Engineering Maths IV

Nov-Dec 2022

(COITAI)

Time (3 hours) **Max Marks: 80**

Note: (1) Question No. 1 is Compulsory

(2) Answer any three questions from Q.2 to Q.6

(3) Figures to the right indicate full marks

1. (a) If $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$ then find the eigen values of $6A^{-1} + A^3 + 2I$ (5)

Solution:

$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 4 \\ 0 & 3-\lambda \end{vmatrix} = 0$

$(2-\lambda)(3-\lambda) - 0 = 0$

$\lambda = 2,3$

The eigen values of A is 2,3

The eigen values of A^{-1} is $2^{-1}, 3^{-1}$ i.e. $\frac{1}{2}, \frac{1}{3}$

The eigen values of $6A^{-1}$ is $6\left(\frac{1}{2}\right), 6\left(\frac{1}{3}\right)$ i.e. 3,2

The eigen values of A^3 is $2^3, 3^3$ i.e. 8,27

The eigen values of I is 1,1

The eigen values of $2I$ is 2,2

Thus, the eigen values of $6A^{-1} + A^3 + 2I$ is

$3 + 8 + 2 \,;\, 2 + 27 + 2$

i.e. 13; 31

1. (b) Evaluate $\int_0^{1+i} (x^2 + iy)dz$ along the path(i) $y = x$ (ii) $y = x^2$ (5)

Solution:

Let $I = \int_0^{1+i} (x^2 + iy)(dx + idy)$

(i) Along the path $y = x$

$y = x$

$dy = dx$

The integral becomes,

$I = \int_0^1 (x^2 + ix)(dx + idx)$

$I = (1 + i) \int_0^1 (x^2 + ix)dx$

$I = (1 + i) \left[\frac{x^3}{3} + i\frac{x^2}{2}\right]_0^1$

$I = (1 + i)\left(\frac{1}{3} + \frac{i}{2}\right)$

$\boxed{I = \frac{-1+5i}{6}}$

(ii) Along the path $y = x^2$

$y = x^2$

$dy = 2x\,dx$

The integral becomes,

$I = \int_0^1 (x^2 + ix^2)(dx + i2xdx)$

$I = \int_0^1 (1 + i)x^2(1 + 2xi)dx$

$I = (1 + i) \int_0^1 (x^2 + 2ix^3)dx$

$I = (1 + i) \left[\frac{x^3}{3} + 2i\frac{x^4}{4}\right]_0^1$

$I = (1 + i)\left(\frac{1}{3} + \frac{i}{2}\right)$

$\boxed{I = \frac{-1+5i}{6}}$

1. (c) Write the dual of the following problem (5)

Maximise $z = 3x_1 + 10x_2 + 2x_3$

subject to $2x_1 + 3x_2 + 2x_3 \leq 8$

$3x_1 - 2x_2 + 4x_3 = 4$

$x_1, x_2, x_3 \geq 0$

Solution:

Primal,

Maximise $z = 3x_1 + 10x_2 + 2x_3$

Subject to $2x_1 + 3x_2 + 2x_3 \leq 8$

$3x_1 - 2x_2 + 4x_3 \leq 4$

$3x_1 - 2x_2 + 4x_3 \geq 4$

$x_1, x_2, x_3 \geq 0$

Primal,

Maximise $z = 3x_1 + 10x_2 + 2x_3$

Subject to $2x_1 + 3x_2 + 2x_3 \leq 8$

$3x_1 - 2x_2 + 4x_3 \leq 4$

$-3x_1 + 2x_2 - 4x_3 \leq -4$

$x_1, x_2, x_3 \geq 0$

Its dual,

Minimise $w = 8y_1 + 4y_2' - 4y_2''$

Subject to $2y_1 + 3y_2' - 3y_2'' \geq 3$

$3y_1 - 2y_2' + 2y_2'' \geq 10$

$2y_1 + 4y_2' - 4y_2'' \geq 2$

$y_1, y_2', y_2'' \geq 0$

Its dual,

Minimise $w = 8y_1 + 4y_2$

Subject to $2y_1 + 3y_2 \geq 3$

$3y_1 - 2y_2 \geq 10$

$2y_1 + 4y_2 \geq 2$

$y_1 \geq 0$ and y_2 is unrestricted

1. (d) A certain drug administered to 12 patients resulted in the following changes of blood pressure: 5, 2, 8, -1, 3, 0, 6, -2, 1, 5, 0, 4
Can we conclude that drug increases the blood pressure?
Solution:

x	x^2
5	25
2	4
8	64
-1	1
3	9
0	0
6	36
-2	4
1	1
5	25
0	0
4	16
Total = 31	Total = 185

