Z Transform

Weight Distribution of Types

Comp/IT/AI

Type	Name	May 2018	Nov 2018	May 2019	Nov 2019	May 2022	Nov 2022	May 2023	Dec 2023	May 2024	Dec 2024
I	Definition	06	---	06	06	---	---	06	11	05	11
II	Properties	---	---	---	---	07	06	---	---	06	---
III	Convolution	---	06	---	---	---	06	06	---	---	---
IV	Binomial Theorem	04	---	08	---	---	---	---	---	---	----
V	Partial Fractions	04	06	---	06	10	06	06	06	06	06
VI	Convolution					02	---	---	---	---	---
Total Marks		**14**	**12**	**14**	**12**	**19**	**18**	**18**	**17**	**17**	**17**

Type I: Based on Definition

1. If $f(k) = \{2^0, 2^1, 2^2, 2^3, \dots \dots \dots .\}$find $Z\{f(k)\}$

Solution:

We have,

$f(k) = \{2^0, 2^1, 2^2, 2^3, \dots \dots \dots .\}$

$f(k) = 2^k, k \geq 0$

By definition,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k).z^{-k}$

$Z\{2^k\} = \sum_{0}^{\infty} 2^k.z^{-k}$

$Z\{2^k\} = 2^0.z^{-0} + 2^1.z^{-1} + 2^2.z^{-2} + 2^3.z^{-3} + \cdots \dots \dots$

$Z\{2^k\} = 1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots \dots.$

$Z\{2^k\} = \left[1 - \frac{2}{z}\right]^{-1}$

$Z\{2^k\} = \left[\frac{z-2}{z}\right]^{-1}$

$Z\{2^k\} = \frac{z}{z-2}$

2. Find the Z transform of

(i) $f(k) = 1, k \geq 0, |z| > 1$

Solution:

$f(k) = 1, k \geq 0$

By definition,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\{1\} = \sum_{0}^{\infty} 1. z^{-k}$

$Z\{1\} = z^0 + z^{-1} + z^{-2} + z^{-3} + \cdots \ldots \ldots$

$Z\{1\} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \ldots.$

$Z\{1\} = \left[1 - \frac{1}{z}\right]^{-1}$

$Z\{1\} = \left[\frac{z-1}{z}\right]^{-1}$

$Z\{1\} = \frac{z}{z-1}$

(ii) $f(k) = a^k, k \geq 0, |z| > a$

Solution:

We have,

$f(k) = a^k, k \geq 0$

By definition,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\{a^k\} = \sum_{0}^{\infty} a^k . z^{-k}$

$Z\{a^k\} = a^0 z^0 + a^1 . z^{-1} + a^2 . z^{-2} + a^3 . z^{-3} + \cdots \ldots \ldots$

$Z\{a^k\} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots \ldots \ldots$

$Z\{a^k\} = \left[1 - \frac{a}{z}\right]^{-1}$

$Z\{a^k\} = \left[\frac{z-a}{z}\right]^{-1}$

$Z\{a^k\} = \frac{z}{z-a}$

(iii) $f(k) = \frac{1}{2^k}, k \geq 0, |2z| > 1$

Solution:

We have,

$f(k) = \frac{1}{2^k}, k \geq 0$

By definition,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\left\{\frac{1}{2^k}\right\} = \sum_{0}^{\infty} \frac{1}{2^k} z^{-k}$

$Z\left\{\frac{1}{2^k}\right\} = \frac{z^0}{2^0} + \frac{z^{-1}}{2^1} + \frac{z^{-2}}{2^2} + \frac{z^{-3}}{2^3} + \cdots \ldots.$

$Z\left\{\frac{1}{2^k}\right\} = 1 + \frac{1}{2z} + \frac{1}{2^2 z^2} + \frac{1}{2^3 z^3} + \cdots \ldots..$

$Z\left\{\frac{1}{2^k}\right\} = \left[1 - \frac{1}{2z}\right]^{-1}$

$Z\left\{\frac{1}{2^k}\right\} = \left[\frac{2z-1}{2z}\right]^{-1}$

$Z\left\{\frac{1}{2^k}\right\} = \frac{2z}{2z-1}$

3. Find the Z transform of $f(k) = b^k, k < 0$

Solution:

We have,

$f(k) = b^k, k < 0$

By definition,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\{b^k\} = \sum_{-\infty}^{-1} b^k z^{-k}$

$Z\{b^k\} = \cdots \ldots \ldots \ldots + b^{-3} z^{--3} + b^{-2} z^{--2} + b^{-1} z^{--1}$

$Z\{b^k\} = \frac{z}{b} + \frac{z^2}{b^2} + \frac{z^3}{b^3} + \cdots \ldots \ldots \ldots$

$Z\{b^k\} = \frac{z}{b}\left[1 + \frac{z}{b} + \frac{z^2}{b^2} + \cdots \ldots\right]$

$Z\{b^k\} = \frac{z}{b}\left[1 - \frac{z}{b}\right]^{-1}$

$Z\{b^k\} = \frac{z}{b}\left[\frac{b-z}{b}\right]^{-1}$

$Z\{b^k\} = \frac{z}{b}\left[\frac{b}{b-z}\right]$

$Z\{b^k\} = \frac{z}{b-z}$

4. Find the Z transform of $f(k) = a^{|k|}$

Solution:

$f(k) = a^{|k|}$

By definition,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\{a^{|k|}\} = \sum_{-\infty}^{\infty} a^{|k|} z^{-k}$

$Z\{a^{|k|}\} = \sum_{-\infty}^{-1} a^{-k} z^{-k} + \sum_{0}^{\infty} a^{k} z^{-k}$

$Z\{a^{|k|}\} = [\ldots\ldots. + a^3z^3 + a^2z^2 + a^1z^1] + [a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots.]$

$Z\{a^{|k|}\} = [az + a^2z^2 + a^3z^3 + \cdots \ldots..] + \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots..\right]$

$Z\{a^{|k|}\} = az[1 + az + a^2z^2 + \cdots.] + \left[1 - \frac{a}{z}\right]^{-1}$

$Z\{a^{|k|}\} = az[1 - az]^{-1} + \left[\frac{z-a}{z}\right]^{-1}$

$Z\{a^{|k|}\} = \frac{az}{1-az} + \frac{z}{z-a}$

$Z\{a^{|k|}\} = \frac{az(z-a)+z(1-az)}{(1-az)(z-a)}$

$Z\{a^{|k|}\} = \frac{az^2-a^2z+z-az^2}{(1-az)(z-a)}$

$Z\{a^{|k|}\} = \frac{z-a^2z}{(1-az)(z-a)}$

5. Find the Z transform of Unit impulse function

Solution:

We have, Unit Impulse function

$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & otherwise \end{cases}$

By definition,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\{\delta(k)\} = \sum_{-\infty}^{-1} 0. z^{-k} + \sum_{0}^{0} 1. z^{-k} + \sum_{1}^{\infty} 0. z^{-k}$

$Z\{\delta(k)\} = z^0 = 1$

6. Find the Z transform of Discrete Unit Step Function

Solution:

We have, Discrete Unit Step function

$$U(k) = \begin{cases} 1 & k \geq 0 \\ 0 & otherwise \end{cases}$$

By definition,

$$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$$
$$Z\{U(k)\} = \sum_{-\infty}^{-1} 0.z^{-k} + \sum_{0}^{\infty} 1.z^{-k}$$
$$Z\{U(k)\} = z^0 + z^{-1} + z^{-2} + z^{-3} + \cdots \ldots.$$
$$Z\{U(k)\} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \ldots$$
$$Z\{U(k)\} = \left[1 - \frac{1}{z}\right]^{-1}$$
$$Z\{U(k)\} = \left[\frac{z-1}{z}\right]^{-1}$$
$$Z\{U(k)\} = \frac{z}{z-1}$$

7. Find the Z transform of $f(k) = \begin{cases} 5^k & k < 0 \\ 3^k & k \geq 0 \end{cases}$

Solution:

$$f(k) = \begin{cases} 5^k & k < 0 \\ 3^k & k \geq 0 \end{cases}$$

By definition,

$$Z\{f(k)\} = \sum_{-\infty}^{-1} 5^k z^{-k} + \sum_{0}^{\infty} 3^k z^{-k}$$
$$Z\{f(k)\} = [\ldots\ldots\ldots + 5^{-3}z^3 + 5^{-2}z^2 + 5^{-1}z^1] + [3^0 z^0 + 3^1 z^{-1} + 3^2 z^{-2} + \cdots.]$$
$$Z\{f(k)\} = \left[\frac{z}{5} + \frac{z^2}{5^2} + \frac{z^3}{5^3} + \cdots..\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \cdots..\right]$$
$$Z\{f(k)\} = \frac{z}{5}\left[1 + \frac{z}{5} + \frac{z^2}{5^2} + \cdots.\right] + \left[1 - \frac{3}{z}\right]^{-1}$$
$$Z\{f(k)\} = \frac{z}{5}\left[1 - \frac{z}{5}\right]^{-1} + \left[\frac{z-3}{z}\right]^{-1}$$
$$Z\{f(k)\} = \frac{z}{5}\left[\frac{5-z}{5}\right]^{-1} + \frac{z}{z-3}$$
$$Z\{f(k)\} = \frac{z}{5}\left[\frac{5}{5-z}\right] + \frac{z}{z-3}$$
$$Z\{f(k)\} = \frac{z}{5-z} + \frac{z}{z-3}$$
$$Z\{f(k)\} = \frac{z^2 - 3z + 5z - z^2}{(5-z)(z-3)}$$
$$Z\{f(k)\} = \frac{2z}{(5-z)(z-3)}$$

8. Find the Z transform of $f(k) = c^k \cos \alpha k$, $k \geq 0$ hence find $\cos \alpha k$

Solution:

We have,

$$f(k) = c^k \cos \alpha k$$

$$f(k) = c^k \left[\frac{e^{i\alpha k}+e^{-i\alpha k}}{2}\right] \qquad \because \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$f(k) = \frac{\left(ce^{i\alpha}\right)^k+\left(ce^{-i\alpha}\right)^k}{2}$$

$$f(k) = \frac{1}{2}\left[\left(ce^{i\alpha}\right)^k + \left(ce^{-i\alpha}\right)^k\right]$$

$$Z\{f(k)\} = \frac{1}{2}\left[Z\left\{\left(ce^{i\alpha}\right)^k\right\} + Z\left\{\left(ce^{-i\alpha}\right)^k\right\}\right]$$

$$Z\{f(k)\} = \frac{1}{2}\left[\frac{z}{z-ce^{i\alpha}} + \frac{z}{z-ce^{-i\alpha}}\right]$$

$$Z\{f(k)\} = \frac{1}{2}\left[\frac{z^2-cze^{-i\alpha}+z^2-cze^{i\alpha}}{z^2-cze^{-i\alpha}-cze^{i\alpha}+c^2e^{i\alpha}e^{-i\alpha}}\right]$$

$$Z\{f(k)\} = \frac{1}{2}\left[\frac{2z^2-cz\left(e^{-i\alpha}+e^{i\alpha}\right)}{z^2-cz\left(e^{-i\alpha}+e^{i\alpha}\right)+c^2}\right]$$

$$Z\{f(k)\} = \frac{1}{2}\left[\frac{2z^2-cz(2\cos\alpha)}{z^2-cz(2\cos\alpha)+c^2}\right] \qquad \because e^{i\theta}+e^{-i\theta} = 2\cos\theta$$

$$Z\{f(k)\} = \frac{2}{2}\left[\frac{z^2-cz\cos\alpha}{z^2-2cz\cos\alpha+c^2}\right]$$

Thus,

$$Z\{c^k \cos\alpha k\} = \frac{z^2-cz\cos\alpha}{z^2-2cz\cos\alpha+c^2}$$

Put $c = 1$,

$$Z\{\cos\alpha k\} = \frac{z^2-z\cos\alpha}{z^2-2z\cos\alpha+1}$$

9. Find the Z transform of $f(k) = c^k \sin \alpha k\,, k \geq 0$ hence find $\sin \alpha k$

Solution:

We have,

$$f(k) = c^k \sin \alpha k$$

$$f(k) = c^k \left[\frac{e^{i\alpha k} - e^{-i\alpha k}}{2i}\right] \qquad \because \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$f(k) = \left[\frac{\left(ce^{i\alpha}\right)^k - \left(ce^{-i\alpha}\right)^k}{2i}\right]$$

$$Z\{f(k)\} = \frac{1}{2i}\left[Z\left\{\left(ce^{i\alpha}\right)^k\right\} - Z\left\{\left(ce^{-i\alpha}\right)^k\right\}\right]$$

$$Z\{f(k)\} = \frac{1}{2i}\left[\frac{z}{z - ce^{i\alpha}} - \frac{z}{z - ce^{-i\alpha}}\right]$$

$$Z\{f(k)\} = \frac{1}{2i}\left[\frac{z^2 - cze^{-i\alpha} - z^2 + cze^{i\alpha}}{z^2 - cz\,e^{-i\alpha} - cz\,e^{i\alpha} + c^2 e^{i\alpha} e^{-i\alpha}}\right]$$

$$Z\{f(k)\} = \frac{1}{2i}\left[\frac{cz\,e^{i\alpha} - cz\,e^{-i\alpha}}{z^2 - cz\left(e^{-i\alpha} + e^{i\alpha}\right) + c^2}\right]$$

$$Z\{f(k)\} = \frac{1}{2i}\left[\frac{cz\left(e^{i\alpha} - e^{-i\alpha}\right)}{z^2 - cz\left(e^{i\alpha} + e^{-i\alpha}\right) + c^2}\right] \qquad \because e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$Z\{f(k)\} = \frac{1}{2i}\left[\frac{cz(2i\sin\alpha)}{z^2 - cz\,(2\cos\alpha) + c^2}\right] \qquad \because e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

Thus,

$$Z\{c^k \sin \alpha k\} = \frac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2}$$

Put $c = 1$

$$Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$$

10. Find the Z transform of $f(k) = \sin\left(\frac{k\pi}{4} + a\right), k \geq 0$

Solution:

We have,

$$f(k) = \sin\left(\frac{k\pi}{4} + a\right)$$

$$f(k) = \sin\frac{k\pi}{4}\cos a + \cos\frac{k\pi}{4}\sin a$$

$$Z\{f(k)\} = \cos a\ Z\left\{\sin\frac{k\pi}{4}\right\} + \sin a\ Z\left\{\cos\frac{k\pi}{4}\right\}$$

$$Z\{f(k)\} = \cos a\ \left[\frac{z\sin\frac{\pi}{4}}{z^2 - 2z\cos\frac{\pi}{4} + 1}\right] + \sin a\ \left[\frac{z^2 - z\cos\frac{\pi}{4}}{z^2 - 2z\cos\frac{\pi}{4} + 1}\right]$$

$$Z\{f(k)\} = \frac{\cos a\ z\left(\sin\frac{\pi}{4}\right) + \sin a\left(z^2 - z\cos\frac{\pi}{4}\right)}{z^2 - 2z\left(\frac{1}{\sqrt{2}}\right) + 1}$$

$$Z\{f(k)\} = \frac{z\cos a\sin\frac{\pi}{4} + z^2\sin a - z\sin a\cos\frac{\pi}{4}}{z^2 - \sqrt{2}z + 1}$$

$$Z\{f(k)\} = \frac{z^2\sin a + z\left(\sin\frac{\pi}{4}\cos a - \cos\frac{\pi}{4}\sin a\right)}{z^2 - \sqrt{2}z + 1}$$

$$Z\{f(k)\} = \frac{z^2\sin a + z\sin\left(\frac{\pi}{4} - a\right)}{z^2 - \sqrt{2}z + 1}$$

11. Find the Z transform of $f(k) = a^k, k \geq 0$
[M24/CompITAI/5M]
Solution:
We have,
$f(k) = a^k, k \geq 0$
By definition,
$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$
$Z\{a^k\} = \sum_{0}^{\infty} a^k . z^{-k}$
$Z\{a^k\} = a^0 z^0 + a^1 . z^{-1} + a^2 . z^{-2} + a^3 . z^{-3} + \cdots \dots \dots$
$Z\{a^k\} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots \dots \dots$
$Z\{a^k\} = \left[1 - \frac{a}{z}\right]^{-1}$
$Z\{a^k\} = \left[\frac{z-a}{z}\right]^{-1}$
$Z\{a^k\} = \frac{z}{z-a}$

12. Find the Z transform of $f(k) = a^{-k}, k \geq 0$
[D24/CompITAI/5M]
Solution:
We have,
$f(k) = a^{-k}, k \geq 0$
By definition,
$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$
$Z\{a^{-k}\} = \sum_{0}^{\infty} a^{-k} . z^{-k}$
$Z\{a^{-k}\} = a^0 z^0 + a^{-1} . z^{-1} + a^{-2} . z^{-2} + a^{-3} . z^{-3} + \cdots \dots \dots$
$Z\{a^{-k}\} = 1 + \frac{1}{az} + \frac{1}{a^2 z^2} + \frac{1}{a^3 z^3} + \cdots \dots \dots$
$Z\{a^{-k}\} = \left[1 - \frac{1}{az}\right]^{-1}$
$Z\{a^{-k}\} = \left[\frac{az-1}{az}\right]^{-1}$
$Z\{a^{-k}\} = \frac{az}{az-1}$

13. Find the Z transform of $f(k) = b^k, k \geq 0$
[M16/CompIT/6M]
Solution:
We have,
$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$
$Z\{b^k\} = \sum_{0}^{\infty} b^k z^{-k}$
$= b^0 z^0 + b^1 z^{-1} + b^2 z^{-2} + b^3 z^{-3} + \cdots \ldots \ldots$
$= 1 + \frac{b}{z} + \frac{b^2}{z^2} + \frac{b^3}{z^3} + \cdots \ldots$
$= \left[1 - \frac{b}{z}\right]^{-1}$
$= \left[\frac{z-b}{z}\right]^{-1}$
$Z\{b^k\} = \frac{z}{z-b}$

14. Find the Z transform of $f(k) = 3^k, k \geq 0$
Solution:
We have,
$f(k) = 3^k, k \geq 0$
By definition,
$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$
$Z\{3^k\} = \sum_{0}^{\infty} 3^k . z^{-k}$
$Z\{3^k\} = 3^0 z^0 + 3^1 . z^{-1} + 3^2 . z^{-2} + 3^3 . z^{-3} + \cdots \ldots \ldots$
$Z\{3^k\} = 1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots \ldots \ldots$
$Z\{3^k\} = \left[1 - \frac{3}{z}\right]^{-1}$
$Z\{3^k\} = \left[\frac{z-3}{z}\right]^{-1}$
$Z\{3^k\} = \frac{z}{z-3}$

15. Find the Z transform of $f(k) = 2^k, k < 0$

Solution:

We have,

$f(k) = 2^k, k < 0$

By definition,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\{2^k\} = \sum_{-\infty}^{-1} 2^k z^{-k}$

$Z\{2^k\} = \cdots \dots \dots \dots + 2^{-3} z^{--3} + 2^{-2} z^{--2} + 2^{-1} z^{--1}$

$Z\{2^k\} = \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots \dots \dots \dots$

$Z\{2^k\} = \frac{z}{2}\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \cdots \dots\right]$

$Z\{2^k\} = \frac{z}{2}\left[1 - \frac{z}{2}\right]^{-1}$

$Z\{2^k\} = \frac{z}{2}\left[\frac{2-z}{2}\right]^{-1}$

$Z\{2^k\} = \frac{z}{2}\left[\frac{2}{2-z}\right]$

$Z\{2^k\} = \frac{z}{2-z}$

16. Find the Z transform of $f(k) = a^k, k < 0$

[D23/CompITAI/5M]

Solution:

We have,

$f(k) = a^k, k < 0$

By definition,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\{a^k\} = \sum_{-\infty}^{-1} a^k z^{-k}$

$Z\{a^k\} = \cdots \dots \dots \dots + a^{-3} z^{--3} + a^{-2} z^{--2} + a^{-1} z^{--1}$

$Z\{a^k\} = \frac{z}{a} + \frac{z^2}{a^2} + \frac{z^3}{a^3} + \cdots \dots \dots \dots$

