Complex Integration

Weight Distribution of Types

MechCivil

Type	Name	Nov 2017	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
I	Cartesian form	---	---	---	---	---	---	05	05	---	---	---
II	Polar form	---	05	---	---	05	---	---	---	05	05	---
Total Marks		00	05	00	00	05	00	05	05	05	05	00

Comp/IT/AI

Type	Name	Nov 2017	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
I	Cartesian form	---	05	05	05	---	05	05	---	05	05	05
II	Polar form	05	---	---	---	05	---	---	05	---	---	---
Total Marks		05	05	05	05	05	05	05	05	05	05	05

Extc

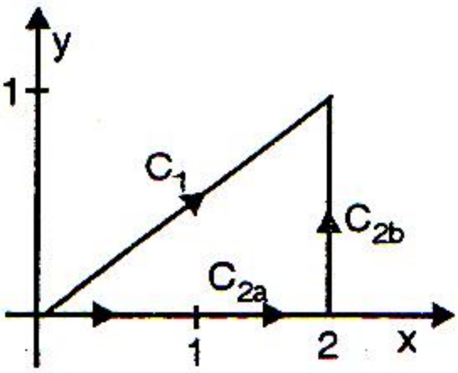
Type	Name	Nov 2017	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
I	Cartesian form	---	06	---	---	---	05	05	05	---	05	05
II	Polar form	---	---	05	---	---	---	---	---	---	---	---
Total Marks		00	06	05	00	00	05	05	05	00	05	05

Elect

Type	Name	Nov 2017	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
I	Cartesian form	---	---	06	---	05	---	05	---	---	---	05
II	Polar form	---	---	---	---	---	05	---	---	---	---	---
Total Marks		00	00	06	00	05	05	05	00	00	00	05

Type I: Cartesian form

1. Integrate $f(z) = z^2$ along the two paths from $z = 0$ to $z = 2 + i$ as shown in figure.

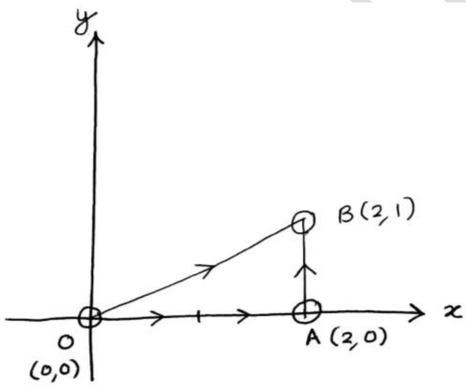


Solution:

$I = \int f(z)dz$

$I = \int z^2 dz$

$I = \int (x + iy)^2 (dx + i\, dy)$



(a) Along OB

$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$\frac{y-0}{1-0} = \frac{x-0}{2-0}$

$\frac{y}{1} = \frac{x}{2}$

$2y = x$

$x = 2y$

$dx = 2dy$

Thus, the integral becomes,

$I = \int (2y + iy)^2 (2dy + i\, dy)$

$I = \int_0^1 (2+i)^2 y^2 (2+i) dy$

$I = (2+i)^3 \int_0^1 y^2 dy \qquad \because \int x^n dx = \frac{x^{n+1}}{n+1}$

$I = (2 + 11i) \left[\frac{y^3}{3}\right]_0^1$

$I = (2 + 11i) \left[\frac{1^3}{3} - \frac{0^3}{3}\right]$

$I = \frac{2+11i}{3}$

(b) Along OAB

(i) along OA

$y = 0$ (x axis)

$dy = 0$

Thus, integral becomes

$I_1 = \int (x+0)^2 (dx+0)$

$I_1 = \int_0^2 x^2 dx$

$I_1 = \left[\frac{x^3}{3}\right]_0^2$

$I_1 = \frac{8}{3}$

(ii) Along AB

$x = 2$

$dx = 0$

Thus, integral becomes

$I_2 = \int (2+iy)^2 (0 + i\,dy)$

$I_2 = \int_0^1 (2+iy)^2 . i\,dy$ $\quad\quad \int x^n dx = \frac{x^{n+1}}{n+1}; \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)\times a}$

$I_2 = i\left[\frac{(2+iy)^3}{3\times i}\right]_0^1$

$I_2 = i\left[\frac{(2+i)^3}{3i} - \frac{(2+0)^3}{3i}\right]$

$I_2 = i\left[\frac{(2+i)^3}{3i} - \frac{8}{3i}\right]$

$I_2 = i\left[\frac{11}{3} + 2i\right]$

$I_2 = -2 + \frac{11}{3}i$

Thus,

$I = I_1 + I_2 = \frac{8}{3} + \left(-2 + \frac{11}{3}i\right)$

$I = \frac{2}{3} + \frac{11}{3}i$

2. Evaluate $\int_0^{1+i} (x^2 + iy)dz$ along the path(i) $y = x$ (ii) $y = x^2$
[N14/ElexExtcElectBiomInst/5M][N22/CompITAI/5M]
Is the line integral independent of the path?
[M22/Chem/5M][N22/Chem/6M]
Solution:
Let $I = \int_0^{1+i}(x^2 + iy)(dx + idy)$
(i) Along the path $y = x$

$$y = x$$
$$dy = dx$$

The integral becomes,

$$I = \int_0^1 (x^2 + ix)(dx + idx)$$
$$I = (1 + i)\int_0^1 (x^2 + ix)dx$$
$$I = (1 + i)\left[\frac{x^3}{3} + i\frac{x^2}{2}\right]_0^1$$
$$I = (1 + i)\left(\frac{1}{3} + \frac{i}{2}\right)$$
$$I = \frac{-1 + 5i}{6}$$

(ii) Along the path $y = x^2$

$$y = x^2$$
$$dy = 2x\,dx$$

The integral becomes,

$$I = \int_0^1 (x^2 + ix^2)(dx + i2xdx)$$
$$I = \int_0^1 (1 + i)x^2(1 + 2xi)dx$$
$$I = (1 + i)\int_0^1 (x^2 + 2ix^3)dx$$
$$I = (1 + i)\left[\frac{x^3}{3} + 2i\frac{x^4}{4}\right]_0^1$$
$$I = (1 + i)\left(\frac{1}{3} + \frac{i}{2}\right)$$
$$I = \frac{-1 + 5i}{6}$$

Now, consider
$f(z) = x^2 + iy$
where,

$u = x^2$	$v = y$
$u_x = 2x$	$v_x = 0$
$u_y = 0$	$v_y = 1$

We see that, $u_x \neq v_y$
Thus, $f(z)$ is not an analytic function and therefore the line integral is not independent of the path of integration

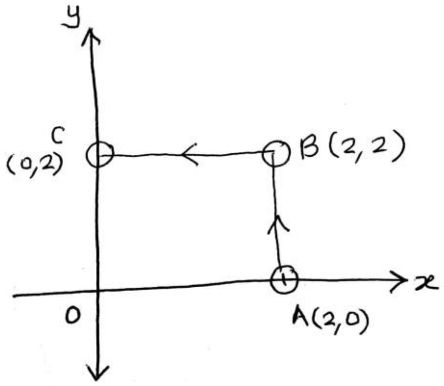
3. Evaluate the line integral $\int_c (z^2+3z)dz$ along a straight line from $(2,0)$ to $(2,2)$ and then from $(2,2)$ to $(0,2)$

Solution:

$I=\int f(z)dz$

$I=\int_c (z^2+3z)dz$

$I=\int_c [(x+iy)^2+3(x+iy)][dx+idy]$



(i) Along AB

$x=2$

$dx=0$

Thus, integral becomes

$I_1=\int_c [(2+iy)^2+3(2+iy)][0+idy]$

$I_1=\int_0^2 [(2+iy)^2+3(2+iy)]i\,dy \qquad \because \int (ax+b)^n dx=\frac{(ax+b)^{n+1}}{(n+1)\times a}$

$I_1=i\left[\frac{(2+iy)^3}{3\times i}+\frac{3(2+iy)^2}{2\times i}\right]_0^2$

$I_1=i\left[\frac{(2+2i)^3}{3i}+\frac{3(2+2i)^2}{2i}-\frac{(2)^3}{3i}-\frac{3(2)^2}{2i}\right]$

$I_1=-14+\frac{52}{3}i$

(ii) Along BC

$y=2$

$dy=0$

Thus, integral becomes

$I_2=\int_c [(x+i2)^2+3(x+i2)][dx+0]$

$I_2=\int_2^0 [(x+2i)^2+3(x+2i)]dx$

$I_2=\left[\frac{(x+2i)^3}{3}+\frac{3(x+2i)^2}{2}\right]_2^0$

$I_2=\left[\frac{(2i)^3}{3}+\frac{3(2i)^2}{2}-\frac{(2+2i)^3}{3}-\frac{3(2+2i)^2}{2}\right]$

