Engineering Maths IV

Nov-Dec 2024

(COITAI)

Time (3 hours) **Max Marks: 80**

Note: (1) Question No. 1 is Compulsory

(2) Answer any three questions from Q.2 to Q.6

(3) Figures to the right indicate full marks

1. (a) If $A=\begin{bmatrix}-1&2&38\\0&2&37\\0&0&-2\end{bmatrix}$ find the Eigen values of $A^3+5A+8I$ (5)

Solution:

$A=\begin{bmatrix}-1&2&38\\0&2&37\\0&0&-2\end{bmatrix}, |A|=4$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}-1-\lambda&2&38\\0&2-\lambda&37\\0&0&-2-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[-1+2-2]\lambda^2+\left[\begin{vmatrix}2&37\\0&-2\end{vmatrix}+\begin{vmatrix}-1&38\\0&-2\end{vmatrix}+\begin{vmatrix}-1&2\\0&2\end{vmatrix}\right]\lambda-4=0$

$\lambda^3+\lambda^2-4\lambda-4=0$

$\lambda=-1,2,-2$

The eigen values of A is $-1,2,-2$

The eigen values of A^3 is $(-1)^3,(2)^3,(-2)^3$ i.e. $-1,8,-8$

The eigen values of $5A$ is $5(-1),5(2),5(-2)$ i.e. $-5,10,-10$

The eigen values of $8I$ is $8(1),8(1),8(1)$ i.e. $8,8,8$

Thus, the eigen values of $A^3+5A+8I$ is

$-1-5+8;8+10+8;-8-10+8$

i.e. $2,26,-10$

1. (b) Integrate the function $f(z) = x^2 + ixy$ from $A(1,1)$ to $B(2,4)$ along $y = x^2$ (5)

Solution:

$I = \int f(z)dz = \int_{(1,1)}^{(2,4)} (x^2 + ixy)(dx + idy)$

Along the curve,

$y = x^2$

$dy = 2x\,dx$

The integral becomes,

$I = \int_1^2 \left(x^2 + ix(x^2)\right)(dx + i\,2xdx)$

$I = \int_1^2 (x^2 + ix^3)(1 + 2ix)dx$

$I = \int_1^2 (x^2 + 2ix^3 + ix^3 + 2i^2x^4)dx$

$I = \int_1^2 (x^2 - 2x^4) + i3x^3.dx$

$I = \left[\frac{x^3}{3} - \frac{2x^5}{5} + 3i\frac{x^4}{4}\right]_1^2$

$\boxed{I = -\frac{151}{15} + \frac{45}{4}i}$

1. (c) Find the Z transform of $f(k) = a^{-k}, k \geq 0$ (5)

Solution:

We have,

$f(k) = a^{-k}, k \geq 0$

By definition,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$

$Z\{a^{-k}\} = \sum_0^{\infty} a^{-k}.z^{-k}$

$Z\{a^{-k}\} = a^0z^0 + a^{-1}.z^{-1} + a^{-2}.z^{-2} + a^{-3}.z^{-3} + \cdots \ldots \ldots$

$Z\{a^{-k}\} = 1 + \frac{1}{az} + \frac{1}{a^2z^2} + \frac{1}{a^3z^3} + \cdots \ldots \ldots$

$Z\{a^{-k}\} = \left[1 - \frac{1}{az}\right]^{-1}$

$Z\{a^{-k}\} = \left[\frac{az-1}{az}\right]^{-1}$

$\boxed{Z\{a^{-k}\} = \frac{az}{az-1}}$

1. (d) If a random variable X follows Poisson distribution such that
$P(X = 1) = 2P(X = 2)$ Find mean and variance of the distribution. (5)

Solution:

By Poisson distribution,

$P(X = r) = \frac{e^{-m}.m^r}{r!}$

It is given that,

$P(X = 1) = 2P(X = 2)$

$\frac{e^{-m}.m^1}{1!} = 2.\frac{e^{-m}.m^2}{2!}$

$m = m^2$

$\boxed{m = 1}$

2. (a) Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (6)

