Engineering Maths IV

May-June 2023

(COITAI)

Time (3 hours) **Max Marks: 80**

Note: (1) Question No. 1 is Compulsory

(2) Answer any three questions from Q.2 to Q.6

(3) Figures to the right indicate full marks

1. (a) If $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ then find the Eigen values of $4A^{-1} + A^3 + I$ (5)

Solution:

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, |A| = 45$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [-2+1+0]\lambda^2 + \left[\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}\right]\lambda - 45 = 0$

$\lambda^3 + 1\lambda^2 - 21\lambda - 45 = 0$

$\lambda = -3, -3, 5$

The eigen values of A is $-3, -3, 5$

The eigen values of A^{-1} is $\frac{1}{-3}, \frac{1}{-3}, \frac{1}{5}$

The eigen values of $4A^{-1}$ is $4\left(\frac{1}{-3}\right), 4\left(\frac{1}{-3}\right), 4\left(\frac{1}{5}\right)$ i.e. $-\frac{4}{3}, -\frac{4}{3}, \frac{4}{5}$

The eigen values of A^3 is $(-3)^3, (-3)^3, 5^3$ i.e. $-27, -27, 125$

The eigen values of I is $1, 1, 1$

Thus, the eigen values of $4A^{-1} + A^3 + I$ is

$-\frac{4}{3} - 27 + 1, -\frac{4}{3} - 27 + 1, \frac{4}{5} + 125 + 1$

i.e. $-\frac{82}{3}, -\frac{82}{3}, \frac{634}{5}$

1. (b) Evaluate $\int_c |z| dz$, where C is the left half of unit circle $|z| = 1$ from $z = -i$ to $z = i$ (5)

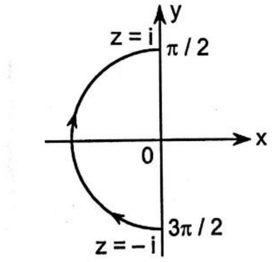
Solution:

Let $I = \int |z| dz$

Put, $z = r\,e^{i\theta}$

$dz = r\,e^{i\theta}\,i\,d\theta$

Here, $|z| = r = 1$



For left half of circle from $z = -i$ *to* $z = i$, θ varies from $\frac{3\pi}{2}$ to $\frac{\pi}{2}$

$$I = \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} 1.1.\,e^{i\theta} i\,d\theta$$

$$I = i\left[\frac{e^{i\theta}}{i}\right]_{\frac{3\pi}{2}}^{\frac{\pi}{2}}$$

$$I = e^{i\frac{\pi}{2}} - e^{i\frac{3\pi}{2}}$$

$$I = \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] - \left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right]$$

$$I = [0 + i] - [0 - i]$$

$$\boxed{I = 2i}$$

1. (c) Maximise $z = x_1 + 3x_2 + 3x_3$
subject to $x_1 + 2x_2 + 3x_3 = 4$
$2x_1 + 3x_2 + 5x_3 = 7$
$x_1, x_2, x_3 \geq 0$

Find all the basic solutions to the above problem. Which of them are basic feasible, non-degenerate, infeasible basic and optimal solution? (5)

Solution:

No	Non-basic var = 0	Basic var	Equations & solutions	Is the solution feasible?	Is the solution degenerate?	Value of z	Is the solution optimal?
1	$x_3 = 0$	x_1, x_2	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $x_1 = 2, x_2 = 1$	Yes	No	5	Yes
2	$x_2 = 0$	x_1, x_3	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $x_1 = 1, x_3 = 1$	Yes	No	4	No
3	$x_1 = 0$	x_2, x_3	$2x_2 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$ $x_2 = -1, x_3 = 2$	No	No	3	No

1. (d) Tests made on breaking strength of 10 pieces of a metal wire gave the following results: 578, 572, 570, 568, 572, 570, 570, 572, 596, and 584 in kgs. Test if the breaking strength of the metal wire can be assumed to be 577 kg. (5)

