Probability

Vocabulary, Rules, Complimentary Events, Addition and Multiplication Rules,

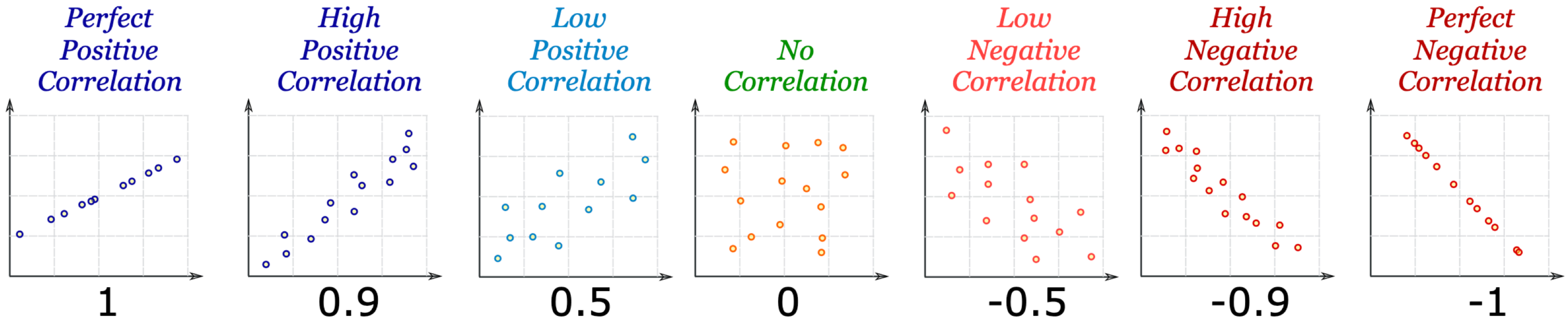
Conditional Probability, Total Probability Rule, Baye’s Rule, Probability Tree

Instructor: Qasim Ali

Recap Correlation and Regression

Correlation

Calculate the correlation of two datasets



$$r = \frac{\sum(x - \overline{x}) - (\sum y - \overline{y})}{\sqrt{\sum(x - \overline{x})^2 \sum(y - \overline{y})^2}}$$

$$r = \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2}\ \sqrt{n(\sum y^2) - (\sum y)^2}}$$

x: Data point for 1st variable

y: Data point for the second variable

n: Sample size

$\overline{x}$ and $\overline{y}$ are respective mean values of x and y

Coefficient of Determination = r^2, a positive value

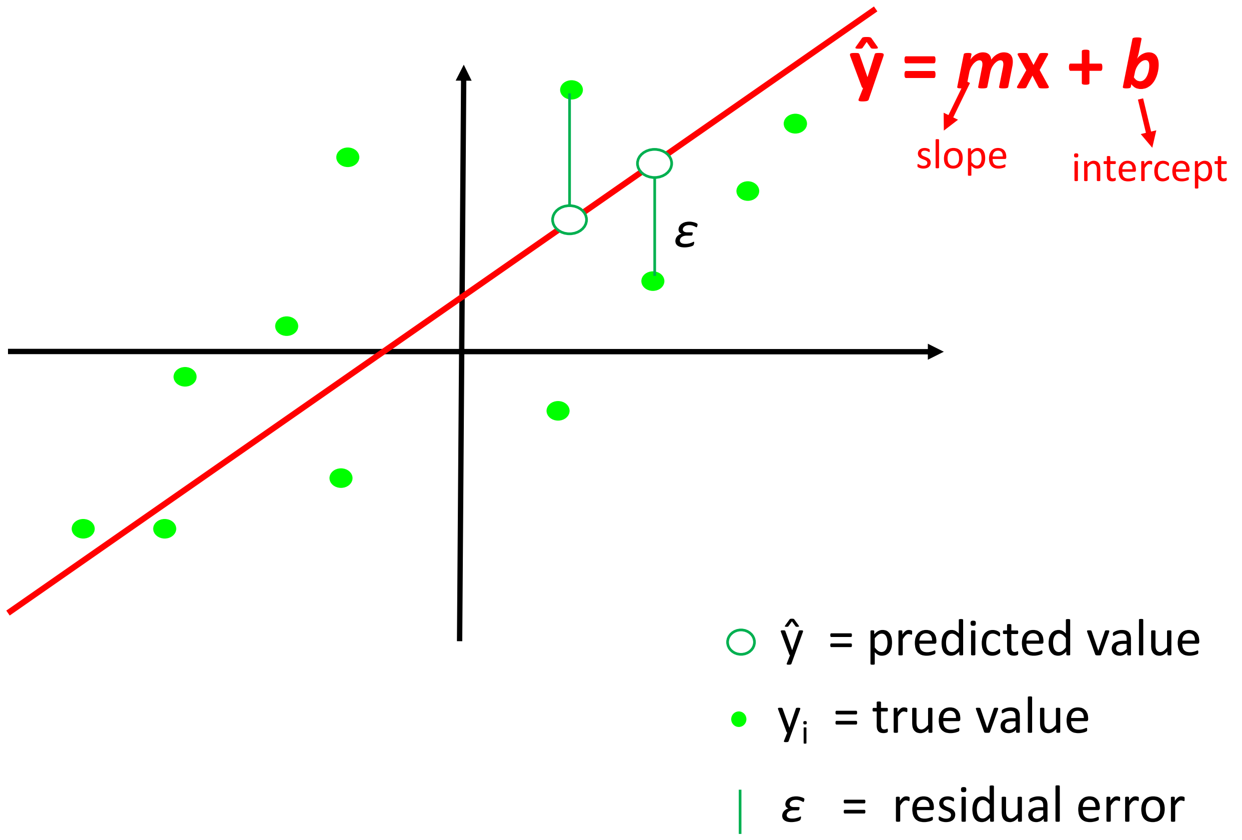
Regression

Best Fit

- Correlation tells you if there is an association between x and y but it doesn’t describe the relationship or doesn’t allow you to predict one variable from the other.

- Aim of linear regression is to fit a straight line, $\hat{y} = mx + b$, to data that gives best prediction of y for any value of x.

