{45}$

$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 12$ the eigen vector is $X_1 = [1,2,1]'$

(ii) For $\lambda = -3$, $[A - \lambda I]X = 0$ gives

$\begin{bmatrix} 1 & 5 & 4 \\ 5 & 10 & 5 \\ 4 & 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + 5x_2 + 4x_3 = 0$

$5x_1 + 10x_2 + 5x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 5 & 4 \\ 10 & 5 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 5 \\ 5 & 10 \end{vmatrix}}$

$\frac{x_1}{-15} = -\frac{x_2}{-15} = \frac{x_3}{-15}$

$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = -3$ the eigen vector is $X_2 = [1, -1, 1]'$

(iii) For $\lambda = -6$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 4 & 5 & 4 \\ 5 & 13 & 5 \\ 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x_1 + 5x_2 + 4x_3 = 0$

$5x_1 + 13x_2 + 5x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 5 & 4 \\ 13 & 5 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 4 & 4 \\ 5 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & 5 \\ 5 & 13 \end{vmatrix}}$

$\frac{x_1}{-27} = -\frac{x_2}{0} = \frac{x_3}{27}$

$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = -6$ the eigen vector is $X_3 = [-1, 0, 1]'$

3. Find the eigen values and eigen vectors of $A=\begin{bmatrix}4 & 6 & 6\\ 1 & 3 & 2\\ -1 & -4 & -3\end{bmatrix}$

[M18/N18/Comp/6M]

Solution:

$A=\begin{bmatrix}4 & 6 & 6\\ 1 & 3 & 2\\ -1 & -4 & -3\end{bmatrix}, |A|=-4$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}4-\lambda & 6 & 6\\ 1 & 3-\lambda & 2\\ -1 & -4 & -3-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[4+3-3]\lambda^2+\left[\begin{vmatrix}3 & 2\\ -4 & -3\end{vmatrix}+\begin{vmatrix}4 & 6\\ -1 & -3\end{vmatrix}+\begin{vmatrix}4 & 6\\ 1 & 3\end{vmatrix}\right]\lambda-(-4)=0$

$\lambda^3-4\lambda^2-\lambda+4=0$

$\lambda=1,-1,4$

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$\begin{bmatrix}3 & 6 & 6\\ 1 & 2 & 2\\ -1 & -4 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$3x_1+6x_2+6x_3=0$

$-x_1-4x_2-4x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}6 & 6\\ -4 & -4\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}3 & 6\\ -1 & -4\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}3 & 6\\ -1 & -4\end{vmatrix}}$

$\frac{x_1}{0}=-\frac{x_2}{-6}=\frac{x_3}{-6}$

$\frac{x_1}{0}=\frac{x_2}{1}=\frac{x_3}{-1}$

Hence, corresponding to $\lambda=1$ the eigen vector is $X_1=[0,1,-1]'$

(ii) For $\lambda=-1, [A-\lambda I]X=0$ gives

$\begin{bmatrix}5 & 6 & 6\\ 1 & 4 & 2\\ -1 & -4 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$5x_1+6x_2+6x_3=0$

$x_1+4x_2+2x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}6 & 6\\ 4 & 2\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}5 & 6\\ 1 & 2\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}5 & 6\\ 1 & 4\end{vmatrix}}$

$\frac{x_1}{-12}=-\frac{x_2}{4}=\frac{x_3}{14}$

$\frac{x_1}{6} = \frac{x_2}{2} = \frac{x_3}{-7}$

Hence, corresponding to $\lambda = -1$ the eigen vector is $X_2 = [6,2,-7]'$

(iii) For $\lambda = 4$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0x_1 + 6x_2 + 6x_3 = 0$

$x_1 - x_2 + 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 6 & 6 \\ -1 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 6 \\ 1 & -1 \end{vmatrix}}$$

$\frac{x_1}{18} = -\frac{x_2}{-6} = \frac{x_3}{-6}$

$\frac{x_1}{3} = \frac{x_2}{1} = \frac{x_3}{-1}$

Hence, corresponding to $\lambda = 4$ the eigen vector is $X_3 = [3,1,-1]'$

4. Determine the eigen values and eigen vectors of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

[M15/ChemBiot/6M]

Solution:

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}, |A| = 0$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 4 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [8+7+3]\lambda^2 + \left[\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}\right]\lambda - 0 = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda - 15)(\lambda - 3) = 0$$

$$\lambda = 0,3,15$$

(i) For $\lambda = 0, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x_1}{10} = -\frac{x_2}{-20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 0$ the eigen vector is $X_1 = [1,2,2]'$

(ii) For $\lambda = 3, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0$$

$$2x_1 - 4x_2 + 0x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -4 & 0 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 5 & 2 \\ 2 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ 2 & -4 \end{vmatrix}}$$

$\frac{x_1}{8} = -\frac{x_2}{-4} = \frac{x_3}{-8}$

$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_2 = [2,1,-2]'$

(iii) For $\lambda = 15$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 - 8x_2 - 4x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$

$\frac{x_1}{40} = -\frac{x_2}{40} = \frac{x_3}{20}$

$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 15$ the eigen vector is $X_3 = [2,-2,1]'$

5. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

[M16/ChemBiot/6M][M19/Chem/6M][N19/Inst/6M][D23/CompIT/6M]

Solution:

$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}, |A| = 36$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [3+5+3]\lambda^2 + \left[\begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}\right]\lambda - 36 = 0$

$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$\lambda = 2,3,6$

(i) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 - x_2 + x_3 = 0$

$-x_1 + 3x_2 - x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}$

$\frac{x_1}{-2} = -\frac{x_2}{0} = \frac{x_3}{2}$

$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_1 = [-1,0,1]'$

(ii) For $\lambda = 3, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0x_1 - x_2 + x_3 = 0$

$-x_1 + 2x_2 - x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$

$\frac{x_1}{-1} = -\frac{x_2}{1} = \frac{x_3}{-1}$

$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_2 = [1,1,1]'$

(iii) For $\lambda = 6, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 - x_2 + x_3 = 0$

$-x_1 - x_2 - x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$

$\frac{x_1}{2} = -\frac{x_2}{4} = \frac{x_3}{2}$

$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 6$ the eigen vector is $X_3 = [1, -2, 1]'$

6. Find the Eigen values and Eigen vectors of the matrix $A=\begin{bmatrix}8&-8&-2\\4&-3&-2\\3&-4&1\end{bmatrix}$

[N13/Chem/5M][N16/M17/MechCivil/8M][N18/MTRX/8M]

Solution:

$A=\begin{bmatrix}8&-8&-2\\4&-3&-2\\3&-4&1\end{bmatrix}, |A|=6$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}8-\lambda&-8&-2\\4&-3-\lambda&-2\\3&-4&1-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[8-3+1]\lambda^2+\left[\begin{vmatrix}-3&-2\\-4&1\end{vmatrix}+\begin{vmatrix}8&-2\\3&1\end{vmatrix}+\begin{vmatrix}8&-8\\4&-3\end{vmatrix}\right]\lambda-6=0$

$\lambda^3-6\lambda^2+11\lambda-6=0$

$\lambda=1,2,3$

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$\begin{bmatrix}7&-8&-2\\4&-4&-2\\3&-4&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$7x_1-8x_2-2x_3=0$

$4x_1-4x_2-2x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}-8&-2\\-4&-2\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}7&-2\\4&-2\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}7&-8\\4&-4\end{vmatrix}}$

$\frac{x_1}{8}=-\frac{x_2}{-6}=\frac{x_3}{4}$

$\frac{x_1}{4}=\frac{x_2}{3}=\frac{x_3}{2}$

Hence, corresponding to $\lambda=1$ the eigen vector is $X_1=[4,3,2]'$

(ii) For $\lambda=2, [A-\lambda I]X=0$ gives

$\begin{bmatrix}6&-8&-2\\4&-5&-2\\3&-4&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$6x_1-8x_2-2x_3=0$

$4x_1-5x_2-2x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}-8&-2\\-5&-2\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}6&-2\\4&-2\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}6&-8\\4&-5\end{vmatrix}}$

$\frac{x_1}{6}=-\frac{x_2}{-4}=\frac{x_3}{2}$

$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [3,2,1]'$

(iii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 8x_2 - 2x_3 = 0$

$4x_1 - 6x_2 - 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$\frac{x_1}{4} = -\frac{x_2}{-2} = \frac{x_3}{2}$

$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [2,1,1]'$

7. Find the eigen values and eigen vectors of $A=\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$.

[N15/ChemBiot/6M][N17/M18/N18/Chem/6M][M18/Biot/6M]

Solution:

$$A=\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}, |A|=6$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}2-\lambda & -1 & 1\\ 1 & 2-\lambda & -1\\ 1 & -1 & 2-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[2+2+2]\lambda^2+\left[\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}+\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}+\begin{vmatrix}2 & -1\\ 1 & 2\end{vmatrix}\right]\lambda-6=0$$

$$\lambda^3-6\lambda^2+11\lambda-6=0$$

$$(\lambda-1)(\lambda-2)(\lambda-3)=0$$

$$\lambda=1,2,3$$

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$x_1-x_2+x_3=0$$

$$x_1+x_2-x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-1 & 1\\ 1 & -1\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}}$$

$$\frac{x_1}{0}=-\frac{x_2}{-2}=\frac{x_3}{2}$$

$$\frac{x_1}{0}=\frac{x_2}{1}=\frac{x_3}{1}$$

Hence, corresponding to $\lambda=1$ the eigen vector is $X_1=[0,1,1]'$

(ii) For $\lambda=2, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$0x_1-x_2+x_3=0$$

$$x_1+0x_2-x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-1 & 1\\ 0 & -1\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}0 & 1\\ 1 & -1\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}0 & -1\\ 1 & 0\end{vmatrix}}$$

$\frac{x_1}{1} = -\frac{x_2}{-1} = \frac{x_3}{1}$

$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [1,1,1]'$

(iii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 - x_2 + x_3 = 0$

$x_1 - x_2 - x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix}}$$

$\frac{x_1}{2} = -\frac{x_2}{0} = \frac{x_3}{2}$

$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [1,0,1]'$

8. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

[N22/MechCivil/6M]

Solution:

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, |A| = -2$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$

$(1-\lambda)(-1-\lambda)(2-\lambda) = 0$

$\lambda = 1, -1, 2$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 0 & 2 & 3 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0x_1 + 2x_2 + 3x_3 = 0$

$0x_1 - 2x_2 + x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 3 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 2 \\ 0 & -2 \end{vmatrix}}$

$\frac{x_1}{8} = -\frac{x_2}{0} = \frac{x_3}{0}$

$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$

Hence, corresponding to $\lambda = 1$ the eigen vector is $X_1 = [1,0,0]'$

(ii) For $\lambda = -1, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$2x_1 + 2x_2 + 3x_3 = 0$

$0x_1 + 0x_2 + x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 2 \\ 0 & 0 \end{vmatrix}}$

$\frac{x_1}{2} = -\frac{x_2}{2} = \frac{x_3}{0}$

$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$

Hence, corresponding to $\lambda = -1$ the eigen vector is $X_2 = [1,-1,0]'$

(iii) For $\lambda = 2$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -1 & 2 & 3 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_2 + 3x_3 = 0$$

$$0x_1 - 3x_2 + x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 2 \\ 0 & -3 \end{vmatrix}}$$

$$\frac{x_1}{11} = -\frac{x_2}{-1} = \frac{x_3}{3}$$

$$\frac{x_1}{11} = \frac{x_2}{1} = \frac{x_3}{3}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_3 = [11,1,3]'$

9. Find the eigen values and eigen vectors of $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

[N19/Biot/6M]

Solution:

$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}, |A| = 6$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [-1+4+3]\lambda^2 + \left[\begin{vmatrix} 4 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} -1 & -2 \\ -3 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 4 \\ -3 & 4 \end{vmatrix}\right]\lambda - 6 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$(\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\lambda = 1,2,3$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x_1 + 4x_2 - 2x_3 = 0$

$-3x_1 + 3x_2 + 0x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 4 & -2 \\ 3 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -2 \\ -3 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 4 \\ -3 & 3 \end{vmatrix}}$

$\frac{x_1}{6} = -\frac{x_2}{-6} = \frac{x_3}{6}$

$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 1$ the eigen vector is $X_1 = [1,1,1]'$

(ii) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} -3 & 4 & -2 \\ -3 & 2 & 0 \\ -3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x_1 + 4x_2 - 2x_3 = 0$

$-3x_1 + 2x_2 + 0x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 4 & -2 \\ 2 & 0 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & -2 \\ -3 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 4 \\ -3 & 2 \end{vmatrix}}$

$\frac{x_1}{4} = -\frac{x_2}{-6} = \frac{x_3}{6}$

$\frac{x_1}{2} = \frac{x_2}{3} = \frac{x_3}{3}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [2,3,3]'$

(iii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -4 & 4 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4x_1 + 4x_2 - 2x_3 = 0$

$-3x_1 + x_2 + 0x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & -2 \\ 1 & 0 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -4 & -2 \\ -3 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -4 & 4 \\ -3 & 1 \end{vmatrix}}$$

$\frac{x_1}{2} = -\frac{x_2}{-6} = \frac{x_3}{8}$

$\frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{4}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [1,3,4]'$

10. Find the eigen values and eigen vectors of $A=\begin{bmatrix}1 & 2 & 0\\ 2 & 1 & -6\\ 2 & -2 & 3\end{bmatrix}$

[M23DSE/MechCivil/6M]

Solution:

$A=\begin{bmatrix}1 & 2 & 0\\ 2 & 1 & -6\\ 2 & -2 & 3\end{bmatrix}, |A|=-45$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}1-\lambda & 2 & 0\\ 2 & 1-\lambda & -6\\ 2 & -2 & 3-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[1+1+3]\lambda^2+\left[\begin{vmatrix}1 & -6\\ -2 & 3\end{vmatrix}+\begin{vmatrix}1 & 0\\ 2 & 3\end{vmatrix}+\begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix}\right]\lambda--45=0$

$\lambda^3-5\lambda^2-9\lambda+45=0$

$(\lambda+3)(\lambda-3)(\lambda-5)=0$

$\lambda=-3,3,5$

(i) For $\lambda=-3$, $[A-\lambda I]X=0$ gives

$\begin{bmatrix}4 & 2 & 0\\ 2 & 4 & -6\\ 2 & -2 & 6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$4x_1+2x_2+0x_3=0$

$2x_1+4x_2-6x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}2 & 0\\ 4 & -6\end{vmatrix}}=\frac{-x_2}{\begin{vmatrix}4 & 0\\ 2 & -6\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}4 & 2\\ 2 & 4\end{vmatrix}}$

$\frac{x_1}{-12}=-\frac{x_2}{-24}=\frac{x_3}{12}$

$\frac{x_1}{-1}=\frac{x_2}{2}=\frac{x_3}{1}$

Hence, corresponding to $\lambda=-3$ the eigen vector is $X_1=[-1,2,1]'$

(ii) For $\lambda=3$, $[A-\lambda I]X=0$ gives

$\begin{bmatrix}-2 & 2 & 0\\ 2 & -2 & -6\\ 2 & -2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$-2x_1+2x_2+0x_3=0$

$2x_1-2x_2-6x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}2 & 0\\ -2 & -6\end{vmatrix}}=\frac{-x_2}{\begin{vmatrix}-2 & 0\\ 2 & -6\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}-2 & 2\\ 2 & -2\end{vmatrix}}$

$\frac{x_1}{-12} = -\frac{x_2}{12} = \frac{x_3}{0}$
$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_2 = [1,1,0]'$

(iii) For $\lambda = 5$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & -6 \\ 2 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4x_1 + 2x_2 + 0x_3 = 0$
$2x_1 - 4x_2 - 6x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 0 \\ -4 & -6 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -4 & 0 \\ 2 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -4 & 2 \\ 2 & -4 \end{vmatrix}}$$

$\frac{x_1}{-12} = -\frac{x_2}{24} = \frac{x_3}{12}$
$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,2,-1]'$

11. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -6 \\ 2 & 2 & 2 \end{bmatrix}$

[J24/MechCivil/6M]

Solution

Question wrong. Because eigen values comes out to be $-1, \frac{5}{2} \pm 3.4278i$. Eigen values are turning out to be complex (unreal).

Correct matrix should be $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -6 \\ 2 & -2 & 3 \end{bmatrix}$

12. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

[N16/CompIT/5M]

Solution:

$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, |A| = 8$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [2+2+2]\lambda^2 + \left[\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}\right]\lambda - 8 = 0$

$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

$\lambda = 2,2,2$

For $\lambda = 2, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0x_1 + x_2 + 0x_3 = 0$

$0x_1 + 0x_2 + x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}}$

$\frac{x_1}{1} = -\frac{x_2}{0} = \frac{x_3}{0}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_1 = X_2 = X_3 = [1,0,0]'$

13. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 1 & -6 & 4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & -6 & 4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

$$|A| = 0$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -6 & 4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diag]\lambda^2 + [sum\ of\ minors\ of\ diag]\lambda - |A| = 0$$

$$\lambda^3 - [1+4-3]\lambda^2 + \left[\begin{vmatrix} 4 & 2 \\ -6 & -3 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}\right]\lambda - 0 = 0$$

$$\lambda^3 - 2\lambda^2 + [-12+12-3-0+4-0]\lambda = 0$$

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\lambda = 0,1,1$$

(i) For $\lambda = 0$

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 1 & -6 & 4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 6x_2 + 4x_3 = 0$$

$$0x_1 + 4x_2 + 2x_3 = 0$$

Solving by Crammers rule,

$$\frac{x_1}{\begin{vmatrix} -6 & 4 \\ 4 & 2 \end{vmatrix}} = \frac{(-x_2)}{\begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}}$$

$$\frac{x_1}{-28} = \frac{-x_2}{2} = \frac{x_3}{4}$$

$$\frac{x_1}{-14} = \frac{x_2}{-1} = \frac{x_3}{2}$$

$$X_1 = \begin{bmatrix} -14 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 1 \\ -2 \end{bmatrix}$$

(ii) For $\lambda = 1$

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 0 & -6 & 4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0x_1 - 6x_2 + 4x_3 = 0$
$0x_1 + 3x_2 + 2x_3 = 0$
Solving by Crammers Rule,

$\frac{x_1}{-12-12} = \frac{-x_2}{0} = \frac{x_3}{0}$

$\frac{x_1}{-24} = \frac{x_2}{0} = \frac{x_3}{0}$

$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{0}$

$$X_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

14. Find eigen values and eigen vectors of the matrix A where $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

[M14/ChemBiot/8M][N14/ChemBiot/5M][N14/ElexExtcElectBiomInst/6M]

Solution:

$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}, |A| = 7$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [2+3+4]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}\right]\lambda - 7 = 0$

$\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$

$(\lambda - 1)(\lambda - 1)(\lambda - 7) = 0$

$\lambda = 1,1,7$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 - 2R_1, R_3 - 3R_1$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore x_1 + x_2 + x_3 = 0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

Let $x_3 = t$ & $x_2 = s$

$\therefore x_1 = -s - t$

$\therefore X = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s-t \\ s+0t \\ 0s+t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} -t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Hence, corresponding to $\lambda = 1$ the eigen vectors are

$X_1 = [-1,1,0]'$ & $X_2 = [-1,0,1]'$

(ii) For $\lambda = 7, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-5x_1 + x_2 + x_3 = 0$
$2x_1 - 4x_2 + 2x_3 = 0$
Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -5 & 1 \\ 2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -5 & 1 \\ 2 & -4 \end{vmatrix}}$

$\frac{x_1}{6} = -\frac{x_2}{-12} = \frac{x_3}{18}$

$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$

Hence, corresponding to $\lambda = 7$ the eigen vector is $X_3 = [1,2,3]'$

15. Find eigen values and eigen vectors of matrix $\begin{bmatrix}1 & -6 & -4\\ 0 & 4 & 2\\ 0 & -6 & -3\end{bmatrix}$

[M22/Chem/5M]

Solution:

$A = \begin{bmatrix}1 & -6 & -4\\ 0 & 4 & 2\\ 0 & -6 & -3\end{bmatrix}, |A| = 0$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix}1-\lambda & -6 & -4\\ 0 & 4-\lambda & 2\\ 0 & -6 & -3-\lambda\end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [1+4-3]\lambda^2 + \left[\begin{vmatrix}4 & 2\\ -6 & -3\end{vmatrix} + \begin{vmatrix}1 & -4\\ 0 & -3\end{vmatrix} + \begin{vmatrix}1 & -6\\ 0 & 4\end{vmatrix}\right]\lambda - 32 = 0$

$\lambda^3 - 2\lambda^2 + \lambda = 0$

$\lambda(\lambda - 1)(\lambda - 1) = 0$

$\lambda = 0,1,1$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$\begin{bmatrix}0 & -6 & -4\\ 0 & 3 & 2\\ 0 & -6 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

By $R_2 + \frac{1}{2}R_1, R_3 - R_1$

$\begin{bmatrix}0 & -6 & -4\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\therefore 0x_1 - 6x_2 - 4x_3 = 0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

Let $x_1 = 1, x_2 = 2 \therefore x_3 = -3$

Let $x_1 = 1, x_2 = -2 \therefore x_3 = 3$

Hence, corresponding to $\lambda = 1$ the eigen vectors are

$X_1 = [1,2,-3]' \& X_2 = [1,-2,3]'$

(ii) For $\lambda = 0, [A - \lambda I]X = 0$ gives

$\begin{bmatrix}1 & -6 & -4\\ 0 & 4 & 2\\ 0 & -6 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$x_1 - 6x_2 - 4x_3 = 0$

$0x_1 + 4x_2 + 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-6 & -4\\ 4 & 2\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}1 & -4\\ 0 & 2\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}1 & -6\\ 0 & 4\end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{2} = \frac{x_3}{4}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 0$ the eigen vector is $X_3 = [2, -1, 2]'$

16. Find the eigen values and eigen vectors of $A=\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}$

[N22/Elect/8M]

Solution:

Solution:

$A=\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}, |A|=4$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}2-\lambda&-1&1\\-1&2-\lambda&-1\\1&-1&2-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[2+2+2]\lambda^2+\left[\begin{vmatrix}2&-1\\-1&2\end{vmatrix}+\begin{vmatrix}2&1\\1&2\end{vmatrix}+\begin{vmatrix}2&-1\\-1&2\end{vmatrix}\right]\lambda-4=0$

$\lambda^3-6\lambda^2+9\lambda-4=0$

$(\lambda-1)(\lambda-1)(\lambda-4)=0$

$\lambda=1,1,4$

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$\begin{bmatrix}1&-1&1\\-1&1&-1\\1&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

By R_2+R_1, R_3-R_1

$\begin{bmatrix}1&-1&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore x_1-x_2+x_3=0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ vectors to be formed

Let $x_3=t$ & $x_2=s$

$\therefore x_1=s-t$

$\therefore X=\begin{bmatrix}s-t\\s\\t\end{bmatrix}=\begin{bmatrix}s-t\\s+0t\\0s+t\end{bmatrix}=\begin{bmatrix}s\\s\\0s\end{bmatrix}+\begin{bmatrix}-t\\0t\\t\end{bmatrix}=s\begin{bmatrix}1\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}$

Hence, corresponding to $\lambda=1$ the eigen vectors are

$X_1=[1,1,0]'$ & $X_2=[-1,0,1]'$

(ii) For $\lambda = 4, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - x_2 + x_3 = 0$$

$$-x_1 - 2x_2 - x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -2 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{3} = -\frac{x_2}{3} = \frac{x_3}{3}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 4$ the eigen vector is $X_3 = [1, -1, 1]'$

17. Find the eigen values and eigen vectors of $A=\begin{bmatrix}2&2&1\\1&3&1\\1&2&2\end{bmatrix}$

[M15/ElexExtcElectBiomInst/6M][N19/Comp/6M][M23DSE/ElectECSExtc/6M]

Solution:

$A=\begin{bmatrix}2&2&1\\1&3&1\\1&2&2\end{bmatrix}, |A|=5$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}2-\lambda&2&1\\1&3-\lambda&1\\1&2&2-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[2+3+2]\lambda^2+\left[\begin{vmatrix}3&1\\2&2\end{vmatrix}+\begin{vmatrix}2&1\\1&2\end{vmatrix}+\begin{vmatrix}2&2\\1&3\end{vmatrix}\right]\lambda-5=0$

$\lambda^3-7\lambda^2+11\lambda-5=0$

$(\lambda-1)(\lambda-1)(\lambda-5)=0$

$\lambda=1,1,5$

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$\begin{bmatrix}1&2&1\\1&2&1\\1&2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

By R_2-R_1, R_3-R_1

$\begin{bmatrix}1&2&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore x_1+2x_2+x_3=0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ vectors to be formed

Let $x_3=t$ & $x_2=s$

$\therefore x_1=-2s-t$

$\therefore X=\begin{bmatrix}-2s-t\\s\\t\end{bmatrix}=\begin{bmatrix}-2s-t\\s+0t\\0s+t\end{bmatrix}=\begin{bmatrix}-2s\\s\\0s\end{bmatrix}+\begin{bmatrix}-t\\0t\\t\end{bmatrix}=s\begin{bmatrix}-2\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}$

Hence, corresponding to $\lambda=1$ the eigen vectors are

$X_1=[-2,1,0]'$ & $X_2=[-1,0,1]'$

(ii) For $\lambda=5, [A-\lambda I]X=0$ gives

$\begin{bmatrix}-3&2&1\\1&-2&1\\1&2&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$-3x_1 + 2x_2 + x_3 = 0$
$x_1 - 2x_2 + x_3 = 0$
Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{-4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,1,1]'$

18. Find the eigen values and eigen vectors of $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

[N15/N22/CompIT/6M][N16/ChemBiot/6M][N22/Extc/6M]
[J23/M23DSE/ElectECSExtc/6M]

Solution:

$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}, |A| = 4$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [4+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}\right]\lambda - 4 = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$

$\lambda = 1,2,2$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 + 2x_3 = 0$

$-x_1 - 5x_2 - 3x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ -5 & -3 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ -1 & -5 \end{vmatrix}}$

$\frac{x_1}{4} = -\frac{x_2}{-1} = \frac{x_3}{-3}$

Hence, corresponding to $\lambda = 1$ the eigen vector is $X_1 = [4,1,-3]'$

(ii) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$2x_1 + 6x_2 + 6x_3 = 0$

$-x_1 - 5x_2 - 4x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 6 & 6 \\ -5 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 2 & 6 \\ -1 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 6 \\ -1 & -5 \end{vmatrix}}$

$\frac{x_1}{6} = -\frac{x_2}{-2} = \frac{x_3}{-4}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [3,1,-2]'$

19. Find the eigen values and eigen vectors of $A=\begin{bmatrix}3 & 10 & 5\\ -2 & -3 & -4\\ 3 & 5 & 7\end{bmatrix}$

[M19/Comp/6M][J23/MechCivil/8M][J23/ElectECSExtc/6M][M24/CompIT/6M]

Solution:

$A=\begin{bmatrix}3 & 10 & 5\\ -2 & -3 & -4\\ 3 & 5 & 7\end{bmatrix}, |A|=12$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}3-\lambda & 10 & 5\\ -2 & -3-\lambda & -4\\ 3 & 5 & 7-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[3-3+7]\lambda^2+\left[\begin{vmatrix}-3 & -4\\ 5 & 7\end{vmatrix}+\begin{vmatrix}3 & 5\\ 3 & 7\end{vmatrix}+\begin{vmatrix}3 & 10\\ -2 & -3\end{vmatrix}\right]\lambda-12=0$

$\lambda^3-7\lambda^2+16\lambda-12=0$

$(\lambda-3)(\lambda-2)(\lambda-2)=0$

$\lambda=3,2,2$

(i) For $\lambda=3, [A-\lambda I]X=0$ gives

$\begin{bmatrix}0 & 10 & 5\\ -2 & -6 & -4\\ 3 & 5 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$0x_1+10x_2+5x_3=0$

$-2x_1-6x_2-4x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}10 & 5\\ -6 & -4\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}0 & 5\\ -2 & -4\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}0 & 10\\ -2 & -6\end{vmatrix}}$

$\frac{x_1}{-10}=-\frac{x_2}{10}=\frac{x_3}{20}$

$\frac{x_1}{1}=\frac{x_2}{1}=\frac{x_3}{-2}$

Hence, corresponding to $\lambda=3$ the eigen vector is $X_1=[1,1,-2]'$

(ii) For $\lambda=2, [A-\lambda I]X=0$ gives

$\begin{bmatrix}1 & 10 & 5\\ -2 & -5 & -4\\ 3 & 5 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$x_1+10x_2+5x_3=0$

