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Aurélien Alfonsi

Affine Diffusions and Related Processes: Simulation, Theory and Applications

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Preface

The development of affine processes in modelling has shadowed the expansion of financial mathematics ever since the pioneering works of Black and Scholes [20] and Merton [106] in the 1970s. These processes have various desirable features, the main one being an explicit description of their marginal laws as a function of their parameters. This property plays a key role in enabling the fitting of affine models to market data within a limited computational time, which has made them popular for the pricing and hedging of derivatives. Surprisingly, up until the late 1990s, there were very few works on the simulation of these affine processes; this can be partly explained by the fact that the two simplest affine diffusions can be sampled exactly by using either a Gaussian or a noncentral chi-square distribution, despite the simulation of the latter is rather time consuming. It is, however, important to generate samples of these affine diffusions to calculate pathwise expectations by a Monte Carlo method, which is required, for example, in order to price exotic options. It can also be useful for portfolio management to test strategies on simulated scenarios or to assess risk.

The main goal of this book is to present recent developments with respect to the simulation of affine diffusions. It aims to present the latest research on the exact and approximation schemes, with a strong emphasis on high-order approximation schemes for the weak error. This approach proves to be very tractable and powerful for affine diffusions. In fact, it allows implementation of a "divide and conquer" strategy to construct second-order approximation schemes for multidimensional affine diffusions, which reduces this problem to the construction of second-order approximation schemes for real valued affine diffusions. So that it is self-contained, the book provides some general background on the approximation of diffusions. It also presents the main properties of affine diffusions together with the mathematical tools that are used to handle them. Furthermore, to motivate the study of the different diffusions and to communicate the need to simulate them, the book presents models, mostly arising from finance, that use them. The last chapter focuses on the simulation of some diffusions of the Wright-Fisher type. These diffusions are related to affine diffusions and are widely used in biology for gene frequency models.

This book will be of interest for researchers working on numerical probability or developing models related to affine diffusions. In addition, it will provide material for preparation of classes on numerical probability and finance. It should also be useful for practitioners in finance who are involved with the simulation of processes. The book is intended to be accessible for Masters and Ph.D. students. It basically requires a good knowledge of stochastic calculus for diffusions. There are many excellent references on this topic, and the book refers as much as possible to the work of Karatzas and Shreve [83] in order to help the reader. Lastly, some exercises are presented in the early chapters. These exercises are generally meant to allow the reader to practice the mathematical arguments that have just been developed and to test his or her understanding.

I would like to thank Springer Milan and the Board of the B&SS – Bocconi & Springer Series for the invitation to write this monograph and for their encouragement. In this series, two other recent books have dealt with probability and finance, and both have some connections with the present one. The monograph [112] from Pascucci offers a very nice overview on mathematical finance. It presents the necessary knowledge in stochastic calculus, the main models in finance, and the essential numerical methods. The book by Baldeaux and Platen [14] focuses on more advanced and recent developments in mathematical finance. It covers a wide spectrum of methods, models, and applications. In particular, it presents most of the affine models in finance that are considered in this book. In comparison with these two books, the present one has a more specialized focus and provides complementary reading on the exact and approximated simulation methods for these affine models.

A significant part of this book concerning the matrix-valued diffusions relies on the work of the Ph.D. thesis of Abdelkoddousse Ahdida. I thank him for our fruitful collaboration on this topic. I also thank Ernesto Palidda for his remarks on the early chapters and the referees for their valuable comments and useful feedback. Despite this, and my care, it would be miraculous if this book were to be free of typographical errors and various mistakes. I would therefore appreciate any feedback from readers by email, and will keep an updated list of errata on my web page. I would like to thank my colleagues from CERMICS, the INRIA MathRisk project, and the Chaire "Risques Financiers" for the numerous stimulating discussions and meetings that we have had together. Finally, writing this book has stolen some of my spare time and I am grateful to my wife for her support and to my children for bringing me back to real life.

Champs-sur-Marne, France — Aurélien Alfonsi
December 2014

Contents

Notations

Notations for Real Matrices

- For $d, d' \in \mathbb{N}^*$, $\mathcal{M}_d(\mathbb{R})$ denotes the real d square matrices and $\mathcal{M}_{d\times d'}(\mathbb{R})$ the real matrices with d rows and d' columns.
- For $x \in \mathcal{M}_d(\mathbb{R})$, $x^\top$, $\mathrm{adj}(x)$, $\det(x)$, $\mathrm{Tr}(x)$ and $\mathrm{Rk}(x)$ are, respectively, the transpose, the adjugate, the determinant, the trace, and the rank of x.
- $\mathcal{S}_d(\mathbb{R})$, $\mathcal{S}_d^+(\mathbb{R})$, $\mathcal{S}_d^{+,*}(\mathbb{R})$, and $\mathcal{G}_d(\mathbb{R})$ denote, respectively, the set of symmetric, symmetric positive semidefinite, symmetric positive definite, and non-singular matrices.
- The set of orthogonal matrices is denoted by $\mathcal{O}_d(\mathbb{R})$, i.e.

$$\mathcal{O}_d(\mathbb{R}) = \{o \in \mathcal{M}_{\boldsymbol{d}}(\mathbb{R}),\ oo^\top = I_d\}.$$

- The set of correlation matrices is denoted by $\mathfrak{C}_d(\mathbb{R})$:

$$\mathfrak{C}_d(\mathbb{R}) = \left\{x \in \mathcal{S}_{\boldsymbol{d}}^+(\mathbb{R}), \forall 1 \le i \le d,\ x_{i,i} = 1\right\}.$$

  We also define $\mathfrak{C}_d^*(\mathbb{R}) = \mathfrak{C}_d(\mathbb{R}) \cap \mathcal{G}_d(\mathbb{R})$, the set of the invertible correlation matrices.
- For $x \in \mathcal{S}_d^+(\mathbb{R})$, $\sqrt{x}$ denotes the unique symmetric positive semidefinite matrix such that $(\sqrt{x})^2 = x$.
- The identity matrix is denoted by I_d and we set for $n \le d$, $I_d^n = (\mathbb{1}_{i=j\le n})_{1\le i,j\le d}$ and $e_d^n = (\mathbb{1}_{i=j=n})_{1\le i,j\le d}$, so that $I_d^n = \sum_{i=1}^n e_d^i$. We also set for $1 \le i, j \le d$, $e_d^{i,j} = (\mathbb{1}_{k=i,l=j})_{1\le k,l\le d}$ and $e_d^{\{i,j\}} = e_d^{i,j} + \mathbb{1}_{i\ne j} e_d^{j,i}$.
- For $x \in \mathcal{S}_{\boldsymbol{d}}(\mathbb{R})$, we denote by $x_{\{i,j\}}$ the value of $x_{i,j}$, so that $x = \sum_{1\le i\le j\le d} x_{\{i,j\}}(e_d^{i,j} + \mathbb{1}_{i\ne j} e_d^{j,i})$. We use both notations in the paper: notation $(x_{i,j})_{1\le i,j\le d}$ is more convenient for matrix calculations while $(x_{\{i,j\}})_{1\le i\le j\le d}$ is preferred to emphasize that we work on symmetric matrices.

- For $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$, $diag(\lambda_1, \ldots, \lambda_d)$ denotes the diagonal matrix which is defined by

$$diag(\lambda_1, \ldots, \lambda_d)_{i,j} = \mathbb{1}_{i=j}\lambda_i, \ 1 \le i, j \le d.$$

- For $x \in \mathcal{S}_d^+(\mathbb{R})$ such that $x_{i,i} > 0$ for all $1 \le i \le d$, we define $\mathbf{p}(x) \in \mathfrak{C}_d(\mathbb{R})$ by

$$(\mathbf{p}(x))_{i,j} = \frac{x_{i,j}}{\sqrt{x_{i,i}x_{j,j}}}, \ 1 \le i, j \le d.$$

- For $x \in \mathcal{S}_d(\mathbb{R})$ and $1 \le i \le d$, we denote by $x^{[i]} \in \mathcal{S}_{d-1}(\mathbb{R})$ the matrix obtained from x by deleting the ith line and the ith row. Namely, $x^{[i]}$ is defined by

$$x^{[i]}_{k,l} = x_{k+\mathbb{1}_{k\ge i}, l+\mathbb{1}_{l\ge i}}, \ 1 \le k, l \le d-1.$$

  We also denote by $x^i \in \mathbb{R}^{d-1}$ the vector defined by $x^i_k = x_{i,k}$ for $1 \le k < i$ and $x^i_k = x_{i,k+1}$ for $i \le k \le d-1$. Notice that for $x \in \mathfrak{C}_d(\mathbb{R})$, we have $(x - xe^i_d x)^{[i]} = x^{[i]} - x^i(x^i)^T$.

Real Valued Random Variables (Tables 1 and 2)

Table 1 Discrete probability distributions

Name and notation	Parameters	Distribution
Bernoulli $\mathcal{B}(p)$	$p \in [0,1]$	$\mathbb{P}(X=1) = p = 1 - \mathbb{P}(X=0)$
Binomial $\mathcal{B}(n,p)$	$n \in \mathbb{N}^*, p \in [0,1]$	$0 \le k \le n, \mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$
Poisson $\mathcal{P}(\lambda)$	$\lambda > 0$	$k \in \mathbb{N}, \mathbb{P}(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$

Table 2 Continuous probability distributions

Name and notation	Parameters	Density
Normal $\mathcal{N}(m, \sigma^2)$	$m \in \mathbb{R}, \sigma > 0$	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-m)^2}{2\sigma^2})$
Uniform $\mathcal{U}(a,b)$	$a < b$	$\frac{1}{b-a}\mathbb{1}_{[a,b]}(x)$
Exponential $\mathcal{E}(\lambda)$	$\lambda > 0$	$\mathbb{1}_{\{x>0\}}\lambda e^{-\lambda x}$
Gamma $\Gamma(a,\theta)$	$a, \theta > 0$	$\frac{\theta^a}{\Gamma(a)} x^{a-1} e^{-\theta x}\mathbb{1}_{\{x>0\}}$
Beta $\beta(a,b)$	$a, b > 0$	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}$
Chi sq. $\chi^2(n)$	$n \in \mathbb{N}^*$	$\frac{(1/2)^{n/2}}{\Gamma(n/2)} x^{n/2-1} e^{-x/2}\mathbb{1}_{\{x>0\}}$

Miscellaneous

- For $k \in \mathbb{C}$, we define $\zeta_k(t) = \int_0^t e^{-ks} ds = \begin{cases} \frac{1}{k}(1 - e^{-kt}) \text{ if } k \neq 0, \\ t \text{ if } k = 0. \end{cases}$
- For $u \in \mathbb{C}$, $\Re(u)$ (resp. $\Im(u)$) denotes its real (imaginary) part.
- For $x \in \mathbb{R}$, $x^+ = \max(x, 0)$.
- Partial derivatives. For $f : \mathbb{R}^d \to \mathbb{R}$ and $1 \leq i \leq d$, we denote by $\partial_i f$ the partial derivative with respect to the ith coordinate. Similarly, for $f : \mathcal{M}_{\boldsymbol{d}}(\mathbb{R}) \to \mathbb{R}$ (resp. $\mathcal{S}_{\boldsymbol{d}}(\mathbb{R}) \to \mathbb{R}$) and $1 \leq i, j \leq d$, $\partial_{i,j}$ (resp. $\partial_{\{i,j\}}$) denote the partial derivative in the direction of $e_{i,j}$ (resp. $e_{\{i,j\}}$).

Chapter 1
Real Valued Affine Diffusions

This chapter gives a first contact with general affine diffusions by presenting the ones that take real values. We will see that these diffusions are basically of two types, and are either a Ornstein-Uhlenbeck process or a Cox-Ingersoll-Ross process. Thus, the two first sections of this chapter study these processes and present their main properties. The third section defines what are affine diffusions and characterize them by the mean of the infinitesimal generator. The last section is devoted to the application of these processes for the interest rate modelling. A quick introduction is given on the financial framework, and we present the main pricing formulas that have made the use of these processes popular.

Throughout this chapter, $(\Omega, \mathcal{F}, \mathbb{P})$ will denote the probability space and $(\mathcal{F}_t)_{t\geq 0}$ a filtration on it. We assume that this filtered probability space is large enough to carry all the needed random variables. In particular, we consider a standard real Brownian motion W which is assumed to be $(\mathcal{F}_t)$-adapted.

1.1 The Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck process is probably the simplest example of Affine diffusion. Let us consider three real parameters a, k and σ. We define X^x as the unique solution of the following Stochastic Differential Equation (SDE):

$$X_t^x = x + \int_0^t (a - kX_s^x)ds + \sigma W_t,\ x \in \mathbb{R}, t \geq 0. \tag{1.1}$$

The drift and diffusion coefficients of this SDE are affine and thus Lipschitz continuous, which ensures that X^x exists and is unique. In this particular case, existence and uniqueness can however be directly obtained as follows. We set

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$Y_t = e^{kt}X_t^x$ and get by Itô's formula that $dY_t = ae^{kt}dt + \sigma e^{kt}dW_t$, which gives immediately

$$\begin{aligned} X_t^x &= e^{-kt}\left[x + a\int_0^t e^{ks}ds + \sigma\int_0^t e^{ks}dW_s\right] \\ &= xe^{-kt} + a\zeta_k(t) + \sigma\int_0^t e^{-k(t-s)}dW_s, \end{aligned}$$

with $\zeta_k(t) = \int_0^t e^{-ks}ds = \begin{cases} \frac{1}{k}(1-e^{-kt}) \text{ if } k \neq 0, \\ t \text{ if } k = 0. \end{cases}$

Thus, we get that X^x is a Gaussian process, and its law is then fully characterized by its mean $\mathbb{E}[X_t^x] = xe^{-kt} + a\zeta_k(t)$ and its covariance function defined for $s \leq t$ by

$$\begin{aligned} Cov(X_s^x, X_t^x) &= \sigma^2\mathbb{E}\left[\int_0^t e^{-k(t-u)}dW_u \int_0^s e^{-k(s-u)}dW_u\right] \\ &= \sigma^2 e^{-k(t+s)}\int_0^s e^{2ku}du = \sigma^2 e^{-k(t-s)}\zeta_{2k}(s). \end{aligned}$$

In particular, the law of X_t^x is $\mathcal{N}(e^{-kt}x + a\zeta_k(t), \sigma^2\zeta_{2k}(t))$ and we have:

$$\forall u \in \mathbb{C}, \mathbb{E}\left[\exp(uX_t^x)\right] = \exp\left(e^{-kt}ux + a\zeta_k(t)u + \frac{\sigma^2u^2}{2}\zeta_{2k}(t)\right). \tag{1.2}$$

When $k > 0$, we get that the Ornstein-Uhlenbeck process is ergodic. In fact, we have $\zeta_k(t) \underset{t\to+\infty}{\to} 1/k$ and then

$$\mathbb{E}\left[\exp(uX_t^x)\right] \underset{t\to+\infty}{\to} \exp\left(\frac{a}{k}u + \frac{\sigma^2u^2}{4k}\right).$$

Thus, X_t^x converges in law to $\mathcal{N}(\frac{a}{k}, \frac{\sigma^2}{2k})$ when $t \to +\infty$.

Now, we can go further and calculate explicitly the joint law of $(X_t^x, \int_0^t X_s^x ds)$ through its Laplace transform. In fact, this is a Gaussian vector, and its law is thus characterized by its expectation and its covariance matrix. We have

$$\begin{aligned} \mathbb{E}\left[\int_0^t X_s^x ds\right] &= x\zeta_k(t) + a\int_0^t \zeta_k(s)ds, \\ Cov\left(X_t^x, \int_0^t X_s^x ds\right) &= \int_0^t Cov(X_s^x, X_t^x)ds = \frac{\sigma^2}{2}\zeta_k(t)^2, \\ Var\left(\int_0^t X_s^x ds\right) &= 2\int_0^t\int_0^s Cov(X_u^x, X_s^x)duds = \sigma^2\int_0^t \zeta_k(s)^2 ds. \end{aligned}$$

and get for $u, v \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(uX_t^x + v\int_0^t X_s^x ds\right)\right] \\
&= \exp\left(x\left[ue^{-kt} + v\zeta_k(t)\right] + ua\zeta_k(t) + va\int_0^t \zeta_k(s)ds \right. \\
&\quad \left. + \frac{\sigma^2}{2}\left(u^2\zeta_{2k}(t) + uv\zeta_k(t)^2 + v^2\int_0^t \zeta_k(s)^2 ds\right)\right). \qquad (1.3)
\end{aligned}$$

Notice that the integrals above can be calculated explicitly as follows

$$\int_0^t \zeta_k(s)ds = \frac{1}{k}(t - \zeta_k(t)), \int_0^t \zeta_k(s)^2 ds = \frac{1}{k^2}(t - \zeta_k(t)) - \frac{1}{2k}\zeta_k(t)^2 \text{ for } k \neq 0, \qquad (1.4)$$

and $\int_0^t \zeta_k(s)ds = t^2/2$, $\int_0^t \zeta_k(s)^2 ds = t^3/3$ for $k = 0$.

1.2 The Cox-Ingersoll-Ross Process

1.2.1 Definition and Existence

Let $a \geq 0$, $k \in \mathbb{R}$ and $\sigma > 0$. The Cox-Ingersoll-Ross process is defined by the following SDE:

$$X_t^x = x + \int_0^t (a - kX_s^x)ds + \int_0^t \sigma\sqrt{X_s^x}dW_s, \; x \in \mathbb{R}_+, t \geq 0. \qquad (1.5)$$

When $k = 0$ and $\sigma = 2$, this process is also known in the literature as the squared Bessel process with dimension a. Here, we exclude the degenerated case $\sigma = 0$ that corresponds to the linear ODE $X_t^x = xe^{-kt} + a\zeta_k(t)$. We also note that the case $\sigma < 0$ leads to the same SDE as (1.5), if we replace the Brownian motion W by $-W$.

A first natural question is to wonder if such a stochastic differential equation admits indeed a unique strong solution. Standard results on strong uniqueness of SDEs usually assume that the drift and the diffusion coefficient are locally Lipschitz (see Karatzas and Shreve [83], Theorem 2.5, p. 287). Here, the square-root that appears in the diffusion part is not Lipschitz at the neighborhood of 0. However, we still have strong existence and uniqueness in this case. This is well known since the work of Yamada and Watanabe for suitable one-dimensional SDEs (see Karatzas and Shreve [83], Proposition 2.13, p. 291). Since the Cox-Ingersoll-Ross process plays a crucial role throughout this book, we give here a proof of this result.

Theorem 1.2.1 *There exists a unique nonnegative continuous process X^x that solves* (1.5).

To prove this theorem, we first consider the following SDE for $x \geq 0$

$$X_t = x + \int_0^t (a - kX_s)ds + \int_0^t \sigma \sqrt{|X_s|}dW_s, \ t \geq 0. \tag{1.6}$$

The only difference with (1.5) is that the diffusion coefficient is well-defined for any values since we do not know a priori that X stays nonnegative. We can now state a strong uniqueness result for (1.6).

Proposition 1.2.2 *Let* $\tilde{X}_t = x + \int_0^t (a - k\tilde{X}_s)ds + \int_0^t \sigma \sqrt{|\tilde{X}_s|}dW_s$ *be another solution of* (1.6). *Then, X and $\tilde{X}$ are indistinguishable, i.e.* $\mathbb{P}(X_t = \tilde{X}_t, \forall t \geq 0) = 1$.

Proof The first step consists in constructing smooth approximations of the absolute value that are called Yamada functions. We set $b_n = \exp\left(-\frac{n(n+1)}{2}\right)$ so that $b_n = e^{-n}b_{n-1}$. Let $n \geq 1$. Since $\int_{b_n}^{b_{n-1}}(1/x)dx = n$, we can find a continuous function $g_n : \mathbb{R} \to \mathbb{R}$ with compact support in (b_n, b_{n-1}) such that

$$\forall x \in \mathbb{R}, 0 \leq g_n(x) \leq \mathbb{1}_{x \in (b_n, b_{n-1})} \frac{2}{nx}, \text{ and } \int_{\mathbb{R}} g_n(x)dx = 1.$$

We define $G_n(x) = \int_0^x g_n(y)dy$ the antiderivative of g_n and clearly have

$$\forall x \in \mathbb{R}, \ \mathbb{1}_{x \in (b_{n-1}, +\infty)} \leq G_n(x) \leq \mathbb{1}_{x \in (b_n, +\infty)}.$$

Last, we set

$$\psi_n(x) = \int_0^{|x|} G_n(y)dy. \tag{1.7}$$

The function ψ_n satisfies $|\psi_n(x)| \leq |x|$, $|\psi_n'(x)| \leq 1$ and is $\mathcal{C}^2$ since $\psi_n''(x) = g_n(|x|)$. Besides, the Lebesgue's dominated convergence theorem gives $\lim_{n \to +\infty} |\psi_n(x)| = |x|$.

We are now in position to prove the claimed result. Itô's formula gives:

$$\begin{aligned}
\psi_n(\tilde{X}_t - X_t) = &- k \int_0^t (\tilde{X}_s - X_s)\psi_n'(\tilde{X}_s - X_s)ds \\
&+ \sigma \int_0^t \left(\sqrt{|\tilde{X}_s|} - \sqrt{|X_s|}\right) \psi_n'(\tilde{X}_s - X_s)dW_s \\
&+ \frac{\sigma^2}{2} \int_0^t \left(\sqrt{|\tilde{X}_s|} - \sqrt{|X_s|}\right)^2 g_n(|\tilde{X}_s - X_s|)ds.
\end{aligned}$$

Since the drift and diffusion coefficients have a sublinear growth, it is well-known that X and $\tilde{X}$ have uniformly bounded moments ([83], Problem 3.15, p. 306) and thus the stochastic integral has a null expectation. Since $|\sqrt{|\tilde{x}|}-\sqrt{|x|}| \leq \sqrt{|\tilde{x}-x|}$, $|\tilde{x}-x|g_n(|\tilde{x}-x|) \leq \frac{2}{n}$ and $|\psi_n'(x)| \leq 1$, we get

$$\mathbb{E}\left[\psi_n(\tilde{X}_t - X_t)\right] \leq |k| \int_0^t \mathbb{E}\left[|\tilde{X}_s - X_s|\right] ds + \frac{\sigma^2}{n} t.$$

Then, Lebesgue's theorem gives $\mathbb{E}\left[\psi_n(\tilde{X}_t - X_t)\right] \underset{n\to+\infty}{\to} \mathbb{E}\left[|\tilde{X}_t - X_t|\right]$ and thus

$$\mathbb{E}\left[|\tilde{X}_t - X_t|\right] \leq |k| \int_0^t \mathbb{E}\left[|\tilde{X}_s - X_s|\right] ds.$$

Gronwall's lemma gives that $\mathbb{E}\left[|\tilde{X}_t - X_t|\right] = 0$ and thus $\mathbb{P}(\tilde{X}_t = X_t) = 1$ for any $t \geq 0$. Since both processes $\tilde{X}$ and X are continuous, they are indistinguishable ([83], Problem 1.5, p. 2). □

We can now obtain a strong existence and uniqueness result for the SDE (1.6). On the one hand, $x \mapsto a - kx$ and $x \mapsto \sqrt{|x|}$ are both continuous functions, which gives that (1.6) admits a weak solution ([83], Theorem 4.22, p. 323 and [78], Theorems 2.3 and 2.4, p. 173 and 177). On the other hand, the pathwise uniqueness obtained in Proposition 1.2.2 gives the existence and uniqueness of a strong solution of (1.6), as a consequence of the Yamada-Watanabe theorem ([83], Corollary 3.23, p. 310).

Now, it remains to prove that the solution X of (1.6) remains nonnegative when the initial value x is nonnegative, so that we can remove the absolute value in the square root. Let us first give an heuristic explanation of this property. Since X is continuous, it must cross the zero to go in the negative values. However, when the process value is zero, the diffusion part vanishes and the drift equals $a \geq 0$ and brings back the solution in the nonnegative values. To prove this result formally, we can use Theorem 5 of Doss and Lenglart [45] that gives a general comparison result for continuous semimartingales. We can also use a comparison theorem for one-dimensional SDEs ([83], Proposition 2.18, p. 293) by observing that 0 is the solution of (1.6) when $x = a = 0$. Here, we give a direct proof that uses the Yamada function ψ_n defined by (1.7). In fact, Itô's formula gives

$$\begin{aligned}\psi_n(X_t) =& \psi_n(x) + \int_0^t \psi_n'(X_s)(a - kX_s)ds + \int_0^t \sigma\psi_n'(X_s)\sqrt{|X_s|}dW_s \\ &+ \int_0^t \frac{\sigma^2}{2} g_n(|X_s|)|X_s|ds.\end{aligned}$$

Taking the expectation and using that $\psi_n'(x) \le 1$ and $g_n(x)x \le 2/n$, we get

$$\mathbb{E}[\psi_n(X_t)] \le \psi_n(x) + \int_0^t (a - k\mathbb{E}[X_s \psi_n'(X_s)])ds + \frac{\sigma^2 t}{n}.$$

Since $\psi_n(z) \underset{n\to+\infty}{\to} |z|$, $z\psi_n'(z) \underset{n\to+\infty}{\to} |z|$ and $|\psi_n(z)| \vee |z\psi_n'(z)| \le |z|$, we get by Lebesgue's theorem that

$$\mathbb{E}[|X_t|] \le x + \int_0^t (a - k\mathbb{E}[|X_s|])ds.$$

Taking the expectation of (1.6), we have $\mathbb{E}[X_t] = x + \int_0^t (a - k\mathbb{E}[X_s])ds$ and thus

$$\mathbb{E}[|X_t|] - \mathbb{E}[X_t] \le -k \int_0^t (\mathbb{E}[|X_s|] - \mathbb{E}[X_s])ds.$$

Gronwall's lemma gives then $\mathbb{E}[|X_t|] = \mathbb{E}[X_t]$ and thus $\mathbb{P}(X_t \ge 0) = 1$ for any $t \ge 0$. This concludes the proof of Theorem 1.2.1.

Exercise 1.2.3 Up to now, we have explained and shown why the assumption $a \ge 0$ ensures that the CIR process stays nonnegative. A natural question is to wonder what happens in the case $a < 0$, which we consider in this exercise. Let $x \ge 0$. Thanks to Proposition (1.2.2), we know that the SDE

$$X_t = x + \int_0^t (a - kX_s)ds + \int_0^t \sigma\sqrt{|X_s|}dW_s,\ t \ge 0$$

admits a unique strong solution. The goal of this exercise is to show that

$$\forall T > 0,\ \mathbb{P}(X_T < 0) > 0.$$

1. Prove that if $X_t \le 0$ for some $t > 0$, then $X_{t+t'} \le 0$ for any $t' \ge 0$ (Observe that $(-X_{t+t'})_{t'\ge 0}$ is then a CIR process starting from $-X_t$).
2. We assume by contradiction that $\exists T > 0,\ \mathbb{P}(X_T < 0) = 0$. Deduce that $\forall t \in [0, T],\ X_t \ge 0$ almost surely.
3. Let $u \le 0$. We consider the functions ϕ_u and ψ_u that are respectively defined by (1.12) and (1.11), and we set $M_t = \exp(\phi_u(T - t) + \psi_u(T - t)X_t)$, for $t \in [0, T]$. Show that $(M_t)_{t\in[0,T]}$ is a martingale and that

$$\mathbb{E}[\exp(uX_T)] = \exp(\phi_u(T) + \psi_u(T)x) \underset{u\to-\infty}{\to} +\infty.$$

   Deduce that for any $T > 0$, $\mathbb{P}(X_T < 0) > 0$.

1.2.2 *Characteristic and Probability Density Functions*

This paragraph presents rather classical results on the CIR process. The main ones on the characteristic function and the probability density function dates back to 1951 with the article [52] by Feller. The characteristic function of $(X_t^x, \int_0^t X_s^x ds)$ is also well-known, see e.g. Lamberton and Lapeyre [93]. Here, we explain in detail how to obtain the characteristic function. The same arguments will be used later on to calculate the characteristic function of other multidimensional affine diffusions, and it is easier to get familiar with them in dimension one. Also, a particular effort is made to describe precisely the set of convergence of the characteristic function. This is possible thanks to the explicit formulas and leads to a nice result on the moment explosion in the Heston model presented by Andersen and Piterbarg [13], see Corollary 4.2.2 in Sect. 4.2.1.

The Characteristic Function

Proposition 1.2.4 *Let X^x denote the solution of* (1.5). *The characteristic function of X_t^x is well-defined on*

$$\{u \in \mathbb{C}, \mathbb{E}[|\exp(uX_t^x)|] < \infty\} = \left\{u \in \mathbb{C}, \Re(u) < \frac{2}{\sigma^2 \zeta_k(t)}\right\} \tag{1.8}$$

and is given by:

$$\mathbb{E}[\exp(uX_t^x)] = \left(1 - \frac{\sigma^2}{2} u\zeta_k(t)\right)^{-\frac{2a}{\sigma^2}} \exp\left(\frac{ue^{-kt}}{1 - \frac{\sigma^2}{2} u\zeta_k(t)} x\right), \tag{1.9}$$

with $\zeta_k(t) = \begin{cases} \frac{1}{k}(1 - e^{-kt}) \text{ if } k \neq 0, \\ t \text{ if } k = 0. \end{cases}$

Remark 1.2.5 Proposition 1.2.4 characterizes the law of the CIR process. In fact, we have for $0 \leq t_1 \leq t_2$, and $u_1, u_2 \leq 0$,

$$\begin{aligned} &\mathbb{E}[\exp(u_1 X_{t_1}^x + u_2 X_{t_2}^x)] = \mathbb{E}[\mathbb{E}[\exp(u_1 X_{t_1}^x + u_2 X_{t_2}^x)|\mathcal{F}_{t_1}]] \\ &= \left(1 - \frac{\sigma^2}{2} u_2 \zeta_k(t_2 - t_1)\right)^{-\frac{2a}{\sigma^2}} \mathbb{E}\left[\exp\left(\left(u_1 + \frac{u_2 e^{-k(t_2 - t_1)}}{1 - \frac{\sigma^2}{2} u_2 \zeta_k(t_2 - t_1)}\right) X_{t_1}^x\right)\right] \\ &= \left(1 - \frac{\sigma^2}{2} u_2 \zeta_k(t_2 - t_1)\right)^{-\frac{2a}{\sigma^2}} \left(1 - \frac{\sigma^2}{2} \tilde{u}_1 \zeta_k(t_1)\right)^{-\frac{2a}{\sigma^2}} \exp\left(\frac{\tilde{u}_1 e^{-kt_1}}{1 - \frac{\sigma^2}{2} \tilde{u}_1 \zeta_k(t_1)} x\right), \end{aligned}$$

with $\tilde{u}_1 = u_1 + \frac{u_2 e^{-k(t_2-t_1)}}{1-\frac{\sigma^2}{2}u_2\zeta_k(t_2-t_1)}$. By iterating this argument, we can calculate explicitly $\mathbb{E}[\exp(\sum_{i=1}^n u_i X_{t_i}^x)]$ for $t_1 \leq \cdots \leq t_n$ and $u_1, \dots, u_n \leq 0$. Therefore, the law of $(X_{t_1}^x, \dots, X_{t_n}^x)$ is characterized, which gives by another mean the uniqueness in law of the CIR process. This argument can be generalized for affine processes to show their weak uniqueness.

Proof We first assume that u is a nonpositive real number. Let us first assume that we can write $\mathbb{E}[\exp(uX_t^x)] = \exp(\phi_u(t) + \psi_u(t)x)$ for some smooth functions $\phi_u(t)$ and $\psi_u(t)$. This equality at time $t = 0$ necessarily gives $\phi_u(0) = 0$ and $\psi_u(0) = u$. Then, the Markov property gives for $0 \leq t \leq T$

$$\mathbb{E}[\exp(uX_T^x)|\mathcal{F}_t] = \exp(\phi_u(T-t) + \psi_u(T-t)X_t^x),$$

since X is a time homogeneous diffusion. We set $M_t = \mathbb{E}[\exp(uX_T^x)|\mathcal{F}_t]$. This is clearly a $(\mathcal{F}_t)$-martingale for $t \in [0, T]$, and Itô's formula gives

$$\begin{aligned} dM_t =& M_t\Bigg[\Big(-\phi_u'(T-t) - \psi_u'(T-t)X_t^x + \psi_u(T-t)(a - kX_t^x) \\ &+ \frac{\sigma^2}{2}\psi_u(T-t)^2 X_t^x\Big) dt + \psi_u(T-t)\sigma\sqrt{X_t^x}dW_t\Bigg]. \end{aligned}$$

Necessarily, the drift term must vanish almost surely

$$-\phi_u'(T-t)+a\psi_u(T-t)+\left(-\psi_u'(T-t) - k\psi_u(T-t) + \frac{\sigma^2}{2}\psi_u(T-t)^2\right) X_t^x = 0,$$

which leads to the following system of Ordinary Differential Equations (ODE):

$$\begin{cases} -\phi_u'(t) + a\psi_u(t) = 0 \\ -\psi_u'(t) - k\psi_u(t) + \frac{\sigma^2}{2}\psi_u(t)^2 = 0 \end{cases}, \ t \geq 0. \tag{1.10}$$

The ODE characterizing ψ_u is autonomous. This is a Riccati differential equation that can be solved explicitly. In fact, $g = 1/\psi_u$ solves the linear differential equation $g' - kg + \frac{\sigma^2}{2} = 0$, which gives $g(t) = e^{kt}(g(0) - \frac{\sigma^2}{2}\zeta_k(t))$. Since $\psi_u(0) = u$, we get

$$\psi_u(t) = \frac{e^{-kt}}{\frac{1}{u} - \frac{\sigma^2}{2}\zeta_k(t)} = \frac{ue^{-kt}}{1 - \frac{\sigma^2}{2}u\zeta_k(t)}. \tag{1.11}$$

We remark that the denominator is positive and greater than one for any $t \geq 0$ since $u \leq 0$ and $\zeta_k(t) \geq 0$. This gives in particular that $\psi_u(t) \leq 0$. Since $\zeta_k'(t) = e^{-kt}$ and $\phi_u(0) = 0$, we easily obtain from (1.10)

$$\phi_u(t) = -\frac{2a}{\sigma^2}\log\left(1 - \frac{\sigma^2}{2}u\zeta_k(t)\right). \tag{1.12}$$

This leads to the formula at the right-hand-side of (1.9). However, it still remains to prove the equality. Up to now, we have postulated a special form for the characteristic function and then derived necessary conditions on it. The solution that we have found could then be different from the true characteristic function. This is not the case thanks to a Feynman-Kac type argument. In fact, the functions ψ_u and ϕ_u defined by (1.11) and (1.12) solve (1.10). Thus, Itô's formula applied to $M_t = \exp(\phi_u(T-t) + \psi_u(T-t)X_t^x)$ gives

$$\exp(uX_T^x) = \exp(\phi_u(T) + \psi_u(T)x) + \int_0^T \psi_u(T-t)M_t\sigma\sqrt{X_t^x}dW_t.$$

Since $\phi_u(t) \leq 0$ and $\psi_u(t) \leq 0$, we have $0 \leq M_t \leq 1$, and the stochastic integral has a null expectation, which gives (1.9).

It remains to determine the set of convergence of the characteristic function. On the one hand, X_t^x is a nonnegative random variable which gives that $\{u \in \mathbb{R}, \mathbb{E}[\exp(uX_t^x)] < \infty\}$ is an interval containing the negative real numbers. Besides, it is well-known that the characteristic function is analytic on the interior of its set of convergence (see Widder [123], p. 240 or Filipović [53], Lemma 10.8). On the other hand, the left-hand-side of (1.9) is an analytic function for $u \in (-\infty, \frac{2}{\sigma^2\zeta_k(t)})$. Since both functions coincides on $\mathbb{R}_-$, we must have (1.9) for $u \in (-\infty, \frac{2}{\sigma^2\zeta_k(t)})$. Then, the monotone convergence theorem gives when $u \to \frac{2}{\sigma^2\zeta_k(t)}$ that

$$\mathbb{E}\left[\exp\left(\frac{2}{\sigma^2\zeta_k(t)}X_t^x\right)\right] = \infty.$$

This gives $\mathbb{E}[|\exp(uX_t^x)|] = \mathbb{E}[\exp(\Re(u)X_t^x)] = \infty$ if, and only if $\Re(u) \geq \frac{2}{\sigma^2\zeta_k(t)}$. Last, we use again that both sides of (1.9) are analytic functions coinciding on $(-\infty, \frac{2}{\sigma^2\zeta_k(t)})$ to conclude that (1.9) holds when $\Re(u) < \frac{2}{\sigma^2\zeta_k(t)}$. □

This proof contains all the key arguments to obtain the characteristic function for more general affine process. First, we postulate that the characteristic function has the following structure $\mathbb{E}[\exp(uX_t^x)] = \exp(\phi_u(t) + \psi_u(t)x)$. Writing that $\mathbb{E}[\exp(uX_T^x)|\mathcal{F}_t]$ should be a martingale, we then derive necessary conditions on ϕ_u and ψ_u that lead to a system of ODEs. In the best case, this system can be solved explicitly. Otherwise, ϕ_u and ψ_u are in general still unique by the Cauchy-Lipschitz theorem. In both cases, one has to be careful about possible explosion in finite time, which is obviously easier to analyze with explicit formulas. The last step consists in checking that the solution obtained is indeed the characteristic function, using a Feynman-Kac type argument as above. We propose here two exercises that allow to repeat and practice these arguments.

Exercise 1.2.6 This exercise proposes to obtain the characteristic function of the Ornstein-Uhlenbeck (1.1) process by using the same arguments as the proof of Proposition (1.2.4).

1. Let $u \in \mathbb{C}$ and assume that we can write $\mathbb{E}[\exp(uX_t^x)] = \exp(\phi_u(t) + \psi_u(t)x)$. Show that we necessarily have

$$\begin{cases} -\phi_u'(t) + a\psi_u(t) + \frac{\sigma^2}{2}\psi_u(t)^2 = 0 \\ -\psi_u'(t) - k\psi_u(t) = 0 \end{cases}, \ t \geq 0. \tag{1.13}$$

Then, deduce that $\psi_u(t) = e^{-kt}u$ and $\phi_u(t) = a\zeta_k(t)u + \frac{\sigma^2 u^2}{2}\zeta_{2k}(t)$.
2. Let $T > 0$. For $t \in [0, T]$, we define $M_t = \exp(\phi_u(T-t) + \psi_u(T-t)X_t^x)$. Show that

$$\exp(uX_T^x) = \exp(\phi_u(T) + \psi_u(T)x) + \int_0^T \sigma\psi_u(T-t)M_t dW_t,$$

$\mathbb{E}\left[\int_0^T \psi_u(T-t)^2 M_t^2 dt\right] < \infty$ and deduce that (1.2) hold.

Exercise 1.2.7 Let X^x be the CIR process defined by (1.5). The goal of this exercise is to calculate the characteristic function of $(X_t^x, \int_0^t X_s^x ds)$. This result will be useful later on for the pricing of some options in the CIR model, see Sect. 1.4.3. It is also related to the calculation of the stock price law in the Heston model in Sect. 4.2.1. Let $u, v \leq 0$ and assume that we can write

$$\mathbb{E}\left[\exp\left(uX_t^x + v\int_0^t X_s^x ds\right)\right] = \exp(\phi_{u,v}(t) + \psi_{u,v}(t)x)$$

for some smooth functions $\phi_{u,v}$ and $\psi_{u,v}$.

1. Let $T > 0$ and, for $t \in [0, T]$, $M_t = \mathbb{E}\left[\exp\left(uX_T^x + v\int_0^T X_s^x ds\right)|\mathcal{F}_t\right]$. Show that

$$M_t = \exp\left(v\int_0^t X_s^x ds\right)\exp(\phi_{u,v}(T-t) + \psi_{u,v}(T-t)X_t^x).$$

Deduce that $\phi_{u,v}$ and $\psi_{u,v}$ should then solve

$$\begin{cases} -\phi_{u,v}'(t) + a\psi_{u,v}(t) = 0 \\ -\psi_{u,v}'(t) - k\psi_{u,v}(t) + \frac{\sigma^2}{2}\psi_{u,v}(t)^2 + v = 0 \end{cases}, \ t \geq 0, \tag{1.14}$$

with $\phi_{u,v}(0) = 0$ and $\psi_{u,v}(0) = u$.
2. Let $\gamma_v = \sqrt{k^2 - 2\sigma^2 v}$ and $\psi_0 = \frac{k+\gamma_v}{\sigma^2}$ be the nonnegative root of

$$-k\psi_0 + \frac{\sigma^2}{2}\psi_0^2 + v = 0.$$

Show that $\tilde{\psi}(t) = \psi_{u,v}(t) - \psi_0$ solves $-\tilde{\psi}'(t) + \gamma_v \tilde{\psi}(t) + \frac{\sigma^2}{2}\tilde{\psi}(t)^2 = 0$ with $\tilde{\psi}(0) = u - \psi_0$. Deduce that

$$\psi_{u,v}(t) = \psi_0 + \frac{(u-\psi_0)e^{\gamma_v t}}{1 - \frac{\sigma^2}{2}(u-\psi_0)\zeta_{-\gamma_v}(t)}, \ \phi_{u,v}(t)$$
$$= a\psi_0 t - \frac{2a}{\sigma^2}\log\left(1 - \frac{\sigma^2}{2}(u-\psi_0)\zeta_{-\gamma_v}(t)\right).$$

3. Check that we indeed have

$$\mathbb{E}\left[\exp\left(uX_t^x + v\int_0^t X_s^x ds\right)\right] \tag{1.15}$$
$$= \left(\frac{e^{\frac{\gamma_v + k}{2}t}}{1 - \frac{\sigma^2}{2}(u-\psi_0)\zeta_{-\gamma_v}(t)}\right)^{\frac{2a}{\sigma^2}} \exp\left(x\left[\psi_0 + \frac{(u-\psi_0)e^{\gamma_v t}}{1 - \frac{\sigma^2}{2}(u-\psi_0)\zeta_{-\gamma_v}(t)}\right]\right).$$

Remark 1.2.8 For the Cox-Ingersoll-Ross process, it is even possible to derive the joint distribution of $(X_t^x, \int_0^t X_s^x ds, \int_0^t 1/X_s^x ds)$. As explained by Hurd and Kuznetsov [77], Craddock and Lennox [33] or in the recent book of Baldeaux and Platen [14], the Laplace transform of this triplet can be given in closed form by the means of confluent hypergeometric functions.

We want to characterize the set of convergence of the characteristic function (1.15):

$$\mathcal{D}_t = \left\{(u,v) \in \mathbb{R},\ \mathbb{E}\left[\exp\left(uX_t^x + v\int_0^t X_s^x ds\right)\right] < \infty\right\}.$$

A priori, this set could also depend on the initial value x. We will see that this is not the case. It is a well known result that the set of convergence of the characteristic function is convex. This is a consequence of Hölder's inequality: if $\lambda \in (0,1)$ and $(u_1, v_1), (u_2, v_2) \in \mathcal{D}_t$, we have

$$\mathbb{E}\left[e^{\lambda\left(u_1 X_t^x + v_1\int_0^t X_s^x ds\right)} e^{(1-\lambda)\left(u_2 X_t^x + v_2 \int_0^t X_s^x ds\right)}\right]$$
$$\leq \mathbb{E}\left[e^{u_1 X_t^x + v_1 \int_0^t X_s^x ds}\right]^{\frac{1}{\lambda}} \mathbb{E}\left[e^{u_2 X_t^x + v_2 \int_0^t X_s^x ds}\right]^{\frac{1}{1-\lambda}} < \infty.$$

By convention, we define $\sqrt{z} = \sqrt{\rho}e^{i\theta/2}$ for $z = \rho e^{i\theta} \in \mathbb{C}$ with $\rho \geq 0$ and $\theta \in (-\pi, \pi]$, and then have $\gamma_v = i\sqrt{2\sigma^2 v - k^2}$ when $v > \frac{k^2}{2\sigma^2}$. By simple calculations, we get the following identities:

$$e^{-\frac{\gamma_v t}{2}}\left[1-\frac{\sigma^2}{2}(u-\psi_0)\zeta_{-\gamma_v}(t)\right]=\cosh\left(\gamma_v\frac{t}{2}\right)-(\sigma^2u-k)\frac{\sinh\left(\gamma_v\frac{t}{2}\right)}{\gamma_v}, \tag{1.16}$$

$$\psi_{u,v}(t)=\frac{u\cosh\left(\gamma_v\frac{t}{2}\right)+(2v-ku)\frac{\sinh\left(\gamma_v\frac{t}{2}\right)}{\gamma_v}}{\cosh\left(\gamma_v\frac{t}{2}\right)-(\sigma^2u-k)\frac{\sinh\left(\gamma_v\frac{t}{2}\right)}{\gamma_v}}. \tag{1.17}$$

We now observe that

$$\begin{aligned}\cosh\left(\gamma_v\frac{t}{2}\right)&=\sum_{j=0}^{\infty}\frac{1}{(2j)!}\left(\frac{t}{2}\right)^{2j}(k^2-2\sigma^2v)^j,\\ \frac{\sinh\left(\gamma_v\frac{t}{2}\right)}{\gamma_v}&=\sum_{j=0}^{\infty}\frac{1}{(2j+1)!}\left(\frac{t}{2}\right)^{2j+1}(k^2-2\sigma^2v)^j,\end{aligned} \tag{1.18}$$

and deduce that the right-hand side of (1.15) is an analytic function with respect to (u,v) on $\{(u,v)\in\mathbb{R},\ s.t.\ \cosh\left(\gamma_v\frac{t}{2}\right)-(\sigma^2u-k)\frac{\sinh\left(\gamma_v\frac{t}{2}\right)}{\gamma_v}\neq 0\}$. Since the characteristic function is analytic on the interior of $\mathcal{D}_t$ and (1.15) holds for $u,v\le 0$, we get by analytic continuation that

$$\mathcal{D}_t=\left\{(u,v)\in\mathbb{R},\ s.t.\ \forall\alpha\in[0,1],\cosh\left(\gamma_{\alpha v}\frac{t}{2}\right)-(\sigma^2\alpha u-k)\frac{\sinh\left(\gamma_{\alpha v}\frac{t}{2}\right)}{\gamma_{\alpha v}}>0\right\}, \tag{1.19}$$

and that (1.15) holds on $\mathcal{D}_t$. We use here that $(0,0)$ is in the interior of $\mathcal{D}_t$ and the convexity of $\mathcal{D}_t$, so that $(u,v)\in\mathcal{D}_t$ if, and only if $(\alpha u,\alpha v)\in\mathcal{D}_t$ for any $\alpha\in[0,1]$. However, the choice of $(0,0)$ is arbitrary and we could have chosen any other point in the interior of $\mathcal{D}_t$. The following proposition gives a more convenient characterization of $\mathcal{D}_t$.

Proposition 1.2.9 *The set of convergence of* (1.15) *is given by*

$$\begin{aligned}\mathcal{D}_t=&\left\{(u,v)\in\mathbb{R},\ s.t.\ v\le\frac{k^2}{2\sigma^2},\frac{2}{\zeta_{-\gamma_v}(t)}>\sigma^2u-(k+\gamma_v)\right\}\\ &\bigcup\left\{(u,v)\in\mathbb{R},\ s.t.\ v>\frac{k^2}{2\sigma^2},\forall s\in[0,t],\right.\\ &\left.\cos\left(|\gamma_v|\frac{s}{2}\right)-\frac{\sigma^2u-k}{|\gamma_v|}\sin\left(|\gamma_v|\frac{s}{2}\right)>0\right\}.\end{aligned} \tag{1.20}$$

Proof We give two proofs of this result, the first one is based on an analytic continuation argument while the second one relies more on the affine property and the explosion of (1.14).

We remark that $(\bar{u}, \bar{v}) = \left(\frac{k}{\sigma^2}, \frac{k^2}{\sigma^2}\right) \in \mathcal{D}_t$. Let $\alpha \in [0, 1]$. We have $\gamma_{\alpha\bar{v}} = |k|\sqrt{1-\alpha}$ and

$$\cosh\left(\gamma_{\alpha\bar{v}}\frac{t}{2}\right) - (\sigma^2\alpha\bar{u} - k)\frac{\sinh\left(\gamma_{\alpha\bar{v}}\frac{t}{2}\right)}{\gamma_{\alpha\bar{v}}} = \cosh\left(\frac{t}{2}|k|\sqrt{1-\alpha}\right) + \frac{k}{|k|}\sqrt{1-\alpha}\sinh\left(\frac{t}{2}|k|\sqrt{1-\alpha}\right) > 0,$$

which gives $(\bar{u}, \bar{v}) \in \mathcal{D}_t$ from (1.19). Since $\mathcal{D}_t$ is convex, we know that $(u, v) \in \mathcal{D}_t$ if, and only if $(\alpha u + (1-\alpha)\bar{u}, \alpha v + (1-\alpha)\bar{v}) \in \mathcal{D}_t$ for any $\alpha \in [0, 1]$. We have $\gamma_{\alpha v+(1-\alpha)\bar{v}} = \sqrt{\alpha}\gamma_v$ and $\sigma^2(\alpha u + (1-\alpha)\bar{u}) - k = \alpha(\sigma^2 u - k)$, and we get by using the same analytic continuation argument that

$$\begin{aligned}\mathcal{D}_t = &\Bigg\{(u, v) \in \mathbb{R},\ s.t.\ \forall\alpha \in [0, 1], \\ &\quad \cosh\left(\gamma_v\frac{\sqrt{\alpha}t}{2}\right) - \sqrt{\alpha}(\sigma^2 u - k)\frac{\sinh\left(\gamma_v\frac{\sqrt{\alpha}t}{2}\right)}{\gamma_v} > 0\Bigg\} \\ = &\left\{(u, v) \in \mathbb{R},\ s.t.\ \forall s \in [0, t], \cosh\left(\gamma_v\frac{s}{2}\right) - \frac{s}{t}(\sigma^2 u - k)\frac{\sinh\left(\gamma_v\frac{s}{2}\right)}{\gamma_v} > 0\right\}.\end{aligned}$$

It still remains to prove that this set is the same as (1.20). In the case $v \le \frac{k^2}{2\sigma^2}$, we have:

$$\forall s \in (0, t],\ \coth\left(\gamma_v\frac{s}{2}\right) > \frac{s}{t}\frac{\sigma^2 u - k}{\gamma_v} \iff \coth\left(\gamma_v\frac{t}{2}\right) > \frac{\sigma^2 u - k}{\gamma_v}.$$

This is obvious when $\sigma^2 u - k \le 0$ since the hyperbolic cotangent is positive. When $\sigma^2 u - k > 0$, this is true since the left hand side is decreasing and the right hand side increasing with respect to s. When $v > \frac{k^2}{2\sigma^2}$, we have

$$\cosh\left(\gamma_v\frac{s}{2}\right) - \frac{s}{t}(\sigma^2 u - k)\frac{\sinh\left(\gamma_v\frac{s}{2}\right)}{\gamma_v} = \cos\left(|\gamma_v|\frac{s}{2}\right) - \frac{s}{t}(\sigma^2 u - k)\frac{\sin\left(|\gamma_v|\frac{s}{2}\right)}{|\gamma_v|},$$

and this function is clearly negative for $s = 2\pi/|\gamma_v|$. Thus, the condition can hold only if $|\gamma_v|\frac{t}{2} < \pi$, which we consider now. We have

$$\forall s \in (0, t],\ \cot\left(|\gamma_v|\frac{s}{2}\right) > \frac{s}{t}\frac{\sigma^2 u - k}{|\gamma_v|} \iff \forall s \in (0, t], \cot\left(|\gamma_v|\frac{s}{2}\right) > \frac{\sigma^2 u - k}{|\gamma_v|}.$$

This equivalence is obvious when $\sigma^2 u - k > 0$ since, once again, the left hand side is decreasing and the right hand side increasing with respect to s. When $\sigma^2 u - k \leq 0$, only the implication "$\Longleftarrow$" has to be proved. The inequality is obvious when $s \in (0, \pi/|\gamma_v|)$. When $s \in [\pi/|\gamma_v|, t)$ and $t > \pi/|\gamma_v|$, we have $s = \frac{t-s}{t-\pi/|\gamma_v|}\frac{\pi}{|\gamma_v|} + \frac{s-\pi/|\gamma_v|}{t-\pi/|\gamma_v|}t$ and we use the convexity of $-\cot(x)$ on $x \in [\pi/2, \pi)$ to get

$$\cot\left(|\gamma_v|\frac{s}{2}\right) \geq \frac{s-\pi/|\gamma_v|}{t-\pi/|\gamma_v|}\cot\left(|\gamma_v|\frac{t}{2}\right) \geq \frac{s}{t}\cot\left(|\gamma_v|\frac{t}{2}\right) > \frac{s}{t}\frac{\sigma^2 u - k}{|\gamma_v|}.$$

Let us now give a second proof and define for $t > 0$ and $s \in [0, t]$,

$$M_s = \left(\frac{e^{k(t-s)/2}}{\cosh\left(\gamma_v \frac{t-s}{2}\right) - (\sigma^2 u - k)\frac{\sinh\left(\gamma_v \frac{t-s}{2}\right)}{\gamma_v}}\right)^{\frac{2a}{\sigma^2}} \times$$
$$\exp\left(X_s^x \frac{u\cosh\left(\gamma_v \frac{t-s}{2}\right) + (2v - ku)\frac{\sinh\left(\gamma_v \frac{t-s}{2}\right)}{\gamma_v}}{\cosh\left(\gamma_v \frac{t-s}{2}\right) - (\sigma^2 u - k)\frac{\sinh\left(\gamma_v \frac{t-s}{2}\right)}{\gamma_v}} + v\int_0^s X_{s'}^x ds'\right).$$

We set

$$\tilde{\mathcal{D}}_t = \left\{(u, v) \in \mathbb{R},\ s.t.\ \forall s \in [0, t], \cosh\left(\gamma_v \frac{s}{2}\right) - (\sigma^2 u - k)\frac{\sinh\left(\gamma_v \frac{s}{2}\right)}{\gamma_v} > 0\right\}.$$

By Itô calculus, we check that for $(u, v) \in \tilde{\mathcal{D}}_t$, M is a positive local martingale and thus a supermartingale. We get

$$\mathbb{E}[M_T] = \mathbb{E}\left[\exp\left(uX_t^x + v\int_0^t X_s^x ds\right)\right] \leq M_0 < \infty,$$

and thus $\tilde{\mathcal{D}}_t \subset \mathcal{D}_t$. By the analytic continuation argument, formula (1.15) necessarily holds on $\tilde{\mathcal{D}}_t$. Let us consider now $(u, v) \notin \tilde{\mathcal{D}}_t$ and define

$$t_{\exp} = \inf\left\{s \geq 0, \cosh\left(\gamma_v \frac{s}{2}\right) - (\sigma^2 u - k)\frac{\sinh\left(\gamma_v \frac{s}{2}\right)}{\gamma_v} = 0\right\} \in (0, t].$$

By continuity, the function $s \mapsto \cosh\left(\gamma_v \frac{s}{2}\right) - (\sigma^2 u - k)\frac{\sinh\left(\gamma_v \frac{s}{2}\right)}{\gamma_v}$ is positive for $s < t_{\exp}$, and its derivative is necessarily negative for an increasing sequence (s_n) such that $s_n \underset{n\to+\infty}{\to} t_{\exp}$, and then

$$(k^2 - 2\sigma^2 v)\frac{\sinh\left(\gamma_v \frac{s_n}{2}\right)}{\gamma_v} - (\sigma^2 u - k)\cosh\left(\gamma_v \frac{s_n}{2}\right) < 0.$$

Since $(u, v) \in \tilde{\mathcal{D}}_{s_n}$, the tower property of the conditional expectation gives

$$\mathbb{E}\left[\exp\left(uX_t^x + v\int_0^t X_s^x ds\right)\right] = \mathbb{E}\left[\mathbb{E}\left[\exp\left(uX_t^x + v\int_0^t X_s^x ds\right)\Big|\mathcal{F}_{t-s_n}\right]\right]$$

$$= \mathbb{E}\left[\left(\frac{e^{ks_n/2}}{\cosh\left(\gamma_v \frac{s_n}{2}\right) - (\sigma^2 u - k)\frac{\sinh\left(\gamma_v \frac{s_n}{2}\right)}{\gamma_v}}\right)^{\frac{2a}{\sigma^2}} \times \right.$$

$$\left. \exp\left(X_{t-s_n}^x \frac{u\cosh\left(\gamma_v \frac{s_n}{2}\right) + (2v - ku)\frac{\sinh\left(\gamma_v \frac{s_n}{2}\right)}{\gamma_v}}{\cosh\left(\gamma_v \frac{s_n}{2}\right) - (\sigma^2 u - k)\frac{\sinh\left(\gamma_v \frac{s_n}{2}\right)}{\gamma_v}} + v\int_0^{t-s_n} X_s^x ds\right)\right].$$

We now observe that

$$u\cosh\left(\gamma_v \frac{s_n}{2}\right) + (2v - ku)\frac{\sinh\left(\gamma_v \frac{s_n}{2}\right)}{\gamma_v} = \frac{k}{\sigma^2}\left[\cosh\left(\gamma_v \frac{s_n}{2}\right) - (\sigma^2 u - k)\frac{\sinh\left(\gamma_v \frac{s_n}{2}\right)}{\gamma_v}\right]$$

$$-\frac{1}{\sigma^2}\left[(k^2 - 2\sigma^2 v)\frac{\sinh\left(\gamma_v \frac{s_n}{2}\right)}{\gamma_v} - (\sigma^2 u - k)\cosh\left(\gamma_v \frac{s_n}{2}\right)\right] > 0,$$

and the exponential is thus greater than 1. Letting $n \to +\infty$, we get by Fatou's lemma that $\mathbb{E}\left[\exp\left(uX_t^x + v\int_0^t X_s^x ds\right)\right] = +\infty$ and thus $\mathcal{D}_t = \tilde{\mathcal{D}}_t$. When $v \leq \frac{k^2}{2\sigma^2}$, we have

$$\forall s \in (0, t],\ \coth\left(\gamma_v \frac{s}{2}\right) > \frac{\sigma^2 u - k}{\gamma_v} \iff \coth\left(\gamma_v \frac{t}{2}\right) > \frac{\sigma^2 u - k}{\gamma_v},$$

so that $\tilde{\mathcal{D}}_t$ is well the set given by (1.20). □

Let us now make some comments on Proposition 1.2.9. A remarkable property is that the set of convergence does not depend on the initial value x. This is a standard fact for affine processes. As we observe from the second proof, the set of convergence is characterized by the explosion time of the ODE (1.14) that determines the characteristic function. Here, this ODE only depends on (u, v) and the CIR parameters. From the first proof, we also see that using only the analytic continuation argument is less convenient than in dimension one. In dimension one, there is only one possible direction to extend the function. In larger dimension, the set of convergence is still convex, but different choices of direction are possible, which leads to different but equivalent ways to write the set of convergence. This is why one should prefer the second proof, and keep in mind the two following facts.

- Suppose that one has on some set an explicit formula for a characteristic function. Suppose that this formula is an analytic function, and well defined outside this set. Then, the only possible value for the characteristic function outside this set, if it is defined, is still given by the same formula.

- For an affine diffusion, the characteristic function will always be determined by a Riccati differential equation as (1.14). Its set of convergence can be determined by an analysis of the explosion time of this differential equation.

Remark 1.2.10 Clearly, $\exp\left(uX_t^x + v\int_0^t X_s^x ds\right)$ is integrable if, and only if $(u, v) \in \bar{\mathcal{D}}_t$ where $\bar{\mathcal{D}}_t = \{(u, v) \in \mathbb{C},\ (\Re(u), \Re(v)) \in \mathcal{D}_t\}$. Thus, it would be tempting to extend formula (1.15) to $\bar{\mathcal{D}}_t$. Unfortunately, the left hand side of (1.15) is not an analytic function on $\bar{\mathcal{D}}_t$. From (1.17) and (1.18), we get that $\psi_{u,v}(t)$ is an analytic function of (u, v). From (1.16) and (1.18), we get

$$\phi_{u,v}(t) = \frac{ak}{\sigma^2}t - \frac{2a}{\sigma^2}\log\left(e^{-\frac{\gamma_v t}{2}}\left[1 - \frac{\sigma^2}{2}(u - \psi_0)\zeta_{-\gamma_v}(t)\right]\right), \tag{1.21}$$

and the argument in the logarithm is an analytic function. However, the complex logarithm is usually defined by $\log(\rho e^{i\theta}) = \log(\rho) + i\theta$ for $\rho > 0$ and $\theta \in (-\pi, \pi]$. It is thus an analytic function on $\mathbb{C}\setminus\mathbb{R}_-$, but is discontinuous in $\mathbb{R}_-$. More precisely, it is analytic modulo $2i\pi$ and formula (1.21) is correct, up to a multiple of $2i\pi \times \frac{2a}{\sigma^2}$, to get $\mathbb{E}\left[\exp\left(uX_t^x + v\int_0^t X_s^x ds\right)\right] = \exp(\phi_{u,v}(t) + \psi_{u,v}(t)x)$. In particular formula (1.15) is correct for $(u, v) \in \mathbb{C}$ such that $(\Re(u), \Re(v)) \in \mathcal{D}_t$ when $\frac{2a}{\sigma^2} \in \mathbb{N}$. Otherwise, one has to get back to (1.14) and use the formula

$$\phi_{u,v}(t) = a\int_0^t \psi_{u,v}(s)ds, \tag{1.22}$$

which is analytic with respect to (u, v). In practice, we can either compute (1.22) by numerical integration or calculate its value from (1.21) by counting how many times the function $s \in [0, t] \mapsto e^{-\frac{\gamma_v s}{2}}\left[1 - \frac{\sigma^2}{2}(u - \psi_0)\zeta_{-\gamma_v}(s)\right]$ crosses $\mathbb{R}_-^*$, see Lord and Kahl [99]. This point will be discussed later on for the Heston model in Sect. 4.2.1.

The Probability Density Function of X_t^x

In fact, the characteristic function obtained in (1.9) is well-known and is the one of a non-centered chi-square distribution, up to a multiplicative constant. More precisely, the chi-square distribution with $\nu > 0$ degrees of freedom and noncentrality parameter $d \geq 0$ has the following density

$$\sum_{i=0}^{\infty} \frac{e^{-d/2}(d/2)^i}{i!} \frac{1/2}{\Gamma(i+\nu)}\left(\frac{z}{2}\right)^{i+\nu-1} e^{-z/2},\ z > 0,$$

where

$$\Gamma(\alpha) = \int_0^\infty z^{\alpha-1}e^{-z}dz,\ \alpha > 0,$$

is the Gamma Euler function and satisfies $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. We have the following result.

Proposition 1.2.11 *Let X^x be the CIR process defined by (1.5) and $t > 0$.*
When $a > 0$, the density of X_t^x is given by

$$p(t, x, z) = \sum_{i=0}^{\infty} \frac{e^{-d_t x/2}(d_t x/2)^i}{i!} \frac{c_t/2}{\Gamma(i + \frac{2a}{\sigma^2})} \left(\frac{c_t z}{2}\right)^{i-1+\frac{2a}{\sigma^2}} e^{-c_t z/2},\ z > 0 \tag{1.23}$$

where $c_t = \frac{4}{\sigma^2 \zeta_k(t)}$ and $d_t = c_t e^{-kt}$. Thus $c_t X_t$ follows a chi-square law with degree $\frac{2a}{\sigma^2}$ and noncentrality $d_t x$.
When $a = 0$, X_t^x is distributed according to the probability measure:

$$e^{-d_t x/2}\delta_0(dz) + \sum_{i=1}^{\infty} \frac{e^{-d_t x/2}(d_t x/2)^i}{i!} \frac{c_t/2}{\Gamma(i)} \left(\frac{c_t z}{2}\right)^{i-1} e^{-c_t z/2}dz,\ z \geq 0,$$

where $\delta_0(dz)$ denote the Dirac mass at 0 and dz the Lebesgue measure.

Proof We first consider the case $a > 0$. Let $u \leq 0$. By a simple change of variable, we have for any $i \in \mathbb{N}$

$$\int_0^\infty z^{i-1+\frac{2a}{\sigma^2}} e^{-(c_t/2-u)z}dz = \frac{1}{(c_t/2-u)^{i+\frac{2a}{\sigma^2}}} \Gamma\left(i + \frac{2a}{\sigma^2}\right),$$

since $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1}e^{-z}dz$ for $\alpha > 0$. Then, we obtain

$$\begin{aligned}
\int_0^\infty p(t, x, z)e^{uz}dz &= \sum_{i=0}^{\infty} \frac{e^{-d_t x/2}(d_t x/2)^i}{i!} \left(\frac{c_t}{c_t - 2u}\right)^{i+\frac{2a}{\sigma^2}} \\
&= \left(\frac{c_t}{c_t - 2u}\right)^{\frac{2a}{\sigma^2}} \exp\left[\frac{d_t}{2}x\left(\frac{c_t}{c_t - 2u} - 1\right)\right] \\
&= \left(\frac{c_t}{c_t - 2u}\right)^{\frac{2a}{\sigma^2}} \exp\left[uxe^{-kt}\left(\frac{c_t}{c_t - 2u}\right)\right],
\end{aligned}$$

and we get back (1.9) since $\frac{c_t}{c_t-2u} = \frac{1}{1-\frac{\sigma^2}{2}u\zeta_k(t)}$.

For $a = 0$, we observe that $\int_0^\infty e^{-d_t x/2}e^{uz}\delta_0(dz) = e^{-d_t x/2}$. We can therefore repeat exactly the same calculation. □

Remark 1.2.12 For sake of consistency, we have used in Proposition 1.2.11 the parametrization coming from the CIR process. Let Y be a random variable distributed according to a chi-square distribution with $\nu > 0$ degrees of freedom and noncentrality parameter $d \geq 0$. Then, the same calculation gives

$$\mathbb{E}\left[\exp\left(u\frac{Y}{c}\right)\right] = \left(\frac{1}{1-2u/c}\right)^{\nu} \exp\left[d\frac{u/c}{1-2u/c}\right],$$

for $u \leq 0$ and even $u < c/2$.

Last, as for the Ornstein-Uhlenbeck process, we see from the characteristic function (1.9) or the density (1.23) that the process is ergodic when $k > 0$. In fact, we have in this case

$$\lim_{t\to+\infty} \mathbb{E}[\exp(uX_t^x)] = \left(1 - \frac{\sigma^2}{2k}u\right)^{-\frac{2a}{\sigma^2}}, \tag{1.24}$$

For $a \neq 0$, this is the characteristic function of the Gamma law with density

$$\mathbb{1}_{z>0}\frac{2k}{\sigma^2\Gamma(\frac{2a}{\sigma^2})}\left(\frac{2kz}{\sigma^2}\right)^{\frac{2a}{\sigma^2}-1} e^{-\frac{2kz}{\sigma^2}},$$

while for $a = 0$, this is the Dirac mass in 0.

1.2.3 A Nice Connection Between Ornstein-Uhlenbeck and Cox-Ingersoll-Ross Processes

Let $x \geq 0$, $k \in \mathbb{R}$, $\sigma > 0$, and $p \in \mathbb{N}^*$ independent standard Brownian motions $W^1, \dots, W^p$. We define the Ornstein-Uhlenbeck processes

$$dY_t^i = -\frac{k}{2}Y_t^i dt + \frac{\sigma}{2}dW_t^i, Y_0^i = \sqrt{x/p}, \text{ for } 1 \leq i \leq p.$$

We set $X_t = \sum_{i=1}^p (Y_t^i)^2$ and have by Itô's formula:

$$dX_t = \left(p\frac{\sigma^2}{4} - kX_t\right)dt + \sigma\sum_{i=1}^p Y_t^i dW_t^i. \tag{1.25}$$

Since $\langle \sum_{i=1}^p Y_t^i dW_t^i \rangle = X_t dt$, we know by Theorem 4.2, p. 170 of [83] that there is a real Brownian motion W such that $\sum_{i=1}^p Y_t^i dW_t^i = \sqrt{X_t}dW_t$. Thus, X is a Cox-Ingersoll-Ross process starting from x with parameters $a = p\frac{\sigma^2}{4}$, k and σ.

We note that we can easily calculate the characteristic function of X_t in this case. In fact, easy calculations give that for $Y \sim \mathcal{N}(m, \varsigma^2)$, $u < 1/(2\varsigma^2)$,

$$\mathbb{E}[\exp(uY^2)] = \int_{\mathbb{R}} \frac{e^{-\frac{1}{2}y^2(1-2u\varsigma^2)}}{\sqrt{2\pi}} e^{2um\varsigma y + um^2} dy$$
$$= \frac{e^{um^2}}{\sqrt{1-2u\varsigma^2}} \int_{\mathbb{R}} \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} e^{\frac{2um\varsigma}{\sqrt{1-2u\varsigma^2}} y} dy = \frac{1}{\sqrt{1-2u\varsigma^2}} e^{\frac{um^2}{1-2u\varsigma^2}}.$$

From (1.2) we have $Y_t^1 \sim \mathcal{N}\left(e^{-kt/2}\sqrt{x/p}, (\sigma/2)^2 \zeta_k(t)\right)$ and thus for $u < 2/[\sigma^2 \zeta_k(t)]$,

$$\mathbb{E}[\exp(uX_t)] = \mathbb{E}\left[\exp\left(u(Y_t^1)^2\right)\right]^p = \frac{1}{\left(1 - u\frac{\sigma^2}{2}\zeta_k(t)\right)^{p/2}} \exp\left(\frac{e^{-kt}x}{1 - u\frac{\sigma^2}{2}\zeta_k(t)}\right), \tag{1.26}$$

since $Y_t^1, \ldots, Y_t^p$ are independent and identically distributed. We get back (1.9) for the particular case $a = p\frac{\sigma^2}{4}$.

Exercise 1.2.13 Let $x_1, x_2 \geq 0, a_1, a_2 \geq 0, k \in \mathbb{R}$ and $\sigma > 0$. We consider the CIR processes

$$X_t^i = x_i + \int_0^t (a_i - kX_s^i)ds + \int_0^t \sigma \sqrt{X_s^i} dW_s^i, \ i \in \{1, 2\},$$

where W^1 and W^2 are two independent Brownian motions.

1. Let $X_t = X_t^1 + X_t^2$, for $t \geq 0$. Show that there exists a standard Brownian motion W such that

$$X_t = (x_1 + x_2) + \int_0^t ((a_1 + a_2) - kX_s)ds + \int_0^t \sigma \sqrt{X_s} dW_s.$$

2. We now assume $a_1 = a_2 = a$ and $x_1 = x_2 = x$ for some $a, x \geq 0$, and consider $m > x$. Using the convention $\inf \emptyset = +\infty$, we define the stopping times

$$\tau_{0,m}^i = \inf\{t \geq 0, X_t^i = 0 \text{ or } X_t^i \geq m\}, i \in \{1, 2\},$$
$$\tau_{0,2m} = \inf\{t \geq 0, X_t = 0 \text{ or } X_t \geq 2m\}.$$

Show that $\tau_{0,2m} \geq \tau_{0,m}^1 \wedge \tau_{0,m}^2$. Observing that $\tau_{0,m}^1$ and $\tau_{0,m}^2$ are independent and identically distributed, show that

$$\mathbb{P}(\tau_{0,2m} < +\infty) = 1 \implies \mathbb{P}(\tau_{0,m}^1 < +\infty) = 1. \tag{1.27}$$

Exercise 1.2.14 We consider the CIR processes

$$\begin{aligned} X_t^x &= x + \int_0^t (a - kX_s^x)ds + \int_0^t \sigma\sqrt{X_s^x}dW_s, \\ \tilde{X}_t^x &= x + at + \int_0^t \sigma\sqrt{\tilde{X}_s^x}dW_s, \ x \in \mathbb{R}_+, t \geq 0. \end{aligned}$$

The goal of this exercise is to prove the well-known identity $(e^{-kt}\tilde{X}^x_{\zeta_{-k}(t)})_{t\geq 0} \overset{law}{=} (X_t^x)_{t\geq 0}$. We recall that $\zeta_{-k}(t) = \frac{e^{kt}-1}{k}$ is a nondecreasing function and can be seen here as a change of time: $\tilde{X}^x_{\zeta_{-k}(t)}$ is the value of the process $\tilde{X}^x$ at time $\zeta_{-k}(t)$.

1. We set $M_t = W_{\zeta_{-k}(t)}$. By using the Dambis-Dubins-Schwarz theorem (see [83], Theorem 4.6 and Proposition 4.8, pp. 174–176), show that

$$\tilde{X}^x_{\zeta_{-k}(t)} = x + a\zeta_{-k}(t) + \sigma\int_0^t \sqrt{\tilde{X}^x_{\zeta_{-k}(s)}}dM_s.$$

2. Show that $\bar{X}_t^x = e^{-kt}\tilde{X}^x_{\zeta_{-k}(t)}$ satisfies

$$\bar{X}_t^x = x + \int_0^t (a - k\bar{X}_s^x)ds + \sigma\int_0^t \sqrt{\bar{X}_s^x}e^{-ks/2}dM_s.$$

   Show that $(\int_0^t e^{-ks/2}dM_s, t \geq 0)$ is a Brownian motion and conclude.
3. By using the characteristic function (1.9), check the (simpler) identity $e^{-kt}\tilde{X}^x_{\zeta_{-k}(t)} \overset{law}{=} X_t^x$ on the marginal laws.

1.2.4 The Feller Condition

For $m \geq 0$, we introduce the stopping time $\tau_m = \inf\{t \geq 0, X_t^x = m\}$ with the standard convention $\inf\emptyset = +\infty$. For $m, m' \geq 0$, we set $\tau_{m,m'} = \min(\tau_m, \tau_{m'})$. The goal of this subsection is to study τ_0 and determine under which conditions the CIR never reaches zero. In fact, we already know from Theorem 1.2.1 that the process X^x is nonnegative and we want to get necessary and sufficient conditions under which it is positive. We have the following result.

Proposition 1.2.15 *Let $x > 0$ and $\tau_0 = \inf\{t \geq 0, X_t^x = 0\}$ with $\inf\emptyset = +\infty$. Then, $\tau_0 = +\infty$ a.s. if, and only if*

$$2a \geq \sigma^2. \tag{1.28}$$

When $\sigma^2 > 2a$, we have $\tau_0 < \infty$ a.s. if, and only if $k \geq 0$.

The condition (1.28) is well known as the Feller condition in the literature. In fact, Proposition 1.2.15 can be easily deduced from the Feller's test for explosions (see, Theorem 5.29, p. 348 in [83]) since $\tau_0 = \inf\{t \geq 0, X_t^x \notin (0, +\infty)\}$. Here, we give a direct proof of this result for the CIR case.

We introduce the scale function of the CIR process which is defined by

$$x > 0, \; s(x) = \int_1^x e^{\frac{2k}{\sigma^2} y} y^{-\frac{2a}{\sigma^2}} dy. \tag{1.29}$$

It is $\mathcal{C}^\infty$ and satisfies $(a - kx)s'(x) + \frac{1}{2}\sigma^2 x s''(x) = 0$. The function s is increasing. We set

$$s(0+) = \lim_{x \to 0+} s(x) \text{ and } s(+\infty) = \lim_{x \to +\infty} s(x),$$

and have

$$s(0^+) = -\infty \iff \sigma^2 \leq 2a, \tag{1.30}$$

$$s(+\infty) = +\infty \iff k > 0 \text{ or } k = 0, \sigma^2 \geq 2a. \tag{1.31}$$

We have the following classical result.

