# Chap 7. Push down automata (PDA)

✓ [PDA is used for recognizing context free language which is generated by context free grammar]

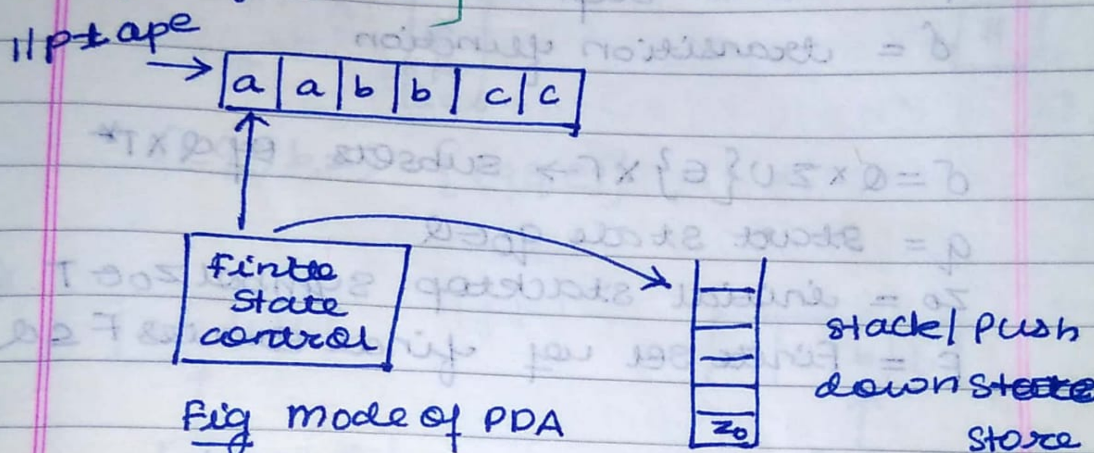



Fig mode of PDA

✓ Components of PDA

PDA consists of finite set of states, I/P tape, read head & a stack

✓ working of PDA

Depending on the state, I/P symbol of stack top symbol

✓ PDA can change the state / remain in same state

PDA moves head to right of current cell

PDA can perform some stack operation

$$\boxed{M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)}$$

where

$Q$ = finite set of states

$\Sigma$ = I/P alphabet

$\Gamma$ = stack alphabet

$\delta$ = transition function

$\delta = Q \times \Sigma \cup \{\epsilon\} \times \Gamma \rightarrow$ subsets of $Q \times \Gamma^*$

