AI 2002
Artificial Intelligence

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Logic

$\alpha \models \beta$ if and only if $M(\alpha) \subseteq M(\beta)$

Logic --- Valid

- A sentence is **valid** if it is ***true*** in *all models (interpretations).*
- **Valid Sentences:**

$$\text{true}, \neg \text{false}, P \vee \neg P$$

- Valid sentences are also known as **tautologies**—they are *necessarily true.*
- From the definition of entailment, we can derive the **deduction theorem,** for example

For any sentences α and β, $\alpha \models \beta$ if and only if the sentence $(\alpha \Rightarrow \beta)$ is valid.

Logic --- Satisfiability

- A sentence is **satisfiable** if it is ***true*** in, or satisfied by, *some* model.
- A sentence is **satisfiable** if and only if it's ***true*** in at least *one interpretation*.
- Satisfiable Sentence:
  - true, P, $\neg P$
- A sentence is **unsatisfiable** if and only if it's truth value is ***false*** in *all interpretation*.
  - $\neg$ true, false, $P \wedge \neg P$

Examples

P	Q	¬ P	P ∧ Q	P ∨ Q	P → Q	Q → P	P ↔ Q
f	f	t	f	f	t	t	t
f	t	t	f	t	t	f	f
t	f	f	f	t	f	t	f
t	t	f	t	t	t	t	t

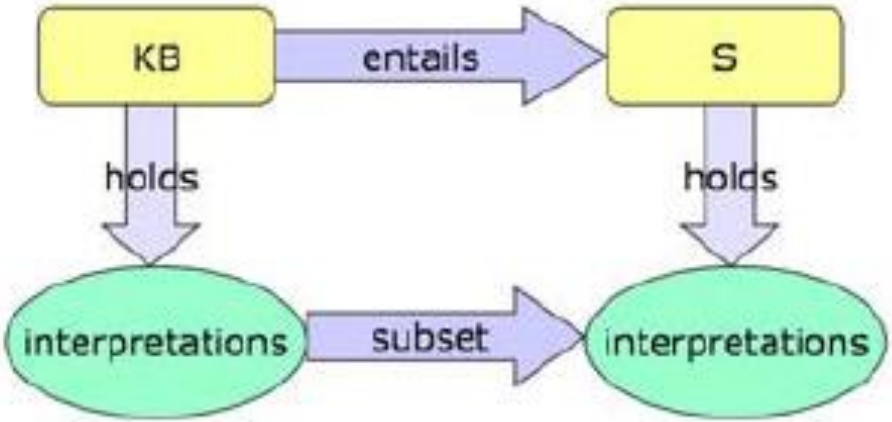
Sentences	**Valid/ Satisfiability?**	**Interpretation**
Smoke → Smoke	Valid	
Smoke ∨ ¬Smoke	Valid	
Smoke → Fire	Satisfiable	Smoke = **t** Fire = **f,** "smoke implies fire"
(s → f) → (¬s → ¬f)	Satisfiable	s = **f**, f = **t** (s → f) = **t**, (¬s → ¬f) = **f** "Smoke implies fire **implies** not smoke implies not fire."
(s → f) → (¬f → ¬s)	Valid	Contrapositive "smoke implies fire **implies** not fire implies not smoke"

Logic --- Satisfiability

- The problem of satisfiability of a sentence is very **related** to the **constraint satisfaction problem**, that has to **find a legal assignment** that can satisfy the constraint,
- Similarly, we have to *find interpretation in such as way that sentence holds that interpretation*.
- Can be solved the problem using the **brute-force method** of enumerating all possible interpretations.

Entailment

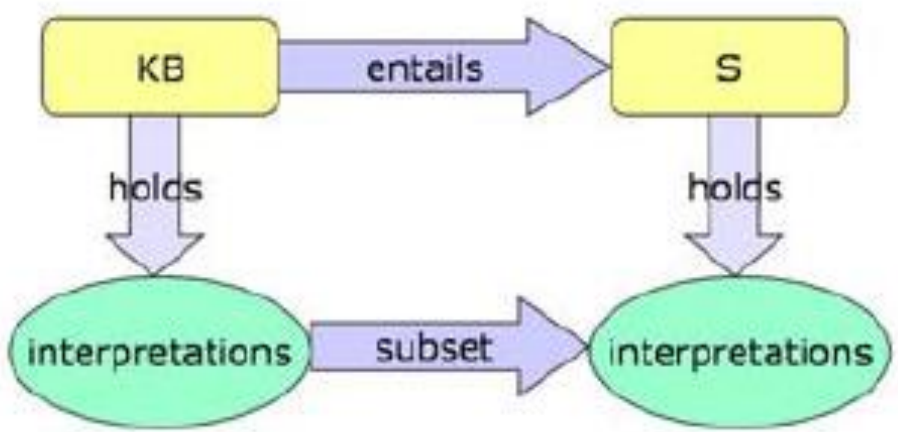
- Entailment that signifies a **relationship** between a knowledge base and another sentence.
- If every interpretation that satisfies the KB also satisfies the conclusion (sentence), we'll say that the KB **"entails" the conclusion (sentence)**.



