AI 2002
Artificial Intelligence

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First Order Logic

Propositional Logic

- Propositional logic is a **declarative language.**
  - its *semantics* is based on a ***truth relation*** ***between sentences and possible worlds***.
- Propositional logic **allows partial information using disjunction & negation** (unlike most data structures and databases).
- Propositional logic has a third property that is **compositionality**. Propositional logic is compositional, i.e., *the meaning of a sentence is a function of the meaning of its parts*.
  - For example: The meaning of $\boldsymbol{B_{1,1} \wedge P_{1,2}}$ is derived from the meaning of $\boldsymbol{B_{1,1}}$ and of $\boldsymbol{P_{1,2}}$

Propositional Logic

- Meaning in propositional logic is **context-independent** (unlike natural language, where meaning depends on context).
- Propositional logic has very **limited expressive power** (unlike natural language)

For example: cannot say

"Pits cause Breezes in adjacent squares".

we have to write a separate rule about breezes and pits for **EACH** square.

First-order Logic

- The propositional logic assumes the world contains ***facts*** while,
- The **first-order logic** (like natural language) assumes the world contains, ***objects***, ***relations,*** and ***functions***.

Objects:

- The nouns and noun phrases refer to **objects**
  - *In Wumpus-world*, the object examples are (squares, pits, and Wumpus)
  - The people, houses, numbers, theories, baseball games, wars, centuries, etc.

First-order Logic

Relations:

- The verbs and verb phrases refer to **relations**
  - *In Wumpus-world*, the relation examples are (is breezy, is adjacent to, shoots)
  - red, round, bogus, prime, brother of, bigger than, inside, part of, has colour, occurred after, owns, etc.

Functions:

- Some of these relations are **functions**—*relations in which there is only **ONE** "value" for a given "input"*.
  - father of, best friend, third inning of, one more than, end of

FOL Motivation

- The statements that cannot be made in propositional logics but can be expressed with FOL.
- First-order logic can also express facts about ***some* or *all*** of the objects in the universe.

1. When you paint a block with green paint, it becomes green.
   - In proposition logic, one would need a statement about **every single block** … for every single aspect of the situation, ***"if this block is black and I paint it, it becomes green"*** and ***"if block # 5 is red and I paint it, it becomes green"***
2. When you sterilize the jar, all the bacteria are dead.
   - In FOL, we can talk about all the bacteria without naming them explicitly.

Logics

Logics

Ontological commitment

- what exists in world
- what it assumes about the ***nature of reality (facts)***.
- Mathematically, this commitment is expressed through the ***nature of the formal models*** with respect to which the *truth of sentences is defined*.
  - For example, propositional logic assumes that there are facts that either hold or do not hold in the world.

Logics

Epistemological commitment

- the ***possible states of knowledge*** that it allows with respect to each fact.

- In both propositional and first order logic, a sentence represents a fact and the agent either believes the sentence to be <u>true</u> or <u>false</u> or has <u>no opinion</u>.

- Thus, the possible values are:

true/false/unknown

Types of Logic

Temporal logic

- assumes that **facts hold at particular *times*** and
- those times (which may be points or intervals) are ordered.

Probability theory

- Systems using **probability theory** can have any ***degree of belief***, ranging from 0 (total disbelief) to 1 (total belief).
  - For example, a probabilistic wumpus-world agent might believe that the wumpus is in [1,3] with probability 0.75

Types of Logic

Fuzzy logic

Fuzzy logic has a **degree of truth** between 0 and 1.

For example, the sentence **"Vienna is a large city"** might be true in our world only to a degree of 0.6 in fuzzy logic.

Language	Ontological Commitment (What exists in the world)	Epistemological Commitment (What an agent believes about facts)
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief $\in [0, 1]$
Fuzzy logic	facts with degree of truth $\in [0, 1]$	known interval value