$-2x_1-5x_2-4x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}10 & 5\\ -5 & -4\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}1 & 5\\ -2 & -4\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}1 & 10\\ -2 & -5\end{vmatrix}}$

$$\frac{x_1}{-15} = -\frac{x_2}{6} = \frac{x_3}{15}$$
$$\frac{x_1}{5} = \frac{x_2}{2} = \frac{x_3}{-5}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [5,2,-5]'$

20. Find the eigen values and eigen vectors of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

[N18/Inst/6M][M22/ElectECSExtc/5M]

Solution:

$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}, |A| = 32$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [6+3+3]\lambda^2 + \left[\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}\right]\lambda - 32 = 0$

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$

$\lambda = 2,2,8$

(i) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 + \frac{1}{2}R_1, R_3 - \frac{1}{2}R_1$

$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore 4x_1 - 2x_2 + 2x_3 = 0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

Let $x_3 = t$ & $x_2 = s$, $\therefore x_1 = \frac{s}{2} - \frac{t}{2}$

$\therefore X = \begin{bmatrix} \frac{s}{2} - \frac{t}{2} \\ s \\ t \end{bmatrix} = \begin{bmatrix} \frac{s}{2} - \frac{t}{2} \\ s + 0t \\ 0s + t \end{bmatrix} = \begin{bmatrix} \frac{s}{2} \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} -\frac{t}{2} \\ 0t \\ t \end{bmatrix} = \frac{s}{2}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{t}{2}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Hence, corresponding to $\lambda = 2$ the eigen vectors are

$X_1 = [1,2,0]'$ & $X_2 = [-1,0,2]'$

(ii) For $\lambda = 8, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x_1 - 2x_2 + 2x_3 = 0$
$-2x_1 - 5x_2 - 1x_3 = 0$
Solving the above equations by Crammers rule, we get
$\frac{x_1}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$
$\frac{x_1}{12} = -\frac{x_2}{6} = \frac{x_3}{6}$
$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$
Hence, corresponding to $\lambda = 8$ the eigen vector is $X_3 = [2, -1, 1]'$

21. Find the eigen values and eigen vectors of $A=\begin{bmatrix}3 & -1 & 1\\ -1 & 3 & -1\\ 1 & -1 & 3\end{bmatrix}$

[M22/CompIT/5M]

Solution:

$A=\begin{bmatrix}3 & -1 & 1\\ -1 & 3 & -1\\ 1 & -1 & 3\end{bmatrix}, |A|=20$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}3-\lambda & -1 & 1\\ -1 & 3-\lambda & -1\\ 1 & -1 & 3-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[3+3+3]\lambda^2+\left[\begin{vmatrix}3 & -1\\ -1 & 3\end{vmatrix}+\begin{vmatrix}3 & 1\\ 1 & 3\end{vmatrix}+\begin{vmatrix}3 & -1\\ -1 & 3\end{vmatrix}\right]\lambda-20=0$

$\lambda^3-9\lambda^2+24\lambda-20=0$

$\lambda=2,2,5$

(i) For $\lambda=2$, $[A-\lambda I]X=0$ gives

$\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

By R_2+R_1, R_3-R_1

$\begin{bmatrix}1 & -1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$x_1-x_2+x_3=0$

The rank (r) of the matrix is 1 and no of unknowns (n) is 3

Thus, $n-r=3-1=2$ vectors to be formed

Let $x_3=t$ & $x_2=s$

$\therefore x_1=s-t$

$\therefore X=\begin{bmatrix}s-t\\ s\\ t\end{bmatrix}=\begin{bmatrix}s-t\\ s+0t\\ 0s+t\end{bmatrix}=\begin{bmatrix}s\\ s\\ 0s\end{bmatrix}+\begin{bmatrix}-t\\ 0t\\ t\end{bmatrix}=s\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$

Hence, corresponding to $\lambda=2$ the eigen vectors are

$X_1=[1,1,0]'$ & $X_2=[-1,0,1]'$

(ii) For $\lambda=5$, $[A-\lambda I]X=0$ gives

$\begin{bmatrix}-2 & -1 & 1\\ -1 & -2 & -1\\ 1 & -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$-2x_1 - x_2 + x_3 = 0$
$-x_1 - 2x_2 - x_3 = 0$
Solving the above equations by Crammers rule, we get
$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -2 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix}}$
$\frac{x_1}{3} = -\frac{x_2}{3} = \frac{x_3}{3}$
$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$
Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1, -1, 1]'$

22. Find the eigen values and eigen vectors of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

[M22/Elex/5M][N22/Elex/6M][N23/ElectECSExtc/6M][N24/ElectECSExtc/8M] [D24/CompIT/6M]

Solution:

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, |A| = 45$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [-2+1+0]\lambda^2 + \left[\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}\right]\lambda - 45 = 0$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$(\lambda - 5)(\lambda + 3)(\lambda + 3) = 0$

$\lambda = 5, -3, -3$

(i) For $\lambda = -3, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 - 2R_1, R_3 + R_1$

$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore x_1 + 2x_2 - 3x_3 = 0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

Let $x_3 = t$ & $x_2 = s, \therefore x_1 = -2s + 3t$

$\therefore X = \begin{bmatrix} -2s+3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2s+3t \\ s+0t \\ 0s+t \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} 3t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Hence, corresponding to $\lambda = -3$ the eigen vectors are

$X_1 = [-2,1,0]'$ & $X_2 = [3,0,1]'$

(ii) For $\lambda = 5, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-7x_1 + 2x_2 - 3x_3 = 0$
$2x_1 - 4x_2 - 6x_3 = 0$
Solving the above equations by Crammers rule, we get
$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$
$\frac{x_1}{-24} = -\frac{x_2}{48} = \frac{x_3}{24}$
$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$
Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,2,-1]'$

23. Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

[J24/ElectECSExtc/6M]

Solution:

$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, |A| = 5$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [1+1+1]\lambda^2 + \left[\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\right]\lambda - 5 = 0$

$\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$

$\lambda = 5, -1, -1$

(i) For $\lambda = -1, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 - R_1, R_3 - R_1$

$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore 2x_1 + 2x_2 + 2x_3 = 0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

The Geometric Multiplicity of $\lambda = -1$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3 = t$ & $x_2 = s$

$\therefore x_1 = -s - t$

$\therefore X = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s-t \\ s+0t \\ 0s+t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} -t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Hence, corresponding to $\lambda = 1$ the eigen vectors are

$X_1 = [-1,1,0]'$ & $X_2 = [-1,0,1]'$

(ii) For $\lambda = 5$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4x_1 + 2x_2 + 2x_3 = 0$$

$$2x_1 - 4x_2 + 2x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ -4 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -4 & 2 \\ 2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -4 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{12} = -\frac{x_2}{-12} = \frac{x_3}{12}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,1,1]'$

24. If $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$ then find the eigen values of $6A^{-1} + A^3 + 2I$

[M17/N22/CompIT/5M][N18/N19/Extc/5M]

Solution:

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 4 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(3-\lambda) - 0 = 0$$

$$\lambda = 2,3$$

The eigen values of A is 2,3

The eigen values of A^{-1} is $2^{-1}, 3^{-1}$ i.e. $\frac{1}{2}, \frac{1}{3}$

The eigen values of $6A^{-1}$ is $6\left(\frac{1}{2}\right), 6\left(\frac{1}{3}\right)$ i.e. 3,2

The eigen values of A^3 is $2^3, 3^3$ i.e. 8,27

The eigen values of I is 1,1

The eigen values of $2I$ is 2,2

Thus, the eigen values of $6A^{-1} + A^3 + 2I$ is

$3 + 8 + 2$; $2 + 27 + 2$

i.e. 13; 31

25. If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, find eigen values and eigen vectors for the matrix $A^2 - 2A + I + adjA$.

Solution:

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$|A| = 36$$

The characteristic equation is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diag]\lambda^2 + [sum\ of\ minors\ of\ diag]\lambda - |A| = 0$$

$$\lambda^3 - [3+5+3]\lambda^2 + \left[\begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}\right]\lambda - 36 = 0$$

$$\lambda^3 - 11\lambda^2 + [15 - 1 + 9 - 1 + 15 - 1]\lambda - 36 = 0$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 6,3,2$$

Eigen values of A is 6,3,2

Matrix	Eigen Values		
A	6	3	2
A^2	36	9	4
$2A$	12	6	4
I	1	1	1
$adj\ A$	$\frac{\|A\|}{\lambda} = \frac{36}{6} = 6$	$\frac{36}{3} = 12$	$\frac{36}{2} = 18$
$A^2 - 2A + I + adjA$	$36 - 12 + 1 + 6 = 31$	$9 - 6 + 1 + 12 = 16$	$4 - 4 + 1 + 18 = 19$

Thus eigen values of $A^2 - 2A + I + adjA$ is 31,16,19

Eigen Vectors of $A^2 - 2A + I + adjA$ is same as that of A

(i) For $\lambda = 2$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + 3x_2 - x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}$$

$$\frac{x_1}{-2} = -\frac{x_2}{0} = \frac{x_3}{2}$$

$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_1 = [-1,0,1]'$

(ii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0x_1 - x_2 + x_3 = 0$

$-x_1 + 2x_2 - x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$\frac{x_1}{-1} = -\frac{x_2}{1} = \frac{x_3}{-1}$

$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_2 = [1,1,1]'$

(iii) For $\lambda = 6$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 - x_2 + x_3 = 0$

$-x_1 - x_2 - x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$

$\frac{x_1}{2} = -\frac{x_2}{4} = \frac{x_3}{2}$

$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 6$ the eigen vector is $X_3 = [1,-2,1]'$

26. Find the value of μ which satisfy the equation $A^{100}X = \mu X$ where $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}$

[N14/ElexExtcElectBiomInst/5M]

Solution:

$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}, |A| = 0$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & -2-\lambda & -2 \\ 1 & 1 & 0-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [2 - 2 + 0]\lambda^2 + \left[\begin{vmatrix} -2 & -2 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix}\right]\lambda - 0 = 0$

$\lambda^3 + 0\lambda^2 - 1\lambda = 0$

$\lambda(\lambda^2 - 1) = 0$

$\lambda(\lambda - 1)(\lambda + 1) = 0$

$\lambda = 0, 1, -1$

We know that, if λ is an eigen value of A and X is an eigen vector then

$[A - \lambda I]X = 0$

$AX - \lambda X = 0$

$AX = \lambda X$

Multiplying by A on both sides,

$A(AX) = A(\lambda X)$

$A^2X = \lambda(AX)$

$A^2X = \lambda(\lambda X)$

$A^2X = \lambda^2 X$ i.e. λ^2 is an eigen value of A^2

Similarly,

$A^3X = \lambda^3 X$

$A^4X = \lambda^4 X$ ……… $A^{100}X = \lambda^{100}X$

It is given that, $A^{100}X = \mu X$

Thus,

$\mu = \lambda^{100}$

Since, $\lambda = 0, 1, -1$

$\mu = 0^{100}, 1^{100}, (-1)^{100}$

$\mu = 0, 1, 1$

27. If $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find characteristic roots and characteristic vectors of $A^3 + I$.

[M15/MechCivil/5M]

Solution:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}, |A| = 5$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+3+2]\lambda^2 + \left[\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}\right]\lambda - 5 = 0$$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$(\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$$

$$\lambda = 1,1,5$$

(i) For $\lambda = 1$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1, R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 + 2x_2 + x_3 = 0$$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

Let $x_3 = t$ & $x_2 = s$

$$\therefore x_1 = -2s - t$$

$$\therefore X = \begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2s - t \\ s + 0t \\ 0s + t \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} -t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, corresponding to $\lambda = 1$ the eigen vectors are

$X_1 = [-2,1,0]'$ & $X_2 = [-1,0,1]'$

(ii) For $\lambda = 5$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 + 2x_2 + x_3 = 0$
$x_1 - 2x_2 + x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{-4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,1,1]'$

The characteristic roots of A is 1,1,5

The characteristic roots of A^3 is $1^3, 1^3, 5^3$ i.e. 1,1,125

The characteristic roots of I is 1,1,1

Thus, the characteristic roots of $A^3 + I$ is

$1 + 1$; $1 + 1$; $125 + 1$

i.e. 2, 2, 126

The characteristic vectors of $A^3 + I$ is same as that of A

28. Find eigen values and eigen vectors of A^3 where $A=\begin{bmatrix}2&1&1\\2&3&2\\3&3&4\end{bmatrix}$

[M16/ElexExtcElectBiomInst/6M]

Solution:

$A=\begin{bmatrix}2&1&1\\2&3&2\\3&3&4\end{bmatrix}, |A|=7$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}2-\lambda&1&1\\2&3-\lambda&2\\3&3&4-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[2+3+4]\lambda^2+\left[\begin{vmatrix}3&2\\3&4\end{vmatrix}+\begin{vmatrix}2&1\\3&4\end{vmatrix}+\begin{vmatrix}2&1\\2&3\end{vmatrix}\right]\lambda-7=0$

$\lambda^3-9\lambda^2+15\lambda-7=0$

$(\lambda-1)(\lambda-1)(\lambda-7)=0$

$\lambda=1,1,7$

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$\begin{bmatrix}1&1&1\\2&2&2\\3&3&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

By R_2-2R_1, R_3-3R_1

$\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore x_1+x_2+x_3=0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ vectors to be formed

Let $x_3=t$ & $x_2=s$

$\therefore x_1=-s-t$

$\therefore X=\begin{bmatrix}-s-t\\s\\t\end{bmatrix}=\begin{bmatrix}-s-t\\s+0t\\0s+t\end{bmatrix}=\begin{bmatrix}-s\\s\\0s\end{bmatrix}+\begin{bmatrix}-t\\0t\\t\end{bmatrix}=s\begin{bmatrix}-1\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}$

Hence, corresponding to $\lambda=1$ the eigen vectors are

$X_1=[-1,1,0]'$ & $X_2=[-1,0,1]'$

(ii) For $\lambda=7, [A-\lambda I]X=0$ gives

$\begin{bmatrix}-5&1&1\\2&-4&2\\3&3&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$-5x_1 + x_2 + x_3 = 0$
$2x_1 - 4x_2 + 2x_3 = 0$
Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -5 & 1 \\ 2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -5 & 1 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{6} = -\frac{x_2}{-12} = \frac{x_3}{18}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$$

Hence, corresponding to $\lambda = 7$ the eigen vector is $X_3 = [1,2,3]'$
Thus, the eigen values of A is 1,1,7
And, the eigen values of A^3 is $1^3, 1^3, 7^3$ i.e. 1,1,343
Eigen vectors of A^3 is same as that of A

29. Find the eigen values and eigen vectors of A^3 where $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

[N15/ElexExtcElectBiomInst/6M]

Solution:

$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}, |A| = 4$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [4+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}\right]\lambda - 4 = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$

$\lambda = 1,2,2$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 + 2x_3 = 0$

$-x_1 - 5x_2 - 3x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ -5 & -3 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ -1 & -5 \end{vmatrix}}$

$\frac{x_1}{4} = -\frac{x_2}{-1} = \frac{x_3}{-3}$

Hence, corresponding to $\lambda = 1$ the eigen vector is $X_1 = [4,1,-3]'$

(ii) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$2x_1 + 6x_2 + 6x_3 = 0$

$-x_1 - 5x_2 - 4x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 6 & 6 \\ -5 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 2 & 6 \\ -1 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 6 \\ -1 & -5 \end{vmatrix}}$

$\frac{x_1}{6} = -\frac{x_2}{-2} = \frac{x_3}{-4}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [3,1,-2]'$
Thus, the eigen values of A is 1,2,2
And, the eigen values of A^3 is $1^3, 2^3, 2^3$ i.e. 1,8,8
Eigen vectors of A^3 is same as that of A

30. Find the eigen values of $A^3 - 3A^2 + A$ where $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

[M19/Elect/5M]

Solution:

$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}, |A| = -4$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [4+3-3]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}\right]\lambda + 4 = 0$

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$(\lambda - 1)(\lambda + 1)(\lambda - 4) = 0$

$\lambda = 1, -1, 4$

Thus, the eigen values of A is $1, -1, 4$

Thus, the eigen values of A^3 is $1^3, (-1)^{3,4^3}$

Thus, the eigen values of A^2 is $1^2, (-1)^2, 4^2$

Thus, the eigen values of $A^3 - 3A^2 + A$ is

$1^3 - 3(1)^2 + 1; (-1)^3 - 3(-1)^2 + (-1); 4^3 - 3(4)^2 + 4$

i.e. $-1; -5; 20$

31. Find the characteristic roots of A and $A^2 + I$ where $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

[M16/MechCivil/5M]

Solution:

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, |A| = 45$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [-2+1+0]\lambda^2 + \left[\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}\right]\lambda - 45 = 0$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$(\lambda - 5)(\lambda + 3)(\lambda + 3) = 0$

$\lambda = 5, -3, -3$

The characteristic roots of A is $5, -3, -3$

The characteristic roots of A^2 is $5^2, (-3)^2, (-3)^2$ i.e. 25,9,9

The characteristic roots of I is 1,1,1

Thus, the characteristic roots of $A^2 + I$ is

$25 + 1\ ; 9 + 1\ ; 9 + 1$

i.e. $26\ ; 10\ ; 10$

32. If $A=\begin{bmatrix}3 & 1 & -1\\ 2 & 2 & -1\\ 2 & 3 & 0\end{bmatrix}$, find characteristics roots and vectors of A^2

[N13/Biot/6M]

Solution:

$$A=\begin{bmatrix}3 & 1 & -1\\ 2 & 2 & -1\\ 2 & 2 & 0\end{bmatrix}, |A|=4$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}3-\lambda & 1 & -1\\ 2 & 2-\lambda & -1\\ 2 & 2 & 0-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[3+2+0]\lambda^2+\left[\begin{vmatrix}2 & -1\\ 2 & 0\end{vmatrix}+\begin{vmatrix}3 & -1\\ 2 & 0\end{vmatrix}+\begin{vmatrix}3 & 1\\ 2 & 2\end{vmatrix}\right]\lambda-4=0$$

$$\lambda^3-5\lambda^2+8\lambda-4=0$$

$$(\lambda-1)(\lambda-2)(\lambda-2)=0$$

$$\lambda=1,2,2$$

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}2 & 1 & -1\\ 2 & 1 & -1\\ 2 & 2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$2x_1+x_2-x_3=0$$

$$2x_1+2x_2-x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}1 & -1\\ 2 & -1\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}2 & -1\\ 2 & -1\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}2 & 1\\ 2 & 2\end{vmatrix}}$$

$$\frac{x_1}{1}=-\frac{x_2}{0}=\frac{x_3}{2}$$

Hence, corresponding to $\lambda=1$ the eigen vector is $X_1=[1,0,2]'$

(ii) For $\lambda=2, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}1 & 1 & -1\\ 2 & 0 & -1\\ 2 & 2 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$x_1+x_2-x_3=0$$

$$2x_1+0x_2-x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}1 & -1\\ 0 & -1\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}1 & -1\\ 2 & -1\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}1 & 1\\ 2 & 0\end{vmatrix}}$$

$$\frac{x_1}{-1}=-\frac{x_2}{1}=\frac{x_3}{-2}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [1,1,2]'$
Thus, the characteristic roots of A is 1,2,2
And, the characteristic roots of A^2 is $1^2, 2^2, 2^2$ i.e. $1,4,4$
Characteristic vectors of A^2 is same as that of A

33. Find eigen values and eigen vectors of A^2+2I where $A=\begin{bmatrix}8 & -8 & -2\\4 & -3 & -2\\3 & -4 & 1\end{bmatrix}$

[N17/ElexExtcElectBiomInst/6M]

Solution:

$A=\begin{bmatrix}8 & -8 & -2\\4 & -3 & -2\\3 & -4 & 1\end{bmatrix}, |A|=6$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}8-\lambda & -8 & -2\\4 & -3-\lambda & -2\\3 & -4 & 1-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[8-3+1]\lambda^2+\left[\begin{vmatrix}-3 & -2\\-4 & 1\end{vmatrix}+\begin{vmatrix}8 & -2\\3 & 1\end{vmatrix}+\begin{vmatrix}8 & -8\\4 & -3\end{vmatrix}\right]\lambda-6=0$

$\lambda^3-6\lambda^2+11\lambda-6=0$

$\lambda=1,2,3$

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$\begin{bmatrix}7 & -8 & -2\\4 & -4 & -2\\3 & -4 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$7x_1-8x_2-2x_3=0$

$4x_1-4x_2-2x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}-8 & -2\\-4 & -2\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}7 & -2\\4 & -2\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}7 & -8\\4 & -4\end{vmatrix}}$

$\frac{x_1}{8}=-\frac{x_2}{-6}=\frac{x_3}{4}$

$\frac{x_1}{4}=\frac{x_2}{3}=\frac{x_3}{2}$

Hence, corresponding to $\lambda=1$ the eigen vector is $X_1=[4,3,2]'$

(ii) For $\lambda=2, [A-\lambda I]X=0$ gives

$\begin{bmatrix}6 & -8 & -2\\4 & -5 & -2\\3 & -4 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$6x_1-8x_2-2x_3=0$

$4x_1-5x_2-2x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}-8 & -2\\-5 & -2\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}6 & -2\\4 & -2\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}6 & -8\\4 & -5\end{vmatrix}}$

$\frac{x_1}{6}=-\frac{x_2}{-4}=\frac{x_3}{2}$

$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [3,2,1]'$

(iii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 8x_2 - 2x_3 = 0$

$4x_1 - 6x_2 - 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$\frac{x_1}{4} = -\frac{x_2}{-2} = \frac{x_3}{2}$

$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [2,1,1]'$

The eigen values of A is 1,2,3

The eigen values of A^2 is $1^2, 2^2, 3^2$ i.e. 1,4,9

The eigen values of I is 1,1,1

The eigen values of $2I$ is 2,2,2

Thus, the eigen values of $A^2 + 2I$ is

$1 + 2\,; 4 + 2; 9 + 2$

i.e. 3; 6; 11

the eigen vectors of $A^2 + 2I$ is same as that of A

34. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ find eigen values and eigen vectors of $A^2 - 2A + I$

[M18/M19/Inst/6M][M19/Biom/6M]

Solution:

$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}, |A| = 24$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 3 & 4 \\ 0 & 4-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [2+4+3]\lambda^2 + \left[\begin{vmatrix} 4 & 2 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix}\right]\lambda - 24 = 0$

$\lambda^3 - 9\lambda^2 + 26\lambda - 24 = 0$

$\lambda = 2,3,4$

(i) For $\lambda = 2$, $[A - \lambda I]X = 0$ gives

$\begin{bmatrix} 0 & 3 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0x_1 + 3x_2 + 4x_3 = 0$

$0x_1 + 2x_2 + 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 0 & 4 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 3 \\ 0 & 2 \end{vmatrix}}$

$\frac{x_1}{-2} = -\frac{x_2}{0} = \frac{x_3}{0}$

$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_1 = [1,0,0]'$

(ii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$\begin{bmatrix} -1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-x_1 + 3x_2 + 4x_3 = 0$

$0x_1 + x_2 + 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -1 & 4 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix}}$

$\frac{x_1}{2} = -\frac{x_2}{-2} = \frac{x_3}{-1}$

$\frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{-1}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_2 = [2,2,-1]'$

(iii) For $\lambda = 4$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -2 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1 + 3x_2 + 4x_3 = 0$

$0x_1 + 0x_2 + 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -2 & 4 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 3 \\ 0 & 0 \end{vmatrix}}$$

$\frac{x_1}{6} = -\frac{x_2}{-4} = \frac{x_3}{0}$

$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{0}$

Hence, corresponding to $\lambda = 4$ the eigen vector is $X_3 = [3,2,0]'$

The eigen values of A is 2,3,4

The eigen values of A^2 is $2^2, 3^2, 4^2$ i.e. 4,9,16

The eigen values of $2A$ is 4,6,8

The eigen values of I is 1,1,1

Thus, the eigen values of $A^2 - 2A + I$ is

$4 - 4 + 1$; $9 - 6 + 1$; $16 - 8 + 1$

i.e. 1; 4; 9

the eigen vectors of $A^2 - 2A + I$ is same as that of A

35. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ find eigen values of $A^2 - 2A + I$ & eigen values of $adjA$

[N22/Elect/5M]

Solution:

$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}, |A| = 24$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 3 & 4 \\ 0 & 4-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [2+4+3]\lambda^2 + \left[\begin{vmatrix} 4 & 2 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix}\right]\lambda - 24 = 0$

$\lambda^3 - 9\lambda^2 + 26\lambda - 24 = 0$

$\lambda = 2,3,4$

The eigen values of A is 2,3,4

The eigen values of A^2 is $2^2, 3^2, 4^2$ i.e. 4,9,16

The eigen values of $2A$ is 4,6,8

The eigen values of I is 1,1,1

Thus, the eigen values of $A^2 - 2A + I$ is

$4 - 4 + 1\ ; 9 - 6 + 1; 16 - 8 + 1$

i.e. 1; 4; 9

the eigen values of $adjA$ is $\frac{|A|}{\lambda}$

i.e. $\frac{24}{2}, \frac{24}{3}, \frac{24}{4}$

i.e. 12, 8, 6

36. If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ find eigen values and eigen vectors of $A^2 + 2A + I$

[N19/Elect/6M]

Solution:

$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}, |A| = 0$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 8-\lambda & -6 & 4 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [8+7+3]\lambda^2 + \left[\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}\right]\lambda - 0 = 0$

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda - 15)(\lambda - 3) = 0$

$\lambda = 0,3,15$

(i) For $\lambda = 0, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 7x_2 - 4x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$

$\frac{x_1}{10} = -\frac{x_2}{-20} = \frac{x_3}{20}$

$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$

Hence, corresponding to $\lambda = 0$ the eigen vector is $X_1 = [1,2,2]'$

(ii) For $\lambda = 3, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$5x_1 - 6x_2 + 2x_3 = 0$

$2x_1 - 4x_2 + 0x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -4 & 0 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 5 & 2 \\ 2 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ 2 & -4 \end{vmatrix}}$

$\frac{x_1}{8} = -\frac{x_2}{-4} = \frac{x_3}{-8}$

$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_2 = [2,1,-2]'$

(iii) For $\lambda = 15$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 - 8x_2 - 4x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$\frac{x_1}{40} = -\frac{x_2}{40} = \frac{x_3}{20}$

$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 15$ the eigen vector is $X_3 = [2,-2,1]'$

The eigen values of A is $0,3,15$

The eigen values of A^2 is $0^2, 3^2, 15^2$ i.e. $0,9,225$

The eigen values of $2A$ is $0,6,30$

The eigen values of I is $1,1,1$

Thus, the eigen values of $A^2 + 2A + I$ is

$0 + 0 + 1\ ; 9 + 6 + 1; 225 + 30 + 1$

i.e. $1; 16; 256$

the eigen vectors of $A^2 + 2A + I$ is same as that of A

37. If $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ then find the eigen values of $4A^{-1} + 3A + 2I$

[M15/CompIT/5M][M19/Chem/5M]

Solution:

$A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}, |A| = 4$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 0 \\ 2 & 4-\lambda \end{vmatrix} = 0$

$(1-\lambda)(4-\lambda) - 0 = 0$

$\lambda = 1,4$

The eigen values of A is $1,4$

The eigen values of A^{-1} is $1^{-1}, 4^{-1}$ i.e. $1, \frac{1}{4}$

The eigen values of $4A^{-1}$ is $4(1), 4\left(\frac{1}{4}\right)$ i.e. $4,1$

The eigen values of $3A$ is $3(1), 3(4)$ i.e. $3,12$

The eigen values of I is $1,1$

The eigen values of $2I$ is $2,2$

Thus, the eigen values of $4A^{-1} + 3A + 2I$ is

$4 + 3 + 2$; $1 + 12 + 2$ i.e. 9; 15

38. If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$, find eigen values of $A^3 - 3A^2 + 5A$
[J23/MechCivil/5M]
Solution:
$A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}, |A| = -9$
The characteristic equation,
$|A - \lambda I| = 0$
$\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$
$\lambda^2 - (sum\ of\ diagonals)\lambda + |A| = 0$
$\lambda^2 - (-1+1)\lambda - 9 = 0$
$\lambda^2 - 9 = 0$
$\lambda = 3, -3$
The eigen values of A is $3, -3$
The eigen values of A^3 is $3^3, (-3)^3$ i.e. $27, -27$
The eigen values of A^2 is $3^2, (-3)^2$ i.e. 9,9
The eigen values of $3A^2$ is $3(9), 3(9)$ i.e. 27,27
The eigen values of $5A$ is $5(3)$,$5(-3)$ i.e. $15, -15$
Thus, the eigen values of $A^3 - 3A^2 + 5A$ is
$27 - 27 + 15$; $-27 - 27 - 15$ i.e. $15; -69$