$I_2=-\frac{2}{3}-20i$

Thus, $I=I_1+I_2=-14+\frac{52}{3}i-\frac{2}{3}-20i=-\frac{44}{3}-\frac{8}{3}i$

4. Evaluate $\int_{1-i}^{2+i}(2x+iy+1)dz$, along
(i) The straight line joining $(1-i) to\ (2+i)$
(ii) $x=t+1, y=2t^2-1$
[M14/N16/ChemBiot/6M]
Solution:
Let $I=\int_{1-i}^{2+i}(2x+iy+1)(dx+idy)$
(i) Along the straight line joining $(1-i)$ to $(2+i)$
Equation of straight line,

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$
$$\frac{y-(-1)}{1-(-1)}=\frac{x-1}{2-1}$$
$$\frac{y+1}{2}=(x-1)$$
$$y+1=2x-2$$
$$y=2x-3$$
$$dy=2dx$$

The integral becomes,

$$I=\int_1^2(2x+i(2x-3)+1)(dx+i2dx)$$
$$I=\int_1^2(2x+i2x-3i+1)(1+2i)dx$$
$$I=(1+2i)\left[2\frac{x^2}{2}+2i\frac{x^2}{2}-3ix+x\right]_1^2$$
$$I=(1+2i)(4+4i-6i+2-1-i+3i-1)$$
$$I=(1+2i)(4)$$
$$I=4+8i$$

(ii) Along the path

$$x=t+1, y=2t^2-1$$
$$dx=dt\ ,\ dy=4tdt$$

When $x=1, t=0$ and when $x=2, t=1$
The integral becomes,

$$I=\int_0^1(2(t+1)+i(2t^2-1)+1)(dt+i4tdt)$$
$$I=\int_0^1(2t+2+2it^2-i+1)(1+4it)dt$$
$$I=\int_0^1[(2t+3)+i(2t^2-1)](1+4it)dt$$
$$I=\int_0^1[2t+3+8it^2+12it+2it^2+8i^2t^3-i-4i^2t]dt$$
$$I=\int_0^1[(-8t^3+6t+3)+i(10t^2+12t-1)]dt$$
$$I=\left[\left(-\frac{8t^4}{4}+\frac{6t^2}{2}+3t\right)+i\left(\frac{10t^3}{3}+\frac{12t^2}{2}-t\right)\right]_0^1$$
$$I=(-2+3+3)+i\left(\frac{10}{3}+6-1\right)$$
$$I=4+\frac{25}{3}i$$

5. Evaluate $\int_0^{1+i} \overline{z}^2\, dz$ along the line $y = x$

Solution:

$I = \int_0^{1+i} \overline{z}^2\, dz$

$I = \int_0^{1+i} (x - iy)^2 (dx + idy)$

Along the line,

$y = x$

$dy = dx$

Thus, integral becomes

$I = \int_0^1 (x - ix)^2 (dx + idx)$

$I = \int_0^1 (1 - i)^2 x^2 (1 + i) dx$

$I = (1 - i)^2 (1 + i) \int_0^1 x^2 dx$

$I = (-2i)(1 + i) \left[\frac{x^3}{3}\right]_0^1$

$I = (2 - 2i) \left[\frac{1}{3} - 0\right]$

$I = \frac{2-2i}{3}$

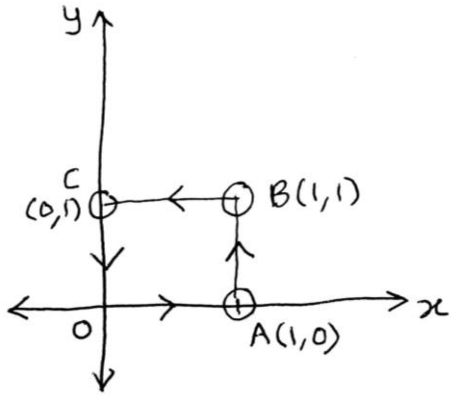
6. Evaluate $\int_c |z|^2\, dz$ where C is the boundary of the square C with vertices $(0,0),(1,0),(1,1),(0,1)$.

Solution:

$I = \int_c |z|^2 dz$

$I = \int_c \left(\sqrt{x^2+y^2}\right)^2 (dx + idy)$

$I = \int_c (x^2+y^2)(dx+idy)$



(i) Along OA

$y = 0$ (X axis)

$dy = 0$

Thus, integral becomes

$I_1 = \int_c \left(\sqrt{x^2+0}\right)^2 (dx+0)$

$I_1 = \int_0^1 x^2 dx$

$I_1 = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

(ii) Along AB

$x = 1$

$dx = 0$

Thus, integral becomes

$I_2 = \int_c \left(\sqrt{1+y^2}\right)^2 (0+idy)$

$I_2 = \int_0^1 (1+y^2)(idy)$

$I_2 = i\left[y + \frac{y^3}{3}\right]_0^1$

$I_2 = \frac{4}{3} i$

(iii) Along BC

$y = 1$

$dy = 0$

Thus, integral becomes

$I_3 = \int_c \left(\sqrt{x^2+1^2}\right)^2 (dx+0)$

$I_3 = \int_1^0 (x^2 + 1)dx$

$I_3 = \left[\frac{x^3}{3} + x\right]_1^0$

$I_3 = -\frac{4}{3}$

(iv) Along CO

$x = 0$ (Y axis)

$dx = 0$

Thus, integral becomes

$I_4 = \int_c \left(\sqrt{0 + y^2}\right)^2 (0 + idy)$

$I_4 = \int_1^0 y^2 i\, dy$

$I_4 = i\left[\frac{y^3}{3}\right]_1^0$

$I_4 = -\frac{i}{3}$

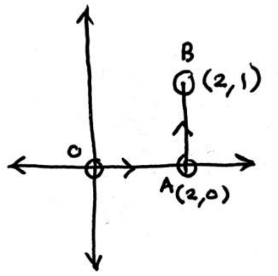
Thus,

$I = I_1 + I_2 + I_3 + I_4$

$I = \frac{1}{3} + \frac{4}{3}i - \frac{4}{3} - \frac{i}{3}$

$I = -1 + i$

7. Evaluate $\int_0^{2+i} \bar{z}^2\, dz$ along
(i) $y = \frac{x}{2}$
(ii) The real axis to 2 and then vertically to $2 + i$
[N18/Elex/6M]
Solution:
$I = \int f(z)dz = \int_{(0,0)}^{(2,1)} (x - iy)^2 (dx + idy)$
(i) Along the path $y = \frac{x}{2}$
$x = 2y$
$dx = 2dy$
The integral becomes,
$I = \int_0^1 (2y - iy)^2 (2dy + idy)$
$I = \int_0^1 y^2 (2 - i)^2 (2 + i) dy$
$I = (10 - 5i) \left[\frac{y^3}{3}\right]_0^1 = \frac{10-5i}{3}$
(ii) Along the real axis to 2 and then vertically to $2 + i$



(a) Along OA
$y = 0$
$dy = 0$
The integral becomes,
$I_1 = \int_0^2 x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$
(b) Along AB
$x = 2$
$dx = 0$
The integral becomes
$I_2 = \int_0^1 (2 - iy)^2 (idy)$
$I_2 = i\left[\frac{(2-iy)^3}{-3i}\right]_0^1$
$I_2 = \frac{(2-i)^3}{-3} - \frac{8}{-3}$
$I_2 = 2 + \frac{11}{3} i$
Thus, $I = I_1 + I_2 = \frac{8}{3} + 2 + \frac{11}{3} i = \frac{14}{3} + \frac{11}{3} i$

8. Evaluate $\int_0^{2+i} z^2\,dz$ along the line joining the point $z_1 = 0$ and $z_2 = 2+i$
[N17/MTRX/5M]
Solution:
Let $I = \int_0^{2+i} z^2\,dz$
$I = \int_0^{2+i} (x+iy)^2(dx+idy)$
Along the straight line joining (0,0) and (2,1)
Equation of straight line,
$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$
$\frac{y-0}{1-0} = \frac{x-0}{2-0}$
$\frac{y}{1} = \frac{x}{2}$
$x = 2y$
$dx = 2dy$
The integral becomes,
$I = \int_0^1 (2y+iy)^2(2dy+idy)$
$I = \int_0^1 y^2(2+i)^2(2+i)\,dy$
$I = (2+i)^3\left[\frac{y^3}{3}\right]_0^1$
$I = (2+11i)\left[\frac{1}{3}\right]$
$I = \frac{2+11i}{3}$

