Cauchy's Theorem

Weight Distribution of Types

MechCivil

Year	May 2017	Nov 2017	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
Total Marks	06	06	00	06	06	00	09	06	06	06	06	08

Comp

Year	May 2017	Nov 2017	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
Total Marks	06	00	06	06	06	06	07	00	00	06	00	06

Extc

Year	May 2017	Nov 2017	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
Total Marks	08	05	06	06	06	05	07	06	00	05	06	00

Elect

Year	May 2017	Nov 2017	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
Total Marks	08	05	11	00	06	00	02	06	12	11	11	05

1. The value of the integral $\int_C \frac{1}{z-1} dz$ where C is $|z-1| = 2$ is
[M22/MechCivil/2M]
Solution:
$I = \int_C \frac{1}{z-1} dz$ where C is $|z-1| = 2$
We see that $z = 1$ lies inside C
By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$
$I = \int_C \frac{1}{z-1} dz = 2\pi i\, f(1) = 2\pi i(1) = 2\pi i$

2. Evaluate $\oint_C \frac{e^{3z}}{z-i} dz$ where C is the curve $|z-2|+|z+2|=6$

Solution:

$I = \int \frac{e^{3z}}{z-i} dz$

Here, $z = z_0 = i$

C is $|z-2|+|z+2|=6$

LHS $= |z-2|+|z+2|$

LHS $= |i-2|+|i+2|$

LHS $= \sqrt{(-2)^2+(1)^2}+\sqrt{(2)^2+(1)^2}$

LHS $= \sqrt{5}+\sqrt{5} = 2\sqrt{5} = 4.472 < 6 < RHS$

Thus, $z = i$ is inside C

By CIF,

$\int \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

$\int \frac{e^{3z}}{z-i} dz = 2\pi i f(i) = 2\pi i\, [e^{3z}]_{z=i} = 2\pi i\left[e^{3i}\right]$

3. Evaluate $\int_C \frac{\sinh z}{z-\frac{\pi i}{2}} dz$ where C is the circle $|z-2|=3$

Solution:

$I = \int_C \frac{\sinh z}{z-\frac{\pi i}{2}} dz$

Here, $z = z_0 = \frac{\pi i}{2}$

C is $|z-2|=3$

LHS $= |z-2| = \left|\frac{\pi i}{2}-2\right| = \sqrt{(-2)^2+\left(\frac{\pi}{2}\right)^2} = 2.543 < 3 < RHS$

Thus, $z = \frac{\pi i}{2}$ is inside C

By CIF,

$\int \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

$$\int \frac{\sinh z}{z-\frac{\pi i}{2}} dz = 2\pi i\, f\left(\frac{\pi i}{2}\right) = 2\pi i[\sinh z]_{z=\frac{\pi i}{2}} = 2\pi i\left[\sinh\frac{\pi i}{2}\right]$$

$$= 2\pi i\,\left[i\sin\frac{\pi}{2}\right]$$

$$= 2\pi i^2(1)$$

$$= 2\pi(-1)$$

$$= -2\pi$$

4. Evaluate $\int_c \frac{e^{2z}}{(z+1)^4} dz$, where C is the
(i) circle $|z-1|=3$
[M16/ChemBiot/6M][N16/M17/CompIT/6M][M18/Extc/6M][M18/Biot/5M][M19/Chem/6M]
(ii) circle $|z|=0.5$
Solution:
$I=\int_c \frac{e^{2z}}{(z+1)^4} dz$ where C is $|z-1|=3$
Here, $z=z_0=-1$
(i) C is $|z-1|=3$
We see that $z=-1$ lies inside $C: |z-1|=3$
By CIF,
$\int_c \frac{f(z)}{(z-z_0)^n} dz=\frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$
$\int_c \frac{e^{2z}}{(z+1)^4} dz=\frac{2\pi i}{3!} f'''(-1)$
Here, $f(z)=e^{2z}$
$\therefore f'(z)=2e^{2z}$
$\therefore f''(z)=4e^{2z}$
$\therefore f'''(z)=8e^{2z}$
$\therefore f'''(-1)=8e^{-2}$
Thus,
$I=\int_c \frac{e^{2z}}{(z+1)^4} dz=\frac{2\pi i}{3!}\ 8e^{-2}=\frac{8\pi i e^{-2}}{3}$
(ii) C is $|z|=0.5$
We see that $z=-1$ lies outside C
By CIT,
$\int_c \frac{e^{2z}}{(z+1)^4} dz=0$

5. Evaluate $\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ where C is $|z| = 1$.

[M14/MechCivil/6M][N18/Elex/5M][N18/N22/Extc/6M][M19/Elect/6M][M22/Extc/5M]

Solution:

$I = \int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ where C is $|z| = 1$

We see that $z = \frac{\pi}{6} = \frac{3.14}{6} = 0.52$ lies inside $C: |z| = 1$

By CIF, $\int_C \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$I = \int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!} f''\left(\frac{\pi}{6}\right)$

Here, $f(z) = \sin^6 z$

$\therefore f'(z) = 6\sin^5 z \cdot \cos z$

$\therefore f''(z) = 6\sin^5 z\,[-\sin z] + 6\cos z\,[5\sin^4 z \cdot \cos z]$

$f''(z) = -6\sin^6 z + 30\sin^4 z\cos^2 z$

$\therefore f''\left(\frac{\pi}{6}\right) = -6\left(\frac{1}{2}\right)^6 + 30\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2$

$\therefore f''\left(\frac{\pi}{6}\right) = \frac{21}{16}$

Thus,

$I = \int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!} \cdot \frac{21}{16} = \frac{21\pi i}{16}$

6. Evaluate $\int_c \frac{1}{z} dz$ where c is unit circle $|z| = 1$

[D24/ElectECS/5M]

Solution:

$I = \int_C \frac{1}{z} dz$ where C is $|z| = 1$

We see that $z = 0$ lies inside C

By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

$I = \int_C \frac{1}{z} dz = 2\pi i\, f(0) = 2\pi i(1) = 2\pi i$

7. Evaluate $\int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^n} dz$ where c is the circle $|z| = 1$ for $n = 1, n = 3$

[N18/M19/Comp/6M]

Solution:

$I = \int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^n} dz$ where C is $|z| = 1$

We see that $z = \frac{\pi}{6} = \frac{3.14}{6} = 0.52$ lies inside $C: |z| = 1$

For $n = 1$,

$I = \int_c \frac{\sin^6 z}{z-\frac{\pi}{6}} dz = 2\pi i\, f\left(\frac{\pi}{6}\right) = 2\pi i \sin^6\left(\frac{\pi}{6}\right) = 2\pi i \left[\frac{1}{2}\right]^6 = \frac{\pi i}{32}$

For $n = 3$,

By CIF, $\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$I = \int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!} f''\left(\frac{\pi}{6}\right)$

Here, $f(z) = \sin^6 z$

$\therefore f'(z) = 6\sin^5 z \cdot \cos z$

$\therefore f''(z) = 6\sin^5 z\,[-\sin z] + 6\cos z\,[5\sin^4 z \cdot \cos z]$

$f''(z) = -6\sin^6 z + 30\sin^4 z \cos^2 z$

$\therefore f''\left(\frac{\pi}{6}\right) = -6\left(\frac{1}{2}\right)^6 + 30\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2$

$\therefore f''\left(\frac{\pi}{6}\right) = \frac{21}{16}$

Thus,

$I = \int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!} \cdot \frac{21}{16} = \frac{21\pi i}{16}$

8. Evaluate $\int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{2}\right)^3} dz$ where c is the circle $|z| = 2$

[N14/ChemBiot/7M][N19/Chem/6M]

Solution:

$I = \int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{2}\right)^3} dz$ where C is $|z| = 2$

We see that $z = \frac{\pi}{2} = \frac{3.14}{2} = 1.57$ lies inside $C: |z| = 2$

By CIF, $\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$I = \int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{2}\right)^3} dz = \frac{2\pi i}{2!} f''\left(\frac{\pi}{2}\right)$

Here, $f(z) = \sin^6 z$

$\therefore f'(z) = 6\sin^5 z \cdot \cos z$

$\therefore f''(z) = 6\sin^5 z\,[-\sin z] + 6\cos z\,[5\sin^4 z \cdot \cos z]$

$f''(z) = -6\sin^6 z + 30\sin^4 z\cos^2 z$

$\therefore f''\left(\frac{\pi}{2}\right) = -6(1)^6 + 30(1)^4(0)^2$

$\therefore f''\left(\frac{\pi}{2}\right) = -6$

Thus, $I = \int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{2}\right)^3} dz = \frac{2\pi i}{2!} \cdot (-6) = -6\pi i$

9. Evaluate $\int_C \frac{\sin 2z}{\left(z+\frac{\pi}{3}\right)^4} dz$ where $C: |z| = 2$

[N22/MechCivil/6M]

Solution:

$I = \int_C \frac{\sin 2z}{\left(z+\frac{\pi}{3}\right)^4} dz$ where C is $|z| = 2$

We see that $z = -\frac{\pi}{3} = -\frac{3.14}{3} = -1.04$ lies inside $C: |z| = 2$

By CIF, $\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$I = \int_C \frac{\sin 2z}{\left(z+\frac{\pi}{3}\right)^4} dz = \frac{2\pi i}{3!} f'''\left(-\frac{\pi}{3}\right)$

Here, $f(z) = \sin 2z$

$\therefore f'(z) = \cos 2z \times 2$

$\therefore f''(z) = -\sin 2z \times 4$

$\therefore f'''(z) = -\cos 2z \times 8$

$\therefore f'''\left(-\frac{\pi}{3}\right) = -\cos\left(-\frac{2\pi}{3}\right) \times 8$

$\therefore f'''\left(-\frac{\pi}{3}\right) = 4$

Thus, $I = \int_C \frac{\sin 2z}{\left(z+\frac{\pi}{3}\right)^4} dz = \frac{2\pi i}{3!} \cdot (4) = \frac{4\pi i}{3}$

10. The value of $\int_c \frac{\sin z}{z^6} dz$ where C is $|z| = 1$

[M22/MechCivil/2M]

Solution:

$I = \int_c \frac{\sin z}{z^6} dz$ where C is $|z| = 1$

We see that $z = 0$ lies inside $C: |z| = 1$

By CIF, $\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$I = \int_c \frac{\sin z}{z^6} dz = \frac{2\pi i}{5!} f^{v}(0)$

Here, $f(z) = \sin z$

$\therefore f'(z) = \cos z$

$\therefore f''(z) = -\sin z$

$\therefore f'''(z) = -\cos z$

$\therefore f^{iv}(z) = \sin z$

$\therefore f^{v}(z) = \cos z$

$\therefore f^{v}(0) = \cos 0 = 1$

Thus,

$I = \int_c \frac{\sin z}{z^6} dz = \frac{2\pi i}{5!} \cdot (1) = \frac{\pi i}{60}$

11. Evaluate the integral $\int_c \frac{\sin^2 z}{z^3} dz$ where C is $|z| = 1$ using Cauchy Integral formula

[N22/Elex/6M]

Solution:

$I = \int_c \frac{\sin^2 z}{z^3} dz$ where C is $|z| = 1$

We see that $z = 0$ lies inside $C: |z| = 1$

By CIF, $\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$I = \int_c \frac{\sin^2 z}{z^3} dz = \frac{2\pi i}{2!} f''(0)$

