NOV-2023

Paper / Subject Code: 50921 / Engineering Mathematics-III

S.E. (IT) (Sem-III) (CBCGS) (R-19) (C Scheme) (IT & Comps)

(Time: 3 Hours) **Max. Marks: 80**

N.B. (1) Question No. 1 is compulsory.
(2) Answer any three questions from Q.2 to Q.6.
(3) Use of Statistical Tables permitted.
(4) Figures to the right indicate full marks

Q1.

(a) Find the Laplace transform of $\frac{\cos 2t \sin t}{e^{t}}$ [5]

(b) Find k such that $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{kx}{y}$ is analytic [5]

(c) Calculate the Spearman's rank correlation coefficient R [5]

X : 10, 12, 18, 18, 15, 40.
Y : 12, 18, 25, 25, 50, 25.

(d) Find the inverse Laplace transform of $\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$. [5]

Q2.

(a) A continuous random variable has probability density function

$$f(x) = k(x - x^2), \quad 0 \le x \le 1.$$
$$f(x) = 0 \quad \text{otherwise}$$

Find k, mean and variance. [6]

(b) Find the Laplace transform of $e^{-3t}\int_0^t u \sin 3u\, du$. [6]

(c) Obtain the Fourier series to represent $f(x) = x^2$ in $(0, 2\pi)$

Hence show that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}$ .... [8]

Q3.

(a) If the imaginary part of the analytic function $w = u + iv = f(z)$ is [6]

$V = x^2 - y^2 + \frac{x}{x^2+y^2}$, then show that $u = -2xy + \frac{y}{x^2+y^2}$.

(b) Find inverse Laplace transform of $\frac{2s^2-6s+5}{(s^3-6s^2+11s-6)}$ [6]

(c) Fit a second-degree parabolic curve and estimate y when x = 10

x : 1, 2, 3, 4, 5, 6, 7, 8, 9,
y : 2, 6, 7, 8, 10, 11, 11, 10, 9. [8]

Q4.

(a) Obtain the Fourier series to represent $f(x) = x^3$ in $(-\pi, \pi)$. [6]

(b) Find (i) the equation of the lines of Regression (ii) coefficient of correlation for the following data

X : 65, 66, 67, 67, 68, 69, 70, 72.
Y : 67, 68, 65, 66, 72, 72, 69, 71. [6]

(c) Prove that $\int_0^\infty e^{-\sqrt{2}t}\frac{\sin t \sinh t}{t}\, dt = \frac{\pi}{8}$. [8]

Q5.

(a) Find the orthogonal trajectories of the family of curves $x^3y - xy^3 = c$. [6]

(b) Find the moment generating function of the distribution

X	:	-2	3	1
P(X = x)	:	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

hence find first four central moments . [6]

(c) Obtain the half range cosine series of $f(x) = x$ in $(0, 2)$

Hence show that $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4}$ .... [8]

Q6.(a) Using convolution theorem Find the inverse Laplace transform of $\left[\frac{s^2}{(s^2+2^2)^2}\right]$ [6]

(b) The probability density function of a random variable X is

X	:	1	2	3	4	5	6	7
P (X =x)	:	k	2k	3k	k^2	$k^2 + k$	$2k^2$	$4k^2$

Find k , $p(X < 5)$, $P(X > 5)$ [6]

(c) If $v = 3x^2y + 6xy - y^3$,show that v is harmonic function

And find the corresponding analytic function . [8]

~~Q1)a)~~ EM III (Q.P. 37858) ESE SOLUTION (NOV-2023)
(Comp & IT)

Q1)a) To find: $L\left\{\frac{\cos 2t \sin t}{e^t}\right\}$

Consider $L\{\cos 2t \sin t\} = \frac{1}{2} L(\sin 3t - \sin t)$

$$= \frac{1}{2}\left(\frac{3}{s^2+9} - \frac{1}{s^2+1}\right) = F(s)$$

By first shifting property,

$$L\left\{\frac{\cos 2t \sin t}{e^t}\right\} = F(s+1)$$

$$= \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+2s+10} - \frac{1}{s^2+2s+2}\right]$$

Q1) b) Given $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{kx}{y}\right)$
$= u+iv$

$\therefore u = \frac{1}{2}\log(x^2+y^2)$ & $v = \tan^{-1}\left(\frac{kx}{y}\right)$

As $f(z) = u+iv$ is analytic

$\therefore$ Cauchy Riemann Equations are satisfied.