9. Integrate $x^2 + ixy$ along the line from $A(1,1)$ to $B(2,4)$
[N19/Inst/5M]
Solution:
$I = \int f(z)dz = \int_{(1,1)}^{(2,4)}(x^2 + ixy)(dx + idy)$
Along the line AB,
$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$
$\frac{y-1}{4-1} = \frac{x-1}{2-1}$
$\frac{y-1}{3} = \frac{x-1}{1}$
$y - 1 = 3x - 3$
$y = 3x - 2$
$dy = 3dx$
The integral becomes,
$I = \int_1^2 \left(x^2 + ix(3x-2)\right)(dx + i3dx)$
$I = \int_1^2 \left(x^2 + i(3x^2 - 2x)\right)(1+3i)dx$
$I = (1+3i)\left[\frac{x^3}{3} + i\left(\frac{3x^3}{3} - \frac{2x^2}{2}\right)\right]_1^2$
$I = (1+3i)\left[\left(\frac{8}{3} + i(8-4)\right) - \left(\frac{1}{3} + i(1-1)\right)\right]$
$I = (1+3i)\left[\frac{7}{3} + 4i\right]$
$I = -\frac{29}{3} + 11i$

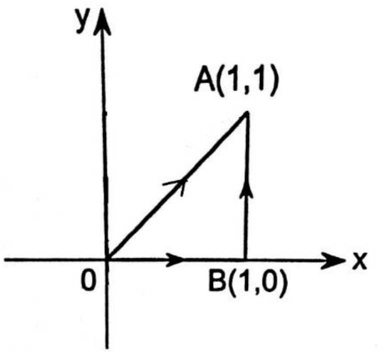
10. Integrate $x^2 + ixy$ from $A(1,1)$ to $B(2,4)$ along the curve $x = t, y = t^2$
[M17/MechCivil/5M][M19/Comp/5M]
Solution:
$I = \int f(z)dz = \int_{(1,1)}^{(2,4)}(x^2 + ixy)(dx + idy)$
Along the curve,
$x = t$ $\quad$ $y = t^2$
$dx = dt$ $\quad$ $dy = 2tdt$
When $x = 1, t = 1$ and when $x = 2, t = 2$
The integral becomes,
$I = \int_1^2 \left(t^2 + it(t^2)\right)(dt + i2tdt)$
$I = \int_1^2 (t^2 + it^3)(1 + 2it)dt$
$I = \int_1^2 (t^2 + 2it^3 + it^3 + 2i^2t^4)dt$
$I = \int_1^2 (t^2 - 2t^4) + i3t^3.dt$
$I = \left[\frac{t^3}{3} - \frac{2t^5}{5} + 3i\frac{t^4}{4}\right]_1^2 = -\frac{151}{15} + \frac{45}{4}i$

11. Integrate function $f(z) = x^2 + ixy$ from $A(1,1)$ to $B(2,4)$ along $y = x^2$
[D24/CompIT/5M]
Solution:
$I = \int f(z)dz = \int_{(1,1)}^{(2,4)} (x^2 + ixy)(dx + idy)$
Along the curve,
$y = x^2$
$dy = 2x\, dx$
The integral becomes,
$I = \int_1^2 \left(x^2 + ix(x^2)\right)(dx + i\, 2xdx)$
$I = \int_1^2 (x^2 + ix^3)(1 + 2ix)dx$
$I = \int_1^2 (x^2 + 2ix^3 + ix^3 + 2i^2x^4)dx$
$I = \int_1^2 (x^2 - 2x^4) + i3x^3 . dx$
$I = \left[\frac{x^3}{3} - \frac{2x^5}{5} + 3i\frac{x^4}{4}\right]_1^2$
$I = -\frac{151}{15} + \frac{45}{4}i$

12. Evaluate $\int \bar{z}dz$ from $z = 0$ to $z = 4 + 2i$ along the curve $z = t^2 + it$
[N14/ChemBiot/5M]
Solution:
Let $I = \int_0^{4+2i} \bar{z}dz = \int_0^{4+2i} (x - iy)(dx + idy)$
Along the curve $z = t^2 + it = x + iy$
$x = t^2 , y = t$
$dx = 2tdt, dy = dt$
When $x = 0, t = 0$ and when $x = 4, t = 2$
The integral becomes,
$I = \int_0^2 (t^2 - it)(2tdt + idt)$
$I = \int_0^2 (t^2 - it)(2t + i)dt$
$I = \int_0^2 [(2t^3 + it^2 - 2it^2 - i^2t)]dt$
$I = \left[\frac{2t^4}{4} + \frac{it^3}{3} - \frac{2it^3}{3} + \frac{t^2}{2}\right]_0^2$
$I = 8 + \frac{8i}{3} - \frac{16i}{3} + 2$
$I = 10 - \frac{8}{3}i$

13. Evaluate the integral $\int \overline{z} dz$ along a straight line from $z = 0$ to $z = 4 + 2i$
[N22/Elex/6M]
Solution:
Let $I = \int_0^{4+2i} \overline{z} dz$
$I = \int_0^{4+2i} (x - iy)(dx + idy)$
Along the line AB,
where $A = (0,0)$ and $B = (4,2)$
$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$
$\frac{y-0}{2-0} = \frac{x-0}{4-0}$
$\frac{y}{2} = \frac{x}{4}$
$2y = x$
$x = 2y$
$dx = 2dy$
The integral becomes,
$I = \int_0^2 (2y - iy)(2dy + idy)$
$I = \int_0^2 y(2 - i)(2 + i)dy$
$I = (2 - i)(2 + i) \int_0^2 y\, dy$
$I = (2^2 - i^2) \left[\frac{y^2}{2}\right]_0^2$
$I = 5[2]$
$I = 10$

14. Evaluate $\int_0^{1+i} \bar{z}\, dz$ along the real axis from $z = 0$ to $z = 1$ then vertically to $1 + i$
[N22/Elect/5M]
Solution:
$I = \int_0^{1+i} \bar{z}\, dz = \int_0^{1+i} (x - iy)(dx + idy)$



Along the path from $z = 0$ to $z = 1$ and then along the path from $z = 1$ to $z = 1 + i$
(i) Along the line OB from $z = 0$ to $z = 1$
$y = 0$ (X axis)
$dy = 0$
The integral becomes,
$I_1 = \int_0^1 (x)(dx)$
$I_1 = \left[\frac{x^2}{2}\right]_0^1$
$I_1 = \frac{1}{2}$
(ii) Along the line BA from $z = 1$ to $z = 1 + i$
$x = 1$
$dx = 0$
The integral becomes,
$I_2 = \int_0^1 (1 - iy)(idy)$
$I_2 = i\left[y - i\frac{y^2}{2}\right]_0^1$
$I_2 = i\left[1 - \frac{i}{2}\right]$
$I_2 = \frac{1}{2} + i$
Thus,
$I = I_1 + I_2$
$I = \frac{1}{2} + \frac{1}{2} + i$
$I = 1 + i$

15. Evaluate $\int zdz$ from $z = 0$ to $z = 1 + i$ along the curve $z = t^2 + it$
[N18/Comp/5M]
Solution:
Let $I = \int_0^{1+i} zdz$
$I = \int_0^{1+i}(x + iy)(dx + idy)$
Along the curve $z = t^2 + it = x + iy$
$x = t^2 , y = t$
$dx = 2tdt, dy = dt$
When $x = 0, t = 0$ and when $x = 4, t = 2$
The integral becomes,
$I = \int_0^1(t^2 + it)(2tdt + idt)$
$I = \int_0^1(t^2 + it)(2t + i)dt$
$I = \int_0^1[(2t^3 + it^2 + 2it^2 + i^2t)]dt$
$I = \left[\frac{2t^4}{4} + \frac{it^3}{3} + \frac{2it^3}{3} - \frac{t^2}{2}\right]_0^1$
$I = \frac{1}{2} + \frac{i}{3} + \frac{2i}{3} - \frac{1}{2}$
$I = i$

16. Evaluate $\int zdz$ from $z = 0$ to $z = 1 + i$ along the curve $y = x$
[M22/Elex/2M]
Solution:
Let $I = \int_0^{1+i} zdz$
$I = \int_0^{1+i}(x + iy)(dx + idy)$
Along the curve
$y = x$
$dy = dx$
The integral becomes,
$I = \int_0^1(x + ix)(dx + idx)$
$I = \int_0^1 x(1 + i)(1 + i)dx$
$I = (1 + i)^2 \int_0^1 x\, dx$
$I = 2i\left[\frac{x^2}{2}\right]_0^1$
$I = i$