$Z\{a^k\} = \frac{z}{a}\left[1 + \frac{z}{a} + \frac{z^2}{a^2} + \cdots \dots\right]$

$Z\{a^k\} = \frac{z}{a}\left[1 - \frac{z}{a}\right]^{-1}$

$Z\{a^k\} = \frac{z}{a}\left[\frac{a-z}{a}\right]^{-1}$

$Z\{a^k\} = \frac{z}{a}\left[\frac{a}{a-z}\right]$

$Z\{a^k\} = \frac{z}{a-z}$

17. Find the Z transform of $f(k) = \left(\frac{1}{2}\right)^{|k|}$

Solution:

We have,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\left\{\left(\frac{1}{2}\right)^{|k|}\right\} = \sum_{-\infty}^{\infty} \left(\frac{1}{2}\right)^{|k|} z^{-k}$

$= \sum_{-\infty}^{-1} \left(\frac{1}{2}\right)^{-k} z^{-k} + \sum_{0}^{\infty} \left(\frac{1}{2}\right)^{k} z^{-k}$

$= \left[\dots \dots . + \left(\frac{1}{2}\right)^{3} z^{3} + \left(\frac{1}{2}\right)^{2} z^{2} + \left(\frac{1}{2}\right)^{1} z^{1}\right] + \left[\left(\frac{1}{2}\right)^{0} z^{0} + \left(\frac{1}{2}\right)^{1} z^{-1} + \left(\frac{1}{2}\right)^{2} z^{-2} + \cdots\right]$

$= \left[\frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots . .\right] + \left[1 + \frac{1}{2z} + \frac{1}{2^2 z^2} + \cdots \dots\right]$

$= \frac{z}{2}\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \cdots .\right] + \left[1 + \frac{1}{2z} + \frac{1}{2^2 z^2} + \cdots . .\right]$

$= \frac{z}{2}\left[1 - \frac{z}{2}\right]^{-1} + \left[1 - \frac{1}{2z}\right]^{-1}$

$= \frac{z}{2}\left[\frac{2-z}{2}\right]^{-1} + \left[\frac{2z-1}{2z}\right]^{-1}$

$= \frac{z}{2}\left[\frac{2}{2-z}\right] + \left[\frac{2z}{2z-1}\right]$

$= \frac{z}{2-z} + \frac{2z}{2z-1}$

18. Find the Z transform of $f(k) = \left(\frac{1}{3}\right)^{|k|}$

[N14/CompIT/5M]

Solution:

We have,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\left\{\left(\frac{1}{3}\right)^{|k|}\right\} = \sum_{-\infty}^{\infty} \left(\frac{1}{3}\right)^{|k|} z^{-k}$

$= \sum_{-\infty}^{-1} \left(\frac{1}{3}\right)^{-k} z^{-k} + \sum_{0}^{\infty} \left(\frac{1}{3}\right)^{k} z^{-k}$

$= \left[\dots \dots . + \left(\frac{1}{3}\right)^{3} z^{3} + \left(\frac{1}{3}\right)^{2} z^{2} + \left(\frac{1}{3}\right)^{1} z^{1}\right] + \left[\left(\frac{1}{3}\right)^{0} z^{0} + \left(\frac{1}{3}\right)^{1} z^{-1} + \left(\frac{1}{3}\right)^{2} z^{-2} + \cdots\right]$

$= \left[\frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots . .\right] + \left[1 + \frac{1}{3z} + \frac{1}{3^2 z^2} + \cdots \dots\right]$

$= \frac{z}{3}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots .\right] + \left[1 + \frac{1}{3z} + \frac{1}{3^2 z^2} + \cdots . .\right]$

$= \frac{z}{3}\left[1 - \frac{z}{3}\right]^{-1} + \left[1 - \frac{1}{3z}\right]^{-1}$

$= \frac{z}{3}\left[\frac{3-z}{3}\right]^{-1} + \left[\frac{3z-1}{3z}\right]^{-1}$

$= \frac{z}{3}\left[\frac{3}{3-z}\right] + \left[\frac{3z}{3z-1}\right]$

$= \frac{z}{3-z} + \frac{3z}{3z-1}$

19. Find the Z transform of $f(k) = \left(\frac{1}{4}\right)^{|k|}$

[M18/Comp/6M]

Solution:

We have,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\left\{\left(\frac{1}{4}\right)^{|k|}\right\} = \sum_{-\infty}^{\infty} \left(\frac{1}{4}\right)^{|k|} z^{-k}$

$= \sum_{-\infty}^{-1} \left(\frac{1}{4}\right)^{-k} z^{-k} + \sum_{0}^{\infty} \left(\frac{1}{4}\right)^{k} z^{-k}$

$= \left[\ldots\ldots + \left(\frac{1}{4}\right)^3 z^3 + \left(\frac{1}{4}\right)^2 z^2 + \left(\frac{1}{4}\right)^1 z^1\right] + \left[\left(\frac{1}{4}\right)^0 z^0 + \left(\frac{1}{4}\right)^1 z^{-1} + \left(\frac{1}{4}\right)^2 z^{-2} + \cdots\right]$

$= \left[\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \cdots..\right] + \left[1 + \frac{1}{4z} + \frac{1}{4^2 z^2} + \cdots\ldots\right]$

$= \frac{z}{4}\left[1 + \frac{z}{4} + \frac{z^2}{4^2} + \cdots.\right] + \left[1 + \frac{1}{4z} + \frac{1}{4^2 z^2} + \cdots..\right]$

$= \frac{z}{4}\left[1 - \frac{z}{4}\right]^{-1} + \left[1 - \frac{1}{4z}\right]^{-1}$

$= \frac{z}{4}\left[\frac{4-z}{4}\right]^{-1} + \left[\frac{4z-1}{4z}\right]^{-1}$

$= \frac{z}{4}\left[\frac{4}{4-z}\right] + \left[\frac{4z}{4z-1}\right]$

$= \frac{z}{4-z} + \frac{4z}{4z-1}$

20. Find the Z transform of $f(k) = a^{|k|}$ and hence find the Z transform of $f(k) = \left(\frac{1}{2}\right)^{|k|}$
[N13/CompIT/6M]
Solution:
We have,
$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$
$Z\{a^{|k|}\} = \sum_{-\infty}^{\infty} a^{|k|} z^{-k}$
$= \sum_{-\infty}^{-1} a^{-k} z^{-k} + \sum_{0}^{\infty} a^{k} z^{-k}$
$= [\dots\dots + (a)^3 z^3 + (a)^2 z^2 + (a)^1 z^1] + [(a)^0 z^0 + (a)^1 z^{-1} + (a)^2 z^{-2} + \cdots]$
$= [az + a^2 z^2 + a^3 z^3 + \cdots..] + \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots \dots\right]$
$= az[1 + az + a^2 z^2 + \cdots.] + \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots \dots\right]$
$= az[1 - az]^{-1} + \left[1 - \frac{a}{z}\right]^{-1}$
$= \frac{az}{1-az} + \left[\frac{z-a}{z}\right]^{-1}$
$\therefore Z\{a^{|k|}\} = \frac{az}{1-az} + \frac{z}{z-a}$
Put $a = \frac{1}{2}$,
$Z\left\{\left(\frac{1}{2}\right)^{|k|}\right\} = \frac{\left(\frac{1}{2}\right)z}{1-\left(\frac{1}{2}\right)z} + \frac{z}{z-\frac{1}{2}}$
$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{2z}{2z-1}$
$= \frac{z}{2-z} + \frac{2z}{2z-1}$

21. Find the Z transform of $f(k) = \begin{cases} 4^k & for \quad k < 0 \\ 3^k & for \quad k \geq 0 \end{cases}$

[N17/N19/Comp/6M]

Solution:

We have,

$$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$$
$$Z\{f(k)\} = \sum_{-\infty}^{-1} 4^k z^{-k} + \sum_{0}^{\infty} 3^k z^{-k}$$
$$= [\dots\dots.. + 4^{-3}z^3 + 4^{-2}z^2 + 4^{-1}z^1] + [3^0 z^0 + 3^1 z^{-1} + 3^2 z^{-2} + \cdots \dots\dots.]$$
$$= \left[\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \cdots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \cdots \dots..\right]$$
$$= \frac{z}{4}\left[1 + \frac{z}{4} + \frac{z^2}{4^2} + \cdots\right] + \left[1 - \frac{3}{z}\right]^{-1}$$
$$= \frac{z}{4}\left[1 - \frac{z}{4}\right]^{-1} + \left[\frac{z-3}{z}\right]^{-1}$$
$$= \frac{z}{4}\left[\frac{4-z}{4}\right]^{-1} + \frac{z}{z-3}$$
$$= \frac{z}{4}\left[\frac{4}{4-z}\right] + \frac{z}{z-3}$$
$$= \frac{z}{4-z} + \frac{z}{z-3}$$
$$= \frac{z^2-3z+4z-z^2}{(4-z)(z-3)}$$
$$= \frac{z}{(4-z)(z-3)}$$

22. Find the Z transform of $f(k) = \begin{cases} 3^k & for \quad k < 0 \\ 2^k & for \quad k \geq 0 \end{cases}$

[M19/Comp/6M]

Solution:

$$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$$
$$Z\{f(k)\} = \sum_{-\infty}^{-1} 3^k z^{-k} + \sum_{0}^{\infty} 2^k z^{-k}$$
$$= [\dots\dots.. + 3^{-3}z^3 + 3^{-2}z^2 + 3^{-1}z^1] + [2^0 z^0 + 2^1 z^{-1} + 2^2 z^{-2} + \cdots \dots\dots.]$$
$$= \left[\frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots\right] + \left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \cdots \dots..\right]$$
$$= \frac{z}{3}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots\right] + \left[1 - \frac{2}{z}\right]^{-1}$$
$$= \frac{z}{3}\left[1 - \frac{z}{3}\right]^{-1} + \left[\frac{z-2}{z}\right]^{-1}$$
$$= \frac{z}{3}\left[\frac{3-z}{3}\right]^{-1} + \frac{z}{z-2}$$
$$= \frac{z}{3}\left[\frac{3}{3-z}\right] + \frac{z}{z-2}$$
$$= \frac{z}{3-z} + \frac{z}{z-2}$$
$$= \frac{z^2-2z+3z-z^2}{(3-z)(z-2)}$$
$$= \frac{z}{(3-z)(z-2)}$$

23. Find the Z transform of $f(k) = \begin{cases} b^k & for \quad k < 0 \\ a^k & for \quad k \geq 0 \end{cases}$

[M23/CompIT/6M]

Solution:

We have,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\{f(k)\} = \sum_{-\infty}^{-1} b^k z^{-k} + \sum_{0}^{\infty} a^k z^{-k}$

$= [\ldots \ldots .. + b^{-3}z^3 + b^{-2}z^2 + b^{-1}z^1] + [a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \cdots \ldots \ldots .]$

$= \left[\frac{z}{b} + \frac{z^2}{b^2} + \frac{z^3}{b^3} + \cdots\right] + \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots \ldots ..\right]$

$= \frac{z}{b}\left[1 + \frac{z}{b} + \frac{z^2}{b^2} + \cdots\right] + \left[1 - \frac{a}{z}\right]^{-1}$

$= \frac{z}{b}\left[1 - \frac{z}{b}\right]^{-1} + \left[\frac{z-a}{z}\right]^{-1}$

$= \frac{z}{b}\left[\frac{b-z}{b}\right]^{-1} + \frac{z}{z-a}$

$= \frac{z}{b}\left[\frac{b}{b-z}\right] + \frac{z}{z-a}$

$= \frac{z}{b-z} + \frac{z}{z-a}$

$= \frac{z^2 - az + bz - z^2}{(b-z)(z-a)}$

$= \frac{bz - az}{(b-z)(z-a)}$

24. Find the Z transform of $f(k) = \begin{cases} -\left(-\frac{1}{4}\right)^k for k < 0 \\ \left(-\frac{1}{5}\right)^k for k \geq 0 \end{cases}$

Solution:

We have,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\{f(k)\} = \sum_{-\infty}^{-1} -\left(-\frac{1}{4}\right)^k z^{-k} + \sum_{0}^{\infty} \left(-\frac{1}{5}\right)^k z^{-k}$

$= \left[\ldots .. - \left(-\frac{1}{4}\right)^{-3} z^3 - \left(-\frac{1}{4}\right)^{-2} z^2 - \left(-\frac{1}{4}\right)^{-1} z^1\right] + \left[\left(-\frac{1}{5}\right)^0 z^0 + \left(-\frac{1}{5}\right)^1 z^{-1} + \left(-\frac{1}{5}\right)^2 z^{-2} + \cdots \ldots .\right]$

$= -[(-4)z + (-4)^2 z^2 + (-4)^3 z^3 + \cdots] + \left[1 + \left(-\frac{1}{5z}\right) + \left(-\frac{1}{5z}\right)^2 + \cdots \ldots ..\right]$

$= 4z[1 + (-4z) + (-4z)^2 + \cdots] + \left[1 - \left(-\frac{1}{5z}\right)\right]^{-1}$

$= 4z[1 - (-4z)]^{-1} + \left[\frac{5z+1}{5z}\right]^{-1}$

$= 4z[1 + 4z]^{-1} + \frac{5z}{5z+1}$

$= \frac{4z}{1+4z} + \frac{5z}{5z+1}$

25. Find the Z transform of $f(k) = c^k \sinh \alpha k\,, k \geq 0$

Solution:

$f(k) = c^k \sinh \alpha k$

$f(k) = c^k \left[\frac{e^{\alpha k} - e^{-\alpha k}}{2}\right]$ $\quad\because \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$

$f(k) = \left[\frac{(ce^{\alpha})^k - (ce^{-\alpha})^k}{2}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[Z\{(ce^{\alpha})^k\} - Z\{(ce^{-\alpha})^k\}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{z}{z - ce^{\alpha}} - \frac{z}{z - ce^{-\alpha}}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{z^2 - cze^{-\alpha} - z^2 + cze^{\alpha}}{z^2 - cz\,e^{-\alpha} - cz\,e^{\alpha} + c^2 e^{\alpha} e^{-\alpha}}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{cz\,e^{\alpha} - cz\,e^{-\alpha}}{z^2 - cz(e^{-\alpha} + e^{\alpha}) + c^2}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{cz(e^{\alpha} - e^{-\alpha})}{z^2 - cz(e^{\alpha} + e^{-\alpha}) + c^2}\right]$ $\quad\because e^{\theta} + e^{-\theta} = 2\cosh\theta$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{cz(2\sinh\alpha)}{z^2 - cz\,(2\cosh\alpha) + c^2}\right]$ $\quad\because e^{\theta} - e^{-\theta} = 2\sinh\theta$

Thus,

$Z\{c^k \sinh \alpha k\} = \frac{cz\sinh\alpha}{z^2 - 2cz\cosh\alpha + c^2}$

26. Show that $Z\{\cos \alpha k\} = \frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}$

Solution:

$f(k) = \cos \alpha k$

$f(k) = \left[\frac{e^{i\alpha k} + e^{-i\alpha k}}{2}\right]$ $\quad\because \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$f(k) = \frac{\left(e^{i\alpha}\right)^k + \left(e^{-i\alpha}\right)^k}{2}$

$f(k) = \frac{1}{2}\left[\left(e^{i\alpha}\right)^k + \left(e^{-i\alpha}\right)^k\right]$

$Z\{f(k)\} = \frac{1}{2}\left[Z\left\{\left(e^{i\alpha}\right)^k\right\} + Z\left\{\left(e^{-i\alpha}\right)^k\right\}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{z}{z - e^{i\alpha}} + \frac{z}{z - e^{-i\alpha}}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{z^2 - ze^{-i\alpha} + z^2 - ze^{i\alpha}}{z^2 - ze^{-i\alpha} - ze^{i\alpha} + e^{i\alpha}e^{-i\alpha}}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{2z^2 - z\left(e^{-i\alpha} + e^{i\alpha}\right)}{z^2 - z\left(e^{-i\alpha} + e^{i\alpha}\right) + 1}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{2z^2 - z(2\cos\alpha)}{z^2 - z(2\cos\alpha) + 1}\right]$ $\quad\because e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$Z\{f(k)\} = \frac{2}{2}\left[\frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}\right]$

Thus,

$Z\{\cos \alpha k\} = \left[\frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}\right]$

27. Show that $Z\{\cos 2k\} = \frac{z^2-(\cos 2)z}{z^2-2(\cos 2)z+1}$

Solution:

$f(k) = \cos 2k$

$f(k) = \left[\frac{e^{i2k}+e^{-i2k}}{2}\right]$ $\quad\because \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$f(k) = \frac{(e^{2i})^k+(e^{-2i})^k}{2}$

$f(k) = \frac{1}{2}\left[\left(e^{2i}\right)^k + \left(e^{-2i}\right)^k\right]$

$Z\{f(k)\} = \frac{1}{2}\left[Z\left\{\left(e^{2i}\right)^k\right\} + Z\left\{\left(e^{-2i}\right)^k\right\}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{z}{z-e^{2i}} + \frac{z}{z-e^{-2i}}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{z^2-ze^{-2i}+z^2-ze^{2i}}{z^2-ze^{-2i}-ze^{2i}+e^{2i}e^{-2i}}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{2z^2-z\left(e^{-2i}+e^{2i}\right)}{z^2-z\left(e^{-2i}+e^{2i}\right)+1}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{2z^2-z(2\cos 2)}{z^2-z(2\cos 2)+1}\right]$ $\quad\because e^{i\theta}+e^{-i\theta} = 2\cos\theta$

$Z\{f(k)\} = \frac{2}{2}\left[\frac{z^2-z\cos 2}{z^2-2z\cos 2+1}\right]$

Thus, $Z\{\cos 2k\} = \left[\frac{z^2-z\cos 2}{z^2-2z\cos 2+1}\right]$

28. Find the Z transform of $\cos k\frac{\pi}{2}$

Solution:

$f(k) = \cos\frac{\pi}{2}k$

$f(k) = \left[\frac{e^{i\frac{\pi}{2}k}+e^{-i\frac{\pi}{2}k}}{2}\right]$ $\quad\because \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$f(k) = \frac{\left(e^{i\frac{\pi}{2}}\right)^k+\left(e^{-i\frac{\pi}{2}}\right)^k}{2}$

$f(k) = \frac{1}{2}\left[\left(e^{i\frac{\pi}{2}}\right)^k + \left(e^{-i\frac{\pi}{2}}\right)^k\right]$ $\quad\because e^{\pm i\theta} = \cos\theta \pm i\sin\theta$

$f(k) = \frac{1}{2}\left[\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^k + \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^k\right]$

$f(k) = \frac{1}{2}[(i)^k + (-i)^k]$

$Z\{f(k)\} = \frac{1}{2}[Z\{(i)^k\} + Z\{(-i)^k\}]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{z}{z-i} + \frac{z}{z--i}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{z^2+iz+z^2-iz}{(z-i)(z+i)}\right]$

$Z\{f(k)\} = \frac{1}{2}\left[\frac{2z^2}{z^2-i^2}\right] = \frac{z^2}{z^2+1}$

29. Find the Z transform of $f(k) = \cos\left(\frac{k\pi}{3} + \alpha\right)\ , k \geq 0$

Solution:

We have,

$$f(k) = \cos\left(\frac{k\pi}{3} + \alpha\right)$$

$$f(k) = \cos\frac{k\pi}{3}\cos\alpha - \sin\frac{k\pi}{3}\sin\alpha$$

$$Z\{f(k)\} = \cos\alpha\ Z\left\{\cos\frac{k\pi}{3}\right\} - \sin\alpha\ Z\left\{\sin\frac{k\pi}{3}\right\}$$

$$Z\{f(k)\} = \cos\alpha\ \left[\frac{z^2 - z\cos\frac{\pi}{3}}{z^2 - 2z\cos\frac{\pi}{3} + 1}\right] - \sin\alpha\ \left[\frac{z\sin\frac{\pi}{3}}{z^2 - 2z\cos\frac{\pi}{3} + 1}\right]$$

$$Z\{f(k)\} = \frac{\cos\alpha\left(z^2 - z\cos\frac{\pi}{3}\right) - \sin\alpha\ z\left(\sin\frac{\pi}{3}\right)}{z^2 - 2z\left(\frac{1}{2}\right) + 1}$$

$$Z\{f(k)\} = \frac{z^2\cos\alpha - z\cos\alpha\cos\frac{\pi}{3} - z\sin\alpha\sin\frac{\pi}{3}}{z^2 - z + 1}$$

$$Z\{f(k)\} = \frac{z^2\cos\alpha - z\left(\cos\frac{\pi}{3}\cos\alpha + \sin\frac{\pi}{3}\sin\alpha\right)}{z^2 - z + 1}$$

$$Z\{f(k)\} = \frac{z^2\cos\alpha - z\cos\left(\frac{\pi}{3} - \alpha\right)}{z^2 - z + 1}$$

30. Find $Z\{f(k)\}$ where $f(k) = \cos\left(\frac{k\pi}{4} + a\right)$ where $k \geq 0$

[D23/CompITAI/6M]

Solution:

We have,

$$f(k) = \cos\left(\frac{k\pi}{4} + \alpha\right)$$

$$f(k) = \cos\frac{k\pi}{4}\cos\alpha - \sin\frac{k\pi}{4}\sin\alpha$$

$$Z\{f(k)\} = \cos\alpha\ Z\left\{\cos\frac{k\pi}{4}\right\} - \sin\alpha\ Z\left\{\sin\frac{k\pi}{4}\right\}$$

$$Z\{f(k)\} = \cos\alpha\ \left[\frac{z^2 - z\cos\frac{\pi}{4}}{z^2 - 2z\cos\frac{\pi}{4} + 1}\right] - \sin\alpha\ \left[\frac{z\sin\frac{\pi}{4}}{z^2 - 2z\cos\frac{\pi}{4} + 1}\right]$$

$$Z\{f(k)\} = \frac{\cos\alpha\left(z^2 - z\cos\frac{\pi}{4}\right) - \sin\alpha\ z\left(\sin\frac{\pi}{4}\right)}{z^2 - 2z\left(\frac{1}{\sqrt{2}}\right) + 1}$$

$$Z\{f(k)\} = \frac{z^2\cos\alpha - z\cos\alpha\cos\frac{\pi}{4} - z\sin\alpha\sin\frac{\pi}{4}}{z^2 - \sqrt{2}z + 1}$$

$$Z\{f(k)\} = \frac{z^2\cos\alpha - z\left(\cos\frac{\pi}{4}\cos\alpha + \sin\frac{\pi}{4}\sin\alpha\right)}{z^2 - \sqrt{2}z + 1}$$

$$Z\{f(k)\} = \frac{z^2\cos\alpha - z\cos\left(\frac{\pi}{4} - \alpha\right)}{z^2 - \sqrt{2}z + 1}$$

31. Find the Z-transform of $\cos\left(\frac{\pi}{4}+k\alpha\right)$ where $k \geq 0$

[D24/CompITAI/6M]

Solution:

We have,

$$f(k)=\cos\left(\frac{\pi}{4}+k\alpha\right)$$

$$f(k)=\cos\frac{\pi}{4}\cos k\alpha-\sin\frac{\pi}{4}\sin k\alpha$$

$$Z\{f(k)\}=\cos\frac{\pi}{4}\ Z\{\cos k\alpha\}-\sin\frac{\pi}{4}\ Z\{\sin k\alpha\}$$

$$Z\{f(k)\}=\frac{1}{\sqrt{2}}\left[\frac{z^2-z\cos\alpha}{z^2-2z\cos\alpha+1}\right]-\frac{1}{\sqrt{2}}\left[\frac{z\sin\alpha}{z^2-2z\cos\alpha+1}\right]$$

$$Z\{f(k)\}=\frac{1}{\sqrt{2}}.\frac{z^2-z\cos\alpha-z\sin\alpha}{z^2-2z\cos\alpha+1}$$

$$Z\{f(k)\}=\frac{z^2-z\cos\alpha-z\sin\alpha}{\sqrt{2}(z^2-2z\cos\alpha+1)}$$

32. Find the Z transform of $f(k)=\sin(3k+5)$

Solution:

Consider,

$$\sin(3k+5)=\sin 3k\cos 5+\cos 3k\sin 5$$

$$Z\{\sin(3k+5)\}=\cos 5\ Z\{\sin 3k\}+\sin 5\ Z\{\cos 3k\}$$

$$Z\{\sin(3k+5)\}=\cos 5\left[\frac{z\sin 3}{z^2-2z\cos 3+1}\right]+\sin 5\left[\frac{z^2-z\cos 3}{z^2-2z\cos 3+1}\right]$$

$$Z\{\sin(3k+5)\}=\frac{z\sin 3\cos 5+z^2\sin 5-z\sin 5\cos 3}{z^2-2z\cos 3+1}$$

$$Z\{\sin(3k+5)\}=\frac{z^2\sin 5+z(\sin 3\cos 5-\cos 3\sin 5)}{z^2-2z\cos 3+1}$$

$$Z\{\sin(3k+5)\}=\frac{z^2\sin 5+z\sin(3-5)}{z^2-2z\cos 3+1}$$

$$Z\{\sin(3k+5)\}=\frac{z^2\sin 5+z\sin(-2)}{z^2-2z\cos 3+1}$$

$$Z\{\sin(3k+5)\}=\frac{z^2\sin 5-z\sin 2}{z^2-2z\cos 3+1}$$

Type II: Based on property

1. Find $Z[2^k \sin(3k+2)],\ k \geq 0$

Solution:

$$\sin(3k+2) = \sin 3k \cos 2 + \cos 3k \sin 2$$
$$Z\{\sin(3k+2)\} = \cos 2\ \ Z\{\sin 3k\} + \sin 2\ \ Z\{\cos 3k\}$$
$$Z\{\sin(3k+2)\} = \cos 2\ \left[\frac{z\sin 3}{z^2-2z\cos 3+1}\right] + \sin 2\ \left[\frac{z^2-z\cos 3}{z^2-2z\cos 3+1}\right]$$
$$Z\{\sin(3k+2)\} = \frac{z\sin 3\cos 2+z^2\sin 2-z\sin 2\cos 3}{z^2-2z\cos 3+1}$$
$$Z\{\sin(3k+2)\} = \frac{z^2\sin 2+z(\sin 3\cos 2-\cos 3\sin 2)}{z^2-2z\cos 3+1}$$
$$Z\{\sin(3k+2)\} = \frac{z^2\sin 2+z\sin(3-2)}{z^2-2z\cos 3+1}$$
$$Z\{\sin(3k+2)\} = \frac{z^2\sin 2+z\sin 1}{z^2-2z\cos 3+1}$$

By change of scale property,

$$Z\{2^k\sin(3k+2)\} = \frac{\left(\frac{z}{2}\right)^2\sin 2+\left(\frac{z}{2}\right)\sin 1}{\left(\frac{z}{2}\right)^2-2\left(\frac{z}{2}\right)\cos 3+1} \times \frac{4}{4}$$
$$Z\{2^k\sin(3k+2)\} = \frac{z^2\sin 2+2z\sin 1}{z^2-4z\cos 3+4}$$

2. Find $Z\left[3^k\cos\left(k\frac{\pi}{2}+\frac{\pi}{4}\right)\right], k \geq 0$

Solution:

$$\cos\left(\frac{k\pi}{2}+\frac{\pi}{4}\right) = \cos\frac{k\pi}{2}\cos\frac{\pi}{4} - \sin\frac{k\pi}{2}\sin\frac{\pi}{4}$$
$$\cos\left(\frac{k\pi}{2}+\frac{\pi}{4}\right) = \cos\frac{k\pi}{2}\ .\frac{1}{\sqrt{2}} - \sin\frac{k\pi}{2}\ .\frac{1}{\sqrt{2}}$$
$$Z\left\{\cos\left(\frac{k\pi}{2}+\frac{\pi}{4}\right)\right\} = \frac{1}{\sqrt{2}}\ Z\left\{\cos\frac{k\pi}{2} - \sin\frac{k\pi}{2}\right\}$$
$$Z\left\{\cos\left(\frac{k\pi}{2}+\frac{\pi}{4}\right)\right\} = \frac{1}{\sqrt{2}}\left[\frac{z^2-z\cos\frac{\pi}{2}}{z^2-2z\cos\frac{\pi}{2}+1} - \frac{z\sin\frac{\pi}{2}}{z^2-2z\cos\frac{\pi}{2}+1}\right]$$
$$Z\left\{\cos\left(\frac{k\pi}{2}+\frac{\pi}{4}\right)\right\} = \frac{1}{\sqrt{2}}\left[\frac{z^2-z(0)-z(1)}{z^2-2z(0)+1}\right]$$
$$Z\left\{\cos\left(\frac{k\pi}{2}+\frac{\pi}{4}\right)\right\} = \frac{1}{\sqrt{2}}\left[\frac{z^2-z}{z^2+1}\right]$$

By change of scale property,

$$Z\left\{3^k\cos\left(\frac{k\pi}{2}+\frac{\pi}{4}\right)\right\} = \frac{1}{\sqrt{2}}\left[\frac{\left(\frac{z}{3}\right)^2-\frac{z}{3}}{\left(\frac{z}{3}\right)^2+1}\right] \times \frac{9}{9}$$
$$Z\left\{3^k\cos\left(\frac{k\pi}{2}+\frac{\pi}{4}\right)\right\} = \frac{1}{\sqrt{2}}\left[\frac{z^2-3z}{z^2+9}\right]$$

3. Find Z transform of $2^k \sinh 3k\,, k \geq 0$
[M17/CompIT/6M]
Solution:
We know that,
$Z\{a^k\} = \frac{z}{z-a}$
Now,
$$Z\{\sinh 3k\} = Z\left\{\frac{e^{3k}-e^{-3k}}{2}\right\}$$
$$= \frac{1}{2}\; Z\{e^{3k} - e^{-3k}\}$$
$$= \frac{1}{2}\left[\frac{z}{z-e^3} - \frac{z}{z-e^{-3}}\right]$$
$$= \frac{1}{2}\left[\frac{z^2-ze^{-3}-z^2+ze^3}{z^2-e^3z-e^{-3}z+1}\right]$$
$$= \frac{1}{2}\left[\frac{z(e^3-e^{-3})}{z^2-z(e^3+e^{-3})+1}\right]$$
$$= \frac{1}{2}\left[\frac{z(2\sinh 3)}{z^2-z(2\cosh 3)+1}\right]$$
$$Z\{\sinh 3k\} = \frac{z\sinh 3}{z^2-2z\cosh 3+1}$$
Now, by Change of scale property $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$
$$Z\{2^k \sinh 3k\} = \frac{\frac{z}{2}\sinh 3}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cosh 3+1} = \frac{2z\sinh 3}{z^2-4z\cosh 3+4}$$

4. Find $Z[2^k \cos(3k+2)], k \geq 0$
[N15/CompIT/6M][M22/CompITAI/5M][N22/M24/CompITAI/6M]
Solution:
$$Z\{\cos(3k+2)\} = Z\{\cos 3k \cos 2 - \sin 3k \sin 2\}$$
$$Z\{\cos(3k+2)\} = \cos 2\;\; Z\{\cos 3k\} - \sin 2\;\; Z\{\sin 3k\}$$
$$Z\{\cos(3k+2)\} = \cos 2\;\left[\frac{z(z-\cos 3)}{z^2-2z\cos 3+1}\right] - \sin 2\;\left[\frac{z\sin 3}{z^2-2z\cos 3+1}\right]$$
By $Z\{\cos\alpha k\} = \frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}, Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2-2z\cos\alpha+1}$
$$\therefore Z\{\cos(3k+2)\} = \frac{z^2\cos 2 - z\cos 2\cos 3 - z\sin 2\sin 3}{z^2-2z\cos 3+1}$$
$$Z\{\cos(3k+2)\} = \frac{z^2\cos 2 - z(\cos 2\cos 3 + \sin 2\sin 3)}{z^2-2z\cos 3+1}$$
$$Z\{\cos(3k+2)\} = \frac{z^2\cos 2 - z\cos 1}{z^2-2z\cos 3+1}$$
Now, by Change of scale property $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$
$$Z\{2^k\cos(3k+2)\} = \frac{\left(\frac{z}{2}\right)^2\cos 2 - \left(\frac{z}{2}\right)\cos 1}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cos 3+1}$$
$$Z\{2^k\cos(3k+2)\} = \frac{z^2\cos 2 - 2z\cos 1}{z^2-4z\cos 3+4}$$

5. Find the Z transform of $e^{3k}\sin 2k$

Solution:

We know that,

$Z\{a^k\} = \frac{z}{z-a}$

Now,

$$Z\{\sin 2k\} = Z\left\{\frac{e^{i2k}-e^{-i2k}}{2i}\right\}$$
$$= \frac{1}{2i}\, Z\{e^{i2k} - e^{-i2k}\}$$
$$= \frac{1}{2i}\left[\frac{z}{z-e^{2i}} - \frac{z}{z-e^{-2i}}\right]$$
$$= \frac{1}{2i}\left[\frac{z^2-ze^{-2i}-z^2+ze^{2i}}{z^2-e^{2i}z-e^{-2i}z+1}\right]$$
$$= \frac{1}{2i}\left[\frac{z\left(e^{2i}-e^{-2i}\right)}{z^2-z\left(e^{2i}+e^{-2i}\right)+1}\right]$$
$$= \frac{1}{2i}\left[\frac{z(2i\sin 2)}{z^2-z(2\cos 2)+1}\right]$$
$$Z\{\sin 2k\} = \frac{z\sin 2}{z^2-2z\cos 2+1}$$

Now, by Change of scale property $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

$$Z\{e^{3k}\sin 2k\} = Z\{(e^3)^k \sin 2k\}$$
$$Z\{e^{3k}\sin 2k\} = \frac{\frac{z}{e^3}\sin 2}{\left(\frac{z}{e^3}\right)^2 - 2\left(\frac{z}{e^3}\right)\cos 2+1} = \frac{e^3 z\sin 2}{z^2-2e^3z\cos 2+e^6}$$

6. Find $Z[3^k \sinh \alpha k],\ k \ge 0$

Solution:

We know that,

$Z\{a^k\} = \frac{z}{z-a}$

Now,

$$Z\{\sinh \alpha k\} = Z\left\{\frac{e^{\alpha k}-e^{-\alpha k}}{2}\right\}$$
$$= \frac{1}{2}\, Z\{e^{\alpha k} - e^{-\alpha k}\}$$
$$= \frac{1}{2}\left[\frac{z}{z-e^{\alpha}} - \frac{z}{z-e^{-\alpha}}\right]$$
$$= \frac{1}{2}\left[\frac{z^2-ze^{-\alpha}-z^2+ze^{\alpha}}{z^2-e^{\alpha}z-e^{-\alpha}z+1}\right]$$
$$= \frac{1}{2}\left[\frac{z(e^{\alpha}-e^{-\alpha})}{z^2-z(e^{\alpha}+e^{-\alpha})+1}\right]$$
$$= \frac{1}{2}\left[\frac{z(2\sinh \alpha)}{z^2-z(2\cosh \alpha)+1}\right]$$
$$Z\{\sinh \alpha k\} = \frac{z\sinh \alpha}{z^2-2z\cosh \alpha+1}$$

Now, by Change of scale property $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

$$Z\{3^k \sinh \alpha k\} = \frac{\frac{z}{3}\sinh \alpha}{\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right)\cosh \alpha + 1} = \frac{3z \sinh \alpha}{z^2 - 6z\cosh \alpha + 9}$$

7. Find $Z\left[3^k \sin\left(k\frac{\pi}{2} + \frac{\pi}{4}\right)\right], k \geq 0$

Solution:

$$\sin\left(\frac{k\pi}{2} + \frac{\pi}{4}\right) = \sin\frac{k\pi}{2}\cos\frac{\pi}{4} + \cos\frac{k\pi}{2}\sin\frac{\pi}{4}$$

$$\sin\left(\frac{k\pi}{2} + \frac{\pi}{4}\right) = \sin\frac{k\pi}{2} \cdot \frac{1}{\sqrt{2}} + \cos\frac{k\pi}{2} \cdot \frac{1}{\sqrt{2}}$$

$$Z\left\{\sin\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\} = \frac{1}{\sqrt{2}} Z\left\{\sin\frac{k\pi}{2} + \cos\frac{k\pi}{2}\right\}$$

$$Z\left\{\sin\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\} = \frac{1}{\sqrt{2}}\left[\frac{z\sin\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1} + \frac{z^2 - z\cos\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1}\right]$$

$$Z\left\{\sin\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\} = \frac{1}{\sqrt{2}}\left[\frac{z(1) + z^2 - z(0)}{z^2 - 2z(0) + 1}\right]$$

$$Z\left\{\sin\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\} = \frac{1}{\sqrt{2}}\left[\frac{z^2 + z}{z^2 + 1}\right]$$

By change of scale property,

$$Z\left\{3^k \sin\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\} = \frac{1}{\sqrt{2}}\left[\frac{\left(\frac{z}{3}\right)^2 + \frac{z}{3}}{\left(\frac{z}{3}\right)^2 + 1}\right] \times \frac{9}{9}$$

$$Z\left\{3^k \sin\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\} = \frac{1}{\sqrt{2}}\left[\frac{z^2 + 3z}{z^2 + 9}\right]$$

8. Find $Z[k.e^{-ak}], k \geq 0$

Solution:

We know that,

$$Z\{a^k\} = \frac{z}{z-a}$$

Thus,

$$Z\{e^{-ak}\} = Z\{(e^{-a})^k\} = \frac{z}{z - e^{-a}}$$

$$Z\{k\, e^{-ak}\} = -z\frac{d}{dz}\left[\frac{z}{z - e^{-a}}\right]$$

$$Z\{k\, e^{-ak}\} = -z\left[\frac{(z - e^{-a})[1] - z[1-0]}{(z - e^{-a})^2}\right]$$

$$Z\{k\, e^{-ak}\} = -z\left[\frac{z - e^{-a} - z}{(z - e^{-a})^2}\right]$$

$$Z\{k\, e^{-ak}\} = -z\left[\frac{-e^{-a}}{(z - e^{-a})^2}\right]$$

$$Z\{k\, e^{-ak}\} = \frac{z\, e^{-a}}{(z - e^{-a})^2}$$

9. Find $Z[k^2 e^{-ak}], k \geq 0$

Solution:

We know that,

$$Z\{a^k\} = \frac{z}{z-a}$$

Thus,

$$Z\{e^{-ak}\} = Z\{(e^{-a})^k\} = \frac{z}{z-e^{-a}}$$

$$Z\{k\, e^{-ak}\} = -z\frac{d}{dz}\left[\frac{z}{z-e^{-a}}\right]$$

$$Z\{k\, e^{-ak}\} = -z\left[\frac{(z-e^{-a})[1]-z[1-0]}{(z-e^{-a})^2}\right]$$

$$Z\{k\, e^{-ak}\} = -z\left[\frac{z-e^{-a}-z}{(z-e^{-a})^2}\right]$$

$$Z\{k\, e^{-ak}\} = -z\left[\frac{-e^{-a}}{(z-e^{-a})^2}\right]$$

$$Z\{k\, e^{-ak}\} = \frac{z\, e^{-a}}{(z-e^{-a})^2}$$

Now,

$$Z\{k.\, k\, e^{-ak}\} = -z\frac{d}{dz}\left[\frac{z\, e^{-a}}{(z-e^{-a})^2}\right]$$

$$Z\{k^2 e^{-ak}\} = -z\left[\frac{(z-e^{-a})^2[1.e^{-a}]-ze^{-a}[2(z-e^{-a})]}{(z-e^{-a})^4}\right]$$

$$Z\{k^2 e^{-ak}\} = -z(z-e^{-a})\left[\frac{(z-e^{-a})e^{-a}-ze^{-a}(2)}{(z-e^{-a})^4}\right]$$

$$Z\{k^2 e^{-ak}\} = -z\left[\frac{z\, e^{-a}-e^{-2a}-2z\, e^{-a}}{(z-e^{-a})^3}\right]$$

$$Z\{k^2 e^{-ak}\} = \frac{-z(-z\, e^{-a}-e^{-2a})}{(z-e^{-a})^3}$$

$$Z\{k^2 e^{-ak}\} = \frac{z^2 e^{-a}+z\, e^{-2a}}{(z-e^{-a})^3}$$

10. Find the Z transform of $f(k) = k^2 - 2k + 3,\ k \geq 0$

Solution:

We know that,

$Z\{a^k\} = \frac{z}{z-a}$

Put $a = 1$

$Z\{1^k\} = Z\{1\} = \frac{z}{z-1}$

Thus ,

$Z\{k.1\} = -z\frac{d}{dz}\left[\frac{z}{z-1}\right]$

$Z\{k\} = -z\ \left[\frac{(z-1)[1]-z[1-0]}{(z-1)^2}\right]$

$Z\{k\} = -z\left[\frac{z-1-z}{(z-1)^2}\right]$

$Z\{k\} = \frac{z}{(z-1)^2}$

Also,

$Z\{k.k\} = -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$

$Z\{k^2\} = -z\ \left[\frac{(z-1)^2[1]-z[2(z-1)]}{(z-1)^4}\right]$

$Z\{k^2\} = -z(z-1)\left[\frac{(z-1)-2z}{(z-1)^4}\right]$

$Z\{k^2\} = -z\left[\frac{-z-1}{(z-1)^3}\right]$

$Z\{k^2\} = \frac{z^2+z}{(z-1)^3}$

Now,

$f(k) = k^2 - 2k + 3$

$Z\{f(k)\} = Z\{k^2\} - 2Z\{k\} + 3Z\{1\}$

$Z\{f(k)\} = \frac{z^2+z}{(z-1)^3} - 2\left[\frac{z}{(z-1)^2}\right] + 3\left[\frac{z}{z-1}\right]$

$Z\{f(k)\} = \frac{z^2+z-2z(z-1)+3z(z-1)^2}{(z-1)^3}$

$Z\{f(k)\} = \frac{z^2+z-2z^2+2z+3z(z^2-2z+1)}{(z-1)^3}$

$Z\{f(k)\} = \frac{3z^3+z^2-2z^2-6z^2+z+2z+3z}{(z-1)^3}$

$Z\{f(k)\} = \frac{3z^3-7z^2+6z}{(z-1)^3}$

11. Find the Z transform of $f(k) = a^{k-1}, k \geq 0$

Solution:

We know that,

$Z\{a^k\} = \frac{z}{z-a}$

By shifting property,

$Z\{a^{k-1}\} = z^{-1}.\frac{z}{z-a} = \frac{1}{z-a}$

12. Find $Z[k^2 a^{k-1}], k \geq 0$

Solution:

We know that,

$Z\{a^k\} = \frac{z}{z-a}$

By shifting property,

$Z\{a^{k-1}\} = z^{-1}.\frac{z}{z-a} = \frac{1}{z-a}$

Now,

$Z\{k\ a^{k-1}\} = -z\frac{d}{dz}\left[\frac{1}{z-a}\right]$

$Z\{k\ a^{k-1}\} = -z\left[\frac{(z-a)[0]-1[1-0]}{(z-a)^2}\right]$

$Z\{k\ a^{k-1}\} = -z\left[\frac{-1}{(z-a)^2}\right]$

$Z\{k\ a^{k-1}\} = \frac{z}{(z-a)^2}$

Now,

$Z\{k\ k\ a^{k-1}\} = -z\frac{d}{dz}\left[\frac{z}{(z-a)^2}\right]$

$Z\{k^2 a^{k-1}\} = -z\left[\frac{(z-a)^2[1]-z[2(z-a)]}{(z-a)^4}\right]$

$Z\{k^2 a^{k-1}\} = -z(z-a)\left[\frac{z-a-2z}{(z-a)^4}\right]$

$Z\{k^2 a^{k-1}\} = -z\left[\frac{-z-a}{(z-a)^3}\right]$

$Z\{k^2 a^{k-1}\} = \frac{z^2+az}{(z-a)^3}$

13. Find $Z[k^2 a^{k-1}\ U(k - 1)], k \geq 0$

[M16/CompIT/6M]

Solution:

By definition,

$$U(k) = \begin{cases} 1 & for\ k \geq 0 \\ 0 & for\ k < 0 \end{cases}$$

$$Z\{U(k)\} = \sum_0^{\infty} 1. z^{-k}$$

$$= [z^0 + z^{-1} + z^{-2} + z^{-3} + \cdots \ldots..]$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \ldots$$

$$= \left[1 - \frac{1}{z}\right]^{-1}$$

$$= \frac{1}{1-\frac{1}{z}}$$

$$\therefore Z\{U(k)\} = \frac{z}{z-1}$$

By Change of Scale, $Z\{a^k U(k)\} = \frac{\frac{z}{a}}{\frac{z}{a}-1} = \frac{z}{z-a}$

By Shifting Property,

$$Z\{a^{k-1}U(k-1)\} = z^{-1}.\frac{z}{z-a} = \frac{1}{z-a}$$

By Multiplication by k,

$$Z\{k\ a^{k-1}U(k-1)\} = -z\frac{d}{dz}\left[\frac{1}{z-a}\right]$$

$$= -z\left[-\frac{1}{(z-a)^2}\right]$$

$$Z\{k\ a^{k-1}U(k-1)\} = \frac{z}{(z-a)^2}$$

By Multiplication by k,

$$Z\{k^2 a^{k-1}U(k-1)\} = -z\frac{d}{dz}\left[\frac{z}{(z-a)^2}\right]$$

$$= -z\left[\frac{(z-a)^2[1]-z[2(z-a)]}{(z-a)^4}\right]$$

$$= -z\left[\frac{z-a-2z}{(z-a)^3}\right]$$

$$= -\frac{z(-z-a)}{(z-a)^3}$$

$$Z\{k^2 a^{k-1}U(k-1)\} = \frac{z(z+a)}{(z-a)^3}$$

14. Find the Z transform of $\delta(k-n)$ where $\delta(k) = \begin{cases} 1 & k=0 \\ 0 & otherwise \end{cases}$

Solution:

$\delta(k) = \begin{cases} 1 & k=0 \\ 0 & otherwise \end{cases}$

$Z\{\delta(k)\} = \sum_0^0 1.z^{-k}$

$Z\{\delta(k)\} = z^0 = 1$

By shifting property,

$Z\{\delta(k-n)\} = z^{-n}.1 = \frac{1}{z^n}$

15. Find $Z[(k+1)a^k], k \geq 0$

Solution:

$$\begin{aligned} Z\{(k+1)a^k\} &= Z\{k\,a^k + a^k\} \\ &= (-z)\frac{d}{dz}[Z\{a^k\}] + Z\{a^k\} \\ &= (-z)\frac{d}{dz}\left[\frac{z}{z-a}\right] + \frac{z}{z-a} \\ &= (-z)\left[\frac{(z-a)(1)-z(1-0)}{(z-a)^2}\right] + \frac{z}{z-a} \\ &= (-z)\left[\frac{-a}{(z-a)^2}\right] + \frac{z}{z-a} \\ &= \frac{az}{(z-a)^2} + \frac{z}{z-a} \\ &= \frac{az+z(z-a)}{(z-a)^2} \\ &= \frac{z^2}{(z-a)^2} \end{aligned}$$

16. Find $Z\{2^k k^2\}$

Solution:

We know that,

$Z\{a^k\} = \frac{z}{z-a}$

Thus,

$Z\{2^k\} = \frac{z}{z-2}$

$Z\{k\ 2^k\} = -z\frac{d}{dz}\left[\frac{z}{z-2}\right]$

$Z\{k\ 2^k\} = -z\left[\frac{(z-2)[1]-z[1-0]}{(z-2)^2}\right]$

$Z\{k\ 2^k\} = -z\left[\frac{z-2-z}{(z-2)^2}\right]$

$Z\{k\ 2^k\} = -z\left[\frac{-2}{(z-2)^2}\right]$

$Z\{k\ 2^k\} = \frac{2z}{(z-2)^2}$

Now,

$Z\{k.k\ 2^k\} = -z\frac{d}{dz}\left[\frac{2z}{(z-2)^2}\right]$

$Z\{k^2 2^k\} = -z\left[\frac{(z-2)^2[2]-2z[2(z-2)]}{(z-2)^4}\right]$

$Z\{k^2 2^k\} = -z(z-2)\left[\frac{(z-2)2-2z(2)}{(z-2)^4}\right]$

$Z\{k^2 2^k\} = -z\left[\frac{2z-4-4z}{(z-2)^3}\right]$

$Z\{k^2 2^k\} = \frac{-z(-2z-4)}{(z-2)^3}$

$Z\{k^2 2^k\} = \frac{2z(z+2)}{(z-2)^3}$

17. Find $Z[k2^k + k3^k], k \geq 0$

Solution:

$Z\{k\ 2^k + k\ 3^k\} = Z\{k\ 2^k\} + Z\{k\ 3^k\}$

$= (-z)\frac{d}{dz}\left[\frac{z}{z-2}\right] + (-z)\frac{d}{dz}\left[\frac{z}{z-3}\right]$

$= (-z)\left[\frac{(z-2)[1]-z[1-0]}{(z-2)^2}\right] + (-z)\left[\frac{(z-3)[1]-z[1-0]}{(z-3)^2}\right]$

$= -z\left[\frac{z-2-z}{(z-2)^2}\right] - z\left[\frac{z-3-z}{(z-3)^2}\right]$

$= \frac{2z}{(z-2)^2} + \frac{3z}{(z-3)^2}$

18. Find the Z transform of (i) $4^k\delta(k-1)$ and (ii) $U(k-1)$
where $\delta(k)=\begin{cases}1 & k=0\\ 0 & otherwise\end{cases}$ and $U(k)=\begin{cases}1 & k\geq 0\\ 0 & otherwise\end{cases}$

Solution:

$\delta(k)=\begin{cases}1 & k=0\\ 0 & otherwise\end{cases}$

$Z\{\delta(k)\}=\sum_0^0 1.z^{-k}$

$Z\{\delta(k)\}=z^0=1$

By shifting property,

$Z\{\delta(k-1)\}=z^{-1}.1=\frac{1}{z}$

By change of scale property,

$Z\{4^k\delta(k-1)\}=\frac{1}{\frac{z}{4}}=\frac{4}{z}$

Now,

$U(k)=\begin{cases}1 & for\ k\geq 0\\ 0 & for\ k<0\end{cases}$

$$\begin{aligned}Z\{U(k)\} &= \sum_0^\infty 1.z^{-k}\\ &= [z^0+z^{-1}+z^{-2}+z^{-3}+\cdots \ldots\ldots]\\ &= 1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots \ldots.\\ &= \left[1-\frac{1}{z}\right]^{-1}\\ &= \frac{1}{1-\frac{1}{z}}\end{aligned}$$

$\therefore Z\{U(k)\}=\frac{z}{z-1}$

By shifting property,

$Z\{U(k-1)\}=z^{-1}.\frac{z}{z-1}=\frac{1}{z-1}$

19. According to Time shifting property of z transform, if $X(z)$ is the z transform of $x(n)$ then what is the z transform of $x(n-k)$?

[M22/CompITAI/2M] Ans. $z^{-k}X(z)$

Type III: Convolution Theorem

1. State convolution Theorem for z transform hence if
   $f(k) = U(k)$ & $g(k) = 2^k U(k)$, find $Z\{f(k) * g(k)\}$

   Solution:

   If $Z\{f(k)\} = F(z)$ and $Z\{g(k) = G(z)$

   Then by Convolution Theorem,

   $Z\{f(k) * g(k)\} = F(z).G(z)$

   By definition,

   $$U(k) = \begin{cases} 1 & for\ k \geq 0 \\ 0 & for\ k < 0 \end{cases}$$

   $$Z\{U(k)\} = \sum_0^\infty 1.z^{-k}$$
   $$= [z^0 + z^{-1} + z^{-2} + z^{-3} + \cdots \ldots..]$$
   $$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \ldots$$
   $$= \left[1 - \frac{1}{z}\right]^{-1}$$
   $$= \frac{1}{1-\frac{1}{z}}$$
   $$\therefore Z\{U(k)\} = \frac{z}{z-1}$$

   By Change of Scale,

   $$Z\{2^k U(k)\} = \frac{\frac{z}{2}}{\frac{z}{2}-1} = \frac{z}{z-2}$$

   Now,

   $$Z\{f(k)\} = Z\{U(k)\}$$
   $$F(z) = \frac{z}{z-1}$$

   Also,

   $$Z\{g(k)\} = Z\{2^k U(k)\}$$
   $$G(z) = \frac{z}{z-2}$$

   By Convolution Theorem,

   $$Z\{f(k) * g(k)\} = F(z).G(z) = \frac{z}{z-1}.\frac{z}{z-2} = \frac{z^2}{(z-1)(z-2)}$$

2. State convolution Theorem for z transform hence if $f(k) = \frac{1}{3^k}, k \geq 0$ and $g(k) = \frac{1}{4^k}, k \geq 0$, find $Z\{f(k) * g(k)\}$

[M15/CompIT/5M]

Solution:

If $Z\{f(k)\} = F(z)$ and $Z\{g(k) = G(z)$

Then by Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z).G(z)$

We have,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$

$Z\left\{\frac{1}{3^k}\right\} = \sum_{0}^{\infty} \frac{1}{3^k}.z^{-k}$

$= \frac{z^0}{3^0} + \frac{z^{-1}}{3^1} + \frac{z^{-2}}{3^2} + \frac{z^{-3}}{3^3} + \cdots..$

$= 1 + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \cdots ....$

$= \left[1 - \frac{1}{3z}\right]^{-1}$

$= \frac{1}{1-\frac{1}{3z}}$

$Z\left\{\frac{1}{3^k}\right\} = \frac{3z}{3z-1}$

$\therefore Z\{f(k)\} = Z\left\{\frac{1}{3^k}\right\}$

$\therefore F(z) = \frac{3z}{3z-1}$

Similarly,

$Z\left\{\frac{1}{4^k}\right\} = \frac{4z}{4z-1}$

$\therefore Z\{g(k)\} = Z\left\{\frac{1}{4^k}\right\}$

$\therefore G(z) = \frac{4z}{4z-1}$

By Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z).G(z) = \frac{3z}{3z-1}.\frac{4z}{4z-1} = \frac{12z^2}{(3z-1)(4z-1)}$

3. State convolution Theorem for z transform hence if $f(k) = 4^k U(k) \& g(k) = 5^k U(k)$, find $Z\{f(k) * g(k)\}$
[M14/M23/CompIT/6M]
Solution:
If $Z\{f(k)\} = F(z)$ and $Z\{g(k) = G(z)$
Then by Convolution Theorem,
$Z\{f(k) * g(k)\} = F(z).G(z)$
By definition,
$$U(k) = \begin{cases} 1 & for\ k \geq 0 \\ 0 & for\ k < 0 \end{cases}$$
$$Z\{U(k)\} = \sum_0^\infty 1.z^{-k}$$
$$= [z^0 + z^{-1} + z^{-2} + z^{-3} + \cdots \ldots\ldots]$$
$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \ldots.$$
$$= \left[1 - \frac{1}{z}\right]^{-1}$$
$$= \frac{1}{1-\frac{1}{z}}$$
$$\therefore Z\{U(k)\} = \frac{z}{z-1}$$
By Change of Scale,
$$Z\{a^k U(k)\} = \frac{\frac{z}{a}}{\frac{z}{a}-1} = \frac{z}{z-a}$$
Now,
$$Z\{f(k)\} = Z\{4^k U(k)\}$$
$$F(z) = \frac{z}{z-4}$$
Also,
$$Z\{g(k)\} = Z\{5^k U(k)\}$$
$$G(z) = \frac{z}{z-5}$$
By Convolution Theorem,
$$Z\{f(k) * g(k)\} = F(z).G(z) = \frac{z}{z-4}.\frac{z}{z-5} = \frac{z^2}{(z-4)(z-5)}$$

4. State convolution Theorem for z transform hence if $f(k) = \frac{1}{2^k}, k \geq 0$ &$g(k) = \cos k\pi, k \geq 0$, find $Z\{f(k) * g(k)$

[N18/Comp/6M][N22/CompITAI/6M]

Solution:

If $Z\{f(k)\} = F(z)$ and $Z\{g(k) = G(z)$

Then by Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z).G(z)$

We have,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$

$Z\left\{\frac{1}{2^k}\right\} = \sum_{0}^{\infty} \frac{1}{2^k}.z^{-k}$

$= \frac{z^0}{2^0} + \frac{z^{-1}}{2^1} + \frac{z^{-2}}{2^2} + \frac{z^{-3}}{2^3} + \cdots ..$

$= 1 + \frac{1}{2z} + \frac{1}{(2z)^2} + \frac{1}{(2z)^3} + \cdots ....$

$= \left[1 - \frac{1}{2z}\right]^{-1}$

$= \frac{1}{1-\frac{1}{2z}}$

$Z\left\{\frac{1}{2^k}\right\} = \frac{2z}{2z-1}$

$\therefore Z\{f(k)\} = Z\left\{\frac{1}{2^k}\right\}$

$\therefore F(z) = \frac{2z}{2z-1}$

Also,

We know that,

$Z\{a^k\} = \frac{z}{z-a}$

Now,

$Z\{\cos k\pi\} = Z\left\{\frac{e^{\pi ik}+e^{-\pi ik}}{2}\right\}$

$= \frac{1}{2} Z\{e^{\pi ik} + e^{-\pi k}\}$

$= \frac{1}{2}\left[\frac{z}{z-e^{\pi i}} + \frac{z}{z-e^{-\pi i}}\right]$

$= \frac{1}{2}\left[\frac{z^2-ze^{-\pi i}+z^2-ze^{\pi i}}{z^2-e^{\pi i}z-e^{-\pi i}z+1}\right]$

$= \frac{1}{2}\left[\frac{2z^2-z(e^{\pi i}-e^{-\pi i})}{z^2-z(e^{\pi i}+e^{-\pi i})+1}\right]$

$= \frac{1}{2}\left[\frac{2z^2-z(2\cos\pi)}{z^2-z(2\cos\pi)+1}\right]$

$Z\{\cos k\pi\} = \frac{z^2+z}{z^2+2z+1} = \frac{z(z+1)}{(z+1)^2} = \frac{z}{z+1}$

$G(z) = \frac{z}{z+1}$

By Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z).G(z) = \frac{2z}{2z-1}.\frac{z}{z+1} = \frac{2z^2}{(2z-1)(z+1)}$

5. State convolution Theorem for z transform hence if $f(k) = 3^k, k \geq 0$ & $g(k) = 4^k, k \geq 0$, find $Z\{f(k) * g(k)\}$

Solution:

If $Z\{f(k)\} = F(z)$ and $Z\{g(k) = G(z)$

Then by Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z). G(z)$

We have,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$

$Z\{3^k\} = \sum_{0}^{\infty} 3^k . z^{-k}$

$= 3^0z^0 + 3^1z^{-1} + 3^2z^{-2} + 3^3z^{-3} + \cdots ..$

$= 1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots ....$

$= \left[1 - \frac{3}{z}\right]^{-1}$

$= \frac{1}{1-\frac{3}{z}}$

$Z\{3^k\} = \frac{z}{z-3}$

$\therefore Z\{f(k)\} = Z\{3^k\}$

$\therefore F(z) = \frac{z}{z-3}$

Similarly,

$Z\{4^k\} = \frac{z}{z-4}$

$\therefore Z\{g(k)\} = Z\{4^k\}$

$\therefore G(z) = \frac{z}{z-4}$

By Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z). G(z) = \frac{z}{z-3} . \frac{z}{z-4} = \frac{z^2}{(z-3)(z-4)}$

6. Find $Z[f(k)]$ where $f(k) = \frac{1}{2^k} * \frac{1}{3^k}$

Solution:

If $Z\{f(k)\} = F(z)$ and $Z\{g(k) = G(z)$

Then by Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z).G(z)$

We have,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$

$Z\left\{\frac{1}{2^k}\right\} = \sum_0^{\infty} \frac{1}{2^k}.z^{-k}$

$= \frac{z^0}{2^0} + \frac{z^{-1}}{2^1} + \frac{z^{-2}}{2^2} + \frac{z^{-3}}{2^3} + \cdots ..$

$= 1 + \frac{1}{2z} + \frac{1}{(2z)^2} + \frac{1}{(2z)^3} + \cdots ....$

$= \left[1 - \frac{1}{2z}\right]^{-1}$

$= \frac{1}{1-\frac{1}{2z}}$

$Z\left\{\frac{1}{2^k}\right\} = \frac{2z}{2z-1}$

$\therefore Z\{f(k)\} = Z\left\{\frac{1}{2^k}\right\}$

$\therefore F(z) = \frac{2z}{2z-1}$

Similarly,

$Z\left\{\frac{1}{3^k}\right\} = \frac{3z}{3z-1}$

$\therefore Z\{g(k)\} = Z\left\{\frac{1}{3^k}\right\}$

$\therefore G(z) = \frac{3z}{3z-1}$

By Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z).G(z) = \frac{2z}{2z-1}.\frac{3z}{3z-1} = \frac{6z^2}{(2z-1)(3z-1)}$

Type IV: Binomial Expansion (Inverse Z transform)

1. $Z^{-1}\left[\frac{1}{z-a}\right]$ for $|z| > a$ and for $|z| < a$

Solution:

Let $F(z) = \frac{1}{z-a}$

(a) $|z| > a$ (ROC – Region of Convergence)

Here,

$$F(z) = \frac{1}{z-a}$$
$$F(z) = \frac{1}{z\left(1-\frac{a}{z}\right)}$$
$$F(z) = \frac{1}{z}.\left[1-\frac{a}{z}\right]^{-1}$$
$$F(z) = \frac{1}{z}\left[1+\frac{a}{z}+\frac{a^2}{z^2}+\frac{a^3}{z^3}+\cdots.\right]$$
$$F(z) = \frac{1}{z}+\frac{a}{z^2}+\frac{a^2}{z^3}+\frac{a^3}{z^4}+\cdots\ldots..$$
$$F(z) = a^0.z^{-1}+a^1.z^{-2}+a^2.z^{-3}+a^3.z^{-4}+\cdots\ldots\ldots..$$

Coefficient of $z^{-k} = a^{k-1}, k \geq 1$

Thus,

$$Z^{-1}\left[\frac{1}{z-a}\right] = a^{k-1}, k \geq 1 \text{ for } |z| > a$$

(b) $|z| < a$

Here,

$$F(z) = \frac{1}{-a+z}$$
$$F(z) = \frac{1}{-a\left(1-\frac{z}{a}\right)}$$
$$F(z) = -\frac{1}{a}.\left[1-\frac{z}{a}\right]^{-1}$$
$$F(z) = -\frac{1}{a}\left[1+\frac{z}{a}+\frac{z^2}{a^2}+\frac{z^3}{a^3}+\cdots\ldots\right]$$
$$F(z) = -\frac{1}{a}-\frac{z}{a^2}-\frac{z^2}{a^3}-\frac{z^3}{a^4}-\cdots\ldots\ldots.$$
$$F(z) = -a^{-1}.z^0-a^{-2}.z^1-a^{-3}.z^2-a^{-4}.z^3-\cdots\ldots..$$

Coefficient of $z^{k} = -a^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -a^{-(-k+1)}, -k \geq 0$

Coefficient of $z^{-k} = -a^{k-1}, k \leq 0$

Thus,

$$Z^{-1}\left[\frac{1}{z-a}\right] = -a^{k-1}, k \leq 0 \text{ for } |z| < a$$

2. $Z^{-1}\left[\frac{z}{z-a}\right]$for $|z| < a$ and for $|z| > a$

Solution:

We have,

$F(z) = \frac{z}{z-a}$

(a) $|z| < a$

Here,

$F(z) = \frac{z}{-a+z}$

$F(z) = \frac{z}{-a\left(1-\frac{z}{a}\right)}$

$F(z) = \frac{z}{-a}.\left[1-\frac{z}{a}\right]^{-1}$

$F(z) = -\frac{z}{a}\left[1+\frac{z}{a}+\frac{z^2}{a^2}+\frac{z^3}{a^3}+\cdots.\right]$

$F(z) = -\frac{z}{a}-\frac{z^2}{a^2}-\frac{z^3}{a^3}-\frac{z^4}{a^4}-\cdots$ ….

$F(z) = -a^{-1}.z^1 - a^{-2}.z^2 - a^{-3}.z^3 - a^{-4}.z^4 - \cdots$ … ….

Coefficient of $z^k = -a^{-k}, k \geq 1$

Coefficient of $z^{-k} = -a^{-(-k)}, -k \geq 1$

Coefficient of $z^{-k} = -a^k, k \leq -1$

Thus,

$Z^{-1}\left[\frac{z}{z-a}\right] = -a^k, k \leq -1$

(b) $|z| > a$

Here,

$F(z) = \frac{z}{z-a}$

$F(z) = \frac{z}{z\left(1-\frac{a}{z}\right)}$

$F(z) = \left[1-\frac{a}{z}\right]^{-1}$

$F(z) = 1+\frac{a}{z}+\frac{a^2}{z^2}+\frac{a^3}{z^3}+\cdots$ …

$F(z) = a^0.z^0 + a^1.z^{-1} + a^2.z^{-2} + a^3.z^{-3} + \cdots$ ….

Coefficient of $z^{-k} = a^k, k \geq 0$

Thus,

$Z^{-1}\left[\frac{z}{z-a}\right] = a^k, k \geq 0$

3. $Z^{-1}\left[\frac{z}{z-5}\right]$for $|z| < 5$

[M19/Comp/4M]

Solution:

Let $F(z) = \frac{z}{z-5}$

$F(z) = \frac{z}{-5+z}$

$F(z) = \frac{z}{(-5)\left(1-\frac{z}{5}\right)}$

$F(z) = -\frac{z}{5}\left[1-\frac{z}{5}\right]^{-1}$

$F(z) = -\frac{z}{5}\left[1+\frac{z}{5}+\frac{z^2}{5^2}+\cdots\ldots.\right]$

$F(z) = -\frac{z}{5}-\frac{z^2}{5^2}-\frac{z^3}{5^3}-\cdots\ldots$

$F(z) = -5^{-1}z^1 - 5^{-2}z^2 - 5^{-3}z^3 - \cdots \ldots.$

Thus, coefficient of $z^k = -5^{-k}$ $, k > 0$

$\therefore$ coefficient of $z^{-k} = -5^k, k < 0$

Thus, $Z^{-1}\left[\frac{z}{z-5}\right] = -5^k \quad , \quad k < 0$

4. $Z^{-1}\left[\frac{1}{(z-a)^2}\right]$ for
(a)$|z| < a$
[N17/Comp/4M]
(b) $|z| > a$
Solution:
Let $F(z) = \frac{1}{(z-a)^2}$
(a) for $|z| < a$
$F(z) = \frac{1}{(-a+z)^2}$
$F(z) = \frac{1}{(-a)^2\left(1-\frac{z}{a}\right)^2}$
$F(z) = \frac{1}{a^2}\left[1-\frac{z}{a}\right]^{-2}$
$F(z) = \frac{1}{a^2}\left[1+2\frac{z}{a}+3\frac{z^2}{a^2}+4\frac{z^3}{a^3}+\cdots\ldots.\right]$
$F(z) = \frac{1}{a^2}+\frac{2z}{a^3}+\frac{3z^2}{a^4}+\frac{4z^4}{a^5}+\cdots.$
$F(z) = 1.a^{-2}z^0+2.a^{-3}z^1+3a^{-4}z^2+4a^{-5}z^3+\cdots\ldots$
Thus, coefficient of $z^k = (k+1)a^{-(k+2)}, k \geq 0$
$\therefore$ coefficient of $z^{-k} = (-k+1)a^{-(-k+2)}, k \leq 0$
Thus, $Z^{-1}\left[\frac{1}{(z-a)^2}\right] = -(k-1)a^{k-2}\ ,\ \ k \leq 0$
(b) for $|z| > a$
Here,
$F(z) = \frac{1}{(z-a)^2}$
$F(z) = \frac{1}{z^2\left(1-\frac{a}{z}\right)^2}$
$F(z) = \frac{1}{z^2}.\left[1-\frac{a}{z}\right]^{-2}$
$F(z) = \frac{1}{z^2}.\left[1+2\frac{a}{z}+3\frac{a^2}{z^2}+4\frac{a^3}{z^3}+\cdots..\right]$
$F(z) = \frac{1}{z^2}+\frac{2a}{z^3}+\frac{3a^2}{z^4}+\frac{4a^3}{z^5}+\cdots\ldots$
$F(z) = 1.a^0.z^{-2}+2.a^1.z^{-3}+3a^2z^{-4}+4a^3z^{-5}+\cdots\ldots$
Coefficient of $z^{-k} = (k-1)a^{k-2}, k \geq 2$
Thus,
$Z^{-1}\left[\frac{1}{(z-a)^2}\right] = (k-1)a^{k-2}, k \geq 2$

5. $Z^{-1}\left[\frac{1}{(z-5)^2}\right]$ for $|z| < 5$

[M18/Comp/4M]

Solution:

Let $F(z) = \frac{1}{(z-5)^2}$

$F(z) = \frac{1}{(-5+z)^2}$

$F(z) = \frac{1}{(-5)^2\left(1-\frac{z}{5}\right)^2}$

$F(z) = \frac{1}{5^2}\left[1-\frac{z}{5}\right]^{-2}$

$F(z) = \frac{1}{5^2}\left[1+2\frac{z}{5}+3\frac{z^2}{5^2}+4\frac{z^3}{5^3}+\cdots\ldots.\right]$

$F(z) = \frac{1}{5^2}+\frac{2z}{5^3}+\frac{3z^2}{5^4}+\frac{4z^4}{5^5}+\cdots.$

$F(z) = 1.5^{-2}z^0 + 2.5^{-3}z^1 + 3.5^{-4}z^2 + 4.5^{-5}z^3 + \cdots\ldots.$

Thus, coefficient of $z^k = (k+1)5^{-(k+2)}, k \geq 0$

$\therefore$ coefficient of $z^{-k} = (-k+1)5^{-(-k+2)}, k \leq 0$

Thus, $Z^{-1}\left[\frac{1}{(z-5)^2}\right] = -(k-1)5^{k-2}\quad , \quad k \leq 0$

6. $Z^{-1}\left[\frac{1}{(z-1)^2}\right]$ for $|z| < 1$ and $|z| > 1$

Solution:

Let $F(z) = \frac{1}{(z-1)^2}$

(a) for $|z| < 1$

$F(z) = \frac{1}{(-1+z)^2}$

$F(z) = \frac{1}{(-1)^2(1-z)^2}$

$F(z) = [1-z]^{-2}$

$F(z) = [1 + 2z + 3z^2 + 4z^3 + \cdots \dots .]$

$F(z) = 1.z^0 + 2.z^1 + 3.z^2 + 4.z^3 + \cdots \dots.$

Thus, coefficient of $z^k = (k+1)\ , k \geq 0$

$\therefore$ coefficient of $z^{-k} = (-k+1), k \leq 0$

Thus, $Z^{-1}\left[\frac{1}{(z-1)^2}\right] = -(k-1)\ ,\ \ k \leq 0$

(b) for $|z| > 1$

Here,

$F(z) = \frac{1}{(z-1)^2}$

$F(z) = \frac{1}{z^2\left(1-\frac{1}{z}\right)^2}$

$F(z) = \frac{1}{z^2}.\left[1 - \frac{1}{z}\right]^{-2}$

$F(z) = \frac{1}{z^2}.\left[1 + 2\frac{1}{z} + 3\frac{1}{z^2} + 4\frac{1}{z^3} + \cdots ..\right]$

$F(z) = \frac{1}{z^2} + \frac{2}{z^3} + \frac{3}{z^4} + \frac{4}{z^5} + \cdots \dots$

$F(z) = 1.z^{-2} + 2.z^{-3} + 3.z^{-4} + 4.z^{-5} + \cdots \dots$

Coefficient of $z^{-k} = (k-1), k \geq 2$

Thus,

$Z^{-1}\left[\frac{1}{(z-1)^2}\right] = (k-1), k \geq 2$

7. $Z^{-1}\left[\frac{1}{(z-1)^2}\right]$ for $|z|>1$

[M19/Comp/4M]

Solution:

Let $F(z)=\frac{1}{(z-1)^2}$

$F(z)=\frac{1}{(z)^2\left(1-\frac{1}{z}\right)^2}$

$F(z)=\frac{1}{z^2}\left[1-\frac{1}{z}\right]^{-2}$

$F(z)=\frac{1}{z^2}\left[1+2.\frac{1}{z}+3.\frac{1}{z^2}+4.\frac{1}{z^3}+\cdots\ldots.\right]$

$F(z)=\frac{1}{z^2}+\frac{2}{z^3}+\frac{3}{z^4}+\frac{4}{z^5}+\cdots.$

$F(z)=1.z^{-2}+2.z^{-3}+3.z^{-4}+4.z^{-5}+\cdots\ldots.$

Thus, coefficient of $z^{-k}=(k-1),\ k\geq 2$

Thus, $Z^{-1}\left[\frac{1}{(z-1)^2}\right]=(k-1)\ \ ,\ \ k\geq 2$

8. $Z^{-1}\left[\frac{1}{(z-5)^3}\right]$for $|z|>5$

Solution:

Here,

$F(z)=\frac{1}{(z-5)^3}$

$F(z)=\frac{1}{z^3\left(1-\frac{5}{z}\right)^3}$

$F(z)=\frac{1}{z^3}\left[1-\frac{5}{z}\right]^{-3}$ $\qquad \boldsymbol{(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots.}$

$F(z)=\frac{1}{z^3}\left[1+\left(-\frac{5}{z}\right)\right]^{-3}$

$F(z)=\frac{1}{z^3}\left[1+(-3)\left(-\frac{5}{z}\right)+\frac{(-3)(-3-1)}{2!}\left(-\frac{5}{z}\right)^2+\frac{(-3)(-3-1)(-3-2)}{3!}\left(-\frac{5}{z}\right)^3+..\right]$

$F(z)=\frac{1}{z^3}+\frac{3.5}{z^4}+\frac{6.5^2}{z^5}+\frac{10.5^3}{z^6}+\cdots\ldots\ldots$

$F(z)=1.5^0.z^{-3}+3.5^1.z^{-4}+6.5^2.z^{-5}+10.5^3.z^{-6}+\cdots\ldots..$

Coefficient of $z^{-k}=\left(\frac{(k-1)(k-2)}{2}\right).5^{k-3},k\geq 3$

Thus,

$Z^{-1}\left[\frac{1}{(z-5)^3}\right]=\frac{(k-1)(k-2)}{2}.5^{k-3},k\geq 3$

9. $Z^{-1}\left[\frac{1}{(z-5)^3}\right]$for $|z|<5$

[N16/CompIT/6M]

Solution:

Let $F(z)=\frac{1}{(z-5)^3}$

$F(z)=\frac{1}{(-5+z)^3}$

$F(z)=\frac{1}{(-5)^3\left(1-\frac{z}{5}\right)^3}$

$F(z)=-\frac{1}{5^3}\left[1-\frac{z}{5}\right]^{-3}$

$F(z)=-\frac{1}{5^3}\left[1+(-3)\left(-\frac{z}{5}\right)+\frac{(-3)(-3-1)}{2!}\left(-\frac{z}{5}\right)^2+\cdots\ldots.\right]$

$F(z)=-\frac{1}{5^3}\left[1+3\frac{z}{5}+6\frac{z^2}{5^2}+10\frac{z^3}{5^3}+\cdots..\right]$

$F(z)=-\frac{1}{5^3}-3\frac{z}{5^4}-6\frac{z^2}{5^5}-10\frac{z^3}{5^6}-\cdots\ldots$

$F(z)=-1.z^0.5^{-3}-3.z^1.5^{-4}-6.z^2.5^{-5}-10.z^3.5^{-6}-\cdots\ldots.$

Thus, coefficient of $z^k=-\frac{(k+1)(k+2)}{2}\ 5^{-(k+3)}, k\geq 0$

$\therefore$ coefficient of $z^{-k}=-\frac{(-k+1)(-k+2)}{2}\ 5^{-(-k+3)}, k\leq 0$

Thus, $Z^{-1}\left[\frac{1}{(z-5)^3}\right]=-\frac{(k-1)(k-2)}{2}\ 5^{k-3}\quad,\quad k\leq 0$

Type V: Partial Fractions (Inverse Z transform)

1. $Z^{-1}\left[\frac{1}{(z-3)(z-2)}\right]$ if ROC is $|z| > 3$

[N17/Comp/4M]

Solution:

$F(z) = \frac{1}{z-3} - \frac{1}{z-2}$

$F(z) = \frac{1}{z\left[1-\frac{3}{z}\right]} - \frac{1}{z\left[1-\frac{2}{z}\right]}$

$F(z) = \frac{1}{z}\left[1 - \frac{3}{z}\right]^{-1} - \frac{1}{z}\left[1 - \frac{2}{z}\right]^{-1}$

$F(z) = \frac{1}{z}\left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots ..\right] - \frac{1}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots ..\right]$

$F(z) = \left[\frac{1}{z} + \frac{3}{z^2} + \frac{3^2}{z^3} + \frac{3^3}{z^4} + \cdots ...\right] + \left[-\frac{1}{z} - \frac{2}{z^2} - \frac{2^2}{z^3} - \frac{2^3}{z^4} - \cdots ...\right]$

$F(z) = [3^0 z^{-1} + 3^1 z^{-2} + 3^2 z^{-3} - \cdots .] + [-2^0 z^{-1} - 2^1 z^{-2} - 2^2 z^{-3} + \cdots]$

From first series,

Coefficient of $z^{-k} = 3^{(k-1)}, k > 0$

From second series,

Coefficient of $z^{-k} = -2^{k-1}, k > 0$

Thus,

$Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = 3^{k-1} - 2^{k-1}, k > 0$

2. Find inverse Z transform of $F(z)=\frac{1}{(z-2)(z-3)}$ for i) $|z|<2$ ii) $|z|>3$

[D24/CompITAI/6M]

Solution:

We have,

$F(z)=\frac{1}{(z-3)(z-2)}$

Let $\frac{1}{(z-3)(z-2)}=\frac{A}{z-3}+\frac{B}{z-2}$

$1=A(z-2)+B(z-3)$

Comparing the coefficients, we get

$A+B=0$

$-2A-3B=1$

On solving, we get $A=1,B=-1$

$F(z)=\frac{1}{z-3}-\frac{1}{z-2}$

(i) $|z|<2$

$F(z)=\frac{1}{-3+z}-\frac{1}{-2+z}$

$F(z)=\frac{1}{-3\left[1-\frac{z}{3}\right]}-\frac{1}{-2\left[1-\frac{z}{2}\right]}$

$F(z)=-\frac{1}{3}\left[1-\frac{z}{3}\right]^{-1}+\frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$

$F(z)=-\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots..\right]+\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots..\right]$

$F(z)=\left[-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\frac{z^3}{3^4}-\cdots\ldots\right]+\left[\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\frac{z^3}{2^4}+\cdots\ldots\right]$

$F(z)=[-3^{-1}z^0-3^{-2}z^1-3^{-3}z^2-\cdots.]+[2^{-1}z^0+2^{-2}z^1+2^{-3}z^2+\cdots]$

From first series,

Coefficient of $z^k=-3^{-(k+1)},k\geq 0$

Coefficient of $z^{-k}=-3^{k-1},k\leq 0$

From second series,

Coefficient of $z^k=2^{-(k+1)},k\geq 0$

Coefficient of $z^{-k}=2^{k-1},k\leq 0$

Thus, $Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\}=2^{k-1}-3^{k-1},k\leq 0$

(ii) $|z|>3$

$F(z)=\frac{1}{z-3}-\frac{1}{z-2}$

$F(z)=\frac{1}{z\left[1-\frac{3}{z}\right]}-\frac{1}{z\left[1-\frac{2}{z}\right]}$

