University of Mumbai

Examination First Half 2022

Examinations Commencing from 3rd June 2022

Program: **Computer Engineering**

Curriculum Scheme: Rev2019

Examination: SE Semester III

Course Code: CSC301 and Course Name: Engineering Mathematics-III

Time: 2hour 30 minutes Max. Marks: 80

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Q1.	**Choose the correct option for following questions. All the Questions are compulsory and carry equal marks**
1.	In the Fourier series of f(x) $=\sqrt{1-\cos x}$ in $(0,2\pi)$ the value of a_0 is
Option A:	$\frac{2\sqrt{3}}{\pi}$
Option B:	$\frac{6\sqrt{2}}{\pi}$
Option C:	$\frac{2\sqrt{2}}{\pi}$
Option D:	$\frac{2\sqrt{2}}{4\pi}$
2.	The formula of complex form of Fourier series for function f(x) in $(-l,l)$ is
Option A:	$\sum_{-\infty}^{\infty} C_n e^{inx}$ where $C_n=\frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx$
Option B:	$\sum_{-\infty}^{\infty} C_n e^{in\pi x/l}$ where $C_n=\frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx$
Option C:	$\sum_{-\infty}^{\infty} C_n e^{inx}$ where $C_n=\frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx$
Option D:	$\sum_{-\infty}^{\infty} C_n e^{ix}$ where $C_n=\frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx$
3.	Evaluate $\int_0^{\infty} e^{-3t}t^5\,dt$.
Option A:	$\frac{60}{s^5}$
Option B:	$\frac{120}{s^6}$
Option C:	$\frac{120}{729}$
Option D:	$\frac{60}{729}$
4.	If $f(z)=u+iv$ is analytic then
Option A:	u is harmonic but v may or may not be harmonic.
Option B:	v is harmonic but u may or may not be harmonic.
Option C:	u and v both need not be harmonic.
Option D:	u and v both harmonic.

Here $l=\pi$

$a_0=\frac{1}{\pi}\int_0^{2\pi}\sqrt{1-\cos x}\,dx$

$=\frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\,dx$

$=\frac{\sqrt{2}}{\pi}\left[-\cos\frac{x}{2}\times 2\right]_0^{2\pi}$

$\Rightarrow a_0=-\frac{2\sqrt{2}}{\pi}\left[(-1)^n-1\right]$

No option is correct

Correct option : B

$\int_0^{\infty} e^{-3t}t^5\,dt = L\{t^5\}_{s=3}$

$=\left[\frac{\Gamma(6)}{s^6}\right]_{s=3}$

$=\frac{5!}{3^6}=\frac{120}{729}$

Correct option : C

Correct option : D

C:	
Option D:	u and v both harmonic.
5.	If Var(X) = 4 then Var(3x+5) is
Option A:	12
Option B:	20
Option C:	26
Option D: ✓	36

$Var(3x+5) = 3^2 Var(x) + 0$

$= 9 \times 4 = 36$

$\boxed{\text{correct option : D}}$

6.	If X has the following probability distribution X: 0 1 2 $P(X = x)$: k 2k 5k Then the value of k is
Option A:	1/6
Option B:	0
Option C:	1/3
Option D: ✓	1/8

$\Sigma p_x = 1 \Rightarrow k + 2k + 5k = 1$

$\Rightarrow 8k = 1$

$\Rightarrow \boxed{k = \frac{1}{8}}$

$\boxed{\text{Correct option : D}}$

7.	Find Inverse L.T. of $\frac{3}{9s^2-16}$.
Option A:	$\frac{1}{4}\sinh\left(\frac{3t}{4}\right)$
Option B:	$\frac{1}{4}\sin\left(\frac{3t}{4}\right)$
Option C: ✓	$\frac{1}{4}\sinh\left(\frac{4t}{3}\right)$
Option D:	$\frac{1}{4}\sin\left(\frac{4t}{3}\right)$

$L^{-1}\left\{\frac{3}{9s^2-16}\right\} = \frac{1}{9} L^{-1}\left\{\frac{3}{s^2-\frac{16}{9}}\right\}$

$= \frac{3}{9} L^{-1}\left\{\frac{1}{s^2-\frac{16}{9}}\right\}$

$= \frac{1}{3} \times \frac{1}{\frac{4}{3}} \sinh\left(\frac{4}{3}\right)t = \frac{1}{4}\sinh \frac{4t}{3}$

$\boxed{\text{Correct option : C}}$

8.	$L^{-1}\left[\frac{1}{s(s+4)}\right]$ is
Option A:	$\frac{1}{4}(e^{-4t} - 1)$
Option B: ✓	$\frac{1}{4}(1 - e^{-4t})$
Option C:	$(e^{-4t} - 1)$
Option D:	$\frac{1}{4}(e^{-4t} + 1)$