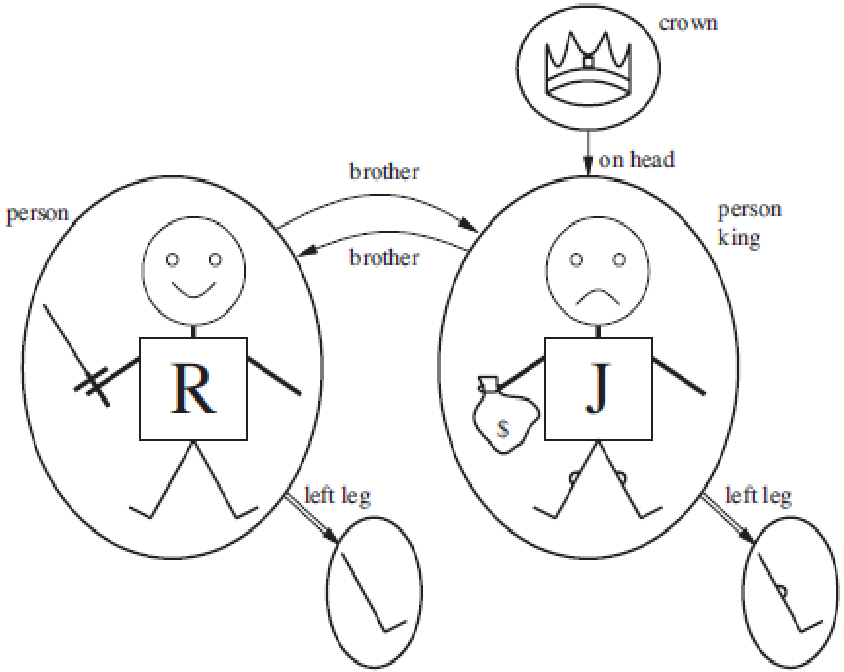
First-order Logic Models

Models for First-order Logic

- Models for first-order logic have **objects** in them.

Domain

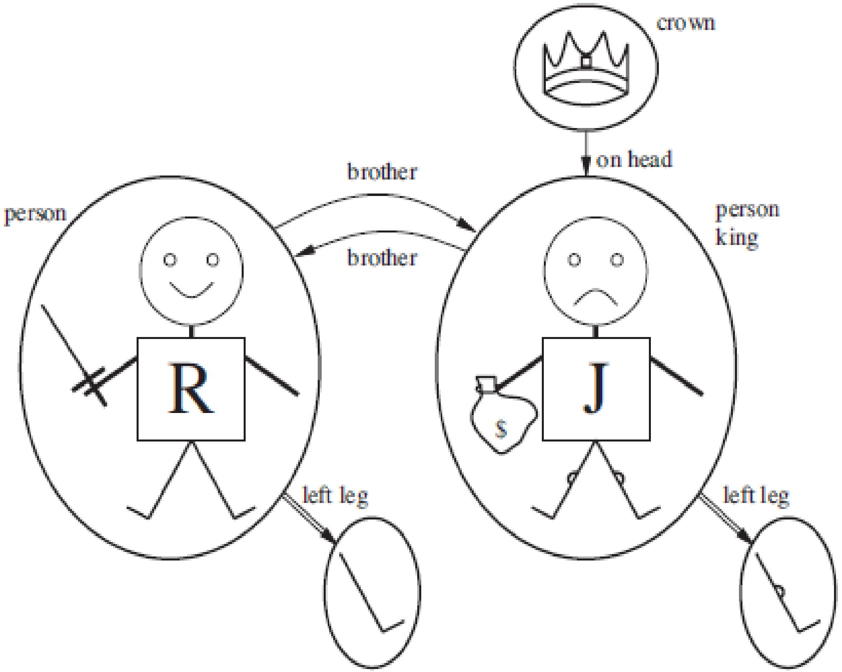
- The **domain** of a model is the ***set of objects*** or ***domain elements*** it contains.
- The domain is required to be ***nonempty***—*every possible world must contain at least one object.*



The **first-order logic** assumes the world contains, ***objects***, ***relations*** and ***functions***.