$n = 12$

$\overline{x} = \frac{\sum x}{n} = \frac{31}{12} = 2.5833$

$\sigma = \sqrt{\frac{\sum x^2}{n} - (\overline{x})^2} = \sqrt{\frac{185}{12} - (2.5833)^2} = 2.9569$

(i) Null Hypothesis: $\mu = 0$ (no change)
Alternative Hypothesis: $\mu \neq 0$

(ii) Test statistic:

$$t = \left|\frac{\overline{x}-\mu}{\frac{\sigma}{\sqrt{n-1}}}\right| = \left|\frac{2.5833-0}{\frac{2.9569}{\sqrt{12-1}}}\right| = 2.897$$

(iii) L.O.S.: $\alpha = 0.05$ (assumed)

(iv) Degree of freedom: $\emptyset = n - 1 = 12 - 1 = 11$

(v) Critical value: $t_\alpha = 2.201$

(vi) Decision: Since, the calculated value of t is more than the critical value, null hypothesis is rejected. Thus, there is rise in BP.

2. (a) Using Cauchy's residue theorem evaluate $\int_C \frac{1-2z}{z(z-1)(z-2)}dz$ where C is $|z| = 1.5$ (6)

Solution:

We have, $f(z) = \frac{1-2z}{z(z-1)(z-2)}$

For singularity,

$z(z-1)(z-2) = 0$

$\therefore z = 0, z = 1, z = 2$

We see that $z = 0$ and $z = 1$ both lies inside $C: |z| = 1.5$ and hence are simple poles.

Residue of $f(z)$ at $(z = 0) = \lim_{z\to 0}(z-0)f(z)$

$$= \lim_{z\to 0}(z-0)\frac{1-2z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 0}\frac{1-2z}{(z-1)(z-2)}$$

$$= \frac{1-0}{(0-1)(0-2)}$$

$$= \frac{1}{2}$$

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}(z-1)\frac{1-2z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 1}\frac{1-2z}{z(z-2)}$$

$$= \frac{1-2(1)}{1(1-2)}$$

$$= \frac{-1}{-1} = 1$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$

$\int_c \frac{1-2z}{z(z-1)(z-2)}dz = 2\pi i\left[\frac{1}{2}+1\right]$

$\int_c \frac{1-2z}{z(z-1)(z-2)}dz = 2\pi i\left[\frac{3}{2}\right]$

$\boxed{\int_c \frac{1-2z}{z(z-1)(z-2)}dz = 3\pi i}$

2. (b) Verify Cayley-Hamilton theorem and find A^{-1} for $A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$. Hence find $2A^3 - A^2 - 35A - 44I$ (6)

Solution:

$A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$

The characteristic equation, $|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 8 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)(1-\lambda) - 16 = 0$

$\lambda^2 - 2\lambda - 15 = 0$

By C-H theorem, $A^2 - 2A - 15I = 0$

L.H.S. $= A^2 - 2A - 15I$

$= \begin{bmatrix} 17 & 16 \\ 4 & 17 \end{bmatrix} - 2\begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} - 15\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} =$ R.H.S

Thus, Cayley Hamilton theorem is verified

Now,

$A^2 - 2A - 15I = 0$

Pre-multiplying by A^{-1}, we get

$A - 2I - 15A^{-1} = 0$

$15A^{-1} = A - 2I$

$15A^{-1} = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$15A^{-1} = \begin{bmatrix} -1 & 8 \\ 2 & -1 \end{bmatrix}$

$\boxed{A^{-1} = \frac{1}{15}\begin{bmatrix} -1 & 8 \\ 2 & -1 \end{bmatrix}}$

Dividing $2\lambda^3 - \lambda^2 - 35\lambda - 44$ by $\lambda^2 - 2\lambda - 15$

$2\lambda + 3$

$\lambda^2 - 2\lambda - 15 \,|\, 2\lambda^3 - \lambda^2 - 35\lambda - 44$

$2\lambda^3 - 4\lambda^2 - 30\lambda$

$- \quad + \quad +$

$3\lambda^2 - 5\lambda - 44$

$3\lambda^2 - 6\lambda - 45$

$- \quad + \quad +$

$\lambda + 1$

Thus,

$2A^3 - A^2 - 35A - 44I = (2A + 3I)(A^2 - 2A - 15I) + A + I$

$2A^3 - A^2 - 35A - 44I = (2A + 3I)(0) + A + I$

$2A^3 - A^2 - 35A - 44I = A + I$

$$2A^3 - A^2 - 35A - 44I = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{2A^3 - A^2 - 35A - 44I = \begin{bmatrix} 2 & 8 \\ 2 & 2 \end{bmatrix}}$$