Here, $f(z) = \sin^2 z$

$\therefore f'(z) = 2 \sin z \times \cos z$

$f'(z) = \sin 2z$

$\therefore f''(z) = \cos 2z \times 2$

$\therefore f''(0) = \cos(0) \times 2 = 2$

Thus,

$I = \int_c \frac{\sin^2 z}{z^3} dz = \frac{2\pi i}{2!} [2] = 2\pi i$

12. Evaluate the integral $\int_C \frac{\cos^2 z}{z^5} dz$ where C is $|z| = 1$ using Cauchy Integral formula
[M23/ElectECS/6M]
Solution:
$I = \int_C \frac{\cos^2 z}{z^5} dz$ where C is $|z| = 1$
We see that $z = 0$ lies inside $C: |z| = 1$
By CIF, $\int_C \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$
$I = \int_C \frac{\cos^2 z}{z^5} dz = \frac{2\pi i}{4!} f^{iv}(0)$
Here, $f(z) = \cos^2 z$
$\therefore f'(z) = 2\cos z \times -\sin z$
$f'(z) = -\sin 2z$
$\therefore f''(z) = -\cos 2z \times 2$
$\therefore f'''(z) = \sin 2z \times 4$
$\therefore f^{iv}(z) = \cos 2z \times 8$
$\therefore f^{iv}(0) = \cos(0) \times 8 = 8$
Thus,
$I = \int_C \frac{\cos^2 z}{z^5} dz = \frac{2\pi i}{24}[8] = \frac{2\pi i}{3}$

13. Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$, where C is $|z| = 2$
[M16/ElexExtcElectBiomInst/5M]
Solution:
$I = \int_C \frac{e^{2z}}{(z+1)^4} dz$ where C is $|z| = 2$
We see that $z = -1$ lies inside $C: |z| = 2$
By CIF, $\int_C \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$
$I = \int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} f'''(-1)$
Here, $f(z) = e^{2z}$
$\therefore f'(z) = 2e^{2z}$
$\therefore f''(z) = 4e^{2z}$
$\therefore f'''(z) = 8e^{2z}$
$\therefore f'''(-1) = 8e^{-2}$
Thus, $I = \int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!}\ 8e^{-2} = \frac{8\pi i e^{-2}}{3}$

14. Evaluate $\int_c \frac{e^{2z}}{(z-1)^4} dz$, where C is $|z| = 2$

[M16/CompIT/6M]

Solution:

$I = \int_c \frac{e^{2z}}{(z-1)^4} dz$ where C is $|z| = 2$

We see that $z = 1$ lies inside $C: |z| = 2$

By CIF, $\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$I = \int_c \frac{e^{2z}}{(z-1)^4} dz = \frac{2\pi i}{3!} f'''(1)$

Here, $f(z) = e^{2z}$

$\therefore f'(z) = 2e^{2z}$

$\therefore f''(z) = 4e^{2z}$

$\therefore f'''(z) = 8e^{2z}$

$\therefore f'''(1) = 8e^{2}$

Thus, $I = \int_c \frac{e^{2z}}{(z-1)^4} dz = \frac{2\pi i}{3!}\ 8e^2 = \frac{8\pi i e^2}{3}$

15. Evaluate $\int_c \frac{e^{2z}}{(z-a)^4} dz$, where C is $|z| = 2$

[N18/Chem/6M]

Solution:

$I = \int_c \frac{e^{2z}}{(z-a)^4} dz$ where C is $|z| = 2$

We assume that $z = a$ lies inside $C: |z| = 2$

By CIF, $\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$I = \int_c \frac{e^{2z}}{(z-1)^4} dz = \frac{2\pi i}{3!} f'''(1)$

Here, $f(z) = e^{2z}$

$\therefore f'(z) = 2e^{2z}$

$\therefore f''(z) = 4e^{2z}$

$\therefore f'''(z) = 8e^{2z}$

$\therefore f'''(a) = 8e^{2a}$

Thus, $I = \int_c \frac{e^{2z}}{(z-a)^4} dz = \frac{2\pi i}{3!}\ 8e^{2a} = \frac{8\pi i e^{2a}}{3}$

16. Evaluate $\int_c \frac{3z^2+2z-2}{(z-1)(z-2)} dz$ where C is (i) $|z| = \frac{1}{2}$ (ii) $|z| = \frac{3}{2}$ (iii) $|z| = 3$

[N19/Elex/6M]

Solution:

$I = \int_c \frac{3z^2+2z-2}{(z-1)(z-2)} dz$

Put $(z-1)(z-2) = 0$, we get $z = 1, z = 2$

(i) $C: |z| = \frac{1}{2}$

We see that $z = 1, z = 2$ lies outside C

By Cauchy's Integral Theorem,

$I = \int_c \frac{3z^2+2z-2}{(z-1)(z-2)} dz = 0$

(ii) $C: |z| = \frac{3}{2}$

We see that $z = 1$ lies inside C and $z = 2$ lies outside C

Now,

$I = \int_c \frac{3z^2+2z-2}{(z-1)(z-2)} dz$

$I = \int_c \frac{\frac{3z^2+2z-2}{z-2}}{z-1} dz$

By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

$I = 2\pi i\, f(1)$

$I = 2\pi i \left[\frac{3(1)^2+2(1)-2}{1-2}\right]$

$I = 2\pi i \left[\frac{3}{-1}\right]$

$I = -6\pi i$

(iii) $C: |z| = 3$

We see that $z = 1$ and $z = 2$ both lies inside C

Consider,

$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$1 = A(z-2) + B(z-1)$

Putting $z = 1$, we get $A = -1$

Putting $z = 2$, we get $B = 1$

$I = \int_c \frac{3z^2+2z-2}{(z-1)(z-2)} dz$

$I = \int_c \frac{-(3z^2+2z-2)}{(z-1)} dz + \int_c \frac{3z^2+2z-2}{(z-2)} dz$

By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

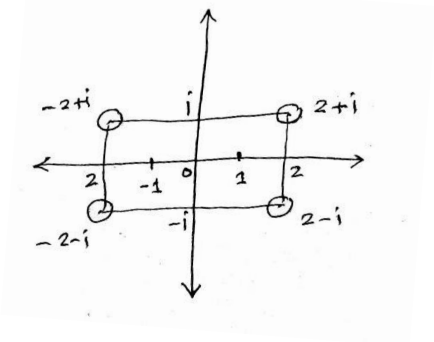
$I = 2\pi i\, f(1) + 2\pi i f(2)$

$I = 2\pi i[-(3(1)^2 + 2(1) - 2)] + 2\pi i[3(2)^2 + 2(2) - 2]$

$I = 2\pi i[-3] + 2\pi i[14]$

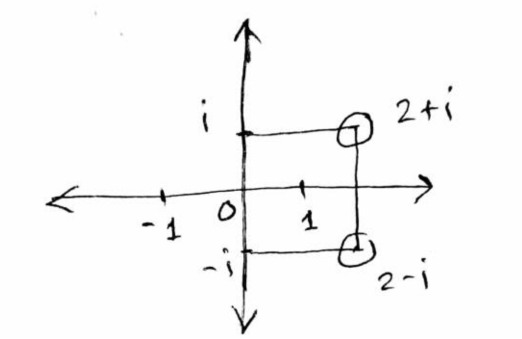
$I = -6\pi i + 28\pi i$
$I = 22\pi i$

17. Evaluate $\int_c \frac{\cos \pi z}{z^2-1} dz$ where c is
(i) a rectangle with vertices at $2 \pm i \& -2 \pm i$
(ii) a square with vertices $\pm i \& \ 2 \pm i$
Solution:
$I = \int_c \frac{\cos \pi z}{z^2-1} dz$
$I = \int \frac{\cos \pi z}{(z+1)(z-1)} dz$
Here, $z = z_0 = -1, z = z_0 = 1$
(i) a rectangle with vertices $2 + i, 2 - i, -2 + i, -2 - i$



We see that $z = -1$ and $z = 1$ lies inside C
$I = \int \frac{\cos \pi z}{(z+1)(z-1)} dz = \int \cos \pi z \left(\frac{1}{(z+1)(z-1)}\right) dz$
Let $\frac{1}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}$
$1 = A(z-1) + B(z+1)$
Put $z = -1$
$1 = A(-1-1) + B(-1+1)$
$1 = A(-2) + 0$
$1 = -2A$
$A = -\frac{1}{2}$
Put $z = 1$
$1 = A(1-1) + B(1+1)$
$1 = 0 + B(2)$
$1 = 2B$
$B = \frac{1}{2}$
Thus,
$I = \int \cos \pi z \left(\frac{-\frac{1}{2}}{z+1} + \frac{\frac{1}{2}}{z-1}\right) dz$
$I = -\frac{1}{2}\int \frac{\cos \pi z}{z+1} dz + \frac{1}{2}\int \frac{\cos \pi z}{z-1} dz$
By CIF,
$I = -\frac{1}{2}. 2\pi i\, f(-1) + \frac{1}{2}. 2\pi i f(1)$
$I = -\pi i[\cos(-\pi)] + \pi i[\cos(\pi)]$
$I = -\pi i(-1) + \pi i(-1)$
$I = 0$

(ii) a square with vertices $i, -i, 2+i, 2-i$



We see that $z=-1$ lies outside C and $z=1$ lies inside C

$I = -\frac{1}{2}\int \frac{\cos \pi z}{z+1} dz + \frac{1}{2}\int \frac{\cos \pi z}{z-1} dz$

By CIT and CIF,

$I = 0 + \frac{1}{2}.2\pi i f(1)$

$I = 0 + \pi i[\cos(\pi)]$

$I = 0 + \pi i(-1)$

$I = -\pi i$

18. Evaluate $\int_c \frac{z-3}{z^2+2z+5}dz$ where c is the circle (i) $|z|=1$ (ii) $|z+1-i|=2$
(iii) $|z+1+i|=2$
[N22/Elect/6M]
Solution:
$I=\int_c \frac{z-3}{z^2+2z+5}dz$
If $z^2+2z+5=0$, we get $z=-1+2i$, $z=-1-2i$
(i) $C:|z|=1$
For $z=-1+2i$
L.H.S $=|-1+2i|=\sqrt{(-1)^2+2^2}=\sqrt{5}>1>$ R.H.S
$\therefore z=-1+2i$ lies outside $C:|z|=1$
For $z=-1-2i$
L.H.S $=|-1-2i|=\sqrt{(-1)^2+(-2)^2}=\sqrt{5}>1>$ R.H.S
$\therefore z=-1-2i$ lies outside $C:|z|=1$
By Cauchy's Integral Theorem,
$I=\int_c \frac{z-3}{z^2+2z+5}dz=0$
(ii) $C:|z+1-i|=2$
For $z=-1+2i$
L.H.S $=|-1+2i+1-i|=|i|=1<2<$ R.H.S
$\therefore z=-1+2i$ lies inside $C:|z+1-i|=2$
For $z=-1-2i$
L.H.S $=|-1-2i+1-i|=|-3i|=3>2>$ R.H.S
$\therefore z=-1-2i$ lies outside $C:|z+1-i|=2$
Now,
$I=\int_c \frac{z-3}{(z+1-2i)(z+1+2i)}dz$
$I=\int_c \frac{\frac{z-3}{z+1+2i}}{z+1-2i}dz$
By CIF, $\int_c \frac{f(z)}{z-z_0}dz=2\pi i\, f(z_0)$
$I=2\pi i\, f(-1+2i)$
$I=2\pi i\left[\frac{-1+2i-3}{-1+2i+1+2i}\right]$
$I=2\pi i\left[\frac{-4+2i}{4i}\right]$
$I=\pi(-2+i)$
(iii) $C:|z+1+i|=2$
For $z=-1+2i$
L.H.S $=|-1+2i+1+i|=|3i|=3>2>$ R.H.S
$\therefore z=-1+2i$ lies outside $C:|z+1+i|=2$
For $z=-1-2i$
L.H.S $=|-1-2i+1+i|=|-i|=1<2<$ R.H.S

