Probability Distribution

Binomial Probability Distribution, Normal Probability Distribution

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Recap Probability

Probability

Simple and Compound Events

General Principles

- The probability of an impossible event is zero (Probability of rolling a 7 on a die is 0).
- The probability of a certain event is one (Probability of getting head or tail by tossing a coin is 1)
- The sum of the **probabilities** of all possible outcomes is 1
- Range of possible probabilities is: $\mathbf{0 \leq \mathit{P} \leq 1}$

<u>Rules for Simple Events</u>

- Rule of Relative Frequency Approximation
- Rule of Classical Approach to Probability
- Rule of Subjective Probabilities

<u>Rules for Compound Events</u>

- Addition Rule
- Multiplication Rule
- Total Probability Rule
- Bayes' Rule

Probability Distribution

- Assigns a probability to each possible outcome of an experiment
- An outcome is a possible value of a random variable
- A **random variable** is a variable (typically represented by x) that has a single numerical value for each outcome of a statistical experiment

Discrete Probability Distribution

Either a finite or a countable number of values

Examples: Binomial Distribution

Flip a coin two times. This simple statistical experiment can have four possible outcomes: HH, HT, TH, and TT (where H - head and T-tail). Now, let the variable x represent the number of heads that result from this experiment. The variable x can take on the values 0, 1, or 2.

Continuous Probability Distribution

Infinitely many values in continuous scale

Examples: Normal Distribution

Fire department mandates that all fire fighters must weigh between 150 and 250 pounds. The weight of a fire fighter would be an example of a continuous variable since a fire fighter's weight could take on any value between 150 and 250 pounds.

Probability Distribution

Discrete Probability Distribution

- Let's flip a coin two times. Each flip can bring outcome H: Head or T: Tail

There are four possible outcomes: HH, HT, TH, and TT

Number of heads x	Probability $P(x)$
0	0.25
1	0.5
2	0.25

$$P(x = 0) = \frac{1}{4} = 0.25$$

$$P(x = 1) = \frac{2}{4} = 0.5$$

$$P(x = 2) = \frac{1}{4} = 0.25$$

Number of tails y	Probability $P(y)$
0	0.25
1	0.5
2	0.25

$$P(y = 0) = \frac{1}{4} = 0.25$$

$$P(y = 1) = \frac{2}{4} = 0.5$$

$$P(y = 2) = \frac{1}{4} = 0.25$$

Requirements for a Probability Distribution

$$\sum P(x) = 1 \qquad \text{where } x \text{ assumes all possible, distinct values}$$

$$0 \leq P(x) \leq 1 \qquad \text{for every value of } x$$

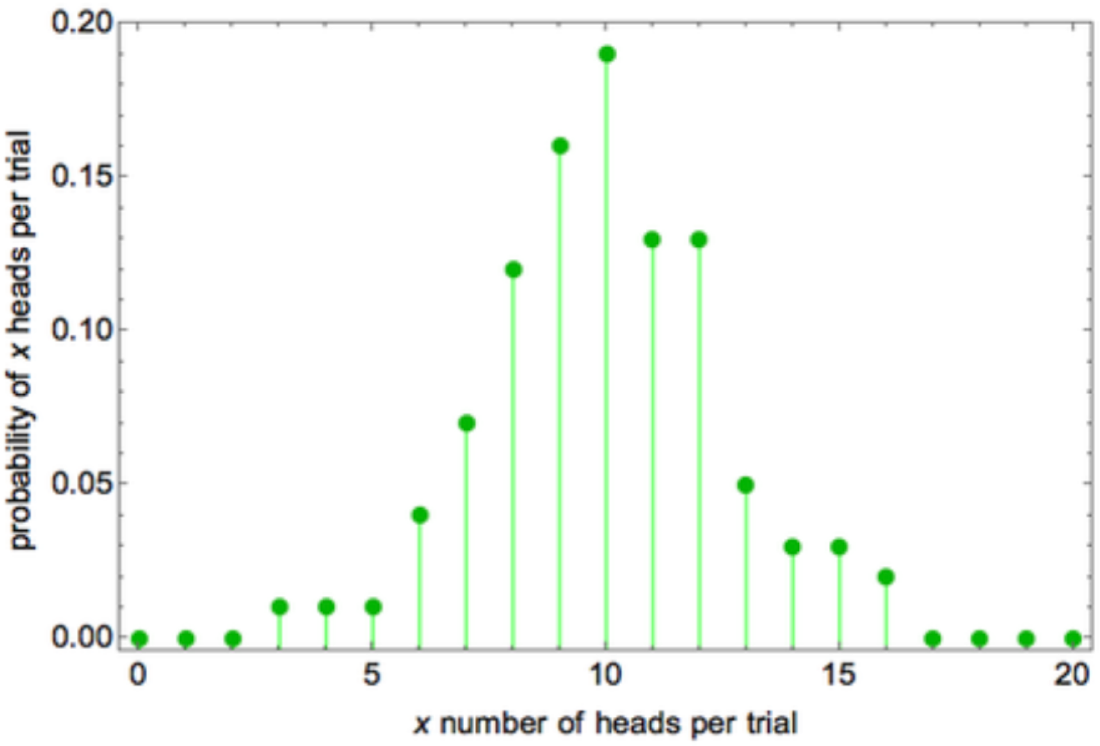
Binomial Probability Distribution

A Discrete Probability Distribution

The word “binomial” literally means “two numbers.” A binomial distribution has only two possible outcomes, **success** and **failure**.

Criteria for Binomial probability distribution:

1. The procedure has a *fixed number of trials* (such as 7 coin tosses, or 20 responses to survey question).
2. The trials must be *independent*. (The outcome of any individual trial does not affect the probabilities in the other trials.)
3. The outcome of each trial must be classifiable into one of *two possible categories (“success” or “failure”).*
4. The probabilities must remain *constant* for each trial.



Binomial Probability Distribution

Example: Tossing a coin three times in each trial:

Question 1: What are the possible outcomes?

There are eight possible outcomes

Question 2: What are the desired outcomes?

Two “head” in each trial.

Binomial Probability Distribution

Example: Tossing a coin three times in each trial:

[HHH, HHT, HTH, HTT, HHT, HTH, THH, TTT]

Question 3: What is the probability of getting two heads?

