Residues

Weight Distribution of Types

MechCivil

Type	Name	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
I	Calculation of Residues	---	---	---	---	---	---	---	---	---	---
II	Cauchy Residue Thm	---	---	---	04	---	---	---	---	---	---
Total Marks		**00**	**00**	**00**	**04**	**00**	**00**	**00**	**00**	**00**	**00**

Comp/IT/AI

Type	Name	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
I	Calculation of Residues	---	---	---	---	02	---	---	---	---	---
II	Cauchy Residue Thm	---	---	---	---	05	06	06	---	08	---
Total Marks		**00**	**00**	**00**	**00**	**07**	**06**	**06**	**00**	**08**	**00**

Extc

Type	Name	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
I	Calculation of Residues	---	---	---	---	02	---	--	---	---	---
II	Cauchy Residue Thm	---	06	06	06	---	06	06	06	06	06
Total Marks		**00**	**06**	**06**	**06**	**02**	**06**	**06**	**06**	**06**	**06**

Elect

Type	Name	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
I	Calculation of Residues	---	---	---	---	05	---	---	---	---	---
II	Cauchy Residue Thm	---	---	06	06	05	---	05	06	06	06
Total Marks		**00**	**00**	**06**	**06**	**10**	**00**	**05**	**06**	**06**	**06**

Type I: Calculation of Residues at Poles

1. Determine the pole of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$and also find the residue at each pole.
[N13/Chem/6M]
Solution:
We have, $f(z) = \frac{z^2}{(z-1)^2(z+2)}$
For singularity,
$(z-1)^2(z+2) = 0$
$\therefore z = 1,1,-2$
$\therefore z = -2$ is a simple pole and $z = 1$ is a pole of order 2
Residue of $f(z)$ at $(z = -2) = \lim_{z \to -2}(z+2)f(z)$

$$= \lim_{z \to -2}(z+2)\frac{z^2}{(z-1)^2(z+2)}$$
$$= \lim_{z \to -2}\frac{z^2}{(z-1)^2}$$
$$= \frac{(-2)^2}{(-2-1)^2}$$
$$= \frac{4}{9}$$

Residue of $f(z)$ at $(z = 1) = \frac{1}{1!}\lim_{z \to 1}\frac{d}{dz}[(z-1)^2 f(z)]$

$$= \lim_{z \to 1}\frac{d}{dz}\left[(z-1)^2\frac{z^2}{(z-1)^2(z+2)}\right]$$
$$= \lim_{z \to 1}\frac{d}{dz}\left[\frac{z^2}{z+2}\right]$$
$$= \lim_{z \to 1}\left[\frac{(z+2)(2z)-z^2(1)}{(z+2)^2}\right]$$
$$= \frac{(1+2)(2)-1^1}{(1+2)^2}$$
$$= \frac{5}{9}$$

2. Find the poles and calculate the residues at them for $f(z) = \frac{z}{(z-1)(z+2)^2}$

[N15/ChemBiot/6M]

Solution:

We have, $f(z) = \frac{z}{(z-1)(z+2)^2}$

For singularity,

$(z-1)(z+2)^2 = 0$

$\therefore z = 1, z = -2, -2$

$\therefore z = 1$ is a simple pole and $z = -2$ is a pole of order 2

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}(z-1)\frac{z}{(z-1)(z+2)^2}$$

$$= \lim_{z\to 1}\frac{z}{(z+2)^2}$$

$$= \frac{1}{(1+2)^2}$$

$$= \frac{1}{9}$$

Residue of $f(z)$ at $(z = -2) = \frac{1}{1!}\lim_{z\to -2}\frac{d}{dz}[(z+2)^2 f(z)]$

$$= \lim_{z\to -2}\frac{d}{dz}\left[(z+2)^2\frac{z}{(z-1)(z+2)^2}\right]$$

$$= \lim_{z\to -2}\frac{d}{dz}\left[\frac{z}{(z-1)}\right]$$

$$= \lim_{z\to -2}\left[\frac{(z-1)(1)-(z)(1)}{(z-1)^2}\right]$$

$$= \frac{(-2-1)-(-2)}{(-2-1)^2}$$

$$= -\frac{1}{9}$$

3. Find the residues of the following functions at their poles: (i) $\frac{1}{(z^2+1)^3}$ (ii) $z^2 e^{\frac{1}{z}}$

Solution:

(i) let $f(z) = \frac{1}{(z^2+1)^3}$

For singularity, put $(z^2+1)^3 = 0$

$$z^2 = -1$$
$$z = \sqrt{-1}$$
$$z = \pm i$$
$$z = \pm i, \pm i, \pm i$$

Thus, $z = i$ is a pole of order 3 and $z = -i$ is also a pole order 3

Res of $f(z)$ at pole of order m,

$$R_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \right]$$
$$R_1 = \frac{1}{2!} \lim_{z \to i} \left[\frac{d^2}{dz^2} (z-i)^3 . \frac{1}{(z+i)^3 (z-i)^3} \right]$$
$$R_1 = \frac{1}{2} \lim_{z \to i} \left[\frac{d^2}{dz^2} \frac{1}{(z+i)^3} \right]$$
$$R_1 = \frac{1}{2} \lim_{z \to i} \left[\frac{d^2}{dz^2} (z+i)^{-3} \right]$$
$$R_1 = \frac{1}{2} \lim_{z \to i} \left[\frac{d}{dz} (-3(z+i)^{-4}) \right]$$
$$R_1 = \frac{1}{2} \lim_{z \to i} [-3 \times -4(z+i)^{-5}]$$
$$R_1 = \frac{1}{2} [12(i+i)^{-5}]$$
$$R_1 = 6(2i)^{-5}$$
$$R_1 = \frac{6}{(2i)^5} = \frac{6}{(2i)^2 \times (2i)^3} = -\frac{3i}{16}$$

Thus, Residue at $z = i$ is $\frac{-3i}{16}$

Thus, Residue of $z = -i$ is $\frac{+3i}{16}$

(ii) $z^2 e^{\frac{1}{z}}$

Let $f(z) = z^2 e^{\frac{1}{z}}$

Note:

$$\boldsymbol{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \ldots ..}$$
$$\boldsymbol{e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \ldots ..}$$
$$\boldsymbol{sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \ldots ..}$$
$$\boldsymbol{cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \ldots ..}$$

$$f(z) = z^2 \left[1 + \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^4}{4!} + \cdots \ldots . \right]$$

$$f(z) = z^2 + \frac{z^2}{z} + \frac{z^2}{2!\times z^2} + \frac{z^2}{3!\times z^3} + \frac{z^2}{4!\times z^4} + \cdots \ldots$$

$$f(z) = z^2 + z + \frac{1}{2} + \frac{1}{6z} + \frac{1}{24z^2} + \cdots \ldots \ldots$$

The above is a Laurent's expansion about $z = 0$

Residue =coefficient of $\frac{1}{z}$ in a Laurent's Series

$R = \frac{1}{6}$ and $z = 0$ is an isolated essential singularity.

4. Prove that the sum of residues of the function $f(z) = \frac{e^z}{z^2+a^2}$ is $\frac{\sin a}{a}$

Solution:

$f(z) = \frac{e^z}{z^2+a^2}$

For singularity,

Put $z^2 + a^2 = 0$

$z^2 = -a^2$

$z = \sqrt{-a^2}$

$z = \pm ia$

$z = ai, z = -ai$

Thus, $z = ai, z = -ai$ are simple poles

$R_1 = \lim_{z \to z_0} (z - z_0) f(z)$

$R_1 = \lim_{z \to ai} (z - ai) \frac{e^z}{z^2+a^2}$

$R_1 = \lim_{z \to ai} \frac{(z-ai)e^z}{(z+ai)(z-ai)}$

$R_1 = \lim_{z \to ai} \frac{e^z}{z+ai}$

$R_1 = \frac{e^{ai}}{2ai}$

Thus, residue of $f(z)$ at $z = ai$ is $R_1 = \frac{e^{ai}}{2ai}$

Thus, residue of $f(z)$ at $z = -ai$ is $R_2 = \frac{e^{-ai}}{-2ai}$

Therefore,

Sum of residues $= R_1 + R_2$

$= \frac{e^{ai}}{2ai} + \frac{e^{-ai}}{-2ai}$

$= \frac{e^{ai}}{2ai} - \frac{e^{-ai}}{2ai}$

$= \frac{e^{ai}-e^{-ai}}{2ai}$ $\qquad \because \frac{e^{i\theta}-e^{-i\theta}}{2i} = \sin\theta$