$\therefore u_x = v_y$ & $u_y = -v_x$

$$\therefore \frac{1}{2}\times\frac{1}{x^2+y^2}\times 2x = \frac{1}{1+\frac{k^2x^2}{y^2}}\times -\frac{kx}{y^2} \quad \& \quad \frac{1}{2}\times\frac{1}{x^2+y^2}\times 2y = \frac{-1}{1+\frac{k^2x^2}{y^2}}\times\frac{k}{y}$$

$$\therefore \frac{x}{x^2+y^2} = \frac{-kx}{k^2x^2+y^2} \quad \& \quad \frac{y}{x^2+y^2} = \frac{-ky}{k^2x^2+y^2}$$

$$\therefore k^2x^3+xy^2 = -kx^3-kxy^2 \quad \& \quad k^2x^2y+y^3 = -ky^3-kx^2y$$

Comparing the coefficients

$k^2=-k$ & $-k=1$ $\qquad$ $k^2=-k$ & $-k=1$

$\therefore k^2+k=0$ & $k=-1$

$k(k+1)=0$

$k=0$ or $k=-1$

$$\boxed{\therefore k=-1}$$

Q1) c) Given

X	10	12	18	18	15	40
Y	12	18	25	25	50	25

Spearman's Rank correlation co-efficient is given by

$$(R \text{ or } \rho) = 1 - \frac{6[\Sigma d^2 + \Sigma C.F]}{n(n^2-1)}$$

X	Y	r_x	r_y	$d = r_x - r_y$	d^2
10	12	1	1	0	0
12	18	2	2	0	0
18	25	4.5	4	0.5	0.25
18	25	4.5	4	0.5	0.25
15	50	3	6	-3	9
40	25	6	4	2	4

$$\Sigma d^2 = 13.5$$

correction factor (C.F)

Series	Rank	Repetition (m)	correction factor $C.F. = \frac{m(m^2-1)}{12}$
X	4.5	2	$C.F = \frac{2(4-1)}{12} = 0.5$
Y	4	3	$C.F = \frac{3(9-1)}{12} = 2$

$$\therefore \Sigma C.F = 2.5$$

$$\therefore \rho = 1 - \frac{6[13.5 + 2.5]}{6(6^2-1)}$$

$$= 1 - \frac{\cancel{6} \times 16}{\cancel{6} \times 35}$$

$$\therefore \boxed{\rho = 0.5428}$$

Q1)d)

$$L^{-1}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$$

$$= L^{-1}\left\{\log(s^2+a^2) - \log(s^2+b^2)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]\right\}$$ {By derivative Rule}

$$= -\frac{1}{t} L^{-1}\left\{\frac{1}{s^2+a^2} \times 2s - \frac{1}{s^2+b^2} \times 2s\right\}$$

$$= -\frac{1}{t} \times 2 \times L^{-1}\left\{\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right\}$$

$$= -\frac{2}{t}\left\{L^{-1}\left(\frac{s}{s^2+a^2}\right) - L^{-1}\left(\frac{s}{s^2+b^2}\right)\right\}$$

$$= -\frac{2}{t}\left[\cos at - \cos bt\right]$$

Q2) a) $f(x) = \begin{cases} k(x-x^2), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

is probability density function

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^1 k(x-x^2)\,dx = 1$

$\Rightarrow k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$

$\Rightarrow k\left(\frac{1}{2} - \frac{1}{3}\right) = 1$

$\Rightarrow k/6 = 1 \quad \boxed{\therefore k = 6}$

Mean $= E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x \times 6(x-x^2)\,dx$

$= 6\int_0^1 x^2 - x^3\,dx$

$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left[\frac{1}{3} - \frac{1}{4} - 0\right]$