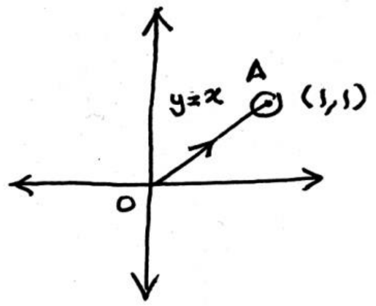
17. Evaluate the complex line integral $\int_0^{1+i}(x^2 - iy)dz$ along the straight line from $z = 0$ to $z = 1 + i$

[M23/Extc/5M]

Solution:

$I = \int_0^{1+i}(x^2 - iy)dz = \int_{(0,0)}^{(1,1)}(x^2 - iy)(dx + idy)$

Along the line OA from $z = 0$ to $z = 1 + i$



Along the line OA,

$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$\frac{y-0}{1-0} = \frac{x-0}{1-0}$

$y = x$

$dy = dx$

The integral becomes,

$I = \int_0^1(x^2 - ix)(dx + idx)$

$I = \int_0^1(x^2 - ix)(1 + i)dx$

$I = (1 + i)\left[\frac{x^3}{3} - \frac{ix^2}{2}\right]_0^1$

$I = (1 + i)\left(\frac{1}{3} - \frac{i}{2}\right)$

$I = \frac{5-i}{6}$

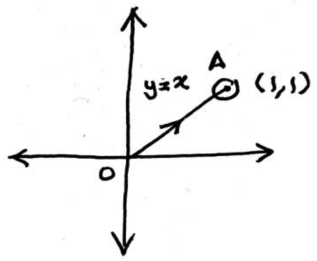
18. Evaluate the complex line integral $\int_0^{1+i}(x-y+ix^2)dz$ along the straight line from $z=0$ to $z=1+i$

[N18/Elect/5M][N18/Chem/5M][N22/MechCivil/5M][N22/Extc/5M] [D24/ElectECS/6M]

Solution:

$I=\int f(z)dz=\int_{(0,0)}^{(1,1)}(x-y+ix^2)(dx+idy)$

Along the line OA from $z=0$ to $z=1+i$



Along the line OA,

$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$

$\frac{y-0}{1-0}=\frac{x-0}{1-0}$

$y=x$

$dy=dx$

The integral becomes,

$I=\int_0^1(x-x+ix^2)(dx+idx)$

$I=\int_0^1 ix^2(1+i)dx$

$I=(i+i^2)\left[\frac{x^3}{3}\right]_0^1$

$I=\frac{-1+i}{3}$

19. Evaluate the integral $\int_0^{1+i}(x-y+ix^2)dz$ along the parabola $y^2=x$
[M24/CompITAI/5M]
Solution:
$I=\int f(z)dz=\int_{(0,0)}^{(1,1)}(x-y+ix^2)(dx+idy)$
Along the parabola,
$x=y^2$
$dx=2y\,dy$
The integral becomes,
$I=\int_0^1(y^2-y+iy^4)(2ydy+i\,dy)$
$I=\int_0^1(y^2-y+i\,y^4)(2y+i)dy$
$I=\int_0^1(2y^3+iy^2-2y^2-iy+2iy^5+i^2y^4)dy$
$I=\left[\frac{2y^4}{4}+\frac{iy^3}{3}-\frac{2y^3}{3}-\frac{iy^2}{2}+\frac{2iy^6}{6}-\frac{y^5}{5}\right]_0^1$
$I=-\frac{11}{30}+\frac{i}{6}$

20. Evaluate the complex line integral $\int_0^{1+i}(x-y+ix^2)dz$ along

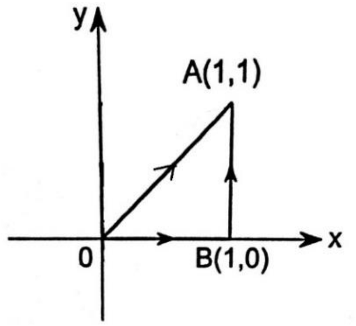
(a) the straight line from $z=0$ to $z=1+i$

(b) the real axis from $z=0$ to $z=1$ & then along a line parallel to the imaginary axis from $z=1$ to $z=1+i$

[M22/Extc/5M]

Solution:

$I=\int f(z)dz=\int_{(0,0)}^{(1,1)}(x-y+ix^2)(dx+idy)$



(a) Along the line OA from $z=0$ to $z=1+i$

Along the line OA,

$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$

$\frac{y-0}{1-0}=\frac{x-0}{1-0}$

$y=x$

$dy=dx$

The integral becomes,

$I=\int_0^1(x-x+ix^2)(dx+idx)$

$I=\int_0^1 ix^2(1+i)dx$

$I=(i+i^2)\left[\frac{x^3}{3}\right]_0^1$

$I=\frac{-1+i}{3}$

(b) Along the path from $z=0$ to $z=1$ and then along the path from $z=1$ to $z=1+i$

(i) Along the line OB from $z=0$ to $z=1$

$y=0$ (X axis)

$dy=0$

The integral becomes,

$I_1=\int_0^1(x+ix^2)(dx)$

$I_1=\left[\frac{x^2}{2}+\frac{ix^3}{3}\right]_0^1$

$I_1=\frac{1}{2}+\frac{i}{3}$

(ii) Along the line BA from $z = 1$ to $z = 1 + i$

$$x = 1$$
$$dx = 0$$

The integral becomes,

$$I_2 = \int_0^1 (1 - y + i)(idy)$$
$$I_2 = i\left[y - \frac{y^2}{2} + iy\right]_0^1$$
$$I_2 = i\left[1 - \frac{1}{2} + i\right]$$
$$I_2 = -1 + \frac{i}{2}$$

Thus,

$$I = I_1 + I_2$$
$$I = \frac{1}{2} + \frac{i}{3} - 1 + \frac{i}{2}$$
$$I = -\frac{1}{2} + \frac{5i}{6}$$

21. Evaluate $\int f(z)dz$ along the parabola $y = 2x^2$ from $z = 0$ to $z = 3 + 18i$ where $f(z) = x^2 - 2iy$

[N14/MechCivil/6M][N15/MechCivil/5M]

Solution:

Let $I = \int_0^{3+18i} f(z)dz$

$$I = \int_0^{3+18i} (x^2 - 2iy)(dx + idy)$$

Along the path $y = 2x^2$

$$dy = 4x\, dx$$

The integral becomes,

$$I = \int_0^3 (x^2 - 4ix^2)(dx + i4xdx)$$
$$I = \int_0^3 (1 - 4i)x^2(1 + 4xi)dx$$
$$I = (1 - 4i)\int_0^3 (x^2 + 4ix^3)dx$$
$$I = (1 - 4i)\left[\frac{x^3}{3} + 4i\frac{x^4}{4}\right]_0^3$$
$$I = (1 - 4i)\left(\frac{27}{3} + 81i\right)$$
$$I = 333 + 45i$$

22. Evaluate $\int_0^{1+i}(x^2 - iy)dz$ along the path (i) $y = x$ (ii) $y = x^2$

[M15/MechCivil/5M][M22/CompITAI/5M][N22/MTRX/6M]

Solution:

Let $I = \int_0^{1+i}(x^2 - iy)(dx + idy)$

(i) Along the path $y = x$

$$y = x$$
$$dy = dx$$

The integral becomes,

$$I = \int_0^1 (x^2 - ix)(dx + idx)$$
$$I = (1 + i)\int_0^1 (x^2 - ix)dx$$
$$I = (1 + i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$
$$I = (1 + i)\left(\frac{1}{3} - \frac{i}{2}\right)$$
$$I = \frac{5 - i}{6}$$

(ii) Along the path $y = x^2$

$$y = x^2$$
$$dy = 2x\,dx$$

The integral becomes,

$$I = \int_0^1 (x^2 - ix^2)(dx + i2xdx)$$
$$I = \int_0^1 (1 - i)x^2(1 + 2xi)dx$$
$$I = (1 - i)\int_0^1 (x^2 + 2ix^3)dx$$
$$I = (1 - i)\left[\frac{x^3}{3} + 2i\frac{x^4}{4}\right]_0^1$$
$$I = (1 - i)\left(\frac{1}{3} + \frac{i}{2}\right)$$
$$I = \frac{5 + i}{6}$$

23. Evaluate $\int_0^{1+i}(x^2 - iy)dz$ along the path $y = x$
[N15/CompIT/5M][M22/Elex/5M]
Solution:
Let $I = \int_0^{1+i}(x^2 - iy)(dx + idy)$
Along the path $y = x$
$$dy = dx$$
The integral becomes,
$$I = \int_0^{1}(x^2 - ix)(dx + idx)$$
$$I = (1 + i)\int_0^{1}(x^2 - ix)dx$$
$$I = (1 + i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$
$$I = (1 + i)\left(\frac{1}{3} - \frac{i}{2}\right)$$
$$I = \frac{5 - i}{6}$$