Solution:

x	$d=(x-A)$	d^2
578	3	9
572	-3	9
570	-5	25
568	-7	49
572	-3	9
570	-5	25
570	-5	25
572	-3	9
596	21	441
584	9	81
Total = 5752	Total = 2	Total = 682

$n=10$

$\overline{x}=\frac{\sum x}{n}=\frac{5752}{10}=575.2\ (A=575)$

$\overline{d}=\frac{\sum d}{n}=\frac{2}{10}=0.2$

$\sigma=\sqrt{\frac{\sum d^2}{n}-\left(\overline{d}\right)^2}=\sqrt{\frac{682}{10}-(0.2)^2}=8.256$

(i) Null Hypothesis: $\mu=577$

Alternative Hypothesis: $\mu\neq 577$

(ii) Test statistic:

$$t=\left|\frac{\overline{x}-\mu}{\frac{\sigma}{\sqrt{n-1}}}\right|=\left|\frac{575.2-577}{\frac{8.256}{\sqrt{10-1}}}\right|=0.654$$

(iii) L.O.S.: $\alpha=0.05$

(iv) Degree of freedom: $\emptyset=n-1=10-1=9$

(v) Critical value: $t_\alpha=2.262$

(vi) Decision: Since, the calculated value of t is less than the critical value, null hypothesis is accepted. Thus, the mean breaking strength of the metal wire can be assumed as 577 kg.

2. (a) Using Cauchy's residue theorem, evaluate $\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2}\,dz$ where C is $|z|=1$ (6)

Solution:

$I=\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2}\,dz$

For singularity or pole,

Put $z^4+5z^3+6z^2=0$

$z^2(z^2+5z+6)=0$

$z^2=0, z^2+5z+6=0$

$z=0,0, z=-3, z=-2$

C is $|z|=1$

We see that $z=0$ is only inside C. thus $z=0$ is a pole of order 2

Residue of $f(z)$ at $(z=0)=\frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left[(z-0)^2.\frac{(z+4)^2}{z^2(z^2+5z+6)}\right]$

$=\lim_{z\to 0}\frac{d}{dz}\left[\frac{(z+4)^2}{z^2+5z+6}\right]$

$=\lim_{z\to 0}\left[\frac{(z^2+5z+6)\{2(z+4)\}-(z+4)^2\{2z+5+0\}}{(z^2+5z+6)^2}\right]$

$=-\frac{8}{9}$

By CRT,

$\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2}\,dz=2\pi i\, R=2\pi i\left[-\frac{8}{9}\right]$

$\boxed{\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2}\,dz=-\frac{16\pi i}{9}}$

2. (b) Find $Z\{f(k) * g(k)\}$ if $f(k) = 4^k U(k),\ g(k) = 5^k U(k)$ (6)

Solution:

If $Z\{f(k)\} = F(z)$ and $Z\{g(k) = G(z)$

Then by Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z). G(z)$

By definition,

$$U(k) = \begin{cases} 1 & for\ k \geq 0 \\ 0 & for\ k < 0 \end{cases}$$

$$Z\{U(k)\} = \sum_0^\infty 1. z^{-k}$$
$$= [z^0 + z^{-1} + z^{-2} + z^{-3} + \cdots \ldots..]$$
$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \ldots.$$
$$= \left[1 - \frac{1}{z}\right]^{-1}$$
$$= \frac{1}{1-\frac{1}{z}}$$

$$\therefore Z\{U(k)\} = \frac{z}{z-1}$$

By Change of Scale,

$$Z\{a^k U(k)\} = \frac{\frac{z}{a}}{\frac{z}{a}-1} = \frac{z}{z-a}$$

Now,

$Z\{f(k)\} = Z\{4^k U(k)\}$

$F(z) = \frac{z}{z-4}$

Also,

$Z\{g(k)\} = Z\{5^k U(k)\}$

$G(z) = \frac{z}{z-5}$

By Convolution Theorem,

$$Z\{f(k) * g(k)\} = F(z). G(z) = \frac{z}{z-4}.\frac{z}{z-5} = \frac{z^2}{(z-4)(z-5)}$$

2. (c) Solve the following L.P.P. by simplex method (8)

