EM - III

SE IT/comp R-2019 paper solution

Dec - 2021

MCQ

1. Laplace transform of $e^{-5t}(t^2 + \sin 2t)$

$\rightarrow$ (a) $\dfrac{2}{(s+3)^2} + \dfrac{2}{(s+5)^2 + 2^2}$

2. If $L(F(t)) = \dfrac{3s}{s^2+1}$, then $L\{F(2t)\}$ at $s=1$ is

$\rightarrow$ (a) $3/5$

3) Inverse Laplace transform of $\cancel{f(s)}\ \dfrac{1}{s^2+4}$ is

$\rightarrow$ (a) $\int_0^t \cos 2u \, du$

4) Inverse Laplace transform of $f(s) = \dfrac{6e^{-5s}}{(s+4)^4}$ is

$\rightarrow$ (a) $f(t) = \begin{cases} 0 & 0<t<5 \\ e^{-2(t-5)}(t-5)^3 & t>5 \end{cases}$

5) If $f(z) = u(x,y) + iv(x,y)$, is analytic then $f'(z)$ is equal

$\rightarrow$ (b) $\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

6) The value of 'm' so that $2x - x^2 + my^2$ is harmonic,

$\rightarrow$ (c) 1

7) The value of coefficient of correlation lies between

$\rightarrow$ (d) -1 to 1

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8) The rank correlation coefficient of following data

X	23	25	27	29	31	33
Y	43	45	47	49	51	53

→ (c) 1

9) Expansion of fourier series of $f(x) = x$ in $(-1, 1)$ is

→ (d) $f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x$

10) What sh would be the expectation of the number of failures preceding the first success in an infinite series of independent trials with the constant probability of success p and failure q

→ (b) $\frac{q}{p}$

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Q.2 Solve any four

A] Find laplace transform of $e^{-3t} t \sqrt{1-\sin 2t}$

$\to L\{\sqrt{1-\sin 2t}\} = L\{\sqrt{(\sin t \mp \cos t)^2}\}$

$= \frac{1}{s^2+1} \mp \frac{s}{s^2+1}$

$\therefore L\{t\sqrt{1-\sin 2t}\} = (-1)\left[\frac{d}{ds}\left(\frac{1}{s^2+1} \mp \frac{s}{s^2+1}\right)\right]$

$= -\left[\frac{-1}{(s^2+1)^2}(2s) \mp \left(\frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right)\right]$

$= \frac{2s}{(s^2+1)^2} \mp \frac{s^2+1}{(s^2+1)^2}$

$L\{e^{-3t} t\sqrt{1-\sin 2t}\} = \frac{-(s+3)^2 \mp 2(s+3) \mp 1}{((s+3)^2+1)^2} = \frac{-(s^2+4s+2)}{(s^2+6s+10)^2}$

B] Find inverse laplace Transform of $\frac{5s^2-15s-11}{(s+1)(s-2)^2}$

$\to \frac{5s^2-15s-11}{(s+1)(s-2)^2} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$

$= A(s-2)^2 + B(s+1)(s-2) + C(s+1)$

$= A(s^2-4s+4) + B(s^2-s-2) + C(s+1)$

$5s^2-15s-11 = (A+B)s^2 + (-4A-B+C)s + (4A-2B+C)$

$\Rightarrow A+B=5, \quad -4A-B+C=-15, \quad 4A-2B+C=-11$

$-4A-B+C=-15$

$-4A+2B-C=+11$

$-8A+B=-4$

$A+B=5$

$-9A=-9 \Rightarrow \boxed{A=1} \Rightarrow \boxed{B=4}$

$\boxed{C=-7}$

$-8A+B=-4$

$+\ 5A+5B=25$

$-3A=21 \quad \boxed{A=-7}$

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$$\therefore \frac{5s^2-15s-11}{(s+1)(s-2)^2} = \frac{1}{s+1} + \frac{4}{s-2} - \frac{7}{(s-2)^2}$$

$$\therefore L^{-1}\left\{ \qquad \right\} = e^{-t} + 4e^{2t} - 7e^{2t} L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^{-t} + 4e^{2t} - 7e^{2t}\frac{t}{1!}$$

c) Expand Fourier series for $f(x) = \frac{1}{2}(\pi - x)$ in $(0, 2\pi)$

$\rightarrow$ $f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$

where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \boxed{0}$

$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$, $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$