$\boxed{E(X) = 0.5}$

Variance $= E(X^2) - [E(X)]^2$ — ①

Now $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2 \times 6(x-x^2)\,dx$

$= 6\int_0^1 x^3 - x^4\,dx$

$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$

$= 6\left[\frac{1}{4} - \frac{1}{5} - 0\right]$

$= \frac{3}{10}$

$\therefore \boxed{E(X^2) = 0.3}$

$\therefore$ from ①, $V(X) = 0.3 - (0.5)^2$

$\boxed{V(X) = 0.75}$

Q Q2) b) To find: $L\left\{e^{-3t}\int_0^t u\sin 3u\,du\right\}$

Consider $L(\sin 3u) = \dfrac{3}{s^2+9} = F_1(s)$

By Multiplication by t^n,

$$L(u\sin 3u) = (-1)^1 \frac{d}{ds}F_1(s) = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= -3\left[\frac{-1}{(s^2+9)^2}\times 2s\right]$$

$$\therefore L(u\sin 3u) = \frac{6s}{(s^2+9)^2} = F_2(s)$$

By Laplace Transform of Integrals,

$$L\left(\int_0^t u\sin 3u\,du\right) = \frac{1}{s}F_2(s) = \frac{1}{s}\times\frac{6s}{(s^2+9)^2}$$

$$\therefore L\left(\int_0^t u\sin 3u\,du\right) = \frac{6}{(s^2+9)^2} = F_3(s)$$

By first shifting property of L·T,

$$L\left(e^{-t}\int_0^t u\sin 3u\,du\right) = F_3(s+1)$$

$$= \frac{6}{((s+1)^2+9)^2}$$

$$= \frac{6}{(s^2+2s+10)^2}$$ //

Q2) c) To find Fourier Series of $f(x)=x^2$ in $(0,2\pi)$

Comparing $(0,2\pi)$ with $(c, c+2l)$

$c=0$ & $c+2l=2\pi$ $\boxed{\therefore l=\pi}$

$\therefore$ Fourier series is given by,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))$$

where, $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}x^2\,dx$

$$=\frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{2\pi}=\frac{1}{3\pi}(8\pi^3-0)$$

$$\therefore \boxed{a_0=\frac{8}{3}\pi^2}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos(nx)\,dx=\frac{1}{\pi}\int_0^{2\pi}x^2\cos(nx)\,dx$$

$$=\frac{1}{\pi}\left[(x^2)\left(\frac{\sin(nx)}{n}\right)-(2x)\left(\frac{-\cos(nx)}{n^2}\right)+(2)\left(-\frac{\sin(nx)}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[\frac{1}{n}(0-0)+\frac{2}{n^2}\left(2\pi(-1)^{2n}-0\right)-\frac{2}{n^3}(0-0)\right]$$

$$=\frac{1}{\cancel{\pi}}\times\frac{4\cancel{\pi}}{n^2}\times(1)=\frac{4}{n^2}$$

$$\boxed{\therefore a_n=\frac{4}{n^2}}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin(nx)\,dx=\frac{1}{\pi}\int_0^{2\pi}x^2\sin(nx)\,dx$$

$$=\frac{1}{\pi}\left[(x^2)\left(\frac{-\cos(nx)}{n}\right)-(2x)\left(-\frac{\sin(nx)}{n^2}\right)+(2)\left(\frac{+\cos(nx)}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[-\frac{1}{n}\left(4\pi^2(-1)^{2n}-0\right)+\frac{2}{n^2}(0-0)+\frac{2}{n^3}\left((-1)^{2n}-1\right)\right]$$

$$=\frac{1}{\pi}\left[-\frac{4\pi^2}{n}+\frac{2}{n^3}(1-1)\right]=-\frac{4\pi}{n}$$

$$\boxed{\therefore b_n=\frac{-4\pi}{n}}$$

$\therefore$ Fourier series is,

$$x^2=\frac{4}{3}\pi^2+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos(nx)+\left(-\frac{4\pi}{n}\right)\sin(nx)\right).$$