Maximise $z = 3x_1 + 2x_2 + 5x_3$

subject to $x_1 + 2x_2 + x_3 \leq 430$

$3x_1 + 2x_3 \leq 460$

$x_1 + 4x_2 \leq 420$

$x_1, x_2, x_3 \geq 0$

Solution:

Max $z - 3x_1 - 2x_2 - 5x_3 + 0s_1 + 0s_2 + 0s_3 = 0$

s.t. $x_1 + 2x_2 + x_3 + s_1 + 0s_2 + 0s_3 = 430$

$3x_1 + 0x_2 + 2x_3 + 0s_1 + s_2 + 0s_3 = 460$

$x_1 + 4x_2 + 0x_3 + 0s_1 + 0s_2 + s_3 = 420$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

Simplex table,

Iteration No.	Basic Var	Coefficient of						RHS	Ratio	Formula
		x_1	x_2	x_3	s_1	s_2	s_3			
0	z	-3	-2	**-5**	0	0	0	0	-	$X + \frac{5}{2}Y$
s_2 leaves x_3 enters	s_1	1	2	**1**	1	0	0	430	430	$X - \frac{1}{2}Y$
	s_2	**3**	**0**	**2**	**0**	**1**	**0**	**460**	230	$Y \div 2$
	s_3	1	4	**0**	0	0	1	420	-	-
1	z	9/2	**-2**	0	0	5/2	0	1150	-	$X + Y$
s_1 leaves x_2 enters	s_1	**-1/2**	**2**	**0**	**1**	**-1/2**	**0**	**200**	100	$Y \div 2$
	x_3	3/2	**0**	1	0	1/2	0	230	-	-
	s_3	1	**4**	0	0	0	1	430	107.5	$X - 2Y$
2	z	4	0	0	1	2	0	1350		
	x_2	-1/4	1	0	1/2	-1/4	0	100		
	x_3	3/2	0	1	0	1/2	0	230		
	s_3	2	0	0	-2	1	1	30		

Thus, the solution is

$\boxed{x_1 = 0, x_2 = 100, x_3 = 230, z_{max} = 1350}$

3. (a) Theory predicts that the proportion of beans in the four groups A, B, C, D should be 9: 3: 3: 1. In an experiment among 1600 beans the numbers in the four groups were 882, 313, 287 and 118. Does the experimental result support the theory? (6)
(Given that critical value of chi-square 3 d.f. and 5% L.O.S. is 7.81)

Solution:

(i) Null Hypothesis: The experimental result support the theory
Alternative Hypothesis: The experimental result does not support the theory

(ii) Test Statistic:

O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
882	$\frac{9}{16} \times 1600 = 900$	-18	324	324/900
313	$\frac{3}{16} \times 1600 = 300$	13	169	169/300
287	$\frac{3}{16} \times 1600 = 300$	-13	169	169/300
118	$\frac{1}{16} \times 1600 = 100$	18	324	324/100
Total				4.7267

(iii) Degree of freedom: $\emptyset = n - 1 = 3$

(iv) L.O.S: $\alpha = 0.05$

(v) Critical value: $\chi^2_\alpha = 7.815$

(vi) Decision: since, the calculated value is less than the critical value, null hypothesis is accepted. Thus, The experimental result support the theory

3. (b) Obtain Taylor's and Laurent's expansions of $f(z) = \frac{z-1}{z^2-2z-3}$ (6)

Solution:

We have, $f(z) = \frac{z-1}{z^2-2z-3} = \frac{z-1}{(z-3)(z+1)}$

Let $\frac{z-1}{(z-3)(z+1)} = \frac{A}{z-3} + \frac{B}{z+1}$

$z - 1 = A(z+1) + B(z-3)$

Comparing the coefficients, we get

$A + B = 1$

$A - 3B = -1$

On solving, we get

$A = \frac{1}{2}, B = \frac{1}{2}$

$f(z) = \frac{\frac{1}{2}}{z-3} + \frac{\frac{1}{2}}{z+1}$

For ROC,

Put $z - 3 = 0, z + 1 = 0$

$z = 3, z = -1$

$|z| = 3, |z| = 1$

The Region of Convergence are (i) $|z| < 1$ (ii) $1 < |z| < 3$ (iii) $|z| > 3$

The Taylors series is given by

(i) $|z| < 1$

$f(z) = \frac{\frac{1}{2}}{-3+z} + \frac{\frac{1}{2}}{1+z}$

$f(z) = \frac{\frac{1}{2}}{-3\left[1-\frac{z}{3}\right]} + \frac{\frac{1}{2}}{1+z}$