Solution:

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, |A| = 45$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [-2+1+0]\lambda^2 + \left[\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}\right]\lambda - 45 = 0$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$(\lambda - 5)(\lambda + 3)(\lambda + 3) = 0$

$\lambda = 5, -3, -3$

(i) For $\lambda = -3$, $[A - \lambda I]X = 0$ gives

$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 - 2R_1, R_3 + R_1$

$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore x_1 + 2x_2 - 3x_3 = 0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n - r = 3 - 1 = 2$ vectors to be formed

Let $x_3 = t$ & $x_2 = s, \therefore x_1 = -2s + 3t$

$\therefore X = \begin{bmatrix} -2s+3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2s+3t \\ s+0t \\ 0s+t \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0s \end{bmatrix} + \begin{bmatrix} 3t \\ 0t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Hence, corresponding to $\lambda = -3$ the eigen vectors are

$X_1 = [-2,1,0]'$ & $X_2 = [3,0,1]'$

(ii) For $\lambda = 5$, $[A - \lambda I]X = 0$ gives

$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-7x_1 + 2x_2 - 3x_3 = 0$

$2x_1 - 4x_2 - 6x_3 = 0$

Solving the above equations by Crammers rule, we get

$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$

$$\frac{x_1}{-24} = -\frac{x_2}{48} = \frac{x_3}{24}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,2,-1]'$

2. (b) Find the Z-transform of $\cos\left(\frac{\pi}{4} + k\alpha\right)$ where $k \geq 0$ (6)

Solution:

We have,

$$f(k) = \cos\left(\frac{\pi}{4} + k\alpha\right)$$

$$f(k) = \cos\frac{\pi}{4}\cos k\alpha - \sin\frac{\pi}{4}\sin k\alpha$$

$$Z\{f(k)\} = \cos\frac{\pi}{4}\ Z\{\cos k\alpha\} - \sin\frac{\pi}{4}\ Z\{\sin k\alpha\}$$

$$Z\{f(k)\} = \frac{1}{\sqrt{2}}\left[\frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}\right] - \frac{1}{\sqrt{2}}\left[\frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}\right]$$

$$Z\{f(k)\} = \frac{1}{\sqrt{2}} . \frac{z^2 - z\cos\alpha - z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$$

$$\boxed{Z\{f(k)\} = \frac{z^2 - z\cos\alpha - z\sin\alpha}{\sqrt{2}(z^2 - 2z\cos\alpha + 1)}}$$

2. (c) Use dual simplex method to solve the LPP

Minimise $z = 2x_1 - x_2 + 3x_3$

Subject to $3x_1 - x_2 + 3x_3 \leq 7$

$2x_1 - 4x_2 \geq 12$

$x_1, x_2, x_3 \geq 0$ (8)

Solution:

The standard form,

Min $z = 2x_1 - x_2 + 3x_3 + 0s_1 + 0s_2$

$z - 2x_1 + x_2 - 3x_3 + 0s_1 + 0s_2 = 0$

s.t. $3x_1 - x_2 + 3x_3 + s_1 + 0s_2 = 7$

$-2x_1 + 4x_2 + 0x_3 + 0s_1 + s_2 = -12$

Simplex table,

Iteration No.	Basic Var	Coefficient of					RHS	Formula
		x_1	x_2	x_3	s_1	s_2		
0	z	-2	**1**	-3	0	0	0	$X - \frac{1}{4}Y$
s_2 leaves x_2 enters	s_1	3	**-1**	3	1	0	7	$X + \frac{1}{4}Y$
	s_2	**-2**	**4**	**0**	**0**	**1**	**-12**	$Y \div 4$
Ratio		1	1/4	-	-	-	-	
1	z	**-3/2**	0	-3	0	-1/4	3	$X - 3Y$
s_1 leaves x_2 enters	s_1	**5/2**	0	3	1	1/4	4	$X + 5Y$
	x_2	**-1/2**	**1**	**0**	**0**	**1/4**	**-3**	$-2Y$
Ratio		3	-	-	-	-	-	
2	z	0	-3	-3	0	-1	12	
x_2 leaves x_1 enters	s_1	**0**	**5**	**3**	**1**	**3/2**	**-11**	
	x_1	1	-2	0	0	-1/2	6	
Ratio		-	-	-	-	-	-	

