DSE CSC/ITC 301 Engineering Mathematics - III (COMP, IT)
Solutions: DSE End Semester Exam, Jan 2023

Q. 1(a) :Find Laplace of $e^{2t} + 4t^3 - \sin 2t \cos 3t$
Solution: Let $f(t) = e^{2t} + 4t^3 - \sin 2t \cos 3t$

$$\begin{aligned} Then\ L\{f(t)\} &= \frac{1}{s-2} + 4\frac{\Gamma(4)}{s^4} - L\{\frac{1}{2}[\sin 5t - \sin t]\} \\ &= \frac{1}{s-2} + 4\frac{3!}{s^4} - \frac{1}{2}[\frac{5}{s^2+25} - \frac{1}{s^2+1}] \\ &= \frac{1}{s-2} + \frac{24}{s^4} - \frac{1}{2}[\frac{5}{s^2+25} - \frac{1}{s^2+1}] \end{aligned}$$

Q. 1(b) : Find the Fourier series of $f(x) = x, \qquad -\pi < x < \pi$
Solution: Since, $f(x) = x$ is an odd function and $l = \pi$
$\therefore a_0 = a_n = 0$
Let $f(x) = x = \sum_{n=1}^{\infty} b_n \sin nx$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx dx \\ b_n &= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[\pi\left(\frac{-\cos n\pi}{n}\right) - 0\right] \\ b_n &= \frac{2(-1)^{n+1}}{n} \end{aligned}$$

$\therefore f(x) = x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$

Q. 1(c) : Calculate the Spearman's rank correlation coefficient for the following data:

X	12	17	22	27	32
Y	113	119	117	115	121

Solution: We have the Spearman's rank correlation coefficient to be :
(Since values (and hence the ranks) are not repeated)
$\rho_{xy} = 1 - 6\frac{\sum d^2}{n(n^2-1)}$
Here $n = 5$ and We get the following table of ranks:

X	Y	Rank in X r_x	Rank in Y r_y	$d = r_x - r_y$	d^2
12	113	5	5	0	0
17	119	4	2	2	4
22	117	3	3	0	0
27	115	2	4	-2	4
32	121	1	1	0	0
					$\sum d^2 = 8$

Therefore the Spearman's rank correlation coefficient is

$$\begin{aligned}\rho(= R) &= 1 - 6\left(\frac{\sum d^2}{n(n^2-1)}\right)\\ &= 1 - 6\left(\frac{8}{5\times 24}\right)\\ \Rightarrow \rho &= 0.6\end{aligned}$$

Q. 1(d) : Find the constants a, b, c, d, e such that the following function is analytic: $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$

Solution:

Let $f(z) = u + iv = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$

$\therefore u = ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2$ and $v = 6x^3y + exy^3 - 4xy$

$\Rightarrow u_x = 4ax^3 + 2bxy^2 + 2dx$ and $u_y = 2bx^2y + 4cy^3 - 4y$

And $v_x = 12x^2y - ey^3 + 4y$ and $v_y = 4x^3 - 3exy^2 + 4x$

$f(z)$ is analytic $\Rightarrow u_x = v_y$ and $u_y = -v_x$

$\therefore$

$$\begin{aligned}4ax^3 + 2bxy^2 + 2dx &= 4x^3 - 3exy^2 + 4x\\ \Rightarrow 4a = 4\ 2b = -3e; and &\quad 2d = 4\\ \Rightarrow a = 1, d = 2 &\quad e = -2b/3\end{aligned}$$

Similarly

$$\begin{aligned}u_y &= -v_x\\ \Rightarrow 2bx^2y + 4cy^3 - 4y &= -(12x^2y - ey^3 + 4y)\\ \Rightarrow 2b = -12, &\quad 4c = e \Rightarrow c = e/4\\ \Rightarrow b = -6\ \ \Rightarrow e = -(2\cdot(-6))/3 = 4 &\quad and\ c = 1\end{aligned}$$

$\therefore a = 1,\ b = -6\ \ c = 1\ \ d = 2$ and $e = 4$

Q. 2(a) : Determine whether the function $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$ is analytic and if so, find its derivative.