39. If $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ find eigen values of $adjA$
[M23/MechCivil/5M]
Solution:
$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, |A| = 8$
The characteristic equation,
$|A - \lambda I| = 0$
$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$
$\lambda^2 - (sum\ of\ diagonals)\lambda + |A| = 0$
$\lambda^2 - (3+3)\lambda + 8 = 0$
$\lambda^2 - 6\lambda + 8 = 0$
$\lambda = 2,4$
The eigen values of A is 2,4
The eigen values of $adjA$ is $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}$ i.e. $\frac{8}{2}, \frac{8}{4}$ i.e. 4,2

40. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$ find the eigen values of $A^3 + 5A + 8I$

[N14/M24/CompIT/5M][J24/N24/ElectECSExtc/5M]

Solution:

$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}, |A| = 6$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -1-\lambda & 2 & 3 \\ 0 & 3-\lambda & 5 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [-1+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 5 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix}\right]\lambda - 6 = 0$

$\lambda^3 - 0\lambda^2 - 7\lambda - 6 = 0$

$\lambda = -1, -2, 3$

The eigen values of A is $-1, -2, 3$

The eigen values of A^3 is $(-1)^3, (-2)^3, 3^3$ i.e $-1, -8, 27$

The eigen values of $5A$ is $5(-1), 5(-2), 5(3)$ i.e. $-5, -10, 15$

The eigen values of I is 1,1,1

The eigen values of $8I$ is 8,8,8

Thus, the eigen values of $A^3 + 5A + 8I$ is

$-1 + (-5) + 8;\ \ -8 + (-10) + 8;\ \ \ 27 + 15 + 8$

i.e. $2, -10, 50$

41. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$ find the eigen values of $A^3 + 5A + 8I + A^{-1}$

[M23/ElectECSExtc/5M]

Solution:

$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}, |A| = 6$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -1-\lambda & 2 & 3 \\ 0 & 3-\lambda & 5 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [-1+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 5 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix}\right]\lambda - 6 = 0$

$\lambda^3 - 0\lambda^2 - 7\lambda - 6 = 0$

$\lambda = -1, -2, 3$

The eigen values of A is $-1, -2, 3$

The eigen values of A^3 is $(-1)^3, (-2)^3, 3^3$ i.e $-1, -8, 27$

The eigen values of $5A$ is $5(-1), 5(-2), 5(3)$ i.e. $-5, -10, 15$

The eigen values of I is 1,1,1

The eigen values of $8I$ is 8,8,8

The eigen values of A^{-1} is $\frac{1}{-1}, \frac{1}{-2}, \frac{1}{3}$

Thus, the eigen values of $A^3 + 5A + 8I + A^{-1}$ is

$-1 + (-5) + 8 - 1;\ \ -8 + (-10) + 8 - \frac{1}{2};\ \ \ 27 + 15 + 8 + \frac{1}{3}$

i.e. $1, -\frac{21}{2}, \frac{151}{3}$

42. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ find the eigen values of $3A^2 - 2A + 5I$

[M19/Elex/5M]

Solution:

$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}, |A| = 6$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & -2 & 3 \\ 0 & 3-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [1+3+2]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix}\right]\lambda - 6 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1,2,3$

The eigen values of A is 1,2,3

The eigen values of A^2 is $1^2, 2^2, 3^2$ i.e 1,4,9

The eigen values of I is 1,1,1

Thus, the eigen values of $3A^2 - 2A + 5I$ is

$3(1) - 2(1) + 5(1);\ \ 3(2^2) - 2(2) + 5(1);\ \ 3(3^2) - 2(3) + 5(1)$

i.e. 6, 13,26

43. Find the eigen values of $5A^2 - 6A + I$ where $A = \begin{bmatrix} -1 & 5 & 9 \\ 0 & -3 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

[N19/Elex/5M]

Solution:

$A = \begin{bmatrix} -1 & 5 & 9 \\ 0 & -3 & 4 \\ 0 & 0 & 2 \end{bmatrix}, |A| = 6$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -1-\lambda & 5 & 9 \\ 0 & -3-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [-1-3+2]\lambda^2 + \left[\begin{vmatrix} -3 & 4 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 9 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 5 \\ 0 & -3 \end{vmatrix}\right]\lambda - 6 = 0$

$\lambda^3 + 2\lambda^2 - 5\lambda - 6 = 0$

$\lambda = -1, 2, -3$

The eigen values of A is $-1, 2, -3$

The eigen values of A^2 is $(-1)^2, 2^2, (-3)^2$ i.e 1,4,9

The eigen values of $5A^2$ is 5,20,45

The eigen values of 6A is $-6, 12, -18$

The eigen values of I is 1,1,1

Thus, the eigen values of $5A^2 - 6A + I$ is

$5 - (-6) + 1;\ 20 - 12 + 1;\ 45 - (-18) + 1$

i.e. $12, 9, 64$

44. If $A=\begin{bmatrix}-1 & 0 & 0\\ 2 & -3 & 0\\ 1 & 4 & -2\end{bmatrix}$ find the eigen values of A^2+I

[N19/Chem/5M]

Solution:

$A=\begin{bmatrix}-1 & 0 & 0\\ 2 & -3 & 0\\ 1 & 4 & -2\end{bmatrix}, |A|=-6$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}-1-\lambda & 0 & 0\\ 2 & -3-\lambda & 0\\ 1 & 4 & -2-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[-1-3-2]\lambda^2+\left[\begin{vmatrix}-3 & 0\\ 4 & -2\end{vmatrix}+\begin{vmatrix}-1 & 0\\ 1 & -2\end{vmatrix}+\begin{vmatrix}-1 & 0\\ 2 & -3\end{vmatrix}\right]\lambda+6=0$

$\lambda^3+6\lambda^2+11\lambda+6=0$

$\lambda=-1,-2,-3$

The eigen values of A is $-1,-2,-3$

The eigen values of A^2 is $(-1)^2,(-2)^2,(-3)^2$ i.e 1,4,9

The eigen values of I is 1,1,1

Thus, the eigen values of A^2+I is

$1+1;\ 4+1;\ 9+1$

i.e. 2,5,10

45. Find the eigen values of $A^2 - 5A + 4I$ if $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & -2 \end{bmatrix}$

[M23DSE/J24/MechCivil/5M]

Solution:

$A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & -2 \end{bmatrix}, |A| = -6$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -1-\lambda & 0 & 0 \\ 2 & -3-\lambda & 0 \\ 1 & 4 & -2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [-1-3-2]\lambda^2 + \left[\begin{vmatrix} -3 & 0 \\ 4 & -2 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 2 & -3 \end{vmatrix}\right]\lambda + 6 = 0$

$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

$\lambda = -1, -2, -3$

The eigen values of A is $-1, -2, -3$

The eigen values of A^2 is $(-1)^2, (-2)^2, (-3)^2$ i.e. 1,4,9

The eigen values of $5A$ is $5(-1), 5(-2), 5(-3)$ i.e. $-5, -10, -15$

The eigen values of I is 1,1,1

The eigen values of $4I$ is 4,4,4

Thus, the eigen values of $A^2 - 5A + 4I$ is

$1 - (-5) + 4;\ 4 - (-10) + 4;\ 9 - (-15) + 4$

i.e. 10,18,28

46. If $A=\begin{bmatrix}-1 & 0 & 0\\ 2 & -3 & 0\\ 1 & 4 & 1\end{bmatrix}$ then find the eigen values of A^3

[M23DSE/ElectECSExtc/5M]

Solution:

$A=\begin{bmatrix}-1 & 0 & 0\\ 2 & -3 & 0\\ 1 & 4 & 1\end{bmatrix}, |A|=3$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}-1-\lambda & 0 & 0\\ 2 & -3-\lambda & 0\\ 1 & 4 & 1-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[-1-3+1]\lambda^2+\left[\begin{vmatrix}-3 & 0\\ 4 & 1\end{vmatrix}+\begin{vmatrix}-1 & 0\\ 1 & 1\end{vmatrix}+\begin{vmatrix}-1 & 0\\ 2 & -3\end{vmatrix}\right]\lambda-3=0$

$\lambda^3+3\lambda^2-\lambda-3=0$

$\lambda=-3,-1,1$

The eigen values of A is $-3,-1,1$

The eigen values of A^3 is $(-3)^3,(-1)^3,(1)^3$ i.e. $-27,-1,1$

47. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 5 & 1 \end{bmatrix}$ find eigen values of $A^2 + I$

[M22/Elex/2M]

Solution:

$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 5 & 1 \end{bmatrix}, |A| = 2$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 2-\lambda & 0 \\ 3 & 5 & 1-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [1+2+1]\lambda^2 + \left[\begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}\right]\lambda - 2 = 0$

$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$

$\lambda = 2,1,1$

The eigen values of A is 2,1,1

The eigen values of A^2 is $2^2, 1^2, 1^2$ i.e. 4,1,1

The eigen values of I is $1, 1, 1$

Thus, the eigen values of $A^2 + I$ is $4 + 1, 1 + 1, 1 + 1$ i.e. 5,2,2

48. Find the eigen values of $A^3 - 2A^2 + I$ where $A = \begin{bmatrix} 4 & 1 & -1 \\ 6 & 3 & -5 \\ 6 & 2 & -2 \end{bmatrix}$

[N19/MechCivil/5M]

Solution:

$A = \begin{bmatrix} 4 & 1 & -1 \\ 6 & 3 & -5 \\ 6 & 2 & -2 \end{bmatrix}, |A| = 4$

The characteristic equation,

$|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 1 & -1 \\ 6 & 3-\lambda & -5 \\ 6 & 2 & -2-\lambda \end{vmatrix} = 0$$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [4+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & -5 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} 4 & -1 \\ 6 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 6 & 3 \end{vmatrix}\right]\lambda - 4 = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$\lambda = 1,2,2$

The eigen values of A is 1,2,2

The eigen values of A^3 is $1^3, 2^3, 2^3$ i.e 1,8,8

The eigen values of A^2 is $1^2, 2^2, 2^2$ i.e. 1,4,4

The eigen values of $2A^2$ is 2,8,8

The eigen values of I is 1,1,1

Thus, the eigen values of $A^3 - 2A^2 + I$ is

$1 - 2 + 1;\ 8 - 8 + 1;\ 8 - 8 + 1$ i.e. 0,1,1

49. Find the eigen values and eigen vectors of $A^3 - 2A + I$ if $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

[M18/Elect/6M]

Solution:

$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}, |A| = 5$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [2+3+2]\lambda^2 + \left[\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}\right]\lambda - 5 = 0$

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$(\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$

$\lambda = 1,1,5$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 - R_1, R_3 - R_1$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore x_1 + 2x_2 + x_3 = 0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

Let $x_3 = t$ & $x_2 = s$

$\therefore x_1 = -2s - t$

$\therefore X = \begin{bmatrix} -2s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2s-t \\ s+0t \\ 0s+t \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} -t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Hence, corresponding to $\lambda = 1$ the eigen vectors are

$X_1 = [-2,1,0]'$ & $X_2 = [-1,0,1]'$

(ii) For $\lambda = 5, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x_1 + 2x_2 + x_3 = 0$
$x_1 - 2x_2 + x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$

$\frac{x_1}{4} = -\frac{x_2}{-4} = \frac{x_3}{4}$

$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,1,1]'$

The eigen values of A is 1,1,5

The eigen values of A^3 is $(1)^3, (1)^3, (5)^3$ i.e 1,1,125

The eigen values of $2A$ is $2(1), 2(1), 2(5)$ i.e. 2,2,10

The eigen values of I is 1,1,1

Thus, the eigen values of $A^3 - 2A + I$ is

$1 - 2 + 1; 1 - 2 + 1; 125 - 10 + 1$ i.e. 0,0,116

Eigen vectors of $A^3 - 2A + I$ is same as that of A

50. Find the eigen values of adjoint of $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

[N15/MechCivil/5M][N17/CompIT/5M][M22/CompIT/2M]

Solution:

$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}, |A| = 6$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [2+2+2]\lambda^2 + \left[\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}\right]\lambda - 6 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$(\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$

$\lambda = 1,3,2$

The eigen values of A is 1,3,2

The eigen values of $adjA$ is $3 \times 2, 1 \times 2, 1 \times 3$ i.e 6,2,3

51. Find the eigen values of $A^2 + 2I$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 5 & 3 \end{bmatrix}$ and I is the Identity matrix of order 3

[M16/CompIT/5M][N23/MechCivil/5M]

Solution:

$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 5 & 3 \end{bmatrix}, |A| = -6$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & -2-\lambda & 0 \\ 3 & 5 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [1 - 2 + 3]\lambda^2 + \left[\begin{vmatrix} -2 & 0 \\ 5 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix}\right]\lambda - (-6) = 0$

$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 1)(\lambda + 2)(\lambda - 3) = 0$

$\lambda = 1, -2, 3$

The eigen values of A is $1, -2, 3$

The eigen values of A^2 is $1^2, 2^2, (-3)^2$ i.e. 1,4,9

The eigen values of I is 1,1,1

The eigen values of $2I$ is 2,2,2

Thus, the characteristic roots of $A^2 + 2I$ is

$1 + 2$; $4 + 2$; $9 + 2$ i.e. 3 ; 6 ; 11

52. If λ is an eigen value of matrix A, then prove that λ^n is an eigen value of A^n and hence find the eigen values for $A^2 - 2A + 5I$ where $A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

[M18/MechCivil/5M]

Solution:

Consider,

$$[A - \lambda I]X = 0$$
$$AX - \lambda X = 0$$
$$AX = \lambda X$$

Where λ is an eigen value of A & X is the corresponding eigen vector

Now, Multiplying the above equation by A,

$$A(AX) = A(\lambda X)$$
$$A^2X = \lambda(AX)$$
$$A^2X = \lambda(\lambda X)$$
$$A^2X = \lambda^2 X$$

$\therefore$ $\lambda$is an eigen value of A^2

Similarly, $A^3X = \lambda^3X \dots\dots\dots\dots\dots\dots A^nX = \lambda^nX$

$\therefore$ λ^n is an eigen value of A^n corresponding to the same eigen vectors X

We have,

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}, |A| = 12$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & -2 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+2+3]\lambda^2 + \left[\begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}\right]\lambda - 12 = 0$$

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

$$\lambda = 2,2,3$$

The eigen values of A is 2,2,3

The eigen values of A^2 is $(2)^2, (2)^2, 3^2$ i.e 4,4,9

The eigen values of $2A$ is $2(2), 2(2), 2(3)$ i.e. 4,4,6

The eigen values of $5I$ is 5,5,5

Thus, the eigen values of $A^2 - 2A + 5I$ is

$4 - 4 + 5; 4 - 4 + 5; 9 - 6 + 5$ i.e. $5,5,8$

53. Find Eigen Values of $A^2 - 2A + I$ where $A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

[N22/Extc/5M]

Solution:

$A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}, |A| = 6$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 1 & -2 \\ 0 & 1-\lambda & 4 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [2+1+3]\lambda^2 + \left[\begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\right]\lambda - 6 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1,2,3$

The eigen values of A is 1,2,3

The eigen values of A^2 is $(1)^2, (2)^2, 3^2$ i.e. 1,4,9

The eigen values of $2A$ is $2(1), 2(2), 2(3)$ i.e. 2,4,6

The eigen values of I is 1, 1, 1

Thus, the eigen values of $A^2 - 2A + I$ is

$1 - 2 + 1; 4 - 4 + 1; 9 - 6 + 1$ i.e. 0,1, 4

54. Find Eigen Values of $A^2 - 2A + I$ where $A = \begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & -3 \end{bmatrix}$

[J23/ElectECSExtc/5M]

Solution:

$A = \begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & -3 \end{bmatrix}, |A| = -6$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & -1 & -2 \\ 0 & 1-\lambda & 4 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [2+1-3]\lambda^2 + \left[\begin{vmatrix} 1 & 4 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}\right]\lambda + 6 = 0$

$\lambda^3 - 0\lambda^2 - 7\lambda + 6 = 0$

$\lambda = 1, 2, -3$

The eigen values of A is $1, 2, -3$

The eigen values of A^2 is $(1)^2, (2)^2, (-3)^2$ i.e. $1, 4, 9$

The eigen values of $2A$ is $2(1), 2(2), 2(-3)$ i.e. $2, 4, -6$

The eigen values of I is $1, 1, 1$

Thus, the eigen values of $A^2 - 2A + I$ is

$1 - 2 + 1; 4 - 4 + 1; 9 + 6 + 1$ i.e. $0, 1, 16$

55. If $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ then find the Eigen values of $4A^{-1} + A^3 + I$

[M23/CompIT/5M]

Solution:

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, |A| = 45$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [-2+1+0]\lambda^2 + \left[\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}\right]\lambda - 45 = 0$

$\lambda^3 + 1\lambda^2 - 21\lambda - 45 = 0$

$\lambda = -3, -3, 5$

The eigen values of A is $-3, -3, 5$

The eigen values of A^{-1} is $\frac{1}{-3}, \frac{1}{-3}, \frac{1}{5}$

The eigen values of $4A^{-1}$ is $4\left(\frac{1}{-3}\right), 4\left(\frac{1}{-3}\right), 4\left(\frac{1}{5}\right)$ i.e. $-\frac{4}{3}, -\frac{4}{3}, \frac{4}{5}$

The eigen values of A^3 is $(-3)^3, (-3)^3, 5^3$ i.e. $-27, -27, 125$

The eigen values of I is $1,1,1$

Thus, the eigen values of $4A^{-1} + A^3 + I$ is

$-\frac{4}{3} - 27 + 1, -\frac{4}{3} - 27 + 1, \frac{4}{5} + 125 + 1$

i.e. $-\frac{82}{3}, -\frac{82}{3}, \frac{634}{5}$

56. Given $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, find the eigen values of A. Also find eigen values of $4A^{-1}$ and eigen vector of $A^2 - 4I$

[M22/CompIT/5M]

Solution:

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, |A| = 45$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [-2+1+0]\lambda^2 + \left[\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}\right]\lambda - 45 = 0$$

$$\lambda^3 + 1\lambda^2 - 21\lambda - 45 = 0$$

$$\lambda = -3, -3, 5$$

(i) For $\lambda = -3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - 2R_1, R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 + 2x_2 - 3x_3 = 0$$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

Let $x_3 = t$ & $x_2 = s$, $\therefore x_1 = -2s + 3t$

$$\therefore X = \begin{bmatrix} -2s+3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2s+3t \\ s+0t \\ 0s+t \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} 3t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Hence, corresponding to $\lambda = -3$ the eigen vectors are

$X_1 = [-2,1,0]'$ & $X_2 = [3,0,1]'$

(ii) For $\lambda = 5$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 + 2x_2 - 3x_3 = 0$
$2x_1 - 4x_2 - 6x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$

$\frac{x_1}{-24} = -\frac{x_2}{48} = \frac{x_3}{24}$

$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,2,-1]'$

The eigen values of A is $-3, -3, 5$

The eigen values of A^{-1} is $\frac{1}{-3}, \frac{1}{-3}, \frac{1}{5}$

The eigen values of $4A^{-1}$ is $4\left(\frac{1}{-3}\right), 4\left(\frac{1}{-3}\right), 4\left(\frac{1}{5}\right)$ i.e. $-\frac{4}{3}, -\frac{4}{3}, \frac{4}{5}$

The eigen vectors of $A^2 - 4I$ is same as that of A

57. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ then find eigen values of $A^{-1} + A^2$

[N23/ElectECSExtc/5M]

Solution:

$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}, |A| = 6$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 0 \\ 1 & 4 & 2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [1+3+2]\lambda^2 + \left[\begin{vmatrix} 3 & 0 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}\right]\lambda - 6 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1,2,3$

The eigen values of A is $1,2,3$

The eigen values of A^{-1} is $1, \frac{1}{2}, \frac{1}{3}$

The eigen values of A^2 is $1^2, 2^2, 3^2$ i.e. $1,4,9$

Thus, the eigen values of $A^{-1} + A^2$ is

$1 + 1; \frac{1}{2} + 4; \frac{1}{3} + 9$

i.e. $2, \frac{9}{2}, \frac{28}{3}$

58. If $A=\begin{bmatrix}-1 & 2 & 38\\ 0 & 2 & 37\\ 0 & 0 & -2\end{bmatrix}$ find the Eigen values of $A^3+5A+8I$

[D24/CompIT/5M]

Solution:

$A=\begin{bmatrix}-1 & 2 & 38\\ 0 & 2 & 37\\ 0 & 0 & -2\end{bmatrix}, |A|=4$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}-1-\lambda & 2 & 38\\ 0 & 2-\lambda & 37\\ 0 & 0 & -2-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[-1+2-2]\lambda^2+\left[\begin{vmatrix}2 & 37\\ 0 & -2\end{vmatrix}+\begin{vmatrix}-1 & 38\\ 0 & -2\end{vmatrix}+\begin{vmatrix}-1 & 2\\ 0 & 2\end{vmatrix}\right]\lambda-4=0$

$\lambda^3+\lambda^2-4\lambda-4=0$

$\lambda=-1,2,-2$

The eigen values of A is $-1,2,-2$

The eigen values of A^3 is $(-1)^3,(2)^3,(-2)^3$ i.e. $-1,8,-8$

The eigen values of $5A$ is $5(-1),5(2),5(-2)$ i.e. $-5,10,-10$

The eigen values of $8I$ is $8(1),8(1),8(1)$ i.e. $8,8,8$

Thus, the eigen values of $A^3+5A+8I$ is

$-1-5+8; 8+10+8; -8-10+8$ i.e. $2,26,-10$

59. If $A = [a_{ij}]$ is a matrix of order 3×3 such that $a_{ij} = \begin{cases} 1 \; if \; i \neq j \\ 0 \; if \; i = j \end{cases}$

Find an eigen value of i) A ii) adjoint of A iii) $A^2 - 2A + I$

[N24/MechCivil/5M]

Solution:

$A = [a_{ij}]$ where $a_{ij} = \begin{cases} 1 \; if \; i \neq j \\ 0 \; if \; i = j \end{cases}$

$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, |A| = 2$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [0+0+0]\lambda^2 + \left[\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\right]\lambda - 2 = 0$

$\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

$\lambda = 2, -1, -1$

The eigen values of A is $2, -1, -1$

The eigen values of adj A is $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}$ i.e. $\frac{2}{2}, \frac{2}{-1}, \frac{2}{-1}$ i.e. $1, -2, -2$

The eigen values of A^2 is $(2)^2, (-1)^2, (-1)^2$ i.e. $4,1,1$

The eigen values of $2A$ is $2(2), 2(-1), 2(-1)$ i.e. $4, -2, -2$

The eigen values of I is $1,1,1$

Thus, the eigen values of $A^2 - 2A + I$ is

$4 - 4 + 1; 1 - (-2) + 1; 1 - (-2) + 1$ i.e. $1, 4, 4$

60. If $A = \begin{bmatrix} x & 4x \\ 2 & y \end{bmatrix}$ has eigen values 5 and – 1 then find values of x and y.

[M14/CompIT/5M]

Solution:

$A = \begin{bmatrix} x & 4x \\ 2 & y \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} x - \lambda & 4x \\ 2 & y - \lambda \end{vmatrix} = 0$

$(x - \lambda)(y - \lambda) - 8x = 0$

$xy - x\lambda - y\lambda + \lambda^2 - 8x = 0$

$\lambda^2 - (x + y)\lambda + (xy - 8x) = 0$

The eigen values are the roots of the above equation which is given as $5 \& -1$.

Sum of roots $= (x + y)$

$5 + (-1) = x + y$

$x + y = 4$

$y = 4 - x$

Product of roots $= (xy - 8x)$

$(5) \times (-1) = xy - 8x$

$xy - 8x = -5$

$x(y - 8) + 5 = 0$

But $y = 4 - x$

$x(4 - x - 8) + 5 = 0$

$x(-x - 4) + 5 = 0$

$x^2 + 4x - 5 = 0$

$\therefore x = -5,1$

$\therefore y = 9,3$

61. Find the matrix $A_{2\times2}$ whose eigen values are 4 and 1 and their corresponding eigen vectors are $v_1 = \begin{bmatrix}1\\2\end{bmatrix}$ and $v_2 = \begin{bmatrix}-2\\1\end{bmatrix}$

[M23/ElectECSExtc/6M]

Solution:

Let $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}_{2\times2}$

We know that, if λ is the eigen value and v is the eigen vector of the matrix A then

$[A - \lambda I]v = 0$

$Av - \lambda v = 0$

$Av = \lambda v$

$\therefore Av_1 = \lambda v_1$

$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = 4\begin{bmatrix}1\\2\end{bmatrix}$

$a + 2b = 4$ ……… (1)

$c + 2d = 8$ ……… (2)

$\therefore Av_2 = \lambda v_2$

$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} = 1\begin{bmatrix}-2\\1\end{bmatrix}$

$-2a + b = -2$ ……… (3)

$-2c + d = 1$ …………. (4)

Solving (1) & (3), we get $a = \frac{8}{5}, b = \frac{6}{5}$

Solving (2) & (4), we get $c = \frac{6}{5}, d = \frac{17}{5}$

Thus, $A = \begin{bmatrix}\frac{8}{5} & \frac{6}{5}\\ \frac{6}{5} & \frac{17}{5}\end{bmatrix} = \frac{1}{5}\begin{bmatrix}8 & 6\\6 & 17\end{bmatrix}$

62. Show that the Eigen values of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is of unit modulus
[N22/Chem/5M]
Solution:
$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $|A| = \cos^2\theta + \sin^2\theta = 1$
Characteristic equation,
$|A - \lambda I| = 0$
$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$
$\lambda^2 - (sum\ of\ diagonals)\lambda + |A| = 0$
$\lambda^2 - (cos\theta + cos\theta)\lambda + 1 = 0$
$\lambda^2 - 2cos\theta\lambda + 1 = 0$
$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$\lambda = \frac{-(-2cos\theta) \pm \sqrt{(-2cos\theta)^2 - 4(1)(1)}}{2(1)}$
$\lambda = \frac{2cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$
$\lambda = \frac{2cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{2}$
$\lambda = cos\theta \pm \sqrt{-\sin^2\theta} = cos\theta \pm i\ sin\theta$
$|\lambda| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$

63. Sum of eigen values of 3 x 3 matrix is 6 and the product of eigen values is also 6. If one of the eigen value is 1, find other two eigen values. Clearly state the result which you use.
[N18/M19/Comp/5M]
Solution:
Let the three eigen values of $\lambda_1, \lambda_2, \lambda_3$
Given that,
$\lambda_1 + \lambda_2 + \lambda_3 = 6$ ………….. (1)
$\lambda_1.\lambda_2.\lambda_3 = 6$ ………………… (2)
Let $\lambda_3 = 1$ (given)
$\therefore \lambda_1 + \lambda_2 = 5$
$\therefore \lambda_1.\lambda_2 = 6$
On solving the two equations, we get
$\lambda_1(5 - \lambda_1) = 6$
$5\lambda_1 - \lambda_1^2 = 6$
$\lambda_1^2 - 5\lambda_1 + 6 = 0$
$\lambda_1 = 2,3$ and $\lambda_2 = 3,2$

64. If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ find the sum and product of eigen values of A

[D23/CompIT/5M]

Solution:

$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}, |A| = 0$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 8-\lambda & -6 & 4 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [8+7+3]\lambda^2 + \left[\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}\right]\lambda - 0 = 0$

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda - 15)(\lambda - 3) = 0$

$\lambda = 0,3,15$

Sum of eigen values $= 18$

Product of eigen values $= 0$

65. If the product of 2 eigen values of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16, find the third eigen value.