The solution is unbounded

3. (a) Evaluate $\int_c \frac{z+8}{z^2+5z+6} dz$ where C is a circle $|z| = 5$ (6)

Solution:

$I = \int_c \frac{z+8}{z^2+5z+6} dz$

If $z^2 + 5z + 6 = 0$, we get $z = -2, z = -3$

For C: $|z| = 5$

We see that $z = -2$ and $z = -3$ both lies inside C

Consider,

$\frac{1}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$1 = A(z+3) + B(z+2)$

Putting $z = -2$, we get $A = 1$

Putting $z = -3$, we get $B = -1$

$I = \int_c \frac{z+8}{(z+2)(z+3)} dz$

$I = \int_c \frac{z+8)}{(z+2)} dz + \int_c \frac{-(z+8)}{(z+3)} dz$

By CIF, $\int_c \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

$I = 2\pi i\, f(-2) + 2\pi i f(-3)$

$I = 2\pi i[-2+8] + 2\pi i[-(-3+8)]$

$I = 2\pi i[6-5]$

$\boxed{I = 2\pi i}$

3. (b) Verify Cayley Hamilton theorem and hence find A^{-1} & A^4 where $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ (6)

Solution:

$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, |A| = 3$

The characteristic equation,

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

$\lambda^3 - [sum\ of\ diagonals]\lambda^2 + [sum\ of\ minors\ of\ diagonals]\lambda - |A| = 0$

$\lambda^3 - [2+1+2]\lambda^2 + \left[\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\right]\lambda - 3 = 0$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley Hamilton theorem,

$A^3 - 5A^2 + 7A - 3I = 0$

Consider,

$A^2 = A.A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$

L.H.S. $= A^3 - 5A^2 + 7A - 3I$

$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$ R.H.S.

Thus, Cayley Hamilton theorem is verified

Now,

$A^3 - 5A^2 + 7A - 3I = 0$

Pre-multiplying by A^{-1}, we get

$A^2 - 5A + 7I - 3A^{-1} = 0$

$3A^{-1} = A^2 - 5A + 7I$

$3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$3A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

$$A^{-1}=\frac{1}{3}\begin{bmatrix}2 & -1 & -1\\ 0 & 3 & 0\\ -1 & -1 & 2\end{bmatrix}$$

Also, $A^3-5A^2+7A-3I=0$

Pre-multiplying by A, we get

$$A^4-5A^3+7A^2-3A=0$$

$$A^4=5A^3-7A^2+3A$$

$$A^4=5\begin{bmatrix}14 & 13 & 13\\ 0 & 1 & 0\\ 13 & 13 & 14\end{bmatrix}-7\begin{bmatrix}5 & 4 & 4\\ 0 & 1 & 0\\ 4 & 4 & 5\end{bmatrix}+3\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$$

$$A^4=\begin{bmatrix}41 & 40 & 40\\ 0 & 1 & 0\\ 40 & 40 & 41\end{bmatrix}$$

3. (c) Solve the LPP by Big M method

Maximise $z = x_1 + 2x_2 + 3x_3 - x_4$

subject to $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$ (8)

Solution:

The standard form,

Max $z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2 - MA_3$

Max $z - x_1 - 2x_2 - 3x_3 + x_4 + MA_1 + MA_2 + MA_3 = 0$ ……(1)

s.t. $x_1 + 2x_2 + 3x_3 + A_1 = 15$ ……(2)