Models for First-order Logic

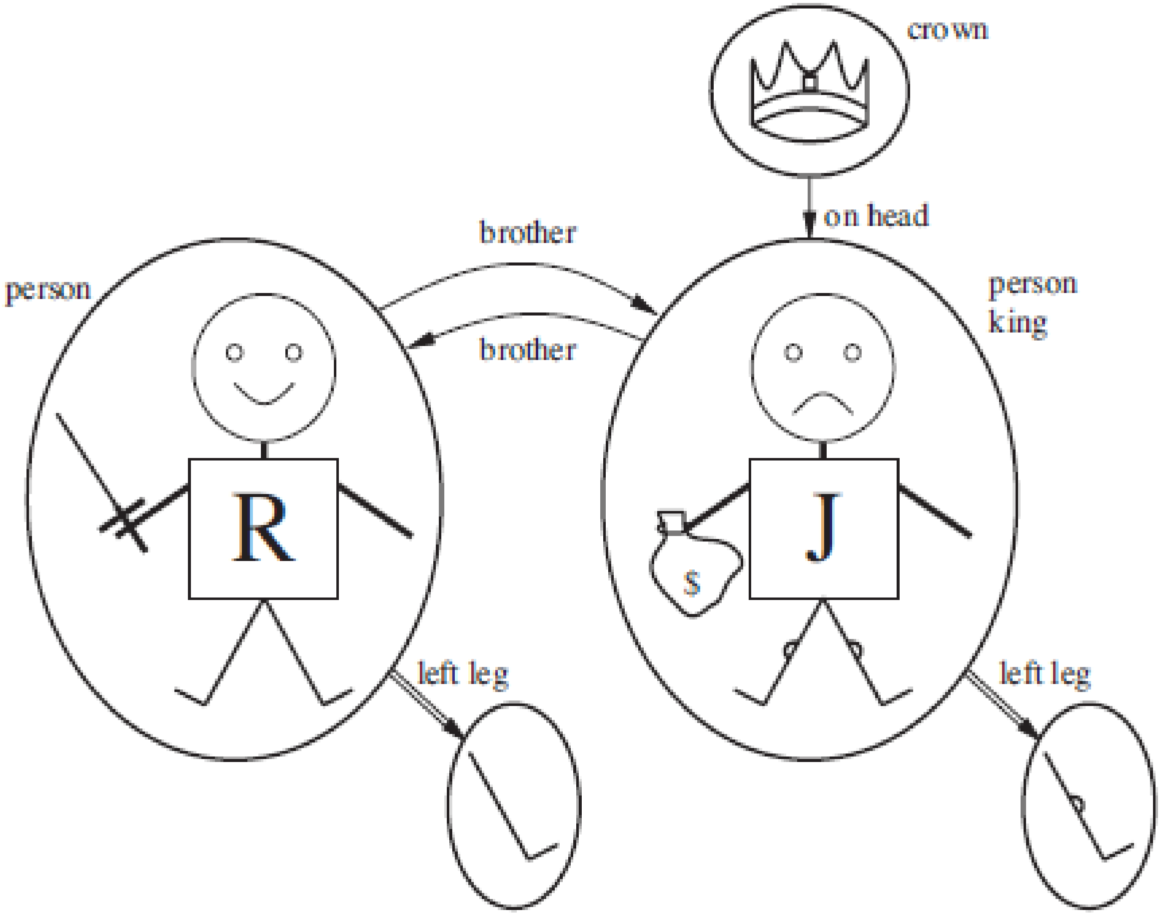
- Mathematically speaking, it doesn't matter ***what*** *these objects are,*
- ***all that matters is <u>how many</u> there are in each particular model.***



Example

<u>A model contains</u>

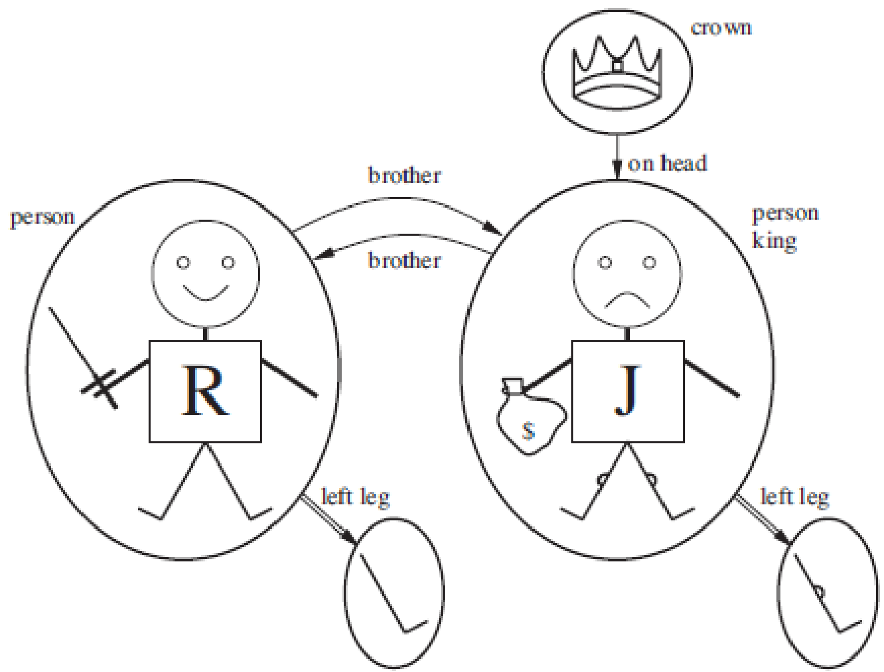
- Five objects,
- Two binary relations,
  - "brother"
  - "on head"
- Three unary relations,
  - "person"
  - "king"
  - "crown"
- One unary function,
  - "left leg"



