Lab (5) Notes

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1 Review

1.1 Optimization

Optimization focuses on **finding the best solution** among a list of feasible solutions by minimizing or maximizing an *objective function* based on the values of decision variables.

1.2 Unconstrained Optimization using 2nd-order Partial Derivative

STEP 1: We solve $f'(x) = 0$ to obtain the critical points (or extreme points).

STEP 2: We then find $f''(x)$ for all critical points and compare it to 0.

This approach can be **computationally expensive and numerically unstable**, especially for high-dimensional problems, as it requires computing and storing multiple second-order derivatives.

1.3 Unconstrained Optimization using Hessian Matrix

STEP 1: Find the ***critical points*** by setting $\nabla(f) = 0$.

STEP 2: Find the ***Hessian matrix*** $H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$

STEP 3: Classify the critical points as follows:

- If $|H| < 0$ then the point is **saddle.**
- If $|H| > 0$ then check the value of $\frac{\partial^2 f}{\partial x_1^2}$. **If positive** the point is **local minimum. If negative** the point is **local maximum.**

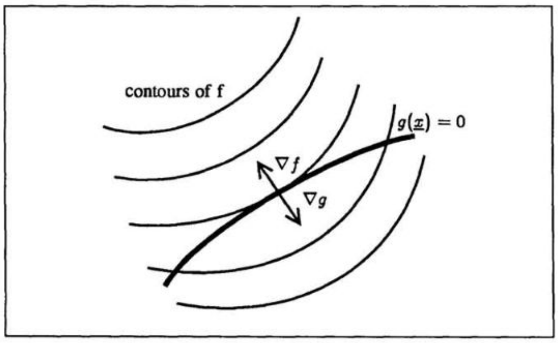
2 Constrained Optimization

2.1 Equality Constrained Optimization Using *Lagrange Multiplier*

The relationship between the gradient of the function and gradients of the constraints rather naturally leads to a reformulation of the original problem, known as the **Lagrangian function**.

$\mathscr{L}(x, \lambda) = f(x) + \lambda g(x)$

where λ is the Lagrange multiplier for the constraint $g(x) = 0$



2.2 Example

Minimize $f(X) = x_1 + 2x_2$, where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

S.t. $x_1^2 + x_2^2 = 4$

STEP 1: Put the problem in the standard form, RHS should be 0.

$g(X) = x_1^2 + x_2^2 - 4$

STEP 2: Formulate the Lagrangian function

$\mathscr{L}(x, \lambda) = f(x) + \lambda g(x)$

$\mathscr{L}(X, \lambda) = x_1 + 2x_2 + \lambda\left(x_1^2 + x_2^2 - 4\right)$

STEP 3: Set all partial derivatives to 0, including the partial derivative with respect to λ.

$$\frac{\partial \mathscr{L}}{\partial x_1} = 2\lambda x_1 + 1 = 0$$

$$\frac{\partial \mathscr{L}}{\partial x_2} = 2\lambda x_2 + 2 = 0$$

$$\frac{\partial \mathscr{L}}{\partial \lambda} = x_1^2 + x_2^2 - 4 = 0$$

STEP 4: Solve for X and λ to find critical points

	$\frac{-4}{\sqrt{5}}$	$\frac{4}{\sqrt{5}}$
$\frac{-2}{\sqrt{5}}$	$\left(\frac{-2}{\sqrt{5}}, \frac{-4}{\sqrt{5}}\right)$	$\left(\frac{-2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$

$\frac{2}{\sqrt{5}}$	$\left(\frac{2}{\sqrt{5}}, \frac{-4}{\sqrt{5}}\right)$	$\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$

STEP 5: Find the point that minimizes $f(x_1, x_2)$

	$\frac{-4}{\sqrt{5}}$	$\frac{4}{\sqrt{5}}$
$\frac{-2}{\sqrt{5}}$	$-2\sqrt{5}$	$\frac{6}{\sqrt{5}}$
$\frac{2}{\sqrt{5}}$	$\frac{-6}{\sqrt{5}}$	$2\sqrt{5}$

```
% Define f(x), g(x) and Larangian Fucntion L
syms x1 x2 lambda;
f = x1 + 2*x2;
g = x1^2 + x2^2 -4;
L= f+ lambda * g
```

L = $x_1 + 2x_2 + \lambda\left(x_1^2 + x_2^2 - 4\right)$

```
% Find pratial derivative for x1 and set it to 0.
dL_dx1 = diff(L, x1) == 0
```

dL_dx1 = $2\lambda x_1 + 1 = 0$

```
% Find pratial derivative for x2 and set it to 0.
dL_dx2 = diff(L, x2) == 0
```

dL_dx2 = $2\lambda x_2 + 2 = 0$

```
% Find pratial derivative for λ and set it to 0.
dL_dlambda = diff(L, lambda) == 0
```

dL_dlambda = $x_1^2 + x_2^2 - 4 = 0$

```
solutions = solve(dL_dx1, dL_dx2, dL_dlambda);
cPoints = table2array(combinations(solutions.x1, solutions.x2))
```