[M19/MTRX/5M]

Solution:

$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}, |A| = 32$

If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values then

$|A| = \lambda_1.\lambda_2.\lambda_3$

It is given that $\lambda_1.\lambda_2 = 16$

$32 = 16\lambda_3$

$\therefore \lambda_3 = 2$

66. Find the eigen values of $Adj\ A$ and $24A^{-1}+2A-I$ where $A=\begin{bmatrix}1&2&3&-2\\0&2&4&6\\0&0&4&-5\\0&0&0&6\end{bmatrix}$

[M18/Elex/5M]

Solution:

$$A=\begin{bmatrix}1&2&3&-2\\0&2&4&6\\0&0&4&-5\\0&0&0&6\end{bmatrix}, |A|=1\times2\times4\times6=48$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}1-\lambda&2&3&-2\\0&2-\lambda&4&6\\0&0&4-\lambda&-5\\0&0&0&6-\lambda\end{vmatrix}=0$$

$$(1-\lambda)(2-\lambda)(4-\lambda)(6-\lambda)=0$$

$$\lambda=1,2,4,6$$

The eigen values of A is 1,2,4,6

The eigen values of $adjA$ is $\frac{|A|}{1},\frac{|A|}{2},\frac{|A|}{4},\frac{|A|}{6}$ i.e. 48,24,12,8

The eigen values of A^{-1} is $1^{-1},2^{-1},4^{-1},6^{-1}$ i.e $1,\frac{1}{2},\frac{1}{4},\frac{1}{6}$

The eigen values of $24A^{-1}$ is $24(1),24\left(\frac{1}{2}\right),24\left(\frac{1}{4}\right),24\left(\frac{1}{6}\right)$ i.e. 24,12,6,4

The eigen values of $2A$ is $2(1),2(2),2(4),2(6)$ i.e. 2,4,8,12

The eigen values of I is 1,1,1,1

Thus, the eigen values of $24A^{-1}+2A-I$ is

$$24+2-1; 12+4-1; 6+8-1; 4+12-1$$

i.e. 25; 15; 13; 15

67. Eigen values of Hermitian matrix is always
[M22/Chem/2M] Ans. purely real

68. If eigen values of a matrix is 1,2,3 then Eigen values of A^2 is
[M22/Chem/2M] Ans. 1,4,9

69. If the matrix $A = \begin{bmatrix} \alpha & 1 \\ 2 & 3 \end{bmatrix}$ has 1 as an eigen value the trace A is

[M22/ElectECSExtc/2M]

Solution:

$A = \begin{bmatrix} \alpha & 1 \\ 2 & 3 \end{bmatrix}$

$|A| = 3\alpha - 2$

Let one of the eigen value be λ and other eigen value is given as 1

We know that,

Sum of eigen values = sum of diagonal elements

$\lambda + 1 = \alpha + 3$

$\lambda - \alpha = 2$ …….. (1)

Product of eigen values $= |A|$

$\lambda \times 1 = 3\alpha - 2$

$\lambda - 3\alpha = -2$ ….. (2)

Solving (1) & (2), we get

$\lambda = 4, \alpha = 2$

Trace = sum of diagonals = 5

70. The characteristic polynomial of matrix $A = \begin{bmatrix} 0 & 0 & -c \\ 1 & 0 & -b \\ 0 & 1 & -a \end{bmatrix}$ is

[M22/ElectECSExtc/2M]

Solution:

$A = \begin{bmatrix} 0 & 0 & -c \\ 1 & 0 & -b \\ 0 & 1 & -a \end{bmatrix}, |A| = 0 - 0 - c[1 - 0] = -c$

The characteristic equation,

$|A - \lambda I| = 0$

$$\begin{vmatrix} 0-\lambda & 0 & -c \\ 1 & 0-\lambda & -b \\ 0 & 1 & -a-\lambda \end{vmatrix} = 0$$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [0 + 0 - a]\lambda^2 + \left[\begin{vmatrix} 0 & -b \\ 1 & -a \end{vmatrix} + \begin{vmatrix} 0 & -c \\ 0 & -a \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\right]\lambda + c = 0$

$\lambda^3 + a\lambda^2 + b\lambda + c = 0$

Thus, the characteristic polynomial is $f(x) = x^3 + ax^2 + bx + c$

Type II: Transformation and Diagonalising of Matrix

1. Show that the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ is diagonalizable. Hence find diagonal and transforming matrix.

Solution:

$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$|A| = 36$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diag]\lambda^2 + [sum\ of\ minors\ of\ diag]\lambda - |A| = 0$

$\lambda^3 - [3+5+3]\lambda^2 + \left[\begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}\right]\lambda - 36 = 0$

$\lambda^3 - 11\lambda^2 + [15 - 1 + 9 - 1 + 15 - 1]\lambda - 36 = 0$

$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$\lambda = 6,3,2$

Since the eigen values are distinct, matrix A is diagonalisable

(i) For $\lambda = 6$

$[A - \lambda I]X = 0$

$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x_1 - x_2 + x_3 = 0$

$-x_1 - x_2 - x_3 = 0$

Solving by Crammers rule,

$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$

$\frac{x_1}{2} = \frac{-x_2}{4} = \frac{x_3}{2}$

$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$

$X_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(ii) For $\lambda = 3$

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0x_1 - x_2 + x_3 = 0$

$-x_1 + 2x_2 - x_3 = 0$

Solving by Crammers rule,

$\frac{x_1}{1-2} = \frac{-x_2}{0+1} = \frac{x_3}{0-1}$

$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$

$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(iii) For $\lambda = 2$

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - x_2 + x_3 = 0$

$-x_1 + 3x_2 - x_3 = 0$

Solving by Crammers rule,

$\frac{x_1}{-2} = \frac{-x_2}{0} = \frac{x_3}{2}$

$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$

$$X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

2. Show that the matrix $A=\begin{bmatrix}4 & 2 & -2\\ -5 & 3 & 2\\ -2 & 4 & 1\end{bmatrix}$ is diagonalisable. Find the diagonal matrix D and the transforming matrix P.

[N17/MechCivil/6M][J24/ElectECSExtc/6M]

Solution:

$$A=\begin{bmatrix}4 & 2 & -2\\ -5 & 3 & 2\\ -2 & 4 & 1\end{bmatrix}, |A|=10$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}4-\lambda & 2 & -2\\ -5 & 3-\lambda & 2\\ -2 & 4 & 1-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[4+3+1]\lambda^2+\left[\begin{vmatrix}3 & 2\\ 4 & 1\end{vmatrix}+\begin{vmatrix}4 & -2\\ -2 & 1\end{vmatrix}+\begin{vmatrix}4 & 2\\ -5 & 3\end{vmatrix}\right]\lambda-10=0$$

$$\lambda^3-8\lambda^2+17\lambda-10=0$$

$$\lambda=1,2,5$$

Since the eigen values are distinct, the matrix A is diagonalisable

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}3 & 2 & -2\\ -5 & 2 & 2\\ -2 & 4 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$3x_1+2x_2-2x_3=0$$

$$-5x_1+2x_2+2x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}2 & -2\\ 2 & 2\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}3 & -2\\ -5 & 2\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}3 & 2\\ -5 & 2\end{vmatrix}}$$

$$\frac{x_1}{8}=-\frac{x_2}{-4}=\frac{x_3}{16}$$

$$\frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{4}$$

Hence, corresponding to $\lambda=1$ the eigen vector is $X_1=[2,1,4]'$

(ii) For $\lambda=2, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}2 & 2 & -2\\ -5 & 1 & 2\\ -2 & 4 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$2x_1+2x_2-2x_3=0$$

$$-5x_1+x_2+2x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 2 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 2 \\ -5 & 1 \end{vmatrix}}$$

$$\frac{x_1}{6} = -\frac{x_2}{-6} = \frac{x_3}{12}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [1,1,2]'$

(iii) For $\lambda = 5$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_2 - 2x_3 = 0$$

$$-5x_1 - 2x_2 + 2x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -1 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 2 \\ -5 & -2 \end{vmatrix}}$$

$$\frac{x_1}{0} = -\frac{x_2}{-12} = \frac{x_3}{12}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [0,1,1]'$

Thus, the matrix $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

3. Show that the matrix $A=\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}$ is diagonalizable. Hence find diagonal and transforming matrix
[M14/N14/MechCivil/8M][M15/MechCivil/6M][M15/N22/CompIT/8M]
[N22/Chem/6M][N22/Elect/8M][N24/ElectECSExtc/8M]

Solution:

$$A=\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}, |A|=0$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}8-\lambda&-6&4\\-6&7-\lambda&-4\\2&-4&3-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[8+7+3]\lambda^2+\left[\begin{vmatrix}7&-4\\-4&3\end{vmatrix}+\begin{vmatrix}8&2\\2&3\end{vmatrix}+\begin{vmatrix}8&-6\\-6&7\end{vmatrix}\right]\lambda-0=0$$

$$\lambda^3-18\lambda^2+45\lambda=0$$

$$\lambda(\lambda-15)(\lambda-3)=0$$

$$\lambda=0,3,15$$

Since the eigen values are distinct, the matrix A is diagonalisable

(i) For $\lambda=0, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$8x_1-6x_2+2x_3=0$$

$$-6x_1+7x_2-4x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-6&2\\7&-4\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}8&2\\-6&-4\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}8&-6\\-6&7\end{vmatrix}}$$

$$\frac{x_1}{10}=-\frac{x_2}{-20}=\frac{x_3}{20}$$

$$\frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{2}$$

Hence, corresponding to $\lambda=0$ the eigen vector is $X_1=[1,2,2]'$

(ii) For $\lambda=3, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}5&-6&2\\-6&4&-4\\2&-4&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$5x_1-6x_2+2x_3=0$$

$$2x_1-4x_2+0x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-6 & 2\\ -4 & 0\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}5 & 2\\ 2 & 0\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}5 & -6\\ 2 & -4\end{vmatrix}}$$

$$\frac{x_1}{8} = -\frac{x_2}{-4} = \frac{x_3}{-8}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_2 = [2,1,-2]'$

(iii) For $\lambda = 15$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix}-7 & -6 & 2\\ -6 & -8 & -4\\ 2 & -4 & -12\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-6 & 2\\ -8 & -4\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}-7 & 2\\ -6 & -4\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}-7 & -6\\ -6 & -8\end{vmatrix}}$$

$$\frac{x_1}{40} = -\frac{x_2}{40} = \frac{x_3}{20}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 15$ the eigen vector is $X_3 = [2,-2,1]'$

Thus, the matrix $A = \begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$ is diagonalised to $D = \begin{bmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 15\end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$

4. Show that the matrix $A=\begin{bmatrix}8 & -8 & -2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{bmatrix}$ is diagonalizable. Hence find diagonal and transforming matrix
[N13/Chem/7M][N14/ChemBiot/7M][N15/ElexExtcElectBiomInst/8M] [M16/ChemBiot/8M][M16/M17/CompIT/8M][N17/M18/Biot/6M][M18/Inst/8M] [N18/MechCivil/6M][N18/Biom/8M][M19/Elect/8M][M19/Chem/8M][N19/Chem/6M]

Solution:

$A=\begin{bmatrix}8 & -8 & -2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{bmatrix}, |A|=6$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}8-\lambda & -8 & -2\\ 4 & -3-\lambda & -2\\ 3 & -4 & 1-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[8-3+1]\lambda^2+\left[\begin{vmatrix}-3 & -2\\ -4 & 1\end{vmatrix}+\begin{vmatrix}8 & -2\\ 3 & 1\end{vmatrix}+\begin{vmatrix}8 & -8\\ 4 & -3\end{vmatrix}\right]\lambda-6=0$

$\lambda^3-6\lambda^2+11\lambda-6=0$

$\lambda=1,2,3$

Since the eigen values are distinct, the matrix A is diagonalisable

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$\begin{bmatrix}7 & -8 & -2\\ 4 & -4 & -2\\ 3 & -4 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$7x_1-8x_2-2x_3=0$

$4x_1-4x_2-2x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}-8 & -2\\ -4 & -2\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}7 & -2\\ 4 & -2\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}7 & -8\\ 4 & -4\end{vmatrix}}$

$\frac{x_1}{8}=-\frac{x_2}{-6}=\frac{x_3}{4}$

$\frac{x_1}{4}=\frac{x_2}{3}=\frac{x_3}{2}$

Hence, corresponding to $\lambda=1$ the eigen vector is $X_1=[4,3,2]'$

(ii) For $\lambda=2, [A-\lambda I]X=0$ gives

$\begin{bmatrix}6 & -8 & -2\\ 4 & -5 & -2\\ 3 & -4 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$6x_1-8x_2-2x_3=0$

$4x_1-5x_2-2x_3=0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-8 & -2\\ -5 & -2\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}6 & -2\\ 4 & -2\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}6 & -8\\ 4 & -5\end{vmatrix}}$$

$$\frac{x_1}{6} = -\frac{x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [3,2,1]'$

(iii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix}5 & -8 & -2\\ 4 & -6 & -2\\ 3 & -4 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$5x_1 - 8x_2 - 2x_3 = 0$$

$$4x_1 - 6x_2 - 2x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-8 & -2\\ -6 & -2\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}5 & -2\\ 4 & -2\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}5 & -8\\ 4 & -6\end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{-2} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [2,1,1]'$

Thus, the matrix $A = \begin{bmatrix}8 & -8 & -2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{bmatrix}$ is diagonalised to $D = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix}4 & 3 & 2\\ 3 & 2 & 1\\ 2 & 1 & 1\end{bmatrix}$

5. Find a matrix P that diagonalises the matrix $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$ and determine $P^{-1}AP$

[N16/ChemBiot/8M]

Solution:

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}, |A| = 6$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [-1+4+3]\lambda^2 + \left[\begin{vmatrix} 4 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} -1 & -2 \\ -3 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 4 \\ -3 & 4 \end{vmatrix}\right]\lambda - 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1,3,2$$

Since the eigen values are distinct, the matrix A is diagonalisable

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 4x_2 - 2x_3 = 0$$

$$-3x_1 + 3x_2 + 0x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & -2 \\ 3 & 0 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -2 & -2 \\ -3 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 4 \\ -3 & 3 \end{vmatrix}}$$

$$\frac{x_1}{6} = -\frac{x_2}{-6} = \frac{x_3}{6}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 1$ the eigen vector is $X_1 = [1,1,1]'$

(ii) For $\lambda = 3, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -4 & 4 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4x_1 + 4x_2 - 2x_3 = 0$$

$$-3x_1 + x_2 + 0x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & -2 \\ 1 & 0 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -4 & -2 \\ -3 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -4 & 4 \\ -3 & 1 \end{vmatrix}}$$

$\frac{x_1}{2} = -\frac{x_2}{-6} = \frac{x_3}{8}$

$\frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{4}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_2 = [1,3,4]'$

(iii) For $\lambda = 2$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -3 & 4 & -2 \\ -3 & 2 & 0 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 + 4x_2 - 2x_3 = 0$

$-3x_1 + 2x_2 + 0x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & -2 \\ 2 & 0 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & -2 \\ -3 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 4 \\ -3 & 2 \end{vmatrix}}$$

$\frac{x_1}{4} = -\frac{x_2}{-6} = \frac{x_3}{6}$

$\frac{x_1}{2} = \frac{x_2}{3} = \frac{x_3}{3}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_3 = [2,3,3]'$

Thus, the matrix $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ by the transformation $P^{-1}AP = D$ where $P = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 3 \\ 1 & 4 & 3 \end{bmatrix}$

6. Show that the matrix $A=\begin{bmatrix}11 & -4 & -7\\ 7 & -2 & -5\\ 10 & -4 & -6\end{bmatrix}$ is similar to a diagonal matrix. Find the diagonal matrix and the transforming matrix.

[N18/Elex/8M]

Solution:

$A=\begin{bmatrix}11 & -4 & -7\\ 7 & -2 & -5\\ 10 & -4 & -6\end{bmatrix}, |A|=0$

The characteristic equation,

$|A-\lambda I|=0$

$$\begin{vmatrix}11-\lambda & -4 & -7\\ 7 & -2-\lambda & -5\\ 10 & -4 & -6-\lambda\end{vmatrix}=0$$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[11-2-6]\lambda^2+\left[\begin{vmatrix}-2 & -5\\ -4 & -6\end{vmatrix}+\begin{vmatrix}11 & -7\\ 10 & -6\end{vmatrix}+\begin{vmatrix}11 & -4\\ 7 & -2\end{vmatrix}\right]\lambda-0=0$

$\lambda^3-3\lambda^2+2\lambda=0$

$\lambda=0,1,2$

Since the eigen values are distinct, the matrix A is diagonalisable

(i) For $\lambda=0, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}11 & -4 & -7\\ 7 & -2 & -5\\ 10 & -4 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$11x_1-4x_2-7x_3=0$

$7x_1-2x_2-5x_3=0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-4 & -7\\ -2 & -5\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}11 & -7\\ 7 & -5\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}11 & -4\\ 7 & -2\end{vmatrix}}$$

$$\frac{x_1}{6}=-\frac{x_2}{-6}=\frac{x_3}{6}$$

$$\frac{x_1}{1}=\frac{x_2}{1}=\frac{x_3}{1}$$

Hence, corresponding to $\lambda=0$ the eigen vector is $X_1=[1,1,1]'$

(ii) For $\lambda=1, [A-\lambda I]X=0$ gives

$$\begin{bmatrix}10 & -4 & -7\\ 7 & -3 & -5\\ 10 & -4 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$10x_1-4x_2-7x_3=0$

$7x_1-3x_2-5x_3=0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-4 & -7\\ -3 & -5\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}10 & -7\\ 7 & -5\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}10 & -4\\ 7 & -3\end{vmatrix}}$$

$$\frac{x_1}{-1} = -\frac{x_2}{-1} = \frac{x_3}{-2}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 1$ the eigen vector is $X_2 = [1, -1, 2]'$

(iii) For $\lambda = 2$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix}9 & -4 & -7\\ 7 & -4 & -5\\ 10 & -4 & -8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$9x_1 - 4x_2 - 7x_3 = 0$$

$$7x_1 - 4x_2 - 5x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-4 & -7\\ -4 & -5\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}9 & -7\\ 7 & -5\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}9 & -4\\ 7 & -4\end{vmatrix}}$$

$$\frac{x_1}{-8} = -\frac{x_2}{4} = \frac{x_3}{-8}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_3 = [2, 1, 2]'$

Thus, the matrix $A = \begin{bmatrix}11 & -4 & -7\\ 7 & -2 & -5\\ 10 & -4 & -6\end{bmatrix}$ is diagonalised to $D = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix}1 & 1 & 2\\ 1 & -1 & 1\\ 1 & 2 & 2\end{bmatrix}$

7. Check whether the given matrix A is diagonalizable, diagonalise if it is, where

$$A = \begin{bmatrix} 8 & 4 & 3 \\ -8 & -3 & -4 \\ -2 & -2 & 1 \end{bmatrix}$$

[N19/MechCivil/6M]

Solution:

$$A = \begin{bmatrix} 8 & 4 & 3 \\ -8 & -3 & -4 \\ -2 & -2 & 1 \end{bmatrix}, |A| = 6$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & 4 & 3 \\ -8 & -3-\lambda & -4 \\ -2 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [8-3+1]\lambda^2 + \left[\begin{vmatrix} -3 & -4 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 8 & 3 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 8 & 4 \\ -8 & -3 \end{vmatrix}\right]\lambda - 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1,2,3$$

Since the eigen values are distinct, the matrix A is diagonalisable

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 7 & 4 & 3 \\ -8 & -4 & -4 \\ -2 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$7x_1 + 4x_2 + 3x_3 = 0$$

$$-8x_1 - 4x_2 - 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & 3 \\ -4 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 7 & 3 \\ -8 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & 4 \\ -8 & -4 \end{vmatrix}}$$

$$\frac{x_1}{-4} = -\frac{x_2}{-4} = \frac{x_3}{4}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 1$ the eigen vector is $X_1 = [-1,1,1]'$

(ii) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 6 & 4 & 3 \\ -8 & -5 & -4 \\ -2 & -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$6x_1 + 4x_2 + 3x_3 = 0$$

$$-8x_1 - 5x_2 - 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & 3 \\ -5 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 6 & 3 \\ -8 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 6 & 4 \\ -8 & -5 \end{vmatrix}}$$

$$\frac{x_1}{-1} = -\frac{x_2}{0} = \frac{x_3}{2}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [-1,0,2]'$

(iii) For $\lambda = 3, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 5 & 4 & 3 \\ -8 & -6 & -4 \\ -2 & -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x_1 + 4x_2 + 3x_3 = 0$$

$$-8x_1 - 6x_2 - 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & 3 \\ -6 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 5 & 3 \\ -8 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & 4 \\ -8 & -6 \end{vmatrix}}$$

$$\frac{x_1}{2} = -\frac{x_2}{4} = \frac{x_3}{2}$$

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [1,-2,1]'$

Thus, the matrix $A = \begin{bmatrix} 8 & 4 & 3 \\ -8 & -3 & -4 \\ -2 & -2 & 1 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 2 & 1 \end{bmatrix}$

8. Is the matrix $A=\begin{bmatrix}3&1&4\\0&2&6\\0&0&5\end{bmatrix}$ diagonalisable? If so find the diagonal form of A and transforming matrix of A

[M23DSE/J24/MechCivil/8M]

Solution:

$$A=\begin{bmatrix}3&1&4\\0&2&6\\0&0&5\end{bmatrix}, |A|=30$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}3-\lambda&1&4\\0&2-\lambda&6\\0&0&5-\lambda\end{vmatrix}=0$$

$$(3-\lambda)(2-\lambda)(5-\lambda)=0$$

$$\lambda=3,2,5$$

Since the eigen values are distinct, the matrix A is diagonalisable

(i) For $\lambda=3$, $[A-\lambda I]X=0$ gives

$$\begin{bmatrix}0&1&4\\0&-1&6\\0&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$0x_1+x_2+4x_3=0$$

$$0x_1-x_2+6x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}1&4\\-1&6\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}0&4\\0&6\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}0&1\\0&-1\end{vmatrix}}$$

$$\frac{x_1}{10}=-\frac{x_2}{0}=\frac{x_3}{0}$$

$$\frac{x_1}{1}=\frac{x_2}{0}=\frac{x_3}{0}$$

Hence, corresponding to $\lambda=3$ the eigen vector is $X_1=[1,0,0]'$

(ii) For $\lambda=2$, $[A-\lambda I]X=0$ gives

$$\begin{bmatrix}1&1&4\\0&0&6\\0&0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x_1+x_2+4x_3=0$$

$$0x_1+0x_2+6x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}1&4\\0&6\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}1&4\\0&6\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}1&1\\0&0\end{vmatrix}}$$

$$\frac{x_1}{6}=-\frac{x_2}{6}=\frac{x_3}{0}$$

$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [1, -1, 0]'$

(iii) For $\lambda = 5$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1 + x_2 + 4x_3 = 0$

$0x_1 - 3x_2 + 6x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 1 & 4 \\ -3 & 6 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -2 & 4 \\ 0 & 6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix}}$$

$\frac{x_1}{18} = -\frac{x_2}{-12} = \frac{x_3}{6}$

$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [3, 2, 1]'$

Thus, the matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ is diagonalized to $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

9. Find the Eigen values and Eigen vectors of $A=\begin{bmatrix}\frac{37}{60} & \frac{17}{60} & \frac{17}{60}\\ \frac{1}{5} & \frac{7}{10} & \frac{1}{5}\\ \frac{1}{12} & -\frac{1}{12} & \frac{5}{12}\end{bmatrix}$. And show that it is diagonalisable matrix and find its transforming matrix and the diagonal form.
[M23/ElectECSExtc/8M]

Solution:

$A=\begin{bmatrix}\frac{37}{60} & \frac{17}{60} & \frac{17}{60}\\ \frac{1}{5} & \frac{7}{10} & \frac{1}{5}\\ \frac{1}{12} & -\frac{1}{12} & \frac{5}{12}\end{bmatrix}, |A|=\frac{3}{20}$

The characteristic equation,

$|A-\lambda I|=0$

$$\begin{vmatrix}\frac{37}{60}-\lambda & \frac{17}{60} & \frac{17}{60}\\ \frac{1}{5} & \frac{7}{10}-\lambda & \frac{1}{5}\\ \frac{1}{12} & -\frac{1}{12} & \frac{5}{12}-\lambda\end{vmatrix}=0$$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$$\lambda^3-\left[\frac{37}{60}+\frac{7}{10}+\frac{5}{12}\right]\lambda^2+\left[\begin{vmatrix}\frac{7}{10} & \frac{1}{5}\\ -\frac{1}{12} & \frac{5}{12}\end{vmatrix}+\begin{vmatrix}\frac{37}{60} & \frac{17}{60}\\ \frac{1}{12} & \frac{5}{12}\end{vmatrix}+\begin{vmatrix}\frac{37}{60} & \frac{17}{60}\\ \frac{1}{5} & \frac{7}{10}\end{vmatrix}\right]\lambda-\frac{3}{20}=0$$

$\lambda^3-\frac{26}{15}\lambda^2+\frac{11}{12}\lambda-\frac{3}{20}=0$

$\left(\lambda-\frac{1}{3}\right)\left(\lambda-\frac{1}{2}\right)\left(\lambda-\frac{9}{10}\right)=0$

$\lambda=\frac{1}{3},\frac{1}{2},\frac{9}{10}$

Since the eigen values are distinct, the matrix A is diagonalisable

(i) For $\lambda=\frac{1}{3}$, $[A-\lambda I]X=0$ gives

$$\begin{bmatrix}\frac{17}{60} & \frac{17}{60} & \frac{17}{60}\\ \frac{1}{5} & \frac{11}{30} & \frac{1}{5}\\ \frac{1}{12} & -\frac{1}{12} & \frac{1}{12}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\frac{17}{60}x_1+\frac{17}{60}x_2+\frac{17}{60}x_3=0$

$\frac{1}{5}x_1+\frac{11}{30}x_2+\frac{1}{5}x_3=0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}\frac{17}{60} & \frac{17}{60}\\ \frac{11}{30} & \frac{1}{5}\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}\frac{17}{60} & \frac{17}{60}\\ \frac{1}{5} & \frac{1}{5}\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}\frac{17}{60} & \frac{17}{60}\\ \frac{1}{5} & \frac{11}{30}\end{vmatrix}}$$

$$\frac{x_1}{-\frac{17}{360}} = -\frac{x_2}{0} = \frac{x_3}{\frac{17}{360}}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = \frac{1}{3}$ the eigen vector is $X_1 = [-1,0,1]'$

(ii) For $\lambda = \frac{1}{2}$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix}\frac{7}{60} & \frac{17}{60} & \frac{17}{60}\\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5}\\ \frac{1}{12} & -\frac{1}{12} & \frac{-1}{12}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\frac{7}{60}x_1 + \frac{17}{60}x_2 + \frac{17}{60}x_3 = 0$$

$$\frac{1}{5}x_1 + \frac{1}{5}x_2 + \frac{1}{5}x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}\frac{17}{60} & \frac{17}{60}\\ \frac{1}{5} & \frac{1}{5}\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}\frac{7}{60} & \frac{17}{60}\\ \frac{1}{5} & \frac{1}{5}\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}\frac{7}{60} & \frac{17}{60}\\ \frac{1}{5} & \frac{1}{5}\end{vmatrix}}$$

$$\frac{x_1}{0} = -\frac{x_2}{-\frac{1}{30}} = \frac{x_3}{-\frac{1}{30}}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{-1}$$

Hence, corresponding to $\lambda = \frac{1}{2}$ the eigen vector is $X_1 = [0,1,-1]'$

(iii) For $\lambda = \frac{9}{10}$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix}\frac{-17}{60} & \frac{17}{60} & \frac{17}{60}\\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5}\\ \frac{1}{12} & -\frac{1}{12} & -\frac{29}{60}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$-\frac{17}{60}x_1 + \frac{17}{60}x_2 + \frac{17}{60}x_3 = 0$$

$$\frac{1}{5}x_1 - \frac{1}{5}x_2 + \frac{1}{5}x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}\frac{17}{60} & \frac{17}{60}\\ -\frac{1}{5} & \frac{1}{5}\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}-\frac{17}{60} & \frac{17}{60}\\ \frac{1}{5} & \frac{1}{5}\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}-\frac{17}{60} & \frac{17}{60}\\ \frac{1}{5} & -\frac{1}{5}\end{vmatrix}}$$

$$\frac{x_1}{\frac{17}{150}} = -\frac{x_2}{-\frac{17}{150}} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

Hence, corresponding to $\lambda = \frac{9}{10}$ the eigen vector is $X_1 = [1,1,0]'$

Thus, the matrix $A = \begin{bmatrix} \frac{37}{60} & \frac{17}{60} & \frac{17}{60} \\ \frac{1}{5} & \frac{7}{10} & \frac{1}{5} \\ \frac{1}{12} & -\frac{1}{12} & \frac{5}{12} \end{bmatrix}$ is diagonalized to $D = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{9}{10} \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

10. Show that the matrix $A=\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}$ is diagonalizable. Hence find diagonal and transforming matrix
[M14/N14/N15/M23/CompIT/8M][N14/N17/ElexExtcElectBiomInst/8M]
[N17/M18/N18/Chem/6M][M18/N22/Elex/8M][M18/MechCivil/6M][N18/Extc/6M]
[N18/Biot/6M][M19/N19/N22/MTRX/8M][N19/N22/Extc/8M][N19/Inst/8M]
[J23/N23/ElectECSExtc/8M][N23/MechCivil/8M][M24/CompIT/6M]

Solution:

$A=\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}, |A|=3$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}-9-\lambda & 4 & 4\\ -8 & 3-\lambda & 4\\ -16 & 8 & 7-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[-9+3+7]\lambda^2+\left[\begin{vmatrix}3 & 4\\ 8 & 7\end{vmatrix}+\begin{vmatrix}-9 & 4\\ -16 & 7\end{vmatrix}+\begin{vmatrix}-9 & 4\\ -8 & 3\end{vmatrix}\right]\lambda-3=0$

$\lambda^3-\lambda^2-5\lambda-3=0$

$(\lambda+1)(\lambda+1)(\lambda-3)=0$

$\lambda=-1,-1,3$

The Algebraic Multiplicity of $\lambda=-1$ is 2 and that of $\lambda=3$ is 1

(i) For $\lambda=-1, [A-\lambda I]X=0$ gives

$\begin{bmatrix}-8 & 4 & 4\\ -8 & 4 & 4\\ -16 & 8 & 8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

By R_2-R_1, R_3-2R_1

$\begin{bmatrix}-8 & 4 & 4\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\therefore -8x_1+4x_2+4x_3=0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ vectors to be formed

The Geometric Multiplicity of $\lambda=1$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3=t$ & $x_2=s$

$\therefore x_1=\frac{s}{2}+\frac{t}{2}$

$$\therefore X = \begin{bmatrix} \frac{s}{2} + \frac{t}{2} \\ s \\ t \end{bmatrix} = \begin{bmatrix} \frac{s}{2} + \frac{t}{2} \\ s + 0t \\ 0s + t \end{bmatrix} = \begin{bmatrix} \frac{s}{2} \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} \frac{t}{2} \\ 0t \\ t \end{bmatrix} = \frac{s}{2}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{t}{2}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Hence, corresponding to $\lambda = -1$ the eigen vectors are

$X_1 = [1,2,0]'$ & $X_2 = [1,0,2]'$

(ii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-12x_1 + 4x_2 + 4x_3 = 0$$

$$-8x_1 + 0x_2 + 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & 4 \\ 0 & 4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -12 & 4 \\ -8 & 4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -12 & 4 \\ -8 & 0 \end{vmatrix}}$$

$$\frac{x_1}{16} = -\frac{x_2}{-16} = \frac{x_3}{32}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [1,1,2]'$

Thus, the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$

11. Test whether the matrix $A=\begin{bmatrix}2&-1&1\\2&2&-1\\1&2&-1\end{bmatrix}$ is diagonalizable.

Solution:

$$A=\begin{bmatrix}2&-1&1\\2&2&-1\\1&2&-1\end{bmatrix}$$

$|A|=1$

The characteristic eqn,

$|A-\lambda I|=0$

$$\begin{vmatrix}2-\lambda&-1&1\\2&2-\lambda&-1\\1&2&-1-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diag]\lambda^2+[sum\ of\ minors\ of\ diag]\lambda-|A|=0$$

$$\lambda^3-[2+2-1]\lambda^2+\left[\begin{vmatrix}2&-1\\2&-1\end{vmatrix}+\begin{vmatrix}2&1\\1&-1\end{vmatrix}+\begin{vmatrix}2&-1\\2&2\end{vmatrix}\right]\lambda-1=0$$

$$\lambda^3-3\lambda^2+[-2+2-2-1+4+2]\lambda-1=0$$

$$\lambda^3-3\lambda^2+3\lambda-1=0$$

$\lambda=1,1,1$

The Algebraic multiplicity for $\lambda=1$ is 3

For $\lambda=1$

$[A-\lambda I]X=0$

$$\begin{bmatrix}1&-1&1\\2&1&-1\\1&2&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

By R_2-2R_1,R_3-R_1

$$\begin{bmatrix}1&-1&1\\0&3&-3\\0&3&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

By R_3-R_2

$$\begin{bmatrix}1&-1&1\\0&3&-3\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Here, $r=2,n=3$

$GM=n-r=3-2=1$ vector to be formed

But $AM\neq GM$

Matrix A is not diagonalizable

12. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 0 \\ \frac{1}{2} & 2 \end{bmatrix}$ prove that both A and B are not diagonalizable but AB is diagonalizable.