$\qquad \| \qquad\qquad\qquad\qquad \|$

$\qquad \boxed{0} \qquad\qquad\qquad \boxed{1/n}$

D) find constants a, b, c, d and e if
$(ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$
is analytic

$\rightarrow$ $u = ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2$

$\therefore u_x = 4ax^3 + 2bxy^2 + 2dx$

$u_y = 2bx^2y + 4cy^3 - 4y$

$V = 4x^3y - exy^3 + 4xy$

$\therefore V_{\cancel{y}} = 12x^2y - \cancel{3exy} \; ey^3 + 4y$

$\therefore V_y = 4x^3 - 3exy^2 + 4x$

Since $u_x = V_y$ $\qquad\qquad$ $u_y = -V_x$

$\boxed{a=1}$, $2b = -3e$, $\boxed{d=2}$ $\qquad$ $\boxed{b=-6}$, $4c = e$

$-12 = -3e$ $\qquad\qquad\qquad\qquad$ $\boxed{c=1}$

$\boxed{e=4}$

E) Ten students got the following percentage of marks in mathematics and statistics

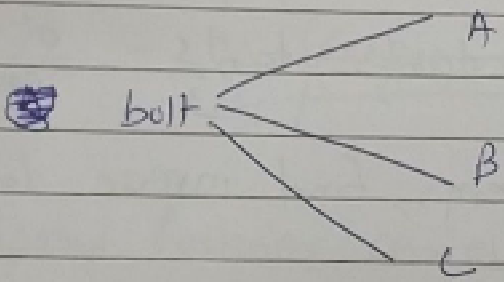
maths	78	36	98	25	75	82	90	62	65	39
Stats	84	51	91	60	68	62	86	58	53	47

Calculate the coef of correlation

→

$$r = \frac{\frac{\Sigma xy}{n} - \frac{\Sigma x}{n} \cdot \frac{\Sigma y}{n}}{\sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} \sqrt{\frac{\Sigma y^2}{n} - \left(\frac{\Sigma y}{n}\right)^2}} = 0.7804$$

F)



A: bolt is manufactured by machine A
B: bolt is manufactured by B
C: bolt is manufactured by C
$\therefore P(A) = 1/2$, $P(B) = 1/4$, $P(C) = 1/4$

E: bolt is defective
$P(E|A) = 0.03$ $P(E|B) = 0.03$, $P(E|C) = 0.05$

$\therefore P(E) = P(E|A) \cdot P(A) + P(E|B) \cdot P(B) + P(E|C) \cdot P(C)$
$= 0.03 \times 1/2 + 0.03 \times 1/4 + 0.05 \times 1/4$

$= 0.035$

Q3) A) By Using Laplace Transform, evaluate $\int_0^\infty \frac{\sin 2t + \sin 3t}{te^t}$

$\rightarrow \int_0^\infty e^{-t} \frac{\sin 2t + \sin 3t}{t} dt = L\left\{ \frac{\sin 2t + \sin 3t}{t} \right\}\Big|_{s=1}$

Now $L\{\sin 2t + \sin 3t\} = \frac{2}{s^2+4} + \frac{3}{s^2+9}$

$\therefore L\left\{ \frac{\sin 2t + \sin 3t}{t} \right\} = \int_s^\infty \left[\frac{2}{s^2+4} + \frac{3}{s^2+9} \right] ds$

$= \left[2\left(\frac{1}{2} \tan^{-1}\frac{s}{2}\right) + 3 \cdot \frac{1}{3} \tan^{-1}\frac{s}{3} \right.$

$= \left[\tan^{-1}\infty + \tan^{-1}\infty - \tan^{-1}\frac{s}{2} - \tan^{-1}\frac{s}{3} \right.$