$\therefore z = -1 - 2i$ lies inside $C: |z + 1 + i| = 2$

Now,

$I = \int_c \frac{z-3}{(z+1-2i)(z+1+2i)} dz$

$I = \int_c \frac{\frac{z-3}{z+1-2i}}{z+1+2i} dz$

By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

$I = 2\pi i\, f(-1 - 2i)$

$I = 2\pi i\, \left[\frac{-1-2i-3}{-1-2i+1-2i}\right]$

$I = 2\pi i\, \left[\frac{-4-2i}{-4i}\right]$

$I = \pi(2 + i)$

19. Evaluate $\int_c \frac{z+3}{2z^2+3z-2} dz$ where c is the circle $|z - i| = 2$

[N18/Biot/6M][M23/D23/MechCivil/6M]

Solution:

$I = \int_c \frac{z+3}{2z^2+3z-2} dz\,; C\ is\ |z - i| = 2$

Let $2z^2 + 3z - 2 = 0$

$z = \frac{1}{2}, z = -2$

We have to check for $z = \frac{1}{2}$ and $z = -2$, whether they are inside C or outside C

For $z = \frac{1}{2}$,

L.H.S. $= \left|\frac{1}{2} - i\right| = \sqrt{\left(\frac{1}{2}\right)^2 + (-1)^2} = \sqrt{\frac{5}{4}} < 2 <$ R.H.S

$\therefore z = \frac{1}{2}$ is a point inside C

For $z = -2$,

L.H.S. $= |-2 - i| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5} > 2 >$ R.H.S

$\therefore z = -2$ is a point outside C

Now,

$I = \int_c \frac{z+3}{2z^2+3z-2} dz$

$I = \int_c \frac{z+3}{(2z-1)(z+2)} dz$

$I = \int_c \frac{\frac{z+3}{z+2}}{(2z-1)} dz$

$I = \frac{1}{2} \int_c \frac{\frac{z+3}{z+2}}{(z-\frac{1}{2})} dz$

By Cauchy's Integral Formula,

$I = \frac{1}{2} . 2\pi i\, \left[\frac{\frac{1}{2}+3}{\frac{1}{2}+2}\right] = \frac{7}{5} \pi i$

20. Using Cauchy Integral formula, evaluate $\int_c \frac{12z-7}{(z-1)^2(2z+3)} dz$ where $C: |z+i| = \sqrt{3}$

[M14/CompIT/6M][M22/MechCivil/5M]

Solution:

We have,

$I = \int_c \frac{12z-7}{(z-1)^2(2z+3)} dz; C\ is\ |z+i| = \sqrt{3}$

We have to check for $z = 1$ and $z = -\frac{3}{2}$, whether they are inside C or outside C

For $z = 1$,

L.H.S. $= |1+i| = \sqrt{1^2+1^2} = \sqrt{2} < \sqrt{3} <$ R.H.S

$\therefore z = 1$ is a point inside $C: |z+i| = \sqrt{3}$

For $z = -\frac{3}{2}$,

L.H.S. $= \left|-\frac{3}{2}+i\right| = \sqrt{\left(-\frac{3}{2}\right)^2 + 1^2} = \sqrt{\frac{13}{4}} > \sqrt{3} >$ R.H.S

$\therefore z = -\frac{3}{2}$ is a point outside $C: |z+i| = \sqrt{3}$

Now,

$I = \int_c \frac{12z-7}{(z-1)^2(2z+3)} dz$

$I = \int_c \frac{\frac{12z-7}{2z+3}}{(z-1)^2} dz$

By Cauchy's Integral Formula, $\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$I = \frac{2\pi i}{1!} f'(1)$

Here,

$f(z) = \frac{12z-7}{2z+3}$

$f'(z) = \frac{(2z+3)(12)-(12z-7)(2)}{(2z+3)^2}$

$\therefore f'(1) = 2$

Thus,

$I = \int_c \frac{12z-7}{(z-1)^2(2z+3)} dz = 2\pi i(2) = 4\pi i$

21. Evaluate $\int_c \frac{1}{(z^3-1)^2} dz$, where C is $|z-1| = 1$

[N16/MechCivil/6M]

Solution:

$I = \int_c \frac{1}{(z^3-1)^2} dz$

If $(z^3 - 1) = 0$

$(z-1)(z^2+z+1) = 0$

We get,

$z = 1, z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

Here, $C: |z-1| = 1$

For $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

L.H.S $= \left|-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i - 1\right|$

$= \left|-\frac{3}{2} \pm \frac{\sqrt{3}}{2} i\right|$

$= \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\pm\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{3} > 1 >$ R.H.S

$\therefore z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ lies outside $C: |z-1| = 1$

And, we see that $z = 1$ lies inside C

Now,

$I = \int_c \frac{1}{(z-1)^2(z^2+z+1)^2} dz$

$I = \int_c \frac{\frac{1}{(z^2+z+1)^2}}{(z-1)^2} dz$

By CIF, $\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$I = \frac{2\pi i}{1!} f'(1)$

Here, $f(z) = \frac{1}{(z^2+z+1)^2}$

$f'(z) = -\frac{2}{(z^2+z+1)^3} \times (2z+1)$

$f'(1) = -\frac{2}{9}$

$I = 2\pi i \left[-\frac{2}{9}\right] = -\frac{4\pi i}{9}$

22. Evaluate $\int_c \frac{z+2}{(z-3)(z-4)} dz$, where C is the circle $|z| = 1$

[N19/Comp/6M]

Solution:

$I = \int_c \frac{z+2}{(z-3)(z-4)} dz$

We see that, $z = 3$ and $z = 4$ both lies outside $C: |z| = 1$

By Cauchy's Integral Theorem,

$I = \int_c \frac{z+2}{(z-3)(z-4)} dz = 0$

23. Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)} dz$ where C is circle $|z| = 3$

[N19/Biot/5M][M24/Extc/6M]

Solution:

$I = \int_C \frac{e^{2z}}{(z-1)(z-2)} dz$

If $(z-1)(z-2) = 0$, we get $z = 1, z = 2$

For $C: |z| = 3$

We see that $z = 1$ and $z = 2$ both lies inside C

Consider,

$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$1 = A(z-2) + B(z-1)$

Putting $z = 1$, we get $A = -1$

Putting $z = 2$, we get $B = 1$

$I = \int_c \frac{e^{2z}}{(z-1)(z-2)} dz$

$I = \int_c \frac{-(e^{2z})}{(z-1)} dz + \int_c \frac{e^{2z}}{(z-2)} dz$

By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

$I = 2\pi i\, f(1) + 2\pi i f(2)$

$I = 2\pi i[-(e^2)] + 2\pi i[e^4]$

$I = 2\pi i[e^4 - e^2]$

24. Evaluate $\int_C \frac{2z+3}{(z+2)(z-3)} dz$ where C is circle $|z| = 3$

[M22/Elect/2M]

Solution:

$I = \int_C \frac{2z+3}{(z+2)(z-3)} dz$

If $(z+2)(z-3) = 0$, we get $z = -2, z = 3$

For C: $|z| = 3$

We see that $z = -2$ lies inside C and $z = 3$ lies on C

$I = \int_C \frac{2z+3}{(z+2)(z-3)} dz$

$I = \int_C \frac{\frac{2z+3}{z-3}}{(z+2)} dz$

By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

$I = 2\pi i\, f(-2)$

$I = 2\pi i \left[\frac{2(-2)+3}{-2-3}\right]$

$I = 2\pi i \left[-\frac{1}{-5}\right]$

$I = \frac{2\pi i}{5}$

25. Evaluate $\int_C \frac{z-1}{z^2+3z+2} dz$, $C: |z| = \frac{3}{2}$

[M24/ElectECS/5M]

Solution:

$I = \int_C \frac{z-1}{z^2+3z+2} dz = \int_C \frac{z-1}{(z+1)(z+2)} dz$

If $(z+2)(z+1) = 0$, we get $z = -2, z = -1$

For C: $|z| = \frac{3}{2}$

We see that $z = -2$ lies outside C and $z = -1$ lies inside C

$I = \int_C \frac{z-1}{(z+1)(z+2)} dz$

$I = \int_C \frac{\frac{z-1}{z+2}}{(z+1)} dz$

By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

$I = 2\pi i\, f(-1)$

$I = 2\pi i \left[\frac{-1-1}{-1+2}\right]$

$I = 2\pi i[-2]$

$I = -4\pi i$

26. Evaluate $\int_c \frac{z+8}{z^2+5z+6} dz$ where C is a circle $|z| = 5$
[D24/CompIT/6M]
Solution:
$I = \int_c \frac{z+8}{z^2+5z+6} dz$
If $z^2 + 5z + 6 = 0$, we get $z = -2, z = -3$
For C: $|z| = 5$
We see that $z = -2$ and $z = -3$ both lies inside C
Consider,
$\frac{1}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$
$1 = A(z+3) + B(z+2)$
Putting $z = -2$, we get $A = 1$
Putting $z = -3$, we get $B = -1$
$I = \int_c \frac{z+8}{(z+2)(z+3)} dz$
$I = \int_c \frac{z+8)}{(z+2)} dz + \int_c \frac{-(z+8)}{(z+3)} dz$
By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$
$I = 2\pi i\, f(-2) + 2\pi i f(-3)$
$I = 2\pi i[-2+8] + 2\pi i[-(-3+8)]$
$I = 2\pi i[6-5]$
$I = 2\pi i$

27. Evaluate $\int_c \frac{1}{4(z^2+1)} dz$ where C is the circle $|z| = 2$
[M18/Biot/6M]
Solution:
$I = \int_c \frac{1}{4(z^2+1)} dz\, ; C\ is\ |z| = 2$
$I = \frac{1}{4} \int_c \frac{1}{(z+i)(z-i)} dz$
We see that, $z = -i$ and $z = i$ both lies inside C: $|z| = 2$
Let $\frac{1}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i}$
$1 = A(z-i) + B(z+i)$
On solving, we get $A = \frac{i}{2}$ and $B = -\frac{i}{2}$
$I = \frac{1}{4} \int \frac{\frac{i}{2}}{z+i} - \frac{\frac{i}{2}}{z-i} dz$
$I = \frac{1}{4}[2\pi i\, f(-i) - 2\pi i\, f(i)]$
By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$
$I = \frac{1}{4}\left[2\pi i\ \left[\frac{i}{2}\right] - 2\pi i \left[\frac{\mathrm{i}}{2}\right]\right] = 0$

28. Evaluate $\int_c \frac{\sin \pi z+\cos \pi z}{z^2+z} dz\,; C\ is\ |z| = 4$ using Cauchy Integral Formula.
[N13/Biot/6M][M18/Elect/6M]
Solution:
$I = \int_c \frac{\sin \pi z+\cos \pi z}{z^2+z} dz\,; C\ is\ |z| = 4$
$I = \int_c \frac{\sin \pi z+\cos \pi z}{z(z+1)} dz$
We see that, $z = 0$ and $z = -1$ both lies inside $C: |z| = 4$
Let $\frac{1}{z(z+1)} = \frac{A}{z} + \frac{B}{z+1}$
$1 = A(z+1) + Bz$
On solving, we get
$A = 1$ and $B = -1$
$I = \int \frac{\sin \pi z+\cos \pi z}{z} - \frac{\sin \pi z+\cos \pi z}{z+1} dz$
$I = 2\pi i\, f(0) - 2\pi i\, f(-1)$ By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$
$I = 2\pi i\, [\sin 0 + \cos 0] - 2\pi i[\sin(-\pi) + \cos(-\pi)]$
$I = 2\pi i[0+1] - 2\pi i[0-1]$
$I = 2\pi i + 2\pi i$
$I = 4\pi i$