$f(z) = -\frac{1}{6}\left[1 - \frac{z}{3}\right]^{-1} + \frac{1}{2}[1+z]^{-1}$

$f(z) = -\frac{1}{6}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots.\right] + \frac{1}{2}[1 - z + z^2 - z^3 + \cdots..]$

The first Laurent's Series is given by

(ii) $1 < |z| < 3$

$f(z) = \frac{\frac{1}{2}}{-3+z} + \frac{\frac{1}{2}}{z+1}$

$f(z) = \frac{\frac{1}{2}}{-3\left[1-\frac{z}{3}\right]} + \frac{\frac{1}{2}}{z\left[1+\frac{1}{z}\right]}$

$f(z) = -\frac{1}{6}\left[1 - \frac{z}{3}\right]^{-1} + \frac{1}{2z}\left[1 + \frac{1}{z}\right]^{-1}$

$f(z) = -\frac{1}{6}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots.\right] + \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots..\right]$

The second Laurent's Series is given by

(iii) $|z| > 3$

$f(z) = \frac{\frac{1}{2}}{z-3} + \frac{\frac{1}{2}}{z+1}$

$$f(z) = \frac{\frac{1}{2}}{z\left[1-\frac{3}{z}\right]} + \frac{\frac{1}{2}}{z\left[1+\frac{1}{z}\right]}$$

$$f(z) = \frac{1}{2z}\left[1-\frac{3}{z}\right]^{-1} + \frac{1}{2z}\left[1+\frac{1}{z}\right]^{-1}$$

$$f(z) = \frac{1}{2z}\left[1+\frac{3}{z}+\frac{3^2}{z^2}+\frac{3^3}{z^3}+\cdots.\right] + \frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots..\right]$$

3. (c) Use the method of Lagrange's multipliers to solve the following N.L.P.P. (8)

Optimise $z = 6x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $4x_1 + 3x_2 = 16$

$3x_1 + 5x_2 = 15$

$x_1, x_2 \geq 0$

Solution:

Let $f = 6x_1 + 8x_2 - x_1^2 - x_2^2$

$h_1 = 4x_1 + 3x_2 - 16$

$h_2 = 3x_1 + 5x_2 - 15$

Lagrangian function,

$L = f - \lambda_1 h_1 - \lambda_2 h_2$

$L = (6x_1 + 8x_2 - x_1^2 - x_2^2) - \lambda_1(4x_1 + 3x_2 - 16) - \lambda_2(3x_1 + 5x_2 - 15)$

Consider,

$L_{x_1} = 0$ gives $6 - 2x_1 - 4\lambda_1 - 3\lambda_2 = 0$

$L_{x_2} = 0$ gives $8 - 2x_2 - 3\lambda_1 - 5\lambda_2 = 0$

$L_{\lambda_1} = 0$ gives $-(4x_1 + 3x_2 - 16) = 0$ i.e. $4x_1 + 3x_2 = 16$ ……. (1)

$L_{\lambda_2} = 0$ gives $-(3x_1 + 5x_2 - 15) = 0$ i.e. $3x_1 + 5x_2 = 15$ ……. (2)

Solving (1) & (2),

$x_1 = \frac{35}{11}, x_2 = \frac{12}{11}$

$$H_B = \begin{bmatrix} O & P \\ P' & Q \end{bmatrix}$$

Where, $P = \begin{bmatrix} h_{1_{x_1}} & h_{1_{x_2}} \\ h_{2_{x_1}} & h_{2_{x_2}} \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix}$