Solution: We have $u = \frac{1}{2}\log(x^2+y^2)$ and $v = \tan^{-1}\left(\frac{y}{x}\right)$

$u_x = \frac{\partial u}{\partial x} = \frac{1}{2}\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$, $u_y = \frac{\partial u}{\partial y} = \frac{1}{2}\frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2}$

$v_x = \frac{\partial v}{\partial x} = \frac{1}{1+(y/x)^2}\frac{-y}{x^2} = \frac{-y}{x^2+y^2}$, $v_y = \frac{\partial v}{\partial y} = \frac{1}{1+(y/x)^2}\frac{1}{x} = \frac{x}{x^2+y^2}$

$\therefore u_x = v_y$ and $u_y = -v_x$

Also all partial derivatives are continuous.

Therefore $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$ is analytic.

$f'(z) = u_x + iv_x = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}$.

Q. 2(b) : A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6
P(X=x)	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find (i) k, (ii) $P(X<4)$, (iii) $P(3<X\le 6)$

Solution: We have,

$$\begin{aligned} \sum p_x &= 1 \\ \Rightarrow k+3k+5k+7k+9k+11k+13k &= 1 \\ i.e\ 49k &= 1 \\ \Rightarrow k &= \frac{1}{49} \end{aligned}$$

Hence the probability distribution of X is :

X	0	1	2	3	4	5	6
P(X=x)	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$

Therefore

$$\begin{aligned} P(X<4) &= P(X=0,1,2,3) \\ &= \frac{1}{49}+\frac{3}{49}+\frac{5}{49}+\frac{7}{49} \\ i.e\ P(X<4) &= \frac{16}{49} \end{aligned}$$

And

$$\begin{aligned} P(3<X\le 6) &= P(X=4,5,6) \\ &= \frac{9}{49}+\frac{11}{49}+\frac{13}{49} \\ i.e\ P(3<X\le 6) &= \frac{33}{49} \end{aligned}$$

Q. 2(c) : Evaluate $\int_0^\infty e^{-2t}\ t\cos t\ dt$
Solution: We have, by the definition of Laplace Transforms
$\int_0^\infty e^{-2t}\ t\cos t\ dt = L\{t\cos t\}_{s=2}$
Now

$$\begin{aligned}
L\{t\cos t\} &= -\frac{d}{ds}(\frac{s}{s^2+1}) \\
&= -(\frac{1(s^2+1)-s(2s)}{(s^2+1)^2}) \\
\Rightarrow L\{t\cos t\} &= \frac{s^2-1}{(s^2+1)^2} \\
\Rightarrow \int_0^\infty e^{-2t}\ t\cos t\ dt &= \frac{2^2-1}{(2^2+1)^2} \\
\Rightarrow \int_0^\infty e^{-2t}\ t\cos t\ dt &= \frac{3}{25}
\end{aligned}$$

Q. 3(a) : Find the Fourier series of $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}, \qquad -\pi < x < \pi$
Solution: Here $f(x)$ is an even function because
$f(-x) = \frac{\pi^2}{12} - \frac{(-x)^2}{4} = \frac{\pi^2}{12} - \frac{x^2}{4} = f(x)$
$\therefore b_n = 0$
Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$
$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx = \frac{2}{\pi}\int_0^\pi \left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)dx = \frac{2}{\pi}\left[\frac{\pi^2}{12}x - \frac{x^3}{12}\right]_0^\pi = 0$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx = \frac{2}{\pi}\int_0^\pi \left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\cos nx dx \\
a_n &= \frac{2}{\pi}\left[\frac{\pi^2}{12}\frac{\sin nx}{n} - x^2\left(\frac{\sin nx}{n}\right) + (2x)\left(\frac{-\cos nx}{n^2}\right) - (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi \\
a_n &= \frac{2}{\pi}\left[0 - 0 + 2\pi\left(\frac{-\cos n\pi}{n^2}\right) - 0\right] \\
a_n &= \frac{4(-1)^{n+1}}{n^2}
\end{aligned}$$

$\therefore f(x) = \frac{\pi^2}{12} - \frac{x^2}{4} = 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$

Q. 3(b) :A continuous random variable X has the following probability density function, $f(x) = k(x - x^2), 0 < x < 1$
Find k, mean and the variance.