$2x_1 + x_2 + 5x_3 + A_2 = 20$ ……(3)

$x_1 + 2x_2 + x_3 + x_4 + A_3 = 10$ ……. (4)

Multiplying eqn (2), (3), (4) by M and subtracting all with eqn (1), we get

$z + (-1 - 4M)x_1 + (-2 - 5M)x_2 + (-3 - 9M) + (1 - M)x_4 + 0A_1 + 0A_2 + 0A_3 = -45M$

Simplex table,

Iteration No.	Basic Var	Coefficient of							RHS	Ratio	Formula
		x_1	x_2	x_3	x_4	A_1	A_2	A_3			
0	z	-1-4M	-2-5M	**-3-9M**	1-M	0	0	0	-45M	-	$X - \frac{-3-9M}{5} Y$
A_2 leaves x_3 enters	A_1	1	2	**3**	0	1	0	0	15	5	$X - \frac{3}{5} Y$
	A_2	**2**	**1**	**5**	**0**	**0**	**1**	**0**	**20**	4	$Y \div 5$
	A_3	1	2	**1**	1	0	0	1	10	10	$X - \frac{1}{5} Y$
1	z	1/5 -2M/5	**-7/5-16M/5**	0	1-M	0		0	12-9M	-	$X - \frac{-7-16M}{7} Y$
A_1 leaves x_2 enters	A_1	**-1/5**	**7/5**	**0**	**0**	**1**		**0**	**3**	15/7	$\frac{5}{7} Y$
	x_3	2/5	**1/5**	1	0	0		0	4	20	$X - \frac{1}{7} Y$
	A_3	3/5	**9/5**	0	1	0		1	6	10/3	$X - \frac{9}{7} Y$
2	z	**-6M/7**	0	0	1-M			0	15-15M/7	-	$X + MY$
A_3 leaves x_1 enters	x_2	**-1/7**	1	0	0			0	15/7	-	$X + \frac{1}{6} Y$
	x_3	**3/7**	0	1	0			0	25/7	25/3	$X - \frac{1}{2} Y$
	A_3	**6/7**	**0**	**0**	**1**			**1**	**15/7**	5/2	$\frac{7}{6} Y$
3	z	0	0	0	1				15		
	x_2	0	1	0	1/6				5/2		
	x_3	0	0	1	-1/2				5/2		
	x_1	1	0	0	7/6				5/2		

Thus,

$$\boxed{x_1 = x_2 = x_3 = \frac{5}{2}, x_4 = 0, z_{max} = 15}$$

4. (a) Find inverse Z transform of $F(z) = \frac{1}{(z-2)(z-3)}$ for i) $|z| < 2$ ii) $|z| > 3$ (6)

Solution:

We have,

$F(z) = \frac{1}{(z-3)(z-2)}$

Let $\frac{1}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2}$

$1 = A(z-2) + B(z-3)$

Comparing the coefficients, we get

$A + B = 0$

$-2A - 3B = 1$

On solving, we get $A = 1, B = -1$

$F(z) = \frac{1}{z-3} - \frac{1}{z-2}$

(i) $|z| < 2$

$F(z) = \frac{1}{-3+z} - \frac{1}{-2+z}$

$F(z) = \frac{1}{-3\left[1-\frac{z}{3}\right]} - \frac{1}{-2\left[1-\frac{z}{2}\right]}$

$F(z) = -\frac{1}{3}\left[1-\frac{z}{3}\right]^{-1} + \frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$

$F(z) = -\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots..\right] + \frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots..\right]$

$F(z) = \left[-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\frac{z^3}{3^4}-\cdots\ldots\right] + \left[\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\frac{z^3}{2^4}+\cdots\ldots\right]$

$F(z) = [-3^{-1}z^0 - 3^{-2}z^1 - 3^{-3}z^2 - \cdots.] + [2^{-1}z^0 + 2^{-2}z^1 + 2^{-3}z^2 + \cdots]$