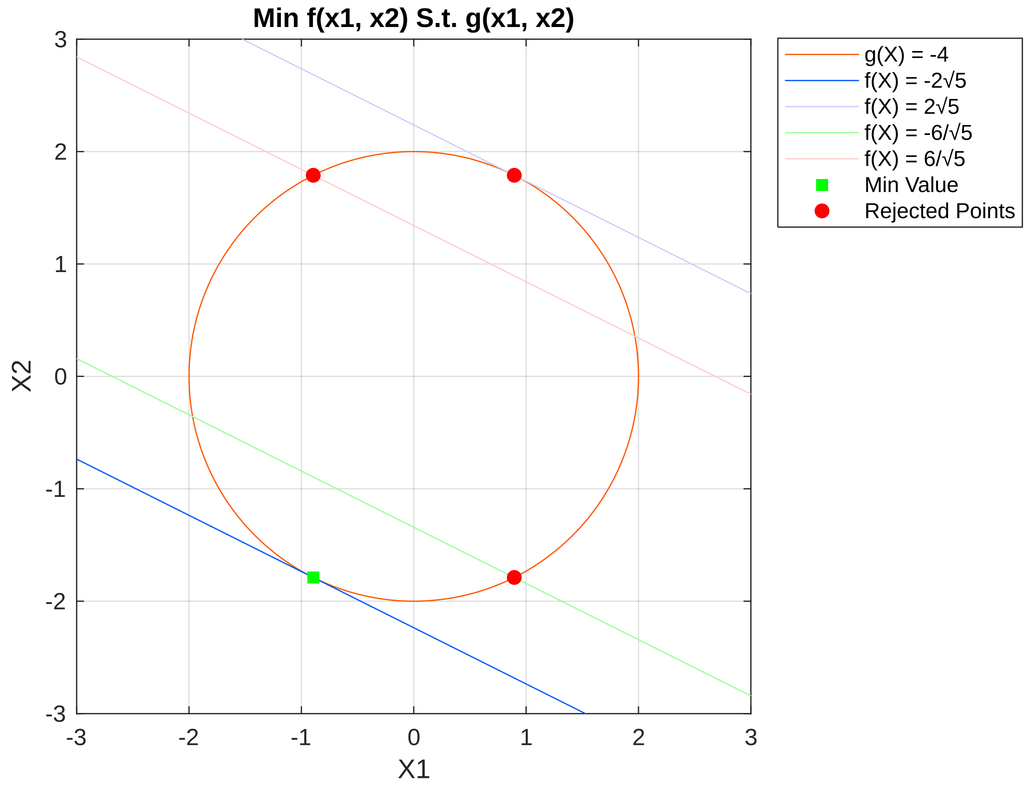
cPoints =

$$\begin{pmatrix} -\frac{2\sqrt{5}}{5} & -\frac{4\sqrt{5}}{5} \\ -\frac{2\sqrt{5}}{5} & \frac{4\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{4\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & \frac{4\sqrt{5}}{5} \end{pmatrix}$$

```
f1 = subs(f, [x1, x2], cPoints(1, :))
```

f1 = $-2\sqrt{5}$

```
f2 = subs(f, [x1, x2], cPoints(2, :));
f3 = subs(f, [x1, x2], cPoints(3, :));
f4 = subs(f, [x1, x2], cPoints(4, :));
min = cPoints(1, :);
fimplicit(g,  "Color","#ff5500");
hold on
fimplicit(f - f1, "Color","#0055ff");
fimplicit(f - f4, "Color", "#ccf")
fimplicit(f - f3, "Color", "#9f9")
fimplicit(f - f2, "Color", "#fcc")
scatter(min(1), min(2),"gs", "filled")
scatter(-2/sqrt(5), 4/sqrt(5), "ro", "filled")
scatter(2/sqrt(5), -4/sqrt(5), "ro", "filled")
scatter(2/sqrt(5), 4/sqrt(5), "ro", "filled")
grid on
axis equal
ylim([-3, 3])
xlim([-3, 3])
legend("g(X) = -4", ...
    "f(X) = -2√5", ...
    "f(X) = 2√5", ...
    "f(X) = -6/√5", ...
    "f(X) = 6/√5" ,...
    "Min Value", ...
    "Rejected Points",...
    "Location", "northeastoutside")
xlabel("X1")
ylabel("X2")
title("Min f(x1, x2) S.t. g(x1, x2)")
hold off
```



2.3 Inequality Constrained Optimization *with KKT*

The simplest way to handle inequality constraints is to convert them to equality constraints ***using slack variables and then use the Lagrange theory.***

For $g(X) \leq 0$

$$g(X) + S^2 = 0,$$
$$S^2 = -g(X) \longrightarrow (1)$$

The Lagrangian function will be: $\mathscr{L}(X, \lambda) = f(X) + \lambda[g(X) + S^2]$.