Models for First-order Logic

Tuple:

- A tuple is a **collection of objects** arranged in a fixed order and is written with angle brackets surrounding the objects.

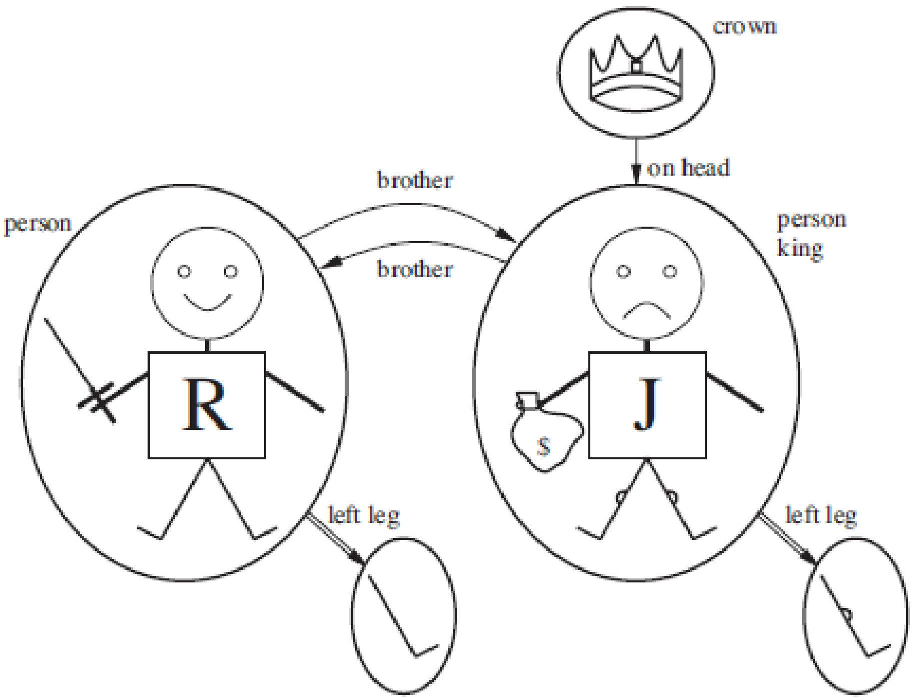


Models for First-order Logic

Tuple Example:

- The *"brotherhood"* relation in the model" is the set: **{<Richard the Lionheart, King John>, <King John, Richard the Lionheart>}.**
- The crown is on King John's head, so the *"on head"* relation contains just one tuple,