$= \frac{1}{a}\left[\frac{e^{ai}-e^{-ai}}{2i}\right]$

$= \frac{1}{a}[\sin a]$

$= \frac{\sin a}{a}$

5. Determine the nature of the poles & find sum of residues at each pole $\frac{z}{az^2+bz+c}$
[M19/Chem/5M]
Solution:
We have, $f(z) = \frac{z}{az^2+bz+c} = \frac{z}{a\left(z^2+\frac{b}{a}z+\frac{c}{a}\right)}$
For singularity,
$a\left(z^2+\frac{b}{a}z+\frac{c}{a}\right) = 0$
Let $z = \alpha$ and $z = \beta$ be the roots of the above equation and hence are simple poles
Residue of $f(z)$ at $(z = \alpha) = \lim_{z\to\alpha}(z-\alpha)f(z)$
$= \lim_{z\to\alpha}(z-\alpha)\frac{z}{a(z-\alpha)(z-\beta)}$
$= \lim_{z\to\alpha}\frac{z}{a(z-\beta)} = \frac{\alpha}{a(\alpha-\beta)}$
$\therefore$ Residue of $f(z)$ at $(z = \beta) = \frac{\beta}{a(\beta-\alpha)} = -\frac{\beta}{a(\alpha-\beta)}$
Sum of Residues $= \frac{\alpha}{a(\alpha-\beta)} + \frac{-\beta}{a(\alpha-\beta)}$
$= \frac{\alpha-\beta}{a(\alpha-\beta)} = \frac{1}{a}$

6. The function $f(z) = \frac{z^2}{(z+2)(z-1)^2}$ has
[M22/Elex/2M]
Solution:
We have, $f(z) = \frac{z^2}{(z+2)(z-1)^2}$
For singularity,
$(z+2)(z-1)^2 = 0$
$\therefore z = -2, z = 1, 1$
$\therefore f(z) = \frac{z^2}{(z+2)(z-1)^2}$ has simple pole at $z = -2$ & pole of order 2 at $z = 1$

7. If $f(z) = \frac{3z^2+z}{z^2-1}$ then residue of $f(z)$ at $z = -1$ is

[M22/CompITAI/2M]

Solution:

We have, $f(z) = \frac{3z^2+z}{z^2-1}$

For singularity,

$z^2 - 1 = 0$

$(z+1)(z-1) = 0$

$\therefore z = 1$ is a simple pole and $z = -1$ is also a simple pole

Residue of $f(z)$ at $(z = -1) = \lim_{z\to -1} (z+1)f(z)$

$$= \lim_{z\to -1} (z+1)\frac{3z^2+z}{(z+1)(z-1)}$$

$$= \lim_{z\to -1} \frac{3z^2+z}{z-1}$$

$$= \frac{3(-1)^2+(-1)}{\mathbf{-1-1}} = \frac{2}{-2} = -1$$

8. The function $f(z) = \frac{2}{(z+5)^3(z-2)^4}$ has poles at $z = -5$ of order ___ and $z = 2$ of order _

[M22/Extc/2M]

Solution:

The function $f(z) = \frac{2}{(z+5)^3(z-2)^4}$ has poles at $z = -5$ of order 3 and $z = 2$ of order 4

9. Find the residues at their poles $f(z) = \frac{z}{(z+3)(z-1)^2}$

[M22/Elect/5M]

Solution:

We have, $f(z) = \frac{z}{(z+3)(z-1)^2}$

For singularity,

$(z-1)^2(z+3) = 0$

$\therefore z = 1,1,-3$

$\therefore z = -3$ is a simple pole and $z = 1$ is a pole of order 2

Residue of $f(z)$ at $(z = -3) = \lim_{z \to -3} (z+3) f(z)$

$$= \lim_{z \to -3} (z+3) \frac{z}{(z+3)(z-1)^2}$$

$$= \lim_{z \to -3} \frac{z}{(z-1)^2}$$

$$= \frac{-3}{(-3-1)^2} = -\frac{3}{16}$$

Residue of $f(z)$ at $(z = 1) = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} [(z-1)^2 f(z)]$

$$= \lim_{z \to 1} \frac{d}{dz} \left[(z-1)^2 \frac{z}{(z+3)(z-1)^2}\right]$$

$$= \lim_{z \to 1} \frac{d}{dz} \left[\frac{z}{z+3}\right]$$

$$= \lim_{z \to 1} \left[\frac{(z+3)(1) - z(1)}{(z+3)^2}\right]$$

$$= \frac{(1+3)(1) - 1}{(1+3)^2} = \frac{3}{16}$$

10. Find the residues of the function $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2}$ at their poles.

[N15/MechCivil/6M]

Solution:

We have, $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2}$

For singularity,

$(z-1)(z-2)^2 = 0$

$\therefore z = 1, z = 2,2$

$\therefore z = 1$ is a simple pole and $z = 2$ is a pole of order 2

Residue of $f(z)$ at $(z = 1) = \lim_{z \to 1} (z-1) f(z)$

$$= \lim_{z \to 1} (z-1) \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2}$$

$$= \lim_{z \to 1} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)^2}$$

$$= \frac{\sin \pi + \cos \pi}{(1-2)^2}$$

$$= \frac{0-1}{(-1)^2} = -1$$

Residue of $f(z)$ at $(z = 2) = \frac{1}{1!} \lim_{z \to 2} \frac{d}{dz} [(z-2)^2 f(z)]$

$$= \lim_{z \to 2} \frac{d}{dz} \left[(z-2)^2 \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2} \right]$$

$$= \lim_{z \to 2} \frac{d}{dz} \left[\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} \right]$$

$$= \lim_{z \to 2} \left[\frac{(z-1)(\cos \pi z^2 \times 2\pi z - \sin \pi z^2 \times 2\pi z) - (\sin \pi z^2 + \cos \pi z^2)(1)}{(z-1)^2} \right]$$

$$= \left[\frac{(1)(4\pi \cos 4\pi - 4\pi \sin 4\pi) - (\sin 4\pi + \cos 4\pi)}{(2-1)^2} \right]$$

$$= 4\pi - 1$$

11. Determine the nature of poles of the following functions and find the residue of each pole $f(z)=\frac{\sin\pi z}{(z-1)^2(z-2)}$.

[N17/N18/Biot/6M]

Solution:

We have, $f(z)=\frac{\sin\pi z}{(z-1)^2(z-2)}$

For singularity,

$(z-1)^2(z-2)=0$

$\therefore z=1,1, z=2$

$\therefore z=2$ is a simple pole and $z=1$ is a pole of order 2

Residue of $f(z)$ at $(z=2)=\lim_{z\to 2}(z-2)f(z)$

$$=\lim_{z\to 2}(z-2)\frac{\sin\pi z}{(z-1)^2(z-2)}$$

$$=\lim_{z\to 2}\frac{\sin\pi z}{(z-1)^2}$$

$$=\frac{\sin 2\pi}{(2-1)^2}$$

$$=0$$

Residue of $f(z)$ at $(z=1)=\frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}[(z-1)^2 f(z)]$

$$=\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{\sin\pi z}{(z-1)^2(z-2)}\right]$$

$$=\lim_{z\to 1}\frac{d}{dz}\left[\frac{\sin\pi z}{(z-2)}\right]$$

$$=\lim_{z\to 1}\left[\frac{(z-2)(\cos\pi z\times\pi)-(\sin\pi z)(1)}{(z-2)^2}\right]$$

$$=\left[\frac{(-1)(\pi\cos\pi)-(\sin\pi)}{(-1)^2}\right]$$

$$=\pi$$

12. Find the sum of residues at singular points of $f(z) = \frac{z-4}{z(z-1)(z-2)}$

[M17/ElexExtcElectBiomInst/6M]

Solution:

We have, $f(z) = \frac{z-4}{z(z-1)(z-2)}$

For singularity,

$z(z-1)(z-2) = 0$

$\therefore z = 0, z = 1, z = 2$

$\therefore$ all are simple poles

Residue of $f(z)$ at $(z = 0) = \lim_{z\to 0}(z-0)f(z)$

$= \lim_{z\to 0}(z-0)\frac{z-4}{z(z-1)(z-2)}$

$= \lim_{z\to 0}\frac{z-4}{(z-1)(z-2)}$

$= \frac{-4}{(-1)(-2)}$

$= -2$

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)f(z)$

$= \lim_{z\to 1}(z-1)\frac{z-4}{z(z-1)(z-2)}$

$= \lim_{z\to 1}\frac{z-4}{z(z-2)}$

$= \frac{-3}{(1)(-1)}$

$= 3$

Residue of $f(z)$ at $(z = 2) = \lim_{z\to 2}(z-2)f(z)$

$= \lim_{z\to 2}(z-2)\frac{z-4}{z(z-1)(z-2)}$

$= \lim_{z\to 2}\frac{z-4}{z(z-1)}$

$= \frac{-2}{(2)(1)}$

$= -1$

Sum of residues $= -2 + 3 - 1 = 0$

13. Find the sum of residues at singular points of $f(z) = \frac{z}{(z-1)^2(z^2-1)}$

[N14/ChemBiot/7M][N16/ElexExtcElectBiomInst/6M]

Solution:

We have, $f(z) = \frac{z}{(z-1)^2(z^2-1)}$

For singularity,

$(z-1)^2(z^2-1) = 0$

$(z-1)^2(z-1)(z+1) = 0$

$\therefore z = 1,1,1,-1$

$\therefore z = -1$ is a simple pole and $z = 1$ is a pole of order 3

Residue of $f(z)$ at $(z = -1) = \lim_{z\to -1}(z+1)f(z)$

$$= \lim_{z\to -1}(z+1)\frac{z}{(z-1)^2(z^2-1)}$$
$$= \lim_{z\to -1}\frac{z}{(z-1)^3}$$
$$= \frac{-1}{(-1-1)^3}$$
$$= \frac{1}{8}$$