From first series,

Coefficient of $z^k = -3^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -3^{k-1}, k \leq 0$

From second series,

Coefficient of $z^k = 2^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = 2^{k-1}, k \leq 0$

Thus,

$$\boxed{Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = 2^{k-1} - 3^{k-1}, k \leq 0}$$

(ii) $|z| > 3$

$F(z) = \frac{1}{z-3} - \frac{1}{z-2}$

$F(z) = \frac{1}{z\left[1-\frac{3}{z}\right]} - \frac{1}{z\left[1-\frac{2}{z}\right]}$

$F(z) = \frac{1}{z}\left[1-\frac{3}{z}\right]^{-1} - \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$

$F(z) = \frac{1}{z}\left[1+\frac{3}{z}+\frac{3^2}{z^2}+\frac{3^3}{z^3}+\cdots..\right] - \frac{1}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots..\right]$

$F(z) = \left[\frac{1}{z}+\frac{3}{z^2}+\frac{3^2}{z^3}+\frac{3^3}{z^4}+\cdots\ldots\right] + \left[-\frac{1}{z}-\frac{2}{z^2}-\frac{2^2}{z^3}-\frac{2^3}{z^4}-\cdots\ldots\right]$

$F(z) = [3^0 z^{-1} + 3^1 z^{-2} + 3^2 z^{-3} - \cdots .] + [-2^0 z^{-1} - 2^1 z^{-2} - 2^2 z^{-3} + \cdots]$

From first series,

Coefficient of $z^{-k} = 3^{(k-1)}, k > 0$

From second series,

Coefficient of $z^{-k} = -2^{k-1}, k > 0$

Thus,

$$\boxed{Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = 3^{k-1} - 2^{k-1}, k > 0}$$

4. (b) A certain drug administered to 12 patients resulted in the following changes in their blood pressure: 5, 2, 8, -1, 3, 0, 6, -2, 1, 5, 0, 4. Can we conclude that the drug increase the blood pressure? (6)

Solution:

x	x^2
5	25
2	4
8	64
-1	1
3	9
0	0
6	36
-2	4
1	1
5	25
0	0
4	16
Total = 31	Total = 185

$n = 12$

$\overline{x} = \frac{\sum x}{n} = \frac{31}{12} = 2.5833$

$\sigma = \sqrt{\frac{\sum x^2}{n} - (\overline{x})^2} = \sqrt{\frac{185}{12} - (2.5833)^2} = 2.9569$

(i) Null Hypothesis: $\mu = 0$ (no change)
Alternative Hypothesis: $\mu \neq 0$

(ii) Test statistic:

$$t = \left|\frac{\overline{x}-\mu}{\frac{\sigma}{\sqrt{n-1}}}\right| = \left|\frac{2.5833-0}{\frac{2.9569}{\sqrt{12-1}}}\right| = 2.897$$

(iii) L.O.S.: $\alpha = 0.05$ (assumed)

(iv) Degree of freedom: $\emptyset = n - 1 = 12 - 1 = 11$

(v) Critical value: $t_{\alpha} = 2.201$

(vi) Decision: Since, the calculated value of t is more than the critical value, null hypothesis is rejected. Thus, there is rise in BP.

4. (c) Find all possible Laurent's series expansion of the function $f(z) = \frac{1}{(z+1)(z-2)}$ about $z = 0$ indicating the region of convergence in each case (8)

Solution:

We have, $f(z) = \frac{1}{(z+1)(z-2)}$

Let $\frac{1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}$

$1 = A(z-2) + B(z+1)$

Comparing the coefficients, we get

$A + B = 0$

$-2A + B = 1$

On solving, we get

$A = -\frac{1}{3}, B = \frac{1}{3}$

$f(z) = \frac{-\frac{1}{3}}{z+1} + \frac{\frac{1}{3}}{z-2}$

(i) $|z| < 1$

$f(z) = \frac{-\frac{1}{3}}{1+z} + \frac{\frac{1}{3}}{-2+z}$

$f(z) = \frac{-\frac{1}{3}}{[1+z]} + \frac{\frac{1}{3}}{-2\left[1-\frac{z}{2}\right]}$