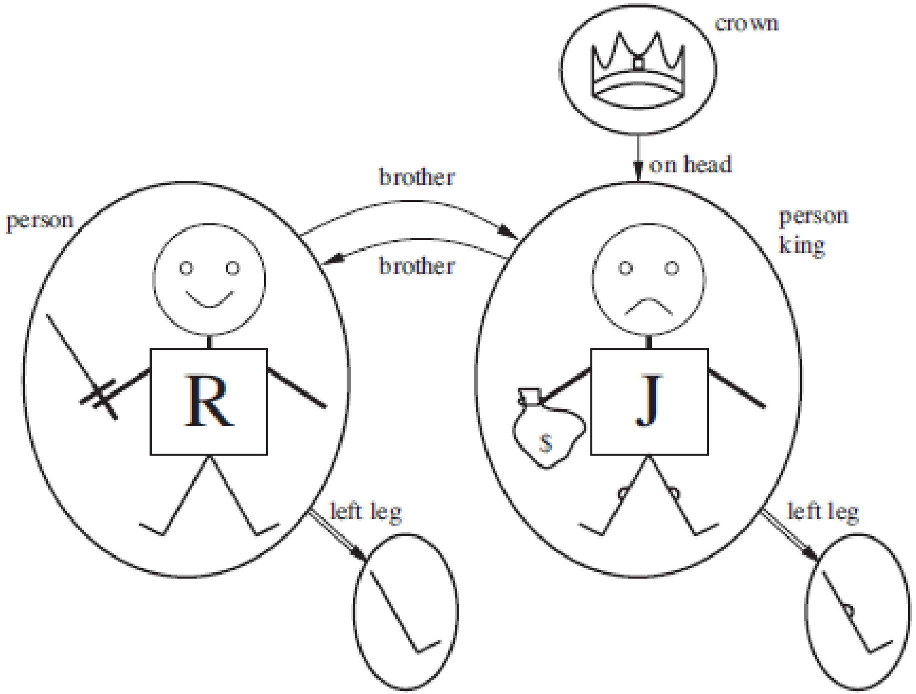
  <the crown, King John>.



Models for First-order Logic

Example:

- The **"brother"** and **"on head"** relations are ***binary relations***.
- The model also contains ***unary relations***, or ***properties***:
  - The **"person"** property is true of both Richard and John;
  - The **"king"** property is true only of John,
  - The **"crown"** property is true only of the crown



FOL Symbols and Interpretations

FOL Symbols and Interpretations

Symbol:

- The basic syntactic elements of first-order logic are the symbols that *stand for objects, relations, and functions*
- The symbols will begin with UPPERCASE letters.
- The symbols come in three kinds:

1. **constant symbol** --- *which stands for objects*, like Richard and John
2. **predicate symbol** --- *which stands for relations*, like Brother, OnHead, Person, King, and Crown
3. **function symbol** --- *which stands for functions*, like LeftLeg

Syntax of FOL: Basic Elements

Constants	$KingJohn,\ 2,$
Predicates	$Brother,\ >, \ldots$
Functions	$Sqrt,\ LeftLegOf, \ldots$
Variables	$x,\ y,\ a,\ b, \ldots$
Connectives	$\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftrightarrow$
Equality	$=$
Quantifiers	$\forall\ \exists$

FOL Symbols and Interpretations

- **Interpretation** specifies exactly which objects, relations, and functions are referred to by the constant, predicate, and function symbols.

Arity:

- Each predicate and function symbol comes with an arity that ***fixes the number of arguments***.

FOL Symbols and Interpretations

Term:

- A term is a logical expression that refers to an object.
- *A term may contain:*
    1. **Constant symbol**: Fred, Japan, Bacterium 39
    2. **Variables:** a,b, x
    3. **Functional symbols** are applied to one or more terms. F(x), Mother-of(John)

- A term with no variables is called a **ground term**.

FOL Sentences

Sentence

1. A **predicate symbol** may be applied to terms. **On(a, b), Sister(Jane, John), Sister(Mother-of(Jane), Jen)**
2. $\boldsymbol{term_1 = term_2}$
3. A **functional symbol** may be applied to one or more terms. **F(x)**, **Mother-of(John)**.
4. If $\boldsymbol{v}$ is a variable and $\boldsymbol{S}$ is a sentence, then
   - $(\forall \boldsymbol{v}\ \boldsymbol{S})$ and $(\exists \boldsymbol{v}\ \boldsymbol{S})$ are sentences too.

FOL Sentences

Atomic sentence

- (or atom for short) is formed from a predicate symbol optionally followed by a parenthesized list of terms, such as

Brother (Richard , John)

- Atomic sentences can have **complex terms** as arguments.

Married(Father (Richard), Mother (John))

FOL Sentences

Complex sentence

- We can use **logical connectives** to construct more complex sentences, with the same syntax and semantics as in propositional calculus.

Example

- There are four sentences,

$$\neg Brother(LeftLeg(Richard), John)$$
$$Brother(Richard, John) \land Brother(John, Richard)$$
$$King(Richard) \lor King(John)$$
$$\neg King(Richard) \Rightarrow King(John) .$$

Quantifiers

Quantifiers - Universal Quantifier (∀)

- **"All kings are persons"** is written in first-order logic as,

$$\forall x \;\; King(x) \Rightarrow Person(x)$$

- **∀** is usually pronounced **"For all . . ."**
- *Intuitively, the sentence* **∀x P**, *where* **P** *is any logical expression, says that* **P** *is true for every object* **x**.
- More precisely, **∀x P** *is true in a given model if* **P** *is true in* ***ALL*** *possible* ***extended interpretations*** *constructed from the interpretation given in the model,*
  - where each extended interpretation specifies a domain element to which x refers.