Now to prove that,

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

put $x = \pi$ in fourier series

$$\therefore \pi^2 = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos(n\pi) - \frac{4\pi}{n}\sin(n\pi)\right)$$

$$\pi^2 - \frac{4}{3}\pi^2 = \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n - 0 \qquad \{\text{As } \sin(n\pi) = 0\}$$

$$\frac{-\pi^2}{3} = \frac{4}{1^2}(-1) + \frac{4}{2^2}(1) + \frac{4}{3^2}(-1) + \frac{4}{4^2}(1) + \cdots$$

$$\therefore +\frac{\pi^2}{3} = 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$$

{Multiply by -1}

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

23) a) $f(z) = u + iv$

where $v = x^2 - y^2 + \frac{x}{x^2+y^2}$

To show that, $u = -2xy + \frac{y}{x^2+y^2}$

we know that,

$$f'(z) = v_y + i v_x \quad \text{—} \ (1)$$

$\therefore v_y = -2y + x \times \frac{-1}{(x^2+y^2)^2} \times 2y$

& $v_x = 2x + \frac{(x^2+y^2)(1) - (x)2x}{(x^2+y^2)^2}$

$\therefore v_x = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2}$

$\therefore$ By Milne Thompson substitution, $(x = z \ \& \ y = 0)$

$v_y(z,0) = 0$ & $v_x(z,0) = 2z + \frac{0 - z^2}{(0^2+z^2)^2}$

$\therefore v_x(z,0) = 2z - \frac{1}{z^2}$

$\therefore$ from Eq^n (1)

$$f'(z) = v_y(z,0) + i v_x(z,0)$$

$\therefore f'(z) = 0 + i\left(2z - \frac{1}{z^2}\right)$

Integrating w.r.t. z,

$f(z) = i \int 2z - \frac{1}{z^2} \, dz$

$\therefore f(z) = i\left(\cancel{2}\frac{z^2}{\cancel{2}} - \frac{z^{-2+1}}{-2+1}\right) + c$

$\therefore f(z) = i\left(z^2 + \frac{1}{z}\right) + c$

put $z = x + iy$,

$\therefore f(z) = i\left((x+iy)^2 + \frac{1}{x+iy}\right) + c$

$= i\left(x^2 - y^2 + 2xyi + \frac{x - iy}{x^2+y^2}\right) + c$

$= ix^2 - iy^2 - 2xy + \frac{ix}{x^2+y^2} + \frac{y}{x^2+y^2} + c$

$f(z) = \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + c = u + iv$

$\therefore u = -2xy + \frac{y}{x^2+y^2}$

Q3) b) To find : $L^{-1}\left\{\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$

consider $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6} = \dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$

$\therefore \dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \dfrac{A}{(s-1)} + \dfrac{B}{(s-2)} + \dfrac{C}{(s-3)}$

$\therefore 2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$

put $s=1$, $1 = A(-1)(-2)$ $\therefore \boxed{A = \frac{1}{2}}$

put $s=2$, $1 = B(1)(-1)$ $\boxed{\therefore B = -1}$

put $s=3$, $5 = C(2)(1)$ $\boxed{\therefore C = \frac{5}{2}}$

$\therefore L^{-1}\left\{\dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right\} = L^{-1}\left\{\dfrac{1/2}{(s-1)} + \dfrac{-1}{(s-2)} + \dfrac{5/2}{(s-3)}\right\}$

$= \frac{1}{2}L^{-1}\left\{\dfrac{1}{s-1}\right\} - L^{-1}\left\{\dfrac{1}{s-2}\right\} + \frac{5}{2}L^{-1}\left\{\dfrac{1}{s-3}\right\}$

$= \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}$ //.