24. Evaluate $\int_0^{1+i}(x^2 + iy)dz$ along the path $y = x$
[M16/CompIT/5M][N16/CompIT/6M][M22/Elex/2M]
Solution:
Let $I = \int_0^{1+i}(x^2 + iy)(dx + idy)$
Along the path $y = x$
$$dy = dx$$
The integral becomes,
$$I = \int_0^{1}(x^2 + ix)(dx + idx)$$
$$I = (1 + i)\int_0^{1}(x^2 + ix)dx$$
$$I = (1 + i)\left[\frac{x^3}{3} + i\frac{x^2}{2}\right]_0^1$$
$$I = (1 + i)\left(\frac{1}{3} + \frac{i}{2}\right)$$
$$I = \frac{-1 + 5i}{6}$$

25. Evaluate $\int_0^{1+i}(y+ix^2)dz$ along the parabola $y=x^2$
[M23/MechCivil/5M]
Solution:
Let $I=\int_0^{1+i}(y+ix^2)(dx+idy)$
Along the path $y=x^2$
$$dy=2xdx$$
The integral becomes,
$$I=\int_0^1(x^2+ix^2)(dx+i2xdx)$$
$$I=\int_0^1 x^2(1+i)(1+2ix)dx$$
$$I=(1+i)\int_0^1(x^2+2ix^3)dx$$
$$I=(1+i)\left[\frac{x^3}{3}+2i\frac{x^4}{4}\right]_0^1$$
$$I=(1+i)\left(\frac{1}{3}+\frac{i}{2}\right)$$
$$I=\frac{-1+5i}{6}$$

26. Evaluate $\int_0^{3+i}|z|^2dz$ along the line $3y=x$
[M19/MTRX/5M]
Solution:
Let $I=\int_0^{3+i}|z|^2dz$
$$I=\int_0^{3+i}\left(\sqrt{x^2+y^2}\right)^2(dx+idy)$$
$$I=\int_0^{3+i}(x^2+y^2)(dx+idy)$$
Along the path $3y=x$
$$x=3y$$
$$dx=3dy$$
The integral becomes,
$$I=\int_0^1(9y^2+y^2)(3dy+idy)$$
$$I=\int_0^1(10y^2)(3+i)dy$$
$$I=(3+i)\left[10\frac{y^3}{3}\right]_0^1$$
$$I=(3+i)\left[\frac{10}{3}\right]$$
$$I=\frac{30+10i}{3}$$

27. Evaluate $\int_0^{1+2i} z^2 dz$ along the curve $2x^2 = y$

[N13/Chem/5M][M18/Comp/5M]

Solution:

Let $I = \int_0^{1+2i} (x+iy)^2(dx+idy)$

Along the path $y = 2x^2$

$$y = 2x^2$$
$$dy = 4xdx$$

The integral becomes,

$$I = \int_0^1 (x + i2x^2)^2(dx + i4xdx)$$
$$I = \int_0^1 (x^2 + 4ix^3 + 4i^2x^4)(1 + 4ix)dx$$
$$I = \int_0^1 (x^2 + 4ix^3 + 4ix^3 + 16i^2x^4 + 4i^2x^4 + 16i^3x^5)dx$$
$$I = \int_0^1 \big((x^2 - 20x^4) + i(8x^3 - 16x^5)\big)dx$$
$$I = \left[\left(\frac{x^3}{3} - 20\frac{x^5}{5}\right) + i\left(\frac{8x^4}{4} - 16\frac{x^6}{6}\right)\right]_0^1$$
$$I = -\frac{11}{3} - \frac{2}{3}i$$

28. Evaluate $\int_0^{1+i} z^2 dz$ along (i) line $y = x$ (ii) parabola $x^2 = y$
[N19/Elect/5M]
Solution:
Let $I = \int_0^{1+i} (x + iy)^2 (dx + idy)$
(i) Along the path $y = x$

$$y = x$$
$$dy = dx$$

The integral becomes,

$$I = \int_0^1 (x + ix)^2 (dx + idx)$$
$$I = (1 + i)^3 \int_0^1 x^2 dx$$
$$I = (1 + i)^3 \left[\frac{x^3}{3}\right]_0^1$$
$$I = (1 + i)^3 \left(\frac{1}{3}\right)$$
$$I = \frac{-2+2i}{3}$$

(ii) Along the path $x^2 = y$

$$y = x^2$$
$$dy = 2x\, dx$$

The integral becomes,

$$I = \int_0^{1+i} (x + ix^2)^2 (dx + i2xdx)$$
$$I = \int_0^1 (x^2 + 2ix^3 + i^2x^4)(1 + i2x)dx$$
$$I = \int_0^1 (x^2 + i2x^3 + i2x^3 + 4i^2x^4 + i^2x^4 + 2i^3x^5)dx$$
$$I = \int_0^1 (-5x^4 + x^2 + 4ix^3 - 2ix^5)dx$$
$$I = \left[-\frac{5x^5}{5} + \frac{x^3}{3} + \frac{4ix^4}{4} - \frac{2ix^6}{6}\right]_0^1$$
$$I = -1 + \frac{1}{3} + i - \frac{i}{3}$$
$$I = -\frac{2}{3} + \frac{2i}{3}$$

29. Evaluate $\int_0^{1+i} z^2 dz$ along (i) line $y = x$ (ii) parabola $x = y^2$. Is line integral independent of path? Explain.
[N13/M16/MechCivil/6M][M18/Extc/6M][N18/Biot/6M]
Solution:
Let $I = \int_0^{1+i} (x + iy)^2 (dx + idy)$
(i) Along the path $y = x$

$$y = x$$
$$dy = dx$$

The integral becomes,

$$I = \int_0^1 (x + ix)^2 (dx + idx)$$
$$I = (1 + i)^3 \int_0^1 x^2 dx$$
$$I = (1 + i)^3 \left[\frac{x^3}{3}\right]_0^1$$
$$I = (1 + i)^3 \left(\frac{1}{3}\right)$$
$$I = \frac{-2+2i}{3}$$

(ii) Along the path $x = y^2$

$$x = y^2$$
$$dx = 2y\,dy$$

The integral becomes,

$$I = \int_0^1 (y^2 + iy)^2 (2ydy + idy)$$
$$I = \int_0^1 (y^4 + 2iy^3 + i^2y^2)(2y + i)dy$$
$$I = \int_0^1 (2y^5 + iy^4 + 4iy^4 + 2i^2y^3 - 2y^3 - iy^2)dy$$
$$I = \left[\frac{2y^6}{6} + \frac{iy^5}{5} + \frac{4iy^5}{5} - \frac{2y^4}{4} - \frac{2y^4}{4} - \frac{iy^3}{3}\right]_0^1$$
$$I = \frac{1}{3} + \frac{i}{5} + \frac{4i}{5} - \frac{1}{2} - \frac{1}{2} - \frac{i}{3}$$
$$I = -\frac{2}{3} + \frac{2}{3}i$$

We have,
$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + i2xy$
where,

$u = x^2 - y^2$	$v = 2xy$
$u_x = 2x$	$v_x = 2y$
$u_y = -2y$	$v_y = 2x$

We see that, $u_x = v_y$ and $u_y = -v_x$
Thus, $f(z) = z^2$ is an analytic function and therefore the line integral is independent of the path of integration

30. Evaluate $\int_0^{1+i} z^2 dz$ along the parabola $x = y^2$
[M17/ElexExtcElectBiomInst/5M]
Solution:
Let $I = \int_0^{1+i} (x + iy)^2 (dx + idy)$
Along the path $x = y^2$

$$x = y^2$$
$$dx = 2y\, dy$$

The integral becomes,

$$I = \int_0^1 (y^2 + iy)^2 (2ydy + idy)$$
$$I = \int_0^1 (y^4 + 2iy^3 + i^2y^2)(2y + i)dy$$
$$I = \int_0^1 (2y^5 + iy^4 + 4iy^4 + 2i^2y^3 - 2y^3 - iy^2)dy$$
$$I = \left[\frac{2y^6}{6} + \frac{iy^5}{5} + \frac{4iy^5}{5} - \frac{2y^4}{4} - \frac{2y^4}{4} - \frac{iy^3}{3}\right]_0^1$$
$$I = \frac{1}{3} + \frac{i}{5} + \frac{4i}{5} - \frac{1}{2} - \frac{1}{2} - \frac{i}{3}$$
$$I = -\frac{2}{3} + \frac{2}{3}i$$