2. (c) Solve by Simplex method (8)

Maximise $z = 4x_1 + 10x_2$

subject to $2x_1 + x_2 \le 50$

$2x_1 + 5x_2 \le 100$

$2x_1 + 3x_2 \le 90$

$x_1, x_2 \ge 0$

Solution:

Max $z - 4x_1 - 10x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

s.t. $2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 50$

$2x_1 + 5x_2 + 0s_1 + s_2 + 0s_3 = 100$

$2x_1 + 3x_2 + 0s_1 + 0s_2 + s_3 = 90$

$x_1, x_2, s_1, s_2, s_3 \ge 0$

Simplex table,

Iteration No.	Basic Var	Coefficient of					RHS	Ratio	Formula
		x_1	x_2	s_1	s_2	s_3			
0	z	-4	**-10**	0	0	0	0	-	$X + 2Y$
s_2 leaves x_2 enters	s_1	2	**1**	1	0	0	50	$\frac{50}{1} = 50$	$X - \frac{1}{5}Y$
	s_2	**2**	**5**	**0**	**1**	**0**	**100**	$\frac{100}{5} = 20$	$\frac{Y}{5}$
	s_3	2	**3**	0	0	1	90	$\frac{90}{3} = 30$	$X - \frac{3}{5}Y$
1	z	0	0	0	2	0	200		
	s_1	$\frac{8}{5}$	0	1	$-\frac{1}{5}$	0	30		
	x_2	$\frac{2}{5}$	1	0	$\frac{1}{5}$	0	20		
	s_3	$\frac{4}{5}$	0	0	$-\frac{3}{5}$	1	30		

Thus, the solution is

$$\boxed{x_1 = 0, x_2 = 20, z_{max} = 200}$$

3. (a) Based on the following determine if there is a relation between literacy and smoking:

	Smokers	Non-smokers
Literates	83	57
Illiterates	45	68

(Given that Critical value of chi-square 1 d.f. and 5% L.O.S. is 3.841) (6)

Solution:

The observed frequency table as given:

	Smokers	Non-smokers	Total
Literates	83	57	140
Illiterates	45	68	113
Total	128	125	253

(i) Null Hypothesis: There is no relation between literacy and smoking
Alternative Hypothesis: There is a relation between literacy and smoking

(ii) Test Statistic:

O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
83	$\frac{140\times128}{253}=71$	12	144	144/71
57	$\frac{125\times140}{253}=69$	-12	144	144/69
45	$\frac{128\times113}{253}=57$	-12	144	144/57
68	$\frac{125\times113}{253}=56$	12	144	144/56
Total				9.213

(iii) Degree of freedom: $\emptyset=(r-1)(c-1)=(2-1)(2-1)=1$

(iv) L.O.S: $\alpha=0.05$

(v) Critical value: $\chi^2_\alpha=3.841$

(vi) Decision: since, the calculated value is more than the critical value, null hypothesis is rejected. Thus, there is a relation between literacy and smoking

3. (b) Obtain Laurent's series expansion of $f(z) = \frac{1}{z^2+4z+3}$ when
(i) $|z| < 1$ (ii) $1 < |z| < 3$ (iii) $|z| > 3$ (6)

Solution:

We have, $f(z) = \frac{1}{z^2+4z+3} = \frac{1}{(z+1)(z+3)}$

Let $\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$

$1 = A(z+3) + B(z+1)$

Comparing the coefficients, we get

$A + B = 0$

$3A + B = 1$

On solving, we get

$A = \frac{1}{2}, B = -\frac{1}{2}$

$f(z) = \frac{\frac{1}{2}}{z+1} - \frac{\frac{1}{2}}{z+3}$

(i) $|z| < 1$

$f(z) = \frac{\frac{1}{2}}{1+z} - \frac{\frac{1}{2}}{3+z}$

$f(z) = \frac{1}{2}[1+z]^{-1} - \frac{\frac{1}{2}}{3\left[1+\frac{z}{3}\right]}$

$f(z) = \frac{1}{2}[1+z]^{-1} - \frac{1}{6}\left[1+\frac{z}{3}\right]^{-1}$

$f(z) = \frac{1}{2}[1 - z + z^2 - z^3 + \cdots.] - \frac{1}{6}\left[1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right]$

(ii) $1 < |z| < 3$

$f(z) = \frac{\frac{1}{2}}{z+1} - \frac{\frac{1}{2}}{3+z}$

$f(z) = \frac{\frac{1}{2}}{z\left[1+\frac{1}{z}\right]} - \frac{\frac{1}{2}}{3\left[1+\frac{z}{3}\right]}$

