Engineering Maths IV

May-June 2024

(COITAI)

Time (3 hours) **Max Marks: 80**

Note: (1) Question No. 1 is Compulsory

(2) Answer any three questions from Q.2 to Q.6

(3) Figures to the right indicate full marks

1. (a) If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$ find Eigen values of $A^3 + 5A + 8I$ (5)

Solution:

$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}, |A| = 6$

The characteristic equation,

$|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 2 & 3 \\ 0 & 3-\lambda & 5 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [-1+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 5 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix}\right]\lambda - 6 = 0$

$\lambda^3 - 0\lambda^2 - 7\lambda - 6 = 0$

$\lambda = -1, -2, 3$

The eigen values of A is $-1, -2, 3$

The eigen values of A^3 is $(-1)^3, (-2)^3, 3^3$ i.e $-1, -8, 27$

The eigen values of $5A$ is $5(-1), 5(-2), 5(3)$ i.e. $-5, -10, 15$

The eigen values of I is 1,1,1

The eigen values of $8I$ is 8,8,8

Thus, the eigen values of $A^3 + 5A + 8I$ is

$-1 + (-5) + 8;\ \ -8 + (-10) + 8;\ \ 27 + 15 + 8$

i.e. $2, -10, 50$

1. (b) Evaluate the integral $\int_0^{1+i}(x-y+ix^2)dz$ along the parabola $y^2=x$ (5)

Solution:

$I=\int f(z)dz=\int_{(0,0)}^{(1,1)}(x-y+ix^2)(dx+idy)$

Along the parabola,

$x=y^2$

$dx=2y\,dy$

The integral becomes,

$I=\int_0^1(y^2-y+iy^4)(2ydy+i\,dy)$

$I=\int_0^1(y^2-y+i\,y^4)(2y+i)dy$

$I=\int_0^1(2y^3+iy^2-2y^2-iy+2iy^5+i^2y^4)dy$

$I=\left[\frac{2y^4}{4}+\frac{iy^3}{3}-\frac{2y^3}{3}-\frac{iy^2}{2}+\frac{2iy^6}{6}-\frac{y^5}{5}\right]_0^1$

$I=-\frac{11}{30}+\frac{i}{6}$

1. (c) Find the Z transform of $f(k)=a^k, k\geq 0$ (5)

Solution:

We have,

$f(k)=a^k, k\geq 0$

By definition,

$Z\{f(k)\}=\sum_{-\infty}^{\infty}f(k)z^{-k}$

$Z\{a^k\}=\sum_0^{\infty}a^k.z^{-k}$

$Z\{a^k\}=a^0z^0+a^1.z^{-1}+a^2.z^{-2}+a^3.z^{-3}+\cdots\ldots\ldots$

$Z\{a^k\}=1+\frac{a}{z}+\frac{a^2}{z^2}+\frac{a^3}{z^3}+\cdots\ldots\ldots$

$Z\{a^k\}=\left[1-\frac{a}{z}\right]^{-1}$

$Z\{a^k\}=\left[\frac{z-a}{z}\right]^{-1}$

$Z\{a^k\}=\frac{z}{z-a}$

1. (d) Maximise $z = x_1 + 3x_2 + 3x_3$
subject to $x_1 + 2x_2 + 3x_3 = 4$
$2x_1 + 3x_2 + 5x_3 = 7$
$x_1, x_2, x_3 \geq 0$
Find all basic solutions. Which of them are basic feasible and optimal basic feasible solutions? (5)

Solution:

No	Non-basic var = 0	Basic var	Equations & solutions	Is the solution feasible?	Is the solution degenerate?	Value of z	Is the solution optimal?
1	$x_3 = 0$	x_1, x_2	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $x_1 = 2, x_2 = 1$	Yes	No	5	Yes
2	$x_2 = 0$	x_1, x_3	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $x_1 = 1, x_3 = 1$	Yes	No	4	No
3	$x_1 = 0$	x_2, x_3	$2x_2 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$ $x_2 = -1, x_3 = 2$	No	No	3	No

2. (a) Verify Cayley-Hamilton theorem for the matrix A where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$. And hence find A^{-1} and A^4 (6)

Solution:

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}, |A| = 40$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [1-1-1]\lambda^2 + \left[\begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}\right]\lambda - 40 = 0$

$\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$

By Cayley Hamilton theorem,

$A^3 + A^2 - 18A - 40I = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$

L.H.S. $= A^3 + A^2 - 18A - 40I$

$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - 18\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = R.H.S.