Let $F(s) = \frac{1}{s+4} \Rightarrow L^{-1}\{F(s)\} = f(t) = e^{-4t}$

$\Rightarrow f(u) = e^{-4u}$

Let $G(s) = \frac{1}{s} \Rightarrow L^{-1}\{G(s)\} = g(t) = 1$

$\Rightarrow g(t-u) = 1$

$\boxed{\text{Correct option : B}}$

By convolution thm

$L^{-1}\left\{\frac{1}{s(s+4)}\right\} = L^{-1}\{F(s)G(s)\}$

$= \int_0^t f(u)\, g(t-u)\, du$

$= \int_0^t e^{-4u} \cdot 1\, du = \left[\frac{e^{-4u}}{-4}\right]_0^t = -\frac{1}{4}\left[e^{-4t} - 1\right] = \frac{1}{4}\left[1 - e^{-4t}\right]$

9.	Find the Laplace transform of $\frac{\sin t}{t}$

9.	Find the Laplace transform of $\frac{sint}{t}$	$L\{\frac{\sin t}{t}\} = \int_s^\infty \frac{1}{s^2+1} ds = \tan^{-1}(s)\Big]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s) = \cot^{-1}(s)$
Option A: ✓	$cot^{-1}s$	
Option B:	$cot^{-1}t$	$\boxed{\text{Correct option : A}}$
Option C:	$tan^{-1}s$	
Option D:	$tan^{-1}t$	
10.	Find $L[(sin3t)(sin5t)]$	$L\{\sin 3t \sin 5t\} = \frac{1}{2} L\{\cos(2t) - \cos 8t\} = \frac{1}{2}\left[\frac{s}{s^2+4} - \frac{1}{s^2+64}\right]$
Option A:	$\frac{1}{2}\left[\frac{s}{s^2+4} + \frac{1}{s^2+64}\right]$	
Option B:	$\frac{1}{2}\left[\frac{s}{s^2-4} - \frac{1}{s^2-64}\right]$	
Option C:	$\frac{1}{2}\left[\frac{s}{s^2-4} - \frac{s}{s^2-64}\right]$	
Option D: ✓	$\frac{1}{2}\left[\frac{s}{s^2+4} - \frac{s}{s^2+64}\right]$	$\boxed{\text{Correct option : D}}$

Q2	**Solve any Four out of Six**	**5 marks each**
A	If $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\sqrt{s}}.e^{-1/(4s)}$, find $L\{\sin 2\sqrt{t}\}$.	
B	If $v = 3x^2y + 6xy - y^3$, show that v is harmonic function and find the corresponding analytic function.	
C	If the mean of the following distribution is 16. Find m, n and variance. X : 8, 12, 16, 20, 24 $P(X)$: 1/8 m n 1/4 1/12	
D	Evaluate the Fourier coefficients a_0 and a_n of $f(x) = \frac{1}{2}(\pi - x)$ in $(0, 2\pi)$.	
E	Find $L^{-1}\left(\log\left(1+\frac{a}{s}\right)\right)$.	
F	The Regression lines of a sample are x +6y = 6 and 3x+2y = 10. Find the coefficient of correlation between x and y.	
Q3	**Solve any Four out of Six**	**5 marks each**
A	Find the inverse Laplace transform of $\frac{s+29}{(s+4)(s^2+9)}$	
B	Calculate the value of rank correlation coefficient from the following data regarding marks of 6 students in Statistics and Mathematics in a test: $Marks: Statistics$: 40, 42, 45, 35, 36, 39 $Marks: Mathematics$: 46, 43, 44, 39, 40, 43	
C	By using Laplace transform, prove that $\int_0^\infty e^{-t}.\frac{\sin^2 t}{t} dt = \frac{1}{4}\log 5$	

D	Evaluate the Fourier coefficients a_0 and b_3 of $f(x) = x$ in $(0, 2\pi)$.
E	Show that the function, $f(z) = sinhz$ is analytic and find $f'(z)$ in terms of z.
F	The probability density function of a random variable X is <table><tr><td>X</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr><tr><td>P(X=x)</td><td>k</td><td>3k</td><td>5k</td><td>7k</td><td>9k</td><td>11k</td><td>13k</td></tr></table>Find P(X<4), P(3<x≤ 6).
Q4	**Solve any Four out of Six** **5 marks each**
A	Find the Fourier series for $f(x)$ in $(0,2\pi)$ where $f(x) = \begin{cases} x, & 0 < x \le \pi \\ 2\pi - x, & \pi \le x < 2\pi \end{cases}$
B	Using convolution theorem, find the inverse Laplace transform of $\frac{1}{(s-2)^4(s+3)}$
C	State true or false with justification. "If two lines of regression are $x+3y-5=0$ and $4x+3y-8=0$, then the correlation coefficient is $+0.5$".
D	Find $L(t\, e^{-3t}\, cos2t\, cos3t)$
E	A continuous random variable has the following probability density function $f(x) = \begin{cases} \frac{x}{4} + k, & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$ Evaluate k and P(1≤ $X \le 2$)
F	From the following data calculate Karl Pearson's coefficient of correlation (r) between X and Y. <table><tr><td>X</td><td>18</td><td>20</td><td>34</td><td>52</td><td>12</td></tr><tr><td>Y</td><td>39</td><td>23</td><td>35</td><td>18</td><td>46</td></tr></table>