Chance of getting two heads = Three out of eight

Binomial Probability Distribution

Example: Tossing a coin three times in each trial:

[HHH, HHT, HTH, HTT, HHT, HTH, THH, TTT]

Question: What is the probability of getting "head"?

$$P(Three\ Heads) = P(\boldsymbol{HHH}) = \mathbf{1/8}$$

$$P(Two\ Heads) = P(\boldsymbol{HHT}) + P(\boldsymbol{HTH}) + P(\boldsymbol{THH}) = 1/8 + 1/8 + 1/8 = \mathbf{3/8}$$

$$P(One\ Head) = P(\boldsymbol{HTT}) + P(\boldsymbol{THT}) + P(\boldsymbol{TTH}) = 1/8 + 1/8 + 1/8 = \mathbf{3/8}$$

$$P(Zero\ Heads) = P(\boldsymbol{TTT}) = \mathbf{1/8}$$

Binomial Probability Distribution

Example: Tossing a coin three times in each trial:

[HHH, HHT, HTH, HTT, HHT, HTH, THH, TTT]

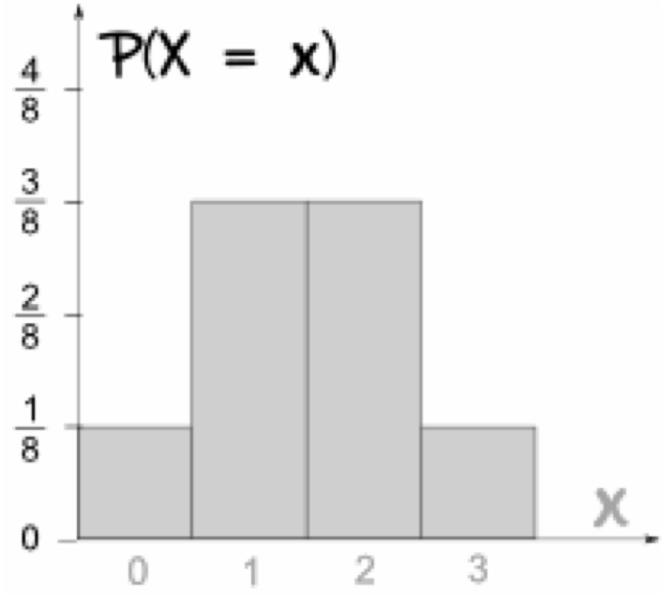
Let *x* is a random variable and counts the number of heads occurred in a three toss coin

$P(3) = \mathbf{1/8}$

$P(2) = \mathbf{3/8}$

$P(1) = \mathbf{3/8}$

$P(0) = \mathbf{1/8}$



It is symmetrical!

Binomial Probability Distribution

In the following questions, determine whether the given experiments are binomial. For those that are not binomial, identify at least one requirement that is not satisfied.

- Rolling a fair die 50 times to see 2 as outcome.

  Answer: Yes. Because the outcomes could be either '2' or not '2'.

- Tossing an unbiased coin 200 times.

  Answer: Yes

- Surveying 1000 people in Toronto by asking each one how he or she voted in the last federal elections

  Answer: No. Because outcomes can be more than 2.

- Sampling (with replacement) a randomly selected group of 12 different tires from the population of 30 tires where 5 are defective

  Answer: Yes, it is binomial

Binomial Probability Distribution

Notation for Binomial Distributions

Let S and F (success and failure) denote the two possible categories of all outcomes;

p and q will denote the probabilities of S and F

$$P(S) = p \qquad (p = \textit{probability of a success})$$
$$P(F) = 1 - p = q \qquad (q = \textit{probability of a failure})$$

n *denotes the fixed number of trials.*

k *denotes a specific number of successes in* n trials *, so* k *can be any whole number between 0 and* n *, inclusive.*

p *denotes the probability of success in one of the* n *trials.*

q *denotes the probability of failure in one of the* n *trials.*

$P(x = k)$ *denotes the probability of getting exactly* k *successes in* n *trials.*

Binomial Probability Distribution

Notation for Binomial Distributions

$n =$ *Fixed number of trials.*

$k =$ *Number of successes among n trials*

$p =$ *probability of success in any one trial*

$q =$ *probability of failure in any one trials* $(q = 1 - p)$

$$P(x = k) = \frac{n!}{(n-k)!\ \ k!} \times p^k \times q^{n-k}$$

$$where\ \ n = 1 \times 2 \times 3 \times \cdots \times n$$

$$for\, k = 0\ ,\ 1\ ,\ 2\ ,\ \ldots\ ,\ n$$

Binomial Probability Distribution

Notation for Binomial Distributions

$n =$ *Fixed number of trials.*

$k =$ *Number of successes among* n *trials*

$p =$ *probability of success in any one trial*

$q =$ *probability of failure in any one trials* $(q = 1 - p)$

$$P(x = k) = \frac{n!}{(n-k)!\ k!} \times p^k \times q^{n-k}$$

where $n! = 1 \times 2 \times 3 \times \cdots \times n$

for $k = 0\ ,\ 1\ ,\ 2\ ,\ \ldots\ ,\ n$

Example: There are 10 trials, and the probability of success in any one trial is $p = 0.6$. Calculate the probability of getting 3 successes. Suppose that x has the binomial distribution.

Given $n = 10,\ k = 3$ and $p = 0.6$. We need to find $P(x = 3)$. Since $p = 0.6$, we get

$$q = 1 - p = 1 - 0.6 = 0.4$$

Binomial Probability Distribution

Notation for Binomial Distributions

Example: There are 10 trials, and the probability of success in any one trial is $p = 0.6$. Calculate the probability of getting 3 successes. Suppose that x has the binomial distribution.

Given $n = 10$, $k = 3$ and $p = 0.6$. We need to find $P(x = 3)$. Since $p = 0.6$, we get

$$q = 1 - p = 1 - 0.6 = 0.4$$

Then,

$$P(x = 3) = \frac{10!}{(10-3)!\ 3!} \times 0.6^3 \times 0.4^{10-3} = \frac{10!}{7!\ 3!} \times 0.6^3 \times 0.4^7$$

We calculate $\frac{10!}{7!\ 3!}$ separately:

$$\frac{10!}{7!\ 3!} = \frac{1\times2\times\ldots\times7\times8\times9\times10}{(1\times2\times\ldots\times7)\times(1\times2\times3)} = \frac{8\times9\times10}{6} = 120$$

$$P(x = 3) = 120\times0.6^3\times0.4^7 = 0.0424673 \approx 0.0425 \text{ or } 4.25\%$$

Binomial Probability Distribution

Notation for Binomial Distributions

Example: A pair of fair dice is rolled 10 times. Let X be the number of rolls in which we see at least one 2. If we look at the chart below, we can see the number of times a 2 shows up when rolling 2 dice.

+	1	2	3	4	5	6
1	1,1	2,1	3,1	4,1	5,1	6,1
2	1,2	2,2	3,2	4,2	5,2	6,2
3	1,3	2,3	3,3	4,3	5,3	6,3
4	1,4	2,4	3,4	4,4	5,4	6,4
5	1,5	2,5	3,5	4,5	5,5	6,5
6	1,6	2,6	3,6	4,6	5,6	6,6

Binomial Probability Distribution

Notation for Binomial Distributions

Example: A pair of fair dice is rolled 10 times. Let X be the number of rolls in which we see at least one 2. If we look at the chart below, we can see the number of times a 2 shows up when rolling 2 dice.