$= \pi - \tan^{-1}\frac{s}{2} - \tan^{-1}\frac{s}{3}$ at $s = 1$

ans $= \frac{3\pi}{4}$

B) Using convolution theorem, find inverse Laplace transform of $\frac{s}{(s^2+1)(s^2+4)}$

$\rightarrow$ 1) $\phi(s) = \frac{s}{s^2+4} \cdot \frac{1}{s^2+4}$

$F(s) \cdot G(s)$

1) $L^{-1}\{F(s)\} = \cos t = f(t)$

2) $L^{-1}\{G(s)\} = \frac{1}{2}\sin 2t = g(t)$

3) $f(u) = \cos u$

$g(t-u) = \frac{1}{2}\sin 2(t-u)$

4) By convolution th^m

$L^{-1}\{\phi(s)\} = \int_0^t f(u)\, g(t-u)\, du$

$= \frac{1}{2}\int_0^t \cos u \sin 2(t-u)\, du$

$$= \frac{1}{2}\int_0^t \frac{1}{2}\left[\sin(2t-2u+u) + \sin(2t-2u-2u)\right] du$$

$$= \frac{1}{4}\left[\cancel{\frac{\cos 2t}{2}(\sin 2t)\cdot u} + \cancel{\sin}\frac{(2t-3u)}{-3}\right.$$

$$= \frac{1}{4}\left[\frac{+\cos(2t-u)}{\cancel{(\cos(2t))\cdot u}\ -1} - \frac{\cos(2t-3u)}{-3}\right]_0^t$$

$$= \frac{1}{4}\left\{\left[\cancel{(\sin 2t)}^{\cos t}\cdot t + \frac{\cos(-2t)}{-3}\right] - \left[\cos 2t - \frac{\cos 2t}{-3}\right]\right\}$$

$$= \frac{1}{4}\left\{\cancel{(\sin 2t)\cdot t}^{\cos t} - \frac{\cos 2t}{3} - 4\frac{\cos 2t}{3}\right\}$$

$$\cancel{\cos t} - \frac{\cos 2t}{3} = \frac{1}{4}\left\{\cancel{(\sin 2t)\cdot t - \frac{\cos 2t}{2}}\right\} //$$

c) Expand Fourier series for $f(x) = 1-x^2$ in $(-1,1)$

$\boxed{l=1}$

$\rightarrow$ $f(x) = 1-x^2$ is even function

$$\therefore f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{1} + \sum b_n \sin\frac{n\pi x}{1}$$

$$a_0 = \frac{2}{1}\int_0^1 (1-x^2)\,dx = 4/3$$

$$a_n = \frac{2}{1}\int_0^1 (1-x^2)\cos\frac{n\pi x}{1}\,dx$$

$$= 2\int_0^1 (1-x^2)\cos n\pi x\,dx$$

$$= 2\left[(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + (-2)\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right.$$

$$= 2\left[\left(0 - 2\frac{\cos n\pi}{n\pi} + 0\right) - \left(0 + 0 + 0\right)\right.$$

$$= \frac{-4(-1)^n}{(n\pi)^2}$$

$b_n = 0$

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(d) Find the analytic function $f(z) = u+iv$ in terms of z, if $v = \dfrac{\sinh 2y}{\cosh 2y + \cos 2x}$

$\rightarrow V_x = \sinh 2y \left(-\dfrac{1}{(\cosh 2y + \cos 2x)^2} (-2\sin 2x) \right)$

$V_x(z,0) = 0$

& $V_y = \dfrac{(\cosh 2y + \cos 2x)(2\cosh 2y) - (\sinh 2y)(2\sinh 2y)}{(\cosh 2y + \cos 2x)^2}$

$\therefore V_y(z,0) = \dfrac{(1 + \cos 2z)(2) - 0}{(1+\cos 2z)^2}$

$= \dfrac{2}{(1+\cos 2z)}$

Now $f'(z) = u_x + iV_x$

$= V_y + iV_x$

$= \dfrac{2}{(1+\cos 2z)}$

$\therefore f(z) = 2\int \dfrac{1}{2\cos^2 z}\, dz + C$

$= \int \sec^2 z\, dz + C$

$\boxed{f(z) = \tan z + C}$

5) Obtain the equations of lines of regression for the following data

X	65	66	67	67	68	69	70	72
Y	67	68	65	68	72	72	69	71

$Y = 0.6667X + 23.6667$

$X = 0.54545Y + 30.3636$

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=) A random variable X has the following probability distribution