Lemma 1.2.16 *Let X^x denote the CIR process (1.5) and consider $\underline{m}, \overline{m}$ such that*

$$0 < \underline{m} < x < \overline{m} < \infty.$$

Then, we have

$$\mathbb{P}(\tau_{\underline{m}} < \tau_{\overline{m}}) = \frac{s(\overline{m}) - s(x)}{s(\overline{m}) - s(\underline{m})}. \tag{1.32}$$

Proof Since the process X^x is continuous, we know that $X_t^x \in [\underline{m}, \overline{m}]$ for $t \leq \tau_{\underline{m},\overline{m}}$ and we have by Itô's formula:

$$s(X^x_{t \wedge \tau_{\underline{m},\overline{m}}}) = s(x) + \int_0^{t \wedge \tau_{\underline{m},\overline{m}}} \sigma \sqrt{X_u^x} s'(X_u^x) dW_u. \tag{1.33}$$

We remark that $s(X^x_{t \wedge \tau_{\underline{m},\overline{m}}})$ is a bounded martingale and thus converges almost surely when $t \to +\infty$ (see e.g. Theorem 3.15, p. 17 in [83]). Setting $c_{\underline{m},\overline{m}} = \min_{x \in [\underline{m},\overline{m}]} x s'(x)^2 > 0$, we get in particular that

$$\begin{aligned} c_{\underline{m},\overline{m}} \sigma^2 \mathbb{E}\left[\tau_{\underline{m},\overline{m}}\right] \leq & \mathbb{E}\left[\int_0^{\tau_{\underline{m},\overline{m}}} \sigma^2 X_u^x s'(X_u^x)^2 du\right] \\ & = \lim_{t \to +\infty} \mathbb{E}\left[(s(X^x_{t \wedge \tau_{\underline{m},\overline{m}}}) - s(x))^2\right] < +\infty, \end{aligned}$$

and thus $\mathbb{P}(\tau_{\underline{m},\overline{m}} < +\infty) = 1$. We deduce that

$$s(X^x_{\tau_{\underline{m},\overline{m}}}) = s(x) + \int_0^{\tau_{\underline{m},\overline{m}}} \sigma\sqrt{X^x_u}\, s'(X^x_u) dW_u.$$

Since the map $x \mapsto \sigma\sqrt{x}s'(x)$ is bounded on $x \in [\underline{m}, \overline{m}]$, we get

$$s(x) = \mathbb{E}[s(X^x_{\tau_{\underline{m},\overline{m}}})] = s(\underline{m})\mathbb{P}(\tau_{\underline{m}} < \tau_{\overline{m}}) + s(\overline{m})\mathbb{P}(\tau_{\overline{m}} < \tau_{\underline{m}}),$$

which leads to (1.32). □

We are now in position to prove Proposition 1.2.15. We consider an increasing (resp. decreasing) sequence $(\overline{m}_n)_{n\in\mathbb{N}} \in (x, +\infty)^{\mathbb{N}}$ (resp. $(\underline{m}_n)_{n\in\mathbb{N}} \in (0, x)^{\mathbb{N}}$) such that $\overline{m}_n \underset{n\to+\infty}{\to} +\infty$ (resp. $\underline{m}_n \underset{n\to+\infty}{\to} 0$). Clearly, $\tau_{\overline{m}_n}$ and $\tau_{\underline{m}_n}$ are increasing sequences of stopping times that converges almost surely. We get that $\lim_{n\to+\infty} \tau_{\underline{m}_n} = \tau_0$ and $\lim_{n\to+\infty} \tau_{\overline{m}_n} = +\infty$ almost surely, since we have $\mathbb{P}(\tau_{\overline{m}_n} \le T) = \mathbb{P}(\max_{t\in[0,T]} X^x_t \ge \overline{m}_n) \underset{n\to+\infty}{\to} 0$.

We first focus on the case $\sigma^2 \le 2a$. We clearly have $\mathbb{P}(\tau_0 < \tau_{\overline{m}}) \le \mathbb{P}(\tau_{\underline{m}} < \tau_{\overline{m}})$ for any $\underline{m} \in (0, x)$ and we get $\mathbb{P}(\tau_0 < \tau_{\overline{m}}) = 0$ from (1.30) and (1.32) by letting $\underline{m} \to 0^+$. Now, the dominated convergence theorem gives

$$\mathbb{P}(\tau_0 < +\infty) = \lim_{n\to+\infty} \mathbb{P}(\tau_0 < \tau_{\overline{m}_n}) = 0.$$

We now consider the case $\sigma^2 > 2a$. From (1.30) and (1.33), we obtain that $s(X^x_{\tau_{\underline{m}_n,\overline{m}}})$ is a bounded discrete-time martingale with respect to the filtration $\mathcal{G}_n = \mathcal{F}_{\tau_{\underline{m}_n,\overline{m}}}$. Thus, it converges almost surely and we have

$$\begin{aligned}
c_{0,\overline{m}}\sigma^2\mathbb{E}\left[\tau_{0,\overline{m}}\right] &= c_{0,\overline{m}}\sigma^2 \lim_{n\to+\infty} \mathbb{E}\left[\tau_{\underline{m}_n,\overline{m}}\right] \\
&\le \lim_{n\to+\infty} \mathbb{E}\left[\int_0^{\tau_{\underline{m}_n,\overline{m}}} \sigma^2 X^x_u s'(X^x_u)^2 du\right] \\
&= \lim_{n\to+\infty} \mathbb{E}\left[(s(X^x_{\tau_{\underline{m}_n,\overline{m}}}) - s(x))^2\right] < +\infty,
\end{aligned}$$

with $c_{0,\overline{m}} = \min_{x\in[0,\overline{m}]} xs'(x)^2$. For $\sigma^2 \in (2a, 4a]$, we have $c_{0,\overline{m}} > 0$ and get $\mathbb{P}(\tau_{0,\overline{m}} < +\infty) = 1$. Using (1.27), we then deduce that $\mathbb{P}(\tau_{0,\overline{m}} < +\infty) = 1$ for $\sigma^2 \in (4a, 8a]$. By induction, (1.27) gives $\mathbb{P}(\tau_{0,\overline{m}} < +\infty) = 1$ for $\sigma^2 \in (2^k a, 2^{k+1}a]$, $k \in \mathbb{N}$ and thus for any $\sigma^2 > 2a$. Therefore $s(X^x_{\tau_{0,\overline{m}}}) = \lim_{n\to+\infty} s(X^x_{\tau_{\underline{m}_n,\overline{m}}})$ and the dominated convergence theorem gives $s(x) = \mathbb{E}[s(X^x_{\tau_{0,\overline{m}}})]$, and we get

$$\mathbb{P}(\tau_0 < \tau_{\overline{m}}) = \frac{s(\overline{m}) - s(x)}{s(\overline{m}) - s(0+)}. \tag{1.34}$$

We finally obtain for $\sigma^2 > 2a$:

$$\mathbb{P}(\tau_0 < +\infty) = \lim_{n\to+\infty} \mathbb{P}(\tau_0 < \tau_{\overline{m}_n}) = \begin{cases} 1 \text{ if } k \geq 0, \\ \frac{s(+\infty)-s(x)}{s(+\infty)-s(0+)} \in (0,1) \text{ otherwise.} \end{cases} \tag{1.35}$$

Remark 1.2.17 It would be tempting to deduce (1.34) directly from (1.32), letting $\underline{m} \to 0$. In fact, we have thanks to the continuity of the paths that

$$\begin{aligned}\bigcap_{n\in\mathbb{N}} \{\tau_{\underline{m}_n} < \tau_{\overline{m}}\} &= \left\{ \inf_{t\in[0,\tau_{\overline{m}})} X_t^x = 0 \right\} \\ &= \{\tau_0 < \tau_{\overline{m}}\} \cup \left(\{\tau_0 = \tau_{\overline{m}} = +\infty\} \cap \left\{ \inf_{t\geq 0} X_t^x = 0 \right\} \right),\end{aligned}$$

the union being disjoint. When $\sigma^2 > 2a$, we precisely show here that $\mathbb{P}(\tau_0 = \tau_{\overline{m}} = +\infty) = \mathbb{P}(\tau_{0,\overline{m}} = +\infty) = 0$ to get $\mathbb{P}(\tau_0 < \tau_{\overline{m}}) = \lim_{n\to+\infty} \mathbb{P}(\tau_{\underline{m}_n} < \tau_{\overline{m}})$.

Exercise 1.2.18 In this exercise, we propose to prove by other means that $\mathbb{P}(\tau_0 = +\infty) = 1$ when $x > 0$ and $\sigma^2 \leq 2a$.

1. Show that for $t \in [0, \tau_0)$, we have $X_t^x = x \exp\left(\int_0^t \frac{a-\sigma^2/2}{X_s} ds - kt + \int_0^t \frac{\sigma}{\sqrt{X_s}} dW_s \right)$. Deduce that $X_t^x \geq x \exp(-kt + M_t)$, with $M_t = \int_0^t \frac{\sigma}{\sqrt{X_s}} dW_s$.
2. We assume by a way of contradiction that $\mathbb{P}(\tau_0 < \infty) > 0$. Then, show that $\mathbb{1}_{\{\tau_0<\infty\}} M_{\min(t,\tau_0)} \underset{t\to+\infty}{\to} -\mathbb{1}_{\{\tau_0<\infty\}}\infty$, almost surely. By using the Dambis-Dubins-Schwarz theorem, get the contradiction and conclude. We recall that for a standard Brownian motion B, we have $\limsup_{t\to+\infty} B_t = -\liminf_{t\to+\infty} B_t = \infty$ almost surely.

The fact that a continuous martingale cannot converge almost surely to ∞ (or $-\infty$) because it is necessary oscillating is known in the literature as the McKean argument.

1.3 Definition and Characterization of Affine Diffusions

Let us now consider a rather general time-homogeneous real valued SDE. More precisely, we consider $\mathbb{D}$ an interval of $\mathbb{R}$ and continuous functions $b : \mathbb{D} \to \mathbb{R}$ and $\sigma : \mathbb{D} \to \mathbb{R}_+$ such that the following SDE

$$X_t^x = x + \int_0^t b(X_s^x) ds + \int_0^t \sigma(X_s^x) dW_s, \ t \geq 0 \tag{1.36}$$

admits a unique weak solution that takes values in $\mathbb{D}$, i.e. $\mathbb{P}(\forall t \geq 0, X_t^x \in \mathbb{D}) = 1$. This condition ensures from Theorem 4.20, p. 322 in [83] that X satisfies the strong Markov property. The infinitesimal generator of this diffusion is given by:

$$L = b(x)\partial_x + \frac{\sigma^2(x)}{2}\partial_x^2. \tag{1.37}$$

We now focus on the characteristic function of the marginal law of X_t^x. A process X is said to be affine if its characteristic function can be written as follows:

$$\forall x \in \mathbb{D}, u \in i\mathbb{R}, t \geq 0,\ \mathbb{E}[\exp(uX_t^x)] = \exp(\phi_u(t) + \psi_u(t)x), \tag{1.38}$$

where $\phi_u, \psi_u : \mathbb{R}_+ \to \mathbb{C}$ are continuous functions. Then, we necessarily have in this case

$$\phi_u(0) = 0 \text{ and } \psi_u(0) = u.$$

Proposition 1.3.1 *Suppose that for some $u \in i\mathbb{R}^*$, there are $\mathcal{C}^1$ functions ϕ_u and ψ_u such that:*

$$\forall x \in \mathbb{D}, t \geq 0,\ \mathbb{E}[\exp(uX_t^x)] = \exp(\phi_u(t) + \psi_u(t)x).$$

Then, $b(x)$ and $\sigma^2(x)$ must be affine functions of $x \in \mathbb{D}$, i.e. the infinitesimal generator (1.37) is affine with respect to x.

This result is well-known in the literature and can be found for example in Filipović [53].

Proof For $0 \leq t \leq T$, we have by using the Markov property at time t

$$\mathbb{E}[\exp(uX_T^x)|\mathcal{F}_t] = \exp(\phi_u(T-t) + \psi_u(T-t)X_t^x),$$

since X is a time homogeneous diffusion. The left-hand side is clearly a $(\mathcal{F}_t)$-martingale for $t \in [0, T]$ that we denote M_t. Thanks to the $\mathcal{C}^1$ assumption on ϕ_u, ψ_u, we can apply Itô's formula to the right hand side and get:

$$\begin{aligned} dM_t = & M_t\Bigg[\bigg(-\phi_u'(T-t) - \psi_u'(T-t)X_t^x + \psi_u(T-t)b(X_t^x) \\ & + \frac{1}{2}\psi_u(T-t)^2\sigma^2(X_t^x)\bigg)dt + \psi_u(T-t)\sigma(X_t^x)dW_t\Bigg]. \end{aligned}$$

We see that M_t is a Doléans-Dade exponential and cannot vanish. We therefore obtain that $-\phi_u'(T-t)-\psi_u'(T-t)X_t^x+\psi_u(T-t)b(X_t^x)+\frac{1}{2}\psi_u(T-t)^2\sigma^2(X_t^x) = 0$, $\mathbb{P}$-a.s. and dt-a.e. Since this quantity is continuous with respect to (t, X_t^x), we get by letting $t \to 0$ that

$$\psi_u(T)b(x) + \frac{1}{2}\psi_u(T)^2\sigma^2(x) = \phi_u'(T) + \psi_u'(T)x, \quad x \in \mathbb{D}, T \geq 0.$$

Now, if there are $T_1 < T_2$ such that the matrix $\begin{bmatrix} \psi_u(T_1) & \psi_u(T_1)^2 \\ \psi_u(T_2) & \psi_u(T_2)^2 \end{bmatrix}$ is invertible, then the claim holds since

$$\begin{bmatrix} b(x) \\ \frac{1}{2}\sigma^2(x) \end{bmatrix} = \begin{bmatrix} \psi_u(T_1) & \psi_u(T_1)^2 \\ \psi_u(T_2) & \psi_u(T_2)^2 \end{bmatrix}^{-1} \begin{bmatrix} \phi_u'(T_1) + \psi_u'(T_1)x \\ \phi_u'(T_2) + \psi_u'(T_2)x \end{bmatrix}.$$

Otherwise, this means that for any $T \geq 0$, $\psi_u(T)^2 = c\psi_u(T)$ for some constant $c \in \mathbb{R}$. This immediately gives that ψ_u is constant and thus $\psi_u \equiv u$ from the initial condition. Then, we get $ub(x) + \frac{1}{2}u^2\sigma^2(x) = \phi_u'(T)$ for any $T > 0$, and both sides are necessarily constant. Thus, imaginary and real parts are constants which gives respectively that b and σ^2 are constant and in particular affine. □

Thanks to Proposition 1.3.1, real valued affine diffusion are easily identified. In fact, there are α, β such that $\sigma(x)^2 = \alpha x + \beta$ for any $x \in \mathbb{D}$. We can put aside the case $\alpha = \beta = 0$ that leads to a linear ODE. If $\alpha = 0$, we get the Ornstein-Uhlenbeck process, and we necessarily have $\mathbb{D} = \mathbb{R}$ since the support of a Gaussian random variable is the full real line. Otherwise, we have $\alpha \neq 0$. Since $\sigma(x)^2 \geq 0$, the domain $\mathbb{D}$ is necessarily included in $[\frac{-\beta}{\alpha}, +\infty)$ when $\alpha > 0$, and in $(-\infty, \frac{-\beta}{\alpha}]$ when $\alpha < 0$. We set then $\tilde{X}_t = \alpha X_t^x + \beta$, $\tilde{b}(x) = b(\frac{x-\beta}{\alpha})$ and $\tilde{W}_t = \frac{\alpha}{|\alpha|}W_t$, and get

$$d\tilde{X}_t = \alpha\tilde{b}(\tilde{X}_t) + |\alpha|\sqrt{\tilde{X}_t}d\tilde{W}_t.$$

Since X_t^x is assumed to be well defined for any $t \geq 0$, $\tilde{X}_t$ is also well defined. This implies that $\alpha\tilde{b}(0) \geq 0$ from Exercise 1.2.3, which gives that $\tilde{X}$ is a CIR process. Thus, any real valued affine diffusion is either an Ornstein-Uhlenbeck process or an affine transform of a Cox-Ingersoll-Ross process.

Remark 1.3.2 We have fully characterized here the real affine diffusions. More general affine processes exist, if we allow jumps. For example, let us consider N^x an homogeneous Poisson process with jump rate λ starting from $x \in \mathbb{R}$ at time 0. We have for $u \in \mathbb{C}$,

$$\mathbb{E}\left[\exp(u(N_t^x))\right] = \exp(-\lambda t)\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}\exp(u(x+k)) = \exp\left(ux + \lambda t(e^u - 1)\right).$$

It satisfies (1.38) and is thus affine. Affine time-homogeneous Markov processes can be characterized in the same way through their infinitesimal generator that should be again affine with respect to x. This is shown in Theorem 2.7 of Duffie et al. [47] for general vector valued affine Markov processes.

Up to now, we only have considered time homogeneous diffusion. Unless specified, we will mainly work with homogeneous diffusion in this book. However, it is natural to extend the result of Proposition 1.3.1 when the coefficients are time-dependent. We consider now functions $b : \mathbb{R}_+ \times \mathbb{D} \to \mathbb{R}$ and $\sigma : \mathbb{R}_+ \times \mathbb{D} \to \mathbb{R}_+$ that are continuous with respect to the second variable, such that the following SDE

$$X_t^{s,x} = x + \int_s^t b(r, X_r^x)dr + \int_s^t \sigma(r, X_r^x)dW_r, \ t \geq s,$$

admits for any $s \geq 0$, $x \in \mathbb{D}$ a unique weak solution that takes values in $\mathbb{D}$. The infinitesimal generator at time t of this diffusion is

$$L_t = b(t,x)\partial_x + \frac{\sigma^2(t,x)}{2}\partial_x^2.$$

Again, we focus on the characteristic function, and say that the process X is affine if we have

$$\forall x \in \mathbb{D}, u \in i\mathbb{R}, 0 \leq t \leq T, \ \mathbb{E}[\exp(uX_T^{t,x})] = \exp(\phi_u(t,T) + \psi_u(t,T)x), \tag{1.39}$$

where $\phi_u, \psi_u : \{(t,T), \ 0 \leq t \leq T\} \mapsto \mathbb{C}$ are continuous functions. We remark that we necessarily have

$$\forall t \geq 0, \ \phi_u(t,t) = 0, \ \psi_u(t,t) = u.$$

Clearly, this definition is compatible with the one given in the time homogeneous case, since the left hand side of (1.39) only depends of (t,T) through $T - t$ when b and σ are not time-dependent. We have the analogous characterization of affine processes.

Proposition 1.3.3 *Suppose that for some $u \in i\mathbb{R}^*$, there are continuous functions ϕ_u and ψ_u that are $\mathcal{C}^1$ with respect to the first variable such that:*

$$\forall x \in \mathbb{D}, T \geq t \geq 0, \ \mathbb{E}[\exp(uX_T^{t,x})] = \exp(\phi_u(t,T) + \psi_u(t,T)x).$$

Then, for any $t \geq 0$, $b(t,x)$ and $\sigma^2(t,x)$ must be affine functions of $x \in \mathbb{D}$, and the infinitesimal generator L_t is affine with respect to x.

Proof We use the same arguments as in the time homogeneous case. First, we consider for $t \leq s \leq T$, $M_s = \mathbb{E}[\exp(uX_T^{t,x})|\mathcal{F}_s] = \exp(\phi_u(s,T) + \psi_u(s,T)X_s^{t,x})$. This is a martingale, and we get by Itô's formula

$$\partial_t\phi_u(s,T) + \partial_t\psi_u(s,T)X_s^{t,x} + \psi_u(s,T)b(s,X_s^{t,x}) + \frac{1}{2}\psi_u(s,T)^2\sigma^2(s,X_s^{t,x}) = 0, \ \mathbb{P}-a.s., \ ds-a.e.$$

Letting $s \to t$, we get $\partial_t \phi_u(t,T) + \partial_t \psi_u(t,T)x + \psi_u(t,T)b(t,x) + \frac{1}{2}\psi_u(t,T)^2\sigma^2(t,x) = 0$ for any $0 \le t \le T$, and we conclude as in the proof of Proposition 1.3.1. □

Let us give some examples of time-inhomogeneous affine diffusions. Let $a, k : \mathbb{R} \to \mathbb{R}$ and $\varsigma : \mathbb{R} \to \mathbb{R}_+$ be piecewise continuous functions. Then, taking $b(t,x) = a(t) - k(t)x$ and $\sigma(t,x) = \varsigma(t)$ gives a time-dependent extension of the Ornstein-Uhlenbeck process. When a is a nonnegative function, taking $b(t,x) = a(t) - k(t)x$ and $\sigma(t,x) = \varsigma(t)\sqrt{x}$ also gives a time-dependent extension of the Cox-Ingersoll-Ross process. Let us note that we could also consider in this case $b(t,x) = a(t) - k(t)x$ and $\sigma(t,x) = \varsigma(t)(\mathbb{1}_{t<1}\sqrt{x} + \mathbb{1}_{t\ge 1})$. Thus, in the time inhomogeneous setting, real affine diffusions are no longer either of Ornstein-Uhlenbeck type or of CIR type, but may along the time switch from a CIR regime to an Ornstein-Uhlenbeck regime.

1.4 Application to Interest Rate Modelling

Affine diffusions have been used early in quantitative finance, in particular to model the interest rates. In this section, we present the pioneering works of Vasicek [122] and Cox-Ingersoll-Ross [31, 32] in this field that propose among the most well-known interest short rate models. To do so, we first need to give a brief introduction on interest rates. Our goal is not to give a full and up to date account on this topic. In particular, we do not deal with the counterparty risk that has become an important issue since the subprime crisis. Our goal is rather to give the elementary background that is necessary to understand why affine processes have been widely used in this area. We refer to Brigo and Mercurio [22] or Andersen and Piterbarg [12] for a wide presentation on interest rate models and Filipović [53] for a focus on interest rate models with affine processes.

1.4.1 Short Rate Models and Interest Rates Contracts in a Nutshell

Short rate models assume the existence of an $(\mathcal{F}_t)$-adapted process $(r_t, t \ge 0)$ that describes the instantaneous interest rate. This means that it is possible to invest $X \in \mathbb{R}$ units of cash at time t and get surely $Xr_t dt$ units of cash on the infinitesimal period $[t, t+dt]$. Said differently, there is a riskless asset called the "bank account" with value

$$B_t = \exp\left(\int_0^t r_s ds\right) \tag{1.40}$$

that can be bought or sold without restriction at this price. Borrowing money corresponds to a short position ($X < 0$) in this asset while lending money corresponds to a long position ($X > 0$). For $0 \leq t \leq T$, we define the discount factor

$$D(t,T) = \exp\left(-\int_t^T r_s ds\right) = B_t/B_T$$

between times t and T. This is the amount that should be invested at time t in the bank account to get at time T a unit of cash. Unless assuming deterministic rates, this amount is a priori random and unknown at time t, because $D(t,T)$ is $\mathcal{F}_T$-measurable and depends on the path $(r_s, s \in [t,T])$.

Since the seminal works of Black and Scholes [20] and Merton [106], one of the main assumption used in quantitative finance for hedging and pricing derivatives is the absence of arbitrage opportunity. Let us explain what this means. To do so, we need to introduce self-financing portfolios. We consider a frictionless market where n risky assets are traded. We denote by $S_t^1, \dots, S_t^n$ their market prices at time t, and we assume that we can sell or buy any quantity of these assets at these prices. We set $S_t^0 = B_t$. A portfolio is fully described by an $(\mathcal{F}_t)$-adapted process φ that takes values in $\mathbb{R}^{n+1}$: the quantity φ_t^i counts the number of the i-th asset in the portfolio at time t. Thus, its value is given by

$$V_t(\varphi) = \sum_{i=0}^n \varphi_t^i S_t^i.$$

A portfolio is self-financing when it is managed independently, without supplying or consuming cash. If we assume that the portfolio is held constant on the infinitesimal period $[t, t+dt)$ and then rebalanced at time $t+dt$, the self-financing condition leads to $\sum_{i=0}^n \varphi_{t+dt}^i S_{t+dt}^i = \sum_{i=0}^n \varphi_t^i S_{t+dt}^i$ and thus $V_{t+dt}(\varphi) = \sum_{i=0}^n \varphi_t^i S_{t+dt}^i$. Therefore, the self-financing condition can be written as follows:

$$dV_t(\varphi) = \sum_{i=0}^n \varphi_t^i dS_t^i.$$

We say that the market contains an arbitrage opportunity if we can find a self-financing portfolio φ such that $V_0(\varphi) = 0$ and $V_t(\varphi) \geq 0$ with $\mathbb{P}(V_t(\varphi) > 0) > 0$ for some $t > 0$. Such portfolios would allow to earn money out of nothing, and we assume that they do not exist. It is well known from Harrison and Pliska [74] that this assumption is equivalent, in the discrete time setting, to the existence of a martingale measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ under which the discounted assets are $(\mathcal{F}_t)$-martingales. Such a probability measure $\tilde{\mathbb{P}}$ is called a martingale probability measure. Things get trickier in continuous time. The existence of a martingale probability measure $\tilde{\mathbb{P}}$ still rules out arbitrage opportunity (see [74]), but the converse is no longer true. One has to make a slightly stronger assumption than the absence of arbitrage opportunity called the "No Free Lunch with Vanishing Risk" to get the existence of an equivalent martingale probability measure (see Delbaen and Schachermayer [41, 42] for a full account on this topic). Here, we will directly assume that the original probability measure $\mathbb{P}$ is a martingale measure, which means that the discounted traded assets

$$\tilde{S}_t^i = S_t^i / B_t,\ i = 1, \dots, n,$$

are $(\mathcal{F}_t)$-martingales. This gives $\mathbb{E}[\tilde{S}_T^i | \mathcal{F}_t] = \tilde{S}_t^i$ for $t \leq T$, and thus:

$$S_t^i = \mathbb{E}[D(t,T) S_T^i | \mathcal{F}_t]. \tag{1.41}$$

Let us now describe some traded financial products on the interest rates. The simplest one is the zero-coupon bond. A zero-coupon bond with maturity $T > 0$ is a product that pays a unit of cash at time T. We denote by $P(t,T)$ its price at time $t \leq T$. We have $P(T,T) = 1$ and get from (1.41)

$$P(t,T) = \mathbb{E}[D(t,T)|\mathcal{F}_t] = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\middle|\mathcal{F}_t\right]. \tag{1.42}$$

We are now in position to introduce the simply compounded rates. To do so, we consider the following contract between a borrower and a lender. At time t, the lender pays a unit of cash to the borrower and recovers at time T his loan, plus the amount $X(T-t)$ as a reward, where the rate X is fixed at time t. We exclude here and in the sequel any kind of default. The market practice is to quote the rate X which is fair for both parts. We denote by $L(t,T)$ this simply compounded rate that is determined by

$$\mathbb{E}[-1 + (1 + (T-t)L(t,T))D(t,T)|\mathcal{F}_t] = 0,\ i.e.\ L(t,T) = \frac{1 - P(t,T)}{(T-t)P(t,T)}. \tag{1.43}$$

Any other rate would give an arbitrage for the borrower or for the lender. For example, if $X < L(t,T)$, the borrower could then buy $1/P(t,T)$ zero coupon bonds at time t and get at time T the amount

$$\frac{1}{P(t,T)} - (1 + X(T-t)) > \frac{1}{P(t,T)} - (1 + L(t,T)(T-t)) = 0.$$

We now introduce some other standard products such as bonds, swaps, caps and floors. To do so, we consider a time schedule $T_0 < T_1 < \cdots < T_l$, and we assume that the current time t is before the beginning of these contracts, i.e. $t \leq T_0$. We consider a contract between two parts that is defined as follows. The borrower receives a unit of cash at time T_0, that he has to get back at time T_l. Besides, he pays to the lender interest (or coupons) $X_i(T_i - T_{i-1})$ at times T_i for $i = 1, \dots, l$. We say that the contract has a fixed rate when the rate X_i is chosen constant (i.e.

$X_i = K$ for some $K > 0$). The fair fixed rate at time t is characterized by

$$\mathbb{E}\left[-D(t,T_0) + \sum_{i=1}^{l} K(T_i - T_{i-1})D(t,T_i) + D(t,T_l)\middle|\mathcal{F}_t\right] = 0,$$

which gives

$$K = \frac{1 - P(t,T_l)}{\sum_{i=1}^{l}(T_i - T_{i-1})P(t,T_i)}. \tag{1.44}$$

This rate is called the swap rate. With the same kind of argument as for the simply compounded rate, any other choice would lead to an arbitrage for the borrower or for the lender. When the rate is fixed at time $t < T_0$, the contract above is a Forward Rate Agreement since both parts agrees for a rate on the period $[T_0, T_l]$ before T_0. When the rate is fixed at the beginning of the contract T_0, this is a bond.

A floating rate note is a contract with variable coupons that corresponds to the choice $X_i = L(T_{i-1}, T_i)$ in the contract described above. The floating rate note is a fair contract for both parts. In fact, we have by using that $D(t,T_i) = D(t,T_{i-1})D(T_{i-1},T_i)$ and the tower property of the conditional expectation

$$\begin{aligned}
&\mathbb{E}\left[D(t,T_i)L(T_{i-1},T_i)(T_i - T_{i-1})\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[D(t,T_{i-1})\frac{1-P(T_{i-1},T_i)}{P(T_{i-1},T_i)}\mathbb{E}\left[D(T_{i-1},T_i)\middle|\mathcal{F}_{T_{i-1}}\right]\middle|\mathcal{F}_t\right] \\
&= P(t,T_{i-1}) - P(t,T_i).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\mathbb{E}\left[-D(t,T_0) + \sum_{i=1}^{l} L(T_{i-1},T_i)(T_i - T_{i-1})D(t,T_i) + D(t,T_l)\middle|\mathcal{F}_t\right] \\
&\qquad = -P(t,T_0) + \sum_{i=1}^{l}\big(P(t,T_{i-1}) - P(t,T_i)\big) + P(t,T_l) = 0,
\end{aligned}$$

and the contract is fair. A swap is a contract between two parts that agree to pay each other respectively $K(T_i - T_{i-1})$ and $L(T_{i-1},T_i)(T_i - T_{i-1})$ at times T_i, $i = 1,\dots,l$. For example, the two parts could be a borrower with a fixed rate decided at time t and a borrower with a floating rate note, where both want to change for the other interest regime. Since the floating rate note is already fair, the swap rate which is fair for both parts is the same as the fair fixed rate for the Forward Rate Agreement, and it is given by Eq. (1.44). Last, we introduce caps (resp. floors) which are contracts that pay $(T_i - T_{i-1})(L(T_{i-1},T_i) - K)^+$ (resp. $(T_i - T_{i-1})(K - L(T_{i-1},T_i))^+$) at times T_i, $i = 1,\dots,l$, against an initial payment at time t when the contract is bought. The rate K is called the cap (resp. floor) rate. Thus, a borrower with a fair floating rate who has bought a cap has to pay at each time

$$(T_i - T_{i-1})[L(T_{i-1}, T_i) - (L(T_{i-1}, T_i) - K)^+] = (T_i - T_{i-1}) \min(K, L(T_{i-1}, T_i))$$

and is thus protected against a rise of the interest rates. Similarly, a lender with a fair floating rate can hedge against a fall of the interest rates by buying a floor. When there is only one period in the schedule (i.e. $l = 1$) these products are called caplets and floorlets. The price of a floorlet at time t is given by

$$\begin{aligned}
&\mathbb{E}[D(t, T_1)(T_1 - T_0)(K - L(T_0, T_1))^+ | \mathcal{F}_t] \\
&= \mathbb{E}\left[D(t, T_0) \left(K(T_1 - T_0) - \frac{1 - P(T_0, T_1)}{P(T_0, T_1)} \right)^+ \mathbb{E}[D(T_0, T_1) | \mathcal{F}_{T_0}] \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}\left[D(t, T_0) \Big([1 + K(T_1 - T_0)] P(T_0, T_1) - 1 \Big)^+ \middle| \mathcal{F}_t \right] \\
&= [1 + K(T_1 - T_0)] \times \mathbb{E}\left[D(t, T_0) \left(P(T_0, T_1) - \frac{1}{1 + K(T_1 - T_0)} \right)^+ \middle| \mathcal{F}_t \right],
\end{aligned} \tag{1.45}$$

where we used the tower property of the conditional expectation for the first equality. Thus, the price of a floorlet is directly related to the price of a call option on the zero coupon bond with maturity T_0 and strike $\frac{1}{1+K(T_1 - T_0)}$. An analogous formula holds between the prices of the floorlet and the put option on the zero coupon bond. In practice, put and call options on zero coupon bonds are not significantly traded. Formula (1.45) is rather useful to price floorlets and caplets that are much more traded.

1.4.2 The Vasicek Model

The Vasicek model [122] assumes that the short interest rate follows an Ornstein-Uhlenbeck process, with the following parametrization:

$$r_t = r_0 + \int_0^t k(\theta - r_t)dt + \sigma W_t.$$

The parameters $k,\ \theta, \sigma$ and r_0 are assumed to be positive. Thus, the process r is mean reverting towards θ and has a stationary law, $\mathcal{N}(\theta, \frac{\sigma^2}{2k})$. We know that r is a Gaussian process. This gives in particular that r_t can take negative values with some positive probability, while interest rates are in practice usually positive. From (1.3) and (1.4), we can easily calculate the price of a zero-coupon bond:

$$P(t, T) = P^{\text{Vas}}(r_t, T - t),$$

with

$$P^{\mathrm{Vas}}(r,t) = A^{\mathrm{Vas}}(t)\exp\left(-rB^{\mathrm{Vas}}(t)\right), \quad r \geq 0,\ t \geq 0, \tag{1.46}$$

$$A^{\mathrm{Vas}}(t) = \exp\left[\left(\frac{\sigma^2}{2k^2} - \theta\right)(t - \zeta_k(t)) - \frac{\sigma^2}{4k}\zeta_k(t)^2\right] \text{ and } B^{\mathrm{Vas}}(t) = \zeta_k(t).$$

Now, we would like to calculate the price of a floorlet. From (1.45), it boils down to calculate the price of a call option on zero coupon bond. Without loss of generality, we assume that the current time is $t = 0$. Thus, the price of a call option on the zero coupon bond between T_0 and T_1 is given by

$$C(T_0, T_1, K) = \mathbb{E}[D(0, T_0)(P^{\mathrm{Vas}}(r_{T_0}, T_1 - T_0) - K)^+].$$

We now observe that the function $r \mapsto P^{\mathrm{Vas}}(r, T_1 - T_0)$ is decreasing and

$$P^{\mathrm{Vas}}(r_{T_0}, T_1 - T_0) \geq K \iff r_{T_0} \leq r^\star(T_1 - T_0),$$
$$\text{with } r^\star(\tau) = \frac{1}{B^{\mathrm{Vas}}(\tau)}\log\left(\frac{A^{\mathrm{Vas}}(\tau)}{K}\right),\ \tau > 0.$$

We then obtain

$$\begin{aligned}
C(T_0, T_1, K) =& \mathbb{E}[D(0, T_0)(P^{\mathrm{Vas}}(r_{T_0}, T_1 - T_0) - K)\mathbb{1}_{r_{T_0} \leq r^\star(T_1 - T_0)}] \\
=& \mathbb{E}[D(0, T_0)\mathbb{E}[D(T_0, T_1)|\mathcal{F}_{T_0}]\mathbb{1}_{r_{T_0} \leq r^\star(T_1 - T_0)}] \\
& - K\mathbb{E}[D(0, T_0)\mathbb{1}_{r_{T_0} \leq r^\star(T_1 - T_0)}] \\
=& P^{\mathrm{Vas}}(r_0, T_1)\mathbb{P}^{T_1}(r_{T_0} \leq r^\star(T_1 - T_0)) \\
& - KP^{\mathrm{Vas}}(r_0, T_0)\mathbb{P}^{T_0}(r_{T_0} \leq r^\star(T_1 - T_0)),
\end{aligned} \tag{1.47}$$

where $\mathbb{P}^T$ denotes, for $T \geq 0$, the T-forward probability measure which is defined on $\mathcal{F}_T$ by

$$\left.\frac{d\mathbb{P}^T}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \frac{D(0, T)}{P(0, T)}. \tag{1.48}$$

To evaluate (1.47), we are interested in the distribution of r_t under the probability $\mathbb{P}^T$, for $t \leq T$. We have the following result.

Lemma 1.4.1 *Let $0 \leq t \leq T$. The law of r_t under the T-forward probability measure is a normal random variable $\mathcal{N}(m_{t,T}, \Sigma_t^2)$ with*

$$m_{t,T} = r_0 e^{-kt} + \left(k\theta - \frac{\sigma^2}{k}\right)\zeta_k(t) + \frac{\sigma^2}{k}e^{-k(T-t)}\zeta_{2k}(t) \textit{ and } \Sigma_t^2 = \sigma^2\zeta_{2k}(t).$$

For $x \in \mathbb{R}$, we set $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$ which is the cumulative distribution function of the standard Gaussian variable. Thanks to (1.47) and Lemma 1.4.1, we get the following explicit form for the call on zero coupon bond:

$$\begin{aligned} C(T_0, T_1, K) = & P^{\mathrm{Vas}}(r_0, T_1)\Phi\left(\frac{r^\star(T_1 - T_0) - m_{T_0,T_1}}{\Sigma_{T_0}}\right) \\ & - KP^{\mathrm{Vas}}(r_0, T_0)\Phi\left(\frac{r^\star(T_1 - T_0) - m_{T_0,T_0}}{\Sigma_{T_0}}\right). \end{aligned} \tag{1.49}$$

It is then easy to get from (1.45) a pricing formula for any floorlet. Of course, similar calculations can be made for any put option on zero coupon bond, and any caplet.

Proof of Lemma 1.4.1 We consider the martingale

$$M_t = \mathbb{E}\left[\frac{D(0,T)}{P(0,T)}\middle| \mathcal{F}_t\right] = \frac{e^{-\int_0^t r_s ds} P^{\mathrm{Vas}}(r_t, T-t)}{P^{\mathrm{Vas}}(r_0, T)}, \ t \in [0, T],$$

and have by Itô's formula

$$\begin{aligned} dM_t &= -\sigma M_t B^{\mathrm{Vas}}(T-t) dW_t, \\ i.e.\ M_t &= \exp\left(-\int_0^t \sigma B^{\mathrm{Vas}}(T-u) dW_u - \frac{1}{2}\sigma^2 \int_0^t \left(B^{\mathrm{Vas}}(T-u)\right)^2 du\right). \end{aligned}$$

Then, we know from Girsanov theorem that

$$W_t^T = W_t + \sigma \int_0^t B^{\mathrm{Vas}}(T-u) du, \ t \in [0, T]$$

is a Brownian motion under $\mathbb{P}^T$, and we have $dr_s = [k(\theta - r_s) - \sigma^2 B^{\mathrm{Vas}}(T-s)]ds + \sigma dW_s^T$. This is a time inhomogeneous affine diffusion. Setting $y_t = e^{kt}(\theta - r_t)$, we get $y_t = y_0 - \int_0^t \sigma e^{ks} dW_s^T + \sigma^2 \int_0^t e^{ks} B^{\mathrm{Vas}}(T-s) ds$ and then

$$r_t = r_0 e^{-kt} + k\theta\zeta_k(t) - \sigma^2 \int_0^t B^{\mathrm{Vas}}(T-s) e^{k(s-t)} ds + \sigma \int_0^t e^{k(s-t)} dW_s^T,$$

which gives the claim after some basic calculations. □

Exercise 1.4.2 The goal of this exercise is to prove again Lemma 1.4.1 by calculating directly the Laplace transform

$$u \in \mathbb{R}, \ g(u) = \mathbb{E}^T[\exp(ur_t)] = \frac{\mathbb{E}[\exp(ur_t - \int_0^T r_s ds)]}{P^{\mathrm{Vas}}(r_0, T)}.$$

First, show that

$$g(u) = \frac{A^{\mathrm{Vas}}(T-t)}{P^{\mathrm{Vas}}(r_0, T)} \mathbb{E}\left[\exp\left((u - B^{\mathrm{Vas}}(T-t)) r_t - \int_0^t r_s ds\right)\right].$$

Using (1.3), deduce then that:

$$g(u) = \exp\left(\frac{1}{2}u^2\sigma^2\zeta_{2k}(t) + u\left(r_0e^{-kt} + k\theta\zeta_k(t)\right.\right.$$
$$\left.\left. - \sigma^2\left[\frac{1}{2}\zeta_k(t)^2 + B^{\mathrm{Vas}}(T-t)\zeta_{2k}(t)\right]\right)\right),$$

and get back that $r_t \sim \mathcal{N}(m_{t,T}, \Sigma_t^2)$ under $\mathbb{P}^T$.

1.4.3 *The Cox-Ingersoll-Ross Model*

Cox et al. [31, 32] have proposed the following model for the short interest rate:

$$r_t = r_0 + \int_0^t k(\theta - r_t)dt + \sigma\int_0^t \sqrt{r_s}dW_s.$$

The parameters $k,\ \theta,\ \sigma$ and r_0 are assumed to be positive, so that the process is mean reverting toward θ hand has a stationary law which is the Gamma law with density $\mathbb{1}_{z>0}\frac{2k}{\sigma^2\Gamma(\frac{2k\theta}{\sigma^2})}\left(\frac{2kz}{\sigma^2}\right)^{\frac{2a}{\sigma^2}-1}e^{-\frac{2kz}{\sigma^2}}$. Contrary to the Vasicek model, the nonnegativity of the interest rates is ensured by this diffusion. From (1.15), we easily obtain the price of a zero coupon

$$P(t,T) = P^{\mathrm{CIR}}(r_t, T-t),$$

with

$$P^{\mathrm{CIR}}(r,t) = A^{\mathrm{CIR}}(t)\exp\left(-rB^{\mathrm{CIR}}(t)\right),\quad r \geq 0,\ t \geq 0, \tag{1.50}$$

$$A^{\mathrm{CIR}}(t) = \left(\frac{2\gamma e^{\frac{\gamma+k}{2}t}}{\gamma - k + (\gamma+k)e^{\gamma t}}\right)^{\frac{2k\theta}{\sigma^2}},\quad B^{\mathrm{CIR}}(t) = \frac{2(e^{\gamma t}-1)}{\gamma - k + (\gamma+k)e^{\gamma t}}, \tag{1.51}$$

where $\gamma = \sqrt{k^2 + 2\sigma^2}$.

We notice in particular that $B^{\mathrm{CIR}}(t) \geq 0$. We are now interested in pricing a floorlet at time 0 with starting and ending maturities T_0 and T_1. Repeating the same calculations as for the Vasicek model, we get:

$$\begin{aligned} C(T_0, T_1, K) =& \mathbb{E}[D(0,T_0)(P^{\mathrm{CIR}}(r_{T_0}, T_1 - T_0) - K)^+] \\ =& P^{\mathrm{CIR}}(r_0, T_1)\mathbb{P}^{T_1}(r_{T_0} \leq r^\star(T_1 - T_0)) \\ & - KP^{\mathrm{CIR}}(r_0, T_0)\mathbb{P}^{T_0}(r_{T_0} \leq r^\star(T_1 - T_0)), \end{aligned}$$

with $r^\star(\tau) = \frac{1}{B^{\mathrm{CIR}}(\tau)}\log\left(\frac{A^{\mathrm{CIR}}(\tau)}{K}\right)$, $\tau > 0$. We are thus again interested in the law of r_t under the T-forward measure. We proceed as in Exercise 1.4.2 and calculate

$g(u) = \mathbb{E}^T[\exp(ur_t)]$ for $u \le 0$. From (1.15), we have

$$\begin{aligned} g(u) &= \frac{\exp\left(r_0 B^{\mathrm{CIR}}(T)\right)}{A^{\mathrm{CIR}}(T)} \mathbb{E}\left[\exp\left(ur_t - \int_0^T r_s ds\right)\right] \\ &= \frac{A^{\mathrm{CIR}}(T-t)\exp\left(r_0 B^{\mathrm{CIR}}(T)\right)}{A^{\mathrm{CIR}}(T)} \mathbb{E}\left[\exp\left([u - B^{\mathrm{CIR}}(T-t)]r_t - \int_0^t r_s ds\right)\right] \\ &= \frac{A^{\mathrm{CIR}}(T-t)\tilde{A}_u(t)}{A^{\mathrm{CIR}}(T)} \exp(r_0[B^{\mathrm{CIR}}(T) + \tilde{B}_u(t)]), \end{aligned}$$

with

$$\tilde{A}_u(t) = \left(\frac{e^{\frac{\gamma+k}{2}t}}{1 + \frac{\sigma^2}{2}(B^{\mathrm{CIR}}(T-t) - u + \frac{k+\gamma}{\sigma^2})\frac{e^{\gamma t}-1}{\gamma}} \right)^{\frac{2k\theta}{\sigma^2}},$$

$$\tilde{B}_u(t) = \frac{k+\gamma}{\sigma^2} + \frac{(u - B^{\mathrm{CIR}}(T-t) - \frac{k+\gamma}{\sigma^2})e^{\gamma t}}{1 - \frac{\sigma^2}{2}(u - B^{\mathrm{CIR}}(T-t) - \frac{k+\gamma}{\sigma^2})\frac{e^{\gamma t}-1}{\gamma}}.$$

An important thing to notice before doing calculations is that $g(0) = 1$ for any $r_0 > 0$, and thus $\frac{A^{\mathrm{CIR}}(T-t)\tilde{A}_u(t)}{A^{\mathrm{CIR}}(T)}$ (resp. $B^{\mathrm{CIR}}(T) + \tilde{B}_u(t)$) is equal to 1 (resp. 0) when $u = 0$.

On the one hand, we have

$$\begin{aligned} \left(\frac{A^{\mathrm{CIR}}(T)}{A^{\mathrm{CIR}}(T-t)\tilde{A}_u(t)}\right)^{\frac{\sigma^2}{2k\theta}} &= \frac{\gamma - k + (\gamma+k)e^{\gamma(T-t)}}{\gamma - k + (\gamma+k)e^{\gamma T}} \times \\ &\quad \left[1 + \frac{\sigma^2}{2}\left(B^{\mathrm{CIR}}(T-t) - u + \frac{k+\gamma}{\sigma^2}\right)\frac{e^{\gamma t}-1}{\gamma}\right] \\ &= 1 - 2\frac{u}{c_{t,T}}, \end{aligned}$$

with

$$\begin{aligned} c_{t,T} &= \frac{4\gamma}{\sigma^2(e^{\gamma t}-1)} \frac{\gamma - k + (\gamma+k)e^{\gamma T}}{\gamma - k + (\gamma+k)e^{\gamma(T-t)}} \\ &= \frac{4\gamma}{\sigma^2(e^{\gamma t}-1)} \left(1 + \frac{(\gamma+k)e^{\gamma(T-t)}(e^{\gamma t}-1)}{\gamma - k + (\gamma+k)e^{\gamma(T-t)}}\right) \\ &= \frac{4\gamma}{\sigma^2(e^{\gamma t}-1)} + \frac{2\gamma(\gamma+k)}{\sigma^2} \frac{2e^{\gamma(T-t)}}{\gamma - k + (\gamma+k)e^{\gamma(T-t)}} \\ &= 2\left(\frac{2\gamma}{\sigma^2(e^{\gamma t}-1)} + \frac{\gamma+k}{\sigma^2} + B^{CIR}(T-t)\right). \end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
B^{\mathrm{CIR}}(T)+\tilde{B}_u(t) &= B^{\mathrm{CIR}}(T)+\frac{k+\gamma}{\sigma^2}+\frac{\frac{4\gamma e^{\gamma t}}{\sigma^2(e^{\gamma t}-1)}(u-B^{\mathrm{CIR}}(T-t)-\frac{k+\gamma}{\sigma^2})}{c_{t,T}-2u}\\
&= \frac{4\gamma e^{\gamma t}}{\sigma^2(e^{\gamma t}-1)}\left(1-2\frac{B^{\mathrm{CIR}}(T-t)+\frac{k+\gamma}{\sigma^2}}{c_{t,T}}\right)\frac{u}{c_{t,T}-2u}\\
&= d_{t,T}\frac{u}{c_{t,T}-2u}, \text{ with } d_{t,T}=\frac{e^{\gamma t}}{c_{t,T}}\left(\frac{4\gamma}{\sigma^2(e^{\gamma t}-1)}\right)^2,
\end{aligned}
$$

by using the identity $\frac{\tilde{a}+u}{\tilde{c}-2u}=\frac{\tilde{a}}{\tilde{c}}+\left(1+2\frac{\tilde{a}}{\tilde{c}}\right)\frac{u}{\tilde{c}-2u}$ for the second equality. This finally gives

$$
g(u)=\left(\frac{1}{1-2u/c_{t,T}}\right)^{\frac{2k\theta}{\sigma^2}}\exp\left(r_0 d_{t,T}\frac{u/c_{t,T}}{1-2u/c_{t,T}}\right).
$$

From Remark 1.2.12, we get that $c_{t,T}r_t$ follows under $\mathbb{P}^T$ a chi-square distribution with $\frac{2k\theta}{\sigma^2}$ degrees of freedom and noncentrality $d_{t,T}r_0$. Let us denote by $\chi^2(x;\nu,d)$ the cumulative distribution function of a chi-square distribution with $\nu>0$ degrees of freedom and noncentrality $d\geq 0$. We finally get

$$
\begin{aligned}
C(T_0,T_1,K) = P^{\mathrm{CIR}}(r_0,T_1)\chi^2\left(c_{T_0,T_1}r^\star(T_1-T_0);\frac{2k\theta}{\sigma^2},d_{T_0,T_1}r_0\right)\\
-KP^{\mathrm{CIR}}(r_0,T_0)\chi^2\left(c_{T_0,T_0}r^\star(T_1-T_0);\frac{2k\theta}{\sigma^2},d_{T_0,T_0}r_0\right).
\end{aligned}
$$

Of course, a similar formula holds for puts on zero coupon bond and we thus get explicit formulas for any floorlet or caplet within the CIR model.

Even if all those calculations may seem tedious and cumbersome, the remarkable point is that they can be carried out and give an explicit formula for any caplet and floorlet prices. From a numerical point of view, calculating these prices boils down to calculate the cumulative distribution function of a noncentral chi-square distribution and is quite instantaneous. Getting fast pricing method is really important in practice, especially to calibrate the parameters to market data, since calibration usually requires an intensive use of the pricing routines. Thus, these formulas partly explain why these affine models are widely used in finance. Maybe, other diffusions would have been more relevant for modelling the short rate, but the lack of explicit formula for basic financial products has made them not suitable for a practical use.

Chapter 2
An Introduction to Simulation Schemes for SDEs

Let us start this chapter by a general motivation for having simulation schemes. To fix the ideas, we consider a continuous process $(X_t, t \in [0, T])$ that takes values in $\mathbb{R}^d$ and a function $F : \mathcal{C}([0, T], \mathbb{R}^d) \to \mathbb{R}$ such that $\mathbb{E}[|F(X_t, t \in [0, T])|] < \infty$. We suppose that we want to calculate $\mathbb{E}[F(X_t, t \in [0, T])]$. Unless for very particular functions F and processes X, this expectation cannot be calculated explicitly, and one has to use numerical methods to approximate it. Again, in some particular cases this expectation can be seen as the solution of a PDE, or can be computed by Fourier inversion, which leads to appropriate numerical approximations. However, these methods are generally struck by the curse of dimensionality and in many circumstances, there is no other choice than to use a Monte-Carlo method. To do so, one has to generate $K \in \mathbb{N}^*$ independent paths $(X^k_t, t \in [0, T])$, $1 \le k \le K$, that are distributed according to the law of $(X_t, t \in [0, T])$. We use the following estimator

$$\frac{1}{K}\sum_{k=1}^{K} F(X^k_t, t \in [0, T])$$

to approximate $\mathbb{E}[F(X_t, t \in [0, T])]$. The law of large numbers ensures that the estimator converges when $K \to +\infty$. Besides, if $\mathbb{E}[F(X_t, t \in [0, T])^2] < \infty$, the Central Limit Theorem gives a confidence interval whose size is proportional to $1/\sqrt{K}$.

Thus, the Monte-Carlo method motivates the need to simulate the process X. Of course, it is not possible in practice to generate full continuous paths. At best, we can only generate the process for a finite number of times. For a time horizon $T > 0$ and $n \in \mathbb{N}^*$, we consider then the regular time grid $t_i = \frac{iT}{n}$, $0 \le i \le n$. To implement in practice the Monte-Carlo method, we have to make two further approximations. On the one hand, we have to generate K independent samples $(\hat{X}^k_{t_i}, 0 \le i \le n)$ that approximate the law of $(X_{t_i}, 0 \le i \le n)$. On the other hand, we have to approximate

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the function $F(x(t), t \in [0, T])$ by another one $\hat{F}(x(t_i), 0 \le i \le n)$. Finally, we use the following estimator of $\mathbb{E}[F(X_t, t \in [0, T])]$:

$$\frac{1}{K}\sum_{k=1}^{K} \hat{F}(\hat{X}^k_{t_i}, 0 \le i \le n).$$

Let $\hat{X}_{t_i} = \hat{X}^1_{t_i}$ for $0 \le i \le n$. We can then decompose the estimation error as follows:

$$\begin{aligned}
&\frac{1}{K}\sum_{k=1}^{K} \hat{F}(\hat{X}^k_{t_i}, 0 \le i \le n) - \mathbb{E}[F(X_t, t \in [0, T])] \\
&\quad = \frac{1}{K}\sum_{k=1}^{K} \hat{F}(\hat{X}^k_{t_i}, 0 \le i \le n) - \mathbb{E}[\hat{F}(\hat{X}_{t_i}, 0 \le i \le n)] && \text{(Monte-Carlo error)} \\
&\quad + \mathbb{E}[\hat{F}(\hat{X}_{t_i}, 0 \le i \le n)] - \mathbb{E}[\hat{F}(X_{t_i}, 0 \le i \le n)] && \text{(approximation of } X \text{ by } \hat{X}) \\
&\quad + \mathbb{E}[\hat{F}(X_{t_i}, 0 \le i \le n)] - \mathbb{E}[F(X_t, t \in [0, T])]. && \text{(approximation of } F \text{ by } \hat{F})
\end{aligned}$$

If $F(x(t), t \in [0, T]) = f(x(T))$ for some function f, we can take as well $\hat{F}(x(t_i), 0 \le i \le n) = f(x(T))$ and eliminate the approximation error of F by $\hat{F}$. Also, if one is able to generate exact samples $(X^k_{t_i}, 0 \le i \le n)$ of the process, the second term of the error disappears.

In this chapter, we are concerned with the approximation of $(X_{t_i}, 0 \le i \le n)$ by $(\hat{X}_{t_i}, 0 \le i \le n)$. Broadly speaking, there are two points of view to analyse the quality of an approximation. The first one, usually called the strong error, focuses on estimating how far is $(\hat{X}_{t_i}, 0 \le i \le n)$ from $(X_{t_i}, 0 \le i \le n)$ on the same event. The second one, usually called the weak error, focuses on estimating how far is the law of $(\hat{X}_{t_i}, 0 \le i \le n)$ from the one of $(X_{t_i}, 0 \le i \le n)$. Of course, there are many different possible criteria to quantify these distances. We will mainly use the two following ones.

Definition 2.0.1 An approximation scheme $(\hat{X}_{t_i}, 0 \le i \le n)$ for the process X is said to have a strong error of order $\nu > 0$ if

$$\exists C > 0, \forall n \in \mathbb{N}^*,\ \mathbb{E}\left[\max_{0 \le i \le n} \|\hat{X}_{t_i} - X_{t_i}\|\right] \le \frac{C}{n^\nu}.$$

It has a weak error of order $\nu > 0$ if for any $\mathcal{C}^\infty$ function $f : \mathbb{R}^d \to \mathbb{R}$ with compact support,

$$\exists C > 0, \forall n \in \mathbb{N}^*,\ |\mathbb{E}[f(\hat{X}_T)] - \mathbb{E}[f(X_T)]| \le \frac{C}{n^\nu}.$$

Since a $\mathcal{C}^\infty$ function with compact support is Lipschitz, we observe that the weak order of convergence is equal or higher than the strong order of convergence.

In this book, we will focus on the case where X is the solution of a time homogeneous Stochastic Differential Equation. Namely, we consider the following diffusion process

$$\boxed{X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma(X_s^x)dW_s, \ t \geq 0,} \tag{2.1}$$

that takes values in a domain $\mathbb{D} \subset \mathbb{R}^d$. Here, W is a standard Brownian motion of dimension d_W. The functions $b : \mathbb{D} \to \mathbb{R}^d$ and $\sigma : \mathbb{D} \to \mathcal{M}_{d \times d_W}(\mathbb{R})$ take respectively their values in $\mathbb{R}^d$ and in the set of matrices with d rows and d_W columns, and we make the following sublinear growth assumption:

$$\boxed{\exists K > 0, \forall x \in \mathbb{D}, \|b(x)\| + \|\sigma(x)\| \leq K(1 + \|x\|). \qquad (2.2)}$$

This gives the uniform boundedness of any moments (see Karatzas and Shreve [83], Problem 3.15, p. 306)

$$\forall p, T > 0, \exists C > 0, \forall x \in \mathbb{D}, \ \mathbb{E}\left[\max_{t \in [0,T]} \|X_t^x\|^p\right] \leq C(1 + \|x\|^p),$$

and excludes explosion in finite time. We assume that for any starting point $x \in \mathbb{D}$, there is a unique weak solution for the SDE (2.1) such that

$$\forall x \in \mathbb{D}, \ \mathbb{P}(\forall t \geq 0, X_t^x \in \mathbb{D}) = 1, \tag{2.3}$$

which means that it is well defined for any $t \geq 0$ and stays in the domain $\mathbb{D}$. It satisfies then the strong Markov property (Theorem 4.20, p. 322 in Karatzas and Shreve [83]). In this book, we will consider different domains $\mathbb{D}$ such as the set of semidefinite positive matrices or the set of correlation matrices. For example, a natural choice for the CIR process is to consider $\mathbb{D} = \mathbb{R}_+$.

Remark 2.0.2 It is still possible to consider a time-dependent diffusion as a time homogeneous diffusion. Let us consider a $\mathbb{D}$-valued diffusion

$$X_t^x = x + \int_0^t b(s, X_s^x)ds + \int_0^t \sigma(s, X_s^x)dW_s.$$

Then the process $((t, X_t^x), t \geq 0)$ is solution of the homogeneous SDE

$$d(t, X_t^x) = (0, x) + \int_0^t (1, b(s, X_s^x))ds + \int_0^t (0, \sigma(s, X_s^x)dW_s).$$

The chapter is structured as follows. First, we present the Euler-Maruyama scheme and its main properties. This is probably the simplest discretization scheme, and certainly the most popular one. However, as we will see through this book, the

Euler-Maruyama is not well suited for Affine diffusions. Also, more generally, one may wish to get approximation schemes that have better convergence properties than the Euler-Maruyama scheme. For sake of completeness, we present briefly strong approximations. However, we will not use this approach in this book and will rather work with weak approximations to sample Affine diffusions. Thus, we explain in this chapter how it is possible to construct discretization schemes that have a weak error convergence of order 2. In particular, we present the scheme of Ninomiya and Victoir [109] that is a general second order scheme.

2.1 The Euler-Maruyama Scheme

The Euler-Maruyama scheme for the SDE (2.1) on the regular time grid $t_i = iT/n$ is defined as follows:

$$\hat{X}_{t_0} = x,\ \hat{X}_{t_{i+1}} = \hat{X}_{t_i} + b(\hat{X}_{t_i})\frac{T}{n} + \sigma(\hat{X}_{t_i})(W_{t_{i+1}} - W_{t_i}),\ 1 \le i \le n-1. \tag{2.4}$$

We can also consider its time continuous extension that we define by

$$\hat{X}_t = \hat{X}_{t_i} + b(\hat{X}_{t_i})(t - t_i) + \sigma(\hat{X}_{t_i})(W_t - W_{t_i}),\ \text{when } t \in [t_i, t_{i+1}). \tag{2.5}$$

Note that the Euler scheme $\hat{X}$ depends on n through the discretization grid and should be denoted by $\hat{X}^n$ to recall this dependency. Unless necessary when we consider simultaneously two different time discretizations, we will prefer to use the light notation $\hat{X}$.

From the definition of the Euler scheme (2.4), we see that if $\sigma(\hat{X}_{t_i})\sigma(\hat{X}_{t_i})^\top$ is invertible for some i, then $\hat{X}_{t_{i+1}}$ can take any value in $\mathbb{R}^d$ because the density of the Gaussian increment is positive on $\mathbb{R}^d$. Thus, if the domain $\mathbb{D}$ is strictly included in $\mathbb{R}^d$, the Euler scheme is no longer well-defined: with some positive probability, $b(\hat{X}_{t_{i+1}})$ and $\sigma(\hat{X}_{t_{i+1}})$ are not defined and $\hat{X}_{t_{i+2}}$ cannot be defined. To get round this problem, it is possible to work with extensions $\tilde{b} : \mathbb{R}^d \to \mathbb{R}^d$ and $\tilde{\sigma} : \mathbb{R}^d \to \mathcal{M}_{d \times d_W}(\mathbb{R})$ such that

$$\forall x \in \mathbb{D},\ \tilde{b}(x) = b(x) \text{ and } \tilde{\sigma}(x) = \sigma(x),$$

and consider the Euler scheme (2.4) with $\tilde{b}$ and $\tilde{\sigma}$ instead of b and σ. A possible choice is simply to take $\tilde{b}(x) = 0$ and $\tilde{\sigma}(x) = 0$ for $x \notin \mathbb{D}$. We will discuss other possible choices in the CIR case (see Sect. 3.2). Here, we will directly assume that $\mathbb{D} = \mathbb{R}^d$ throughout this section on the Euler scheme, so that $\hat{X}$ is well defined.

We now present well known results on the strong and weak error for the Euler scheme.

2.1.1 The Strong Error

Let $\|.\|$ denote a norm on $\mathbb{R}^d$. The first technical lemma shows that the moments of the Euler scheme are uniformly bounded with respect to the time-step. Then, we prove the main result on the strong error.

Lemma 2.1.1 *Let (2.2) hold. Then, for any $p > 0$ there is a constant $C > 0$ that depends on p, T and the sublinear growth constant K such that*

$$\forall n \in \mathbb{N}^*, \mathbb{E}\left[\sup_{t\in[0,T]} \|\hat{X}_t\|^p\right] \le C(1 + \|x\|^p).$$

Proof First, we remark that if the upper bound above holds for some p, then it also holds for any $p' \in (0, p]$ from Hölder's inequality. Thus, we can assume without loss of generality that $p \ge 2$.

For $t \in [0, T]$, we set $\tau_t = \max\{t_i,\ t_i \le t\}$ and can write the Euler scheme as follows:

$$\hat{X}_t = x + \int_0^t b(\hat{X}_{\tau_s})ds + \int_0^t \sigma(\hat{X}_{\tau_s})dW_s. \tag{2.6}$$

Let $t' \in [0, T]$. We have $\|\hat{X}_t\|^p \le 3^{p-1}\Big(\|x\|^p + \left\|\int_0^t b(\hat{X}_{\tau_s})ds\right\|^p + \left\|\int_0^t \sigma(\hat{X}_{\tau_s})dW_s\right\|^p\Big)$ and then by Jensen's inequality

$$\sup_{t\in[0,t']} \|\hat{X}_t\|^p \le 3^{p-1}\left(\|x\|^p + T^{p-1}\int_0^{t'} \left\|b(\hat{X}_{\tau_s})\right\|^p ds + \sup_{t\in[0,t']}\left\|\int_0^t \sigma(\hat{X}_{\tau_s})dW_s\right\|^p\right).$$

Burkholder-Davis-Gundy and Jensen inequalities give

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,t']}\left\|\int_0^t \sigma(\hat{X}_{\tau_s})dW_s\right\|^p\right] &\le C_p\mathbb{E}\left[\left(\int_0^{t'} \|\sigma(\hat{X}_{\tau_s})\|^2 ds\right)^{p/2}\right] \\ &\le C_p T^{p/2-1}\int_0^{t'} \mathbb{E}[\|\sigma(\hat{X}_{\tau_s})\|^p]ds.\end{aligned}$$

From (2.2), we get that there is a constant $C > 0$ (depending on p and T and the sublinear growth constant) such that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,t']}\|\hat{X}_t\|^p\right] &\le C\left(\|x\|^p + 1 + \int_0^{t'} \mathbb{E}[\|\hat{X}_{\tau_t}\|^p]dt\right) \\ &\le C\left(\|x\|^p + 1 + \int_0^{t'} \mathbb{E}\left[\sup_{s\in[0,t]}\|\hat{X}_s\|^p\right]dt\right).\end{aligned}$$

This holds for any $t' \in [0, T]$, and Gronwall's lemma then gives

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\hat{X}_t\|^p\right] \leq C\,(\|x\|^p + 1)\, e^{CT},$$

which is the desired result. □

Theorem 2.1.2 (Kanagawa [82]) *Assume that b and σ are Lipschitz, i.e.*

$$\exists K > 0, \forall x, y \in \mathbb{R}^d, \|b(x) - b(y)\| + \|\sigma(x) - \sigma(y)\| \leq K\|x - y\|.$$

Then, the strong error of the Euler scheme is of order $1/2$. *More precisely, we have*

$$\forall p > 0, \exists C > 0, \forall n \in \mathbb{N}^*,\ \mathbb{E}\left[\max_{t\in[0,T]} \|\hat{X}^n_t - X^x_t\|^p\right]^{\frac{1}{p}} \leq \frac{C}{n^{1/2}}.$$

Unless in the special case where σ is a constant function, this rate is optimal. In fact, Kurtz and Protter [91, 92] have shown that $(\sqrt{n}(\hat{X}^n_t - X_t), t \in [0, T])$ converges in law towards some non degenerate process and therefore $\mathbb{E}\left[\max_{t\in[0,T]} \|\sqrt{n}(\hat{X}^n_t - X_t)\|^p\right]^{\frac{1}{p}}$ converges towards a positive constant.

Proof Again, it is sufficient from Hölder's inequality to prove the claim for $p > 0$ large enough, and we assume without loss of generality that $p \geq 2$.

By subtracting the diffusion (2.1) to the Euler scheme (2.6) and taking the norm, we get

$$\begin{aligned}\|\hat{X}_t - X^x_t\|^p &\leq 2^{p-1}\left(\left\|\int_0^t b(\hat{X}_{\tau_s}) - b(X^x_s)ds\right\|^p + \left\|\int_0^t \sigma(\hat{X}_{\tau_s}) - \sigma(X^x_s)dW_s\right\|^p\right)\\ &\leq 2^{p-1}\left(t^{p-1}\int_0^t K^p\|\hat{X}_{\tau_s} - X^x_s\|^p ds + \sup_{s\in[0,t]}\left\|\int_0^s \sigma(\hat{X}_{\tau_u}) - \sigma(X^x_u)dW_u\right\|^p\right),\end{aligned}$$

by Jensen's inequality. Since the right hand side is nondecreasing with respect to t, we get

$$\begin{aligned}\sup_{s\in[0,t]} \|\hat{X}_s - X^x_s\|^p \leq 2^{p-1}\Bigg(&t^{p-1}\int_0^t K^p\|\hat{X}_{\tau_s} - X^x_s\|^p ds\\ &+ \sup_{s\in[0,t]}\left\|\int_0^s \sigma(\hat{X}_{\tau_u}) - \sigma(X^x_u)dW_u\right\|^p\Bigg).\end{aligned}$$

From the Burkholder-Davis-Gundy inequality, and by using again the Lipschitz property and Jensen's inequality, we get that there is a constant $C_p > 0$ such that

$$\mathbb{E}\left[\left\|\sup_{s\in[0,t]}\int_0^s \sigma(\hat{X}_{\tau_u}) - \sigma(X_u^x)dW_u\right\|^p\right] \leq C_p\mathbb{E}\left[\left(\int_0^t \|\sigma(\hat{X}_{\tau_s}) - \sigma(X_s^x)\|^2 ds\right)^{p/2}\right]$$
$$\leq C_p t^{p/2-1}K^p\mathbb{E}\left[\int_0^t \|\hat{X}_{\tau_s} - X_s^x\|^p ds\right].$$

We now take $t \in [0, T]$. Using the triangle inequality, we get that there is a constant $C > 0$ depending on p, K and T such that

$$\mathbb{E}\left[\sup_{s\in[0,t]}\|\hat{X}_s - X_s^x\|^p\right] \leq C\mathbb{E}\left[\int_0^t \|\hat{X}_s - X_s^x\|^p + \|\hat{X}_s - \hat{X}_{\tau_s}\|^p ds\right]$$
$$\leq C\left(\mathbb{E}\left[\int_0^t \sup_{u\in[0,s]}\|\hat{X}_u - X_u^x\|^p ds\right] + \mathbb{E}\left[\int_0^t \|\hat{X}_s - \hat{X}_{\tau_s}\|^p ds\right]\right).$$

Besides, we have

$$\mathbb{E}\left[\|\hat{X}_s - \hat{X}_{\tau_s}\|^p\right] \leq 2^{p-1}\mathbb{E}\left[(s-\tau_s)^p\,\|b(\hat{X}_{\tau_s})\|^p + \mathbb{E}\left[\|\sigma(\hat{X}_{\tau_s})(W_s - W_{\tau_s})\|^p\middle|\mathcal{F}_{\tau_s}\right]\right]$$
$$\leq C\left(\frac{T}{n}\right)^{p/2}$$

for some constant $C > 0$ by using Lemma 2.1.1 and the sublinear growth assumption (2.2), which is also a consequence of the Lipschitz property. Plugging this inequality into the previous one yields to

$$\mathbb{E}\left[\sup_{s\in[0,t]}\|\hat{X}_s - X_s^x\|^p\right] \leq C\left(\mathbb{E}\left[\int_0^t \sup_{u\in[0,s]}\|\hat{X}_u - X_u^x\|^p ds\right] + \left(\frac{T}{n}\right)^{p/2}\right).$$

We conclude by Gronwall's lemma that $\mathbb{E}\left[\sup_{s\in[0,t]}\|\hat{X}_s - X_s^x\|^p\right] \leq Ce^{CT}\left(\frac{T}{n}\right)^{p/2}$.

□

2.1.2 The Weak Error

We first introduce some notations. For a $\mathcal{C}^\infty$ function $f : \mathbb{R}^d \to \mathbb{R}$ and $1 \leq i \leq d$, we denote by $\partial_i f(x)$ the partial derivative of f with respect to the ith coordinate x_i. More generally, for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, we set $\partial_\alpha f(x) = \partial_1^{\alpha_1}\ldots\partial_d^{\alpha_d} f(x)$. We state now the main result.

Theorem 2.1.3 (Talay and Tubaro [120]) *Let us assume that b and σ are $\mathcal{C}^\infty$ functions whose derivatives of any order are bounded, i.e.*

$$\forall 1 \le i \le d, 1 \le j \le d_W,\ \forall \alpha \in \mathbb{N}^d \setminus \{0\}, \exists C_\alpha > 0, |\partial_\alpha b_i(x)| + |\partial_\alpha \sigma_{ij}(x)| \le C_\alpha.$$

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function such that

$$\forall \alpha \in \mathbb{N}^d, \exists C_\alpha > 0, e_\alpha \in \mathbb{N}^*,\ \forall x \in \mathbb{R}^d, \|\partial_\alpha f(x)\| \le C_\alpha(1 + \|x\|^{e_\alpha}).$$

Then, there is a constant $C > 0$ such that

$$\forall n \in \mathbb{N}^*, |\mathbb{E}[f(X_T^x)] - \mathbb{E}[f(\hat{X}_T^n)]| \le C/n. \tag{2.7}$$

Besides, for any $\nu \in \mathbb{N}^*$, there are constants $c_1, \dots, c_\nu \in \mathbb{R}$ such that

$$\mathbb{E}[f(\hat{X}_T^n)] = \mathbb{E}[f(X_T^x)] + \frac{c_1}{n} + \cdots + \frac{c_\nu}{n^\nu} + O\left(\frac{1}{n^{\nu+1}}\right). \tag{2.8}$$

We admit this result, and will only give a partial proof of (2.7), see Theorem 2.3.8. Let us make some comments on this. First, we observe that the order of the weak convergence is one. It is then strictly better than the order of the strong convergence which is equal to one half. A Taylor expansion around X_T^x gives $f(\hat{X}_T^n) = f(X_T^x) + f'(X_T^x)(\hat{X}_T^n - X_T^x) + O((\hat{X}_T^n - X_T^x)^2)$, and then from Theorem 2.1.2,

$$\mathbb{E}[f(\hat{X}_T^n)] = \mathbb{E}[f(X_T^x)] + \mathbb{E}[f'(X_T^x)(\hat{X}_T^n - X_T^x)] + O(1/n).$$

Thus, Theorem 2.1.3 gives that $\mathbb{E}[f'(X_T^x)(\hat{X}_T^n - X_T^x)] = O(1/n)$. Roughly speaking, the term of order $O(n^{-1/2})$ has in fact an expectation of order $O(n^{-1})$.

Besides, thanks to the error expansion (2.8), it is possible to accelerate the convergence by using the Romberg (or Richardson) extrapolation:

$$2\mathbb{E}[f(\hat{X}_T^{2n})] - \mathbb{E}[f(\hat{X}_T^n)] = \mathbb{E}[f(X_T^x)] + O\left(\frac{1}{n^2}\right).$$

This estimator converges thus faster toward $\mathbb{E}[f(X_T^x)]$ at a rate of order 2. In practice, one has then to run a Monte-Carlo algorithm that samples the discretization scheme for two different time-steps (here T/n and $T/(2n)$). This approach has been studied in detail by Kebaier [85] to determine the optimal number of samples to generate for each time step. A generalization of this approach called the Multilevel Monte-Carlo algorithm has been proposed by Giles [61]. It involves more than two different time steps.

A possible drawback of Theorem 2.1.3 is that it only applies to smooth functions f. In practice, a convergence of order one is observed for functions that are less regular. In this direction, Bally and Talay [16] have shown that $\mathbb{E}[f(\hat{X}_T^n)] =$

$\mathbb{E}[f(X_T^x)] + \frac{c_1}{n} + O(1/n^2)$ when f is a bounded measurable function under the same smoothness assumption on b and σ and an Hörmander type condition. Guyon [73] has even shown that this expansion makes sense when f is a tempered distribution, provided that σ satisfies an uniform ellipticity condition.

However, the Euler scheme is often used in practice to calculate pathwise expectations while the weak error expansion above only focuses on expectations that involve the final value X_T^x. Thus, it would be interesting to study the convergence rate of $\mathbb{E}[F(\hat{X}_t^n, t \in [0,T])]$ towards $\mathbb{E}[F(X_t, t \in [0,T])]$, under some assumptions on $F : \mathcal{C}([0,T], \mathbb{R}^d) \to \mathbb{R}$. Unless for particular cases of F such as $F(x(t), t \in [0,T]) = \int_0^T g(x(s))ds$ for some function $g : \mathbb{R}^d \to \mathbb{R}$, very few results are known on this convergence. Of course, when F is a Lipschitz function with respect to the sup norm, i.e.

$$\forall x, y \in \mathcal{C}([0,T], \mathbb{R}^d), |F(x) - F(y)| \le [F]_{Lip} \max_{t \in [0,T]} \|x(t) - y(t)\|,$$

we can apply Theorem 2.1.2 with $p = 1$ and get

$$\exists C > 0, \forall n \in \mathbb{N}^*, \ \left|\mathbb{E}[F(\hat{X}_t^n, t \in [0,T])] - \mathbb{E}[F(X_t, t \in [0,T])]\right| \le \frac{C[F]_{Lip}}{\sqrt{n}}.$$

On the other hand, we know from the weak error estimates above that the convergence order cannot be better than 1, since the constant c_1 in the expansion (2.8) is generally different from zero. Thus, we know from these results that

$$\exists C, c > 0, \ \frac{c}{n} \le \sup_{\substack{F:\mathcal{C}([0,T],\mathbb{R}^d)\to\mathbb{R}, \\ [F]_{Lip}\le 1}} \left|\mathbb{E}[F(\hat{X}^n)] - \mathbb{E}[F(X)]\right| \le \frac{C}{n^{1/2}}.$$

Recently, Alfonsi et al. [10] have shown that

$$\forall \varepsilon > 0, \exists C > 0, \ \sup_{\substack{F:\mathcal{C}([0,T],\mathbb{R}^d)\to\mathbb{R}, \\ [F]_{Lip}\le 1}} \left|\mathbb{E}[F(\hat{X}^n)] - \mathbb{E}[F(X)]\right| \le \frac{C}{n^{2/3-\varepsilon}},$$

when $d = 1$ and assuming that $\underline{\sigma} \le \sigma(x)$ for some $\underline{\sigma} > 0$. This shows that the convergence rate for pathwise expectation is strictly better than the convergence rate given by the strong error. However, it still remains many open questions. In particular, is the rate of order $2/3 - \varepsilon$ optimal or could we prove an order of convergence closer to 1? Also, is it possible to extend this result to functions F that are less regular than Lipschitz?

2.1.3 Beyond the Euler Scheme: Strong and Weak High Order Approximations

The Euler-Maruyama discretization scheme is simple and easy to implement in practice. However, one may wish to use other discretization schemes to accelerate the convergence. To do so, two different approaches exist in the literature. The first one consists in finding a discretization scheme that has a better strong convergence order, while the second one focuses on getting a better weak convergence order. Of course, these objectives are not disconnected since the weak order of convergence is at least better than the strong order of convergence. However, in the first case, one has to find a scheme such that for any given Brownian path $(W_t, t \in [0,T])$, $(\hat{X}_t, t \in [0,T])$ is close to $(X_t^x, t \in [0,T])$. In the second case, we only require that the law of $(\hat{X}_t, t \in [0,T])$ is close to the law of $(X_t^x, t \in [0,T])$. In particular, this relaxes the need to define the discretization scheme $\hat{X}$ that corresponds to a given Brownian path $(W_t, t \in [0,T])$, and we can define it intrinsically. When the aim of the simulation is to run a crude Monte-Carlo algorithm, only the law of $(\hat{X}_t, t \in [0,T])$ matters. Getting accurate schemes for the weak error is then sufficient for this use.