$$\frac{\partial \mathscr{L}}{\partial S} = 2\lambda S = 0, \lambda S = 0$$
$$\lambda S^2 = \lambda S * S = 0 * S = 0$$
$$\text{from } (1),\ \lambda S^2 = 0$$
$$-\lambda g(X) = 0$$
$$\lambda g(X) = 0 \longrightarrow (2)$$

Equation (2) states that either $\lambda = 0$ or $g(X) = 0$

2.4 Example

Minimize $f(X) = (x_1 - 1)^2 + (x_2 - 3)^2$ where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

S.t. $\begin{array}{l} x_1 + x_2 \leq 2, \\ x_2 \geq x_1 \end{array}$

STEP 0: Put the problem in the standard form.

$$\mathscr{L}(X, \lambda_1, \lambda_2) = f(X) + \lambda_1 g_1(X) + \lambda_2 g_2(X)$$

$$f(X) = (x_1 - 1)^2 + (x_2 - 3)^2$$

$$g_1(X) = x_1 + x_2 - 2$$

$$g_2(X) = x_1 - x_2$$

$$\lambda_1 g_1(X) = 0 \longleftarrow \text{KKT}$$

$$\lambda_2 g_2(X) = 0 \longleftarrow \text{KKT}$$

STEP 1: Find the Lagrangian

$$\mathscr{L}(X, \lambda_1, \lambda_2) = (x_1 - 1)^2 + (x_2 - 3)^2 + \lambda_1[x_1 + x_2 - 2] + \lambda_2[x_1 - x_2]$$

where $\lambda_1, \lambda_2 \geq 0$ because we need $g_i(X) \leq 0$

STEP 2: Set partial derivatives to 0.

$$\frac{\partial \mathscr{L}}{\partial x_1} = 2(x_1 - 1) + \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial \mathscr{L}}{\partial x_2} = 2(x_2 - 3) + \lambda_1 - \lambda_2 = 0$$

STEP 3: Apply KKT

$$\lambda_1(x_1 + x_2 - 2) = 0$$

$$\lambda_2(x_1 - x_2) = 0$$

STEP 4: Solve the system and <u>**check**</u> solutions on original constraints.

Case 1: $\lambda_1 = 0, \lambda_2 = 0$

$$2(x_1 - 1) = 0,$$

$$x_1 = 1$$

$$2(x_2 - 2) = 0$$

$$x_2 = 3$$

Rejected.

Case 2: $\lambda_1 = 0, \lambda_2 \neq 0$

$$\because \lambda_2 \neq 0$$

$$\because x_1 - x_2 = 0,$$

$$x_1 = x_2$$

$\because\ \lambda_1 = 0$

$\therefore\ \dfrac{\partial \mathscr{L}}{\partial x_1} = 2(x_1 - 1) + 0 + \lambda_2\ = 0,$

$2(x_1 - 1) +\ \lambda_2 = 0\,,$

$\dfrac{\partial \mathscr{L}}{\partial x_2} = 2(x_2 - 3) + 0 - \lambda_2\ = 0,$

$2(x_2 - 3) - \lambda_2 = 0$

$4x_1 + -8 = 0,$

$x_1 = x_2 = 2$

Rejected.

Case 3: $\lambda_1 \neq 0, \lambda_2 = 0$

$\because\ \lambda_1\ \neq 0$

$\therefore\ x_1 + x_2 - 2 = 0,$

$x_1 = 2 - x_2$

$\because\ \lambda_2 = 0$

$\therefore\ \dfrac{\partial \mathscr{L}}{\partial x_1} = 2(x_1 - 1) + \lambda_1 + 0\ = 0,$

$2(x_1 - 1) + \lambda_1 = 0,$

$\dfrac{\partial \mathscr{L}}{\partial x_2} = 2(x_2 - 3) + \lambda_1 - 0\ = 0,$

$2(x_2 - 3) + \lambda_1 = 0$

$2 - 2x_2 + \lambda_1 = 0$

$-6 + 2x_2 + \lambda_1 = 0$

$\lambda_1 = 2$

$x_1 = 0$

$x_2 = 2$

Accepted.

The point (0, 2) minimizes $f(X) = (x_1 - 1)^2 + (x_2 - 3)^2$ to 2

No need to check case 4.

```
c = @(x,y) (x-1).^2 + (y-3).^2 - 2;
r = @(x,y) (x + y - 2);
s = @(x, y)(x - y);
fimplicit(c)
hold on
fimplicit(r)
hold on
```

```
fimplicit(s)
axis equal
scatter(0, 2, "gs", "filled")
scatter(1, 3, "ro", "filled")
scatter(2, 2, "ro", "filled")
grid on
xticks(-5:1:5)
xlim([-2,5])
ylim([0, 5])
xlabel("x1")
ylabel("x2")
title("Min f(x1, x2) S.t. g1(x1, x2) and g2(x1, x2)")
legend("f(X)", ...
    "g1(X)", ...
    "g2(X)", ...
    "Min Value", ...
    "Rejected Points", ...
    "Location", ...
    "northeastoutside")
hold off
```