+	1	2	3	4	5	6
1	1,1	2,1	3,1	4,1	5,1	6,1
2	1,2	2,2	3,2	4,2	5,2	6,2
3	1,3	2,3	3,3	4,3	5,3	6,3
4	1,4	2,4	3,4	4,4	5,4	6,4
5	1,5	2,5	3,5	4,5	5,5	6,5
6	1,6	2,6	3,6	4,6	5,6	6,6

What is the probability of seeing at least one 2 in any one roll of the pair of dice?

The probability of seeing at least one 2 in any one roll of the pair of dice is:

$$P(X) = \frac{11}{36} = 0.306$$

Binomial Probability Distribution

Notation for Binomial Distributions

Example: A pair of fair dice is rolled 10 times. Let X be the number of rolls in which we see at least one 2. If we look at the chart below, we can see the number of times a 2 shows up when rolling 2 dice.

+	1	2	3	4	5	6
1	1,1	2,1	3,1	4,1	5,1	6,1
2	1,2	2,2	3,2	4,2	5,2	6,2
3	1,3	2,3	3,3	4,3	5,3	6,3
4	1,4	2,4	3,4	4,4	5,4	6,4
5	1,5	2,5	3,5	4,5	5,5	6,5
6	1,6	2,6	3,6	4,6	5,6	6,6

What is the probability that in exactly half of the 10 rolls, we see at least one 2?

The probability of seeing at least one 2 in exactly 5 of the 10 rolls is calculated as follows:

$$p = 0.306$$
$$q = 1 - 0.306 = 0.694$$
$$n = 10$$
$$\text{k} = 5$$

$$P(X = k) = {}_nC_k \times p^k \times q^{(n-k)}$$

$$P(X = 5) = {}_{10}C_5 \times p^5 \times q^{(10-5)}$$

$$P(X = 5) = {}_{10}C_5 \times (0.306)^5 \times 0.694^{(10-5)}$$

$$P(X = 5) = 252 \times 0.00268 \times 0.161$$

$$P(X = 5) = 0.109$$

Therefore, the probability of rolling at least one 2 exactly 5 times when 2 dice are rolled 10 times is 10.9%.

Binomial Probability Distribution

Binomial Distribution Table

You can use the binomial distribution table to find the probability of obtaining at most k successes during n trials when the probability of success on each trial is p.

$$P(x \le k) = \sum_{i=0}^{k} P(x = i) = P(x = 0) + P(x = 1) + \cdots + P(x = k)$$

See binomial table

Example: Five percent of the population of Toronto speaks Italian. If you randomly select 25 Toronto residents,

a. What is the probability that at most 3 of them speak Italian?

Binomial Probability Distribution

Binomial Distribution Table

Example: Five percent of the population of Toronto speaks Italian. If you randomly select 25 Toronto residents,

a. What is the probability that at most 3 of them speak Italian?

Given $n = 25,\ k = 3$ and $p = 0.05$. We have to find the probability $P(x \leq 3)$.

To answer this question, we can look up the value in the binomial distribution table that corresponds to $n = 25$, $k = 3$ and $p = 0.05$:

Using the table: $P(x \leq 3) = 0.\ 9659$

$n = 25$

	p							
k	0.01	0.05	0.10	0.20	0.25	0.30	0.40	0.50
0	0.7778	0.2774	0.0718	0.0038	0.0008	0.0001	0.0000	0.0000
1	0.9742	0.6424	0.2712	0.0274	0.0070	0.0016	0.0001	0.0000
2	0.9980	0.8729	0.5371	0.0982	0.0321	0.0090	0.0004	0.0000
3	0.999[illegible]	0.9659	0.7636	0.2340	0.0962	0.0332	0.0024	0.0001
4	1.0000	0.9928	0.9020	0.4207	0.2137	0.0905	0.0095	0.0005
5	1.0000	0.9988	0.9666	0.6167	0.3783	0.1935	0.0294	0.0020

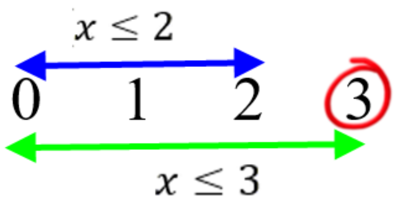
Binomial Probability Distribution

Binomial Distribution Table

Example: Five percent of the population of Toronto speaks Italian. If you randomly select 25 Toronto residents,

a. What is the probability that at most 3 of them speak Italian? $\boldsymbol{P(x \leq 3) = 0.9659}$

b. What is the probability that exactly 3 of them speak Italian?



$n = 25$

	p							
k	0.01	0.05	0.10	0.20	0.25	0.30	0.40	0.50
0	0.7778	0.2774	0.0718	0.0038	0.0008	0.0001	0.0000	0.0000
1	0.9742	0.6424	0.2712	0.0274	0.0070	0.0016	0.0001	0.0000
2	0.9980	0.8729	0.5371	0.0982	0.0321	0.0090	0.0004	0.0000
3	0.9999	0.9659	0.7636	0.2340	0.0962	0.0332	0.0024	0.0001
4	1.0000	0.9928	0.9020	0.4207	0.2137	0.0905	0.0095	0.0005

$$P(x = 3) = \frac{25!}{(25-3)!\ 3!} \times 0.05^3 \times 0.95^{25-3}$$

$$= \frac{25!}{22!\ 3!} \times 0.05^3 \times 0.95^{22} = 0.0930$$

Binomial Probability Distribution

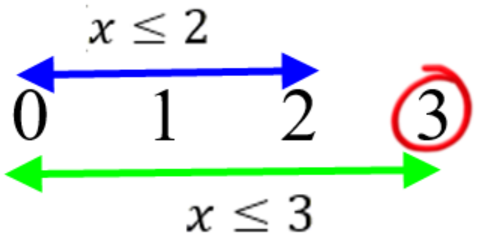
Binomial Distribution Table

Example: Five percent of the population of Toronto speaks Italian. If you randomly select 25 Toronto residents,

a. What is the probability that at most 3 of them speak Italian? $P(x \leq 3) = 0.9659$

b. What is the probability that exactly 3 of them speak Italian? $P(x = 3) = 0.0930$

c. What is the probability that more than 3 of them speak Italian?