2.2 Strong Approximations

The main way to construct strong discretization schemes is to use iterated stochastic Taylor expansion. To fix the ideas, let us assume for a while that the dimension $d = 1$ and that the coefficients b and σ are smooth. We have:

$$\forall t \in [t_i, t_{i+1}], X_t^x = X_{t_i}^x + \int_{t_i}^t b(X_s^x)ds + \int_{t_i}^t \sigma(X_s^x)dW_s, \tag{2.9}$$

and the Euler scheme (2.5) can be seen as the approximation where X_s^x is replaced by $X_{t_i}^x$. We can however use Itô's formula to get:

$$\begin{aligned} f(X_s^x) &= f(X_{t_i}^x) + \int_{t_i}^s f'(X_u^x)b(X_u^x) + \frac{1}{2}f''(X_u^x)\sigma^2(X_u^x)du \\ &\quad + \int_{t_i}^s f'(X_u^x)\sigma(X_u^x)dW_u \\ &\approx f(X_{t_i}^x) + \left[f'(X_{t_i}^x)b(X_{t_i}^x) + \frac{1}{2}f''(X_{t_i}^x)\sigma^2(X_{t_i}^x)\right](s - t_i) \\ &\quad + f'(X_{t_i}^x)\sigma(X_{t_i}^x)(W_s - W_{t_i}). \end{aligned} \tag{2.10}$$

Using this approximation for $f \in \{b, \sigma\}$ in (2.9), this suggests the following discretization scheme for $t \in [t_i, t_{i+1}]$:

$$
\begin{aligned}
\hat{X}_t &= \hat{X}_{t_i} + b(\hat{X}_{t_i})(t - t_i) + \sigma(\hat{X}_{t_i})(W_t - W_{t_i}) \\
&+ \frac{1}{2}\left[b'(X^x_{t_i})b(X^x_{t_i}) + \frac{1}{2}b''(X^x_{t_i})\sigma^2(X^x_{t_i})\right](t - t_i)^2 \\
&+ b'(X^x_{t_i})\sigma(X^x_{t_i})\int_{t_i}^{t}(W_s - W_{t_i})ds \\
&+ \left[\sigma'(X^x_{t_i})b(X^x_{t_i}) + \frac{1}{2}\sigma''(X^x_{t_i})\sigma^2(X^x_{t_i})\right]\int_{t_i}^{t}(s - t_i)dW_s \\
&+ \sigma'(X^x_{t_i})\sigma(X^x_{t_i})\int_{t_i}^{t}(W_s - W_{t_i})dW_s.
\end{aligned}
$$

We can see this scheme as the addition of some corrective terms to the Euler scheme. If one wants to achieve a strong convergence order equal to 1, many of these terms are useless. Roughly speaking, $W_{t_{i+1}} - W_{t_i}$ is of order $O(n^{-1/2})$ and therefore $\int_{t_i}^{t}(W_s - W_{t_i})dW_s = \frac{1}{2}[(W_t - W_{t_i})^2 - (t - t_i)]$ is of order $O(n^{-1})$ while the other corrective terms are of order $O(n^{-3/2})$ or $O(n^{-2})$ and can be neglected. This leads to the well known Milstein's scheme:

$$
\hat{X}_t = \hat{X}_{t_i} + b(\hat{X}_{t_i})(t - t_i) + \sigma(\hat{X}_{t_i})(W_t - W_{t_i}) + \frac{\sigma(\hat{X}_{t_i})\sigma'(\hat{X}_{t_i})}{2}[(W_t - W_{t_i})^2 - (t - t_i)]. \tag{2.11}
$$

We know from Milstein [107] that this scheme has a strong order of convergence equal to 1, when b and σ are $\mathcal{C}^2$ with bounded derivatives. It is even possible to derive schemes with any higher order of convergence by using again Itô's Formula to the integrands in (2.10), and repeating this procedure up to reach the desired strong convergence order. We refer to Kloeden and Platen [89], Chap. 10, for a detailed description of this approach.

Unfortunately, these high order schemes can be barely implemented in practice for dimensions $d_W \geq 2$. In fact, unless under some rather restrictive commutativity conditions on σ (see [89], p. 348), Milstein's scheme already requires to sample the joint law of $\left((W_{t_{i+1}} - W_{t_i})_k, \int_{t_i}^{t_{i+1}}(W_t - W_{t_i})_k d(W_t)_l\right)_{1 \leq k,l \leq d_W}$. Since $\int_{t_i}^{t_{i+1}}(W_t - W_{t_i})_k d(W_t)_k = \frac{1}{2}\left[(W_{t_{i+1}} - W_{t_i})_k^2 - T/n\right]$ and $(W_{t_{i+1}} - W_{t_i})_k(W_{t_{i+1}} - W_{t_i})_l = \int_{t_i}^{t_{i+1}}(W_t - W_{t_i})_k d(W_t)_l + \int_{t_i}^{t_{i+1}}(W_t - W_{t_i})_l d(W_t)_k$ for $k \neq l$, it is in fact sufficient to sample the law of

$$
\left((W_{t_{i+1}} - W_{t_i})_k, \int_{t_i}^{t_{i+1}}(W_t - W_{t_i})_k d(W_t)_l\right)_{1 \leq k < l \leq d_W}.
$$

Gaines and Lyons [58] have given a method to do this in dimension $d_W = 2$. Doing this exactly in higher dimension is still an open problem. A possible way to get round this problem is to find accurate approximations of these laws that can be sampled. The cubature formulas proposed by Lyons and Victoir [102] give a practical way to achieve this goal. Developing this topic would be far beyond the scope of this book, and we refer to the books of Lyons et al. [103] and Friz and Victoir [55] for a detailed presentation on Rough Paths.

2.3 Weak Approximations

This section is devoted to weak approximation schemes and the analysis of the weak error. Contrary to the strong approximation schemes, we will no longer expect that the increments $\hat{X}_{t_{i+1}} - \hat{X}_{t_i}$ are close to $X_{t_{i+1}} - X_{t_i}$ for a given Brownian realization. We will only focus on their similarity in law. When X is the solution of the SDE (2.1) or more generally is a $\mathbb{D}$-valued time homogeneous Markov process, the conditional law of $X_{t_{i+1}}$ given $(X_{t_0}, \dots, X_{t_i})$ is the same as the conditional law of $X_{t_{i+1}}$ given X_{t_i}. Besides, this law only depends on the time step T/n. Thus, it is natural to consider an approximation scheme $\hat{X}$ that has the same property and generate $\hat{X}_{t_{i+1}}$ according to a law that only depends on T/n and $\hat{X}_{t_i}$. This motivates the following definition.

Definition 2.3.1 A family of transition probabilities $(\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})$ on $\mathbb{D}$ is such that $\hat{p}_x(t)$ is a probability law on $\mathbb{D}$ for $t > 0$ and $x \in \mathbb{D}$. We will denote by $\hat{X}_t^x$ a random variable distributed according to the probability law $\hat{p}_x(t)(dz)$.

An approximation scheme with transition probabilities $(\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})$ is a sequence $(\hat{X}_{t_i}, 0 \le i \le n)$ of $\mathbb{D}$-valued random variables such that for any $1 \le i \le n$, $\hat{p}_{\hat{X}_{t_i}}(T/n)(dz)$ is the law of $\hat{X}_{t_{i+1}}$ conditional to $(\hat{X}_{t_0}, \dots, \hat{X}_{t_i})$, i.e.

$$\mathbb{E}[f(\hat{X}_{t_{i+1}})|\hat{X}_{t_0}, \dots, \hat{X}_{t_i}] = \int_{\mathbb{R}^d} f(z)\hat{p}_{\hat{X}_{t_i}}(T/n)(dz),$$

for any bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$.

A discretization scheme $(\hat{X}_{t_i}, 0 \le i \le n)$ is thus entirely determined by its initial value and its transition probabilities. Since the initial value is quite always taken equal to the initial value of the SDE, we will call with a slight abuse of language "scheme" either the whole path $(\hat{X}_{t_i}, 0 \le i \le n)$, its transition probabilities $\hat{p}_x(t)(dz)$ or even the random variable $\hat{X}_t^x$. Within this framework, the Euler-Maruyama scheme corresponds to take $\hat{p}_x(t)(dz)$ as the probability measure of a Gaussian vector with mean $x + b(x)t$ and covariance $t\sigma(x)\sigma(x)^\top$.

We now present a framework to analyze the weak error that has been proposed in Alfonsi [8] and is convenient for affine diffusions.

2.3.1 The Weak Error Analysis

We first introduce some notations. We recall that the domain $\mathbb{D}$, where the SDE (2.1) takes its values, is a subset of $\mathbb{R}^d$. For $1 \le i \le d$, ∂_i is the partial differential operator with respect to the i-th coordinate x_i. For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we denote by $\partial_\alpha = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$ the operator that differentiates α_i times with respect to x_i, and we set $|\alpha| = \sum_{l=1}^d \alpha_l$. We introduce the following functional space

$$\begin{aligned}\mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D}) = \big\{&f \in \mathcal{C}^\infty(\mathbb{D}, \mathbb{R}), \forall \alpha \in \mathbb{N}^d, \exists C_\alpha > 0, e_\alpha \in \mathbb{N}^*, \forall x \in \mathbb{D},\\ &|\partial_\alpha f(x)| \le C_\alpha(1 + \|x\|^{e_\alpha})\big\},\end{aligned}$$

where $\|.\|$ is a given norm on $\mathbb{R}^d$. This is the space of smooth functions whose any derivatives have a polynomial growth. It contains in particular all the smooth functions with a compact support.

Definition 2.3.2 We will say that $(C_\alpha, e_\alpha)_{\alpha \in \mathbb{N}^d}$ is a good sequence for $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$ if one has $\forall x \in \mathbb{D}, |\partial_\alpha f(x)| \le C_\alpha(1 + \|x\|^{e_\alpha})$.

We now make further assumptions on the coefficients of the SDE (2.1). We assume that $b : \mathbb{D} \to \mathbb{R}^d$ and $\sigma : \mathbb{D} \to \mathcal{M}_{d \times d_W}(\mathbb{R})$ are such that

$$\forall 1 \le i \le d, 1 \le j \le d_W, x \in \mathbb{D} \mapsto b_i(x), x \in \mathbb{D} \mapsto (\sigma(x)\sigma^\top(x))_{i,j} \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D}). \tag{2.12}$$

This assumption is automatically satisfied if $b(x)$ and $\sigma(x)\sigma^\top(x)$ are affine functions of x, which corresponds exactly to the case of affine diffusion (see Proposition 4.1.2). The infinitesimal generator associated to the SDE (2.1) is given by

$$f \in \mathcal{C}^2(\mathbb{D}, \mathbb{R}),\ Lf(x) = \sum_{i=1}^d b_i(x)\partial_i f(x) + \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d (\sigma(x)\sigma^\top(x))_{i,j}\partial_i \partial_j f(x). \tag{2.13}$$

In fact, from a straightforward application of Itô's formula, we have

$$\frac{\mathbb{E}[f(X_t^x)] - f(x)}{t} \underset{t \to 0+}{\to} Lf(x).$$

Thanks to the regularity assumptions made on b and σ, we observe that $\mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$ is stable by the differential operator L. Namely if $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$, then all the iterated functions $L^k f(x)$ are well defined on $\mathbb{D}$ and belong to $\mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$ for any $k \in \mathbb{N}$.

Definition 2.3.3 We will say that a differential operator L satisfies the required assumptions on $\mathbb{D}$ if it is defined by (2.13) for some functions $b(x)$ and $\sigma(x)$ that satisfies (2.12), (2.2) and (2.3).

To study the weak error, we will focus on the asymptotic behavior of

$$\mathbb{E}[f(\hat{X}_t^x)] - \mathbb{E}[f(X_t^x)],$$

when $t \to 0^+$, for $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$. Heuristically, the smaller is this quantity, the better would be the approximation scheme $\hat{X}_t^x$. However, we need to make this more precise and introduce the following definition.

Definition 2.3.4 A function $(f, t, x) \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D}) \times \mathbb{R}_+^* \times \mathbb{D} \mapsto Rf(t, x) \in \mathbb{R}$ is a remainder of order $\nu \in \mathbb{N}$ if for any function $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$ with a good sequence $(C_\alpha, e_\alpha)_{\alpha\in\mathbb{N}^d}$, there exist positive constants C, E, and η depending only on $(C_\alpha, e_\alpha)_{\alpha\in\mathbb{N}^d}$ such that

$$\forall t \in (0, \eta), \forall x \in \mathbb{D}, |Rf(t, x)| \le Ct^\nu(1 + \|x\|^E).$$

We will say that $\hat{X}_t^x$ is a potential weak νth-order scheme for the operator L if $(f, t, x) \mapsto \mathbb{E}[f(X_t^x)] - \mathbb{E}[f(\hat{X}_t^x)]$ is a remainder of order $\nu + 1$.

From a mathematical point of view, the important thing here is to notice that the estimate on the remainder is assumed to be the same for all function $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$ sharing the same good sequence. With this definition, we notice that the exact simulation scheme $\hat{X}_t^x = X_t^x$ is a potential weak νth-order scheme for L for any $\nu \in \mathbb{N}$. We have listed some rather immediate properties in the following exercise.

Exercise 2.3.5 Let $(f, t, x) \mapsto Rf(t, x)$ be a remainder of order ν and L an operator satisfying the required assumptions. Show that

1. For any $k, p \in \mathbb{N}$, $(f, t, x) \mapsto t^k L^p f(x)$ is a remainder of order k.
2. For any $\nu' \in \{0, \dots \nu\}$, $(f, t, x) \mapsto Rf(t, x)$ is a remainder of order ν'.
3. For any $\lambda, \tilde{\lambda} \in \mathbb{R}$ and $\tilde{R}$ remainder of order ν, $(f, t, x) \mapsto \lambda Rf(t, x) + \tilde{\lambda}\tilde{R} f(t, x)$ is a remainder of order ν.

Lemma 2.3.6 *Let L be an operator that satisfies the required assumptions on $\mathbb{D}$. Let $\hat{X}_t^x$ be a scheme such that*

$$\exists \eta, C, E > 0, \forall t \in (0, \eta), \ \|\hat{X}_t^x - X_t^x\| \le C(1 + \|x\|^E)t^{\nu+1}, \ a.s.$$

Then, $\hat{X}_t^x$ is a potential weak νth-order scheme.

Proof Let $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$. We have $f(\hat{X}_t^x) - f(X_t^x) = \int_0^1 \sum_{i=1}^d \partial_i f(X_t^x + h(\hat{X}_t^x - X_t^x))(\hat{X}_t^x - X_t^x)_i dh$ and therefore there is are constant $C', E' > 0$ such that for $t \in (0, \eta)$,

$$|f(\hat{X}_t^x) - f(X_t^x)| \le C' \sup_{h\in[0,1]} (1 + \|X_t^x + h(\hat{X}_t^x - X_t^x)\|^{E'}) \times C(1 + \|x\|^E)t^{\nu+1}.$$

Using the triangle inequality and the assumption, the first term of the right hand side is bounded by $C'(1 + \|X_t^x\|^{E'})$ for some new constants that we still denote by $C' > 0$ and $E' > 0$ for simplicity. Thanks to (2.2), moments are uniformly bounded and we get $|\mathbb{E}[f(\hat{X}_t^x)] - \mathbb{E}[f(X_t^x)]| \leq \mathbb{E}\left[|f(\hat{X}_t^x)] - \mathbb{E}[f(X_t^x)]|\right] \leq C'(1 + \|x\|^{E'})t^{\nu+1}$ for some constants $C' > 0$ and $E' > 0$, which gives the claim. □

Remark 2.3.7 Let $f \in \mathcal{C}^{\infty}_{\mathbf{pol}}(\mathbb{D})$ and an operator L satisfying the required assumptions. Thanks to Itô's formula and the boundedness of the moments of X^x, we have $\mathbb{E}[f(X_t^x)] = f(x) + \int_0^t \mathbb{E}[Lf(X_s^x)]ds$. Since $Lf \in \mathcal{C}^{\infty}_{\mathbf{pol}}(\mathbb{D})$, the same formula holds for $\mathbb{E}[Lf(X_s^x)]$, and we get $\mathbb{E}[f(X_t^x)] = f(x) + tLf(x) + \int_0^t \int_0^s \mathbb{E}[L^2 f(X_u^x)]duds = f(x) + tLf(x) + \int_0^t (t-s)\mathbb{E}[L^2 f(X_s^x)]ds$. Iterating this, we get that

$$\forall \nu \in \mathbb{N}, \forall t \geq 0,\ \mathbb{E}[f(X_t^x)] = \sum_{k=0}^{\nu} \frac{t^k}{k!} L^k f(x) + \int_0^t \frac{(t-s)^\nu}{\nu!} \mathbb{E}[L^{\nu+1} f(X_s^x)]ds.$$

Thanks to the sublinear growth condition (2.2), we have bounds on the moments of X_t^x, i.e. $\forall q \in \mathbb{N}^*, \exists C_q > 0, \forall t \in [0,1], \mathbb{E}[\|X_t^x\|^q] \leq C_q(1+x^q)$. Since $f \in \mathcal{C}^{\infty}_{\mathbf{pol}}(\mathbb{D})$ and L satisfies the required assumptions, there are constants $C > 0$ and $q \in \mathbb{N}^*$ depending only on a good sequence of f such that $|L^{\nu+1} f|(x) \leq C(1 + \|x\|^q)$. We deduce that

$$\left| \mathbb{E}[f(X_t^x)] - \sum_{k=0}^{\nu} \frac{t^k}{k!} L^k f(x) \right| \leq \frac{t^{\nu+1}}{(\nu+1)!} C(1 + C_q(1 + \|x\|^q)).$$

Thus, $\tilde{R}f(t,x) = \mathbb{E}[f(X_t^x)] - \sum_{k=0}^{\nu} \frac{t^k}{k!} L^k f(x)$ is a remainder of order $\nu + 1$, and $\hat{X}_t^x$ is a potential weak νth-order scheme for L if, and only if

$$Rf(t,x) = \mathbb{E}[f(\hat{X}_t^x)] - \sum_{k=0}^{\nu} \frac{t^k}{k!} L^k f(x) \text{ is a remainder of order } \nu + 1.$$

We are now in position to state the key result for the weak error analysis. This is in fact a direct consequence of the weak error analysis proposed by Talay and Tubaro [120] for the Euler scheme.

Theorem 2.3.8 *Let L be an operator that satisfies the required assumptions on $\mathbb{D}$. Let $(\hat{X}_{t_i}, 0 \leq i \leq n)$ be a discretization scheme with transition probabilities $\hat{p}_x(t)(dz)$ on $\mathbb{D}$ that starts from $\hat{X}_{t_0} = x \in \mathbb{D}$. We assume that*

(i) *$f : \mathbb{D} \to \mathbb{R}$ is a function such that $u(t,x) = \mathbb{E}[f(X_{T-t}^x)]$ is defined on $[0,T] \times \mathbb{D}$, $\mathcal{C}^\infty$, solves $\forall t \in [0,T], \forall x \in \mathbb{D}, \partial_t u(t,x) = -Lu(t,x)$, and satisfies:*

$$\forall l \in \mathbb{N}, \alpha \in \mathbb{N}^d, \exists C_{l,\alpha}, e_{l,\alpha} > 0, \forall x \in \mathbb{D}, t \in [0,T],$$
$$|\partial_t^l \partial_\alpha u(t,x)| \leq C_{l,\alpha}(1 + \|x\|^{e_{l,\alpha}}). \qquad (2.14)$$

(ii) The scheme $\hat{p}_x(t)$ is a potential weak νth-order discretization scheme for the operator L, and has uniformly bounded moments, i.e.

$$\forall q \in \mathbb{N}^*, \exists n_q \in \mathbb{N}^*, \sup_{n \geq n_q, 0 \leq i \leq n} \mathbb{E}[\|\hat{X}_{t_i}\|^q] < \infty. \tag{2.15}$$

Then, there is $K > 0$ and $n_0 \in \mathbb{N}$, such that

$$\forall n \geq n_0, \ |\mathbb{E}[f(\hat{X}_{t_n})] - \mathbb{E}[f(X_T^x)]| \leq K/n^\nu.$$

Before proving this result, let us make some comments on it. The first assumption (i) only brings on the diffusion itself. It is true or not, but it does not depend on how clever we could be to construct an approximation scheme. When $\mathbb{D} = \mathbb{R}^d$, $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$, b and σ are $\mathcal{C}^\infty$ with bounded derivatives, Talay [119] has shown that (i) is automatically satisfied. The second assumption (ii) only brings on the approximation scheme. The assumption (2.15) on the moments is usually satisfied. In fact we know from the sublinear growth assumption (2.2) that the continuous diffusion has uniformly bounded moments. Since the approximation scheme is meant to be close to the diffusion, we can expect this property to be satisfied otherwise the moments themselves would not be accurately approximated. Thus, the main assumption that is required on the approximation scheme in order to obtain a weak error of order ν is to have that $(f, t, x) \mapsto \mathbb{E}[f(X_t^x)] - \mathbb{E}[f(\hat{X}_t^x)]$ is a remainder of order $\nu + 1$. This justifies our denomination of "potential" weak νth-order discretization scheme.

Proof of Theorem 2.3.8 Following Talay and Tubaro [120], we write the weak error as follows

$$\begin{aligned}\mathbb{E}[f(\hat{X}_{t_n})] - \mathbb{E}[f(X_T^x)] &= \mathbb{E}[u(T, \hat{X}_{t_n}) - u(0, \hat{X}_{t_0})] \\ &= \sum_{i=0}^{n-1} \mathbb{E}[u(t_{i+1}, \hat{X}_{t_{i+1}}) - u(t_i, \hat{X}_{t_i})].\end{aligned} \tag{2.16}$$

From the Taylor expansion of u at the point $(t_{i+1}, \hat{X}_{t_i})$, $\partial_t u = -Lu$ and (2.14), we obtain

$$\begin{aligned}&\left| u(t_i, \hat{X}_{t_i}) - \left[u(t_{i+1}, \hat{X}_{t_i}) + \sum_{k=1}^{\nu} \frac{1}{k!} \left(\frac{T}{n}\right)^k L^k u(t_{i+1}, \hat{X}_{t_i}) \right] \right| \\ &\quad \leq \frac{(T/n)^{\nu+1}}{(\nu+1)!} C_{\nu+1,0}(1 + \|\hat{X}_{t_i}\|^{e_{\nu+1,0}}).\end{aligned}$$

On the other hand, we notice from (2.14) that all the functions $x \mapsto u(t, x)$ for $t \in [0, T]$ belong to $\mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$ and share the same good sequence $(C_{0,\alpha}, e_{0,\alpha})_\alpha$. From assumption (*ii*) and Remark 2.3.7, we get that there are positive constants $C,\ E,\ n_0$ that depend on $(C_{0,\alpha}, e_{0,\alpha})_\alpha$ such that for $n \geq n_0$,

$$\mathbb{E}[u(t_{i+1}, \hat{X}_{t_{i+1}})|\hat{X}_{t_i}] = u(t_{i+1}, \hat{X}_{t_i}) + \sum_{k=1}^{\nu} \frac{1}{k!}\left(\frac{T}{n}\right)^k L^k u(t_{i+1}, \hat{X}_{t_i}) + Ru(t_{i+1}, .)(\hat{X}_{t_i}),$$

with

$$\forall 0 \leq i \leq n,\ \forall x \in \mathbb{D},\ |Ru(t_{i+1}, .)(x)| \leq C(T/n)^{\nu+1}(1 + \|x\|^E).$$

Gathering the both previous expansions, we get

$$|\mathbb{E}[u(t_{i+1}, \hat{X}_{t_{i+1}}) - u(t_i, \hat{X}_{t_i})|\hat{X}_{t_i}]| \leq \left(\frac{T}{n}\right)^{\nu+1}\Bigg(C(1 + \|\hat{X}_{t_i}\|^E) \\ + \frac{C_{\nu+1,0}}{(\nu+1)!}(1 + \|\hat{X}_{t_i}\|^{e_{\nu+1,0}})\Bigg).$$

Since the scheme has uniformly bounded moments, we know that for $q \in \{e_{\nu+1,0}, E\}$,

$$\kappa(q) = \sup_{n \geq n_q, 0 \leq i \leq n} \mathbb{E}[\|\hat{X}_{t_i}\|^q] < \infty.$$

We get $|\mathbb{E}[u(t_{i+1}, \hat{X}_{t_{i+1}}) - u(t_i, \hat{X}_{t_i})]| \leq \frac{K}{n^{\nu+1}}$ for $n \geq \max(n_{e_{\nu+1,0}}, n_E)$ by Jensen's lemma, with $K = T^{\nu+1}\left(\frac{C_{\nu+1,0}}{(\nu+1)!}(1 + \kappa(e_{\nu+1,0})) + C(1 + \kappa(E))\right)$. We finally get the claim from (2.16). □

The Euler-Maruyama Scheme

We now illustrate Theorem 2.3.8 on the case of the Euler-Maruyama scheme. On the one hand, we will check that it has uniformly bounded moments. On the other hand, we will prove that it is a potential first order scheme. Therefore, Theorem 2.3.8 will give a weak error of order 1 for any test function f satisfying the condition (2.14). We recall that this condition is known to be satisfied for $f \in \mathcal{C}^\infty_{\textbf{pol}}(\mathbb{D})$, when $\mathbb{D} = \mathbb{R}^d$ and the coefficients b and σ are $\mathcal{C}^\infty$ with bounded derivatives, see Talay [119]. This will prove the result (2.7) of Theorem 2.1.3.

Lemma 2.3.9 *Let $q \in \mathbb{N}^*$. We assume that the scheme $\hat{X}^x_t$ is such that*

$$\exists \eta > 0, \forall t \in (0, \eta), \exists C_q > 0, \forall x \in \mathbb{D},\ \mathbb{E}[\|\hat{X}^x_t\|^q] \leq \|x\|^q(1 + C_q t) + C_q t. \tag{2.17}$$

Then, $\mathbb{E}[\sup_{0 \leq i \leq n} \|\hat{X}_{t_i}\|^q] \leq e^{C_q T}(1 + \|\hat{X}_{t_0}\|^q)$ *for* $n > T/\eta$.

Proof Let $n > T/\eta$. Clearly, we have $\mathbb{E}[\|\hat{X}_{t_{i+1}}\|^q] \leq (1 + C_q T/n)\mathbb{E}[\|\hat{X}_{t_i}\|^q] + C_q T/n$. We define $u_0 = \|\hat{X}_{t_0}\|^q$ and $u_{i+1} = (1 + C_q T/n)u_i + C_q T/n$ for $0 \leq i \leq n-1$. We have $\mathbb{E}[\|\hat{X}_{t_{i+1}}\|^q] - u_{i+1} \leq (1 + C_q T/n)(\mathbb{E}[\|\hat{X}_{t_i}\|^q] - u_i)$, and therefore $\mathbb{E}[\|\hat{X}_{t_i}\|^q] \leq u_i$ for any $0 \leq i \leq n$. Since $u_i = (1 + C_q T/n)^i(u_0 + 1) - 1$ and $(1 + C_q T/n)^i \leq e^{C_q T}$, we get that

$$\max_{0\leq i\leq n} u_i \leq e^{C_q T}(1 + \|\hat{X}_{t_0}\|^q).$$

□

For the Euler scheme we can consider $\hat{X}_t^x = x + b(x)t + \sigma(x)W_t$. Let us assume for a while that $d = d_W = 1$. Then, the binomial theorem gives $|\hat{X}_t^x|^{2q} = \sum_{p=0}^{2q} \binom{2q}{p}(x + b(x)t)^{2q-p}(\sigma(x)W_t)^p$. We deduce for $t \in [0, 1]$ that,

$$\begin{aligned}
\mathbb{E}[|\hat{X}_t^x|^{2q}] &= \sum_{p=0}^{q} \binom{2q}{2p}(x + b(x)t)^{2q-2p} t^p \frac{(2p)!}{2^p(p)!}(\sigma(x))^{2p} \\
&\leq (|x| + |b(x)|t)^{2q} + t\sum_{p=1}^{q} \binom{2q}{2p}\frac{(2p)!}{2^p(p)!}(|x| + |b(x)|)^{2q-2p}(\sigma(x))^{2p} \\
&\leq |x|^{2q} + t\left(\sum_{p=1}^{2q}\binom{2q}{p}|b(x)|^p|x|^{2q-p}\right. \\
&\quad \left. + \sum_{p=1}^{q}\binom{2q}{2p}\frac{(2p)!}{2^p(p)!}(|x| + |b(x)|)^{2q-2p}(\sigma(x))^{2p}\right).
\end{aligned}$$

From the sublinear growth assumption (2.2), the bracket can be bounded by $C_{2q}(1 + |x|^{2q})$ for some constant $C_{2q} > 0$. Thus, Lemma 2.3.9 gives that $\sup_{n>T} \mathbb{E}[\sup_{0\leq i\leq n} \|\hat{X}_{t_i}\|^{2q}] < \infty$ and then $\sup_{n>T} \mathbb{E}[\sup_{0\leq i\leq n} \|\hat{X}_{t_i}\|^{2q}] < \infty$ for any $q \in \mathbb{N}^*$.

In the multidimensional case, we can choose the norm thanks to the equivalence of norms, and we consider $\|x\|_{2q} = (\sum_{i=1}^d |x_i|^{2q})^{1/(2q)}$. We have

$$\|\hat{X}_t^x\|_{2q}^{2q} = \sum_{k=1}^{d}\left(x_k + b_k(x)t + \sum_{l=1}^{d_W}\sigma_{kl}(x)W_t^k\right)^{2q},$$

and we can proceed as before to get the boundedness of the moments.

Now, we want to check that the Euler scheme is a potential first order scheme. Let $f \in \mathcal{C}^{\infty}_{\mathbf{pol}}(\mathbb{R}^d)$. Itô's formula gives that

$$f(\hat{X}^x_t) = f(x) + \int_0^t \sum_{k=1}^d \partial_k f(\hat{X}^x_s) b_k(x) + \frac{1}{2}\sum_{k,l=1}^d \partial_k \partial_l f(\hat{X}^x_s)(\sigma(x)\sigma^\top(x))_{k,l} ds$$
$$+ \int_0^t \sum_{k=1}^d \partial_k f(\hat{X}^x_s)(\sigma(x) dW_s)_k.$$

Taking the expectation, we get

$$\mathbb{E}[f(\hat{X}^x_t)] - f(x) = \int_0^t \sum_{k=1}^d \mathbb{E}[\partial_k f(\hat{X}^x_s)] b_k(x)$$
$$+ \frac{1}{2}\sum_{k,l=1}^d \mathbb{E}[\partial_k \partial_l f(\hat{X}^x_s)](\sigma(x)\sigma^\top(x))_{k,l} ds. \tag{2.18}$$

Let (C_α, e_α) be a good sequence of f. We set $c = 2\max_{|\alpha|\le 2} C_\alpha$ and $e = \max_{|\alpha|\le 2} e_\alpha$. Since $C_\alpha(1 + \|x\|^{e_\alpha}) \le C_\alpha(2 + \|x\|^e)$, we get

$$\forall 1 \le k, l \le d, |\partial_k f(x)| \le c(1 + \|x\|^e) \text{ and } |\partial_{k,l} f(x)| \le c(1 + \|x\|^e),$$

and the constants c and e only depends on a good sequence of f. Let $\eta > 0$ be such that $\forall t \in (0, \eta)$, $\mathbb{E}[\|\hat{X}^x_t\|^e] \le 2\|x\|^e$. We then have

$$\forall t \in (0, \eta),\ |\mathbb{E}[f(\hat{X}^x_t)] - f(x)| \le tc(1 + \|x\|^e) \times$$
$$\left[\sum_{k=1}^d |b_k(x)| + \frac{1}{2}\sum_{k,l=1}^d |(\sigma(x)\sigma^\top(x))_{k,l}|\right].$$

Using the sublinear growth condition (2.2), we deduce that $(f, t, x) \mapsto \mathbb{E}[f(\hat{X}^x_t)] - f(x)$ is a remainder of order 1. We now observe that $\partial_k f, \partial_k \partial_l f \in \mathcal{C}^{\infty}_{\mathbf{pol}}(\mathbb{R}^d)$ and that a good sequence for these functions can be explicitly obtained from a good sequence of f. Therefore, there are constants $c > 0$, $e \in \mathbb{N}^*$, and $\eta > 0$ that only depend on a good sequence of f such that

$$\forall g \in \{\partial_k f, \partial_k \partial_l f,\ 1 \le k, l \le d\},\ \forall t \in (0, \eta),\ |\mathbb{E}[g(\hat{X}^x_t)] - g(x)| \le tc(1 + \|x\|^e). \tag{2.19}$$

Now, we rewrite (2.18) as follows

$$\begin{aligned}\mathbb{E}[f(\hat{X}_t^x)] =& f(x) + tLf(x) + \sum_{k=1}^{d} b_k(x) \int_0^t \mathbb{E}[\partial_k f(\hat{X}_s^x) - \partial_k f(x)]ds \\ &+ \frac{1}{2}\sum_{k,l=1}^{d} (\sigma(x)\sigma^\top(x))_{k,l} \int_0^t \mathbb{E}[\partial_k\partial_l f(\hat{X}_s^x) - \partial_k\partial_l f(x)]ds,\end{aligned}$$

and get from (2.19)

$$\begin{aligned}|\mathbb{E}[f(\hat{X}_t^x)] - [f(x) + tLf(x)]| \le& t^2 c(1 + \|x\|^e) \times \\ &\left[\sum_{k=1}^{d} |b_k(x)| + \frac{1}{2}\sum_{k,l=1}^{d} |(\sigma(x)\sigma^\top(x))_{k,l}|\right],\end{aligned}$$

which gives that $(f, t, x) \mapsto \mathbb{E}[f(\hat{X}_t^x)] - [f(x) + tLf(x)]$ is a remainder of order 2. Using Remark 2.3.7, we conclude that the Euler scheme is a potential first order scheme.

2.3.2 Composition of Approximation Schemes and Operator Splitting

The aim of this section is to present a general way to construct by recursion approximation schemes. Let us assume that we can write the infinitesimal generator (2.13) as $L = L_1 + L_2$, where L_i is the infinitesimal generator of the SDE $dX_t^i = b_i(X_t^i)dt + \sigma_i(X_t^i)dW_t$. We assume that L_1 and L_2 satisfy the required assumptions, see Definition 2.3.3. We suppose that we already know two corresponding potential weak ν-th order schemes $\hat{X}_t^{1,x}$ and $\hat{X}_t^{2,x}$ that take values in the domain $\mathbb{D}$. We will explain now how to construct from these schemes an approximation scheme for the SDE (2.1) with infinitesimal generator L that is

1. A potential weak second order scheme if $\nu \ge 2$.
2. A potential weak ν-th order schemes if in addition the operators commute, i.e. $L_1 L_2 = L_2 L_1$.

We first give an important definition for that purpose.

Definition 2.3.10 Let us consider two transition probabilities $\hat{p}_x^1(t)(dz)$ and $\hat{p}_x^2(t)(dz)$ on $\mathbb{D}$. Then, we define the composition $\hat{p}^2(t_2) \circ \hat{p}_x^1(t_1)(dz)$ by

$$\hat{p}^2(t_2) \circ \hat{p}_x^1(t_1)(dz) = \int_{\mathbb{D}} \hat{p}_y^2(t_2)(dz)\hat{p}_x^1(t_1)(dy).$$

This amounts to first use the scheme 1 with a time step t_1 and then the scheme 2 with a time step t_2 with independent samples. We denote by $\hat{X}_{t_2}^{2,\hat{X}_{t_1}^{1,x}}$ a random variable with the law $\hat{p}^2(t_2) \circ \hat{p}_x^1(t_1)(dz)$.

More generally, if one has m transition probabilities $\hat{p}_x^1, \dots, \hat{p}_x^m$ on $\mathbb{D}$, we define

$$\hat{p}^m(t_m) \circ \cdots \circ \hat{p}_x^1(t_1)(dz) = \hat{p}^m(t_m) \circ \left(\hat{p}^{m-1}(t_{m-1}) \circ \cdots \circ \hat{p}_x^1(t_1)\right)(dz).$$

Remark 2.3.11 The criterion (2.17) that gives estimates on the moments is easy to use with the scheme composition. Let $\hat{X}_t^{1,x}$ and $\hat{X}_t^{2,x}$ be two schemes that satisfy (2.17), and we denote by C_1 and C_2 the respective constants. Then we have, for $t \in (0,1)$ small enough, $\mathbb{E}[\|\hat{X}_t^{2,\hat{X}_t^{1,x}}\|^q|\hat{X}_t^{1,x}] \le (1+C_2t)\|\hat{X}_t^{1,x}\|^q + C_2t$ and thus

$$\mathbb{E}[\|\hat{X}_t^{2,\hat{X}_t^{1,x}}\|^q] \le (1+C_1t)(1+C_2t)\|x\|^q + (C_2+C_1(1+C_2t))t \le (1+Ct)\|x\|^q + Ct,$$

with $C = C_1 + C_2 + C_1C_2$. Therefore the scheme $\hat{X}_t^{2,\hat{X}_t^{1,x}}$ also satisfies the criterion (2.17).

Proposition 2.3.12 *Let L_1 and L_2 be two operators that satisfy the required assumptions on $\mathbb{D}$. Let $\hat{p}_x^1(t)(dz)$ and $\hat{p}_x^2(t)(dz)$ be respectively potential weak νth-order discretization schemes on $\mathbb{D}$ for L_1 and L_2. Then, for $\lambda_1, \lambda_2 > 0$, $\hat{p}^2(\lambda_2 t) \circ \hat{p}_x^1(\lambda_1 t)(dz)$ is such that for $f \in \mathcal{C}_{\mathbf{pol}}^{\infty}(\mathbb{D})$:*

$$\mathbb{E}[f(\hat{X}_{\lambda_2 t}^{2,\hat{X}_{\lambda_1 t}^{1,x}})] = \sum_{l_1+l_2 \le \nu} \frac{\lambda_1^{l_1}\lambda_2^{l_2}}{l_1!l_2!} t^{l_1+l_2} L_1^{l_1} L_2^{l_2} f(x) + Rf(t,x)$$

where $Rf(t,x)$ is a remainder of order $\nu+1$.

The proof of Proposition 2.3.12 is postponed at the end of this section. Its result is both simple and crucial. It says to us that a potential scheme of order ν with a time step t acts as an operator $I + tL + \cdots + \frac{t^\nu}{\nu!}L^\nu + rem(t^{\nu+1})$ on f, where $rem(t^{\nu+1})$ is an operator such that $rem(t^{\nu+1})f(x)$ is a remainder of order $\nu+1$ in the sense of Definition 2.3.4. The composition of two schemes is thus simply the composition of their operators (in the reverse order) because

$$\sum_{l_1+l_2 \le \nu} \frac{\lambda_1^{l_1}\lambda_2^{l_2}}{l_1!l_2!} t^{l_1+l_2} L_1^{l_1} L_2^{l_2}$$
$$= \left[I + \lambda_1 t L_1 + \cdots + \frac{(\lambda_1 t)^\nu}{\nu!} L_1^\nu\right]\left[I + \lambda_2 t L_2 + \cdots + \frac{(\lambda_2 t)^\nu}{\nu!} L_2^\nu\right] + rem(t^{\nu+1}).$$

Thus, we can do some formal calculations to study the weak error of schemes obtained by composition. For example, when $L_1L_2 = L_2L_1$, we recognize a

Cauchy product and get $\sum_{l_1+l_2\le\nu}\frac{\lambda_1^{l_1}\lambda_2^{l_2}}{l_1!l_2!}t^{l_1+l_2}L_1^{l_1}L_2^{l_2} = \sum_{k\le\nu}\frac{t^k}{k!}(\lambda_1 L_1+\lambda_2 L_2)^k$. This gives the following result.

Corollary 2.3.13 *Under the assumptions of Proposition 2.3.12, if besides $L_1L_2 = L_2L_1$, then $\hat{p}^2(t)\circ\hat{p}^1_x(t)(dz)$ is a potential weak νth-order approximation scheme for L_1+L_2.*

When the operators are not commuting, it is still possible to construct by recursion second order schemes as follows. We set $S_i(t) = I + tL_i + \frac{t^2}{2}L_i + rem(t^3)$. By formal calculations, we have the following identities:

$$\frac{1}{2}(S_1(t)S_2(t)+S_2(t)S_1(t)) = I + t(L_1+L_2) + \frac{t^2}{2}(L_1+L_2)^2 + rem(t^3),$$
$$S_1(t/2)S_2(t)S_1(t/2) = I + t(L_1+L_2) + \frac{t^2}{2}(L_1+L_2)^2 + rem(t^3).$$

The second identity is known in the literature as the Strang splitting [118] and has been introduced for the numerical approximation of Ordinary Differential Equations. These identities lead to the following key result.

Corollary 2.3.14 *We make the assumptions of Proposition 2.3.12 with $\nu = 2$. Let B be an independent Bernoulli variable with parameter $1/2$. Then, the schemes*

$$\hat{X}^x_t = B\hat{X}_t^{2,\hat{X}_t^{1,x}} + (1-B)\hat{X}_t^{1,\hat{X}_t^{2,x}} \text{ and } \hat{X}_{t/2}^{1,\hat{X}_t^{2,\hat{X}_{t/2}^{1,x}}}$$

with respective transition probabilities $\frac{1}{2}[\hat{p}^2(t)\circ\hat{p}^1_x(t)(dz)+\hat{p}^1(t)\circ\hat{p}^2_x(t)(dz)]$ and $\hat{p}^1(t/2)\circ\hat{p}^2(t)\circ\hat{p}^1_x(t/2)(dz)$ are potential second-order schemes for L_1+L_2.

Thus, if we have at our disposal potential second order schemes for some elementary diffusions, we can by this technique construct potential second order schemes for more intricate ones. From a computational point of view, it is also very easy to implement the schemes obtained by scheme composition. In fact, it is enough to implement the routines for sampling the elementary diffusions that are then called several times when sampling the more intricate diffusions.

We now give two exercises that are rather direct applications of Proposition 2.3.12. The first one shows that it is possible to draw only one Bernoulli variable when we want to extend the first construction of Corollary 2.3.14 with m schemes. The second one shows that the basic composition of potential first order schemes is still a potential first order scheme. This result has however a limited practical interest. On the one hand the Euler scheme already gives a first order scheme. On the other hand, Corollary 2.3.14 gives a second order scheme without further assumptions on the operators and with a similar computational cost.

Exercise 2.3.15 Let $\hat{p}^1_x, \dots, \hat{p}^m_x$ be m potential second order discretization schemes on $\mathbb{D}$ for the operators $L_1, \dots, L_m$ that satisfy the required assumption. Then, show that

$$\frac{1}{2}\left(\hat{p}^m(t) \circ \dots \circ \hat{p}^2(t) \circ \hat{p}^1_x(t) + \hat{p}^1(t) \circ \hat{p}^2(t) \circ \dots \circ \hat{p}^m_x(t)\right)$$

is a potential second order scheme for $L_1 + \dots + L_m$.

Exercise 2.3.16 Let $\hat{p}^1_x, \hat{p}^2_x$ be potential first order discretization schemes on $\mathbb{D}$ for the operators L_1, L_2 that satisfy the required assumption. Show that

$$\hat{p}^2(t) \circ \hat{p}^1_x(t)$$

is a potential first order scheme for $L_1 + L_2$.

Proof of Proposition 2.3.12 By assumption, the schemes $\hat{X}^{1,x}_t$ and $\hat{X}^{2,x}_t$ are potential νth-order schemes. For $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$ and $k \le \nu + 1$, we set for $i \in \{1, 2\}$

$$R^i_k f(t, x) = \mathbb{E}[f(\hat{X}^{i,x}_t)] - \sum_{j=0}^{k-1} \frac{t^j}{j!} L^i_j f(x),$$

which is a remainder of order k from Remark 2.3.7 and Exercise 2.3.5. We have

$$\mathbb{E}[f(\hat{X}^{2,\hat{X}^{1,x}_{\lambda_1 t}}_{\lambda_2 t})|\hat{X}^{1,x}_{\lambda_1 t}] = f(\hat{X}^{1,x}_{\lambda_1 t}) + \sum_{k=1}^{\nu} \frac{1}{k!}\lambda_2^k t^k L_2^k f(\hat{X}^{1,x}_{\lambda_1 t}) + R^2_{\nu+1} f(\lambda_2 t, \hat{X}^{1,x}_{\lambda_1 t})$$

and then

$$\mathbb{E}[f(\hat{X}^{2,\hat{X}^{1,x}_{\lambda_1 t}}_{\lambda_2 t})] = \sum_{l_1+l_2 \le \nu} \frac{\lambda_1^{l_1}\lambda_2^{l_2}}{l_1! l_2!} t^{l_1+l_2} L_1^{l_1} L_2^{l_2} f(x) + Rf(t, x),$$

where $Rf(t, x) = \mathbb{E}[R^2_{\nu+1} f(\lambda_2 t, \hat{X}^{1,x}_{\lambda_1 t})] + \sum_{k=0}^{\nu} \frac{1}{k!}\lambda_2^k t^k R^1_{\nu+1-k} L_2^k f(\lambda_1 t, x)$. Since $R^1_{\nu+1-k} L_2^k f(\lambda_1 t, x)$ is a remainder of order $\nu + 1 - k$, it is easy to get that the sum is a remainder of order $\nu + 1$ by using Exercise 2.3.5. It remains thus to prove that $(f, t, x) \mapsto \mathbb{E}[R^2_{\nu+1} f(\lambda_2 t, \hat{X}^{1,x}_{\lambda_1 t})]$ is a remainder of order $\nu + 1$ to get the claim, which we do now.

Since $R^2_{\nu+1}$ is a remainder of order $\nu + 1$, we know that there are constants $\eta, C, E > 0$ depending on a good sequence of f such that

$$\forall t \in (0, \eta/\lambda_2),\ |R^2_{\nu+1} f(\lambda_2 t, \hat{X}^{1,x}_{\lambda_1 t})| \le C\lambda_2^{\nu+1} t^{\nu+1}(1 + \|\hat{X}^{1,x}_{\lambda_1 t}\|^E).$$

Let n_E be the smallest integer such that $E \leq 2n_E$. There is a constant depending on E such that $\|x\|^E \leq c\Phi_{n_E}(x)$, with $\Phi_n(x) = 1 + x_1^{2n} + \cdots + x_d^{2n}$. Thus, there are constants $\eta, C > 0$ and $n_E \in \mathbb{N}$ depending on a good sequence of f such that

$$\forall t \in (0, \eta/\lambda_2),\ |R^2_{\nu+1} f(\lambda_2 t, \hat{X}^{1,x}_{\lambda_1 t})| \leq C\lambda_2^{\nu+1} t^{\nu+1}(1 + \Phi_{n_E}(\hat{X}^{1,x}_{\lambda_1 t})).$$

We have $\Phi_{n_E} \in \mathcal{C}^{\infty}_{\mathbf{pol}}(\mathbb{D})$ and Φ_{n_E} only depends on f through a good sequence of f. Besides $\hat{X}^{1,x}$ is a potential νth-order scheme. It is thus of order 0, which gives that there exists constants $\eta', C', E' > 0$ depending on a good sequence of f such that

$$\forall t \in (0, \eta'/\lambda_1),\ \mathbb{E}[\Phi_{n_E}(\hat{X}^{1,x}_{\lambda_1 t})] \leq C'(1 + \|x\|^{E'}),$$

and therefore

$$\forall t \in (0, \frac{\eta}{\lambda_2} \wedge \frac{\eta'}{\lambda_1}),\ |\mathbb{E}[R^2_{\nu+1} f(\lambda_2 t, \hat{X}^{1,x}_{\lambda_1 t})]| \leq C\lambda_2^{\nu+1} t^{\nu+1}(1 + C'(1 + \|x\|^{E'})),$$

which gives that $(f, t, x) \mapsto \mathbb{E}[R^2_{\nu+1} f(\lambda_2 t, \hat{X}^{1,x}_{\lambda_1 t})]$ is a remainder of order $\nu+1$. □

2.3.3 *The Ninomiya-Victoir Scheme*

We are now in position to present the scheme proposed by Ninomiya and Victoir [109]. As we will see, this is a potential second order scheme under suitable assumptions on the coefficients. The strength of their approach is to reduce the problem to the numerical approximation of Ordinary Differential Equations (ODE).

We consider an operator L that satisfies the required assumptions on $\mathbb{D}$. It is thus defined by (2.13) for some smooth coefficients b and σ. Besides, we assume that the following operators

$$V_0 f(x) = \sum_{i=1}^{d} b_i(x)\partial_i f(x) - \frac{1}{2}\sum_{k=1}^{d_W}\sum_{i,j=1}^{d} \partial_j \sigma_{i,k}(x)\sigma_{j,k}(x)\partial_i f(x)$$

$$V_k f(x) = \sum_{i=1}^{d} \sigma_{i,k}(x)\partial_i f, \text{ for } k = 1, \ldots, d_W$$

are well defined on $\mathbb{D}$ and that V_0 and $\frac{1}{2}V_k^2$ for $k = 1, \ldots, d_W$ satisfy the required assumptions on the same domain $\mathbb{D}$. Then, we have the following identity

$$L = V_0 + \frac{1}{2}\sum_{k=1}^{d_W} V_k^2,$$

since $V_k^2 f(x) = \sum_{i,j=1}^d \sigma_{j,k}(x)[\partial_j \sigma_{i,k}(x)\partial_i f(x) + \sigma_{i,k}(x)\partial_j \partial_i f(x)]$. We observe that for $k \in \{0, \dots, d_W\}$,

$$V_k f(x) = v_k(x).\nabla f(x),$$

with $(v_0(x))_i = b_i(x) - \frac{1}{2}\sum_{k=1}^{d_W}\sum_{j=1}^d \partial_j \sigma_{i,k}(x)\sigma_{j,k}(x)$ and $(v_k(x))_i = \sigma_{i,k}(x)$ for $k \geq 1$. Since all the operators V_0, $\frac{1}{2}V_k^2$ for $1 \leq k \leq d_W$ and L satisfy the required assumptions, we get that

$$\exists K > 0, \|v_k(x)\| \leq K(1 + \|x\|),\ 0 \leq k \leq d_W.$$

We then consider the following ODEs:

$$X_0(0, x) = x \in \mathbb{D},\ \frac{dX_0(t, x)}{dt} = v_0(X_0(t, x)),\ t \geq 0,$$
$$X_k(0, x) = x \in \mathbb{D},\ \frac{dX_k(t, x)}{dt} = v_k(X_k(t, x)),\ t \in \mathbb{R}.$$

Since $X_0(t, x)$ is the ODE associated to V_0 that satisfy the required assumptions, we have $X_0(t, x) \in \mathbb{D}$ for all $t \geq 0$. Also, let B denote a one-dimensional Brownian motion. For $1 \leq i \leq d$, we have

$$\frac{d^2(X_k(t, x))_i}{dt^2} = \nabla\sigma_{i,k}(X_k(t, x)).\frac{dX_k(t, x)}{dt} = \sum_{j=1}^d \partial_j \sigma_{i,k}(X_k(t, x))\sigma_{j,k}(X_k(t, x)). \tag{2.20}$$

By Itô's formula, we get that $X_k(B_t, x)$ is the solution of the SDE with infinitesimal generator $\frac{1}{2}V_k^2$. Since this operator satisfy the requires assumptions, we have $X_k(B_t, x) \in \mathbb{D}$ and thus $X_k(u, x) \in \mathbb{D}$ for any $u \in \mathbb{R}$, $x \in \mathbb{D}$. Since $X_0(t, x)$ and $X_k(B_t, x)$ are respectively exact schemes for the operators V_0 and $\frac{1}{2}V_k^2$, they are in particular potential second order schemes, and we get the following result by applying Proposition 2.3.12 and Exercise 2.3.15.

Theorem 2.3.17 (Ninomiya-Victoir) *Under the above framework, we denote by $\hat{p}_x^0(t)(dz)$ (resp. $\hat{p}_x^k(t)(dz)$) the law of $X_0(t, x)$ (resp. $X_k(\sqrt{t}N, x)$ where $N \sim \mathcal{N}(0, 1)$, for $k = 1, \dots, d_W$).*

Then, $\hat{p}_x^0(t)(dz)$ (resp. $\hat{p}_x^k(t)(dz)$) is an exact scheme for V_0 (resp. $\frac{1}{2}V_k^2$) and thus a potential νth-order scheme. Moreover

$$\frac{1}{2}\left(\hat{p}^0(t/2) \circ \hat{p}^m(t) \circ \cdots \circ \hat{p}^1(t) \circ \hat{p}_x^0(t/2) + \hat{p}^0(t/2) \circ \hat{p}^1(t) \circ \cdots \circ \hat{p}^m(t) \circ \hat{p}_x^0(t/2)\right)$$

is well-defined and is a potential second order scheme on $\mathbb{D}$ for L.

Now, we would like to apply Theorem 2.3.8 and get conditions under which we have indeed a weak error of order 2. We first want to check that the condition on the moments holds. Since the operators V_0 and $\frac{1}{2}V_k^2$ for $1 \le k \le d_W$ satisfy the required assumptions, we know that their corresponding diffusion coefficients have a sublinear growth. We get that there is $K > 0$ such that $\|v_k(x)\| \le K(1 + \|x\|)$ for $k = 0, \dots, d_W$. We deduce from $X_k(t,x) = x + \int_0^t v_k(X_k(s,x))ds$ that

$$\|X_k(t,x)\| \le \|x\| + \int_0^{|t|} K(1 + \|X_k(s,x)\|)ds.$$

Gronwall's lemma then gives

$$\forall t \in \mathbb{R}, \|X_k(t,x)\| \le (\|x\| + K|t|)e^{K|t|}. \tag{2.21}$$

Besides, since $\frac{1}{2}V_k^2$ satisfy the required assumptions when $k \ge 1$, we get from (2.20) that $\|\frac{d^2X_k(t,x)}{dt^2}\| \le K(1 + \|X_k(t,x)\|)$. We set $r_k(t,x) = X_k(t,x) - x - v_k(x)t$ and get by a Taylor expansion at time 0 that

$$\begin{aligned}\forall t \in \mathbb{R}, \|r_k(t,x)\| &\le \int_0^{|t|} (|t| - s)K(1 + (\|x\| + Ks)e^{Ks})ds \\ &\le Kt^2(1 + (\|x\| + K|t|)e^{K|t|}). \end{aligned} \tag{2.22}$$

Let $G \sim \mathcal{N}(0,1)$ a standard normal variable and $q \in \mathbb{N}$. When $X_k(t,x)$ is real valued, the multinomial formula gives

$$\mathbb{E}[(X_k(\sqrt{t}G,x))^{2q}] = \sum_{i_1+i_2+i_3=2q} \frac{i_1!i_2!i_3!}{(2q)!}\mathbb{E}\left[x^{i_1}(\sqrt{t}Gv_k(x))^{i_2}r_k(\sqrt{t}G,x)^{i_3}\right].$$

In this sum, the term $(i_1,i_2,i_3) = (2q,0,0)$ gives x^{2q}, the term $(i_1,i_2,i_3) = (2q-1,1,0)$ has a null expectation, and all the other terms can be bounded by $Ct(1+x^{2q})$ for $t \in [0,1]$ thanks to (2.22). When $X_k(t,x)$ takes values in $\mathbb{R}^d$, we work with the norm $\|x\|_{2q} = \left(\sum_{i=1}^d x_i^{2q}\right)^{\frac{1}{2q}}$ and get that there is a constant C such that

$$\mathbb{E}[\|X_k(\sqrt{t}G,x)\|_{2q}^{2q}] \le \|x\|_{2q}^{2q} + Ct(1 + \|x\|_{2q}^{2q}).$$

This gives from Lemma 2.3.9, Remark 2.3.11 and Lemma 2.3.9 that the Ninomiya and Victoir scheme has bounded moments.

On the other hand, Talay [119] has shown that the condition (i) of Theorem 2.3.8 holds when $\mathbb{D} = \mathbb{R}^d$, b and σ are $\mathcal{C}^\infty$ with bounded derivatives, and $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$. This gives the following claim.

Corollary 2.3.18 *Suppose that $\mathbb{D} = \mathbb{R}^d$, $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$, b and σ are $\mathcal{C}^\infty$ with bounded derivatives. Let $\hat{X}$ denote the Ninomiya and Victoir scheme defined by Theorem 2.3.17. Then, there is $K > 0$ and $n_0 \in \mathbb{N}$, such that*

$$\forall n \geq n_0, \ |\mathbb{E}[f(\hat{X}_{t_n})] - \mathbb{E}[f(X_T^x)]| \leq K/n^2.$$

Here, we focus on bounding the weak error. However, a natural question is to know if we could have a more precise convergence result, and if we could get as for the Euler scheme an expansion of the weak error. This would then allow to use extrapolation techniques in order to speed up the convergence. This question has been investigated by Fujiwara [56] and Oshima et al. [111]. They have explained how to construct extrapolations of any order to approximate $\mathbb{E}[f(X_T^x)]$.

Further Developments on the Ninomiya and Victoir Scheme

We note that in Theorem 2.3.17, the schemes for V_0 and $\frac{1}{2}V_k^2$ with $1 \leq k \leq d_W$ are exact. However, it would have been sufficient to have second order schemes to get at the end a potential second order scheme. We now discuss some possible extensions of this result. Let $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$. We have

$$f(X_k(t,x)) = f(x) + \int_0^t v_k(X_k(s,x)).f(X_k(s,x))ds = f(x) + \int_0^t V_k f(X_k(s,x))ds.$$

By iterating this formula, we get for any $l \in \mathbb{N}$

$$f(X_k(t,x)) = f(x) + tV_k f(x) + \dots + \frac{t^l}{l!}V_k^l f(x) + \int_0^t \frac{(t-s)^l}{l!} V_k^{l+1} f(X_k(s,x))ds. \tag{2.23}$$

We now consider a random variable Y that matches the first $2\nu + 1$ moments of the normal variable, i.e.

$$\mathbb{E}[Y^{2m}] = \frac{(2m)!}{2^m m!}, \ \mathbb{E}[Y^{2l+1}] = 0, \ m \in \{0, \dots, \nu\}, \tag{2.24}$$

and such that $\mathbb{E}[e^{c|Y|}] < \infty$ for any $c > 0$. Applying formula (2.23) with $l = 2\nu + 1$ and taking the expectation gives for $t \geq 0$,

$$\begin{aligned} \mathbb{E}[f(X_k(\sqrt{t}Y, x))] = {} & f(x) + \frac{t}{2}V_k^2 f(x) + \dots + \frac{t^\nu}{\nu!}(\frac{1}{2}V_k^2)^\nu f(x) \\ & + \mathbb{E}\left[\int_0^{\sqrt{t}Y} \frac{(\sqrt{t}Y - s)^{2\nu+1}}{(2\nu+1)!} V_k^{2\nu+2} f(X_k(s,x))ds\right]. \end{aligned} \tag{2.25}$$

Since $\frac{1}{2}V_k^2$ satisfies the required assumption on $\mathbb{D}$, we have $V_k^{2\nu+2}f(x) \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$. There are constants $C, E > 1$ that depend on a good sequence of f such that $\|V_k^{2\nu+2}f(x)\| \le C(1+\|x\|^E)$. We have by (2.21)

$$\left|\int_0^{\sqrt{t}Y} \frac{(\sqrt{t}Y-s)^{2\nu+1}}{(2\nu+1)!} V_k^{2\nu+2} f(X_k(s,x))ds\right|$$
$$\le \frac{t^{\nu+1}|Y|^{2\nu+2}}{(2\nu+1)!}C(1+(\|x\|+K\sqrt{t}|Y|)^E e^{KE\sqrt{t}|Y|})$$

and remark that for $t \in (0,1)$,

$$\begin{aligned} C\mathbb{E}[|Y|^{2\nu+2}(1+(\|x\|+K\sqrt{t}|Y|)^E e^{KE\sqrt{t}|Y|})] &\le C\mathbb{E}[|Y|^{2\nu+2} \\ &\quad (1+2^{E-1}(\|x\|^E+K^E|Y|^E)e^{KE|Y|})] \\ &\le C''(1+\|x\|^E) \end{aligned}$$

for a constant C'' that depends on f only through a good sequence. From (2.25) and Remark 2.3.7, we get the following proposition.

Proposition 2.3.19 *Let Y be a random variable satisfying (2.24) and $\mathbb{E}[e^{c|Y|}] < \infty$ for any $c > 0$. Then, $X_k(\sqrt{t}Y, x)$ defines a potential νth-order scheme for $\frac{1}{2}V_k^2$.*

Corollary 2.3.20 *Let Y be a random variable that satisfies (2.24) with $\nu = 2$ and $\mathbb{E}[e^{c|Y|}] < \infty$ for any $c > 0$. Let $\hat{p}_x^0(t)(dz)$ (resp. $\hat{p}_x^k(t)(dz)$) the law of $X_0(t,x)$ (resp. $X_k(\sqrt{t}Y, x)$ for $k = 1, \dots, d_W$). Then,*

$$\frac{1}{2}\left(\hat{p}^0(t/2)\circ\hat{p}^m(t)\circ\cdots\circ\hat{p}^1(t)\circ\hat{p}_x^0(t/2)+\hat{p}^0(t/2)\circ\hat{p}^1(t)\circ\cdots\circ\hat{p}^m(t)\circ\hat{p}_x^0(t/2)\right) \tag{2.26}$$

is well-defined and is a potential second order scheme on $\mathbb{D}$ for L.

In practice, we will mainly work with bounded random variables Y which gives immediately that $\mathbb{E}[e^{c|Y|}] < \infty$ for any $c > 0$. We will in fact mainly use the two following ones

$$\mathbb{P}(Y=\sqrt{3}) = \mathbb{P}(Y=-\sqrt{3}) = \frac{1}{6},\ \mathbb{P}(Y=0) = 2/3, \tag{2.27}$$

and

$$\mathbb{P}(Y = \sqrt{3+\sqrt{6}}) = \mathbb{P}(Y = -\sqrt{3+\sqrt{6}}) = \frac{\sqrt{6}-2}{4\sqrt{6}}, \tag{2.28}$$
$$\mathbb{P}(Y = \sqrt{3-\sqrt{6}}) = \mathbb{P}(Y = -\sqrt{3-\sqrt{6}}) = \frac{1}{2} - \frac{\sqrt{6}-2}{4\sqrt{6}},$$

that matches respectively the five and seven first moments of the standard Normal variable. These random variables can be easily obtained from the Gauss-Hermite quadrature formulas. Discrete approximations may be interesting also because they are in principle faster to sample than a Normal random variable. We have observed on our machine that sampling the random variable (2.27) is approximately 2.5 faster than sampling a standard random variable. Of course, this observation depends on the hardware and how are at the end implemented the corresponding sampling routines, and may be different from a machine to another one.

Up to now, we have assumed that we were able to solve exactly the ODEs satisfied by $X_k(t,x)$. Of course this is not always possible, and one has then to use approximation schemes. We now assume that we have $\mathbb{D}$-valued approximations $\hat{X}_k(t,x)$ that are as follows: the scheme $\hat{X}_0(t,x)$ is a potential second order scheme for V_0, $\hat{X}_k(t,x)$ with $k \geq 1$ is a potential fifth order scheme for V_k. This means by Remark 2.3.7 that

$$R_0 f(t,x) = f(\hat{X}_0(t,x)) - \sum_{l=0}^{2} \frac{t^l}{l!} V_0^l f(x) \text{ and } R_k f(t,x) = f(\hat{X}_k(t,x)) - \sum_{l=0}^{5} \frac{t^l}{l!} V_0^l f(x)$$

are remainder of order 3 and 6. Here, in addition to Definition 2.3.4 we assume for $k \geq 1$ that there are positive constants C, E, and η depending only on a good sequence of f such that

$$\forall t \in (-\eta,\eta), \forall x \in \mathbb{D}, |R_k f(t,x)| \leq Ct^6(1+\|x\|^E),$$

i.e. the bound is also valid for small negative values of t. Now, we consider Y a bounded random variable that satisfies (2.24) with $\nu = 2$. We get

$$\mathbb{E}[f(\hat{X}_k(\sqrt{t}Y,x))] = f(x) + t\frac{1}{2}V_k^2 + \frac{t^2}{2}\left(\frac{1}{2}V_k^2\right)^2 + \mathbb{E}[R_k f(\sqrt{t}Y,x)].$$

Since R_k is a remainder of order 6, and Y is bounded, we have for $t \in (0, \eta/\|Y^2\|_\infty)$

$$|\mathbb{E}[R_k f(\sqrt{t}Y,x)]| \leq Ct^3\mathbb{E}[Y^6](1+\|x\|^E).$$

Therefore, $(f,t,x) \mapsto \mathbb{E}[R_k f(\sqrt{t}Y,x)]$ is a remainder of order 3. Thus, $\hat{X}_k(\sqrt{t}Y,x)$ is a potential second order scheme for $\frac{1}{2}V_k^2$. If $\hat{p}_x^0(t)(dz)$ (resp.

$\hat{p}_x^k(t)(dz)$) denotes the law of $\hat{X}_0(t,x)$ (resp. $\hat{X}_k(\sqrt{t}Y,x)$ for $k = 1,\dots,d_W$), we get that

$$\frac{1}{2}\left(\hat{p}^0(t/2)\circ\hat{p}^m(t)\circ\cdots\circ\hat{p}^1(t)\circ\hat{p}_x^0(t/2)+\hat{p}^0(t/2)\circ\hat{p}^1(t)\circ\cdots\circ\hat{p}^m(t)\circ\hat{p}_x^0(t/2)\right)$$

is a potential second order scheme on $\mathbb{D}$ for L. Numerical approximations of ODEs such as Runge-Kutta methods are rather well investigated. The combination of these schemes with the operator splitting given by Ninomiya and Victoir allow thus to get second order schemes for general SDEs.

Chapter 3
Simulation of the CIR Process

Up to now, computers are only able to do deterministic tasks and they cannot generate true random numbers. To sample random numbers, they run deterministic sequences called pseudorandom number generators that produce a sequence of real numbers in $[0, 1]$ that behaves like a sequence of independent random variables that are distributed uniformly on $[0, 1]$. Different families of pseudorandom number generators exist. It is important to use generators that have a large period, such as the Mersenne twister. In fact, running a Monte-Carlo algorithm to compute pathwise expectations may use intensively the generator. The convergence of the Monte-Carlo algorithm is degraded when the amount of pseudorandom numbers used is close or larger than the period.

Once we have chosen a suitable pseudorandom number generator, we have at our hand a sequence of numbers that looks like independent uniform random variables on $[0, 1]$. If we want to sample any other random variable, we have to find a way to do this by the mean of uniform random variables. For example, the Box-Muller transform says that

$$\sqrt{-2\log(U)}\cos(2\pi V) \text{ and } \sqrt{-2\log(U)}\sin(2\pi V)$$

are independent standard normal variables when $U, V \sim \mathcal{U}([0, 1])$ are independent. Thus, it is easy to sample Brownian increments. Here, we are particularly interested in sampling discrete paths of affine diffusions. From the first chapter, we know that a time homogeneous real affine diffusion is either an Ornstein-Uhlenbeck process or a Cox-Ingersoll-Ross process. The Ornstein-Uhlenbeck process is Gaussian and can be thus exactly simulated on any given time grid by using (1.2). Thus, we focus in this chapter on the simulation of the CIR process

$$X_t^x = x + \int_0^t (a - kX_s^x)ds + \int_0^t \sigma\sqrt{X_s^x}dW_s, \ x, t \geq 0.$$

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We make a survey on the different simulation methods. We first explain how it is possible to draw a random variable that follows a noncentral chi-squared distribution. Then, we present approximation schemes for the CIR. We first introduce schemes that may be seen as modifications of the Euler-Maruyama scheme. Unfortunately, these schemes turn out to be efficient only for some parameters of the CIR process. Then, we give high order weak approximations that are accurate without restriction on the parameters.

At this stage, it may seem weird to study approximations of the CIR process while it is possible to sample it exactly. In fact, generating exactly the CIR increments requires more computation time than using approximation schemes. Besides, it is in many cases unnecessary because one has to do elsewhere approximations. For example, if the CIR is only a part of a more intricate diffusion, one may have to approximate anyway the whole diffusion. This can also be the approximation of a continuous payoff by a discrete one. Thus, it is a good thing to have at our disposal both exact and high order approximation schemes. The exact scheme is useful to calculate some particular expectations for which there is no longer any discretization error. The high order schemes are faster to sample and are thus better suited to calculate general kind of expectations.

3.1 Exact Simulation Methods

The goal of this section is to sample a random variable X_t^x according to the probability density function (1.23) that we repeat here:

$$p(t,x,z) = \sum_{i=0}^{\infty} \frac{e^{-d_t x/2}(d_t x/2)^i}{i!} \frac{c_t/2}{\Gamma(i+\frac{2a}{\sigma^2})} \left(\frac{c_t z}{2}\right)^{i-1+\frac{2a}{\sigma^2}} e^{-c_t z/2}, \; z > 0, \tag{3.1}$$

with $c_t = \frac{4}{\sigma^2 \zeta_k(t)}$, $d_t = c_t e^{-kt}$. We can easily describe this law by means of a Poisson and a Gamma distributions. We recall that $N \sim \mathcal{P}(\lambda)$ is Poisson random variable with parameter $\lambda > 0$ if

$$\forall k \in \mathbb{N},\; \mathbb{P}(N = k) = e^{-\lambda}\frac{\lambda^k}{k!},$$

and that $\mathbb{E}[N] = \lambda$. To generate this variable from a uniform random variable $U \sim \mathcal{U}([0,1])$, we use the inverse transform sampling that gives that $\sum_{l \in \mathbb{N}^*} l \mathbb{1}_{E_{l-1} \le U < E_l} \sim \mathcal{P}(\lambda)$, with $E_l = e^{-\lambda}\sum_{k=0}^{l} \frac{\lambda^k}{k!}$. In practice, we first sample U and calculate successively the sums E_l until it exceeds U, which is made by a while loop. When λ is small, generating a Poisson random variable is cheap, since the while loop stops quickly. This can instead be rather time consuming when λ gets large, which flattens the probability distribution function and increases the average number of sums to compute. To correct this, different sampling methods have been proposed, and we refer to the authoritative book of Devroye [44] for a survey. However, we have used in our numerical experiments the inversion transform sampling.

The Gamma distribution $\Gamma(\alpha, \beta)$ with shape $\alpha > 0$ and rate $\beta > 0$ has the following density

$$\frac{\beta^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\beta z}, \; z > 0. \tag{3.2}$$

Methods to sample Gamma random variable can be found in Devroye [44], Fishman [54] and Glasserman [62]. For sake of completeness, we repeat in Appendix B the simulation methods stated in [62].

Now, we can state the following proposition that enables us to draw the marginal laws of a CIR process by sampling one Poisson and one Gamma random variables.

Proposition 3.1.1 *Let N be a Poisson random variable with parameter $d_t x/2$ and Z be a random variable such that the conditional law of Z given N is $\Gamma\left(N + \frac{2a}{\sigma^2}, \frac{c_t}{2}\right)$. Then, Z and X_t^x have the same law.*

Proof Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable bounded variable. We have

$$\mathbb{E}[f(Z)|N] = \int_0^\infty \frac{\left(\frac{c_t}{2}\right)^{N+\frac{2a}{\sigma^2}}}{\Gamma(N + \frac{2a}{\sigma^2})} z^{N+\frac{2a}{\sigma^2}-1} e^{-\frac{c_t}{2}z} dz,$$

and deduce $\mathbb{E}[f(Z)] = \int_0^\infty f(z)p(t,x,z)dz$, which gives the claim. □

In practice, sampling Poisson random variables may be time consuming. When $\frac{4a}{\sigma^2} > 1$, we can use the following method that requires to sample one Normal and one Gamma random variables. Let us consider W^1 and W^2 two independent Brownian motions and

$$X_t^1 = x + \int_0^t (\frac{\sigma^2}{4} - kX_s^1)ds + \int_0^t \sigma\sqrt{X_s^1}dW_s^1,$$
$$X_t^2 = \int_0^t (a - \frac{\sigma^2}{4} - kX_s^2)ds + \int_0^t \sigma\sqrt{X_s^2}dW_s^2.$$

From Exercise 1.2.13, we know that $(X_t^1 + X_t^2)_{t\geq 0}$ and $(X_t^x)_{t\geq 0}$ have the same law, and in particular $X_t^1 + X_t^2$ is distributed according to the probability density function $p(t,x,z)$. We note that $X^2 \equiv 0$ in the special case $a = \frac{\sigma^2}{4}$. From (1.26) with $p = 1$, we get the following result.

Proposition 3.1.2 *Suppose that* $\frac{4a}{\sigma^2} \geq 1$. *Let* $G \sim \mathcal{N}(0,1)$. *We consider an independent random variable* $Z \sim \Gamma\left(\frac{2a}{\sigma^2} - \frac{1}{2}, \frac{c_t}{2}\right)$ *when* $\frac{4a}{\sigma^2} > 1$ *and set* $Z = 0$ *if* $\sigma^2 = 4a$. *Then,* $(e^{-kt/2}\sqrt{x} + (\sigma/2)\sqrt{\zeta_k(t)}G)^2 + Z$ *and* X_t^x *have the same law.*

Recently, Shao [115] has proposed an analogous method when $\frac{4a}{\sigma^2} < 1$ which avoids to sample a Poisson random variable. To do so, we observe that

$$\frac{d}{dx}[e^{d_t x/2} p(t,x,z)] = \frac{d_t}{2} \sum_{i=1}^{\infty} \frac{(d_t x/2)^{i-1}}{(i-1)!} \frac{c_t/2}{\Gamma(i + \frac{2a}{\sigma^2})} \left(\frac{c_t z}{2}\right)^{i-1+\frac{2a}{\sigma^2}} e^{-c_t z/2}$$
$$= \frac{d_t}{2} e^{d_t x/2} \tilde{p}(t,x,z), \tag{3.3}$$

where $\tilde{p}(t,x,z)$ is a transition density of a CIR process $\tilde{X}_t^x$ with parameters $\tilde{a} = a + \frac{\sigma^2}{2}$, k and σ. We note that $\tilde{a} \geq \frac{\sigma^2}{2}$ and therefore the law of $\tilde{X}_t^x$ can be sampled by using Proposition 3.1.2. By integrating (3.3), we get

$$p(t,x,z) = e^{-d_t x/2} p(t,0,z) + \int_0^x \frac{d_t}{2} e^{d_t(\xi - x)/2} \tilde{p}(t,\xi,z) d\xi$$
$$= e^{-d_t x/2} p(t,0,z) + \int_{e^{-d_t x/2}}^1 \tilde{p}\left(t, \frac{2}{d_t}\log(u) + x, z\right) du,$$

which gives the following proposition.

Proposition 3.1.3 *Let* $U \sim \mathcal{U}([0,1])$ *and* $Z \sim \Gamma\left(\frac{2a}{\sigma^2}, \frac{c_t}{2}\right)$ *be independent random variables that are independent from* $\tilde{X}$. *Then,* $\mathbb{1}_{U \leq e^{-d_t x/2}} Z + \mathbb{1}_{U > e^{-d_t x/2}} \tilde{X}_t^{x + \frac{2}{d_t}\log(U)}$ *and* X_t^x *have the same law.*

3.2 Discretization Schemes

We present in this section an overview of the different discretization schemes for the CIR process. As in Chap. 2, we consider a time horizon $T > 0$ and, for $n \in \mathbb{N}^*$ the regular time discretization $t_i = iT/n$, $i \in \{0, \ldots, n\}$. For the CIR process, the Euler-Maruyama scheme should satisfy

$$\hat{X}_{t_0} = x,\ \hat{X}_{t_{i+1}} = \hat{X}_{t_i} + (a - k\hat{X}_{t_i})\frac{T}{n} + \sigma\sqrt{\hat{X}_{t_i}}(W_{t_{i+1}} - W_{t_i}),\ 1 \leq i \leq n-1.$$

Unfortunately, this scheme is not well defined. In fact, the Gaussian increments may lead the scheme to negative values with some positive probability, and the square-root is then no longer defined. Also, if we consider the Milstein scheme (2.11)

$$\hat{X}_{t_{i+1}} = \hat{X}_{t_i} + (a - k\hat{X}_{t_i})\frac{T}{n} + \sigma\sqrt{\hat{X}_{t_i}}(W_{t_{i+1}} - W_{t_i}) + \frac{\sigma^2}{4}[(W_{t_{i+1}} - W_{t_i})^2 - \frac{T}{n}] \tag{3.4}$$

$$= \left(\sqrt{\hat{X}_{t_i}} + \frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i})\right)^2 + (a - \frac{\sigma^2}{4} - k\hat{X}_{t_i})\frac{T}{n}.$$

This scheme is well defined for $k \le 0$ and $\sigma^2 \le 4a$. Otherwise, we may have $\hat{X}_{t_i} \ge 0$ and $a - \frac{\sigma^2}{4} - k\hat{X}_{t_i} < 0$, $\hat{X}_{t_{i+1}}$ may then take negative value with some positive probability. Therefore, the Milstein scheme is not almost surely well defined. The schemes that we present in this section can be seen as corrections to the Euler-Maruyama and Milstein schemes.