$f(z) = \frac{1}{2z}\left[1+\frac{1}{z}\right]^{-1} - \frac{1}{6}\left[1+\frac{z}{3}\right]^{-1}$

$f(z) = \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots.\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right]$

(iii) $|z| > 3$

$f(z) = \frac{\frac{1}{2}}{z+1} - \frac{\frac{1}{2}}{z+3}$

$f(z) = \frac{\frac{1}{2}}{z\left[1+\frac{1}{z}\right]} - \frac{\frac{1}{2}}{z\left[1+\frac{3}{z}\right]}$

$f(z) = \frac{1}{2z}\left[1+\frac{1}{z}\right]^{-1} - \frac{1}{2z}\left[1+\frac{3}{z}\right]^{-1}$

$f(z) = \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots.\right] - \frac{1}{2z}\left[1 - \frac{3}{z} + \frac{3^2}{z^2} - \frac{3^3}{z^3} + \cdots\right]$

3. (c) Using the method of Lagrangian multipliers solve the following N.L.P.P. (8)

Optimize $z = x_1^2 + x_2^2 + x_3^2$

subject to $x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$

$x_1, x_2, x_3 \geq 0$

Solution:

Let $f = x_1^2 + x_2^2 + x_3^2$

Let $h_1 = x_1 + x_2 + 3x_3 - 2$ & $h_2 = 5x_1 + 2x_2 + x_3 - 5$

Consider the Lagrangian function,

$L = f - \lambda_1 h_1 - \lambda_2 h_2$

$L = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$

Now,

$L_{x_1} = 0 \Rightarrow 2x_1 - \lambda_1 - 5\lambda_2 = 0$ i.e. $x_1 = \frac{\lambda_1 + 5\lambda_2}{2}$

$L_{x_2} = 0 \Rightarrow 2x_2 - \lambda_1 - 2\lambda_2 = 0$ i.e. $x_2 = \frac{\lambda_1 + 2\lambda_2}{2}$

$L_{x_3} = 0 \Longrightarrow 2x_3 - 3\lambda_1 - \lambda_2 = 0$ i.e. $x_3 = \frac{3\lambda_1 + \lambda_2}{2}$

$L_{\lambda_1} = 0 \Rightarrow -(x_1 + x_2 + 3x_3 - 2) = 0 \Rightarrow x_1 + x_2 + 3x_3 = 2$ …(1)

$L_{\lambda_2} = 0 \Rightarrow -(5x_1 + 2x_2 + x_3 - 5) = 0 \Rightarrow 5x_1 + 2x_2 + x_3 = 5$ …(2)

Putting x_1, x_2, x_3 in eqn (1) we get

$\left(\frac{\lambda_1 + 5\lambda_2}{2}\right) + \left(\frac{\lambda_1 + 2\lambda_2}{2}\right) + 3\left(\frac{3\lambda_1 + \lambda_2}{2}\right) = 2$

$11\lambda_1 + 10\lambda_2 = 4$ ….. (3)

Putting x_1, x_2, x_3 in eqn (2) we get

$5\left(\frac{\lambda_1 + 5\lambda_2}{2}\right) + 2\left(\frac{\lambda_1 + 2\lambda_2}{2}\right) + \left(\frac{3\lambda_1 + \lambda_2}{2}\right) = 5$

$10\lambda_1 + 30\lambda_2 = 10$ ….. (4)

Solving (3) & (4), we get

$\lambda_1 = \frac{2}{23}, \lambda_2 = \frac{7}{23}$

Thus, $x_1 = \frac{37}{46}, x_2 = \frac{8}{23}, x_3 = \frac{13}{46}$

Now, hessian matrix

$$H^B = \begin{bmatrix} O & P \\ P' & Q \end{bmatrix}$$

where, $P = \begin{bmatrix} h_{1_{x_1}} & h_{1_{x_2}} & h_{1_{x_3}} \\ h_{2_{x_1}} & h_{2_{x_2}} & h_{2_{x_3}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 1 \end{bmatrix}$

$$Q = \begin{bmatrix} L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore H^B = \begin{bmatrix} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