31. Evaluate $\int_0^{2+i} z^2 dz$ along the line $x = 2y$
[N19/Elex/5M]
Solution:
Let $I = \int_0^{2+i} z^2\, dz$

$$I = \int_0^{2+i} (x + iy)^2 (dx + idy)$$

Along the straight line

$$x = 2y$$
$$dx = 2dy$$

The integral becomes,

$$I = \int_0^1 (2y + iy)^2 (2dy + idy)$$
$$I = \int_0^1 y^2 (2 + i)^2 (\, 2 + i)\, dy$$
$$I = (2 + i)^3 \left[\frac{y^3}{3}\right]_0^1$$
$$I = (2 + 11i)\left[\frac{1}{3}\right]$$
$$I = \frac{2+11i}{3}$$

32. Integrate the function $f(z) = z^2$ from $A(0,0)$ to $B(1,1)$ along straight line AB **[D23/CompITAI/5M]**

Solution:

$I = \int z^2 dz$

$I = \int (x + iy)^2 (dx + idy)$

Along the line AB,

$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$\frac{y-0}{1-0} = \frac{x-0}{1-0}$

$y = x$

$dy = dx$

The integral becomes,

$I = \int_0^1 (x + ix)^2 (dx + idx)$

$I = \int_0^1 x^2 (1 + i)^2 (1 + i) dx$

$I = (1 + i)^3 \left[\frac{x^3}{3}\right]_0^1$

$I = (-2 + 2i) \left[\frac{1}{3}\right]$

$I = \frac{-2+2i}{3}$

33. Evaluate the complex line integral $\int_C (y - x - 3x^2 i)dz$, C is the straight line from $z = 0$ to $z = 1 + i$

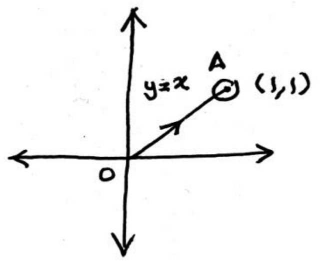
[M24/D24/Extc/5M]

Solution:

$I = \int_C (y - x - 3x^2 i)dz$

$I = \int (y - x - 3x^2 i)(dx + idy)$

Along the line OA from $z = 0$ to $z = 1 + i$



Along the line OA,

$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$\frac{y-0}{1-0} = \frac{x-0}{1-0}$

$y = x$

$dy = dx$

The integral becomes,

$I = \int_0^1 (x - x - 3x^2 i)(dx + idx)$

$I = \int_0^1 (-3x^2 i)(1 + i)dx$

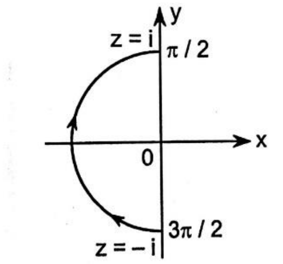
$I = i(1 + i)\left[\frac{-3x^3}{3}\right]_0^1$

$I = (-1 + i)[-1]$

$I = 1 - i$

Type II: Polar form

1. Evaluate $\int_c |z|dz$, where C is the left half of unit circle $|z| = 1$from $z = -i$ to $z = i$
[N13/Biot/5M][M15/ElexExtcElectBiomInst/5M][N19/MTRX/5M][M23/CompIT/5M]
Solution:
Let $I = \int |z|dz$
Put, $z = r\, e^{i\theta}$
$dz = r\, e^{i\theta}\, i\, d\theta$
Here, $|z| = r = 1$



For left half of circle from $z = -i$ to $z = i$, θ varies from $\frac{3\pi}{2}$ to $\frac{\pi}{2}$

$$I = \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} 1.1.\, e^{i\theta} i\, d\theta$$

$$I = i\,\left[\frac{e^{i\theta}}{i}\right]_{\frac{3\pi}{2}}^{\frac{\pi}{2}}$$

$$I = e^{i\frac{\pi}{2}} - e^{i\frac{3\pi}{2}}$$

$$I = \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] - \left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right]$$

$$I = [0 + i] - [0 - i]$$

$$I = 2i$$

2. Evaluate $\int_c \frac{2z+3}{z} dz$ where c is
(i) upper half of the circle $|z| = 2$
[M18/M19/Biom/5M][M19/Inst/5M]
(ii) lower half of the circle $|z| = 2$
Solution:
Let $\int_c \frac{2z+3}{z} dz$
Put, $z = r\, e^{i\theta}$
$dz = r\, e^{i\theta}\, i\, d\theta$
Here, $|z| = r = 2$
For the upper half of circle, θ varies from 0 to π

$$I = \int_0^{\pi} \frac{(4e^{i\theta}+3)}{2e^{i\theta}} 2.e^{i\theta} i\, d\theta$$

$$I = i\int_0^{\pi} \left(4e^{i\theta} + 3\right) d\theta$$

$$I = i\ \left[4\frac{e^{i\theta}}{i} + 3\theta\right]_0^{\pi}$$

$$I = i\left[\frac{4e^{i\pi}}{i} + 3\pi - \frac{4e^0}{i} - 0\right]$$

$$I = i\left[\frac{4(\cos\pi + i\sin\pi)}{i} + 3\pi - \frac{4}{i}\right]$$

$$I = i\left[-\frac{4}{i} + 3\pi - \frac{4}{i}\right]$$

$$I = i\ \left[-\frac{8}{i} + 3\pi\right]$$

$$I = -8 + 3\pi i$$

For the lower half of circle, θ varies from π to 2π

$$I = \int_{\pi}^{2\pi} \frac{(4e^{i\theta}+3)}{2e^{i\theta}} 2.e^{i\theta} i\, d\theta$$

$$I = i\int_{\pi}^{2\pi} \left(4e^{i\theta} + 3\right) d\theta$$

$$I = i\ \left[4\frac{e^{i\theta}}{i} + 3\theta\right]_{\pi}^{2\pi}$$

$$I = i\left[\frac{4e^{i2\pi}}{i} + 6\pi - \frac{4e^{i\pi}}{i} - 3\pi\right]$$

$$I = i\left[\frac{4(\cos 2\pi + i\sin 2\pi)}{i} + 3\pi - \frac{4(\cos\pi + i\sin\pi)}{i}\right]$$

$$I = i\left[\frac{4}{i} + 3\pi + \frac{4}{i}\right]$$

$$I = i\ \left[\frac{8}{i} + 3\pi\right]$$

$$I = 8 + 3\pi i$$

3. Evaluate $\int_c (z^2 + 3z)dz$ along the circle $|z| = 2$ from $(2,0)$ to $(0,2)$

Solution:

$$I = \int (z^2 + 3z)dz$$

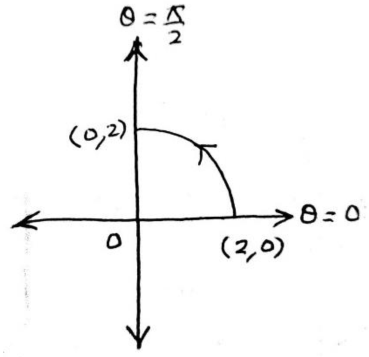
$$I = \int \left(\left(r\, e^{i\theta}\right)^2 + 3r\, e^{i\theta}\right) r\, e^{i\theta}.\, i\, d\theta$$

But $r = |z| = 2$

$$I = \int \left(\left(2e^{i\theta}\right)^2 + 6e^{i\theta}\right) 2e^{i\theta} i\, d\theta$$

$$I = \int \left(4e^{2i\theta} + 6e^{i\theta}\right) 2e^{i\theta} i\, d\theta$$

$$I = i \int (8\, e^{3i\theta} + 12\, e^{2i\theta})\, d\theta$$



$$I = i \int_0^{\frac{\pi}{2}} (8\, e^{3i\theta} + 12\, e^{2i\theta})\, d\theta$$

$$I = i \left[8 \frac{e^{3i\theta}}{3i} + 12 \frac{e^{2i\theta}}{2i} \right]_0^{\frac{\pi}{2}}$$

$$I = i \left[\left(\frac{8e^{\frac{3\pi}{2}i}}{3i} + \frac{12e^{i\pi}}{2i} \right) - \left(\frac{8e^0}{3i} + \frac{12e^0}{2i} \right) \right]$$

$$I = i \left[\frac{8\left(\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2}\right)}{3i} + \frac{12(\cos\pi + i\sin\pi)}{2i} - \frac{8}{3i} - \frac{12}{2i} \right]$$

$$I = i \left[\frac{8(0-i)}{3i} + \frac{12(-1+0)}{2i} - \frac{8}{3i} - \frac{6}{i} \right]$$

$$I = i \left[-\frac{8}{3} - \frac{6}{i} - \frac{8}{3i} - \frac{6}{i} \right]$$

$$I = i \left[-\frac{8}{3} + \frac{44}{3} i \right]$$

$$I = -\frac{44}{3} - \frac{8}{3} i$$

4. Show that $\int_c \log z\, dz = 2\pi i$, where C is the unit circle in the z – plane.
[M15/N17/CompIT/5M][N15/M16/ChemBiot/5M][M18/M19/Elex/5M] [M18/Inst/5M][N18/Extc/5M][N18/Biom/5M] [N19/Chem/5M]
Solution:
Let $I = \int \log z \; dz$
Put, $z = r\, e^{i\theta}$
$dz = r\, e^{i\theta}\; i\, d\theta$
Here, $|z| = r = 1$ & $\overline{z} = re^{-i\theta}$
For the entire circle, θ varies from 0 to 2π

$$I = \int_0^{2\pi} \left(\log\left(1.e^{i\theta}\right)\right) 1.e^{i\theta} i\; d\theta$$

$$I = i \int_0^{2\pi} i\theta.e^{i\theta} d\theta$$

$$I = i^2 \; \left[\theta\left(\frac{e^{i\theta}}{i}\right) - (1)\left(\frac{e^{i\theta}}{i^2}\right)\right]_0^{2\pi}$$