3.2.1 Implicit Euler Schemes

The implicit Euler scheme or backward Euler method is a standard scheme to approximate ordinary differential equations. When applying this idea to SDEs, one has to take care about the stochastic integral and compensate suitably with a quadratic variation term. Let us illustrate this on the CIR case. We have $\langle d\sqrt{X_t^x}, dW_t\rangle = \frac{\sigma}{2}dt$ and therefore $(\sqrt{X_{t_{i+1}}^x} - \sqrt{X_{t_i}^x})(W_{t_{i+1}} - W_{t_i}) \approx \frac{\sigma}{2}\frac{T}{n}$. This suggests the following scheme

$$\hat{X}_{t_{i+1}} = \hat{X}_{t_i} + \left(a - k\hat{X}_{t_{i+1}} - \frac{\sigma^2}{2}\right)\frac{T}{n} + \sigma\sqrt{\hat{X}_{t_{i+1}}}(W_{t_{i+1}} - W_{t_i}),\ 1 \le i \le n-1,$$

that has been proposed in Brigo and Alfonsi [21]. We see that $\sqrt{\hat{X}_{t_{i+1}}}$ appears as a root of a second degree polynomial. When $\sigma^2 < 2a$, $1 + kT/n > 0$ and $\hat{X}_{t_i} \ge 0$, there is only one positive root which defines $\sqrt{\hat{X}_{t_{i+1}}}$. Then, the scheme is well defined when $\sigma^2 \ge 2a$ and n is large enough, and we get:

$$\hat{X}_{t_{i+1}} = \left(\frac{\sigma(W_{t_{i+1}} - W_{t_i}) + \sqrt{\sigma^2(W_{t_{i+1}} - W_{t_i})^2 + 4(\hat{X}_{t_i} + (a - \frac{\sigma^2}{2})\frac{T}{n})(1 + k\frac{T}{n})}}{2(1 + k\frac{T}{n})}\right)^2. \tag{3.5}$$

This scheme has been studied in Alfonsi [7]. It is shown that the scheme (3.5) converges strongly, i.e. $\mathbb{E}\left[\max_{0\le i\le n}|\hat{X}_{t_i}-X^x_{t_i}|\right] \underset{n\to+\infty}{\to} 0$, and has a weak error of order 1.

Following the same idea, Alfonsi [7] has proposed another implicit scheme that is obtained from the SDE satisfied by $\sqrt{X^x_t}$. When the process X^x_t is away from zero, which is the case when $x>0$ and $\sigma^2\ge 2a$, we have by Itô's formula:

$$d\sqrt{X^x_t}=\left(\frac{a-\sigma^2/4}{2\sqrt{X^x_t}}-k\sqrt{X^x_t}\right)dt+\frac{\sigma}{2}dW_t.$$

This dynamics suggests the following implicit scheme:

$$\sqrt{\hat{X}_{t_{i+1}}}=\sqrt{\hat{X}_{t_i}}+\left(\frac{a-\sigma^2/4}{2\sqrt{\hat{X}_{t_{i+1}}}}-\frac{k}{2}\sqrt{\hat{X}_{t_{i+1}}}\right)\frac{T}{n}+\frac{\sigma}{2}(W_{t_{i+1}}-W_{t_i}).$$

Multiplying this equation by $\sqrt{\hat{X}_{t_{i+1}}}$, we observe once again that $\sqrt{\hat{X}_{t_{i+1}}}$ is a root of a second degree polynomial function. When $\sigma^2<4a$, $1+kT/2n>0$ and $\hat{X}_{t_i}\ge 0$, there is only one positive root which defines $\sqrt{\hat{X}_{t_{i+1}}}$. We get the following value

$$\hat{X}_{t_{i+1}}=\left(\frac{\frac{\sigma}{2}(W_{t_{i+1}}-W_{t_i})+\sqrt{\hat{X}_{t_i}}+\sqrt{(\frac{\sigma}{2}(W_{t_{i+1}}-W_{t_i})+\sqrt{\hat{X}_{t_i}})^2+4(1+\frac{kT}{2n})\frac{a-\sigma^2/4}{2}\frac{T}{n}}}{2(1+\frac{kT}{2n})}\right)^2. \tag{3.6}$$

The numerical behaviour of this scheme for the weak and the strong convergence is discussed in [7]. It is observed that the strong convergence rate downgrades as σ increases, as for many other schemes. This is not fully surprising if we keep in mind Proposition 1.2.15 and take a look to Fig. 3.1. The larger is σ, the more the CIR process may spend time around zero where the square-root is non Lipschitz and its derivative is blowing up. When $X^x_{t_i}$ is close to zero, even if the error the error $\hat{X}_{t_i}-X^x_{t_i}$ is small, the error between the square-roots $\sqrt{\hat{X}_{t_i}}-\sqrt{X^x_{t_i}}$ is significantly larger and is propagated. This explains heuristically why the strong convergence rate gets lower when σ gets higher.

The theoretical study of the strong convergence for the scheme (3.6) has recently been investigated by Dereich et al. [43], followed by Alfonsi [9] and Neuenkirch and Szpruch [108]. They have obtained the following results.

Theorem 3.2.1 *Let $x > 0$, $2a > \sigma^2$ and $T > 0$. Then, for all $p \in [1, \frac{2a}{\sigma^2})$, there is a constant $K_p > 0$ such that for any $n \geq \frac{T}{2}\max(-k, 0)$,*

$$\left(\mathbb{E}\left[\max_{0\leq i\leq n} |\hat{X}_{t_i} - X^x_{t_i}|^p\right]\right)^{1/p} \leq K_p\sqrt{\frac{T}{n}}.$$

If in addition $a > \sigma^2$, then we have for all $p \in [1, \frac{4a}{3\sigma^2})$,

$$\left(\mathbb{E}\left[\max_{0\leq i\leq n} |\hat{X}_{t_i} - X^x_{t_i}|^p\right]\right)^{1/p} \leq K_p\frac{T}{n}.$$

The fact that the scheme (3.6) may have a strong order rate of convergence equal to 1 under suitable conditions on the coefficients is not a blind chance. In fact, if we consider that the Brownian increment is of order $\sqrt{T/n}$ and make an expansion of (3.6) up to order T/n, we get

$$\hat{X}_{t_{i+1}} \approx \hat{X}_{t_i}\left(1 - k\frac{T}{n}\right) + \sigma\sqrt{\hat{X}_{t_i}}(W_{t_{i+1}} - W_{t_i}) + \frac{\sigma^2}{4}(W_{t_{i+1}} - W_{t_i})^2 + \left(a - \frac{\sigma^2}{4}\right)\frac{T}{n}.$$

Thus, we get up to terms of order $(T/n)^{3/2}$ the same expansion as the Milstein scheme (3.4), which is known to have a strong convergence of order 1 under suitable conditions on the SDE coefficients that are however not satisfied by the CIR.

In addition to the convergence results, the implicit schemes that we have presented above have some nice properties. First, they are naturally positive. Besides, they have the monotonicity property: $\hat{X}_{t_{i+1}}$ is an increasing function of $\hat{X}_{t_i}$. This property is also satisfied by the CIR process, since we have $X^x_t \leq X^{x'}_t$ for all $t \geq 0$ if $0 \leq x \leq x'$ by [83], Proposition 2.18, p. 293. However, they have the drawback of being only defined for some range of parameters, namely $\sigma^2 \leq 2a$ for (3.5) and $\sigma^2 \leq 4a$ for (3.6). Of course, we can try to extend these schemes, and it is proposed in [7] to set $\hat{X}_{t_{i+1}} = 0$ when the discriminant of the second degree polynomial function in $\sqrt{\hat{X}_{t_{i+1}}}$ is negative. However, the numerical study on the convergence made in [7] show that these corrections have a degraded convergence when $\sigma^2 \gg 4a$. Before concluding this section on implicit schemes, we mention that Kahl and Jäckel [81] have also proposed an implicit scheme which is derived from the Milstein scheme (3.4). As the other schemes, it is not well suited when the volatility coefficient is large.

3.2.2 Modified Explicit Euler Schemes

We have seen that the Euler-Maruyama scheme is not well-defined for the CIR process, because it would require to calculate the square-root of a negative real number. To correct this, Delbaen and Deelstra [40] have proposed the following scheme

$$\hat{X}_{t_0} = x,\ \hat{X}_{t_{i+1}} = \hat{X}_{t_i} + (a - k\hat{X}_{t_i})\frac{T}{n} + \sigma\sqrt{\left(\hat{X}_{t_i}\right)^+}(W_{t_{i+1}} - W_{t_i}),\ 1 \le i \le n-1. \tag{3.7}$$

Doing so, the scheme may take negative values but it is still well defined. They have shown the strong convergence of their scheme. In the same way, Higham and Mao [76] have considered the scheme

$$\hat{X}_{t_0} = x,\ \hat{X}_{t_{i+1}} = \hat{X}_{t_i} + (a - k\hat{X}_{t_i})\frac{T}{n} + \sigma\sqrt{|\hat{X}_{t_i}|}(W_{t_{i+1}} - W_{t_i}), \tag{3.8}$$

while Lord et al. [101] have proposed

$$\hat{X}_{t_0} = x,\ \hat{X}_{t_{i+1}} = \hat{X}_{t_i} + (a - k(\hat{X}_{t_i})^+)\frac{T}{n} + \sigma\sqrt{\left(\hat{X}_{t_i}\right)^+}(W_{t_{i+1}} - W_{t_i}). \tag{3.9}$$

Again, it has been shown in the respective papers that these schemes converge strongly. To preserve the nonnegativity, Berkaoui et al. [19] have considered the following scheme

$$\hat{X}_{t_0} = x,\ \hat{X}_{t_{i+1}} = \left|\hat{X}_{t_i} + (a - k\hat{X}_{t_i})\frac{T}{n} + \sigma\sqrt{\hat{X}_{t_i}}(W_{t_{i+1}} - W_{t_i})\right|. \tag{3.10}$$

They have shown under rather restrictive conditions on the coefficients that it has a strong convergence of order $1/2$. Some of these schemes have been studied numerically in [7]. It is observed that when σ is small enough, typically $\sigma^2 \le 2a$, these schemes have a weak error of order one and a strong error of order $1/2$. This is exactly the convergence rate that has been obtained for the Euler-Maruyama scheme in Theorems 2.1.2 and 2.1.3. When σ is getting large, say $\sigma^2 \gg 4a$, we observe again that the convergence of all these schemes is degraded. However, as observed by Lord et al. [101], the schemes (3.7) and (3.9) behave better than the schemes (3.8) and (3.10). We can understand this if we take a look at Fig. 3.1. We see that when σ gets large, the CIR process spends more time close to zero. It seems even to be stuck in the neighbourhood of zero when σ is really large. When the scheme takes a negative value, the absolute value in (3.8) and (3.10) produces a noise that pushes the scheme away from zero. Instead, the positive part in (3.7) and (3.9) cancels the noise when the scheme gets negative, which better reproduces the behaviour of the CIR process.

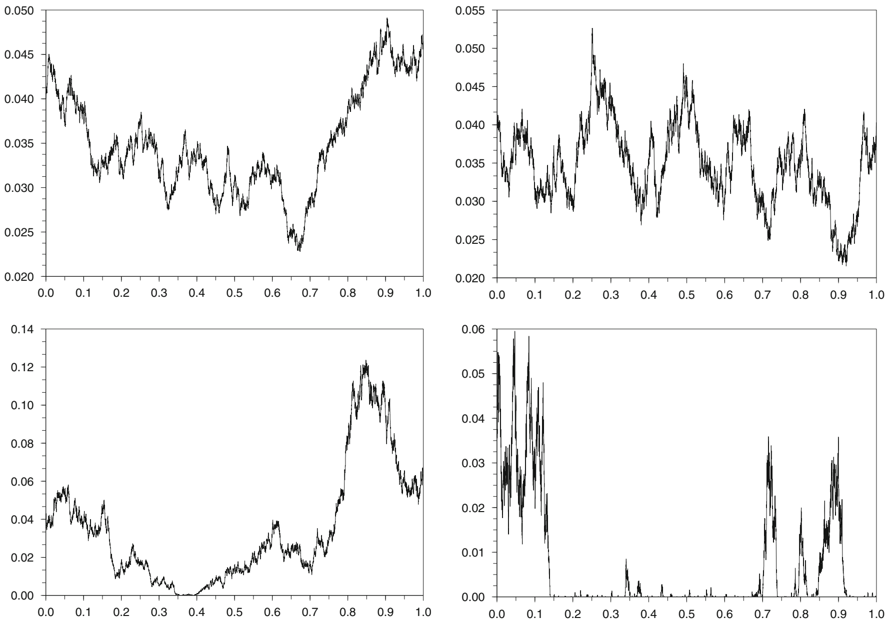

Fig. 3.1 Examples of paths of $(X_t^x)_{t\in[0,1]}$ with $x = 0.04$, $k = 0.5$, $a = 0.02$ and $\sigma = 0.1$ (*upper left*), $\sigma = 0.2$ (*upper right*), $\sigma = 0.4$ (*lower left*) and $\sigma = 1$ (*lower right*). This corresponds respectively to $\sigma^2 = a/2$, $\sigma^2 = 2a$, $\sigma^2 = 8a$ and $\sigma^2 = 25a$

The schemes (3.7)–(3.10) can be seen as modifications of the Euler scheme. It would be also natural to do the same thing with the Milstein scheme. Thus, Alfonsi [7] has considered the scheme

$$\hat{X}_{t_{i+1}} = \left(\left(1 - \frac{kT}{2n}\right)\sqrt{\hat{X}_{t_i}} + \frac{\sigma(W_{t_{i+1}} - W_{t_i})}{2(1 - \frac{kT}{2n})}\right)^2 + (a - \sigma^2/4)\frac{T}{n}, \tag{3.11}$$

which is well defined for $\sigma^2 \leq 4a$ and $n > kT/2$. Up to terms of order $(T/n)^{3/2}$ this scheme has the same expansion as the Milstein scheme. A strong convergence rate of order 1 is observed numerically in [7] for the scheme (3.11) when σ is small enough. Theoretically, the strong convergence is proved as well as the following weak error expansion

$$\mathbb{E}[f(\hat{X}_{t_n})] = \mathbb{E}[f(X_T^x)] + \frac{c_1}{n} + \cdots + \frac{c_\nu}{n^\nu} + O\left(\frac{1}{n^{\nu+1}}\right),$$

for any $\nu \in \mathbb{N}$ and some constants $c_1, \ldots, c_\nu$ when $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{R}_+)$. Of course, it is possible to extend this scheme to the case $\sigma^2 \geq 4a$ by taking for example the positive part in the square-root in the left hand side of (3.11). However, as the other schemes, its convergence gets worse when $\sigma \gg 4a$.

3.3 Weak Order Schemes

We have presented the Ninomiya and Victoir scheme in Chap. 2, Sect. 2.3. Under suitable assumptions, we have shown that this is a second order scheme for the weak error. Even tough these assumptions are not satisfied by the CIR process, it is natural to investigate if this scheme may work. To do so, we split the infinitesimal generator of the CIR process as follows

$$Lf(x) = (a - kx)\partial_x f(x) + \frac{1}{2}\sigma^2 x \partial_x^2 f(x) = V_0 f(x) + \frac{1}{2}V_1^2 f(x), \tag{3.12}$$

with $V_0 f(x) = (a - kx - \frac{\sigma^2}{4})f'(x)$ and $V_1 f(x) = \sigma\sqrt{x} f'(x)$. Each part can be solved explicitly. On the one hand,

$$X_0(t, x) = xe^{-kt} + (a - \sigma^2/4)\zeta_k(t)$$

solves the ODE $\frac{d}{dt}X_0(t, x) = a - kX_0(t, x) - \frac{\sigma^2}{4}$; we recall that $\zeta_k(t) = \frac{1-e^{-kt}}{k}$ when $k \neq 0$ and $\zeta_0(t) = t$ otherwise. On the other hand, we know by Itô's formula that $\frac{1}{2}V_1^2$ is the infinitesimal generator of the process $X_1(W_t, x)$ with

$$X_1(t, x) = (\sqrt{x} + \frac{\sigma}{2}t)^2.$$

The Strang splitting $X_0(t/2, X_1(W_t, X_0(t/2, x)))$ (see Corollary 2.3.14) gives the Ninomiya and Victoir scheme for the CIR:

$$\hat{X}_t^x = e^{-\frac{kt}{2}}\left(\sqrt{(a - \sigma^2/4)\zeta_k(t/2) + e^{-\frac{kt}{2}}x} + \frac{\sigma}{2}W_t\right)^2 + (a - \sigma^2/4)\zeta_k(t/2).$$

Here, we use the notations introduced in Chap. 2. The law of $\hat{X}_t^x$ describes how is sampled the scheme starting from x with a time step t. On the regular time grid, this corresponds to the scheme

$$\hat{X}_{t_{i+1}} = X_0\left(\frac{T}{2n}, X_1\left(W_{t_{i+1}} - W_{t_i}, X_0\left(\frac{T}{2n}, \hat{X}_{t_i}\right)\right)\right),$$

which has been studied by Ninomiya and Victoir [109]. This scheme is well defined when $\sigma^2 \leq 4a$, and it is a potential second order scheme by Theorem 2.3.17. This is indeed a second order scheme thanks to the following result.

Proposition 3.3.1 *For* $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{R})$*, the function* $u(t, x) = \mathbb{E}[f(X_{T-t}^x)]$ *is well defined on* $[0, T] \times \mathbb{R}_+$*,* $\mathcal{C}^\infty$ *and solves the PDE*

$$t \in [0, T], x \geq 0, \ \partial_t u(t, x) = -(a - kx)\partial_x u(t, x) - \frac{\sigma^2}{2}x\partial_x^2 u(t, x).$$

Besides, it satisfies

$$\forall l \in \mathbb{N}, m \in \mathbb{N}, \exists C_{l,m}, e_{l,m} > 0, \forall x \in \mathbb{R}_+, t \in [0,T],\ |\partial_t^l \partial_x^m u(t,x)| \leq C_{l,m}(1 + x^{e_{l,m}}). \tag{3.13}$$

The proof of this result is postponed to Sect. 3.3.5. In enables us to apply Theorem 2.3.8, and we get that the Ninomiya and Victoir scheme is a second order scheme when $\sigma^2 \leq 4a$.

Unfortunately, the Ninomiya and Victoir scheme is no longer defined when $\sigma^2 > 4a$. Let us show this and consider $x \geq 0$. When x is close to zero, $X_0(t/2, x)$ is negative and $X_1(W_t, X_0(t/2, x))$ is not defined. When x is larger, $X_1(W_t, X_0(t/2, x))$ is well defined but can be close to zero with some positive probability due to the Gaussian law. Thus, we still have $\mathbb{P}(\hat{X}_t^x < 0) > 0$ in this case. Once the scheme is fallen into negative values, it is not well defined at the next time-step. Now, we present weak high order schemes that have been proposed in Alfonsi [8] to correct this problem. They are well defined for any range of parameters and achieve second and third orders of convergence.

3.3.1 A Second Order Scheme

As noticed in Corollary 2.3.20, the weak convergence rate of the Ninomiya and Victoir scheme is not modified if we replace the Gaussian increments by random variables that match their five first moments. In particular, it is possible to take random variables that have a compact support. This simple remark enables us to correct the Ninomiya and Victoir scheme when the initial value is large enough. Let Y be any bounded random variable that matches the five first moments of the standard normal variable. We consider the corrected scheme

$$\boxed{\begin{aligned} \hat{X}_t^x &= X_0(t/2, X_1(\sqrt{t}Y, X_0(t/2, x))) \qquad (3.14) \\ &= e^{-\frac{kt}{2}} \left(\sqrt{(a - \sigma^2/4)\zeta_k(t/2) + e^{-\frac{kt}{2}} x} + \frac{\sigma}{2}\sqrt{t}Y \right)^2 + (a - \sigma^2/4)\zeta_k(t/2). \end{aligned}}$$

Again, this scheme is well defined and nonnegative when $\sigma^2 \leq 4a$. Now when $\sigma^2 > 4a$, $\hat{X}_t^x$ is also well defined and nonnegative if x is large enough. Namely, if $\mathbb{P}(|Y| \leq A) = 1$ for some $A > 0$, the scheme (3.14) is well defined and nonnegative if

$$x \geq e^{\frac{kt}{2}} \left((\frac{\sigma^2}{4} - a)\zeta_k(t/2) + \left[\sqrt{e^{\frac{kt}{2}}[(\frac{\sigma^2}{4} - a)\zeta_k(t/2)]} + \frac{\sigma}{2}A\sqrt{t} \right]^2 \right).$$

In fact, this condition implies $X_0(t/2,x) \geq \left[\sqrt{e^{\frac{kt}{2}}[(\frac{\sigma^2}{4}-a)\zeta_k(t/2)]} + \frac{\sigma}{2}A\sqrt{t}\right]^2$ and then $X_1(\sqrt{t}Y, X_0(t/2,x)) \geq e^{\frac{kt}{2}}[(\frac{\sigma^2}{4}-a)\zeta_k(t/2)]$ which eventually gives

$$X_0(t/2, X_1(\sqrt{t}Y, X_0(t/2,x))) \geq 0.$$

To fix the ideas and lighten notations, we will consider from now the random variable Y defined by (2.27), even though it would be possible to construct a second order scheme for any other bounded random variable Y that matches the five first moments of the standard normal variable. The scheme (3.14) is then well defined as soon as $x \geq \mathbf{K}_2(t)$, with

$$\mathbf{K}_2(t) = \mathbb{1}_{\{\sigma^2>4a\}}e^{\frac{kt}{2}}\left((\frac{\sigma^2}{4}-a)\zeta_k(t/2)+\left[\sqrt{e^{\frac{kt}{2}}[(\frac{\sigma^2}{4}-a)\zeta_k(t/2)]}+\frac{\sigma}{2}\sqrt{3t}\right]^2\right). \tag{3.15}$$

For what follows, an important thing to notice is the asymptotic behaviour of this threshold when the time step goes to zero. We have for $\sigma^2 > 4a$

$$\mathbf{K}_2(t) \underset{t\to 0}{\sim} \left[\frac{1}{2}(\frac{\sigma^2}{4}-a) + (\sqrt{\frac{1}{2}(\frac{\sigma^2}{4}-a)}+\frac{\sigma}{2}\sqrt{3})^2\right]t, \tag{3.16}$$

the region where this scheme cannot be used has a size which is asymptotically proportional to the time step.

Now, we would like to construct an approximation scheme when $x \in [0, \mathbf{K}_2(t))$. To do so, we claim that it is sufficient to get a nonnegative scheme $\hat{X}_t^x$ that is such that

$$\begin{cases} \forall i \in \{1,2\},\ \mathbb{E}[(\hat{X}_t^x)^i] = \mathbb{E}[(X_t^x)^i], \\ \forall q \in \mathbb{N}^*, \exists C_q > 0, \forall t \in [0,1], x \in [0, \mathbf{K}_2(t)),\ \mathbb{E}[(\hat{X}_t^x)^q] \leq C_q t^q. \end{cases} \tag{3.17}$$

Of course, this property is satisfied by the exact scheme X_t^x, which we left as an exercise for the reader.

Exercise 3.3.2 Show by induction on $q \in \mathbb{N}^*$ that for any $q \in \mathbb{N}^*$ there is $C_q > 0$ such that $\forall t \in [0,1], x \in [0, \mathbf{K}_2(t)),\ \mathbb{E}[(X_t^x)^q] \leq C_q t^q$.

Let $f \in C^\infty_{\mathbf{pol}}(\mathbb{R}_+)$. The Taylor formula around 0 gives $f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \int_0^x \frac{(x-y)^2}{2} f^{(3)}(y)dy$. Since $f \in C^\infty_{\mathbf{pol}}(\mathbb{R}_+)$, there are constants $C > 0$, $q \in \mathbb{N}^*$ such that $|f^{(3)}(x)| \leq C(1+x^q)$ for any $x \geq 0$, and therefore

$$\left|\int_0^x \frac{(x-y)^2}{2} f^{(3)}(y)dy\right| \leq \frac{C}{2}x^3(1+x^q).$$

By (3.17) and Exercise 3.3.2, we deduce that for $x \in [0, \mathbf{K}_2(t))$,

$$\left|\mathbb{E}[f(\hat{X}_t^x)] - \mathbb{E}[f(X_t^x)]\right| \le \tilde{C}t^3,$$

and this constant $\tilde{C}$ only depends on a good sequence on f, namely on the polynomial growth coefficients of its third derivative. We have thus obtained the following result.

Proposition 3.3.3 *Let $\hat{X}_t^x$ be a nonnegative scheme that satisfies* (3.17)*. Then, it is a potential second order scheme for $x \in [0, \mathbf{K}_2(t))$.*

Remark 3.3.4 From the proof of Proposition 3.3.3, we see that it is not necessary to fit exactly the first two moments in order to get a potential second order scheme. It would have been sufficient to have $|\mathbb{E}[(\hat{X}_t^x)^i] - \mathbb{E}[(X_t^x)^i]| \le Ct^3$ for $i \in \{1, 2\}$ and $x \in [0, \mathbf{K}_2(t))$. We will not use this remark for the CIR process because the moments are known explicitly. However, it can be interesting if one would like to extend this scheme construction to similar processes for which moments are not explicit, for example $dX_t = (a - k(X_t)^\alpha)dt + \sigma\sqrt{X_t}dW_t$ with $X_0, a, k, \sigma, \alpha > 0$ and $\alpha \neq 1$.

Let us now construct a scheme that achieves (3.17). We consider a scheme $\hat{X}_t^x$ that takes two possible values $0 \le x_-(t, x) < x_+(t, x)$ with respective probabilities $1 - \pi(t, x)$ and $\pi(t, x)$. For $q \in \mathbb{N}$, we denote by $\tilde{u}_q(t, x) = \mathbb{E}((X_t^x)^q)$ the qth moment of X_t^x. We want to have

$$\begin{cases} \pi(t,x)x_+(t,x) + (1 - \pi(t,x))x_-(t,x) = \tilde{u}_1(t,x) \\ \pi(t,x)x_+(t,x)^2 + (1 - \pi(t,x))x_-(t,x)^2 = \tilde{u}_2(t,x). \end{cases}$$

Some calculations give

$$\boxed{\tilde{u}_1(t,x) = xe^{-kt} + a\zeta_k(t) \text{ and } \tilde{u}_2(t,x) = \tilde{u}_1(t,x)^2 + \sigma^2\zeta_k(t)[a\zeta_k(t)/2 + xe^{-kt}].} \tag{3.18}$$

Let us define $\gamma_\pm(t, x) = \frac{x_\pm(t,x)}{\tilde{u}_1(t,x)}$. We want to solve

$$\begin{cases} \pi(t,x)\gamma_+(t,x) + (1 - \pi(t,x))\gamma_-(t,x) = 1 \\ \pi(t,x)\gamma_+(t,x)^2 + (1 - \pi(t,x))\gamma_-(t,x)^2 = \frac{\tilde{u}_2(t,x)}{\tilde{u}_1(t,x)^2}. \end{cases} \tag{3.19}$$

We arbitrarily take $\gamma_+(t, x) = 1/(2\pi(t, x))$ and $\gamma_-(t, x) = 1/(2(1 - \pi(t, x)))$ which ensures the first equation and the positivity of the random variable when $\pi(t, x) \in (0, 1)$. Then, we obtain from the last equation

$$\pi^2(t,x) - \pi(t,x) + \tilde{u}_1(t,x)^2/(4\tilde{u}_2(t,x)) = 0.$$

The discriminant is $\Delta(t,x) = 1 - \tilde{u}_1(t,x)^2/\tilde{u}_2(t,x) \in [0,1]$, and we take

$$\pi(t,x) = \frac{1-\sqrt{\Delta(t,x)}}{2} \tag{3.20}$$

to have $\gamma_+ > \gamma_-$. $0 \le \pi(t,x) \le 1/2$. Besides, we get $\tilde{u}_2(t,x)/\tilde{u}_1(t,x)^2 \le 1 + \sigma^2/(2a)$ from (3.18) since $\tilde{u}_1(t,x)^2 \ge a^2\zeta_k(t)^2 + 2ax\zeta_k(t)e^{-kt}$. Therefore, $\Delta(t,x) \ge 1 - 1/(1+\sigma^2/(2a))$ and we get $0 < \pi_{\min} = (1 - \sqrt{1-1/(1+\sigma^2/(2a))})/2 \le \pi(t,x) \le 1/2$. From (3.16), there is a constant $C > 0$ that depends on the CIR parameters such that $\tilde{u}_1(t,x) \le Ct$ for $x \in [0, \mathbf{K}_2(t)]$ and $t \le 1$. Therefore, we get $0 \le \hat{X}_t^x \le \frac{C}{2\pi_{\min}} t$ and $\hat{X}_t^x$ satisfies (3.17). From Corollary 2.3.20, Proposition 3.3.3, Theorem 2.3.8 and Proposition 3.3.1, we deduce the following result.

Proposition 3.3.5 *Let $U \sim \mathcal{U}([0,1])$. The scheme $\hat{X}_t^x$ defined by (3.14) for $x \ge \mathbf{K}_2(t)$ and by $\hat{X}_t^x = \mathbb{1}_{\{U \le \pi(t,x)\}} \frac{\tilde{u}_1(t,x)}{2\pi(t,x)} + \mathbb{1}_{\{U > \pi(t,x)\}} \frac{\tilde{u}_1(t,x)}{2(1-\pi(t,x))}$ for $x \in [0, \mathbf{K}_2(t))$ is a second order scheme for the weak error.*

Let us now make some comments on this scheme. A first natural question is to wonder if we could have made the same construction to get simply a first order scheme. Of course, the answer is positive, and it is in fact much easier. We have left it as an exercise, since it repeats exactly the same arguments used above. Another natural question is then to know if we can construct schemes of higher order. By using a trick, we will propose a third order scheme for the CIR. In fact, the difficulty is not to sample the CIR close to zero, but rather to find high order scheme since there is no general construction of weak νth order schemes for $\nu \ge 3$.

Exercise 3.3.6 The aim of this exercise is to construct a first order scheme for the CIR process. We first consider the case $k = 0$ and the modified Euler scheme

$$x \ge 0,\ \hat{X}_t^x = x + at + \sigma\sqrt{tx}Y = \left(\sqrt{x} + \frac{\sigma}{2}\sqrt{t}Y\right)^2 + \left(a - \frac{\sigma^2}{4}\right)t, \tag{3.21}$$

where $\mathbb{P}(Y=1) = \mathbb{P}(Y=-1) = 1/2$.

1. Show that $\hat{X}_t^x$ is nonnegative for $x \ge \mathbf{K}_1(t)$, with $\mathbf{K}_1(t) = \mathbb{1}_{\{\sigma^2 > 4a\}} \frac{\sigma + \sqrt{\sigma^2 - 4a}}{2} t$, and that this is a potential first order scheme.
2. Show that $\hat{X}_t^x = x + at = \tilde{u}_1(t,x)$ (yes, it is deterministic!) is a potential first order scheme for $x \in [0, \mathbf{K}_1(t))$. Deduce that the scheme obtained by (3.21) for $x \ge \mathbf{K}_1(t)$ and $\hat{X}_t^x = x + at$ for $x \in [0, \mathbf{K}_1(t))$ achieves a weak error of order one for test functions $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{R}_+)$.
3. By using Exercise 2.3.16, show that $e^{-kt}\hat{X}_t^x$ is then a first order scheme for the CIR process when $k \in \mathbb{R}$.

Algorithm 3.1: Algorithm for the second-order scheme of the CIR with time-step $t > 0$, U (resp. Y) being sampled uniformly on $[0, 1]$ (resp. as (2.27)).

Input: $x \geq 0, a, \sigma \geq 0, k \in \mathbb{R}$ and $t > 0$.
Output: X, sampled according to the second order scheme of Proposition 3.3.5.
if $(x \geq \mathbf{K}_2(t))$ **then**

$$X = e^{-\frac{kt}{2}} \left(\sqrt{(a - \sigma^2/4)\zeta_k(t/2) + e^{-\frac{kt}{2}} x} + \frac{\sigma}{2} \sqrt{t} Y \right)^2 + (a - \sigma^2/4)\zeta_k(t/2),$$

else

$\pi = \frac{1 - \sqrt{1 - \tilde{u}_1(t,x)^2/\tilde{u}_2(t,x)}}{2}$,

if $(U < \pi)$ **then**

$X = \frac{\tilde{u}_1(t,x)}{2\pi}$,

else

$X = \frac{\tilde{u}_1(t,x)}{2(1-\pi)}$,

end

end

Let us now focus on the question of sampling the random variables Y. It is easy to sample a discrete random variable from a uniform random variable. In fact, it is even possible in principle to sample a sequence of independent and identically distributed random variables Y from only one uniform random variable, as recalled by the following exercise. If we use this idea in practice, this raises again the question of the period of this pseudorandom generator. Instead, we can use this exercise to sample a few numbers (say from 2 to 5) of random variables Y for each uniform random variable. In practice, this accelerates the Monte-Carlo algorithm since computing a pseudorandom is in principle more time consuming than computing the function ψ below.

Exercise 3.3.7 Let $U_0 \sim \mathcal{U}([0, 1])$. We define $y(u) = -\sqrt{3}\mathbb{1}_{u \leq \frac{1}{6}} + \sqrt{3}\mathbb{1}_{\frac{1}{6} < u \leq \frac{1}{3}}$ and $(u) = 6u\mathbb{1}_{u \leq \frac{1}{6}} + 6\left(u - \frac{1}{6}\right)\mathbb{1}_{\frac{1}{6} < u \leq \frac{1}{3}} + \frac{3}{2}\left(u - \frac{1}{3}\right)\mathbb{1}_{\frac{1}{3} < u}$. We define for $n \in \mathbb{N}$,

$$U_{n+1} = \psi(U_n) \text{ and } Y_n = \psi(U_n).$$

Show that Y_0 is a discrete random variable distributed according to (2.27). Show that $U_1 \sim \mathcal{U}([0, 1])$ and is independent from Y_0. Deduce that $Y_0, \ldots, Y_n, \ldots$ is a sequence of independent random variables distributed according to (2.27).

3.3.2 The Quadratic-Exponential (QE) Scheme

The QE scheme proposed by Andersen [11] is a scheme that matches the two first moments of the CIR process. These moments have been calculated in (3.18), and we set

$$\psi(t,x) = \frac{\tilde{u}_2(t,x) - \tilde{u}_1(t,x)^2}{\tilde{u}_1(t,x)^2} = \frac{\sigma^2 \zeta_k(t)[a\zeta_k(t)/2 + xe^{-kt}]}{(xe^{-kt} + a\zeta_k(t))^2}.$$

We can easily check that this function is decreasing with respect to $x \in \mathbb{R}_+$, and goes from $\psi(t,0) = \frac{\sigma^2}{2a}$ to $\psi(t,+\infty) = 0$. Let $N \sim \mathcal{N}(0,1)$ be a standard normal variable. When $\psi(t,x) \leq 2$, we obtain easily that the scheme

$$\hat{X}_t^x = \alpha(t,x)(\beta(t,x)+N)^2 \text{ with } \begin{cases} \beta(t,x)^2 = \frac{2}{\psi(t,x)} - 1 + \sqrt{\frac{2}{\psi(t,x)}}\sqrt{\frac{2}{\psi(t,x)} - 1} \\ \alpha(t,x) = \frac{\tilde{u}_1(t,x)}{1+\beta(t,x)^2} \end{cases} \tag{3.22}$$

is well defined and satisfies $\mathbb{E}[(\hat{X}_t^x)^i] = \mathbb{E}[(X_t^x)^i]$ for $i \in \{1,2\}$. Similarly, let $U \sim \mathcal{U}(0,1)$ be a uniform random variable. Then, the scheme

$$\hat{X}_t^x = -\mathbb{1}_{U \leq 1-p(t,x)} \frac{1}{\gamma(t,x)} \log\left(\frac{U}{1-p(t,x)}\right) \text{ with } \begin{cases} p(t,x) = \frac{\psi(t,x)-1}{\psi(t,x)+1} \\ \gamma(t,x) = \frac{2}{\tilde{u}_1(t,x)(1+\psi(t,x))} \end{cases} \tag{3.23}$$

is well defined when $\psi(t,x) \geq 1$ and also matches the two first moments of the CIR process. Andersen [11] then defines the scheme by (3.22) when $\psi(t,x) \leq \psi_c$ and by (3.22) when $\psi(t,x) \geq \psi_c$ for some critical value $\psi_c \in [1,2]$. In practice, he recommends to take $\psi_c = 3/2$. Since $\psi(t,x)$ is decreasing with respect to x, the scheme (3.22) is used away from 0 while the scheme (3.23) may be used when the discretization is close to zero. This is very similar to the second order scheme given by Proposition 3.3.5. Also, since $\psi(t,x) \leq \frac{\sigma^2}{2a}$, we observe that the scheme (3.23) is never used if $\sigma^2 \leq 2\psi_c a$.

Let us now discuss the weak error convergence of the QE scheme. Up to now, there are no dedicated studies on this. There are only numerical tests that show a rather good convergence, see Sect. 4.2.5 for the Heston model. Here, we discuss more generally on the weak convergence of schemes that matches the two first moments of a given diffusion. The construction of the second order scheme near zero (see Proposition 3.3.3) may let think that this is enough to match the two first moments in order to get a potential second order scheme. This is not the case. To get convinced, let us consider a scheme that matches the two first moments of a diffusion, i.e. $\mathbb{E}[(\hat{X}_t^x)^i] = \mathbb{E}[(X_t^x)^i]$ for $i \in \{1,2\}$. By a Taylor expansion around x, we have $f(z) = f(x) + f'(x)(z-x) + \frac{1}{2}f''(x)(z-x)^2 + \int_x^z \frac{(z-y)^2}{2} f^{(3)}(y)dy$ and get

$$\mathbb{E}[f(\hat{X}_t^x)] = \mathbb{E}[f(X_t^x)] + \mathbb{E}\left[\int_x^{\hat{X}_t^x} \frac{(\hat{X}_t^x - y)^2}{2} f^{(3)}(y)dy - \int_x^{X_t^x} \frac{(X_t^x - y)^2}{2} f^{(3)}(y)dy\right].$$

Algorithm 3.2: Algorithm for the QE scheme with time-step $t > 0$, U (resp. N) being sampled uniformly on $[0, 1]$ (resp. as a standard normal variable).

Input: $x \geq 0, a, \sigma \geq 0, k \in \mathbb{R}$ and $t > 0$.
Output: X, sampled according to the QE scheme.
if $(\psi(t, x) \leq 3/2)$ **then**
  $X = \alpha(t, x)(\beta(t, x) + N)^2$,
else
  if $U \leq 1 - p(t, x)$ **then**
    $X = -\frac{1}{\gamma(t,x)} \log\left(\frac{U}{1-p(t,x)}\right)$,
  else
    $X = 0$,
  end
end

To have a potential second order scheme, the second term of the right hand side should be in $O(t^3)$ which is not true in general. Without further assumptions, it is typically in $O(t^{3/2})$, which means that this is even not a potential first order scheme. For example, if one takes $f(x) = x^3$, $X_t^x = x + W_t$ and $\hat{X}_t^x = x + \sqrt{t}Y$ with

$$\mathbb{P}(Y = -\frac{1}{\sqrt{2}}) = \frac{2}{3},\ \mathbb{P}(Y = \sqrt{2}) = \frac{1}{3},$$

we have $\mathbb{E}[(\hat{X}_t^x)^i] = \mathbb{E}[(X_t^x)^i]$ for $i \in \{1, 2\}$ and $\mathbb{E}[(\hat{X}_t^x)^3] = \mathbb{E}[(X_t^x)^3] + t^{3/2}\mathbb{E}[Y^3]$, with $\mathbb{E}[Y^3] = 1/\sqrt{2} \neq 0$.

3.3.3 A Third Order Scheme

The construction of the third order scheme follows the same line as the construction of the second order scheme. On the one hand, we will construct a third order scheme $\hat{X}_t^x$ that is well defined and nonnegative for $x \geq \mathbf{K}_3(t)$, with $\mathbf{K}_3(t) \underset{t\to 0}{=} O(t)$. On the other hand, by using the same argument as above, it is sufficient to have when $x \in [0, \mathbf{K}_3(t))$ a nonnegative random variable $\hat{X}_t^x$ such that

$$\begin{cases} \forall i \in \{1, 2, 3\},\ \mathbb{E}[(\hat{X}_t^x)^i] = \mathbb{E}[(X_t^x)^i], \\ \forall q \in \mathbb{N}^*, \exists C_q > 0, \forall t \in [0, 1], x \in [0, \mathbf{K}_3(t)),\ \mathbb{E}[(\hat{X}_t^x)^q] \leq C_q t^q \end{cases}$$

to get a potential third order scheme. As explained in Alfonsi [8], these conditions are achieved by the random variable

$$\hat{X}_t^x = \mathbb{1}_{\{U \leq \pi(t,x)\}} x_+(t, x) + \mathbb{1}_{\{U > \pi(t,x)\}} x_-(t, x), \tag{3.24}$$

with $U \sim \mathcal{U}([0,1])$, $x_\pm(t,x) = \frac{s(t,x)\pm\sqrt{\Delta(t,x)}}{2}$ and $\pi(t,x) = \frac{\tilde{u}_1(t,x)-x_-(t,x)}{x_+(t,x)-x_-(t,x)}$, where $s(t,x) = \frac{\tilde{u}_3(t,x)-\tilde{u}_1(t,x)\tilde{u}_2(t,x)}{\tilde{u}_2(t,x)-\tilde{u}_1(t,x)^2}$, $p(t,x) = \frac{\tilde{u}_1(t,x)\tilde{u}_3(t,x)-\tilde{u}_2(t,x)^2}{\tilde{u}_2(t,x)-\tilde{u}_1(t,x)^2}$ and $\Delta(t,x) = s^2(t,x) - 4p(t,x) > 0$. We recall that the first and second moments have been given in equation (3.18), and that the third moment is given by

$$\tilde{u}_3(t,x) = \tilde{u}_1(t,x)\tilde{u}_2(t,x)+\sigma^2\zeta_k(t)\left[2x^2e^{-2kt} + \zeta_k(t)\left(a+\frac{\sigma^2}{2}\right)(3xe^{-kt}+a\zeta_k(t))\right].$$

We now focus on getting a third order scheme away from zero. We first consider the case $k = 0$, which is not a restriction thanks to Exercise 1.2.14. In this case, we observe that the operators V_0 and V_1 introduced in (3.12) satisfy

$$\frac{1}{2}\left(V_0V_1^2 - V_1^2V_0\right) = \frac{\sigma^2}{2}\left(a - \frac{\sigma^2}{4}\right)\partial_x^2.$$

We set $L_1 = V_0$ and $L_2 = \frac{1}{2}V_1^2$ if $\sigma^2 \le 4a$, $L_1 = \frac{1}{2}V_1^2$ and $L_2 = V_0$ otherwise. We also set $L_3 = \frac{\sigma}{\sqrt{2}}\sqrt{\left|a - \frac{\sigma^2}{4}\right|}\partial_x$ and then have

$$L_1L_2 - L_2L_1 = L_3^2.$$

We consider the formal series $S_i(t) = I + tL_i + \frac{t^2}{2}L_i^2 + \frac{t^3}{6}L_i^3 + \dots$ where the dots represent terms of order higher than 4. By simple calculations, we obtain

$$\begin{aligned}&\frac{1}{6}\sum_{\varepsilon\in\{-1,1\}}[S_2(t)S_1(t)S_3(\varepsilon t) + S_2(t)S_3(\varepsilon t)S_1(t) + S_3(\varepsilon t)S_2(t)S_1(t)]\\ &= I + t(L_1+L_2) + \frac{t^2}{2}(L_1+L_2)^2 + \frac{t^3}{6}(L_1+L_2)^3 + \dots . \end{aligned} \tag{3.25}$$

Therefore, if one has potential third order schemes for L_1, L_2 and L_3, we can get a potential third order schemes for the CIR process by using Proposition 2.3.12 provided that the composition is well defined. We recall that $X_0(t,x) = xe^{-kt} + (a-\sigma^2/4)\zeta_k(t)$ solves exactly the ODE associated to V_0, while

$$\tilde{X}(t,x) = x + t\frac{\sigma}{\sqrt{2}}\sqrt{\left|a - \frac{\sigma^2}{4}\right|}$$

solves the one associated to L_3. Last, we know from Proposition 2.3.19 that

$$X_1(\sqrt{t}Y,x) = \left(\sqrt{x} + \frac{\sigma}{2}\sqrt{t}Y\right)^2,$$

with Y given by (2.28) is a potential third order scheme for $\frac{1}{2}V_1^2$. We consider random variables ε and ξ that are independent and uniformly distributed on $\{-1,1\}$ and $\{1,2,3\}$. Following (3.25), we define for $\sigma^2 \leq 4a$ (resp. $\sigma^2 > 4a$)

$$\hat{X}_t^{x,k=0} = \begin{cases} \tilde{X}(\varepsilon t, X_0(t, X_1(\sqrt{t}Y, x))) \text{ (resp. } \tilde{X}(\varepsilon t, X_1(\sqrt{t}Y, X_0(t,x)))) \text{ if } \xi = 1, \\ X_0(t, \tilde{X}(\varepsilon t, X_1(\sqrt{t}Y, x))) \text{ (resp. } X_1(\sqrt{t}Y, \tilde{X}(\varepsilon t, X_0(t,x)))) \text{ if } \xi = 2, \\ X_0(t, X_1(\sqrt{t}Y, \tilde{X}(\varepsilon t, x))) \text{ (resp. } X_1(\sqrt{t}Y, X_0(t, \tilde{X}(\varepsilon t, x)))) \text{ if } \xi = 3. \end{cases} \tag{3.26}$$

It is checked in [8] that this scheme is well defined and nonnegative if $x \geq \mathbf{K}_3(t)$, where

$$\mathbf{K}_3(t) = \zeta_{-k}(t) \left[\mathbf{1}_{\{4a/3<\sigma^2<4a\}} \left(\sqrt{\frac{\sigma^2}{4} - a + \frac{\sigma}{\sqrt{2}}\sqrt{a - \frac{\sigma^2}{4}}} + \frac{\sigma}{2}\sqrt{3+\sqrt{6}} \right)^2 \right. \tag{3.27}$$

$$+\mathbf{1}_{\{\sigma^2 \leq 4a/3\}} \frac{\sigma}{\sqrt{2}} \sqrt{a - \sigma^2/4}$$

$$\left. +\mathbf{1}_{\{4a<\sigma^2\}} \left[\frac{\sigma^2}{4} - a + \left(\sqrt{\frac{\sigma}{\sqrt{2}}\sqrt{\frac{\sigma^2}{4} - a}} + \frac{\sigma}{2}\sqrt{3+\sqrt{6}} \right)^2 \right] \right]$$

is calculated with $k = 0$. When k is arbitrary, it is also checked that the scheme

$$\hat{X}_t^x = e^{-kt} \hat{X}_{\zeta_{-k}(t)}^{x,k=0} \tag{3.28}$$

is well defined and nonnegative for $x \geq \mathbf{K}_3(t)$ and is a potential third order scheme for the CIR process. These two verifications are rather easy and are left as an exercise for the reader. We finally get from Theorem 2.3.8 and Proposition 3.3.1 the following result.

Proposition 3.3.8 *The scheme $\hat{X}_t^x$ defined by* (3.28) *for $x \geq \mathbf{K}_3(t)$ and by* (3.24) *for $x \in [0, \mathbf{K}_3(t))$ is a third order scheme for the weak error.*

Let us note that for convenience, we have written the scheme by using three random variables ε, ξ and Y. Since these variables are discrete and independent, they can be sampled by using only one pseudorandom number.

Algorithm 3.3: Algorithm for the third-order scheme starting from x with a time-step t. The variables U, ε and ζ are independent and sampled uniformly on $[0,1]$, $\{-1,1\}$ and $\{1,2,3\}$. The variable Y is independent and sampled according to (2.28).

Input: $x \geq 0, a, \sigma \geq 0, k \in \mathbb{R}$ and $t > 0$.
Output: X, sampled according to the third order scheme of Proposition 3.3.8.
Function $X_0(x)$: **return** $x + (a - \sigma^2/4)\zeta_{-k}(t)$;
Function $X_1(x)$: **return** $(\sqrt{x} + \sigma\sqrt{\zeta_{-k}(t)}Y/2)^2$;
Function $\tilde{X}(x)$: **return** $x + \frac{\sigma}{\sqrt{2}}\sqrt{|a - \sigma^2/4|}\varepsilon\zeta_{-k}(t)$;
if $(x \geq \mathbf{K}_3(t))$ **then**
  if $(\zeta = 1)$ **then**
    if $(\sigma^2 \leq 4a)$ **then**
      $X = \tilde{X}(X_0(X_1(x)))$,
    else
      $X = \tilde{X}(X_1(X_0(x)))$,
    end
  end
  if $(\zeta = 2)$ **then**
    if $(\sigma^2 \leq 4a)$ **then**
      $X = X_0(\tilde{X}(X_1(x)))$,
    else
      $X = X_1(\tilde{X}(X_0(x)))$,
    end
  end
  if $(\zeta = 3)$ **then**
    if $(\sigma^2 \leq 4a)$ **then**
      $X = X_0(X_1(\tilde{X}(x)))$,
    else
      $X = X_1(X_0(\tilde{X}(x)))$,
    end
  end
  $X = Xe^{-kt}$,
else
  $s = \frac{\tilde{u}_3(t,x) - \tilde{u}_1(t,x)\tilde{u}_2(t,x)}{\tilde{u}_2(t,x) - \tilde{u}_1(t,x)^2}$, $p = \frac{\tilde{u}_1(t,x)\tilde{u}_3(t,x) - \tilde{u}_2(t,x)^2}{\tilde{u}_2(t,x) - \tilde{u}_1(t,x)^2}$, $\delta = \sqrt{s^2 - 4p}$, $\pi = \frac{\tilde{u}_1 - (s-\delta)/2}{\delta}$,
  if $(U < \pi)$ **then**
    $X = (s + \delta)/2$,
  else
    $X = (s - \delta)/2$,
  end
end

3.3.4 A Second Order Scheme for the CIR Process with Time-Dependent Parameters

In this paragraph, we want to illustrate that the general construction of potential second order schemes given by Corollary 2.3.14 is very tractable with time dependent parameters. Namely, we will consider the following SDE

$$X_t^x = x + \int_0^t (a(s) - k(s)X_s^x)ds + \int_0^t \sigma(s)\sqrt{X_s^x}dW_s, \ x, t \geq 0, \tag{3.29}$$

with $a : \mathbb{R}_+ \to \mathbb{R}_+$ and $k, \sigma : \mathbb{R}_+ \to \mathbb{R}$. It is rather natural to consider such diffusions since it preserves the affine property and offers a much wider parametrization. For example, this process has been considered for applications in finance by Maghsoodi [104] to model interest rates and by Benhamou et al. [18] in equity. Now, we use the trick explained in Remark 2.0.2 and see this time-inhomogeneous SDE as the following two-dimensional time-homogeneous SDE:

$$\begin{cases} X_t^x & = x + \int_0^t (a(Y_s) - k(Y_s)X_s^x)ds + \int_0^t \sigma(Y_s)\sqrt{X_s^x}dW_s, \ x, t \geq 0. \\ Y_t & = t. \end{cases}$$

Its infinitesimal generator is given by $L = L_1 + L_2$, where

$$L_1 = (a(y) - k(y)x)\partial_x + \frac{\sigma(y)^2}{2}\partial_x^2, \ L_2 = \partial_y.$$

To get into the framework of Sect. 2.3, we assume that the functions $a, \sigma^2 : \mathbb{R}_+ \to \mathbb{R}_+$ and $k : \mathbb{R}_+ \to \mathbb{R}$ belong to $\mathcal{C}_{\mathbf{pol}}^\infty(\mathbb{R}_+)$. We see that L_1 is the infinitesimal generator of the SDE

$$\begin{cases} dX_t = (a(Y_t) - k(Y_t)X_t)dt + \sigma(Y_t)\sqrt{X_t}dW_t \\ dY_t = 0, \end{cases}$$

which is a CIR process with frozen parameters $a(Y_0)$, $k(Y_0)$ and $\sigma(Y_0)$. Thus, the second and third order schemes that we have obtained for the CIR process naturally give second and third order schemes for L_1. On the other hand, L_2 is the operator associated to $dX_t = 0$ and $dY_t = 1$, which is straightforward to solve. Therefore, we can apply the second construction of Corollary 2.3.14 also known as Strang's splitting to obtain a potential second order scheme for (3.29). This scheme simply amounts to use on the time step $[t_i, t_{i+1}]$ the scheme for the CIR with frozen parameters at time $\frac{t_i+t_{i+1}}{2}$, i.e. with constant parameters $a(\frac{t_i+t_{i+1}}{2})$, $k(\frac{t_i+t_{i+1}}{2})$ and $\sigma(\frac{t_i+t_{i+1}}{2})$.

3.3.5 Study of the Cauchy Problem for the CIR

The goal of this paragraph is to prove Proposition 3.3.1 on the regularity of the Cauchy problem, also known as the Kolmogorov equation or the fundamental equation. Let $f \in \mathcal{C}^{\infty}_{\mathbf{pol}}(\mathbb{R})$ and $\tilde{u}(t,x) = \mathbb{E}[f(X^x_t)]$. For $t, x \in [0,T] \times \mathbb{R}_+$, we have $\tilde{u}(t,x) = u(T-t,x)$, and we will prove the estimate on $\tilde{u}$. From Itô's formula, we get $\tilde{u}(t,x) = f(x) + \int_0^t \mathbb{E}[Lf(X^x_s)]ds$ and thus $\partial_t \tilde{u}(t,x) = \mathbb{E}[Lf(X^x_t)]$. We have $Lf(x) = (a-kx)f'(x) + \frac{\sigma^2}{2}xf''(x) \in \mathcal{C}^{\infty}_{\mathbf{pol}}(\mathbb{R})$, and we get by iterating that

$$\partial^l_t \tilde{u}(t,x) = \mathbb{E}[L^l f(X^x_t)].$$

Since $L^l f \in \mathcal{C}^{\infty}_{\mathbf{pol}}(\mathbb{R})$, we see that it is sufficient to prove the estimates (3.13) only for the derivatives with respect to x. To do so, we can directly work with the explicit formula for the transition density (1.23) of the CIR process. This approach has been used in [7]. Here, we give another proof which takes advantage from a remarkable formula for the space derivative of the Laplace transform.

The idea of the proof is to write f as the Fourier transform of its Fourier transform. To do so, let us assume for a while that f belongs to the Schwarz space of rapidly decreasing functions, which means that $\sup_{x \in \mathbb{R}} |x^k f^{(l)}(x)| < \infty$ for any $k, l \in \mathbb{N}$. We introduce the Fourier transform of f,

$$v \in \mathbb{R}, \ \mathcal{F}(f)(v) = \int_{\mathbb{R}} e^{ivx} f(x)dx,$$

which is also a rapidly decreasing function and is in particular bounded. We get from the Fourier inversion theorem that $f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ivx}\mathcal{F}(f)(v)dv$, which leads to the following identity

$$\tilde{u}(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathbb{E}[e^{-ivX^x_t}]\mathcal{F}(f)(v)dv$$

by using Fubini's theorem. We now use the explicit formula for the characteristic function (1.8) and set

$$\phi^{x,t}_{a,k,\sigma}(v) = \mathbb{E}[e^{-ivX^x_t}] = \left(1 + i\frac{\sigma^2}{2}v\zeta_k(t)\right)^{-\frac{2a}{\sigma^2}} \exp\left(-\frac{ive^{-kt}}{1 + i\frac{\sigma^2}{2}v\zeta_k(t)}x\right),$$

which is the characteristic function of the CIR process at time t, starting from x, with parameters a, k and σ. By differentiating with respect to x, we get

$$\partial_x \phi^{x,t}_{a,k,\sigma}(v) = -ive^{-kt}\phi^{x,t}_{a+\frac{\sigma^2}{2},k,\sigma}(v). \tag{3.30}$$

We recall that $\mathcal{F}(f')(v) = -iv\mathcal{F}(f)(v)$. Then, by using Lebesgue's theorem, we get

$$\partial_x \tilde{u}(t,x) = -\frac{1}{2\pi}\int_{\mathbb{R}} ive^{-kt}\phi^{x,t}_{a+\frac{\sigma^2}{2},k,\sigma}(v)\mathcal{F}(f)(v)dv$$
$$= e^{-kt}\frac{1}{2\pi}\int_{\mathbb{R}}\phi^{x,t}_{a+\frac{\sigma^2}{2},k,\sigma}(v)\mathcal{F}(f')(v)dv = e^{-kt}\mathbb{E}[f'(X^{x,1}_t)],$$

where $X^{x,m}_t = x + \int_0^t (a + m\frac{\sigma^2}{2} - k\tilde{X}^x_s)ds + \int_0^t \sigma\sqrt{\tilde{X}^x_s}dW_s$. Since f' is also a rapidly decreasing function, we can iterate the argument and get

$$\partial^m_x \tilde{u}(t,x) = e^{-mkt}\mathbb{E}[f^{(m)}(X^{x,m}_t)], \tag{3.31}$$

which gives $|\partial^m_x \tilde{u}(t,x)| \leq C_m$, where $C_m = e^{m|k|T}\sup_{x\in\mathbb{R}}|f^{(m)}(x)|$. We note however that if we would be able to prove (3.31) for $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{R})$, we would have $|f^{(m)}(x)| \leq C_m(1+x^{e_m})$ for $x \in \mathbb{R}_+$ and thus $|\partial^m_x \tilde{u}(t,x)| \leq \tilde{C}_m(1+x^{e_m})$ for some constant $\tilde{C}_m$ depending on m and the CIR parameters.

Now, we want to show that (3.31) holds when $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{R})$. Let $h : \mathbb{R} \to \mathbb{R}$ be the function defined by $h(x) = 0$ if $x \notin (0,1)$ and $h(x) = \exp\left(\frac{1}{x(x+1)}\right)$ if $x \in (0,1)$. This is a $\mathcal{C}^\infty$ function, and we define the cutoff function

$$\vartheta(x) = \frac{\int_{-\infty}^x h(y)dy}{\int_{\mathbb{R}} h(y)dy}. \tag{3.32}$$

The function ϑ is $\mathcal{C}^\infty$, nonnegative, nondecreasing and such that $\vartheta(x) = 0$ for $x \leq -1$ and $\vartheta(x) = 1$ for $x \geq 0$. Besides, $\vartheta \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{R})$ since all its derivatives have a compact support on $[0,1]$. Let $\rho \in (0, \frac{2}{\sigma^2\zeta_k(t)})$, so that $\mathbb{E}[|\exp((\rho - iv)X^x_t)|] < \infty$ for any $v \in \mathbb{R}$ by (1.8). We define

$$f_\rho(x) = \vartheta(x)f(x)e^{-\rho x}, x \in \mathbb{R},$$

that belongs to the space of rapidly decreasing functions since $\vartheta f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{R})$ and $\vartheta(x)f(x) = 0$ if $x \leq -1$. Then, we have $f_\rho(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ivx}\mathcal{F}(f_\rho)(v)dv$ and therefore

$$\tilde{u}(t,x) = \mathbb{E}[e^{\rho X^x_t} f_\rho(X^x_t)] = \frac{1}{2\pi}\int_{\mathbb{R}}\mathbb{E}[e^{(\rho-iv)X^x_t}]\mathcal{F}(f_\rho)(v)dv,$$

by Fubini's theorem. Similarly as before, we define $\tilde{\phi}^{x,t}_{a,k,\sigma}(v) = \mathbb{E}[e^{(\rho-iv)X^x_t}]$ and have

$$\partial_x\tilde{\phi}^{x,t}_{a,k,\sigma}(v) = (\rho - iv)e^{-kt}\tilde{\phi}^{x,t}_{a+\frac{\sigma^2}{2},k,\sigma}(v). \tag{3.33}$$

Using again Lebesgue's theorem and $\mathcal{F}(f'_\rho)(v) = -iv\mathcal{F}(f_\rho)(v)$, we get

$$\begin{aligned}\partial_x \tilde{u}(t,x) &= e^{-kt}\frac{1}{2\pi}\int_{\mathbb{R}} \tilde{\Phi}^{x,t}_{a+\frac{\sigma^2}{2},k,\sigma}(v)\mathcal{F}(\rho f_\rho + f'_\rho)(v)dv \\ &= e^{-kt}\mathbb{E}[e^{\rho X_t^{x,1}}(\rho f_\rho(X_t^{x,1}) + f'_\rho(X_t^{x,1}))] = e^{-kt}\mathbb{E}[f'(X_t^{x,1})],\end{aligned}$$

since $e^{\rho x}(\rho f_\rho(x) + f'_\rho(x)) = f(x)$ on $x \geq 0$. Then, we get $\partial_x^m \tilde{u}(t,x) = e^{-kmt}\mathbb{E}[f^{(m)}(X_t^{x,m})]$ for $m \in \mathbb{N}$ and the desired bounds on $\tilde{u}$.

Now, it remains to prove that $\partial_t \tilde{u}(t,x) = (a-kx)\partial_x \tilde{u}(t,x) + \frac{\sigma^2}{2}x\partial_x^2\tilde{u}(t,x)$. Let $\varepsilon > 0$. By the Markov property, we have $\tilde{u}(t+\varepsilon,x) = \mathbb{E}[\mathbb{E}[f(X^x_{t+\varepsilon})|X^x_\varepsilon]] = \mathbb{E}[\tilde{u}(t,X^x_\varepsilon)]$. By Itô's formula,

$$\begin{aligned}\tilde{u}(t,X^x_\varepsilon) = \tilde{u}(t,x) &+ \int_0^\varepsilon [(a-kX_s^x)\partial_x\tilde{u}(t,X_s^x) + \frac{\sigma^2}{2}X_s^x\partial_x^2\tilde{u}(t,X_s^x)]ds \\ &+ \int_0^\varepsilon \sigma\sqrt{X_s^x}\partial_x\tilde{u}(t,X_s^x)dW_s.\end{aligned}$$

Thanks to the estimates on $\partial_x\tilde{u}$, we get that the stochastic integral has a null expectation. Since $X_\varepsilon^x \underset{\varepsilon\to0+}{\to} x$ almost surely, we get that

$$\frac{\tilde{u}(t+\varepsilon,x)-\tilde{u}(t,x)}{\varepsilon} \underset{\varepsilon\to0+}{\to} (a-kx)\partial_x\tilde{u}(t,x) + \frac{\sigma^2}{2}x\partial_x^2\tilde{u}(t,x).$$

3.4 Numerical Results

This section takes back the numerical experiments that were made in the article [8]. We focus on the convergence of the second and third order schemes for the CIR given by Propositions 3.3.5 and 3.3.8 and the modified Euler scheme (3.9). In particular, we will consider an example with parameters such that $\sigma^2 \gg 4a$, for which few existing discretization schemes are accurate. In order to underline the importance of the threshold $\mathbf{K}_2(t)$, we will consider different schemes. Schemes 1 and 2 are respectively the second and the third order schemes that we have stated in Propositions 3.3.5 and 3.3.8. Their simulations are plotted in solid line in Fig. 3.2. We consider also three distortions of the second-order scheme that illustrate the importance of the choice of $\mathbf{K}_2(t)$, the threshold around which we switch between the schemes. First, we see from the proof of Proposition 3.3.5 that any other threshold $\tilde{\mathbf{K}}(t)$ greater than $\mathbf{K}_2(t)$ such that $\tilde{\mathbf{K}}(t) \underset{t\to0}{=} O(t)$ would have led to another second-order scheme. Instead, if one takes a threshold smaller than $\mathbf{K}_2(t)$ and forces nonnegativity by taking positive parts, it is not clear mathematically that we get a second order scheme. We can however wonder if this is just a mathematical

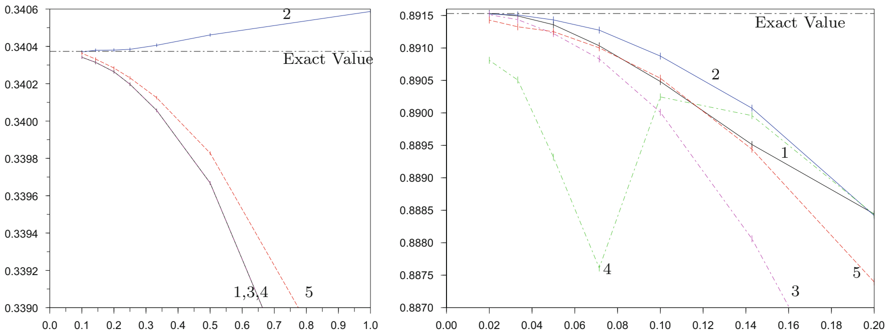

Fig. 3.2 $\mathbb{E}(\exp(-\hat{X}_1))$ in function of the time step $1/n$ with $x_0 = 3/2$, $k = 1/2$, $a = 1/2$ and $\sigma = 0.8$ (*left*) and $x_0 = 0.3$, $k = 0.1$, $a = 0.04$ and $\sigma = 2$ (*right*). The width of each point gives the precision up to two standard deviations

restriction or if it leads indeed to a worse scheme. We thus consider the following schemes:

3. Second order scheme of Proposition 3.3.5, with switching threshold $3\mathbf{K}_2(t)/2$.
4. Second order scheme of Proposition 3.3.5, with switching threshold $\mathbf{K}_2(t)/2$, forcing nonnegativity with positive parts.

Last, the way to obtain $\mathbf{K}_2(t)$ is closely linked with the support of Y, the moment-matching random variable that we have chosen for N. Taking a bounded random variable was important to prove the convergence of our scheme, but once again, we can wonder if it is of numerical importance and we consider the following scheme:

5. Second order scheme of Proposition 3.3.5, with $N \sim \mathcal{N}(0, 1)$ instead of Y, forcing nonnegativity with positive parts.

In Fig. 3.2, we have set $T = 1$ and plotted the values of $\mathbb{E}(\exp(-\hat{X}_1))$ in function of the time step $1/n$ for two choices of parameters: $\sigma^2 < 4a$ (left) and $\sigma^2 \gg 4a$ (right). The first set of parameters is such that $\sigma^2 < 4a$, and the schemes are most of the time largely above the switching threshold, which explain that we observe no differences between the schemes 1, 3 and 4. For the same reason, the scheme 5 has also a qualitatively quadratic convergence and is even slightly better than scheme 1. Last, the third order scheme 2 converges here much better than the other schemes, giving in that case a five digit precision from $n = 5$.

The second set of parameters such that $\sigma^2 \gg 4a$ is more interesting to discuss the choice of the threshold, because the schemes are often around its value. First, we observe that the convergence of the schemes 1 and 2 is compatible with the theoretical results, and the third order scheme 2 converges more quickly to the right value than the second order scheme 1. Then, the scheme 3 converges as expected with a quadratic speed. Nonetheless with respect to scheme 1, the convergence has been slightly downgraded with the increasing of the threshold. Thus, even if

Table 3.1 $x_0 = 3/2$, $k = 1/2$, $a = 1/2$ and $\sigma = 0.8$

n	1	2	3	4	5	7	10
$\mathbb{E}(\exp(-\hat{X}_1))$	0.3864	0.36836	0.35924	0.35442	0.35151	0.34822	0.3458

Table 3.2 $x_0 = 0.3$, $k = 0.1$, $a = 0.04$, and $\sigma = 2$

n	5	7	10	14	20	30	50
$\mathbb{E}(\exp(-\hat{X}_1))$	0.80636	0.82799	0.84635	0.85974	0.8704	0.87883	0.88522

theoretically any switching threshold $\tilde{\mathbf{K}}(t)$ greater than $\mathbf{K}_2(t)$ s.t. $\tilde{\mathbf{K}}(t) \underset{t \to 0}{=} O(t)$ gives a second order scheme, it seems better to take the smaller one possible as in scheme 1. The erratic behaviour of scheme 4 is sufficient to convince that our choice of $\mathbf{K}_2(t)$ is not just a convenient choice for the proofs, but has a real impact on the convergence. Last, the convergence of scheme 5 is also worse when the time-step gets smaller than the scheme 1 and 3 for the following reason. The threshold $\mathbf{K}_2(t)$ has been calculated for a random variable Y that takes value in $[-\sqrt{3}, \sqrt{3}]$, which is of course not satisfied by a standard Gaussian variable.

To illustrate that most of the usual schemes are not accurate for large values of σ, we have also calculated the same expectations with the Full Truncation scheme (3.9). The corresponding values are outside Fig. 3.2 and are given in Tables 3.1 and 3.2.

It is important to notice here that for the second set of parameters, the number of samples for the Monte-Carlo method to get a precision up to four digits is about 10^8. Therefore, when $\sigma^2 \gg 4a$, the choice of the scheme is really crucial to make calculations within limited time or computational means.

Chapter 4
The Heston Model and Multidimensional Affine Diffusions

In Chap. 1, we have presented the real valued affine diffusions. Basically, these diffusions are either the Ornstein-Uhlenbeck process or the Cox-Ingersoll-Ross process. This chapter presents the general framework for affine diffusions in a multidimensional context. In the first section, we give the definition and the main properties of affine diffusions. Then, we present two examples of vector valued affine processes that are of practical use in finance. Section 4.2 focuses on the Heston model [75], which is a very popular stochastic volatility model for the stock value. Section 4.3 presents the affine term structure model for the short interest rate, as stated by Dai and Singleton [39]. This model embeds the Vasicek and Cox-Ingersoll-Ross models presented in Sect. 1.4.

4.1 Definition and Properties of Affine Diffusions

We consider the same framework as in Chap. 2. Namely, we focus on the general time homogeneous SDE

$$X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma(X_s^x)dW_s,\ t \geq 0, \tag{4.1}$$

that is well defined on a domain $\mathbb{D} \subset \mathbb{R}^d$. We recall that W denotes a standard Brownian motion of dimension d_W. The functions $b : \mathbb{D} \to \mathbb{R}^d$ and $\sigma : \mathbb{D} \to \mathcal{M}_{d\times d_W}(\mathbb{R})$ are assumed to be continuous with a sublinear growth (2.2), and are such that for any starting point $x \in \mathbb{D}$, there is a unique weak solution for the SDE (4.1). Besides, this solution is assumed to stay in the domain $\mathbb{D}$, which means that $\forall x \in \mathbb{D},\ \mathbb{P}(\forall t \geq 0, X_t^x \in \mathbb{D}) = 1$. We recall that these assumptions ensure

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that X satisfies the strong Markov property. Also, the infinitesimal generator of this diffusion is given by

$$f \in \mathcal{C}^2(\mathbb{D}, \mathbb{R}),\ Lf(x) = \sum_{i=1}^{d} b_i(x)\partial_i f(x) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}(\sigma(x)\sigma^\top(x))_{i,j}\partial_i\partial_j f(x). \tag{4.2}$$

We now focus on the characteristic function of the marginal law of X_t^x. The diffusion X is said to be affine if its characteristic function can be written as follows:

$$\forall x \in \mathbb{D}, u \in i\mathbb{R}^d, t \geq 0,\ \mathbb{E}[\exp(u^\top X_t^x)] = \exp(\phi_u(t) + \psi_u(t)^\top x), \tag{4.3}$$

where $\phi_u : \mathbb{R}_+ \to \mathbb{C}$ and $\psi_u : \mathbb{R}_+ \to \mathbb{C}^d$ are continuous functions. Since $X_0^x = x$, we necessarily have

$$\phi_u(0) = 0 \text{ and } \psi_u(0) = u.$$

Remark 4.1.1 We consider here affine diffusions that are thus naturally continuous. When dealing with more general Markov processes X, a stochastic continuity assumption is included in the definition of affine processes, see Definition 2.1 of Cuchiero et al. [35].

Proposition 4.1.2 *Suppose that for all $u \in i\mathbb{R}^d$, there are $\mathcal{C}^1$ functions ϕ_u and ψ_u such that:*

$$\forall x \in \mathbb{D}, t \geq 0,\ \mathbb{E}[\exp(u^\top X_t^x)] = \exp(\phi_u(t) + \psi_u(t)^\top x).$$

Then, $x \in \mathbb{D} \mapsto b(x)$ and $x \in \mathbb{D} \mapsto \sigma(x)\sigma^\top(x)$ must be affine functions of $x \in \mathbb{D}$, i.e. the infinitesimal generator (4.2) is affine with respect to x.

Proof The proof follows the same lines as the one of Proposition 1.3.1 in dimension one. Thus, we consider for $T > 0$ and $t \in [0, T]$ the martingale $M_t = \mathbb{E}[\exp(u^\top X_T^x)|\mathcal{F}_t]$. Since X is a time homogeneous Markov process, we have

$$M_t = \exp(\phi_u(T-t) + \psi_u(T-t)^\top X_t^x).$$

By Itô's formula, we then get

$$\begin{aligned} dM_t =& M_t\Big[\Big(-\phi_u'(T-t) - \psi_u'(T-t)^\top X_t^x + \psi_u(T-t)^\top b(X_t^x) \\ &+ \frac{1}{2}\psi_u(T-t)^\top \sigma(X_t^x)\sigma(X_t^x)^\top \psi_u(T-t)\Big)dt + \psi_u(T-t)^\top \sigma(X_t^x)dW_t\Big]. \end{aligned}$$

Again, the drift vanishes dt almost everywhere and $\mathbb{P}$ almost surely and is continuous with respect to (t, X_t^x). Letting $t \to 0$, we obtain

$$\psi_u(T)^\top b(x) + \frac{1}{2}\psi_u(T)^\top \sigma(x)\sigma(x)^\top \psi_u(T) = \phi_u'(T) + \psi_u'(T)^\top x \tag{4.4}$$

for any $x \in \mathbb{D}$ and $T \geq 0$. Now, we proceed differently from the proof of Proposition 1.3.1. For $1 \leq k \leq d$, we define $e_k \in i\mathbb{R}^d$ by $(e_k)_j = \mathbb{1}_{j=k} i$, and we use the identity above at time $T = 0$ for $u = e_k$, and $u = e_k + e_l$ for $1 \leq k, l \leq d$. Since $\psi_u(0) = u$, we obtain

$$ib_k(x) - \frac{1}{2}(\sigma(x)\sigma(x)^\top)_{k,k} = \phi_{e_k}'(0) + \psi_{e_k}'(0)^\top x,$$
$$2ib_k(x) - 2(\sigma(x)\sigma(x)^\top)_{k,k} = \phi_{2e_k}'(0) + \psi_{2e_k}'(0)^\top x,$$

and, for $l \neq k$,

$$ib_k(x) + ib_l(x) - \frac{1}{2}(\sigma(x)\sigma(x)^\top)_{k,k} - \frac{1}{2}(\sigma(x)\sigma(x)^\top)_{l,l} - (\sigma(x)\sigma(x)^\top)_{k,l}$$
$$= \phi_{e_k+e_l}'(0) + \psi_{e_k+e_l}'(0)^\top x.$$

We deduce that $(\sigma(x)\sigma(x)^\top)_{k,k} = 2\phi_{e_k}'(0) - \phi_{2e_k}'(0) + (2\psi_{e_k}'(0) - \psi_{2e_k}'(0))^\top x$ is an affine function of x, and then that $b_k(x)$ is also an affine function of x. Last, we conclude that

$$(\sigma(x)\sigma(x)^\top)_{k,l} = \phi_{e_k}'(0) + \phi_{e_l}'(0) - \phi_{e_k+e_l}'(0) + (\psi_{e_k}'(0) + \psi_{e_l}'(0) - \psi_{e_k+e_l}'(0))^\top x$$

is affine with respect to x. □

Remark 4.1.3 In the one dimensional case, we have only assumed in Proposition 1.3.1 that the affine structure of the characteristic function (4.3) holds for one $u \neq 0$. Then, we have used in the proof that (4.4) holds for two different times to conclude. Instead, we have assumed here that (4.3) holds for any $u \in i\mathbb{R}^d$. In fact, we only use in the proof this assumption on $\frac{d(d+1)}{2} + d$ vectors u, which is precisely the number of distinct coordinates of $b(x)$ and $(\sigma(x)\sigma(x)^\top)$. To use the same argument as in dimension 1, we would have to consider (4.4) at $\frac{d(d+1)}{2} + d$ different times and analyse the invertibility of the system, but this point is no longer obvious.