29. Evaluate $\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle $|z| = 3$
[M22/CompITAI/5M]
Solution:
$I = \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)} dz\,; C\ is\ |z| = 3$
We see that, $z = 1$ and $z = 2$ both lies inside $C: |z| = 3$
Let $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$
$1 = A(z-2) + B(z-1)$
Putting $z = 1$, we get $A = -1$
Putting $z = 2$, we get $B = 1$
$I = \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)} dz$
$I = \int_C \frac{-(\sin \pi z^2+\cos \pi z^2)}{(z-1)} dz + \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)} dz$
$I = 2\pi i\, f(1) + 2\pi i\, f(2)$ By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$
$I = 2\pi i\, [-(\sin \pi + \cos \pi)] + 2\pi i[\sin 4\pi + \cos 4\pi]$
$I = 2\pi i[1] + 2\pi i[1]$
$I = 2\pi i + 2\pi i$
$I = 4\pi i$

30. Evaluate $\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle $|z| = 4$

[M22/Chem/5M][N22/Chem/6M]

Solution:

$I = \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)} dz \,; C\ is\ |z| = 4$

We see that, $z = 1$ and $z = 2$ both lies inside $C: |z| = 4$

Let $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$1 = A(z-2) + B(z-1)$

Putting $z = 1$, we get $A = -1$

Putting $z = 2$, we get $B = 1$

$I = \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)} dz$

$I = \int_C \frac{-(\sin \pi z^2+\cos \pi z^2)}{(z-1)} dz + \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)} dz$

$I = 2\pi i\ f(1) + 2\pi i\ f(2)$ By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\ f(z_0)$

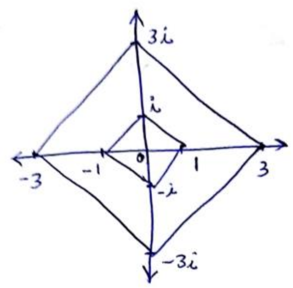
$I = 2\pi i\ [-(\sin \pi + \cos \pi)] + 2\pi i[\sin 4\pi + \cos 4\pi]$

$I = 2\pi i[1] + 2\pi i[1]$

$I = 2\pi i + 2\pi i$

$I = 4\pi i$

31. Evaluate $\int_c \frac{\sin z}{4z^2-8iz}dz$, c consists of the boundaries of the squares with vertices $\pm 3, \pm 3i$ (anticlockwise) and $\pm 1, \pm i$ (clockwise).
[M15/ChemBiot/6M]
Solution:
$I = \int_c \frac{\sin z}{4z^2-8iz}dz$
$I = \int_c \frac{\sin z}{4z(z-2i)}dz$
C consists of the boundaries of the squares with vertices $\pm 3, \pm 3i$ and $\pm 1, \pm i$ as shown below



We see that, $z = 0$ is a point outside C and $z = 2i$ is a point inside C
Now,
$I = \int_c \frac{\sin z}{4z(z-2i)}dz$
$I = \int_c \frac{\frac{\sin z}{4z}}{z-2i}dz$
By CIF, $\int_C \frac{f(z)}{(z-z_0)}dz = 2\pi i f(z_0)$
$I = 2\pi i\ f(2i)$
$I = 2\pi i\ \left[\frac{\sin 2i}{4(2i)}\right]$
$I = \frac{\pi}{4}\sin 2i$

32. If C is the rectangle formed by the lines $x = \pm 2, y = \pm\frac{1}{2}$ then evaluate the integral $\int_C \frac{2z}{z^4-1}dz$

[M23/ElectECS/6M]

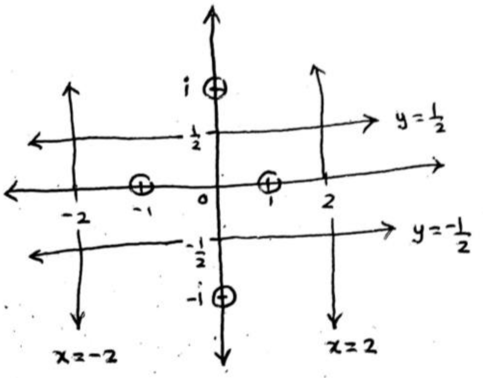
Solution:

$I = \int_C \frac{2z}{z^4-1}dz = \int_c \frac{2z}{(z^2-1)(z^2+1)}dz$

Put $(z^2-1)(z^2+1) = 0$

$z^2 = 1, z^2 = -1$

$z = \pm 1, z = \pm i$



We see that $z = 1, z = -1$ lies inside C and $z = i, z = -i$ lies outside C

$I = \int_c \frac{2z}{(z^2-1)(z^2+1)}dz$

$I = \int_c \frac{\frac{2z}{z^2+1}}{(z^2-1)}dz$

$I = \int_c \frac{\frac{2z}{z^2+1}}{(z-1)(z+1)}dz$

Let $\frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$

$1 = A(z+1) + B(z-1)$

Putting $z = 1$, we get $A = \frac{1}{2}$

Putting $z = -1$, we get $B = -\frac{1}{2}$

$I = \int_c \frac{2z}{(z^2+1)}\left(\frac{1}{(z-1)(z+1)}\right) dz$

$I = \int_c \frac{2z}{(z^2+1)}\left(\frac{\frac{1}{2}}{z-1} + \frac{-\frac{1}{2}}{z+1}\right) dz$

$I = \int_c \frac{\frac{z}{z^2+1}}{(z-1)}dz - \int_c \frac{\frac{z}{z^2+1}}{(z+1)}dz$

By CIF,

$I = 2\pi i\, f(1) - 2\pi i f(-1)$

$I = 2\pi i \left(\frac{1}{1^2+1}\right) - 2\pi i\left(\frac{-1}{(-1)^2+1}\right)$

$I = 2\pi i$

33. Evaluate $\int_c \frac{z^2-2z+4}{z^2-1} dz$ where c is $|z-1| = 1$
[N17/ElexExtcElectBiomInst/5M][M18/Elect/5M]
Solution:
$I = \int_c \frac{z^2-2z+4}{z^2-1} dz = \int_c \frac{z^2-2z+4}{(z-1)(z+1)} dz$
We see that $z = 1$ is a point inside C: $|z-1| = 1$ and $z = -1$ is outside C
$I = \int_c \frac{\frac{z^2-2z+4}{(z+1)}}{z-1} dz$
By CIF,
$I = 2\pi i\, f(1)$
$I = 2\pi i \left(\frac{(1)^2-2(1)+4}{1+1}\right)$
$I = 2\pi i \left(\frac{3}{2}\right)$
$I = 3\pi i$

34. Evaluate $\int_c \frac{3z^2+z}{z^2-1} dz$ where c is $|z-1| = 1$
[M24/MechCivil/6M]
Solution:
$I = \int_c \frac{3z^2+z}{z^2-1} dz = \int_c \frac{3z^2+z}{(z-1)(z+1)} dz$
We see that $z = 1$ is a point inside C: $|z-1| = 1$ and $z = -1$ is outside C
$I = \int_c \frac{\frac{3z^2+z}{(z+1)}}{z-1} dz$
By CIF,
$I = 2\pi i\, f(1)$
$I = 2\pi i \left(\frac{3(1)^2+1}{1+1}\right)$
$I = 2\pi i \left(\frac{4}{2}\right)$
$I = 4\pi i$

35. Evaluate $\int_C \frac{\cos \pi z}{z^2-1} dz$ where C is the circle $|z| = \frac{1}{2}$
[M22/CompITAI/2M]
Solution:
We have,
$I = \int_C \frac{\cos \pi z}{z^2-1} dz = \int_c \frac{\cos \pi z}{(z-1)(z+1)} dz$
We see that $z = 1$ and $z = -1$ both lies outside C: $|z| = \frac{1}{2}$
By CIT
$I = \int_C \frac{\cos \pi z}{z^2-1} dz = 0$

36. Evaluate $\int_c \frac{z^2+3}{z^2-1} dz$ where C is (i) $|z-1|=1$ (ii) $|z+1|=1$
[D24/MechCivil/8M]
Solution:
$I = \int_c \frac{z^2+3}{z^2-1} dz = \int_c \frac{z^2+3}{(z-1)(z+1)} dz$
(i) C is $|z-1|=1$
We see that $z=1$ is a point inside C: $|z-1|=1$ and $z=-1$ is outside C
$I = \int_c \frac{\frac{z^2+3}{(z+1)}}{z-1} dz$
By CIF,
$I = 2\pi i\, f(1)$
$I = 2\pi i\, \left(\frac{1^2+3}{1+1}\right)$
$I = 2\pi i\, \left(\frac{4}{2}\right)$
$I = 4\pi i$
(ii) C is $|z+1|=1$
We see that $z=-1$ is a point inside C: $|z-1|=1$ and $z=1$ is outside C
$I = \int_c \frac{\frac{z^2+3}{(z-1)}}{z+1} dz$
By CIF,
$I = 2\pi i\, f(-1)$
$I = 2\pi i\, \left(\frac{(-1)^2+3}{-1-1}\right)$
$I = 2\pi i\, \left(\frac{4}{-2}\right)$
$I = -4\pi i$

37. State and Prove Cauchy's integral formula for the simply connected region and hence evaluate $\int \frac{z+6}{z^2-4} dz$, $|z-2| = 5$

[N15/ElexExtcElectBiomInst/6M]

Solution:

Statement: If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R and if z_0 is any point within C, then

$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz$i.e.$\int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$

Since $f\ (z)$ is analytic inside and on C, $\frac{f(z)}{z-z_0}$ is also analytic inside and on C except at $z = z_0$. We draw a small curve C_1 around z_0 with centre at $z = z_0$ and radius r lying wholly inside C.

Now, $\frac{f(z)}{z-z_0}$ is analytic in the region enclosed between the curves C and C_1.

Hence, by Cauchy's extended theorem,

$\int_C \frac{f(z)}{(z-z_0)} dz = \int_{C_1} \frac{f(z)}{(z-z_0)} dz$ ……………………………(1)

Putting $z - z_0 = re^{i\theta}, dz = rie^{i\theta} d\theta$, we get on C_1,

$\int_{C_1} \frac{f(z)}{(z-z_0)} dz = \int_{C_1} \frac{f(z_0+re^{i\theta})}{re^{i\theta}} .rie^{i\theta} d\theta = \int_{C_1} f\left(z_0 + re^{i\theta}\right) id\theta$

As $r \to 0$, the circle tends to the point z_0,

$\int_{c_1} f\left(z_0 + re^{i\theta}\right) id\theta = \int_{c_1} f(z_0)\, id\theta = if(z_0) \int_{c_1} d\theta$

$= if(z_0) \int_0^{2\pi} d\theta$

$= 2\pi i f(z_0)$

Hence from (1), we get

$\int_C \frac{f(z)}{(z-z_0)} dz = 2\pi i f(z_0)$

We have,

$I = \int \frac{z+6}{z^2-4} dz = \int \frac{z+6}{(z+2)(z-2)} dz$

We have to check whether $z = 2$ and $z = -2$ is a point inside or outside $C: |z - 2| = 5$