$$\alpha \models \beta \text{ if and only if } M(\alpha) \subseteq M(\beta)$$

Entailment

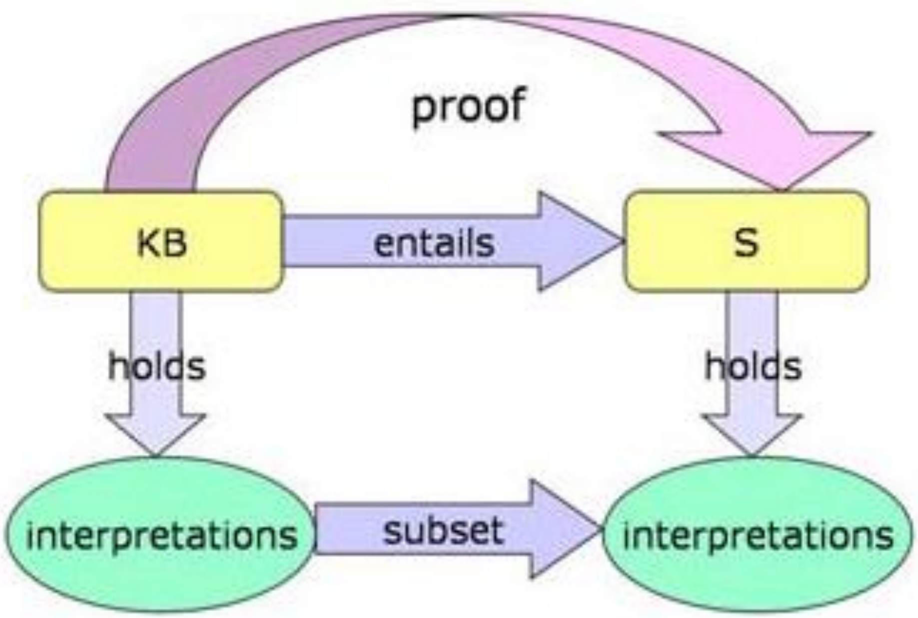
- We have to enumerate all interpretations
- Those interpretations are selected where all elements of $\boldsymbol{KB}$ are true.
- Check if the sentance $\boldsymbol{S}$ is true for all those interpretations.



Too many intpretations in general!

Entailment and Proof

- Proof is a way to test whether a KB entails a sentence without enumerating too many interpretations.
- Proof is basically a sequence of sentences.
- Proof is a chain of conclusions that leads to the desired goal.



Inference

- *If an inference algorithm* $\boldsymbol{i}$ *can derive* $\boldsymbol{\alpha}$ *from* $\boldsymbol{KB}$, we write

$$KB \vdash_i \alpha$$

- which is pronounced "**$\boldsymbol{\alpha}$ is derived from $\boldsymbol{KB}$ by $\boldsymbol{i}$**" or "**$\boldsymbol{i}$ derives $\boldsymbol{\alpha}$ from $\boldsymbol{KB}$**."

Inference Rules and Proofs

Some of Inference rules can be can be described as,

Modus Ponens

- If the sentences α → β and α are known to be true, then ***modus ponens*** lets us infer β.

$$\begin{array}{c} \alpha \to \beta \\ \alpha \\ \hline \beta \end{array}$$

Modus Tollens

- Under the inference rule ***modus tollens***, if α → β is known to be true and β is known to be false, we can infer ¬α.

$$\begin{array}{c} \alpha \to \beta \\ \neg \beta \\ \hline \neg \alpha \end{array}$$

Inference Rules and Proofs

And Elimination:

- And elimination allows us to infer the truth of either of the conjuncts from the truth of a conjunctive sentence. For instance, $\alpha \land \beta$ lets us conclude α and β are true.

$$\frac{\alpha \land \beta}{\alpha}$$

And Introduction:

- And introduction lets us infer the truth of a conjunction from the truth of its conjuncts. For instance, if α and β are true, then $\alpha \land \beta$ is true.

$$\frac{\begin{array}{c}\alpha \\ \beta\end{array}}{\alpha \land \beta}$$

Inference Rules and Proofs

- Inference rules can be applied to derive a proof.

$$\frac{\alpha \rightarrow \beta \qquad \alpha}{\beta}$$

Modus ponens

$$\frac{\alpha \rightarrow \beta \qquad \neg \beta}{\neg \alpha}$$

Modus tolens

$$\frac{\alpha \qquad \beta}{\alpha \wedge \beta}$$

And-introduction

$$\frac{\alpha \wedge \beta}{\alpha}$$

And-elimination

Example

- Consider there are 3 sentences in our knowledge base, $P \land Q$, $P \rightarrow R$ and $(Q \land R) \rightarrow S$. **We have to prove S**.