Q3) c) The second degree parabolic curve is given by

$$y = a + bx + cx^2$$

$\therefore$ Normal Equations are,

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Here, $n = 9$ $\quad \Sigma x = 45$

$\Sigma y = 74$ $\quad \Sigma x^2 = 285$

$\Sigma xy = 421$ $\quad \Sigma x^3 = 2025$

$\Sigma x^2 y = 2771$ $\quad \Sigma x^4 = 15333$

$\therefore$ Normal Eq^{n}s are

$$74 = 9a + 45b + 285c$$

$$421 = 45a + 285b + 2025c$$

$$2771 = 285a + 2025b + 15333c$$

Solving simultaneously,

$$a = -\frac{13}{14} = -0.9285 \quad b = 3.523 \quad c = -\frac{247}{924} = -0.2673$$

$\therefore$ Eq^n of parabola is,

$$y = (-0.9285) + (3.523)x + (-0.2673)x^2$$

To estimate y, put $x = 10$

$$\therefore y = (-0.9285) + 3.523 \times 10 - 0.2673 \times 10^2$$

$$\boxed{\therefore y = 7.5715}$$

Q4) a) To find Fourier series ~~of~~ of $f(x) = x^3$ in $(-\pi, \pi)$

consider $f(-x) = (-x)^3 = -x^3 = -f(x)$

$\therefore f(x)$ is odd (i.e $a_0 = 0$ & $a_n = 0$)

$\therefore$ Fourier series is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^3 \sin(nx)\,dx$$

$$= \frac{2}{\pi}\Big[(x^3)\left(\frac{-\cos(nx)}{n}\right) - (3x^2)\left(-\frac{\sin(nx)}{n^2}\right) + (6x)\left(\frac{\cos(nx)}{n^3}\right) - (6)\left(\frac{\sin(nx)}{n^4}\right)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\Big[-\frac{1}{n}(\pi^3(-1)^n - 0) + \frac{3}{n^2}(0-0) + \frac{6}{n^3}(\pi(-1)^n - 0) - \frac{6}{n^4}(0-0)\Big]$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3(-1)^n}{n} + \frac{6\pi(-1)^n}{n^3}\right]$$

$$b_n = \frac{2(-1)^n}{n}\left[\frac{6}{n^2} - \pi^2\right]$$

$\therefore$ Fourier series is,

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n}\left(\frac{6}{n^2} - \pi^2\right)\sin(nx)$$

4) b) line of Regression Y on X is given by

$$(y-\bar{y}) = b_{yx}(x-\bar{x}) \quad \text{--- (1)}$$

& line of Regression X on Y is given by

$$(x-\bar{x}) = b_{xy}(y-\bar{y}) \quad \text{--- (2)}$$

where, $\cancel{b_{xy}}$ $b_{yx} = \dfrac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$

& $b_{xy} = \dfrac{n\sum xy - \sum x \sum y}{n\sum y^2 - (\sum y)^2}$

Here, $n = 8$, $\sum x = 544$, $\sum y = 550$, $\bar{x} = 68$

$\sum xy = 37426$, $\sum x^2 = 37028$, $\sum y^2 = 37864$, $\bar{y} = 68.75$

$\therefore b_{yx} = \dfrac{8(37426)-(544)(550)}{8(37028)-(544)^2}$

$\therefore b_{yx} = \dfrac{13}{18} = 0.7222$

& $b_{xy} = \dfrac{8(37426)-(544)(550)}{8(37864)-(550)^2}$

$\therefore b_{xy} = \dfrac{52}{103} = 0.5048$

$\therefore$ line of Regression Y on X is, (from 1)

$y - 68.75 = (0.7222)(x-68)$

$\therefore y = 0.7222x - 68 \times 0.7222 + 68.75$

$\boxed{\therefore y = 0.7222x + 19.6404}$

& line of Regression X on Y is (from 2)

$x - 68 = (0.5048)(y - 68.75)$

$\therefore x = 0.5048y - 68.75 \times 0.5048 + 68$

$\therefore \boxed{x = 0.5048y + 33.295}$

wkt, coefficient of correlation 'r' is given by,

$r^2 = b_{xy} \times b_{yx} \quad \therefore r^2 = 0.7222 \times 0.5048 = 0.3645$