$n = 25$

	p							
k	0.01	0.05	0.10	0.20	0.25	0.30	0.40	0.50
0	0.7778	0.2774	0.0718	0.0038	0.0008	0.0001	0.0000	0.0000
1	0.9742	0.6424	0.2712	0.0274	0.0070	0.0016	0.0001	0.0000
2	0.9980	0.8729	0.5371	0.0982	0.0321	0.0090	0.0004	0.0000
3	0.999[illegible]	0.9659	0.7636	0.2340	0.0962	0.0332	0.0024	0.0001
4	1.0000	0.9928	0.9020	0.4207	0.2137	0.0905	0.0095	0.0005
5	1.0000	0.9988	0.9666	0.6167	0.3783	0.1935	0.0294	0.0020

The event complementary to "more than 3 of them speak Italian" is "less than or equal to 3 speak Italian". Using the complementary events rule and the binomial distribution table, we get

$$P(x > 3) = 1 - P(x \leq 3) = 1 - 0.9659 = 0.0341$$

Binomial Probability Distribution

Binomial Distribution Table

Example: Five percent of the population of Toronto speaks Italian. If you randomly select 25 Toronto residents,

a. What is the probability that at most 3 of them speak Italian? $\boldsymbol{P(x \leq 3) = 0.\ 9659}$
b. What is the probability that exactly 3 of them speak Italian? $\boldsymbol{P(x\ =\ 3)\ =\ 0.0930}$
c. What is the probability that more than 3 of them speak Italian? $\boldsymbol{P(x > 3)\ =\ 0.0341}$
d. What is the probability that at least 4 of them speak Italian?

$P(x \geq 4) = ?$

$x < 4$ $\qquad$ $x \geq 4$

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \ldots \quad 25$

$x \leq 3$

Now we can apply the binomial distribution table:

$$P(x \geq 4) = 1 - P(x < 4) = 1 - P(x \leq 3)$$

$$= 1 - 0.\ 9659\ =\ \ 0.0341$$

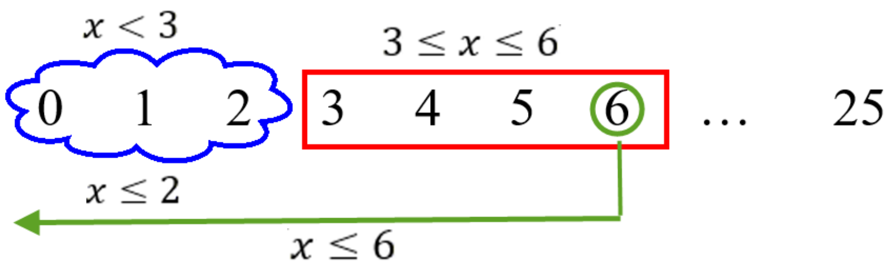
Binomial Probability Distribution

Binomial Distribution Table

Example: Five percent of the population of Toronto speaks Italian. If you randomly select 25 Toronto residents,

a. What is the probability that at most 3 of them speak Italian? $\boldsymbol{P(x \leq 3) = 0.9659}$
b. What is the probability that exactly 3 of them speak Italian? $\boldsymbol{P(x = 3) = 0.0930}$
c. What is the probability that more than 3 of them speak Italian? $\boldsymbol{P(x > 3) = 0.0341}$
d. What is the probability that at least 4 of them speak Italian? $\boldsymbol{P(x \geq 4) = 0.0341}$
e. What is the probability that at least 3 and at most 6 of them speak Italian?

We need to calculate $P(3 \leq x \leq 6)$.



Using the diagram below, we get

$$P(3 \leq x \leq 6) = P(x \leq 6) - P(x < 3) = P(x \leq 6) - P(x \leq 2)$$

Now we can apply the binomial distribution table:

$$P(3 \leq x \leq 6) = P(x \leq 6) - P(x < 3) = P(x \leq 6) - P(x \leq 2) = 0.9998 - 0.8729 = 0.1269$$

Binomial Probability Distribution

Using Excel

BINOM.DIST function finds the binomial distribution probability

$$=BINOM.DIST(number\_s, trials, probability\_s, cumulative)$$

where number_s $= k$, trials $= n$, probability_s $= p$, and cumulative is a switch that's set to either the logical value TRUE (if we want to calculate cumulative probability $P(x \leq k)$) or the logical value FALSE (if you want to calculate the exact probability $P(x = k)$).

Example: Five percent of the population of Toronto speaks Italian. If you randomly select 25 Toronto residents,

a. What is the probability that at most 3 of them speak Italian?

```
=BINOM.DIST(3,25,0.05,TRUE) = 0.965909
```

Binomial Probability Distribution

Using Excel

Example: Five percent of the population of Toronto speaks Italian. If you randomly select 25 Toronto residents,

a. What is the probability that at most 3 of them speak Italian?
b. What is the probability that exactly 3 of them speak Italian?

Solution: Since we have to find the probability $P(x = 3)$, we can use the following function Note that in this case the logical value would be `FALSE` .

```
=BINOM.DIST(3,25,0.05,FALSE)
```

We get,

```
0.093016
```

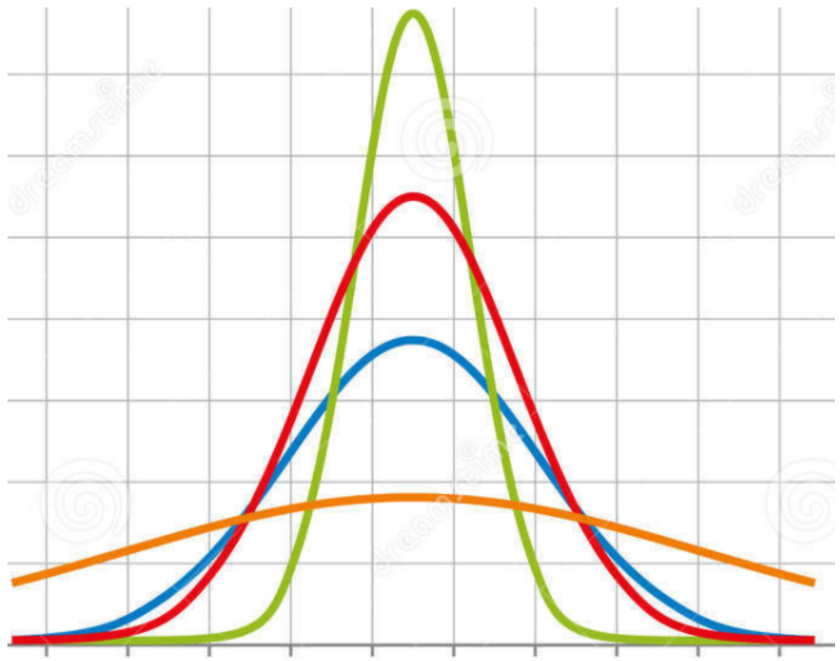
Again, it is the same answer $P(x = 3) = 0.0930$

Normal Probability Distribution

Normal Probability Distribution

Normal distribution, also known as the **Gaussian distribution**, is a probability distribution that is symmetric about the mean (or average, which is the maximum of the graph), showing that data near the mean are more frequent in occurrence than data far from the mean.

A continuous random variable has a **normal distribution** if this distribution has a graph that is symmetric and bell shaped.