Residue of $f(z)$ at $(z = 1) = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}[(z-1)^3 f(z)]$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left[(z-1)^3\frac{z}{(z-1)^3(z+1)}\right]$$
$$= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left[\frac{z}{z+1}\right]$$
$$= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left[\frac{(z+1)(1)-z(1)}{(z+1)^2}\right]$$
$$= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left[\frac{1}{(z+1)^2}\right]$$
$$= \frac{1}{2}\lim_{z\to 1} -\frac{2}{(z+1)^3}$$
$$= \frac{1}{2}.-\frac{2}{(1+1)^3}$$
$$= -\frac{1}{8}$$

Sum of Residues $= \frac{1}{8} - \frac{1}{8} = 0$

14. The distance between z_0 and the nearest singularity of $f(z)$ is called as

[M22/Chem/2M] Ans. Radius of convergence

Type II: Cauchy's Residue Theorem

1. Using Cauchy's residue theorem, evaluate $\int_c \frac{12z-7}{(z-1)^2(2z+3)} dz$ where C is the circle

(i) $|z| = \frac{1}{2}$ (ii) $|z+i| = \sqrt{3}$

Solution:

We have, $I = \int \frac{12z-7}{(z-1)^2(2z+3)} dz$

For singularity,

Put $(z-1)^2(2z+3) = 0$

$z = 1,1,-\frac{3}{2}$

(i) $|z| = \frac{1}{2}$

We see that $z = 1, z = -\frac{3}{2}$ both lies outside C hence they are not poles

By CIT,

$\int \frac{12z-7}{(z-1)^2(2z+3)} dz = 0$

(ii) $|z+i| = \sqrt{3}$

For $z = 1$, LHS = $|1+i| = \sqrt{1^2+1^2} = \sqrt{2} < \sqrt{3} < RHS$

Thus, $z = 1$ lies inside C hence $z = 1$ is a pole of order 2

For $z = -\frac{3}{2}$, LHS = $\left|-\frac{3}{2}+i\right| = \sqrt{\left(-\frac{3}{2}\right)^2 + 1^2} = 1.80 > \sqrt{3} > RHS$

Thus, $z = -\frac{3}{2}$ lies outside C hence $z = -\frac{3}{2}$ is not a pole

Res of $f(z)$ at $z = 1$,

$R = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$

$R = \frac{1}{1!} \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 . \frac{12z-7}{(z-1)^2(2z+3)}\right]$

$R = \lim_{z\to 1} \frac{d}{dz}\left[\frac{12z-7}{2z+3}\right]$

$R = \lim_{z\to 1}\left[\frac{(2z+3)[12-0]-(12z-7)[2+0]}{(2z+3)^2}\right]$

$R = 2$

By CRT,

$\int \frac{12z-7}{(z-1)^2(2z+3)} dz = 2\pi i\,[R] = 2\pi i[2] = 4\pi i$

2. Using Cauchy's residue theorem, evaluate $\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2}dz$ where C is $|z|=1$
[M23/CompIT/6M]
Solution:
$I=\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2}dz$
For singularity or pole,
Put $z^4+5z^3+6z^2=0$
$z^2(z^2+5z+6)=0$
$z^2=0, z^2+5z+6=0$
$z=0,0, z=-3, z=-2$
C is $|z|=1$
We see that $z=0$ is only inside C. thus $z=0$ is a pole of order 2
Residue of $f(z)$ at $(z=0)=\frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left[(z-0)^2.\frac{(z+4)^2}{z^2(z^2+5z+6)}\right]$
$=\lim_{z\to 0}\frac{d}{dz}\left[\frac{(z+4)^2}{z^2+5z+6}\right]$
$=\lim_{z\to 0}\left[\frac{(z^2+5z+6)\{2(z+4)\}-(z+4)^2\{2z+5+0\}}{(z^2+5z+6)^2}\right]$
$=-\frac{8}{9}$
By CRT,
$\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2}dz=2\pi i\, R=2\pi i\left[-\frac{8}{9}\right]=-\frac{16\pi i}{9}$

3. Evaluate $\oint_C e^{-\frac{1}{z}}\sin\left(\frac{1}{z}\right)dz$ where C is $|z|=1$
Solution:
We have, $I=\int e^{-\frac{1}{z}}\sin\left(\frac{1}{z}\right)dz$
Here,
$f(z)=e^{-\frac{1}{z}}\sin\left(\frac{1}{z}\right)=\left[1-\frac{1}{z}+\frac{\left(\frac{1}{z}\right)^2}{2!}-\frac{\left(\frac{1}{z}\right)^3}{3!}+\cdots\right]\left[\frac{1}{z}-\frac{\left(\frac{1}{z}\right)^3}{3!}+\frac{\left(\frac{1}{z}\right)^5}{5!}-\cdots..\right]$
$f(z)=\left[1-\frac{1}{z}+\frac{1}{2z^2}-\frac{1}{6z^3}+\cdots.\right]\left[\frac{1}{z}-\frac{1}{6z^3}+\frac{1}{120z^5}-\cdots.\right]$
$f(z)=\frac{1}{z}-\frac{1}{6z^3}+\frac{1}{120z^5}-\frac{1}{z^2}+\frac{1}{6z^4}-\frac{1}{120z^6}+\frac{1}{2z^3}-\frac{1}{12z^5}+\cdots\ldots..$
$f(z)=\frac{1}{z}-\frac{1}{z^2}-\frac{1}{6z^3}+\frac{1}{2z^3}+\frac{1}{6z^4}+\cdots\ldots\ldots..$
$f(z)=\frac{1}{z}-\frac{1}{z^2}+\frac{1}{3z^3}+\frac{1}{6z^4}+\cdots\ldots..$
Residues = coefficient of $\frac{1}{z}=1$
By CRT,
$\int e^{-\frac{1}{z}}\sin\left(\frac{1}{z}\right)dz=2\pi i\,[R]=2\pi i[1]=2\pi i$

4. Evaluate $\int_c z^6 e^{-\frac{1}{z}} dz\,; c: |z| = 1$

Solution:

We have, $I = \int_c z^6 e^{-\frac{1}{z}} dz$

Here,

$$f(z) = z^6 e^{-\frac{1}{z}}$$

$$f(z) = z^6\left[1 - \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^4}{4!} - \frac{\left(\frac{1}{z}\right)^5}{5!} + \frac{\left(\frac{1}{z}\right)^6}{6!} - \frac{\left(\frac{1}{z}\right)^7}{7!} + \cdots \ldots.\right]$$

$$f(z) = z^6\left[1 - \frac{1}{z} + \frac{1}{2z^2} - \frac{1}{6z^3} + \frac{1}{24z^4} - \frac{1}{120z^5} + \frac{1}{720z^6} - \frac{1}{5040z^7} + \cdots \ldots\right]$$

$$f(z) = z^6 - z^5 + \frac{z^4}{2} - \frac{z^3}{6} + \frac{z^2}{24} - \frac{z}{120} + \frac{1}{720} - \frac{1}{5040z} + \cdots \ldots \ldots$$

Residues = coefficient of $\frac{1}{z} = -\frac{1}{5040}$

By CRT,

$$\int_c z^6 e^{-\frac{1}{z}} dz = 2\pi i\,[R] = 2\pi i\left[-\frac{1}{5040}\right] = -\frac{\pi i}{2520}$$

5. Using Cauchy's residue theorem, evaluate $\int_c \operatorname{cosec} z\ dz$ where c is $|z| = 1$

Solution:

$$I = \int \operatorname{cosec} z\ dz = \int \frac{1}{\sin z} dz$$

For singularity,

Put $\sin z = 0$

$$z = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots \ldots \ldots ..$$

C is $|z| = 1$

We see that, $z = 0$ is inside C hence it is a simple pole.

$$R = \lim_{z \to z_0} (z - z_0) f(z)$$

$$R = \lim_{z \to 0} (z - 0).\frac{1}{sinz}$$

$$R = \lim_{z \to 0} \frac{z}{\sin z}\left[\frac{0}{0}\right]$$

Applying L-Hospital rule,

$$R = \lim_{z \to 0} \frac{1}{\cos z}$$

$$R = \frac{1}{\cos 0} = \frac{1}{1} = 1$$

By CRT,

$$\int \operatorname{cosec} z\ dz = 2\pi i[R] = 2\pi i[1] = 2\pi i$$

6. Evaluate $\oint_c \tan z\, dz$ where c is (i) is the circle $|z| = 2$ (ii) is the circle $|z| = 1$.

Solution:

$I = \int \tan z\ dz = \int \frac{\sin z}{\cos z} dz$

For singularity,

Put $\cos z = 0$

$$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots\dots\dots$$

(i) C is $|z| = 2$

We see that $z = \frac{\pi}{2}, z = -\frac{\pi}{2}$ both lies inside C hence they are simple poles

$R_1 = \lim_{z\to z_0}(z - z_0)f(z)$

$R_1 = \lim_{z\to\frac{\pi}{2}}\left(z - \frac{\pi}{2}\right).\frac{\sin z}{\cos z}$

$R_1 = \lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)\sin z}{\cos z}$

Applying L-Hospital rule,

$R_1 = \lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)[\cos z]+\sin z[1-0]}{-\sin z}$