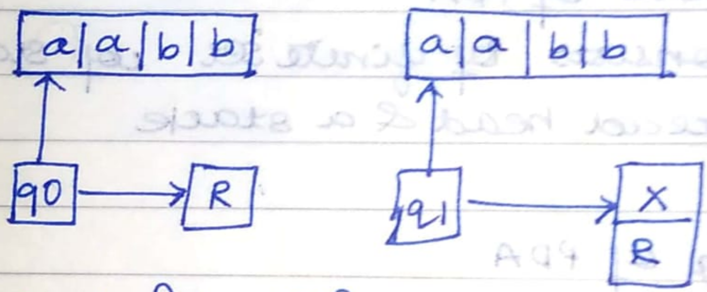
$q$ = start state $q_0 \in Q$

$Z_0$ = initial stacktop symbol $Z_0 \in \Gamma$

$F$ = Finite set of final states $F \subseteq Q$

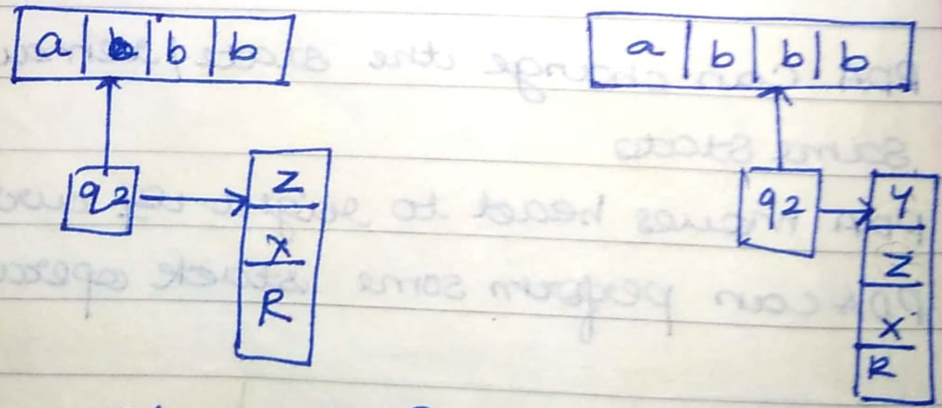
Some eg of transition function

①



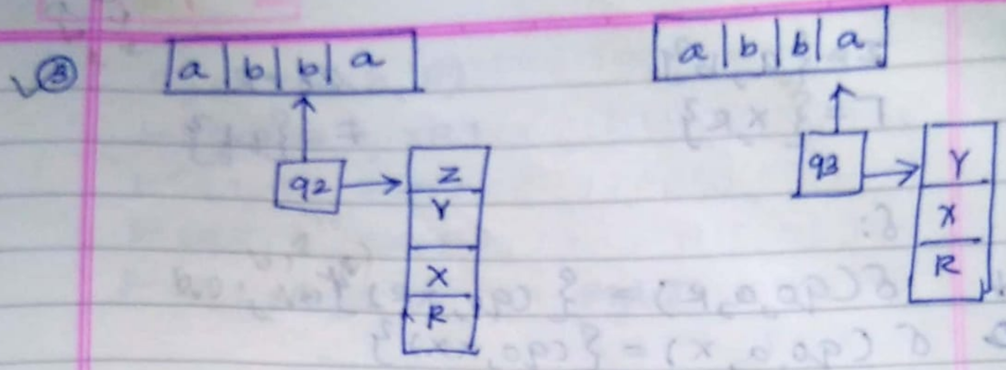

$\delta(q_0, a, R) = \{(q_1, XR)\}$

②



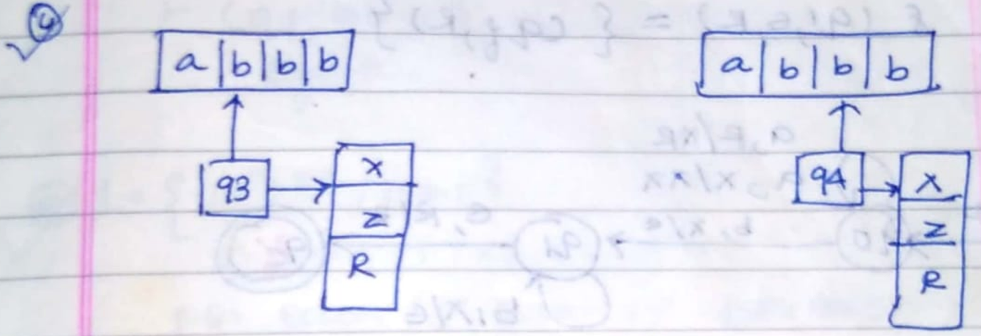

$\delta(q_2, b, Z) = \{(q_2, YZ)\}$

③




$\delta(q_2, b, z) = \{(q_3, \epsilon)\}$

④




$\delta(q_3, b, x) = \{(q_4, x)\}$

① Design a PDA for recognising
$L = \{a^n b^n \mid n \geq 1\}$

Solution <u>Step 1</u> Theory

<u>Step 2</u> Logic
For each 'a' push 1X for each 'b' pop 1X.