$\therefore r = \pm 0.6037$

$\boxed{\therefore r = 0.6037}$ {As b_{xy} & b_{yx} both are +ve}

(Q4)c) w.k.t,

$$\int_0^\infty e^{-\sqrt{2}t}\,\frac{\sin t \sinh t}{t}\,dt = L\left\{\frac{\sin t \sinh t}{t}\right\}\Bigg|_{s=\sqrt{2}} \quad \text{--- (1)}$$

consider, $L\left\{\frac{\sin t \sinh t}{t}\right\} = L\left\{\frac{\sin t}{t}\times\left(\frac{e^t - e^{-t}}{2}\right)\right\}$

$$\cancel{\frac{1}{2}L\{} = \frac{1}{2}L\left\{\frac{\sin t\, e^t}{t} - \frac{\sin t\, e^{-t}}{t}\right\}$$

$$L\left\{\frac{\sin t \sinh t}{t}\right\} = \frac{1}{2}\left[L\left(e^t\frac{\sin t}{t}\right) - L\left(e^{-t}\frac{\sin t}{t}\right)\right] \quad \text{--- (2)}$$

consider, $L(\sin t) = \frac{1}{s^2+1} = F_1(s)$

$\therefore$ By division by 't' property,

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty F_1(s)\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left[\tan^{-1}(s)\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$L\left(\frac{\sin t}{t}\right) = \frac{\pi}{2} - \tan^{-1}(s) = F_2(s)$$

from (2), $L\left\{\frac{\sin t \sinh t}{t}\right\} = \frac{1}{2}\left[F_2(s-1) - F_2(s+1)\right]$ {By first shifting property}

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s-1) - \frac{\pi}{2} + \tan^{-1}(s+1)\right]$$

$$\therefore L\left\{\frac{\sin t \sinh t}{t}\right\} = \frac{1}{2}\left[\tan^{-1}(s+1) - \tan^{-1}(s-1)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{(s+1)-(s-1)}{1+(s+1)(s-1)}\right)\right]$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2}{1+s^2-1}\right)$$

$$L\left\{\frac{\sin t \sinh t}{t}\right\} = \frac{1}{2}\tan^{-1}\left(\frac{2}{s^2}\right) = F_3(s)$$

from (1)

$$\int_0^\infty e^{-\sqrt{2}t}\,\frac{\sin t \sinh t}{t}\,dt = F_3(\sqrt{2}) = \frac{1}{2}\tan^{-1}\left(\frac{2}{(\sqrt{2})^2}\right)$$

$$= \frac{1}{2}\tan^{-1}(1)$$

$$= \frac{1}{2}\times\frac{\pi}{4}$$

$$\therefore \int_0^\infty e^{-\sqrt{2}t}\,\frac{\sin t \sinh t}{t}\,dt = \frac{\pi}{8}$$

Q5) a) Given family of curves is $x^3y - xy^3 = c$

wkt, If $f(z) = u + iv$ is analytic then,

$u(x,y) = a$ & $v(x,y) = b$ are orthogonal trajectories of eachother.

Taking $u(x,y) = x^3y - xy^3$

wkt,

$$v(x,y) = \int_{y \text{ is const}} -u_y \, dx + \int u_x (\text{free from } x) \, dy \quad \text{—(i)}$$

Now, $u_x = 3x^2y - y^3$ & $u_y = x^3 - 3xy^2$

$$\therefore v(x,y) = \int -x^3 + 3xy^2 \, dx + \int -y^3 \, dy$$

$$= -\frac{x^4}{4} + 3\frac{x^2}{2}y^2 - \frac{y^4}{4}$$

$\therefore$ orthogonal trajectory is

$$-\frac{x^4}{4} + \frac{3}{2}x^2y^2 - \frac{y^4}{4} = b$$

i.e $x^4 - 6x^2y^2 + y^4 = b$

Q5) b) Given distribution is

x	-2	3	1
$P(x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

$$\text{mean} = E(X) = \overline{X} = \Sigma x p_x = -2\times\frac{1}{3} + 3\times\frac{1}{2} + 1\times\frac{1}{6}$$

$$\therefore \text{mean}, \overline{x} = 1$$

$\therefore$ Moment generating function is

$$M_{\overline{x}}(t) = E\left(e^{t(x-\overline{x})}\right) = E\left(e^{t(x-1)}\right)$$

$$= \Sigma\, e^{t(x-1)} p_x$$

$$= e^{t(-2-1)}\times\frac{1}{3} + e^{t(3-1)}\times\frac{1}{2} + e^{t(1-1)}\times\frac{1}{6}$$

$$\therefore M_{\overline{x}}(t) = \frac{e^{-3t}}{3} + \frac{e^{2t}}{2} + \frac{1}{6}$$