Solution:

Consider,

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$|A| = 1$

Characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$\lambda^2 - [sum\ of\ diag]\lambda + |A| = 0$

$\lambda^2 - [1+1]\lambda + 1 = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda = 1,1$

Algebraic multiplicity for $\lambda = 1$ is 2

For $\lambda = 1$

$[A - \lambda I]X = 0$

$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Here, $r = 1, n = 2$

$GM = n - r = 2 - 1 = 1$

But $AM \neq GM$

Matrix A is not diagonalizable

Consider,

$B = \begin{bmatrix} 2 & 0 \\ \frac{1}{2} & 2 \end{bmatrix}$

$|B| = 4$

Characteristic equation,

$|B - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 0 \\ \frac{1}{2} & 2-\lambda \end{vmatrix} = 0$

$\lambda^2 - [sum\ of\ diag]\lambda + |A| = 0$

$\lambda^2 - [2+2]\lambda + 4 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$\lambda = 2,2$

Algebraic multiplicity for $\lambda = 2$ is 2

For $\lambda = 2$

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here, $r = 1, n = 2$

$GM = n - r = 2 - 1 = 1$

But $AM \neq GM$

Matrix B is not diagonalizable

Consider,

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ \frac{1}{2} & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2+1 & 0+4 \\ 0+\frac{1}{2} & 0+2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 4 \\ \frac{1}{2} & 2 \end{bmatrix}$$

$|AB| = 6 - 2 = 4$

Characteristic equation,

$|AB - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 4 \\ \frac{1}{2} & 2-\lambda \end{vmatrix} = 0$$

$\lambda^2 - [sum\ of\ diag]\lambda + |A| = 0$

$\lambda^2 - [3+2]\lambda + 4 = 0$

$\lambda^2 - 5\lambda + 4 = 0$

$\lambda = 1,4$

Since eigen values are distinct, matrix AB is diagonalizable

13. Show that $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ is diagonalizable. Find the transforming and diagonal form.

[M16/N22/N24/MechCivil/8M][N18/Inst/8M][N19/Elex/8M][M22/MechCivil/5M]

Solution:

$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}, |A| = 5$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [2+3+2]\lambda^2 + \left[\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}\right]\lambda - 5 = 0$

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$(\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$

$\lambda = 1,1,5$

The Algebraic Multiplicity of $\lambda = 1$ is 2 and that of $\lambda = 5$ is 1

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 - R_1, R_3 - R_1$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore x_1 + 2x_2 + x_3 = 0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

The Geometric Multiplicity of $\lambda = 1$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3 = t$ & $x_2 = s$, $\therefore x_1 = -2s - t$

$\therefore X = \begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2s - t \\ s + 0t \\ 0s + t \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} -t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Hence, corresponding to $\lambda = 1$ the eigen vectors are

$X_1 = [-2,1,0]'$ & $X_2 = [-1,0,1]'$

(ii) For $\lambda = 5, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + 2x_2 + x_3 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{-4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,1,1]'$

Thus, the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

14. Show that the matrix $A=\begin{bmatrix}1 & -6 & -4\\0 & 4 & 2\\0 & -6 & -3\end{bmatrix}$ is similar to a diagonal matrix. Also find the transforming matrix and diagonal matrix.

[M16/ElexExtcElectBiomInst/8M][M19/Elex/8M]

Solution:

$A=\begin{bmatrix}1 & -6 & -4\\0 & 4 & 2\\0 & -6 & -3\end{bmatrix}, |A|=0$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}1-\lambda & -6 & -4\\0 & 4-\lambda & 2\\0 & -6 & -3-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[1+4-3]\lambda^2+\left[\begin{vmatrix}4 & 2\\-6 & -3\end{vmatrix}+\begin{vmatrix}1 & -4\\0 & -3\end{vmatrix}+\begin{vmatrix}1 & -6\\0 & 4\end{vmatrix}\right]\lambda-32=0$

$\lambda^3-2\lambda^2+\lambda=0$

$\lambda(\lambda-1)(\lambda-1)=0$

$\lambda=0,1,1$

The Algebraic Multiplicity of $\lambda=1$ is 2 and that of $\lambda=0$ is 1

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$\begin{bmatrix}0 & -6 & -4\\0 & 3 & 2\\0 & -6 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

By $R_2+\frac{1}{2}R_1, R_3-R_1$

$\begin{bmatrix}0 & -6 & -4\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore 0x_1-6x_2-4x_3=0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ vectors to be formed

The Geometric Multiplicity of $\lambda=1$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_1=1, x_2=2 \therefore x_3=-3$

Let $x_1=1, x_2=-2 \therefore x_3=3$

Hence, corresponding to $\lambda=1$ the eigen vectors are

$X_1=[1,2,-3]'\ \&\ X_2=[1,-2,3]'$

(ii) For $\lambda = 0, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - 6x_2 - 4x_3 = 0$

$0x_1 + 4x_2 + 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -6 & -4 \\ 4 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{2} = \frac{x_3}{4}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 0$ the eigen vector is $X_3 = [2, -1, 2]'$

Thus, the matrix $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -2 & -1 \\ -3 & 3 & 2 \end{bmatrix}$

15. Show that the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is diagonalizable. Hence find diagonal and transforming matrix

[M15/ElexExtcElectBiomInst/8M][M22/CompIT/5M][J23/MechCivil/8M]

Solution:

$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}, |A| = 32$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [6+3+3]\lambda^2 + \left[\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}\right]\lambda - 32 = 0$

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$

$\lambda = 2,2,8$

The Algebraic Multiplicity of $\lambda = 2$ is 2 and that of $\lambda = 8$ is 1

(i) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 + \frac{1}{2}R_1, R_3 - \frac{1}{2}R_1$

$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore 4x_1 - 2x_2 + 2x_3 = 0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

The Geometric Multiplicity of $\lambda = 2$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3 = t$ & $x_2 = s$, $\therefore x_1 = \frac{s}{2} - \frac{t}{2}$

$\therefore X = \begin{bmatrix} \frac{s}{2} - \frac{t}{2} \\ s \\ t \end{bmatrix} = \begin{bmatrix} \frac{s}{2} - \frac{t}{2} \\ s + 0t \\ 0s + t \end{bmatrix} = \begin{bmatrix} \frac{s}{2} \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} -\frac{t}{2} \\ 0t \\ t \end{bmatrix} = \frac{s}{2}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{t}{2}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Hence, corresponding to $\lambda = 2$ the eigen vectors are

$X_1 = [1,2,0]'$ & $X_2 = [-1,0,2]'$

(ii) For $\lambda = 8$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 - 5x_2 - 1x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

$$\frac{x_1}{12} = -\frac{x_2}{6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 8$ the eigen vector is $X_3 = [2, -1, 1]'$

Thus, the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$

16. Is the matrix $A=\begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix}$ diagonalisable? If so, find diagonal form and transforming matrix.

[M14/ElexExtcElectBiomInst/8M][N17/CompIT/8M][M19/Extc/8M]

Solution:

$A=\begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix}, |A|=3$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}2-\lambda&1&1\\1&2-\lambda&1\\0&0&1-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[2+2+1]\lambda^2+\left[\begin{vmatrix}2&1\\0&1\end{vmatrix}+\begin{vmatrix}2&1\\0&1\end{vmatrix}+\begin{vmatrix}2&1\\1&2\end{vmatrix}\right]\lambda-3=0$

$\lambda^3-5\lambda^2+7\lambda-3=0$

$(\lambda-1)(\lambda-1)(\lambda-3)=0$

$\lambda=1,1,3$

The Algebraic Multiplicity of $\lambda=1$ is 2 and that of $\lambda=3$ is 1

(i) For $\lambda=1, [A-\lambda I]X=0$ gives

$\begin{bmatrix}1&1&1\\1&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

By R_2-R_1

$\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore x_1+x_2+x_3=0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ vectors to be formed

The Geometric Multiplicity of $\lambda=1$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3=t$ & $x_2=s$

$\therefore x_1=-s-t$

$\therefore X=\begin{bmatrix}-s-t\\s\\t\end{bmatrix}=\begin{bmatrix}-s-t\\s+0t\\0s+t\end{bmatrix}=\begin{bmatrix}-s\\s\\0s\end{bmatrix}+\begin{bmatrix}-t\\0t\\t\end{bmatrix}=s\begin{bmatrix}-1\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}$

Hence, corresponding to $\lambda = 1$ the eigen vectors are
$X_1 = [-1,1,0]'$ & $X_2 = [-1,0,1]'$

(ii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 + x_3 = 0$
$x_1 - x_2 + x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{2} = -\frac{x_2}{-2} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [1,1,0]'$

Thus, the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

17. Determine whether matrix $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$ is diagonalizable, if yes diagonalise it

[N18/Elect/8M]

Solution:

$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}, |A| = 4$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [4+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}\right]\lambda - 4 = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$

$\lambda = 1,2,2$

The Algebraic Multiplicity of $\lambda = 1$ is 1 and that of $\lambda = 2$ is 2

(i) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 - \frac{1}{2}R_1, R_3 + \frac{1}{2}R_1$

$\begin{bmatrix} 2 & 6 & 6 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_3 - R_2$

$\begin{bmatrix} 2 & 6 & 6 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The rank (r) of the matrix is 2 and number of unknowns (n) is 3

Thus, $n - r = 3 - 2 = 1$ vectors to be formed

The Geometric Multiplicity of $\lambda = 2$ is 1

Since, Algebraic Multiplicity $\neq$ Geometric Multiplicity,

matrix A is not diagonalizable.

18. Show that the given matrix $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$ is diagonalisable and hence find diagonal form and transforming matrix
[D23/CompIT/6M]
Solution:
$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}, |A| = 4$
The characteristic equation,
$|A - \lambda I| = 0$
$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$
$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$
$\lambda^3 - [4+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}\right]\lambda - 4 = 0$
$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$
$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$
$\lambda = 1,2,2$
The Algebraic Multiplicity of $\lambda = 1$ is 1 and that of $\lambda = 2$ is 2
(i) For $\lambda = 2$, $[A - \lambda I]X = 0$ gives
$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
By $R_2 - \frac{1}{2}R_1, R_3 + \frac{1}{2}R_1$
$\begin{bmatrix} 2 & 6 & 6 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
By $R_3 - R_2$
$\begin{bmatrix} 2 & 6 & 6 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
The rank (r) of the matrix is 2 and number of unknowns (n) is 3
Thus, $n - r = 3 - 2 = 1$ vectors to be formed
The Geometric Multiplicity of $\lambda = 2$ is 1
Since, Algebraic Multiplicity $\neq$ Geometric Multiplicity,
matrix A is not diagonalizable.

19. Determine whether matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ is diagonalizable, if yes diagonalise it

[N19/Elect/8M]

Solution:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, |A| = 5$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [1+1+1]\lambda^2 + \left[\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\right]\lambda - 5 = 0$$

$$\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$$

$$\lambda = 5, -1, -1$$

The Algebraic Multiplicity of $\lambda = 5$ is 1 and that of $\lambda = -1$ is 2

(i) For $\lambda = -1$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1, R_3 - R_1$

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 2x_1 + 2x_2 + 2x_3 = 0$$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

The Geometric Multiplicity of $\lambda = -1$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3 = t$ & $x_2 = s$

$$\therefore x_1 = -s - t$$

$$\therefore X = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s-t \\ s+0t \\ 0s+t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} -t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, corresponding to $\lambda = 1$ the eigen vectors are

$X_1 = [-1,1,0]'$ & $X_2 = [-1,0,1]'$

(ii) For $\lambda = 5, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4x_1 + 2x_2 + 2x_3 = 0$$

$$2x_1 - 4x_2 + 2x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ -4 & 2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -4 & 2 \\ 2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -4 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{12} = -\frac{x_2}{-12} = \frac{x_3}{12}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,1,1]'$

Thus, the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

20. Show that the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ is diagonalizable. Hence find diagonal and transforming matrix.

[M19/MechCivil/6M][M23DSE/ElectECSExtc/8M]

Solution:

$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}, |A| = 20$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [3+3+3]\lambda^2 + \left[\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}\right]\lambda - 20 = 0$

$\lambda^3 - 9\lambda^2 + 24\lambda - 20 = 0$

$\lambda = 2,2,5$

The Algebraic Multiplicity of $\lambda = 5$ is 1 and that of $\lambda = 2$ is 2

(i) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 + R_1, R_3 - R_1$

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 - x_2 + x_3 = 0$

The rank (r) of the matrix is 1 and no of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

The Geometric Multiplicity of $\lambda = 1$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3 = t$ & $x_2 = s$

$\therefore x_1 = s - t$

$\therefore X = \begin{bmatrix} s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} s-t \\ s+0t \\ 0s+t \end{bmatrix} = \begin{bmatrix} s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} -t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Hence, corresponding to $\lambda = 1$ the eigen vectors are $X_1 = [1,1,0]'$ & $X_2 = [-1,0,1]'$

(ii) For $\lambda = 5, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - x_2 + x_3 = 0$$

$$-x_1 - 2x_2 - x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -2 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{3} = -\frac{x_2}{3} = \frac{x_3}{3}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1, -1, 1]'$

Thus, the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

21. Show that the matrix $A=\begin{bmatrix}4 & 1 & -1\\ 2 & 5 & -2\\ 1 & 1 & 2\end{bmatrix}$ is diagonalizable. Hence find diagonal and transforming matrix.

[M18/Extc/8M]

Solution:

$A=\begin{bmatrix}4 & 1 & -1\\ 2 & 5 & -2\\ 1 & 1 & 2\end{bmatrix}, |A|=45$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}4-\lambda & 1 & -1\\ 2 & 5-\lambda & -2\\ 1 & 1 & 2-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[4+5+2]\lambda^2+\left[\begin{vmatrix}5 & -2\\ 1 & 2\end{vmatrix}+\begin{vmatrix}4 & -1\\ 1 & 2\end{vmatrix}+\begin{vmatrix}4 & 1\\ 2 & 5\end{vmatrix}\right]\lambda-45=0$

$\lambda^3-11\lambda^2+39\lambda-45=0$

$\lambda=3,3,5$

The Algebraic Multiplicity of $\lambda=3$ is 2 and that of $\lambda=5$ is 1

(i) For $\lambda=3,[A-\lambda I]X=0$ gives

$\begin{bmatrix}1 & 1 & -1\\ 2 & 2 & -2\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

By R_2-2R_1, R_3+R_1

$\begin{bmatrix}1 & 1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\therefore x_1+x_2-x_3=0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ vectors to be formed

The Geometric Multiplicity of $\lambda=3$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3=t$ & $x_2=s$, $\therefore x_1=-s+t$

$\therefore X=\begin{bmatrix}-s+t\\ s\\ t\end{bmatrix}=\begin{bmatrix}-s+t\\ s+0t\\ 0s+t\end{bmatrix}=\begin{bmatrix}-s\\ s\\ 0s\end{bmatrix}+\begin{bmatrix}t\\ 0t\\ t\end{bmatrix}=s\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$

Hence, corresponding to $\lambda=3$ the eigen vectors are

$X_1=[-1,1,0]'$ & $X_2=[1,0,1]'$

(ii) For $\lambda = 5, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 - x_3 = 0$

$2x_1 + 0x_2 - 2x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -1 & -1 \\ 2 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix}}$$

$$\frac{x_1}{-2} = -\frac{x_2}{4} = \frac{x_3}{-2}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,2,1]'$

Thus, the matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

22. Determine whether the matrix $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ is diagonalizable, if yes diagonalise it

[M18/M19/Biom/8M][M19/Inst/8M][D24/CompIT/6M]

Solution:

$A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}, |A|=45$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[-2+1+0]\lambda^2+\left[\begin{vmatrix}1 & -6\\ -2 & 0\end{vmatrix}+\begin{vmatrix}-2 & -3\\ -1 & 0\end{vmatrix}+\begin{vmatrix}-2 & 2\\ 2 & 1\end{vmatrix}\right]\lambda-45=0$

$\lambda^3+1\lambda^2-21\lambda-45=0$

$\lambda=-3,-3,5$

The Algebraic Multiplicity of $\lambda=-3$ is 2 and that of $\lambda=5$ is 1

(i) For $\lambda=-3$, $[A-\lambda I]X=0$ gives

$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

By R_2-2R_1, R_3+R_1

$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\therefore x_1+2x_2-3x_3=0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ vectors to be formed

The Geometric Multiplicity of $\lambda=-3$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3=t$ & $x_2=s$, $\therefore x_1=-2s+3t$

$\therefore X=\begin{bmatrix}-2s+3t\\ s\\ t\end{bmatrix}=\begin{bmatrix}-2s+3t\\ s+0t\\ 0s+t\end{bmatrix}=\begin{bmatrix}-2s\\ s\\ 0s\end{bmatrix}+\begin{bmatrix}3t\\ 0t\\ t\end{bmatrix}=s\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}3\\ 0\\ 1\end{bmatrix}$

Hence, corresponding to $\lambda=-3$ the eigen vectors are

$X_1=[-2,1,0]'$ & $X_2=[3,0,1]'$

(ii) For $\lambda = 5, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1 + 2x_2 - 3x_3 = 0$$

$$2x_1 - 4x_2 - 6x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{-24} = -\frac{x_2}{48} = \frac{x_3}{24}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1, 2, -1]'$

Thus, matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

23. Is the matrix $A=\begin{bmatrix}2 & 0 & 2\\0 & 2 & 1\\0 & 0 & 1\end{bmatrix}$ diagonalisable? If so find the diagonal form of A and transforming matrix of A

[M23/MechCivil/8M]

Solution:

$A=\begin{bmatrix}2 & 0 & 2\\0 & 2 & 1\\0 & 0 & 1\end{bmatrix}, |A|=4$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}2-\lambda & 0 & 2\\0 & 2-\lambda & 1\\0 & 0 & 1-\lambda\end{vmatrix}=0$

$(2-\lambda)(2-\lambda)(1-\lambda)$

$\lambda=2,2,1$

The Algebraic Multiplicity of $\lambda=2$ is 2 and that of $\lambda=1$ is 1

(i) For $\lambda=2, [A-\lambda I]X=0$ gives

$\begin{bmatrix}0 & 0 & 2\\0 & 0 & 1\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

By $R_2-\frac{1}{2}2R_1, R_3+\frac{1}{2}R_1$

$\begin{bmatrix}0 & 0 & 2\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore 0x_1+0x_2+2x_3=0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ vectors to be formed

The Geometric Multiplicity of $\lambda=3$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_1=1$ & $x_2=1, \therefore x_3=0$

Let $x_1=1$ & $x_2=-1, \therefore x_3=0$

Hence, corresponding to $\lambda=2$ the eigen vectors are

$X_1=[1,1,0]'$ & $X_2=[1,-1,0]'$

(ii) For $\lambda=1, [A-\lambda I]X=0$ gives

$\begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$x_1 + 0x_2 + 2x_3 = 0$
$0x_1 + x_2 + x_3 = 0$
Solving the above equations by Crammers rule, we get
$\frac{x_1}{\begin{vmatrix}0 & 2\\1 & 1\end{vmatrix}} = -\frac{x_2}{\begin{vmatrix}1 & 2\\0 & 1\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}}$
$\frac{x_1}{-2} = -\frac{x_2}{1} = \frac{x_3}{1}$
$\frac{x_1}{-2} = \frac{x_2}{-1} = \frac{x_3}{1}$
Hence, corresponding to $\lambda = 1$ the eigen vector is $X_3 = [-2, -1, 1]'$
Thus, the matrix $A = \begin{bmatrix}2 & 0 & 2\\0 & 2 & 1\\0 & 0 & 1\end{bmatrix}$ is diagonalized to $D = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}$ by the transformation
$M^{-1}AM = D$ where $M = \begin{bmatrix}1 & 1 & -2\\1 & -1 & -1\\0 & 0 & 1\end{bmatrix}$

Type III: Cayley-Hamilton Theorem

1. Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and hence find A^{-1}, A^{-2}, A^4

Solution:

$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$|A| = 5$

Characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diag]\lambda^2 + [sum\ of\ minors\ of\ diag]\lambda - |A| = 0$

$\lambda^3 - [1-1-1]\lambda^2 + \left[\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}\right]\lambda - 5 = 0$

$\lambda^3 + \lambda^2 + [1-0-1-0-1-4]\lambda - 5 = 0$

$\lambda^3 + \lambda^2 - 5\lambda - 5 = 0$

By C-H theorem,

$A^3 + A^2 - 5A - 5I = 0$

Now,

$A^2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

LHS $= A^3 + A^2 - 5A - 5I$

$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = RHS

Hence, verified

Now,

$A^3 + A^2 - 5A - 5I = 0$

Multiplying by A^{-1}

$A^2 + A - 5I - 5A^{-1} = 0$

$A^2 + A - 5I = 5A^{-1}$

$$5A^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$5A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}}$$

And,

$A^3 + A^2 - 5A - 5I = 0$

Multiplying by A^{-2},

$A + I - 5A^{-1} - 5A^{-2} = 0$

$A + I - 5A^{-1} = 5A^{-2}$

$$5A^{-2} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$5A^{-2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\boxed{A^{-2} = \frac{1}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}}$$

Also,

$A^3 + A^2 - 5A - 5I = 0$

Multiplying by A,

$A^4 + A^3 - 5A^2 - 5A = 0$

$A^4 = -A^3 + 5A^2 + 5A$

$$A^4 = -\begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + 5\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\boxed{A^4 = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

2. State Cayley-Hamilton theorem & verify the same for $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

[N15/CompIT/5M]

Solution:

Cayley-Hamilton Theorem: Every square matrix satisfies its characteristic equation.