<u>Step 3</u> Implementation
$m = (Q, \Sigma, T, \delta, q_0, z_0, F)$

$Q = \{q_0, q_1, q_f\}$ $\quad q_0 = q_0$

$\Sigma = \{a, b\}$ $\quad Z_0 = R$

$\Gamma = \{x, R\}$ $\quad F = \{q_f\}$

$\delta$:

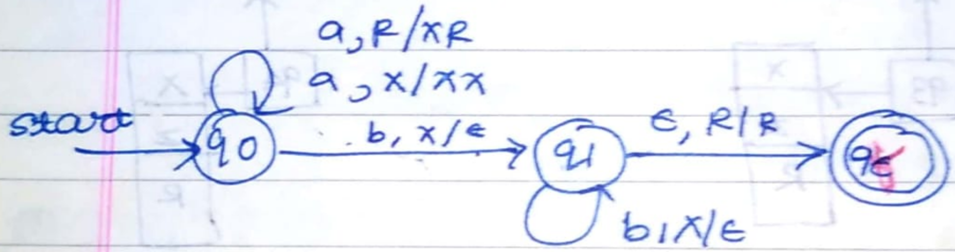
$\delta(q_0, a, R) = \{(q_0, xR)\}$

$\delta(q_0, a, x) = \{(q_0, xx)\}$

$\delta(q_0, b, x) = \{(q_1, \epsilon)\}$

$\delta(q_1, b, x) = \{(q_1, \epsilon)\}$

$\delta(q_1, \epsilon, R) = \{(q_f, R)\}$




stack

**Step 4** $(q_0, aaabbb, R)$

$\vdash (q_0, aabbb, xR)$

$\vdash (q_0, abbb, xxR)$

$\vdash (q_0, bbb, xxxR)$

$\vdash (q_1, bb, xxR)$

$\vdash (q_1, \epsilon, R)$

$\vdash (q_f, \epsilon, R)$

accept

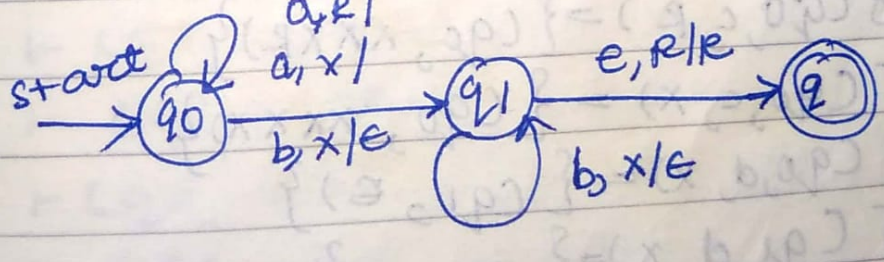
(2) $(q_0, aaab, R)$
$\vdash (q_0, aab, xR)$
$\vdash (q_0, ab, xxR)$
$\vdash (q_0, b, xxxR)$
$\vdash (q_1, \epsilon, xxR)$ Reject

(3) $(q_0, abab, R)$
$\vdash (q_0, bab, xR)$
$\vdash (q_1, ab, R)$
Reject

(P) $L = \{0^n 1^{2n} \mid n \geq 1\}$
$000111111$
For each '0' push 2x for each '1' pop 1x

$\delta$:

$\delta(q_0, 0, R) = \{(q_0, xxR)\}$
$\delta(q_0, 0, x) = \{(q_0, xxx)\}$
$\delta(q_0, 1, x) = \{(q_1, \epsilon)\}$
$\delta(q_1, 1, x) = \{(q_1, \epsilon)\}$
$\delta(q_1, \epsilon, R) = \{(q_f, R)$



$(q_0, 000111111, R)$

$\vdash (q_0, 00111111, XXR)$

$\vdash (q_0, 0111111, XXXXR)$

$\vdash (q_0, 111111, XXXXXXR)$

$\vdash (q_1, 11111, XXXXXR)$

$\vdash (q_1, 1111, XXXXR)$

$\vdash (q_2, 111, XXXR)$

$\vdash (q_1, 11, XXR)$

$\vdash (q_1, 1, XF)$

$\vdash (q_1, 1, XF)$

$\vdash (q_1, 1, XF)$

$\vdash (q_1, \epsilon, R)$

$\vdash (q_f, f, R)$ accept

(8) $L = \{ c^n j^{3n} \mid n \geq 1 \}$

Solution <u>logic</u>:-

X

$\delta$ :