$f(z) = -\frac{1}{3}[1+z]^{-1} - \frac{1}{6}\left[1-\frac{z}{2}\right]^{-1}$

$f(z) = -\frac{1}{3}[1 - z + z^2 - z^3 + \cdots.] - \frac{1}{6}\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right]$

(ii) $1 < |z| < 2$

$f(z) = \frac{-\frac{1}{3}}{z+1} + \frac{\frac{1}{3}}{-2+z}$

$f(z) = \frac{-\frac{1}{3}}{z\left[1+\frac{1}{z}\right]} + \frac{\frac{1}{3}}{-2\left[1-\frac{z}{2}\right]}$

$f(z) = -\frac{1}{3z}\left[1+\frac{1}{z}\right]^{-1} - \frac{1}{6}\left[1-\frac{z}{2}\right]^{-1}$

$f(z) = -\frac{1}{3z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots.\right] - \frac{1}{6}\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right]$

(iii) $|z| > 2$

$f(z) = \frac{-\frac{1}{3}}{z+1} + \frac{\frac{1}{3}}{z-2}$

$f(z) = \frac{-\frac{1}{3}}{z\left[1+\frac{1}{z}\right]} + \frac{\frac{1}{3}}{z\left[1-\frac{2}{z}\right]}$

$f(z) = -\frac{1}{3z}\left[1+\frac{1}{z}\right]^{-1} + \frac{1}{3z}\left[1-\frac{2}{z}\right]^{-1}$

$f(z) = -\frac{1}{3z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots.\right] + \frac{1}{3z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots\right]$

5. (a) Determine all basic solutions to the following problem.

Maximise $z = x_1 - 2x_2 + 4x_3$

subject to $x_1 + 2x_2 + 3x_3 = 7$

$3x_1 + 4x_2 + 6x_3 = 15$

$x_1, x_2, x_3 \geq 0$ (6)

Solution:

No	Non-basic var = 0	Basic var	Equations & solutions	Is the solution feasible?	Is the solution degenerate?	Value of z	Is the solution optimal?
1	$x_3 = 0$	x_1, x_2	$x_1 + 2x_2 = 7$ $3x_1 + 4x_2 = 15$ $x_1 = 1, x_2 = 3$	Yes	No	-5	No
2	$x_2 = 0$	x_1, x_3	$x_1 + 3x_3 = 7$ $3x_1 + 6x_3 = 15$ $x_1 = 1, x_3 = 2$	Yes	No	9	Yes
3	$x_1 = 0$	x_2, x_3	$2x_2 + 3x_3 = 7$ $4x_2 + 6x_3 = 15$ Unbounded solution	-	-	-	-

5. (b) If X is a normal variate with mean 10 & s.d. 4. Find (i) $P(5 \leq X \leq 18)$ (ii) $P(X \leq 12)$ (6)

Solution:

$\mu = 10, \sigma = 4$

$z = \frac{x-\mu}{\sigma} = \frac{x-10}{4}$

(i) $P(5 \leq x \leq 18) = P\left(\frac{5-10}{4} \leq z \leq \frac{18-10}{4}\right)$

$= P(-1.25 \leq z \leq 2)$

$= A(2) - A(-1.25)$

$= A(2) + A(1.25)$

$= 0.4772 + 0.3944$

$= 0.8716$

(ii) $P(x \leq 12) = P\left(z \leq \frac{12-10}{4}\right)$

$= P(z \leq 0.5)$

$= A(0.5) - A(-\infty)$

$= A(\infty) + A(0.5)$

$= 0.5 + 0.1915$

$= 0.6915$

5. (c) Solve the N.L.P.P.