Quantifiers - Universal Quantifier (∀)

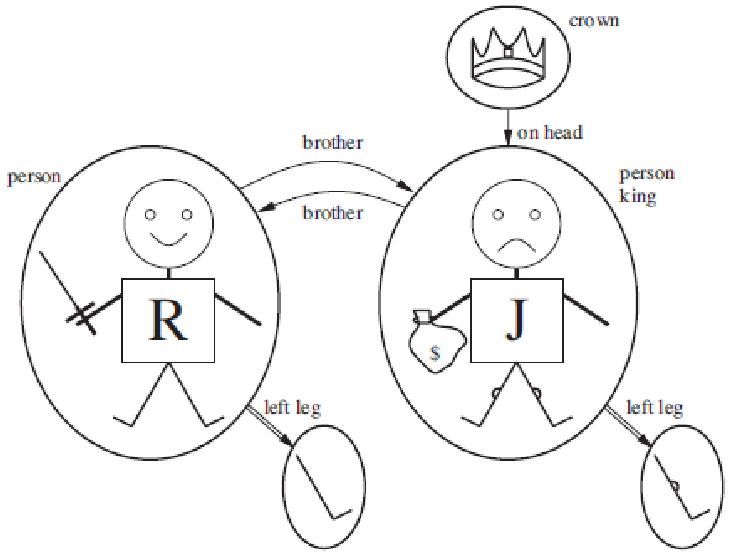
- **"All kings are persons"** is written in first-order logic as,

$$\forall x \;\; King(x) \Rightarrow Person(x)$$

- **∀** is usually pronounced **"For all . . ."**
- **Example:** **"For all x, if x is a king, then x is a person."**

We can extend the interpretation in **five** ways:

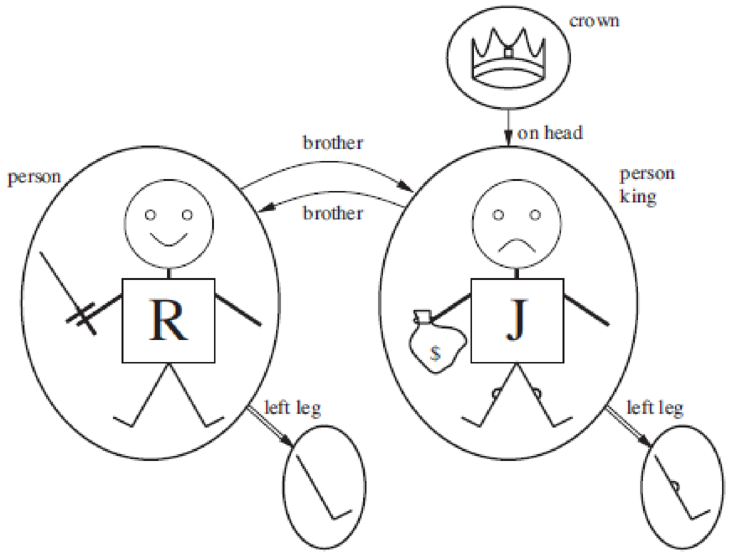
$$x \rightarrow \text{Richard the Lionheart,}$$
$$x \rightarrow \text{King John,}$$
$$x \rightarrow \text{Richard's left leg,}$$
$$x \rightarrow \text{John's left leg,}$$
$$x \rightarrow \text{the crown.}$$



Quantifiers - Universal Quantifier (∀)

- The **universally quantified sentence** is equivalent to asserting the following five sentences:

Richard the Lionheart is a king $\Rightarrow$ Richard the Lionheart is a person.
King John is a king $\Rightarrow$ King John is a person.
Richard's left leg is a king $\Rightarrow$ Richard's left leg is a person.
John's left leg is a king $\Rightarrow$ John's left leg is a person.
The crown is a king $\Rightarrow$ the crown is a person.