$\delta(q_0, c, R) = \{(q_0, XXXR)\}$

$\delta(q_0, c, X) = \{(q_0, XXXX)\}$

$\delta(q_0, d, X) = \{(q_1, \epsilon)\}$

$\delta(q_1, d, X) = \{(q_1, \epsilon)\}$

$\delta(q_1, \epsilon, R) = \{(q_f, R)\}$

① $L = \{ (ab)^n c^n \mid n \geq 1 \}$

ababab ccc

Imp

- For each 'ab' push 1 x

For each 'c' pop 1 x

$\delta(q0, a, R) = \{(q1, R)\}$

$\delta(q1, b, R) = \{(q0, xR)\}$

$\delta(q0, a, x) = \{(q1, x)\}$

$\delta(q1, b, x) = \{(q0, xx)\}$

$\delta(q0, c, x) = \{(q2, \epsilon)\}$

$\delta(q2, c, x) = \{(q2, \epsilon)\}$

$\delta(q2, \epsilon, R) = \{(q_f, R)\}$

(q0, abababccc R)

$\vdash$ (q1, bababccc, R)

$\vdash$ (q0, ababccc, xR)

$\vdash$ (q1, babccc, xR)

$\vdash$ (q0, abccc, xxR)

$\vdash$ (q1, bccc, xxR)

$\vdash$ (q0, ccc, xxxR)

$\vdash$ (q2, cc, xxR)

$\vdash$ (q2, c, xR)

$\vdash$ (q2, $\epsilon$, R)

$\vdash$ ($q_f$, $\epsilon$, R) accept

2) $L = \{ (213)^n 4^n \mid n \geq 1 \}$

$\delta(q_0, 2, R) = \{(q_1, R)\}$
$\delta(q_1, 1, R) = \{(q_2, R)\}$
$\delta(q_2, 3, R) = \{(q_0, xR)\}$

$\delta(q_0, 2, x) = \{(q_1, x)\}$
$\delta(q_1, 1, x) = \{(q_2, x)\}$
$\delta(q_2, 3, x) = \{q_0, xx)\}$
$\delta(q_0, 4, x) = \{(q_3, \epsilon)\}$
$\delta(q_3, 4, x) = \{(q_3, \epsilon)\}$
$\delta(q_3, \epsilon, R) = \{(q_f, R)\}$

3) $L = \{ (bdac)^n e^n \mid n \geq 1 \}$

$\delta(q_0, b, R) = \{(q_1, R)\}$
$\delta(q_1, d, R) = \{(q_2, R)\}$
$\delta(q_2, a, R) = \{(q_3, R)\}$
$\delta(q_3, c, R) = \{(q_0, xR)\}$

$\delta(q_0, b, \cancel{R}x) = \{(q_1, x)\}$
$\delta(q_1, d, \cancel{R}x) = \{(q_2, x)\}$
$\delta(q_2, a, \cancel{R}x) = \{(q_3, x)\}$
$\delta(q_3, c, \cancel{R}x) = \{(q_0, xx)\}$

$\delta(q_0, e, x) = \{(q_4, \epsilon)\}$
$\delta(q_4, e, x) = \{q_4, \epsilon)\}$
$\delta(q_4, \epsilon, R) = \{(q_f, R)\}$