Solution: Since $f(x)$ is a pdf, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\ dx &= 1 \\
\Rightarrow \int_0^1 k(x - x^2)\ dx &= 1 \\
\Rightarrow k(\frac{x^2}{2} - \frac{x^3}{3})|_0^1 &= 1 \\
\Rightarrow k(\frac{1}{2} - \frac{1}{3}) &= 1 \\
\Rightarrow k(\frac{1}{6} &= 1 \\
\Rightarrow k &= 6 \\
\Rightarrow\ pdf,\ f(x) &= 6(x - x^2)\ 0 < x < 1
\end{aligned}$$

Now

$$\begin{aligned}
Mean\ = E(X) &= \int_{-\infty}^{\infty} xf(x)\ dx \\
\Rightarrow E(X) &= \int_0^1 6(x^2 - x^3)\ dx \\
&= 6(\frac{x^3}{3} - \frac{x^4}{4})\ |_0^1 \\
&= 6(\frac{1}{3} - \frac{1}{4}) \\
\Rightarrow E(X) &= \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\ dx \\
\Rightarrow E(X^2) &= \int_0^1 6(x^3 - x^4)\ dx \\
&= 6(\frac{x^4}{4} - \frac{x^5}{5})\ |_0^1 \\
&= 6(\frac{1}{4} - \frac{1}{5}) \\
\Rightarrow E(X^2) &= \frac{3}{10}
\end{aligned}$$

Hence

$$\begin{aligned}
Var(X) &= E(X^2) - (E(X))^2 \\
&= \frac{3}{10} - \frac{1}{4} \\
\Rightarrow\ Var(X) &= \frac{1}{20}
\end{aligned}$$

Q. 3(c) : Find the inverse Laplace transform of $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$

Solution:

$$\begin{aligned}
L^{-1}\left(\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right) &= L^{-1}\left(\frac{s^2+2s+1+2}{(s^2+2s+1+4)(s^2+2s+1+1)}\right)\\
&= L^{-1}\left(\frac{(s+1)^2+2}{((s+1)^2+4)((s+1)^2+1)}\right)\\
&= e^{-t}L^{-1}\left(\frac{s^2+2}{(s^2+4)(s^2+1)}\right)\\
&= e^{-t}L^{-1}\left(\frac{s^2}{(s^2+4)(s^2+1)}\right)+e^{-t}L^{-1}\left(\frac{2}{(s^2+4)(s^2+1)}\right)
\end{aligned}$$

Consider $\dfrac{s^2}{(s^2+4)(s^2+1)} = \dfrac{As+B}{s^2+4}+\dfrac{Cs+D}{s^2+1}$

After solving we will get $A=0, \quad B=\dfrac{4}{3}, \quad C=0, \quad D=-\dfrac{1}{3}$

$\therefore \dfrac{s^2}{(s^2+4)(s^2+1)} = \dfrac{4}{3(s^2+4)}-\dfrac{1}{3(s^2+1)}$

$\therefore L^{-1}\left(\dfrac{s^2}{(s^2+4)(s^2+1)}\right) = \dfrac{4}{3}L^{-1}\left(\dfrac{1}{s^2+4}\right)-\dfrac{1}{3}L^{-1}\left(\dfrac{1}{s^2+1}\right) = \dfrac{2}{3}\sin 2t-\dfrac{1}{3}\sin t$

Now $\dfrac{2}{(s^2+4)(s^2+1)} = \dfrac{As+B}{s^2+4}+\dfrac{Cs+D}{s^2+1}$

After solving we will get $A=0, \quad B=-\dfrac{2}{3}, \quad C=0, \quad D=\dfrac{2}{3}$

$\therefore \dfrac{2}{(s^2+4)(s^2+1)} = -\dfrac{1}{3(s^2+4)}+\dfrac{1}{3(s^2+1)}$

$\therefore L^{-1}\left(\dfrac{2}{(s^2+4)(s^2+1)}\right) = -\dfrac{1}{3}L^{-1}\left(\dfrac{1}{s^2+4}\right)+\dfrac{1}{3}L^{-1}\left(\dfrac{1}{s^2+1}\right) = -\dfrac{1}{6}\sin 2t+\dfrac{1}{3}\sin t$

$L^{-1}\left(\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right) = e^{-t}\left(\dfrac{2}{3}\sin 2t-\dfrac{1}{3}\sin t+-\dfrac{1}{6}\sin 2t+\dfrac{1}{3}\sin t\right)$