For $z = 2$

L.H.S $= |2-2| = 0 < 5 <$ R.H.S

$\therefore z = 2$ is a point inside C

For $z = -2$

L.H.S $= |-2-2| = |-4| = 4 < 5 <$ R.H.S

$\therefore z = -2$ is a point inside C

Let $\frac{1}{(z+2)(z-2)} = \frac{A}{z+2} + \frac{B}{z-2}$

$1 = A(z-2) + B(z+2)$

Comparing the coefficients, we get

$A + B = 0$

$-2A + 2B = 1$

On solving, we get
$A = -\frac{1}{4}, B = \frac{1}{4}$
Now,
$I = -\int \frac{\frac{z+6}{4}}{z+2} dz + \int \frac{\frac{z+6}{4}}{z-2} dz$
$I = -2\pi i\, f(-2) + 2\pi i\, f(2)$
$I = -2\pi i \left[\frac{-2+6}{4}\right] + 2\pi i \left[\frac{2+6}{4}\right]$
$I = -2\pi i + 4\pi i$
$I = 2\pi i$

38. Evaluate $\int_c \frac{z+6}{z^2-4} dz$ where c (i) $|z| = 1$ (ii) $|z-2| = 1$
[N19/Extc/5M]
Solution:
We have,
$I = \int \frac{z+6}{z^2-4} dz = \int \frac{z+6}{(z+2)(z-2)} dz$
(i) $C: |z| = 1$
We have to check whether $z = 2$ and $z = -2$ is a point inside or outside C
We see that $z = 2, z = -2$ lies outside C
By Cauchy's Integral Theorem,
$I = \int \frac{z+6}{z^2-4} dz = 0$
(ii) $C: |z-2| = 1$
We see that $z = 2$ lies inside C and $z = -2$ lies outside C
$I = \int \frac{z+6}{(z+2)(z-2)} dz$
$I = \int \frac{\frac{z+6}{z+2}}{z-2} dz$
$I = 2\pi i\, f(2)$
$I = 2\pi i \left[\frac{2+6}{2+2}\right]$
$I = 4\pi i$

39. Evaluate $\int_c \frac{z+4}{z^2+2z+5} dz$ where c is the circle $|z+1+i| = 2$

[M16/ElexExtcElectBiomInst/6M]

Solution:

$I = \int_c \frac{z+4}{z^2+2z+5} dz$

If $z^2 + 2z + 5 = 0$, we get

$z = -1 + 2i, z = -1 - 2i$

Here, $C: |z+1+i| = 2$

For $z = -1 + 2i$

L.H.S $= |-1 + 2i + 1 + i| = |3i| = 3 > 2 >$ R.H.S

$\therefore z = -1 + 2i$ lies outside $C: |z+1+i| = 2$

For $z = -1 - 2i$

L.H.S $= |-1 - 2i + 1 + i| = |-i| = 1 < 2 <$ R.H.S

$\therefore z = -1 - 2i$ lies inside $C: |z+1+i| = 2$

Now,

$I = \int_c \frac{z+4}{(z+1-2i)(z+1+2i)} dz$

$I = \int_c \frac{\frac{z+4}{z+1-2i}}{z+1+2i} dz$

By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

$I = 2\pi i\, f(-1-2i)$

$I = 2\pi i \left[\frac{-1-2i+4}{-1-2i+1-2i}\right]$

$I = 2\pi i \left[\frac{3-2i}{-4i}\right]$

$I = \frac{\pi}{2}(-3+2i)$

40. Evaluate $\int_c \frac{z+4}{z^2+2z+5}\,dz$ where c is the circle $|z+1|=1$

[N17/Biot/6M]

Solution:

$I=\int_c \frac{z+4}{z^2+2z+5}\,dz$

If $z^2+2z+5=0$, we get

$z=-1+2i, z=-1-2i$

Here, $C: |z+1|=1$

For $z=-1+2i$

L.H.S $= |-1+2i+1| = |2i| = 2 > 1 >$ R.H.S

$\therefore z=-1+2i$ lies outside $C: |z+1|=1$

For $z=-1-2i$

L.H.S $= |-1-2i+1| = |-2i| = 2 > 1 >$ R.H.S

$\therefore z=-1-2i$ lies outside $C: |z+1|=1$

By Cauchy's Integral Theorem,

$I=\int_c \frac{z+4}{z^2+2z+5}\,dz = 0$

41. Evaluate $\int_c \frac{z+3}{z^2+2z+5}dz$ where c is the circle (i) $|z| = 1$ (ii) $|z+1-i| = 2$
[N14/CompIT/6M][M15/MechCivil/8M][N22/MTRX/6M]
Solution:
$I = \int_c \frac{z+3}{z^2+2z+5}dz$
If $z^2 + 2z + 5 = 0$, we get
$z = -1 + 2i, z = -1 - 2i$
(i) $C: |z| = 1$
For $z = -1 + 2i$
L.H.S = $|-1 + 2i| = \sqrt{(-1)^2 + 2^2} = \sqrt{5} > 1 >$ R.H.S
$\therefore z = -1 + 2i$ lies outside $C: |z| = 1$
For $z = -1 - 2i$
L.H.S = $|-1 - 2i| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5} > 1 >$ R.H.S
$\therefore z = -1 - 2i$ lies outside $C: |z| = 1$
By Cauchy's Integral Theorem,
$I = \int_c \frac{z+3}{z^2+2z+5}dz = 0$
(ii) $C: |z + 1 - i| = 2$
For $z = -1 + 2i$
L.H.S = $|-1 + 2i + 1 - i| = |i| = 1 < 2 <$ R.H.S
$\therefore z = -1 + 2i$ lies inside $C: |z + 1 - i| = 2$
For $z = -1 - 2i$
L.H.S = $|-1 - 2i + 1 - i| = |-3i| = 3 > 2 >$ R.H.S
$\therefore z = -1 - 2i$ lies outside $C: |z + 1 - i| = 2$
Now,
$I = \int_c \frac{z+3}{(z+1-2i)(z+1+2i)}dz$
$I = \int_c \frac{\frac{z+3}{z+1+2i}}{z+1-2i}dz$
By CIF, $\int_c \frac{f(z)}{z-z_0}dz = 2\pi i\, f(z_0)$
$I = 2\pi i\, f(-1 + 2i)$
$I = 2\pi i \left[\frac{-1+2i+3}{-1+2i+1+2i}\right]$
$I = 2\pi i \left[\frac{2+2i}{4i}\right]$
$I = \pi(1 + i)$

42. State and prove Cauchy Integral formula and hence evaluate $\int_C \frac{z+2}{z^3-2z^2} dz$ where C is $|z-2-i| = 2$

[M15/CompIT/6M][N15/ChemBiot/6M]

Solution:

Statement: If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R and if z_0 is any point within C, then

$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0} dz$i.e.$\int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$

Since $f\ (z)$ is analytic inside and on C, $\frac{f(z)}{z-z_0}$ is also analytic inside and on C except at $z = z_0$. We draw a small curve C_1 around z_0 with centre at $z = z_0$ and radius r lying wholly inside C.

Now, $\frac{f(z)}{z-z_0}$ is analytic in the region enclosed between the curves C and C_1.

Hence, by Cauchy's extended theorem,

$\int_C \frac{f(z)}{(z-z_0)} dz = \int_{C_1} \frac{f(z)}{(z-z_0)} dz$ ……………………………(1)

Putting $z - z_0 = re^{i\theta}, dz = rie^{i\theta} d\theta$, we get on C_1,

$\int_{C_1} \frac{f(z)}{(z-z_0)} dz = \int_{C_1} \frac{f(z_0+re^{i\theta})}{re^{i\theta}}.rie^{i\theta} d\theta = \int_{C_1} f\left(z_0 + re^{i\theta}\right) id\theta$

As $r \to 0$, the circle tends to the point z_0,

$\int_{c_1} f\left(z_0 + re^{i\theta}\right) id\theta = \int_{c_1} f(z_0)\, id\theta = if(z_0) \int_{c_1} d\theta$

$= if(z_0) \int_0^{2\pi} d\theta$

$= 2\pi i f(z_0)$

Hence from (1), we get

$\int_C \frac{f(z)}{(z-z_0)} dz = 2\pi i f(z_0)$

We have,

$I = \int_c \frac{z+2}{z^3-2z^2} dz = \int_c \frac{z+2}{z^2(z-2)} dz$

We have to check whether $z = 0$ and $z = 2$ is a point inside or outside C

For $z = 0$

L.H.S $= |0-2-i| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5} > 2 >$ R.H.S

$\therefore z = 0$ is a point outside C

For $z = 2$

L.H.S $= |2-2-i| = |-i| = 1 < 2 <$ R.H.S

$\therefore z = 2$ is a point inside C

Now,

$I = \int_c \frac{z+2}{z^2(z-2)} dz$

$I = \int_c \frac{\frac{z+2}{z^2}}{z-2} dz$

$I = 2\pi i\, f(2)$

$I = 2\pi i \left[\frac{2+2}{2^2}\right]$

$I = 2\pi i$

43. Evaluate $\int_C \frac{z+3}{2z^2+2z+5} dz$ where C is the circle $|z - i| = 2$

[M22/MTRX/5M]

Solution:

$I = \int_C \frac{z+3}{2z^2+2z+5} dz = \int_C \frac{z+3}{2\left(z^2+z+\frac{5}{2}\right)} dz = \int_C \frac{z+3}{2\left(z+\frac{1}{2}+\frac{3i}{2}\right)\left(z+\frac{1}{2}-\frac{3i}{2}\right)} dz$

For $z = -\frac{1}{2} - \frac{3i}{2}$

LHS $= \left|-\frac{1}{2} - \frac{3i}{2} - i\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{5}{2}\right)^2} = 2.55 > 2 >$ RHS

$\therefore z = -\frac{1}{2} - \frac{3i}{2}$ lies outside C

For $z = -\frac{1}{2} + \frac{3i}{2}$

LHS $= \left|-\frac{1}{2} + \frac{3i}{2} - i\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 0.71 < 2 <$ RHS

$\therefore z = -\frac{1}{2} + \frac{3i}{2}$ lies inside C

$I = \int_C \frac{z+3}{2\left(z+\frac{1}{2}+\frac{3i}{2}\right)\left(z+\frac{1}{2}-\frac{3i}{2}\right)} dz$

$I = \int_C \frac{\frac{z+3}{2\left(z+\frac{1}{2}+\frac{3i}{2}\right)}}{\left(z+\frac{1}{2}-\frac{3i}{2}\right)} dz$

By CIF

$I = 2\pi i\, f\left(-\frac{1}{2} + \frac{3i}{2}\right)$

$I = 2\pi i \left[\frac{-\frac{1}{2}+\frac{3i}{2}+3}{2\left(-\frac{1}{2}+\frac{3i}{2}+\frac{1}{2}+\frac{3i}{2}\right)}\right]$

$I = 2\pi i \left[\frac{1}{4} - \frac{5i}{12}\right]$

$I = \pi i \left(\frac{1}{2} - \frac{5i}{6}\right)$

44. Evaluate $\int_c \frac{z+1}{z^3-2z^2} dz$ where C is the circle (i) $|z| = 1$ (ii) $|z-2-i| = 2$
[N13/Chem/7M]
Solution:
$I = \int_c \frac{z+1}{z^3-2z^2} dz = \int_c \frac{z+1}{z^2(z-2)} dz$
(i) We see that $z = 0$ is a point inside C: $|z| = 1$ and $z = 2$ is outside C
$I = \int_c \frac{\frac{z+1}{(z-2)}}{z^2} dz$
By CIF, $\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$
$I = \frac{2\pi i}{1!} f'(0)$
Here, $f(z) = \frac{(z+1)}{z-2}$
$f'(z) = \frac{(z-2)(1)-(z+1)(1)}{(z-2)^2}$
$f'(0) = -\frac{3}{4}$
$I = \int_c \frac{\frac{z+1}{(z-2)}}{z^2} dz = 2\pi i \left(-\frac{3}{4}\right) = -\frac{3\pi i}{2}$
(ii) Now, we have to check whether $z = 0$ and $z = 2$ is a point inside or outside $C: |z-2-i| = 2$
For $z = 0$
L.H.S $= |0-2-i| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5} > 2 >$ R.H.S
$\therefore z = 0$ is a point outside C
For $z = 2$
L.H.S $= |2-2-i| = |-i| = 1 < 2 <$ R.H.S
$\therefore z = 2$ is a point inside C
Now,
$I = \int_c \frac{z+1}{z^2(z-2)} dz$
$I = \int_c \frac{\frac{z+1}{z^2}}{z-2} dz$
$I = 2\pi i\, f(2)$
$I = 2\pi i \left[\frac{2+1}{2^2}\right]$
$I = \frac{3\pi i}{2}$