Let us now assume that we are under the assumption of Proposition 4.1.2. Therefore, there are vectors $\beta_0, \ldots, \beta_d \in \mathbb{R}^d$ and symmetric matrices $\alpha_0, \ldots, \alpha_d \in \mathcal{S}_d(\mathbb{R})$ such that

$$\forall x \in \mathbb{D},\ b(x) = \beta_0 + \sum_{i=1}^{d} \beta_i x_i,\ \sigma(x)\sigma(x)^\top = \alpha_0 + \sum_{i=1}^{d} \alpha_i x_i. \tag{4.5}$$

If $\mathbb{D} = \mathbb{R}_+^{d'} \times \mathbb{R}^{d-d'}$ for some $0 \le d' \le d$, we necessarily have $\alpha_0, \dots, \alpha_{d'} \in \mathcal{S}_d^+(\mathbb{R})$ and $\alpha_i = 0$ for $i > d'$ since $\sigma(x)\sigma(x)^\top$ is a semidefinite positive matrix for any $x \in \mathbb{D}$. Then, we can rewrite Eq. (4.4) as follows

$$\psi_u(T)^\top \beta_0 + \frac{1}{2}\psi_u(T)^\top \alpha_0 \psi_u(T) + \sum_{i=1}^d \left(\psi_u(T)^\top \beta_i + \frac{1}{2}\psi_u(T)^\top \alpha_i \psi_u(T)\right) x_i$$

$$= \phi_u'(T) + \sum_{i=1}^d (\psi_u'(T))_i x_i, \; x \in \mathbb{D}, T \ge 0. \tag{4.6}$$

Then, we easily see that (4.6) is satisfied if one has for $t \ge 0$,

$$\begin{cases} \phi_u'(t) = \psi_u(t)^\top \beta_0 + \frac{1}{2}\psi_u(t)^\top \alpha_0 \psi_u(t) \\ (\psi_u'(t))_i = \psi_u(t)^\top \beta_i + \frac{1}{2}\psi_u(t)^\top \alpha_i \psi_u(t), \; 1 \le i \le d. \end{cases} \tag{4.7}$$

Conversely, let us assume that $\mathbb{D}$ is not contained in an hyperplane of $\mathbb{R}^d$, which means that there is $x \in \mathbb{D}$ and $\varepsilon > 0$ such that the ball centered in x with radius $\varepsilon > 0$ is included in $\mathbb{D}$, i.e.

$$B(x, \varepsilon) = \{y \in \mathbb{R}^d, \|x - y\| \le \varepsilon\} \subset \mathbb{D}.$$

This condition is satisfied for example when the rank of $\sigma(x)$ is equal to d. In this case, (4.6) is equivalent to (4.7).

One would like to characterize more precisely the affine diffusions. At this stage, two questions naturally arise. First, what are the domains $\mathbb{D}$ on which it is possible to define affine diffusions. Clearly, if we have two domains $\mathbb{D}_1$ and $\mathbb{D}_2$ on which affine diffusions are well-defined, we can construct an affine diffusion on $\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_2$ by considering two independent copies of these processes. Therefore, one has to wonder what are the elementary domains on which affine diffusions may be defined. This issue has been tackled by Grasselli and Tebaldi [71]. They have shown that the possible domains are symmetric cones related to the algebraic notion of Euclidean Jordan Algebra. Second, once a domain $\mathbb{D}$ has been given, one would like to classify the different affine diffusions that are well defined on $\mathbb{D}$. For example, when $\mathbb{D} = \mathbb{R}_+$, we have seen that the CIR process is the only one affine diffusion on $\mathbb{D}$. When $\mathbb{D} = \mathbb{R}_+^{d_1} \times \mathbb{R}^{d_2}$, Duffie, Filipović and Schachermayer have given a full description of affine diffusions and affine processes, see Theorem 2.7 of [47]. Formerly, Dai and Singleton [39] have proposed a specification of Affine diffusions that we will present in Sect. 4.3. However, their specification is not exhaustive unless d_1 or d_2 belong to $\{0, 1\}$, as shown by Cheridito et al. [29].

4.2 The Heston Model

The Heston model [75] is a model for the dynamic of a stock price S_t. In this section, we make the assumption of a constant short interest rate $r \geq 0$. As in Sect. 1.4, we assume that the probability measure $\mathbb{P}$ is a martingale measure, which means that the discounted assets are martingales with respect to the filtration that describes the market information. When proposing a model for S under P, one should have in particular that $\tilde{S}_t = e^{-rt}S_t$ is a martingale, which we will check for the Heston model in Corollary 4.2.2.

When the stock price is time continuous and positive, the volatility of the stock at time t is usually defined by $\frac{1}{\tilde{S}_t}\sqrt{\frac{d\langle \tilde{S}_t\rangle}{dt}}$, and the main issue is to propose a model for this quantity. In the celebrated Black-Scholes model [20], the volatility is assumed to be constant. A very popular extension of this model is the local volatility model which assumes that the volatility at time t is given by $\sigma(t, S_t)$, where $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a deterministic function. However, empirical evidences from financial market data show that the volatility cannot be written as a deterministic function of the stock price, see for example Dumas et al. [48]. Even though it is related to the stock price, the volatility has an intrinsic source of noise. This motivates the introduction of the so called stochastic volatility models such as the Heston model, where the volatility has its own randomness. For a detailed account on volatility modeling, we refer to the authoritative book of Gatheral [59].

The Heston model is a very popular stochastic volatility model. Let $W = (W^1, W^2)$ denote a two dimensional standard Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by W. The Heston model assumes that (S, V) solves the following SDE

$$\begin{cases} S_t = S_0 + \int_0^t rS_u du + \int_0^t \sqrt{V_u} S_u(\rho dW_u^1 + \sqrt{1-\rho^2}dW_u^2) \\ V_t = V_0 + \int_0^t (a - kV_u)du + \sigma \int_0^t \sqrt{V_u} dW_u^1 \end{cases}, \ t \geq 0, \tag{4.8}$$

with the initial condition $V_0, S_0 \geq 0$. We assume that $a > 0$, $\sigma > 0$, $k \in \mathbb{R}$ and $\rho \in [-1, 1]$. The volatility process is thus a CIR process. We exclude here the case $a = 0$ because S_t would then become a riskless asset as soon as V reaches zero. We know from Theorem 1.2.1 that pathwise uniqueness holds for V. Then, S is a Doléans-Dade exponential and therefore satisfies:

$$S_t = S_0 \exp\left(rt + \int_0^t \sqrt{V_u}(\rho dW_u^1 + \sqrt{1-\rho^2}dW_u^2) - \frac{1}{2}\int_0^t V_u du\right), t \geq 0.$$

This gives in particular the pathwise uniqueness for (V, S). Thus, the assumptions that we have made on the parameters ensure that the SDE (4.8) is well defined. However, for financial application, it is often assumed in addition that $k > 0$ in order to work with a stationary process. This condition gives that the volatility process is mean reverting toward $\frac{a}{k}$ and ergodic, see Eq. (1.24). Thus, the parameters have a clear practical meaning: $\frac{a}{k}$ is the average volatility, k describes the speed of the mean reversion while σ tunes the randomness of the volatility. Besides, one often observe in practice a negative instantaneous correlation between the stock and its volatility, i.e. $\rho \leq 0$. From an heuristic point of view, this means that the volatility mostly increases when the stock value decreases. In the sequel, we will make no assumption on k and ρ since it is not required for our mathematical analysis of the model.

For mathematical reasons, it will be more convenient to work with $X_t = \log(S_t)$, and we exclude the meaningless case $S_0 = 0$. By Itô's formula, we easily get the SDE satisfied by (X, V),

$$\begin{cases} X_t^x = X_0^x + \int_0^t (r - \frac{1}{2}V_u)du + \int_0^t \sqrt{V_u}(\rho dW_u^1 + \sqrt{1-\rho^2}dW_u^2) \\ V_t = V_0 + \int_0^t (a - kV_u)du + \sigma \int_0^t \sqrt{V_u}dW_u^1 \end{cases}, \ t \geq 0. \tag{4.9}$$

Its infinitesimal generator is given by

$$\begin{aligned} Lf(x,v) &= (r - \frac{v}{2})\partial_x f(x,v) + (a - kv)\partial_v f(x,v) \\ &\quad + \frac{\sigma^2}{2} v \partial_v^2 f(x,v) + v \partial_x^2 f(x,v) + \rho\sigma v \partial_x \partial_v f(x,v) \end{aligned}$$

and is affine with respect to (x, v).

4.2.1 The Characteristic Function

The next proposition determines the joint law of (S_t, V_t) through the characteristic function. For reasons that will be clear later, we consider a slight generalization of the dynamic of $X_t^x = \log(S_t)$.

Proposition 4.2.1 *Let $\lambda \in \mathbb{R}$, $x_1 \in \mathbb{R}$ and $x_2 \geq 0$. Let X^x be the solution of*

$$\begin{cases} (X_t^x)_1 = x_1 + \int_0^t (r - \lambda (X_s^x)_2)ds + \int_0^t \sqrt{(X_s^x)_2}(\rho dW_s^1 + \sqrt{1-\rho^2}dW_s^2) \\ (X_t^x)_2 = x_2 + \int_0^t (a - k(X_s^x)_2)ds + \sigma \int_0^t \sqrt{(X_s^x)_2}dW_s^1 \end{cases}, \ t \geq 0. \tag{4.10}$$

Then, its characteristic function is given by

$$u \in i\mathbb{R}^2, \ \boxed{\mathbb{E}[\exp(u^\top X_t^x)] = \exp(\phi_u(t) + \psi_u(t)^\top x),} \tag{4.11}$$

where

$$\phi_u(t) = \left(ru_1 + a(\Psi - \frac{2\sqrt{\Delta}}{\sigma^2}) \right) t - \frac{2a}{\sigma^2} \log \left(\frac{\exp(-\sqrt{\Delta}t) - g}{1-g} \right), \tag{4.12}$$

$$(\psi_u(t))_1 = u_1,\ (\psi_u(t))_2 = u_2 + (\Psi - u_2) \frac{1 - \exp(\sqrt{\Delta}t)}{1 - g\exp(\sqrt{\Delta}t)}, \tag{4.13}$$

with $\Delta = (\rho\sigma u_1 - k)^2 - \sigma^2(u_1^2 - 2\lambda u_1)$, $\Psi = \frac{k-\rho\sigma u_1+\sqrt{\Delta}}{\sigma^2}$ *and* $g = \frac{k-\rho\sigma u_1+\sqrt{\Delta}-\sigma^2 u_2}{k-\rho\sigma u_1-\sqrt{\Delta}-\sigma^2 u_2}$. *These formulas are valid when* $\Delta \neq 0$, *considering that* $\frac{\exp(-\sqrt{\Delta}t)-g}{1-g} = 1$ *and* $\frac{1-\exp(\sqrt{\Delta}t)}{1-g\exp(\sqrt{\Delta}t)} = 0$ *when the denominator of* g *is zero.*

When $\Delta = 0$, *one has the following formulas*

$$\phi_u(t) = (ru_1 + a\Psi)\, t - \frac{2a}{\sigma^2} \log \left(1 + \frac{\sigma^2}{2} t(\Psi - u_2) \right), \tag{4.14}$$

$$(\psi_u(t))_1 = u_1,\ (\psi_u(t))_2 = u_2 + (\Psi - u_2)^2 \frac{\sigma^2 t}{2 + \sigma^2 t(\Psi - u_2)}. \tag{4.15}$$

Below, we give a direct proof of this Proposition. However, it is possible to calculate the Laplace transform of X_t^x by using the Laplace transform of $\left((X_t^x)_2, \int_0^t (X_s^x)_2 ds\right)$ obtained in (1.15). In fact, we have

$$(X_t^x)_1 = x_1 + rt - \lambda \int_0^t (X_s^x)_2 ds + \frac{\rho}{\sigma} \left[(X_t^x)_2 - x_2 - at + k \int_0^t (X_s^x)_2 ds \right]$$
$$+ \int_0^t \sqrt{(X_s^x)_2} \sqrt{1-\rho^2} dW_s^2,$$

and therefore we get for $u \in \mathbb{R}^2$,

$$\begin{aligned} &\mathbb{E}[\exp(u^\top X_t^x)] = \mathbb{E}\left[\mathbb{E}[\exp(u^\top X_t^x)|(W_s^1, s \in [0,t])]\right] \\ &= \exp\left(u_1[x_1 + rt - \frac{\rho}{\sigma}(x_2 + at)] \right) \\ &\times \mathbb{E}\left[\exp\left([u_2 + \frac{\rho}{\sigma}u_1](X_t^x)_2 + [(k\frac{\rho}{\sigma} - \lambda)u_1 + (1-\rho^2)\frac{u_1^2}{2}] \int_0^t (X_s^x)_2 ds \right) \right] \end{aligned} \tag{4.16}$$

by using the tower property of the conditional expectation. This formula is valid for any $u \in \mathbb{R}^2$, and the value of the expectation can be $+\infty$. However, if we want to extend this formula to $u \in \mathbb{C}$, one has to be careful with the complex logarithm, see Remark 1.2.10. The goal of the proof of Proposition 4.2.1 is precisely to prove that formula (4.12) is correct when $u \in i\mathbb{R}^2$, as it has been pointed out by Lord and Kahl [100]. This was not the case of the original formula given in Heston's paper [75], which has been discussed and corrected by several articles including Kahl and Jäckel [80] and Albrecher et al. [6].

Equation (4.16) enables us to characterize the finiteness of the moment in the Heston model, which gives some asymptotic properties of the volatility smile, see Lee [94]. In fact, we know from Proposition 1.2.9 that the expectation in (4.16) is finite if, and only if

$$\left(u_2 + \frac{\rho}{\sigma}u_1, (k\frac{\rho}{\sigma} - \lambda)u_1 + (1-\rho^2)\frac{u_1^2}{2}\right) \in \mathcal{D}_t,$$

where $\mathcal{D}_t$ is the set defined by (1.20). In particular, we get for $\lambda = \frac{1}{2}$, $u_1 = p \in \mathbb{R}$ and $u_2 = 0$ the following corollary.

Corollary 4.2.2 *For $p \in \mathbb{R}$, we set $v(p) = (k\frac{\rho}{\sigma} - \frac{1}{2})p + (1-\rho^2)\frac{p^2}{2}$. We also define $\bar{\gamma}_v = \sqrt{|k^2 - 2\sigma^2 v|}$ for $v \in \mathbb{R}$. In the Heston model, the moment of order p is finite at time $t > 0$, i.e. $\mathbb{E}[S_t^p] < \infty$, if, and only if one of these three condition holds.*

1. *$v(p) \le \frac{k^2}{2\sigma^2}$ and $\rho\sigma p \le k + \bar{\gamma}_{v(p)}$.*
2. *$v(p) \le \frac{k^2}{2\sigma^2}$, $\rho\sigma p > k + \bar{\gamma}_{v(p)}$ and $t < \frac{1}{\bar{\gamma}_{v(p)}} \log\left(1 + \frac{2\bar{\gamma}_{v(p)}}{\rho\sigma p - (k + \bar{\gamma}_{v(p)})}\right)$.*
3. *$v(p) > \frac{k^2}{2\sigma^2}$, and $t < \frac{2}{\bar{\gamma}_{v(p)}} \arctan\left(\frac{\bar{\gamma}_{v(p)}}{\rho\sigma p - k}\right) + \pi \mathbb{1}_{\{\rho\sigma p - k < 0\}}$.*

In particular, we have $v(1) \le \frac{k^2}{2\sigma^2}$ and $\rho\sigma \le k + \bar{\gamma}_{v(1)}$ which gives

$$\forall t \ge 0,\ \mathbb{E}[S_t] = e^{rt} S_0. \tag{4.17}$$

This result has been stated by Andersen and Piterbarg, see Proposition 3.1 in [13]. Then, Glasserman and Kim [63], Keller-Ressel [86] and Mayerhofer and Keller-Ressel [87] have extended this result on moment explosion to more general affine processes.

Proof This is a direct consequence of Proposition 1.2.9 once we have observed that the function ζ is positive. For $p = 1$, we check that $v(1) = \frac{k\rho}{\sigma} - \frac{1}{2}\rho^2 \le \frac{k^2}{2\sigma^2}$ and $\rho\sigma \le k + \sqrt{(\rho\sigma - k)^2}$, which gives (4.17) by using again Proposition 1.2.9 and formula (4.16) with $\lambda = \frac{1}{2}$, $u_1 = 1$ and $u_2 = 0$. □

Proof of Proposition 4.2.1 We proceed as in the proof of Proposition 1.2.4, and we assume that (4.11) holds for some smooth functions. Let $T > 0$. We know that $M_t = \mathbb{E}[\exp(u^\top X_T^x)|\mathcal{F}_t]$ is a martingale, and we have by the Markov property that $M_t = \exp(\phi_u(T-t) + \psi_u(T-t)^\top X_t^x)$. By Itô's formula, we get that

$$\begin{aligned}
&-\phi_u'(T-t) - \psi_u'(T-t)^\top X_t^x + (r - \lambda(X_t^x)_2)(\psi_u(T-t))_1 \\
&+ (a - k(X_t^x)_2)(\psi_u(T-t))_2 + \frac{1}{2}(X_t^x)_2(\psi_u(T-t))_1^2 \\
&+ \rho\sigma(X_t^x)_2(\psi_u(T-t))_1(\psi_u(T-t))_2 + \frac{\sigma^2}{2}(X_t^x)_2(\psi_u(T-t))_2^2 = 0.
\end{aligned}$$

This leads to

$$\begin{cases} -\phi_u'(T-t) + r(\psi_u(T-t))_1 + a(\psi_u(T-t))_2 = 0 \\ -(\psi_u'(T-t))_1 = 0 \\ -(\psi_u'(T-t))_2 - \lambda(\psi_u(T-t))_1 - k(\psi_u(T-t))_2 \\ \qquad +\frac{1}{2}(\psi_u(T-t))_1^2 + \rho\sigma(\psi_u(T-t))_1(\psi_u(T-t))_2 + \frac{\sigma^2}{2}(\psi_u(T-t))_2^2 = 0. \end{cases} \tag{4.18}$$

Besides, we have the initial condition $\phi(0) = 0$ and $\psi_u(0) = u$. This gives $(\psi_u(t))_1 = u_1$ and then

$$-(\psi_u')_2 + \frac{1}{2}u_1^2 - \lambda u_1 + (\rho\sigma u_1 - k)(\psi_u)_2 + \frac{\sigma^2}{2}(\psi_u)_2^2 = 0. \tag{4.19}$$

This is a Riccati differential equation. Let $\Delta = (\rho\sigma u_1 - k)^2 - \sigma^2(u_1^2 - 2\lambda u_1)$ and $\Psi = \frac{k - \rho\sigma u_1 + \sqrt{\Delta}}{\sigma^2}$ be a root of $\frac{1}{2}u_1^2 - \lambda u_1 + (\rho\sigma u_1 - k)\Psi + \frac{\sigma^2}{2}\Psi^2 = 0$.

We set $\tilde{\psi} = (\psi_u)_2 - \Psi$ and get

$$-\tilde{\psi}' + \sqrt{\Delta}\tilde{\psi} + \frac{\sigma^2}{2}\tilde{\psi}^2 = 0,$$

since $\rho\sigma u_1 - k + \sigma^2\Psi = \sqrt{\Delta}$. Let us assume now that $\Delta \neq 0$. We get $\left(\frac{1}{\tilde{\psi}} + \frac{\sigma^2}{2\sqrt{\Delta}}\right)' + \sqrt{\Delta}\left(\frac{1}{\tilde{\psi}} + \frac{\sigma^2}{2\sqrt{\Delta}}\right) = 0$, and thus

$$\frac{1}{(\psi_u(t))_2 - \Psi} = \left(\frac{\sigma^2}{2\sqrt{\Delta}} + \frac{1}{u_2 - \Psi}\right)\exp(-\sqrt{\Delta}t) - \frac{\sigma^2}{2\sqrt{\Delta}},\ t \geq 0.$$

This leads to

$$(\psi_u(t))_2 = \Psi + \frac{2\sqrt{\Delta}(\Psi - u_2)}{\left(\sigma^2(\Psi - u_2) - 2\sqrt{\Delta}\right)\exp(-\sqrt{\Delta}t) - \sigma^2(\Psi - u_2)}.$$

By using that $\sigma^2(\Psi - u_2) = k - \rho\sigma u_1 + \sqrt{\Delta} - \sigma^2 u_2$, we obtain (4.13). When $\Delta = 0$, we have $\frac{1}{(\psi_u(t))_2 - \Psi} = \frac{1}{u_2 - \Psi} - \frac{\sigma^2}{2}t$. This leads to (4.15), which is also the limit of (4.13) when $\Delta \to 0$. Then, we have from (4.18),

$$\phi_u(t) = ru_1 t + a\int_0^t (\psi_u(s))_2 ds. \tag{4.20}$$

We now prove that the real value of $(\psi_u(t))_2$ is always nonpositive. First, let us assume that $\rho \in (-1, 1)$. For $u \neq 0$, we have from (4.19) that $\Re((\psi_u'(0))_2) = \frac{1}{2}u_1^2 + \rho\sigma u_1 u_2 + \frac{\sigma^2}{2}u_2^2 < 0$ since $|\sigma u_1 u_2| \leq \frac{1}{2}|u_1|^2 + \frac{\sigma^2}{2}|u_2|^2$. Therefore, we have

$\bar{t} := \inf\{t \geq 0,\ \Re((\psi_u(t))_2) > 0\} > 0$. We use the convention $\inf \emptyset = +\infty$. If $\bar{t} < \infty$, we have $\Re((\psi_u(\bar{t}))_2) = 0$ and get similarly that $\Re((\psi_u'(\bar{t}))_2) < 0$. This leads to a contradiction. By continuity of $(\psi_u(t))_2$ with respect to ρ, we get that $\Re((\psi_u(t))_2) \leq 0$ for any $t \geq 0$ also for $\rho \in [-1, 1]$. From (4.20), we get

$$\forall t \geq 0,\ \Re(\phi_u(t)) \leq 0.$$

Now, we show that we have indeed (4.11) and set, for $s \in [0, t]$, $M_s = \exp(\phi_u(t-s) + \psi_u(t-s)^\top X_s^x)$. We have $|M_s| = \exp(\Re(\phi_u(t-s)) + \Re((\psi_u(t-s))_2)^\top (X_s^x)_2) \leq 1$, and by Itô's formula, we get

$$dM_s = M_s \sqrt{(X_s^x)_2}\psi_u(t-s)^\top \begin{bmatrix} \rho dW_s^1 + \sqrt{1-\rho^2} dW_s^2 \\ \sigma dW_s^1 \end{bmatrix}.$$

Thus, M is a square integrable martingale, which gives $M_0 = \mathbb{E}[M_t]$ and thus (4.11).

It remains to calculate $\phi_u(t)$. When $\Delta \neq 0$, we have from (4.20) and (4.13),

$$\begin{aligned}
\phi_u(t) &= (ru_1 + au_2)t + ag^{-1}(\Psi - u_2) \int_0^t \frac{\exp(-\sqrt{\Delta}s) - 1}{g^{-1}\exp(-\sqrt{\Delta}s) - 1} ds \\
&= (ru_1 + au_2 + ag^{-1}(\Psi - u_2))t \\
&\quad + \frac{a(\Psi - u_2)(1 - g^{-1})}{\sqrt{\Delta}} \int_0^t \frac{g^{-1}\sqrt{\Delta}\exp(-\sqrt{\Delta}s)}{g^{-1}\exp(-\sqrt{\Delta}s) - 1} ds \\
&= \left(ru_1 + a(\Psi - \frac{2\sqrt{\Delta}}{\sigma^2}) \right) t + \frac{2a}{\sigma^2} \int_0^t \frac{g^{-1}\sqrt{\Delta}\exp(-\sqrt{\Delta}s)}{g^{-1}\exp(-\sqrt{\Delta}s) - 1} ds, \qquad (4.21)
\end{aligned}$$

since $1 - g^{-1} = \frac{2\sqrt{\Delta}}{\sigma^2(\Psi - u_2)}$. Let us recall now that the complex logarithm is usually defined by $\log(\rho e^{i\theta}) = \log(\rho) + i\theta$ for $\rho > 0$ and $\theta \in (-\pi, \pi)$, and is therefore discontinuous on $\mathbb{R}_-$. To obtain (4.12) from (4.21), one has to check that

$$\forall t \geq 0,\ \frac{\exp(-\sqrt{\Delta}t) - g}{1 - g} \notin \mathbb{R}_-.$$

From (4.13), we have

$$(\psi_u(t))_2 = \Psi + (\Psi - u_2)\frac{g - 1}{\exp(-\sqrt{\Delta}t) - g}. \qquad (4.22)$$

Therefore, $\frac{\exp(-\sqrt{\Delta}t) - g}{1 - g}$ cannot be zero since $\psi_u(t)$ is well defined for any $t \geq 0$. We note that $\Re(\Delta) = k^2 + (1-\rho^2)\sigma^2|u_1|^2 \geq 0$, which gives $\Re(\sqrt{\Delta}) \geq \sqrt{\Re(\Delta)} \geq |k|$ and then $\Re(\Psi) \geq 0$ (we recall that the square root of a complex number has a nonnegative real part). If $\Re(\Psi) > 0$, $\frac{\exp(-\sqrt{\Delta}t)-g}{1-g}$ cannot be a negative real number, otherwise we would have $\Re((\psi_u(t))_2) = \Re(\psi)\left(1 - \frac{\exp(-\sqrt{\Delta}t)-g}{1-g}\right) > 0$. If $\Re(\Psi) = 0$, we have $\Re(\sqrt{\Delta}) = |k| = -k > 0$. Besides, we necessarily have $\Im(\sqrt{\Delta}) = 0$ since we would have $\Re(\sqrt{\Delta}) > \sqrt{\Re(\Delta)}$ otherwise. Therefore, we have $\sqrt{\Delta} = -k$. Then, we have $g = \frac{z}{2k+z}$ with $z = -\rho\sigma u_1 - \sigma^2 u_2 \in i\mathbb{R}$ and get that

$$\frac{\exp(-\sqrt{\Delta}t) - g}{1-g} = \frac{1}{2k}[(2k+z)\exp(kt) - z]$$

has a positive real part.

Last, when $\Delta = 0$ we necessarily have $k = 0$ and thus $\Psi \in i\mathbb{R}$, which gives that $\Re\left(1 + \frac{\sigma^2}{2}t(\Psi - u_2)\right) = 1$, which validates the logarithm in (4.14). □

4.2.2 *Pricing Formulas for the European Options*

In this section, we focus on getting pricing formulas for the European call and put options in the Heston model. Let $T > 0$ and $K > 0$ be respectively a maturity and a strike. The European call option with cash delivery pays $(S_T - K)^+$ at time T, while the put option pays $(K - S_T)^+$. We recall that $x^+ = \max(x, 0)$ for $x \in \mathbb{R}$. Thus, their price are respectively given by

$$C(T, K) = \mathbb{E}[e^{-rT}(S_T - K)^+],\ \ P(T, K) = \mathbb{E}[e^{-rT}(K - S_T)^+].$$

We now define the probability measure

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = e^{-rT}\frac{S_T}{S_0} = \exp\left(\int_0^t \sqrt{V_u}(\rho dW_u^1 + \sqrt{1-\rho^2}dW_u^2) - \frac{1}{2}\int_0^t V_u du\right).$$

This is well a probability measure from (4.17). Also, we have the Call Put parity

$$C(T, K) - P(T, K) = e^{-rT}(\mathbb{E}[S_T] - K) = S_0 - Ke^{-rT},$$

and we only then focus on the Call prices. By using that $(S_T - K)^+ = \mathbb{1}_{S_T \geq K}(S_T - K)$, we obtain

$$\begin{aligned} C(T, K) &= S_0\tilde{\mathbb{P}}(S_T \geq K) - Ke^{-rT}\mathbb{P}(S_T \geq K) \\ &= S_0\tilde{\mathbb{P}}(X_T \geq \log(K)) - Ke^{-rT}\mathbb{P}(X_T \geq \log(K)). \end{aligned} \tag{4.23}$$

Thus, we are interested in calculating the law of S_T under the probability measure $\tilde{\mathbb{P}}$. From the Girsanov theorem, we know that $(\tilde{W}_t^1, \tilde{W}_t^2)_{t\in[0,T]}$ defined by

$$\tilde{W}_t^1 = W_t^1 - \rho \int_0^t \sqrt{V_u} du, \ \tilde{W}_t^2 = W_t^2 - \sqrt{1-\rho^2} \int_0^t \sqrt{V_u} du,$$

is a standard two dimensional Brownian motion under $\tilde{\mathbb{P}}$. Then, we have from (4.9)

$$\begin{aligned} dX_t &= (r - \frac{1}{2} V_t)dt + \sqrt{V_t} \left(\rho [d\tilde{W}_t^1 + \rho\sqrt{V_t} dt] + \sqrt{1-\rho^2}[d\tilde{W}_t^2 + \sqrt{1-\rho^2}\sqrt{V_t} dt] \right) \\ &= (r + \frac{1}{2} V_t)dt + \sqrt{V_t}(\rho d\tilde{W}_t^1 + \sqrt{1-\rho^2} d\tilde{W}_t^2), \\ dV_t &= (a - kV_t)dt + \sigma\sqrt{V_t}\rho[d\tilde{W}_t^1 + \rho\sqrt{V_t} dt] \\ &= (a - (k - \rho\sigma)V_t)dt + \sigma\sqrt{V_t}\rho d\tilde{W}_t^1. \end{aligned}$$

Therefore, (X, V) follows under $\tilde{\mathbb{P}}$ the same SDE as under $\mathbb{P}$ with $\tilde{\lambda} = -1/2$ and $\tilde{k} = k - \rho\sigma$ instead of $\lambda = 1/2$ and k. The characteristic function of (X_T, V_T) is then given by Proposition 4.2.1 with these parameters. We are now in position to apply classical results on the Fourier inversion.

Theorem 4.2.3 *Let X be a real random variable and $\Phi(v) = \mathbb{E}[e^{ivX}]$ denote its characteristic function. Then, we have*

$$\forall x \in \mathbb{R}, \ \mathbb{P}(X \le x) + \mathbb{P}(X < x) = 1 - \frac{2}{\pi} \lim_{m\to 0^+ M\to+\infty} \int_m^M \Re\left(\frac{e^{-ivx}\Phi(v)}{iv}\right) dv.$$

Besides, if $\int_{\mathbb{R}} |\Phi(v)|dv < \infty$, X has a density which is given by

$$p(x) = \int_{\mathbb{R}} e^{-ivx}\Phi(v)dv.$$

In particular, we then have

$$\forall x \in \mathbb{R}, \ \mathbb{P}(X \ge x) = \frac{1}{2} + \frac{1}{\pi} \lim_{m\to 0^+ M\to+\infty} \int_m^M \Re\left(\frac{e^{-ivx}\Phi(v)}{iv}\right) dv. \tag{4.24}$$

The first result of this theorem can be found in Gurland [72] and is often called the Gil-Pelaez inversion formula. The second result is a classical result on the inverse Fourier transform. Also, we note that when X is integrable, $\Phi(v) = 1 + iv\mathbb{E}[X] + o(v)$, and the integral (4.24) is proper in $0+$.

We want to apply this result to the Heston model and focus on the characteristic function of X_T. Thus, we set

$$\boxed{\Phi(v) = \mathbb{E}[e^{ivX_T}] \text{ and } \tilde{\Phi}(v) = \tilde{\mathbb{E}}[e^{ivX_T}],} \tag{4.25}$$

and use (4.11) with $u_1 = iv$ and $u_2 = 0$. When $\rho \in (-1, 1)$, we observe that $\Delta \underset{|v|\to+\infty}{\sim} \sigma^2(1-\rho^2)v^2$, which gives that g and then $(\psi_u(T))_2$ converges, as well as the logarithm in (4.12). Since we have assumed $a > 0$, we get that $\Re(\phi_u(T)) \underset{|v|\to+\infty}{\sim} -\frac{a|v|T\sqrt{1-\rho^2}}{\sigma}$. This gives that Φ and $\tilde{\Phi}$ decay exponentially with v. We can also check that they decay exponentially with $\sqrt{|v|}$ when $\rho \in \{-1, 1\}$. They are in particular integrable and by Theorem 4.2.3, X_T has a density under $\mathbb{P}$ and $\tilde{\mathbb{P}}$. Besides, we get the following result.

Corollary 4.2.4 *The price of a European call option in the Heston model is given by*

$$\boxed{\begin{aligned} C(T,K) =& S_0 \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\left(\frac{e^{-iv\log(K)}\tilde{\Phi}(v)}{iv} \right) dv \right) \\ & - Ke^{-rT} \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\left(\frac{e^{-iv\log(K)}\Phi(v)}{iv} \right) dv \right), \end{aligned}}$$

where $\Phi(v)$ (resp. $\tilde{\Phi}(v)$) is defined by (4.25) and can be obtained by using (4.11) with $u_1 = iv$, $u_2 = 0$, k and $\lambda = 1/2$ (resp. $k - \rho\sigma$ and $\lambda = -1/2$).

To calculate in practice $C(T, K)$, it is possible either to use (4.23) with Gram-Charlier expansions or Corollary 4.2.4 and use numerical integration or quadrature formulas to approximate the integrals.

4.2.3 Pricing with the Fast Fourier Transform

In this section, we present a pricing method that has been popularized by Carr and Madan [26]. It relies on a well-known algorithm called the Fast Fourier Transform which is widely used in digital signal processing. We now briefly this algorithm as well as the discrete Fourier transform. Other efficient Numerical methods based on the Fourier transform exist, see Chap. 15 of Pascucci [112].

Let $N \in \mathbb{N}^*$ and $x \in \mathbb{C}^N$. Then, the discrete Fourier transform of x is the vector $\hat{x} \in \mathbb{C}^N$, which is defined by

$$\hat{x}_l = \sum_{j=1}^{N} \exp\left(-\frac{2i\pi}{N}(j-1)(l-1) \right) x_j, \ 1 \le l \le N. \tag{4.26}$$

The application $x \mapsto \hat{x}$ is linear and invertible, and we have

$$x_j = \frac{1}{N} \sum_{l=1}^{N} \exp\left(\frac{2i\pi}{N}(j-1)(l-1)\right) \hat{x}_l, \ 1 \le j \le N.$$

If we calculate naively $\hat{x}$ from x, one has to calculate N different sums, which requires $O(N^2)$ operations. In 1965, Cooley and Tukey [30] have proposed a divide and conquer algorithm that calculate $\hat{x}$ from x with only $O(N\log(N))$ operations. Then, several improvements have been made on this algorithm, and different implementations can be now downloaded for free. One of the most famous is probably the FFTW that can be downloaded at http://www.fftw.org/.

The strength of the method proposed by Carr and Madan [26] is to compute simultaneously all the call option prices at the same maturity T for different strikes. To do so, they consider for $k \in \mathbb{R}$,

$$c_T(k) = \exp(\alpha k) C(T, e^k).$$

The parameter $\alpha \in \mathbb{R}$ is chosen to get that $c_T(k)$ is integrable. By dominated convergence, we have $\lim_{k \to -\infty} C(T, e^k) = S_0$ from (4.17) and therefore α has to be positive. Besides, since we know from Corollary (4.2.2) that $\mathbb{E}[S_T^{1+p}] < \infty$ for some $p > 0$ since the set of convergence $\mathcal{D}_T$ is an open set that contains 1. Now, let us admit for a while the following inequality given by Lee [95]

$$K \ge 0, x \ge 0, \ (x-K)^+ \le \frac{x^{p+1}}{(p+1)K^p}\left(\frac{p}{p+1}\right)^p. \tag{4.27}$$

Then, we have $C(T, e^k) \le e^{-pk} \frac{\mathbb{E}[S_T^{p+1}]}{(p+1)} \left(\frac{p}{p+1}\right)^p$ and c_T is integrable on $\mathbb{R}$ if $0 < \alpha < p$. It is even square integrable thanks to the exponential decay. Using again that $\mathcal{D}_T$ is an open set, we get that c_T is square integrable if

$$\alpha > 0 \text{ and } \mathbb{E}[S_T^{1+\alpha}] < \infty. \tag{4.28}$$

Let us now prove (4.27) and denote $g_{K,p}(x)$ its right hand side. The function $g_{K,p}$ is nonnegative and convex. It is then above its tangent at $x = \frac{p+1}{p}K$. Since $g'_{K,p}(\frac{p+1}{p}K) = 1$, $g_{K,p}(\frac{p+1}{p}K) = \frac{K}{p}$, this gives (4.27).

Once α is chosen such that (4.28), we can calculate the Fourier transform of c_T. Let us denote $q_T(x)$ the density of the law of X_T under $\mathbb{P}$. We have for $v \in \mathbb{R}$,

$$\begin{aligned}\hat{c}_T(v) &= \int_{\mathbb{R}} e^{ivk} c_T(k)dk = e^{-rT} \int_{\mathbb{R}} e^{ivk} \int_k^{+\infty} e^{\alpha k}(e^x - e^k) q_T(x)dx \\ &= e^{-rT} \int_{\mathbb{R}} \left(\int_{-\infty}^{x} e^{(\alpha+iv)k}(e^x - e^k)dk \right) q_T(x)dx\end{aligned}$$

$$= e^{-rT} \int_{\mathbb{R}} \left(\frac{1}{\alpha + iv} - \frac{1}{\alpha + 1 + iv} \right) e^{(\alpha+1+iv)x} q_T(x) dx$$
$$= \frac{e^{-rT}}{(\alpha + iv)(\alpha + iv + 1)} \mathbb{E}[e^{(\alpha+1+iv)X_T}],$$

where we have used the Fubini theorem from the first to the second line. The value of $\mathbb{E}[e^{(\alpha+1+iv)X_T}]$ is given explicitly by using (4.16), (1.15) and Remark 1.2.10. We can also use the formulas of Proposition 4.2.1 with $u_2 = 0$ and $u_1 = \alpha + 1 + iv$, that are correct up to a multiple of $2i\pi \times \frac{2a}{\sigma^2}$ for ϕ_u, see again Remark 1.2.10. Thus, the function $\hat{c}_T$ is known explicitly. It is also square integrable, and we have by the Fourier inversion theorem

$$c_T(k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ivk} \hat{c}_T(v) dv = \frac{1}{\pi} \Re \left(\int_0^\infty e^{-ivk} \hat{c}_T(v) dv \right), \tag{4.29}$$

since the conjugate of $e^{-ivk}\hat{c}_T(v)$ is $e^{ivk}\hat{c}_T(-v)$. Formula (4.29) is very interesting from a computational point of view since it gives the call prices as the Fourier inverse transform of the explicit function $\hat{c}_T(v)$. By using the same kind of arguments, Lewis [97] and Lipton [98] have proposed a similar formula for the call option price that formally corresponds to the choice $\alpha = -1/2$.

Now, we have to approximate the integral in (4.29) to calculate $c_T(k)$ and thus $C(T, e^k) = e^{-\alpha k} c_T(k)$. To do so, we have to truncate the integral and then use an approximation. Here, for the sake of simplicity we will use Riemann sums even though better approximations such as the Simpson's rule can be used since the function $\hat{c}_T$ is smooth. We consider a discretization step $\Delta > 0$ and truncate the integral on $[0, N\Delta]$ for some $N \in \mathbb{N}^*$. This leads to the approximation:

$$c_T(k) \approx \frac{\Delta}{\pi} \Re \left(\sum_{j=1}^{N} \exp\left(-i(j-1)\Delta k\right) \hat{c}_T((j-1)\Delta) \right).$$

To fit in the framework of the discrete Fourier Transform, we have to consider a grid for the strikes, and we set

$$k_l = \frac{2\pi}{\Delta N}(l-1) + k_0,$$

for some $k_0 \in \mathbb{R}$. Typically, k_0 can be set such that $e^{k_{\lfloor N/2 \rfloor}} = S_0$ in order to center this grid around the strike at the money. We get

$$c_T(k_l) \approx \frac{\Delta}{\pi} \Re \left(\sum_{j=1}^{N} \exp\left(-\frac{2i\pi}{N}(j-1)(l-1)\right) \exp\left(-i(j-1)\Delta k_0\right) \hat{c}_T((j-1)\Delta) \right), \tag{4.30}$$

and the vector $(c_T(k_l))_{1\le l\le N}$ can be calculated by using the FFT algorithm on the vector $\big(\exp(-i(j-1)\Delta k_0)\,\hat{c}_T((j-1)\Delta)\big)_{1\le j\le N}$. Thus, it is possible to calculate with only one FFT the call prices for different strikes. In practice, we see that the strike grid is rather constrained by the choice of the integral truncation and discretization and one often has to make some interpolation to get the price at a given strike. Here, we do not discuss how accurate is the approximation (4.30). We refer to Lee [95] for an analysis of the discretization and the truncation error. Last, we observe that the method works in principle for any choice of α satisfying (4.28). However, numerically some choices may be more appropriate than others, and we refer again to Lee [95] and Kienitz and Wetterau [88] for a discussion on the choice of α.

4.2.4 Simulation Schemes for the Heston Model

A Potential Second Order Scheme for $(\log(S_t), V_t)$

We have presented in Sect. 3.3 a second and even a third order scheme for the CIR process. Once we have at our disposal a second order scheme for the CIR process, it is really easy to construct a potential second order scheme for the process X defined by (4.9). We recall that the case $\lambda = 1/2$ corresponds to the SDE satisfied by $(\log(S_t), V_t)$. In fact, we will construct a potential second order scheme for

$$\left((X_t)_1, (X_t)_2, \int_0^t (X_t)_2 ds, \int_0^t e^{(X_t)_1} ds\right)^\top .$$

We still denote by X this process, so that $(X_t)_3$ and $(X_t)_4$ denote respectively the third and the fourth coordinate. For financial applications, simulating $(X_t)_4$ is interesting to calculate the price of Asian options that brings on the average price over a given period.

To construct a potential second order scheme, we will again use the composition of schemes. We then have to split in an appropriate way the infinitesimal generator of X. The infinitesimal generator is given by

$$Lf = (r-\frac{x_2}{2})\partial_1 f + (a-kx_2)\partial_2 f + \frac{\sigma^2}{2}x_2\partial_2^2 f + \frac{1}{2}x_2\partial_1^2 f + \rho\sigma x_2\partial_1\partial_2 f + x_2\partial_3 + e^{x_1}\partial_4.$$

We split this operator as $L = L_1 + L_2$, with

$$\begin{aligned}
L_1 f &= (r-\frac{x_2}{2})\partial_1 f + (a-kx_2)\partial_2 f + \frac{\sigma^2}{2}x_2\partial_2^2 f + \frac{1}{2}\rho^2 x_2\partial_1^2 f + \rho\sigma x_2\partial_1\partial_2 f \\
&\quad + x_2\partial_3 + e^{x_1}\partial_4, \\
L_2 f &= \frac{1}{2}(1-\rho^2)x_2\partial_1^2 f.
\end{aligned}$$

The SDE associated to L_1 can be written as

$$\boxed{\begin{cases} d(X_t)_1 = (r - \frac{1}{2}(X_t)_2)dt + \rho\sqrt{(X_t)_2}dW_t^1 \\ d(X_t)_2 = (a - k(X_t)_2)dt + \sigma\sqrt{(X_t)_2}dW_t^1 \\ d(X_t)_3 = (X_t)_2 dt \\ d(X_t)_4 = e^{(X_t)_1}dt. \end{cases}}$$

We notice that $(X_t)_1$ can be solved explicitly by the mean of $(X_t)_2$ and $(X_t)_3$ since we have

$$(X_t)_1 = (X_0)_1 + rt - \frac{1}{2}((X_t)_3 - (X_0)_3) + \frac{\rho}{\sigma}\left[(X_t)_2 - (X_0)_2 - at + k((X_t)_3 - (X_0)_3)\right].$$

Therefore, we can get a second order approximation of the first coordinate from a potential second order scheme for the second and third coordinates. Thus, we use the second (or the third) order scheme for the CIR process $(X_t)_2$ starting from $(X_0)_2$, which we denote by $(\hat{X}_t)_2$. Then, we use a trapezoidal rule for the third coordinate, which amounts to use the second construction of Corollary 2.3.14. We then calculate the first coordinate by the formula above. Last, we use again the trapezoidal rule for the fourth coordinate, and finally get the following potential second order scheme for L_1:

$$\hat{X}_t := \begin{pmatrix} (X_0)_1 + \left(r - a\frac{\rho}{\sigma}\right)t + \left(\frac{k\rho}{\sigma} - \frac{1}{2}\right)((\hat{X}_t)_3 - (X_0)_3) + \frac{\rho}{\sigma}\left[(\hat{X}_t)_2 - (X_0)_2\right] \\ (\hat{X}_t)_2 \\ (X_0)_3 + \frac{(X_0)_2 + (\hat{X}_t)_2}{2}t \\ (X_0)_4 + \frac{e^{(X_0)_1} + e^{(\hat{X}_t)_1}}{2}t \end{pmatrix}. \tag{4.31}$$

Equation (4.31) is autoreferenced, but once we have taken a scheme $(\hat{X}_t)_2$ for the CIR, $(\hat{X}_t)_3$, then $(\hat{X}_t)_1$ and then $(\hat{X}_t)_4$ can be deduced. The SDE associated to L_2 is a Brownian motion and is exactly solved by

$$\left((X_0)_1 + \sqrt{(X_0)_2}\sqrt{1-\rho^2}W_t^2, (X_0)_2, (X_0)_3, (X_0)_4\right)^\top. \tag{4.32}$$

Then, by using Corollary 2.3.14, we get from (4.31) and (4.32) a potential second order scheme for the Heston model.

If we want to prove by using Theorem 2.3.8 that this scheme leads to a weak error of order 2 for $\mathbb{E}[f(\log(S_T), V_T)]$ with $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{R} \times \mathbb{R}_+)$, one would have to check first the boundedness of the moments. This should hold since the SDE satisfied by $(\log(S_t), V_t)$ has sublinear coefficients and therefore bounded moments and we may expect that scheme has the same property. Then, one has to study the Kolmogorov equation related to the $(\log(S_T), V_T)$ and get the estimates needed in Theorem 2.3.8. This issue has been investigated very recently in the Ph.D. thesis of Gabrielli [57]. However, for some applications, one may wish to have a weak error of order 2 for

Algorithm 4.1: Algorithm for the potential second-order scheme with time-step $t > 0$, $B \sim \mathcal{B}(1/2)$ and $N \sim \mathcal{N}(0,1)$ are sampled independently.

Input: $x \in \mathbb{R}^4$, $a \geq 0$, $\sigma > 0$, $k \in \mathbb{R}$, $\rho \in [-1,1]$ and $t > 0$.
Output: X, potential second order scheme described above.
Function $Sch_1(x)$: Sample $\hat{x}_2$ by using either Algorithms 3.1 or 3.3 starting from x_2 with time step $t > 0$.
$\hat{x}_3 = x_3 + \frac{x_2+\hat{x}_2}{2} t$,
$\hat{x}_1 = x_1 + \left(r - a\frac{\rho}{\sigma}\right) t + \left(\frac{k\rho}{\sigma} - \frac{1}{2}\right)(\hat{x}_3 - x_3) + \frac{\rho}{\sigma}[\hat{x}_2 - x_2]$,
$\hat{x}_4 = x_4 + \frac{e^{x_1}+e^{\hat{x}_1}}{2} t$,
return $\hat{x}$;
Function $Sch_2(x)$: **return** $\left(x_1 + \sqrt{x_2}\sqrt{1-\rho^2}\sqrt{t}N, x_2, x_3, x_4\right)$;
if $(B = 1)$ **then**
| $X = Sch_1(Sch_2(x))$,
else
| $X = Sch_2(Sch_1(x))$,
end

$\mathbb{E}[f(S_T, V_T)]$ with $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{R}_+ \times \mathbb{R}_+)$. In this case, it is not clear how to apply Theorem 2.3.8. In fact, the moments of S_T may explode by Corollary 4.2.2 and we may expect that the same hold for the simulation scheme. Thus, it is not possible to analyze the weak error by using the arguments of Theorem 2.3.8. It is still an open issue to prove a second order convergence in this setting.

Simulations with the Potential Second Order Scheme for the Heston Model

We now test numerically the scheme described above, with the first construction of Corollary 2.3.14. This amounts to use first (4.31) and then (4.32) with probability $1/2$, and first (4.32) and then (4.31) with probability $1/2$. We take back simulations that have been presented in [8]. We will denote scheme 1 (resp. 2) the scheme that uses for $(\hat{X}_t)_2$ the second (resp. third) order scheme for the CIR. Since we use scheme composition given by Corollary 2.3.14, we may hope at the best that these both schemes have a second order of convergence. Nonetheless, we would like to see numerically if there is some interest to use the third-order scheme for the CIR instead of the second-order one. Last, for comparison, we introduce the following scheme which coincides for the second coordinates to the scheme (3.9):

$$\hat{X}^x_t = \begin{pmatrix} x_1 + (r - x_2^+/2)t + \sqrt{x_2^+}(\rho W_t^1 + \sqrt{1-\rho^2} W_t^2) \\ x_2 + (a - kx_2^+)t + \sigma\sqrt{x_2^+} W_t^1 \\ x_3 + x_2 t \\ x_4 + e^{x_1} t \end{pmatrix}. \tag{4.33}$$

This is the scheme 3.

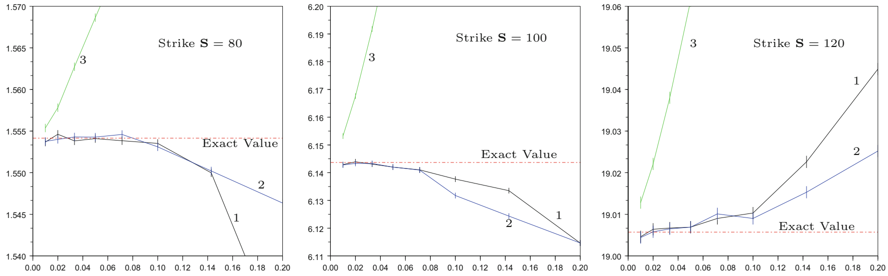

Fig. 4.1 $\mathbb{E}[e^{-r}(\mathbf{S} - e^{(\hat{X}_1)_1})^+]$ in function of the time-step $1/n$ with $X_0^2 = 0.04$, $k = 0.5$, $a = 0.02$, $\sigma = 0.4$, $r = 0.02$, $S_0 = e^{X_0^1} = 100$ and $\rho = -0.5$. Point width gives 95 % confidence interval

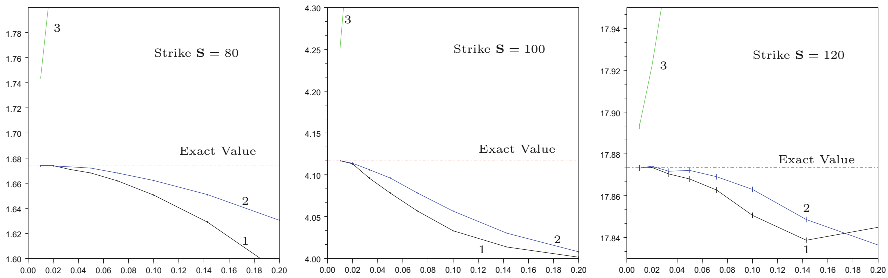

Fig. 4.2 $\mathbb{E}[e^{-r}(\mathbf{S} - e^{(\hat{X}_1)_1})^+]$ in function of the time-step $1/n$ with $X_0^2 = 0.04$, $k = 0.5$, $a = 0.02$, $\sigma = 1$, $r = 0.02$, $S_0 = e^{X_0^1} = 100$ and $\rho = -0.8$. Point width gives 95 % confidence interval

In all the simulations, we have fixed $T = 1$. To test the schemes, we have calculated European put prices for different strikes with rather high values of σ in Figs. 4.1 and 4.2. It is hard to say qualitatively from the curves that the convergence is indeed quadratic for the schemes 1 and 2. Nonetheless in the European put case we can compare the value obtained with the exact value. For example in Fig. 4.1, for a time step $1/50$ and for each strike, the exact value is in the two standard deviations window of which width is between 0.5×10^{-3} and 1.5×10^{-3} according to the strike value. Therefore, the bias is not much big as $(1/50)^2 = 0.4 \times 10^{-3}$ and the convergence quality is not far from being the one of a true second-order scheme. In comparison, the scheme 3 has in that case a rather linear convergence and is still far from the exact value for $n = 50$. Last, we observe that schemes 1 and 2 give similar convergence orders. In Fig. 4.1 where σ is not that big, the difference between the schemes is not really significant. Instead, in Fig. 4.2, when the volatility of the volatility is really high ($\sigma^2 \gg 4a$), the use of the third-order scheme for the CIR in scheme 2 allows to reduce the bias with respect to the scheme 1.

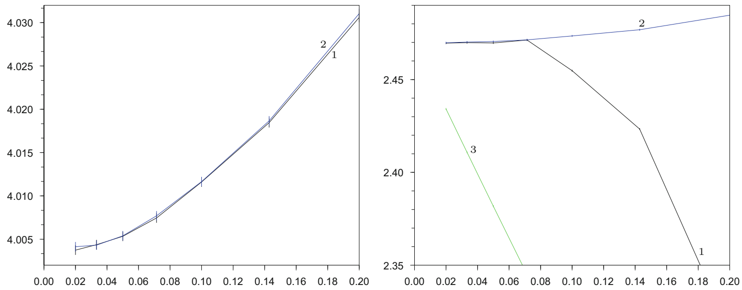

Fig. 4.3 $\mathbb{E}[e^{-r}(100-(\hat{X}_1)_4)^+]$ (*left*) and $\mathbb{E}[e^{-r}\mathbf{1}_{(\hat{X}_1)_3>a/k}((\hat{X}_1)_4-e^{(\hat{X}_1)_1})^+]$ (*right*) in function of the time step $1/n$ with $X_0^2=0.04$, $k=0.5$, $a=0.02$, $\sigma=0.2$, $r=0.02$, $e^{X_0^1}=100$ and $\rho=-0.3$. Point width gives the two standard deviation precision

Table 4.1 Results for the scheme 3

n	5	7	10	14	20	30	50
$\mathbb{E}[e^{-r}(100-(\hat{X}_{t_n^n})_4)^+]$	4.6189	4.4427	4.3108	4.2235	4.1570	4.1062	4.0646

Parameters as in Fig. 4.3. Precision up to two standard deviations: 5×10^{-4}

We have also plotted in Fig. 4.3 the prices of an Asian put and of an exotic option that gives the right to earn the difference between the average stock and the stock when the realized variance is above a certain level. We have chosen here a rather low value of σ ($\sigma^2<4a$). Thus, the CIR process does not spend much time near 0 and the convergence observed for the schemes 1 and 2 is qualitatively parabolic in function of the time-step. For the exotic option considered here, we also notice that the scheme 2 gives minor bias than scheme 1 for large time-steps. In comparison and to underline the importance of the method chosen, we have put in Table 4.1 the values obtained with the scheme 3 for the Asian option, because they could not have been plotted on the same scale. For that scheme, the convergence is in that case quasi-linear.

4.2.5 *Pricing and Simulation with PREMIA*

PREMIA is a software that gathers many computational routines for quantitative finance. It is developed by the MathRisk team which gathers research scientists from INRIA, Ecole des Ponts ParisTech and the University Paris-Est of Marne la Vallée. PREMIA is a good tool to compare different pricing methods. It can be downloaded for free, and an on-line version can be used at https://quanto.inria.fr Here, we use this on-line version of PREMIA and present different snapshots. We calculate a put option price in the Heston model, and compare different simulation schemes. Of course, we focus here on some particular algorithms and options: we

Fig. 4.4 Pricing of an European Put option in the Heston model by using the Carr-Madan method with $r = 0$, $S_0 = 100$, strike $K = 100$, $V_0 = 0.04$, $k = 0.3$, $a = 0.3 \times 0.04$, $\sigma = 0.9$, $r = 0$ and $\rho = -0.5$

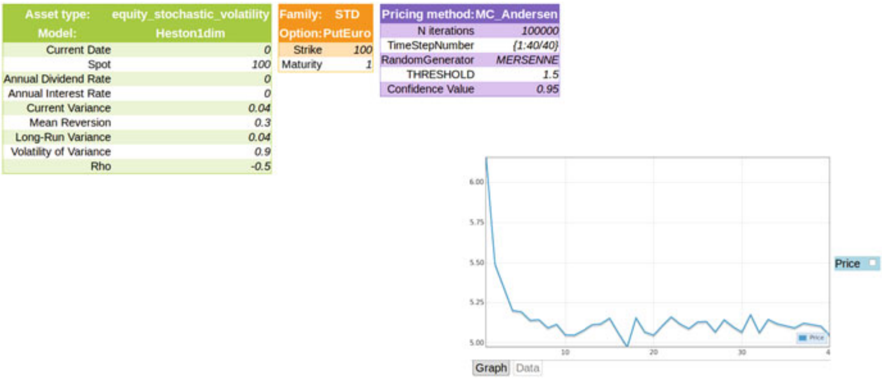

Fig. 4.5 Pricing of the same option by using Andersen's QE scheme. Price in function of the number of time-steps

invite the reader to test by himself the different numerical methods on the PREMIA website.

In Fig. 4.4, we calculate the European put option by using the Carr-Madan method. Then, we draw for different discretization schemes the MC estimation with 10^5 samples in function of the time-step. Here, we have chosen parameters where $\sigma^2 \geq 4a$, which is somehow the most difficult for the scheme convergence. In Fig. 4.5 is plotted the convergence for Andersen's scheme [11]. Essentially, this scheme consists in using the QE scheme $\hat{V}_t$ (see Sect. 3.3.2) for V and consider for X the following one

$$\hat{X}_t = \hat{X}_0 + \left(r - a\frac{\rho}{\sigma}\right)t + \left(\frac{k\rho}{\sigma} - \frac{1}{2}\right)\frac{\hat{V}_t + \hat{V}_0}{2}t + \frac{\rho}{\sigma}\left[\hat{V}_t - \hat{V}_0\right]$$
$$+ \sqrt{t\frac{\hat{V}_t + \hat{V}_0}{2}}N + corrMG(t),$$

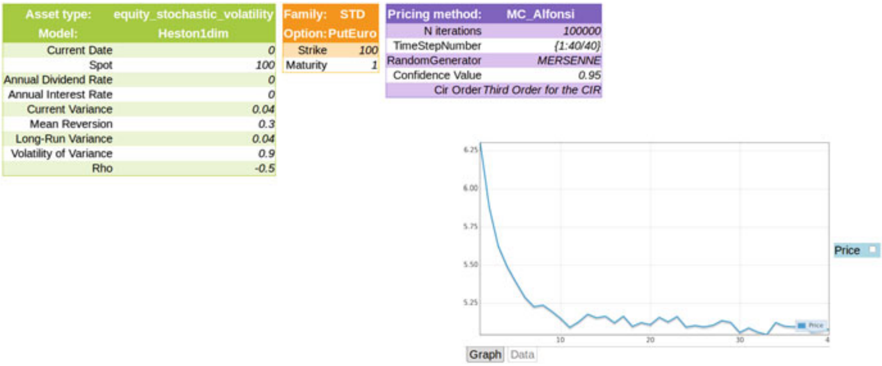

Fig. 4.6 Pricing of the same option by using the potential second order scheme. Price in function of the number of time-steps

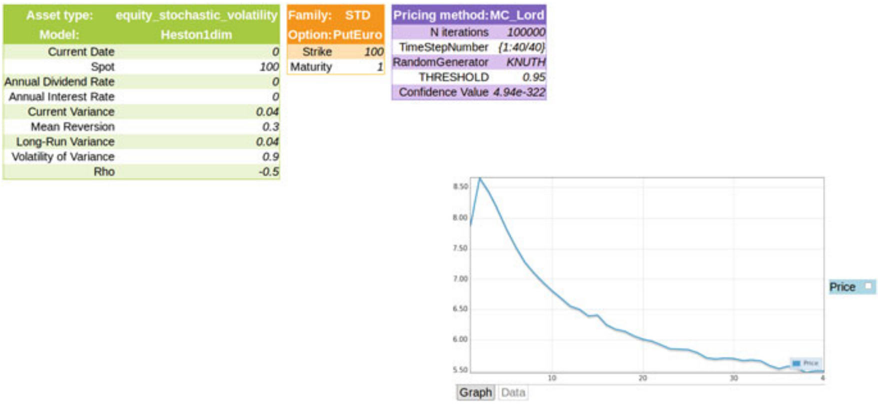

Fig. 4.7 Pricing of the scheme (3.9) for the CIR. Price in function of the number of time-steps

where N is a standard normal variable and $corrMG(t)$ is a deterministic correction term that gives exactly $\mathbb{E}[e^{-rt}e^{\hat{X}_t}] = e^{\hat{X}_0}$. Without this correcting term, this approximation has the same form as the one given for the potential second order scheme, see Eq. (4.31). In Fig. 4.6 is plotted the convergence for the potential second order scheme described above, as the composition of the schemes (4.31) and (4.32). Last, in Fig. 4.7 is plotted the convergence of the scheme given by Eq. (4.33) and based on the Full Truncation scheme (3.9) of Lord et al. [101]. We observe that the potential second order scheme and Andersen's scheme are around the true price after 10 time-steps, and then we mainly observe the noise of the MC method. Instead, the Euler modified scheme in Fig. 4.7 has still not converged after 40

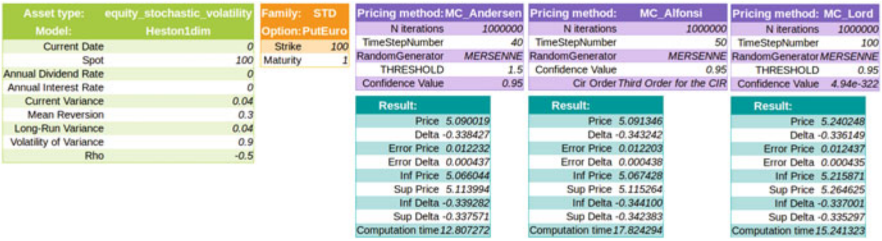

Fig. 4.8 Pricing comparison between the schemes. Same parameters as in Fig. 4.4

Fig. 4.9 Pricing comparison between the potential second order scheme and Andersen's scheme on a European put option. Parameters: $r = 0$, $S_0 = 100$, strike $K = 50$, $V_0 = 0.04$, $k = 0.1$, $a = 0.1 \times 0.04$, $\sigma = 1$, $r = 0$ and $\rho = -0.95$

time-steps. To be more precise, we have indicated in Fig. 4.8 roughly the number of time-steps required to have a precision of 10^{-2} on the price. On this example, Andersen's scheme requires about 40 time-steps, the potential second order scheme about 50 time-steps, while the Euler modified scheme is still 10^{-1} above the true price after 100 time-steps. This illustrates again the need to choose an appropriate discretization scheme in the Heston model when the volatility of the volatility is high.

From Fig. 4.8, it may seem that Andersen's scheme converges slightly better than the potential second order scheme. However, we can find examples where we observe the opposite conclusion. In Fig. 4.9 is given the case of an out-of-the-money put option where the potential second order scheme is more precise. Thus, these two schemes leads to roughly the same precision with a similar computational effort. From a practical point of view, an interesting thing given by the composition of potential second order schemes is its tractability: it is easy to reuse a potential second order scheme in order to construct potential second order schemes for much elaborated SDEs.

4.2.6 The Exact Simulation Method and Derivative Schemes

Proposition 4.2.1 gives the joint law of (X_t, V_t) through its characteristic function. Since this law is explicitly known, one may wonder if we can also get a method to simulate it exactly. This is the goal of the Broadie and Kaya method that we present now.

The Broadie and Kaya [23] Method

From equation (4.9), we can easily get

$$X_t = X_0 + \left(r - \frac{a\rho}{\sigma}\right)t + \int_0^t (\frac{k\rho}{\sigma} - \frac{1}{2})V_u du + \frac{\rho}{\sigma}(V_t - V_0) + \sqrt{1-\rho^2}\int_0^t \sqrt{V_u} dW_u^2. \tag{4.34}$$

Since the Brownian motion W^2 is independent from V, we get that the conditional law of X_t given V_t and $\int_0^t V_u du$ is a normal variable with mean

$$X_0 + \left(r - \frac{a\rho}{\sigma}\right)t + \int_0^t (\frac{k\rho}{\sigma} - \frac{1}{2})V_u du + \frac{\rho}{\sigma}(V_t - V_0)$$

and variance $(1-\rho^2)\int_0^t V_u du$. Therefore, sampling exactly (X_t, V_t) is possible if one is able to sample exactly $(V_t, \int_0^t V_u du)$. Again, this joint law is known explicitly through its characteristic function, see (1.15). We have already explained in Sect. 3.1 how to generate exactly the random variable V_t. We are then interested in simulating a random variable that follows the conditional law of $\int_0^t V_s ds$ given V_t. The characteristic function of this law is also known explicitly, see Pitman and Yor [113] and Broadie and Kaya [23]. Namely, we have

$$\begin{aligned} u \in \mathbb{R},\ &\mathbb{E}\left[\exp\left(iu\int_0^t V_s ds\right) \Big| V_t\right] \\ &= \frac{e^{\frac{k}{2}t}\zeta_k(t)}{e^{\frac{\gamma_{iu}}{2}t}\zeta_{\gamma_{iu}}(t)} \exp\left(\frac{V_0 + V_t}{\sigma^2}\left[\frac{1+e^{-kt}}{\zeta_k(t)} - \frac{1+e^{-\gamma_{iu}t}}{\zeta_{\gamma_{iu}}(t)}\right]\right) \\ &\times \frac{I_{\frac{2a}{\sigma^2}-1}\left(\sqrt{V_0 V_t}\frac{4}{\sigma^2 e^{\frac{\gamma_{iu}}{2}t}\zeta_{\gamma_{iu}}(t)}\right)}{I_{\frac{2a}{\sigma^2}-1}\left(\sqrt{V_0 V_t}\frac{4}{\sigma^2 e^{\frac{k}{2}t}\zeta_k(t)}\right)}, \end{aligned} \tag{4.35}$$

where $I_\nu(x) = \sum_{n=0}^\infty \frac{(x/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)}$ with $\nu > -1$ denotes the modified Bessel function of the first kind, $\gamma_{iu} = \sqrt{k^2 - 2\sigma^2 iu}$, $\zeta_z(t) = \frac{1-e^{-zt}}{z}$ for $z \in \mathbb{C}^*$ and $\zeta_0(t) = t$ otherwise. Then, Broadie and Kaya propose to use the inverse transform sampling method. Let $\Psi(u)$ denote the characteristic function (4.35). From (4.24), we have for $x \in \mathbb{R}$,

$$\begin{aligned}\mathbb{P}\left(\int_0^t V_s ds \leq x \middle| V_t\right) &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Re\left(\frac{e^{-iux}\Psi(u)}{iu}\right) du \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin(ux)}{u}\Re(\Psi(u))du \\ &\quad -\frac{1}{\pi}\int_0^\infty \frac{\cos(ux)}{u}\Im(\Psi(u))du.\end{aligned}$$

We have $\mathbb{P}\left(\int_0^t V_s ds \leq x \middle| V_t\right) = \mathbb{P}\left(\int_0^t V_s ds \leq x \middle| V_t\right) + \mathbb{P}\left(\int_0^t V_s ds \leq -x \middle| V_t\right)$ for $x \geq 0$ since $\int_0^t V_s ds$ is nonnegative. This yields to

$$\mathbb{P}\left(\int_0^t V_s ds \leq x \middle| V_t\right) = \frac{2}{\pi}\int_0^\infty \frac{\sin(ux)}{u}\Re(\Psi(u))du.$$

Broadie and Kaya [23] propose the following approximation

$$\frac{\Delta x}{\pi} + \frac{2}{\pi}\sum_{j=1}^{N}\frac{\sin(j\Delta x)}{j}\Re(\Psi(j\Delta)),$$

which amounts to use a trapezoidal rule with a discretization step Δ and a truncation. They explain how to chose N and Δ in order to achieve a given precision. To sum up, the Broadie and Kaya method works as follows.

1. Sample exactly V_t as explained in Sect. 3.1.
2. Simulate $\int_0^t V_s ds$ by inverse transform sampling, using the approximation above with N and $1/\Delta$ large enough.
3. Sample X_t as a normal variable with mean $X_0 + \left(r - \frac{a\rho}{\sigma}\right) t + \int_0^t (\frac{k\rho}{\sigma} - \frac{1}{2}) V_u du + \frac{\rho}{\sigma}(V_t - V_0)$ and variance $(1-\rho^2)\int_0^t V_u du$.

The Glasserman and Kim Method [64]

The exact simulation method proposed by Broadie and Kaya is unfortunately rather time consuming as indicated by Fig. 4.10, which has led us to run only 5,000 scenarios because the on-line version of PREMIA limits the computation time to 30 s. This is mainly due to the second step of the method. The inverse transform sampling requires many evaluations of the conditional cumulative distribution function. Smith [116] has proposed an approximation of the Broadie and Kaya algorithm that reduces the computational cost. Glasserman and Kim have proposed

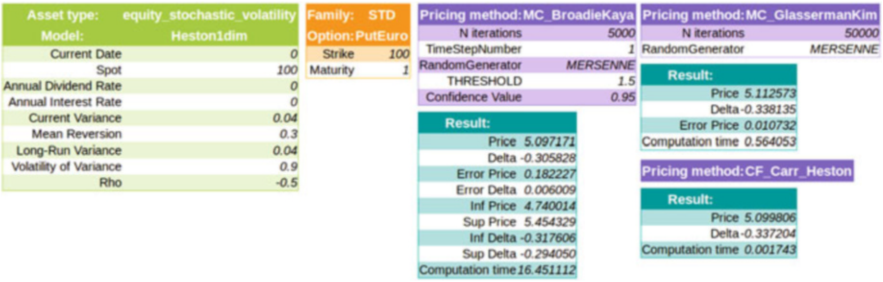

Fig. 4.10 Comparison between the methods by Broadie and Kaya, and Glasserman and Kim. Same parameters as in Fig. 4.4

another way to sample the conditional law of $\int_0^t V_s ds$ given V_t. It relies on the following result, which is proved in [64].

Proposition 4.2.5 *Let* $\lambda_n = \frac{16\pi^2 n^2}{\sigma^2 t(k^2t^2+4\pi^2n^2)}$, $\alpha_n = \frac{k^2t^2+4\pi^2n^2}{2\sigma^2 t}$. *We consider independent random variables* $E_{j,n}$, G_n, G'_n, *and* N_n *for* $j, n \in \mathbb{N}$, *that are such that* $E_{j,n} \sim \mathcal{E}(1)$, $G_n \sim \Gamma(\frac{2a}{\sigma^2}, 1)$, $G'_n \sim \Gamma(2, 1)$ *and* $N_n \sim \mathcal{P}((V_0 + V_t)\lambda_n)$. *We consider*

$$X_1 = \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \sum_{j=1}^{N_n} E_{j,n}, \ X_2 = \sum_{n=1}^{\infty} \frac{1}{\alpha_n} G_n, \ Z_1 = \sum_{n=1}^{\infty} \frac{1}{\alpha_n} G'_n,$$

and $Z_2, Z_3, \ldots$ *independent random variables distributed as* Z_1. *Let* η *be an independent Bessel random variable with parameters* $\nu = \frac{2a}{\sigma^2} - 1$ *and* $z = \frac{2k/\sigma^2}{\sinh(kt/2)}\sqrt{V_0 V_t}$, *i.e.* $\mathbb{P}(\eta = n) = \frac{(z/2)^{2n+\nu}}{I_\nu(z) n! \Gamma(n+\nu+1)}$ *for* $n \in \mathbb{N}$. *Then, we have*

$$x \in \mathbb{R}, \ \mathbb{P}\left(\int_0^t V_s ds \le x \middle| V_t\right) = \mathbb{P}\left(X_1 + X_2 + \sum_{n=1}^{\eta} Z_n \le x\right).$$

Obviously, it is not possible to simulate these infinite sums of random variables. Glasserman and Kim [64] explain how to truncate these series to get an accurate approximation. Figure 4.10 obtained with PREMIA shows that this method drastically reduces the computational time with respect to the method of Broadie and Kaya. Exact simulation methods are competitive with respect to discretization schemes when sampling the Heston process at few dates. However, if one has to sample trajectories to price pathwise options, an easy cross-multiplication between Figs. 4.10 and 4.8 shows that the Glasserman and Kim method requires about 30 times more computational time than the potential second order scheme or Andersen's scheme. Therefore, these simulation schemes have to be preferred to calculate pathwise expectations with a Monte-Carlo algorithm.

4.3 Affine Term Structure Short Rate Models (ATSM)

In Sect. 1.4, we have presented the Vasicek and Cox-Ingersoll-Ross models that are based on one-dimensional affine diffusion. It is then rather natural to consider extensions of these models that are based on multi-dimensional affine diffusions, in order to get richer dynamics and possibly better fit the market. Let us consider an affine process X_t^x on $\mathbb{D} = \mathbb{R}_+^{d'} \times \mathbb{R}^{d-d'}$ that solves (4.1) with b and σ defined by (4.5). Then, we assume that the short rate is an affine function of X_t^x, i.e.

$$t \geq 0,\ r_t = \lambda_0 + \lambda^\top X_t^x, \tag{4.36}$$

with $\lambda^\top = (\lambda_1, \dots, \lambda_d)$. Then, using the same argument as in the proof of Proposition 4.1.2, we obtain that the price of the zero-coupon bond between t and T satisfies

$$\mathbb{E}\left[\exp\left(-\int_t^T (\lambda_0 + \lambda^\top X_s^x) ds \middle| \mathcal{F}_t\right)\right] = \exp\left(\phi(T-t) + \psi(T-t)^\top X_t^x\right),$$

where ϕ and ψ solve

$$\begin{cases} \phi'(t) = -\lambda_0 + \psi(t)^\top \beta_0 + \frac{1}{2}\psi(t)^\top \alpha_0 \psi(t) \\ (\psi'(t))_i = -\lambda_i + \psi(t)^\top \beta_i + \frac{1}{2}\psi(t)^\top \alpha_i \psi(t),\ 1 \leq i \leq d' \\ (\psi'(t))_i = -\lambda_i + \psi(t)^\top \beta_i,\ d'+1 \leq i \leq d, \end{cases}$$

with the initial condition $\phi(0) = \psi(0) = 0$. Such an affine model for the short rate are also known as factor models. In principle, the parameters of the model (α_i, β_i and λ_i) are fixed and are supposed to be valid on a long period while the factor x is adjusted to fit the market data on zero-coupon bonds and some other options. There is a wide literature on these affine term structure models. To mention a few, El Karoui and Lacoste [49] have considered the Gaussian framework where $d' = 0$, Duffie and Kan [46] and then Dai and Singleton [39] have proposed some more explicit parametrization of the affine process for $\mathbb{D} = \mathbb{R}_+^{d'} \times \mathbb{R}^{d-d'}$. We refer to Chap. 9 of Baldeaux and Platen [14] for a simultaneous presentation of [39, 46]. Here, we present the framework of [39].

4.3.1 The Dai and Singleton Parametrization

Dai and Singleton consider the following affine diffusion

$$dX_t^x = (A - KX_t^x)dt + \Sigma\sqrt{D_t}dW_t, \tag{4.37}$$

where $A \in \mathbb{R}^d$, $K, \Sigma \in \mathcal{S}_d^+(\mathbb{R})$, D_t is a diagonal matrix such that $(D_t)_{ii} = \gamma_{i0} + \sum_{j=1}^d \gamma_{ij}(X_t^x)_j$, and $(W_t, t \geq 0)$ is a standard d-dimensional Brownian motion. The matrix $\sqrt{D_t}$ is simply the diagonal matrix such that $(\sqrt{D_t})_{ii} = \sqrt{(D_t)_{ii}}$. We consider here the following canonical parametrization that ensures that the process is well-defined on the domain $\mathbb{D} = \mathbb{R}_+^{d'} \times \mathbb{R}^{d-d'}$, see Definition 1 of [39] :

1. $A, X_0 \in \mathbb{D}$, $\Sigma = I_d$.
2. $(K_{ij})_{1 \leq i \leq d', d'+1 \leq j \leq d} = 0$ and $K_{ij} \leq 0$ for $1 \leq i, j \leq d', i \neq j$.
3. For $1 \leq i \leq d'$, $\gamma_{ii} \geq 0$ and $\gamma_{ij} = 0$ for $j \neq i$.
4. For $d' + 1 \leq i \leq d$, $\gamma_{ij} \geq 0$ for $0 \leq j \leq d'$ and $\gamma_{ij} = 0$ for $d' + 1 \leq j \leq d$.

To make the link with the parametrization (4.5), this amounts to have $\sigma(x)\sigma(x)^\top = \alpha_0 + \sum_{i=1}^{d'} \alpha_i x_i$ with α_i is the diagonal matrix such that $(\alpha_i)_{jj} = \gamma_{ij}$ for $0 \leq i \leq d'$. Then, more general admissible affine diffusions can be obtained from these canonical affine processes by affine transformations, diffusion rescaling and Brownian rotation. We refer to Appendix A of [39] for further details.