By $C_4 - C_3, C_5 - 3C_1$

$$H^B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & -3 & -14 \\ 1 & 5 & 2 & -2 & -6 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

$$\Delta = 1 \begin{vmatrix} 0 & 0 & -3 & -14 \\ 1 & 5 & -2 & -6 \\ 1 & 2 & 2 & 0 \\ 3 & 1 & 0 & 2 \end{vmatrix} = -3 \begin{vmatrix} 1 & 5 & -6 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} - (-14) \begin{vmatrix} 1 & 5 & -2 \\ 1 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix}$$

$$\Delta = -3(24) + 14(38) = 460$$

Since, Δ is positive, it is a minima

$$\therefore z_{min} = \left(\frac{37}{46}\right)^2 + \left(\frac{8}{23}\right)^2 + \left(\frac{13}{46}\right)^2$$

$$\boxed{z_{min} = \frac{39}{46}}$$

4. (a) Using the method of Lagrange's multipliers, solve the following N.L.P.P. (6)

Optimise $z = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

subject to $x_1 + x_2 + x_3 = 7$

$x_1, x_2, x_3 \geq 0$

Solution:

Let $f = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

and $h = x_1 + x_2 + x_3 - 7$

Consider the Lagrangian function,

$L = f - \lambda h$

$L = \left(x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3\right) - \lambda(x_1 + x_2 + x_3 - 7)$

$L_{x_1} = 0 \Rightarrow 2x_1 - 10 - \lambda = 0 \Rightarrow x_1 = \frac{\lambda+10}{2}$

$L_{x_2} = 0 \Rightarrow 2x_2 - 6 - \lambda = 0 \Rightarrow x_2 = \frac{\lambda+6}{2}$

$L_{x_3} = 0 \Rightarrow 2x_3 - 4 - \lambda = 0 \Rightarrow x_3 = \frac{\lambda+4}{2}$

$L_\lambda = 0 \Rightarrow -(x_1 + x_2 + x_3 - 7) = 0$

$x_1 + x_2 + x_3 = 7$

$\frac{\lambda+10}{2} + \frac{\lambda+6}{2} + \frac{\lambda+4}{2} = 7$

$\frac{3\lambda+20}{2} = 7$

$\lambda = -2$

$\therefore x_1 = 4, x_2 = 2, x_3 = 1$

Now, hessian matrix,

$$H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} & h_{x_3} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -4$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$\Delta_4 = -4 - 4 - 4 = -12$

Since both Δs are negative, it is a minima

$\therefore z_{min} = (4)^2 + (2)^2 + (1)^2 - 10(4) - 6(2) - 4(1)$

$\boxed{z_{min} = -35}$

4. (b) Find the inverse Z transform of $\frac{1}{z^2-3z+2}$ if ROC is (i) $|z|<1$ (ii) $|z|>2$ (6)

Solution:

We have,

$F(z)=\frac{1}{z^2-3z+2}=\frac{1}{(z-1)(z-2)}$

Let $\frac{1}{(z-1)(z-2)}=\frac{A}{z-1}+\frac{B}{z-2}$

$1=A(z-2)+B(z-1)$

Comparing the coefficients, we get

$A+B=0$

$-2A-B=1$

On solving, we get $A=-1, B=1$

$F(z)=\frac{-1}{z-1}+\frac{1}{z-2}$

(i) $|z|<1$

$F(z)=\frac{-1}{-1+z}+\frac{1}{-2+z}$

$F(z)=\frac{-1}{-[1-z]}+\frac{1}{-2\left[1-\frac{z}{2}\right]}$

$F(z)=[1-z]^{-1}-\frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$

$F(z)=[1+z+z^2+z^3+\cdots..]-\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots..\right]$

$F(z)=[1+z+z^2+z^3+\cdots..]+\left[-\frac{1}{2}-\frac{z}{2^2}-\frac{z^2}{2^3}-\frac{z^3}{2^4}+\cdots\ldots\right]$

$F(z)=[z^0+z^1+z^2-\cdots.]+[-2^{-1}z^0-2^{-2}z^1-2^{-3}z^2+\cdots]$

From first series,

Coefficient of $z^k=1, k\geq 0$

Coefficient of $z^{-k}=1, k\leq 0$

From second series,

Coefficient of $z^k=-2^{-(k+1)}, k\geq 0$

Coefficient of $z^{-k}=-2^{k-1}, k\leq 0$

Thus,

$$\boxed{Z^{-1}\left\{\frac{1}{(z-1)(z-2)}\right\}=1-2^{k-1}, k\leq 0}$$

(ii) $|z|>2$

$F(z)=\frac{-1}{z-1}+\frac{1}{z-2}$

$F(z)=\frac{-1}{z\left[1-\frac{1}{z}\right]}+\frac{1}{z\left[1-\frac{2}{z}\right]}$

$F(z)=-\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}+\frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$

$F(z)=-\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots..\right]+\frac{1}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots..\right]$