Q2	Solve any Four out of Six	5 marks each
A	If $L\{\sin\sqrt{t}\}=\frac{\sqrt{\pi}}{2s\sqrt{s}}.e^{-1/(4s)}$, find $L\{\sin 2\sqrt{t}\}$.	
B	If $v=3x^2y+6xy-y^3$, show that v is harmonic function and find the corresponding analytic function.	
C	If the mean of the following distribution is 16. Find m, n and variance. X : 8, 12, 16, 20, 24 $P(X)$: 1/8 m n 1/4 1/12	
D	Evaluate the Fourier coefficients a_0 and a_n of $f(x)=\frac{1}{2}(\pi-x)$ in $(0,2\pi)$.	
E	Find $L^{-1}\left(\log\left(1+\frac{a}{s}\right)\right)$.	
F	The Regression lines of a sample are x +6y = 6 and 3x+2y = 10. Find the coefficient of correlation between x and y.	

A	If $L\{\sin\sqrt{t}\}=\frac{\sqrt{\pi}}{2s\sqrt{s}}.e^{-1/(4s)}$, find $L\{\sin 2\sqrt{t}\}$.

Solⁿ: We have $F(s) = L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\sqrt{s}}\, e^{-\frac{1}{4s}}$

By change of scale property

$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$ [1 M]

$\sin 2\sqrt{t} = \sin\sqrt{4t} \Rightarrow$ scale $a=4$ [2M]

$\therefore L\{\sin 2\sqrt{t}\} = \frac{1}{4} \frac{\sqrt{\pi}}{\cancel{2}\frac{s}{\cancel{4}_2}\sqrt{\frac{s}{4}}}\, e^{-\frac{1}{\cancel{4}\frac{s}{\cancel{4}}}}$

$= \frac{1}{\cancel{4}} \times \frac{\cancel{2\times 2}}{s\sqrt{s}}\, e^{-\frac{1}{s}}$

$\Rightarrow L\{\sin 2\sqrt{t}\} = \frac{e^{-\frac{1}{s}}}{s\sqrt{s}}$ [5M]

B	If $v=3x^2y+6xy-y^3$, show that v is harmonic function and find the corresponding analytic function.

Solⁿ: $v = 3x^2y+6xy-y^3$

$\Rightarrow v_x = 6xy+6y$ — ①; $v_y = 3x^2+6x-3y^2$ — ②

$v_{xx} = 6y$ $\quad v_{yy} = -6y$

$\Rightarrow v_{xx}+v_{yy} = 0$

i.e. v satisfies the Laplace's eqⁿ.

$\therefore$ v is harmonic [2M]

Let $f(z) = u+iv$ be the required analytic fⁿ

Now, $v_x(z,0) = 0$ (using ①)

[put $x=z$, $y=0$]

and $v_y(z,0) = 3z^2+6z$ [using ②] [3M]

Now $f(z) = u+iv$

$\Rightarrow f'(z) = u_x+iv_x$

$= v_y + i v_x$ (by C-R eqⁿˢ)

$= v_y(z,0)+i\, v_x(z,0)$ [By Milne Thompson Method] [4M]

$\Rightarrow f'(z) = 3z^2+6z$

$\Rightarrow f(z) = \int(3z^2+6z)\,dz$

$= \frac{3z^3}{3} + \frac{6z^2}{2} + c$

$\Rightarrow f(z) = z^3+3z^2+c$ [5M]

is the required analytic fⁿ

C	If the mean of the following distribution is 16. Find m, n and variance. X : 8, 12, 16, 20, 24 $P(X)$: 1/8 m n 1/4 1/12

Solⁿ: $\Sigma p_x = 1$

$\Rightarrow \frac{1}{8}+m+n+\frac{1}{4}+\frac{1}{12} = 1$

$\Rightarrow m+n = 1-\left(\frac{1}{8}+\frac{1}{4}+\frac{1}{12}\right) = 1-\left(\frac{3+6+2}{24}\right) = \frac{13}{24}$

i.e. $m+n = \frac{13}{24}$ — ① [1M]