X	-2	-1	0	1	2	3
P	0.1	K	0.1	2K	0.2	3K

(i) find the constant k (ii) find the mean & variance of X

=) $\sum P_i = 1$

$\therefore 0.1 + k + 0.1 + 2k + 0.2 + 3k = 1 \Rightarrow \boxed{k = 0.1}$

$\therefore$ X	-2	-1	0	1	2	3
P	0.1	0.1	0.1	0.2	0.2	0.3

) mean $= \sum xp = -2\times0.1 + -1\times0.1 + 0\times0.1 + 1\times0.2 + 2\times0.2 + 3\times0.3$

$= -0.2 - 0.1 + 0 + 0.2 + 0.4 + 0.9$

$= 1.2$

Now

$E(X^2) = \sum x^2 p = 4\times0.1 + 1\times0.1 + 0\times0.1 + 1\times0.2 + 4\times0.2 + 9\times0.3$

$= 0.4 + 0.1 + 0.2 + 0.8 + 2.7$

$= 0.5 + 3.1$

$= 4.2$

Variance $= E(x^2) - (E(x))^2 = 4.2 - (1.2)^2 = 4.2 - 1.44$

$= 2.76$

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Q4)

A) Find Laplace Transform of $\int_0^t e^{-2u}\cos^2 u \, du$

$\rightarrow \cos^2 t = \frac{1+\cos 2t}{2}$

$\therefore L\{\cos^2 t\} = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)$

$L\{e^{-2t}\cos^2 t\} = \frac{1}{2}\left(\frac{1}{s+2} + \frac{s+2}{(s+2)^2+4}\right)$

$L\{\int_0^t e^{-2u}\cos^2 u \, du\} = \frac{1}{2s}\left(\frac{1}{s+2} + \frac{s+2}{s^2+4s+8}\right)$ //

B) Find inverse laplace Transform of $\frac{1}{s}\log\sqrt{\frac{s^2+9}{s^2+16}}$

$\rightarrow L^{-1}\left\{\frac{1}{s}\log\sqrt{\frac{s^2+9}{s^2+16}}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s}\log s^2+9 - \frac{1}{s}\log s^2+16\right\}$

$= \frac{1}{2}\left(\frac{-1}{t}\right)L^{-1}\left\{\frac{d}{ds}\left[\frac{1}{s}\log s^2+9 - \frac{1}{s}\log s^2+16\right]\right\}$

$= \frac{-1}{2t}L^{-1}\left\{\left[\frac{1}{s}\frac{1}{s^2+9}(2s) + \left(\frac{-1}{s^2}\right)\log(s^2+9)\right] - \left[\frac{1}{s}\frac{1}{s^2+16}(2s) + \left(\frac{-1}{s^2}\right)\log(s^2+16)\right]\right\}$

$= \frac{-1}{2t}L^{-1}\left\{\frac{2}{s^2+9} - \right.$

c) Find the half range cosine series for $f(x) = (x-1)^2;\ 0<x<1$

$\rightarrow$ $f(x) = \frac{a_0}{2} \sum a_n \cos \frac{n\pi x}{l}$ $\boxed{l=1}$

Where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = 2\int_0^1 (x^2-2x+1)\,dx = 2\left[\frac{x^3}{3} - \frac{2x^2}{2} + x\right]_0^1$

$= 2\left\{\frac{1}{3} - 0\right\} = 2/3$

$a_n = \frac{2}{l}\int_0^l f(x) \cdot \cos \frac{n\pi x}{l}\,dx$

$= 2\int_0^1 (x-1)^2 \cos n\pi x\,dx$

$= 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + 2\left(\frac{-\sin(n\pi x)}{(n\pi)^3}\right)\right]_0^1$