$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$

$(1-\lambda)(2-\lambda) - 6 = 0$

$\lambda^2 - 3\lambda - 4 = 0$

By C-H theorem,

$A^2 - 3A - 4I = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 6 & 10 \end{bmatrix}$

L.H.S. $= A^2 - 3A - 4I$

$= \begin{bmatrix} 7 & 9 \\ 6 & 10 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ = R.H.S

Thus, Cayley Hamilton theorem is verified

3. Use Cayley Hamilton theorem for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ to find A^3 and A^{-1}

[N22/MechCivil/5M]

Solution:

$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$

$(1-\lambda)(3-\lambda) - 8 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

By C-H theorem,

$A^2 - 4A - 5I = 0$

Multiplying by A

$A^3 - 4A^2 - 5A = 0$

$A^3 = 4A^2 + 5A$

$A^3 = 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$A^3 = 4\begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} + \begin{bmatrix} 5 & 20 \\ 10 & 15 \end{bmatrix}$

$A^3 = \begin{bmatrix} 41 & 84 \\ 42 & 83 \end{bmatrix}$

Also,

$A^2 - 4A - 5I = 0$

Multiplying by A^{-1}

$A - 4I - 5A^{-1} = 0$

$5A^{-1} = A - 4I$

$5A^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$5A^{-1} = \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

4. Find characteristic equation of A where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ Show that the matrix A satisfies the characteristic equation and hence find A^{-1} and A^4
[M16/ChemBiot/6M][M19/Extc/6M][D23/M24/CompIT/6M]
Solution:
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}, |A| = 40$$
The characteristic equation,
$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$
$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$
$$\lambda^3 - [1-1-1]\lambda^2 + \left[\begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}\right]\lambda - 40 = 0$$
$$\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$$
By Cayley Hamilton theorem,
$$A^3 + A^2 - 18A - 40I = 0$$
Consider,
$$A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$
$$A^3 = A^2.A = \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$
L.H.S. $= A^3 + A^2 - 18A - 40I$
$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - 18\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.H.S.}$$
Thus, Cayley Hamilton theorem is verified
Now,
$$A^3 + A^2 - 18A - 40I = 0$$
Pre-multiplying by A^{-1}, we get
$$A^2 + A - 18I - 40A^{-1} = 0$$
$$40A^{-1} = A^2 + A - 18I$$

$$40A^{-1} = \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

Also,

$$A^3 + A^2 - 18A - 40I = 0$$

Pre-multiplying by$A$, we get

$$A^4 + A^3 - 18A^2 - 40A = 0$$

$$A^4 = -A^3 + 18A^2 + 40A$$

$$A^4 = -\begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + 18\begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + 40\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$$

5. Show that the matrix $A=\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}$ satisfies Cayley Hamilton theorem where a,b,c are positive real nos. hence find A^{-1} if its exists

[M18/Extc/6M]

Solution:

$$A=\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}, |A| = 0 - c(0-ab) - b(ca-0) = 0$$

The characteristic equation,

$$|A-\lambda I| = 0$$

$$\begin{vmatrix}0-\lambda & c & -b\\ -c & 0-\lambda & a\\ b & -a & 0-\lambda\end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [0+0+0]\lambda^2 + \left[\begin{vmatrix}0 & a\\ -a & 0\end{vmatrix} + \begin{vmatrix}0 & -b\\ b & 0\end{vmatrix} + \begin{vmatrix}0 & c\\ -c & 0\end{vmatrix}\right]\lambda - 0 = 0$$

$$\lambda^3 - 0\lambda^2 + (a^2+b^2+c^2)\lambda - 0 = 0$$

$$\lambda^3 + (a^2+b^2+c^2)\lambda = 0$$

By Cayley Hamilton theorem,

$$A^3 + (a^2+b^2+c^2)A = 0$$

Now,

$$A^2 = A.A = \begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}$$

$$A^2 = \begin{bmatrix}-c^2-b^2 & ab & ac\\ ab & -c^2-a^2 & bc\\ ac & bc & -b^2-a^2\end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix}-c^2-b^2 & ab & ac\\ ab & -c^2-a^2 & bc\\ ac & bc & -b^2-a^2\end{bmatrix}\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}$$

$$A^3 = \begin{bmatrix}-abc+abc & -c^3-b^2c-a^2c & bc^2+b^3+a^2b\\ c^3+a^2c+b^2c & abc-abc & -ab^2-ac^2-a^3\\ -bc^2-b^3-a^2b & ac^2+ab^2+a^3 & -abc+abc\end{bmatrix}$$

$$A^3 = \begin{bmatrix}0 & -c(a^2+b^2+c^2) & b(a^2+b^2+c^2)\\ c(a^2+b^2+c^2) & 0 & -a(a^2+b^2+c^2)\\ -b(a^2+b^2+c^2) & a(a^2+b^2+c^2) & 0\end{bmatrix}$$

$$A^3 = -(a^2+b^2+c^2)\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}$$

$$A^3 = -(a^2+b^2+c^2)A$$

$A^3 + (a^2 + b^2 + c^2)A = 0$

Hence, Cayley Hamilton theorem is verified

A^{-1} does not exist as the matrix is singular ($|A| = 0$)

6. If $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, then prove that $A^{-1} = A^2 - 5A + 9I$

[N17/MechCivil/5M]

Solution:

$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}, |A| = 1$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [1+3+1]\lambda^2 + \left[\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}\right]\lambda - 1 = 0$

$\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$

By Cayley Hamilton theorem,

$A^3 - 5A^2 + 9A - I = 0$

Now,

$A^3 - 5A^2 + 9A - I = 0$

Pre-multiplying by A^{-1}, we get

$A^2 - 5A + 9I - A^{-1} = 0$

$A^{-1} = A^2 - 5A + 9I$

7. Verify that the matrix $A=\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$ satisfies the characteristic equation, hence find A^{-2}

[N14/ElexExtcElectBiomInst/6M]

Solution:

$A=\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}, |A|=5$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}1-\lambda & 2 & 0\\ 2 & -1-\lambda & 0\\ 0 & 0 & -1-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[1-1-1]\lambda^2+\left[\begin{vmatrix}-1 & 0\\ 0 & -1\end{vmatrix}+\begin{vmatrix}1 & 0\\ 0 & -1\end{vmatrix}+\begin{vmatrix}1 & 2\\ 2 & -1\end{vmatrix}\right]\lambda-5=0$

$\lambda^3+\lambda^2-5\lambda-5=0$

By Cayley Hamilton theorem,

$A^3+A^2-5A-5I=0$

Consider,

$A^2=A.A=\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}=\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 1\end{bmatrix}$

$A^3=A^2.A=\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}=\begin{bmatrix}5 & 10 & 0\\ 10 & -5 & 0\\ 0 & 0 & -1\end{bmatrix}$

L.H.S. $=A^3+A^2-5A-5I$

$=\begin{bmatrix}5 & 10 & 0\\ 10 & -5 & 0\\ 0 & 0 & -1\end{bmatrix}+\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 1\end{bmatrix}-5\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}-5\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=$ R.H.S.

Thus, Cayley Hamilton theorem is verified

Now,

$A^3+A^2-5A-5I=0$

Pre-multiplying by A^{-1}, we get

$A^2+A-5I-5A^{-1}=0$

$5A^{-1}=A^2+A-5I$

$$5A^{-1}=\begin{bmatrix}5&0&0\\0&5&0\\0&0&1\end{bmatrix}+\begin{bmatrix}1&2&0\\2&-1&0\\0&0&-1\end{bmatrix}-5\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$A^{-1}=\frac{1}{5}\begin{bmatrix}1&2&0\\2&-1&0\\0&0&-5\end{bmatrix}$$

Also,

$$A^3+A^2-5A-5I=0$$

Pre-multiplying by A^{-2}, we get

$$A+I-5A^{-1}-5A^{-2}=0$$

$$5A^{-2}=A+I-5A^{-1}$$

$$5A^{-2}=\begin{bmatrix}1&2&0\\2&-1&0\\0&0&-1\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1&2&0\\2&-1&0\\0&0&-5\end{bmatrix}$$

$$A^{-2}=\frac{1}{5}\begin{bmatrix}1&0&0\\0&1&0\\0&0&5\end{bmatrix}$$

8. Show that the matrix A satisfies the characteristic equation and hence find A^{-1} where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

[N18/Comp/6M]

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}, |A| = 40$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [1-1-1]\lambda^2 + \left[\begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}\right]\lambda - 40 = 0$$

$$\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$$

By Cayley Hamilton theorem,

$$A^3 + A^2 - 18A - 40I = 0$$

Consider,

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$

L.H.S. $= A^3 + A^2 - 18A - 40I$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - 18\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.H.S.}$$

Thus, Cayley Hamilton theorem is verified

Now,

$$A^3 + A^2 - 18A - 40I = 0$$

Pre-multiplying by A^{-1}, we get

$$A^2 + A - 18I - 40A^{-1} = 0$$

$$40A^{-1} = A^2 + A - 18I$$

$$40A^{-1} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 18 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{40} \begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

9. Show that the matrix $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ satisfies Cayley Hamilton theorem and hence find A^{-1} if exists.

[M16/ElexExtcElectBiomInst/6M][M18/MechCivil/6M]

Solution:

$A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}, |A|=4$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}2-\lambda & -1 & 1\\ -1 & 2-\lambda & -1\\ 1 & -1 & 2-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[2+2+2]\lambda^2+\left[\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}+\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}+\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}\right]\lambda-4=0$

$\lambda^3-6\lambda^2+9\lambda-4=0$

By Cayley Hamilton theorem,

$A^3-6A^2+9A-4I=0$

Consider,

$A^2=A.A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$

$A^3=A^2.A=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}$

L.H.S. $=A^3-6A^2+9A-4I$

$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-6\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}+9\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}-4\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ = R.H.S.

Thus, Cayley Hamilton theorem is verified

Now,

$A^3-6A^2+9A-4I=0$

Pre-multiplying by A^{-1}, we get

$A^2-6A+9I-4A^{-1}=0$

$4A^{-1}=A^2-6A+9I$

$$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

10. Verify Cayley-Hamilton theorem for the matrix A and hence find A^{-1} and A^4 where
$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$
[J24/ElectECSExtc/8M]
Solution:
$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}, |A| = 4$
The characteristic equation,
$|A - \lambda I| = 0$
$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$
$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$
$\lambda^3 - [2+2+2]\lambda^2 + \left[\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}\right]\lambda - 4 = 0$
$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$
By Cayley Hamilton theorem,
$A^3 - 6A^2 + 9A - 4I = 0$
Consider,
$A^2 = A.A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$
$A^3 = A^2.A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$
L.H.S. $= A^3 - 6A^2 + 9A - 4I$
$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = R.H.S.
Thus, Cayley Hamilton theorem is verified
Now,
$A^3 - 6A^2 + 9A - 4I = 0$
Pre-multiplying by A^{-1}, we get
$A^2 - 6A + 9I - 4A^{-1} = 0$
$4A^{-1} = A^2 - 6A + 9I$

$$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Also,

$$A^3 - 6A^2 + 9A - 4I = 0$$

Pre-multiplying by$A$, we get

$$A^4 - 6A^3 + 9A^2 - 4A = 0$$

$$A^4 = 6A^3 - 9A^2 + 4A$$

$$A^4 = 6\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 9\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 4\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix}$$

11. Show that the matrix $A=\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ satisfies Cayley Hamilton theorem and hence find A^{-1}

[N19/Elect/6M]

Solution:

$A=\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}, |A|=6$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}2-\lambda & -1 & 1\\ 1 & 2-\lambda & -1\\ 1 & -1 & 2-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[2+2+2]\lambda^2+\left[\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}+\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}+\begin{vmatrix}2 & -1\\ 1 & 2\end{vmatrix}\right]\lambda-6=0$

$\lambda^3-6\lambda^2+11\lambda-6=0$

By Cayley Hamilton theorem,

$A^3-6A^2+11A-6I=0$

Consider,

$A^2=A.A=\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}=\begin{bmatrix}4 & -5 & 5\\ 3 & 4 & -3\\ 3 & -5 & 6\end{bmatrix}$

$A^3=A^2.A=\begin{bmatrix}4 & -5 & 5\\ 3 & 4 & -3\\ 3 & -5 & 6\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}=\begin{bmatrix}8 & -19 & 19\\ 7 & 8 & -7\\ 7 & -19 & 20\end{bmatrix}$

L.H.S. $=A^3-6A^2+11A-6I$

$=\begin{bmatrix}8 & -19 & 19\\ 7 & 8 & -7\\ 7 & -19 & 20\end{bmatrix}-6\begin{bmatrix}4 & -5 & 5\\ 3 & 4 & -3\\ 3 & -5 & 6\end{bmatrix}+11\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}-6\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=$ R.H.S.

Thus, Cayley Hamilton theorem is verified

Now,

$A^3-6A^2+11A-6I=0$

Pre-multiplying by A^{-1}, we get

$A^2-6A+11I-6A^{-1}=0$

$6A^{-1}=A^2-6A+11I$

$$6A^{-1} = \begin{bmatrix} 4 & -5 & 5 \\ 3 & 4 & -3 \\ 3 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 1 & -1 \\ -3 & 3 & 3 \\ -3 & 1 & 5 \end{bmatrix}$$

12. Show that the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ satisfies Cayley Hamilton theorem and hence find A^{-1}

[N19/Comp/6M]

Solution:

$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}, |A| = 35$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [1+2+1]\lambda^2 + \left[\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}\right]\lambda - 35 = 0$

$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

By Cayley Hamilton theorem,

$A^3 - 4A^2 - 20A - 35I = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$

L.H.S. $= A^3 - 4A^2 - 20A - 35I$

$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = R.H.S.

Thus, Cayley Hamilton theorem is verified

Now,

$A^3 - 4A^2 - 20A - 35I = 0$

Pre-multiplying by A^{-1}, we get

$A^2 - 4A - 20I - 35A^{-1} = 0$

$35A^{-1} = A^2 - 4A - 20I$

$$35A^{-1} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

13. Show that the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ satisfies Cayley Hamilton theorem and hence find A^{-1}

[N18/Inst/6M][M19/Comp/6M]

Solution:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}, |A| = 6$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+2+2]\lambda^2 + \left[\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}\right]\lambda - 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

By Cayley Hamilton theorem,

$$A^3 - 6A^2 + 11A - 6I = 0$$

Consider,

$$A^2 = A.A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix}$$

L.H.S. $= A^3 - 6A^2 + 11A - 6I$

$$= \begin{bmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix} + 11\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.H.S.}$$

Thus, Cayley Hamilton theorem is verified

Now,

$$A^3 - 6A^2 + 11A - 6I = 0$$

Pre-multiplying by A^{-1}, we get

$$A^2 - 6A + 11I - 6A^{-1} = 0$$

$$6A^{-1} = A^2 - 6A + 11I$$

$$6A^{-1} = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix} - 6\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

14. Verify Cayley Hamilton theorem for the matrix A and hence find A^{-1} where
$$A=\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}$$
[M14/N16/ChemBiot/5M][N17/ElexExtcElectBiomInst/6M][M18/Comp/6M]
Solution:
$$A=\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix},|A|=1$$
The characteristic equation,
$$|A-\lambda I|=0$$
$$\begin{vmatrix}1-\lambda&2&-2\\-1&3-\lambda&0\\0&-2&1-\lambda\end{vmatrix}=0$$
$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$
$$\lambda^3-[1+3+1]\lambda^2+\left[\begin{vmatrix}3&0\\-2&1\end{vmatrix}+\begin{vmatrix}1&-2\\0&1\end{vmatrix}+\begin{vmatrix}1&2\\-1&3\end{vmatrix}\right]\lambda-1=0$$
$$\lambda^3-5\lambda^2+9\lambda-1=0$$
By Cayley Hamilton theorem,
$$A^3-5A^2+9A-I=0$$
Consider,
$$A^2=A.A=\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}=\begin{bmatrix}-1&12&-4\\-4&7&2\\2&-8&1\end{bmatrix}$$
$$A^3=A^2.A=\begin{bmatrix}-1&12&-4\\-4&7&2\\2&-8&1\end{bmatrix}\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}=\begin{bmatrix}-13&42&-2\\-11&9&10\\10&-22&-3\end{bmatrix}$$
L.H.S. $=A^3-5A^2+9A-I$
$$=\begin{bmatrix}-13&42&-2\\-11&9&10\\10&-22&-3\end{bmatrix}-5\begin{bmatrix}-1&12&-4\\-4&7&2\\2&-8&1\end{bmatrix}+9\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$
$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=\text{R.H.S.}$$
Thus, Cayley Hamilton theorem is verified
Now,
$$A^3-5A^2+9A-I=0$$
Pre-multiplying by A^{-1}, we get
$$A^2-5A+9I-A^{-1}=0$$
$$A^{-1}=A^2-5A+9I$$

$$A^{-1} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

15. Verify Cayley Hamilton theorem for the matrix A and hence find A^{-1} and A^4 where
$A=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$

[M15/ElexExtcElectBiomInst/8M][M18/Biot/8M][N22/Extc/6M]
[N23/N24/ElectECSExtc/6M]

Solution:

$A=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}, |A|=1$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}1-\lambda & 2 & -2\\ -1 & 3-\lambda & 0\\ 0 & -2 & 1-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[1+3+1]\lambda^2+\left[\begin{vmatrix}3 & 0\\ -2 & 1\end{vmatrix}+\begin{vmatrix}1 & -2\\ 0 & 1\end{vmatrix}+\begin{vmatrix}1 & 2\\ -1 & 3\end{vmatrix}\right]\lambda-1=0$

$\lambda^3-5\lambda^2+9\lambda-1=0$

By Cayley Hamilton theorem,

$A^3-5A^2+9A-I=0$

Consider,

$A^2=A.A=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}=\begin{bmatrix}-1 & 12 & -4\\ -4 & 7 & 2\\ 2 & -8 & 1\end{bmatrix}$

$A^3=A^2.A=\begin{bmatrix}-1 & 12 & -4\\ -4 & 7 & 2\\ 2 & -8 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}=\begin{bmatrix}-13 & 42 & -2\\ -11 & 9 & 10\\ 10 & -22 & -3\end{bmatrix}$

L.H.S. $=A^3-5A^2+9A-I$

$=\begin{bmatrix}-13 & 42 & -2\\ -11 & 9 & 10\\ 10 & -22 & -3\end{bmatrix}-5\begin{bmatrix}-1 & 12 & -4\\ -4 & 7 & 2\\ 2 & -8 & 1\end{bmatrix}+9\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=$ R.H.S.

Thus, Cayley Hamilton theorem is verified

Now,

$A^3-5A^2+9A-I=0$

Pre-multiplying by A^{-1}, we get

$A^2-5A+9I-A^{-1}=0$

$A^{-1}=A^2-5A+9I$

$$A^{-1}=\begin{bmatrix}-1 & 12 & -4\\ -4 & 7 & 2\\ 2 & -8 & 1\end{bmatrix}-5\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}+9\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}$$

Also,

$$A^3-5A^2+9A-I=0$$

Pre-multiplying by$A$, we get

$$A^4-5A^3+9A^2-A=0$$

$$A^4=5A^3-9A^2+A$$

$$A^4=5\begin{bmatrix}-13 & 42 & -2\\ -11 & 9 & 10\\ 10 & -22 & -3\end{bmatrix}-9\begin{bmatrix}-1 & 12 & -4\\ -4 & 7 & 2\\ 2 & -8 & 1\end{bmatrix}+\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$$

$$A^4=\begin{bmatrix}-55 & 104 & 24\\ -20 & -15 & 32\\ 32 & -40 & -23\end{bmatrix}$$

16. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find A^{-1} & A^4

[M17/ElexExtcElectBiomInst/6M][D24/CompIT/6M]

Solution:

$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, |A| = 3$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [2+1+2]\lambda^2 + \left[\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\right]\lambda - 3 = 0$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley Hamilton theorem,

$A^3 - 5A^2 + 7A - 3I = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$

L.H.S. $= A^3 - 5A^2 + 7A - 3I$

$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$ R.H.S.

Thus, Cayley Hamilton theorem is verified

Now,

$A^3 - 5A^2 + 7A - 3I = 0$

Pre-multiplying by A^{-1}, we get

$A^2 - 5A + 7I - 3A^{-1} = 0$

$3A^{-1} = A^2 - 5A + 7I$

$$3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Also, $A^3 - 5A^2 + 7A - 3I = 0$

Pre-multiplying by A, we get

$$A^4 - 5A^3 + 7A^2 - 3A = 0$$

$$A^4 = 5A^3 - 7A^2 + 3A$$

$$A^4 = 5\begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 7\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 3\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 41 & 40 & 40 \\ 0 & 1 & 0 \\ 40 & 40 & 41 \end{bmatrix}$$

17. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find A^{-1}

[N18/MechCivil/6M][M18/M19/Biom/6M][N18/N22/Elect/6M][M19/Inst/6M]

Solution:

$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, |A| = 3$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [2+1+2]\lambda^2 + \left[\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\right]\lambda - 3 = 0$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley Hamilton theorem,

$A^3 - 5A^2 + 7A - 3I = 0$

$A^2 = A.A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$

L.H.S. $= A^3 - 5A^2 + 7A - 3I$

$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = R.H.S.

Thus, Cayley Hamilton theorem is verified

Now,

$A^3 - 5A^2 + 7A - 3I = 0$

Pre-multiplying by A^{-1}, we get

$A^2 - 5A + 7I - 3A^{-1} = 0$

$3A^{-1} = A^2 - 5A + 7I$

$3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$3A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

18. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

[N19/Chem/6M]

Solution:

$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, |A| = 3$

The characteristic equation, $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+1+2]\lambda^2 + \left[\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\right]\lambda - 3 = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$A^2 = A.A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

L.H.S. $= A^3 - 5A^2 + 7A - 3I$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.H.S.}$$

Thus, Cayley Hamilton theorem is verified

19. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$ and hence find A^{-1}

[N18/Elex/6M]

Solution:

$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}, |A| = 11$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [4+1+1]\lambda^2 + \left[\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}\right]\lambda - 11 = 0$

$\lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0$

By Cayley Hamilton theorem,

$A^3 - 6A^2 + 6A - 11I = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 125 & 84 & -12 \\ 36 & 23 & 0 \\ 48 & 30 & -7 \end{bmatrix}$

L.H.S. $= A^3 - 6A^2 + 6A - 11I$

$= \begin{bmatrix} 125 & 84 & -12 \\ 36 & 23 & 0 \\ 48 & 30 & -7 \end{bmatrix} - 6\begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} + 6\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} - 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$ R.H.S.

Thus, Cayley Hamilton theorem is verified

Now, $A^3 - 6A^2 + 6A - 11I = 0$

Pre-multiplying by A^{-1}, we get

$A^2 - 6A + 6I - 11A^{-1} = 0$

$11A^{-1} = A^2 - 6A + 6I$

$11A^{-1} = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} - 6\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$11A^{-1} = \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{11}\begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

20. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find A^{-1}

[M18/Inst/6M][N18/Biom/6M]

Solution:

$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}, |A| = 14$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 2 & -1 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [3+2+2]\lambda^2 + \left[\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix}\right]\lambda - 14 = 0$

$\lambda^3 - 7\lambda^2 + 17\lambda - 14 = 0$

By Cayley Hamilton theorem,

$A^3 - 7A^2 + 17A - 14I = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 9 & -5 \\ 0 & 4 & 0 \\ 5 & 6 & 3 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 8 & 9 & -5 \\ 0 & 4 & 0 \\ 5 & 6 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 19 & 29 & -18 \\ 0 & 8 & 0 \\ 18 & 25 & 1 \end{bmatrix}$

L.H.S. $= A^3 - 7A^2 + 17A - 14I$

$= \begin{bmatrix} 19 & 29 & -18 \\ 0 & 8 & 0 \\ 18 & 25 & 1 \end{bmatrix} - 7\begin{bmatrix} 8 & 9 & -5 \\ 0 & 4 & 0 \\ 5 & 6 & 3 \end{bmatrix} + 17\begin{bmatrix} 3 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 14\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = R.H.S.

Thus, Cayley Hamilton theorem is verified

Now, $A^3 - 7A^2 + 17A - 14I = 0$

Pre-multiplying by A^{-1}, we get

$A^2 - 7A + 17I - 14A^{-1} = 0$

$14A^{-1} = A^2 - 7A + 17I$

$14A^{-1} = \begin{bmatrix} 8 & 9 & -5 \\ 0 & 4 & 0 \\ 5 & 6 & 3 \end{bmatrix} - 7\begin{bmatrix} 3 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 17\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$14A^{-1} = \begin{bmatrix} 4 & -5 & 2 \\ 0 & 7 & 0 \\ -2 & -1 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 4 & -5 & 2 \\ 0 & 7 & 0 \\ -2 & -1 & 6 \end{bmatrix}$$

21. Verify Cayley Hamilton theorem for matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

[M22/Chem/5M]

Solution:

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}, |A| = 1$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [1+3+1]\lambda^2 + \left[\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}\right]\lambda - 1 = 0$$

$$\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

By Cayley Hamilton theorem,

$$A^3 - 5A^2 + 9A - I = 0$$

Consider,

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

L.H.S. $= A^3 - 5A^2 + 9A - I$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.H.S.}$$

Thus, Cayley Hamilton theorem is verified

22. Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

[M22/Elex/5M]

Solution:

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, |A| = 45$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [-2+1+0]\lambda^2 + \left[\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}\right]\lambda - 45 = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

By Cayley Hamilton theorem,

$$A^3 + A^2 - 21A - 45I = 0$$

Consider,

$$A^2 = A.A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -8 & 38 & -57 \\ 38 & 49 & -114 \\ -19 & -38 & 30 \end{bmatrix}$$

L.H.S. $= A^3 + A^2 - 21A - 45I$

$$= \begin{bmatrix} -8 & 38 & -57 \\ 38 & 49 & -114 \\ -19 & -38 & 30 \end{bmatrix} + \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix} - 21\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - 45\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.H.S.}$$

Thus, Cayley Hamilton theorem is verified

23. Find the eigen values of matrix $A=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}$ and show that matrix satisfies the characteristic equation

[N22/Elex/6M]

Solution:

$A=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}, |A|=40$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}1-\lambda & 2 & 3\\ 2 & -1-\lambda & 4\\ 3 & 1 & -1-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[1-1-1]\lambda^2+\left[\begin{vmatrix}-1 & 4\\ 1 & -1\end{vmatrix}+\begin{vmatrix}1 & 3\\ 3 & -1\end{vmatrix}+\begin{vmatrix}1 & 2\\ 2 & -1\end{vmatrix}\right]\lambda-40=0$

$\lambda^3+\lambda^2-18\lambda-40=0$

Eigen Values are imaginary

By Cayley Hamilton theorem,

$A^3+A^2-18A-40I=0$

Consider,

$A^2=A.A=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}=\begin{bmatrix}14 & 3 & 8\\ 12 & 9 & -2\\ 2 & 4 & 14\end{bmatrix}$

$A^3=A^2.A=\begin{bmatrix}14 & 3 & 8\\ 12 & 9 & -2\\ 2 & 4 & 14\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}=\begin{bmatrix}44 & 33 & 46\\ 24 & 13 & 74\\ 52 & 14 & 8\end{bmatrix}$

L.H.S. $=A^3+A^2-18A-40I$

$=\begin{bmatrix}44 & 33 & 46\\ 24 & 13 & 74\\ 52 & 14 & 8\end{bmatrix}+\begin{bmatrix}14 & 3 & 8\\ 12 & 9 & -2\\ 2 & 4 & 14\end{bmatrix}-18\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}-40\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=$ R.H.S.

Thus, Cayley Hamilton theorem is verified

24. Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

[M23/CompIT/6M]

Solution:

$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}, |A| = 4$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [4+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}\right]\lambda - 4 = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

By Cayley Hamilton theorem,

$A^3 - 5A^2 + 8A - 4I = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix} = \begin{bmatrix} 16 & 12 & 24 \\ 5 & 5 & 8 \\ -7 & -11 & -12 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 16 & 12 & 24 \\ 5 & 5 & 8 \\ -7 & -11 & -12 \end{bmatrix}\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix} = \begin{bmatrix} 52 & 12 & 72 \\ 17 & 5 & 24 \\ -27 & -15 & -40 \end{bmatrix}$

L.H.S. $= A^3 - 5A^2 + 8A - 4I$

$= \begin{bmatrix} 52 & 12 & 72 \\ 17 & 5 & 24 \\ -27 & -15 & -40 \end{bmatrix} - 5\begin{bmatrix} 16 & 12 & 24 \\ 5 & 5 & 8 \\ -7 & -11 & -12 \end{bmatrix} + 8\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$ R.H.S.

Thus, Cayley Hamilton theorem is verified

25. Verify Cayley Hamilton Theorem for $A=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}$ and hence evaluate $2A^4-5A^3-7A+6I$

[M15/ChemBiot/6M][M19/N23/MechCivil/6M][N22/Chem/6M]

Solution:

$A=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}1-\lambda & 2\\ 2 & 2-\lambda\end{vmatrix}=0$

$(1-\lambda)(2-\lambda)-4=0$

$\lambda^2-3\lambda-2=0$

By C-H theorem, $A^2-3A-2I=0$

Consider, $A^2=A.A=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\ 6 & 8\end{bmatrix}$

L.H.S. $=A^2-3A-2I$

$=\begin{bmatrix}5 & 6\\ 6 & 8\end{bmatrix}-3\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=$ R.H.S

Thus, Cayley Hamilton theorem is verified

Dividing $2\lambda^4-5\lambda^3-7\lambda+6$ by $\lambda^2-3\lambda-2$

$$
\begin{array}{r|l}
 & 2\lambda^2+\lambda+7 \\
\cline{2-2}
\lambda^2-3\lambda-2 & 2\lambda^4-5\lambda^3-7\lambda+6 \\
 & 2\lambda^4-6\lambda^3-4\lambda^2 \\
 & -\quad +\quad + \\
\cline{2-2}
 & \lambda^3+4\lambda^2-7\lambda+6 \\
 & \lambda^3-3\lambda^2-2\lambda \\
 & -\quad +\quad + \\
\cline{2-2}
 & 7\lambda^2-5\lambda+6 \\
 & 7\lambda^2-21\lambda-14 \\
 & +\quad + \\
\cline{2-2}
 & 16\lambda+20
\end{array}
$$

Thus,

$2A^4-5A^3-7A+6I=(2A^2+A+7I)(A^2-3A-2I)+16A+20I$

$2A^4-5A^3-7A+6I=(2A^2+A+7I)(0)+16A+20I$

$2A^4-5A^3-7A+6I=16A+20I$

$2A^4-5A^3-7A+6I=16\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}+20\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}36 & 32\\ 32 & 52\end{bmatrix}$

26. Verify Cayley Hamilton theorem and find A^{-1} for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Hence find $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A.