Also, mean $= 16$

$\Rightarrow \Sigma x p_x = 16$

i.e. $8\times\frac{1}{8}+12m+16n+20\times\frac{1}{4}+24\times\frac{1}{12} = 16$

i.e. $12m+16n = 16-(1+5+2) = 8$ [2M]

i.e. $12m+16n = 8$ — ②

Solving ① & ② we get $m=\frac{1}{6}=0.167$, $n=\frac{3}{8}=0.375$ [3M]

Hence the prob. distⁿ of X is:

X:	8	12	16	20	24
$P(X=x)$:	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{12}$

Now $E(x^2) = \sum_x x^2 p_x = 64\times\frac{1}{8}+144\times\frac{1}{6}+256\times\frac{3}{8}+400\times\frac{1}{4}+576\times\frac{1}{12}$

$= 8+24+96+100+48$

i.e. $E(x^2) = 276$ [4M]

$\text{Var}(x) = E(x^2)-\{E(x)\}^2$

$= 276-(16)^2$

$\Rightarrow \text{Var}(x) = 20$ [5M]

D	Evaluate the Fourier coefficients a_0 and a_n of $f(x)=\frac{1}{2}(\pi-x)$ in $(0,2\pi)$.

Solⁿ: We have the Fourier Series of $f(x)$ to be:

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ [1M]

Here, $l = \pi$

and $a_0 = \frac{1}{l}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\,dx$

$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[2\pi^2 - \frac{\cancel{4}^{2}\pi^2}{\cancel{2}}\right] = 0$

$\Rightarrow \boxed{a_0 = 0}$ [3M]

and $a_n = \frac{1}{l}\int_0^{2\pi} f(x)\cos\frac{n\pi x}{l}\,dx$

$= \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\cos nx\,dx$

$= \frac{1}{2\pi}\left[(\pi-x)\left[\frac{\sin nx}{n}\right]-(-1)\left[-\frac{\cos nx}{n^2}\right]\right]_0^{2\pi}$

$\Rightarrow a_n = \frac{1}{2\pi}\left[0-\frac{1}{n^2}[1-1]\right]$

$\Rightarrow \boxed{a_n = 0}$ [5M]

E	Find $L^{-1}\left(\log\left(1+\frac{a}{s}\right)\right)$.

Solⁿ: $L^{-1}\left(\log\left(1+\frac{a}{s}\right)\right) = L^{-1}\left\{\log\left(\frac{s+a}{s}\right)\right\}$ [1M]

$= L^{-1}\left[\log(s+a)-\log s\right]$ [2M]

$= -\frac{1}{t}\left[L^{-1}\left[\left\{\frac{1}{s+a}\right\}-\frac{1}{s}\right]\right]$ [3M]

$\left[\because L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}\right]$ [4M]

i.e. $L^{-1}\left\{\log\left(1+\frac{a}{s}\right)\right\} = -\frac{1}{t}\left[e^{-at}-1\right]$ [5M]

F	The Regression lines of a sample are x +6y = 6 and 3x+2y = 10. Find the coefficient of correlation between x and y.

Solⁿ: Let the regression line of y on x be:

$x+6y = 6$

$\Rightarrow y = -\frac{1}{6}x+1$

$\Rightarrow b_{yx} = -\frac{1}{6}$ [regression coefficient of y on x] [1M]

Let the regression line of x on y be:

$3x+2y = 10$

$\Rightarrow x = -\frac{2}{3}y+\frac{10}{3}$

$\Rightarrow b_{xy} = -\frac{2}{3}$ [regression coefficient of x on y] [2M]

Now $r^2 = b_{yx}\, b_{xy}$ [3M]

$\Rightarrow r^2 = -\frac{1}{6}\times-\frac{2}{3}$

i.e. $r^2 = \frac{1}{9} < 1 \Rightarrow$ our assumption is right [4M]

$\Rightarrow r = -\frac{1}{3}$ ($\because b_{xy}, b_{yx} < 0 \Rightarrow r<0$)

$\therefore$ the correlation coefficient $r = -\frac{1}{3}$ [5M]

Q3	Solve any Four out of Six	5 marks each
A	Find the inverse Laplace transform of $\frac{s+29}{(s+4)(s^2+9)}$	
B	Calculate the value of rank correlation coefficient from the following data regarding marks of 6 students in Statistics and Mathematics in a test: *Marks : Statistics* : 40, 42, 45, 35, 36, 39 *Marks : Mathematics* : 46, 43, 44, 39, 40, 43	
C	By using Laplace transform, prove that $\int_0^\infty e^{-t}.\frac{\sin^2 t}{t}dt = \frac{1}{4}\log 5$	
D	Evaluate the Fourier coefficients a_0 and b_3 of $f(x) = x$ in $(0, 2\pi)$.	
E	Show that the function, $f(z) = \sinh z$ is analytic and find $f'(z)$ in terms of z.	
F	The probability density function of a random variable X is X: 0, 1, 2, 3, 4, 5, 6 P(X=x): k, 3k, 5k, 7k, 9k, 11k, 13k Find P(X<4), P(3<x≤ 6).	