Thus, Cayley Hamilton theorem is verified

Now,

$A^3 + A^2 - 18A - 40I = 0$

Pre-multiplying by A^{-1}, we get

$A^2 + A - 18I - 40A^{-1} = 0$

$40A^{-1} = A^2 + A - 18I$

$40A^{-1} = \begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$

Also,

$A^3 + A^2 - 18A - 40I = 0$

Pre-multiplying by$A$, we get

$A^4 + A^3 - 18A^2 - 40A = 0$

$A^4 = -A^3 + 18A^2 + 40A$

$$A^4 = -\begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + 18\begin{bmatrix} 14 & 3 & -2 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + 40\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$$

2. (b) The means of two random samples of size 9 and 7 are 196.42 & 198.82 respectively. The sums of the squares of the deviations from the means are 26.94 and 18.73 respectively. Can the samples be considered to have been drawn from the same population? (6)

Solution:

$n_1 = 9, n_2 = 7$

$\overline{x}_1 = 196.42, \overline{x}_2 = 198.82$

$\sum(x_1 - \overline{x}_1)^2 = 26.94, \sum(x_2 - \overline{x}_2)^2 = 18.73$

$\sigma_1 = \sqrt{\frac{\sum(x_1-\overline{x}_1)^2}{n_1}} = \sqrt{\frac{26.94}{9}} = 1.7301, \sigma_2 = \sqrt{\frac{\sum(x_2-\overline{x}_2)^2}{n_2}} = \sqrt{\frac{18.73}{7}} = 1.6358$

(i) Null Hypothesis: $\mu_1 = \mu_2$

Alternative Hypothesis: $\mu_1 \neq \mu_2$

(ii) Test statistic:

$s_p = \sqrt{\frac{n_1\sigma_1^2+n_2\sigma_2^2}{n_1+n_2-2}} = \sqrt{\frac{9(1.7301)^2+7(1.6358)^2}{9+7-2}} = 1.806$

$S.E. = s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 1.806\sqrt{\frac{1}{9} + \frac{1}{7}} = 0.9102$

$t = \left|\frac{\overline{x}_1-\overline{x}_2}{S.E.}\right| = \left|\frac{196.42-198.82}{0.9102}\right| = 2.637$

(iii) L.O.S.: $\alpha = 0.05$

(iv) Degree of freedom: $\emptyset = (n_1 - 1) + (n_2 - 1) = 8 + 6 = 14$

(v) Critical value: $t_\alpha = 2.145$

(vi) Decision: Since, the calculated value of t is more than the critical value, null hypothesis is rejected.

Thus, the samples cannot be regarded as drawn from the same populations

2. (c) Solve the L.P.P. by using simplex method (8)

Maximise $z = 3x_1 + 2x_2$

subject to $3x_1 + 2x_2 \leq 18$

$0 \leq x_1 \leq 4$

$0 \leq x_2 \leq 6$

$x_1, x_2 \geq 0$

Solution:

Max $z - 3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

s.t. $3x_1 + 2x_2 + s_1 + 0s_2 + 0s_3 = 18$

$x_1 + 0x_2 + 0s_1 + s_2 + 0s_3 = 4$

$0x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 6$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