Let, rth central moment is given by,

$$\mu_r = \frac{d^r}{dt^r} M_{\overline{x}}(t)\Big|_{t=0}$$

$$\therefore \mu_0 = M_{\overline{x}}(t)\Big|_{t=0} = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1$$

$$\mu_1 = \frac{d}{dt} M_{\overline{x}}(t)\Big|_{t=0} = \frac{-3e^{-3t}}{3} + \frac{2e^{2t}}{2} + 0\Big|_{t=0}$$

$$\therefore \boxed{\mu_1 = 1}$$

$$\mu_2 = \frac{d^2}{dt^2} M_x(t)\Big|_{t=0} = \frac{9e^{-3t}}{3} + \frac{4e^{2t}}{2}\Big|_{t=0}$$

$$\boxed{\therefore \mu_2 = 3+2 = 5}$$

$$\mu_3 = \frac{d^3}{dt^3} M_x(t)\Big|_{t=0} = \frac{-27e^{-3t}}{3} + \frac{8e^{2t}}{2}\Big|_{t=0}$$

$$\boxed{\therefore \mu_3 = -9+4 = -5}$$

$$\mu_4 = \frac{d^4}{dt^4} M_x(t)\Big|_{t=0} = \frac{81e^{-3t}}{3} + \frac{16e^{2t}}{2}\Big|_{t=0}$$

$$\therefore \boxed{\mu_4 = 27+8 = 35}$$

Q5) c) Half Range cosine series of $f(x) = x$ in $(0,2)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

where, $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 x\,dx = \left.\frac{x^2}{2}\right|_0^2 = 2 - 0 = 2$

$$\boxed{\therefore a_0 = 2}$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$\cancel{= \left[(x)\left(\frac{\sin(nx)}{n}\right) - (1)\left(-\frac{\cos(nx)}{n^2}\right)\right]_0^2}$$

$$\cancel{= \left[\frac{2}{n}(0-0) + \frac{1}{n^2}\left(\right.\right.}$$

$$\cancel{= \left[(x)\left(\frac{\sin(n\pi x)}{n\pi}\right) - (1)\left(-\frac{\cos(n\pi x)}{n^2\pi^2}\right)\right]_0^2}$$

$$= \left[(x)\left(\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right) - (1)\left(-\frac{2^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right)\right]_0^2$$

$$= \left[\frac{2}{n\pi}(0-0) + \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)\right]$$

$$\therefore \boxed{a_n = \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)}$$

$\therefore$ Half Range cosine series is, $x = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{2}\right)$ — (1)

for deduction, use Parseval's Identity,

$$\frac{2}{2}\int_0^2 [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$L.H.S = \int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

$$R.H.S = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 = \frac{(2)^2}{2} + \sum_{n=1}^{\infty}\left\{\frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]\right\}^2$$

$$= 2 + \frac{16}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\left((-1)^n - 1\right)^2$$

As L.H.S = R.H.S $\therefore \frac{8}{3} = 2 + \frac{16}{\pi^4}\left(\sum_{n=1}^{\infty}\frac{((-1)^n - 1)^2}{n^4}\right)$

$$\therefore \left(\frac{8}{3} - 2\right)\times\frac{\pi^4}{16} = \frac{4}{1^4} + \frac{0}{2^4} + \frac{4}{3^4} + \frac{0}{4^4} + \frac{4}{5^4}$$

$$\therefore \frac{\cancel{2}}{3}\times\frac{\pi^4}{\cancel{16}\,8} = 4\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right)$$

$$\therefore \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

Q6) a) To find: $L^{-1}\left\{\frac{s^2}{(s^2+2^2)^2}\right\}$

Consider $\frac{s^2}{(s^2+2^2)^2} = \frac{s}{s^2+2^2} \times \frac{s}{s^2+2^2} = F(s) \times G(s)$