Optimise $z = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$

subject to $x_1 + x_2 + x_3 = 10$

$x_1, x_2, x_3 \geq 0$ (8)

Solution:

Let $f = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$

and $h = x_1 + x_2 + x_3 - 10$

Consider the Lagrangian function,

$L = f - \lambda h$

$L = \left(12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23\right) - \lambda(x_1 + x_2 + x_3 - 10)$

$L_{x_1} = 0 \Rightarrow 12 - 2x_1 - \lambda = 0 \Rightarrow x_1 = \frac{12-\lambda}{2}$

$L_{x_2} = 0 \Rightarrow 8 - 2x_2 - \lambda = 0 \Rightarrow x_2 = \frac{8-\lambda}{2}$

$L_{x_3} = 0 \Rightarrow 6 - 2x_3 - \lambda = 0 \Rightarrow x_3 = \frac{6-\lambda}{2}$

$L_\lambda = 0 \Rightarrow -(x_1 + x_2 + x_3 - 10) = 0$

$x_1 + x_2 + x_3 = 10$

$\frac{12-\lambda}{2} + \frac{8-\lambda}{2} + \frac{6-\lambda}{2} = 10$

$\frac{26-3\lambda}{2} = 10$

$\lambda = 2$

$\therefore x_1 = 5, x_2 = 3, x_3 = 1$

Now, hessian matrix, $H = \begin{bmatrix} 0 & h_{x_1} & h_{x_2} & h_{x_3} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}$

$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 4$

$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \\ 1 & 0 & 0 \end{vmatrix}$

$\Delta_4 = -4 - 4 - 4 = -12$

Since both Δ_3 is positive and Δ_4 is negative, it is a maxima

$\therefore z_{max} = 12(5) + 8(3) + 6(1) - (5)^2 - (3)^2 - (1)^2 - 23$

$\boxed{z_{max} = 32}$

6. (a) Show that given matrix is diagonalizable and hence find diagonal form and transforming matrix where $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ (6)

Solution:

$A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}, |A|=45$

The characteristic equation,

$|A-\lambda I|=0$

$\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=0$

$\lambda^3-[sum\ of\ diagonals]\lambda^2+[sum\ of\ minors\ of\ diagonals]\lambda-|A|=0$

$\lambda^3-[-2+1+0]\lambda^2+\left[\begin{vmatrix}1 & -6\\ -2 & 0\end{vmatrix}+\begin{vmatrix}-2 & -3\\ -1 & 0\end{vmatrix}+\begin{vmatrix}-2 & 2\\ 2 & 1\end{vmatrix}\right]\lambda-45=0$

$\lambda^3+1\lambda^2-21\lambda-45=0$

$\lambda=-3,-3,5$

The Algebraic Multiplicity of $\lambda=-3$ is 2 and that of $\lambda=5$ is 1

(i) For $\lambda=-3$, $[A-\lambda I]X=0$ gives

$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

By R_2-2R_1, R_3+R_1

$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\therefore x_1+2x_2-3x_3=0$

The rank (r) of the matrix is 1 and number of unknowns (n) is 3

Thus, $n-r=3-1=2$ vectors to be formed

The Geometric Multiplicity of $\lambda=-3$ is 2

Since, Algebraic Multiplicity = Geometric Multiplicity,

matrix A is diagonalizable.