Step	Formula	Derivations
1	$P \land Q$	Given
2	$P \rightarrow R$	Given
3	$(Q \land R) \rightarrow S$	Given
4	P	1 And-Elimination
5	R	2,4 Modes Ponens
6	Q	1 And-Elimination
7	$Q \land R$	6,5 And-Introduction
8	S	3,7 Modes Ponens

Example ... 2

- Consider there are 5 sentences in our knowledge base as given below.
- **We have to prove R**.

Step	Formula	Derivations
1	$P \rightarrow Z$	Given
2	$(P \wedge R) \rightarrow S$	Given
3	$(Q \wedge Z) \rightarrow R$	Given
4	$Z \rightarrow Q$	Given
5	$P \wedge S$	Given
6	P	5 And-Elimination
7	Z	6,1 Modes Ponens
8	Q	4,7 Modes Ponens
9	$Q \wedge Z$	7,8 And introduction
10	R	3,9 Modes Ponens

Propositional Resolution

Propositional Resolution

- Resolution is one method for **automated theorem proving**.
- It helps logical agents **to reason** about the world.
- It helps logical agents **to prove new theorems**, and therefore helps them **to add to their knowledge**.
- Resolution requires all sentences to be first written in a special form - **conjunctive normal form**.

Propositional Resolution

- Input a knowledge base and an expression.
- It **negates the expression**, adds that to the knowledge base, and finds a **contradiction** if one exists
- If it **finds** a contradiction, then the negated statement is **false**.
- Therefore, the original statement **must be true.**

Conjunctive Normal Form

- A series of "conjunctions" (clauses joined together by "∧").

$$\underbrace{(A \vee B \vee \neg C)} \wedge \underbrace{(B \vee D)} \wedge (\neg A) \wedge (B \vee C)$$

clauses

- A, B, ¬C are **literals**.
- Inside the brackets, we only have "∨(OR)", "¬ (NOT)" symbols.
- *There must be no "implies" (→) symbols anywhere*.

Conjunctive Normal Form

- **Resolution algorithm** ‘resolves’ clauses
- Each pair of clauses that contains **complementary literals** is resolved. Complementary literals have the property that one negates the other
  - P, ¬P
  - the clause $B_{1,1} \vee \neg B_{1,1} \vee P_{1,2}$ is equivalent to $True \ \vee \ P_{1,2}$ which is equivalent to $True$. Deducing that $True$ is true is not helpful.
- Therefore, *any clause in which two complementary literals appear can be discarded*.
- Each clause is a requirement that must be satisfied.
- It can be satisfied in multiple ways

Converting into CNF

Converting into CNF

- The first step is **to eliminate single and double arrows** using their definitions.

$$(\alpha \Leftrightarrow \beta) \equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination}$$

$$(\alpha \Rightarrow \beta) \equiv (\neg\alpha \vee \beta) \quad \text{implication elimination}$$

- The next step is to **drive in negation**, De Morgan Laws

$$\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta) \quad \text{De Morgan}$$
$$\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta) \quad \text{De Morgan}$$

- The third step is to **distribute OR over AND**;

$$(\alpha \vee (\beta \wedge \gamma)) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \quad \text{distributivity of } \vee \text{ over } \wedge$$

Converting into CNF … Example

$$(A \vee B) \rightarrow (C \rightarrow D)$$

1. Eliminate arrows

$$\neg(A \vee B) \vee (\neg C \vee D)$$

2. Drive in negations

$$(\neg A \wedge \neg B) \vee (\neg C \vee D)$$

3. Distribute

$$(\neg A \vee \neg C \vee D) \wedge (\neg B \vee \neg C \vee D)$$

Resolution

- Resolution Rule:

Resolution rule:

$$\frac{\alpha \vee \beta \qquad \neg\beta \vee \gamma}{\alpha \vee \gamma}$$

- Wumpus-world Example

$$\frac{P_{1,1} \vee P_{3,1}, \qquad \neg P_{1,1} \vee \neg P_{2,2}}{P_{3,1} \vee \neg P_{2,2}}$$

Resolution

Propositional Resolution:

- Convert all sentances into CNF
- Negate the desired conclusion (converted to CNF)
- Apply resolution rule until either

1. Derive false (contradiction)
   - The conclusion follows from the axioms
2. Can not apply anymore,
   - The conclusion can not be proved from axioms

Propositional Resolution is sound and complete.