Normal Probability Distribution

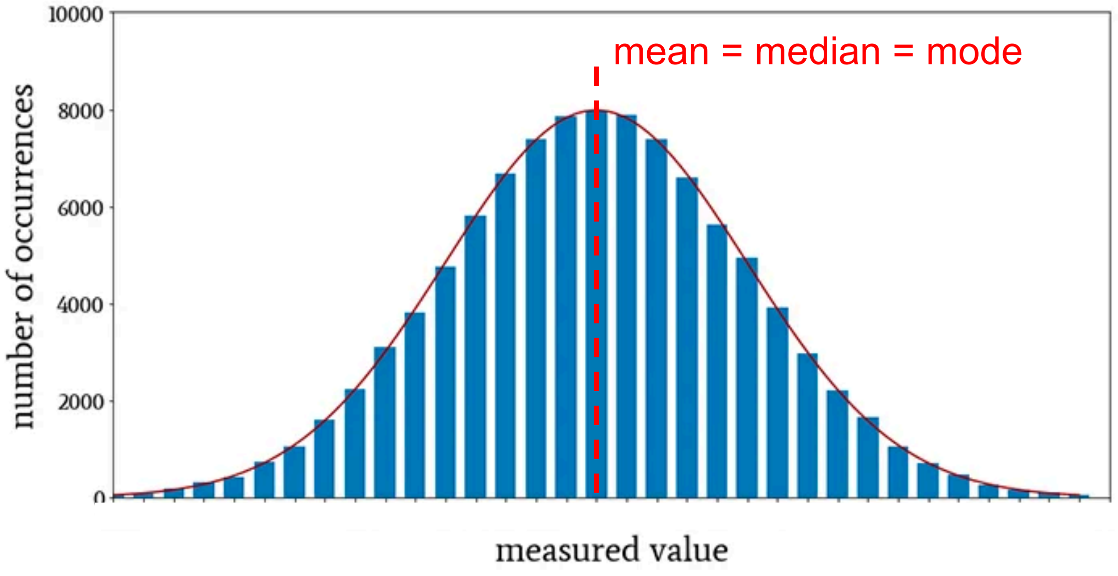
- The total area under the normal curve is equal to 1.

- The equation for the distribution looks like this:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ,$$

- where μ is the population mean , $and \ \ \sigma$ is the standard deviation.

- The **mean**, **median**, **and mode** of a **normal distribution** are equal.



- The area under the **normal curve** (sum of all probability values) is equal to 1.

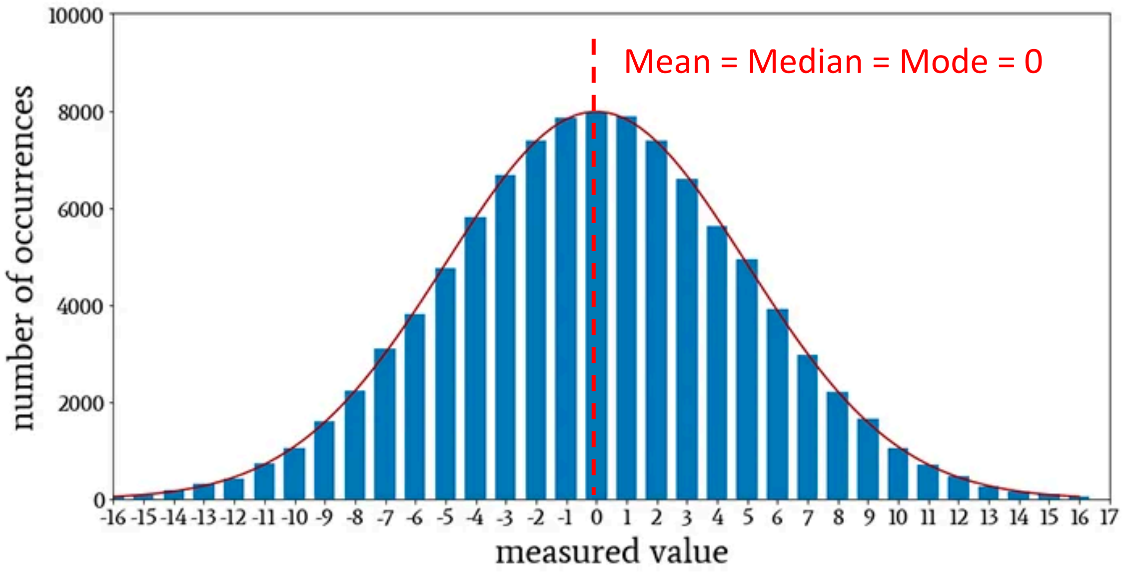
Normal Probability Distribution

Standard Normal Distribution

- The total area under the normal curve is equal to 1.
- The equation for the distribution looks like this:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} ,$$

- where $\mu = 0 , \ and \ \sigma = 1$
- Standard Normal Distribution is a special case of Normal Distribution.



Normal Probability Distribution
Calculate using Table values

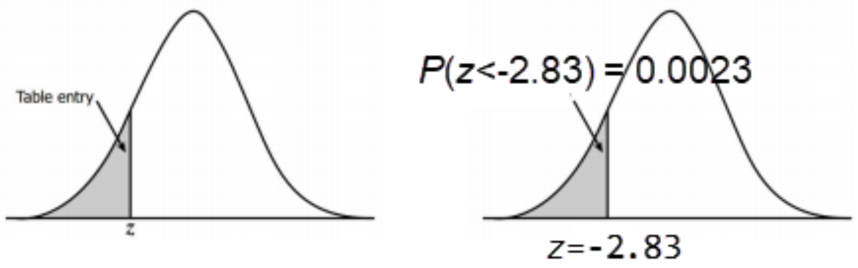
- Normal Distribution uses standard normal distribution tables to calculate the probability of specific values of a random variable

- We find z-score and match its value in the standard normal distribution table.

$$P(x \leq a) = P\left(z < \frac{a-\mu}{\sigma}\right)$$

- **Example:** $z < -2.83$
- In the first column of the table, find -2.8
- In the first row of the table, find 0.03
- Find the common value in the table

Standard Normal Distribution z-scores



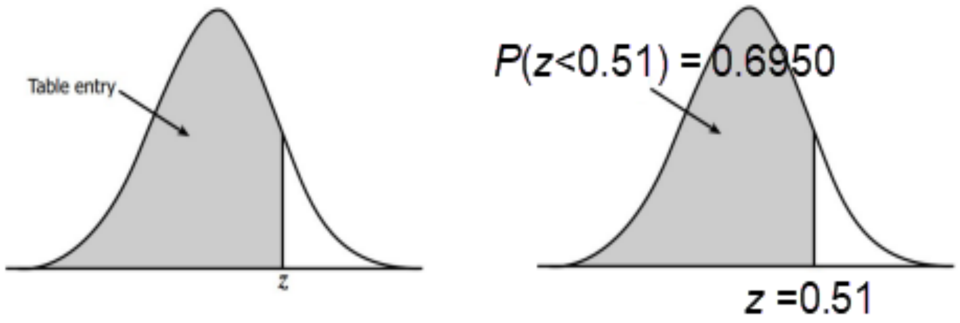
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	[illegible]	$\boxed{0.0023}$	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036