Quantifiers - Universal Quantifier (∀)

- Asserting the universally quantified sentence is **equivalent** to asserting a *whole list of individual implications*.
- The **implication is true whenever its premise is false**—*regardless of the truth of the conclusion.*

Richard the Lionheart is a king $\Rightarrow$ Richard the Lionheart is a person.
King John is a king $\Rightarrow$ King John is a person.
Richard's left leg is a king $\Rightarrow$ Richard's left leg is a person.
John's left leg is a king $\Rightarrow$ John's left leg is a person.
The crown is a king $\Rightarrow$ the crown is a person.

Quantifiers – Existential Quantifiers (∃)

- **"King John has a crown on his head"**, we write

$$\exists x \;\; Crown(x) \wedge OnHead(x, John)$$

- **∃x** is pronounced **"There exists an x such that . . ."** or **"For some x . . ."**.
- *Intuitively, the sentence* **∃x P** *says that* **P** *is true for at least one object* **x**.
- More precisely, ***∃x P*** *is true in a given model if* ***P*** *is true in* ***at least one extended interpretation*** *that assigns* ***x*** *to a domain element*.

Quantifiers – Existential Quantifiers (∃)

- That is, at least one of the following is true:

Richard the Lionheart is a crown $\wedge$ Richard the Lionheart is on John's head;
King John is a crown $\wedge$ King John is on John's head;
Richard's left leg is a crown $\wedge$ Richard's left leg is on John's head;
John's left leg is a crown $\wedge$ John's left leg is on John's head;
The crown is a crown $\wedge$ the crown is on John's head.

- The **fifth assertion is true in the model**, so the original existentially quantified sentence is true in the model.

Quantifiers – Existential Quantifiers (∃)

- Notice that, by the definition, the sentence would also be true in a model in which **King John was wearing two crowns**.

- There is a variant of the existential quantifier, usually written **$\exists^1$** or **$\exists!$**, that means

"There exists exactly one."

Quantifiers

- Typically $\rightarrow$ is the main connective with $\forall$.

- Typically, $\wedge$ is the main connective with $\exists$.

Quantifiers - Universal Quantifier (∀)

Common Mistake:

- Using ∧ as the main connective with ∀.

Everyone at Berkeley is smart:

$$\forall x \;\; At(x, Berkeley) \;\Rightarrow\; Smart(x)$$

$$\forall x \;\; At(x, Berkeley) \wedge Smart(x)$$

means "Everyone is at Berkeley and everyone is smart"

Quantifiers – Existential Quantifiers (∃)

Common Mistake:

- Using → as the main connective with ∃

Someone at Stanford is smart:

$\exists x \;\; At(x, Stanford) \wedge Smart(x)$

$$\exists x \;\; At(x, Stanford) \;\Rightarrow\; Smart(x)$$

is true if there is anyone who is not at Stanford!

The **implication is true whenever its premise is false**—*regardless of the truth of the conclusion.*

Nested Quantifiers

- For example, **"Brothers are siblings"** can be written as

$$\boxed{\forall x\ \forall y}\ Brother(x,y) \Rightarrow Sibling(x,y)\ .$$

- **Consecutive quantifiers** of the same type can be written as one quantifier with several variables.
- For example, to say that siblinghood is a symmetric relationship, we can write,

$$\boxed{\forall x, y}\ Sibling(x,y) \Leftrightarrow Sibling(y,x)$$

Nested Quantifiers

- The <u>**order of quantification**</u> **is very important**. For example: **"Everybody loves somebody"** means that for every person, there is someone that person loves:

$$\forall x \;\; \exists y \;\; Loves(x, y)$$

- On the other hand, to say **"There is someone who is loved by everyone"** we write

$$\exists y \;\; \forall x \;\; Loves(x, y)$$

Nested Quantifiers

- *Some confusion may arise when two quantifiers are used with the same variable name.*
- Consider the sentence

$$\forall x \ \ (Crown(x) \lor (\exists x \ \ Brother(Richard, x)))$$

- Here the **x** in **Brother (Richard, x)** is *existentially quantified.*
- ***The <u>rule</u> is that the variable belongs to the innermost quantifier that mentions it; then it will not be subject to any other quantification.***

$$\exists z \ Brother(Richard, z).$$

Reading Material

- **Artificial Intelligence,** A Modern Approach

  Stuart J. Russell and Peter Norvig
  - **Chapter 8.**