① $L = \{ a^n (bcd)^n \mid n \geq 1 \}$

aaabcdbcdbcd

For each 'a' push 1x

For every 'bcd' pop 1x

$\delta(q_0, a, R) = \{(q_0, xR)\}$

$\delta(q_0, a, x) = \{q_0, xx)\}$

$\delta(q_0, b, x) = \{(q_1, x)\}$

$\delta(q_1, c, x) = \{(q_2, x)\}$

$\delta(q_2, d, x) = \{(q_3, \epsilon)\}$

$\delta(q_3, b, x) = \{(q_1, x)\}$

$\delta(q_3, \epsilon, R) = \{(q_f, R)\}$

② $L = \{ 0^n (2143)^n \mid n \geq 1 \}$

$\delta(q_0, 0, R) = \{(q_0, xR)\}$

$\delta(q_0, 0, x) = \{(q_0, xx)\}$

$\delta(q_0, 2, x) = \{(q_1, x)\}$

$\delta(q_1, 1, x) = \{(q_2, x)\}$

$\delta(q_2, 4, x) = \{(q_3, x)\}$

$\delta(q_3, 3, x) = \{(q_4, \epsilon)\}$

$\delta(q_4, 2, x) = \{(q_1, x)\}$

$\delta(q_4, \epsilon, R) = \{(q_f, R)\}$

③ $L = \{ (ab)^n (cdef)^n \mid n \geq 1 \}$

$\delta(q0, a, R) = \{(q1, R)\}$
$\delta(q1, b, R) = \{q0, xR\}$

$\delta(q0, a, x) = \{(q1, x)\}$
$\delta(q1, b, x) = \{q0, xx\}$

$\delta(q0, c, x) = \{(q2, x)\}$
$\delta(q2, d, x) = \{(q3, x)\}$
$\delta(q3, e, x) = \{(q4, x)\}$
$\delta(q4, f, x) = \{(q5, \epsilon)\}$
$\delta(q5, c, x) = \{(q2, x)\}$
$\delta(q5, \epsilon, R) = \{(qf, R)\}$

☆ PDA design methods

① PDA by final state method

$(q0, w, z) \vdash^* (q, \epsilon, x) \quad q \in F \quad x \in T^*$

② PDA by Null / Empty state method

$(q0, w, z) \vdash^* (q, \epsilon, \epsilon) \quad q \in Q \quad F = \{ \}$

① $L = \{ a^n b^{n+1} \mid n \geq 1 \}$

For each 'a' push 1x
Bypass 1st b
For each 'b' pop 1x

$\delta(q_0, a, R) = \{(q_0, xR)\}$
$\delta(q_0, a, x) = \{(q_0, xx)\}$
$\delta(q_0, b, x) = \{(q_1, x)\}$
$\delta(q_1, b, x) = \{(q_1, \epsilon)\}$
$\delta(q_1, \epsilon, R) = \{(q_f, R)\}$ = FSM
$= \{(q_1, \epsilon)\}$ = NSM

② $L = \{ a^n b^m a^n \mid m, n \geq 1 \}$

aaa bbbbbbbbbaaa

For each 'a' push 1x Bypass all b's
for each 'a' pop 1x

$\delta(q_0, a, R) = \{(q_0, xR)\}$
$\delta(q_0, a, x) = \{(q_0, xx)\}$
$\delta(q_0, b, x) = \{(q_1, x)\}$
$\delta(q_1, b, x) = \{(q_1, x)\}$
$\delta(q_1, a, x) = \{(q_2, \epsilon)\}$
$\delta(q_2, a, x) = \{(q_2, \epsilon)\}$
$\delta(q_2, \epsilon, R) = \{(q_2, \epsilon)\}$

③ $L = \{ a^n b^m c^m d^n \mid m, n \geq 1 \}$

logic :- $1a \rightarrow$ push 1x, $1d \rightarrow$ pop 1x | $1b \rightarrow$ push 1y, $1c \rightarrow$ pop 1y

**Solution**

$\delta(q0, a, R) = \{q0, xR\}$
$\delta(q0, a, x) = \{q0, xx\}$

$\delta(q0, b, x) = \{q1, yx\}$
$\delta(q1, b, Y) = \{q1, yy\}$

$\delta(q1, c, y)\} = \{q2, \epsilon\}$
$\delta(q2, c, y)\} = \{(q2, \epsilon)\}$