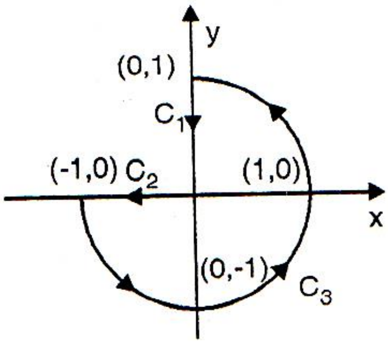
$$I = i^2 \left[2\pi\left(\frac{e^{i2\pi}}{i}\right) - \left(\frac{e^{i2\pi}}{i^2}\right) - 0 + \frac{e^0}{i^2}\right]$$

$$I = i^2 \left[2\pi\left(\frac{\cos 2\pi + i\sin 2\pi}{i}\right) - \frac{\cos 2\pi + i\sin 2\pi}{i^2} + \frac{1}{i^2}\right]$$

$$I = i^2 \left[2\pi\left(\frac{1}{i}\right) - \frac{1}{i^2} + \frac{1}{i^2}\right]$$

$$I = 2\pi i$$

5. Integrate $f(z) = z$ around the closed contour show in figure



Solution:

$I = \int f(z)dz$

$I = \int z\, dz$

(i) Along C_1

$I = \int (x + iy)(dx + idy)$

$x = 0$

$dx = 0$

$I_1 = \int (0 + iy)(0 + idy)$

$I_1 = \int_1^0 (iy)i\, dy$

$I_1 = i^2 \left[\frac{y^2}{2}\right]_1^0$

$I_1 = -1\left[0 - \frac{1}{2}\right]$

$I_1 = \frac{1}{2}$

(ii) along C_2

$I = \int (x + iy)(dx + idy)$

$y = 0$

$dy = 0$

$I_2 = \int (x + 0)(dx + 0)$

$I_2 = \int_0^{-1} x\, dx$

$I_2 = \left[\frac{x^2}{2}\right]_0^{-1}$

$I_2 = \frac{(-1)^2}{2} - 0$

$I_2 = \frac{1}{2}$

(iii) along C_3

$I = \int z\, dz$

$I = \int r\, e^{i\theta} r\, e^{i\theta} i\, d\theta$

But $r = |z| = 1$

$I_3 = \int e^{2i\theta} i\, d\theta$

$I_3 = i \int_{\pi}^{\frac{5\pi}{2}} e^{2i\theta} d\theta$

$I_3 = i\left[\frac{e^{2i\theta}}{2i}\right]_{\pi}^{\frac{5\pi}{2}}$

$I_3 = \frac{e^{i5\pi}}{2} - \frac{e^{i2\pi}}{2}$

$I_3 = \frac{\cos 5\pi + i\sin 5\pi}{2} - \frac{\cos 2\pi + i\sin 2\pi}{2}$

$I_3 = -\frac{1}{2} - \frac{1}{2} = -1$

Thus, $I = I_1 + I_2 + I_3 = \frac{1}{2} + \frac{1}{2} - 1 = 0$

6. Evaluate $\int \bar{z}dz$ over C where C is the upper half of the circle $r = 1$.
[M14/MechCivil/5M][M22/MTRX/2M]
Solution:
Let $I = \int \bar{z}dz$
Put, $z = r\, e^{i\theta}$
$dz = r\, e^{i\theta}\, i\, d\theta$
Here, $|z| = r = 1$ & $\bar{z} = re^{-i\theta}$
For upper half of circle, θ varies from 0 to π
$I = \int_0^{\pi} 1.e^{-i\theta}.1.e^{i\theta} i\, d\theta$
$I = i\int_0^{\pi} d\theta$
$I = i\,[\theta]_0^{\pi}$
$I = \pi i$

7. Evaluate $\int_C \bar{z}dz$ where C is unit circle $|z| = 2$
[M24/MechCivil/5M]
[Note: The question was wrongly asked.
A unit circle has $|z| = 1$, it cannot have $|z| = 2$]
Solution:
Let $I = \int \bar{z}dz$
Put, $z = r\, e^{i\theta}$
$dz = r\, e^{i\theta}\, i\, d\theta$
Here, $|z| = r = 1$ & $\bar{z} = re^{-i\theta}$
For complete circle, θ varies from 0 to 2π
$I = \int_0^{2\pi} 1.e^{-i\theta}.1.e^{i\theta} i\, d\theta$
$I = i\int_0^{2\pi} d\theta$
$I = i\,[\theta]_0^{2\pi}$
$I = 2\pi i$

8. Evaluate $\int_c (\overline{z} + 2z)dz$ along the circle c: $x^2 + y^2 = 1$

[M14/CompIT/5M][N16/MechCivil/5M]

Solution:

Let $I = \int (\overline{z} + 2z)dz$

Put, $z = r\, e^{i\theta}$

$dz = r\, e^{i\theta}\, i\, d\theta$

Here, $|z| = r = 1$ & $\overline{z} = re^{-i\theta}$

For the entire circle, θ varies from 0 to 2π

$I = \int_0^{2\pi} \left(1. e^{-i\theta} + 2.1. e^{i\theta}\right) 1. e^{i\theta} i\, d\theta$

$I = i \int_0^{2\pi} \left(1 + 2e^{2i\theta}\right) d\theta$

$I = i\, \left[\theta + 2\frac{e^{2i\theta}}{2i}\right]_0^{2\pi}$

$I = i\left[2\pi + \frac{e^{i4\pi}}{i} - 0 - \frac{e^0}{i}\right]$

$I = i\left[2\pi + \frac{\cos 4\pi + i \sin 4\pi}{i} - \frac{1}{i}\right]$

$I = 2\pi i$

9. Evaluate $\int_c (\overline{z} + 2z)dz$ where C is (i) upper half of the circle $|z| = 2$ (ii) lower half of the circle $|z| = 2$

[N18/Inst/5M]

Solution:

Let $I = \int_c (\overline{z} + 2z)dz$

Put, $z = r\, e^{i\theta}$

$dz = r\, e^{i\theta}\, i\, d\theta$

Here, $|z| = r = 2$ & $\overline{z} = re^{-i\theta}$

For the upper half of circle, θ varies from 0 to π

$$I = \int_0^{\pi} \left(2.e^{-i\theta} + 2.2e^{i\theta}\right) 2.e^{i\theta} i\, d\theta$$

$$I = i\int_0^{\pi} \left(4 + 8e^{2i\theta}\right) d\theta$$

$$I = i\left[4\theta + 8\frac{e^{i2\theta}}{2i}\right]_0^{\pi}$$

$$I = i\left[4\pi + 8\frac{e^{i2\pi}}{2i} - 0 - 8\frac{e^0}{2i}\right]$$

$$I = i\left[4\pi + \frac{8(\cos 2\pi + i\sin 2\pi)}{2i} - \frac{8}{2i}\right]$$

$$I = i\left[4\pi + \frac{8}{2i} - \frac{8}{2i}\right] = 4\pi i$$

For the lower half of circle, θ varies from π to 2π

$$I = i\left[4\theta + 8\frac{e^{i2\theta}}{2i}\right]_{\pi}^{2\pi}$$

$$I = i\left[8\pi + 8\frac{e^{i4\pi}}{2i} - 4\pi - 8\frac{e^{i2\pi}}{2i}\right]$$

$$I = i\left[4\pi + \frac{8(\cos 4\pi + i\sin 4\pi)}{2i} - \frac{8(\cos 2\pi + i\sin 2\pi)}{2i}\right]$$