- This will be the line that minimizes distance between data and fitted line, i.e. the residuals



Regression

How to find the best fit

Step 1: For each (x, y) calculate x^2 and xy

Step 2: Calculate Slope $\boldsymbol{m}$

Step 3: Calculate Intercept $\boldsymbol{b}$

Step 4: Assemble the equation of a line:

Step 5: Find the (x, y) points and plot line $\boldsymbol{y = mx + b}$

$$b = \frac{(\sum y)(\sum x^2) - (\sum x)(\sum xy)}{n(\sum x^2) - (\sum x)^2}$$

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

x: Data point for 1st variable
y: Data point for the second variable
n: Sample size

Probability

Probability

Probability is based on experiments and trials in which there is an uncertain outcome between 0 and 1.

Simple Event

A single outcome (only one "answer")

Examples:

1. The probability of rolling a 3 on a die
2. The probability of drawing the ace of hearts from a deck of cards.
3. The probability of tossing a head with a penny.

$$Probability\ of\ event = \frac{number\ of\ times\ event\ occurs}{total\ possible\ outcomes}$$

$number\ of\ times\ it\ \textbf{occurs}\ is\ exactly\ 1$

Compound Event

Combination of **two or more** simple events

Examples:

1. The probability of rolling an even number less than 5.
2. The probability of drawing a red ace from a deck.
3. The probability of tossing three pennies and getting at least 2 heads.

$$Probability\ of\ event = \frac{number\ of\ times\ event\ occurs}{total\ possible\ outcomes}$$

$number\ of\ times\ it\ \textbf{occurs}\ is\ more\ than\ 1$

Notations in Probability

P denotes a Probability

- *A*, *B*, and *C* denote a specific event
- *P*(*A*) means a probability that an event *A* occurs.
- *P*($\overline{A}$) means a probability that an event *A* does not occur.
- *P*(*A* and *B*) means a probability that both events *A* and *B* occur together.
- *P*(*A* or *B*) means a probability that at least one of the two events *A and B occurs*
- *P*(*A* | B) means a probability that an event A occurs given that an event *B occurs*

Rules in Probability

Probabilities are defined based on some rules or principles.

General Principles

- The probability of an impossible event is zero (Probability of rolling a 7 on a die is 0).
- The probability of a certain event is one (Probability of getting head or tail by tossing a coin is 1)
- The sum of the **probabilities** of all possible outcomes is 1
- Range of possible probabilities is: $\mathbf{0 \leq \mathit{P} \leq 1}$

Rules for Simple Events

- Rule of Relative Frequency Approximation
- Rule of Classical Approach to Probability
- Rule of Subjective Probabilities

Rules for Compound Events

- Addition Rule
- Multiplication Rule
- Total Probability Rule
- Bayes' Rule

Probability

In the following examples, lets consider that there are exactly three possible outcomes A, B and C.

- The sum of all possible outcomes is:

$$P(A) + P(B) + P(C) \;\; = ?$$

Answer: 1

- The sum of an outcome and its compliments is:

$$P(A) + P(\overline{A}) = ?$$

Answer: 1

Simple Event

Simple Event

Relative Frequency Approximation of Probability

Conduct (or observe) a procedure a large number of times and count the number of times that event A actually occurs.

$\boldsymbol{P(A)}$ is *estimated* as follows:

$$P(A) = \frac{\text{number of times } A \text{ occurred}}{\text{number of times trial was repeated}}$$

For example, if you observed 100 passing cars and found that 23 of them were red, the relative frequency would be 23/100.

You can get a more accurate result in surveys of events if you carry out a large number of trials or survey a large number of people.

Simple Event

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Simple Event

Relative Frequency Approximation of Probability

Problem: Two air companies, A-Airlines and B-Airlines, operate out of Lester Pearson airport either with direct flights or as partners with other carriers. Recently, A-Airlines canceled or delayed 73 flights out of 993 total flights due to technical issues. Then, B-Airlines canceled or delayed 110 flights out of 2087 for the same reason. Which company looks more reliable?

Let A means A-Airlines canceled or delayed a flight; B means B-Airlines canceled or delayed a flight.

$$\boldsymbol{P}(\boldsymbol{A}) = \frac{\text{Total number of flights canceled by A} - \text{Airlines}}{\text{Total number of A} - \text{Airlines flights}} = \frac{73}{993} = 0.0735 \text{ or } 7.35\%$$

$$\boldsymbol{P}(\boldsymbol{B}) = \frac{\text{Total number of flights canceled by B} - \text{Airlines}}{\text{Total number of B} - \text{Airlines flights}} = \frac{110}{2087} = 0.0527 \text{ or } 5.27\%$$

Answer: B-Airlines is more reliable.

Simple Event

Rule of Classical Approach to Probability

Assume that a given experiment has n different simple events, each of which has an *equal chance* of occurring. If event A can occur in s of these n ways, then

$$P(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{number of different simple events}} = \frac{s}{n}$$

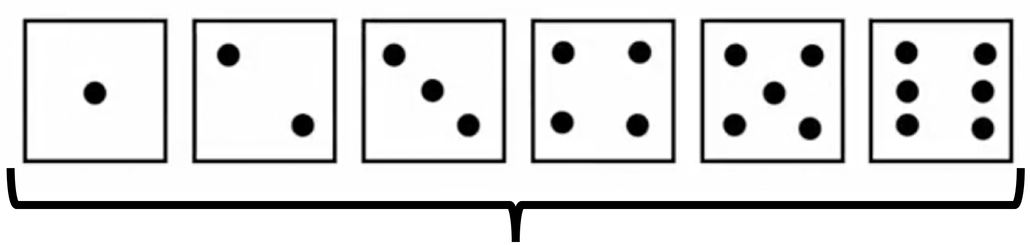
Procedure/Experiment	Simple Events	Sample Space
Roll a cube with six colored faces	R – getting red; Y – getting yellow; B – getting blue; G – getting green; O – getting orange; P – getting purple	{R, Y, B, G, O, P} There are 6 simple events.
Toss a coin	The upward face of the coin is H – head; T – tail.	{H, T } There are 2 simple events.
Play a hockey game	W – win; L – lose: T – tie	{W, L, T } There are 3 simple events.
Record student evaluation of a statistics course	A – excellent; B – very good; C – good; D – fair; E – poor	{A, B, C, D, E} There are 5 simple events.
Draw a card from a deck of playing cards	For example, KD – drawing king of diamonds; S_7 – drawing seven of spades;	There are 52 simple events.