Simplex table,

Iteration No.	Basic Var	Coefficient of					RHS	Ratio	Formula
		x_1	x_2	s_1	s_2	s_3			
0	z	**-3**	-2	0	0	0	0	-	$X + 3Y$
s_2 leaves x_1 enters	s_1	**3**	2	1	0	0	18	$\frac{18}{3} = 6$	$X - 3Y$
	s_2	**1**	**0**	**0**	**1**	**0**	**4**	$\frac{4}{1} = 4$	-
	s_3	**0**	1	0	0	1	6	-	-
1	z	0	**-2**	0	3	0	12	-	$X + Y$
s_1 leaves x_2 enters	s_1	**0**	**2**	**1**	**-3**	**0**	**6**	$\frac{6}{2} = 3$	$\frac{Y}{2}$
	x_1	1	**0**	0	1	0	4	-	-
	s_3	0	**1**	0	0	1	6	$\frac{6}{1} = 6$	$X - \frac{1}{2}Y$
2	z	0	0	1	0	0	18		
	x_2	0	1	1/2	-3/2	0	3		
	x_1	1	0	0	1	0	4		
	s_3	0	0	-1/2	3/2	1	3		

Thus, the solution is

$\boxed{x_1 = 4, x_2 = 3, z_{max} = 18}$

3. (a) Find the Laurent's series for $f(z) = \frac{4z+3}{z(z-3)(z+2)}$ valid for $2 < |z| < 3$ (6)

Solution:

We have, $f(z) = \frac{4z+3}{z(z-3)(z+2)}$

Let $\frac{4z+3}{z(z-3)(z+2)} = \frac{A}{z} + \frac{B}{z-3} + \frac{C}{z+2}$

$4z + 3 = A(z-3)(z+2) + Bz(z+2) + Cz(z-3)$

$4z + 3 = A(z^2 - z - 6) + B(z^2 + 2z) + C(z^2 - 3z)$

Comparing the coefficients, we get

$A + B + C = 0$

$-A + 2B - 3C = 4$

$-6A + 0B + 0C = 3$

On solving, we get

$A = -\frac{1}{2}, B = 1, C = -\frac{1}{2}$

$f(z) = \frac{-\frac{1}{2}}{z} + \frac{1}{z-3} - \frac{\frac{1}{2}}{z+2}$

For $2 < |z| < 3$

$f(z) = -\frac{1}{2z} + \frac{1}{-3+z} - \frac{\frac{1}{2}}{z+2}$

$f(z) = -\frac{1}{2z} + \frac{1}{-3\left(1-\frac{z}{3}\right)} - \frac{\frac{1}{2}}{z\left(1+\frac{2}{z}\right)}$

$f(z) = -\frac{1}{2z} - \frac{1}{3}\left[1 - \frac{z}{3}\right]^{-1} - \frac{1}{2z}\left[1 + \frac{2}{z}\right]^{-1}$

$f(z) = -\frac{1}{2z} - \frac{1}{3}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots.\right] - \frac{1}{2z}\left[1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots\right]$

3. (b) Using the method of Lagrange's multipliers, solve the N.L.P.P. (6)

Optimise $z = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$

subject to $x_1 + x_2 + x_3 = 10$

$x_1, x_2, x_3 \geq 0$

Solution:

Let $f = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$

and $h = x_1 + x_2 + x_3 - 10$

Consider the Lagrangian function,

$L = f - \lambda h$

$L = \left(12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23\right) - \lambda(x_1 + x_2 + x_3 - 10)$

$L_{x_1} = 0 \Rightarrow 12 - 2x_1 - \lambda = 0 \Rightarrow x_1 = \frac{12-\lambda}{2}$

$L_{x_2} = 0 \Rightarrow 8 - 2x_2 - \lambda = 0 \Rightarrow x_2 = \frac{8-\lambda}{2}$

$L_{x_3} = 0 \Rightarrow 6 - 2x_3 - \lambda = 0 \Rightarrow x_3 = \frac{6-\lambda}{2}$