$L^{-1}\left(\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right) = \dfrac{e^{-t}}{2}\sin 2t$

Q. 4(a) :Find Laplace Transform of $f(t)$ where $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$

Solution:
We have

$$\begin{aligned}
L[f(t)] &= \int_0^\infty e^{-st}\ f(t)\ dt \\
\Rightarrow L[f(t)] &= \int_0^\pi e^{-st}\ \cos t\ dt + \int_\pi^\infty e^{-st}\ \sin t\ dt \\
&= R.P \int_0^\pi e^{-st}\ e^{it}\ dt + I.P \int_\pi^\infty e^{-st}\ e^{it}\ dt \\
&= R.P \int_0^\pi e^{-t(s-i)}\ dt + I.P \int_\pi^\infty e^{-t(s-i)dt} \\
&= R.Pe^{-t(s-i)}(\frac{-1}{s-i}\ |_0^\pi + I.Pe^{-t(s-i)(\frac{-1}{s-i})}\ |_\pi^\infty \\
&= R.P(e^{(s-i)(-\pi)} - 1)(\frac{-1}{s-i}) + I.P(0 - e^{-\pi(s-i)}(\frac{-1}{s-i}) \\
&= R.P(e^{-s\pi}e^{i\pi} - 1)(\frac{-1(s+i)}{s^2+1}) + I.P(-e^{-s\pi}e^{i\pi})(\frac{-1(s+i)}{s^2+1}) \\
&= R.P(e^{-s\pi}(\cos\pi + i\sin\pi) - 1)(\frac{-s-i}{s^2+1}) + I.P(-e^{-s\pi}(\cos\pi + i\sin\pi)(\frac{-s-i}{s^2+1}) \\
&= R.P(e^{-s\pi}(-1) - 1)(\frac{-s-i}{s^2+1}) + I.P(-e^{-s\pi}(-1)(\frac{-s-i}{s^2+1}) \\
&= \frac{s(e^{-s\pi}+1)}{s^2+1} + \frac{e^{-s\pi}}{s^2+1}
\end{aligned}$$

Q. 4(b) : Find the Karl Pearson's Correlation Coefficient for the following data:

X	65	66	67	67	68	69	70	71
Y	67	68	65	68	72	72	69	71

Solution: We have

$$r = \frac{\sum \frac{xy}{n} - \sum \frac{x}{n} \sum \frac{y}{n}}{\sqrt{\sum \frac{x^2}{n} - (\sum \frac{x}{n})^2}\sqrt{\sum \frac{y^2}{n} - (\sum \frac{y}{n})^2}} \quad \text{i.e } r_{xy} = \frac{n\sum xy - \sum x \sum y}{\sqrt{n\sum x^2 - (\sum x)^2}\sqrt{n\sum y^2 - (\sum y)^2}}$$

For this data:
$\sum x = 544;\ \sum x^2 = 37028;\ \sum y = 552;\ \sum y^2 = 38132;\ \sum xy =;\ n = 8$

Substituting these values in the formula, we get
the Karl Pearson's Coefficient of Correlation $r = 0.6030$

Q. 4(c) :Find the Fourier series of $f(x) = \begin{cases} x, & 0 < x \le \pi \\ 2\pi - x, & \pi \le x < 2\pi \end{cases}$

Solution: The interval is $[0, 2\pi] \implies l = \frac{2\pi}{2} = \pi$
therefore the Fourier series is:
$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

That is, $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\left[\int_0^{\pi} x\ dx + \int_{\pi}^{2\pi}(2\pi - x)\ dx\right] \\
&= \frac{1}{\pi}\left[\frac{x^2}{2}|_0^{\pi} + (2\pi x - \frac{x^2}{2})\right]_{\pi}^{2\pi} \\
&= \frac{1}{\pi}\left[\frac{\pi^2}{2} - 0 + (4\pi^2 - \frac{4\pi^2}{2}) - (2\pi^2 - \frac{\pi^2}{2})\right] \\
\Rightarrow a_0 &= 0
\end{aligned}$$