$R_1 = \frac{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)\left[\cos\frac{\pi}{2}\right]+\sin\frac{\pi}{2}[1]}{-\sin\frac{\pi}{2}}$

$R_1 = -1$

Now,

$R_2 = \lim_{z\to-\frac{\pi}{2}}\left(z - -\frac{\pi}{2}\right).\frac{\sin z}{\cos z}$

$R_2 = \lim_{z\to-\frac{\pi}{2}}\frac{\left(z+\frac{\pi}{2}\right)\sin z}{\cos z}$

Applying L-Hospital rule,

$R_2 = \lim_{z\to-\frac{\pi}{2}}\frac{\left(z+\frac{\pi}{2}\right)[\cos z]+\sin z[1+0]}{-\sin z}$

$R_2 = \frac{\left(-\frac{\pi}{2}+\frac{\pi}{2}\right)\left[\cos\left(-\frac{\pi}{2}\right)\right]+\sin\left(-\frac{\pi}{2}\right)[1]}{-\sin\left(-\frac{\pi}{2}\right)}$

$R_2 = -1$

By CRT,

$\int \tan z\ dz = 2\pi i\ [R_1 + R_2] = 2\pi i[-1 - 1] = -4\pi i$

(ii) C is $|z| = 1$

We see that no points lies inside C and hence there are no poles

By CIT,

$\int \tan z\ dz = 0$

7. Evaluate $\int_c \frac{dz}{\sinh z}$ where c is $x^2 + 2y^2 = 16$ and define simple pole.

Solution:

$I = \int \frac{1}{\sinh z} dz$

For singularity,

Put $\sinh z = 0$

$\frac{e^z - e^{-z}}{2} = 0$

$e^z - e^{-z} = 0$

$e^z = e^{-z}$

$z = -z$

$z + z = 0$

$2z = 0$

$z = 0$ Note that $z = 0$ implies $x = 0, y = 0$

C is $x^2 + 2y^2 = 16$

LHS $= x^2 + 2y^2 = (0)^2 + 2(0)^2 = 0 < 16 < RHS$

Thus, $z = 0$ lies inside C and hence it is a simple pole

$R = \lim_{z \to 0} (z - 0).\frac{1}{\sinh z}$

$R = \lim_{z \to 0} \frac{z}{\sinh z}$

Applying L-hospital rule

$R = \lim_{z \to 0} \frac{1}{\cosh z} = \frac{1}{cosh 0} = \frac{1}{1} = 1$

By CRT,

$\int \frac{1}{\sinh z} dz = 2\pi i R = 2\pi i[1] = 2\pi i$

8. Evaluate by using Residue theorem $\int_c \operatorname{sech} z\, dz$ where c is $|z| = 2$

Solution:

$I = \int \operatorname{sech} z \ dz = \int \frac{1}{\cosh z} dz$

For singularity,

Put $\cosh z = 0$

$\frac{e^z + e^{-z}}{2} = 0$

$e^z + e^{-z} = 0$

$e^z = -e^{-z}$

$e^z = -\frac{1}{e^z}$

$e^{2z} = -1$ $\quad\quad$ $since, -1 + 0i = \cos\pi + i\sin\pi = e^{i\pi}$

$e^{2z} = e^{i\pi}$

$2z = i\pi$

$z = \frac{i\pi}{2}$

C is $|z| = 2$

We see that $z = \frac{i\pi}{2}$ lies inside C and it is a simple pole

$R = \lim_{z\to\frac{\pi i}{2}} \left(z - \frac{\pi i}{2}\right) . \frac{1}{\cosh z}$

$R = \lim_{z\to\frac{\pi i}{2}} \frac{\left(z - \frac{\pi i}{2}\right)}{\cosh z}$

Applying L-Hospital rule,

$R = \lim_{z\to\frac{\pi i}{2}} \frac{1-0}{\sinh z}$

$R = \frac{1}{\sinh\frac{i\pi}{2}} = \frac{1}{i\sin\frac{\pi}{2}} = \frac{1}{i}$ $\quad\quad$ $\because \sinh i\theta = i\sin\theta$

By CRT,

$\int \operatorname{sech} z \ dz = 2\pi i\, R = 2\pi i \left[\frac{1}{i}\right] = 2\pi$

9. Evaluate $\int_c \frac{\cos z}{z} dz$ where C is the ellipse $9x^2 + 4y^2 = 1$

[N17/CompIT/6M][N19/Extc/6M]

Solution:

We have, $f(z) = \frac{\cos z}{z}$

For singularity,

$z = 0$

$z = 0$ is a simple pole

Residue of $f(z)$ at $(z = 0) = \lim_{z \to 0}(z - 0)f(z)$

$$= \lim_{z \to 0}(z - 0)\frac{\cos z}{z}$$

$$= \lim_{z \to 0} \cos z$$

$$= \cos 0$$

$$= 1$$

By Cauchy's Residue Theorem,

$\oint_c \frac{\cos z}{z} dz = 2\pi i\ (1) = 2\pi i$

10. Evaluate $\oint_c \frac{z^2}{(z-1)^2(z+1)} dz$ where c is $|z| = 2$ using residue theorem

[N16/CompIT/6M]

Solution:

We have, $f(z) = \frac{z^2}{(z-1)^2(z+1)}$

For singularity,

$(z-1)^2(z+1) = 0$

$\therefore z = 1,1,-1$

$\therefore z = -1$ is a simple pole and $z = 1$ is a pole of order 2

Residue of $f(z)$ at $(z = -1) = \lim_{z\to-1}(z+1)f(z)$

$$= \lim_{z\to-1}(z+1)\frac{z^2}{(z-1)^2(z+1)}$$

$$= \lim_{z\to-1}\frac{z^2}{(z-1)^2}$$

$$= \frac{(-1)^2}{(-1-1)^2} = \frac{1}{4}$$

Residue of $f(z)$ at $(z = 1) = \frac{1}{1!}\lim_{z\to1}\frac{d}{dz}[(z-1)^2 f(z)]$

$$= \lim_{z\to1}\frac{d}{dz}\left[(z-1)^2\frac{z^2}{(z-1)^2(z+1)}\right]$$

$$= \lim_{z\to1}\frac{d}{dz}\left[\frac{z^2}{z+1}\right]$$

$$= \lim_{z\to1}\left[\frac{(z+1)(2z)-z^2(1)}{(z+1)^2}\right]$$

$$= \frac{4-1}{(1+1)^2}$$

$$= \frac{3}{4}$$

Sum of Residues $= \frac{1}{4} + \frac{3}{4} = 1$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$

$\oint_c \frac{z^2}{(z-1)^2(z+1)} dz = 2\pi i\,(1) = 2\pi i$

11. Evaluate $\oint_c \frac{z^2}{(z-1)^2(z-2)} dz$ where c is $|z| = 2.5$ using Cauchy's residue theorem
[N19/Chem/6M]
Solution:
We have, $f(z) = \frac{z^2}{(z-1)^2(z-2)}$
For singularity,
$(z-1)^2(z-2) = 0$
$\therefore z = 1,1,2$
$\therefore z = 1$ is a pole of order 2 and $z = 2$ is a simple pole
Residue of $f(z)$ at $(z = 2) = \lim_{z\to 2}(z-2)\, f(z)$
$= \lim_{z\to 2}(z-2)\frac{z^2}{(z-1)^2(z-2)}$
$= \lim_{z\to 2}\frac{z^2}{(z-1)^2}$
$= \frac{2^2}{(2-1)^2} = 4$
Residue of $f(z)$ at $(z = 1) = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}[(z-1)^2 f(z)]$
$= \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{z^2}{(z-1)^2(z-2)}\right]$
$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{z^2}{z-2}\right]$
$= \lim_{z\to 1}\left[\frac{(z-2)(2z)-z^2(1)}{(z-2)^2}\right]$
$= \frac{-2-1}{(-1)^2}$
$= -3$
By Cauchy's Residue Theorem,
$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$
$\oint_c \frac{z^2}{(z-1)^2(z-2)} dz = 2\pi i\,(4-3) = 2\pi i$

12. Evaluate $\oint_c \frac{z^2}{(z-1)(z-2)} dz$ where c is circle $|z| = 2.5$ using Cauchy's residue theorem

[M22/Chem/5M]

Solution:

We have, $f(z) = \frac{z^2}{(z-1)(z-2)}$

For singularity,

$(z-1)(z-2) = 0$

$\therefore z = 1,2$

$\therefore z = 1$ is a simple pole and $z = 2$ is a simple pole

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)\, f(z)$

$$= \lim_{z\to 1}(z-1)\frac{z^2}{(z-1)(z-2)}$$

$$= \lim_{z\to 1}\frac{z^2}{(z-2)}$$

$$= \frac{1^2}{(1-2)} = -1$$

Residue of $f(z)$ at $(z = 2) = \lim_{z\to 2}(z-2)\, f(z)$

$$= \lim_{z\to 2}(z-2)\frac{z^2}{(z-1)(z-2)}$$

$$= \lim_{z\to 2}\frac{z^2}{(z-1)}$$

$$= \frac{2^2}{(2-1)} = 4$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\oint_c \frac{z^2}{(z-1)(z-2)} dz = 2\pi i\,(-1+4) = 6\pi i$