[M14/N14/MechCivil/5M][N18/Extc/8M]

Solution:

$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$

$(1-\lambda)(3-\lambda) - 8 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

By C-H theorem,

$A^2 - 4A - 5I = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$

L.H.S. $= A^2 - 4A - 5I$

$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ = R.H.S

Thus, Cayley Hamilton theorem is verified

Now,

$A^2 - 4A - 5I = 0$

Pre-multiplying by A^{-1}, we get

$A - 4I - 5A^{-1} = 0$

$5A^{-1} = A - 4I$

$5A^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$5A^{-1} = \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

Dividing $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10$ by $\lambda^2 - 4\lambda - 5$

$$
\begin{array}{r|l}
 & \lambda^3 - 2\lambda + 3 \\
\hline
\lambda^2 - 4\lambda - 5 & \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 \\
 & \lambda^5 - 4\lambda^4 - 5\lambda^3 \\
 & - \quad + \qquad + \\
\hline
 & -2\lambda^3 + 11\lambda^2 - \lambda - 10 \\
 & -2\lambda^3 + 8\lambda^2 + 10\lambda \\
 & + \qquad - \qquad - \\
\hline
 & 3\lambda^2 - 11\lambda - 10 \\
 & 3\lambda^2 - 12\lambda - 15 \\
 & - \quad + \qquad + \\
\hline
 & \lambda + 5
\end{array}
$$

Thus,

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^3 - 2A + 3I)(A^2 - 4A - 5I) + A + 5I$

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^3 - 2A + 3I)(0) + A + 5I$

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5I$

27. State Cayley-Hamilton theorem. Use it to express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A, when $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

[N13/N19/Biot/5M][N18/MTRX/5M][M22/ElectECSExtc/5M]

Solution:

Cayley-Hamilton Theorem: Every square matrix satisfies its characteristic equation.

We have,

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda) + 1 = 0$$

$$\lambda^2 - 5\lambda + 7 = 0$$

By C-H theorem,

$$A^2 - 5A + 7I = 0$$

Dividing $2\lambda^5 - 3\lambda^4 + \lambda^2 - 4$ by $\lambda^2 - 5\lambda + 7$

$$2\lambda^3 + 7\lambda^2 + 21\lambda + 57$$

$$\lambda^2 - 5\lambda + 7 \,\big|\, 2\lambda^5 - 3\lambda^4 + \lambda^2 - 4$$

$$2\lambda^5 - 10\lambda^4 + 14\lambda^3$$

$$- \quad + \quad -$$

$$7\lambda^4 - 14\lambda^3 + \lambda^2 - 4$$

$$7\lambda^4 - 35\lambda^3 + 49\lambda^2$$

$$- \quad + \quad -$$

$$21\lambda^3 - 48\lambda^2 - 4$$

$$21\lambda^3 - 105\lambda^2 + 147\lambda$$

$$- \quad + \quad -$$

$$57\lambda^2 - 147\lambda - 4$$

$$57\lambda^2 - 285\lambda + 399$$

$$- \quad + \quad -$$

$$138\lambda - 403$$

Thus,

$$2A^5 - 3A^4 + A^2 - 4I = (2A^3 + 7A^2 + 21A + 57I)(A^2 - 5A + 7I) + 138A - 403I$$

$$2A^5 - 3A^4 + A^2 - 4I = (2A^3 + 7A^2 + 21A + 57I)(0) + 138A - 403I$$

$$2A^5 - 3A^4 + A^2 - 4I = 138A - 403I$$

28. Apply C-H Theorem to $A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$ and deduce that $A^8=625I$

[M14/ElexExtcElectBiomInst/6M]

Solution:

$A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}1-\lambda & 2\\ 2 & -1-\lambda\end{vmatrix}=0$

$(1-\lambda)(-1-\lambda)-4=0$

$\lambda^2-5=0$

By C-H theorem,

$A^2-5I=0$

$A^2=5I$

Squaring both the sides, $A^4=25I$

Squaring both the sides, $A^8=625I$

29. Verify Cayley Hamilton Theorem and find A^{54} where $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

[N15/ElexExtcElectBiomInst/6M]

Solution:

$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$(1-\lambda)(-1-\lambda) - 4 = 0$

$\lambda^2 - 5 = 0$

By C-H theorem,

$A^2 - 5I = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

L.H.S. $= A^2 - 5I$

$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ = R.H.S

Thus, Cayley Hamilton theorem is verified

Now,

$A^2 - 5I = 0$

$A^2 = 5I$

Raise to 27 on both sides, we get

$(A^2)^{27} = (5I)^{27}$

$A^{54} = 5^{27} I$

30. If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ find A^4

[M22/Elex/2M]

Solution:

$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$(1-\lambda)(-1-\lambda) - 4 = 0$

$\lambda^2 - 5 = 0$

By C-H theorem,

$A^2 - 5I = 0$

$A^2 = 5I$

Squaring both the sides, $A^4 = 25I$

31. Verify C H Theorem for $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ and hence find $A^{-1}, A^3 - 5A^2$

[N17/M18/N18/Chem/5M]

Solution:

$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)(2-\lambda) - 4 = 0$

$\lambda^2 - 4\lambda = 0$

By C-H theorem,

$A^2 - 4A = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ 4 & 8 \end{bmatrix}$

L.H.S. $= A^2 - 4A$

$= \begin{bmatrix} 8 & 16 \\ 4 & 8 \end{bmatrix} - 4\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ = R.H.S

Thus, Cayley Hamilton theorem is verified

Since $|A| = 0$, A^{-1} does not exist

Now,

$A^2 - 4A = 0$

$A^2 - 4A - A = 0 - A$

$A^2 - 5A = -A$

Multiplying by A, we get

$A^3 - 5A^2 = -A^2 = \begin{bmatrix} -8 & -16 \\ -4 & -8 \end{bmatrix}$

32. Use Cayley Hamilton Theorem to find $2A^4-5A^3-7A+6I$ where $A=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}$

[N16/M17/MechCivil/5M][M18/Biot/5M]

Solution:

$A=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}1-\lambda & 2\\ 2 & 2-\lambda\end{vmatrix}=0$

$(1-\lambda)(2-\lambda)-4=0$

$\lambda^2-3\lambda-2=0$

By C-H theorem, $A^2-3A-2I=0$

Dividing $2\lambda^4-5\lambda^3-7\lambda+6$ by $\lambda^2-3\lambda-2$

$$
\begin{array}{r|l}
 & 2\lambda^2+\lambda+7 \\ \hline
\lambda^2-3\lambda-2 & 2\lambda^4-5\lambda^3-7\lambda+6 \\
 & 2\lambda^4-6\lambda^3-4\lambda^2 \\
 & -\quad +\quad + \\ \hline
 & \lambda^3+4\lambda^2-7\lambda+6 \\
 & \lambda^3-3\lambda^2-2\lambda \\
 & -\quad +\quad + \\ \hline
 & 7\lambda^2-5\lambda+6 \\
 & 7\lambda^2-21\lambda-14 \\
 & +\quad + \\ \hline
 & 16\lambda+20
\end{array}
$$

Thus,

$2A^4-5A^3-7A+6I=(2A^2+A+7I)(A^2-3A-2I)+16A+20I$

$2A^4-5A^3-7A+6I=(2A^2+A+7I)(0)+16A+20I$

$2A^4-5A^3-7A+6I=16A+20I$

$2A^4-5A^3-7A+6I=16\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}+20\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}36 & 32\\ 32 & 52\end{bmatrix}$

33. Verify Cayley Hamilton theorem and find A^{-1} for $A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$.
Hence find $2A^3 - A^2 - 35A - 44I$
[N22/CompIT/6M]
Solution:

$A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$

The characteristic equation, $|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 8 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)(1-\lambda) - 16 = 0$

$\lambda^2 - 2\lambda - 15 = 0$

By C-H theorem, $A^2 - 2A - 15I = 0$

L.H.S. $= A^2 - 2A - 15I$

$= \begin{bmatrix} 17 & 16 \\ 4 & 17 \end{bmatrix} - 2\begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} - 15\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} =$ R.H.S

Thus, Cayley Hamilton theorem is verified

Now,

$A^2 - 2A - 15I = 0$

Pre-multiplying by A^{-1}, we get

$A - 2I - 15A^{-1} = 0$

$15A^{-1} = A - 2I$

$15A^{-1} = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$15A^{-1} = \begin{bmatrix} -1 & 8 \\ 2 & -1 \end{bmatrix}$

$A^{-1} = \frac{1}{15}\begin{bmatrix} -1 & 8 \\ 2 & -1 \end{bmatrix}$

Dividing $2\lambda^3 - \lambda^2 - 35\lambda - 44$ by $\lambda^2 - 2\lambda - 15$

$$
\begin{array}{r|l}
 & 2\lambda + 3 \\
\hline
\lambda^2 - 2\lambda - 15 & 2\lambda^3 - \lambda^2 - 35\lambda - 44 \\
 & 2\lambda^3 - 4\lambda^2 - 30\lambda \\
 & - \quad + \quad + \\
\hline
 & 3\lambda^2 - 5\lambda - 44 \\
 & 3\lambda^2 - 6\lambda - 45 \\
 & - \quad + \quad + \\
\hline
 & \lambda + 1
\end{array}
$$

Thus,

$$2A^3 - A^2 - 35A - 44I = (2A + 3I)(A^2 - 2A - 15I) + A + I$$
$$2A^3 - A^2 - 35A - 44I = (2A + 3I)(0) + A + I$$
$$2A^3 - A^2 - 35A - 44I = A + I$$
$$2A^3 - A^2 - 35A - 44I = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 2 & 2 \end{bmatrix}$$

34. Find the characteristic equation of the matrix A given below and hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

[N13/Chem/7M][N19/Inst/6M][M22/MechCivil/5M]

Solution:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, |A| = 3$$

The characteristic equation, $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+1+2]\lambda^2 + \left[\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\right]\lambda - 3 = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3I = 0$$

Dividing $\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1$ by $\lambda^3 - 5\lambda^2 + 7\lambda - 3$

$$\begin{array}{r|l}
 & \lambda^5 + \lambda \\ \hline
\lambda^3 - 5\lambda^2 + 7\lambda - 3 & \lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1 \\
 & \lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 \\
 & - \quad + \quad - \quad + \\ \hline
 & \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1 \\
 & \lambda^4 - 5\lambda^4 + 7\lambda^2 - 3\lambda \\
 & - \quad + \quad - \quad + \\ \hline
 & \lambda^2 + \lambda + 1
\end{array}$$

Thus,

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = (A^5 + A)(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = (A^5 + A)(0) + A^2 + A + I$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = A^2 + A + I$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

35. Find the characteristic equation of the matrix A given below and hence find the matrix represented by $A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I$ where $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

[N15/MechCivil/6M][N16/ElexExtcElectBiomInst/6M]

Solution:

$$A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

$$|A| = 12$$

The characteristic equation, $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [3 - 3 + 7]\lambda^2 + \left[\begin{vmatrix} -3 & -4 \\ 5 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 10 \\ -2 & -3 \end{vmatrix}\right]\lambda - 12 = 0$$

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

By Cayley Hamilton theorem,

$$A^3 - 7A^2 + 16A - 12I = 0$$

Dividing $\lambda^6 - 6\lambda^5 + 9\lambda^4 + 4\lambda^3 - 12\lambda^2 + 2\lambda - 1$ by $\lambda^3 - 7\lambda^2 + 16\lambda - 12$

$$\begin{array}{r|l} & \lambda^3 + \lambda^2 \\ \hline \lambda^3 - 7\lambda^2 + 16\lambda - 12 & \lambda^6 - 6\lambda^5 + 9\lambda^4 + 4\lambda^3 - 12\lambda^2 + 2\lambda - 1 \\ & \lambda^6 - 7\lambda^5 + 16\lambda^4 - 12\lambda^3 \\ & - \quad + \quad - \quad + \\ \hline & \lambda^5 - 7\lambda^4 + 16\lambda^3 - 12\lambda^2 + 2\lambda - 1 \\ & \lambda^5 - 7\lambda^4 + 16\lambda^3 - 12\lambda^2 \\ & - \quad + \quad - \quad + \\ \hline & 2\lambda - 1 \end{array}$$

Thus,

$$A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I = (A^3 + A^2)(A^3 - 7A^2 + 16A - 12I) + 2A - I$$

$$A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I = (A^3 + A^2)(0) + 2A - I$$

$$A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I = 2A - I$$

$$A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I = 2\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I = \begin{bmatrix} 5 & 20 & 10 \\ -4 & -7 & -8 \\ 6 & 10 & 13 \end{bmatrix}$$

36. Compute $A^9 - 6A^8 + 10A^7 - 3A^6 + A + I$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

[N19/Extc/6M][M23/MechCivil/6M]

Solution:

$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

$|A| = 3$

The characteristic equation, $|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 & 3 \\ -1 & 3-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [1+3+2]\lambda^2 + \left[\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}\right]\lambda - 3 = 0$

$\lambda^3 - 6\lambda^2 + 10\lambda - 3 = 0$

By Cayley Hamilton theorem,

$A^3 - 6A^2 + 10A - 3I = 0$

Dividing $\lambda^9 - 6\lambda^8 + 10\lambda^7 - 3\lambda^6 + \lambda + 1$ by $\lambda^3 - 6\lambda^2 + 10\lambda - 3$

$$\begin{array}{r|l} & \lambda^6 \\ \hline \lambda^3 - 6\lambda^2 + 10\lambda - 3 & \lambda^9 - 6\lambda^8 + 10\lambda^7 - 3\lambda^6 + \lambda + 1 \\ & \lambda^9 - 6\lambda^8 + 10\lambda^7 - 3\lambda^6 \\ & \underline{- \quad + \qquad - \qquad +} \\ & \qquad\qquad\qquad\qquad \lambda + 1 \end{array}$$

Thus,

$A^9 - 6A^8 + 10A^7 - 3A^6 + A + I = (A^6)(A^3 - 6A^2 + 10A - 3I) + A + I$

$A^9 - 6A^8 + 10A^7 - 3A^6 + A + I = (A^6)(0) + A + I$

$A^9 - 6A^8 + 10A^7 - 3A^6 + A + I = A + I$

$A^9 - 6A^8 + 10A^7 - 3A^6 + A + I = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^9 - 6A^8 + 10A^7 - 3A^6 + A + I = \begin{bmatrix} 2 & 2 & 3 \\ -1 & 4 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

37. Verify Cayley Hamilton theorem for matrix A and hence find the matrix represented by $A^5-4A^4-7A^3+11A^2-A-11I$ where $A=\begin{bmatrix}3&-2&3\\10&-3&5\\5&-4&7\end{bmatrix}$

[N19/MechCivil/6M]

Solution:

$A=\begin{bmatrix}3&-2&3\\10&-3&5\\5&-4&7\end{bmatrix}, |A|=12$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}3-\lambda&-2&3\\10&-3-\lambda&5\\5&-4&7-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[3-3+7]\lambda^2+\left[\begin{vmatrix}-3&5\\-4&7\end{vmatrix}+\begin{vmatrix}3&3\\5&7\end{vmatrix}+\begin{vmatrix}3&-2\\10&-3\end{vmatrix}\right]\lambda-12=0$

$\lambda^3-7\lambda^2+16\lambda-12=0$

By Cayley Hamilton theorem,

$A^3-7A^2+16A-12I=0$

Consider,

$A^2=A.A=\begin{bmatrix}3&-2&3\\10&-3&5\\5&-4&7\end{bmatrix}\begin{bmatrix}3&-2&3\\10&-3&5\\5&-4&7\end{bmatrix}=\begin{bmatrix}4&-12&20\\25&-31&50\\10&-25&44\end{bmatrix}$

$A^3=A^2.A=\begin{bmatrix}4&-12&20\\25&-31&50\\10&-25&44\end{bmatrix}\begin{bmatrix}3&-2&3\\10&-3&5\\5&-4&7\end{bmatrix}=\begin{bmatrix}-8&-52&92\\15&-157&270\\-10&-118&208\end{bmatrix}$

L.H.S. $=A^3-7A^2+16A-12I$

$=\begin{bmatrix}-8&-52&92\\15&-157&270\\-10&-118&208\end{bmatrix}-7\begin{bmatrix}4&-12&20\\25&-31&50\\10&-25&44\end{bmatrix}+16\begin{bmatrix}3&-2&3\\10&-3&5\\5&-4&7\end{bmatrix}-12\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ = R.H.S.

Thus, Cayley Hamilton theorem is verified

Dividing $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 11$ by $\lambda^3 - 7\lambda^2 + 16\lambda - 12$

$$
\begin{array}{r|l}
 & \lambda^2 + 3\lambda - 2 \\
\hline
\lambda^3 - 7\lambda^2 + 16\lambda - 12 & \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 11 \\
 & \lambda^5 - 7\lambda^4 + 16\lambda^3 - 12\lambda^2 \\
 & - \quad + \quad - \quad + \\
\hline
 & 3\lambda^4 - 23\lambda^3 + 23\lambda^2 - \lambda - 11 \\
 & 3\lambda^4 - 21\lambda^3 + 48\lambda^2 - 36\lambda \\
 & - \quad + \quad - \quad + \\
\hline
 & -2\lambda^3 - 25\lambda^2 + 35\lambda - 11 \\
 & -2\lambda^3 + 14\lambda^2 - 32\lambda + 24 \\
 & + \quad - \quad + \quad - \\
\hline
 & -39\lambda^2 + 67\lambda - 35
\end{array}
$$

Thus,

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 11I = (A^2 + 3A - 2I)(A^3 - 7A^2 + 16A - 12I) + (-39A^2 + 67A - 35I)$$

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 11I = (A^2 + 3A - 2I)(0) + (-39A^2 + 67A - 35I)$$

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 11I = -39A^2 + 67A - 35I$$

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 11I = -39\begin{bmatrix} 4 & -12 & 20 \\ 25 & -31 & 50 \\ 10 & -25 & 44 \end{bmatrix} + 67\begin{bmatrix} 3 & -2 & 3 \\ 10 & -3 & 5 \\ 5 & -4 & 7 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 11I = \begin{bmatrix} 10 & 334 & -579 \\ -305 & 973 & -1615 \\ -55 & 746 & -1282 \end{bmatrix}$$

38. Using Cayley Hamilton theorem find $A^6 - 12A^5 + 30A^4 + 72A^3 - 207A^2 - 110A + 330I$ where $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

[M23/ElectECSExtc/6M]

Solution:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$|A| = -18$

The characteristic equation, $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 3 & 1 \\ 3 & 1-\lambda & 2 \\ 1 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [2+1+3]\lambda^2 + \left[\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}\right]\lambda + 18 = 0$$

$$\lambda^3 - 6\lambda^2 - 3\lambda + 18 = 0$$

By Cayley Hamilton theorem,

$$A^3 - 6A^2 - 3A + 18I = 0$$

Dividing $\lambda^6 - 12\lambda^5 + 30\lambda^4 + 72\lambda^3 - 207\lambda^2 - 110\lambda + 330$ by $\lambda^3 - 6\lambda^2 - 3\lambda + 18$

$$\lambda^3 - 6\lambda^2 - 3\lambda + 18$$

$$\lambda^3 - 6\lambda^2 - 3\lambda + 18 \,\big|\, \lambda^6 - 12\lambda^5 + 30\lambda^4 + 72\lambda^3 - 207\lambda^2 - 110\lambda + 330$$

$$\lambda^6 - 6\lambda^5 - 3\lambda^4 + 18\lambda^3$$

$$- \quad + \quad + \quad -$$

$$-6\lambda^5 + 33\lambda^4 + 54\lambda^3 - 207\lambda^2 - 110\lambda + 330$$

$$-6\lambda^5 + 36\lambda^4 + 18\lambda^3 - 108\lambda^2$$

$$+ \quad - \quad - \quad +$$

$$-3\lambda^4 + 36\lambda^3 - 99\lambda^2 - 110\lambda + 330$$

$$-3\lambda^4 + 18\lambda^3 + 9\lambda^2 - 54\lambda$$

$$+ \quad - \quad - \quad +$$

$$18\lambda^3 - 108\lambda^2 - 56\lambda + 330$$

$$18\lambda^3 - 108\lambda^2 - 54\lambda + 324$$

$$- \quad + \quad + \quad -$$

$$-2\lambda + 6$$

Thus,

$$A^6 - 12A^5 + 30A^4 + 72A^3 - 207A^2 - 110A + 330I = (A^3 - 6A^2 - 3A + 18I)(A^3 - 6A^2 - 3A + 18I) - 2A + 6I$$

$$A^6 - 12A^5 + 30A^4 + 72A^3 - 207A^2 - 110A + 330I = (A^3 - 6A^2 - 3A + 18I)(0) - 2A + 6I$$

$$A^6 - 12A^5 + 30A^4 + 72A^3 - 207A^2 - 110A + 330I = -2A + 6I$$

$$A^6 - 12A^5 + 30A^4 + 72A^3 - 207A^2 - 110A + 330I = -2\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^6 - 12A^5 + 30A^4 + 72A^3 - 207A^2 - 110A + 330I = \begin{bmatrix} 2 & -6 & -2 \\ -6 & 4 & -4 \\ -2 & -4 & 0 \end{bmatrix}$$

39. If characteristic equation of matrix A of order 3×3 is $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$. Then by Cayley Hamilton theorem A^{-1} is equal to
[M22/MechCivil/2M]
Solution:
Characteristic equation is,
$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$
By Cayley Hamilton theorem,
$A^3 - 3A^2 + 3A - I = 0$
Multiplying by A^{-1}
$A^2 - 3A + 3I - A^{-1} = 0$
$A^{-1} = A^2 - 3A + 3I$

40. If $A = [a_{ij}]$ is a matrix of order 3×3 such that

$$a_{ij} = \begin{cases} 2 & if\ i = j \\ -1 & if\ i + j = 3\ or\ 5 \\ 1 & if\ i + j = 4\ and\ i \neq j \end{cases}$$

Compute $A^9 - 6A^8 - 9A^7 - 4A^6 + A^5 - 12A^4 - 18A^3 - 8A^2 + 2A + I$
[N24/MechCivil/6M]
[Note: Correction is, sign of $9A^7$ must be positive as shown below
$A^9 - 6A^8 + 9A^7 - 4A^6 + A^5 - 12A^4 - 18A^3 - 8A^2 + 2A + I$]
Solution:

$$A = [a_{ij}] \text{ where } a_{ij} = \begin{cases} 2 & if\ i = j \\ -1 & if\ i + j = 3\ or\ 5 \\ 1 & if\ i + j = 4\ and\ i \neq j \end{cases}$$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}, |A| = 4$$

The characteristic equation,
$|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$$\lambda^3 - [2+2+2]\lambda^2 + \left[\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}\right]\lambda - 4 = 0$$

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$
By Cayley Hamilton theorem,
$A^3 - 6A^2 + 9A - 4I = 0$

Dividing $\lambda^9 - 6\lambda^8 + 9\lambda^7 - 4\lambda^6 + \lambda^5 - 12\lambda^4 - 18\lambda^3 - 8\lambda^2 + 2\lambda + 1$ by $\lambda^3 - 6\lambda^2 + 9\lambda - 4$

$$\lambda^6 + \lambda^2 - 6\lambda - 63$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 \,\big|\, \overline{\lambda^9 - 6\lambda^8 + 9\lambda^7 - 4\lambda^6 + \lambda^5 - 12\lambda^4 - 18\lambda^3 - 8\lambda^2 + 2\lambda + 1}$$

$$\lambda^6 - 6\lambda^8 + 9\lambda^7 - 4\lambda^6$$

$$- \quad + \quad - \quad +$$

$$\lambda^5 - 12\lambda^4 - 18\lambda^3 - 8\lambda^2 + 2\lambda + 1$$

$$\lambda^5 - 6\lambda^4 + 9\lambda^3 - 4\lambda^2$$

$$- \quad + \quad - \quad +$$

$$-6\lambda^4 - 27\lambda^3 - 4\lambda^2 + 2\lambda + 1$$

$$-6\lambda^4 + 36\lambda^3 - 54\lambda^2 + 24\lambda$$

$$+ \quad - \quad + \quad -$$

$$-63\lambda^3 + 50\lambda^2 - 22\lambda + 1$$

$$-63\lambda^3 + 378\lambda^2 - 567\lambda + 252$$

$$+ \quad - \quad + \quad -$$

$$-328\lambda^2 + 545\lambda - 251$$

Thus,

$$A^9 - 6A^8 + 9A^7 - 4A^6 + A^5 - 12A^4 - 18A^3 - 8A^2 + 2A + I$$
$$= (A^6 + A^2 - 6A - 63I)(A^3 - 6A^2 + 9A - 4I) - 328A^2 + 545A - 251I$$
$$= (A^6 + A^2 - 6A - 63I)(0) - 328A^2 + 545A - 251I$$
$$= -328A^2 + 545A - 251I$$

Type IV: Functions of a Square matrix

1. Find $A^{20} - 2A^{19} + A$ where $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

Solution:

$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

$|A| = 2 - 0 = 2$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$\lambda^2 - [sum\ of\ diagonals]\lambda + |A| = 0$

$\lambda^2 - [2+1]\lambda + 2 = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda = 2,1$

Method I:

Let $f(A) = \alpha_1 A + \alpha_0 I$

Here, $A^{20} - 2A^{19} + A = \alpha_1 A + \alpha_0 I$

By Cayley Hamilton theorem,

$\lambda^{20} - 2\lambda^{19} + \lambda = \alpha_1 \lambda + \alpha_0$

Put $\lambda = 2$

$2^{20} - 2(2)^{19} + 2 = 2\alpha_1 + \alpha_0$

$2 = 2\alpha_1 + \alpha_0$ ……….. (1)

Put $\lambda = 1$

$1^{20} - 2(1)^{19} + 1 = \alpha_1 + \alpha_0$

$0 = \alpha_1 + \alpha_0$ …………. (2)

Solving (1) & (2), we get

$\alpha_1 = 2, \alpha_0 = -2$

Thus,

$A^{20} - 2A^{19} + A = 2A - 2I$

$= 2\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 6 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 6 \\ 0 & 0 \end{bmatrix}$

Method II (modal matrix method):

(i) For $\lambda = 2$

$[A - \lambda I]X = 0$

$\begin{bmatrix} 0 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$0x_1 + 3x_2 = 0$

$0x_1 = -3x_2$

$\frac{x_1}{-3} = \frac{x_2}{0}$

$X_1 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

(ii) For $\lambda = 1$

$[A - \lambda I]X = 0$

$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$1x_1 + 3x_2 = 0$

$1x_1 = -3x_2$

$\frac{x_1}{-3} = \frac{x_2}{1}$

$X_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Thus, the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} -3 & -3 \\ 0 & 1 \end{bmatrix}$ and $M^{-1} = \frac{1}{|M|}\, adj\, M$

$M^{-1} = \frac{1}{-3}\begin{bmatrix} 1 & 3 \\ 0 & -3 \end{bmatrix}$

Note: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Now,

$f(A) = Mf(D)M^{-1}$

$A^{20} - 2A^{19} + A = M[D^{20} - 2D^{19} + D]M^{-1}$

$$= \begin{bmatrix} -3 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2^{20} - 2(2)^{19} + 2 & 0 \\ 0 & 1^{20} - 2(1)^{19} + 1 \end{bmatrix}\frac{1}{-3}\begin{bmatrix} 1 & 3 \\ 0 & -3 \end{bmatrix}$$

$$= \frac{1}{-3}\begin{bmatrix} -3 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & -3 \end{bmatrix}$$

$$= \frac{1}{-3}\begin{bmatrix} -6 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & -3 \end{bmatrix}$$

$$= \frac{1}{-3}\begin{bmatrix} -6 & -18 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 \\ 0 & 0 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$, find A^{50}

[N14/MechCivil/6M][M16/ElexExtcElectBiomInst/6M][N19/Elect/6M][N19/Inst/5M]

Solution:

$A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 3 \\ -3 & -4-\lambda \end{vmatrix} = 0$

$(2-\lambda)(-4-\lambda) + 9 = 0$

$\lambda^2 + 2\lambda + 1 = 0$

$\lambda = -1, -1$

Let $A^{50} = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$\lambda^{50} = \alpha_1 \lambda + \alpha_0$ ……………(A)

Putting $\lambda = -1$, we get

$1 = -\alpha_1 + \alpha_0$ ………………(1)

Differentiating eqn (A) w.r.t λ, we get

$50\lambda^{49} = \alpha_1$

Putting $\lambda = -1$, we get

$\alpha_1 = -50$

Thus, $\alpha_0 = -49$

Therefore,

$A^{50} = -50A - 49I$

$A^{50} = -50\begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} - 49\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^{50} = \begin{bmatrix} -149 & -150 \\ 150 & 151 \end{bmatrix}$

3. Find A^n where $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$

Solution:

$A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$

$|A| = 15 - 3 = 12$

Characteristic equation,

$\lambda^2 - [sum\ of\ diag]\lambda + |A| = 0$

$\lambda^2 - [5+3]\lambda + 12 = 0$

$\lambda^2 - 8\lambda + 12 = 0$

$\lambda = 6,2$

Let $A^n = \alpha_1 A + \alpha_0 I$

By CH theorem,

$\lambda^n = \alpha_1 \lambda + \alpha_0$

Put $\lambda = 6$

$6^n = 6\alpha_1 + \alpha_0$ …… (1)