$L_\lambda = 0 \Rightarrow -(x_1 + x_2 + x_3 - 10) = 0$

$x_1 + x_2 + x_3 = 10$

$\frac{12-\lambda}{2} + \frac{8-\lambda}{2} + \frac{6-\lambda}{2} = 10$

$\frac{26-3\lambda}{2} = 10$

$\lambda = 2$

$\therefore x_1 = 5, x_2 = 3, x_3 = 1$

Now, hessian matrix,

$$H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} & h_{x_3} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 4$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \\ 1 & 0 & 0 \end{vmatrix}$$

$\Delta_4 = -4 - 4 - 4 = -12$

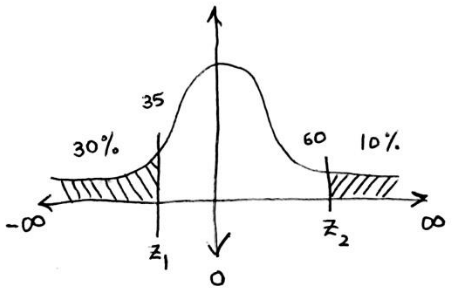
Since both Δ_3 is positive and Δ_4 is negative, it is a maxima

$\therefore z_{max} = 12(5) + 8(3) + 6(1) - (5)^2 - (3)^2 - (1)^2 - 23$

$\boxed{z_{max} = 32}$

3. (c) Marks obtained by students in an examination follow normal distribution. If 30% of the students got below 35 marks and 10% got above 60 marks. Find the mean and standard deviation (8)

Solution:



$A(0\ to\ z_1) = 20\% = 0.20$

From table, $z_1 = -0.52$

$z_1 = \frac{x_1-\mu}{\sigma}$

$-0.52 = \frac{35-\mu}{\sigma}$

$-0.52\sigma + \mu = 35$ ……………….(1)

$A(0\ to\ z_2) = 40\% = 0.40$

From table, $z_2 = 1.28$

$z_2 = \frac{x_2-\mu}{\sigma}$

$1.28 = \frac{60-\mu}{\sigma}$

$1.28\sigma + \mu = 60$ ………………….(2)

Solving (1) & (2), we get

$\boxed{\sigma = 13.88, \mu = 42.22}$

4. (a) Find the Eigen values and Eigen vectors of matrix $A=\begin{bmatrix}3 & 10 & 5\\ -2 & -3 & -4\\ 3 & 5 & 7\end{bmatrix}$ (6)

Solution:

$A=\begin{bmatrix}3 & 10 & 5\\ -2 & -3 & -4\\ 3 & 5 & 7\end{bmatrix}, |A|=12$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}3-\lambda & 10 & 5\\ -2 & -3-\lambda & -4\\ 3 & 5 & 7-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[3-3+7]\lambda^2+\left[\begin{vmatrix}-3 & -4\\ 5 & 7\end{vmatrix}+\begin{vmatrix}3 & 5\\ 3 & 7\end{vmatrix}+\begin{vmatrix}3 & 10\\ -2 & -3\end{vmatrix}\right]\lambda-12=0$

$\lambda^3-7\lambda^2+16\lambda-12=0$

$(\lambda-3)(\lambda-2)(\lambda-2)=0$

$\lambda=3,2,2$

(i) For $\lambda=3$, $[A-\lambda I]X=0$ gives

$\begin{bmatrix}0 & 10 & 5\\ -2 & -6 & -4\\ 3 & 5 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$0x_1+10x_2+5x_3=0$

$-2x_1-6x_2-4x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}10 & 5\\ -6 & -4\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}0 & 5\\ -2 & -4\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}0 & 10\\ -2 & -6\end{vmatrix}}$

$\frac{x_1}{-10}=-\frac{x_2}{10}=\frac{x_3}{20}$

$\frac{x_1}{1}=\frac{x_2}{1}=\frac{x_3}{-2}$

Hence, corresponding to $\lambda=3$ the eigen vector is $X_1=[1,1,-2]'$

(ii) For $\lambda=2$, $[A-\lambda I]X=0$ gives

$\begin{bmatrix}1 & 10 & 5\\ -2 & -5 & -4\\ 3 & 5 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$x_1+10x_2+5x_3=0$