4.3.2 A Potential Second Order Scheme

For simulation purposes, it therefore sufficient to be able to generate paths of the canonical affine processes that satisfy the four properties above. To do so, we calculate the infinitesimal generator of (4.37). It is given by

$$f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D}), Lf = L_A f + L_B f + L_C f, \text{ with} \tag{4.38}$$

$$L_A f = \sum_{i=1}^{d'} \left((A_i - K_{ii}x_i)\partial_i + \frac{\gamma_{ii}}{2} x_i \partial_i^2 \right), L_B f = -\sum_{i=1}^{d} \sum_{j=1}^{d} \tilde{K}_{ij} x_j \partial_i f,$$

$$L_C f = \sum_{i=d'+1}^{d} \left(A_i \partial_i f + \frac{1}{2}(\gamma_{i0} + \sum_{j=1}^{d'} \gamma_{ij} x_j) \partial_i^2 f \right),$$

where $\tilde{K}_{ij} = 0$ if $1 \leq i = j \leq d'$, and $\tilde{K}_{ij} = K_{ij}$ otherwise. We remark that L_A is the operator associated with d' independent CIR processes. Therefore, one gets from Proposition 3.3.5 a potential second-order scheme for L_A by taking d' independent samples. We may even get a third order scheme by using Proposition 3.3.8. We denote by $\hat{p}_x^A(t)$ a second order scheme for L_A. Let $\hat{p}_x^B(t)$ be the Dirac mass in $\exp(-\tilde{K}t)x$: this solves exactly the ODE associated to L_B. Last, the SDE associated to L_C can be solved also exactly and for $x = (x_1, \ldots, x_d)^\top$, we denote $\hat{p}_x^C(t)$ the law of $(x_1(t), \ldots, x_d(t))^\top$ with $x_i(t) = x_i$ for $i \leq d'$ and $x_i(t) = x_i + A_i t + \sqrt{\gamma_{i0} + \sum_{j=1}^{d'} \gamma_{ij} x_j} \times (W_t)_i$ for $i > d'$. We draw the attention on the fact that the

Algorithm 4.2: Potential second-order scheme for the Dai-Singleton model. Starting point x, time-step $t > 0$, $B \sim \mathcal{B}(1/2)$ and $N_i \sim \mathcal{N}(0,1)$. Independent samples.

Input: $x \in \mathbb{R}^d$, $A \in \mathbb{R}^d$, $K \in \mathcal{S}_d^+(\mathbb{R})$ and $t > 0$.
Output: $X \in \mathbb{R}^d$, sampled as described above.
$X = \exp(-\tilde{K}t/2)x$
if $(B = 1)$ **then**
  for $i = 1$ **to** d' **do**
    Use Algorithm 3.1 (or 3.3) with time step t starting from X_i with CIR parameters $(A_i, K_{ii}, \sqrt{\gamma_{ii}})$. Store this value in X_i.
  end
  for $i = d' + 1$ **to** d **do**
    $X_i = X_i + A_i t + \sqrt{\gamma_{i0} + \sum_{j=1}^{d'} \gamma_{ij} X_j} \sqrt{t} N_i,$
  end
else
  for $i = d' + 1$ **to** d **do**
    $X_i = X_i + A_i t + \sqrt{\gamma_{i0} + \sum_{j=1}^{d'} \gamma_{ij} X_j} \sqrt{t} N_i,$
  end
  for $i = 1$ **to** d' **do**
    Use Algorithm 3.1 (or 3.3) with time step t starting from X_i with CIR parameters $(A_i, K_{ii}, \sqrt{\gamma_{ii}})$. Store this value in X_i.
  end
end
$X = \exp(-\tilde{K}t/2)X$.

domain $\mathbb{D}$ is stable for the schemes $\hat{p}_x^A(t)$, $\hat{p}_x^B(t)$ and $\hat{p}_x^C(t)$ for any $t > 0$. We can thus compose them and from Corollary 2.3.14, we get the following result.

Proposition 4.3.1 *The scheme* $\frac{1}{2}\hat{p}^B(t/2) \circ \hat{p}^A(t) \circ \hat{p}^C(t) \circ \hat{p}_x^B(t/2) + \frac{1}{2}\hat{p}^B(t/2) \circ \hat{p}^C(t) \circ \hat{p}^A(t) \circ \hat{p}_x^B(t/2)$ *is a potential second-order scheme for the operator defined in* (4.38) *on* $\mathbb{D}$.

If we want to prove by using Theorem 2.3.8 that the scheme of Proposition 4.3.1 leads to a weak error of order 2, one would have to study the Kolmogorov equation related to the Affine Term Structure model. This issue has been tackled very recently in the Ph.D. thesis of Gabrielli [57].

Chapter 5
Wishart Processes and Affine Diffusions on Positive Semidefinite Matrices

Wishart processes have been first introduced by Bru [24] for some applications in biology on the perturbation of experimental data. Their definition and main mathematical properties are described in her paper [25]. They are also named because, as we will see, their marginal laws follow Wishart distributions. These distributions have been introduced by Wishart [124] in 1928. They arise naturally in statistics when estimating the covariance matrix of a Gaussian vector. Wishart processes belong to the class of affine processes. Recently, Cuchiero et al. [35] have introduced a general framework for affine processes on positive semidefinite matrices $\mathcal{S}_d^+(\mathbb{R})$ that embeds Wishart processes and includes possible jumps. In this chapter, we only consider affine diffusions on $\mathcal{S}_d^+(\mathbb{R})$ and exclude jumps. Namely, we consider the following SDE:

$$X_t^x = x + \int_0^t \left(\overline{\alpha} + B(X_s^x)\right) ds + \int_0^t \left(\sqrt{X_s^x} dW_s a + a^\top dW_s^\top \sqrt{X_s^x}\right). \tag{5.1}$$

Notations for matrices are recalled page xi. Here, and throughout the chapter, $(W_t, t \geq 0)$ denotes a d-by-d square matrix made of independent standard Brownian motions,

$$x, \bar{\alpha} \in \mathcal{S}_d^+(\mathbb{R}),\ a \in \mathcal{M}_d(\mathbb{R}) \text{ and } B \in \mathcal{L}(\mathcal{S}_d(\mathbb{R})) \tag{5.2}$$

is a linear mapping on $\mathcal{S}_d(\mathbb{R})$. Wishart processes correspond to the case where

$$\exists \alpha \geq 0, \bar{\alpha} = \alpha a^\top a \text{ and } \exists b \in \mathcal{M}_d(\mathbb{R}), \forall x \in \mathcal{S}_d(\mathbb{R}), B(x) = bx + xb^\top. \tag{5.3}$$

When $d = 1$, (5.1) is simply the SDE of the Cox-Ingersoll-Ross (CIR) process that has been studied in Chap. 1, and we implicitly assume that $d \geq 2$ throughout

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Chap. 5. The conditions under which the SDE (5.1) admits weak and strong solutions will be given in Sect. 5.1.

The chapter is structured as follows. First, we show the classical results on the existence and uniqueness of (5.1). We also present some basic calculations that are useful when doing stochastic calculus on matrices. Then, we focus on the calculation of the characteristic function and determine precisely for the case of Wishart processes its set of convergence. In the third section, we present different useful identities in laws that allow to make some simplification before tackling the simulation problem. In the next section, we give some recent applications of Wishart processes in financial modelling. In Sect. 5.5, we explain how it is possible to sample exactly Wishart processes by using a remarkable splitting of their infinitesimal generator. Section 5.6 is devoted to high order simulation schemes. We use the same splitting to construct second and third order schemes for Wishart processes and potential second order schemes for general affine diffusions on semidefinite positive matrices. All these results on the simulation have been obtained in Ahdida and Alfonsi [2], and this chapter takes back different parts of this paper.

5.1 Existence and Uniqueness Results

Weak and strong uniqueness of the SDE (5.1) has been studied by Bru [25], Cuchiero et al. [35] and Mayerhofer et al. [105]. The following theorem gathers these results.

Theorem 5.1.1 *If $x \in \mathcal{S}_d^+(\mathbb{R})$, $\bar{\alpha} - (d-1)a^\top a \in \mathcal{S}_d^+(\mathbb{R})$ and B satisfies the following condition*

$$\forall x_1, x_2 \in \mathcal{S}_d^+(\mathbb{R}), \;\; \mathrm{Tr}(x_1 x_2) = 0 \implies \mathrm{Tr}(B(x_1)x_2) \geq 0, \tag{5.4}$$

there is a unique weak solution to the SDE (5.1) in $\mathcal{S}_d^+(\mathbb{R})$. We denote by $AFF_d(x, \overline{\alpha}, B, a)$ the law of $(X_t^x)_{t\geq 0}$ and $AFF_d(x, \overline{\alpha}, B, a; t)$ the marginal law of X_t^x. If we assume moreover that $\bar{\alpha} - (d+1)a^\top a \in \mathcal{S}_d^+(\mathbb{R})$ and $x \in \mathcal{S}_d^{+,*}(\mathbb{R})$, there is a unique strong solution to the SDE (5.1).

Under the parametrization of Wishart processes (5.3), condition (5.4) is satisfied and weak uniqueness holds as soon as $\alpha \geq d-1$. In that case, we denote by $WIS_d(x, \alpha, b, a)$ the law of the Wishart process $(X_t^x)_{t\geq 0}$ and $WIS_d(x, \alpha, b, a; t)$ the law of X_t^x.

When using the notation $AFF_d(x, \overline{\alpha}, B, a)$ or $AFF_d(x, \overline{\alpha}, B, a; t)$ (resp. $WIS_d(x, \alpha, b, a)$ or $WIS_d(x, \alpha, b, a; t)$), we implicitly assume that $\bar{\alpha} - (d-1)a^\top a \in \mathcal{S}_d^+(\mathbb{R})$ (resp. $\alpha \geq d-1$) and B satisfies (5.4) so that weak uniqueness holds.

In the CIR case ($d = 1$), the condition $\alpha \geq d + 1 = 2$ is the Feller condition (1.28) that makes 0 unattainable. Strong uniqueness holds even when $\alpha \geq d - 1 = 0$, which we have obtained in Theorem 1.2.1 by using Yamada functions. Such argument is not so easy to extend, and the question of the strong uniqueness of (5.1) when $\bar{\alpha} - (d-1)a^\top a \in \mathcal{S}_d^+(\mathbb{R})$ but $\bar{\alpha} - (d+1)a^\top a \notin \mathcal{S}_d^+(\mathbb{R})$ is still open. Recently, Graczyk and Małecki [70] have tackled this issue and have shown the strong uniqueness of an SDE, whose solution has the same law as the Wishart process.

Let us first observe that the Wishart parametrization (5.3) satisfies (5.4). Let $x_1, x_2 \in \mathcal{S}_d^+(\mathbb{R})$ such that $\mathrm{Tr}(x_1x_2) = 0$. We have $\mathrm{Tr}(x_1x_2) = \mathrm{Tr}(\sqrt{x_2}\sqrt{x_1}\sqrt{x_1}\sqrt{x_2}) = 0$ and thus $\sqrt{x_2}\sqrt{x_1} = 0$ since $M \in \mathcal{M}_d(\mathbb{R}) \mapsto \sqrt{\mathrm{Tr}(MM^\top)}$ is the Euclidean norm. Multiplying on the left by $\sqrt{x_2}$ and on the right by $\sqrt{x_1}$, we get that

$$x_1, x_2 \in \mathcal{S}_d^+(\mathbb{R}),\ \mathrm{Tr}(x_1x_2) = 0 \iff x_1x_2 = 0. \tag{5.5}$$

Now, it remains to observe that $\mathrm{Tr}(bx_1x_2 + x_1b^\top x_2) = \mathrm{Tr}(bx_1x_2) + \mathrm{Tr}(x_2bx_1) = 2\mathrm{Tr}(bx_1x_2)$ to conclude.

The goal of this section is to prove Theorem 5.1.1. Before doing the proof, we first present some simple results about Itô calculus on matrices. They will be frequently used through this chapter and enables us to calculate the infinitesimal generator associated to affine processes on semidefinite positive matrices.

5.1.1 Itô Calculus on Matrices

Even if there is no particular difficulty, it is important to practice some Itô calculus on matrices in order to be familiar with Wishart processes and matrix-valued processes, see also Chap. 10 of Baldeaux and Platen [14]. We start with a simple lemma, that will be useful to calculate the infinitesimal generator of (5.1).

Lemma 5.1.2 *Let $(\mathcal{F}_t)_{t\geq 0}$ denote the filtration generated by $(W_t, t \geq 0)$. We consider continuous $(\mathcal{F}_t)$-adapted processes $(A_t)_{t\geq 0}$, $(B_t)_{t\geq 0}$ and $(C_t)_{t\geq 0}$ respectively valued in $\mathcal{M}_d(\mathbb{R})$, $\mathcal{M}_d(\mathbb{R})$ and $\mathcal{S}_d(\mathbb{R})$, and a process $(Y_t)_{t\geq 0}$ that admits the following semimartingale decomposition:*

$$dY_t = C_t dt + B_t dW_t A_t + A_t^\top dW_t^\top B_t^\top. \tag{5.6}$$

Then, for $i, j, m, n \in \{1, \dots, d\}$, the quadratic covariation of $(Y_t)_{i,j}$ and $(Y_t)_{m,n}$ is

$$\begin{aligned}\langle d(Y_t)_{i,j}, d(Y_t)_{m,n}\rangle = \big[&(B_tB_t^\top)_{i,m}(A_t^\top A_t)_{j,n} + (B_tB_t^\top)_{i,n}(A_t^\top A_t)_{j,m} \\ &+ (B_tB_t^\top)_{j,m}(A_t^\top A_t)_{i,n} + (B_tB_t^\top)_{j,n}(A_t^\top A_t)_{i,m}\big]\, dt.\end{aligned} \tag{5.7}$$

It is worth to notice that the quadratic covariation given by (5.6) depends on A_t and B_t only through the matrices $A_t^\top A_t$ and $B_tB_t^\top$.

Proof We first calculate explicitly the variations of the $(Y_t)_{i,j}$:

$$\begin{aligned}(dY_t)_{i,j} &= (C_t)_{i,j}dt + (B_t dW_t A_t + A_t^\top dW_t^\top B_t^\top)_{i,j} \\ &= (C_t)_{i,j}dt + \sum_{k,l=1}^d ((B_t)_{i,k}(A_t)_{l,j} + (B_t)_{j,k}(A_t)_{l,i})(dW_t)_{k,l}.\end{aligned}$$

Then, we calculate quadratic covariation

$$\begin{aligned}&\langle d(Y_t)_{i,j}, d(Y_t)_{m,n}\rangle \\ &= \sum_{k,l=1}^d ((B_t)_{i,k}(A_t)_{l,j} + (B_t)_{j,k}(A_t)_{l,i})((B_t)_{m,k}(A_t)_{l,n} + (B_t)_{n,k}(A_t)_{l,m})dt \\ &= \sum_{k,l=1}^d (B_t)_{i,k}(B_t)_{m,k}(A_t)_{l,n}(A_t)_{l,j} + \sum_{k,l=1}^d (B_t)_{i,k}(B_t)_{n,k}(A_t)_{l,m}(A_t)_{l,j} \\ &\quad + \sum_{k,l=1}^d (B_t)_{j,k}(B_t)_{m,k}(A_t)_{l,n}(A_t)_{l,i} + \sum_{k,l=1}^d (B_t)_{j,k}(B_t)_{n,k}(A_t)_{l,m}(A_t)_{l,i} \\ &= (B_t B_t^\top)_{i,m}(A_t^\top A_t)_{n,j} + (B_t B_t^\top)_{i,n}(A_t^\top A_t)_{m,j} \\ &\quad + (B_t B_t^\top)_{j,m}(A_t^\top A_t)_{n,i} + (B_t B_t^\top)_{j,n}(A_t^\top A_t)_{m,i}.\end{aligned}$$

□

In particular, when Y is given by (5.6), we have

$$\begin{aligned}d\mathrm{Tr}(Y_t) &= \mathrm{Tr}(C_t)dt + 2\mathrm{Tr}(A_t^\top B_t dW_t), \\ \langle d\mathrm{Tr}(Y_t)\rangle &= \sum_{i,j=1}^d d\langle (Y_t)_{i,i}, (Y_t)_{j,j}\rangle = 4\sum_{i,j=1}^d (B_t B_t^\top)_{i,j}(A_t^\top A_t)_{i,j} \\ &= 4\mathrm{Tr}(B_t B_t^\top A_t^\top A_t).\end{aligned} \tag{5.8}$$

This calculation enables us to consider changes of probability. Let us take $A_t = \frac{1}{2}I_d$ and $C_t = 0$ and define

$$\mathcal{E}_t = \exp\left(\int_0^t \mathrm{Tr}(B_s dW_s) - \frac{1}{2}\int_0^t \mathrm{Tr}(B_s B_s^\top)ds\right),$$

which is the Doléans-Dade exponential associated to the martingale $M_t = \int_0^t \mathrm{Tr}(B_s dW_s)$. Since $dM_t = \sum_{i,j=1}^d (B_t)_{j,i} d(W_t)_{i,j}$, we have $\langle d(W_t)_{i,j}, dM_t\rangle = (B_t)_{j,i}dt$. If $\mathbb{E}[\mathcal{E}_T] = 1$ for some $T > 0$, then Girsanov's theorem ([83], Theorem 5.1, p. 191) ensures that

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_T$$

defines a change of probability under which

$$\tilde{W}_t = W_t - \int_0^t B_s^\top ds, \; t \in [0, T],$$

is a Brownian motion.

Remark 5.1.3 When $C \equiv 0$, we easily get from Lemma 5.1.2 that

$$\mathrm{Tr}(Y_T^2) - 2\int_0^T \mathrm{Tr}(B_t B_t^\top)\mathrm{Tr}(A_t^\top A_t) + \mathrm{Tr}(A_t^\top A_t B_t B_t^\top) dt$$

is a martingale, since $\mathrm{Tr}(Y_T^2) = \sum_{i,j=1}^d (Y_T)_{i,j}^2$. In particular, we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \mathrm{Tr}((B_t dW_t A_t + A_t^\top dW_t^\top B_t^\top)^2)\right] \\
&= 2\int_0^T \mathbb{E}[\mathrm{Tr}(B_t B_t^\top)\mathrm{Tr}(A_t^\top A_t)] + \mathbb{E}[\mathrm{Tr}(A_t^\top A_t B_t B_t^\top)] dt
\end{aligned}$$

when the right hand side is finite.

5.1.2 The Infinitesimal Generator on $\mathcal{M}_d(\mathbb{R})$ and $\mathcal{S}_d(\mathbb{R})$

Lemma 5.1.2 enables us to calculate easily the infinitesimal generator for the affine process (5.1) which is defined by:

$$\begin{aligned}
&x \in \mathcal{S}_d^+(\mathbb{R}), \\
&L^{\mathcal{M}} f(x) = \lim_{t \to 0^+} \frac{\mathbb{E}[f(X_t^x)] - f(x)}{t} \text{ for } f \in \mathcal{C}^2(\mathcal{M}_d(\mathbb{R}), \mathbb{R}) \\
&\qquad\qquad \text{with bounded derivatives.}
\end{aligned}$$

In fact, we get that the generator of $AFF_d(x, \overline{\alpha}, B, a)$ is given by:

$$
\begin{aligned}
L^{\mathcal{M}} f(x) = & \sum_{i,j=1}^{d} [\overline{\alpha}_{i,j} + (B(x))_{i,j}]\partial_{i,j} f(x) \\
& + \sum_{i,j,m,n=1}^{d} [x_{i,m}(a^\top a)_{j,n} + x_{i,n}(a^\top a)_{j,m} + x_{j,m}(a^\top a)_{i,n} \\
& + x_{j,n}(a^\top a)_{i,m}]\partial_{i,j}\partial_{m,n} f(x).
\end{aligned}
$$

This operator can be written in a more concise manner as follows

$$
\begin{aligned}
L^{\mathcal{M}} = & \operatorname{Tr}([\overline{\alpha} + B(x)]D^{\mathcal{M}}) + \frac{1}{2}\{2\operatorname{Tr}(xD^{\mathcal{M}}a^\top aD^{\mathcal{M}}) \\
& + \operatorname{Tr}(x(D^{\mathcal{M}})^\top a^\top aD^{\mathcal{M}}) + \operatorname{Tr}(xD^{\mathcal{M}}a^\top a(D^{\mathcal{M}})^\top)\}, \qquad (5.9)
\end{aligned}
$$

where $D^{\mathcal{M}} = (\partial_{i,j})_{1\le i,j\le d}$. Since we know that the affine process $(X_t^x)_{t\ge 0}$ takes values in $\mathcal{S}_d^+(\mathbb{R}) \subset \mathcal{S}_d(\mathbb{R})$, we can also look at the infinitesimal generator of this diffusion on $\mathcal{S}_d(\mathbb{R})$, which is defined by:

$x \in \mathcal{S}_d^+(\mathbb{R})$,

$L^{\mathcal{S}} f(x) = \lim_{t\to 0^+} \frac{\mathbb{E}[f(X_t^x)] - f(x)}{t}$ for $f \in \mathcal{C}^2(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$ with bounded derivatives.

For $x \in \mathcal{S}_d(\mathbb{R})$, we denote by $x_{\{i,j\}} = x_{i,j} = x_{j,i}$ the value of the coordinates (i, j) and (j, i), so that $x = \sum_{1\le i\le j\le d} x_{\{i,j\}}(e_d^{i,j} + \mathbb{1}_{i\ne j} e_d^{j,i})$. For $f \in \mathcal{C}^2(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$, we then denote by $\partial_{\{i,j\}} f$ its derivative with respect to $x_{\{i,j\}}$. For $x \in \mathcal{M}_d(\mathbb{R})$, we set $\pi(x) = (x + x^\top)/2$. It is such that $\pi(x) = x$ for $x \in \mathcal{S}_d(\mathbb{R})$, and we have:

$$L^{\mathcal{S}} f(x) = L^{\mathcal{M}} f \circ \pi(x).$$

By the chain rule, we have for $x \in \mathcal{S}_d(\mathbb{R})$, $\partial_{i,j} f \circ \pi(x) = (\mathbb{1}_{i=j} + \frac{1}{2}\mathbb{1}_{i\ne j})\partial_{\{i,j\}} f(x)$ and get from (5.9) the following result.

Proposition 5.1.4 *The infinitesimal generator on $\mathcal{S}_d(\mathbb{R})$ associated to AFF_d $(x, \overline{\alpha}, B, a)$ is given by:*

$$L^{\mathcal{S}} = \operatorname{Tr}([\overline{\alpha} + B(x)]D^{\mathcal{S}}) + 2\operatorname{Tr}(xD^{\mathcal{S}}a^\top aD^{\mathcal{S}}), \qquad (5.10)$$

where $D^{\mathcal{S}}$ is defined by $D^{\mathcal{S}}_{i,j} = (\mathbb{1}_{i=j} + \frac{1}{2}\mathbb{1}_{i\ne j})\partial_{\{i,j\}}$, for $1 \le i, j \le d$.

Of course, the generators $L^{\mathcal{M}}$ and $L^{\mathcal{S}}$ are equivalent: one can be deduced from the other. However, $L^{\mathcal{S}}$ already embeds the fact that the process lies in $\mathcal{S}_d(\mathbb{R})$, which reduces the dimension from d^2 to $d(d+1)/2$ and gives in practice shorter formulas. This is why we will mostly work in this chapter with infinitesimal generators on $\mathcal{S}_d(\mathbb{R})$. Unless it is necessary to make the distinction with $L^{\mathcal{M}}$, we will simply denote $L = L^{\mathcal{S}}$.

5.1.3 Strong Existence and Uniqueness Results

The goal of this section is to prove the results of Theorem 5.1.1 on the strong existence and uniqueness. The results on the weak existence and uniqueness are made in Sect. 5.1.4 in a constructive manner. For $y \in \mathcal{S}_d(\mathbb{R})$, we define $y^+ \in \mathcal{S}_d^+(\mathbb{R})$ by

$$y^+ = odiag(\lambda_1^+, \dots, \lambda_d^+)o^\top, \tag{5.11}$$

where o is an orthogonal matrix such that $y = odiag(\lambda_1, \dots, \lambda_d)o^\top$. We can check easily that this choice does not depend on o and that the mapping $y \in \mathcal{S}_d(\mathbb{R}) \mapsto y^+ \in \mathcal{S}_d^+(\mathbb{R})$ is Lipschitz continuous, see Lemma A.1.3. Following Bru [25], we consider a weak solution of the following SDE

$$X_t^x = x + \int_0^t \left(\overline{\alpha} + B(X_s^x)\right) ds + \int_0^t \left(\sqrt{(X_s^x)^+} dW_s a + a^\top dW_s^\top \sqrt{(X_s^x)^+}\right), \tag{5.12}$$

which is possible by using Ikeda and Watanabe [78], Theorems 2.3, p. 173 and 2.4, p. 177. The matrix square-root is locally Lipschitz on $\mathcal{S}_d^{+,*}(\mathbb{R})$, see for example Proposition 2.1 of van Hemmen and Ando [121] or Eq. (12.13), p. 134 of Rogers and Williams [114] for the analytic property. Therefore, when $x \in \mathcal{S}_d^+(\mathbb{R})$, there is a unique strong solution up to

$$\tau = \inf\{t \geq 0, X_t^x \notin \mathcal{S}_d^{+,*}(\mathbb{R})\},$$

with the standard convention $\inf \emptyset = +\infty$. We want to show that $\tau = +\infty$ almost surely. To do so, we focus on the determinant of X_t^x. First, let us recall that $\forall i, j, m, n \in \{1, \dots, d\}, \ \forall x \in \mathcal{S}_d(\mathbb{R}) \cap \mathcal{G}_d(\mathbb{R})$,

$$\begin{aligned} \partial_{i,j} \det(x) &= (\mathrm{adj}(x))_{i,j} = \det(x) x_{i,j}^{-1}, \ \partial_{m,n}\partial_{i,j}(\det(x)) \\ &= \det(x)(x_{n,m}^{-1} x_{i,j}^{-1} - x_{j,n}^{-1} x_{i,m}^{-1}). \end{aligned} \tag{5.13}$$

We obtain for $t \in [0, \tau)$,

$$\begin{aligned} d(\det(X_t^x)) = \det(X_t^x) \Big[&\mathrm{Tr}\left((X_t^x)^{-1} dX_t^x\right) \\ &+ \frac{1}{2} \sum_{1 \leq i,j,m,n \leq d} [(X_t^x)_{m,n}^{-1} (X_t^x)_{i,j}^{-1} \\ &- (X_t^x)_{n,j}^{-1} (X_t^x)_{i,m}^{-1}] \langle d(X_t^x)_{i,j}, d(X_t^x)_{m,n} \rangle \Big]. \end{aligned}$$

From (5.7), we have

$$\langle d(X_t^x)_{i,j}, d(X_t^x)_{m,n}\rangle = \big[(X_t^x)_{i,m}(a^\top a)_{j,n} + (X_t^x)_{i,n}(a^\top a)_{j,m} + (X_t^x)_{j,m}(a^\top a)_{i,n} + (X_t^x)_{j,n}(a^\top a)_{i,m}\big]\,dt.$$

Since

$$\begin{aligned}
\sum_{1\le i,j,m,n\le d} (X_t^x)^{-1}_{m,n}(X_t^x)^{-1}_{i,j}(X_t^x)_{i,m}(a^\top a)_{j,n} &= \sum_{1\le j,m\le d}\left((X_t^x)^{-1}a^\top a\right)_{m,j}\mathbb{1}_{j=m}\\
&= \mathrm{Tr}((X_t^x)^{-1}a^\top a),\\
\sum_{1\le i,j,m,n\le d} (X_t^x)^{-1}_{m,n}(X_t^x)^{-1}_{i,j}(X_t^x)_{i,n}(a^\top a)_{j,m} &= \sum_{1\le j,n\le d}\left((X_t^x)^{-1}a^\top a\right)_{n,j}\mathbb{1}_{j=n}\\
&= \mathrm{Tr}((X_t^x)^{-1}a^\top a),\\
\sum_{1\le i,j,m,n\le d} (X_t^x)^{-1}_{n,j}(X_t^x)^{-1}_{i,m}(X_t^x)_{i,m}(a^\top a)_{j,n} &= \sum_{1\le j,m\le d}\left((X_t^x)^{-1}a^\top a\right)_{j,j}\\
&= d\,\mathrm{Tr}((X_t^x)^{-1}a^\top a),\\
\sum_{1\le i,j,m,n\le d} (X_t^x)^{-1}_{n,j}(X_t^x)^{-1}_{i,m}(X_t^x)_{i,n}(a^\top a)_{j,m} &= \sum_{1\le m,n\le d}\left((X_t^x)^{-1}a^\top a\right)_{n,m}\mathbb{1}_{m=n}\\
&= \mathrm{Tr}((X_t^x)^{-1}a^\top a),
\end{aligned}$$

we get that

$$\sum_{1\le i,j,m,n\le d}[(X_t^x)^{-1}_{m,n}(X_t^x)^{-1}_{i,j} - (X_t^x)^{-1}_{n,j}(X_t^x)^{-1}_{i,m}]\langle d(X_t^x)_{i,j}, d(X_t^x)_{m,n}\rangle$$
$$= 4\mathrm{Tr}((X_t^x)^{-1}a^\top a) - (2d+2)\mathrm{Tr}((X_t^x)^{-1}a^\top a) = 2(1-d)\mathrm{Tr}((X_t^x)^{-1}a^\top a).$$

This leads to

$$d(\det(X_t^x)) = \det(X_t^x)\Big[\mathrm{Tr}\left((X_t^x)^{-1}\left(\overline{\alpha} + (1-d)a^\top a + B(X_t^x)\right)\right)dt + 2\mathrm{Tr}\left((\sqrt{X_t^x})^{-1}dW_t a\right)\Big],$$

and thus

$$d\log(\det(X_t^x)) = \mathrm{Tr}\left((X_t^x)^{-1}\left(\overline{\alpha} - (d+1)a^\top a + B(X_t^x)\right)\right)dt + 2\mathrm{Tr}\left((\sqrt{X_t^x})^{-1}dW_t a\right),$$

by using the calculation made in (5.8).

We now define $Y_t = \log(\det(X_t^x)) - \int_0^t \mathrm{Tr}\left((X_s^x)^{-1}B(X_s^x)\right) ds$. By Lemma A.1.2, we obtain for $t \in [0, \tau)$

$$Y_t \geq Y_0 + 2\int_0^t \mathrm{Tr}\left((\sqrt{X_s^x})^{-1} dW_s a\right).$$

In the Wishart parametrization (5.3), we have $Y_t = \log(\det(X_t^x)) - 2\mathrm{Tr}(b)t$ and therefore $Y_t \underset{t\to\tau}{\to} -\infty$ on $\{\tau < \infty\}$. We now use the McKean argument: $\int_0^t \mathrm{Tr}\left((\sqrt{X_s^x})^{-1} dW_s a\right)$ is a local martingale and thus a time changed Brownian motion. It cannot go to $-\infty$ without oscillating, and we necessarily have $\tau = +\infty$ almost surely. In the general case where B is a linear map on $\mathcal{S}_d(\mathbb{R})$ satisfying (5.4), we need the following exercise to conclude.

Exercise 5.1.5 Let $x \in \mathcal{S}_d^{+,*}(\mathbb{R})$. There is an orthogonal matrix $o \in \mathcal{O}_d(\mathbb{R})$ and a diagonal matrix $\delta = \sum_{i=1}^d \delta_i e_d^i$ with $\delta_1, \ldots, \delta_d > 0$ such that $x = o\delta o^\top$.

1. Show that $\mathrm{Tr}(x^{-1}B(x)) = \sum_{i,j=1}^d \frac{\delta_j}{\delta_i}\gamma(i,j,o)$, with $\gamma(i,j,o) = \mathrm{Tr}(oe_d^i o^\top B(oe_d^j o^\top))$.
2. By using (5.4), show that $\mathrm{Tr}(x^{-1}B(x)) \geq \min_{o\in\mathcal{O}_d(\mathbb{R})} \sum_{i=1}^d \gamma(i,i,o) =: m > -\infty$.
3. Deduce that $Y_t \leq \log(\det(X_t^x)) - mt$ and conclude.

5.1.4 Weak Existence and Uniqueness

The weak uniqueness is a direct consequence of the results on the characteristic function given in Proposition 5.2.2. The Laplace transform of $(X_{t_1}^x, \ldots, X_{t_n}^x)$ given by

$$\mathbb{E}\left[\exp\left(\sum_{i=1}^n \mathrm{Tr}(v_i X_{t_i}^x)\right)\right],$$

for $-v_1, \ldots, -v_n \in \mathcal{S}_d^+(\mathbb{R})$ is uniquely determined. Thus, there is a unique possible law for the solution of (5.1), but it remains to show that this law exists. The proof that we give here is a constructive one. We first need to characterize the linear ODEs that stay in the domain $\mathcal{S}_d^+(\mathbb{R})$.

Lemma 5.1.6 *Let $a \in \mathcal{S}_d^+(\mathbb{R})$ and consider the linear ODE $x'(t) = a + B(x(t))$ starting from $x(0) = x_0 \in \mathcal{S}_d^+(\mathbb{R})$. Then, for any $t \geq 0$, $x(t) \in \mathcal{S}_d^+(\mathbb{R})$.*

Proof By using the continuity of the solution with respect to the initial condition, it is sufficient to prove that $\forall t \geq 0$, $x(t) \in \mathcal{S}_d^{+,*}(\mathbb{R})$ if $x_0 \in \mathcal{S}_d^{+,*}(\mathbb{R})$. Let us assume now $x_0 \in \mathcal{S}_d^{+,*}(\mathbb{R})$. Let $\tilde{t} = \inf\{t \geq 0, \det(x(t)) = 0\}$. By (5.13), we get

$\partial_t \det(x(t)) = \det(x(t))\mathrm{Tr}[x(t)^{-1}a + B(x(t))]$ on $t \in [0, \tilde{t})$, and therefore

$$\det(x(t)) = \det(x_0) \exp\left(\int_0^t \mathrm{Tr}[x(s)^{-1}\{a + B(x(s))\}]ds\right).$$

We have $x(t) \in \mathcal{S}_d^{+,*}(\mathbb{R})$ for any $t \in [0, \tilde{t})$. Therefore, we get from Lemma A.1.2 and Exercise 5.1.5 that $\det(x(t)) \geq \det(x_0)\exp(mt)$ on $t \in [0, \tilde{t})$ for some $m > -\infty$. We thus necessarily have $\tilde{t} = +\infty$, which concludes the proof. □

Remark 5.1.7 Condition (5.4) is the necessary and sufficient one to have that the solution of the linear differential equation

$$x(t) = a + B(x(t)),\ x(0) = x_0, \tag{5.14}$$

satisfies

$$\forall t \geq 0,\ x(t) \in \mathcal{S}_d^{+}(\mathbb{R})$$

for any $x_0, a \in \mathcal{S}_d^+(\mathbb{R})$. In fact, let us assume that there are $x_1, x_2 \in \mathcal{S}_d^+(\mathbb{R})$ such that $\mathrm{Tr}(x_1 x_2) = 0$ and $\mathrm{Tr}(B(x_1)x_2) < 0$. We note that x_1 and x_2 cannot be equal to zero and therefore cannot be also invertible by using (5.5). We consider the solution of (5.14) with $x_0 = x_1$. There is an orthogonal matrix o such that $x_1 = o^\top diag(0, \ldots, 0, \delta_{i_1+1}, \ldots, \delta_d)o$ for some $i_1 \in \{2, \ldots, d\}$ and $\delta_{i_1+1}, \ldots, \delta_n > 0$. We have

$$0 = \mathrm{Tr}(x_1 x_2) = \mathrm{Tr}(diag(0, \ldots, 0, \delta_{i_1+1}, \ldots, \delta_d)ox_2o^\top) = \sum_{i=i_1+1}^{d} \delta_i (ox_2o^\top)_{i,i},$$

which yields to $(ox_2o^\top)_{i,i} = 0$ for $i > i_1$. Since $ox_2o^\top \in \mathcal{S}_d^+(\mathbb{R})$, we deduce that $(ox_2o^\top)_{i,j} = 0$ when $i > i_1$ or $j > i_1$. We now use the following expansion

$$ox(\varepsilon)o^\top \underset{\varepsilon\to 0}{=} diag(0, \ldots, 0, \delta_{i_1+1}, \ldots, \delta_d) + \varepsilon oB(x_1)o^\top + O(\varepsilon^2).$$

Since $\mathrm{Tr}(oB(x_1)o^\top ox_2o^\top) = \mathrm{Tr}(B(x_1)x_2) < 0$, we get by Lemma A.1.2 that $(oB(x_1)o^\top)_{1\leq i,j\leq i_1}$ is not semidefinite positive. Therefore, $(ox(\varepsilon)o)_{1\leq i,j\leq i_1}$ is not semidefinite positive for ε small enough, which gives that $x(\varepsilon) \notin \mathcal{S}_d^+(\mathbb{R})$.

The existence of Wishart processes with $\alpha = d - 1$ is a direct consequence of Proposition 5.3.4: in this case, it is simply obtained from a matrix-valued Ornstein-Uhlenbeck process. Let us consider now $\overline{\alpha}$ such that $\overline{\alpha} - (d-1)a^\top a \in \mathcal{S}_d^+(\mathbb{R})$ and $B \in \mathcal{L}(\mathcal{S}_d(\mathbb{R}))$ satisfying (5.4). We denote by $\xi(t, x)$ the solution of the linear equation

$$\xi'(t, x) = \overline{\alpha} - (d-1)a^\top a + B(\xi(t, x)),\ \xi(0, x) = x,$$

and we know by Lemma 5.1.6 that $\xi(t,x) \in \mathcal{S}_d^+(\mathbb{R})$ if $x \in \mathcal{S}_d^+(\mathbb{R})$ and $t \geq 0$. Let $N \in \mathbb{N}^*$, $T > 0$ and $t_i^N = iT/N$. We consider the following approximation $(\hat{X}_t^N, t \in [0,T])$.

- We set $\hat{X}_0^N = x \in \mathcal{S}_d^+(\mathbb{R})$.
- For $t \in [t_i, t_{i+1}]$, we consider a (weak) solution $\bar{X}^N$ of the following SDE

$$\bar{X}_t^N = \xi(T/N, \hat{X}_{t_i}^N) + (d-1)a^\top a(t-t_i) + \int_{t_i}^t \left(\sqrt{\bar{X}_s^N} dW_s a + a^\top dW_s^\top \sqrt{\bar{X}_s^N}\right),$$

  and set $\hat{X}_{t_{i+1}}^N = \bar{X}_{t_{i+1}}^N$.
- Last, we set $\hat{X}_t^N = \frac{t-t_i}{T/N}\hat{X}_{t_{i+1}}^N + \frac{t_{i+1}-t}{T/N}\hat{X}_{t_i}^N$ for $t \in [t_i, t_{i+1}]$.

We then proceed in a classical way. First we prove that the sequence $\hat{X}^N$ is tight by using the Kolmogorov criterion. Then, we show that any limit in law of sub-sequences of $\hat{X}^N$ solve the martingale problem associated to the process (Lemma 5.1.6), which gives the weak existence of Lemma 5.1.6. Since we will make an analogous proof in detail for the mean-reverting correlation process (see Sect. 6.4), we leave the completion of this proof as an exercise for the reader.

5.2 The Characteristic Function

As for any multidimensional affine process, the characteristic function of affine diffusions on positive semidefinite matrices can be obtained by solving ODEs. To be more precise, we will prove that $\mathbb{E}[\exp(\mathrm{Tr}(vX_T^x))] = \exp(\phi_v(T) + \mathrm{Tr}(\psi_v(T)x))$ for some $v \in \mathcal{S}_d(\mathbb{R})$, where the functions ϕ_v and ψ_v solve a system of ODEs. The difficulty is that this differential equation may explode in finite time, in which case $\exp(\mathrm{Tr}(vX_T^x))$ is not integrable when T is larger than the explosion time. However, this never happens when $-v \in \mathcal{S}_d^+(\mathbb{R})$ since $X_T^x \in \mathcal{S}_d^+(\mathbb{R})$ and $\mathrm{Tr}(vX_T^x) \leq 0$. Then, one would like to describe more precisely the set of convergence $\{v \in \mathcal{S}_d(\mathbb{R}), \mathbb{E}[\exp(\mathrm{Tr}(vX_T^x))] < \infty\}$ as we have made in dimension 1 for the CIR process, see Proposition 1.2.4. This is done in the particular case of Wishart processes where ϕ_v and ψ_v can be calculated explicitly.

We first focus on general affine diffusions on positive semidefinite matrices and consider v such that $-v \in \mathcal{S}_d^+(\mathbb{R})$, so that $\mathrm{Tr}(vX_t^x) \leq 0$ by Lemma A.1.2. We therefore get $\mathbb{E}[\exp(\mathrm{Tr}(vX_T^x))] \leq 1 < \infty$ for any $T \geq 0$. For a fixed $T > 0$, we consider the martingale $M_t = \mathbb{E}[\exp(\mathrm{Tr}(vX_T^x))|\mathcal{F}_t]$ for $t \in [0,T]$, where $\mathcal{F}_t = \sigma(W_s, s \in [0,t])$, and assume that there are smooth functions $\phi_v : \mathcal{S}_d(\mathbb{R}) \to \mathbb{R}$ and $\psi_v : \mathcal{S}_d(\mathbb{R}) \to \mathcal{S}_d(\mathbb{R})$ such that

$$M_t = \exp(\phi_v(T-t) + \mathrm{Tr}(\psi_v(T-t)X_t^x)).$$

By applying Itô's formula, we get that the drift part of $\frac{dM_t}{M_t}$ is

$$-\phi_v'(T-t) - \mathrm{Tr}(\psi_v'(T-t)X_t^x) + \mathrm{Tr}(\psi_v(T-t)[\overline{\alpha} + B(X_t^x)]) \\ +2\mathrm{Tr}(\psi_v(T-t)a^\top a\psi_v(T-t)X_t^x).$$

Since $B : \mathcal{S}_d(\mathbb{R}) \to \mathcal{S}_d(\mathbb{R})$ is linear, there is an adjoint linear application $B^* : \mathcal{S}_d(\mathbb{R}) \to \mathcal{S}_d(\mathbb{R})$ such that

$$\forall x, y \in \mathcal{S}_d(\mathbb{R}),\ \mathrm{Tr}(xB(y)) = \mathrm{Tr}(B^*(x)y),$$

and therefore $\mathrm{Tr}(\psi_v(T-t)B(X_t^x)) = \mathrm{Tr}(B^*(\psi_v(T-t))X_t^x)$. Since M is a martingale, the drift term of $\frac{dM_t}{M_t}$ should vanish and we get by identifying the constant term and the one in X_t^x:

$$\begin{aligned} \psi_v'(t) &= B^*(\psi_v(t)) + 2\psi_v(t)a^\top a\psi_v(t)\ ;\ \psi_v(0) = v \\ \phi_v'(t) &= \mathrm{Tr}(\overline{\alpha}\psi_v(t)) \qquad\qquad\qquad ;\ \phi_v(0) = 0. \end{aligned} \tag{5.15}$$

Lemma 5.2.1 *Let $v \in \mathcal{S}_d(\mathbb{R})$ such that $-v \in \mathcal{S}_d^+(\mathbb{R})$. Then, the differential equation (5.15) has a unique solution defined for $t \in \mathbb{R}_+$ that satisfies $-\psi_v(t) \in \mathcal{S}_d^+(\mathbb{R})$ for any $t \geq 0$. Besides, ϕ_v is nonincreasing and thus nonpositive.*

Proof Let $t_e = \inf\{t \geq 0, \mathrm{Tr}(\psi_v(t)^2) < \infty\}$ denote the explosion time of the ODE on ψ_v. By the Cauchy-Lipschitz theorem, we know that there is a unique solution ψ_v up to t_e. Then, ϕ_v is clearly defined on $[0, t_e)$. Suppose that we are able to show that for any $t \in [0, t_e)$, $-\psi_v(t) \in \mathcal{S}_d^+(\mathbb{R})$. Then, we would have

$$\partial_t \mathrm{Tr}(\psi_v(t)^2) = \mathrm{Tr}(\psi_v(t)B(\psi_v(t))) + 2\mathrm{Tr}(\psi_v(t)^3 a^\top a) \leq C\mathrm{Tr}(\psi_v(t)^2),$$

by using that $-\psi_v(t)^3 \in \mathcal{S}_d^+(\mathbb{R})$, Lemma A.1.2 and the continuity of B, see Eq. (A.2). This immediately gives that $\mathrm{Tr}(\psi_v(t)^2) \leq \mathrm{Tr}(v^2)e^{Ct}$ by Gronwall's lemma for any $t \in [0, t_e)$. This necessarily implies $t_e = +\infty$.

Thus, it is sufficient to prove that $-\psi_v(t) \in \mathcal{S}_d^+(\mathbb{R})$. Using the continuity with respect to the initial condition, we are going to show that $-\psi_v(t) \in \mathcal{S}_d^{+,*}(\mathbb{R})$ when $-v \in \mathcal{S}_d^{+,*}(\mathbb{R})$. Let $\tilde{t} = \inf\{t \geq 0, \det(\psi_v(t)) = 0\}$. Clearly, $-\psi_v(t) \in \mathcal{S}_d^+(\mathbb{R})$ on $[0, \tilde{t})$ and we have $t_e \geq \tilde{t}$. By (5.13), we get for $t \in [0, \tilde{t})$

$$\partial_t \det(\psi_v(t)) = \det(\psi_v(t))[\mathrm{Tr}(\psi_v(t)^{-1}B(\psi_v(t))) + \mathrm{Tr}(\psi_v(t)a^\top a)].$$

This yields to

$$\det(\psi_v(t)) = \det(v)\exp\left(\int_0^t \mathrm{Tr}(\psi_v(s)^{-1}B(\psi_v(s))) + \mathrm{Tr}(\psi_v(s)a^\top a)ds\right).$$

The right-hand-side can vanish in $\tilde{t}$ only if the integral goes to $-\infty$. By using Exercise 5.1.5, we get that there is $m > -\infty$ such that $\mathrm{Tr}(\psi_v(s)^{-1}B(\psi_v(s)) \geq m$. Therefore, we necessarily have $\tilde{t} = +\infty$, which proves the claim on ψ_v. Then, Lemma A.1.2 gives the monotonicity property of ϕ_v. □

Now, it remains to check that we indeed have

$$\mathbb{E}[\exp(\mathrm{Tr}(vX_T^x))] = \exp(\phi_v(T) + \mathrm{Tr}(\psi_v(T)x)) \tag{5.16}$$

for $T > 0$ and v such that $-v \in \mathcal{S}_d^+(\mathbb{R})$, where ϕ_v and ψ_v are the solutions of (5.15). We proceed as for the CIR process (Proposition 1.2.4) and consider the process $M_t = \exp(\phi_v(T-t) + \mathrm{Tr}(\psi_v(T-t)X_t^x))$ for $t \in [0,T]$. By Itô's formula, we clearly have $dM_t = M_t\mathrm{Tr}(\psi_v(T-t)[\sqrt{X_t^x}dW_t a + a^\top dW_t^\top \sqrt{X_t^x}])$. This is a true martingale since $0 \leq M_t \leq 1$. This yields to $M_0 = \mathbb{E}[M_T]$, which is precisely (5.16). We eventually get the following result.

Proposition 5.2.2 *Let $X_t^x \sim AFF_d(x, \overline{\alpha}, B, a; t)$ and $v \in \mathcal{S}_d(\mathbb{R})$ such that $-v \in \mathcal{S}_d^+(\mathbb{R})$. Then,* (5.16) *holds, where the functions ϕ_v and ψ_v are the solution of* (5.15)*.*

In the case of Wishart processes, it is possible to solve explicitly the ODEs (5.15) by solving a matrix Riccati equation (see Appendix A.4). We give here the closed formula for the Laplace transform. This enables us also to give a more precise description of the set of convergence.

Proposition 5.2.3 *Let $X_t^x \sim WIS_d(x, \alpha, b, a; t)$, $q_t = \int_0^t \exp(sb)a^\top a \exp(sb^\top)ds$ and $m_t = \exp(tb)$. We introduce the set of convergence of the Laplace transform of X_t^x, $\mathcal{D}_{b,a;t} = \{v \in \mathcal{S}_d(\mathbb{R}), \mathbb{E}[\exp(\mathrm{Tr}(vX_t^x))] < \infty\}$. This is a convex open set that is given explicitly by*

$$\mathcal{D}_{b,a;t} = \{v \in \mathcal{S}_d(\mathbb{R}), \forall s \in [0,t], I_d - 2q_s v \in \mathcal{G}_d(\mathbb{R})\}. \tag{5.17}$$

Besides, the Laplace transform of X_t^x is well-defined for $v = v_R + iv_I$ with $v_R \in \mathcal{D}_{b,a;t}$, $v_I \in \mathcal{S}_d(\mathbb{R})$ and is given by:

$$\mathbb{E}[\exp(\mathrm{Tr}(vX_t^x))] = \frac{\exp(\mathrm{Tr}[v(I_d - 2q_t v)^{-1}m_t x m_t^\top])}{\det(I_d - 2q_t v)^{\frac{\alpha}{2}}}. \tag{5.18}$$

The characteristic function corresponds to the case $v_R = 0$ that clearly belongs to $\mathcal{D}_{b,a;t}$, see Lemma A.1.1. Let us observe also that $\rho I_d \in \mathcal{D}_{b,a;t}$ when $\rho > 0$ is small enough. This will help us to study the Cauchy problem related to the Wishart process (Proposition 5.6.2). In the case where $\tilde{X}_t^x \sim WIS_d(x, \alpha, 0, I_d^n; t)$, the formula (5.18) becomes even simpler and we have for $v = v_R + iv_I$ such that $v_R \in \mathcal{D}_{b,a;t}$, $v_I \in \mathcal{S}_d(\mathbb{R})$:

$$\mathbb{E}[\exp(\mathrm{Tr}(v\tilde{X}_t^x))] = \frac{\exp(\mathrm{Tr}[v(I_d - 2tI_d^n v)^{-1}x])}{\det(I_d - 2tI_d^n v)^{\frac{\alpha}{2}}}. \tag{5.19}$$

Proof Let $v \in \mathcal{S}_d(\mathbb{R})$ such that $\forall s \in [0,t]$, $I_d - 2q_s v \in \mathcal{G}_d(\mathbb{R})$. In the case of the Wishart process, the ODE (5.15) can be written as follows:

$$\begin{aligned} \psi_v'(t) &= \psi_v(t)b + b^\top \psi_v(t) + 2\psi_v(t)a^\top a \psi_v(t) \;;\; \psi_v(0) = v \\ \phi_v'(t) &= \alpha \mathrm{Tr}(a^\top a \psi_v(t)) \qquad\qquad\qquad\qquad ;\; \phi_v(0) = 0. \end{aligned}$$

The function ψ_v solves an usual matrix Riccati ODE. By using Theorem A.4.1, we get that $\psi_v(s) = [Z_1(s)v + Z_2(s)][Z_3(s)v + Z_4(s)]^{-1}$ with $\begin{bmatrix} Z_1(s) & Z_2(s) \\ Z_3(s) & Z_4(s) \end{bmatrix} = \exp\left(s \begin{bmatrix} b^\top & 0 \\ -2a^\top a & -b \end{bmatrix}\right)$, for $s \in [0, \bar{t})$, with $\bar{t} = \inf\{t \geq 0, \det(Z_3(s)v + Z_4(s)) = 0\}$. Since $\begin{bmatrix} b^\top & 0 \\ -a^\top a & -b \end{bmatrix}^k = \begin{bmatrix} (b^\top)^k & 0 \\ * & (-b)^k \end{bmatrix}$, we clearly get $Z_1(s) = \exp(sb^\top)$, $Z_2(s) = 0$ and $Z_4(s) = \exp(-sb)$. This yields to $\psi_v(s) = \exp(sb^\top)v[\exp(sb)Z_3(s)v + I_d]^{-1}\exp(sb)$. Since $\frac{d}{ds}[\exp(sb)Z_3(s)] = \exp(sb)[bZ_3(s) - 2a^\top a Z_1(s) - bZ_3(s)] = -2\exp(sb)a^\top a \exp(sb^\top)$, we deduce that $\bar{t} > t$ since $I_d + \exp(sb)Z_3(s)v = I_d - 2q_s v$ is invertible for $s \in [0,t]$ and we eventually get

$$\psi_v(t) = \exp(tb^\top)v(I_d - 2q_t v)^{-1}\exp(tb).$$

Therefore we obtain for $x \in \mathcal{S}_d(\mathbb{R})$,

$$\mathrm{Tr}(\psi_v(t)x) = \mathrm{Tr}\left(v(I_d - 2q_t v)^{-1}\exp(tb)x\exp(tb^\top)\right).$$

As explained by Grasselli and Tebaldi ([71], Sect. 4.2), ϕ_v can also be calculated explicitly as follows. Let us assume for a while that $v \in \mathcal{G}_d(\mathbb{R})$. Then, $\psi_v(s)$ is invertible for $s \in [0,t]$ and we get from (5.13) that

$$\frac{\frac{d}{ds}\det(\psi_v(s))}{\det(\psi_v(s))} = 2\mathrm{Tr}(b) + 2\mathrm{Tr}(a^\top a \psi_v(s)) = 2(\mathrm{Tr}(b) + \frac{1}{\alpha}\phi_v'(s)).$$

We obtain

$$\phi_v(t) = \frac{\alpha}{2}\left[\log\left(\frac{\det(\psi_v(t))}{\det(v)}\right) - 2t\mathrm{Tr}(b)\right] = \frac{\alpha}{2}\log\left(\frac{1}{\det(I_d - 2q_t v)}\right),$$

by using the multiplicativity of the determinant and $\det(\exp(tb)) = \exp(t\mathrm{Tr}(b))$. This yields to

$$\exp(\phi_v(t)) = \frac{1}{\det(I_d - 2q_t v)^{\frac{\alpha}{2}}},$$

and this formula is also valid when $v \notin \mathcal{G}_d(\mathbb{R})$ by using the continuity with respect to v.

Now, it remains to show that (5.18) indeed holds. By Itô calculus, we get that for $s \in (0, t)$:

$$
\begin{aligned}
& d \exp[\phi_v(t-s) + \mathrm{Tr}(\psi_v(t-s) X_s^x)] \\
& = \exp[\phi_v(t-s) + \mathrm{Tr}(\psi_v(t-s) X_s^x)] \mathrm{Tr}[\psi_v(t-s)(\sqrt{X_s^x} dW_s a + a^\top dW_s^\top \sqrt{X_s^x})].
\end{aligned}
\tag{5.20}
$$

Thus, $\exp[\phi_v(t-s) + \mathrm{Tr}(\psi_v(t-s) X_s^x)]$ is a positive local martingale and therefore a supermartingale, which gives that $\mathbb{E}[\exp(\mathrm{Tr}(v X_t^x))] \le \exp[\phi_v(t) + \mathrm{Tr}(\psi_v(t) x)] < \infty$, i.e. $\mathcal{D}_{b,a;t} \subset \tilde{\mathcal{D}}_{x,\alpha,b,a;t}$, where

$$
\begin{aligned}
\mathcal{D}_{b,a;t} &:= \{v \in \mathcal{S}_d(\mathbb{R}), \forall s \in [0,t], I_d - 2q_s v \in \mathcal{G}_d(\mathbb{R})\}, \\
\tilde{\mathcal{D}}_{x,\alpha,b,a;t} &:= \{v \in \mathcal{S}_d(\mathbb{R}), \mathbb{E}[\exp(\mathrm{Tr}(v X_t^x))] < \infty\}.
\end{aligned}
$$

On the other hand, when $-v \in \mathcal{S}_d^{+,*}(\mathbb{R})$, we have already seen that $\exp[\phi_v(t-s) + \mathrm{Tr}(\psi_v(t-s) X_s^x)]$ is a (bounded) martingale, and (5.18) holds from (5.16).

We have just shown that (5.18) holds for $-v \in \mathcal{S}_d^{+,*}(\mathbb{R})$, and we explain now why it holds for $v \in \mathcal{D}_{b,a;t}$ by using a classical argument on analytic functions. We first observe that $\mathcal{D}_{b,a;t}$ is convex. In fact, we have $\det(I_d - 2q_s v) = \det(I_d - 2\sqrt{q_s} v \sqrt{q_s})$, and therefore $\mathcal{D}_{b,a;t} = \{v \in \mathcal{S}_d(\mathbb{R}), \forall s \in [0,t], I_d - 2\sqrt{q_s} v \sqrt{q_s} \in \mathcal{S}_d^{+,*}(\mathbb{R})\}$ which is obviously convex. The Laplace transform $v \mapsto \mathbb{E}[\exp(\mathrm{Tr}(v X_t^x))]$ is an analytic function on $\mathcal{D}_{b,a;t}$ (see for example Lemma 10.8 in Filipović [53]). The right-hand-side of (5.18) is also analytic on $\mathcal{D}_{b,a;t}$ and coincides with the Laplace transform when $-v \in \mathcal{S}_d^{+,*}(\mathbb{R})$. Therefore, (5.18) holds for $v \in \mathcal{D}_{b,a;t}$ since $\mathcal{D}_{b,a;t}$ is convex. Now, we can extend to complex values of v. Indeed, the right-hand-side of (5.18) is well defined for $v = v_R + i v_I$ with $v_R \in \mathcal{D}_{b,a;t}$, thanks to Lemma A.1.1. Since both hand sides are analytic functions of v, (5.18) holds for $v = v_R + i v_I$.

Last, the remainder of the proof consists in showing that $\mathcal{D}_{b,a;t} = \tilde{\mathcal{D}}_{x,\alpha,b,a;t}$, which is unfortunately quite technical. We first consider the case $b = 0$ and assume by a way of contradiction that there is $v \in \tilde{\mathcal{D}}_{x,\alpha,0,a;t} \setminus \mathcal{D}_{0,a;t}$ for some x, α, a and $t > 0$. Let $\tilde{t} = \min\{s \in [0,t], I_d - 2q_s v \notin \mathcal{G}_d(\mathbb{R})\} \in (0,t]$. On the one hand, we have $v \notin \mathcal{D}_{0,a;\tilde{t}}$ and $v \in \mathcal{D}_{0,a;s}$ for $s \in [0, \tilde{t})$. On the other hand, we have by Jensen's inequality:

$$
s \in [0,t],\ \exp(\alpha(t-s)\mathrm{Tr}(v a^\top a)) \exp(\mathrm{Tr}(v X_s^x)) \le \mathbb{E}[\exp(\mathrm{Tr}(v X_t^x)) | \mathcal{F}_s],
$$

which gives $s \in [0,t] \mapsto \exp(-\alpha s \mathrm{Tr}(v a^\top a)) \mathbb{E}[\exp(\mathrm{Tr}(v X_s^x))]$ is nondecreasing and finite. Since (5.18) holds for $s < \tilde{t}$, we get that $\mathbb{E}[\exp(\mathrm{Tr}(v X_{\tilde{t}}^x))] = +\infty$, which leads to a contradiction. Let us now consider the case $b \neq 0$. From Proposition 5.3.2 (which is a consequence of the characteristic function obtained above), we have

$$
v \in \tilde{\mathcal{D}}_{x,\alpha,b,a;t} \iff \theta_t^\top v \theta_t \in \mathcal{D}_{0,I_d^n;t} \iff \forall s \in [0,t], \det(I_d - 2(s/t) q_t v) \neq 0.
$$

In particular, $\tilde{\mathcal{D}}_{x,\alpha,b,a;t}$ is an open set. For $v \in \mathcal{G}_d(\mathbb{R})$, we have $\det(I_d - 2(s/t)q_t v) \neq 0 \iff \det(v^{-1} - 2(s/t)q_t) \neq 0$ (resp. $\det(I_d - 2q_s v) \neq 0 \iff \det(v^{-1} - 2q_s) \neq 0$). Since $sq_t \leq s'q_t$ (resp. $q_s \leq q_{s'}$) for $s \leq s'$, we know from Theorem 8.1.5 in [67] that the (real) eigenvalues of $v^{-1} - 2(s/t)q_t$ (resp. $v^{-1} - 2q_s$) are nonincreasing w.r.t. s. Since they are also continuous, and $v^{-1} - 2(s/t)q_t = v^{-1} - 2q_s$ for $s \in \{0, t\}$, we get that $\forall s \in [0,t], \det(v^{-1} - 2(s/t)q_t) \neq 0 \iff \forall s \in [0,t], \det(v^{-1} - 2q_s) \neq 0$, and thus $\tilde{\mathcal{D}}_{x,\alpha,b,a;t} \cap \mathcal{G}_d(\mathbb{R}) = \mathcal{D}_{b,a;t} \cap \mathcal{G}_d(\mathbb{R})$. Let $v \in \tilde{\mathcal{D}}_{x,\alpha,b,a;t}$. Since $\tilde{\mathcal{D}}_{x,\alpha,b,a;t}$ is an open set, there is $\varepsilon > 0$ such that $v \pm \varepsilon I_d \in \tilde{\mathcal{D}}_{x,\alpha,b,a;t} \cap \mathcal{G}_d(\mathbb{R})$. Since $\mathcal{D}_{b,a;t}$ is convex, $v = (v + \varepsilon I_d + v - \varepsilon I_d)/2 \in \mathcal{D}_{b,a;t}$. □

5.3 Some Useful Identities in Law

This section gives simple but interesting identities in law for affine processes. First, we observe that their infinitesimal generator (5.10) only depend on a through $a^\top a$ and get:

$$AFF_d(x, \bar{\alpha}, B, a) \overset{law}{=} AFF_d(x, \bar{\alpha}, B, \sqrt{a^\top a}). \tag{5.21}$$

Also, it is natural to look at linear transformations of affine processes. Let $q \in \mathcal{G}_d(\mathbb{R})$ and define $B_q \in \mathcal{L}(\mathcal{S}_d(\mathbb{R}))$ by $B_q(x) = (q^\top)^{-1} B(q^\top x q) q^{-1}$. One has easily that B satisfies (5.4) iff B_q satisfies (5.4), and we get

$$AFF_d(x, \bar{\alpha}, B, a) \overset{law}{=} q^\top AFF_d((q^{-1})^\top x q^{-1}, (q^{-1})^\top \bar{\alpha} q^{-1}, B_q, a q^{-1}) q, \tag{5.22}$$

since both processes solve the same martingale problem. An interesting consequence is given in the following proposition: any affine process can be obtained as a linear transformation of an affine process for which $\bar{\alpha}$ is a diagonal matrix and $a = I_d^n$. This identity is interesting for the simulation problem since it allows to focus on a reduced parametrization of the process.

Proposition 5.3.1 *Let $n = \mathrm{Rk}(a)$ be the rank of $a^\top a$. Then, there exist a diagonal matrix $\bar{\delta}$, and a non singular matrix $u \in \mathcal{G}_d(\mathbb{R})$ such that $\bar{\alpha} = u^\top \bar{\delta} u$, and $a^\top a = u^\top I_d^n u$, and we have:*

$$AFF_d(x, \bar{\alpha}, B, a) \overset{law}{=} u^\top AFF_d\left((u^{-1})^\top x u^{-1}, \bar{\delta}, B_u, I_d^n\right) u,$$

where $\forall y \in \mathcal{S}_d(\mathbb{R}),\ B_u(y) = (u^{-1})^\top B(u^\top y u) u^{-1}$.

Proof Once u is given, the identity in law comes directly from (5.22). We give now a constructive proof of the existence of u, which takes back the arguments given by Golub and Van Loan ([67], Theorem 8.7.1). Nonetheless, we explain it entirely since it gives a practical way to calculate u.

Let us consider $\bar{\alpha} + a^\top a \in \mathcal{S}_d^+(\mathbb{R})$. From the extended Cholesky decomposition given in Lemma A.2.1 there is a matrix $v \in \mathcal{G}_d(\mathbb{R})$ such that $v^\top \bar{\alpha} v + v^\top a^\top a v = I_d^r$, where $r = \mathrm{Rk}(\bar{\alpha} + a^\top a)$. Since $v^\top \bar{\alpha} v \in \mathcal{S}_d^+(\mathbb{R})$, $v^\top a^\top a v \in \mathcal{S}_d^+(\mathbb{R})$ and $z^\top I_d^r z = 0$ for $z \in \mathbb{R}^d$ such that $z_1 = \cdots = z_r = 0$, there are $s_1, s_2 \in \mathcal{S}_n^+(\mathbb{R})$ such that:

$$v^\top \bar{\alpha} v = \begin{pmatrix} s_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } v^\top a^\top a v = \begin{pmatrix} s_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let o_2 be an orthogonal matrix such that $o_2^\top s_2 o_2$ is a diagonal matrix. We assume without loss of generality that only the first n elements of this diagonal are positive: $o_2^\top s_2 o_2 = diag(\eta_1, \ldots, \eta_n, 0, \ldots, 0)$. We set $o = \begin{pmatrix} o_2 & 0 \\ 0 & I_{d-r} \end{pmatrix}$ and get $I_d^r = o^\top v^\top \bar{\alpha} v o + o^\top v^\top a^\top a v o$, which gives that $o^\top v^\top \bar{\alpha} v o$ is a diagonal matrix. Thus, we get the desired result by taking $u = diag(\sqrt{\eta_1}, \ldots, \sqrt{\eta_n}, 1, \ldots, 1) o^{-1} v^{-1}$. □

Let us notice however that in the case of Wishart processes, u can directly be obtained by using a single extended Cholesky decomposition, see Lemma A.2.1.

Up to now, we have stated identities for the law of affine processes. Thanks to the explicit characteristic function of Wishart processes, we are also able to get another interesting identity on the marginal laws.

Proposition 5.3.2 *Let $t > 0$, $a, b \in \mathcal{M}_d(\mathbb{R})$ and $\alpha \geq d-1$. Let $m_t = \exp(tb)$, $q_t = \int_0^t \exp(sb) a^\top a \exp(sb^\top) ds$ and $n = \mathrm{Rk}(q_t)$. Then, there is $\theta_t \in \mathcal{G}_d(\mathbb{R})$ such that $q_t = t\theta_t I_d^n \theta_t^\top$, and we have:*

$$WIS_d(x, \alpha, b, a; t) \overset{law}{=} \theta_t WIS_d(\theta_t^{-1} m_t x m_t^\top (\theta_t^{-1})^\top, \alpha, 0, I_d^n; t) \theta_t^\top. \tag{5.23}$$

This proposition plays a crucial role for the exact simulation of Wishart processes. Thanks to (5.23), we can sample any Wishart distribution if we are able to simulate exactly the distribution $WIS_d(x, \alpha, 0, I_d^n; t)$ for any $x \in \mathcal{S}_d^+(\mathbb{R})$. In Sect. 5.5, we focus on this and give a way to sample exactly $WIS_d(x, \alpha, 0, I_d^n; t)$. Let us stress here that we can compute the matrix θ_t by using the extended Cholesky decomposition of q_t/t, as it is explained in the proof below.

Proof We apply Lemma A.2.1 to $q_t/t \in \mathcal{S}_d^+(\mathbb{R})$ and consider (p, c_n, k_n) an extended Cholesky decomposition of q_t/t. We set $\theta_t = p^{-1} \begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix}$. Then, θ_t is invertible and it is easy to check that $q_t = t\theta_t I_d^n \theta_t^\top$. Now, let us observe that for $v \in \mathcal{S}_d(\mathbb{R})$,

$$\begin{aligned}
\det(I_d - 2iq_t v) &= \det(\theta_t(\theta_t^{-1} - 2itI_d^n \theta_t^\top v)) = \det(I_d - 2itI_d^n \theta_t^\top v \theta_t), \\
\mathrm{Tr}[iv(I_d - 2iq_t v)^{-1} m_t x m_t^\top] &= \mathrm{Tr}[i(\theta_t^{-1})^\top \theta_t^\top v(\theta_t \theta_t^{-1} - 2it\theta_t I_d^n \theta_t^\top v \theta_t \theta_t^{-1})^{-1} m_t x m_t^\top] \\
&= \mathrm{Tr}[i\theta_t^\top v \theta_t (I_d - 2itI_d^n \theta_t^\top v \theta_t)^{-1} \theta_t^{-1} m_t x m_t^\top (\theta_t^{-1})^\top].
\end{aligned}$$

Let $X_t^x \sim WIS_d(x, \alpha, b, a; t)$ and $\tilde{X}_t^x \sim WIS_d(x, \alpha, 0, I_d^n; t)$. Then, from (5.18) and (5.19), we get that

$$\begin{aligned}\mathbb{E}[\exp(i\operatorname{Tr}(vX_t^x))] &= \mathbb{E}[\exp(i\operatorname{Tr}(\theta_t^\top v\theta_t \tilde{X}_t^{\theta_t^{-1}m_t x m_t^\top (\theta_t^{-1})^\top}))] \\ &= \mathbb{E}[\exp(i\operatorname{Tr}(v\theta_t \tilde{X}_t^{\theta_t^{-1}m_t x m_t^\top (\theta_t^{-1})^\top}\theta_t^\top))].\end{aligned}$$

□

Remark 5.3.3 The identity in law given by Proposition 5.3.2 can be seen as a generalization to Wishart processes of the well-known identity between CIR and squared Bessel distribution stated in Exercise 1.2.14. In fact, it gives when $d = 1$ that

$$WIS_1(x, \alpha, b, a; t) \overset{law}{=} a^2 \frac{e^{2bt} - 1}{2bt} WIS_1\left(\frac{2btx}{a^2(1 - e^{-2bt})}, \alpha, 0, 1; t\right).$$

We note that for the CIR process, we even have an identity in law between the processes while it only brings here on marginal distributions.

5.3.1 A Connection with Matrix-Valued Ornstein-Uhlenbeck Processes

We have seen in Sect. 1.2.3 that the square of one-dimensional Ornstein-Uhlenbeck processes are CIR process. As mentioned by Bru [25], this law identity can in fact be extended to Wishart processes as follows. We consider a standard Brownian process Z_t in $\mathcal{M}_{d\times d'}(\mathbb{R})$, which means that all the matrix entries are independent standard real Brownian motions. We define an Ornstein-Uhlenbeck process Y on $\mathcal{M}_{d\times d'}(\mathbb{R})$ as follows:

$$dY_t = bY_t dt + adZ_t,\ Y_0 = y_0, \tag{5.24}$$

where $a, b \in \mathcal{M}_d(\mathbb{R})$. By using Itô calculus, we get

$$dY_t Y_t^\top = (bY_t Y_t^\top + Y_t Y_t^\top b^\top + d' aa^\top)dt + Y_t dZ_t^\top a^\top + adZ_t Y_t^\top,$$

since $(Y_t Y_t^\top)_{i,j} = \sum_{k=1}^{d'} (Y_t)_{i,k}(Y_t)_{j,k}$ and

$$\langle d(Y_t Y_t^\top)_{i,j}\rangle = \sum_{k=1}^{d'} \langle d(Y_t)_{i,k}, d(Y_t)_{j,k}\rangle$$

$$
\begin{aligned}
&= \sum_{k=1}^{d'} \langle \sum_{l=1}^{d} a_{i,l} d(Z_t)_{l,k}, \sum_{l=1}^{d} a_{j,l} d(Z_t)_{l,k} \rangle \\
&= d'(aa^\top)_{i,j} dt.
\end{aligned}
$$

Let us set $X_t = Y_t Y_t^\top$ and calculate its quadratic covariations. By repeating the same calculations as in the proof of Lemma 5.1.2 (the only difference is that the Brownian motion is not as square matrix), we also get

$$
\begin{aligned}
\langle d(X_t)_{i,j}, d(X_t)_{m,n} \rangle = (aa^\top)_{i,m}(X_t)_{j,n} + (aa^\top)_{i,n}(X_t)_{j,m} \\
+ (aa^\top)_{j,m}(X_t)_{i,n} + (aa^\top)_{j,n}(X_t^\top)_{i,m}.
\end{aligned}
$$

This is the quadratic covariation of a Wishart process.

Proposition 5.3.4 *Let $a, b \in \mathcal{M}_d(\mathbb{R})$ and Y be the process defined by* (5.24)*. When $d' \geq d - 1$, $X_t = Y_t Y_t^\top$ is distributed as a Wishart process and we have*

$$
X \sim WIS_d(y_0 y_0^\top, d', b, a^\top). \tag{5.25}
$$

When $d' \in \{1, \dots, d-2\}$, we remark that the process X is still defined. This gives a way to define Wishart processes for $\alpha \in \{1, \dots, d-2\}$. In this case, X_t is a matrix of maximal rank d' and is never invertible. The set of α's where the Wishart process can be defined is often referred as the Gindikin set.

5.4 Financial Modelling with Wishart Processes

Wishart processes and affine processes on semidefinite positive matrices have been recently considered in finance. Even if their use is still nowadays at an experimental stage, these processes are rather promising since they provide a wide range of stochastic dynamics while keeping the calculation of the marginal laws explicit. Up to now, Wishart processes have been considered for financial modelling in three main directions. First, we have seen in Sect. 4.3 that affine processes can be used to model the short-interest rate. Dai and Singleton have already considered a wide class of affine processes, but it is somehow natural to broaden their parametrization by using Wishart processes. This approach has been considered for example by Gourieroux and Sufana [69], Gnoatto [66] or Ahdida et al. [4]. Second, Wishart processes have been used to model the stochastic volatility of a single asset. Typically, the trace of the Wishart process is now the volatility of the stock, which extends the Heston model and keeps the affine structure. This modelling approach has been considered by Da Fonseca et al. [38] and Benabid et al. [17]. Numerical experiments on market data are given in Da Fonseca and Grasselli [37]. Third, Wishart processes can be used to model the instantaneous covariance of a basket of assets. This approach, which we present now, can be seen as a way to extend the Heston model to a multidimensional framework.

Let us first consider the model introduced by Gourieroux and Sufana [69]. This is a model for d risky assets $S_t^1, \dots, S_t^d$. We assume that the interest rate r is deterministic and constant. We denote by $(B_t, t \geq 0)$ a standard Brownian motion on $\mathbb{R}^d$ that is independent from $(W_t, t \geq 0)$. Then, we assume the following dynamics for the assets under a risk-neutral measure:

$$t \geq 0, 1 \leq l \leq d,\ S_t^l = S_0^l + r \int_0^t S_u^l du + \int_0^t S_u^l (\sqrt{X_u} dB_u)_l, \tag{5.26}$$

where

$$X_t = X_0 + \int_0^t \left(\alpha a^\top a + bX_u + X_u b^\top\right) du + \int_0^t \left(\sqrt{X_u} dW_u a + a^\top dW_u^\top \sqrt{X_u}\right)$$

is a Wishart process. Here, $(\sqrt{X_u} dB_u)_l$ is simply the lth coordinates of the vector $\sqrt{X_u} dB_u$. We can easily check that the instantaneous quadratic covariation matrix between the log-prices of the assets is X_t. In fact, we have

$$\begin{aligned}
\langle d\log(S_t^k), d\log(S_t^l)\rangle &= \langle (\sqrt{X_t} dB_t)_k, (\sqrt{X_t} dB_t)_l \rangle \\
&= \langle \sum_{i=1}^d (\sqrt{X_t})_{k,i} (dB_t)_i, \sum_{i=1}^d (\sqrt{X_t})_{l,i} (dB_t)_i \rangle \\
&= \left(\sum_{i=1}^d (\sqrt{X_t})_{k,i} (\sqrt{X_t})_{l,i} \right) dt = (X_t)_{k,l} dt.
\end{aligned}$$

This also gives that

$$d\log(S_t^l) = (r - \frac{1}{2}(X_t)_{l,l})dt + (\sqrt{X_t} dB_t)_l.$$

Let Y_t denote the vector of log-prices. We observe that the process (Y_t, X_t) is affine and has the following infinitesimal generator on $\mathbb{R}^d \times \mathcal{M}_d(\mathbb{R})$

$$L = \sum_{i=1}^d (r - \frac{x_{i,i}}{2}) \partial_{y_i} + \sum_{i,j=1}^d x_{i,j} \partial_{y_i} \partial_{y_j} + L^{WIS},$$

where L^{WIS} is the operator given by (5.9). Again, the Laplace transform of (Y_t, X_t) can be obtained by the mean of a system of differential equations, see Proposition 1 of [69].