$F(z)=\frac{1}{z}\left[1-\frac{3}{z}\right]^{-1}-\frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$

$F(z)=\frac{1}{z}\left[1+\frac{3}{z}+\frac{3^2}{z^2}+\frac{3^3}{z^3}+\cdots..\right]-\frac{1}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots..\right]$

$F(z)=\left[\frac{1}{z}+\frac{3}{z^2}+\frac{3^2}{z^3}+\frac{3^3}{z^4}+\cdots\ldots\right]+\left[-\frac{1}{z}-\frac{2}{z^2}-\frac{2^2}{z^3}-\frac{2^3}{z^4}-\cdots\ldots\right]$

$F(z) = [3^0 z^{-1} + 3^1 z^{-2} + 3^2 z^{-3} - \cdots .] + [-2^0 z^{-1} - 2^1 z^{-2} - 2^2 z^{-3} + \cdots]$

From first series,

Coefficient of $z^{-k} = 3^{(k-1)}, k > 0$

From second series,

Coefficient of $z^{-k} = -2^{k-1}, k > 0$

Thus,

$Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = 3^{k-1} - 2^{k-1}, k > 0$

3. $Z^{-1}\left[\frac{1}{(z-3)(z-2)}\right]$ if ROC is (i) $|z|<2$ (ii) $2<|z|<3$ (iii) $|z|>3$

[N13/CompIT/8M]

Solution:

We have,

$F(z)=\frac{1}{(z-3)(z-2)}$

Let $\frac{1}{(z-3)(z-2)}=\frac{A}{z-3}+\frac{B}{z-2}$

$1=A(z-2)+B(z-3)$

Comparing the coefficients, we get

$A+B=0$

$-2A-3B=1$

On solving, we get $A=1, B=-1$

$F(z)=\frac{1}{z-3}-\frac{1}{z-2}$

(i) $|z|<2$

$F(z)=\frac{1}{-3+z}-\frac{1}{-2+z}$

$F(z)=\frac{1}{-3\left[1-\frac{z}{3}\right]}-\frac{1}{-2\left[1-\frac{z}{2}\right]}$

$F(z)=-\frac{1}{3}\left[1-\frac{z}{3}\right]^{-1}+\frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$

$F(z)=-\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots..\right]+\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots..\right]$

$F(z)=\left[-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\frac{z^3}{3^4}-\cdots...\right]+\left[\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\frac{z^3}{2^4}+\cdots...\right]$

$F(z)=[-3^{-1}z^0-3^{-2}z^1-3^{-3}z^2-\cdots.]+[2^{-1}z^0+2^{-2}z^1+2^{-3}z^2+\cdots]$

From first series,

Coefficient of $z^k=-3^{-(k+1)}, k\geq 0$

Coefficient of $z^{-k}=-3^{k-1}, k\leq 0$

From second series,

Coefficient of $z^k=2^{-(k+1)}, k\geq 0$

Coefficient of $z^{-k}=2^{k-1}, k\leq 0$

Thus, $Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\}=2^{k-1}-3^{k-1}, k\leq 0$

(ii) $2<|z|<3$

$F(z)=\frac{1}{-3+z}-\frac{1}{z-2}$

$F(z)=\frac{1}{-3\left[1-\frac{z}{3}\right]}-\frac{1}{z\left[1-\frac{2}{z}\right]}$

$F(z)=-\frac{1}{3}\left[1-\frac{z}{3}\right]^{-1}-\frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$

$F(z)=-\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots..\right]-\frac{1}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots..\right]$

$F(z)=\left[-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\frac{z^3}{3^4}-\cdots...\right]+\left[-\frac{1}{z}-\frac{2}{z^2}-\frac{2^2}{z^3}-\frac{2^3}{z^4}-\cdots...\right]$

$F(z) = [-3^{-1}z^0 - 3^{-2}z^1 - 3^{-3}z^2 - \cdots.] + [-2^0z^{-1} - 2^1z^{-2} - 2^2z^{-3} + \cdots]$

From first series,

Coefficient of $z^k = -3^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -3^{-(-k+1)}, k \leq 0$

i.e. Coefficient of $z^{-k} = -3^{k-1}, k \leq 0$

From second series,

Coefficient of $z^{-k} = -2^{k-1}, k > 0$

Thus,

$Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = \begin{cases} -3^{k-1} & k \leq 0 \\ -2^{k-1} & k > 0 \end{cases}$

(iii) $|z| > 3$

$F(z) = \frac{1}{z-3} - \frac{1}{z-2}$

$F(z) = \frac{1}{z\left[1-\frac{3}{z}\right]} - \frac{1}{z\left[1-\frac{2}{z}\right]}$

$F(z) = \frac{1}{z}\left[1 - \frac{3}{z}\right]^{-1} - \frac{1}{z}\left[1 - \frac{2}{z}\right]^{-1}$

$F(z) = \frac{1}{z}\left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots..\right] - \frac{1}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots..\right]$

$F(z) = \left[\frac{1}{z} + \frac{3}{z^2} + \frac{3^2}{z^3} + \frac{3^3}{z^4} + \cdots...\right] + \left[-\frac{1}{z} - \frac{2}{z^2} - \frac{2^2}{z^3} - \frac{2^3}{z^4} - \cdots...\right]$

$F(z) = [3^0z^{-1} + 3^1z^{-2} + 3^2z^{-3} - \cdots.] + [-2^0z^{-1} - 2^1z^{-2} - 2^2z^{-3} + \cdots]$

From first series,

Coefficient of $z^{-k} = 3^{(k-1)}, k > 0$

From second series,

Coefficient of $z^{-k} = -2^{k-1}, k > 0$

Thus,

$Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = 3^{k-1} - 2^{k-1}, k > 0$

4. Find the inverse Z-transform of $\frac{1}{(z-2)(z-3)}$ if ROC is (i) $|z| < 2$ (ii) $2 < |z| < 3$

[M23/CompIT/6M]

Solution:

We have,

$F(z) = \frac{1}{(z-3)(z-2)}$

Let $\frac{1}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2}$

$1 = A(z-2) + B(z-3)$

Comparing the coefficients, we get

$A + B = 0$

$-2A - 3B = 1$

On solving, we get $A = 1, B = -1$

$F(z) = \frac{1}{z-3} - \frac{1}{z-2}$

(i) $|z| < 2$

$F(z) = \frac{1}{-3+z} - \frac{1}{-2+z}$

$F(z) = \frac{1}{-3\left[1-\frac{z}{3}\right]} - \frac{1}{-2\left[1-\frac{z}{2}\right]}$

$F(z) = -\frac{1}{3}\left[1-\frac{z}{3}\right]^{-1} + \frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$

$F(z) = -\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots..\right] + \frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots..\right]$

$F(z) = \left[-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\frac{z^3}{3^4}-\cdots...\right] + \left[\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\frac{z^3}{2^4}+\cdots...\right]$

$F(z) = [-3^{-1}z^0 - 3^{-2}z^1 - 3^{-3}z^2 - \cdots.] + [2^{-1}z^0 + 2^{-2}z^1 + 2^{-3}z^2 + \cdots]$

From first series,

Coefficient of $z^k = -3^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -3^{k-1}, k \leq 0$

From second series,

Coefficient of $z^k = 2^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = 2^{k-1}, k \leq 0$

Thus, $Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = 2^{k-1} - 3^{k-1}, k \leq 0$

(ii) $2 < |z| < 3$

$F(z) = \frac{1}{-3+z} - \frac{1}{z-2}$

$F(z) = \frac{1}{-3\left[1-\frac{z}{3}\right]} - \frac{1}{z\left[1-\frac{2}{z}\right]}$

$F(z) = -\frac{1}{3}\left[1-\frac{z}{3}\right]^{-1} - \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$

$F(z) = -\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots..\right] - \frac{1}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots..\right]$

$F(z) = \left[-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\frac{z^3}{3^4}-\cdots...\right] + \left[-\frac{1}{z}-\frac{2}{z^2}-\frac{2^2}{z^3}-\frac{2^3}{z^4}-\cdots...\right]$

$F(z) = [-3^{-1}z^{0} - 3^{-2}z^{1} - 3^{-3}z^{2} - \cdots.] + [-2^{0}z^{-1} - 2^{1}z^{-2} - 2^{2}z^{-3} + \cdots]$

From first series,

Coefficient of $z^{k} = -3^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -3^{-(-k+1)}, k \leq 0$

i.e. Coefficient of $z^{-k} = -3^{k-1}, k \leq 0$

From second series,

Coefficient of $z^{-k} = -2^{k-1}, k > 0$

Thus,

$$Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = \begin{cases} -3^{k-1} & k \leq 0 \\ -2^{k-1} & k > 0 \end{cases}$$

5. $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right], |z| > 2$

[M16/N16/CompIT/6M][M22/CompITAI/5M]

Solution:

We have,

$F(z) = \frac{z}{(z-1)(z-2)}$

Let $\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$z = A(z-2) + B(z-1)$

Comparing the coefficients, we get

$A + B = 1$

$-2A - B = 0$

On solving, we get

$A = -1, B = 2$

$F(z) = -\frac{1}{z-1} + \frac{2}{z-2}$

$F(z) = -\frac{1}{z\left[1-\frac{1}{z}\right]} + \frac{2}{z\left[1-\frac{2}{z}\right]}$

$F(z) = -\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} + \frac{2}{z}\left[1-\frac{2}{z}\right]^{-1}$

$F(z) = -\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots..\right] + \frac{2}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots.\right]$

$F(z) = [-z^{-1} - z^{-2} - z^{-3} + \cdots.] + [2^1 z^{-1} + 2^2 z^{-2} + 2^3 z^{-3} + \cdots]$

From first series,

Coefficient of $z^{-k} = -1\ , k > 0$

From second series,

Coefficient of $z^{-k} = 2^k\ , k > 0$

Thus,

$Z^{-1}\left\{\frac{z}{(z-1)(z-2)}\right\} = 2^k - 1, k > 0$

6. Find the inverse z transform of $F(z)=\frac{1}{(z-1)(z-3)}$ for (i) $|z|<1$ (ii) $1<|z|<3$

[D23/CompITAI/6M]

Solution:

We have,

$F(z)=\frac{1}{(z-3)(z-1)}$

Let $\frac{1}{(z-3)(z-1)}=\frac{A}{z-3}+\frac{B}{z-1}$

$1=A(z-1)+B(z-3)$

Comparing the coefficients, we get

$A+B=0$

$-A-3B=1$

On solving, we get $A=\frac{1}{2}, B=-\frac{1}{2}$

$F(z)=\frac{\frac{1}{2}}{z-3}-\frac{\frac{1}{2}}{z-1}$

(i) $|z|<1$

$F(z)=\frac{\frac{1}{2}}{-3+z}-\frac{\frac{1}{2}}{-1+z}$

$F(z)=\frac{\frac{1}{2}}{-3\left[1-\frac{z}{3}\right]}-\frac{\frac{1}{2}}{-[1-z]}$

$F(z)=-\frac{1}{2}.\frac{1}{3}\left[1-\frac{z}{3}\right]^{-1}+\frac{1}{2}[1-z]^{-1}$

$F(z)=-\frac{1}{2}.\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots..\right]+\frac{1}{2}[1+z+z^2+z^3+\cdots..]$

$F(z)=\frac{1}{2}.\left[-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\frac{z^3}{3^4}-\cdots\ldots\right]+\frac{1}{2}[1+z+z^2+z^3+\cdots..]$

$F(z)=\frac{1}{2}[-3^{-1}z^0-3^{-2}z^1-3^{-3}z^2-\cdots.]+\frac{1}{2}[z^0+z^1+z^2+\cdots]$

From first series,

Coefficient of $z^k=\frac{1}{2}.\left(-3^{-(k+1)}\right), k\geq 0$

Coefficient of $z^{-k}=-\frac{3^{k-1}}{2}, k\leq 0$

From second series,

Coefficient of $z^k=\frac{1}{2}, k\geq 0$

Coefficient of $z^{-k}=\frac{1}{2}, k\leq 0$

Thus, $Z^{-1}\left\{\frac{1}{(z-3)(z-1)}\right\}=\frac{1}{2}-\frac{3^{k-1}}{2}, k\leq 0$

(ii) $1<|z|<3$

$F(z)=\frac{\frac{1}{2}}{-3+z}-\frac{\frac{1}{2}}{z-1}$

$F(z)=\frac{\frac{1}{2}}{-3\left[1-\frac{z}{3}\right]}-\frac{\frac{1}{2}}{z\left[1-\frac{1}{z}\right]}$

$F(z) = -\frac{1}{2}.\frac{1}{3}\left[1-\frac{z}{3}\right]^{-1} - \frac{1}{2}.\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}$

$F(z) = -\frac{1}{2}.\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots..\right] - \frac{1}{2}.\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots..\right]$

$F(z) = \frac{1}{2}\left[-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\frac{z^3}{3^4}-\cdots...\right] + \frac{1}{2}\left[-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\cdots...\right]$

$F(z) = \frac{1}{2}[-3^{-1}z^0 - 3^{-2}z^1 - 3^{-3}z^2 - \cdots.] + \frac{1}{2}[-z^{-1} - z^{-2} - z^{-3} + \cdots]$

From first series,

Coefficient of $z^k = \frac{1}{2}.\left(-3^{-(k+1)}\right), k \geq 0$

Coefficient of $z^{-k} = -\frac{3^{-(-k+1)}}{2}, k \leq 0$

i.e. Coefficient of $z^{-k} = -\frac{3^{k-1}}{2}, k \leq 0$

From second series,

Coefficient of $z^{-k} = -\frac{1}{2}, k > 0$

Thus,

$$Z^{-1}\left\{\frac{1}{(z-3)(z-1)}\right\} = \begin{cases} -\frac{3^{k-1}}{2} & k \leq 0 \\ -\frac{1}{2} & k > 0 \end{cases}$$

7. $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right], 1<|z|<2$

Solution:

We have,

$F(z)=\frac{z}{(z-1)(z-2)}$

Let $\frac{z}{(z-1)(z-2)}=\frac{A}{z-1}+\frac{B}{z-2}$

$z=A(z-2)+B(z-1)$

Comparing the coefficients, we get

$A+B=1$

$-2A-B=0$

On solving, we get

$A=-1, B=2$

$F(z)=-\frac{1}{z-1}+\frac{2}{2-z}$

$F(z)=-\frac{1}{z\left[1-\frac{1}{z}\right]}+\frac{2}{2\left[1-\frac{z}{2}\right]}$

$F(z)=-\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}+\frac{2}{2}\left[1-\frac{z}{2}\right]^{-1}$

$F(z)=-\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots..\right]+\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots.\right]$

$F(z)=[-z^{-1}-z^{-2}-z^{-3}+\cdots.]+[2^0z^0+2^{-1}z^1+2^{-2}z^2+\cdots..]$

From first series,

Coefficient of $z^{-k}=-1, k>0$

From second series,

Coefficient of $z^{k}=2^{-k}, k\geq 0$

Coefficient of $z^{-k}=2^{k}, k\leq 0$

Thus,

$$Z^{-1}\left\{\frac{z}{(z-1)(z-2)}\right\}=\begin{cases}2^k & k\leq 0\\ -1 & k>0\end{cases}$$

8. $Z^{-1}\left[\frac{z}{(z-3)(z-2)}\right]$ if ROC is $|z| > 3$

[M18/Comp/4M]

Solution:

We have,

$F(z) = \frac{z}{(z-3)(z-2)}$

Let $\frac{z}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2}$

$z = A(z-2) + B(z-3)$

Comparing the coefficients, we get

$A + B = 1$

$-2A - 3B = 0$

On solving, we get $A = 3, B = -2$

$F(z) = \frac{3}{z-3} - \frac{2}{z-2}$

ROC is $|z| > 3$

$F(z) = \frac{3}{z\left[1-\frac{3}{z}\right]} - \frac{2}{z\left[1-\frac{2}{z}\right]}$

$F(z) = \frac{3}{z}\left[1 - \frac{3}{z}\right]^{-1} - \frac{2}{z}\left[1 - \frac{2}{z}\right]^{-1}$

$F(z) = \frac{3}{z}\left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots..\right] - \frac{2}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots..\right]$

$F(z) = \left[\frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \frac{3^4}{z^4} + \cdots ...\right] + \left[-\frac{2}{z} - \frac{2^2}{z^2} - \frac{2^3}{z^3} - \frac{2^4}{z^4} - \cdots ...\right]$

$F(z) = [3^1 z^{-1} + 3^2 z^{-2} + 3^3 z^{-3} - \cdots.] + [-2^1 z^{-1} - 2^2 z^{-2} - 2^3 z^{-3} + \cdots]$

From first series,

Coefficient of $z^{-k} = 3^k, k > 0$

From second series,

Coefficient of $z^{-k} = -2^k, k > 0$

Thus,

$Z^{-1}\left\{\frac{z}{(z-3)(z-2)}\right\} = 3^k - 2^k, k > 0$

9. Find the inverse Z transform of $\frac{1}{z^2-3z+2}$ if ROC is (i) $|z| < 1$ (ii) $|z| > 2$

[N22/CompITAI/6M]

Solution:

We have,

$F(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)}$

Let $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$1 = A(z-2) + B(z-1)$

Comparing the coefficients, we get

$A + B = 0$

$-2A - B = 1$

On solving, we get $A = -1, B = 1$

$F(z) = \frac{-1}{z-1} + \frac{1}{z-2}$

(i) $|z| < 1$

$F(z) = \frac{-1}{-1+z} + \frac{1}{-2+z}$

$F(z) = \frac{-1}{-[1-z]} + \frac{1}{-2\left[1-\frac{z}{2}\right]}$

$F(z) = [1-z]^{-1} - \frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$

$F(z) = [1 + z + z^2 + z^3 + \cdots..] - \frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots..\right]$

$F(z) = [1 + z + z^2 + z^3 + \cdots..] + \left[-\frac{1}{2} - \frac{z}{2^2} - \frac{z^2}{2^3} - \frac{z^3}{2^4} + \cdots ...\right]$

$F(z) = [z^0 + z^1 + z^2 - \cdots.] + [-2^{-1}z^0 - 2^{-2}z^1 - 2^{-3}z^2 + \cdots]$

From first series,

Coefficient of $z^k = 1, k \geq 0$

Coefficient of $z^{-k} = 1, k \leq 0$

From second series,

Coefficient of $z^k = -2^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -2^{k-1}, k \leq 0$

Thus, $Z^{-1}\left\{\frac{1}{(z-1)(z-2)}\right\} = 1 - 2^{k-1}, k \leq 0$

(ii) $|z| > 2$

$F(z) = \frac{-1}{z-1} + \frac{1}{z-2}$

$F(z) = \frac{-1}{z\left[1-\frac{1}{z}\right]} + \frac{1}{z\left[1-\frac{2}{z}\right]}$

$F(z) = -\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} + \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$

$F(z) = -\frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots..\right] + \frac{1}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots..\right]$

$F(z) = \left[-\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} + \cdots ...\right] + \left[\frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \cdots ...\right]$

$F(z) = [-z^{-1} - z^{-2} - z^{-3} - \cdots.] + [2^0z^{-1} + 2^1z^{-2} + 2^2z^{-3} + \cdots]$

From first series,
Coefficient of $z^{-k} = -1, k > 0$
From second series,
Coefficient of $z^{-k} = 2^{k-1}, k > 0$
Thus,
$Z^{-1}\left\{\frac{1}{(z-1)(z-2)}\right\} = 2^{k-1} - 1, k > 0$

10. Obtain inverse z transform $\frac{z+2}{z^2-2z-3}$, $1 < |z| < 3$

[M22/CompITAI/5M]

Solution:

We have,

$$F(z) = \frac{z+2}{z^2-2z-3} = \frac{z+2}{(z-3)(z+1)}$$

Let $\frac{z+2}{(z-3)(z+1)} = \frac{A}{z-3} + \frac{B}{z+1}$

$$z + 2 = A(z + 1) + B(z - 3)$$

Comparing the coefficients, we get

$$A + B = 1$$

$$A - 3B = 2$$

On solving, we get $A = \frac{5}{4}, B = -\frac{1}{4}$

$$F(z) = \frac{\frac{5}{4}}{z-3} - \frac{\frac{1}{4}}{z+1}$$

For $1 < |z| < 3$

$$F(z) = \frac{\frac{5}{4}}{-3+z} - \frac{\frac{1}{4}}{z+1}$$

$$F(z) = \frac{\frac{5}{4}}{-3\left[1-\frac{z}{3}\right]} - \frac{\frac{1}{4}}{z\left[1+\frac{1}{z}\right]}$$

$$F(z) = -\frac{5}{4}.\frac{1}{3}\left[1-\frac{z}{3}\right]^{-1} - \frac{1}{4}.\frac{1}{z}\left[1+\frac{1}{z}\right]^{-1}$$

$$F(z) = -\frac{5}{4}.\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots..\right] - \frac{1}{4}.\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots..\right]$$

$$F(z) = \left[-\frac{5}{4}.\frac{1}{3} - \frac{5}{4}.\frac{z}{3^2} - \frac{5}{4}.\frac{z^2}{3^3} - \cdots\ldots\right] + \left[-\frac{1}{4}.\frac{1}{z} - \frac{1}{4}.\frac{1}{z^2} - \frac{1}{4}.\frac{1}{z^3} - \cdots\ldots\right]$$

$$F(z) = \left[-\frac{5}{4}3^{-1}z^0 - \frac{5}{4}3^{-2}z^1 - \frac{5}{4}3^{-3}z^2 - \cdots.\right] + \left[-\frac{1}{4}z^{-1} - \frac{1}{4}z^{-2} - \frac{1}{4}z^{-3} + \cdots\right]$$

From first series,

Coefficient of $z^k = -\frac{5}{4}.3^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -\frac{5}{4}.3^{-(-k+1)}, k \leq 0$

i.e. Coefficient of $z^{-k} = -\frac{5.3^{k-1}}{4}, k \leq 0$

From second series,

Coefficient of $z^{-k} = -\frac{1}{4}, k > 0$

Thus,

$$Z^{-1}\left\{\frac{z+2}{(z-3)(z+1)}\right\} = \begin{cases} -\frac{5.3^{k-1}}{4} & k \leq 0 \\ -\frac{1}{4} & k > 0 \end{cases}$$

11. $Z^{-1}\left[\frac{z+2}{z^2-2z+1}\right], |z| > 1$

[M14/CompIT/8M][N15/CompIT/6M]

Solution:

We have,

$$F(z) = \frac{z+2}{z^2-2z+1}$$
$$F(z) = \frac{z+2}{(z-1)^2}$$
$$F(z) = \frac{z-1+1+2}{(z-1)^2}$$
$$F(z) = \frac{z-1}{(z-1)^2} + \frac{3}{(z-1)^2}$$
$$F(z) = \frac{1}{z-1} + \frac{3}{(z-1)^2}$$
$$F(z) = \frac{1}{z\left[1-\frac{1}{z}\right]} + \frac{3}{\left(z\left[1-\frac{1}{z}\right]\right)^2}$$
$$F(z) = \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} + \frac{3}{z^2}\left[1-\frac{1}{z}\right]^{-2}$$
$$F(z) = \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots.\right] + \frac{3}{z^2}\left[1+\frac{2}{z}+\frac{3}{z^2}+\frac{4}{z^3}+\cdots..\right]$$
$$F(z) = \left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots..\right] + \left[\frac{3}{z^2}+\frac{3.2}{z^3}+\frac{3.3}{z^4}+\frac{3.4}{z^5}+\cdots.\right]$$
$$F(z) = [z^{-1}+z^{-2}+z^{-3}+\cdots \ldots] + [3.1z^{-2}+3.2z^{-3}+3.3z^{-4}+\cdots.]$$
$$F(z) = (3.0+1)z^{-1} + (3.1+1)z^{-2} + (3.2+1)z^{-3} + \cdots \ldots \ldots$$

Coefficient of $z^{-k} = (3(k-1)+1), k > 0$

Coefficient of $z^{-k} = 3k-2, k > 0$

Thus,

$$Z^{-1}\left\{\frac{z+2}{z^2-2z+1}\right\} = 3k-2, k > 0$$

12. $Z^{-1}\left[\frac{2z^2-10z+13}{(z-3)^2(z-2)}\right], 2<|z|<3$

Solution:

We have,

$F(z)=\frac{2z^2-10z+13}{(z-3)^2(z-2)}$

Let $\frac{2z^2-10z+13}{(z-3)^2(z-2)}=\frac{A}{z-3}+\frac{B}{(z-3)^2}+\frac{C}{z-2}$

$2z^2-10z+13=A(z-3)(z-2)+B(z-2)+C(z-3)^2$

$2z^2-10z+13=A(z^2-5z+6)+B(z-2)+C(z^2-6z+9)$

Comparing the coefficients, we get

$A+0B+C=2$

$-5A+B-6C=-10$

$6A-2B+9C=13$

On solving, we get

$A=1, B=1, C=1$

$F(z)=\frac{1}{z-3}+\frac{1}{(z-3)^2}+\frac{1}{z-2}$

For ROC, $2<|z|<3$

$F(z)=\frac{1}{-3+z}+\frac{1}{(-3+z)^2}+\frac{1}{z-2}$

$F(z)=\frac{1}{-3\left[1-\frac{z}{3}\right]}+\frac{1}{(-3)^2\left[1-\frac{z}{3}\right]^2}+\frac{1}{z\left(1-\frac{2}{z}\right)}$

$F(z)=-\frac{1}{3}\left[1-\frac{z}{3}\right]^{-1}+\frac{1}{9}\left[1-\frac{z}{3}\right]^{-2}+\frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$

$F(z)=-\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots\right]+\frac{1}{9}\left[1+2\frac{z}{3}+3\frac{z^2}{3^2}+\cdots\right]+\frac{1}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots..\right]$

$F(z)=[-3^{-1}z^0-3^{-2}.z^1-3^{-3}.z^2-3^{-4}z^3 \ldots.]+[1.3^{-2}z^0+2.3^{-3}z^1+3.3^{-4}.z^2+\cdots\ldots.]$
$\quad +[2^0z^{-1}+2^1z^{-2}+2^2z^{-3}+\cdots\ldots.]$

$F(z)=[(1.3^{-2}-3^{-1})z^0+(2.3^{-3}-3^{-2})z^1+(3.3^{-4}-3^{-3})z^2+\cdots..]$
$\quad +[2^0z^{-1}+2^1z^{-2}+2^2z^{-3}+\cdots\ldots.]$

From first series,

Coefficient of $z^k=(k+1).3^{-(k+2)}-3^{-(k+1)}, k\geq 0$

$=[k+1-3]3^{-k-2}$

$=[k-2]3^{-k-2}, k\geq 0$

Coefficient of $z^{-k}=[-k-2]3^{k-2}, k\leq 0$

From second series,

Coefficient of $z^{-k}=2^{k-1}, k>0$

Thus, $Z^{-1}\left\{\frac{2z^2-10z+13}{(z-3)^2(z-2)}\right\}=\begin{cases}[-k-2]3^{k-2} & ,k\leq 0\\ 2^{k-1} & ,k>0\end{cases}$

13. $Z^{-1}\left[\frac{3z^2+2z}{z^2-3z+2}\right]$ for $1<|z|<2$

Solution:

We have,

$F(z)=\frac{3z^2+2z}{(z-1)(z-2)}$

$\frac{F(z)}{z}=\frac{3z+2}{(z-1)(z-2)}$

Let $\frac{3z+2}{(z-1)(z-2)}=\frac{A}{z-1}+\frac{B}{z-2}$

$3z+2=A(z-2)+B(z-1)$

Comparing the coefficients, we get

$A+B=3$

$-2A-B=2$

On solving, we get

$A=-5, B=8$

$\frac{F(z)}{z}=-\frac{5}{z-1}+\frac{8}{z-2}$

$F(z)=-\frac{5z}{z-1}+\frac{8z}{z-2}$

For ROC $1<|z|<2$

$F(z)=-\frac{5z}{z-1}+\frac{8z}{-2+z}$

$F(z)=-\frac{5z}{z\left(1-\frac{1}{z}\right)}+\frac{8z}{-2\left(1-\frac{z}{2}\right)}$

$F(z)=-5\left[1-\frac{1}{z}\right]^{-1}-4z\left[1-\frac{z}{2}\right]^{-1}$

$F(z)=-5\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots..\right]-4z\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots.\right]$

$F(z)=[-5z^0-5z^{-1}-5z^{-2}+\cdots.]+[-4.\,2^0z^1-4.\,2^{-1}z^2-4.\,2^{-2}z^3+]$

From first series,

Coefficient of $z^{-k}=-5\ , k\geq 0$

From second series,

Coefficient of $z^{k}=-4.2^{-(k-1)}\ , k>0$

Coefficient of $z^{-k}=-4.2^{k+1}, k<0$

Coefficient of $z^{-k}=-8.2^{k}, k<0$

Thus,

$$Z^{-1}\left\{\frac{3z^2+2z}{z^2-3z+2}\right\}=\begin{cases}-8.2^k & k<0\\ -5 & k\geq 0\end{cases}$$

14. $Z^{-1}\left[\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}\right]$ for (i) $|z|>1$ (ii) $|z|<\frac{1}{2}$ (iii) $\frac{1}{2}<|z|<1$

Solution:

We have,

$F(z)=\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}$

$\frac{F(z)}{z}=\frac{z}{(z-1)\left(z-\frac{1}{2}\right)}$

Let $\frac{z}{(z-1)\left(z-\frac{1}{2}\right)}=\frac{A}{z-1}+\frac{B}{z-\frac{1}{2}}$

$z=A\left(z-\frac{1}{2}\right)+B(z-1)$

Comparing the coefficients, we get

$A+B=1$

$-\frac{1}{2}A-B=0$

On solving, we get $A=2, B=-1$

$\frac{F(z)}{z}=\frac{2}{z-1}-\frac{1}{z-\frac{1}{2}}$

$F(z)=\frac{2z}{z-1}-\frac{z}{z-\frac{1}{2}}$

(i) $|z|>1$

$F(z)=\frac{2z}{z-1}-\frac{z}{z-\frac{1}{2}}$

$F(z)=\frac{2z}{z\left[1-\frac{1}{z}\right]}-\frac{z}{z\left[1-\frac{1}{2z}\right]}$

$F(z)=2\left[1-\frac{1}{z}\right]^{-1}-\left[1-\frac{1}{2z}\right]^{-1}$

$F(z)=2\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots..\right]-\left[1+\frac{1}{2z}+\frac{1}{2^2z^2}+\frac{1}{2^3z^3}+\cdots..\right]$

$F(z)=[2z^0+2z^{-1}+2z^{-2}+\cdots.]+[-2^0z^0-2^{-1}z^{-1}-2^{-2}z^{-2}+\cdots]$

From first series,

Coefficient of $z^{-k}=2,\quad k\geq 0$

From second series,

Coefficient of $z^{-k}=-2^{-k}, k\geq 0$

Thus, $Z^{-1}\left\{\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}\right\}=2-2^{-k}, k\geq 0$

(ii) $|z|<\frac{1}{2}$

$F(z)=\frac{2z}{-1+z}-\frac{z}{-\frac{1}{2}+z}$

$F(z)=\frac{2z}{-[1-z]}-\frac{z}{-\frac{1}{2}[1-2z]}$

$F(z)=-2z[1-z]^{-1}+2z[1-2z]^{-1}$

$F(z)=-2z[1+z+z^2+z^3+\cdots..]+2z[1+2z+2^2z^2+2^3z^3+\cdots..]$

$F(z) = [-2z - 2z^2 - 2z^3 - \cdots \ldots] + [2z + 2^2z^2 + 2^3z^3 + \cdots \ldots]$

From first series,

Coefficient of $z^k = -2, k > 0$

Coefficient of $z^{-k} = -2, k < 0$

From second series,

Coefficient of $z^k = 2^k, k > 0$

Coefficient of $z^{-k} = 2^{-k}, k < 0$

Thus,

$$Z^{-1}\left\{\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}\right\} = 2^{-k} - 2 \quad , \quad k < 0$$

(iii) $\frac{1}{2} < |z| < 1$

$$F(z) = \frac{2z}{-1+z} - \frac{z}{z-\frac{1}{2}}$$

$$F(z) = \frac{2z}{-[1-z]} - \frac{z}{z\left[1-\frac{1}{2z}\right]}$$

$$F(z) = -2z[1-z]^{-1} - \left[1 - \frac{1}{2z}\right]^{-1}$$

$$F(z) = -2z[1 + z + z^2 + z^3 + \cdots ..] - \left[1 + \frac{1}{2z} + \frac{1}{2^2z^2} + \frac{1}{2^3z^3} + \cdots ..\right]$$

$$F(z) = [-2z - 2z^2 - 2z^3 - \cdots \ldots] + [-2^0z^0 - 2^{-1}z^{-1} - 2^{-2}z^{-2} + \cdots]$$

From first series,

Coefficient of $z^k = -2, k > 0$

Coefficient of $z^{-k} = -2, k < 0$

From second series,

Coefficient of $z^{-k} = -2^{-k}, k \geq 0$

Thus,

$$Z^{-1}\left\{\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}\right\} = \begin{cases} -2 & k < 0 \\ -2^{-k} & k \geq 0 \end{cases}$$

15. $Z^{-1}\left[\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}\right]$ for $\frac{1}{2}<|z|<1$

Solution:

$F(z)=\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}$

$\frac{F(z)}{z}=\frac{z}{(z-1)\left(z-\frac{1}{2}\right)}$

Let $\frac{z}{(z-1)\left(z-\frac{1}{2}\right)}=\frac{A}{z-1}+\frac{B}{z-\frac{1}{2}}$

$z=A\left(z-\frac{1}{2}\right)+B(z-1)$

Comparing the coefficients, we get

$A+B=1$

$-\frac{1}{2}A-B=0$

On solving, we get $A=2, B=-1$

$\frac{F(z)}{z}=\frac{2}{z-1}-\frac{1}{z-\frac{1}{2}}$

$F(z)=\frac{2z}{z-1}-\frac{z}{z-\frac{1}{2}}$

For ROC $\frac{1}{2}<|z|<1$

$F(z)=\frac{2z}{-1+z}-\frac{z}{z-\frac{1}{2}}$

$F(z)=\frac{2z}{-[1-z]}-\frac{z}{z\left[1-\frac{1}{2z}\right]}$

$F(z)=-2z[1-z]^{-1}-\left[1-\frac{1}{2z}\right]^{-1}$

$F(z)=-2z[1+z+z^2+z^3+\cdots..]-\left[1+\frac{1}{2z}+\frac{1}{2^2z^2}+\frac{1}{2^3z^3}+\cdots..\right]$

$F(z)=[-2z-2z^2-2z^3-\cdots\ldots]+[-2^0z^0-2^{-1}z^{-1}-2^{-2}z^{-2}+\cdots]$

From first series,

Coefficient of $z^k=-2, k>0$

Coefficient of $z^{-k}=-2, k<0$

From second series,

Coefficient of $z^{-k}=-2^{-k}, k\geq 0$

Thus,

$Z^{-1}\left\{\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}\right\}=\begin{cases}-2 & k<0\\ -2^{-k} & k\geq 0\end{cases}$

16. Find the inverse z transform of $F(z)=\frac{z^3}{(z-3)(z-2)^2}$ (i) $2<|z|<3$ (ii) $|z|>3$

[N14/CompIT/6M]

Solution:

We have,

$$F(z)=\frac{z^3}{(z-3)(z-2)^2}$$

$$\frac{F(z)}{z}=\frac{z^2}{(z-3)(z-2)^2}$$

Let $\frac{z^2}{(z-3)(z-2)^2}=\frac{A}{z-3}+\frac{B}{z-2}+\frac{C}{(z-2)^2}$

$$z^2=A(z-2)^2+B(z-3)(z-2)+C(z-3)$$

$$z^2=A(z^2-4z+4)+B(z^2-5z+6)+C(z-3)$$

Comparing the coefficients, we get

$$A+B=1$$

$$-4A-5B+C=0$$

$$4A+6B-3C=0$$

On solving, we get

$$A=9, B=-8, C=-4$$

$$\frac{F(z)}{z}=\frac{9}{z-3}-\frac{8}{z-2}-\frac{4}{(z-2)^2}$$

$$F(z)=\frac{9z}{z-3}-\frac{8z}{z-2}-\frac{4z}{(z-2)^2}$$

(i) $2<|z|<3$

$$F(z)=\frac{9z}{-3+z}-\frac{8z}{z-2}-\frac{4z}{(z-2)^2}$$

$$F(z)=\frac{9z}{-3\left[1-\frac{z}{3}\right]}-\frac{8z}{z\left[1-\frac{2}{z}\right]}-\frac{4z}{\left(z\left[1-\frac{2}{z}\right]\right)^2}$$

$$F(z)=-3z\left[1-\frac{z}{3}\right]^{-1}-8\left[1-\frac{2}{z}\right]^{-1}-\frac{4}{z}\left[1-\frac{2}{z}\right]^{-2}$$

$$F(z)=-3z\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots\right]-8\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots\right]-\frac{4}{z}\left[1+\frac{2.2}{z}+\frac{3.2^2}{z^2}+\cdots..\right]$$

$$F(z)=[-3.3^0.z-3.3^{-1}.z^2-3.3^{-2}.z^3-\cdots.]+[-8.2^0.z^0-8.2^1.z^{-1}-8.2^2.z^{-2}-\cdots..]$$
$$+[-4.1.2^0.z^{-1}-4.2.2^1.z^{-2}-4.3.2^2.z^{-3}-\cdots....]$$

$$F(z)=[-3.3^0.z-3.3^{-1}.z^2-3.3^{-2}.z^3-\cdots.]+[(-8.2^0-4.0.2^{-1})z^0+(-8.2^1-4.1.2^0)z^{-1}+(-8.2^2-4.2.2^1)z^{-2}+\cdots..]$$

From first series,

Coefficient of $z^k=-3.3^{-(k-1)}, k>0$

Coefficient of $z^{-k}=-3.3^{k+1}, k<0$

Coefficient of $z^{-k}=-3^{k+2}, k<0$

From second series,

Coefficient of $z^{-k}=(-8.2^k-4.k.2^{k-1}), k\geq 0$

Coefficient of $z^{-k}=-8.2^k-2k.2^k$

Coefficient of $z^{-k}=-(2k+8)2^k, k\geq 0$

Thus,

$$Z^{-1}\left\{\frac{z^3}{(z-3)(z-2)^2}\right\} = \begin{cases} -3^{k+2} & ,k<0 \\ -(2k+8)2^k & ,k\geq 0 \end{cases}$$

(ii) $|z| > 3$

$$F(z) = \frac{9z}{z-3} - \frac{8z}{z-2} - \frac{4z}{(z-2)^2}$$

$$F(z) = \frac{9z}{z\left[1-\frac{3}{z}\right]} - \frac{8z}{z\left[1-\frac{2}{z}\right]} - \frac{4z}{\left(z\left[1-\frac{2}{z}\right]\right)^2}$$

$$F(z) = 9\left[1-\frac{3}{z}\right]^{-1} - 8\left[1-\frac{2}{z}\right]^{-1} - \frac{4}{z}\left[1-\frac{2}{z}\right]^{-2}$$

$$F(z) = 9\ \left[1+\frac{3}{z}+\frac{3^2}{z^2}+\frac{3^3}{z^3}+\cdots\right] - 8\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots\right] - \frac{4}{z}\left[1+\frac{2.2}{z}+\frac{3.2^2}{z^2}+\cdots..\right]$$

$$F(z) = [9.\,3^0.z^0 + 9.\,3^1.z^{-1} + 9.3^2.z^{-2} + \cdots.] + [-8.2^0.z^0 - 8.2^1.z^{-1} - 8.2^2.z^{-2} - \cdots..]$$
$$+[-4.1.2^0.z^{-1} - 4.2.2^1.z^{-2} - 4.3.2^2.z^{-3} - \cdots ....]$$

$$F(z) = [9.\,3^0.z^0 + 9.\,3^1.z^{-1} + 9.3^2.z^{-2} + \cdots.] + [(-8.2^0 - 4.0.2^{-1})z^0 + (-8.2^1 - 4.1.2^0)z^{-1} +$$
$$(-8.2^2 - 4.2.2^1)z^{-2} + \cdots..]$$