$\delta(q2, d, x)\} = (q3, \epsilon)$
$\delta(q3, d, x) = (q3, \epsilon)$
$\delta(q3, \epsilon, x) = \{(q3, \epsilon)\}$

④ $L = \{ x \mid na(x) = nb(x) \}$

**Logic**

ab — stack: x / R

ba — stack: x / R

abba — stack: x / R

CONCLUSION

a $\xrightarrow{push}$ x/R
a $\xrightarrow{pop}$ Y/R
b $\xrightarrow{pop}$ x/R
b $\xrightarrow{push}$ Y/R

$\delta(q_0, a, R) = \{(q_0, xR)\}$
$\delta(q_0, b, R) = \{(q_0, YR)\}$
$\delta(q_0, a, x) = \{(q_0, xx)\}$
$\delta(q_0, b, y) = \{(q_0, yy)\}$
$\delta(q_0, a, y) = \{(q_0, \epsilon)\}$
$\delta(q_0, b, x) = \{(q_0, \epsilon)\}$
$\delta(q_0, \epsilon, R) = \{(q_0, \epsilon)\}$

5) Design PDA ~~to~~ check for well formedness of parenthesis

( ( ) ( ( ) ) )

For each '(' push 1 x
for each ')' pop 1 x

$\delta$:-

$\delta(q_0, (, R) = \{(q_0, xR)\}$
$\delta(q_0, (, x) = \{(q_0, xx)\}$
$\delta(q_0, ), x) = \{(q_0, \epsilon)\}$
$\delta(q_0, \epsilon, R) = \{(q_0, \epsilon)\}$

6) $L = \{ wcw^R \mid w \in \{a,b\}^*, \; w^R - \text{Reverse of } w \}$ odd palin

abaabcbaaba

δ:
1 $\delta(q_0, a, R) = \{(q_0, XR)\}$
2 $\delta(q_0, b, R) = \{q_0, YR)\}$
3 $\delta(q_0, a, X) = \{q_0, XX)\}$
5 $\delta(q_0, b, X) = \{q_0, YX)\}$
6 $\delta(q_0, a, Y) = \{(q_0, XY)\}$
4 $\delta(q_0, b, Y) = \{(q_0, YY)\}$

$\delta(q_0, c, R) = \{(q_1, R)\}$
$\delta(q_0, c, X) = \{(q_1, X)\}$
$\delta(q_0, c, Y) = \{(q_1, Y)\}$
$\delta(q_1, a, X) = \{(q_1, \epsilon)\}$
$\delta(q_1, b, Y) = \{(q_1, \epsilon)\}$
$\delta(q_1, \epsilon, R) = \{(q_1, \epsilon)\}$

② $L = \left\{ ww^R \;/\; \begin{array}{l} w \in (a,b)^* \\ w^R \text{ reverse of } w \end{array} \right\}$

This is a case of NPDA design where NPDA stands for non deterministic PDA

whenever there is a double letter two cases are possible

1) The middle of the string is not reached & hence we continue to push the respective symbols on to the stack

2) middle of the string is reached & hence we start to pop the respective symbols from the stack.

$$\delta(q_0, \epsilon, R) = \{(q_0, \epsilon)\}$$
$$\delta(q_0, a, R) = \{(q_0, XR)\}$$
$$\delta(q_0, b, R) = \{(q_0, YR)\}$$
$$\delta(q_0, a, X) = \{(q_0, XX), (q_1, \epsilon)\}$$
$$\delta(q_0, b, X) = \{(q_0, YX)\}$$
$$\delta(q_0, a, Y) = \{(q_0, XY)\}$$
$$\delta(q_0, b, Y) = \{(q_0, YY), (q_1, \epsilon)\}$$
$$\delta(q_1, a, X) = \{(q_1, \epsilon)\}$$
$$\delta(q_2, b, Y) = \{(q_1, \epsilon)\}$$
$$\delta(q_1, \epsilon, R) = \{(q_1, \epsilon)\}$$