13. Evaluate $\int_c \frac{z+3}{(z-1)(z-4)} dz$ where c is the circle $|z-1| = 2$

[D24/ElectECS/6M]

Solution:

We have, $f(z) = \frac{z+3}{(z-1)(z-4)}$

For singularity,

$(z-1)(z-4) = 0$

$\therefore z = 1,4$

$\therefore z = 1$ is a simple pole as it lies inside $C: |z-1| = 2$ and

$z = 4$ lies outside $C: |z-1| = 2$

Residue of $f(z)$ at $(z = 1) = \lim_{z\to1}(z-1)\, f(z)$

$$= \lim_{z\to1}(z-1)\frac{z+3}{(z-1)(z-4)}$$

$$= \lim_{z\to1}\frac{z+3}{(z-4)}$$

$$= \frac{4}{(1-4)} = -\frac{4}{3}$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$

$\oint_c \frac{z+3}{(z-1)(z-4)} dz = 2\pi i\left(-\frac{4}{3}\right) = -\frac{8\pi i}{3}$

14. Evaluate the following integral by Cauchy's residue theorem $\int_C \frac{e^{2z}}{(z-1)(z-2)} dz$ where C is the circle $|z| = 3$

[N22/Chem/5M]

Solution:

We have, $f(z) = \frac{e^{2z}}{(z-1)(z-2)}$

For singularity,

$(z-1)(z-2) = 0$

$\therefore z = 1,2$

$\therefore z = 1$ is a simple pole and $z = 2$ is a simple pole

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)\, f(z)$

$$= \lim_{z\to 1}(z-1)\frac{e^{2z}}{(z-1)(z-2)}$$

$$= \lim_{z\to 1}\frac{e^{2z}}{(z-2)}$$

$$= \frac{e^2}{(1-2)} = -e^2$$

Residue of $f(z)$ at $(z = 2) = \lim_{z\to 2}(z-2)\, f(z)$

$$= \lim_{z\to 2}(z-2)\frac{e^{2z}}{(z-1)(z-2)}$$

$$= \lim_{z\to 2}\frac{e^{2z}}{(z-1)}$$

$$= \frac{e^4}{(2-1)} = e^4$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\int_C \frac{e^{2z}}{(z-1)(z-2)} dz = 2\pi i\ (-e^2 + e^4) = 2\pi i[e^4 - e^2]$

15. Evaluate $\oint_c \frac{z^2}{(z+1)^2(z-2)} dz$ where c is $|z| = 1.5$

[N19/MechCivil/4M]

Solution:

We have, $f(z) = \frac{z^2}{(z+1)^2(z-2)}$

for singularity,

$(z+1)^2(z-2) = 0$

$\therefore z = -1, -1, 2$

$\therefore z = -1$ is a pole of order 2 and $z = 2$ is outside C

Residue of $f(z)$ at $(z = -1) = \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz}[(z+1)^2 f(z)]$

$$= \lim_{z \to -1} \frac{d}{dz}\left[(z+1)^2 \frac{z^2}{(z+1)^2(z-2)}\right]$$

$$= \lim_{z \to -1} \frac{d}{dz}\left[\frac{z^2}{z-2}\right]$$

$$= \lim_{z \to -1}\left[\frac{(z-2)(2z) - z^2(1)}{(z-2)^2}\right]$$

$$= \frac{6-1}{(-3)^2}$$

$$= \frac{5}{9}$$

By Cauchy's Residue Theorem,

$\int_c f(z) dz = 2\pi i\ [sum\ of\ residues]$

$\oint_c \frac{z^2}{(z+1)^2(z-2)} dz = 2\pi i \left(\frac{5}{9}\right) = \frac{10\pi i}{9}$

16. Using Cauchy's Residue Theorem evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)} dz$ where C is $|z| = 4$

[M19/Extc/6M]

Solution:

We have, $f(z) = \frac{z-1}{(z+1)^2(z-2)}$

For singularity,

$(z+1)^2(z-2) = 0$

$\therefore z = -1, -1, 2$

$\therefore z = 2$ is a simple pole and $z = -1$ is a pole of order 2

Residue of $f(z)$ at $(z = 2) = \lim_{z\to 2}(z-2)f(z)$

$$= \lim_{z\to 2}(z-2)\frac{z-1}{(z+1)^2(z-2)}$$

$$= \lim_{z\to 2}\frac{z-1}{(z+1)^2}$$

$$= \frac{2-1}{(2+1)^2}$$

$$= \frac{1}{9}$$

Residue of $f(z)$ at $(z = -1) = \frac{1}{1!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2 f(z)]$

$$= \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\frac{z-1}{(z+1)^2(z-2)}\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z-1}{z-2}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}\right]$$

$$= \frac{-3+2}{(-1-2)^2}$$

$$= \frac{-1}{9}$$

Sum of Residues $= \frac{1}{9} - \frac{1}{9} = 0$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\oint_c \frac{z-1}{(z+1)^2(z-2)}dz = 2\pi i\ (0) = 0$

17. Using Cauchy's Residue Theorem evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)}dz$ where C is $|z-i| = 2$

[M23/Extc/6M]

Solution:

We have, $f(z) = \frac{z-1}{(z+1)^2(z-2)}$

For singularity,

$(z+1)^2(z-2) = 0$

$\therefore z = -1, -1, 2$

C is $|z-i| = 2$

For $z = -1$, LHS $= |-1-i| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} < 2 <$ RHS

$z = -1$ lies inside C

For $z = 2$, LHS $= |2-i| = \sqrt{(2)^2 + (-1)^2} = \sqrt{5} > 2 >$ RHS

$z = 2$ lies outside C

$\therefore z = -1$ is a pole of order 2

Residue of $f(z)$ at $(z = -1) = \frac{1}{1!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2 f(z)]$

$$= \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\frac{z-1}{(z+1)^2(z-2)}\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z-1}{z-2}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}\right]$$

$$= \frac{-3+2}{(-1-2)^2}$$

$$= \frac{-1}{9}$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\oint_c \frac{z-1}{(z+1)^2(z-2)}dz = 2\pi i\left(-\frac{1}{9}\right) = -\frac{2\pi i}{9}$

18. Evaluate using Cauchy's residue theorem $\int_c \frac{12z-7}{z(2z+1)(z+2)}dz$ where C is $|z| = 1$

[M16/CompIT/6M]

Solution:

We have, $f(z) = \frac{12z-7}{z(2z+1)(z+2)}$

For singularity,

$z(2z+1)(z+2) = 0$

$\therefore z = 0, z = -\frac{1}{2}, z = -2$

We see that $z = 0$ and $z = -\frac{1}{2}$ both lies inside $C: |z| = 1$ and hence are simple poles.

Residue of $f(z)$ at $(z = 0) = \lim_{z\to 0}(z-0)f(z)$

$$= \lim_{z\to 0}(z-0)\frac{12z-7}{z(2z+1)(z+2)}$$

$$= \lim_{z\to 0}\frac{12z-7}{(2z+1)(z+2)}$$

$$= \frac{0-7}{(0+1)(0+2)} = -\frac{7}{2}$$

Residue of $f(z)$ at $\left(z = -\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}}\left(z+\frac{1}{2}\right)f(z)$

$$= \lim_{z\to -\frac{1}{2}}\left(z+\frac{1}{2}\right)\frac{12z-7}{z(2z+1)(z+2)}$$

$$= \lim_{z\to -\frac{1}{2}}\frac{2z+1}{2}.\frac{12z-7}{z(2z+1)(z+2)}$$

$$= \frac{1}{2}\lim_{z\to -\frac{1}{2}}\frac{12z-7}{z(z+2)}$$

$$= \frac{1}{2}.\frac{12\left(-\frac{1}{2}\right)-7}{-\frac{1}{2}\left(-\frac{1}{2}+2\right)}$$

$$= \frac{1}{2}.\frac{-13}{-\frac{3}{4}}$$

$$= \frac{26}{3}$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\int_c \frac{12z-7}{z(2z+1)(z+2)}dz = 2\pi i\ \left[-\frac{7}{2}+\frac{26}{3}\right] = 2\pi i\left[\frac{31}{6}\right] = \frac{31\pi i}{3}$

19. Evaluate using Cauchy's residue theorem $\int_c \frac{2z-1}{z(2z+1)(z+2)}dz$ where C is $|z| = 1$

[M19/Elect/6M][N22/MTRX/8M][M24/CompITAI/8M]

Solution:

We have, $f(z) = \frac{12z-1}{z(2z+1)(z+2)}$

For singularity,

$z(2z+1)(z+2) = 0$

$\therefore z = 0, z = -\frac{1}{2}, z = -2$

We see that $z = 0$ and $z = -\frac{1}{2}$ both lies inside $C: |z| = 1$ and hence are simple poles.