The Gourieroux and Sufana model (5.26) can be seen as an extension of the Heston model when the instantaneous correlation between the assets and their covariance is equal to zero, i.e. $\rho = 0$. However, one would like in practice to let the possibility of a possible dependence between the moves of asset prices and the ones of their covariance. This is of course possible in general, but this is less obvious if one wants to keep the affine structure of (Y, X). Da Fonseca et al. [36] have found a way to do this. We present now their model. They consider a vector $\rho \in \mathbb{R}^d$ such that $\|\rho\|_2^2 = \rho^\top \rho = \sum_{i=1}^d \rho_i^2 \leq 1$ and assume the following dynamic for the log-prices:

$$d(Y_t)_l = (r - \frac{1}{2}(X_t)_{l,l})dt + (\sqrt{X_t}[\sqrt{1 - \|\rho\|_2^2} dB_t + dW_t \rho])_l. \tag{5.27}$$

We observe that $\sqrt{1 - \|\rho\|_2^2} B_t + W_t \rho$ is a standard Brownian motion on $\mathbb{R}^d$, since we have

$$\begin{aligned}
&\langle d(\sqrt{1 - \|\rho\|_2^2} B_t + W_t \rho)_k, d(\sqrt{1 - \|\rho\|_2^2} B_t + W_t \rho)_l \rangle \\
&= \langle \sqrt{1 - \|\rho\|_2^2}(dB_t)_k + \sum_{i=1}^d (dW_t)_{k,i} \rho_i, \sqrt{1 - \|\rho\|_2^2}(dB_t)_l + \sum_{i=1}^d (dW_t)_{l,i} \rho_i \rangle \\
&= \mathbb{1}_{k=l} \left(1 - \|\rho\|_2^2 + \sum_{i=1}^d \rho_i^2 \right) dt = \mathbb{1}_{k=l} dt.
\end{aligned}$$

We therefore have as before $\langle d(Y_t)_k, d(Y_t)_l \rangle = (X_t)_{k,l} dt$, but we now have to calculate the quadratic covariation between the coordinates of Y and X. On the one hand, we have

$$d(X_t)_{i,j} = (\dots)dt + \sum_{k,l=1}^d ((\sqrt{X_t})_{i,k} a_{l,j} + (\sqrt{X_t})_{j,k} a_{l,i})(dW_t)_{k,l},$$

and on the other hand

$$d(Y_t)_m = (\dots)dt + (\sqrt{X_t}[\sqrt{1 - \|\rho\|_2^2} dB_t)_m + \sum_{l,k=1}^d (\sqrt{X_t})_{m,k} (dW_t)_{k,l} \rho_l.$$

We eventually obtain

$$\begin{aligned}
\langle d(X_t)_{i,j}, d(Y_t)_m \rangle &= dt \times \sum_{l,k=1}^d (\sqrt{X_t})_{m,k} ((\sqrt{X_t})_{i,k} a_{l,j} + (\sqrt{X_t})_{j,k} a_{l,i}) \rho_l \\
&= [(a^\top \rho)_j (X_t)_{i,m} + (a^\top \rho)_i (X_t)_{j,m}] dt,
\end{aligned}$$

which is affine with respect to X_t. The Laplace transform of (Y_t, X_t) can again be obtained by the mean of matrix Riccati differential equations, and we refer to [36], Proposition 5, for a precise statement. In particular, we can calculate quite efficiently the Laplace transform of each log-prices, which is interesting to calculate European option prices on single assets, see for example the Carr and Madan method in Sect. 4.2.3. However, one of the interest of the Da Fonseca, Grasselli and Tebaldi model (5.27) is to give a model for the dependence between the stock. In particular, one would like to use this model in order to price options that involve this dependence. This could be for example options that pays at time $T > 0$ in cash the value of $\left(\frac{S_T^1}{S_0^1} - \frac{S_T^2}{S_0^2}\right)^+$, or $\left(\frac{1}{d}\sum_{i=1}^d S_T^i - K\right)^+$, or $\left(\max_{i=1}^d \frac{S_T^i}{S_0^i} - \min_{i=1}^d \frac{S_T^i}{S_0^i}\right)^+$. Unfortunately, unless in the first case which involves only two assets, inverse Fourier method would require to perform d-dimensional integrals, which is expensive from a computation point of view as soon as $d \geq 4$. To avoid this curse of dimensionality, it is more efficient to use a Monte-Carlo method and numerical schemes, even if the payoff only depends on the final time. Examples in dimension 2 where Fourier inversion methods are still competitive are considered in [36].

Remark 5.4.1 Let us assume that the matrix b is diagonal. We have $\langle d(X_t)_{i,i}, d(Y_t)_i \rangle = 2(a^\top \rho)_i (X_t)_{i,i} dy$ and we know from Lemma 5.1.2 that $\langle d(X_t)_{i,i} \rangle = 4(X_t)_{i,i}(a^\top a)_{i,i} dt$. We set

$$\tilde{\rho}_i = \frac{(a^\top \rho)_i}{\sqrt{(a^\top a)_{i,i}}}.$$

We have $\tilde{\rho}_i \in [-1, 1]$ and even $|\tilde{\rho}_i| \leq \|\rho\|_2$ since $(a^\top \rho)_i = \sum_{l=1}^d a_{l,i} \rho_l \leq \sqrt{(a^\top a)_{i,i}} \|\rho\|_2$ by the Cauchy-Schwarz inequality. Therefore, there is a two-dimensional Brownian motion (β^i, γ^i) such that $\langle d\beta_t^i, d\gamma_t^i \rangle = \tilde{\rho}_i dt$ and

$$\begin{aligned} d(X_t)_{i,i} &= [\alpha(a^\top a)_{i,i} + 2b_{i,i}(X_t)_{i,i}]dt + 2\sqrt{(a^\top a)_{i,i}(X_t)_{i,i}}\, d\beta_t^i \\ d(Y_t)_i &= (r - \frac{1}{2}(X_t)_{i,i})dt + \sqrt{(X_t)_{i,i}}\, d\gamma_t^i. \end{aligned}$$

Thus, with this parametrization, each single asset follows the Heston model, see Sect. 4.2.

5.5 Exact Simulation of Wishart Processes

In this section, we present a method to simulate exactly a Wishart process. One remarkable point of this exact simulation method for non-central Wishart distributions is that it works for any $\alpha \geq d - 1$, without any restriction. Wishart distributions have been thoroughly studied in statistics when $\alpha \in \mathbb{N}$ (which is then called the number of degrees of freedom). Exact simulation methods have already been proposed in that case. For instance, Odell and Feiveson [110] and Smith and Hocking [117] have proposed an exact simulation method for central Wishart distributions based on the Bartlett's decomposition. Gleser [65] extends it to any (non-central) Wishart distribution. Bru [25] proposes when $\alpha \in \mathbb{N}$ to sample Wishart processes by using Proposition 5.3.4. In fact, Ornstein-Uhlenbeck processes are can easily be sampled since they are Gaussian processes.

Here, the method relies on the identity in law (5.23) that enables us to focus on the case $b = 0$, $a = I_d^n$. Then, we show a remarkable splitting of the infinitesimal generator as the sum of commuting operators. These operators are associated to a stochastic differential equation that can be solved explicitly on $\mathcal{S}_d^+(\mathbb{R})$, which enables us to sample any Wishart distribution.

5.5.1 A Remarkable Splitting for $WIS_d(x, \alpha, 0, I_d^n)$

The following theorem explains how to split the infinitesimal generator of $WIS_d(x, \alpha, 0, I_d^n)$ as the sum of commutative infinitesimal generators. This result plays a crucial role in the sequel for both for the exact and approximated simulation schemes.

Theorem 5.5.1 *Let L be the generator associated to the Wishart process $WIS_d(x, \alpha, 0, I_d^n)$ and $L_{e_d^i}$ be the generator associated to $WIS_d(x, \alpha, 0, e_d^i)$ for $i \in \{1, \ldots, d\}$. Then, we have*

$$L = \sum_{i=1}^{n} L_{e_d^i} \text{ and } \quad \forall i, j \in \{1, \ldots, d\},\ L_{e_d^i} L_{e_d^j} = L_{e_d^j} L_{e_d^i}. \tag{5.28}$$

Proof From (5.10), we easily get that $L = \sum_{i=1}^n L_{e_d^i}$ since $I_d^n = \sum_{i=1}^n e_d^i$. The commutativity property comes from a tedious but simple calculation that we do now.

From (5.10), we get:

$$L_{e_d^i} = \alpha \partial_{\{i,i\}} + 2x_{\{i,i\}} \partial^2_{\{i,i\}} + 2 \sum_{\substack{1 \le m \le d \\ m \ne i}} x_{\{i,m\}} \partial_{\{i,m\}} \partial_{\{i,i\}} + \frac{1}{2} \sum_{\substack{1 \le m,l \le d \\ m \ne i, l \ne i}} x_{\{m,l\}} \partial_{\{i,m\}} \partial_{\{i,l\}}. \tag{5.29}$$

We want to show that $L_{e_d^i} L_{e_d^j} = L_{e_d^j} L_{e_d^i}$ for $i \ne j$. Up to a permutation of the coordinates, $L_{e_d^i}$ and $L_{e_d^j}$ are the same operators as $L_{e_d^1}$ and $L_{e_d^2}$. It is therefore sufficient to check that $L_{e_d^1} L_{e_d^2} = L_{e_d^2} L_{e_d^1}$. By a straightforward but tedious calculation, we get

$$
\begin{aligned}
&L_{e_d^1} L_{e_d^2} \\
&= \underbrace{\alpha^2 \partial_{\{1,1\}} \partial_{\{2,2\}}}_{(\bar{0})} + \underbrace{2\alpha x_{\{2,2\}} \partial_{\{1,1\}} \partial^2_{\{2,2\}}}_{(1)} + \underbrace{2\alpha \sum_{j \neq 2} x_{\{2,j\}} \partial_{\{1,1\}} \partial_{\{2,2\}} \partial_{\{2,j\}}}_{(2)} \\
&\quad + \frac{\alpha}{2} (\underbrace{\partial^2_{\{1,2\}}}_{(\bar{3})} + \underbrace{\sum_{j \neq 2, k \neq 2} x_{\{j,k\}} \partial_{\{1,1\}} \partial_{\{2,j\}} \partial_{\{2,k\}}}_{(4)}) + \underbrace{2\alpha x_{\{1,1\}} \partial^2_{\{1,1\}} \partial_{\{2,2\}}}_{(1)} \\
&\quad + \underbrace{4 x_{\{1,1\}} x_{\{2,2\}} \partial^2_{\{1,1\}} \partial^2_{\{2,2\}}}_{(\bar{5})} + \underbrace{4 \sum_{j \neq 2} x_{\{1,1\}} x_{\{2,j\}} \partial^2_{\{1,1\}} \partial_{\{2,j\}} \partial_{\{2,2\}}}_{(6)} \\
&\quad + x_{\{1,1\}} (\underbrace{2 \partial_{\{1,1\}} \partial^2_{\{1,2\}}}_{(7)} + \underbrace{\sum_{j \neq 2, k \neq 2} x_{\{j,k\}} \partial^2_{\{1,1\}} \partial_{\{2,j\}} \partial_{\{2,k\}}}_{(8)}) \\
&\quad + \underbrace{2\alpha \sum_{m \neq 1} x_{\{1,m\}} \partial_{\{1,1\}} \partial_{\{1,m\}} \partial_{\{2,2\}}}_{(2)} + \underbrace{4 \sum_{m \neq 1} x_{\{1,m\}} x_{\{2,2\}} \partial_{\{1,1\}} \partial_{\{1,m\}} \partial^2_{\{2,2\}}}_{(6)} \\
&\quad + 4 (\underbrace{\sum_{m \neq 1, j \neq 2} x_{\{1,m\}} x_{\{2,j\}} \partial_{\{1,1\}} \partial_{\{1,m\}} \partial_{\{2,j\}} \partial_{\{2,2\}}}_{(\bar{9})} + \underbrace{x_{\{1,2\}} \partial_{\{1,1\}} \partial_{\{1,2\}} \partial_{\{2,2\}}}_{(\overline{10})}) \\
&\quad + \underbrace{\sum_{m \neq 1, k \neq 2, j \neq 2} x_{\{1,m\}} x_{\{j,k\}} \partial_{\{1,1\}} \partial_{\{1,m\}} \partial_{\{2,j\}} \partial_{\{2,k\}}}_{(11)} + \underbrace{\sum_{m \neq 1, m \neq 2} x_{\{1,m\}} \partial^2_{\{1,2\}} \partial_{\{1,m\}}}_{(12)} \\
&\quad + \underbrace{2 \sum_{m \neq 1, m \neq 2} x_{\{1,m\}} \partial_{\{1,1\}} \partial_{\{1,2\}} \partial_{\{2,m\}}}_{(13)} + \underbrace{x_{\{1,2\}} \partial^3_{\{1,2\}} + 2 x_{\{1,2\}} \partial_{\{1,1\}} \partial_{\{1,2\}} \partial_{\{2,2\}}}_{(\overline{14})} \\
&\quad + \underbrace{\frac{\alpha}{2} \sum_{m \neq 1, l \neq 1} x_{\{m,l\}} \partial_{\{1,m\}} \partial_{\{1,l\}} \partial_{\{2,2\}}}_{(4)} + \underbrace{\sum_{m \neq 1, l \neq 1} x_{\{2,2\}} x_{\{m,l\}} \partial_{\{1,m\}} \partial_{\{1,l\}} \partial^2_{\{2,2\}}}_{(8)} \\
&\quad + \underbrace{\sum_{m \neq 1, l \neq 1, j \neq 2} x_{\{2,j\}} x_{\{m,l\}} \partial_{\{1,m\}} \partial_{\{1,l\}} \partial_{\{2,2\}} \partial_{\{2,j\}}}_{(11)} + \underbrace{2 \sum_{m \neq 1, m \neq 2} x_{\{2,m\}} \partial_{\{1,2\}} \partial_{\{1,m\}} \partial_{\{2,2\}}}_{(13)} \\
&\quad + \underbrace{2 x_{\{2,2\}} \partial^2_{\{1,2\}} \partial_{\{2,2\}}}_{(7)} + \underbrace{\frac{1}{4} \sum_{\substack{m \neq 1, l \neq 1 \\ j \neq 2, k \neq 2}} x_{\{m,l\}} x_{\{j,k\}} \partial_{\{1,m\}} \partial_{\{1,l\}} \partial_{\{2,k\}} \partial_{\{2,j\}}}_{(\overline{15})}
\end{aligned}
$$

$$+ \underbrace{\sum_{\substack{m\neq 1, l\neq 1\\ m\neq 2, l\neq 2}} x_{\{m,l\}}\partial_{\{1,l\}}\partial_{\{1,2\}}\partial_{\{2,m\}}}_{(\overline{16})} + \underbrace{\sum_{m\neq 1, m\neq 2} x_{\{2,m\}}\partial^2_{\{1,2\}}\partial_{\{2,m\}}}_{(12)}.$$

Obviously, we have the same formula for $L_{e_d^2}L_{e_d^1}$ simply by exchanging the index 1 and 2. It is then sufficient to check that the above formula remains unchanged when we exchange the two indices. Each term above is marked with a number. If this number is in the form $(\overline{r})$, then the associated term is symmetric. Otherwise, there exist in the formula its corresponding symmetric term which is marked with the same number. □

Beyond the commutativity property, two other features of (5.28) are important to notice.

- The operators $L_{e_d^i}$ and $L_{e_d^j}$ are the same up to the exchange of coordinates i and j.
- The processes $WIS_d(x,\alpha,0,e_d^i)$ and $WIS_d(x,\alpha,0,I_d^n)$ are well defined on $\mathcal{S}_d^+(\mathbb{R})$ under the same hypothesis, namely $\alpha \geq d-1$ and $x \in \mathcal{S}_d^+(\mathbb{R})$.
- The operator $L_{e_d^i}$ only involves the entries on the ith column and row. The other entries are unchanged by the process $WIS_d(x,\alpha,0,e_d^i)$.

The second property makes possible the composition that we explain now. Let us consider $t > 0$ and $x \in \mathcal{S}_d^+(\mathbb{R})$. We define iteratively:

$$\begin{aligned}
X_t^{1,x} &\sim WIS_d(x,\alpha,0,e_d^1;t),\\
X_t^{2,X_t^{1,x}} &\sim WIS_d(X_t^{1,x},\alpha,0,e_d^2;t),\\
&\dots\\
X_t^{n,\dots X_t^{1,x}} &\sim WIS_d(X_t^{n-1,\dots X_t^{1,x}},\alpha,0,e_d^n;t).
\end{aligned}$$

Thus, conditionally to $X_t^{i-1,\dots X_t^{1,x}}$, $X_t^{i,\dots X_t^{1,x}}$ is sampled according to the distribution at time t of a Wishart process starting from $X_t^{i-1,\dots X_t^{1,x}}$ and with parameters $(\alpha,0,e_d^i)$. We have the following result.

Proposition 5.5.2 *Let $X_t^{n,\dots X_t^{1,x}}$ be defined as above. Then*

$$X_t^{n,\dots X_t^{1,x}} \sim WIS_d(x,\alpha,0,I_d^n;t).$$

Thanks to this proposition, we can generate a sample according to $WIS_d(x,\alpha,0,I_d^n;t)$ as soon as we can simulate $WIS_d(x,\alpha,0,e_d^i;t)$. These laws are the same as $WIS_d(x,\alpha,0,e_d^1;t)$, up to the permutation of the first and ith coordinates. In the next subsection, it is explained how to draw such random variables.

It is really easy to give a formal proof of Proposition 5.5.2. Let $X_t^x \sim WIS_d(x, \alpha, 0, I_d^n; t)$ and f be a smooth function on $\mathcal{S}_d^+(\mathbb{R})$ such that the series below converge absolutely. By iterating Itô's formula, we have that $\mathbb{E}[f(X_t^x)] = \sum_{k=0}^{\infty} t^k L^k f(x)/k!$. Similarly, we also get by using the tower property of the conditional expectation that:

$$\mathbb{E}\left[f(X_t^{n,\dots X_t^{1,x}})\right] = \mathbb{E}\left[\mathbb{E}\left[f(X_t^{n,\dots X_t^{1,x}})|X_t^{n-1,\dots X_t^{1,x}}\right]\right] = \sum_{k_n=0}^{+\infty} \frac{t^{k_n}}{k_n!}\mathbb{E}\left[L_{e_d^n}^{k_n} f(X_t^{n-1,\dots X_t^{1,x}})\right]. \tag{5.30}$$

Simply by repeating this argument, we get that

$$\begin{aligned}\mathbb{E}\left[f(X_t^{n,\dots X_t^{1,x}})\right] &= \sum_{k_1,\dots,k_n=0}^{+\infty} \frac{t^{\sum_{i=1}^n k_i}}{k_1!\dots k_n!} L_{e_d^1}^{k_1} \dots L_{e_d^n}^{k_n} f(x) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!}(L_{e_d^1} + \dots + L_{e_d^n})^k f(x) = \mathbb{E}[f(X_t^x)]. \end{aligned} \tag{5.31}$$

To get the second equality, we identify a Cauchy product and use that the operators $L_{e_d^1}, \dots, L_{e_d^n}$ commute. To make this formal proof correct, one has to check that the series are well defined and can be switched with the expectation, which we do now.

Proof of Proposition 5.5.2 Let $X_t^x \sim WIS_d(x, \alpha, 0, I_d^n; t)$. We will check that for any polynomial function f of the matrix elements, we have $\mathbb{E}[f(X_t^x)] = \mathbb{E}[f(X_t^{n,\dots X_t^{1,x}})]$. Let us consider a polynomial function f of degree m:

$$x \in \mathcal{S}_d(\mathbb{R}), \;\; f(x) = \sum_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \le m} a_\gamma x^\gamma,$$

where $|\gamma| = \sum_{1 \le i \le j \le d} |\gamma_{\{i,j\}}|$ and $x^\gamma = \prod_{1 \le i \le j \le d} x_{\{i,j\}}^{\gamma_{\{i,j\}}}$. Since the operator are affine, it is easy to check that $Lf(x)$ and $L_{e_d^i} f(x)$ are also polynomial functions of degree m. We set:

$$\|f\|_{\mathbb{P}} = \sum_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \le m} |a_\gamma| \text{ and } |L| = \max_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \le m} \|Lx^\gamma\|_{\mathbb{P}},$$

so that $\|L^k f\|_{\mathbb{P}} \le |L|^k \|f\|_{\mathbb{P}}$ for any $k \in \mathbb{N}$. Therefore, the series $\sum_{k=0}^{\infty} t^k L^k f(x)/k!$ converges absolutely. By using $l+1$ times Itô's formula, we get:

$$\mathbb{E}[f(X_t^x)] = \sum_{k=0}^{l} \frac{t^k}{k!} L^k f(x) + \int_0^t \frac{(t-s)^l}{l!} \mathbb{E}[L^{l+1} f(X_s^x)] ds.$$

Wishart processes have bounded moments since the drift and diffusion coefficients have a sublinear growth. Thus, $C = \max_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \le m} \sup_{s \in [0,t]} \mathbb{E}[|\{X_s^x\}^\gamma|] < \infty$ and we obtain that $|\int_0^t \frac{(t-s)^l}{l!} \mathbb{E}[L^{l+1} f(X_s^x)] ds| \le C \| f \|_{\mathbb{P}} (t|L|)^{l+1}/(l+1)! \underset{l \to +\infty}{\to} 0$. Thus, we have $\mathbb{E}[f(X_t^x)] = \sum_{k=0}^{\infty} t^k L^k f(x)/k!$ and similarly we get that

$$\mathbb{E}\left[f(X_t^{n,\dots X_t^{1,x}}) | X_t^{n-1,\dots X_t^{1,x}} \right] = \sum_{k_n=0}^{+\infty} \frac{t^{k_n}}{k_n!} L_{e_d^n}^{k_n} f(X_t^{n-1,\dots X_t^{1,x}}).$$

Now, we remark that

$$\tilde{C} := \max_{\gamma \in \mathbb{N}^{d(d+1)/2}, |\gamma| \le m} \sup_{s \in [0,t]} \max(\mathbb{E}[|\{X_t^{1,x}\}^\gamma|], \dots, \mathbb{E}[|\{X_t^{n,\dots X_t^{1,x}}\}^\gamma|]) < \infty$$

by using once again that Wishart processes have bounded moments. Since

$$\mathbb{E}[|L_{e_d^n}^{k_n} f(X_t^{n-1,\dots X_t^{1,x}})|] \le \tilde{C} \| f \|_{\mathbb{P}} |L_{e_d^n}|^{k_n},$$

we can switch the expectation with the series and get (5.30). Then, since $L_{e_d^n}^{k_n} f(x)$ are polynomial function of degree m, we can iterate this argument and finally get (5.31), which gives the result. □

5.5.2 *Exact Simulation for $WIS_d(x, \alpha, 0, e_d^1; t)$*

For the sake of clarity, we start with the case of $d = 2$ that avoids complexities due to matrix decompositions. We deal with the general case just after.

The Case $d = 2$

We start by writing explicitly the infinitesimal generator $L_{e_2^1}$ of $WIS_2(x, \alpha, 0, e_2^1)$. From (5.10), we get:

$$x \in \mathcal{S}_2^+(\mathbb{R}), \ L_{e_2^1} f(x) = \alpha \partial_{\{1,1\}} f(x) + 2x_{\{1,1\}} \partial_{\{1,1\}}^2 f(x) \\ + 2x_{\{1,2\}} \partial_{\{1,1\}} \partial_{\{1,2\}} f(x) + \frac{x_{\{2,2\}}}{2} \partial_{\{1,2\}}^2 f(x). \quad (5.32)$$

We show now that this operator is in fact associated to an SDE that can be explicitly solved. We will denote by $(Z_t^1, t \ge 0)$ and $(Z_t^2, t \ge 0)$ two independent standard Brownian motions in $\mathbb{R}$.

When $x_{\{2,2\}} = 0$, we also have $x_{\{1,2\}} = 0$ since x is nonnegative. In that case,

$$X_0^x = x, \; d(X_t^x)_{\{1,1\}} = \alpha dt + 2\sqrt{(X_t^x)_{\{1,1\}}} dZ_t^1, \; d(X_t^x)_{\{1,2\}} = 0, \; d(X_t^x)_{\{2,2\}} = 0 \tag{5.33}$$

has the infinitesimal generator (5.32), which is the one of a CIR process (or of a squared Bessel process of dimension α to be more precise). By using an algorithm that samples exactly a non central chi-square distribution (see Sect. 3.1), we can then sample $WIS_2(x, \alpha, 0, e_2^1; t)$ when $x_{\{2,2\}} = 0$.

When $x_{\{2,2\}} > 0$, it easy to check that the SDE

$$\begin{aligned} d(X_t^x)_{\{1,1\}} &= \alpha dt + 2\sqrt{(X_t^x)_{\{1,1\}} - \tfrac{((X_t^x)_{\{1,2\}})^2}{(X_t^x)_{\{2,2\}}}} dZ_t^1 + 2\tfrac{(X_t^x)_{\{1,2\}}}{\sqrt{(X_t^x)_{\{2,2\}}}} dZ_t^2 \\ d(X_t^x)_{\{1,2\}} &= \sqrt{(X_t^x)_{\{2,2\}}} dZ_t^2 \\ d(X_t^x)_{\{2,2\}} &= 0 \end{aligned} \tag{5.34}$$

starting from $X_0^x = x$ has an infinitesimal generator equal to $L_{e_2^1}$. To solve (5.34), we set:

$$(U_t^u)_{\{1,1\}} = (X_t^x)_{\{1,1\}} - \frac{((X_t^x)_{\{1,2\}})^2}{(X_t^x)_{\{2,2\}}}, \; (U_t^u)_{\{1,2\}} = \frac{(X_t^x)_{\{1,2\}}}{\sqrt{x_{\{2,2\}}}}, (U_t^u)_{\{2,2\}} = x_{\{2,2\}}. \tag{5.35}$$

Here, u stands for the initial condition, i.e. $u = U_0^u$. We get by using Itô calculus that

$$d(U_t^u)_{\{1,1\}} = (\alpha - 1)dt + 2\sqrt{(U_t^u)_{\{1,1\}}} dZ_t^1, \; d(U_t^u)_{\{1,2\}} = dZ_t^2 \text{ and } d(U_t^u)_{\{2,2\}} = 0. \tag{5.36}$$

Therefore, $(U_t^u)_{\{1,2\}}$ and $(U_t^u)_{\{1,1\}}$ can be sampled respectively by independent Gaussian and non-central chi-square variables. Then, we can get back X_t^x by inverting (5.35):

$$\begin{aligned} (X_t^x)_{\{1,1\}} = (U_t^u)_{\{1,1\}} + (U_t^u)_{\{1,2\}}^2, \; (X_t^x)_{\{1,2\}} &= (U_t^u)_{\{1,2\}} \sqrt{(U_t^u)_{\{2,2\}}}, \\ (X_t^x)_{\{2,2\}} &= (U_t^u)_{\{2,2\}}. \end{aligned} \tag{5.37}$$

This result gives an interesting way to figure out the dynamics associated to the operator $L_{e_2^1}$, by using a change of variable. It is worth to notice that the CIR process $(U_t^u)_{\{1,1\}}$ is well defined as soon as its degree $\alpha - 1$ is nonnegative, which coincides with the condition under which the Wishart process $WIS_2(x, \alpha, 0, e_2^1)$ is well-defined. Last, we notice that the solution of the operator $L_{e_2^1}$ involves a CIR process in the diagonal term and a Brownian motion in the non diagonal one. A similar structure holds for larger d.

The General Case

We present now a general way to sample exactly $WIS_d(x,\alpha,0,e_d^1;t)$. We first write explicitly from (5.10) the infinitesimal generator of $WIS_d(x,\alpha,0,e_d^1)$ for $x \in \mathcal{S}_d^+(\mathbb{R})$:

$$\begin{aligned} L_{e_d^1} f(x) =& \alpha\partial_{\{1,1\}} f(x) + 2x_{\{1,1\}}\partial^2_{\{1,1\}} f(x) + 2\sum_{\substack{1\le m\le d\\ m\neq 1}} x_{\{1,m\}}\partial_{\{1,m\}}\partial_{\{1,1\}} f(x) \\ &+ \frac{1}{2}\sum_{\substack{1\le m,l\le d\\ m\neq 1,l\neq 1}} x_{\{m,l\}}\partial_{\{1,m\}}\partial_{\{1,l\}} f(x). \end{aligned}$$

As for $d = 2$ we will construct an SDE that has the same infinitesimal generator $L_{e_d^1}$ and that can be solved explicitly. To do so, we need however to use further matrix decomposition results. In the case $d = 2$, we have already noticed that we choose different SDEs whether $x_{2,2} = 0$ or not. Here, the SDE will depend on the rank of the submatrix $(x_{i,j})_{2\le i,j\le d}$, and we set:

$$r = \mathrm{Rk}((x_{i,j})_{2\le i,j\le d}) \in \{0,\dots,d-1\}.$$

First, we consider the case where

$$\begin{aligned} &\exists c_r \in \mathcal{G}_r \text{ lower triangular}, k_r \in \mathcal{M}_{d-1-r\times r}(\mathbb{R}), \\ &(x)_{2\le i,j\le d} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix}\begin{pmatrix} c_r^\top & k_r^\top \\ 0 & 0 \end{pmatrix} =: cc^\top. \end{aligned} \tag{5.38}$$

With a slight abuse of notation, we consider that this decomposition also holds when $r = 0$ with $c = 0$. When $r = d-1$, $c = c_r$ is simply the usual Cholesky decomposition of $(x_{i,j})_{2\le i,j\le d}$. As it is explained in Corollary 5.5.5, we can still get such a decomposition up to a permutation of the coordinates $\{2,\dots,d\}$.

Theorem 5.5.3 *Let us consider $x \in \mathcal{S}_d^+(\mathbb{R})$ such that (5.38) holds. Let $(Z_t^l)_{1\le l\le r+1}$ be a vector of independent standard Brownian motions. Then, the following SDE (convention $\sum_{k=1}^r(\dots) = 0$ when $r = 0$)*

$$\begin{aligned} d(X_t^x)_{\{1,1\}} &= \alpha dt + 2\sqrt{(X_t^x)_{\{1,1\}} - \sum_{k=1}^r \left(\sum_{l=1}^r (c_r^{-1})_{k,l}(X_t^x)_{\{1,l+1\}}\right)^2} dZ_t^1 \\ &\quad +2\sum_{k=1}^r\sum_{l=1}^r (c_r^{-1})_{k,l}(X_t^x)_{\{1,l+1\}} dZ_t^{k+1} \\ d(X_t^x)_{\{1,i\}} &= \sum_{k=1}^r c_{i-1,k} dZ_t^{k+1},\ i = 2,\dots,d \\ d((X_t^x)_{\{l,k\}})_{2\le k,l\le d} &= 0 \end{aligned} \tag{5.39}$$

has a unique strong solution starting from x. *It takes values in* $\mathcal{S}_d^+(\mathbb{R})$ *and has the infinitesimal generator* $L_{e_d^1}$. *Moreover, this solution is given explicitly by:*

$$X_t^x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix} \begin{pmatrix} (U_t^u)_{\{1,1\}} + \sum_{k=1}^{r} ((U_t^u)_{\{1,k+1\}})^2 & ((U_t^u)_{\{1,l+1\}})_{1\le l\le r}^\top & 0 \\ ((U_t^u)_{\{1,l+1\}})_{1\le l\le r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r^\top & k_r^\top \\ 0 & 0 & I_{d-r-1} \end{pmatrix}, \tag{5.40}$$

where

$$\begin{aligned} d(U_t^u)_{\{1,1\}} &= (\alpha - r)dt + 2\sqrt{(U_t^u)_{\{1,1\}}}dZ_t^1, \\ u_{\{1,1\}} &= x_{\{1,1\}} - \sum_{k=1}^{r} (u_{\{1,k+1\}})^2 \ge 0, \\ d((U_t^u)_{\{1,l+1\}})_{1\le l\le r} &= (dZ_t^{l+1})_{1\le l\le r}, \\ (u_{\{1,l+1\}})_{1\le l\le r} &= c_r^{-1}(x_{\{1,l+1\}})_{1\le l\le r}. \end{aligned} \tag{5.41}$$

Once again, we have made a slight abuse of notation when $r = 0$, and (5.40) should be simply read as $X_t^x = \begin{pmatrix} (U_t^u)_{\{1,1\}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in that case. In the statement above, it may seem weird that we use for u and U_t^u the same indexation as the one for symmetric matrices while we only use its first row (or column). The reason is that we can in fact see X_t^x as a function of U_t^u by setting:

$$(U_t^u)_{\{i,j\}} = u_{\{i,j\}} = x_{\{i,j\}} \text{ for } i, j \ge 2 \text{ and } (U_t^u)_{\{1,i\}} = u_{\{1,i\}} = 0 \text{ for } r+1 \le i \le d. \tag{5.42}$$

Thus, (c_r, k_r, I_{d-1}) is an extended Cholesky decomposition of $((U_t^u)_{i,j})_{2\le i,j\le d}$ and can be seen as a function of U_t^u. We get from (5.40) that

$$X_t^x = h(U_t^u), \text{ with } h(u) = \sum_{r=0}^{d-1} \mathbb{1}_{r=\mathrm{Rk}[(u_{i,j})_{2\le i,j\le d}]} h_r(u) \text{ and} \tag{5.43}$$

$$h_r(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r(u) & 0 \\ 0 & k_r(u) & I_{d-r-1} \end{pmatrix}$$
$$\times \begin{pmatrix} u_{\{1,1\}} + \sum_{k=1}^{r} (u_{\{1,k+1\}})^2 & (u_{\{1,l+1\}})^\top_{1\le l\le r} & 0 \\ (u_{\{1,l+1\}})_{1\le l\le r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r(u)^\top & k_r(u)^\top \\ 0 & 0 & I_{d-r-1} \end{pmatrix},$$

where $(c_r(u), k_r(u), I_{d-1})$ is the extended Cholesky decomposition of $(u_{i,j})_{2\le i,j\le d}$ given by some algorithm (e.g. Golub and Van Loan [67], Algorithm 4.2.4). Equation (5.43) will be useful to analyse discretization schemes, see Theorem 5.6.3.

Theorem 5.5.3 enables us to simulate exactly the distribution $WIS_d(x, \alpha, 0, e_d^1; t)$ simply by sampling one non-central chi-square distribution for $(U_t^u)_{\{1,1\}}$ as explained in Sect. 3.1 and r other independent Gaussian random variables. Like in the $d = 2$ case, we notice that the condition which ensures that the CIR process $((U_t^u)_{\{1,1\}}, t \ge 0)$ is well defined for any $r \in \{0, \dots, d-1\}$, namely $\alpha - (d-1) \ge 0$, is the same as the one required for the definition of $WIS_d(x, \alpha, 0, e_d^1)$.

Proof of Theorem 5.5.3 The proof is divided into two parts. First, we prove that the SDE (5.39) has a unique strong solution which is given by (5.40) and is well defined on $\mathcal{S}_d^+(\mathbb{R})$. Second, we show that its infinitesimal generator is equal to the operator $L_{e_d^1}$ defined in (5.32).

First step Let us assume that $(X_t^x)_{t\ge 0}$ is a solution to (5.39). We use the matrix decomposition of $(x_{i,j})_{2\le i,j\le d}$ given by (5.38) and set:

$$(U_t)_{\{1,l+1\}} = \sum_{i=1}^{r} (c_r^{-1})_{l,i} (X_t^x)_{\{1,i+1\}}, \quad l \in \{l, \dots, r\},$$

$$(U_t)_{\{1,1\}} = (X_t^x)_{\{1,1\}} - \sum_{l=1}^{r} \left(\sum_{i=1}^{r} (c_r^{-1})_{l,i} (X_t^x)_{\{1,i+1\}} \right)^2$$
$$= (X_t^x)_{\{1,1\}} - \sum_{l=1}^{r} ((U_t)_{\{1,l+1\}})^2.$$

We get by using Lemma A.2.2 that:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix} \begin{pmatrix} (U_t)_{\{1,1\}} + \sum_{k=1}^{r} ((U_t)_{\{1,k+1\}})^2 & ((U_t)_{\{1,l+1\}})^\top_{1\le l\le r} & 0 \\ ((U_t)_{\{1,l+1\}})_{1\le l\le r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r^\top & k_r^\top \\ 0 & 0 & I_{d-r-1} \end{pmatrix}$$
$$= \begin{pmatrix} (U_t)_{\{1,1\}} + \sum_{k=1}^{r} ((U_t)_{\{1,k+1\}})^2 & ((U_t)_{\{1,l+1\}})^\top_{1\le l\le r} c_r^\top & ((U_t)_{\{1,l+1\}})^\top_{1\le l\le r} k_r^\top \\ c_r ((U_t)_{\{1,l+1\}})_{1\le l\le r} & c_r c_r^\top & c_r k_r^\top \\ k_r ((U_t)_{\{1,l+1\}})_{1\le l\le r} & k_r c_r^\top & 0 \end{pmatrix}$$
$$= X_t^x.$$

Since $\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix}$ is invertible, $X_t^x \in \mathcal{S}_d^+(\mathbb{R})$ if, and only if:

$$\forall z \in \mathbb{R}^d, \quad z^\top \begin{pmatrix} (U_t)_{\{1,1\}} + \sum_{i=1}^r ((U_t)_{\{1,i+1\}})^2 & ((U_t)_{\{1,l\}})_{2\le l\le r+1} & 0 \\ ((U_t)_{\{l,1\}})_{2\le l\le r+1} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} z \tag{5.44}$$

$$= z_1^2 (U_t)_{\{1,1\}} + \sum_{i=1}^r (z_{i+1} + (U_t)_{\{1,i+1\}} z_1)^2 \ge 0, \iff (U_t)_{\{1,1\}} \ge 0.$$

In particular, we get that $(U_0)_{\{1,1\}} = u_{\{1,1\}} \ge 0$ since $x \in \mathcal{S}_d^+(\mathbb{R})$. Now, by Itô calculus, we get from (5.39) that $d(U_t)_{\{1,l+1\}} = \sum_{i=1}^r \sum_{k=1}^r (c_r^{-1})_{l,i} (c_r)_{i,k} dZ_t^{k+1} = dZ_t^{l+1}$ and

$$\begin{aligned} d(U_t)_{\{1,1\}} &= (\alpha - r)dt + 2\sqrt{(U_t)_{\{1,1\}}} dW_t^1 + 2\sum_{l=1}^r \sum_{k=1}^r (c_r^{-1})_{l,k} (X_t)_{\{1,k+1\}} dW_t^{l+1} \\ &\quad - \sum_{l=1}^r 2((U_t)_{\{1,l+1\}}) dW_t^{l+1} \\ &= (\alpha - r)dt + 2\sqrt{(U_t)_{\{1,1\}}} dW_t^1. \end{aligned}$$

Thus, the solution $(X_t^x)_{t\ge 0}$ is necessarily the one given by (5.40) (pathwise uniqueness holds for $((U_t^u)_{\{1,l\}})_{1\le l\le r+1}$, and especially for the CIR diffusion $(U_t^u)_{\{1,1\}}$ since $\alpha \ge d-1 \ge r$). Reciprocally, it is easy to check by Itô calculus that (5.40) solves (5.39).

Second step Now, we want to show that $L_{e_d^1}$ is the infinitesimal operator associated to the process $(X_t^x)_{t\ge 0}$. It is sufficient to compare the drift and the quadratic covariation of the process X_t^x with $L_{e_d^1}$. Since the drift part of $(X_t^x)_{t\ge 0}$ clearly corresponds to the first order of $L_{e_d^1}$, we study directly the quadratic part. From (5.39), we have for $i, j \in \{2, \dots, d\}^2$:

$$\begin{aligned} \langle d(X_t^x)_{\{1,1\}}, d(X_t^x)_{\{1,1\}} \rangle &= 4((X_t^x)_{\{1,1\}} - \sum_{k=1}^r \left[\sum_{l=1}^r (c_r^{-1})_{k,l} (X_t^x)_{\{1,l+1\}}\right]^2 \\ &\quad + \sum_{k=1}^r [\sum_{l=1}^r (c_r^{-1})_{k,l} (X_t^x)_{\{1,l+1\}}]^2) dt \\ &= 4(X_t^x)_{\{1,1\}} dt, \\ \langle d(X_t^x)_{\{1,i\}}, d(X_t^x)_{\{1,j\}} \rangle &= \sum_{k=1}^r (c_r)_{i-1,k} (c_r)_{j-1,k} dt \\ &= (cc^\top)_{i-1,j-1} dt = (X_t^x)_{\{i,j\}} dt. \end{aligned}$$

If $i \le r+1$, we have

$$\begin{aligned} \langle d(X_t^x)_{\{1,1\}}, d(X_t^x)_{\{1,i\}} \rangle &= 2\sum_{k=1}^r \sum_{l=1}^r (c_r)_{i-1,k} (c_r^{-1})_{k,l} (X_t^x)_{\{1,l+1\}} dt \\ &= 2(X_t^x)_{\{1,i\}} dt, \text{ if } i \le r+1. \end{aligned}$$

Otherwise $i > r + 1$, and we have by Lemma A.2.2,

$$\langle d(X_t^x)_{\{1,1\}}, d(X_t^x)_{\{1,i\}}\rangle = 2\sum_{k=1}^{r}\sum_{l=1}^{r}(k_r)_{i-1-r,k}(c_r^{-1})_{k,l}(X_t^x)_{\{1,l+1\}}dt$$

$$= 2\sum_{l=1}^{r}(k_r c_r^{-1})_{i-1-r,l}(X_t^x)_{\{1,l+1\}}dt = 2(X_t^x)_{\{1,i\}}dt.$$

Thus, we deduce that $L_{e_d^1}$ is the infinitesimal generator of $(X_t^x)_{t\geq 0}$. □

Remark 5.5.4 From (5.40), we get easily by a calculation made in (5.44) that $\mathrm{Rk}(X_t^x) = \mathrm{Rk}((x_{i,j})_{2\leq i,j\leq d}) + \mathbb{1}_{(U_t^u)_{\{1,1\}}\neq 0}$, and therefore,

$$\mathrm{Rk}(X_t^x) = \mathrm{Rk}((x_{i,j})_{2\leq i,j\leq d}) + 1, \text{ a.s.}$$

In particular, X_t^x is almost surely positive definite if $x \in \mathcal{S}_d^{+,*}(\mathbb{R})$.

Theorem 5.5.3 assumes that the initial value $x \in \mathcal{S}_d^{+}(\mathbb{R})$ satisfies (5.38). Now, we explain why it is still possible up to a permutation of the coordinates to be in such a case. This relies on the extended Cholesky decomposition which is stated in Lemma A.2.1.

Corollary 5.5.5 *Let $x \in \mathcal{S}_d^{+}(\mathbb{R})$ and (c_r, k_r, p) be an extended Cholesky decomposition of $(x_{i,j})_{2\leq i,j\leq d}$ (Lemma A.2.1). Then, $\pi = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ is a permutation matrix,*

$$WIS_d(x, \alpha, 0, e_d^1) \underset{law}{=} \pi^\top WIS_d(\pi x \pi^\top, \alpha, 0, e_d^1)\pi,$$

and $((\pi x \pi^\top)_{i,j})_{2\leq i,j\leq d} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix}\begin{pmatrix} c_r^\top & k_r^\top \\ 0 & 0 \end{pmatrix}$ *satisfies* (5.38).

Proof The result comes directly from (5.22), since $\pi^\top = \pi^{-1}$ and $\pi e_d^1 \pi^\top = e_d^1$. □

Therefore, by a combination of Corollary 5.5.5 and Theorem 5.5.3, we get a simple way to construct explicitly a process that has the infinitesimal generator $L_{e_d^1}$ for any initial condition $x \in \mathcal{S}_d^{+}(\mathbb{R})$. In particular, this enables us to sample exactly the Wishart distribution $WIS_d(x, \alpha, 0, e_d^1; t)$. Let us discuss now the complexity of this simulation method. The number of operations required by the extended Cholesky decomposition is of order $O(d^3)$. From a computational point of view, the permutation is handled directly and does not require any matrix multiplication so that we can consider without loss of generality that $\pi = I_d$. Since c_r is lower triangular, the calculation of $u_{\{1,i\}}$, $i = 1, \ldots, r+1$ only requires $O(d^2)$ operations. Also, we do not perform in practice the matrix product (5.40), but only compute the values of $X_{\{1,i\}}$ for $i = 1, \ldots, d$, which requires also $O(d^2)$ operations. Last, d samples are at most required. To sum up, it comes out that the complexity of simulating $WIS_d(x, \alpha, 0, e_d^1; t)$ is of order $O(d^3)$.

Algorithm 5.1: Exact simulation of $WIS_d(x, \alpha, 0, e_d^1; t)$.

Input: $x \in \mathcal{S}_d^+(\mathbb{R})$, d, $\alpha \geq d-1$ and $t > 0$.
Output: X, sampled according to $WIS_d(x, \alpha, 0, e_d^1; t)$
Compute the extended Cholesky decomposition (p, k_r, c_r) of $(x_{i,j})_{2 \leq i,j \leq d}$ given by Lemma A.2.1, $r \in \{0, \dots, d-1\}$ (see Golub and Van Loan [67] for an algorithm).
Set $\pi = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, $\tilde{x} = \pi x \pi^T$, $(u_{\{1,l+1\}})_{1 \leq l \leq r} = (c_r)^{-1}(\tilde{x}_{\{1,l+1\}})_{1 \leq l \leq r}$ and $u_{\{1,1\}} = \tilde{x}_{\{1,1\}} - \sum_{k=1}^r (u_{\{1,k+1\}})^2 \geq 0$.
Sample independently r normal variables $G_2, \dots, G_{r+1} \sim \mathcal{N}(0,1)$ and $(U_t^u)_{\{1,1\}}$ as a CIR process at time t starting from $u_{\{1,1\}}$ solving $d(U_t^u)_{\{1,1\}} = (\alpha - r)dt + 2\sqrt{(U_t^u)_{\{1,1\}}}dZ_t^1$ (see Sect. 3.1).
for $l \in \{1, \dots, r\}$ **do**
   $(U_t^u)_{\{1,l+1\}} = u_{\{1,l+1\}} + \sqrt{t} G_{l+1}$
end

$$X = \pi^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix} \begin{pmatrix} (U_t^u)_{\{1,1\}} + \sum_{k=1}^r ((U_t^u)_{\{1,k+1\}})^2 & ((U_t^u)_{\{1,l+1\}})_{1 \leq l \leq r}^T & 0 \\ ((U_t^u)_{\{1,l+1\}})_{1 \leq l \leq r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r^T & k_r^T \\ 0 & 0 & I_{d-r-1} \end{pmatrix} \pi.$$

5.5.3 *Exact Simulation for Wishart Processes*

We have now shown all the mathematical results that enable us to give an exact simulation method for general Wishart processes. This is made in two steps.

First, we know how to sample exactly $WIS_d(x, \alpha, 0, e_d^1; t)$ thanks to Theorem 5.5.3 and Corollary 5.5.5. By a simple permutation of the first and kth coordinates, we are then also able to sample according to $WIS_d(x, \alpha, 0, e_d^k; t)$ for $k \in \{1, \dots, d\}$. Thus, we get by Proposition 5.5.2 an exact simulation method to sample $WIS_d(x, \alpha, 0, I_d^n; t)$. Then, we get an exact simulation scheme for $WIS_d(x, \alpha, b, a; t)$ by using the law identity (5.23).

Algorithm 5.2: Exact simulation for $WIS_d(x, \alpha, 0, I_d^n; t)$.

Input: $x \in \mathcal{S}_d^+(\mathbb{R})$, $n \leq d$, $\alpha \geq d-1$ and $t > 0$.
Output: X, sampled according to $WIS_d(x, \alpha, 0, I_d^n; t)$.
$y = x$.
for $k = 1$ **to** n **do**
   Set $p_{k,1} = p_{1,k} = p_{i,i} = 1$ for $i \notin \{1, k\}$, and $p_{i,j} = 0$ otherwise (permutation of the first and kth coordinates).
   $y = pYp$ where Y is sampled according to $WIS_d(pyp, \alpha, 0, e_d^1; t)$ by using Algorithm 5.1.
end
$X = y$.

Algorithm 5.3: Exact simulation for $WIS_d(x, \alpha, b, a; t)$.

Input: $x \in \mathcal{S}_d^+(\mathbb{R})$, $\alpha \geq d-1$, $a, b \in \mathcal{M}_d(\mathbb{R})$ and $t > 0$.
Output: X, sampled according to $WIS_d(x, \alpha, b, a; t)$.
Calculate $q_t = \int_0^t \exp(sb)a^T a \exp(sb^T)ds$ and (p, c_n, k_n) an extended Cholesky decomposition of q_t/t.
Set $\theta_t = p^{-1}\begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix}$ and $m_t = \exp(tb)$.
$X = \theta_t Y \theta_t^T$, where $Y \sim WIS_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, I_d^n; t)$ is sampled by Algorithm 5.2.

Let us analyse the overall complexity of simulating $WIS_d(x, \alpha, b, a; t)$. Since it basically uses n times the simulation of $WIS_d(x, \alpha, 0, e_d^1; t)$, it requires a complexity of order $O(nd^3)$. Therefore, it is at most of order $O(d^4)$. Somehow, the "bottle-neck" is the extended Cholesky decomposition which requires $O(d^3)$ operations and has to be recalculated for each the simulation of $WIS_d(x, \alpha, 0, e_d^1; t)$. All the other calculations require at most $O(d^2)$ operations.

Remark 5.5.6 When $\alpha \geq 2d - 1$, it is possible to sample $WIS_d(x, \alpha, 0, I_d^n; t)$ in $O(d^3)$ by another mean. If $X_t^1 \sim WIS_d(x, d, 0, I_d^n; t)$ and $X_t^2 \sim WIS_d(0, \alpha - d, 0, I_d^n; t)$ are independent, we can check that $X_t^1 + X_t^2 \sim WIS_d(x, \alpha, 0, I_d^n; t)$. Then, X_t^1 can be sampled by using Proposition 5.6.9 and X_t^2 by using Bartlett's decomposition (5.45) since $X_t^2 \overset{law}{=} t WIS_d(0, \alpha - d, 0, I_d^n; 1)$ from (5.19).

5.5.4 The Bartlett's Decomposition Revisited

Now, we would like to illustrate our exact simulation method on the following particular distribution $WIS_d(0, \alpha, 0, I_d^n; 1)$, which is known in the literature as the central Wishart distribution. In that case, we can perform explicitly the composition $X_1^{n,\dots X_1^{1,0}}$ given by Proposition 5.5.2. We will show by an induction on n that:

$$X_1^{n,\dots X_1^{1,0}} = \begin{pmatrix} (L_{i,j})_{1 \leq i,j \leq n} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (L_{i,j}^\top)_{1 \leq i,j \leq n} & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.45}$$

where $(L_{i,j})_{1 \leq j < i \leq d}$ and $L_{i,i}$ are independent random variables such that $L_{i,j} \sim \mathcal{N}(0, 1)$ and $(L_{i,i})^2 \sim \chi^2(\alpha - i + 1)$, and $L_{i,j} = 0$ for $i < j$. This result is known as the Bartlett's decomposition and dates back to 1933 (see Kshirsagar [90] or Kabe [79]).

For $n = 1$, we know from Theorem 5.5.3 that $(X_1^{1,0})_{1,1} \sim \chi^2(\alpha)$ since $d(X_t^{1,0})_{1,1} = \alpha dt + 2\sqrt{(X_t^{1,0})_{1,1}} dZ_t^1$ with $(X_0^{1,0})_{1,1} = 0$, and all the other elements are equal to 0. Let us assume now that the induction hypothesis is satisfied for $n-1$. Then, we can apply once again Theorem 5.5.3 (up to the permutation of the first and nth coordinates). We have $\mathrm{Rk}(X_1^{n-1,\dots X_1^{1,0}}) = n-1, a.s.$, and the Cholesky decomposition is directly given by $(L_{i,j})_{1\le i,j\le n-1}$. Then, we get from (5.40) that there are independent variables $L_{n,n}^2 \sim \chi^2(\alpha - n + 1)$ and $L_{n,i} \sim \mathcal{N}(0,1)$ for $i \in \{1,\dots,n-1\}$ such that $X_1^{n,\dots X_1^{1,0}} =$

$$\begin{pmatrix} (L_{i,j})_{1\le i,j\le n-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{d-n} \end{pmatrix} \begin{pmatrix} I_{n-1} & (L_{n,i})_{1\le i\le n-1} & 0 \\ (L_{n,i})^\top_{1\le i\le n-1} & \sum_{i=1}^n L_{n,i}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (L_{i,j})^\top_{1\le i,j\le n-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{d-n} \end{pmatrix}.$$

Since

$$\begin{pmatrix} I_{n-1} & (L_{n,i})_{1\le i\le n-1} & 0 \\ (L_{n,i})^\top_{1\le i\le n-1} & \sum_{i=1}^n L_{n,i}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{n-1} & 0 & 0 \\ (L_{n,i})^\top_{1\le i\le n-1} & L_{n,n} & 0 \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} I_{n-1} & (L_{n,i})_{1\le i\le n-1} & 0 \\ 0 & L_{n,n} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we conclude by induction on n.

5.6 High Order Discretization Schemes for Wishart and Semidefinite Positive Affine Processes

We now focus on constructing high order discretization schemes. This will enable us to simulate not only Wishart processes, but also general affine processes. Besides, the discretization schemes that we introduce are in practice faster than the exact simulation scheme, especially if one has to sample entire paths. This will be illustrated in Sect. 5.7.

Up to our knowledge, there are very few papers in the literature that deal with discretization schemes for Wishart processes. Recently, Benabid et al. [17] have proposed a Monte-Carlo method to calculate expectations on Wishart processes which is based on a Girsanov change of probability. Gauthier and Possamaï [60] introduce a moment-matching scheme for Wishart processes. Both methods are well defined under some restrictions on the parameters, and there is no theoretical result on their accuracy. Here, we explain how to get high order schemes for $WIS_d(x, \alpha, 0, e_d^1)$ from the construction given by Theorem 5.5.3. The remarkable splitting (5.28) enables us to get high order schemes for $WIS_d(x, \alpha, 0, I_d^n)$ by using the composition of schemes, see Proposition 2.3.12 and Corollaries 2.3.13 and 2.3.14. From this result, we can construct a second order scheme for any semidefinite positive affine processes and a third order scheme for Wishart processes. Before that, we study the Cauchy problem in the case of Wishart processes in order to check that the assumption (i) of Theorem 2.3.8 is satisfied.

5.6.1 Study of the Cauchy Problem

The study of the Cauchy problem in for the Wishart process is similar to the one that is made in Sect. 3.3.5 for the CIR process. We first prove in Lemma 5.6.1 a remarkable formula for the Laplace transform that extends the formula (3.30) for the CIR. Then, we use the same technique and write the test function as the inverse Fourier transform of its Fourier transform.

Lemma 5.6.1 *Let* $(X_t^x)_{t\geq 0} \underset{Law}{\sim} WIS_d(x,\alpha,b,a)$ *and* $v = v_R + i v_I$ *such that* $v_R \in \mathcal{D}_{b,a;t}$ *and* $v_I \in \mathcal{S}_d(\mathbb{R})$*. We denote by* $\phi(t,\alpha,x,v)$ *the Laplace transform of* X_t^x *given by* (5.18)*, the other parameters* a*,* b *being fixed. Then, the derivative w.r.t* $x_{\{k,l\}}$ *satisfies the following equality*

$$\partial_{\{k,l\}}\phi(t,\alpha,x,v) = \phi(t,\alpha+2,x,v)p_t^{\{k,l\}}(v), \tag{5.46}$$

where $p_t^{\{k,l\}}$ *is a polynomial function of the matrix elements of degree* d *defined by:*

$$p_t^{\{k,l\}}(v) = \mathrm{Tr}\left[v\mathrm{adj}(I_d - 2q_t v)m_t(e_d^{k,l} + \mathbb{1}_{k\neq l}e_d^{l,k})m_t^\top\right] =: \sum_{\gamma\in\mathbb{N}^{\frac{d(d+1)}{2}},|\gamma|\leq d} a_t^{\gamma,\{k,l\}}\overline{v}^\gamma,$$

where

$$\overline{v}^\gamma = \prod_{\{i,j\}} v_{\{i,j\}}^{\gamma_{\{i,j\}}}.$$

Moreover, its coefficients are bounded uniformly in time:

$$\exists K_t > 0,\ \forall s \in [0,t],\quad \max_{\gamma\in\mathbb{N}^{\frac{d(d+1)}{2}},|\gamma|\leq d}\left(|a_s^{\gamma,\{k,l\}}|\right) \leq K_t.$$

Proof We get from (5.18),

$$\begin{aligned}
&\partial_{\{k,l\}}\phi(t,\alpha,x,v)\\
&= \frac{\mathrm{Tr}\left[v\mathrm{adj}(I_d - 2q_t v)m_t(e_d^{k,l} + \mathbb{1}_{k\neq l}e_d^{l,k})m_t^\top\right]}{\det(I_d - 2q_t v)}\\
&\quad\times\frac{\exp\left(\mathrm{Tr}\left[v(I_d - 2q_t v)^{-1}m_t x m_t^\top\right]\right)}{\det(I_d - 2q_t v)^{\frac{\alpha}{2}}}\\
&= \phi(t,\alpha+2,x,v)\mathrm{Tr}\left[v\mathrm{adj}(I_d - 2q_t v)m_t(e_d^{k,l} + \mathbb{1}_{k\neq l}e_d^{l,k})m_t^\top\right].
\end{aligned}$$

Since $s \mapsto \|m_s\|$ and $s \mapsto \|q_s\|$ are continuous functions on $[0,t]$, we obtain the bounds on the polynomial coefficients. □

Proposition 5.6.2 *Let $(X_t^x)_{t\geq 0} \sim WIS_d(x,\alpha,b,a)$ and L the associated generator. Let $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d(\mathbb{R}))$, $x \in \mathcal{S}_d^+(\mathbb{R})$ and $T>0$. Then, $\tilde{u}(t,x) = \mathbb{E}[f(X_t^x)]$ is $\mathcal{C}^\infty$ on $[0,T]\times \mathcal{S}_d^+(\mathbb{R})$, solves $\partial_t \tilde{u}(t,x) = L\tilde{u}(t,x)$ and its derivatives satisfy*

$$\forall l \in \mathbb{N}, \forall n \in \mathbb{N}^{\frac{d(d+1)}{2}}, \exists C_{l,n}, e_{l,n} > 0, \forall x \in \mathcal{S}_d^+(\mathbb{R}),\ \forall t \in [0,T],$$

$$\left| \partial_t^l \prod_{1\leq i \leq j \leq d} \partial_{\{i,j\}}^{n_{\{i,j\}}} \tilde{u}(t,x) \right| \leq C_{l,n}(1+\|x\|^{e_{l,n}}). \tag{5.47}$$

Proof Let $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d(\mathbb{R}))$. First, let us observe that (5.47) is obvious when $l = |n| = 0$ Since we have $\forall l \in \mathbb{N}, L^l f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d(\mathbb{R}))$, and $\partial_t^l \tilde{u}(t,x) = \mathbb{E}(L^l f(X_t^x))$, it is sufficient to prove (5.47) only for the derivatives w.r.t. x.

We first focus on the case $|n| = 1$ and want to show that $\partial_{\{k,l\}}\tilde{u}(t,x)$ satisfies (5.47). The sketch of this proof is to write f as the inverse Fourier transform of its Fourier transform and then use Lemma 5.6.1. Unfortunately, f has not a priori the required integrability to do that, and we have to introduce an auxiliary function f_ρ.

Definition of the New Function f_ρ. Since $\mathcal{D}_{b,a;T}$ given by (5.17) is an open set and $0 \in \mathcal{D}_{b,a;T}$, there is $\rho > 0$ such that $\rho I_d \in \mathcal{D}_{b,a;T}$. Let $\mu : \mathbb{R} \to \mathbb{R}$ be the function such that $\mu(x) = 0$ if $x \leq -1$ or $x \geq 0$, $\mu(x) = \exp(\frac{1}{x(x+1)})$ if $-1 < x < 0$. We have $\mu \in \mathcal{C}^\infty(\mathbb{R})$.

We consider then the cutoff function $\zeta : \mathbb{R} \to \mathbb{R} \in \mathcal{C}^\infty(\mathbb{R})$ defined as $\forall x \in \mathbb{R}$, $\zeta(x) = \frac{\int_{-\infty}^x \mu(y)dy}{\int_{\mathbb{R}} \mu(y)dy}$. It is nondecreasing, such that $0 \leq \zeta(x) \leq 1$, $\zeta(x) = 0$ if $x \leq -1$ and $\zeta(x) = 1$ if $x \geq 0$. Besides, we have $\zeta \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{R})$ since all its derivatives have a compact support. Now, we define a $\vartheta \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d(\mathbb{R}))$ as

$$\begin{aligned} \vartheta : \mathcal{S}_d(\mathbb{R}) &\to \mathbb{R} \\ x &\mapsto \prod_{i=1}^d \zeta(x_{\{i,i\}}) \prod_{i\neq j} \zeta(x_{\{j,j\}}x_{\{i,i\}} - x_{\{i,j\}}^2). \end{aligned}$$

It is important to notice that $0 \leq \vartheta \leq 1$, $\vartheta(x) = 1$ if $x \in \mathcal{S}_d^+(\mathbb{R})$ and $\vartheta(x) = 0$ if there is $i \in \{1,\dots,d\}$ such that $x_{\{i,i\}} < -1$ or $i < j \in \{1,\dots,d\}$ such that $x_{\{i,j\}}^2 > 1 + x_{\{i,i\}}x_{\{i,i\}}$. Let $\gamma \in \mathbb{N}^{d(d-1)/2}$. Since $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d(\mathbb{R}))$, there are constants $K, E > 0$ and $K', E' > 0$ such that, $\forall x \in \mathcal{S}_d(\mathbb{R})$

$$\begin{aligned} |\partial^\gamma(\vartheta f)(x)| &\leq K(1+\|x\|^E)\prod_{i=1}^d \left(1_{\{|x_{\{i,i\}}|>-1\}}\right) \prod_{1\leq i<j\leq d} \left(\mathbb{1}_{\{x_{\{i,j\}}^2 \leq 1 + x_{\{i,i\}}x_{\{j,j\}}\}}\right) \\ &\leq K'(1+\|(x_{\{i,i\}})_{1\leq i\leq d}\|^{E_1})\prod_{i=1}^d \left(1_{\{|x_{\{i,i\}}|>-1\}}\right) \\ &\quad \times \prod_{1\leq i<j\leq d} \left(\mathbb{1}_{\{x_{\{i,j\}}^2 \leq 1 + x_{\{i,i\}}x_{\{j,j\}}\}}\right). \end{aligned}$$

Here, the upper bound only involves the diagonal coefficients. We define

$$x \in \mathcal{S}_d(\mathbb{R}),\ \ f_\rho(x) := \vartheta(x) f(x) \exp(-\mathrm{Tr}(\rho x)),$$

and obtain from the last inequality that f_ρ belongs to the Schwartz space of rapidly decreasing functions since $\rho > 0$. Thus, its Fourier transform also belongs to the Schwartz space and we have

$$f_\rho(x) = \frac{1}{(2\pi)^{\frac{d(d+1)}{2}}} \int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \exp(-\mathrm{Tr}(ivx))\mathcal{F}(f_\rho)(v)dv,$$

where

$$\mathcal{F}(f_\rho)(v) = \int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \exp(\mathrm{Tr}(ivx)) f_\rho(x)dx,$$

and in particular $f_\rho, \mathcal{F}(f_\rho) \in L^1(\mathcal{S}_d(\mathbb{R})) \cap L^\infty(\mathcal{S}_d(\mathbb{R}))$.

A New Representation of $\tilde{u}(t,x)$. We have $f(x) = \exp(\rho\mathrm{Tr}(x)) f_\rho(x)$ for $x \in \mathcal{S}_d^+(\mathbb{R})$, and therefore

$$\begin{aligned}
\tilde{u}(t,x) &= \mathbb{E}[\exp(\mathrm{Tr}(\rho X_t^x)) f_\rho(X_t^x)] \\
&= \frac{1}{(2\pi)^{\frac{d(d+1)}{2}}} \mathbb{E}\left[\int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \exp(\mathrm{Tr}[(-iv + \rho I_d)X_t^x])\mathcal{F}(f_\rho)(v)dv\right] \\
&= \frac{1}{(2\pi)^{\frac{d(d+1)}{2}}} \int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \mathbb{E}[\exp(\mathrm{Tr}[(-iv + \rho I_d)X_t^x])]\mathcal{F}(f_\rho)(v)dv.
\end{aligned}$$

The last equality holds since

$$\int_{\mathbb{R}^{\frac{d(d+1)}{2}}} |\mathbb{E}[\exp(\mathrm{Tr}[(-iv+\rho I_d)X_t^x])]| \times |\mathcal{F}(f_\rho)(v)|dv \leq \phi(t,\alpha,x,\rho I_d)\|\mathcal{F}(f_\rho)\|_1 < \infty.$$

Here, we have used that $\rho I_d \in \mathcal{D}_{b,a;T}$ to get $\phi(t,\alpha,x,\rho I_d) < \infty$.

Derivation with Respect to $x_{\{k,l\}}$, $k,l \in \{1,\dots,d\}$. From Lemma 5.6.1, we have by Lebesgue's theorem

$$\partial_{\{k,l\}}\tilde{u}(t,x) = \frac{1}{(2\pi)^{\frac{d(d+1)}{2}}} \int_{\mathbb{R}^{\frac{d(d+1)}{2}}} \phi(t,\alpha+2,x,-iv+\rho I_d) p_t^{\{k,l\}}(\rho I_d - iv)\mathcal{F}(f_\rho)(v)dv \tag{5.48}$$

since $|\partial^x_{\{k,l\}}\phi(t,\alpha,x,-iv+\rho I_d)\mathcal{F}(f_\rho)(v)| \leq |\phi(t,\alpha+2,x,\rho I_d)||p_t^{\{k,l\}}(\rho I_d - iv)\mathcal{F}(f_\rho)(v)|$ and $p_t^{\{k,l\}}(\rho I_d - iv)\mathcal{F}(f_\rho)(v)$ is a rapidly decreasing function.

Let $1 \le k', l' \le d$. An integration by part gives

$$\int_{\mathbb{R}} (\rho I_d - iv)_{\{k',l'\}} \exp\left(\mathrm{Tr}[x(iv - \rho I_d)]\right) \vartheta(x) f(x) dx_{\{k',l'\}}$$
$$= \left(\frac{\mathbb{1}_{k' \neq l'}}{2} + \mathbb{1}_{k'=l'}\right) \int_{\mathbb{R}} \exp\left(\mathrm{Tr}[x(iv - \rho I_d)]\right) \partial_{\{k',l'\}}(\vartheta(x) f(x)) dx_{\{k',l'\}},$$

and thus

$$(\rho I_d - iv)_{\{k',l'\}} \mathcal{F}(\exp[-\rho \mathrm{Tr}(x)] \vartheta(x) f(x))(v)$$
$$= \left(\frac{\mathbb{1}_{k' \neq l'}}{2} + \mathbb{1}_{k'=l'}\right) \mathcal{F}(\exp[-\rho \mathrm{Tr}(x)] \partial_{\{k',l'\}}[\vartheta(x) f(x)])(v).$$

We set $\varphi(\gamma) = \prod_{1 \le k' \le l' \le d} \left(\frac{\mathbb{1}_{k' \neq l'}}{2} + \mathbb{1}_{k'=l'}\right)^{\gamma_{\{k',l'\}}}$ for $\gamma \in \mathbb{N}^{d(d+1)/2}$ and get by iterating the argument that:

$$\prod_{1 \le k' \le l' \le d} (\rho I_d - iv)_{\{k',l'\}}^{\gamma_{\{k',l'\}}} \mathcal{F}(f_\rho)(v) = \varphi(\gamma) \mathcal{F}\left(\exp\left[-\rho \mathrm{Tr}(x)\right] \partial_\gamma(\vartheta \times f)(x)\right)(v). \tag{5.49}$$

Since $p_t^{\{k,l\}}(\rho I_d - iv) = \sum_{\gamma \in \mathbb{N}^{\frac{d(d+1)}{2}}, |\gamma| \le d} a_t^{\gamma,\{k,l\}} \prod_{1 \le k' \le l' \le d} (\rho I_d - iv)_{\{k',l'\}}^{\gamma_{\{k',l'\}}}$, we get from (5.48) and (5.49):

$$\partial_{\{k,l\}} u(t,x) = \sum_{|\gamma| \le d} a_t^{\gamma,\{k,l\}} \varphi(\gamma) \mathbb{E}\left(\partial_\gamma(f \times \vartheta)(Y_t^x)\right) = \sum_{|\gamma| \le d} a_t^{\gamma,\{k,l\}} \varphi(\gamma) \mathbb{E}\left(\partial_\gamma f(Y_t^x)\right), \tag{5.50}$$

where $(Y_t^x)_{t \ge 0} \underset{Law}{\sim} WIS_d(x, \alpha + 2, b, a)$. Here, we have used that $\partial_\gamma(\vartheta \times f)(y) = \partial_\gamma f(y)$ for $y \in \mathcal{S}_d^+(\mathbb{R})$. From Lemma 5.6.1 $(a_t^{\gamma,\{k,l\}})_{\gamma \in \mathbb{N}^{\frac{d(d+1)}{2}}, |\gamma| \le d}$ is bounded for $t \in [0,T]$, and we get (5.47) when $|n| = 1$ since $\partial_\gamma f \in \mathcal{C}_{\mathbf{pol}}^\infty(\mathcal{S}_d(\mathbb{R}))$. Thanks to (5.50), a derivative of order $|n|$, can be seen as a (bounded) linear combination of derivatives of order $|n| - 1$, and we easily get (5.47) by an induction on $|n|$.

It remains to check that we have indeed $\partial_t \tilde{u}(t,x) = Lu(t,x)$. Let $t, h > 0$. By the Markov property, we have $\tilde{u}(t+h,x) = \mathbb{E}[\tilde{u}(t, X_h^x)]$. From (5.47) and Itô's formula, we get $[\tilde{u}(t+h,x) - u(t,x)]/h \underset{h \to 0+}{\to} Lu(t,x)$. □

A natural question is to wonder if the result of Proposition 5.6.2 could be extended to general affine diffusions on semidefinite positive matrices. It does not seem straightforward to get in this case a similar formula to (5.46), since we no longer have an explicit formula for the characteristic function. It is therefore not clear how to adapt the proof to any affine diffusion on $\mathcal{S}_d^+(\mathbb{R})$. Despite this obstacle, there is no fundamental reason to expect that Proposition 5.6.2 may hold for Wishart processes and not for the other affine diffusions on $\mathcal{S}_d^+(\mathbb{R})$.

5.6.2 High Order Schemes for Wishart Processes

In this paragraph, we will give a way to get weak νth-order schemes for any Wishart processes. The construction is similar to the one used for the exact scheme. First we obtain a νth-order scheme for $WIS_d(x,\alpha,0,e_d^1)$. Then, we get a νth-order scheme for $WIS_d(x,\alpha,0,I_d^n)$ from the splitting (5.28) and Corollary 2.3.13. Last, we use the identity in law (5.23) to get a weak νth-order scheme for any Wishart processes.

Let us start then by introducing a potential weak νth-order scheme for $WIS_d(x,\alpha,0,e_d^1)$. Roughly speaking, we obtain this scheme from the exact scheme given by Theorem 5.5.3 and Corollary 5.5.5 by replacing the Gaussian random variables with moment matching variables and the exact CIR distribution with a sample according to a potential weak ν-th order scheme for the CIR.

Theorem 5.6.3 *Let $x \in \mathcal{S}_d^+(\mathbb{R})$ and (c_r,k_r,p) be an extended Cholesky decomposition of $(x_{i,j})_{2\le i,j\le d}$. We set $\pi = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and $\tilde{x} = \pi x \pi^\top$, so that $(\tilde{x}_{i,j})_{2\le i,j\le d} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix}\begin{pmatrix} c_r^\top & k_r^\top \\ 0 & 0 \end{pmatrix}$. Like in Theorem 5.5.3, we have*

$$u_{\{1,1\}} = \tilde{x}_{\{1,1\}} - \sum_{k=1}^{r}(u_{\{1,k+1\}})^2 \ge 0, \text{ where } (u_{\{1,l+1\}})_{1\le l\le r} = c_r^{-1}(\tilde{x}_{\{1,l+1\}})_{1\le l\le r},$$

and we set $u_{\{1,i\}} = 0$ if $r+2 \le i \le d$ and $u_{\{i,j\}} = \tilde{x}_{\{i,j\}}$ if $i,j \ge 2$. Let $(\hat{G}^i)_{1\le i\le r}$ be a sequence of independent real variables with finite moments of any order such that:

$$\forall i \in \{1,\dots,r\},\ \forall k \le 2\nu+1,\ \mathbb{E}[(\hat{G}^i)^k] = \mathbb{E}[G^k], \text{ where } G \sim \mathcal{N}(0,1).$$

Let h_r be the function defined by (5.43). Let $(\hat{U}_t^u)_{\{1,1\}}$ be sampled independently according to a potential weak νth-order scheme for the CIR process $d(U_t^u)_{\{1,1\}} = (\alpha - r)dt + 2\sqrt{(U_t^u)_{\{1,1\}}}dZ_t^1$ starting from $u_{\{1,1\}}$. We set:

$$(\hat{U}_t^u)_{\{1,i\}} = u_{\{1,i\}} + \sqrt{t}\hat{G}^i,\ 2 \le i \le r+1,\ \ (\hat{U}_t^u)_{\{1,i\}} = 0,\ r+2 \le i \le d,$$
$$(\hat{U}_t^u)_{\{i,j\}} = u_{\{i,j\}} \text{ if } i,j \ge 2.$$

Then, the scheme $\hat{X}_t^x = \pi^\top h_r(\hat{U}_t^u)\pi$ is a potential νth-order scheme for $L_{e_d^1}$ and takes values in $\mathcal{S}_d^+(\mathbb{R})$.

Let us give the idea of the proof. By construction, we have $\hat{X}_t^x \in \mathcal{S}_d^+(\mathbb{R})$ since an analogous formula to (5.40) holds for $\hat{X}_t^x$. The tedious part is to check that it is a potential ν-th order scheme. We know from Theorem 5.5.3, Eq. (5.43) and Corollary 5.5.5 that we have $X_t^x = \pi^\top h_r(U_t^u)\pi$. It is easy to check that $\hat{U}_t^u$ is a potential ν-th order scheme for the operator associated to the diffusion U_t^u. Let us suppose for a while that $h_r(u) \in \mathcal{C}^{\infty}_{\mathbf{pol}}(\mathcal{S}_d(\mathbb{R}))$. Then, $u \mapsto f(\pi^\top h_r(u)\pi)$ is also in $\mathcal{C}^{\infty}_{\mathbf{pol}}(\mathcal{S}_d(\mathbb{R}))$, and for any $f \in \mathcal{C}^{\infty}_{\mathbf{pol}}(\mathcal{S}^+_d(\mathbb{R}))$, there are constants $C, E, \eta > 0$ depending only on a good sequence of f such that:

$$|\mathbb{E}[f(\pi^\top h_r(\hat{U}^u_t)\pi)] - \mathbb{E}[f(X^x_t)]| \le Ct^{\nu+1}(1 + \|x\|^E),$$

which basically gives the desired result. Unfortunately, h_r is not in $\mathcal{C}^{\infty}_{\mathbf{pol}}(\mathcal{S}_d(\mathbb{R}))$. In fact, h_r is only smooth with respect to the coefficients of the first row and the first columns. However, these coefficients are also the only ones that are changed by $\hat{U}^u_t$ (the submatrix $((\hat{U}^u_t)_{i,j})_{2\le i,j\le d} = (u_{i,j})_{2\le i,j\le d}$ is constant), and it comes out that the regularity on h_r is sufficient to get a potential νth-order scheme for $L_{e^1_d}$. The detailed proof of Theorem 5.6.3 that uses this argument is given in Sect. 5.8.1.

Algorithm 5.4: Third (resp. second) order scheme for $WIS_d(x, \alpha, 0, e^1_d; t)$.

Input: $x \in \mathcal{S}^+_d(\mathbb{R})$, d, $\alpha \ge d - 1$ and $t > 0$.
Output: X.
Same as Algorithm 5.1, but using Algorithm 3.3 (resp. 3.1) to sample $(U^u_t)_{\{1,1\}}$ and moment-matching variables (2.28) (resp. (2.27)) instead of Normal variables.

We comment now briefly the practical implementation of Theorem 5.6.3. Second and third order schemes for the CIR process satisfying have been obtained in Propositions 3.3.5 and 3.3.8. We can therefore get second (resp. third) order schemes for $L_{e^1_d}$ by taking any variables that matches the five (resp. the seven) first moments of $\mathcal{N}(0, 1)$. This can be obtained by using the discrete random variables (2.27) and (2.28).

We focus now on the construction of a potential weak νth-order scheme for $WIS_d(x, \alpha, 0, I^n_d)$. Let $\hat{X}^{1,x}_t$ denote a potential weak νth-order scheme for $WIS_d(x, \alpha, 0, e^1_d)$. For $i \in \{2, \dots, d\}$, $WIS_d(x, \alpha, 0, e^i_d)$ and $WIS_d(x, \alpha, 0, e^1_d)$ have the same law up to the permutation of the first and ith coordinate. Let $\pi^{1\leftrightarrow i}$ denote the associated permutation matrix. Then, we easily get that

$$\hat{X}^{i,x}_t = \pi^{1\leftrightarrow i}\hat{X}^{1,\pi^{1\leftrightarrow i}x\pi^{1\leftrightarrow i}}_t\pi^{1\leftrightarrow i}$$

is a potential νth-order scheme for $WIS_d(x, \alpha, 0, e^i_d)$. Last, we get from Theorem 5.5.1 and Corollary 2.3.13 that

$$\hat{X}^{n,\dots\hat{X}^{1,x}_t}_t \text{ is a potential weak } \nu\text{th-order scheme for } WIS_d(x, \alpha, 0, I^n_d). \tag{5.51}$$

Now we are in position to construct a scheme for any Wishart process $WIS_d(x, \alpha, b, a)$ thanks to the identity (5.23). Let $\theta_t \in \mathcal{G}_d(\mathbb{R})$ be such as in Proposition 5.3.2 and $\hat{Y}^y_t$ denote a potential weak νth-order scheme for $WIS_d(y, \alpha, 0, I^n_d)$. Then, we consider the following scheme for $WIS_d(x, \alpha, b, a)$:

$$\hat{X}^x_t = \theta_t\hat{Y}^{\theta_t^{-1}m_t x m_t^\top(\theta_t^{-1})^\top}_t\theta_t^\top. \tag{5.52}$$

Unfortunately, we need to make some technical restrictions on a and b (namely, $a \in \mathcal{G}_d(\mathbb{R})$ or $ba^\top a = a^\top ab$) to show that we get like this a potential νth-order scheme. We however believe that this is rather due to our analysis of the error and that the scheme converges as well without this restriction. We mention in addition that we give in the next section a second order scheme based on Proposition 5.3.1 for which we can make our error analysis for any parameters.