$(q_0, abba, R)$
$\vdash (q_0, bba, XR)$
$\vdash (q_0, ba, YXR)$

$\vdash (q_0, a, YYXR)$
$\vdash (q_0, \epsilon, XYYXR)$

$\vdash (q_1, a, XR)$
$\vdash (q_1, \epsilon, R)$
$\vdash (q_1, \epsilon, \epsilon)$
<u>accept</u>

CFG to PDA

Given CFG $G = (V, T, P, S)$

Define PDA $m = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$ for recognizing CFL generated by CFG $G$

S1 Express CFG in GNF $(A \to a\alpha)$

S2 PDA $m = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$

1. $Q = \{q_0\}$
2. $\Sigma = T'$
3. $T = V'$
4. $q_0 = q_0$
5. $z_0 = S$
6. $F = \{\ \}$
7. $\delta$ :-

* $A \to aBCD$

$\delta(q_0, a, A) = \{(q_0, BCD)\}$

* $A \to a$

$\delta(q_0, a, A) = \{(q_0, \epsilon)\}$

$A \to aY$

① Design PDA for the following
$S \to aSa \mid bSb \mid c$

Solution

Null Store method

S1 $S \to aSC_1 \mid bSC_2 \mid c$
$C_1 \to a$
$C_2 \to b$

S.R selective replacement

S2 PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$

① $Q = \{q_0\}$
② $\Sigma = \{a, b, c\}$
③ $\Gamma = \{S, C_1, C_2\}$
④ $q_0 = q_0$
⑤ $z_0 = S$
⑥ $F = \{\}$

i/p ST

⑦ $S \to aSC_1$ $\delta(q_0, a, S) = \{(q_0, SC_1)\}$
$S \to bSC_2$ $\delta(q_0, b, S) = \{(q_0, SC_2)\}$
$S \to c$ $\delta(q_0, c, S) = \{(q_0, \epsilon)\}$
$C_1 \to a$ $\delta(q_0, a, C_1) = \{(q_0, \epsilon)\}$
$C_2 \to b$ $\delta(q_0, b, C_2) = \{(q_0, \epsilon)\}$

$(q_0, abcba, S)$
$\vdash (q_0, bcba, SC_1)$
$\vdash (q_0, cba, SC_2C_1)$
$\vdash (q_0, ba, C_2C_1)$
$\vdash (q_0, a, C_1)$
$\vdash (q_0, \epsilon, \epsilon)$ accept

June 2011/10marks

Q Design PDA for
$S \to 0BB$
$B \to 0S | 1S | 0$
and test whether $010^4$ belongs to the language

Solution S1 :- (done)

S2 :- PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$

① $Q = \{q_0\}$
② $\Sigma = \{0, 1\}$
③ $\Gamma = \{S, B\}$
④ $q_0 = q_0$
⑤ $z_0 = S$
⑥ $F = \{\}$
⑦ $\delta$ :-

$S \to 0BB$ $\quad \delta(q_0, 0, S) = \{(q_0, BB)\}$
$B \to 0S$ $\quad \delta(q_0, 0, B) = \{(q_0, S)\}$
$B \to 1S$ $\quad \delta(q_0, 1, B) = \{(q_0, S)\}$
$B \to 0$ $\quad \delta(q_0, 0, B) = \{(q_0, \epsilon)\}$

$(q_0, 010000, S)$
$\vdash (q_0, 10000, BB)$
$\vdash (q_0, 0000, SB)$
$\vdash (q_0, 000, BBB)$
$\vdash (q_0, 00, BB)$ $\quad \vdash (q_0, 00, SBB)$
$\vdash (q_0, 0, B)$ $\quad \vdash (q_0, 0, BBBB)$
$\vdash (q_0, \epsilon, \epsilon)$ $\quad$ ~~$\vdash (q_0, 0,$~~