Now

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\ dx = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\ dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\ dx\right] \\
&= \frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n} - (1)\frac{-\cos nx}{n^2}\right]_0^{\pi} + \left[(2\pi - x)\frac{\sin nx}{n} - (-1)\frac{-\cos nx}{n^2}\right]_{\pi}^{2\pi}\right\} \\
&= \frac{1}{\pi}\left\{\left[0 + (1)\frac{1}{n^2}((-1)^n - 1)\right] + \left[0 - \frac{1}{n^2}(1 - (-1)^n)\right]\right\} \\
\Rightarrow a_n &= \frac{1}{\pi}(\frac{2}{n^2}((-1)^n - 1)) \\
\Rightarrow a_n &= \frac{2((-1)^n - 1)}{\pi n^2}
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\ dx = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\ dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\ dx\right] \\
&= \frac{1}{\pi}\left\{\left[x\frac{-\cos nx}{n} - (1)\frac{-\sin nx}{n^2}\right]_0^{\pi} + \left[(2\pi - x)\frac{-\cos nx}{n} - (-1)\frac{-\sin nx}{n^2}\right]_{\pi}^{2\pi}\right\} \\
&= \frac{1}{\pi}\left\{\left[-\frac{1}{n}(\pi(-1)^n - 0) + (1)(0)\right] + \left[0 - \frac{1}{n}(0 - \pi(-1)^n) - \frac{1}{n^2}(0)\right]\right\} \\
\Rightarrow b_n &= 0
\end{aligned}$$

Therefore the Fourier series of the given function is:
$f(x) = \sum\limits_{n=1}^{\infty} \dfrac{2((-1)^n - 1)}{\pi n^2} \cos nx$

Q. 5(a) : Find the inverse Laplace Transform of $\dfrac{s}{(2s+1)^2}$

Solution: $L^{-1}\left\{\dfrac{s}{(2s+1)^2}\right\} = \dfrac{1}{4}L^{-1}\left\{\dfrac{s}{(s+1/2)^2}\right\} = \dfrac{1}{4}L^{-1}\left\{\dfrac{s+1/2-1/2}{(s+1/2)^2}\right\}$

$L^{-1}\left\{\dfrac{s}{(2s+1)^2}\right\} = \dfrac{1}{4}e^{-t/2}L^{-1}\left\{\dfrac{s-1/2}{s^2}\right\} = \dfrac{1}{4}e^{-t/2}\left(L^{-1}\left\{\dfrac{1}{s}\right\} - L^{-1}\left\{\dfrac{1/2}{s^2}\right\}\right)$

$L^{-1}\left\{\dfrac{s}{(2s+1)^2}\right\} = \dfrac{1}{4}e^{-t/2}\left(1 - \dfrac{t}{2}\right)$

Q. 5(b) : Find the Laplace Transform of $t(\frac{\sin t}{e^t})^2$

Solution: We have

$$\begin{aligned} L\{t(\frac{\sin t}{e^t})^2\} &= L\{te^{-2t}\sin^2 t\} \\ &= L\{t\sin^2 t\}|_{s\to s+2} \quad \cdots\cdot (1) \\ Now,\ L\{t\sin^2 t\} &= -\frac{d}{ds}L\{\sin^t\} \\ &= -\frac{d}{ds}L\{\frac{1-\cos 2t}{2}\} \\ &= -\frac{d}{ds}\frac{1}{2}(\frac{1}{s}-\frac{s}{s^2+4}) \\ &= \frac{-1}{2}(\frac{-1}{s^2}-\frac{s^2+4-s(2s)}{(s^2+4)^2}) \\ &= \frac{1}{2}(\frac{1}{s^2}+\frac{4-s^2}{(s^2+4)^2}) \\ \Rightarrow L\{t(\frac{\sin t}{e^t})^2\} &= \frac{1}{2}(\frac{1}{(s+2)^2}+\frac{4-(s+2)^2}{((s+2)^2+4)^2}) \end{aligned}$$

Q. 5(c) : Find the lines of regression for the following data:

X	78	36	98	25	75	82	90	62	65	39
Y	84	51	91	60	68	62	86	58	53	47

Solution:

We have, For this data:

$\sum x = 650;\ \sum x^2 = 47648;\ \sum y = 660;\ \sum y^2 = 45784;\ \sum xy = 45604;\ n = 10$

The lines of regression are:

$$\begin{aligned} (x-\bar{x}) &= \frac{r\sigma_x}{\sigma_y}(y-\bar{y}) \\ and\ (y-\bar{y}) &= \frac{r\sigma_y}{\sigma_x}(x-\bar{x}) \end{aligned}$$