Residue of $f(z)$ at $(z = 0) = \lim_{z\to0}(z-0)f(z)$

$$= \lim_{z\to0}(z-0)\frac{12z-1}{z(2z+1)(z+2)}$$

$$= \lim_{z\to0}\frac{12z-1}{(2z+1)(z+2)}$$

$$= \frac{0-1}{(0+1)(0+2)}$$

$$= -\frac{1}{2}$$

Residue of $f(z)$ at $\left(z = -\frac{1}{2}\right) = \lim_{z\to-\frac{1}{2}}\left(z+\frac{1}{2}\right)f(z)$

$$= \lim_{z\to-\frac{1}{2}}\left(z+\frac{1}{2}\right)\frac{12z-1}{z(2z+1)(z+2)}$$

$$= \lim_{z\to-\frac{1}{2}}\frac{2z+1}{2}\cdot\frac{12z-1}{z(2z+1)(z+2)}$$

$$= \frac{1}{2}\lim_{z\to-\frac{1}{2}}\frac{12z-1}{z(z+2)}$$

$$= \frac{1}{2}\cdot\frac{12\left(-\frac{1}{2}\right)-1}{-\frac{1}{2}\left(-\frac{1}{2}+2\right)} = \frac{1}{2}\cdot\frac{-7}{-\frac{3}{4}}$$

$$= \frac{14}{3}$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$

$\int_c \frac{12z-1}{z(2z+1)(z+2)}dz = 2\pi i\left[-\frac{1}{2}+\frac{14}{3}\right] = 2\pi i\left[\frac{25}{6}\right] = \frac{25\pi i}{3}$

20. Evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle $|z| = 3$

[M15/ElexExtcElectBiomInst/6M]

Solution:

We have, $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

For singularity,

$(z-1)(z-2) = 0$

$\therefore z = 1, z = 2$

We see that $z = 1$ and $z = 2$ both lies inside $C: |z| = 3$ and hence are simple poles.

Residue of $f(z)$ at $(z = 1) = \lim_{z \to 1} (z-1) f(z)$

$$= \lim_{z \to 1} (z-1) \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$$

$$= \lim_{z \to 1} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}$$

$$= \frac{\sin \pi + \cos \pi}{(1-2)}$$

$$= \frac{0-1}{-1} = 1$$

Residue of $f(z)$ at $(z = 2) = \lim_{z \to 2} (z-2) f(z)$

$$= \lim_{z \to 2} (z-2) \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$$

$$= \lim_{z \to 2} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}$$

$$= \frac{\sin 4\pi + \cos 4\pi}{(2-1)}$$

$$= \frac{0+1}{1} = 1$$

By Cauchy's Residue Theorem,

$$\int_c f(z) dz = 2\pi i\, [sum\ of\ residues]$$

$$\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i\, [1 + 1] = 4\pi i$$

21. Using Cauchy's Residue theorem evaluate $\int_c \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)^2(z-2)} dz$ where C is the circle $|z| = 3$

[N18/Elex/6M]

Solution:

We have, $f(z) = \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)^2(z-2)}$

For singularity,

$(z-1)^2(z-2) = 0$

$\therefore z = 1,1, z = 2$

We see that $z = 1$ and $z = 2$ both lies inside $C: |z| = 3$ and hence $z = 1$ is a pole of order 2 and $z = 2$ is a simple pole

Residue of $f(z)$ at $(z = 1) = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}[(z-1)^2 f(z)]$

$$= \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)^2(z-2)}\right]$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{\sin \pi z^2+\cos \pi z^2}{(z-2)}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z-2)(\cos \pi z^2 \times 2\pi z - \sin \pi z^2 \times 2\pi z) - (\sin \pi z^2+\cos \pi z^2)(1)}{(z-2)^2}\right]$$

$$= \frac{(-1)(\cos \pi \times 2\pi) - (\cos \pi)}{(1-2)^2}$$

$$= 2\pi + 1$$

Residue of $f(z)$ at $(z = 2) = \lim_{z\to 2}(z-2)f(z)$

$$= \lim_{z\to 2}(z-2)\frac{\sin \pi z^2+\cos \pi z^2}{(z-1)^2(z-2)}$$

$$= \lim_{z\to 2}\frac{\sin \pi z^2+\cos \pi z^2}{(z-1)^2}$$

$$= \frac{\sin 4\pi + \cos 4\pi}{(2-1)^2} = \frac{0+1}{1} = 1$$

By Cauchy's Residue Theorem,

$$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$\int_c \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)^2(z-2)} dz = 2\pi i\ [2\pi + 1 + 1] = 2\pi i(2\pi + 2)$$

22. Using Cauchy's Residue theorem evaluate $\int_c \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)^2(z-1)} dz$ where C is the circle $|z| = 3$

[D23/ElectECS/6M]

Solution:

We have, $f(z) = \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)^2(z-1)}$

For singularity,

$(z-2)^2(z-1) = 0$

$\therefore z = 2,2, z = 1$

We see that $z = 1$ and $z = 2$ both lies inside $C: |z| = 3$ and hence $z = 2$ is a pole of order 2 and $z = 1$ is a simple pole

Residue of $f(z)$ at $(z = 2) = \frac{1}{1!}\lim_{z\to 2}\frac{d}{dz}[(z-2)^2 f(z)]$

$= \lim_{z\to 2}\frac{d}{dz}\left[(z-2)^2 \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)^2(z-1)}\right]$

$= \lim_{z\to 2}\frac{d}{dz}\left[\frac{\sin \pi z^2+\cos \pi z^2}{(z-1)}\right]$

$= \lim_{z\to 2}\left[\frac{(z-1)(\cos \pi z^2 \times 2\pi z - \sin \pi z^2 \times 2\pi z) - (\sin \pi z^2+\cos \pi z^2)(1)}{(z-1)^2}\right]$

$= \frac{(1)(\cos 4\pi \times 4\pi) - (\cos 4\pi)}{(2-1)^2}$

$= 4\pi - 1$

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)f(z)$

$= \lim_{z\to 1}(z-1)\frac{\sin \pi z^2+\cos \pi z^2}{(z-2)^2(z-1)}$

$= \lim_{z\to 1}\frac{\sin \pi z^2+\cos \pi z^2}{(z-2)^2}$

$= \frac{\sin \pi+\cos \pi}{(1-2)^2} = \frac{0-1}{1} = -1$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$

$\int_c \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)^2(z-1)} dz = 2\pi i\,[4\pi - 1 - 1] = 2\pi i(4\pi - 2)$

23. Evaluate $\int_c \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)(z-3)} dz$ where C is the circle (i) $|z| = 1$ (ii) $|z| = 4$

[N22/Extc/6M]

Solution:

We have, $f(z) = \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)(z-3)}$

For singularity,

$(z-2)(z-3) = 0$

$\therefore z = 2, z = 3$

(i) $|z| = 1$

We see that $z = 2$ and $z = 3$ both lies outside $C: |z| = 1$

By Cauchy's theorem

$\int_c \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)(z-3)} dz = 0$

(ii) $|z| = 4$

We see that $z = 2$ and $z = 3$ both lies inside $C: |z| = 4$ and hence they are simple poles

Residue of $f(z)$ at $(z = 2) = \lim_{z\to 2}(z-2)f(z)$

$= \lim_{z\to 2}(z-2)\frac{\sin \pi z^2+\cos \pi z^2}{(z-2)(z-3)}$

$= \lim_{z\to 2}\frac{\sin \pi z^2+\cos \pi z^2}{(z-3)}$

$= \frac{\sin 4\pi+\cos 4\pi}{(2-3)}$

$= \frac{0+1}{-1} = -1$

Residue of $f(z)$ at $(z = 3) = \lim_{z\to 3}(z-3)f(z)$

$= \lim_{z\to 3}(z-3)\frac{\sin \pi z^2+\cos \pi z^2}{(z-2)(z-3)}$

$= \lim_{z\to 3}\frac{\sin \pi z^2+\cos \pi z^2}{(z-2)}$

$= \frac{\sin 9\pi+\cos 9\pi}{(3-2)}$

$= \frac{0-1}{1} = -1$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\int_c \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)(z-3)} dz = 2\pi i\ [-1-1] = -4\pi i$

24. By using Cauchy's Residue theorem evaluate $\int_c \frac{\sin \pi z+\cos \pi z}{(z-1)(z-2)}dz$ where C is the circle $|z| = 3$

[M19/Inst/6M][M19/Biom/6M]

Solution:

We have, $f(z) = \frac{\sin \pi z+\cos \pi z}{(z-1)(z-2)}$

For singularity,

$(z-1)(z-2) = 0$

$\therefore z = 1, z = 2$

We see that $z = 1$ and $z = 2$ both lies inside $C: |z| = 3$ and hence $z = 1$ and $z = 2$ are simple poles

Residue of $f(z)$ at $(z = 2) = \lim_{z\to 2}(z-2)f(z)$

$$= \lim_{z\to 2}(z-2)\frac{\sin \pi z+\cos \pi z}{(z-1)(z-2)}$$

$$= \lim_{z\to 2}\frac{\sin \pi z+\cos \pi z}{(z-1)}$$

$$= \frac{\sin 2\pi+\cos 2\pi}{1} = \frac{0+1}{1} = 1$$

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}(z-1)\frac{\sin \pi z+\cos \pi z}{(z-1)(z-2)}$$

$$= \lim_{z\to 1}\frac{\sin \pi z+\cos \pi z}{(z-2)}$$

$$= \frac{\sin \pi+\cos \pi}{-1} = \frac{0-1}{-1} = 1$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\int_c \frac{\sin \pi z+\cos \pi z}{(z-1)(z-2)}dz = 2\pi i\ [1+1] = 4\pi i$

