Question Bank in Probability, Distributions

5 marks Questions

1. A discrete random variable X has the following distribution:

X:	0	1	2	3	4	5	6
Y:	k	15k	8k	7k	5k	3k	k

Find k and the mean of X.

2. The probability distribution of a random variable X is given as:

X:	-2	-1	0	1	2	3
P(X=x):	0.1	k	0.2	2k	0.3	k

Find k, the mean of X, and $Var(X)$

3. Find k and mean of the following distribution:

X :	8	12	16	20	24
P(X):	1/8	k	3/8	1/4	1/12

4. A continuous random variable has the pdf
$$f(x) = \begin{cases} \frac{x}{6} + kx, & 0 \leq x \leq 3 \\ 0, \ otherwise \end{cases}$$
find k and $P(1 \leq X \leq 2)$

6 marks Questions

1. In a factory, machines A, B and C produce 30%, 50% and 20% of the total production of an item. Out of their production 80%, 50% and 10% are defective respectively. An item is chosen at random and found to be defective. What is the probability that it was produced by machine A?

2. Three factories A, B and C produce 20%, 70% and 10% of the total production of an item. Out of their production 1%, 30% and 40% are defective respectively. An item is chosen at random and found to be defective. What is the probability that it was produced by factory B?

3. A random variable X has the distribution:

X :	0	1	2	3	4	5	6
P(X):	k	3k	5k	7k	9k	11k	13k

Find k and $P(3 < X \leq 6)$

4. A continuous random variable has the pdf
$$f(x) = \begin{cases} \frac{x}{6} + kx, & 0 \leq x \leq 3 \\ 0, \ otherwise \end{cases}$$
find k and $P(1 \leq X \leq 2)$

5. A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6
P(X=x)	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find (i) k, (ii) $P(X < 4)$, (iii) $P(3 < X \leq 6)$

6. Find the mean and variance for the following distribution:

X	1	3	4	5
P(X=x)	0.4	0.1	0.2	0.3

7. The probability distribution of a random variable X is given by [6]

X:	0	1	2	3	4	5	6
P(X=x):	k	3k	5k	7k	9k	11k	13k

Find (i) k (ii) $P(X < 4)$, $P(3 < X \leq 6)$

8. Two unbiased dice are thrown. If X represents the sum of numbers on the 2 dice,find the probability distribution of X and obtain mean, standard deviation and $P(|X-3| \geq 3)$

9. If a continuous random variable X has the following probability density function, $f(x) = \begin{cases} ke^{-x/4}, x > 0 \\ 0, \ elsewhere \end{cases}$

Find k, mean and the variance.

Solution: Since $f(x)$ is a pdf, we have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\ dx &= 1 \\ \Rightarrow \int_{0}^{\infty} ke^{-x/4}\ dx &= 1 \\ \Rightarrow k\frac{e^{-x/4}}{-1/4}|_0^\infty &= 1 \\ \Rightarrow k(4) &= 1 \\ \Rightarrow k &= \frac{1}{4} \\ \Rightarrow pdf,\ f(x) &= \frac{1}{4}e^{-x/4}, x > 0 \end{aligned}$$

Now

$$\begin{aligned} Mean\ = E(X) &= \int_{-\infty}^{\infty} xf(x)\ dx \\ \Rightarrow E(X) &= \int_{0}^{\infty} x\frac{1}{4}e^{-x/4} \\ &= \frac{1}{4}(x\frac{e^{-x/4}}{-1/4} - (1)\frac{e^{-x/4}}{1/16})\ |_0^\infty \\ \Rightarrow E(X) &= 4 \end{aligned}$$

and

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2f(x)\ dx \\ \Rightarrow E(X^2) &= \int_{0}^{\infty} x^2\frac{1}{4}e^{-x/4} \\ &= \frac{1}{4}(x^2\frac{e^{-x/4}}{-1/4} - (2x)\frac{e^{-x/4}}{1/16} + (2)\frac{e^{-x/4}}{1/64})\ |_0^\infty \\ \Rightarrow E(X^2) &= 32 \end{aligned}$$

Hence

$$\begin{aligned} Var(X) &= E(X^2) - (E(X))^2 \\ &= 32 - 16 \\ \Rightarrow Var(X) &= 16 \end{aligned}$$

10. Find k, if the function,
$f(x) = kx^2(1-x^3), 0 < x < 1$ is a probability density function.

11. A continuous random variable has pdf $f(x) = kx^2, 0 \le x \le 2$.
Obtain k, mean and $P(0.2 < X < 0.5)$.

12. A continuous random variable has pdf $f(x) = kx^2(1-x), 0 \le x \le 1$ Obtain k, mean and variance.

13. A continuous r.v X has pdf $f(x) = kx^2e^{-x},\ x > 0$. Find k, mean and variance.

14. A continuous random variable X has the following probability density function, $f(x) = k(x - x^2), 0 < x < 1$
Find k, mean and the variance.