$F(z)=\left[-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}+\cdots\ldots\right]+\left[\frac{1}{z}+\frac{2}{z^2}+\frac{2^2}{z^3}+\frac{2^3}{z^4}+\cdots\ldots\right]$

$F(z) = [-z^{-1} - z^{-2} - z^{-3} - \cdots.] + [2^0 z^{-1} + 2^1 z^{-2} + 2^2 z^{-3} + \cdots]$

From first series,

Coefficient of $z^{-k} = -1, k > 0$

From second series,

Coefficient of $z^{-k} = 2^{k-1}, k > 0$

Thus,

$$\boxed{Z^{-1}\left\{\frac{1}{(z-1)(z-2)}\right\} = 2^{k-1} - 1, k > 0}$$

4. (c) Show that the matrix $A=\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}$ is diagonalizable. Find the transforming matrix and the diagonal matrix (8)

Solution:

$$A=\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}, |A|=0$$

The characteristic equation,

$$|A-\lambda I|=0$$

$$\begin{vmatrix}8-\lambda&-6&4\\-6&7-\lambda&-4\\2&-4&3-\lambda\end{vmatrix}=0$$

$$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$$

$$\lambda^3-[8+7+3]\lambda^2+\left[\begin{vmatrix}7&-4\\-4&3\end{vmatrix}+\begin{vmatrix}8&2\\2&3\end{vmatrix}+\begin{vmatrix}8&-6\\-6&7\end{vmatrix}\right]\lambda-0=0$$

$$\lambda^3-18\lambda^2+45\lambda=0$$

$$\lambda(\lambda-15)(\lambda-3)=0$$

$$\lambda=0,3,15$$

Since the eigen values are distinct, the matrix A is diagonalisable

(i) For $\lambda=0$, $[A-\lambda I]X=0$ gives

$$\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$8x_1-6x_2+2x_3=0$$

$$-6x_1+7x_2-4x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-6&2\\7&-4\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}8&2\\-6&-4\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}8&-6\\-6&7\end{vmatrix}}$$

$$\frac{x_1}{10}=-\frac{x_2}{-20}=\frac{x_3}{20}$$

$$\frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{2}$$

Hence, corresponding to $\lambda=0$ the eigen vector is $X_1=[1,2,2]'$

(ii) For $\lambda=3$, $[A-\lambda I]X=0$ gives

$$\begin{bmatrix}5&-6&2\\-6&4&-4\\2&-4&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$5x_1-6x_2+2x_3=0$$

$$2x_1-4x_2+0x_3=0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix}-6&2\\-4&0\end{vmatrix}}=-\frac{x_2}{\begin{vmatrix}5&2\\2&0\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}5&-6\\2&-4\end{vmatrix}}$$

$$\frac{x_1}{8}=-\frac{x_2}{-4}=\frac{x_3}{-8}$$

$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_2 = [2,1,-2]'$

(iii) For $\lambda = 15$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 - 8x_2 - 4x_3 = 0$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$\frac{x_1}{40} = -\frac{x_2}{40} = \frac{x_3}{20}$

$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$

Hence, corresponding to $\lambda = 15$ the eigen vector is $X_3 = [2,-2,1]'$

Thus, the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

5. (a) Find $Z\{f(k)*g(k)\}$ if $f(k)=\left(\frac{1}{2}\right)^k$, $g(k)=\cos\pi k$ (6)

Solution:

If $Z\{f(k)\}=F(z)$ and $Z\{g(k)=G(z)$

Then by Convolution Theorem,

$Z\{f(k)*g(k)\}=F(z).G(z)$

We have,

$Z\{f(k)\}=\sum_{-\infty}^{\infty} f(k)z^{-k}$

$$Z\left\{\frac{1}{2^k}\right\}=\sum_{0}^{\infty}\frac{1}{2^k}.z^{-k}$$
$$=\frac{z^0}{2^0}+\frac{z^{-1}}{2^1}+\frac{z^{-2}}{2^2}+\frac{z^{-3}}{2^3}+\cdots..$$
$$=1+\frac{1}{2z}+\frac{1}{(2z)^2}+\frac{1}{(2z)^3}+\cdots....$$
$$=\left[1-\frac{1}{2z}\right]^{-1}$$
$$=\frac{1}{1-\frac{1}{2z}}$$
$$Z\left\{\frac{1}{2^k}\right\}=\frac{2z}{2z-1}$$
$$\therefore Z\{f(k)\}=Z\left\{\frac{1}{2^k}\right\}$$
$$\therefore F(z)=\frac{2z}{2z-1}$$