From first series,

Coefficient of $z^{-k} = 9.3^k, k \geq 0$

From second series,

Coefficient of $z^{-k} = (-8.2^k - 4.\,k.\,2^{k-1})\ , k \geq 0$

Coefficient of $z^{-k} = -8.2^k - 2k.\,2^k$

Coefficient of $z^{-k} = -(2k+8)2^k, k \geq 0$

Thus,

$$Z^{-1}\left\{\frac{z^3}{(z-3)(z-2)^2}\right\} = 9.3^k - (2k+8)2^k\ , k \geq 0$$

17. $Z^{-1}\left[\frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right], \frac{1}{5}<|z|<\frac{1}{4}$

Solution:

We have,

$F(z)=\frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$

Let $\frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}=\frac{A}{z-\frac{1}{4}}+\frac{B}{z-\frac{1}{5}}$

$z=A\left(z-\frac{1}{5}\right)+B\left(z-\frac{1}{4}\right)$

Comparing the coefficients, we get

$A+B=1$

$-\frac{A}{5}-\frac{B}{4}=0$

On solving, we get

$A=5, B=-4$

$F(z)=\frac{5}{z-\frac{1}{4}}-\frac{4}{z-\frac{1}{5}}$

For ROC $\frac{1}{5}<|z|<\frac{1}{4}$

$F(z)=\frac{5}{z-\frac{1}{4}}-\frac{4}{-\frac{1}{5}+z}$

$F(z)=\frac{5}{z\left[1-\frac{1}{4z}\right]}-\frac{4}{-\frac{1}{5}[1-5z]}$

$F(z)=\frac{5}{z}\left[1-\frac{1}{4z}\right]^{-1}+20[1-5z]^{-1}$

$F(z)=\frac{5}{z}\left[1+\frac{1}{4z}+\frac{1}{4^2z^2}+\frac{1}{4^3z^3}+\cdots..\right]+20[1+5z+5^2z^2+5^3z^3+..]$

$F(z)=[5.4^0z^{-1}+5.4^{-1}z^{-2}+5.4^{-2}z^{-3}+\cdots\ldots]+[20.5^0z^0+20.5^1z^1+20.5^2z^2+\cdots\ldots]$

From first series,

Coefficient of $z^{-k}=5.4^{-(k-1)}, k>0$

Coefficient of $z^{-k}=20.4^{-k}, k>0$

Coefficient of $z^{-k}=5.4^{-k+1}, k>0$

From second series,

Coefficient of $z^{k}=20.5^{k}, k\geq 0$

Coefficient of $z^{-k}=20.5^{-k}, k\leq 0$

Coefficient of $z^{-k}=4.5^{-k+1}, k\leq 0$

Thus,

$$Z^{-1}\left\{\frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right\}=\begin{cases}4.5^{-k+1} & k\leq 0\\ 5.4^{-k+1} & k>0\end{cases}$$

18. $Z^{-1}\left[\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}\right], \frac{1}{3} < |z| < \frac{1}{2}$

[M15/CompIT/6M]

Solution:

We have,

$F(z) = \frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$

Let $\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{3}}$

$1 = A\left(z-\frac{1}{3}\right) + B\left(z-\frac{1}{2}\right)$

Comparing the coefficients, we get

$A + B = 0$

$-\frac{A}{3} - \frac{B}{2} = 1$

On solving, we get

$A = 6, B = -6$

$F(z) = \frac{6}{z-\frac{1}{2}} - \frac{6}{z-\frac{1}{3}}$

$F(z) = \frac{6}{-\frac{1}{2}+z} - \frac{6}{z-\frac{1}{3}}$

$F(z) = \frac{6}{-\frac{1}{2}[1-2z]} - \frac{6}{z\left[1-\frac{1}{3z}\right]}$

$F(z) = -12[1-2z]^{-1} - \frac{6}{z}\left[1-\frac{1}{3z}\right]^{-1}$

$F(z) = -12[1+2z+2^2z^2+2^3z^3+\cdots..] - \frac{6}{z}\left[1+\frac{1}{3z}+\frac{1}{3^2z^2}+\frac{1}{3^3z^3}+..\right]$

$F(z) = [-12.2^0.z^0 - 12.2^1.z^1 - 12.2^2.z^2 - \cdots ...] + [-6.3^0.z^{-1} - 6.3^{-1}.z^{-2} - 6.3^{-2}.z^{-3} - \cdots ...]$

From first series,

Coefficient of $z^k = -12.2^k, k \geq 0$

Coefficient of $z^{-k} = -12.2^{-k}, k \leq 0$

From second series,

Coefficient of $z^{-k} = -6.3^{k-1}, k > 0$

Thus,

$Z^{-1}\left\{\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}\right\} = \begin{cases} -12.2^{-k} & k \leq 0 \\ -6.3^{k-1} & k > 0 \end{cases}$

19. $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right]$

Solution:

We have,

$F(z) = \frac{8z^2}{(2z-1)(4z-1)} = \frac{8z^2}{2\left(z-\frac{1}{2}\right)4\left(z-\frac{1}{4}\right)}$

$\frac{F(z)}{z} = \frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$

Let $\frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{4}}$

$z = A\left(z-\frac{1}{4}\right) + B\left(z-\frac{1}{2}\right)$

Comparing the coefficients, we get

$A + B = 1$

$-\frac{A}{4} - \frac{B}{2} = 0$

On solving, we get $A = 2, B = -1$

$\frac{F(z)}{z} = \frac{2}{z-\frac{1}{2}} - \frac{1}{z-\frac{1}{4}}$

$F(z) = \frac{2z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}$

(i) For $|z| < \frac{1}{4}$

$F(z) = \frac{2z}{-\frac{1}{2}+z} - \frac{z}{-\frac{1}{4}+z}$

$F(z) = \frac{2z}{-\frac{1}{2}(1-2z)} - \frac{z}{-\frac{1}{4}(1-4z)}$

$F(z) = -4z[1-2z]^{-1} + 4z[1-4z]^{-1}$

$F(z) = -4z[1+2z+2^2z^2+\cdots..] + 4z[1+4z+4^2z^2+\cdots\ldots.]$

$F(z) = [-4.2^0z^1 - 4.2^1z^2 - 4.2^2z^3 - \cdots.] + [4^1z^1 + 4^2z^2 + 4^3z^3 + \cdots.]$

From I series,

Coefficient of $z^k = -4.2^{k-1}, k > 0$

Coefficient of $z^{-k} = -4.2^{-k-1}, k < 0$

Coefficient of $z^{-k} = -2.2^{-k}, k < 0$

From II series,

Coefficient of $z^k = 4^k, k > 0$

Coefficient of $z^{-k} = 4^{-k}, k < 0$

$Z^{-1}\left\{\frac{8z^2}{(2z-1)(4z-1)}\right\} = 4^{-k} - 2.2^{-k}, k < 0$

(ii) For $\frac{1}{4} < |z| < \frac{1}{2}$,

$F(z) = \frac{2z}{-\frac{1}{2}+z} - \frac{z}{z-\frac{1}{4}}$

$F(z) = \frac{2z}{-\frac{1}{2}(1-2z)} - \frac{z}{z\left(1-\frac{1}{4z}\right)}$

$F(z) = -4z[1-2z]^{-1} - \left[1-\frac{1}{4z}\right]^{-1}$

$F(z) = -4z[1+2z+2^2z^2+\cdots..] - \left[1+\frac{1}{4z}+\frac{1}{4^2z^2}+\cdots....\right]$

$F(z) = [-4.2^0z^1 - 4.2^1z^2 - 4.2^2z^3 - \cdots.] + [-4^0z^0 - 4^{-1}z^{-1} - 4^{-2}z^{-2} + \cdots.]$

From I series,

Coefficient of $z^k = -4.2^{k-1}, k > 0$

Coefficient of $z^{-k} = -4.2^{-k-1}, k < 0$

Coefficient of $z^{-k} = -2.2^{-k}, k < 0$

From II series,

Coefficient of $z^{-k} = -4^{-k}, k \geq 0$

$Z^{-1}\left\{\frac{8z^2}{(2z-1)(4z-1)}\right\} = \begin{cases} -2.2^{-k} & k < 0 \\ -4^{-k} & k \geq 0 \end{cases}$

(iii) For $|z| > \frac{1}{2}$

$F(z) = \frac{2z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}$

$F(z) = \frac{2z}{z\left(1-\frac{1}{2z}\right)} - \frac{z}{z\left(1-\frac{1}{4z}\right)}$

$F(z) = 2\left[1-\frac{1}{2z}\right]^{-1} - \left[1-\frac{1}{4z}\right]^{-1}$

$F(z) = 2\left[1+\frac{1}{2z}+\frac{1}{2^2z^2}+\cdots..\right] - \left[1+\frac{1}{4z}+\frac{1}{4^2z^2}+\cdots....\right]$

$F(z) = [2.2^0z^0 + 2.2^{-1}z^{-1} + 2.2^{-2}z^{-2} + \cdots.] + [-4^0z^0 - 4^{-1}z^{-1} - 4^{-2}z^{-2} + \cdots.]$

From I series,

Coefficient of $z^{-k} = 2.2^{-k}, k \geq 0$

From II series,

Coefficient of $z^{-k} = -4^{-k}, k \geq 0$

$Z^{-1}\left\{\frac{8z^2}{(2z-1)(4z-1)}\right\} = 2.2^{-k} - 4^{-k}, k \geq 0$

20. Find inverse Z transform of $\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}, 3 < |z| < 4$

[M17/N18/Comp/6M][M24/CompITAI/6M]

Solution:

We have,

$F(z) = \frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$

Let $\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{z-4}$

$3z^2 - 18z + 26 = A(z-3)(z-4) + B(z-4)(z-2) + C(z-3)(z-2)$

$3z^2 - 18z + 26 = A(z^2 - 7z + 12) + B(z^2 - 6z + 8) + C(z - 5z + 6)$

Comparing the coefficients, we get

$A + B + C = 3$

$-7A - 6B - 5C = -18$

$12A + 8B + 6C = 26$

On solving, we get

$A = 1, B = 1, C = 1$

$F(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{z-4}$

For $3 < |z| < 4$,

$F(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{-4+z}$

$F(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} + \frac{1}{z\left(1-\frac{3}{z}\right)} + \frac{1}{-4\left(1-\frac{z}{4}\right)}$

$F(z) = \frac{1}{z}\left[1 - \frac{2}{z}\right]^{-1} + \frac{1}{z}\left[1 - \frac{3}{z}\right]^{-1} - \frac{1}{4}\left[1 - \frac{z}{4}\right]^{-1}$

$F(z) = \frac{1}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \cdots\right] + \frac{1}{z}\left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \cdots\right] - \frac{1}{4}\left[1 + \frac{z}{4} + \frac{z^2}{4^2} + \cdots..\right]$

$F(z) = [2^0 z^{-1} + 2^1 z^{-2} + 2^2 z^{-3} + \cdots.] + [3^0 z^{-1} + 3^1 z^{-2} + 3^2 z^{-3} + \cdots.]$
$+[-4^{-1} z^0 - 4^{-2} z^1 - 4^{-3} z^2 - \cdots..]$

From first series,

Coefficient of $z^{-k} = 2^{k-1}, k > 0$

From second series,

Coefficient of $z^{-k} = 3^{k-1}, k > 0$

From third series,

Coefficient of $z^k = -4^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -4^{k-1}, k \leq 0$

Thus,

$$Z^{-1}\left\{\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}\right\} = \begin{cases} -4^{k-1} & k \leq 0 \\ \{2^{k-1} + 3^{k-1}\} & k > 0 \end{cases}$$

21. Find inverse Z transform of $\frac{5z}{(2z-1)(z-3)}, \frac{1}{2} < |z| < 3$

[N19/Comp/6M]

Solution:

We have,

$F(z) = \frac{5z}{(2z-1)(z-3)}$

Let $\frac{5z}{(2z-1)(z-3)} = \frac{A}{2z-1} + \frac{B}{z-3}$

$5z = A(z-3) + B(2z-1)$

Comparing the coefficients, we get

$A + 2B = 5$

$-3A - B = 0$

On solving, we get $A = -1, B = 3$

$F(z) = \frac{-1}{2z-1} + \frac{3}{z-3}$

$F(z) = -\frac{1}{2\left(z-\frac{1}{2}\right)} + \frac{3}{z-3}$

For $\frac{1}{2} < |z| < 3$

$F(z) = -\frac{1}{2\left(z-\frac{1}{2}\right)} + \frac{3}{-3+z}$

$F(z) = -\frac{1}{2z\left(1-\frac{1}{2z}\right)} + \frac{3}{-3\left(1-\frac{z}{3}\right)}$

$F(z) = -\frac{1}{2z}\left[1 - \frac{1}{2z}\right]^{-1} - \left[1 - \frac{z}{3}\right]^{-1}$

$F(z) = -\frac{1}{2z}\left[1 + \frac{1}{2z} + \frac{1}{2^2z^2} + \cdots..\right] - \left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots....\right]$

$F(z) = \left[-\frac{1}{2z} - \frac{1}{2^2z^2} - \frac{1}{2^3z^3} - \cdots..\right] + \left[-1 - \frac{z}{3} - \frac{z^2}{3^2} - \cdots.\right]$

$F(z) = [-2^{-1}z^{-1} - 2^{-2}z^{-2} - 2^{-3}z^{-3} - \cdots] + [-3^0z^0 - 3^{-1}z^1 - 3^{-2}z^2 - ..]$

From I series,

Coefficient of $z^{-k} = -2^{-k}, k > 0$

From II series,

Coefficient of $z^{k} = -3^{-k}, k \geq 0$

Coefficient of $z^{-k} = -3^{k}, k \leq 0$

$Z^{-1}\left\{\frac{5z}{(2z-1)(z-3)}\right\} = \begin{cases} -3^k & k \leq 0 \\ -2^{-k} & k > 0 \end{cases}$

Type VI: Convolution Theorem

1. Find the inverse z transform of $\frac{z^2}{(z-a)(z-b)}$ by convolution method

Solution:

We know that,

$Z^{-1}\left[\frac{z}{z-a}\right]=a^k$ and $Z^{-1}\left[\frac{z}{z-b}\right]=b^k$

By convolution theorem,

$Z^{-1}\{F(z).G(z)\}=\sum_{k=0}^{n}f(k)g(n-k)$

$$
\begin{aligned}
Z^{-1}\left\{\frac{z}{z-a}.\frac{z}{z-b}\right\} &= \sum_{k=0}^{n}a^k b^{n-k} \\
&= \sum_{k=0}^{n}a^k.b^n.b^{-k} \\
&= b^n\sum_{k=0}^{n}\left(\frac{a}{b}\right)^k \\
&= b^n\left[\left(\frac{a}{b}\right)^0+\left(\frac{a}{b}\right)^1+\left(\frac{a}{b}\right)^2+\cdots\ldots\left(\frac{a}{b}\right)^n\right] \\
&= b^n\left[1+\frac{a}{b}+\frac{a^2}{b^2}+\cdots\ldots..\frac{a^n}{b^n}\right] \\
&= b^n.\frac{1\left(\left(\frac{a}{b}\right)^{n+1}-1\right]}{\left(\frac{a}{b}\right)-1} \qquad S_{n+1}=\frac{a(r^{n+1}-1)}{r-1} \\
&= b^n.\frac{\frac{a^{n+1}-b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}}
\end{aligned}
$$

$Z^{-1}\left\{\frac{z}{z-a}.\frac{z}{z-b}\right\}=\frac{a^{n+1}-b^{n+1}}{a-b}$

2. The value of $Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$ is

[M22/CompITAI/2M] Ans. $\frac{a^{n+1}-b^{n+1}}{a-b}$

3. Find the inverse z transform of $\frac{z^2}{(z-1)(2z-1)}$ by convolution method

Solution:

We know that,

$$Z^{-1}\left[\frac{z}{z-1}\right] = 1^k \text{ and } Z^{-1}\left[\frac{z}{2z-1}\right] = \frac{1}{2}Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] = \frac{1}{2}.\left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{k+1}$$

By convolution theorem,

$$Z^{-1}\{F(z).G(z)\} = \sum_{k=0}^{n} f(k)g(n-k)$$

$$Z^{-1}\left\{\frac{z}{z-1}.\frac{z}{2z-1}\right\} = \sum_{k=0}^{n} 1^k\left(\frac{1}{2}\right)^{n-k+1}$$

$$= \sum_{k=0}^{n}\left(\frac{1}{2}\right)^{n+1}.\left(\frac{1}{2}\right)^{-k}$$

$$= \left(\frac{1}{2}\right)^{n+1}\sum_{k=0}^{n}(2)^k$$

$$= \frac{1}{2^{n+1}}.\left[(2)^0 + (2)^1 + (2)^2 + \cdots \ldots (2)^n\right]$$

$$= \frac{1}{2^{n+1}}\left[1 + 2 + 2^2 + \cdots \ldots . . +2^n\right]$$

$$= \frac{1}{2^{n+1}}.\frac{1((2)^{n+1}-1]}{(2)-1} \qquad S_{n+1} = \frac{a(r^{n+1}-1)}{r-1}$$

$$= \frac{1}{2^{n+1}}.(2^{n+1}-1)$$

$$Z^{-1}\left\{\frac{z}{z-1}.\frac{z}{2z-1}\right\} = \frac{2^{n+1}-1}{2^{n+1}}$$

4. Find the inverse z transform of $\frac{z^2}{(z-a)^2}$ by convolution method

Solution:

We know that,

$$Z^{-1}\left[\frac{z}{z-a}\right] = a^k$$

By convolution theorem,

$$Z^{-1}\{F(z).G(z)\} = \sum_{k=0}^{n} f(k)g(n-k)$$

$$Z^{-1}\left\{\frac{z}{z-a}.\frac{z}{z-a}\right\} = \sum_{k=0}^{n} a^k(a)^{n-k}$$

$$= \sum_{k=0}^{n}(a)^n$$

$$= a^n\sum_{k=0}^{n}1$$

$$= a^n.\left[1 + 1 + 1 + \cdots \ldots (n+1)times\right]$$

$$= a^n[n+1]$$

$$Z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\} = a^n(n+1)$$

5. Find the inverse z transform of $\frac{z^3}{(z-1)^3}$ by convolution method

Solution:

We know that,

$$Z^{-1}\left[\frac{z}{z-1}\right] = 1^k$$

By convolution theorem,

$$Z^{-1}\{F(z).G(z)\} = \sum_{k=0}^{n} f(k)g(n-k)$$

$$Z^{-1}\left\{\frac{z}{z-1}.\frac{z}{z-1}\right\} = \sum_{k=0}^{n} 1^k(1)^{n-k}$$

$$= \sum_{k=0}^{n}(1)^n$$

$$= 1^n \sum_{k=0}^{n} 1$$

$$= [1+1+1+\cdots\ldots(n+1)times]$$

$$= [n+1]$$

$$Z^{-1}\left\{\frac{z^2}{(z-1)^2}\right\} = (n+1) = (k+1)$$

By convolution theorem,

$$Z^{-1}\{F(z).G(z)\} = \sum_{k=0}^{n} f(k)g(n-k)$$

$$Z^{-1}\left\{\frac{z^2}{(z-1)^2}.\frac{z}{z-1}\right\} = \sum_{k=0}^{n}(k+1).1^{n-k}$$

$$= 1^n \sum_{k=0}^{n}(k+1)$$

$$= \sum_{k=0}^{n}(k+1)$$

$$= [1+2+3+\cdots\ldots(n+1)]$$

$$= \frac{(n+1)(n+2)}{2}$$

$$Z^{-1}\left\{\frac{z^3}{(z-1)^3}\right\} = \frac{(n+1)(n+2)}{2}$$