$$I = i\left[4\pi + \frac{8}{2i} - \frac{8}{2i}\right] = 4\pi i$$

10. Evaluate $\int_c (z - z^2)dz$ where c is the upper half of the circle $|z| = 1$
[M17/CompIT/5M][N19/Comp/5M]
What is the value for the lower half of the same circle ?
[N14/CompIT/5M]
Solution:
Let $I = \int (z - z^2)dz$
Put, $z = r\, e^{i\theta}$
$dz = r\, e^{i\theta}\, i\, d\theta$
Here, $|z| = r = 1$ & $\overline{z} = re^{-i\theta}$
For the upper half of circle, θ varies from 0 to π

$$I = \int_0^{\pi} \left(1. e^{i\theta} - 1^2. e^{i2\theta}\right) 1. e^{i\theta} i\, d\theta$$

$$I = i \int_0^{\pi} \left(e^{i2\theta} - e^{3i\theta}\right) d\theta$$

$$I = i\, \left[\frac{e^{i2\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_0^{\pi}$$

$$I = i \left[\frac{e^{i2\pi}}{2i} - \frac{e^{i3\pi}}{3i} - \frac{e^0}{2i} + \frac{e^0}{3i}\right]$$

$$I = i \left[\frac{\cos 2\pi + i \sin 2\pi}{2i} - \frac{\cos 3\pi + i \sin 3\pi}{3i} - \frac{1}{2i} + \frac{1}{3i}\right]$$

$$I = i \left[\frac{1}{2i} + \frac{1}{3i} - \frac{1}{2i} + \frac{1}{3i}\right] = \frac{2}{3}$$

For the lower half of circle, θ varies from π to 2π

$$I = i\, \left[\frac{e^{i2\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_{\pi}^{2\pi}$$

$$I = i \left[\frac{e^{i4\pi}}{2i} - \frac{e^{i6\pi}}{3i} - \frac{e^{i2\pi}}{2i} + \frac{e^{i3\pi}}{3i}\right]$$

$$I = i \left[\frac{cos 4\pi + i \sin 4\pi}{2i} - \frac{\cos 6\pi + i \sin 6\pi}{3i} - \frac{\cos 2\pi + i \sin 2\pi}{2i} + \frac{\cos 3\pi + i \sin 3\pi}{3i}\right]$$

$$I = i \left[\frac{1}{2i} - \frac{1}{3i} - \frac{1}{2i} - \frac{1}{3i}\right] = -\frac{2}{3}$$

11. Evaluate $\int_c (z - z^3)dz$ where c is the left half of unit circle from $-i$ to i
[M18/MTRX/5M]
Solution:
Let $I = \int_c \ (z - z^3)dz$
Put, $z = r\, e^{i\theta}$
$dz = r\, e^{i\theta}\ i\, d\theta$
Here, $|z| = r = 1$
For left half of circle from $z = -i$ to $z = i$, θ varies from $\frac{3\pi}{2}$ to $\frac{\pi}{2}$

$$I = \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \left(e^{i\theta} - e^{3i\theta}\right) e^{i\theta} i\, d\theta$$

$$I = i \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \left(e^{2i\theta} - e^{4i\theta}\right) d\theta$$

$$I = i\ \left[\frac{e^{2i\theta}}{2i} - \frac{e^{4i\theta}}{4i}\right]_{\frac{3\pi}{2}}^{\frac{\pi}{2}}$$

$$I = \frac{e^{i\pi}}{2} - \frac{e^{2\pi i}}{4} - \frac{e^{3\pi i}}{2} + \frac{e^{6\pi i}}{4}$$

$$I = \frac{\cos \pi + i \sin \pi}{2} - \frac{\cos 2\pi + i \sin 2\pi}{4} - \frac{\cos 3\pi + i \sin 3\pi}{2} + \frac{\cos 6\pi + i \sin 6\pi}{4}$$

$$I = -\frac{1}{2} - \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 0$$

12. Evaluate $\int_c (z^2 + 3z^{-4})dz$ where c is the upper half of the unit circle from $(1,0)$ to $(-1,0)$
[N18/MTRX/5M]
Solution:
Let $I = \int_c (z^2 + 3z^{-4})dz$
Put, $z = r\, e^{i\theta}$
$dz = r\, e^{i\theta}\ i\, d\theta$
Here, $|z| = r = 1$
For upper half of circle, θ varies from 0 to π

$$I = \int_0^{\pi} \left(e^{2i\theta} + 3e^{-4i\theta}\right) e^{i\theta} i\, d\theta$$

$$I = i \int_0^{\pi} \left(e^{3i\theta} + 3e^{-3i\theta}\right) d\theta$$

$$I = i\ \left[\frac{e^{3i\theta}}{3i} + 3\,\frac{e^{-3i\theta}}{-3i}\right]_0^{\pi}$$

$$I = \frac{e^{3\pi i}}{3} - e^{-3\pi i} - \frac{e^0}{3} + e^0$$

$$I = \frac{\cos 3\pi + i \sin 3\pi}{3} - (\cos 3\pi - i \sin 3\pi) - \frac{1}{3} + 1$$

$$I = -\frac{1}{3} + 1 - \frac{1}{3} + 1 = \frac{4}{3}$$

13. Evaluate $\int_c (z^2 - 2\overline{z} + 1)dz$ where c is the circle $|z| = 1$

[M18/N19/MechCivil/5M]

Solution:

Let $I = \int (z^2 - 2\overline{z} + 1)dz$

Put, $z = r\, e^{i\theta}$

$dz = r\, e^{i\theta}\, i\, d\theta$

Here, $|z| = r = 1$ & $\overline{z} = re^{-i\theta}$

For the entire circle, θ varies from 0 to 2π

$I = \int_0^{2\pi} \left(1. e^{2i\theta} - 2.1. e^{-i\theta} + 1\right) 1. e^{i\theta} i\, d\theta$

$I = i \int_0^{2\pi} \left(e^{i3\theta} - 2 + e^{i\theta}\right) d\theta$

$I = i\, \left[\frac{e^{i3\theta}}{3i} - 2\theta + \frac{e^{i\theta}}{i}\right]_0^{2\pi}$

$I = i\left[\frac{e^{i6\pi}}{3i} - 4\pi + \frac{e^{i2\pi}}{i} - \frac{e^0}{3i} + 0 - \frac{e^0}{i}\right]$

$I = i\left[\frac{\cos 6\pi + i\sin 6\pi}{3i} - 4\pi + \frac{\cos 2\pi + i\sin 2\pi}{i} - \frac{1}{3i} - \frac{1}{i}\right]$

$I = i\left[\frac{1}{3i} - 4\pi + \frac{1}{i} - \frac{1}{3i} - \frac{1}{i}\right]$

$I = -4\pi i$

14. Evaluate $\int z^2 dz$ along the upper half of circle $|z| = 2$

[M22/Elect/5M]

Solution:

Let $I = \int z^2 dz$

Put, $z = r\, e^{i\theta}$

$dz = r\, e^{i\theta}\, i\, d\theta$ and $r = |z| = 2$

For the upper half of circle, θ varies from 0 to π

$I = \int_0^{\pi} \left(2e^{i\theta}\right)^2 2. e^{i\theta} i\, d\theta$

$I = 8i \int_0^{\pi} \left(e^{i3\theta}\right) d\theta$

$I = 8i\, \left[\frac{e^{i3\theta}}{3i}\right]_0^{\pi}$

$I = 8i\left[\frac{e^{i3\pi}}{3i} - \frac{e^0}{3i}\right]$

$I = 8i\left[\frac{\cos 3\pi + i\sin 3\pi}{3i} - \frac{1}{3i}\right]$

$I = 8i\left[\frac{-1}{3i} - \frac{1}{3i}\right]$

$I = -\frac{16}{3}$

15. Evaluate $\int_c z\, dz$ where C is unit circle $|z| = 1$
[D23/MechCivil/5M]
Solution:
Let $I = \int zdz$
Put, $z = r\, e^{i\theta}$
$dz = r\, e^{i\theta}\, i\, d\theta$ and $r = |z| = 1$
For the full circle, θ varies from 0 to 2π

$I = \int_0^{2\pi} e^{i\theta} e^{i\theta} i\, d\theta$

$I = i \int_0^{2\pi} \left(e^{i2\theta}\right) d\theta$

$I = i \left[\frac{e^{i2\theta}}{2i}\right]_0^{2\pi}$

$I = i \left[\frac{e^{i4\pi}}{2i} - \frac{e^{0}}{2i}\right]$

$I = i \left[\frac{\cos 4\pi + i \sin 4\pi}{2i} - \frac{1}{2i}\right]$

$I = i \left[\frac{1}{2i} - \frac{1}{2i}\right]$

$I = 0$