$-2x_1-5x_2-4x_3=0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix}10 & 5\\ -5 & -4\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}1 & 5\\ -2 & -4\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}1 & 10\\ -2 & -5\end{vmatrix}}$

$\frac{x_1}{-15}=-\frac{x_2}{6}=\frac{x_3}{15}$

$\frac{x_1}{5}=\frac{x_2}{2}=\frac{x_3}{-5}$

Hence, corresponding to $\lambda=2$ the eigen vector is $X_2=[5,2,-5]'$

4. (b) Find inverse Z transform of $F(z) = \frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}, 3 < |z| < 4$ (6)

Solution:

We have,

$F(z) = \frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$

Let $\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{z-4}$

$3z^2 - 18z + 26 = A(z-3)(z-4) + B(z-4)(z-2) + C(z-3)(z-2)$

$3z^2 - 18z + 26 = A(z^2 - 7z + 12) + B(z^2 - 6z + 8) + C(z - 5z + 6)$

Comparing the coefficients, we get

$A + B + C = 3$

$-7A - 6B - 5C = -18$

$12A + 8B + 6C = 26$

On solving, we get

$A = 1, B = 1, C = 1$

$F(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{z-4}$

For $3 < |z| < 4$,

$F(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{-4+z}$

$F(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} + \frac{1}{z\left(1-\frac{3}{z}\right)} + \frac{1}{-4\left(1-\frac{z}{4}\right)}$

$F(z) = \frac{1}{z}\left[1 - \frac{2}{z}\right]^{-1} + \frac{1}{z}\left[1 - \frac{3}{z}\right]^{-1} - \frac{1}{4}\left[1 - \frac{z}{4}\right]^{-1}$

$F(z) = \frac{1}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \cdots\right] + \frac{1}{z}\left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \cdots\right] - \frac{1}{4}\left[1 + \frac{z}{4} + \frac{z^2}{4^2} + \cdots..\right]$

$F(z) = [2^0 z^{-1} + 2^1 z^{-2} + 2^2 z^{-3} + \cdots.] + [3^0 z^{-1} + 3^1 z^{-2} + 3^2 z^{-3} + \cdots.]$
$+[-4^{-1} z^0 - 4^{-2} z^1 - 4^{-3} z^2 - \cdots..]$

From first series,

Coefficient of $z^{-k} = 2^{k-1}, k > 0$

From second series,

Coefficient of $z^{-k} = 3^{k-1}, k > 0$

From third series,

Coefficient of $z^k = -4^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -4^{k-1}, k \leq 0$

Thus,

$$\boxed{Z^{-1}\left\{\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}\right\} = \begin{cases} -4^{k-1} & k \leq 0 \\ \{2^{k-1} + 3^{k-1}\} & k > 0 \end{cases}}$$

4. (c) Using the Kuhn-Tucker conditions, solve the N.L.P.P. (8)

Maximise $z = 2x_1^2 - 7x_2^2 + 12x_1x_2$

subject to $2x_1 + 5x_2 \leq 98$

$x_1, x_2 \geq 0$

Solution:

Let $f = 2x_1^2 - 7x_2^2 + 12x_1x_2$

Let $h = 2x_1 + 5x_2 - 98$

Consider, $L = f - \lambda h$

$L = 2x_1^2 - 7x_2^2 + 12x_1x_2 - \lambda(2x_1 + 5x_2 - 98)$

According to Kuhn Tucker conditions,

$L_{x_1} = 0 \Rightarrow 4x_1 + 12x_2 - 2\lambda = 0$ ……………….(1)

$L_{x_2} = 0 \Rightarrow -14x_2 + 12x_1 - 5\lambda = 0$ ……………….(2)

$\lambda h = 0 \Rightarrow \lambda(2x_1 + 5x_2 - 98) = 0$ …………………..(3)

$h \leq 0 \Rightarrow 2x_1 + 5x_2 - 98 \leq 0$ ……………………………….(4)