A Find the inverse Laplace transform of $\frac{s+29}{(s+4)(s^2+9)}$

Sol^n: Let

$$F(s) = \frac{s+29}{(s+4)(s^2+9)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+9}$$ [1M]

$$\Rightarrow s+29 \equiv A(s^2+9) + (Bs+C)(s+4)$$

ie $s+29 = As^2 + 9A + Bs^2 + 4Bs + Cs + 4C$

Comparing like coeffs:

s^2: $0 = A + B \Rightarrow \boxed{A = -B}$ — ①

s: $1 = 4B + C \Rightarrow C = 1 - 4B$ — ②

constant: $29 = 9A + 4C$

$\Rightarrow 29 = -9B + 4(1-4B)$ [using ① & ②]

$\Rightarrow 25 = -25B \Rightarrow \boxed{B=-1} \Rightarrow \boxed{A=1}$

$\Rightarrow \boxed{C = 5}$.

$$\therefore F(s) = \frac{s+29}{(s+4)(s^2+9)} = \frac{1}{s+4} + \frac{-s+5}{s^2+9}$$ [2M]

$$\therefore L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+4}\right\} - L^{-1}\left\{\frac{s}{s^2+9}\right\} + 5L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

ie $L^{-1}\left\{\frac{s+29}{(s+4)(s^2+9)}\right\} = e^{-4t} - \cos 3t + \frac{5}{3}\sin 3t$ [5M]

B Calculate the value of rank correlation coefficient from the following data regarding marks of 6 students in Statistics and Mathematics in a test:

Marks : Statistics : 40, 42, 45, 35, 36, 39

Marks : Mathematics : 46, 43, 44, 39, 40, 43

Sol^n: We first construct the table of ranks

Let X: Marks in Statistics

and Y: Marks in Mathematics

X	Y	Rank in X r_X	Rank in Y r_Y	$d = r_X - r_Y$	d^2
40	46	3	1	2	4
42	43	2	3.5	−1.5	2.25
45	44	1	2	−1	1
35	39	6	6	0	0
36	40	5	5	0	0
39	43	4	3.5	0.5	0.25

$\Sigma d^2 = 7.5$ [2M]

correction factor: (CF)

Repeating rank : 3.5 (Y series)

No. of times repeating : 2

$$CF = \frac{2 \times (2^2-1)}{12} = \frac{1}{2} = 0.5$$ [3M]

Hence the Spearman's rank correlation coefficient is:

$$\rho = 1 - \frac{6[\Sigma d^2 + CF]}{n(n^2-1)}, \quad n = 6$$ [4M]

$$\Rightarrow \rho = 1 - \frac{\cancel{6}[7.5 + 0.5]}{\cancel{6} \times 35}$$

$\Rightarrow \rho = 0.7714$ [5M]

C By using Laplace transform, prove that $\int_0^\infty e^{-t}.\frac{\sin^2 t}{t}dt = \frac{1}{4}\log 5$

Sol^n: We have

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t}dt = L\left\{\frac{\sin^2 t}{t}\right\}_{s=1} \text{ —(*)}$$ [by the definition of Laplace transform] [1M]

Now, $L\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty L\{\sin^2 t\}ds$

$$= \int_s^\infty L\left\{\frac{1-\cos 2t}{2}\right\}ds$$ [3M]

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{1}{2}\log s^2 - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty$$

ie $L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4}\left[0 - \log\left(\frac{s^2}{s^2+4}\right)\right] = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$ —(**) [4M]

(*) & (**)

$$\Rightarrow \int_0^\infty e^{-t}\left(\frac{\sin^2 t}{t}\right)dt = \frac{1}{4}\log 5$$ [5M]

D Evaluate the Fourier coefficients a_0 and b_3 of $f(x) = x$ in $(0, 2\pi)$.