15. A continuous random variable X has the following probability density function,
$f(x) = k(x - x^2), 0 < x < 1$ Find k, mean and $P(0.5 \le X \le 3)$

16. If $f(x)$ is probability density function of a continuous random variable X. find k, mean, variance.
$$f(x) = \begin{cases} kx^2 & 0 \le x \le 1 \\ (2-x)^2 & 1 \le x \le 2 \end{cases}$$

17. If X denotes the outcome when a fair die is tossed, find the moment generating function of X and hence the mean and variance. [6]

Solution: The probability distribution of X is given by:

X	1	2	3	4	5	6
P(X=x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Now the MGF of X is

$$\begin{aligned} M_X(t) &= E(e^{tx}) \\ &= \sum_x e^{tx} p_x \\ &= \frac{1}{6}(e^{t(1)} + e^{t(2)} + e^{t(3)} + e^{t(4)} + e^{t(5)} + e^{t(6)}) \\ i.e\ M_X(t) &= \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}) \qquad \cdots\cdots (*) \end{aligned}$$

Now,the rth raw moment (about the origin) is given by

$\mu'_r = \frac{d^r}{dt^r} M_X(t)|_{(t=0)}$

$$\begin{aligned}
(*) \Rightarrow \frac{d}{dt} M_X(t) &= \frac{1}{6}(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}) \\
\Rightarrow \frac{d}{dt} M_X(t)|_{(t=0)} &= \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = 3.5 \quad \cdots\cdots (1) \\
and \ \frac{d^2}{dt^2} M_X(t) &= \frac{1}{6}(e^t + e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}) \\
\Rightarrow \frac{d^2}{dt^2} M_X(t)|_{(t=0)} &= \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6} = 15.17 \\
\cdots\cdots (2) & \\
(1)\&(2) \Rightarrow \mu'_1 &= 3.5 \ \ and \ \ \mu'_2 = 15.17
\end{aligned}$$

Now

$$\begin{aligned}
variance \ = \mu_2 &= \mu'_2 - (\mu'_1)^2 \\
\Rightarrow variance &= 15.17 - (3.5)^2 \\
i.e \ \ variance &= 2.92
\end{aligned}$$

18. Find moment generating function of the following distribution. hence find mean and variance.

X	1	3	4	5
P(X)	0.4	0.1	0.2	0.3

19. The first 4 moments of a distribution about origin of the random variable X are -1.5, 17, -30 and 108. Compute mean , variance, μ_3 and $\mu_4 = -1.5,$.

20. An unbiased coin is thrown 3 times. If X denotes the absolute difference between the number of heads and the number of tails, find the mgf of X and hence the first moment about the origin and the second moment about the mean.

Solution: The probability distribution of X is given by:

Outcome	{HHH, TTT}	{HHT,HTH,THH,TTH, THT, HTT}
X	0	1
P(X=x)	$\frac{2}{8} = \frac{1}{4}$	$\frac{6}{8} = \frac{3}{4}$

Now the MGF of X is

$$\begin{aligned}
M_X(t) &= E(e^{tx}) \\
&= \sum_x e^{tx} p_x \\
&= e^{t(0)}\frac{1}{4} + e^{t(1)}\frac{3}{4} \\
i.e \ M_X(t) &= \frac{1}{4} + \frac{3}{4}e^t \quad \cdots\cdots (*)
\end{aligned}$$

Now,the rth raw moment (about the origin) is given by

$\mu'_r = \frac{d^r}{dt^r} M_X(t)|_{(t=0)}$

$$\begin{aligned}
(*) \Rightarrow \frac{d}{dt} M_X(t) &= 0 + \frac{3}{4} e^t \\
\Rightarrow \frac{d}{dt} M_X(t)|_{(t=0)} &= \frac{3}{4} \quad \cdots\cdots (1) \\
and \quad \frac{d^2}{dt^2} M_X(t) &= \frac{3}{4} e^t \\
\Rightarrow \frac{d^2}{dt^2} M_X(t)|_{(t=0)} &= \frac{3}{4} \quad \cdots\cdots (2) \\
(1)\&(2) \Rightarrow \mu'_1 &= \frac{3}{4} \quad and \quad \mu'_2 = \frac{3}{4}
\end{aligned}$$

Now

$$\begin{aligned}
\mu_2 &= \mu'_2 - (\mu'_1)^2 \\
\Rightarrow \mu_2 &= \frac{3}{4} - (\frac{3}{4})^2 \\
i.e \quad \mu_2 &= \frac{12-9}{16} = \frac{3}{16}
\end{aligned}$$

Hence the first moment about the origin = Mean $= \mu'_1 = \frac{3}{4}$ and

the second moment about the mean = Variance $= \mu_2 = \frac{3}{16}$