$x_1, x_2, \lambda \geq 0$ ………………(5)

Case I: If $\lambda = 0$

From (1), $4x_1 + 12x_2 = 0$

From (2), $12x_1 - 14x_2 = 0$

$x_1 = 0, x_2 = 0$

$z_{max} = 2(0)^2 - 7(0)^2 + 12(0)(0) = 0$

Case II: If $\lambda \neq 0$

From (1), $4x_1 + 12x_2 - 2\lambda = 0$

From (2), $12x_1 - 14x_2 - 5\lambda = 0$

From (3), $2x_1 + 5x_2 + 0\lambda = 98$

On solving,

$x_1 = 44, x_2 = 2, \lambda = 100$

$z_{max} = 2(44)^2 - 7(2)^2 + 12(44)(2) = 4900$

Thus, the optimal solution is

$\boxed{z_{max} = 4900}$ at $\boxed{x_1 = 44, x_2 = 2}$

5. (a) Show that the matrix $A=\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}$ is diagonalizable. Find the diagonal form D and diagonalising matrix M. (6)

Solution:

$$A=\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}, |A|=3$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}-9-\lambda & 4 & 4\\ -8 & 3-\lambda & 4\\ -16 & 8 & 7-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[-9+3+7]\lambda^2+\left[\begin{vmatrix}3 & 4\\ 8 & 7\end{vmatrix}+\begin{vmatrix}-9 & 4\\ -16 & 7\end{vmatrix}+\begin{vmatrix}-9 & 4\\ -8 & 3\end{vmatrix}\right]\lambda-3=0$$

$$\lambda^3-\lambda^2-5\lambda-3=0$$

$$(\lambda+1)(\lambda+1)(\lambda-3)=0$$

$$\lambda=-1,-1,3$$

The Algebraic Multiplicity of $\lambda=-1$ is 2 and that of $\lambda=3$ is 1

(i) For $\lambda=-1$, $[A-\lambda I]X=0$ gives

$$\begin{bmatrix}-8 & 4 & 4\\ -8 & 4 & 4\\ -16 & 8 & 8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

By R_2-R_1, R_3-2R_1

$$\begin{bmatrix}-8 & 4 & 4\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\therefore -8x_1+4x_2+4x_3=0$$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ vectors to be formed

The Geometric Multiplicity of $\lambda=1$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3=t$ & $x_2=s$

$$\therefore x_1=\frac{s}{2}+\frac{t}{2}$$

$$\therefore X=\begin{bmatrix}\frac{s}{2}+\frac{t}{2}\\ s\\ t\end{bmatrix}=\begin{bmatrix}\frac{s}{2}+\frac{t}{2}\\ s+0t\\ 0s+t\end{bmatrix}=\begin{bmatrix}\frac{s}{2}\\ s\\ 0s\end{bmatrix}+\begin{bmatrix}\frac{t}{2}\\ 0t\\ t\end{bmatrix}=\frac{s}{2}\begin{bmatrix}1\\ 2\\ 0\end{bmatrix}+\frac{t}{2}\begin{bmatrix}1\\ 0\\ 2\end{bmatrix}$$

Hence, corresponding to $\lambda=-1$ the eigen vectors are

$X_1=[1,2,0]'$ & $X_2=[1,0,2]'$

(ii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-12x_1 + 4x_2 + 4x_3 = 0$$

$$-8x_1 + 0x_2 + 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & 4 \\ 0 & 4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -12 & 4 \\ -8 & 4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -12 & 4 \\ -8 & 0 \end{vmatrix}}$$

$$\frac{x_1}{16} = -\frac{x_2}{-16} = \frac{x_3}{32}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [1,1,2]'$

Thus, the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$

5. (b) Find the relative maximum or minimum of the function

$$z = x_1^2 + x_2^2 + x_3^2 - 4x_1 - 8x_2 - 12x_3 + 100 \quad (6)$$

Solution:

Let $f = x_1^2 + x_2^2 + x_3^2 - 4x_1 - 8x_2 - 12x_3 + 100$

$f_{x_1} = 0 \Rightarrow 2x_1 - 4 = 0$ ...(1)

$f_{x_2} = 0 \Rightarrow 2x_2 - 8 = 0$ ...(2)

$f_{x_3} = 0 \Rightarrow 2x_3 - 12 = 0$ ...(3)

Solving (1), (2) and (3), we get

$x_1 = 2, x_2 = 4, x_3 = 6$

Now, Hessian matrix,

$$H = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Delta_1 = 2$

$$\Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$$\Delta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$$

Since, all Δs are positive, it is a minima

$\therefore z_{min} = (2)^2 + (4)^2 + (6)^2 - 4(2) - 8(4) - 12(6) + 100$

$\boxed{\therefore z_{min} = 44}$

5. (c) Evaluate $\oint \frac{2z-1}{z(2z+1)(z+2)} dz$ using Cauchy's residue theorem where C is the circle $|z| = 1$ (8)

Solution:

We have, $f(z) = \frac{12z-1}{z(2z+1)(z+2)}$

For singularity,

$z(2z+1)(z+2) = 0$

$\therefore z = 0, z = -\frac{1}{2}, z = -2$

We see that $z = 0$ and $z = -\frac{1}{2}$ both lies inside $C: |z| = 1$ and hence are simple poles.

Residue of $f(z)$ at $(z = 0) = \lim_{z\to 0}(z-0)f(z)$

$$= \lim_{z\to 0}(z-0)\frac{12z-1}{z(2z+1)(z+2)}$$

$$= \lim_{z\to 0}\frac{12z-1}{(2z+1)(z+2)}$$

$$= \frac{0-1}{(0+1)(0+2)}$$

$$= -\frac{1}{2}$$

Residue of $f(z)$ at $\left(z = -\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}}\left(z+\frac{1}{2}\right)f(z)$

$$= \lim_{z\to -\frac{1}{2}}\left(z+\frac{1}{2}\right)\frac{12z-1}{z(2z+1)(z+2)}$$

$$= \lim_{z\to -\frac{1}{2}}\frac{2z+1}{2}.\frac{12z-1}{z(2z+1)(z+2)}$$

$$= \frac{1}{2}\lim_{z\to -\frac{1}{2}}\frac{12z-1}{z(z+2)}$$

$$= \frac{1}{2}.\frac{12\left(-\frac{1}{2}\right)-1}{-\frac{1}{2}\left(-\frac{1}{2}+2\right)} = \frac{1}{2}.\frac{-7}{-\frac{3}{4}}$$

$$= \frac{14}{3}$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\int_c \frac{12z-1}{z(2z+1)(z+2)}dz = 2\pi i\left[-\frac{1}{2}+\frac{14}{3}\right] = 2\pi i\left[\frac{25}{6}\right]$

$$\boxed{\int_c \frac{12z-1}{z(2z+1)(z+2)}dz = \frac{25\pi i}{3}}$$

6. (a) The number of car accidents in a metropolitan city was found to be 20, 17, 12, 6, 7, 15, 8, 5, 16 and 14 per month respectively. Use χ^2 test to check whether these frequencies are in agreement with that occurrence was the same during 10 months period. Test at 5% level of significance. (6)

Solution:

(i) Null Hypothesis: The accidents was same during 10 months period

Alternative Hypothesis: The accidents was not same during 10 months period

(ii) Test Statistic:

O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
20	12	8	64	64/12
17	12	5	25	25/12
12	12	0	0	0/12
6	12	−6	36	36/12
7	12	−5	25	25/12
15	12	3	9	9/12
8	12	−4	16	16/12
5	12	−7	49	49/12
16	12	4	16	16/12
14	12	2	4	4/12
Total				20.33

(iii) Degree of freedom: $\emptyset = n - 1 = 9$

(iv) L.O.S: $\alpha = 0.05$

(v) Critical value: $\chi^2_\alpha = 16.919$

(vi) Decision: since, the calculated value is more than the critical value, null hypothesis is rejected. Thus, accidents was not same during 10 months period