Sol^n: The Fourier Series of $f(x)$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^\infty b_n \sin\frac{n\pi x}{l}$$

Here $l = \pi$

$$a_0 = \frac{1}{l}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{4\pi^2}{2}\right]$$

$\Rightarrow \boxed{a_0 = 2\pi}$ [2M]

$$b_n = \frac{1}{l}\int_0^{2\pi} f(x)\sin\frac{n\pi x}{l}dx$$

ie $b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin nx\, dx$

$\Rightarrow b_3$ $(n=3)$ $= \frac{1}{\pi}\int_0^{2\pi} x \sin 3x\, dx$ [3M]

$$= \frac{1}{\pi}\left[x\left[-\frac{\cos 3x}{3}\right] - (1)\left[-\frac{\sin 3x}{3^2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{3}[2\pi \cos 6\pi] + 0\right]$$

ie $b_3 = -\frac{2}{3}$ [5M]

E Show that the function, $f(z) = \sinh z$ is analytic and find $f'(z)$ in terms of z.

Sol^n: Let $f(z) = u + iv = \sinh z$

$= \sinh(x+iy)$

$= \sinh x \cos y + i \cosh x \sin y$ [1M]

$\Rightarrow u = \sinh x \cos y$; $v = \cosh x \sin y$

$\Rightarrow u_x = \cosh x \cos y$; $v_x = \sinh x \sin y$

& $u_y = -\sinh x \sin y$; $v_y = \cosh x \cos y$

$\Rightarrow u_x = v_y$ & $u_y = -v_x \Rightarrow$ C-R eq^{ns} are satisfied — I

Also, u, v, u_x, u_y, v_x, v_y are all continuous functions — II [3M]

I and II $\Rightarrow f(z) = \sinh z$ is analytic

Now, $f(z) = \sinh z = u + iv$

$\Rightarrow f'(z) = u_x + iv_x$

$= \cosh x \cos y + i \sinh x \sin y$ [4M]

$= \cosh(x+iy)$

ie $f'(z) = \cosh z$ [5M]

F The probability density function of a random variable X is

X	0	1	2	3	4	5	6
P(X=x)	k	3k	5k	7k	9k	11k	13k

Find P(X<4), P(3<x≤ 6).

Sol^n: $\Sigma p_x = 1$

$\Rightarrow k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

$\Rightarrow \boxed{k = \frac{1}{49}}$ [1M]

$\therefore$ the probability distribution of X is:

X :	0	1	2	3	4	5	6
P(X=x) :	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$

[2M]

Now, $P(X<4) = 1 - P(X \geq 4)$

$= 1 - P(X = 4 \text{ or } 5 \text{ or } 6)$

$= 1 - \left[\frac{9}{49} + \frac{11}{49} + \frac{13}{49}\right]$

ie $P(X<4) = \frac{16}{49} = 0.3265$ [4M]

& $P(3 < X \leq 6) = P(X = 4 \text{ or } 5 \text{ or } 6)$

$= \frac{9}{49} + \frac{11}{49} + \frac{13}{49}$

$= \frac{33}{49} = 0.6735$ [5M]

Q4	Solve any Four out of Six	5 marks each
A	Find the Fourier series for $f(x)$ in $(0,2\pi)$ where $f(x)=\begin{cases} x, & 0<x\le\pi \\ 2\pi-x, & \pi\le x<2\pi \end{cases}$	
B	Using convolution theorem, find the inverse Laplace transform of $\frac{1}{(s-2)^4(s+3)}$	
C	State true or false with justification. "If two lines of regression are $x+3y-5=0$ and $4x+3y-8=0$, then the correlation coefficient is $+0.5$"	
D	Find $L(t\,e^{-3t}\cos 2t\cos 3t)$	
E	A continuous random variable has the following probability density function $f(x)=\begin{cases} \frac{x}{4}+k, & 0\le x\le 2 \\ 0, & elsewhere \end{cases}$ Evaluate k and $P(1\le X\le 2)$	
F	From the following data calculate Karl Pearson's coefficient of correlation (r) between X and Y. (X: 18, 20, 34, 52, 12; Y: 39, 23, 35, 18, 46)	

A	Find the Fourier series for $f(x)$ in $(0,2\pi)$ where $f(x)=\begin{cases} x, & 0<x\le\pi \\ 2\pi-x, & \pi\le x<2\pi \end{cases}$

Soln: The Fourier series for $f(x)$ is:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

Where

$$a_0=\frac{1}{l}\int_0^{2\pi}f(x)\,dx\ ;\ a_n=\frac{1}{l}\int_0^{2\pi}f(x)\cos\frac{n\pi x}{l}\,dx;\ b_n=\frac{1}{l}\int_0^{2\pi}f(x)\sin\frac{n\pi x}{l}\,dx$$

[1M]

Here $l=\pi$

$$\therefore a_0=\frac{1}{\pi}\left[\int_0^{\pi}x\,dx+\int_{\pi}^{2\pi}(2\pi-x)\,dx\right]=\frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi}+\left[2\pi x-\frac{x^2}{2}\right]_{\pi}^{2\pi}\right\}$$

$$=\frac{1}{\pi}\left[\frac{\pi^2}{2}+\left[\cancel{4\pi^2}-\frac{4\pi^2}{2}-\left(\cancel{2\pi^2}-\frac{\pi^2}{2}\right)\right]\right]$$

$$\Rightarrow a_0=\frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^2}{2}\right]\quad\Rightarrow\boxed{a_0=\pi}$$