Simple Event

Rule of Classical Approach to Probability

Experiment: Rolling a die

Possible outcomes:



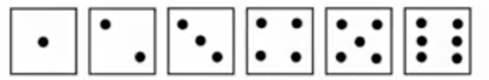
Sample space

Simple Event: Rolling a four

There is only one in six chance that we roll a die and get four as outcome.

$$Probability\ of\ an\ event = \frac{number\ of\ times\ it\ occurs}{total\ possible\ outcomes}$$

$Probability\ of\ rolling\ a\ four =$



$$Probability\ of\ rolling\ a\ four = \frac{1}{6} = 0.1667 = 16.7\%$$

Simple Event

Rule of Subjective Probabilities

An individual's personal judgment or own experience about whether a specific outcome is likely to occur.

Like the two other probabilities we defined, probability values need to satisfy the same general rules:

What is the probability for a particular stock to go up tomorrow?

≤
≤

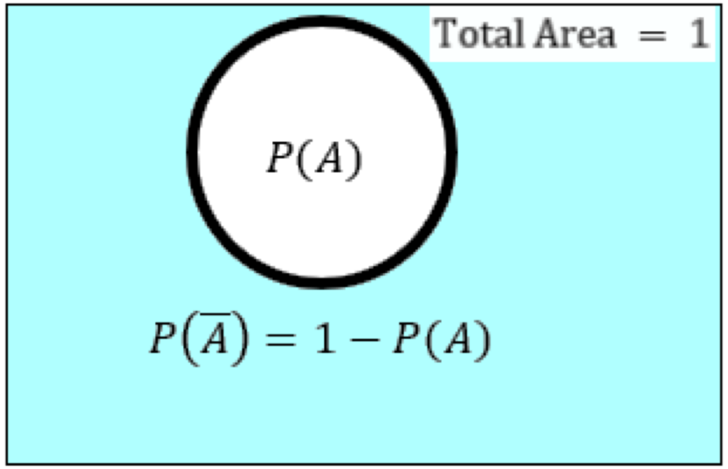
An economic analyst estimates the probability that the economic recession will end this fall as about 0.3.

Simple Event

Complementary Events

The **complement** of event A , denoted by $\overline{A}$ **,** consists of all outcomes in which event *A* does *not* occur.

Diagram for the Complement of Event A



Rule of Complementary Events:

- $P(A)+P(\overline{A})=1$
- $P(\overline{A})=1-P(A)$
- $P(A)=1-P(\overline{A})$

Simple Event

Complementary Events

A recent independent research confirms that, to meet their financial goals, 86% of Canadians have greater confidence in mutual funds than other financial product such as GICs, bonds, and stocks. If you randomly select someone from Canada, what is the probability that he or she does not think that mutual funds would be the most reliable investment?

Solution: Let M – a person trusts mutual funds more than other financial products. Then, $\overline{M}$– a person does not trust mutual funds more than other financial products.
Given $P(M) = 86\% = 0.86$. Thus,

$$P(\overline{M}) = 1 - P(M) = 1 - 0.86 = 0.14$$

Answer: 0.14 or 14%

Simple Event

- Lets consider that there are exactly three possible outcomes A, B and C. The sum of their compliments

$$P(\overline{A}) + P(\overline{B}) + P(\overline{C}) = ?$$

Answer: 2

$$P(\overline{A}) + P(\overline{B}) + P(\overline{C})$$
$$= 1 - P(A) + 1 - P(B) + 1 - P(C)$$
$$= 3 - \big(P(A) + P(B) + P(C)\big) = 3 - 1$$
$$= 2$$

In the following examples, please identify the type of simple event.

1. Classical Approach
2. Subjective Probabilities
3. Relative Frequency

- Roll the given die 100 times (say) and suppose the number of times the outcome 1 is observed is 15. which simple event rule would apply?

Answer: Relative Frequency

- What is the probability for a particular stock to go up tomorrow? Which simple event approach (rule) would apply?

Answer: Subjective approach

Compound Event

Probability

Compound Event

Any event combining two or more simple events.

Rules for Compound Events

- Addition Rule
- Multiplication Rule
- Total Probability Rule
- Bayes' Rule

Compound Event

Joint and Disjoint Probabilities

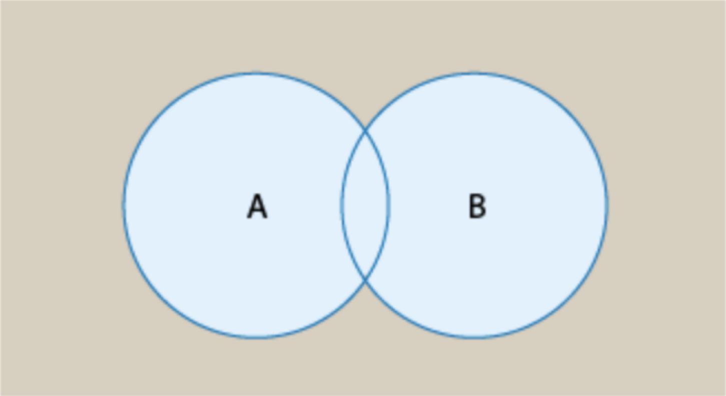
Joint and Disjoint Probabilities

It is possible that event A and event B occurs at the same time.

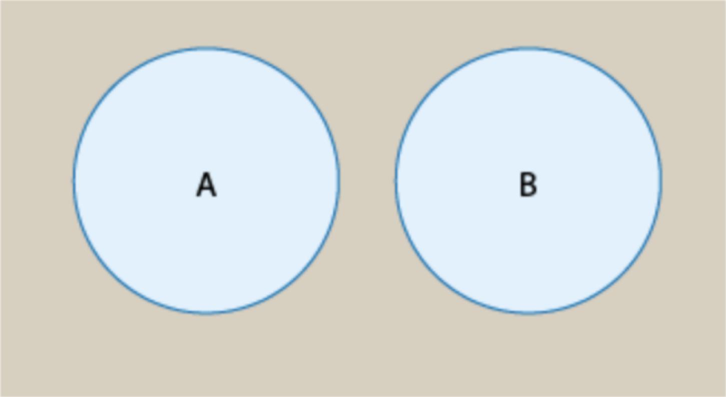
It should be clear from the picture that

- In the first case, where the events are **NOT disjoint,** **P(A and B) ≠ 0**
- In the second case, where the events **ARE disjoint,** **P(A and B) = 0.**

A and B are NOT Disjoint



A and B are Disjoint



Probability

Compound Event

Any event combining two or more simple events.