Normal Probability Distribution

Calculate using Table values

- **Example:** $z < 0.51$
- In the first column of the table, find - 0.5
- In the first row of the table, find 0.01
- Match the values

Standard Normal Distribution z-scores (continued)



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.69[illegible]	$\boxed{0.6950}$	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549

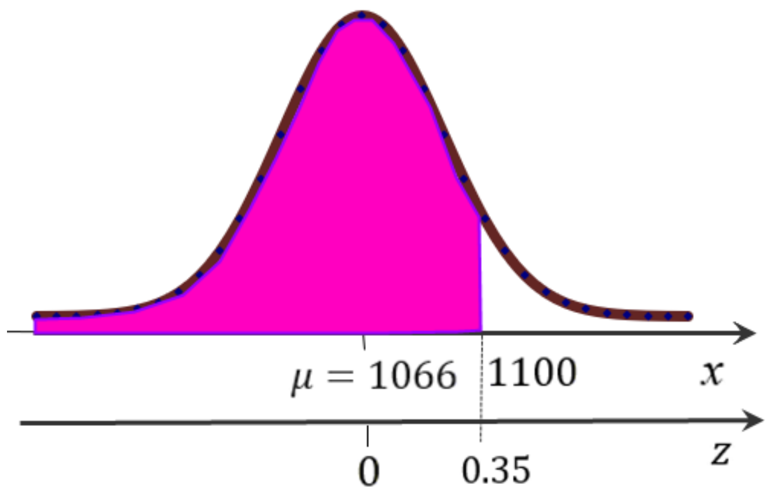
Normal Probability Distribution
Standard Normal Distribution

Example: Suppose that at a gas station the daily demand for gasoline is normally distributed with a mean of 1066 gallons and a standard deviation of 98 gallons.
a) The station manager notes that there is exactly 1100 gallons of gasoline in storage. The manager would like to know the probability that he will have enough gasoline to satisfy today's demand.

Solution: Given $\mu = 1066$ and $\sigma = 98$. We label the demand for gasoline as x, and we want to find the probability $P(x \leq 1100)$ or simply $P(x < 1100)$. For continuous distribution one point would not make any difference.

Next step is to standardize x: $z = \frac{x-\mu}{\sigma} = \frac{1100-1066}{98} = \frac{34}{98} = 0.34694 \approx 0.35$

Then, we get $P(x < 1100) = P(z < 0.35)$ and now we can work with the standard normal distribution.



Normal Probability Distribution

Standard Normal Distribution

Example: Suppose that at a gas station the daily demand for gasoline is normally distributed with a mean of 1066 gallons and a standard deviation of 98 gallons.

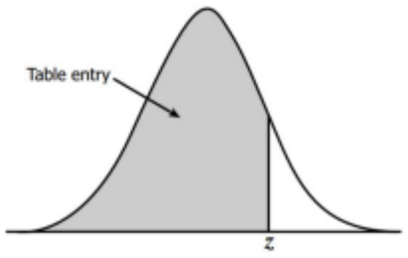
a) The station manager notes that there is exactly 1100 gallons of gasoline in storage. The manager would like to know the probability that he will have enough gasoline to satisfy today's demand.

Using the Standard Normal Distribution Table, we find:

0.3 in the first column and 0.05 in the first row. Next, we find the common value as,

$$P(x < 1100) = P(z < 0.35) = 0.6368$$

Standard Normal Distribution z-scores (continued)



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.535
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.575
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.614
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	$\boxed{0.6368}$	0.6406	0.6443	0.6480	0.651
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.687
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.722
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.754

Normal Probability Distribution

Standard Normal Distribution

Example: Suppose that at a gas station the daily demand for gasoline is normally distributed with a mean of 1066 gallons and a standard deviation of 98 gallons.

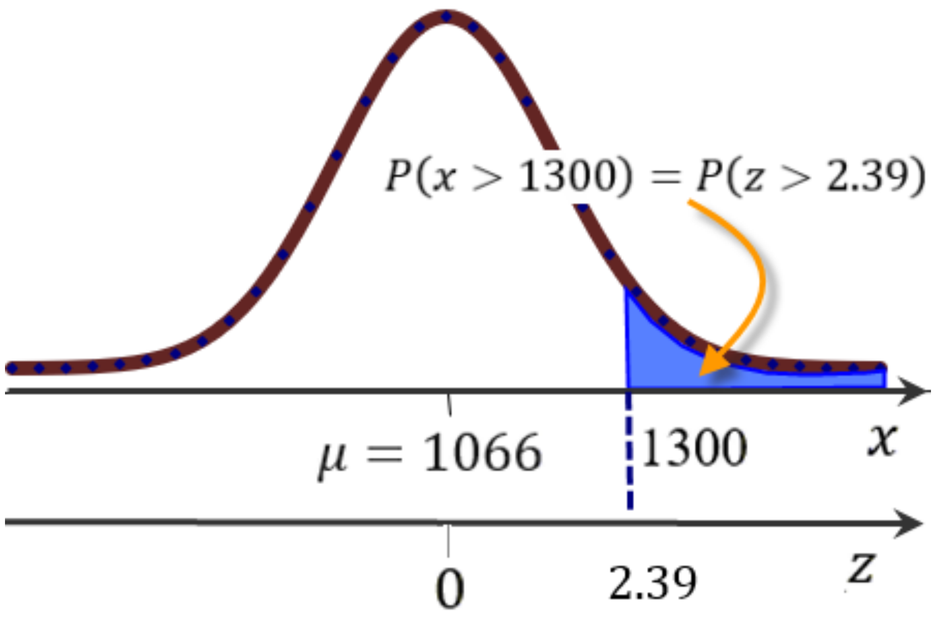
b) The manager would like to know the probability that the demand will exceed 1300 gallons.

Solution: Given $\mu = 1066$ and $\sigma = 98$. We label the demand for gasoline as x, and we want to find the probability $P(x > 1300)$.

Next step is to find the z-score: $z = \frac{x-\mu}{\sigma} = \frac{1300-1066}{98} = \frac{234}{98} = 2.387755 \approx 2.39$

Then, we get $P(x > 1300) = P(z > 2.39)$ and now we can work with the standard normal distribution.

Standard normal distribution table gives values for $z \leq 2.39$



Normal Probability Distribution

Standard Normal Distribution

Example: Suppose that at a gas station the daily demand for gasoline is normally distributed with a mean of 1066 gallons and a standard deviation of 98 gallons.
b) The manager would like to know the probability that the demand will exceed 1300 gallons.