Also,

We know that,

$$Z\{a^k\}=\frac{z}{z-a}$$

Now,

$$Z\{\cos k\pi\}=Z\left\{\frac{e^{\pi ik}+e^{-\pi ik}}{2}\right\}$$
$$=\frac{1}{2}Z\{e^{\pi ik}+e^{-\pi k}\}$$
$$=\frac{1}{2}\left[\frac{z}{z-e^{\pi i}}+\frac{z}{z-e^{-\pi i}}\right]$$
$$=\frac{1}{2}\left[\frac{z^2-ze^{-\pi i}+z^2-ze^{\pi i}}{z^2-e^{\pi i}z-e^{-\pi i}z+1}\right]$$
$$=\frac{1}{2}\left[\frac{2z^2-z\left(e^{\pi i}-e^{-\pi i}\right)}{z^2-z\left(e^{\pi i}+e^{-\pi i}\right)+1}\right]$$
$$=\frac{1}{2}\left[\frac{2z^2-z(2\cos\pi)}{z^2-z(2\cos\pi)+1}\right]$$
$$Z\{\cos k\pi\}=\frac{z^2+z}{z^2+2z+1}=\frac{z(z+1)}{(z+1)^2}=\frac{z}{z+1}$$
$$G(z)=\frac{z}{z+1}$$

By Convolution Theorem,

$$Z\{f(k)*g(k)\}=F(z).G(z)=\frac{2z}{2z-1}.\frac{z}{z+1}=\frac{2z^2}{(2z-1)(z+1)}$$

5. (b) Find the eigen values and eigen vectors of the following matrix (6)

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}, |A| = 4$$

The characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$$

$$\lambda^3 - [4+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}\right]\lambda - 4 = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$$

$$\lambda = 1,2,2$$

(i) For $\lambda = 1, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 2x_3 = 0$$

$$-x_1 - 5x_2 - 3x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ -5 & -3 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ -1 & -5 \end{vmatrix}}$$

$$\frac{x_1}{4} = -\frac{x_2}{-1} = \frac{x_3}{-3}$$

Hence, corresponding to $\lambda = 1$ the eigen vector is $X_1 = [4,1,-3]'$

(ii) For $\lambda = 2, [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 + 6x_2 + 6x_3 = 0$$

$$-x_1 - 5x_2 - 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 6 & 6 \\ -5 & -4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 2 & 6 \\ -1 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 6 \\ -1 & -5 \end{vmatrix}}$$

$$\frac{x_1}{6} = -\frac{x_2}{-2} = \frac{x_3}{-4}$$

Hence, corresponding to $\lambda = 2$ the eigen vector is $X_2 = [3,1,-2]'$

5. (c) Solve by the dual simplex method (8)

Minimize $z = x_1 + x_2$

Subject to $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

$x_1, x_2 \geq 0$

Solution:

The standard form,

Min $z = x_1 + x_2$

$z - x_1 - x_2 + 0s_1 + 0s_2 = 0$

s.t. $-2x_1 - x_2 + s_1 = -2$

$x_1 + x_2 + s_2 = -1$

Simplex table,

Iteration No.	Basic Var	Coefficient of				RHS	Formula
		x_1	x_2	s_1	s_2		
0	z	**-1**	-1	0	0	0	$X - \frac{1}{2}Y$
s_1 leaves x_1 enters	s_1	**-2**	**-1**	**1**	**0**	**-2**	$\frac{Y}{-2}$
	s_2	**1**	1	0	1	-1	$X + \frac{1}{2}Y$
Ratio		$\frac{-1}{-2} = \frac{1}{2}$	$\frac{-1}{-1} = 1$	-	-	-	
1	z	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	
s_2 leaves x_1 enters	x_2	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	
	s_2	**0**	$\frac{1}{2}$	$\frac{1}{2}$	**1**	**-2**	
Ratio		-	-	-	-	-	

Since, there are no positive ratios obtained, the problem has no solution

6. (a) Find $Z[2^k \cos(3k+2)], k \geq 0$ (6)

Solution:

$Z\{\cos(3k+2)\} = Z\{\cos 3k \cos 2 - \sin 3k \sin 2\}$

$Z\{\cos(3k+2)\} = \cos 2\ Z\{\cos 3k\} - \sin 2\ Z\{\sin 3k\}$

$Z\{\cos(3k+2)\} = \cos 2\ \left[\frac{z(z-\cos 3)}{z^2-2z\cos 3+1}\right] - \sin 2\ \left[\frac{z\sin 3}{z^2-2z\cos 3+1}\right]$

By $Z\{\cos\alpha k\} = \frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}, Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2-2z\cos\alpha+1}$