Rules for Compound Events

- **Addition Rule**
- Multiplication Rule
- Total Probability Rule
- Bayes' Rule

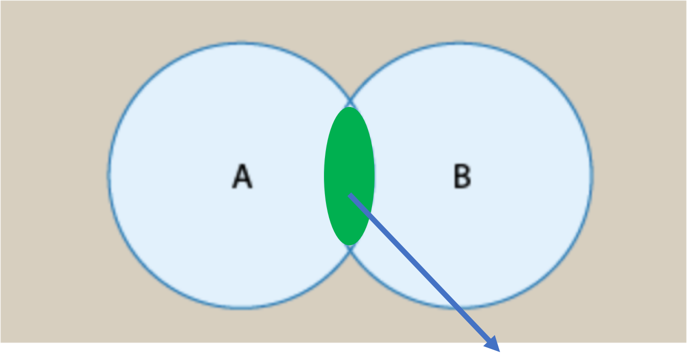
Compound Event

Addition Rule

- Addition rule finds a probability that either of an event occurs or both.
- Key word when using the addition rule is ***OR***.
- Notation for Addition: $P(A \text{ or } B) = P(\text{event } \ A \text{ occurs or } \ \text{event } \ B \ \text{ occurs or they both occur})$

$$P(A \, or \, B) = P(A) + P(B) - P(A \, and \, B)$$

where $P(A \, and \, B) = P(both \ \ A \ \ and \ \ B \ \ occur \, simultaneously)$

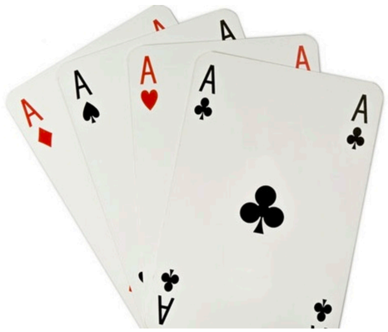


$P(A \, and \, B)$

Compound Event

Addition Rule

Example: If you take out a single card from a regular pack of cards, what is **probability** that the card is either an ace or spade?

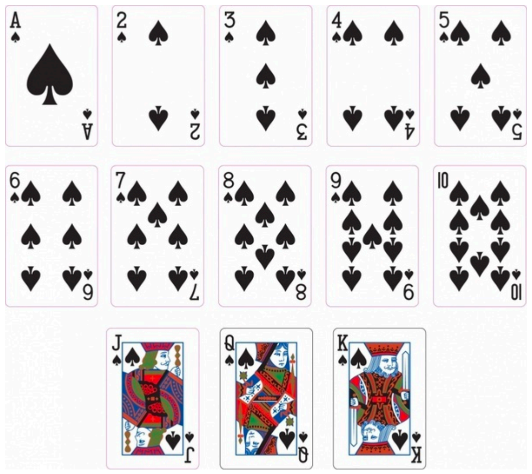


Let X be the event of picking an ace and Y be the event of picking a spade.

$$P(X) = \frac{4}{52}$$

$$P(Y) = \frac{13}{52}$$

The two events are not mutually exclusive, as there is one favorable outcome in which the card can be both an ace and spade.



$$P(X \text{ and } Y) = \frac{1}{52}$$

$$P(X \text{ or } Y) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Probability

Compound Event

Any event combining two or more simple events.

Rules for Compound Events

- Addition Rule
- **Multiplication Rule**
- Total Probability Rule
- Bayes' Rule

Compound Event

Multiplication Rule

- Multiplication rule finds a probability that both event occurs together.
- Key word when using the Multiplication rule is ***AND***.
- Notation for Multiplication: $P(A \text{ and } B) = P(\text{event } A \text{ occurs and event } B \text{ occurs})$

<u>Dependent and Independent events</u>

- $P(A \text{ and } B) = P(A) \times P(B)$ if A and B are **independent (with replacement)**.

- $P(A \text{ and } B) = P(A) \times P(B|A)$ if A and B are **dependent (without replacement)**.

Compound Event

Multiplication rule and Conditional Probability

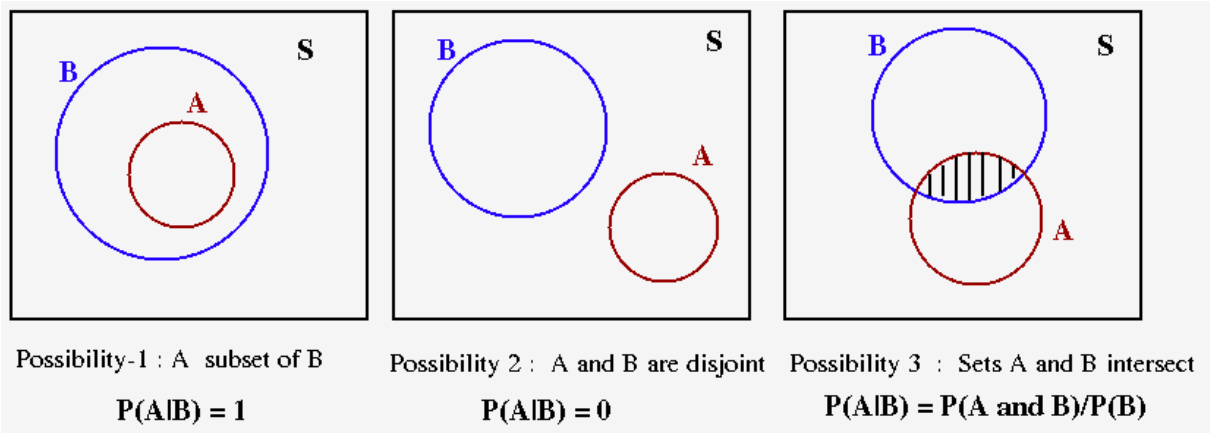
Conditional probability is the **probability** of one event occurring with some relationship to one or more other events.

$$P(A|B) = \textit{ conditional probability of } A \textit{ given that } B \textit{ has occurred}$$
$$= P(A \textit{ and } B)/P(B)$$

For example:

Event A is that it is raining outside, and it has a 0.3 (30%) chance of raining today.