Resolution Example

- Consider we are given "P or Q", "P implies R" and "Q implies R". We would like to conclude R from these three axioms.

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

Steps	Formula	Derivations
1	$P \vee Q$	Given
2	$\neg P \vee R$	Given
3	$\neg Q \vee R$	Given

Resolution Example

- Consider we are given "P or Q", "P implies R" and "Q implies R". We would like to conclude R from these three axioms

1	$P \lor Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

$$\frac{\begin{array}{l} \text{false} \lor R \\ \neg R \lor \text{false} \end{array}}{\text{false} \lor \text{false}}$$

Steps	Formula	Derivations
1	$P \lor Q$	Given
2	$\neg P \lor R$	Given
3	$\neg Q \lor R$	Given
4	$\neg R$	Negated Conclusion
5	$Q \lor R$	1,2 Resolution Rule
6	$\neg P$	2,4
7	$\neg Q$	3,4
8	R	5,7
9	.	4,8

Resolution Example

- Consider we are given "P or Q", "P implies R" and "Q implies R". We would like to conclude R from these three axioms

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

false v R
¬ R v false
false v false

Steps	Formula	Derivations
1	$P \vee Q$	Given
2	$\neg P \vee R$	Given
3	$\neg Q \vee R$	Given
4	$\neg R$	Negated Conclusion
5	$Q \vee R$	1,2 Resolution Rule
~~6~~	~~$\neg P$~~	~~2,4~~
7	$\neg Q$	3,4
8	R	5,7
9	.	4,8

Resolution Example 2

- Consider we are given following table. We would like to conclude R from these three axioms

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Resolution Example 2

- Consider we are given following table. We would like to conclude R from these three axioms

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Step	Formula & Derivations
1	$(P \rightarrow Q) \rightarrow Q$
2	$(\neg P \vee Q) \rightarrow Q$
3	$\neg(\neg P \vee Q) \vee Q$
4	$(P \wedge \neg Q) \vee Q$
5	$(P \vee Q) \wedge (\neg Q \vee Q)$
6	$(P \vee Q)$

Resolution Example 2

- Consider we are given following table. We would like to conclude R from these three axioms

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Step	Formula & Derivations
1	$(P \rightarrow P) \rightarrow R$
2	$(\neg P \lor P) \rightarrow R$
3	$\neg(\neg P \lor P) \lor R$
4	$(P \land \neg P) \lor R$
5	$(P \lor R) \land (\neg P \lor R)$

Resolution Example 2

- Consider we are given following table. We would like to conclude R from these three axioms

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Step	Formula & Derivations
1	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$
2	$(\neg R \vee S) \rightarrow \neg(\neg S \vee Q)$
3	$\neg(\neg R \vee S) \vee \neg(\neg S \vee Q)$
4	$(R \wedge \neg S) \vee (S \wedge \neg Q)$
5	$(R \vee S) \wedge (R \vee \neg Q) \wedge (\neg S \vee S) \wedge (\neg S \vee \neg Q)$
6	$(R \vee S) \wedge (R \vee \neg Q) \wedge (\neg S \vee \neg Q)$

Resolution Example 2

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Step	Formula	Derivations
1	$(P \vee Q)$	
2	$(P \vee R)$	
3	$(\neg P \vee R)$	
4	$(R \vee S)$	
5	$(R \vee \neg Q)$	
6	$(\neg S \vee \neg Q)$	
7	$\neg R$	Negation

Resolution Example 2

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Step	Formula	Derivations
1	$(P \vee Q)$	
2	$(P \vee R)$	
3	$(\neg P \vee R)$	
4	$(R \vee S)$	
5	$(R \vee \neg Q)$	
6	$(\neg S \vee \neg Q)$	
7	$\neg R$	Negation
8	S	4,7
9	$\neg Q$	6,8
10	P	1,9
11	R	3,10
12	.	7,11

Proof Strategies

- **Unit Preference:** Prefer a resolution step involving a unit clause (*clause with one literal*)
  - Produce a shorter clause which is good since we are trying to produce a zero-length clause, that is, a contradiction.
- **Set of support:** Choose *a resolution involving the negated goal* or any clause derived from the negated goal
  - We're trying to produce a contradiction that follows from the negated goal, so these are "relevant" clauses.
  - If a contradiction exists, one can find it using the set-of-support strategy.

Reading Material

- **Artificial Intelligence,** A Modern Approach

  Stuart J. Russell and Peter Norvig

  - **Chapter 7.**