$\therefore Z\{\cos(3k+2)\} = \frac{z^2\cos 2 - z\cos 2\cos 3 - z\sin 2\sin 3}{z^2-2z\cos 3+1}$

$Z\{\cos(3k+2)\} = \frac{z^2\cos 2 - z(\cos 2\cos 3+\sin 2\sin 3)}{z^2-2z\cos 3+1}$

$Z\{\cos(3k+2)\} = \frac{z^2\cos 2 - z\cos 1}{z^2-2z\cos 3+1}$

Now, by Change of scale property $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

$Z\{2^k\cos(3k+2)\} = \frac{\left(\frac{z}{2}\right)^2\cos 2 - \left(\frac{z}{2}\right)\cos 1}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cos 3+1}$

$\boxed{Z\{2^k\cos(3k+2)\} = \frac{z^2\cos 2 - 2z\cos 1}{z^2-4z\cos 3+4}}$

6. (b) If the heights of 500 students is normally distributed with mean 68 inches and standard deviation 4 inches, estimate the number of students having heights: (i) greater than 72 inches (ii) less than 62 inches (iii) between 65 and 71 inches. (6)

Solution:

$N = 500$
$\mu = 68$
$\sigma = 4$
$z = \frac{x-\mu}{\sigma} = \frac{x-68}{4}$

(i) $P(greater\ than\ 72) = P(X > 72)$
$= P\left(z > \frac{72-68}{4}\right)$
$= P(z > 1)$
$= P(1 < z < \infty)$
$= A(\infty) - A(1)$
$= 0.5 - 0.3413$
$= 0.1587$

Expected number of students $= N \times P = 500 \times 0.1587 = 79.35 \approx 79$

(ii) $P(less\ than\ 62) = P(X < 62)$
$= P\left(z < \frac{62-68}{4}\right)$
$= P(z < -1.5)$
$= A(-1.5) - A(-\infty)$
$= -A(1.5) + A(\infty)$
$= A(\infty) - A(1.5)$
$= 0.5 - 0.4332$
$= 0.0668$

Expected number of students $= N \times P = 500 \times 0.0668 = 33.4 \approx 33$

(iii) $P(between\ 65\ and\ 71) = P(65 < X < 71)$
$= P\left(\frac{65-68}{4} < z < \frac{71-68}{4}\right)$
$= P(-0.75 < z < 0.75)$
$= A(0.75) - A(-0.75)$
$= A(0.75) + A(0.75)$
$= 2 \times 0.2734$
$= 0.5468$

Expected number of students $= N \times P = 500 \times 0.5468 = 273.4 \approx 273$

6. (c) Using Kuhn-Tucker conditions, solve the following N.L.P.P. (8)

Maximise $z = 2x_1^2 - 7x_2^2 + 12x_1x_2$

subject to $2x_1 + 5x_2 \leq 98$

$x_1, x_2 \geq 0$

Solution:

Let $f = 2x_1^2 - 7x_2^2 + 12x_1x_2$

Let $h = 2x_1 + 5x_2 - 98$

Consider, $L = f - \lambda h$

$L = 2x_1^2 - 7x_2^2 + 12x_1x_2 - \lambda(2x_1 + 5x_2 - 98)$

According to Kuhn Tucker conditions,

$L_{x_1} = 0 \Rightarrow 4x_1 + 12x_2 - 2\lambda = 0$ ……………….(1)

$L_{x_2} = 0 \Rightarrow -14x_2 + 12x_1 - 5\lambda = 0$ ……………….(2)

$\lambda h = 0 \Rightarrow \lambda(2x_1 + 5x_2 - 98) = 0$ …………………..(3)

$h \leq 0 \Rightarrow 2x_1 + 5x_2 - 98 \leq 0$ ……………………………….(4)

$x_1, x_2, \lambda \geq 0$ ………………(5)

Case I: If $\lambda = 0$

From (1), $4x_1 + 12x_2 = 0$

From (2), $12x_1 - 14x_2 = 0$

$x_1 = 0, x_2 = 0$

$z_{max} = 2(0)^2 - 7(0)^2 + 12(0)(0) = 0$

Case II: If $\lambda \neq 0$

From (1), $4x_1 + 12x_2 - 2\lambda = 0$

From (2), $12x_1 - 14x_2 - 5\lambda = 0$

From (3), $2x_1 + 5x_2 + 0\lambda = 98$

On solving,

$x_1 = 44, x_2 = 2, \lambda = 100$

$z_{max} = 2(44)^2 - 7(2)^2 + 12(44)(2) = 4900$

Thus, the optimal solution is

$\boxed{z_{max} = 4900}$ at $\boxed{x_1 = 44, x_2 = 2}$