Where, $P' = \begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix}$

Where, $Q = \begin{bmatrix} L_{x_1x_1} & L_{x_1x_2} \\ L_{x_2x_1} & L_{x_2x_2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

$$H_B = \begin{bmatrix} 0 & 0 & 4 & 3 \\ 0 & 0 & 3 & 5 \\ 4 & 3 & -2 & 0 \\ 3 & 5 & 0 & -2 \end{bmatrix}$$

$$\Delta = 0\begin{vmatrix} 0 & 3 & 5 \\ 3 & -2 & 0 \\ 5 & 0 & -2 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 & 5 \\ 4 & -2 & 0 \\ 3 & 0 & -2 \end{vmatrix} + 4\begin{vmatrix} 0 & 0 & 5 \\ 4 & 3 & 0 \\ 3 & 5 & -2 \end{vmatrix} - 3\begin{vmatrix} 0 & 0 & 3 \\ 4 & 3 & -2 \\ 3 & 5 & 0 \end{vmatrix}$$

$\Delta = 4[55] - 3[33]$

$\Delta = 121$

Since Δ is positive, its a maxima

$$z_{max} = 6\left(\frac{35}{11}\right) + 8\left(\frac{12}{11}\right) - \left(\frac{35}{11}\right)^2 - \left(\frac{12}{11}\right)^2$$

$$\boxed{z_{max} = 16.5041}$$

4. (a) Fit a Poisson distribution to the following data: (6)

No of deaths	0	1	2	3	4
Frequencies	123	59	14	3	1

Solution:

$N = \sum F = 123 + 59 + 14 + 3 + 1 = 200$

Mean $= \frac{\sum F.x}{\sum F} = \frac{123(0)+59(1)+14(2)+3(3)+1(4)}{200} = 0.5$

$m = 0.5$

By Poisson distribution,

$P(X = r) = \frac{e^{-m}.m^r}{r!} = \frac{e^{-0.5}.(0.5)^r}{r!}$

X	$P(X)$	$F(X) = NP = 200P$
0	0.6065	$121.31 \approx 121$
1	0.3033	$60.65 \approx 61$
2	0.0758	$15.16 \approx 15$
3	0.0126	$2.53 \approx 3$
4	0.0016	$0.32 \approx 0$

4. (b) Find the inverse Z-transform of $\frac{1}{(z-2)(z-3)}$ if ROC is (i) $|z| < 2$ (ii) $2 < |z| < 3$ (6)

Solution:

We have,

$F(z) = \frac{1}{(z-3)(z-2)}$

Let $\frac{1}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2}$

$1 = A(z-2) + B(z-3)$

Comparing the coefficients, we get

$A + B = 0$

$-2A - 3B = 1$

On solving, we get $A = 1, B = -1$

$F(z) = \frac{1}{z-3} - \frac{1}{z-2}$

(i) $|z| < 2$

$F(z) = \frac{1}{-3+z} - \frac{1}{-2+z}$

$F(z) = \frac{1}{-3\left[1-\frac{z}{3}\right]} - \frac{1}{-2\left[1-\frac{z}{2}\right]}$

$F(z) = -\frac{1}{3}\left[1-\frac{z}{3}\right]^{-1} + \frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$

$F(z) = -\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots..\right] + \frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots..\right]$

$F(z) = \left[-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\frac{z^3}{3^4}-\cdots ...\right] + \left[\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\frac{z^3}{2^4}+\cdots ...\right]$

$F(z) = [-3^{-1}z^0 - 3^{-2}z^1 - 3^{-3}z^2 - \cdots.] + [2^{-1}z^0 + 2^{-2}z^1 + 2^{-3}z^2 + \cdots]$

From first series,

Coefficient of $z^k = -3^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -3^{k-1}, k \leq 0$

From second series,

Coefficient of $z^k = 2^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = 2^{k-1}, k \leq 0$

Thus,

$$\boxed{Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = 2^{k-1} - 3^{k-1}, k \leq 0}$$

(ii) $2 < |z| < 3$

$F(z) = \frac{1}{-3+z} - \frac{1}{z-2}$

$F(z) = \frac{1}{-3\left[1-\frac{z}{3}\right]} - \frac{1}{z\left[1-\frac{2}{z}\right]}$