Let $x_3=t$ & $x_2=s$, $\therefore x_1=-2s+3t$

$\therefore X=\begin{bmatrix}-2s+3t\\ s\\ t\end{bmatrix}=\begin{bmatrix}-2s+3t\\ s+0t\\ 0s+t\end{bmatrix}=\begin{bmatrix}-2s\\ s\\ 0s\end{bmatrix}+\begin{bmatrix}3t\\ 0t\\ t\end{bmatrix}=s\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}3\\ 0\\ 1\end{bmatrix}$

Hence, corresponding to $\lambda=-3$ the eigen vectors are

$X_1=[-2,1,0]'$ & $X_2=[3,0,1]'$

(ii) For $\lambda=5$, $[A-\lambda I]X=0$ gives

$\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$-7x_1 + 2x_2 - 3x_3 = 0$
$2x_1 - 4x_2 - 6x_3 = 0$
Solving the above equations by Crammers rule, we get

$$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{-24} = -\frac{x_2}{48} = \frac{x_3}{24}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

Hence, corresponding to $\lambda = 5$ the eigen vector is $X_3 = [1,2,-1]'$

Thus, matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalised to $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ by the transformation $M^{-1}AM = D$ where $M = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

6. (b) Based on the following determine if there is a relation between literacy and smoking:

	Smokers	Non-smokers
Literates	83	57
Illiterates	45	68

(6)

Solution:

The observed frequency table as given:

	Smokers	Non-smokers	Total
Literates	83	57	140
Illiterates	45	68	113
Total	128	125	253

(i) Null Hypothesis: There is no relation between literacy and smoking

Alternative Hypothesis: There is a relation between literacy and smoking

(ii) Test Statistic:

O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
83	$\frac{140\times128}{253}=71$	12	144	144/71
57	$\frac{125\times140}{253}=69$	-12	144	144/69
45	$\frac{128\times113}{253}=57$	-12	144	144/57
68	$\frac{125\times113}{253}=56$	12	144	144/56
Total				9.213

(iii) Degree of freedom: $\emptyset=(r-1)(c-1)=(2-1)(2-1)=1$

(iv) L.O.S: $\alpha=0.05$

(v) Critical value: $\chi^2_\alpha=3.841$

(vi) Decision: since, the calculated value is more than the critical value, null hypothesis is rejected. Thus, there is a relation between literacy and smoking

6. (c) Max $z = 2x_1^2 - 7x_2^2 + 12x_1x_2$ subject to $2x_1 + 5x_2 \leq 98, x_1, x_2 \geq 0$ by K.T. condition (8)

Solution:

Let $f = 2x_1^2 - 7x_2^2 + 12x_1x_2$

Let $h = 2x_1 + 5x_2 - 98$

Consider, $L = f - \lambda h$

$L = 2x_1^2 - 7x_2^2 + 12x_1x_2 - \lambda(2x_1 + 5x_2 - 98)$

According to Kuhn Tucker conditions,

$L_{x_1} = 0 \Rightarrow 4x_1 + 12x_2 - 2\lambda = 0$ ……………….(1)

$L_{x_2} = 0 \Rightarrow -14x_2 + 12x_1 - 5\lambda = 0$ ……………….(2)

$\lambda h = 0 \Rightarrow \lambda(2x_1 + 5x_2 - 98) = 0$ …………………..(3)

$h \leq 0 \Rightarrow 2x_1 + 5x_2 - 98 \leq 0$ ……………………………….(4)

$x_1, x_2, \lambda \geq 0$ ………………(5)

Case I: If $\lambda = 0$

From (1), $4x_1 + 12x_2 = 0$

From (2), $12x_1 - 14x_2 = 0$

$x_1 = 0, x_2 = 0$

$z_{max} = 2(0)^2 - 7(0)^2 + 12(0)(0) = 0$

Case II: If $\lambda \neq 0$

From (1), $4x_1 + 12x_2 - 2\lambda = 0$

From (2), $12x_1 - 14x_2 - 5\lambda = 0$

From (3), $2x_1 + 5x_2 + 0\lambda = 98$

On solving,

$x_1 = 44, x_2 = 2, \lambda = 100$

$z_{max} = 2(44)^2 - 7(2)^2 + 12(44)(2) = 4900$

Thus, the optimal solution is

$\boxed{z_{max} = 4900}$ at $\boxed{x_1 = 44, x_2 = 2}$