25. Evaluate $\int_C \frac{dz}{(z^2-4)(z+4)}$ where C is circle $|z| = 4$ using Cauchy's residue theorem
[M22/Elect/5M]
Solution:
We have, $f(z) = \frac{1}{(z^2-4)(z+4)}$
For singularity,
$(z^2 - 4)(z + 4) = 0$
$\therefore z = -2, 2, -4$
We see that $z = -2$ and $z = 2$ both lies inside $C: |z| = 4$ and hence $z = -2$ and $z = 2$ are simple poles
Residue of $f(z)$ at $(z = 2) = \lim_{z\to 2}(z-2)f(z)$

$$= \lim_{z\to 2}(z-2)\frac{1}{(z-2)(z+2)(z+4)}$$
$$= \lim_{z\to 2}\frac{1}{(z+2)(z+4)}$$
$$= \frac{1}{24}$$

Residue of $f(z)$ at $(z = -2) = \lim_{z\to -2}(z+2)f(z)$

$$= \lim_{z\to -2}(z+2)\frac{1}{(z-2)(z+2)(z+4)}$$
$$= \lim_{z\to -2}\frac{1}{(z-2)(z+4)}$$
$$= \frac{1}{-8}$$

By Cauchy's Residue Theorem,
$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$
$\int_c \frac{1}{(z^2-4)(z+4)}dz = 2\pi i\left[\frac{1}{24} - \frac{1}{8}\right] = -\frac{\pi i}{6}$

26. By using Cauchy's Residue theorem evaluate $\int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{2}\right)^3} dz$ where C is the circle $|z| = 2$

[N19/Elect/6M]

Solution:

We have, $f(z) = \frac{\sin^6 z}{\left(z-\frac{\pi}{2}\right)^3}$

For singularity,

$\left(z - \frac{\pi}{2}\right)^3 = 0$

$\therefore z = \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$

We see that $z = \frac{\pi}{2}$ lies inside C and hence $z = \frac{\pi}{2}$ is a pole of order 3

Residue of $f(z)$ at $\left(z = \frac{\pi}{2}\right) = \frac{1}{2!} \lim_{z \to \frac{\pi}{2}} \frac{d^2}{dz^2} \left[\left(z - \frac{\pi}{2}\right)^3 f(z)\right]$

$$= \frac{1}{2} \lim_{z \to \frac{\pi}{2}} \frac{d^2}{dz^2} \left[\left(z - \frac{\pi}{2}\right)^3 \frac{\sin^6 z}{\left(z-\frac{\pi}{2}\right)^3}\right]$$

$$= \frac{1}{2} \lim_{z \to \frac{\pi}{2}} \frac{d^2}{dz^2} [\sin^6 z]$$

$$= \frac{1}{2} \lim_{z \to \frac{\pi}{2}} \frac{d}{dz} [6 \sin^5 z \times \cos z]$$

$$= \frac{1}{2} \lim_{z \to \frac{\pi}{2}} [6 \sin^5 z\,(-\sin z) + \cos z\,(30 \sin^4 z \times \cos z)]$$

$$= \frac{1}{2}[-6] = -3$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{2}\right)^3} dz = 2\pi i\,[-3] = -6\pi i$

27. Using Cauchy's residue theorem evaluate $\oint_c \frac{z^2+3}{z^2-1} dz$ where C is the circle (i) $|z-1| = 1$ (ii) $|z+1| = 1$

[N16/ElexExtcElectBiomInst/8M]

Solution:

We have, $f(z) = \frac{z^2+3}{z^2-1}$

For singularity,

$z^2 - 1 = 0$

$(z-1)(z+1) = 0$

$\therefore z = 1, -1$

(i) C is $|z-1| = 1$

We see that, $z = 1$ is a simple pole and $z = -1$ lies outside C

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}(z-1)\frac{z^2+3}{(z+1)(z-1)}$$

$$= \lim_{z\to 1}\frac{z^2+3}{z+1}$$

$$= \frac{4}{2} = 2$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\oint_c \frac{z^2+3}{z^2-1} dz = 2\pi i\ (2) = 4\pi i$

(ii) C is $|z+1| = 1$

We see that, $z = -1$ is a simple pole and $z = 1$ lies outside C

Residue of $f(z)$ at $(z = -1) = \lim_{z\to -1}(z+1)f(z)$

$$= \lim_{z\to -1}(z+1)\frac{z^2+3}{(z+1)(z-1)}$$

$$= \lim_{z\to -1}\frac{z^2+3}{z-1}$$

$$= \frac{4}{-2} = -2$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\oint_c \frac{z^2+3}{z^2-1} dz = 2\pi i\ (-2) = -4\pi i$

28. Evaluate using Cauchy's Residue theorem $\oint_c \frac{1-2z}{z(z-1)(z-2)} dz$ where c is circle $|z| = 1.5$

[N15/CompIT/6M][N22/CompITAI/6M]

Solution:

We have, $f(z) = \frac{1-2z}{z(z-1)(z-2)}$

For singularity,

$z(z-1)(z-2) = 0$

$\therefore z = 0, z = 1, z = 2$

We see that $z = 0$ and $z = 1$ both lies inside $C: |z| = 1.5$ and hence are simple poles.

Residue of $f(z)$ at $(z = 0) = \lim_{z\to 0}(z-0)f(z)$

$$= \lim_{z\to 0}(z-0)\frac{1-2z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 0}\frac{1-2z}{(z-1)(z-2)}$$

$$= \frac{1-0}{(0-1)(0-2)}$$

$$= \frac{1}{2}$$

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}(z-1)\frac{1-2z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 1}\frac{1-2z}{z(z-2)}$$

$$= \frac{1-2(1)}{1(1-2)}$$

$$= \frac{-1}{-1} = 1$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\int_c \frac{1-2z}{z(z-1)(z-2)} dz = 2\pi i\ \left[\frac{1}{2} + 1\right] = 2\pi i\left[\frac{3}{2}\right] = 3\pi i$

29. Using Residue theorem, evaluate $\int_c \frac{e^z}{z^2+\pi^2} dz$ where c is $|z| = 4$

[M17/ElexExtcElectBiomInst/6M]

Solution:

We have, $f(z) = \frac{e^z}{z^2+\pi^2}$

For singularity,

$z^2 + \pi^2 = 0$

$(z + \pi i)(z - \pi i) = 0$

$\therefore z = \pm \pi i$

$\therefore z = \pi i$ and $z = -\pi i$ are simple poles

Residue of $f(z)$ at $(z = \pi i) = \lim_{z\to\pi i}(z - \pi i) f(z)$

$$= \lim_{z\to\pi i}(z - \pi i)\frac{e^z}{(z+\pi i)(z-\pi i)}$$

$$= \lim_{z\to\pi i}\frac{e^z}{z+\pi i}$$

$$= \frac{e^{\pi i}}{2\pi i}$$

Residue of $f(z)$ at $(z = -\pi i) = \frac{e^{-\pi i}}{-2\pi i}$

By Cauchy's Residue Theorem,

$\int_c f(z) dz = 2\pi i\ [sum\ of\ residues]$

$\oint_c \frac{e^z}{z^2+\pi^2} dz = 2\pi i \left(\frac{e^{\pi i} - e^{-\pi i}}{2\pi i}\right) = 2i \sin\pi = 0$

30. Using Residue theorem, evaluate $\int_c \frac{e^z}{(z^2+\pi^2)^2} dz$ where c is $|z| = 4$

[N16/MechCivil/6M][N18/Extc/6M]

Solution:

We have, $f(z) = \frac{e^z}{(z^2+\pi^2)^2}$

For singularity,

$(z^2 + \pi^2)^2 = 0$

$(z + \pi i)^2 (z - \pi i)^2 = 0$

$\therefore z = \pm\pi i, \pm\pi i$

$\therefore z = \pi i$ and $z = -\pi i$ are poles of order 2

Residue of $f(z)$ at $(z = \pi i) = \frac{1}{1!} \lim_{z\to\pi i} \frac{d}{dz} [(z - \pi i)^2 f(z)]$

$$= \lim_{z\to\pi i} \frac{d}{dz} \left[(z - \pi i)^2 \frac{e^z}{(z+\pi i)^2 (z-\pi i)^2}\right]$$

$$= \lim_{z\to\pi i} \frac{d}{dz} \left[\frac{e^z}{(z+\pi i)^2}\right]$$

$$= \lim_{z\to\pi i} \left[\frac{(z+\pi i)^2 (e^z) - e^z.2(z+\pi i)}{(z+\pi i)^4}\right]$$

$$= \frac{e^{\pi i}(2\pi i - 2)}{(2\pi i)^3}$$

$$= \frac{(\cos\pi + i\sin\pi)(2\pi i - 2)}{8\pi^3 i^3}$$

$$= \frac{(-1)(2\pi i - 2)}{-8\pi^3 i}$$

$$= \frac{\pi i - 1}{4\pi^3 i}$$

$\therefore$Residue of $f(z)$ at $(z = -\pi i) = \frac{-\pi i - 1}{-4\pi^3 i} = \frac{\pi i + 1}{4\pi^3 i}$