Put $\lambda = 2$

$2^n = 2\alpha_1 + \alpha_0$ ……… (2)

Subtracting (1) & (2),

$6^n - 2^n = 4\alpha_1$

$\alpha_1 = \frac{6^n - 2^n}{4}$

From eqn (1),

$6^n = 6\left[\frac{6^n - 2^n}{4}\right] + \alpha_0$

$6^n - 6\left[\frac{6^n - 2^n}{4}\right] = \alpha_0$

$\frac{4.6^n - 6.6^n + 6.2^n}{4} = \alpha_0$

$\alpha_0 = \frac{6.2^n - 2.6^n}{4}$

Thus,

$A^n = \alpha_1 A + \alpha_0 I$

$A^n = \left(\frac{6^n - 2^n}{4}\right)\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix} + \left(\frac{6.2^n - 2.6^n}{4}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^n = \begin{bmatrix} \frac{5.6^n - 5.2^n}{4} & \frac{3.6^n - 3.2^n}{4} \\ \frac{6^n - 2^n}{4} & \frac{3.6^n - 3.2^n}{4} \end{bmatrix} + \begin{bmatrix} \frac{6.2^n - 2.6^n}{4} & 0 \\ 0 & \frac{6.2^n - 2.6^n}{4} \end{bmatrix}$

$A^n = \begin{bmatrix} \frac{3.6^n + 2^n}{4} & \frac{3.6^n - 3.2^n}{4} \\ \frac{6^n - 2^n}{4} & \frac{6^n + 3.2^n}{4} \end{bmatrix}$

4. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find A^{100}

Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$|A| = -1$

The characteristic equation

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$\lambda^3 - [sum\ of\ diag]\lambda^2 + [sum\ of\ minors\ of\ diag]\lambda - |A| = 0$

$\lambda^3 - [1+0+0]\lambda^2 + \left[\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}\right]\lambda - (-1) = 0$

$\lambda^3 - \lambda^2 + [0 - 1 + 0 - 0 + 0 - 0]\lambda + 1 = 0$

$\lambda^3 - \lambda^2 - \lambda + 1 = 0$

$\lambda = 1, 1, -1$

Let $A^{100} = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$

By CH Theorem,

$\lambda^{100} = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ …………………. (A)

Put $\lambda = 1$

$1 = \alpha_2 + \alpha_1 + \alpha_0$ …….. (1)

Put $\lambda = -1$

$1 = \alpha_2 - \alpha_1 + \alpha_0$ …….. (2)

Diff eqn (A) w.r.t λ,

$100\lambda^{99} = \alpha_2(2\lambda) + \alpha_1(1) + 0$

Put $\lambda = 1$

$100 = 2\alpha_2 + \alpha_1$ ………… (3)

Solving (1), (2), (3),

$\alpha_2 = 50, \alpha_1 = 0, \alpha_0 = -49$

Thus,

$A^{100} = 50A^2 + (0)A - 49I$

$$A^{100} = 50\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 49\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{100} = 50\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} 50 & 0 & 0 \\ 50 & 50 & 0 \\ 50 & 0 & 50 \end{bmatrix} - \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$
$$A^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 50 & 1 & 0 \\ 50 & 0 & 1 \end{bmatrix}$$

5. If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, prove that $A^{50} - 5A^{49} = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$
[M16/MechCivil/6M][N16/ElexExtcElectBiomInst/6M]
Solution:
$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$
The characteristic equation,
$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$
$$(1-\lambda)(3-\lambda) - 8 = 0$$
$$\lambda^2 - 4\lambda - 5 = 0$$
$$\lambda = -1, 5$$
Let $A^{50} - 5A^{49} = \alpha_1 A + \alpha_0 I$
By C-H theorem,
$$\lambda^{50} - 5\lambda^{49} = \alpha_1 \lambda + \alpha_0$$
Putting $\lambda = 5$ and $\lambda = -1$, we get
$0 = 5\alpha_1 + \alpha_0$ ………………(1)
$6 = -\alpha_1 + \alpha_0$ …………(2)
Subtracting (1) & (2), we get
$$6\alpha_1 = -6$$
$$\therefore \alpha_1 = -1$$
And,
$$6 = 1 + \alpha_0$$
$$\alpha_0 = 5$$
Thus,
$$A^{50} - 5A^{49} = -A + 5I$$
$$A^{50} - 5A^{49} = -\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$A^{50} - 5A^{49} = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$$

6. If $A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$, find A^{100}

[M18/M19/Biom/6M][M19/Inst/6M][N19/Comp/5M]

Solution:

$A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 3 \\ -3 & -4-\lambda \end{vmatrix} = 0$

$(2-\lambda)(-4-\lambda) + 9 = 0$

$\lambda^2 + 2\lambda + 1 = 0$

$\lambda = -1, -1$

Let $A^{100} = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$\lambda^{100} = \alpha_1 \lambda + \alpha_0$ ……………(A)

Putting $\lambda = -1$, we get

$1 = -\alpha_1 + \alpha_0$ ………………(1)

Differentiating eqn (A) w.r.t λ, we get

$100\lambda^{99} = \alpha_1$

Putting $\lambda = -1$, we get

$\alpha_1 = -100$

Thus, $\alpha_0 = -99$

Therefore,

$A^{100} = -100A - 99I$

$A^{100} = -100 \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} - 99 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^{50} = \begin{bmatrix} -299 & -300 \\ 300 & 301 \end{bmatrix}$

7. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then show that $A^{50} = \frac{1}{2}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$

[M15/ElexExtcElectBiomInst/6M][M15/N17/MechCivil/6M][N16/CompIT/8M]
[N22/Elex/5M]

Solution:

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)(2-\lambda) - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda = 1, 3$

Let $A^{50} = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$\lambda^{50} = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = 3$ and $\lambda = 1$, we get

$3^{50} = 3\alpha_1 + \alpha_0$ ………………(1)

$1 = \alpha_1 + \alpha_0$ …………(2)

Subtracting (1) & (2), we get

$2\alpha_1 = 3^{50} - 1$

$\therefore \alpha_1 = \frac{3^{50}-1}{2}$

And,

$1 = \frac{3^{50}-1}{2} + \alpha_0$

$1 - \frac{3^{50}-1}{2} = \alpha_0$

$\alpha_0 = \frac{3-3^{50}}{2}$

Thus,

$A^{50} = \frac{3^{50}-1}{2} A + \frac{3-3^{50}}{2} I$

$A^{50} = \frac{3^{50}-1}{2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{3-3^{50}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^{50} = \frac{1}{2}\begin{bmatrix} 2.3^{50}-2 & 3^{50}-1 \\ 3^{50}-1 & 2.3^{50}-2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3-3^{50} & 0 \\ 0 & 3-3^{50} \end{bmatrix}$

$A^{50} = \frac{1}{2}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$

8. If $A=\begin{bmatrix}1 & 4\\ 1 & 1\end{bmatrix}$ find A^7-9A^2+I

[N15/M16/ChemBiot/5M]

Solution:

$A=\begin{bmatrix}1 & 4\\ 1 & 1\end{bmatrix}$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}1-\lambda & 4\\ 1 & 1-\lambda\end{vmatrix}=0$

$(1-\lambda)(1-\lambda)-4=0$

$\lambda^2-2\lambda-3=0$

$\lambda=-1,3$

Let $A^7-9A^2+I=\alpha_1 A+\alpha_0 I$

By C-H theorem,

$\lambda^7-9\lambda^2+1=\alpha_1\lambda+\alpha_0$

Putting $\lambda=3$ and $\lambda=-1$, we get

$2107=3\alpha_1+\alpha_0$ ………………(1)

$-9=-\alpha_1+\alpha_0$ …………(2)

Subtracting (1) & (2), we get

$4\alpha_1=2116$

$\therefore\ \alpha_1=529$

And,

$-9=-529+\alpha_0$

$\alpha_0=520$

Thus,

$A^7-9A^2+I=529A+520I$

$A^7-9A^2+I=529\begin{bmatrix}1 & 4\\ 1 & 1\end{bmatrix}+520\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$A^7-9A^2+I=\begin{bmatrix}1049 & 2116\\ 529 & 1049\end{bmatrix}$

9. Find the characteristic equation of $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ hence find $A^7 + 31A^2 + I$

[N22/MTRX/6M]

Solution:

Solution:

$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)(1-\lambda) - 4 = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$\lambda = -1, 3$

Let $A^7 + 31A^2 + I = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$\lambda^7 + 31\lambda^2 + 1 = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = 3$ and $\lambda = -1$, we get

$2467 = 3\alpha_1 + \alpha_0$ ………………(1)

$31 = -\alpha_1 + \alpha_0$ …………(2)

Subtracting (1) & (2), we get

$4\alpha_1 = 2436$

$\therefore \alpha_1 = 609$

And,

$31 = -609 + \alpha_0$

$\alpha_0 = 640$

Thus,

$A^7 + 31A^2 + I = 609A + 640I$

$A^7 + 31A^2 + I = 609\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} + 640\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^7 + 31A^2 + I = \begin{bmatrix} 1249 & 2436 \\ 609 & 1249 \end{bmatrix}$

10. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find A^{50}

[M18/Elex/6M][N18/Extc/6M]

Solution:

$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, |A| = -1$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [1+0+0]\lambda^2 + \left[\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}\right]\lambda - (-1) = 0$

$\lambda^3 - \lambda^2 - \lambda + 1 = 0$

$\lambda = -1,1,1$

Let $A^{50} = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$

By C-H theorem,

$\lambda^{50} = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ …….. (1)

Putting $\lambda = -1$ in eqn (1), we get

$1 = \alpha_2 - \alpha_1 + \alpha_0$ ………. (A)

Putting $\lambda = 1$ in eqn (1), we get

$1 = \alpha_2 + \alpha_1 + \alpha_0$ ……… (B)

Diff eqn (1) w.r.t λ, we get

$50\lambda^{49} = 2\alpha_2\lambda + \alpha_1$ ……. (2)

Putting $\lambda = -1$ in eqn (2), we get

$-50 = -2\alpha_2 + \alpha_1$ ……. (C)

Putting $\lambda = 1$ in eqn (2), we get

$50 = 2\alpha_2 + \alpha_1$ …….. (D)

Solving (A), (B), (C) & (D), we get

$\alpha_2 = 25, \alpha_1 = 0, \alpha_0 = -24$

Thus, $A^{50} = 25A^2 - 24I$

$A^{50} = 25 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^{50} = 25 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^{50} = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

11. Find e^A& 4^A if $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$ with the help of modal matrix

[N15/ElexExtcElectBiomInst/8M]

Solution:

$$A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} \frac{3}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{3}{2} - \lambda\right)\left(\frac{3}{2} - \lambda\right) - \frac{1}{4} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda = 1, 2$$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{2} + \frac{x_2}{2} = 0$$

$$x_1 = -x_2$$

$$\frac{x_1}{-1} = \frac{x_2}{1}$$

Thus, eigen vectors for $\lambda = 1$ is $[-1,1]'$

(ii) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\frac{x_1}{2} + \frac{x_2}{2} = 0$$

$$x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{1}$$

Thus, eigen vectors for $\lambda = 1$ is $[1,1]'$

Therefore, the matrix $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

And, $M^{-1} = \frac{1}{|M|} adjM = \frac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

Now, $f(A) = Mf(D)M^{-1}$

$e^A = Me^D M^{-1}$

$e^A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$e^A = \frac{1}{2}\begin{bmatrix} -e & e^2 \\ e & e^2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$e^A = \frac{1}{2}\begin{bmatrix} e + e^2 & -e + e^2 \\ -e + e^2 & e + e^2 \end{bmatrix}$

Now, $4^A = \frac{1}{2}\begin{bmatrix} 4 + 4^2 & -4 + 4^2 \\ -4 + 4^2 & 4 + 4^2 \end{bmatrix}$

$4^A = \frac{1}{2}\begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix}$

$4^A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$

12. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, find e^{At}

[M19/MTRX/8M]

Solution:

We have,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{vmatrix} = 0$$

$$(-\lambda)(-\lambda) + 1 = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = i, -i$$

Let $e^{At} = \alpha_1 A + \alpha_0 I$

By C-H theorem, $e^{\lambda t} = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = i$ and $\lambda = -i$, we get

$e^{it} = i\alpha_1 + \alpha_0$ ………………(1)

$e^{-it} = -i\alpha_1 + \alpha_0$ …………(2)

Adding (1) & (2), we get

$$2\alpha_0 = e^{it} + e^{-it}$$

$$\alpha_0 = \frac{e^{it} + e^{-it}}{2} = \cos t$$

Subtracting (1) & (2), we get

$$2\alpha_1 = e^{it} - e^{-it}$$

$$\alpha_1 = \frac{e^{it} - e^{-it}}{2i} = \sin t$$

Thus,

$$e^{At} = \sin t\, A + \cos t\, I$$

$$e^{At} = \sin t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \cos t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

13. If $\beta = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix}$, then show that $e^{\beta} = e^{\alpha}\begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix}$

[N22/Chem/6M]

Solution:

$\beta = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix}$

$|\beta| = \alpha^2 - \alpha^2 = 0$

Characteristic equation,

$|\beta - \lambda I| = 0$

$\begin{vmatrix} \alpha-\lambda & \alpha \\ \alpha & \alpha-\lambda \end{vmatrix} = 0$

$\lambda^2 - [sum\ of\ diag]\lambda + |\beta| = 0$

$\lambda^2 - [\alpha+\alpha]\lambda + 0 = 0$

$\lambda^2 - 2\alpha\lambda = 0$

$\lambda(\lambda - 2\alpha) = 0$

$\lambda = 0, \lambda = 2\alpha$

Let

$e^{\beta} = \alpha_1\beta + \alpha_0 I$

By CH theorem,

$e^{\lambda} = \alpha_1\lambda + \alpha_0$

Put $\lambda = 0$

$e^0 = \alpha_1(0) + \alpha_0$

$\alpha_0 = 1$

Put $\lambda = 2\alpha$

$e^{2\alpha} = \alpha_1(2\alpha) + \alpha_0$

$e^{2\alpha} = 2\alpha_1\alpha + 1$

$e^{2\alpha} - 1 = 2\alpha_1\alpha$

$\frac{e^{2\alpha}-1}{2\alpha} = \alpha_1$

Thus,

$e^{\beta} = \left(\frac{e^{2\alpha}-1}{2\alpha}\right)\beta + 1.I$

$e^{\beta} = \left(\frac{e^{2\alpha}-1}{2\alpha}\right)\begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$e^{\beta} = \begin{bmatrix} \left(\frac{e^{2\alpha}-1}{2}\right) & \left(\frac{e^{2\alpha}-1}{2}\right) \\ \left(\frac{e^{2\alpha}-1}{2}\right) & \left(\frac{e^{2\alpha}-1}{2}\right) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$e^{\beta} = \begin{bmatrix} \frac{e^{2\alpha}-1}{2} + 1 & \frac{e^{2\alpha}-1}{2} + 0 \\ \frac{e^{2\alpha}-1}{2} + 0 & \frac{e^{2\alpha}-1}{2} + 1 \end{bmatrix}$

$$e^{\beta} = \begin{bmatrix} \frac{e^{2\alpha}+1}{2} & \frac{e^{2\alpha}-1}{2} \\ \frac{e^{2\alpha}-1}{2} & \frac{e^{2\alpha}+1}{2} \end{bmatrix}$$

$$e^{\beta} = e^{\alpha} \begin{bmatrix} \frac{e^{\alpha}+e^{-\alpha}}{2} & \frac{e^{\alpha}-e^{-\alpha}}{2} \\ \frac{e^{\alpha}-e^{-\alpha}}{2} & \frac{e^{\alpha}+e^{-\alpha}}{2} \end{bmatrix}$$

$$e^{\beta} = e^{\alpha} \begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix}$$

14. Find 5^A where $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

[N17/ElexExtcElectBiomInst/6M]

Solution:

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$(3-\lambda)(3-\lambda) - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$\lambda = 2, 4$

Let $5^A = \alpha_1 A + \alpha_0 I$

By C-H theorem, $5^\lambda = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = 2$ and $\lambda = 4$, we get

$5^2 = 2\alpha_1 + \alpha_0$ ………………(1)

$5^4 = 4\alpha_1 + \alpha_0$ …………(2)

Solving (1) & (2), we get

$\alpha_1 = 300, \alpha_0 = -575$

Thus,

$5^A = 300A - 575I$

$5^A = 300\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - 575\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$5^A = \begin{bmatrix} 325 & 300 \\ 300 & 325 \end{bmatrix}$

15. Find e^A& 4^A if $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$

[N15/MechCivil/8M][M18/Inst/6M][N18/M19/MechCivil/6M][N18/Biom/6M][N19/Elex/6M]

Solution:

$A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} \frac{3}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} - \lambda \end{vmatrix} = 0$

$\left(\frac{3}{2} - \lambda\right)\left(\frac{3}{2} - \lambda\right) - \frac{1}{4} = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda = 1, 2$

Let $e^A = \alpha_1 A + \alpha_0 I$

By C-H theorem, $e^\lambda = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = 2$ and $\lambda = 1$, we get

$e^2 = 2\alpha_1 + \alpha_0$ ………………(1)

$e = \alpha_1 + \alpha_0$ …………(2)

Subtracting (1) & (2), we get

$\alpha_1 = e^2 - e$

And, $e = e^2 - e + \alpha_0$

$\alpha_0 = 2e - e^2$

Thus,

$e^A = (e^2 - e)A + (2e - e^2)I$

$e^A = (e^2 - e)\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} + (2e - e^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$e^A = \frac{1}{2}\begin{bmatrix} 3e^2 - e & e^2 - e \\ e^2 - e & 3e^2 - e \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4e - 2e^2 & 0 \\ 0 & 4e - 2e^2 \end{bmatrix}$

$e^A = \frac{1}{2}\begin{bmatrix} e + e^2 & -e + e^2 \\ -e + e^2 & e + e^2 \end{bmatrix}$

Putting $e = 4$, we get $4^A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$

16. Find e^A if $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$

[N19/Extc/6M]

Solution:

$A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} \frac{3}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} - \lambda \end{vmatrix} = 0$

$\left(\frac{3}{2} - \lambda\right)\left(\frac{3}{2} - \lambda\right) - \frac{1}{4} = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda = 1, 2$

Let $e^A = \alpha_1 A + \alpha_0 I$

By C-H theorem, $e^\lambda = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = 2$ and $\lambda = 1$, we get

$e^2 = 2\alpha_1 + \alpha_0$ ………………(1)

$e = \alpha_1 + \alpha_0$ …………(2)

Subtracting (1) & (2), we get

$\alpha_1 = e^2 - e$

And, $e = e^2 - e + \alpha_0$

$\alpha_0 = 2e - e^2$

Thus,

$e^A = (e^2 - e)A + (2e - e^2)I$

$e^A = (e^2 - e)\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} + (2e - e^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$e^A = \frac{1}{2}\begin{bmatrix} 3e^2 - e & e^2 - e \\ e^2 - e & 3e^2 - e \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4e - 2e^2 & 0 \\ 0 & 4e - 2e^2 \end{bmatrix}$

$e^A = \frac{1}{2}\begin{bmatrix} e + e^2 & -e + e^2 \\ -e + e^2 & e + e^2 \end{bmatrix}$

17. Calculate e^A and 5^A if $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

[M22/MechCivil/5M]

Solution:

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$(3-\lambda)(3-\lambda) - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$\lambda = 2, 4$

Let $e^A = \alpha_1 A + \alpha_0 I$

By C-H theorem, $e^\lambda = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = 2$ and $\lambda = 4$, we get

$e^2 = 2\alpha_1 + \alpha_0$ ………………(1)

$e^4 = 4\alpha_1 + \alpha_0$ …………(2)

Subtracting (1) & (2), we get

$2\alpha_1 = e^4 - e^2$

$\alpha_1 = \frac{e^4 - e^2}{2}$

And, $e^2 = e^4 - e^2 + \alpha_0$

$\alpha_0 = 2e^2 - e^4$

Thus,

$e^A = \left(\frac{e^4 - e^2}{2}\right) A + (2e^2 - e^4) I$

$e^A = \left(\frac{e^4 - e^2}{2}\right) \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} + (2e^2 - e^4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$e^A = \frac{1}{2} \begin{bmatrix} 3e^4 - 3e^2 & e^4 - e^2 \\ e^4 - e^2 & 3e^4 - 3e^2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2e^2 - e^4 & 0 \\ 0 & 2e^2 - e^4 \end{bmatrix}$

$e^A = \frac{1}{2} \begin{bmatrix} e^4 + e^2 & e^4 - e^2 \\ e^4 - e^2 & e^4 + e^2 \end{bmatrix}$

Thus,

$5^A = \frac{1}{2} \begin{bmatrix} 5^4 + 5^2 & 5^4 - 5^2 \\ 5^4 - 5^2 & 5^4 + 5^2 \end{bmatrix} = \begin{bmatrix} 325 & 300 \\ 300 & 325 \end{bmatrix}$

18. If $A = \begin{bmatrix} \pi & \frac{\pi}{4} \\ 0 & \frac{\pi}{2} \end{bmatrix}$, find $\cos A$

[N18/Elex/5M][N19/MTRX/8M]

Solution:

$A = \begin{bmatrix} \pi & \frac{\pi}{4} \\ 0 & \frac{\pi}{2} \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} \pi - \lambda & \frac{\pi}{4} \\ 0 & \frac{\pi}{2} - \lambda \end{vmatrix} = 0$

$\left(\frac{\pi}{2} - \lambda\right)(\pi - \lambda) - 0 = 0$

$\lambda = \pi, \frac{\pi}{2}$

Let $\cos A = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$\cos \lambda = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = \pi$ and $\lambda = \frac{\pi}{2}$, we get

$-1 = \pi\alpha_1 + \alpha_0$ ………………(1)

$0 = \frac{\pi}{2}\alpha_1 + \alpha_0$ …………(2)

Subtracting (1) & (2), we get

$\frac{\pi}{2}\alpha_1 = -1$

$\therefore \alpha_1 = -\frac{2}{\pi}$

Thus, $\alpha_0 = 1$

$\cos A = -\frac{2}{\pi} A + I$

$\cos A = -\frac{2}{\pi}\begin{bmatrix} \pi & \frac{\pi}{4} \\ 0 & \frac{\pi}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\cos A = \begin{bmatrix} -2 & -\frac{1}{2} \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\cos A = \begin{bmatrix} -1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$

19. If $A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$, then find $\cos^{-1} A$

Solution:

$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$

$|A| = 0.25 - 0.25 = 0$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 0.5-\lambda & 0.5 \\ 0.5 & 0.5-\lambda \end{vmatrix} = 0$

$\lambda^2 - [sum\ of\ diag]\lambda + |A| = 0$

$\lambda^2 - [0.5 + 0.5]\lambda + 0 = 0$

$\lambda^2 - \lambda = 0$

$\lambda(\lambda - 1) = 0$

$\lambda = 0, \lambda = 1$

Let

$\cos^{-1} A = \alpha_1 A + \alpha_0 I$

By CH theorem,

$\cos^{-1} \lambda = \alpha_1 \lambda + \alpha_0$

Put $\lambda = 0$

$\cos^{-1} 0 = \alpha_1(0) + \alpha_0$

$\frac{\pi}{2} = \alpha_0$

Put $\lambda = 1$

$\cos^{-1} 1 = \alpha_1(1) + \alpha_0$

$0 = \alpha_1 + \frac{\pi}{2}$

$\alpha_1 = -\frac{\pi}{2}$

Thus,

$\cos^{-1} A = -\frac{\pi}{2} A + \frac{\pi}{2} I$

$\cos^{-1} A = -\frac{\pi}{2}\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} + \frac{\pi}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\cos^{-1} A = \begin{bmatrix} -\frac{\pi}{4} & -\frac{\pi}{4} \\ -\frac{\pi}{4} & -\frac{\pi}{4} \end{bmatrix} + \begin{bmatrix} \frac{\pi}{2} & 0 \\ 0 & \frac{\pi}{2} \end{bmatrix}$

$\cos^{-1} A = \begin{bmatrix} \frac{\pi}{4} & -\frac{\pi}{4} \\ -\frac{\pi}{4} & \frac{\pi}{4} \end{bmatrix}$

20. If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ then prove that: $3\tan A = A\tan 3$

[M14/MechCivil/6M][M17/ElexExtcElectBiomInst/6M][M18/Elect/6M][M19/Elex/6M][M19/Extc/6M][M22/Chem/5M][M22/ElectECSExtc/5M][M22/Elex/5M]

Solution:

$A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$(-1-\lambda)(1-\lambda) - 8 = 0$

$\lambda^2 - 9 = 0$

$\lambda = 3, -3$

Let $3\tan A = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$3\tan\lambda = \alpha_1\lambda + \alpha_0$

Putting $\lambda = 3$ and $\lambda = -3$, we get

$3\tan 3 = 3\alpha_1 + \alpha_0$ ………………(1)

$-3\tan 3 = -3\alpha_1 + \alpha_0$ …………(2)

Adding (1) & (2), we get

$2\alpha_0 = 0$

$\therefore \alpha_0 = 0$

Thus, $\alpha_1 = \tan 3$

Therefore, $\boxed{3\tan A = A\tan 3}$

21. If $A = \begin{bmatrix} \frac{\pi}{2} & 3\frac{\pi}{2} \\ \pi & \pi \end{bmatrix}$, find $\sin A$

[M14/ElexExtcElectBiomInst/4M]

Solution:

$A = \begin{bmatrix} \frac{\pi}{2} & 3\frac{\pi}{2} \\ \pi & \pi \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} \frac{\pi}{2} - \lambda & \frac{3\pi}{2} \\ \pi & \pi - \lambda \end{vmatrix} = 0$

$\left(\frac{\pi}{2} - \lambda\right)(\pi - \lambda) - \frac{3\pi^2}{2} = 0$

$\lambda^2 - \frac{3\pi}{2}\lambda - \pi^2 = 0$

$\lambda = 2\pi, -\frac{\pi}{2}$

Let $\sin A = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$\sin \lambda = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = 2\pi$ and $\lambda = -\frac{\pi}{2}$, we get

$0 = 2\pi\alpha_1 + \alpha_0$ ………………(1)

$-1 = -\frac{\pi}{2}\alpha_1 + \alpha_0$ …………(2)

Subtracting (1) & (2), we get

$\frac{5\pi}{2}\alpha_1 = 1$

$\therefore \alpha_1 = \frac{2}{5\pi}$ and $\alpha_0 = -\frac{4}{5}$

Therefore,

$\sin A = \frac{2}{5\pi} A - \frac{4}{5} I$

$\sin A = \frac{2}{5\pi}\begin{bmatrix} \frac{\pi}{2} & 3\frac{\pi}{2} \\ \pi & \pi \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\sin A = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{2}{5} \end{bmatrix} - \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & \frac{4}{5} \end{bmatrix}$

$\sin A = \begin{bmatrix} -\frac{3}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{2}{5} \end{bmatrix}$

22. If $A = \begin{bmatrix} \frac{\pi}{2} & \pi \\ 0 & \frac{3\pi}{2} \end{bmatrix}$, prove that $\sin A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

[M18/Comp/5M][N18/Elect/6M][N22/MechCivil/6M]

Solution:

$A = \begin{bmatrix} \frac{\pi}{2} & \pi \\ 0 & \frac{3\pi}{2} \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} \frac{\pi}{2} - \lambda & \pi \\ 0 & \frac{3\pi}{2} - \lambda \end{vmatrix} = 0$

$\left(\frac{\pi}{2} - \lambda\right)\left(\frac{3\pi}{2} - \lambda\right) - 0 = 0$

$\lambda = \frac{\pi}{2}, \frac{3\pi}{2}$

Let $\sin A = \alpha_1 A + \alpha_0 I$

By C-H theorem,

$\sin \lambda = \alpha_1 \lambda + \alpha_0$

Putting $\lambda = \frac{\pi}{2}$ and $\lambda = \frac{3\pi}{2}$, we get

$1 = \frac{\pi}{2}\alpha_1 + \alpha_0$ ………………(1)

$-1 = \frac{3\pi}{2}\alpha_1 + \alpha_0$ …………(2)

Subtracting (1) & (2), we get

$-\pi\alpha_1 = 2$

$\therefore \alpha_1 = -\frac{2}{\pi}$

Thus, $\alpha_0 = 2$

Therefore,

$\sin A = -\frac{2}{\pi}A + 2I$

$\sin A = -\frac{2}{\pi}\begin{bmatrix} \frac{\pi}{2} & \pi \\ 0 & \frac{3\pi}{2} \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\sin A = \begin{bmatrix} -1 & -2 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\sin A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$