Event B is that you will need to go outside, and that has a **probability** of 0.5 (50%).

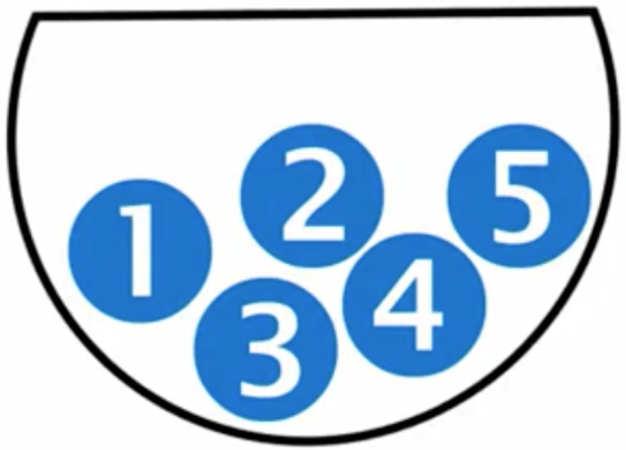


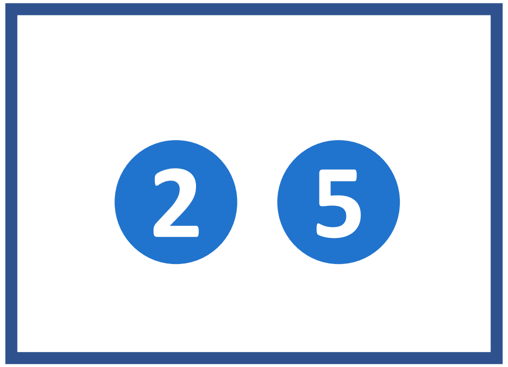
Compound Event

Dependent and Independent Events

Suppose we hold a lottery based on five numbered balls in a bowl. Let’s say you have a lottery ticket and you have chosen numbers 2 and 5.

The draw is about to begin and you look at your card. You know that the odds of it being drawn are 1 in 5.





Compound Event

Dependent and Independent Events

$$P(2) = 0.2$$

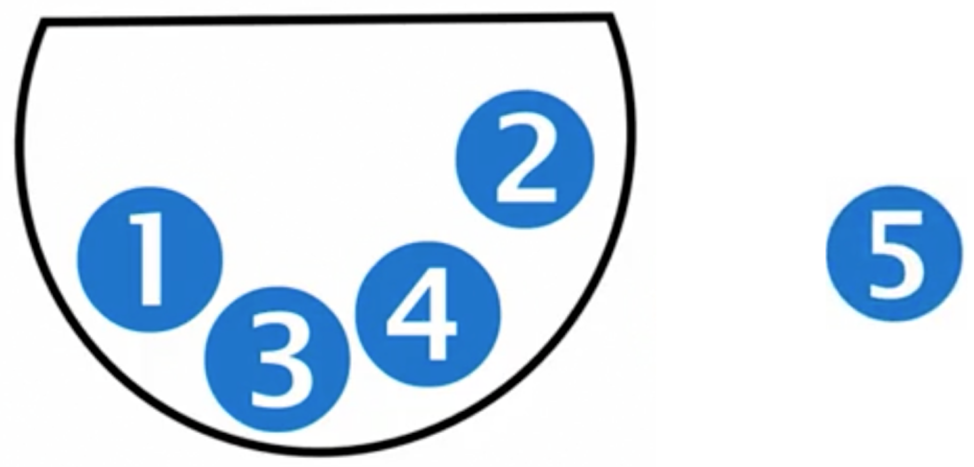
Compound Event

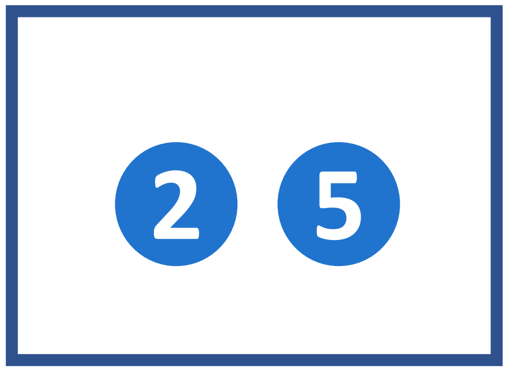
Dependent and Independent Events

In case of independent events, we replace the previous number before taking the next number:

$$P(2) = 0.2$$

$$P(5) = 0.2$$





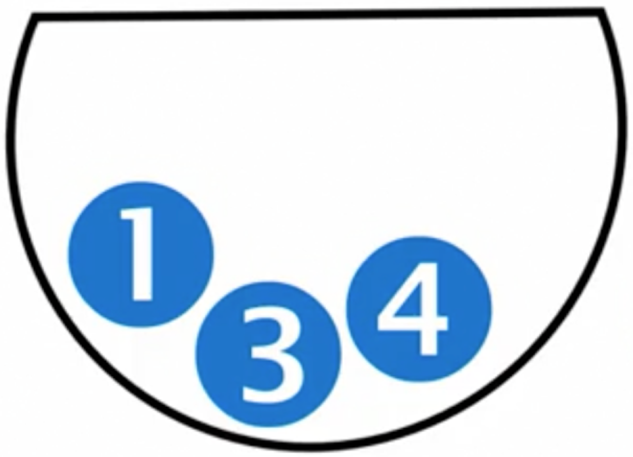
Compound Event

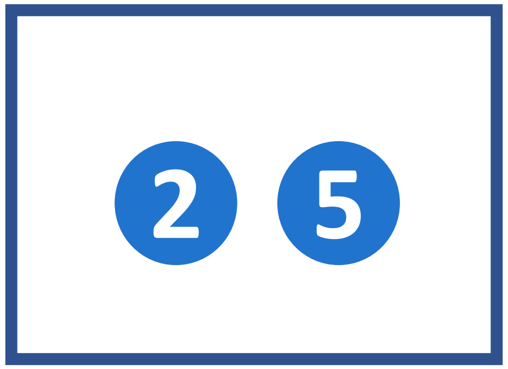
Dependent and Independent Events

In case of dependent events, we do not replace the number and take the second number.

$$P(2) = 0.2$$

$$P(5|2) = 0.25$$





Compound Event

Dependent and Independent Events

Example:

Let us say we have a box with five marbles, two are blue and three are red. We randomly draw one marble and then another. What is the probability of getting two red ones?