6. (b) Find z transform of $[2^k \cos(3k+2)], k \geq 0$ (6)

Solution:

$Z\{\cos(3k+2)\} = Z\{\cos 3k \cos 2 - \sin 3k \sin 2\}$

$Z\{\cos(3k+2)\} = \cos 2 \; Z\{\cos 3k\} - \sin 2 \; Z\{\sin 3k\}$

$Z\{\cos(3k+2)\} = \cos 2 \left[\frac{z(z-\cos 3)}{z^2-2z\cos 3+1}\right] - \sin 2 \left[\frac{z \sin 3}{z^2-2z\cos 3+1}\right]$

By $Z\{\cos \alpha k\} = \frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}, Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2-2z\cos\alpha+1}$

$\therefore Z\{\cos(3k+2)\} = \frac{z^2\cos 2 - z\cos 2\cos 3 - z\sin 2\sin 3}{z^2-2z\cos 3+1}$

$Z\{\cos(3k+2)\} = \frac{z^2\cos 2 - z(\cos 2\cos 3+\sin 2\sin 3)}{z^2-2z\cos 3+1}$

$Z\{\cos(3k+2)\} = \frac{z^2\cos 2 - z\cos 1}{z^2-2z\cos 3+1}$

Now, by Change of scale property $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

$Z\{2^k\cos(3k+2)\} = \frac{\left(\frac{z}{2}\right)^2\cos 2 - \left(\frac{z}{2}\right)\cos 1}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cos 3+1}$

$\boxed{Z\{2^k\cos(3k+2)\} = \frac{z^2\cos 2 - 2z\cos 1}{z^2-4z\cos 3+4}}$

6. (c) Use the dual simplex method to solve the L.P.P.

Minimise $z = 2x_1 + x_2$

subject to $3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$ (8)

Solution:

The standard form,

Min $z - 2x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

s.t. $-3x_1 - x_2 + s_1 + 0s_2 + 0s_3 = -3$

$-4x_1 - 3x_2 + 0s_1 + s_2 + 0s_3 = -6$

$x_1 + 2x_2 + 0s_1 + 0s_2 + s_3 = 3$

Simplex table,

Iteration No.	Basic Var	Coefficient of					RHS	Formula
		x_1	x_2	s_1	s_2	s_3		
0	z	-2	**-1**	0	0	0	0	$X - \frac{1}{3}Y$
s_2 leaves x_2 enters	s_1	-3	**-1**	1	0	0	-3	$X - \frac{1}{3}Y$
	s_2	**-4**	**-3**	**0**	**1**	**0**	**-6**	$\frac{Y}{-3}$
	s_3	1	**2**	0	0	1	3	$X + \frac{2}{3}Y$
Ratio		$\frac{-2}{-4} = \frac{1}{2}$	$\frac{-1}{-3} = \frac{1}{3}$	-	-	-	-	-
1	z	**-2/3**	0	0	-1/3	0	2	$X - \frac{2}{5}Y$
s_1 leaves x_1 enters	s_1	**-5/3**	**0**	**1**	**-1/3**	**0**	**-1**	$-\frac{3}{5}Y$
	x_2	**4/3**	1	0	-1/3	0	2	$X + \frac{4}{5}Y$
	s_3	**-5/3**	0	0	2/3	1	-1	$X - Y$
Ratio		$\frac{-\frac{2}{3}}{-\frac{5}{3}} = \frac{2}{5}$	-	-	$\frac{-\frac{1}{3}}{-\frac{1}{3}} = 1$	-	-	-
2	z	0	0	-2/5	-1/5	0	12/5	
	x_2	1	0	-3/5	1/5	0	3/5	
	x_1	0	1	4/5	-3/5	0	6/5	
	s_3	0	0	-1	1	1	0	

Thus, the solution is

$\boxed{x_1 = \frac{6}{5}, x_2 = \frac{3}{5}, z_{min} = \frac{12}{5}}$