$F(z) = -\frac{1}{3}\left[1-\frac{z}{3}\right]^{-1} - \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$

$F(z) = -\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots..\right] - \frac{1}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots..\right]$

$F(z) = \left[-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\frac{z^3}{3^4}-\cdots\ldots\right] + \left[-\frac{1}{z}-\frac{2}{z^2}-\frac{2^2}{z^3}-\frac{2^3}{z^4}-\cdots\ldots\right]$

$F(z) = [-3^{-1}z^0 - 3^{-2}z^1 - 3^{-3}z^2 - \cdots.] + [-2^0z^{-1} - 2^1z^{-2} - 2^2z^{-3} + \cdots]$

From first series,

Coefficient of $z^k = -3^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -3^{-(-k+1)}, k \leq 0$

i.e. Coefficient of $z^{-k} = -3^{k-1}, k \leq 0$

From second series,

Coefficient of $z^{-k} = -2^{k-1}, k > 0$

Thus,

$$\boxed{Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = \begin{cases} -3^{k-1} & k \leq 0 \\ -2^{k-1} & k > 0 \end{cases}}$$

4. (c) Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable. Find the transforming matrix and the diagonal matrix (8)

Solution:

$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}, |A| = 3$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [-9+3+7]\lambda^2 + \left[\begin{vmatrix} 3 & 4 \\ 8 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -16 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -8 & 3 \end{vmatrix}\right]\lambda - 3 = 0$

$\lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

$(\lambda+1)(\lambda+1)(\lambda-3) = 0$

$\lambda = -1, -1, 3$

The Algebraic Multiplicity of $\lambda = -1$ is 2 and that of $\lambda = 3$ is 1

(i) For $\lambda = -1$, $[A - \lambda I]X = 0$ gives

$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 - R_1, R_3 - 2R_1$

$\begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore -8x_1 + 4x_2 + 4x_3 = 0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

The Geometric Multiplicity of $\lambda = 1$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3 = t$ & $x_2 = s$

$\therefore x_1 = \frac{s}{2} + \frac{t}{2}$

$\therefore X = \begin{bmatrix} \frac{s}{2} + \frac{t}{2} \\ s \\ t \end{bmatrix} = \begin{bmatrix} \frac{s}{2} + \frac{t}{2} \\ s + 0t \\ 0s + t \end{bmatrix} = \begin{bmatrix} \frac{s}{2} \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} \frac{t}{2} \\ 0t \\ t \end{bmatrix} = \frac{s}{2}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{t}{2}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Hence, corresponding to $\lambda = -1$ the eigen vectors are $X_1 = [1,2,0]'$ & $X_2 = [1,0,2]'$

(ii) For $\lambda = 3$, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-12x_1 + 4x_2 + 4x_3 = 0$$

$$-8x_1 + 0x_2 + 4x_3 = 0$$

Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 4 & 4 \\ 0 & 4 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -12 & 4 \\ -8 & 4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -12 & 4 \\ -8 & 0 \end{vmatrix}}$$

$$\frac{x_1}{16} = -\frac{x_2}{-16} = \frac{x_3}{32}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{2}$$

Hence, corresponding to $\lambda = 3$ the eigen vector is $X_3 = [1,1,2]'$

Thus, the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$

5. (a) Using the method of Lagrange's multipliers to solve the following N.L.P.P. (6)

Optimise $z = 4x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $x_1 + x_2 = 4$

$x_1, x_2 \geq 0$

Solution:

Let $f = 4x_1 + 8x_2 - x_1^2 - x_2^2$ and $h = x_1 + x_2 - 4$

Consider the Lagrangian function,

$L = f - \lambda h = (4x_1 + 8x_2 - x_1^2 - x_2^2) - \lambda(x_1 + x_2 - 4)$

$L_{x_1} = 0 \Rightarrow 4 - 2x_1 - \lambda = 0 \Rightarrow 2x_1 + 0x_2 + \lambda = 4$ ……(1)