45. Evaluate $\int_c \frac{z+1}{z^3-2z^2} dz$ where C is the circle $|z| = 1$

[M17/MechCivil/6M]

Solution:

$I = \int_c \frac{z+1}{z^3-2z^2} dz = \int_c \frac{z+1}{z^2(z-2)} dz$

We see that $z = 0$ is a point inside C: $|z| = 1$ and $z = 2$ is outside C

$I = \int_c \frac{\frac{z+1}{(z-2)}}{z^2} dz$

By CIF, $\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$I = \frac{2\pi i}{1!} f'(0)$

Here, $f(z) = \frac{(z+1)}{z-2}$

$f'(z) = \frac{(z-2)(1)-(z+1)(1)}{(z-2)^2}$

$f'(0) = -\frac{3}{4}$

Thus,

$I = \int_c \frac{\frac{z+1}{(z-2)}}{z^2} dz = 2\pi i \left(-\frac{3}{4}\right) = -\frac{3\pi i}{2}$

46. Evaluate $\int_c \frac{z+1}{z^3-2z^2}dz$ where C is the circle (i) $|z|=1$ (ii) $|z-2-i|=2$
(iii) $|z-1-2i|=2$
[M18/Elex/6M]
Solution:

$I=\int_c \frac{z+1}{z^3-2z^2}dz=\int_c \frac{z+1}{z^2(z-2)}dz$

(i) C is $|z|=1$

We see that $z=0$ is a point inside C: $|z|=1$ and $z=2$ is outside C

$I=\int_c \frac{\frac{z+1}{(z-2)}}{z^2}dz$

By CIF, $\int_c \frac{f(z)}{(z-z_0)^n}dz=\frac{2\pi i}{(n-1)!}f^{n-1}(z_0)$

$I=\frac{2\pi i}{1!}\,f'(0)$

Here, $f(z)=\frac{(z+1)}{z-2}$

$f'(z)=\frac{(z-2)(1)-(z+1)(1)}{(z-2)^2}$

$f'(0)=-\frac{3}{4}$

$I=\int_c \frac{\frac{z+1}{(z-2)}}{z^2}dz=2\pi i\left(-\frac{3}{4}\right)=-\frac{3\pi i}{2}$

(ii) C is $|z-2-i|=2$

For $z=0$, L.H.S. $=|-2-i|=\sqrt{(-2)^2+(-1)^2}=\sqrt{5}>$ R.H.S

$\therefore z=0$ is a point outside C

For $z=2$, L.H.S. $=|-i|=1<$ R.H.S

$\therefore z=2$ is a point inside C

$I=\int_c \frac{\frac{z+1}{z^2}}{(z-2)}dz$

By CIF, $I=2\pi i f(2)=2\pi i\left[\frac{2+1}{2^2}\right]=\frac{3\pi i}{2}$

(iii) C is $|z-1-2i|=2$

For $z=0$, L.H.S. $=|-1-2i|=\sqrt{(-1)^2+(-2)^2}=\sqrt{5}>$ R.H.S

$\therefore z=0$ is a point outside C

For $z=2$, L.H.S. $=|1-2i|=\sqrt{1^2+(-2)^2}=\sqrt{5}>$ R.H.S

$\therefore z=2$ is a point outside C

By CIT, $I=\int_c \frac{z+1}{z^3-2z^2}dz=0$

47. Evaluate using Cauchy Integral formula $\int_C \frac{2z^3+z^2+4}{z^4+4z^2} dz, C: |z-2-2i| = 3$

[M24/ElectECS/6M]

Solution:

$I = \int_C \frac{2z^3+z^2+4}{z^4+4z^2} dz = \int_C \frac{2z^3+z^2+4}{z^2(z^2+4)} dz$

$z^2(z^2+4) = 0$ gives $z = 0, z = 2i, -2i$

C is $|z-2-2i| = 3$

For $z = 0$, L.H.S. $= |-2-2i| = \sqrt{(-2)^2+(-2)^2} = \sqrt{8} < 3 <$ R.H.S

$\therefore z = 0$ is a point inside C

For $z = 2i$,

L.H.S. $= |2i-2-2i| = |-2| = 2 < 3 <$ R.H.S

$\therefore z = 2i$ is a point inside C

For $z = -2i$,

L.H.S. $= |-2i-2-2i| = |-2-4i| = \sqrt{(-2)^2+(-4)^2} = \sqrt{20} > 3 >$ R.H.S

$\therefore z = -2i$ is a point outside C

Let $\frac{1}{z^2(z+2i)(z-2i)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z+2i} + \frac{D}{z-2i}$

$1 = Az(z+2i)(z-2i) + B(z+2i)(z-2i) + Cz^2(z-2i) + Dz^2(z+2i)$

Putting $z = 0$, we get

$1 = B(2i)(-2i)$

$B = \frac{1}{4}$

Putting $z = -2i$, we get

$1 = C(-2i)^2(-4i)$

$C = -\frac{i}{16}$

Putting $z = 2i$, we get

$1 = D(2i)^2(4i)$

$D = \frac{i}{16}$

Putting $z = 1, B = \frac{1}{4}, C = -\frac{i}{16}, D = \frac{i}{16}$, we get

$1 = A(1)(1+2i)(1-2i) + \left(\frac{1}{4}\right)(1+2i)(1-2i) - \frac{i}{16}(1-2i) + \frac{i}{16}(1+2i)$

$1 = 5A + \frac{5}{4} - \frac{1}{8} - \frac{i}{16} - \frac{1}{8} + \frac{i}{16}$

$1 = 5A + 1$

$5A = 0$

$A = 0$

Now,

$I = \int_C (2z^3+z^2+4)\left(\frac{\frac{1}{4}}{z^2} + \frac{-\frac{i}{16}}{z+2i} + \frac{\frac{i}{16}}{z-2i}\right) dz$

$I = \frac{1}{4}\int_C \frac{2z^3+z^2+4}{z^2} dz - \frac{i}{16}\int_C \frac{2z^3+z^2+4}{z+2i} dz + \frac{i}{16}\int_C \frac{2z^3+z^2+4}{z-2i} dz$

$I=\frac{1}{4}.\frac{2\pi i}{1!}.\left[\frac{d}{dz}(2z^3+z^2+4)\right]_{z=0}-\frac{i}{16}(0)+\frac{i}{16}.2\pi i[2z^3+z^2+4]_{z=2i}$

$I=\frac{2\pi i}{4}[6z^2+2z]_{z=0}+\frac{2\pi i^2}{16}[-16i]$

$I=2\pi i$

48. Evaluate $\int_c \frac{z}{(z-1)^2(z-2)}dz$, where c is $|z-2|=0.5$

[N16/ElexExtcElectBiomInst/5M]

Solution:

$I=\int_c \frac{z}{(z-1)^2(z-2)}dz$ where C is $|z-2|=0.5$

We see that $z=2$ lies inside C and $z=1$ lies outside C

$I=\int_c \frac{\frac{z}{(z-1)^2}}{z-2}dz$

By C.I.F,

$I=2\pi i\, f(2)$

$I=2\pi i\, \left[\frac{2}{(2-1)^2}\right]$

$I=4\pi i$

49. Evaluate $\int_c \frac{z}{(z+1)^2(z-2)}dz$, where c is $|z|=3$

[M19/MechCivil/6M]

Solution:

$I=\int_c \frac{z}{(z+1)^2(z-2)}dz$ where C is $|z|=3$

We see that $z=2$ lies inside C and $z=-1$ also lies inside C

Let $\frac{1}{(z+1)^2(z-2)}=\frac{A}{z+1}+\frac{B}{(z+1)^2}+\frac{C}{z-2}$

$1=A(z+1)(z-2)+B(z-2)+C(z+1)^2$

$1=A(z^2-z-2)+B(z-2)+C(z^2+2z+1)$

On comparing the coefficients, we get

$A+0B+C=0$

$-A+B+2C=0$

$-2A-2B+C=1$

On solving, we get

$A=-\frac{1}{9}, B=-\frac{1}{3}, C=\frac{1}{9}$

$I=\int_c \frac{-\frac{1}{9}.z}{z+1}dz+\int_c \frac{-\frac{1}{3}.z}{(z+1)^2}dz+\int_c \frac{\frac{1}{9}.z}{z-2}dz$

By C.I.F,

$I=2\pi i\, f(-1)+\frac{2\pi i}{1!}.f'(-1)+2\pi i.f(2)$

$I=2\pi i\, \left[-\frac{1}{9}(-1)\right]+2\pi i\left[-\frac{1}{3}.1\right]+2\pi i\left[\frac{1}{9}.2\right]$

$I=0$

50. Evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)} dz$, where c is $|z| = 3$

[M18/Inst/6M]

Solution:

$I = \int_c \frac{z-1}{(z+1)^2(z-2)} dz$ where C is $|z| = 3$

We see that $z = 2$ lies inside C and $z = -1$ also lies inside C

Let $\frac{1}{(z+1)^2(z-2)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z-2}$

$1 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$

$1 = A(z^2 - z - 2) + B(z-2) + C(z^2 + 2z + 1)$

On comparing the coefficients, we get

$A + 0B + C = 0$

$-A + B + 2C = 0$

$-2A - 2B + C = 1$

On solving, we get

$A = -\frac{1}{9}, B = -\frac{1}{3}, C = \frac{1}{9}$

$I = \int_c \frac{-\frac{1}{9}.(z-1)}{z+1} dz + \int_c \frac{-\frac{1}{3}.(z-1)}{(z+1)^2} dz + \int_c \frac{\frac{1}{9}.(z-1)}{z-2} dz$

By C.I.F,

$I = 2\pi i\, f(-1) + \frac{2\pi i}{1!}.f'(-1) + 2\pi i.f(2)$

$I = 2\pi i \left[-\frac{1}{9}(-1-1)\right] + 2\pi i\left[-\frac{1}{3}.1\right] + 2\pi i\left[\frac{1}{9}.(2-1)\right]$

$I = \frac{4\pi i}{9} - \frac{2\pi i}{3} + \frac{2\pi i}{9} = 0$

51. Evaluate the integral $\int_c \frac{z^2}{(z+2)^2(z-i)} dz$, where c is $|z - i| = 1$

[N22/Elex/5M]

Solution:

$I = \int_c \frac{z^2}{(z+2)^2(z-i)} dz$ where C is $|z - i| = 1$

For $z = -2$, LHS = $|-2 - i| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5} > 1 >$ RHS

$\therefore z = -2$ lies outside C

For $z = i$, LHS = $|i - i| = 0 < 1 <$ RHS

$\therefore z = i$ lies inside C

$I = \int_c \frac{\frac{z^2}{(z+2)^2}}{z-i} dz$

By C.I.F,

$I = 2\pi i\, f(i)$

$I = 2\pi i \left[\frac{i^2}{(i+2)^2}\right]$

$I = \frac{\pi(-8-6i)}{25}$

52. Evaluate $\int_c \frac{dz}{z^3(z+4)}$ where c is the circle $|z| = 2$

[N15/CompIT/6M][N15/ElexExtcElectBiomInst/5M][M18/Comp/6M][M22/Elex/5M]