The answer depends on whether we draw marbles with or without replacement

Case 1 (With Replacement): Events $First\ Red$ and $Second\ Red$ are independent, therefore we apply Multiplication Rule for independent events.

$$P(First\ Red \text{ and } Second\ Red)$$
$$= P(First\ Red) \times P(Second\ Red) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$$
$$== 0.36 \text{ or } 36\%$$

Case 2 (Without Replacement): Now the probability of the event $Second\ Red$ depends on whether $First\ Red$ occurred.

$$P(Second\ red | First\ Red) = \frac{2}{4} = \frac{1}{2} = 0.5$$

$$P\left(Second\ Red \middle| \overline{First\ Red}\right) = \frac{3}{4} = 0.75$$

Therefore, we apply Multiplication Rule for dependent events.

$$P(First\ Red \text{ and } Second\ Red)$$
$$= P(First\ Red) \times P(Second\ Red | First\ Red)$$
$$= \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \frac{3}{10} = 0.30 \text{ or } 30\%$$

Compound Event

Conditional Probability Formula

Conditional Probability Example

Conditional probability can be computed based on the various forms of the Multiplication rule

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(A)\times P(B|A)}{P(B)}$$

Compound Event

Conditional Probability Example

Conditional Probability Example

Example: In a town affected by a season of epidemics, two diseases D1 and D2 are prevalent. It was estimated that 3.2% of the population contracted the disease D1 and 1.6% of the population has contracted both the diseases. Estimate the probability that a person who is affected by the disease D1 to get the disease D2.

Solution:

Using the expression for the conditional probability, we have

$$P(D2|D1) = P(D1\ and\ D2)/P(D1) = \frac{0.016}{0.032} = 0.5$$

Therefore, there is 50% chance that the person who contracted disease D1 will also get the disease D2.

Multiplication Rule Event

Attending classes in a statistics course
Passing a statistics course

Answer: Dependent events

Getting a flat tire on the way to class
Sleeping too late for class

Answer: Independent events

Events A and B, where $P(A) = 0.40$, $P(B) = 0.60$, and $P(A \text{ and } B) = 0.20$

Answer: Dependent events

$P(A \text{ and } B) \neq P(A) \times P(B)$

Events A and B , where $P(A) = 0.90$, $P(B) = 0.80$, and $P(A \text{ and } B) = 0.72$

Answer: Independent events

$P(A \text{ and } B) = P(A) \times P(B)$

Probability

Compound Event

Any event combining two or more simple events.

Rules for Compound Events

- Addition Rule
- Multiplication Rule
- **Total Probability Rule**
- Bayes' Rule

Compound Event

Total Probability Rule

The conditional $P(B|A)$, $P(B|\overline{A})$ and unconditional $P(B)$ probabilities are related in the following way:

For any events A and B, $P(B) = P(B|A)\times P(A) + P(B|\overline{A})\times P(\overline{A})$

Probability

Compound Event

Any event combining two or more simple events.

Rules for Compound Events

- Addition Rule
- Multiplication Rule
- Total Probability Rule
- **Bayes' Rule**

Bayes' Rule

Bayes' Rule is a formula that extends the use of the rule of conditional probabilities to allow *revision* of original probabilities with new information. For any events A and B,

$$P(A|B) = \frac{P(B|A)\times P(A)}{P(B|A)\times P(A) + P(B|\overline{A})\times P(\overline{A})}$$

A is the event we want the probability of, and **B** is the new evidence that is related to A in some way.

P(A|B) is called the **posterior**; this needs to be estimated. For example, "probability of having cancer given that the person is a smoker".

P(B|A) is called the **likelihood**; this is the probability of observing the new evidence, given our initial hypothesis. For example, "probability of being a smoker given that the person has cancer".

P(A) is called the **prior**; this is the probability of our hypothesis without any additional prior information. For example, "probability of having cancer".

Bayes' Rule

A test has been developed to detect a certain disease. It is known that 16% of males over 50 have this disease. The test was given to the males over 50 with the confirmed case of the disease and a positive test result was obtained in 88% of the cases. When the test was given to the males from the same age group who were known to be free of the disease, a positive test result was obtained in 5% of the cases.

1. If the male over 50 is selected at random and the test is administered to him, what is the probability that the test result is positive?

Solution: Let us introduce the following events: D - a patient has the disease; $\overline{D}$ - a patient has no disease; " $+$ "- the test result is positive; " $-$ " - the test result is negative.

Then, $P(D) = 0.16,\ P(\overline{D}) = 1 - 0.16 = 0.84$

$$P(+|D) = 0.88,\ P(+|\overline{D}) = 0.05$$

Using the Total Probability Rule,

$$P(+) = P(+|D)\times P(D) + P(+|\ \overline{D})\times P(\overline{D}) = 0.88\times 0.16 + 0.05\times 0.84 = 0.1828 = 18\%$$

Bayes' Rule

A test has been developed to detect a certain disease. It is known that 16% of males over 50 have this disease. The test was given to the males over 50 with the confirmed case of the disease and a positive test result was obtained in 88% of the cases. When the test was given to the males from the same age group who were known to be free of the disease, a positive test result was obtained in 5% of the cases.