$L_{x_2} = 0 \Rightarrow 8 - 2x_2 - \lambda = 0 \Rightarrow 0x_1 + 2x_2 + \lambda = 8$ …..(2)

$L_{\lambda} = 0 \Rightarrow -(x_1 + x_2 - 4) = 0 \Rightarrow x_1 + x_2 + 0\lambda = 4$ …(3)

Solving (1), (2) and (3), we get

$x_1 = 1, x_2 = 3, \lambda = 2$

Now, hessian matrix,

$$H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 4$$

Since Δ is positive, it is a maxima

$\therefore z_{max} = 4(1) + 8(3) - 1^2 - 3^2$

$\boxed{z_{max} = 18}$

5. (b) Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$ (6)

Solution:

$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}, |A| = 4$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [4+3-2]\lambda^2 + \left[\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}\right]\lambda - 4 = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

By Cayley Hamilton theorem,

$A^3 - 5A^2 + 8A - 4I = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix} = \begin{bmatrix} 16 & 12 & 24 \\ 5 & 5 & 8 \\ -7 & -11 & -12 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 16 & 12 & 24 \\ 5 & 5 & 8 \\ -7 & -11 & -12 \end{bmatrix}\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix} = \begin{bmatrix} 52 & 12 & 72 \\ 17 & 5 & 24 \\ -27 & -15 & -40 \end{bmatrix}$

L.H.S. $= A^3 - 5A^2 + 8A - 4I$

$= \begin{bmatrix} 52 & 12 & 72 \\ 17 & 5 & 24 \\ -27 & -15 & -40 \end{bmatrix} - 5\begin{bmatrix} 16 & 12 & 24 \\ 5 & 5 & 8 \\ -7 & -11 & -12 \end{bmatrix} + 8\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = R.H.S.

Thus, Cayley Hamilton theorem is verified

5. (c) Solve by the dual simplex method (8)

Minimize $z = 6x_1 + x_2$

Subject to $2x_1 + x_2 \geq 3$

$x_1 - x_2 \geq 0$

$x_1, x_2 \geq 0$

Solution:

The standard form,

Min $z = 6x_1 + x_2$

$z - 6x_1 - x_2 + 0s_1 + 0s_2 = 0$

s.t. $-2x_1 - x_2 + s_1 = -3$

$-x_1 + x_2 + s_2 = 0$

Simplex table,

Iteration No.	Basic Var	Coefficient of x_1	x_2	s_1	s_2	RHS	Formula
0	z	-6	**-1**	0	0	0	$X - Y$
s_1 leaves x_2 enters	s_1	**-2**	**-1**	**1**	**0**	**-3**	$-Y$
	s_2	-1	**1**	0	1	0	$X + Y$
Ratio		$\frac{-6}{-2} = 3$	$\frac{-1}{-1} = 1$	-	-	-	
1	z	**-4**	0	-1	0	3	$X - \frac{4}{3}Y$
s_2 leaves x_1 enters	x_2	**2**	1	-1	0	3	$X + \frac{2}{3}Y$
	s_2	**-3**	**0**	**1**	**1**	**-3**	$-\frac{Y}{3}$
Ratio		$\frac{-4}{-3} = 1.33$	-	-	-	-	
2	z	0	0	$-\frac{7}{3}$	$-\frac{4}{3}$	7	
	x_2	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	1	
	x_1	1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	1	

The solution is

$\boxed{x_1 = 1, x_2 = 1, z_{min} = 7}$

6. (a) Find the Z transform of $f(k) = \begin{cases} b^k & for \quad k < 0 \\ a^k & for \quad k \geq 0 \end{cases}$ (6)

Solution:

We have,

$$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$$
$$Z\{f(k)\} = \sum_{-\infty}^{-1} b^k z^{-k} + \sum_{0}^{\infty} a^k z^{-k}$$
$$= [\dots\dots.. + b^{-3}z^3 + b^{-2}z^2 + b^{-1}z^1] + [a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots \dots\dots.]$$
$$= \left[\frac{z}{b} + \frac{z^2}{b^2} + \frac{z^3}{b^3} + \cdots\right] + \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots \dots..\right]$$
$$= \frac{z}{b}\left[1 + \frac{z}{b} + \frac{z^2}{b^2} + \cdots\right] + \left[1 - \frac{a}{z}\right]^{-1}$$
$$= \frac{z}{b}\left[1 - \frac{z}{b}\right]^{-1} + \left[\frac{z-a}{z}\right]^{-1}$$
$$= \frac{z}{b}\left[\frac{b-z}{b}\right]^{-1} + \frac{z}{z-a}$$
$$= \frac{z}{b}\left[\frac{b}{b-z}\right] + \frac{z}{z-a}$$
$$= \frac{z}{b-z} + \frac{z}{z-a}$$
$$= \frac{z^2 - az + bz - z^2}{(b-z)(z-a)}$$
$$= \frac{bz - az}{(b-z)(z-a)}$$

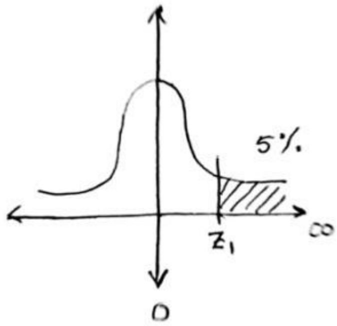
6. (b) The income of a group of 10,000 persons were found to be normally distributed with mean Rs. 520 and standard deviation Rs. 60. Find the lowest income of the richest 500 (6)

Solution:

$\mu = 520,\ \sigma = 60$

$z = \frac{x-\mu}{\sigma} = \frac{x-520}{60}$

The richest 500 out of the total 10000 belongs to the top 5%



$A(0\ to\ z_1) = 45\% = 0.45$

From table, $z_1 = \frac{1.64+1.65}{2} = 1.645$

$z_1 = \frac{x_1-520}{60}$

$1.645 = \frac{x_1-520}{60}$

$\therefore x_1 = 618.7 \approx 619$

Thus, the minimum salary of workers should be Rs. 619/- so that they belong to the top 5% workers.

6. (c) Using Kuhn-Tucker conditions, solve the following N.L.P.P. (8)

Maximise $z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$
subject to $2x_1 + x_2 \leq 5$
$x_1, x_2 \geq 0$

Solution:

Let $f = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$
Let $h = 2x_1 + x_2 - 5$
Consider, $L = f - \lambda h$
$L = 10x_1 + 4x_2 - 2x_1^2 - x_2^2 - \lambda(2x_1 + x_2 - 5)$
According to Kuhn Tucker conditions,
$L_{x_1} = 0 \Rightarrow 10 - 4x_1 - 2\lambda = 0$ ……………….(1)
$L_{x_2} = 0 \Rightarrow 4 - 2x_2 - \lambda = 0$ ……………….(2)
$\lambda h = 0 \Rightarrow \lambda(2x_1 + x_2 - 5) = 0$ …………………..(3)
$h \leq 0 \Rightarrow 2x_1 + x_2 - 5 \leq 0$ ……………………………….(4)
$x_1, x_2, \lambda \geq 0$ ………………(5)
Case I: If $\lambda = 0$
From (1), $x_1 = \frac{5}{2}$
From (2), $x_2 = 2$
We see that eqn (4) is not satisfied
Case II: If $\lambda \neq 0$
From (1), $4x_1 + 0x_2 + 2\lambda = 10$
From (2), $0x_1 + 2x_2 + \lambda = 4$
From (3), $2x_1 + x_2 + 0\lambda = 5$
On solving,
$x_1 = \frac{11}{6}, x_2 = \frac{4}{3}, \lambda = \frac{4}{3}$
$z_{max} = 10\left(\frac{11}{6}\right) + 4\left(\frac{4}{3}\right) - 2\left(\frac{11}{6}\right)^2 - \left(\frac{4}{3}\right)^2 = \frac{91}{6}$
Thus, the optimal solution is $z_{max} = \frac{91}{6}$ at $x_1 = \frac{11}{6}, x_2 = \frac{4}{3}$