[2M]

$$a_n=\frac{1}{\pi}\left\{\int_0^{\pi}x\cos nx\,dx+\int_{\pi}^{2\pi}(2\pi-x)\cos nx\,dx\right\}$$

$$\Rightarrow a_n=\frac{1}{\pi}\left\{\left[x\left[\frac{\sin nx}{n}\right]-(1)\left[-\frac{\cos nx}{n^2}\right]\right]_0^{\pi}+\left[(2\pi-x)\frac{\sin nx}{n}-(-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$\Rightarrow a_n=\frac{1}{\pi}\left\{0+\frac{1}{n^2}\left[(-1)^n-1\right]+0-\frac{1}{n^2}\left(1-(-1)^n\right)\right\}$$

$$\Rightarrow\boxed{a_n=\frac{1}{\pi}\left[\frac{2}{n^2}\left((-1)^n-1\right)\right]}$$

[4M]

$$b_n=\frac{1}{\pi}\left[\int_0^{\pi}x\sin nx\,dx+\int_{\pi}^{2\pi}(2\pi-x)\sin nx\,dx\right]$$

$$=\frac{1}{\pi}\left\{\left[x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}+\left[(2\pi-x)\left[-\frac{\cos nx}{n}\right]-(-1)\left[-\frac{\sin nx}{n^2}\right]\right]_{\pi}^{2\pi}\right\}$$

$$\Rightarrow b_n=\frac{1}{\pi}\left\{\left[-\frac{1}{n}\left[\pi(-1)^n-0\right]+0\right]+\left[-\frac{1}{n}\left[0-\pi(-1)^n\right]+0\right]\right\}$$

$$=\frac{1}{\pi}\left\{-\frac{\pi}{n}(-1)^n+\frac{\pi}{n}(-1)^n\right\}$$

$$\Rightarrow b_n=0$$

$\therefore$ the Fourier series of $f(x)$ is:

$$f(x)=\frac{\pi}{2}+\sum\frac{2}{\pi n^2}\left((-1)^n-1\right)\cos nx$$

[5M]

B	Using convolution theorem, find the inverse Laplace transform of $\frac{1}{(s-2)^4(s+3)}$

Soln: Let $F(s)=\frac{1}{(s-2)^2}$ & $G(s)=\frac{1}{s+3}$

$$\Rightarrow f(t)=L^{-1}\{F(s)\}=e^{2t}L^{-1}\left\{\frac{1}{s^2}\right\}\ \&\ g(t)=L^{-1}\left\{\frac{1}{s+3}\right\}=e^{-3t}$$

$$=e^{2t}\cdot t\qquad\Rightarrow g(t-u)=e^{-3(t-u)}=e^{-3t}\cdot e^{3u}$$

$$\Rightarrow f(u)=e^{2u}\cdot u$$

[2M]

$$\therefore L^{-1}\left\{\frac{1}{(s-2)^2(s+3)}\right\}=L^{-1}\{F(s)\,G(s)\}$$

$$=\int_0^t f(u)\,g(t-u)\,du$$

[3M]

$$=\int_0^t e^{2u}\cdot u\cdot e^{-3t}e^{3u}\,du$$

$$=e^{-3t}\int_0^t u\,e^{5u}\,du$$

$$=e^{-3t}\left[u\frac{e^{5u}}{5}-(1)\frac{e^{5u}}{25}\right]_0^t$$

[4M]

$$=e^{-3t}\left[\frac{1}{5}\left[te^{5t}-0\right]-\frac{1}{25}\left[e^{5t}-1\right]\right]$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(s-2)^2(s+3)}\right\}=\frac{t}{5}e^{2t}-\frac{1}{25}\left[e^{2t}-e^{-3t}\right]$$

[5M]

C	State true or false with justification. "If two lines of regression are $x+3y-5=0$ and $4x+3y-8=0$, then the correlation coefficient is $+0.5$"

Soln: Let the regression line of y on x be

$$x+3y-5=0$$

$$\Rightarrow y=-\frac{1}{3}x+5$$

$\Rightarrow$ the regression coefficient $b_{yx}=-\frac{1}{3}$ [1M]

Let the regression line of x on y be

$$4x+3y-8=0$$

$$\Rightarrow x=-\frac{3}{4}y+2$$

$\Rightarrow$ the regression coefficient $b_{xy}=-\frac{3}{4}$ [2M]