Solution:

$I = \int_c \frac{dz}{z^3(z+4)}$

We see that $z = 0$ is a point inside $C: |z| = 2$ and $z = -4$ is a point outside $C: |z| = 2$

Now,

$I = \int_c \frac{dz}{z^3(z+4)}$

$I = \int_c \frac{\frac{1}{z+4}}{z^3} dz$

By CIF, $\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$I = \frac{2\pi i}{2!} f''(0)$

Here, $f(z) = \frac{1}{z+4}$

$f'(z) = -\frac{1}{(z+4)^2}$

$f''(z) = \frac{2}{(z+4)^3}$

$f''(0) = \frac{2}{(4)^3} = \frac{1}{32}$

Thus, $I = \int_c \frac{dz}{z^3(z+4)} = \frac{2\pi i}{2} . \frac{1}{32} = \frac{\pi i}{32}$

53. Evaluate $\int_c \frac{dz}{z^3(z+4)}$ where c is the circle (i) $|z| = 2$ (ii) $|z-3| = 2$

[M19/Extc/6M]

Solution:

$I = \int_c \frac{dz}{z^3(z+4)}$

(i) C is $|z| = 2$. We see that $z = 0$ is a point inside C and $z = -4$ is a point outside C

Now,

$I = \int_c \frac{dz}{z^3(z+4)}$

$I = \int_c \frac{\frac{1}{z+4}}{z^3} dz$

By CIF, $\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$I = \frac{2\pi i}{2!} f''(0)$

Here, $f(z) = \frac{1}{z+4}$

$f'(z) = -\frac{1}{(z+4)^2}$

$f''(z) = \frac{2}{(z+4)^3}$

$f''(0) = \frac{2}{(4)^3} = \frac{1}{32}$

Thus,

$I = \int_c \frac{dz}{z^3(z+4)} = \frac{2\pi i}{2} . \frac{1}{32} = \frac{\pi i}{32}$

(ii) C is $|z-3| = 2$.

We see that $z = 0$ is a point outside C and $z = -4$ is a point outside C

Thus,

$I = \int_c \frac{dz}{z^3(z+4)} = 0$

54. Evaluate the integral using Cauchy's Integral formula $\int_c \frac{(4-3z)dz}{z(z-1)(z-2)}$ where C is the circle $|z| = \frac{3}{2}$

[M18/Chem/6M]

Solution:

$I = \int_c \frac{(4-3z)dz}{z(z-1)(z-2)}$ where C is $|z| = \frac{3}{2}$

We see that $z = 0, z = 1$ lies inside C and $z = 2$ also lies outside C

Let $\frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$

$1 = A(z-1)(z-2) + Bz(z-2) + Cz(z-1)$

$1 = A(z^2 - 3z + 2) + B(z^2 - 2z) + C(z^2 - z)$

On comparing the coefficients, we get

$A + B + C = 0$

$-3A - 2B - C = 0$

$2A + 0B + 0C = 1$

On solving, we get

$A = \frac{1}{2}, B = -1, C = \frac{1}{2}$

$I = \int_c \frac{\frac{1}{2}(4-3z)}{z} dz + \int_c \frac{-1.(4-3z)}{z-1} dz + \int_c \frac{\frac{1}{2}.(4-3z)}{z-2} dz$

By C.I.F,

$I = 2\pi i\, f(0) + 2\pi i\, f(1) + 2\pi i. f(2)$

$I = 2\pi i \left[\frac{1}{2}(4-0)\right] + 2\pi i[-1(4-3)] + 2\pi i\left[\frac{1}{2}(4-6)\right]$

$I = 4\pi i - 2\pi i - 2\pi i = 0$

55. Evaluate $\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2} dz$ where C is a circle $|z| = 1$

[N19/Inst/6M][D23/ElectECS/6M]

Solution:

$I = \int_C \frac{(z+4)^2}{z^4+5z^3+6z^2} dz = \int_C \frac{(z+4)^2}{z^2(z^2+5z+6)} dz = \int_C \frac{(z+4)^2}{z^2(z+2)(z+3)} dz$

We see that $z = 0$ lies inside C: $|z| = 1$ and $z = -2, z = -3$ both lies outside C

$I = \int_C \frac{\frac{(z+4)^2}{z^2+5z+6}}{z^2} dz$

$I = \frac{2\pi i}{1!} f'(0)$

$I = 2\pi i \frac{d}{dz}\left[\frac{(z+4)^2}{z^2+5z+6}\right]_{z=0}$

$I = 2\pi i \left[\frac{(z^2+5z+6)\big(2(z+4)\big) - (z+4)^2(2z+5)}{(z^2+5z+6)^2}\right]_{z=0}$

$I = 2\pi i \left[\frac{24-80}{36}\right]$

$I = -\frac{16\pi i}{9}$

56. Evaluate $\int_c \frac{1}{(z^2+1)(z^2+4)} dz$ where C is the circle $|z-i| = 1$
[D23/Extc/5M]
Solution:
$I = \int_c \frac{1}{(z^2+1)(z^2+4)} dz$
Put $(z^2+1)(z^2+4) = 0$
$z^2 = -1, z^2 = -4$
$z = \pm i, z = \pm 2i$
C is $|z-i| = 1$
We see that $z = i$ lies inside C and $z = -i, z = 2i, z = -2i$ all lies outside C
$I = \int_c \frac{1}{(z+i)(z-i)(z+2i)(z-2i)} dz$
$I = \int_c \frac{\frac{1}{(z+i)(z+2i)(z-2i)}}{(z-i)} dz$
By CIF,
$I = 2\pi i f(i)$
$I = 2\pi i \left[\frac{1}{(i+i)(i+2i)(i-2i)}\right]$
$I = \frac{\pi}{3}$

57. Evaluate $\int_c \frac{z^2}{(z-1)(z-2)} dz$ where C is the circle $|z-1| = 1$
[D23/CompITAI/6M]
Solution:
$I = \int_c \frac{z^2}{(z-1)(z-2)} dz$
Put $(z-1)(z-2) = 0$
$z = 1, z = 2$
C is $|z-1| = 1$
We see that $z = 1$ lies inside C and $z = 2$ lies on C
$I = \int_c \frac{\frac{z^2}{z-2}}{(z-1)} dz$
By CIF,
$I = 2\pi i f(1)$
$I = 2\pi i \left[\frac{1^2}{(1-2)}\right]$
$I = -2\pi i$

58. Evaluate the integral $\int_C \frac{z^2}{(z-3)^2(z+2)} dz, C: |z+1| = 2$

[D23/ElectECS/5M]

Solution:

$I = \int_c \frac{z^2}{(z-3)^2(z+2)} dz$

Put $(z-3)^2(z+2) = 0$

$z = 3, z = -2$

C is $|z+1| = 2$

We see that $z = -2$ lies inside C and $z = 3$ lies outside C

$I = \int_c \frac{\frac{z^2}{(z-3)^2}}{(z+2)} dz$

By CIF,

$I = 2\pi i f(-2)$

$I = 2\pi i \left[\frac{(-2)^2}{(-2-3)^2}\right]$

$I = \frac{8\pi i}{25}$

59. Find $f(3), f'(1+i), f''(1-i)$ If $f(a) = \int_C \frac{3z^2+11z+7}{z-a} dz$ where C is a circle $|z| = 2$ **[M14/ElexExtcElectBiomInst/6M]**

Solution:

$f(a) = \int_C \frac{3z^2+11z+7}{z-a} dz$

If $z = a$ is a point on C or outside C, then

By Cauchy's Integral Theorem,

$f(a) = \int_C \frac{3z^2+11z+7}{z-a} dz = 0$ ……………………….(1)

If $z = a$ is a point inside C, then

By Cauchy's Integral Formula,

$f(a) = \int_C \frac{3z^2+11z+7}{z-a} dz = 2\pi i\ [3a^2 + 11a + 7]$ …………….(2)

(i) Consider, $z = a = 3$

We see that, $z = a = 3$ is outside the $C: |z| = 2$

Thus, from (1), we get

$f(a) = f(3) = 0$

(ii) Consider, $z = a = 1 + i$

We see that, $z = a = 1 + i$ is inside the $C: |z| = 2$

Thus, from (2), we get

$f(a) = 2\pi i[3a^2 + 11a + 7]$

$f'(a) = 2\pi i[6a + 11]$

$\therefore f'(1+i) = 2\pi i[6(1+i) + 11]$

$\therefore f'(1+i) = 2\pi i[17 + 6i]$

(iii) Consider, $z = a = 1 - i$

We see that, $z = a = 1 - i$ is inside the $C: |z| = 2$

Thus, from (2), we get

$f(a) = 2\pi i[3a^2 + 11a + 7]$

$f'(a) = 2\pi i[6a + 11]$

$f''(a) = 2\pi i[6]$

$\therefore f''(1-i) = 12\pi i$

60. If $\emptyset(\alpha) = \oint_c \frac{ze^z}{z-\alpha} dz$ where C is $|z - 2i| = 3$ find $\emptyset(1), \emptyset'(2), \emptyset(3), \emptyset'(4)$

[N14/ElexExtcElectBiomInst/6M]

Solution:

We have, $\emptyset(\alpha) = \oint_c \frac{ze^z}{z-\alpha} dz$

If $z = \alpha$ is a point on C or outside C, then

By Cauchy's Integral Theorem,

$\emptyset(\alpha) = \oint_c \frac{ze^z}{z-\alpha} dz = 0$ ……………………….(1)

If $z = \alpha$ is a point inside C, then

By Cauchy's Integral Formula,

$\emptyset(\alpha) = \oint_c \frac{ze^z}{z-\alpha} dz = 2\pi i\, [\alpha\, e^{\alpha}]$ …………….(2)

(i) Consider, $z = \alpha = 1$ where C is $|z - 2i| = 3$

L.H.S. = $|1 - 2i| = \sqrt{1^2 + (-2)^2} = \sqrt{5} < 3 <$ R.H.S

$\therefore z = \alpha = 1$ lies inside $C: |z - 2i| = 3$

Thus, from (2), we get

$\emptyset(\alpha) = 2\pi i\, [\alpha\, e^{\alpha}]$

$\therefore \emptyset(1) = 2\pi i\, [1. e^1] = 2\pi i e$

(ii) Consider, $z = \alpha = 2$ where C is $|z - 2i| = 3$

L.H.S. = $|2 - 2i| = \sqrt{2^2 + (-2)^2} = \sqrt{8} < 3 <$ R.H.S

$\therefore z = \alpha = 2$ lies inside $C: |z - 2i| = 3$

Thus, from (2), we get

$\emptyset(\alpha) = 2\pi i\, [\alpha\, e^{\alpha}]$

$\emptyset'(\alpha) = 2\pi i[\alpha(e^{\alpha}) + e^{\alpha}(1)]$

$\emptyset'(\alpha) = 2\pi i\, e^{\alpha}[\alpha + 1]$

$\therefore \emptyset'(2) = 6\pi i\, e^2$

(iii) Consider, $z = \alpha = 3$ where C is $|z - 2i| = 3$

L.H.S. = $|3 - 2i| = \sqrt{3^2 + (-2)^2} = \sqrt{13} > 3 >$ R.H.S

$\therefore z = \alpha = 3$ lies outside $C: |z - 2i| = 3$

Thus, from (1), we get

$\emptyset(\alpha) = \emptyset(3) = 0$

(iv) Consider, $z = \alpha = 4$ where C is $|z - 2i| = 3$

L.H.S. = $|4 - 2i| = \sqrt{4^2 + (-2)^2} = \sqrt{20} > 3 >$ R.H.S

$\therefore z = \alpha = 4$ lies outside $C: |z - 2i| = 3$

Thus, from (1), we get $\emptyset(\alpha) = \emptyset(4) = 0$

$\therefore \emptyset'(4) = 0$

61. If $f(a) = \int_C \frac{3z^2+7z+1}{z-a} dz$ where C is a circle $|z| = 2$, then find (i) $f(-3)$ (ii) $f(i)$ (iii) $f(1-i)$ (iv) $f'(1-i)$