2. If the test result is negative, find the probability that the patient still has the disease.

Solution: Applying Rule of Complimentary Events, we get

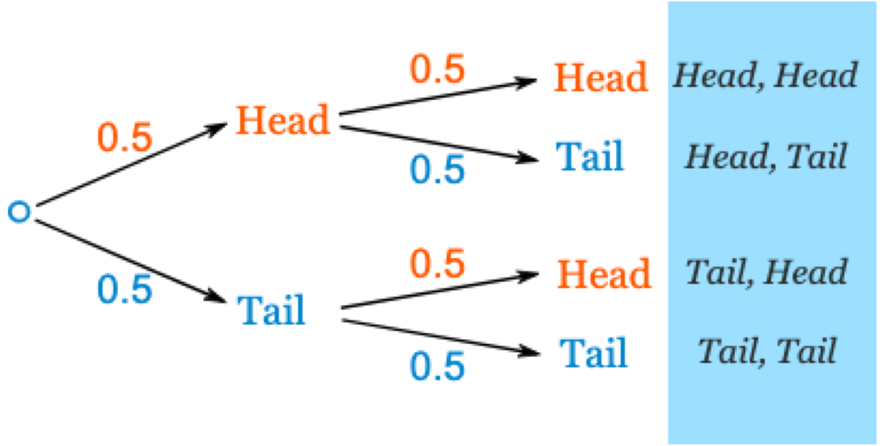
$$P(-|D) = 1 - P(+|D) = 1 - 0.88 = 0.12$$
$$P(-|\overline{D}) = 1 - P(+|\overline{D}) = 1 - 0.05 = 0.95$$

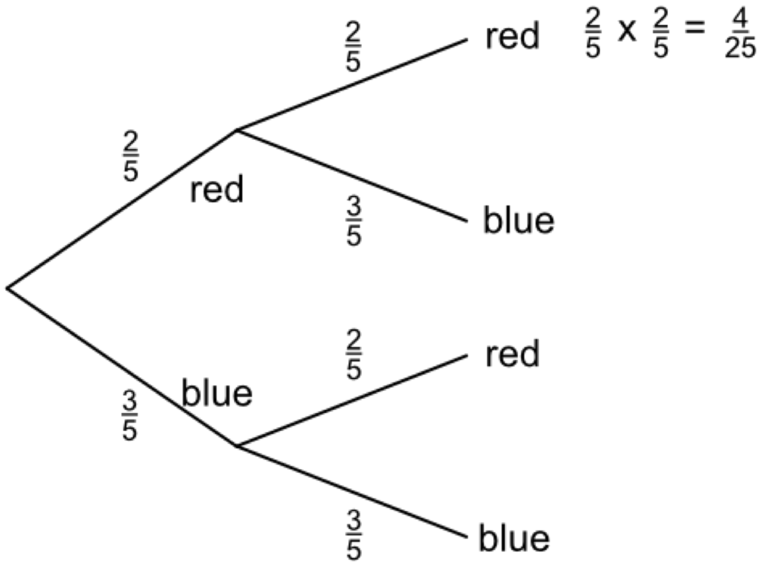
Using the Bayes' Rule,

$$P(D|-) = \frac{P(-|D)\times P(D)}{P(-|D)\times P(D) + P(-|\overline{D})\times P(\overline{D})} = \frac{0.12\times 0.16}{0.12\times 0.16 + 0.95\times 0.84} = \frac{0.0192}{0.8172} =$$
$$= 0.0235 = 2\%$$

Probability Tree

probability tree is a special type of graph used to determine the outcomes of an experiment.





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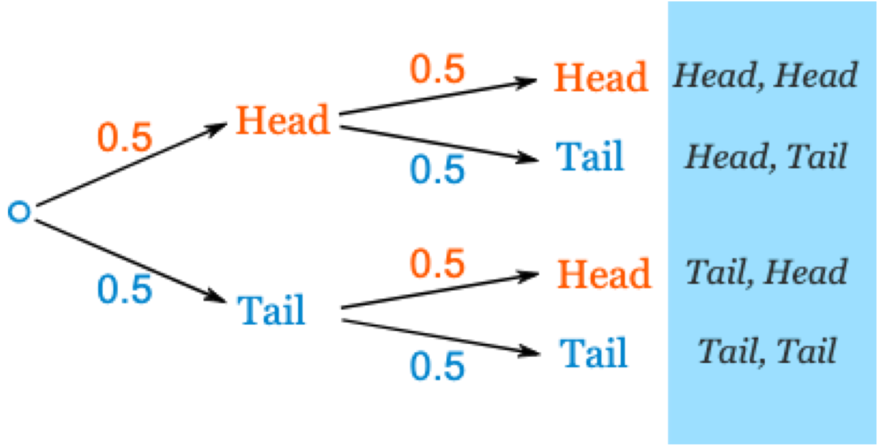
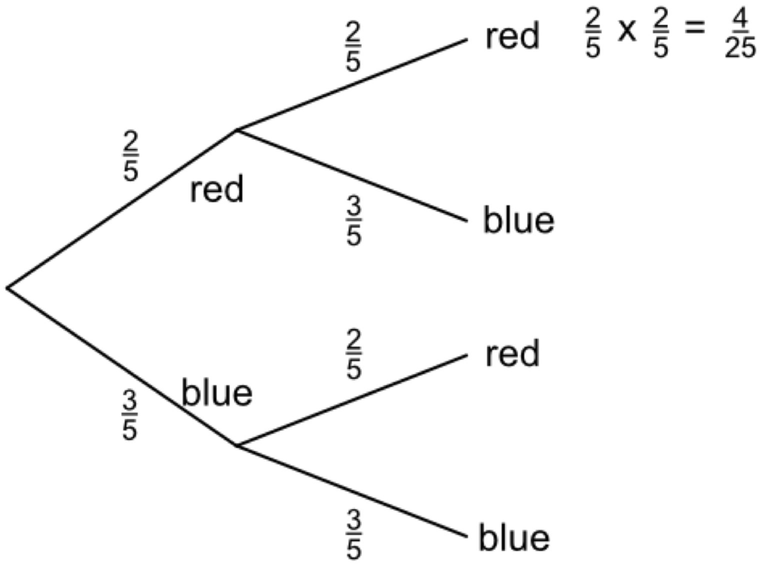
probability tree is a special type of graph used to determine the outcomes of an experiment.