Now, $r^2=b_{yx}b_{xy}$

$$=\left(-\frac{1}{3}\right)\left(-\frac{3}{4}\right)=\frac{1}{4}$$

[3M]

$\Rightarrow r=-\frac{1}{2}$ ie $r=-0.5$ ($\because b_{xy}<0, b_{yx}<0 \Rightarrow r<0$) [4M]

$\therefore$ the given value of r is not correct

& $\therefore$ the answer is False [5M]

D	Find $L(t\,e^{-3t}\cos 2t\cos 3t)$

Soln: $L\{te^{-3t}\cos 2t\cos 3t\}=L\{t\cos 2t\cos 3t\}\Big]_{s\to s+3}$ — (*)

[1M]

Now, $L\{\cos 2t\cos 3t\}=\frac{1}{2}L\{\cos 5t+\cos t\}$

$$=\frac{1}{2}\left\{\frac{s}{s^2+25}+\frac{s}{s^2+1}\right\}$$

[2M]

$$\Rightarrow L\{t\cos 2t\cos 3t\}=-\frac{d}{ds}\left\{\frac{1}{2}\left(\frac{s}{s^2+25}+\frac{s}{s^2+1}\right)\right\}$$

[3M]

$$=-\frac{1}{2}\left[\frac{s^2+25-s(2s)}{(s^2+25)^2}+\frac{s^2+1-s(2s)}{(s^2+1)^2}\right]$$

$$=-\frac{1}{2}\left[\frac{25-s^2}{(s^2+25)^2}+\frac{1-s^2}{(s^2+1)^2}\right]$$

[4M]

ie $L\{t\cos 2t\cos 3t\}=\frac{1}{2}\left[\frac{s^2-25}{(s^2+25)^2}+\frac{s^2-1}{(s^2+1)^2}\right]$ — (**)

(*) & (**) $\Rightarrow$

$$L\{e^{-3t}t\cos 2t\cos 3t\}=\frac{1}{2}\left[\frac{(s+3)^2-25}{[(s+3)^2+25]^2}+\frac{(s+3)^2-1}{[(s+3)^2+1]^2}\right]$$

[5M]

E	A continuous random variable has the following probability density function $f(x)=\begin{cases} \frac{x}{4}+k, & 0\le x\le 2 \\ 0, & elsewhere \end{cases}$ Evaluate k and $P(1\le X\le 2)$

Soln: Since $f(x)$ is a pdf, we have

$$\int_{-\infty}^{\infty}f(x)\,dx=1$$

$$\Rightarrow\int_0^2\left(\frac{x}{4}+k\right)dx=1$$

$$\Rightarrow\left.\frac{x^2}{2\times 4}+kx\right]_0^2=1\Rightarrow\frac{1}{2}+2k=1$$

$$\Rightarrow\boxed{k=\frac{1}{4}}$$

[2M]

$$\therefore f(x)=\begin{cases}\frac{x}{4}+\frac{1}{4}, & 0\le x\le 2\\ 0, & elsewhere\end{cases}$$

Now $P(1\le x\le 2)=\int_1^2 f(x)\,dx$

$$=\int_1^2\left[\frac{x}{4}+\frac{1}{4}\right]dx$$

$$=\left.\frac{x^2}{8}+\frac{x}{4}\right]_1^2$$

$$=\frac{4}{8}-\frac{1}{8}+\frac{2}{4}-\frac{1}{4}$$

$$=\frac{5}{8}$$

$$\Rightarrow P(1\le x\le 2)=\frac{5}{8}=0.625$$

[5M]

F	From the following data calculate Karl Pearson's coefficient of correlation (r) between X and Y.

X	18	20	34	52	12
Y	39	23	35	18	46

Soln: The Karl Pearson's coefficient of correlation

$$r=\frac{Cov(x,y)}{\sigma_x\sigma_y}=\frac{\frac{1}{n}\Sigma xy-\frac{1}{n}\Sigma x\frac{1}{n}\Sigma y}{\sqrt{\frac{1}{n}\Sigma x^2-\left(\frac{1}{n}\Sigma x\right)^2}\sqrt{\frac{1}{n}\Sigma y^2-\left(\frac{1}{n}\Sigma y\right)^2}}\quad -(*)$$

[1M]

Here, $n=5$

$\Sigma x=136$, $\Sigma x^2=4728$

$\Sigma y=161$, $\Sigma y^2=5715$, $\Sigma xy=3840$

Substituting these values in (*) we get [4M]

$$r=\frac{768-(27.2)(32.2)}{\sqrt{945.6-739.84}\sqrt{1143-1036.84}}=\frac{107.84}{\sqrt{205.76}\sqrt{106.16}}$$

$$\Rightarrow r=0.7297$$

[5M]