[M15/ElexExtcElectBiomInst/6M]

Solution:

$f(a) = \int_C \frac{3z^2+7z+1}{z-a} dz$

If $z = a$ is a point on C or outside C, then

By Cauchy's Integral Theorem,

$f(a) = \int_C \frac{3z^2+7z+1}{z-a} dz = 0$ ……………………….(1)

If $z = a$ is a point inside C, then

By Cauchy's Integral Formula,

$f(a) = \int_C \frac{3z^2+7z+1}{z-a} dz = 2\pi i\ [3a^2 + 7a + 1]$ …………….(2)

(i) Consider, $z = a = -3$

We see that, $z = a = -3$ is outside the $C: |z| = 2$

Thus, from (1), we get

$f(a) = f(-3) = 0$

(ii) Consider, $z = a = i$

We see that, $z = a = i$ is inside the $C: |z| = 2$

Thus, from (2), we get

$f(a) = 2\pi i[3a^2 + 7a + 1]$

$\therefore f(i) = 2\pi i[3i^2 + 7i + 1]$

$\therefore f(i) = 2\pi i[-2 + 7i]$

(iii) Consider, $z = a = 1 - i$

We see that, $z = a = 1 - i$ is inside the $C: |z| = 2$

Thus, from (2), we get

$f(a) = 2\pi i[3a^2 + 7a + 1]$

$f(1 - i) = 2\pi i[3(1 + i)^2 + 7(1 + i) + 1]$

$f(1 - i) = 2\pi i[8 - 13i]$

Also,

$f'(a) = 2\pi i[6a + 7]$

$f'(1 - i) = 2\pi i[6(1 - i) + 7]$

$f'(1 - i) = 2\pi i[13 - 6i]$

62. If $f(a)=\int_c \frac{4z^2+z+4}{z-a}dz$ where c is the ellipse $4x^2+9y^2=36$. Find the values of (i) $f(4)$ (ii) $f'(-1)$ (iii) $f"(-i)$

[N18/MechCivil/6M]

Solution:

$f(a)=\int_c \frac{4z^2+z+4}{z-a}dz$

If $z=a$ is a point on C or outside C, then

By Cauchy's Integral Theorem,

$f(a)=\int_c \frac{4z^2+z+4}{z-a}dz=0$ ……………………….(1)

If $z=\alpha$ is a point inside C, then

By Cauchy's Integral Formula,

$f(a)=\int_c \frac{4z^2+z+4}{z-a}dz=2\pi i\ [4a^2+a+4]$ …………….(2)

$f'(a)=2\pi i[8a+1]$ ………………. (3)

$f''(a)=16\pi i$ …………………………. (4)

(i) Consider, $z=a=4$

We see that, $z=a=4$ is outside the $C: 4x^2+9y^2=36$

Thus, from (1), we get

$f(a)=f(4)=0$

(ii) Consider, $z=a=-1$

We see that, $z=a=-1$ is inside the $C: 4x^2+9y^2=36$

Thus, from (3), we get

$f'(a)=2\pi i[8a+1]$

$\therefore f'(-1)=2\pi i[-8+1]=-14\pi i$

(iii) Consider, $z=a=-i$

We see that, $z=a=-i$ is inside the $C: 4x^2+9y^2=36$

Thus, from (4), we get

$f''(a)=16\pi i$

$\therefore f''(-i)=16\pi i$

63. If $f(a) = \int_c \frac{4z^2+z+5}{z-a}\,dz$ where c is $|z| = 2$. Find the values of $f(1), f(i), f'(-1), f''(-i)$
[N18/Inst/6M]
Solution:
$f(a) = \int_c \frac{4z^2+z+5}{z-a}\,dz$
If $z = a$ is a point on C or outside C, then
By Cauchy's Integral Theorem,
$f(a) = \int_c \frac{4z^2+z+5}{z-a}\,dz = 0$ ……………………….(1)
If $z = \alpha$ is a point inside C, then
By Cauchy's Integral Formula,
$f(a) = \int_c \frac{4z^2+z+5}{z-a}\,dz = 2\pi i\,[4a^2 + a + 5]$ …………….(2)
$f'(a) = 2\pi i[8a + 1]$ ………………. (3)
$f''(a) = 16\pi i$ …………………………. (4)
(i) Consider, $z = a = 1$
We see that, $z = a = 1$ is inside the $C: |z| = 2$
Thus, from (2), we get
$f(a) = 2\pi i\,[4a^2 + a + 5]$
$f(1) = 20\pi i$
(ii) Consider, $z = a = i$
We see that, $z = a = i$ is inside the $C: |z| = 2$
Thus, from (2), we get
$f(a) = 2\pi i\,[4a^2 + a + 5]$
$f(i) = \pi(-2 + 2i)$
(iii) Consider, $z = a = -1$
We see that, $z = a = -1$ is inside the $C: |z| = 2$
Thus, from (3), we get
$f'(a) = 2\pi i[8a + 1]$
$\therefore f'(-1) = -14\pi i$
(iv) Consider, $z = a = -i$
We see that, $z = a = -i$ is inside the $C: |z| = 2$
Thus, from (4), we get
$f''(a) = 16\pi i$
$\therefore f''(-i) = 16\pi i$

64. If $\emptyset(\alpha) = \int_c \frac{4z^2+z+5}{z-\alpha} dz$ where c is $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Find the values of $\emptyset(3.5)$, $\emptyset(i)$, $\emptyset'(-1)$, $\emptyset''(-i)$

[N16/ElexExtcElectBiomInst/6M]

Solution:

$\emptyset(\alpha) = \int_c \frac{4z^2+z+5}{z-\alpha} dz$

If $z = \alpha$ is a point on C or outside C, then

By Cauchy's Integral Theorem,

$\emptyset(\alpha) = \int_c \frac{4z^2+z+5}{z-\alpha} dz = 0$ ……………………….(1)

If $z = \alpha$ is a point inside C, then

By Cauchy's Integral Formula,

$\emptyset(\alpha) = \int_c \frac{4z^2+z+5}{z-\alpha} dz = 2\pi i\,[4\alpha^2 + \alpha + 5]$ ……………(2)

(i) Consider, $z = \alpha = 3.5$

We see that, $z = \alpha = 3.5$ is outside the $C: \frac{x^2}{4} + \frac{y^2}{9} = 1$

Thus, from (1), we get

$\emptyset(\alpha) = \emptyset(3.5) = 0$

(ii) Consider, $z = \alpha = i$

We see that, $z = \alpha = i$ is inside the $C: \frac{x^2}{4} + \frac{y^2}{9} = 1$

Thus, from (2), we get

$\emptyset(\alpha) = 2\pi i\,[4\alpha^2 + \alpha + 5]$

$\therefore \emptyset(i) = 2\pi i[4i^2 + i + 5]$

$\therefore \emptyset(i) = 2\pi i[1 + i]$

(iii) Consider, $z = \alpha = -1$

We see that, $z = \alpha = -1$ is inside the $C: \frac{x^2}{4} + \frac{y^2}{9} = 1$

Thus, from (2), we get

$\emptyset(\alpha) = 2\pi i\,[4\alpha^2 + \alpha + 5]$

$\emptyset'(\alpha) = 2\pi i[8\alpha + 1]$

$\therefore \emptyset'(-1) = -14\pi i$

(iv) Consider, $z = \alpha = -i$

We see that, $z = \alpha = -i$ is inside the $C: \frac{x^2}{4} + \frac{y^2}{9} = 1$

Thus, from (2), we get

$\emptyset(\alpha) = 2\pi i\,[4\alpha^2 + \alpha + 5]$

$\emptyset'(\alpha) = 2\pi i[8\alpha + 1]$

$\emptyset''(\alpha) = 2\pi i[8]$

$\therefore \emptyset''(-i) = 16\pi i$

65. If $f(a)=\int_C \frac{3z^2+7z+1}{z-a}dz$ where C: $x^2+y^2=4$, then find (i) $f(-3)$ (ii) $f'(1-i)$ (iii) $f''(1-i)$

[N17/MechCivil/6M]

Solution:

We have, $f(a)=\int_C \frac{3z^2+7z+1}{z-a}dz$

If $z=a$ is a point on C or outside C, then

By Cauchy's Integral Theorem,

$f(a)=\int_C \frac{3z^2+7z+1}{z-a}dz=0$ ……………………….(1)

If $z=a$ is a point inside C, then

By Cauchy's Integral Formula,

$f(a)=\int_C \frac{3z^2+7z+1}{z-a}dz=2\pi i\,[3a^2+7a+1]$ …………….(2)

C : $x^2+y^2=4$ is a circle with centre at $(0,0)$ and radius 2 i.e. $|z|=2$

(i) Consider, $z=a=-3$

We see that, $z=a=-3$ is outside the C: $|z|=2$

Thus, from (1), we get

$f(a)=f(-3)=0$

(ii) Consider, $z=a=1-i$

We see that, $z=a=1-i$ is inside the C: $|z|=2$

Thus, from (2), we get

$f'(a)=2\pi i[6a+7]$

$f'(1-i)=2\pi i[6(1-i)+7]$

$f'(1-i)=2\pi i[13-6i]$

Also,

$f''(a)=2\pi i[6]$

$f''(1-i)=12\pi i$

66. If $f(\alpha) = \int_c \frac{3z^2-z+5}{z-\alpha} dz$ where c is the circle $|z| = 3$ then find $f(1), f^{'}(-1), f^{''}(-i)$
[M19/Elex/6M]
Solution:
$f(\alpha) = \int_c \frac{3z^2-z+5}{z-\alpha} dz$
If $z = \alpha$ is a point on C or outside C, then
By Cauchy's Integral Theorem,
$f(\alpha) = \int_c \frac{3z^2-z+5}{z-\alpha} dz = 0$ ……………………….(1)
If $z = \alpha$ is a point inside C, then
By Cauchy's Integral Formula,
$f(\alpha) = \int_c \frac{3z^2-z+5}{z-\alpha} dz = 2\pi i\ [3\alpha^2 - \alpha + 5]$ …………….(2)
$f'(\alpha) = 2\pi i[6\alpha - 1]$ …………… (3)
$f''(\alpha) = 2\pi i[6]$ ……………………. (4)
(i) Consider, $z = \alpha = 1$
We see that, $z = \alpha = 1$ is inside the $C: |z| = 3$
Thus, from (2), we get
$f(\alpha) = f(1) = 2\pi i[3(1)^2 - 1 + 5] = 14\pi i$
(ii) Consider, $z = \alpha = -1$
We see that, $z = \alpha = -1$ is inside the $C: |z| = 3$
Thus, from (2), we get
$f'(\alpha) = 2\pi i[6(-1) - 1] = -14\pi i$
(iii) Consider, $z = \alpha = -i$
We see that, $z = \alpha = -i$ is inside the $C: |z| = 3$
Thus, from (3), we get
$f''(\alpha) = 2\pi i[6] = 12\pi i$

67. If $f(z)$ is analytic function and $f'(z)$ is continuous at all points inside and on simple closed curve 'C' then
[M22/Extc/2M]
Solution:
By Cauchy's Integral Theorem,
$\oint_C f(z)dz = 0$