$\therefore L^{-1}(F(s)) = L^{-1}\left(\frac{s}{s^2+2^2}\right) = \cos 2t = f(t)$

$L^{-1}(G(s)) = L^{-1}\left(\frac{s}{s^2+2^2}\right) = \cos 2t = g(t)$

By convolution theorem,

$$L^{-1}(F(s) \times G(s)) = \int_0^t f(u) \cdot g(t-u)\, du$$

$$\therefore L^{-1}\left(\frac{s^2}{(s^2+2^2)^2}\right) = \int_0^t \cos(2u) \cos(2(t-u))\, du.$$

$$= \int_0^t \frac{1}{2}\left[\cos(2u+2t-2u) + \cos(2u-2t+2u)\right] du$$

$$= \frac{1}{2}\int_0^t \cos 2t + \cos(4u-2t)\, du$$

$$= \frac{1}{2}\left[(\cos 2t)u + \frac{\sin(4u-2t)}{4}\right]_0^t$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{\sin(2t)}{4} - 0 - \frac{\sin(-2t)}{4}\right]$$

$$L^{-1}\left(\frac{s^2}{(s^2+2^2)^2}\right) = \frac{1}{2}\left(t\cos 2t + \frac{1}{2}\sin 2t\right)$$

Q6, b) wkt, $\sum p_x = 1$

$\therefore k + 2k + 3k + k^2 + k^2 + k + 2k^2 + 4k^2 = 1$

$\therefore 8k^2 + 7k - 1 = 0$

$\therefore \boxed{k = \frac{1}{8}}$ or $k = -1$

As probabilities cannot be negative $\therefore k \neq -1$

$\therefore$ distribution table is

X	1	2	3	4	5	6	7
P(X)	k	2k	3k	k^2	k^2+k	$2k^2$	$4k^2$

X	1	2	3	4	5	6	7
P(X)	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{2}{64}$	$\frac{4}{64}$

Now $P(X<5) = P(1)+P(2)+P(3)+P(4)$

$$= \frac{1}{8}+\frac{2}{8}+\frac{3}{8}+\frac{1}{64}$$

$$= \frac{8}{64}+\frac{16}{64}+\frac{24}{64}+\frac{1}{64} = \frac{39}{64}$$

$\therefore P(X<5) = \frac{39}{64} = 0.7656.$

& $P(X>5) = P(6)+P(7) = \frac{2}{64}+\frac{4}{64} = \frac{6}{64}$

$\therefore P(X>5) = \frac{6}{64} = \frac{3}{32} = 0.09375$

Q6) c) $V = 3x^2y+6xy-y^3$ will be Harmonic

if $V_{xx}+V_{yy}=0$

$V_x = 6xy+6y \quad \therefore V_{xx} = 6y$

$V_y = 3x^2+6x-3y^2 \quad \therefore V_{yy} = -6y$

$\therefore V_{xx}+V_{yy}=0$

$\therefore V$ is Harmonic.

Hence Harmonic conjugate is given by

$u = \int \underset{y \text{ is const}}{V_y}\,dx + \int -V_x(\text{free from } x)\,dy$

$\therefore u = \int 3x^2+6x-3y^2\,dx + \int -6y\,dy$

$= x^3+3x^2-3y^2x - 3y^2$

$\therefore u = x^3-3y^2+3x^2-3xy^2$

wkt, if $f(z) = u+iv$ is analytic then,

u & v are both Harmonic ~~& Har~~

$\therefore f(z) = (x^3-3y^2+3x^2-3xy^2)+i(3x^2y+6xy-y^3)$

Using Milne Thompson substitution

$x=z$ & $y=0$

$f(z) = (z^3-0+3z^2-0)+i(0+0+0)$

$\therefore \boxed{f(z) = z^3-3z^2}$