Using the Standard Normal Distribution Table (see below), we find:

$$P(x < 1300) = P(z < 2.39) = 0.9916$$

Thus,

$$P(x > 1300) = P(z > 2.39)$$
$$= 1 - P(z < 2.39)$$
$$= 1 - 0.9916 = 0.0084.$$

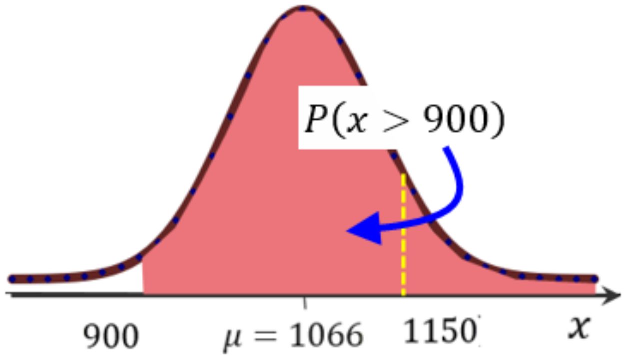
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964

Normal Probability Distribution

Standard Normal Distribution

Example: Suppose that at a gas station the daily demand for gasoline is normally distributed with a mean of 1066 gallons and a standard deviation of 98 gallons.
c) The manager would like to know the probability that the demand will fall between 900 and 1150 gallons.

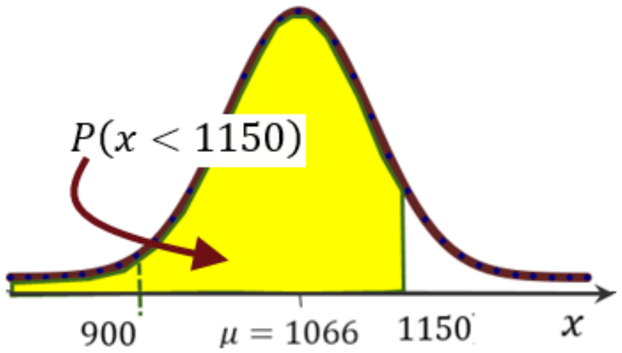
Given $\mu = 1066$ and $\sigma = 98$. We label the demand for gasoline as x, and we want to find the probability $P(900 < x < 1150)$.



Next step is to find the z-scores:

$$z_1 = \frac{x-\mu}{\sigma} = \frac{900-1066}{98} = -\frac{166}{98} = -1.693878 \approx -1.69$$

$$z_2 = \frac{x-\mu}{\sigma} = \frac{1150-1066}{98} = \frac{84}{98} = 0.857143 \approx 0.86$$

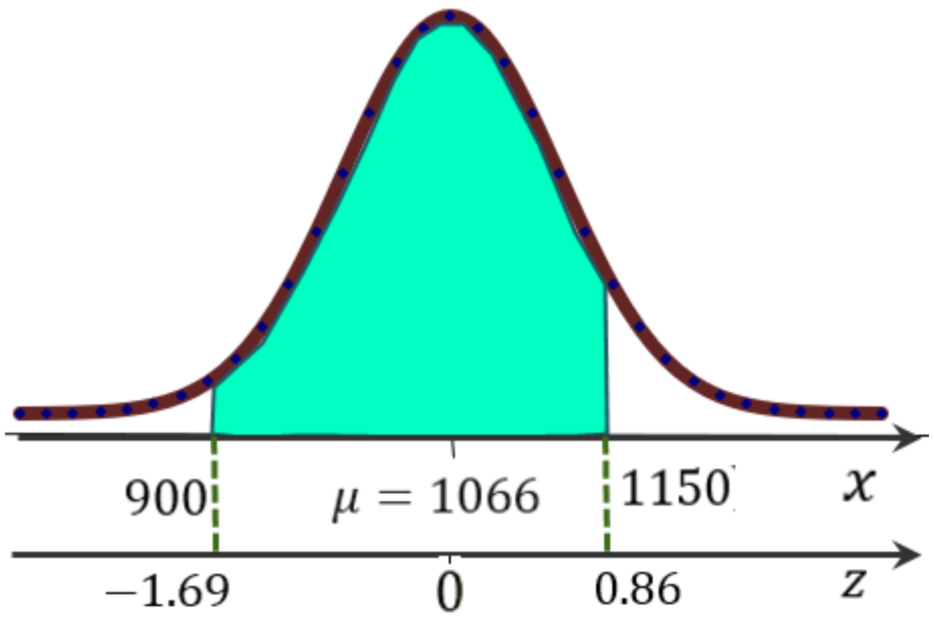


Normal Probability Distribution

Standard Normal Distribution

Example: Suppose that at a gas station the daily demand for gasoline is normally distributed with a mean of 1066 gallons and a standard deviation of 98 gallons.
c) The manager would like to know the probability that the demand will fall between 900 and 1150 gallons.

Then, we get $P(900 < x < 1150) = P(-1.69 < z < 0.86)$
and now we can work with the standard normal distribution.



Using the Standard Normal Distribution Table (see the next page), we find:

$P(900 < x < 1500) = P(-1.69 < z < 0.86)$
$= P(z < 0.86) - P(z < -1.69)$
$= 0.8051 - 0.0455 = 0.7596$

Normal Probability Distribution

Standard Normal Distribution

Example: Suppose that at a gas station the daily demand for gasoline is normally distributed with a mean of 1066 gallons and a standard deviation of 98 gallons.
c) The manager would like to know the probability that the demand will fall between 900 and 1150 gallons.

Then, we get $P(900 < x < 1150) = P(-1.69 < z < 0.86)$
and now we can work with the standard normal distribution.

Using the Standard Normal Distribution Table (see the next page), we find:

$$P(900 < x < 1500) = P(-1.69 < z < 0.86)$$
$$= P(z < 0.86) - P(z < -1.69)$$
$$= 0.8051 - 0.0455 = 0.7596$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559

Normal Probability Distribution

Standard Normal Distribution

Example: Suppose that at a gas station the daily demand for gasoline is normally distributed with a mean of 1066 gallons and a standard deviation of 98 gallons.
c) The manager would like to know the probability that the demand will fall between 900 and 1150 gallons.

Then, we get $P(900 < x < 1150) = P(-1.69 < z < 0.86)$
and now we can work with the standard normal distribution.

Using the Standard Normal Distribution Table (see the next page), we find:

$$P(900 < x < 1500) = P(-1.69 < z < 0.86)$$
$$= P(z < 0.86) - P(z < -1.69)$$
$$= 0.8051 - 0.0455 = 0.7596$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621

Normal Probability Distribution

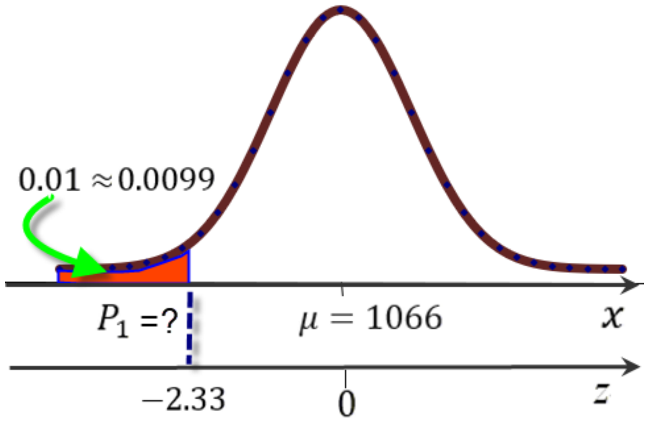
Standard Normal Distribution

Example: Suppose that at a gas station the daily demand for gasoline is normally distributed with a mean of 1066 gallons and a standard deviation of 98 gallons.
d) The manager would like to know what level of demand separates the bottom 1% of the distribution from the top 99%.