We have $r = 0.7804$

and the regression lines are:

$$\begin{aligned} x-65 &= 0.7804(y-66) \\ \Rightarrow x &= 33.44+0.5009 \end{aligned}$$

Q. 6(a) : Find the mean and variance for the following distribution:

X	1	3	4	5
P(X=x)	0.4	0.1	0.2	0.3

Solution: The mean of X is

$$\begin{aligned} E(X) &= \sum_x x p_x \\ &= 1(0.4) + 3(0.1) + 4(0.2) + 5(0.3) \\ &= 0.4 + 0.3 + 0.8 + 1.5 \\ i.e\ E(X) &= 3 \end{aligned}$$

and

$$\begin{aligned} E(X^2) &= \sum_x x^2 p_x \\ &= 1(0.4) + 9(0.1) + 16(0.2) + 25(0.3) \\ &= 0.4 + 0.9 + 3.2 + 7.5 \\ i.e\ E(X^2) &= 12 \end{aligned}$$

Therefore the variance

$$\begin{aligned} Var(x) &= E(X^2) - \{E(X)\}^2 \\ &= 12 - 3^2 \\ \Rightarrow Var(X) &= 3 \end{aligned}$$

Q. 6(b) : Find the inverse Laplace Transform of $\log\left(1 + \dfrac{a^2}{s^2}\right)$.

Solution: $L^{-1}\left\{\log\left(1+\dfrac{a^2}{s^2}\right)\right\} = -\dfrac{1}{t}L^{-1}\left\{\dfrac{d}{ds}\log\left(1+\dfrac{a^2}{s^2}\right)\right\} = -\dfrac{1}{t}L^{-1}\left\{\dfrac{d}{ds}\log\left(\dfrac{s^2+a^2}{s^2}\right)\right\}$

$L^{-1}\left\{\log\left(1+\dfrac{a^2}{s^2}\right)\right\} = -\dfrac{1}{t}L^{-1}\left\{\dfrac{d}{ds}\left(\log(s^2+a^2) - \log s^2\right)\right\} = -\dfrac{1}{t}L^{-1}\left\{\left(\dfrac{2s}{s^2+a^2} - \dfrac{2}{s}\right)\right\}$

$L^{-1}\left\{\log\left(1+\dfrac{a^2}{s^2}\right)\right\} = -\dfrac{1}{t}\left(2\cos at - 2\right) = \dfrac{2}{t}\left(1 - \cos at\right)$

Q. 6(c) : Find the analytic function $f(z) = u+iv$ whose imaginary part is $v = x^2-y^2+\dfrac{x}{x^2+y^2}$.

Solution: $f(z) = u + iv$ where $v = x^2 - y^2 + \dfrac{x}{x^2+y^2}$

Step 1: Differentiate v partially with respect to x & y, we get

$v_x = 2x + \dfrac{x^2+y^2-2x^2}{(x^2+y^2)^2} = 2x + \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$v_y = -2y + \dfrac{-x}{(x^2+y^2)^2}(2y) = -2y - \dfrac{2xy}{(x^2+y^2)^2}$

Step 2: We have $v_x(z,0) = 2z - \dfrac{1}{z^2}$ and $v_y(z,0) = 0$

Step 3:We have $f(z) = u + iv$

$\implies f'(z) = u_x + iv_x = v_y + iv_x$
($\because$ C-R equations $u_x = v_y$)
By Milne-Thompson method
$f'(z) = v_y(z,0) + iv_x(z,0) = 0 + i\left(2z - \dfrac{1}{z^2}\right)$
$\Rightarrow f'(z) = i\left(2z - \dfrac{1}{z^2}\right)$
Step 4:Integrating w.r.t z, we get
$f(z) = \int i\left(2z - \dfrac{1}{z^2}\right)\, dz = i\left(z^2 + \dfrac{1}{z}\right) + c$
$f(z) = i\left((x+iy)^2 + \dfrac{1}{x+iy}\right) + c = i\left(x^2 + 2ixy - y^2 + \dfrac{x-iy}{x^2+y^2}\right) + c$
$\therefore f(z) = \left(-2xy + \dfrac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \dfrac{x}{x^2+y^2}\right) + c$, is the required analytic function