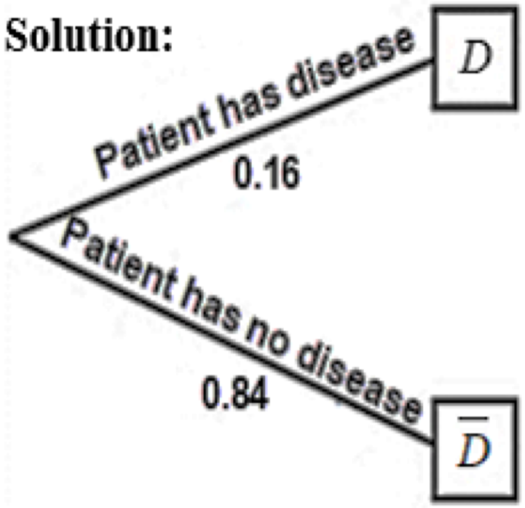




Probability Tree

A test has been developed to detect a certain disease. It is known that 16% of males over 50 have this disease. The test was given to the males over 50 with the confirmed case of the disease and a positive test result was obtained in 88% of the cases. When the test was given to the males from the same age group who were known to be free of the disease, a positive test result was obtained in 5% of the cases.

Solution:



The first set of branches separates males with disease from males without disease. Each branch of the tree is labeled with the probability of the respective outcome. Notice that the probabilities always add up to one when we cover *all* possible outcomes.

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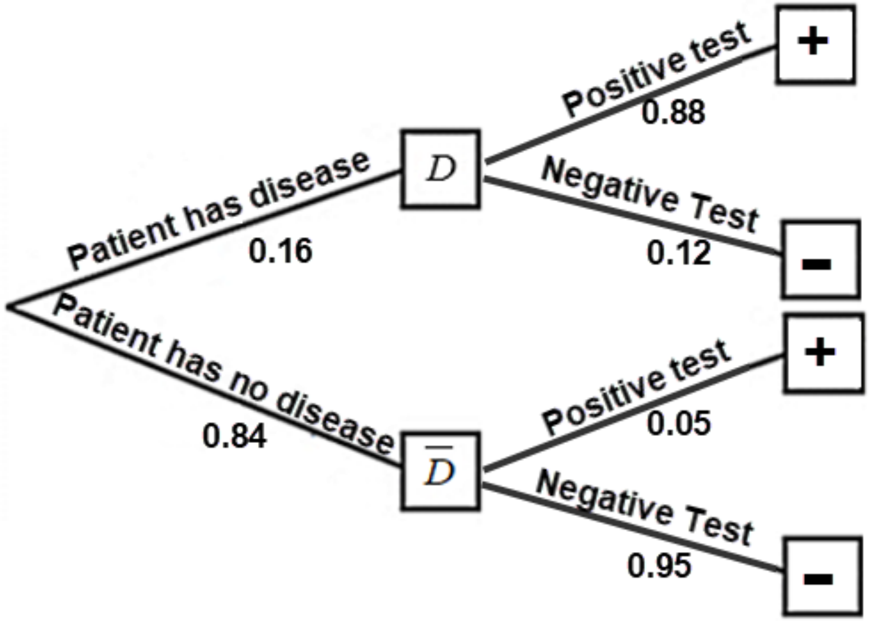

Then, we are also interested in test results. The probability of the test result depends on the disease detection. Since the probabilities are conditional, we draw the alternatives separately on each node. On each second set of branches, we put conditional probabilities depending on what event, D or $\overline{D}$, has occurred. Given that D has occurred (a male has disease), the probability that the test result is positive is $P(+|D) = 0.88$. In case that D has occurred (a male has disease), the probability that the test result is negative is $P(-|D) = 0.12$.

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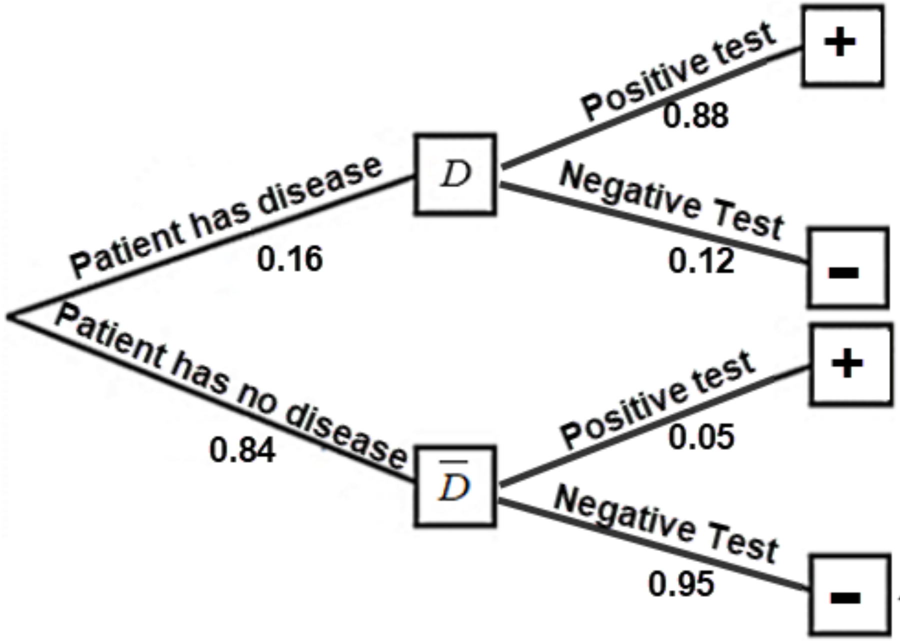

$$P(D \text{ and } +) = 0.16 \times 0.88 = 0.1408$$

$$P(D \text{ and } -) = 0.16 \times 0.12 = 0.0192$$

$$P(\overline{D} \text{ and } +) = 0.84 \times 0.05 = 0.0420$$

$$P(\overline{D} \text{ and } -) = 0.84 \times 0.95 = 0.7980$$

Summary

- ✓ **Probability**
  - ✓ Simple Event
  - ✓ Compound Event
- ✓ **Simple Event**
  - ✓ Rule of Relative Frequency Approximation
  - ✓ Rule of Classical Approach to Probability
  - ✓ Rule of Subjective Probabilities
- ✓ **Compound Event**
  - ✓ Addition Rule
  - ✓ Multiplication Rule
  - ✓ Total Probability Rule
  - ✓ Bayes' Rule