Given $\mu = 1066$ and $\sigma = 98$.
In part a) – c), numbers were given and we were looking for an area, In part d), area is given and we are looking for a number.

Find a specific x value so that 1% or 0.01 of x values are less than this number.

Find $x = P_1$ (like the percentile).



Normal Probability Distribution

Standard Normal Distribution

Example: Suppose that at a gas station the daily demand for gasoline is normally distributed with a mean of 1066 gallons and a standard deviation of 98 gallons.

d) The manager would like to know what level of demand separates the bottom 1% of the distribution from the top 99%.

as $P(x < P_1) = 0.01$, we look for the probability in the Standard Normal Distribution Table

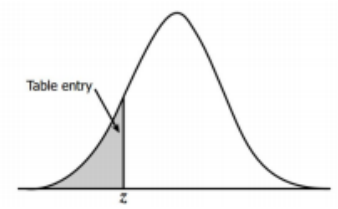
closest to 0.01. It appears that the closest value is 0.0099 . Therefore, z- score corresponding to P_1 is -2.33 .

$$P_1 = \mu + z\sigma$$

$$= 1066 + (-2.33)\times 98$$

$$= 1066 - 2.33\times 98 = 837.66$$

Standard Normal Distribution z-scores



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110

Normal Probability Distribution

Standard Normal Distribution

Example: Suppose that at a gas station the daily demand for gasoline is normally distributed with a mean of 1066 gallons and a standard deviation of 98 gallons.
e) The manager would like to know what level of demand separates the top 5% of the distribution from the bottom 95%.

Here again, we are given with area and we are looking for a number.
The question is to find a specific x value so that 95% or 0.95 of the x values are less than this number.

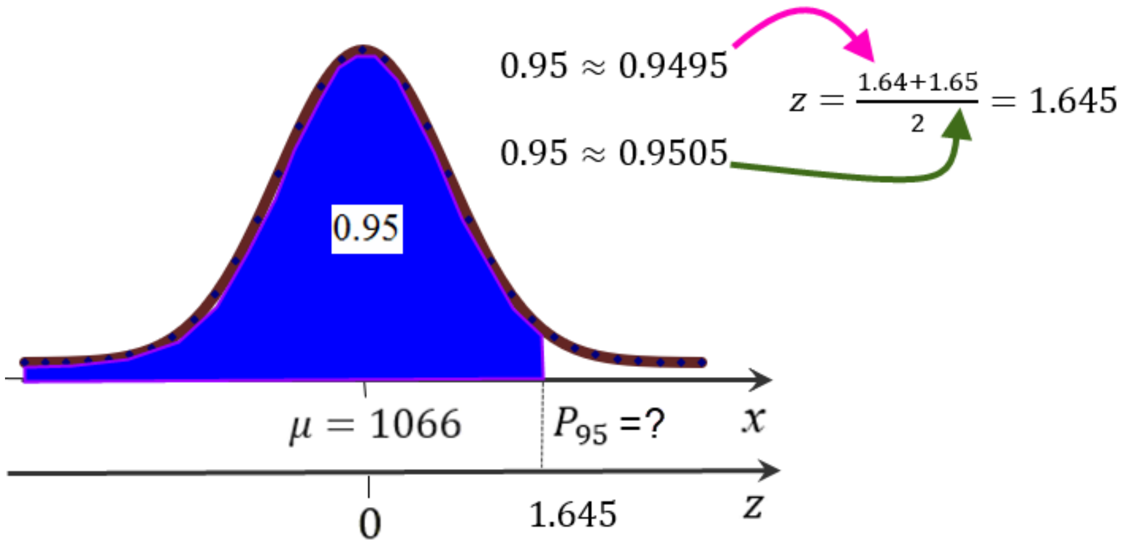
We have to find $x = P_{95}$ (95^{th} Percentile).

As far as $P(x < P_{95}) = 0.95$, we look for the probability

in the Standard Normal Distribution Table closest to 0.95.

Closest values (equally close values) are 0.9495 and 0.9505.

Therefore, z- score corresponding to P_{95} is $\frac{1.64+1.65}{2} = 1.645$.



Normal Probability Distribution

Standard Normal Distribution

Example: Suppose that at a gas station the daily demand for gasoline is normally distributed with a mean of 1066 gallons and a standard deviation of 98 gallons.

e) The manager would like to know what level of demand separates the top 5% of the distribution from the bottom 95%.

Here again, we are given with area and we are looking for a number.

The question is to find a specific x value so that 95% or 0.95 of the x values are less than this number.

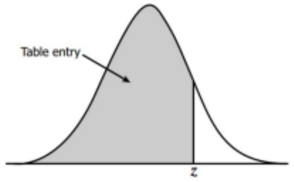
We have to find $x = P_{95}$ (95^{th} Percentile).

As far as $P(x < P_{95}) = 0.95$, we look for the probability in the Standard Normal Distribution Table closest to 0.95.

Closest values (equally close values) are 0.9495 and 0.9505.

Therefore, z- score corresponding to P_{95} is $\frac{1.64+1.65}{2} = 1.645$.

Standard Normal Distribution z-scores (continued)



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	$\boxed{0.9495}$	$\boxed{0.9505}$	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633

Normal Probability Distribution

Standard Normal Distribution

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Here again, we are given with area and we are looking for a number.
The question is to find a specific x value so that 95% or 0.95 of the x values are less than this number.

We have to find $x = P_{95}$ (95[th] Percentile).

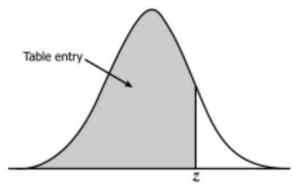
As far as $P(x < P_{95}) = 0.95$, we look for the probability in the Standard Normal Distribution Table closest to 0.95.

Closest values (equally close values) are 0.9495 and 0.9505.

Therefore, z- score corresponding to P_{95} is $\frac{1.64+1.65}{2} = 1.645$.

$P_{95} = \mu + z\sigma = 1066 + 1.645 \times 98 = 1227.21$

Standard Normal Distribution z-scores (continued)



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	$\boxed{0.9495}$	$\boxed{0.9505}$	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633

Summary

- ✓ **Probability Distribution**
  - ✓ Discrete Probability
  - ✓ Continuous Probability
- ✓ **Binomial Probability Distribution**
  - ✓ Reading Table values
  - ✓ Examples
- ✓ **Normal Probability Distribution**
  - ✓ Standard Normal Distribution
  - ✓ Z-score
  - ✓ Example