By Cauchy's Residue Theorem,

$\int_c f(z) dz = 2\pi i\ [sum\ of\ residues]$

$\oint_c \frac{e^z}{(z^2+\pi^2)^2} dz = 2\pi i \left(\frac{\pi i + 1}{4\pi^3 i} + \frac{\pi i - 1}{4\pi^3 i}\right) = 2\pi i \left[\frac{2\pi i}{4\pi^3 i}\right] = \frac{i}{\pi}$

31. Using Cauchy Residue Theorem evaluate $\int_C \frac{e^{2z}}{(z-\pi i)^3} dz$, C is $|z-2i| = 4$

[M24/D24/Extc/6M]

Solution:

We have, $f(z) = \frac{e^{2z}}{(z-\pi i)^3}$

For singularity,

$(z-\pi i)^3 = 0$

$\therefore z = \pi i, \pi i, \pi i$

$\therefore z = \pi i$ is a pole of order 3

Residue of $f(z)$ at $(z = \pi i) = \frac{1}{2!} \lim_{z\to\pi i} \frac{d^2}{dz^2}[(z-\pi i)^3 f(z)]$

$$= \frac{1}{2}. \lim_{z\to\pi i} \frac{d^2}{dz^2}\left[(z-\pi i)^3 \frac{e^{2z}}{(z-\pi i)^3}\right]$$

$$= \frac{1}{2}. \lim_{z\to\pi i} \frac{d^2}{dz^2}[e^{2z}]$$

$$= \frac{1}{2}. \lim_{z\to\pi i}[e^{2z} \times 2^2]$$

$$= \frac{1}{2}.\left[4e^{2\pi i}\right]$$

$$= 2(\cos 2\pi + i \sin 2\pi)$$

$$= 2(1+0)$$

$$= 2$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\oint_c \frac{e^{2z}}{(z-\pi i)^3} dz = 2\pi i\ (2) = 4\pi i$

32. Evaluate the integral $\int_c \frac{1}{(z^2+1)(z^2+4)} dz$, $C: |z-2i| = 2$

[M23/ElectECS/5M]

Solution:

We have, $f(z) = \frac{1}{(z^2+1)(z^2+4)}$

For singularity,

$(z^2+1)(z^2+4) = 0$

$z^2 = -1, z^2 = -4$

$\therefore z = \pm i, z = \pm 2i$

We see that $z = i, 2i$ are simple poles since they lie inside C

Residue of $f(z)$ at $(z = i) = \lim_{z\to i}(z-i)f(z)$

$$= \lim_{z\to i}(z-i).\frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$$

$$= \lim_{z\to i}\frac{1}{(z+i)(z+2i)(z-2i)}$$

$$= \frac{1}{(2i)(3i)(-i)} = -\frac{i}{6}$$

Residue of $f(z)$ at $(z = 2i) = \lim_{z\to 2i}(z-2i)f(z)$

$$= \lim_{z\to 2i}(z-2i).\frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$$

$$= \lim_{z\to 2i}\frac{1}{(z+i)(z-i)(z+2i)}$$

$$= \frac{1}{(3i)(i)(4i)} = -\frac{i}{12}$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\int_c \frac{1}{(z^2+1)(z^2+4)} dz = 2\pi i\left(-\frac{i}{6} - \frac{i}{12}\right) = 2\pi i\left[-\frac{i}{4}\right] = \frac{\pi}{2}$

33. Evaluate $\int_C \frac{4z^2+1}{(2z-3)(z+1)^2} dz, C: |z| = 4$ using Cauchy's residue theorem

[M22/CompITAI/5M]

Solution:

We have, $f(z) = \frac{4z^2+1}{(2z-3)(z+1)^2}$

For singularity,

$(2z-3)(z+1)^2 = 0$

$\therefore z = \frac{3}{2}, -1, -1$

$\therefore z = \frac{3}{2}$ is a simple pole and $z = -1$ is a pole of order 2

Residue of $f(z)$ at $\left(z = \frac{3}{2}\right) = \lim_{z \to \frac{3}{2}} \left(z - \frac{3}{2}\right) f(z)$

$$= \lim_{z \to \frac{3}{2}} \left(z - \frac{3}{2}\right) \frac{4z^2+1}{(2z-3)(z+1)^2}$$

$$= \lim_{z \to \frac{3}{2}} \frac{4z^2+1}{2(z+1)^2}$$

$$= \frac{4\left(\frac{3}{2}\right)^2+1}{2\left(\frac{3}{2}+1\right)^2} = \frac{4}{5}$$

Residue of $f(z)$ at $(z = -1) = \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz}[(z+1)^2 f(z)]$

$$= \lim_{z \to -1} \frac{d}{dz}\left[(z+1)^2 \frac{4z^2+1}{(2z-3)(z+1)^2}\right]$$

$$= \lim_{z \to -1} \frac{d}{dz}\left[\frac{4z^2+1}{(2z-3)}\right]$$

$$= \lim_{z \to -1} \left[\frac{(2z-3)(8z)-(4z^2+1)(2)}{(2z-3)^2}\right]$$

$$= \frac{(-5)(-8)-(5)(2)}{(-5)^2}$$

$$= \frac{30}{25} = \frac{6}{5}$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\oint_c \frac{4z^2+1}{(2z-3)(z+1)^2} dz = 2\pi i \left(\frac{4}{5} + \frac{6}{5}\right) = 2\pi i[2] = 4\pi i$

34. Using residue theorem evaluate $\int_C \frac{3z^2+z}{z^2-1}\,dz$ where C is the circle $|z| = 2$

[M22/Elex/5M]

Solution:

We have, $f(z) = \frac{3z^2+z}{z^2-1}$

For singularity,

$z^2 - 1 = 0$

$(z-1)(z+1) = 0$

$\therefore z = 1, -1$

We see that, $z = 1$ is a simple pole and $z = -1$ is also a simple pole

Residue of $f(z)$ at $(z = 1) = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}(z-1)\frac{3z^2+z}{(z+1)(z-1)}$$

$$= \lim_{z\to 1}\frac{3z^2+z}{z+1}$$

$$= \frac{4}{2} = 2$$

Residue of $f(z)$ at $(z = -1) = \lim_{z\to -1}(z+1)f(z)$

$$= \lim_{z\to -1}(z+1)\frac{3z^2+z}{(z+1)(z-1)}$$

$$= \lim_{z\to -1}\frac{3z^2+z}{z-1}$$

$$= \frac{2}{-2} = -1$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\;[sum\ of\ residues]$

$\oint_c \frac{3z^2+z}{z^2-1}dz = 2\pi i\,(2-1) = 2\pi i$

35. By using Cauchy Residue theorem, evaluate $\int_c \frac{\sin^3 z}{\left(z-\frac{\pi}{6}\right)^2} dz$ where C is the circle $|z| = 2$

[D23/Extc/6M]

Solution:

We have, $f(z) = \frac{\sin^3 z}{\left(z-\frac{\pi}{6}\right)^2}$

For singularity,

$\left(z - \frac{\pi}{6}\right)^2 = 0$

$\therefore z = \frac{\pi}{6}, \frac{\pi}{6}$

We see that $z = \frac{\pi}{6}$ lies inside C and hence $z = \frac{\pi}{6}$ is a pole of order 2

Residue of $f(z)$ at $\left(z = \frac{\pi}{6}\right) = \frac{1}{1!} \lim_{z \to \frac{\pi}{6}} \frac{d}{dz}\left[\left(z - \frac{\pi}{6}\right)^2 f(z)\right]$

$$= \lim_{z \to \frac{\pi}{6}} \frac{d}{dz}\left[\left(z - \frac{\pi}{6}\right)^2 \frac{\sin^3 z}{\left(z-\frac{\pi}{6}\right)^2}\right]$$

$$= \lim_{z \to \frac{\pi}{6}} \frac{d}{dz}[\sin^3 z]$$

$$= \lim_{z \to \frac{\pi}{6}}[3 \sin^2 z \times \cos z]$$

$$= 3\left(\frac{1}{2}\right)^2 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{2}\right)^3} dz = 2\pi i \left[\frac{3\sqrt{3}}{8}\right] = \frac{3\sqrt{3}\pi i}{4}$

36. Evaluate the following integral using Cauchy Residue theorem $\int_C \frac{1}{z^5}.e^{z^2}dz$, $|z| = 1$

[M24/ElectECS/6M]

Solution:

$I = \int_C \frac{1}{z^5}.e^{z^2}dz$

Here,

$f(z) = \frac{e^{z^2}}{z^5}$

$f(z) = \frac{1}{z^5}\left[1 + z^2 + \frac{(z^2)^2}{2!} + \frac{(z^2)^3}{3!} + \frac{(z^2)^4}{4!} + \cdots\right]$

$f(z) = \frac{1}{z^5} + \frac{z^2}{z^5} + \frac{z^4}{2z^5} + \frac{z^6}{6z^5} + \frac{z^8}{24z^5} + \cdots$ …

$f(z) = \frac{1}{z^5} + \frac{1}{z^3} + \frac{1}{2z} + \frac{z}{6} + \frac{z^3}{24} + \cdots$ ..

Residue = coefficient of $\frac{1}{z} = \frac{1}{2}$

By Cauchy's Residue Theorem,

$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$

$\int_C \frac{1}{z^5}.e^{z^2}dz = 2\pi i\ \left[\frac{1}{2}\right] = \pi i$