$= 2\left\{\left[0 - 0 + 0\right] - \left[0 + \frac{2(-1)(-1)^n}{(n\pi)^2} + 0\right]\right\}$

$= \frac{4\,\cancel{(-1)^{n+1}}}{(n\pi)^2}$

D) Find the family of curves orthogonal to the family of curves $x^3y - xy^3 = c$

$\rightarrow$ $u = (x^3y - xy^3)$

$u_x = 3x^2y - y^3$ $\qquad$ $u_y = x^3 - 3xy^2$

$u_x(z,0) = 0$ $\qquad$ $u_y(z,0) = z^3$

$f'(z) = u_x + iv_x$

$= u_x - iu_y$

$= -iz^3$

$f(z) = -i\frac{z^4}{4} + c$

$= \frac{-i}{4}(x+iy)^4 + c$

$= \frac{-i}{4}\left(x^4 + 4x^3iy + 6x^2i^2y^2 + 4xi^3y^3 + i^4y^4\right) + c$

$$= \frac{-i}{4}\left[\left(x^4 - 6x^2y^2 + y^4\right) + i4\left(x^3y - xy^3\right)\right] + c$$

$$= \left(x^3y - xy^3\right) + i\left(-\frac{x^4}{4} + \frac{3x^2y^2}{2} - \frac{y^4}{4}\right) + c$$

orthogonal trajectory is

$$-\frac{x^4}{4} + \frac{3x^2y^2}{2} - \frac{y^4}{4} = k$$

6) Fit a ~~stra~~ straight line of the form $y = a + bx$ to the following data

X	1	3	5	7	8	10
Y	8	12	15	17	18	20

→ $Y - \bar{Y} = r\frac{\sigma_x}{\sigma_y}(X - \bar{X})$

$$Y = 1.3012X + 7.62651$$

F) A random variable X has probability density f^n

$$f(x) = \begin{cases} Kx^2e^{-x} & x > 0,\ K > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find 'K' and hence find mean and variance

→ $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{0}^{\infty} Kx^2e^{-x}\,dx = 1$

$$K\left[x^2\left(\frac{e^{-x}}{-1}\right) - 2x\left(\frac{e^{-x}}{+1}\right) + 2\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty} = 1$$

$$\therefore K\left[(0 - 0 + 0) - (0 - 0 - 2(1))\right] = 1$$

$$\therefore 2k = 1 \Rightarrow \boxed{k = 1/2}$$

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$$\therefore f(x) = \begin{cases} \frac{1}{2} x^2 e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\therefore \text{mean} = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^{\infty} \frac{1}{2} x^3 e^{-x}\, dx$$

$$= \frac{1}{2}\left[x^3\left(\frac{e^{-x}}{-1}\right) - 3x^2\left(\frac{e^{-x}}{1}\right) + 6x\left(\frac{e^{-x}}{-1}\right) - 6\frac{e^{-x}}{1}\right]_0^{\infty}$$

$$= \frac{1}{2}\left\{\left[0 - 0 + 0 - 0\right] - \left[0 - 0 + 0 - 6\right]\right\}$$

$$= \frac{6}{2} = 3$$

Now

$$\int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^{\infty} \frac{x^4 e^{-x}}{2}\, dx = \cancel{\frac{1}{2}\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = -\frac{1}{2}\left[0 - 1\right] = \frac{1}{2}}$$

$$\therefore \text{Variance} = \cancel{E(x^2) - (E(x))^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}}$$

$$= \frac{1}{2}\left[x^4\left(\frac{e^{-x}}{-1}\right) - 4x^3\left(\frac{e^{-x}}{+1}\right) + 12x^2\left(\frac{e^{-x}}{-1}\right) - 24x\left(\frac{e^{-x}}{+1}\right) + 24\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty}$$

$$= \frac{1}{2}\left\{\left[0 - 0 + 0 - 0 + 0\right] - \left[0 - 0 + 0 - 0 - 24\right]\right\}$$

$$= \frac{24}{2}$$

$$= 12$$

$$\text{Variance} = E(X^2) - (E(X))^2 = 12 - 9 = 3$$