Proposition 5.6.4 *Let $t > 0$, $a, b \in \mathcal{M}_d(\mathbb{R})$ and $\alpha \geq d-1$. Let $m_t = \exp(tb)$, $q_t = \int_0^t \exp(sb)a^\top a \exp(sb^\top)ds$ and $n = \mathrm{Rk}(a^\top a)$. We assume that either $a \in \mathcal{G}_d(\mathbb{R})$ or b and $a^\top a$ commute. We define*

- *If $n = d$, θ_t as the (usual) Cholesky decomposition of q_t/t.*
- *If $n < d$, $\theta_t = \sqrt{\frac{1}{t}\int_0^t \exp(sb)\exp(sb^\top)ds}\, p^{-1}\begin{pmatrix} c_n & 0 \\ k_n & I_{d-n}\end{pmatrix}$ where (c_n, k_n, p) is the extended Cholesky decomposition of $a^\top a$ otherwise.*

In both cases, $\theta_t \in \mathcal{G}_d(\mathbb{R})$ and the scheme (5.52) is a potential weak νth-order scheme for $WIS_d(x, \alpha, b, a)$.

The proof of Proposition 5.6.4 is left in Sect. 5.8.2. From Theorem 2.3.8, we finally get the following result by using Propositions 5.6.2 and 5.6.4.

Theorem 5.6.5 *Let $(X_t^x)_{t\geq 0} \sim WIS_d(x, \alpha, b, a)$ such that either $a \in \mathcal{G}_d(\mathbb{R})$ or $a^\top ab = ba^\top a$, and $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d(\mathbb{R}))$. Let $T > 0$, $t_i^N = \frac{iT}{N}$ for $0 \leq i \leq N$ and $(\hat{X}_{t_i^N}, 0 \leq i \leq N)$ be sampled with the scheme defined by Proposition 5.6.4 and Theorem 5.6.3 with the third order scheme for the CIR given by Proposition 3.3.8. Then,*

$$\exists C, N_0 > 0, \forall N \geq N_0, |\mathbb{E}[f(\hat{X}_{t_N^N})] - \mathbb{E}[f(X_T^x)]| \leq C/N^3.$$

Algorithm 5.5: Third (resp. second) order scheme for $WIS_d(x, \alpha, b, a; t)$.

Input: $x \in \mathcal{S}_d^+(\mathbb{R})$, $\alpha \geq d-1$, $a, b \in \mathcal{M}_d(\mathbb{R})$ and $t > 0$.
Output: X.
Same as Algorithm 5.3, using inside Algorithm 5.4 instead of Algorithm 5.1.

5.6.3 Second Order Schemes for Affine Diffusions on $\mathcal{S}_d^+(\mathbb{R})$

We now present a potential second order scheme for general affine diffusions on semidefinite positive matrices, namely for $AFF_d(x, \overline{\alpha}, B, a)$. Thanks to Proposition 5.3.1, there is $u \in \mathcal{G}_d(\mathbb{R})$ and a diagonal matrix $\overline{\delta}$ such that $\overline{\alpha} = u^\top \overline{\delta} u$, $a^\top a = u^\top I_d^n u$ and we have:

$$(u^\top Y_t^{(u^{-1})^\top x u^{-1}} u)_{t\geq 0} \sim AFF_d(x, \overline{\alpha}, B, a), \text{ where } (Y_t^y)_{t\geq 0} \sim AFF_d(y, \overline{\delta}, B_u, I_d^n).$$

Using the same linear transformation, we can get a potential νth-order scheme for $AFF_d(x, \overline{\alpha}, B, a)$ from a potential νth-order scheme for $AFF_d(y, \overline{\delta}, B_u, I_d^n)$ as stated below.

Lemma 5.6.6 *If $\hat{Y}_t^y$ is a potential νth-order scheme for $AFF_d(y, \overline{\delta}, B_u, I_d^n)$, then*

$$u^\top \hat{Y}_t^{(u^{-1})^\top x u^{-1}} u$$

is a potential νth-order scheme for $AFF_d(x, \overline{\alpha}, B, a)$.

Proof Let $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d^+(\mathbb{R}))$. We then have $x \mapsto f(u^\top x u) \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d^+(\mathbb{R}))$. Since u is fixed, there are constants C, η, E depending only on a good sequence of f such that for $t \in (0, \eta)$, $|\mathbb{E}[f(u^\top \hat{Y}_t^{(u^{-1})^\top x u^{-1}} u)] - \mathbb{E}[f(X_t^x)]| = |\mathbb{E}[f(u^\top \hat{Y}_t^{(u^{-1})^\top x u^{-1}} u)] - \mathbb{E}[f(u^\top Y_t^{(u^{-1})^\top x u^{-1}} u)]| \le C t^{\nu+1}(1 + \|(u^{-1})^\top x u^{-1}\|^E) \le C' t^{\nu+1}(1 + \|x\|^E)$, for some constant $C' > C$. □

We now focus on finding a scheme for $AFF_d(y, \overline{\delta}, B_u, I_d^n)$, and we will construct it from the second order scheme for $WIS_d(x, \alpha, 0, I_d^n)$ obtained in (5.51). Since $\overline{\delta}$ is a diagonal matrix such that $\overline{\delta} - (d-1)I_d^n \in \mathcal{S}_d^+(\mathbb{R})$, we have

$$\delta_{\min} := \min_{1 \le i \le n} \overline{\delta}_{i,i} \ge d - 1.$$

We rewrite the infinitesimal generator of Y_t^y as follows:

$$\begin{aligned} L &= \mathrm{Tr}([\overline{\delta} + B_u(x)]D^{\mathcal{S}}) + 2\mathrm{Tr}(x D^{\mathcal{S}} I_d^n D^{\mathcal{S}}) \\ &= \underbrace{\mathrm{Tr}([\overline{\delta} - \delta_{\min} I_d^n + B_u(x)]D^{\mathcal{S}})}_{L_{ODE}} + \underbrace{\delta_{\min}\mathrm{Tr}(I_d^n D^{\mathcal{S}}) + 2\mathrm{Tr}(x D^{\mathcal{S}} I_d^n D^{\mathcal{S}})}_{L_{WIS_d(x, \delta_{\min}, 0, I_d^n)}}. \end{aligned} \tag{5.53}$$

It is the sum of the infinitesimal generator of $WIS_d(x, \delta_{\min}, 0, I_d^n)$ and of the generator of the affine ODE

$$dX_t^{ODE,x} = [\overline{\delta} - \delta_{\min} I_d^n + B_u(X_t^{ODE,x})]dt, \ X_0^{ODE,x} = x \in \mathcal{S}_d^+(\mathbb{R}).$$

We know by Lemma 5.1.6 that $X_t^{ODE,x} \in \mathcal{S}_d^+(\mathbb{R})$ for any $t \ge 0$ since Assumption (5.4) holds for B_u and $\overline{\delta} - \delta_{\min} I_d^n \in \mathcal{S}_d^+(\mathbb{R})$. Besides, this ODE can be solved explicitly since it is linear and we have

$$\begin{aligned} X_t^{ODE,x} &= \exp(tB_u)(x) + \int_0^t \exp(sB_u)(\overline{\delta} - \delta_{\min} I_d^n) ds \\ &= \sum_{k=0}^\infty \frac{t^k}{k!} B_u^{(k)}(x) + \frac{t^{k+1}}{(k+1)!} B_u^{(k)}(\overline{\delta} - \delta_{\min} I_d^n). \end{aligned}$$

Let $\hat{X}_t^x$ denote the potential second order scheme for $WIS_d(x, \delta_{\min}, 0, I_d^n)$ obtained by (5.51) that uses the nested second-order scheme for the CIR given by Proposition 3.3.5. By using Corollary 2.3.14, the schemes

$$\hat{Y}_t^x = X_{t/2}^{ODE, \hat{X}_t^{X_{t/2}^{ODE,x}}} \text{ or } \hat{Y}_t^x = (1-B)\hat{X}_t^{X_t^{ODE,x}} + BX_t^{ODE,\hat{X}_t^x} \text{ with } B \sim \mathcal{B}(1/2), \tag{5.54}$$

are potential second order scheme for $AFF_d(x, \overline{\delta}, B_u, I_d^n)$. In the numerical experiments in Sect. 5.7, we have used $X_{t/2}^{ODE, \hat{X}_t^{X_{t/2}^{ODE,x}}}$ even though the other scheme would have worked as well: it is in fact a computational trade-off between solving a deterministic ODE and drawing a Bernoulli variable. Thanks to Lemma 5.6.6, Proposition 5.6.2 and Theorem 2.3.8, we finally get the following result.

Theorem 5.6.7 *The scheme defined by Lemma 5.6.6 and Eq.* (5.54) *is a potential second order scheme for* $AFF_d(x, \overline{\alpha}, B, a)$. *In the Wishart case* (5.3), *we have for* $f \in \mathcal{C}^{\infty}_{\mathbf{pol}}(\mathcal{S}_d(\mathbb{R}))$:

$$\exists C, N_0 > 0, \forall N \geq N_0, |\mathbb{E}[f(\hat{X}_{t_N^N})] - \mathbb{E}[f(X_T^x)]| \leq C/N^2,$$

where $(\hat{X}_{t_i^N}, 0 \leq i \leq N)$ *is sampled with this scheme on the regular time grid* $t_i^N = \frac{iT}{N}$, $0 \leq i \leq N$.

Algorithm 5.6: Second order scheme for $AFF_d(x, \overline{\alpha}, B, a; t)$.

Input: $x \in \mathcal{S}_d^+(\mathbb{R})$, $B \in \mathcal{L}(\mathcal{S}_d(\mathbb{R}))$ satisfying (5.4), $a \in \mathcal{M}_d(\mathbb{R})$, $\overline{\alpha} \in \mathcal{S}_d^+(\mathbb{R})$ such that $\overline{\alpha} - (d-1)a^\top a \in \mathcal{S}_d^+(\mathbb{R})$ and $t > 0$.

Output: X.

Calculate u such that $\overline{\alpha} = u^\top \overline{\delta} u$, $a^\top a = u^\top I_d^n u$, see Algorithm 8.7.1 of Golub and Van Loan [67].

$X = (u^{-1})^\top x u^{-1}$,

$X = \sum_{k=0}^{\infty} \frac{(t/2)^k}{k!} B_u^{(k)}(X) + \frac{(t/2)^{k+1}}{(k+1)!} B_u^{(k)}(\overline{\delta} - \delta_{\min} I_d^n)$.

Use the second order scheme given by Algorithm 5.4 for $WIS_d(X, \delta_{\min}, 0, I_d^n; t)$ and store the result in X.

$X = \sum_{k=0}^{\infty} \frac{(t/2)^k}{k!} B_u^{(k)}(X) + \frac{(t/2)^{k+1}}{(k+1)!} B_u^{(k)}(\overline{\delta} - \delta_{\min} I_d^n)$,

$X = u^\top X u$.

Remark 5.6.8 Unless in special cases such as $B(x) = \lambda x$, $\lambda \in \mathbb{R}$ where the series in Algorithm 5.6 can be calculated with a real exponential, one has otherwise to truncate the series far enough to achieve a very sharp accuracy. However, this can be quite time consuming. To avoid this, we can replace the exact scheme for L_{ODE} by the following one

$$\bar{X}_t^{ODE,x} = \sum_{k=0}^{2} \frac{t^k}{k!} B_u^{(k)}(x) + \sum_{k=0}^{1} \frac{t^{k+1}}{(k+1)!} B_u^{(k)}(\overline{\delta} - \delta_{\min} I_d^n).$$

We check by using Lemma 2.3.6 that this is a potential second order scheme for L_{ODE} since $\|\bar{X}_t^{ODE,x} - X_t^{ODE,x}\| \leq Ct^3(1 + \|x\|)$ for some constant $C > 0$. However, even though we know that $X_t^{ODE,x} \in \mathcal{S}_d^+(\mathbb{R})$ there is no reason a priori to have $\bar{X}_t^{ODE,x} \in \mathcal{S}_d^+(\mathbb{R})$ and the composition $\bar{X}_{t/2}^{ODE,\bar{X}_t^{\bar{X}_{t/2}^{ODE,x}}}$ may not be well defined. Therefore, we take $\hat{X}_t^{ODE,x} = \left(\bar{X}_t^{ODE,x}\right)^+$ which is semidefinite positive. It satisfies $\|\hat{X}_t^{ODE,x} - X_t^{ODE,x}\| \leq \|\bar{X}_t^{ODE,x} - X_t^{ODE,x}\| \leq Ct^3(1 + \|x\|)$ by Lemma A.1.3 and therefore is also a potential second order scheme for L_{ODE} by Lemma 2.3.6. Thus, $\hat{X}_{t/2}^{ODE,\hat{X}_t^{\hat{X}_{t/2}^{ODE,x}}}$ is well defined and is also a second order scheme for $AFF_d(x, \overline{\alpha}, B, a)$.

5.6.4 A Faster Second Order Scheme for $AFF_d(x, \overline{\alpha}, B, a)$ When $\bar{\alpha} - da^\top a \in \mathcal{S}_d^+(\mathbb{R})$

In this section, we focus on the complexity of the discretization schemes with respect to the dimension d. Up to now, the discretization schemes that we have considered in Theorems 5.6.5 and 5.6.7 have a complexity of $O(d^4)$, as the exact sampling method. Indeed, both schemes rely on the construction (5.51) to sample $WIS_d(x, \alpha, 0, I_d^n)$, which requires n Cholesky decompositions. This requires at most $O(d^4)$ operations. Here, we present a second order scheme whose complexity is $O(d^3)$, provided that $\bar{\alpha} - da^\top a \in \mathcal{S}_d^+(\mathbb{R})$ or $\alpha \geq d$ in the Wishart case. The practical relevance of such a scheme will be illustrated in Sect. 5.7.

To do so, we use the same construction as in Sect. 5.6.3, and we remark that different splitting from (5.53) are possible. In fact, we could have chosen instead $L = \mathrm{Tr}([\overline{\delta} - \beta I_d^n + B_u(x)]D^{\mathcal{S}}) + \beta \mathrm{Tr}(I_d^n D^{\mathcal{S}}) + 2\mathrm{Tr}(xD^{\mathcal{S}} I_d^n D^{\mathcal{S}})$ for any $\beta \in [d-1, \delta_{\min}]$: the first part is the operator of an affine ODE which is well defined on $\mathcal{S}_d^+(\mathbb{R})$ by Lemma 5.1.6 while the second part is the generator of $WIS_d(x, \beta, 0, I_d^n)$. When $\delta_{\min} \geq d$, which is equivalent to $\overline{\alpha} - da^\top a \in \mathcal{S}_d^+(\mathbb{R})$, the following splitting obtained with $\beta = d$

$$L = \underbrace{\mathrm{Tr}([\overline{\delta} - dI_d^n + B_u(x)]D^{\mathcal{S}})}_{\tilde{L}_{ODE}} + \underbrace{d\mathrm{Tr}(I_d^n D^{\mathcal{S}}) + 2\mathrm{Tr}(xD^{\mathcal{S}} I_d^n D^{\mathcal{S}})}_{L_{WIS_d(x,d,0,I_d^n)}} \tag{5.55}$$

is really interesting. Indeed, the process $WIS_d(x, d, 0, I_d^n)$ can be seen as the square of an Ornstein-Uhlenbeck process on matrices and can be simulated very efficiently as follows.

Proposition 5.6.9 *Let $x \in \mathcal{S}_d^+(\mathbb{R})$ and $c \in \mathcal{M}_d(\mathbb{R})$ be such that $c^\top c = x$. We have:*

$$((c + W_t I_d^n)^\top (c + W_t I_d^n), t \geq 0) \underset{law}{=} WIS_d(x, d, 0, I_d^n).$$

If $\hat{G}$ denote a d-by-d matrix with independent elements sampled according to (2.27), $\hat{X}_t^x = (c + \sqrt{t}\hat{G} I_d^n)^\top (c + \sqrt{t}\hat{G} I_d^n)$ is a potential second order scheme for $WIS_d(x, d, 0, I_d^n)$.

Proof The law identity is direct consequence of Proposition 5.3.4 with $d' = d$, $b = 0$ and $a = I_d^n$.

Let us show now that $\hat{X}_t^x$ is a potential second order scheme. We can see $c + \sqrt{t}\hat{G} I_d^n$ as the Ninomiya-Victoir scheme with moment-matching variables (see Theorem 2.3.17) associated to the infinitesimal generator $\frac{1}{2}\sum_{i=1}^d \sum_{j=1}^n \partial_{i,j}^2$ on $\mathcal{M}_d(\mathbb{R})$. Let $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d^+(\mathbb{R}))$. Then, $x \in \mathcal{M}_d(\mathbb{R}) \mapsto f(x^\top x) \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{M}_d(\mathbb{R}))$ and there are constants $C, E, \eta > 0$ depending only on a good sequence of f such that:

$$\begin{aligned}\forall t \in (0, \eta), |\mathbb{E}[f((c+\sqrt{t}\hat{G} I_d^n)^\top (c+\sqrt{t}\hat{G} I_d^n))] - \mathbb{E}[f((c+W_t I_d^n)^\top (c+W_t I_d^n))]| \\ \leq C t^{\nu+1}(1+\|c\|^E).\end{aligned}$$

Let us observe now that the Frobenius norm of c is $\sqrt{\mathrm{Tr}(c^\top c)} = \sqrt{\mathrm{Tr}(x)} \leq \sqrt{d + \mathrm{Tr}(x^2)} \leq \sqrt{d} + \sqrt{\mathrm{Tr}(x^2)}$. Therefore, for any norm, there is a constant $K > 0$ such that $\|c\| \leq K(1 + \|x\|)$, which gives the result. □

To compute $\hat{X}_t^x$, one has to sample d^2 random variables and to make one matrix product, which requires $O(d^3)$ operations. This is faster than the scheme obtained by (5.51). Then we follow the same argument as in Sect. 5.6.3 and set

$$d\tilde{X}_t^{ODE,x} = [\overline{\delta} - d I_d^n + B_u(\tilde{X}_t^{ODE,x})]dt, \ \tilde{X}_0^{ODE,x} = x \in \mathcal{S}_d^+(\mathbb{R}).$$

Again, this linear ODE is well defined on $\mathcal{S}_d^+(\mathbb{R})$ by Lemma 5.1.6 and can be solved explicitly. By Corollary 2.3.14,

$$\hat{Y}_t^x = \tilde{X}_{t/2}^{ODE,\hat{X}_t^{\tilde{X}_{t/2}^{ODE,x}}} \text{ or } \hat{Y}_t^x = (1-B)\hat{X}_t^{\tilde{X}_t^{ODE,x}} + B\tilde{X}_t^{ODE,\hat{X}_t^x} \tag{5.56}$$

are potential second order scheme for $AFF_d(x, \overline{\delta}, B_u, I_d^n)$ that have still a $O(d^3)$ complexity. Thanks to Lemma 5.6.6, Proposition 5.6.2 and Theorem 2.3.8, we get a similar result to Theorem 5.6.7.

Theorem 5.6.10 *Let us assume that $\bar{\alpha} - da^\top a \in \mathcal{S}_d^+(\mathbb{R})$. The scheme defined by Lemma 5.6.6 and Eq. (5.56) is a potential second order scheme for $AFF_d(x, \overline{\alpha}, B, a)$ that requires at most $O(d^3)$ operations. In the Wishart case (5.3), we have for $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d(\mathbb{R}))$:*

$$\exists C, N_0 > 0, \forall N \geq N_0, |\mathbb{E}[f(\hat{X}_{t_N^N})] - \mathbb{E}[f(X_T^x)]| \leq C/N^2.$$

In a recent paper, Baldi and Pisani [15] have proposed a second order scheme that has some similarities to the scheme given by Theorem 5.6.10.

Algorithm 5.7: Fast second order scheme for $AFF_d(x, \overline{\alpha}, B, a; t)$.

Input: $x \in \mathcal{S}_d^+(\mathbb{R})$, $B \in \mathcal{L}(\mathcal{S}_d(\mathbb{R}))$ satisfying (5.4), $a \in \mathcal{S}_d^+(\mathbb{R})$, $\overline{\alpha} \in \mathcal{S}_d^+(\mathbb{R})$ such that $\overline{\alpha} - da^\top a \in \mathcal{S}_d^+(\mathbb{R})$ and $t > 0$.
Output: X.
Calculate u such that $\overline{\alpha} = u^\top \overline{\delta} u$, $a^\top a = u^\top I_d^n u$, see Algorithm 8.7.1 of Golub and Van Loan [67].
$X = (u^{-1})^\top x u^{-1}$,
$X = \sum_{k=0}^\infty \frac{(t/2)^k}{k!} B_u^{(k)}(X) + \frac{(t/2)^{k+1}}{(k+1)!} B_u^{(k)}(\overline{\delta} - \delta_{\min} I_d^n)$.
Calculate c the Cholesky decomposition of X.
Sample $\hat{G}$, a d-by-d matrix with independent elements following the law (2.27), and set $X = (c + \sqrt{t}\hat{G} I_d^n)^\top (c + \sqrt{t}\hat{G} I_d^n)$.
$X = \sum_{k=0}^\infty \frac{(t/2)^k}{k!} B_u^{(k)}(X) + \frac{(t/2)^{k+1}}{(k+1)!} B_u^{(k)}(\overline{\delta} - \delta_{\min} I_d^n)$.
$X = u^\top X u$.

5.7 Numerical Results on the Simulation Methods

The goal of this section is to compare the different simulation methods that we have presented for Wishart processes. We consider a time horizon T and the regular time-grid $t_i^N = iT/N$, for $i = 0, \ldots, N$. In addition, we want to compare our schemes to a standard one. Thus, we will consider the following corrected Euler-Maruyama scheme for $AFF_d(x, \overline{\alpha}, B, a)$:

$$\begin{aligned}
\hat{X}_{t_0^N}^N &= x, \\
\hat{X}_{t_{i+1}^N}^N &= \hat{X}_{t_i^N}^N + (\overline{\alpha} + B(\hat{X}_{t_i^N}^N))\frac{T}{N} + \sqrt{(\hat{X}_{t_i^N}^N)^+}(W_{t_{i+1}^N} - W_{t_i^N})a \\
&\quad + a^\top (W_{t_{i+1}^N} - W_{t_i^N})^\top \sqrt{(\hat{X}_{t_i^N}^N)^+}, 0 \leq i \leq N-1.
\end{aligned} \tag{5.57}$$

The positive part of a symmetric matrix is given by (5.11) and belongs to $\mathcal{S}_d^+(\mathbb{R})$. Its square root is thus well defined. Without this positive part, the scheme above would not be well defined for any realization of W.

First, we compare the time required by the different schemes and the exact simulation. Then, we present numerical results on the convergence of the different schemes. Last, we give an illustration of the simulation schemes to the Gourieroux-Sufana model in finance.

5.7.1 Time Comparison Between the Different Algorithms

As it has already been mentioned, the complexity of the exact scheme as well as the one of the second order scheme (given by Theorem 5.6.7) and the third order scheme (given by Theorem 5.6.5) is in $O(d^4)$ for one time-step. To be more precise, they require $O(d^4)$ operations that mainly corresponds to d Cholesky decompositions, $O(d^2)$ generations of Gaussian (or moment-matching) variables and $O(d)$ generations of noncentral chi-square distributions (or second or third order schemes for the CIR). The time saved by the second and third order schemes with respect to the exact scheme only comes from the generation of random variables. For example, the generation of the moment-matching variables (2.27) and (2.28) is approximately 2.5 faster than the generation of $\mathcal{N}(0,1)$ on the computer that has be used for the simulations. The gain between the second or third order schemes for the CIR given by Propositions 3.3.5 and 3.3.8 and the exact sampling of the CIR given by Glasserman [62] (namely given by Propositions 3.1.1 and 3.1.2) is much greater, but it depends on the parameters of the CIR. When the dimension d gets larger, the absolute gain in time between the discretization schemes and the exact scheme is of course increased. However, the relative gain instead decreases to 1, because more and more time is devoted to matrix operations and Cholesky decompositions that are the same in both cases. Let us now quickly analyse the complexity of the other schemes. The second order scheme given by Theorem 5.6.10 (called "second order bis" later on) has a complexity in $O(d^3)$ operations for one Cholesky decomposition and matrix multiplications, with $O(d^2)$ generations of Gaussian variables. The complexity of the corrected Euler scheme is of the same kind. At each time-step, $O(d^3)$ operations are needed for matrix multiplications and for diagonalizing the matrix in order to compute the square-root of its positive part. However, diagonalizing a symmetric matrix is in practice much longer than computing a Cholesky decomposition even though both algorithms are in $O(d^3)$. Also, one has to sample $O(d^2)$ Gaussian variables for the Brownian increments.

In Table 5.1, we have calculated by a Monte-Carlo method one value of the characteristic function of a Wishart process. It is also known analytically thanks to (5.18), and we have indicated in each case the exact value. We have considered dimensions $d = 3$ and $d = 10$. We have given in each case an example where

Table 5.1 $\mathbb{E}[\exp(-\mathrm{Tr}(iv\hat{X}^N_{t^N_N}))]$ calculated by a Monte-Carlo with 10^6 samples for a Wishart process with $a = I_d$, $b = 0$, $x = 10I_d$, $v = 0.09I_d$ and $T = 1$

	$N = 10$			$N = 30$		
Schemes	R. value	Im. value	Time	R. value	Im. value	Time
Exact (1 step)	−0.526852	−0.227962	12			
Second order bis	−0.526229	−0.228663	41	−0.526486	−0.229078	125
Second order	−0.526577	−0.228923	76	−0.526574	−0.228133	229
Third order	−0.527021	−0.227286	82	−0.527613	−0.228376	244
Exact (N steps)	−0.526963	−0.228303	123	−0.526891	−0.227729	369
Corrected Euler	−0.525627*	−0.233863*	225	−0.525638*	−0.231449*	687
$\alpha = 3.5$, $d = 3$, $\Delta_R = 10^{-3}$, $\Delta_{Im} = 10^{-3}$, $EV = -0.527090 - 0.228251i$						
Exact (1 step)	−0.591579	−0.037651	12			
Second order	−0.590444	−0.037024	77	−0.590808	−0.036487	229
Third order	−0.591234	−0.034847	82	−0.590818	−0.036210	246
Exact (N steps)	−0.591169	−0.036618	174	−0.592145	−0.037411	920
Corrected Euler	−0.589735*	−0.042002*	223	−0.590079*	−0.039937*	680
$\alpha = 2.2$, $d = 3$, $\Delta_R = 0.9 \times 10^{-3}$, $\Delta_{Im} = 1.3 \times 10^{-3}$, $EV = -0.591411 - 0.036346i$						
Exact (1 step)	0.062712	−0.063757	181			
Second order bis	0.064237	−0.063825	921	0.064573	−0.062747	2762
Second order	0.064922	−0.064103	1431	0.063534	−0.063280	4283
Third order	0.064620	−0.064543	1446	0.064120	−0.063122	4343
Exact (N steps)	0.063418	−0.064636	1806	0.063469	−0.064380	5408
Corrected Euler	0.068298*	−0.058491*	2312	0.061732*	−0.056882*	7113
$\alpha = 10.5$, $d = 10$, $\Delta_R = 1.4 \times 10^{-3}$, $\Delta_{Im} = 1.3 \times 10^{-3}$, $EV = 0.063960 - 0.063544i$						
Exact (1 step)	−0.036869	−0.094156	177			
Second order	−0.036246	−0.094196	1430	−0.035944	−0.092770	4285
Third order	−0.035408	−0.093479	1441	−0.036277	−0.093178	4327
Exact (N steps)	−0.036478	−0.092860	1866	−0.036145	−0.093003	6385
Corrected Euler	−0.028685*	−0.094281*	2321	−0.030118*	−0.088988*	7144
$\alpha = 9.2$, $d = 10$, $\Delta_R = 1.4 \times 10^{-3}$, $\Delta_{Im} = 1.4 \times 10^{-3}$, $EV = -0.036064 - 0.093275i$						

The exact value of $\mathbb{E}[\exp(-\mathrm{Tr}(ivX^x_1))]$ is denoted by EV. The starred numbers are those for which the exact value is outside the 95 % confidence interval, and Δ_R (resp. Δ_I) gives the two standard deviations value on the real (resp. imaginary) part

$\alpha \geq d$ and another one where $d - 1 \leq \alpha < d$. We have used the different algorithms presented in this paper: "Second order bis" stands for the scheme given by Theorem 5.6.10 [with the moment-matching variables (2.27)], "second order" stands for the scheme given by Theorem 5.6.7 (with (2.27) and the second order scheme for the CIR given by Proposition 3.3.5), "third order" stands for the scheme given by Theorem 5.6.5 (with (2.28) and the third order scheme for the CIR given by Proposition 3.3.8), and "Corrected Euler" stands for the corrected Euler-Maruyama scheme (5.57). For the exact scheme, we have both considered the cases with one time-step T and N time-steps T/N. Of course, the first case is sufficient to calculate an expectation that only depends on X_T, but the second case would be the time needed to compute pathwise expectations. For each method, we have given the value obtained and the time needed (in seconds) on the computer, a 3,000 MHz CPU.

First, let us mention that the exact value is in each case in the confidence interval except for the corrected Euler scheme. As one can expect, the exact method with one time-step is by far the quickest method to compute an expectation that only depends on the final value. We put aside this case and focus now on the generation of the whole path. We see from Table 5.1 that the second and the third order schemes require roughly the same computation time. As expected, the second order scheme bis is much faster when it is defined (i.e. when $\alpha \geq d$). On the contrary, the Euler scheme is much slower than the second and third order scheme. This is due to the cost of the matrix diagonalization. Let us mention that the time required by the discretization schemes is proportional to N and do not depend on the parameters when the dimension is given. On the contrary, the time needed by the exact scheme may change according to α and can increase considerably when α is close to $d-1$. To be more precise, the exact simulation method for the CIR given by Propositions 3.1.1 and 3.1.2 uses a rejection sampling when the degree of freedom is lower than 1, which corresponds to the case $d-1 \leq \alpha < d$. The rejection rate can in fact be rather high, notably when the time-step gets smaller. For $N = 30$, $d = 3$ and $\alpha = 2.2$, the exact scheme is four times slower than the second order scheme and 2.5 slower than the exact scheme with $\alpha = 3.5$.

Let us draw a conclusion from this time comparison between the different schemes. Obviously, we recommend to use the exact scheme when calculating expectations that depend on one or few dates. Instead, when calculating pathwise expectations of affine processes by Monte-Carlo, we would recommend to use in general the second order bis scheme when $\alpha \geq d$ and the second order (or third order for Wishart processes) when $d-1 \leq \alpha < d$.

5.7.2 Numerical Results on the Convergence

Now, we want to illustrate the theoretical results of convergence obtained in this paper for the different schemes. To do so, we have plotted for each scheme $\mathbb{E}[\exp(-\mathrm{Tr}(i v \hat{X}^N_{t^N_N}))]$ in function of the time step T/N. This expectation is calculated by a Monte-Carlo method. As for the time comparison, we illustrate the convergence for $d = 3$ in Fig. 5.1 and $d = 10$ in Fig. 5.2. Each time, we consider a case where $\alpha \geq d$ and a case where $d-1 \leq \alpha < d$, which is in general tougher. In these figures,

- Scheme 1 denotes the value obtained by the exact scheme with one time-step.
- Scheme 2 stands for the second order scheme given by Theorem 5.6.7.
- Scheme 3 denotes the third order scheme given by Theorem 5.6.5.
- Scheme 4 is the corrected Euler scheme (5.57).

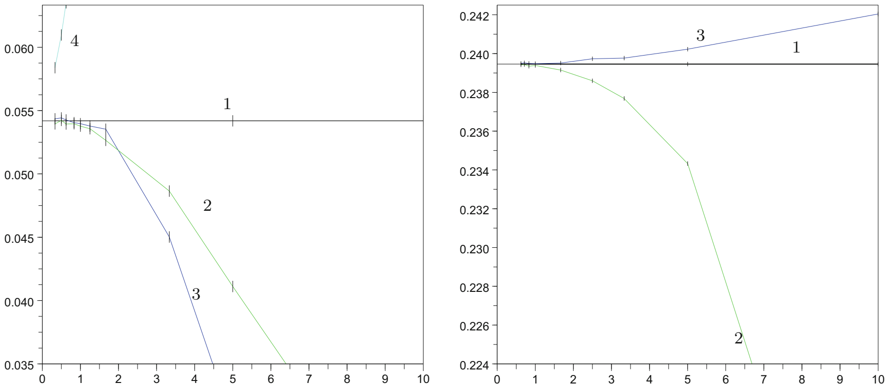

Fig. 5.1 $d = 3$, 10^7 Monte-Carlo samples, $T = 10$. The real value of $\mathbb{E}[\exp(-\mathrm{Tr}(iv\hat{X}^N_{t^N_N}))]$ in function of the time-step T/N. *Left*: $v = 0.05I_d$ and Wishart parameters $x = 0.4I_d$, $\alpha = 4.5$, $a = I_d$ and $b = 0$. Exact value: 0.054277. *Right*: $v = 0.2I_d + 0.04q$ and Wishart parameters $x = 0.4I_d + 0.2q$, $\alpha = 2.22$, $a = I_d$ and $b = -0.5I_d$. Exact value: 0.239836. Here, q is the matrix defined by: $q_{i,j} = \mathbb{1}_{i \neq j}$. The width of each point represents the 95 % confidence interval

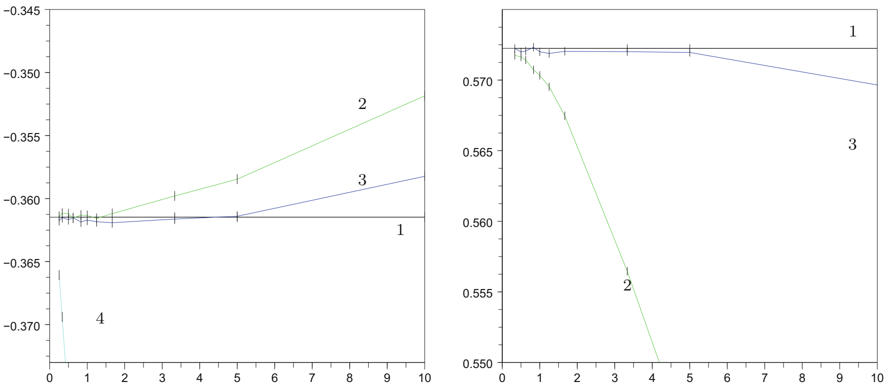

Fig. 5.2 $d = 10$, 10^7 Monte-Carlo samples, $T = 10$. *Left*: imaginary value of $\mathbb{E}[\exp(-\mathrm{Tr}(iv\hat{X}^N_{t^N_N}))]$ with $v = 0.009I_d$ in function of the time-step T/N. Wishart parameters: $x = 0.4I_d$, $\alpha = 12.5$, $b = 0$ and $a = I_d$. Exact value: -0.361586. *Right*: real value of $\mathbb{E}[\exp(-\mathrm{Tr}(iv\hat{X}^N_{t^N_N}))]$ with $v = 0.009I_d$ in function of T/N. Wishart parameters: $x = 0.4I_d$, $\alpha = 9.2$, $b = -0.5I_d$ and $a = I_d$. Exact value 0.572241. The width of each point represents the 95 % confidence interval

Here, we have not plotted the convergence of the second order (bis) scheme given by Theorem 5.6.10 because it would have given almost the same convergence as the other second order scheme.

As expected, we observe in both Figs. 5.1 and 5.2 convergences that fit our theoretical results. Namely, the second order scheme (scheme 2) converges in $O(1/N^2)$ and the third order scheme (scheme 3) converges faster in $O(1/N^3)$.

Table 5.2 Values obtained by the Euler scheme in the numerical experiments of Figs. 5.1 and 5.2

N	2	4	8	10	16	30
Figure 5.1, right	−0.000698	0.000394	0.033193	0.111991	0.185128	0.210201
Figure 5.2, right	0.494752	−0.464121	0.657041	0.643042	0.637585	0.619553

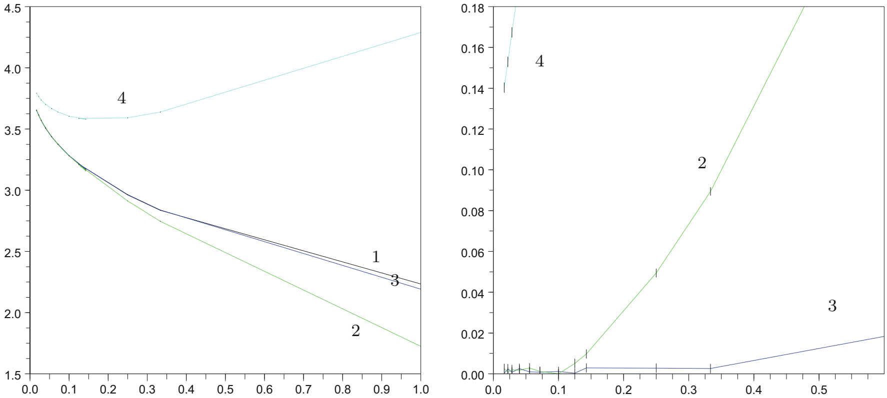

Fig. 5.3 $d = 3$, 10^7 Monte-Carlo samples, $T = 1$. Wishart parameters $x = 0.4I_d + 0.2q$ with $q_{i,j} = \mathbb{1}_{i \neq j}$, $\alpha = 2.2$, $b = 0$ and $a = I_d$. *Left*: $\mathbb{E}[\max_{0 \le k \le N} \mathrm{Tr}(\hat{X}^N_{t_k^N})]$. *Right*: $\mathbb{E}[\max_{0 \le k \le N} \mathrm{Tr}(\hat{X}^N_{t_k^N})] - \mathbb{E}[\max_{0 \le k \le N} \mathrm{Tr}(X^x_{t_k^N})]$ in function of T/N. The width of each point gives the precision up to two standard deviations

In some cases such as Fig. 5.2, scheme 3 already matches the exact value from $N = 2$. Even though it seems to converge at a $O(1/N)$ speed, the corrected Euler scheme is clearly not competitive with respect to the other schemes. In the tough case $d - 1 \le \alpha \le d$, the values obtained by the Euler scheme are in fact outside the figures, and we have put the corresponding values in Table 5.2.

We want to conclude this section by testing numerically the convergence of our schemes when we calculate pathwise expectations. Of course, the theoretical results that we have presented only bring on the weak error. However, we may hope that the schemes converge also quickly when considering more intricate expectations. In Fig. 5.3, we approximate $\mathbb{E}[\max_{0 \le t \le T} \mathrm{Tr}(X^x_t)]$ with the different schemes by computing the maximum on the time-grid. The convergence seems to be roughly in $O(1/\sqrt{N})$ for all the schemes (see Fig. 5.3, left), including the exact scheme. However, the main error seems to come from the approximation of $\max_{0 \le t \le T} \mathrm{Tr}(X^x_t)$ by $\max_{0 \le k \le N} \mathrm{Tr}(X^x_{t_k^N})$. In fact, we have plotted in Fig. 5.3 (right) the difference between $\mathbb{E}[\max_{0 \le k \le N} \mathrm{Tr}(\hat{X}^N_{t_k^N})]$ and $\mathbb{E}[\max_{0 \le k \le N} \mathrm{Tr}(X^x_{t_k^N})]$. Then, we find convergences that are very similar to those obtained for the weak error: schemes 2 and 3 converge at a speed which is respectively compatible with $O(1/N^2)$ and $O(1/N^3)$. Scheme 4 seems also to give a $O(1/N)$ convergence. It would be hasty to draw a global conclusion from this simple example. Nonetheless, the convergence of schemes 2 and 3 is really encouraging on pathwise expectations, if we put aside the problem of approximating a function of $(X_t^x, 0 \leq t \leq T)$ by a function of $(X_{t_k^N}^x, 0 \leq k \leq N)$.

5.7.3 An Application in Finance to the Gourieroux and Sufana Model

We now want to give a possible application of the simulation schemes to the Gourieroux and Sufana model, which is defined by (5.26). To simulate both assets and the Wishart matrix, we proceed as follows. We observe that the generator of (S_t, X_t) can be written as

$$L = L^S + L^X, \text{ where } L^S = \sum_{i=1}^d r s_i \partial_{s_i} + \frac{1}{2} \sum_{i,j=1}^d s_i s_j x_{i,j} \partial_{s_i} \partial_{s_j},$$

and L^X is the generator of the Wishart process $WIS_d(x, \alpha, b, a)$. The operator L^S is associated to the SDE $dS_t^l = rS_t^l + S_t^l(\sqrt{x} dB_t)_l$ that can be solved explicitly. We have indeed $S_t^l = S_0^l \exp[(r - x_{l,l}/2)t + (\sqrt{x} B_t)_l]$. Let us also remark that $\sqrt{x} B_t \overset{law}{=} cB_t$ if we have $cc^\top = x$: both are centered Gaussian vectors with the same covariance matrix. In practice, it is more efficient to use $S_t^l = S_0^l \exp[(r - x_{l,l}/2)t + (cB_t)_l]$ where c is computed with an extended Cholesky decomposition of x rather than calculating $\sqrt{x}$, which requires a diagonalization. Then, we consider the scheme given by Corollary 2.3.14, where we take a second order scheme for $WIS_d(x, \alpha, b, a)$ and the exact scheme for L^S. This construction is known to preserve the second-order convergence. To be consistent with Sect. 5.7.2, this scheme will be denoted by scheme 2 in this paragraph. To compare this scheme with a more basic one, we consider the Euler-Maruyama scheme defined by (5.57) and

$$\hat{S}_{t_0^N}^{l,N} = S_0^l, \; \hat{S}_{t_{i+1}^N}^{l,N} = \hat{S}_{t_i^N}^{l,N} \left(1 + rT/N + \left(\sqrt{\left(\hat{X}_{t_i^N}^N\right)^+} (B_{t_{i+1}^N} - B_{t_i^N})\right)_l \right), 0 \leq i \leq N-1.$$

It is denoted by scheme 4 like in Sect. 5.7.2.

We have plotted in Fig. 5.4 the price of a put option on the maximum of two risky assets ($d = 2$). The Gourieroux and Sufana model is an affine model, and the characteristic function of S_t is explicitly known (see [69]). Thus, it is possible to adapt the method proposed by Carr and Madan explained in Sect. 4.2.3 and to calculate by numerical integration (which is possible for small dimensions) the value of this put option. We have given in Fig. 5.4 the exact value obtained by this method. As one might have guessed, we observe a quadratic convergence for scheme 2 and a linear convergence for scheme 4. The benefit of using scheme 2 is clear since it

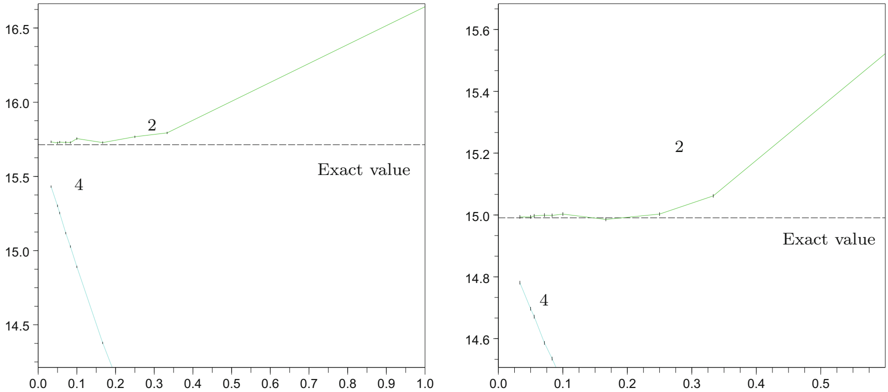

Fig. 5.4 $\mathbb{E}[e^{-rT}(K - \max(\hat{S}^{1,N}_{t_N^N}, \hat{S}^{2,N}_{t_N^N}))^+]$ in function of T/N. $d = 2$, $T = 1$, $K = 120$, $S_0^1 = S_0^2 = 100$, and $r = 0.02$. Wishart parameters: $x = 0.04I_d + 0.02q$ with $q_{i,j} = \mathbb{1}_{i \neq j}$, $a = 0.2I_d$, $b = 0.5I_d$ and $\alpha = 4.5$ (*left*), $\alpha = 1.05$ (*right*). The width of each point gives the precision up to two standard deviations (10^6 Monte-Carlo samples)

already fits with the exact value from $N = 5$ in both cases: its convergence is really satisfactory.

In this paragraph, we only have considered the Gourieroux and Sufana model, and one may wonder if it is possible to construct a potential second order scheme in the extension of this model proposed by Da Fonseca, Grasselli and Tebaldi, see Eq. (5.27). The answer is positive, but the splitting is more sophisticated and reuse the remarkable splitting for Wishart processes. This scheme is given by Ahdida, Alfonsi and Palidda [4]. Like the scheme given by (4.31) and (4.32) for the Heston model, a part of the stock dynamics has to be simulated jointly with the covariance matrix.

5.8 Technical Proofs

5.8.1 Proof of Theorem 5.6.3

Theorem 5.6.3 defines the scheme as $\pi^\top h_r(\hat{U}_t^u)\pi$. We can prove (see later) that $\hat{U}_t^u$ is a potential νth-order scheme for some operator, and it would be then easy to analyze the error if h_r were in $\mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d^+(\mathbb{R}))$. Unfortunately, h_r is only smooth w.r.t. to the coordinates $u_{\{1,1\}}, \dots u_{\{1,d\}}$ and is not a priori in $\mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d^+(\mathbb{R}))$. In fact, this is sufficient to show that $\pi^\top h_r(\hat{U}_t^u)\pi$ is a potential νth-order scheme because these coordinates are the only one that are modified by the scheme. This requires however some further analysis which is made below.

Let $\mathbb{D} \subset \mathbb{R}^\zeta$ be a domain. We introduce for any domain $\tilde{\mathbb{D}} \subset \mathbb{R}^\xi$, $\xi \in \mathbb{N}^*$ the set

$$\mathcal{C}^\infty_{\mathbf{pol}}\big|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}}) = \{f \in \mathcal{C}^\infty(\mathbb{D} \times \tilde{\mathbb{D}}, \mathbb{R}),\ \forall \gamma \in \mathbb{N}^\zeta, \exists C_\gamma > 0,\ e_\gamma \in \mathbb{N}^*,\ \forall (x, \tilde{x}) \in \mathbb{D},$$
$$|\partial_\gamma f(x, \tilde{x})| \le C_\gamma (1 + \|(x, \tilde{x})\|^{e_\gamma})\},$$

where $\|.\|$ is a norm on $\mathbb{R}^{\zeta+\xi}$ and $\partial_\gamma = \partial_1^{\gamma_1}, \dots, \partial_\zeta^{\gamma_\zeta}$ denotes the derivatives w.r.t. the coordinates of $\mathbb{D}$. For $f \in \mathcal{C}^\infty_{\mathbf{pol}}\big|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$, we will say that $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$ is a *good sequence* for f if it is such that $|\partial_\gamma f(x, \tilde{x})| \le C_\gamma (1 + \|(x, \tilde{x})\|^{e_\gamma})$ for any $(x, \tilde{x}) \in \mathbb{D} \times \tilde{\mathbb{D}}$.

Let us now consider an operator L that satisfies the required assumption on $\mathbb{D}$. It is easy to check that all the iterated functions $L^k f$ are well defined and belong to $\mathcal{C}^\infty_{\mathbf{pol}}\big|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$ as soon as $f \in \mathcal{C}^\infty_{\mathbf{pol}}\big|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$. Let us fix $\tilde{x} \in \tilde{\mathbb{D}}$ and consider $\hat{X}^x_t$ sampled according to a potential weak νth-order scheme for L. Since $x \mapsto f(x, \tilde{x})$ belongs to $\mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$, we know by Definition 2.3.4 that there are constants $C_{\tilde{x}}$, $E_{\tilde{x}}$, $\eta_{\tilde{x}}$ such that

$$\forall t \in (0, \eta_{\tilde{x}}),\ \left| \mathbb{E}[f(\hat{X}^x_t)] - \sum_{k=0}^{\nu} \frac{1}{k!} t^k L^k f(x, \tilde{x}) \right| \le C_{\tilde{x}} t^{\nu+1} (1 + \|x\|^{E_{\tilde{x}}}).$$

In practice, one would like instead to get some bounds where the dependence with respect to $\tilde{x}$ is more tractable. This is why we introduce the following definition.

Definition 5.8.1 Let L be an operator that satisfies the required assumption on $\mathbb{D}$. We will say that a potential weak νth-order scheme for L satisfies the immersion property if for any $\tilde{\mathbb{D}} \subset \mathbb{R}^\xi$, $\xi \in \mathbb{N}^*$ and any function $f \in \mathcal{C}^\infty_{\mathbf{pol}}\big|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$ with a good sequence $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$, there exist positive constants C, E, and η depending only on $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^\zeta}$ such that

$$\forall t \in (0, \eta),\ \left| \mathbb{E}[f(\hat{X}^x_t, \tilde{x})] - \sum_{k=0}^{\nu} \frac{1}{k!} t^k L^k f(x, \tilde{x}) \right| \le C t^{\nu+1} (1 + \|(x, \tilde{x})\|^E).$$

In practice, most of the usual schemes satisfy this property. In fact, to prove that a scheme is a potential νth-order scheme, it is common to use a Taylor expansion that gives generally at the same time the immersion property. We illustrate this for the exact scheme below.

Proposition 5.8.2 *Let $\mathbb{D} \subset \mathbb{R}^\zeta$, $b : \mathbb{D} \to \mathbb{R}^\zeta$ and $\sigma : \mathbb{D} \to \mathcal{M}_\zeta(\mathbb{R})$ such that $\|b(x)\| + \|\sigma(x)\| \le C(1 + \|x\|)$ for some $C > 0$, and assume that $Lf(x) = \sum_{i=1}^\zeta b_i(x) \partial_i f(x) + \frac{1}{2} \sum_{i,j=1}^\zeta (\sigma\sigma^\top(x))_{i,j} \partial_i \partial_j f(x)$ satisfies the required assumption on $\mathbb{D}$. Then, for any $\nu \in \mathbb{N}$, the exact scheme is a potential weak νth-order scheme for L and it satisfies the immersion property.*

Proof Let $f \in \mathcal{C}^\infty_{\mathbf{pol}}\big|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$. We know from the sublinear growth condition that we have bounds on the moments of X^x_t: $\forall q \in \mathbb{N}^*, \exists C_q > 0, \forall t \in$

$[0,1]$, $\mathbb{E}[\|(X_t^x, \tilde{x})\|^q] \le C_q(1 + \|(x,\tilde{x})\|^q)$. By iterating Itô's formula, we get then easily for $t \in [0,1]$,

$$\mathbb{E}[f(X_t^x, \tilde{x})] = \sum_{k=0}^{\nu} \frac{t^k}{k!} L^k f(x,\tilde{x}) + \int_0^t \frac{(t-s)^\nu}{\nu!} \mathbb{E}[L^{\nu+1} f(X_s^x, \tilde{x})] ds.$$

Since $L^{\nu+1} f \in \mathcal{C}^{\infty}_{\textbf{pol}}\big|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$, there are constants $C > 0$ and $q \in \mathbb{N}^*$ depending only on a good sequence of f such that $|L^{\nu+1} f(x,\tilde{x})| \le C(1 + \|(x,\tilde{x})\|^q)$. Thus, we deduce that $|\mathbb{E}[f(X_t^x,\tilde{x})] - \sum_{k=0}^{\nu} \frac{t^k}{k!} L^k f(x,\tilde{x})| \le \frac{t^{\nu+1}}{(\nu+1)!} C(1 + C_q(1 + \|(x,\tilde{x})\|^q))$. □

Besides, the immersion property is easily preserved by scheme composition.

Proposition 5.8.3 *Let L_1 and L_2 be two operators that satisfy the required assumptions on $\mathbb{D}$, and assume that $\hat{p}_x^1(t)(dz)$ and $\hat{p}_x^2(t)(dz)$ are respectively potential weak νth-order discretization schemes on $\mathbb{D}$ for these operators that satisfy the immersion property. Let $\lambda_1, \lambda_2 > 0$ and $\hat{X}^{2\circ 1,x}_{\lambda_2 t, \lambda_1 t} \sim \hat{p}^2(\lambda_2 t) \circ \hat{p}_x^1(\lambda_1 t)(dz)$. Let $f \in \mathcal{C}^{\infty}_{\textbf{pol}}\big|_{\mathbb{D}}(\mathbb{D} \times \tilde{\mathbb{D}})$. Then, there are constants C, E, ν that only depend on a good sequence of f such that*

$$\forall t \in (0,\eta), \left| \mathbb{E}[f(\hat{X}^{2\circ 1,x}_{\lambda_2 t, \lambda_1 t}, \tilde{x})] - \sum_{l_1 + l_2 \le \nu} \frac{\lambda_1^{l_1} \lambda_2^{l_2}}{l_1! l_2!} t^{l_1+l_2} L_1^{l_1} L_2^{l_2} f(x,\tilde{x}) \right|$$
$$\le C t^{\nu+1}(1 + \|(x,\tilde{x})\|^E).$$

Therefore,

- *If $L_1 L_2 = L_2 L_1$, $\hat{p}^2(t) \circ \hat{p}_x^1(t)(dz)$ is a potential weak νth-order discretization scheme for $L_1 + L_2$ satisfying the immersion property.*
- *If $\nu \ge 2$, $\hat{p}^2(t/2) \circ \hat{p}^1(t) \circ \hat{p}_x^2(t/2)$ and $\frac{1}{2}\left(\hat{p}^2(t) \circ \hat{p}_x^1(t) + \hat{p}^1(t) \circ \hat{p}_x^2(t)\right)$ are potential weak second order schemes for $L_1 + L_2$ satisfying the immersion property.*

This proposition is a straightforward extension of Proposition 2.3.12 and Corollaries 2.3.13 and 2.3.14, and we do not repeat the proof here. Thanks to this result, we can prove the immersion property of the schemes that are obtained by splitting. The Ninomiya-Victoir scheme [109] which is obtained by a composition of exact schemes naturally satisfies this property. By looking at the proof of Proposition 2.3.19, it still satisfies this property if we replace the Gaussian samples by moment matching variables. Also, we can check that the second and third order schemes for the CIR process given by Propositions 3.3.5 and 3.3.8 satisfy the immersion property.

Corollary 5.8.4 *Let L_1 (resp. L_2) be an operator that satisfies the required assumptions on $\mathbb{D}_1$ (resp. $\mathbb{D}_2$). Let $\hat{X}_t^{1,x_1}$ and $\hat{X}_t^{2,x_2}$ be potential weak νth-order* *schemes for L_1 and L_2 sampled independently. Then, $(\hat{X}_t^{1,x_1}, \hat{X}_t^{2,x_2})$ is a potential weak νth-order schemes on $\mathbb{D}_1 \times \mathbb{D}_2$ that satisfies the immersion property.*

Proof From the immersion property, it is easy to check that $(\hat{X}_t^{1,x_1}, x_2)$ (resp. $(x_1, \hat{X}_t^{2,x_2})$) is a potential νth order scheme for L_1 (resp. L_2) on $\mathbb{D}_1 \times \mathbb{D}_2$ that satisfies the immersion property. The composition of these schemes is simply $(\hat{X}_t^{1,x_1}, \hat{X}_t^{2,x_2})$. Since L_1 and L_2 operate on different domains, we have $L_1 L_2 = L_2 L_1$, which gives the result. □

Proof of Theorem 5.6.3 Let $x \in \mathcal{S}_d^+(\mathbb{R})$ and $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d^+(\mathbb{R}))$ and $r = \mathrm{Rk}((x_{i,j})_{2\le i,j\le d})$. Since the operator $L_{e_d^1}$ satisfies the required assumption, we know that the exact scheme is a potential νth-order scheme (Proposition 5.8.2), and there are constants $C, E, \eta > 0$ depending on a good sequence of f such that

$$\forall t \in (0,\eta), |\mathbb{E}[f(X_t^x)] - \sum_{k=0}^{\nu} \frac{t^k}{k!} L_{e_d^1}^k f(x)| \le C(1 + \|x\|^E). \tag{5.58}$$

On the other hand, we know from Theorem 5.5.3, Eq. (5.43) and Corollary 5.5.5 that we have

$$X_t^x = \pi^\top h_r(U_t^u)\pi,$$

where $(U_t^u)_{\{1,l\}}$ solves the SDEs (5.41) starting from the initial condition $u_{1,l}$ for $1 \le l \le r+1$, and $(U_t^u)_{\{i,j\}} = u_{\{i,j\}}$ for the other coordinates. We have also $\hat{X}_t^x = \pi^\top h_r(\hat{U}_t^u)\pi$ by construction, and it is natural to focus on the function $u \mapsto f(\pi^\top h_r(u)\pi)$.

Let us consider the set

$$\{x \in \mathcal{S}_d(\mathbb{R}),\ s.t.\ (x_{i,j})_{2\le i,j\le d} \in \mathcal{S}_{d-1}^+(\mathbb{R}), x_{1,1} \ge 0\}.$$

It is isomorphic to $(\mathbb{R}_+ \times \mathbb{R}^{d-1}) \times \mathcal{S}_{d-1}^+(\mathbb{R})$ by the map $x \mapsto \big((x_{1,1}, \ldots, x_{1,d}), (x_{i,j})_{2\le i,j\le d}\big)$. We have to notice now that the function h_r defined by (5.43) is such that $h_r \in \mathcal{C}^\infty_{\mathbf{pol}}\big|_{\mathbb{R}_+\times\mathbb{R}^{d-1}}(\mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathcal{S}_{d-1}^+(\mathbb{R}))$. It is indeed a polynomial function with respect to $u_{1,1}, \ldots, u_{1,d}$. Then, it is easy to check that $u \mapsto f(\pi^\top h_r(u)\pi) \in \mathcal{C}^\infty_{\mathbf{pol}}\big|_{\mathbb{R}_+\times\mathbb{R}^{d-1}}(\mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathcal{S}_{d-1}^+(\mathbb{R}))$ since $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d^+(\mathbb{R}))$. Moreover, by the chain rule, we can get a good sequence for this function that only depend on a good sequence of f since π and h_r are fixed.

By assumption, $(\hat{U}_t^u)_{\{1,1\}}$ is a potential νth-order scheme for the operator $(\alpha - r)\partial_{\{1,1\}} + 2u_{\{1,1\}}\partial^2_{\{1,1\}}$ and satisfy the immersion property. The schemes $(\hat{U}_t^u)_{\{1,i\}}$ $(2 \le i \le r+1)$ can be seen as a Ninomiya-Victoir scheme with moment matching variables. They are therefore potential νth-order scheme for the operator $\frac{1}{2}\partial^2_{\{1,i\}}$ (see Theorem 2.3.17) and satisfy the immersion property. Therefore, from Corollary 5.8.4, $((\hat{U}_t^u)_{\{1,1\}}, \ldots, (\hat{U}_t^u)_{\{1,d\}})$ is a potential νth order scheme for

$(\alpha-r)\partial_{\{1,1\}}+2u_{\{1,1\}}\partial^2_{\{1,1\}}+\frac{1}{2}\sum_{i=1}^r \partial^2_{\{1,i\}}$ satisfying the immersion property. Thus, there are constants that we still denote by $C, E, \eta > 0$ depending on a good sequence of f such that:

$$\forall t \in (0,\eta),\ |\mathbb{E}[f(\pi^\top h_r(U_t^u)\pi)] - \mathbb{E}[f(\pi^\top h_r(\hat{U}_t^u)\pi)]| \le C t^{\nu+1}(1+\|u\|^E). \tag{5.59}$$

Now, one has to notice that $\|u\| \le C'(1+\|x\|)$ for some constant $C' > 0$ since we have $u_{\{1,1\}} + \sum_{k=1}^r (u_{\{1,k+1\}})^2 = \tilde{x}_{\{1,1\}}$ and $\tilde{x}$ and x have the same Frobenius norm. We get then the result by gathering (5.58) and (5.59). □

5.8.2 Proof of Proposition 5.6.4

First, let us check that $\theta_t \in \mathcal{G}_d(\mathbb{R})$ is well defined, such that $q_t/t = \theta_t I_d^n \theta_t^\top$ and satisfies:

$$\exists K, \eta > 0, \forall t \in (0,\eta), \max(\|\theta_t\|, \|\theta_t\|^{-1}) \le K. \tag{5.60}$$

When $n = d$, q_t/t is definite positive as a convex combination of definite positive matrices and the usual Cholesky decomposition is well defined. Moreover, (5.60) holds since q_t/t goes to $a^\top a$ which is invertible when $t \to 0^+$. When $n < d$, we have assumed in addition that b and $a^\top a$ commute. Therefore, $q_t = a^\top a(\int_0^t \exp(sb)\exp(sb^\top)ds/t)$. Since $a^\top a$ and $(\int_0^t \exp(sb)\exp(sb^\top)ds/t)$ are positive semidefinite matrices that commute, we have

$$q_t = \sqrt{\frac{1}{t}\int_0^t \exp(sb)\exp(sb^\top)ds}\, a^\top a \sqrt{\frac{1}{t}\int_0^t \exp(sb)\exp(sb^\top)ds}.$$

Once again, $\frac{1}{t}\int_0^t \exp(sb)\exp(sb^\top)ds$ is definite positive as a convex combination of definite positive matrices and we get that $\theta_t = \sqrt{\frac{1}{t}\int_0^t \exp(sb)\exp(sb^\top)ds}\, p^{-1} \begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix} \in \mathcal{G}_d(\mathbb{R})$ satisfies $q_t/t = \theta_t I_d^n \theta_t^\top$ by Lemma A.2.1. Similarly, (5.60) holds since $p^{-1}\begin{pmatrix} c_n & 0 \\ k_n & I_{d-n} \end{pmatrix}$ does not depend on t and $\sqrt{\frac{1}{t}\int_0^t \exp(sb)\exp(sb^\top)ds}$ goes to I_d when $t \to 0^+$.

Let $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d^+(\mathbb{R}))$. Let $X_t^x \sim WIS_d(x,\alpha,b,a;t)$. Since the exact scheme is a potential νth-order scheme, there are constants $C, E, \eta > 0$ depending only on a good sequence of f such that

$$\forall t \in (0,\eta),\ |\mathbb{E}[f(X_t^x)] - \sum_{k=0}^{\nu} \frac{t^k}{k!} L^k f(x)| \le C t^{\nu+1}(1+\|x\|^E). \tag{5.61}$$

On the other hand we have from Proposition 5.3.2,

$$\mathbb{E}[f(\hat{X}_t^x)] - \mathbb{E}[f(X_t^x)] = \mathbb{E}[f(\theta_t \hat{Y}_t^{\theta_t^{-1} m_t x m_t^\top (\theta_t^{-1})^\top} \theta_t^\top)] - \mathbb{E}[f(\theta_t Y_t^{\theta_t^{-1} m_t x m_t^\top (\theta_t^{-1})^\top} \theta_t^\top)]. \tag{5.62}$$

Let us introduce $f_{\theta_t}(y) := f(\theta_t y \theta_t^\top) \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathcal{S}_d^+(\mathbb{R}))$. By the chain rule, we have $\partial_{\{i,j\}} f_{\theta_t}(y) = \mathrm{Tr}[\theta_t(e_d^{i,j} + \mathbb{1}_{i \neq j} e_d^{j,i})\theta_t^\top \partial f(\theta_t y \theta_t^\top)]$, where $(\partial f(x))_{k,l} = (\mathbb{1}_{k=l} + \frac{1}{2}\mathbb{1}_{k \neq l})\partial_{\{k,l\}} f(x)$ and $e_d^{i,j} = (\mathbb{1}_{k=i,l=j})_{1 \le k,l \le d}$. From (5.60), we see that there is a good sequence $(C_\gamma, e_\gamma)_{\gamma \in \mathbb{N}^{d(d+1)/2}}$ that can be obtained from a good sequence of f such that:

$$\forall t \in (0,\eta), \forall y \in \mathcal{S}_d^+(\mathbb{R}),\ |\partial_\gamma f_{\theta_t}(y)| \le C_\gamma(1 + \|y\|^{e_\gamma}).$$

Therefore, we get that there are constants still denoted by $C, E, \eta > 0$ such that

$$\begin{aligned}\forall t \in (0,\eta),\ &\left|\mathbb{E}[f(\theta_t \hat{Y}_t^{\theta_t^{-1} m_t x m_t^\top (\theta_t^{-1})^\top} \theta_t^\top)] - \mathbb{E}[f(\theta_t Y_t^{\theta_t^{-1} m_t x m_t^\top (\theta_t^{-1})^\top} \theta_t^\top)]\right| \\ &\le C t^{\nu+1}(1 + \|\theta_t^{-1} m_t x m_t^\top (\theta_t^{-1})^\top\|^E).\end{aligned} \tag{5.63}$$

From (5.60), we get that there is a constant $K' > 0$ such that $\|\theta_t^{-1} m_t x m_t^\top (\theta_t^{-1})^\top\|^E \le K'\|x\|^E$ for $t \in (0,\eta)$. Thus, we get the result by gathering (5.61)–(5.63).

Chapter 6
Processes of Wright-Fisher Type

The main focus of this book is on affine diffusions and their simulation. In this chapter, we go slightly beyond this scope in the sense that the processes that we consider are not affine. They however have a clear connection with affine processes and belong to the class of Polynomial processes introduced by Cuchiero et al. [34] that also includes Affine diffusions. We first present Wright-Fisher processes that are well known in biology to model the frequency of a gene. These processes are directly related to Cox-Ingersoll-Ross diffusions, and we explain how it is possible to get second order schemes for these processes by reusing the second order schemes that we have developed for the CIR diffusion. This is made in Sect. 6.1. The other sections are devoted to a process of Wright-Fisher type that takes values in the set of correlation matrices that has been introduced by Ahdida and Alfonsi [3]. They take back some parts of the article [3] and generalize in a multidimensional setting the results of Sect. 6.1. First, we present the correlation processes, their main properties and their link with Wishart processes. Then, we explain how to get second order schemes for these processes by reusing the schemes that we have developed in Chap. 5 for Wishart processes.

6.1 Wright-Fisher Processes

Wright-Fisher processes are considered in biology to model the gene frequencies in a population. Originally, the Wright-Fisher model has been stated in discrete time. It is a haploid model of random reproduction, which means that a gene is coded by one chromosome. It considers a population with N individuals and a gene with two different types. At the generation $k \in \mathbb{N}$, the number of genes of first type is described by the variable $\xi_k \in \{0, \dots, N\}$. The Wright-Fisher model assumes that $(\xi_k, k \in \mathbb{N})$ is a Markov chain. In the simplest case with no selection and mutation,

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the transition of this Markov chain is given by

$$\mathbb{P}(\xi_{k+1} = y|\xi_k = x) = \frac{N!}{y!(N-y)!}\left(\frac{x}{N}\right)^y\left(1-\frac{x}{N}\right)^{N-y}, \ x, y \in \{0, \dots, N\}.$$

This transition matrix corresponds to the following mechanism. The parent of each individual of the generation $k+1$ is chosen uniformly among the generation k and independently from the other individuals. The gene is assumed to be transmitted with probability one. Therefore, the probability of an individual of the generation $k+1$ to be of the first type is ξ_k/N, and the conditional law of ξ_{k+1} given ξ_k is a binomial distribution with N trials and a probability of success equal to ξ_k/N. More elaborated transition matrices taking into account mutation and selection are presented in the book of Karlin and Taylor [84], p. 176.

The Wright-Fisher process is valued in $\mathbb{D} = [0, 1]$ and is defined by the following SDE

$$X_t^x = x + \int_0^t (a - kX_t^x)dt + \int_0^t \sigma\sqrt{X_t^x(1-X_t^x)}dW_t, \ t \geq 0, \tag{6.1}$$

with $x \in [0, 1]$. It is well defined if we assume that $0 \leq a \leq k$ and $\sigma \in \mathbb{R}$, see Theorem 6.1.1 below. We exclude the trivial case $\sigma = 0$ which leads to a linear ODE. This process can be seen as the limit process of the Wright-Fisher model in discrete time. In fact, we have for the model above $\mathbb{E}[\xi_{k+1} - \xi_k|\xi_k] = 0$ and $\mathbb{E}[(\xi_{k+1} - \xi_k)^2|\xi_k] = N\frac{\xi_k}{N}\left(1 - \frac{\xi_k}{N}\right)$. We now define $\hat{X}_{k/N} = \frac{\xi_k}{N}$ and $\hat{X}_t = (Nt-k)\hat{X}_{(k+1)/N} + (k+1-Nt)\hat{X}_{k/N}$ for $t \in [k/N, (k+1)/N]$. Then, we have $\mathbb{E}[\hat{X}_{(k+1)/N} - \hat{X}_{k/N}|\hat{X}_{k/N}] = 0$ and

$$\mathbb{E}[(\hat{X}_{(k+1)/N} - \hat{X}_{k/N})^2|\hat{X}_{k/N}] = \frac{1}{N}\hat{X}_{k/N}(1-\hat{X}_{k/N}),$$

and we can show that the process $\hat{X}$ converges in law when $N \to \infty$ to the process (6.1) with $a = k = 0$ and $\sigma = 1$. Nonzero parameters a and k arise when considering mutation and selection. We refer to [84] for the details.

Theorem 6.1.1 *Let* $0 \leq a \leq k$, $\sigma \in \mathbb{R}$ *and* $x \in [0, 1]$. *Then, there is a unique strong solution* X^x *of* (6.1), *and this solution satisfies* $\mathbb{P}(\forall t \geq 0, X_t^x \in [0, 1]) = 1$.

Proof The proof follows the same arguments as the one of Theorem 1.2.1 for the CIR. First, we consider the following SDE

$$X_t = x + \int_0^t (a - kX_t)dt + \int_0^t \sigma\sqrt{|X_t(1-X_t)|}dW_t.$$

The function $x \in \mathbb{R} \mapsto \sqrt{|x(1-x)|}$ is Hölder continuous with exponent $1/2$. We know by using Proposition 2.13, p. 291 of Karatzas and Shreve [83] that there is a unique strong solution X. The claim will follow if we show that for any $t \geq 0$, $X_t \in [0,1]$ almost surely. To do so, we use again the Yamada functions introduced in (1.7). We have by Itô's formula

$$\begin{aligned}\psi_n(X_t) =&\psi_n(x) + \int_0^t \psi_n'(X_s)(a-kX_s)ds + \int_0^t \sigma\psi_n'(X_s)\sqrt{|X_s(1-X_s)|}dW_s \\ &+ \int_0^t \frac{\sigma^2}{2} g_n(|X_s|)|X_s||1-X_s|ds.\end{aligned}$$

Taking the expectation and using that $\psi_n'(x) \leq 1$ and $g_n(x)x \leq 2/n$, we obtain

$$\mathbb{E}[\psi_n(X_t)] \leq \psi_n(x) + \int_0^t (a - k\mathbb{E}[X_s\psi_n'(X_s)])ds + \frac{\sigma^2 t}{n}\int_0^t \mathbb{E}[|1-X_s|]ds.$$

Since $\psi_n(z) \underset{n\to+\infty}{\to} |z|$, $z\psi_n'(z) \underset{n\to+\infty}{\to} |z|$ and $|\psi_n(z)| \vee |z\psi_n'(z)| \leq |z|$, we get by Lebesgue's theorem that

$$\mathbb{E}[|X_t|] \leq x + \int_0^t (a - k\mathbb{E}[|X_s|])ds.$$

Taking the expectation of (1.6), we have $\mathbb{E}[X_t] = x + \int_0^t (a - k\mathbb{E}[X_s])ds$ and thus

$$\mathbb{E}[|X_t|] - \mathbb{E}[X_t] \leq -k\int_0^t (\mathbb{E}[|X_s|] - \mathbb{E}[X_s])ds.$$

Gronwall's lemma gives then $\mathbb{E}[|X_t|] = \mathbb{E}[X_t]$ and thus $\mathbb{P}(X_t \geq 0) = 1$ for any $t \geq 0$. We can now repeat the same argument on the process $1 - X_t$ and get that $\mathbb{P}(1 - X_t \geq 0) = 1$, which concludes the proof. □

We now state the Feller condition for Wright-Fisher processes.

Proposition 6.1.2 *Let* $x \in (0,1)$, $\tau_0 = \inf\{t \geq 0, X_t^x = 0\}$ *and* $\tau_1 = \inf\{t \geq 0, X_t^x = 1\}$ *with* $\inf \emptyset = +\infty$. *Then,* $\min(\tau_0, \tau_1) = +\infty$ *a.s. if, and only if*

$$2a \geq \sigma^2 \text{ and } 2(k-a) \geq \sigma^2. \tag{6.2}$$

This proposition can be deduced from the Feller's test for explosions, see Theorem 5.29, p. 348 in [83]. A direct proof of the sufficient condition is given in Exercise 6.1.3.

6.1.1 Affine Transformations

Let X^x be defined by (6.1). Then, $\bar{X}_t = 1 - X_t^x$ is also a Wright-Fisher process. This is not surprising since $\bar{X}$ represents the proportion of the population of the other gene type. In fact, we have

$$d\bar{X}_t = (k - a - k\bar{X}_t)dt - \sigma\sqrt{\bar{X}_t(1-\bar{X}_t)}dW_t.$$

We notice that the condition on the parameters $0 \leq k - a \leq k$ is well satisfied since we have $0 \leq a \leq k$.

Exercise 6.1.3 We assume $\sigma^2 \leq 2a$. Let $x \in (0, 1]$ and $\tau_0 = \inf\{t \geq 0, X_t^x = 0\}$. The goal of this exercise is to prove that $\mathbb{P}(\tau_0 = +\infty) = 1$ by using the McKean argument.

1. Show that $X_t^x = x\exp\left(\int_0^t \frac{a-\sigma^2/2}{X_s}ds + (\sigma^2/2-k)t + \int_0^t \sigma\sqrt{\frac{1-X_s}{X_s}}dW_s\right)$ for $t \in [0, \tau_0)$. Deduce that $X_t^x \geq x\exp((\sigma^2/2-k)t + M_t)$, with $M_t = \sigma\int_0^t \sqrt{\frac{1-X_s}{X_s}}dW_s$.
2. We assume by way of contradiction that $\mathbb{P}(\tau_0 < \infty) > 0$. Then, show that $M_t \underset{t\to\tau_0}{\to} -\infty$ on $\{\tau_0 < \infty\}$, and conclude.
3. We now assume in addition that $\sigma^2 \leq 2(k-a)$ and $x \in (0, 1)$. Show that $\tau_1 = +\infty$ and deduce the "only if" part of Proposition 6.1.2.

From the modelling of the gene frequency, it is rather natural to consider a normalized process on [0, 1]. However, one may be interested for other applications to consider a similar process on the domain $\mathbb{D} = [\alpha, \beta]$, with $\alpha < \beta$. The following SDE

$$d\tilde{X}_t = (\tilde{a} - \tilde{k}\tilde{X}_t)dt + \tilde{\sigma}\sqrt{(\tilde{X}_t - \alpha)(\beta - \tilde{X}_t)}dW_t,\ \tilde{X}_0 \in [\alpha, \beta], \tag{6.3}$$

is well defined as soon as $\tilde{a} - \tilde{k}\alpha \geq 0$ and $\tilde{a} - \tilde{k}\beta \leq 0$. Such a process can be easily obtained by an affine transform of the Wright-Fisher process (6.1). In fact, the process $\tilde{X}_t = \alpha + (\beta - \alpha)X_t^x$ satisfies

$$\begin{aligned} d\tilde{X}_t &= (\beta - \alpha)\left\{\left(a - k\frac{\tilde{X}_t - \alpha}{\beta - \alpha}\right)dt + \sigma\sqrt{\frac{\tilde{X}_t - \alpha}{\beta - \alpha}\frac{\beta - \tilde{X}_t}{\beta - \alpha}}dW_t\right\} \\ &= a(\beta - \alpha) + k\alpha - k\tilde{X}_t + \sigma\sqrt{(\tilde{X}_t - \alpha)(\beta - \tilde{X}_t)}dW_t. \end{aligned}$$

Therefore, $\tilde{X}$ follows the SDE (6.3) with

$$\tilde{a} = a(\beta - \alpha) + k\alpha,\ \tilde{k} = k,\ \tilde{\sigma} = \sigma. \tag{6.4}$$

In particular, we have $\tilde{a} - \tilde{k}\alpha = a(\beta - \alpha)$, $\tilde{a} - \tilde{k}\beta = (a-k)(\beta - \alpha)$ and therefore

$$\tilde{a} - \tilde{k}\alpha \geq 0 \text{ and } \tilde{a} - \tilde{k}\beta \leq 0 \iff 0 \leq a \leq k.$$

Thus, the processes $\tilde{X}$ and X^x can be obtained from the other one by an affine transformation, and by changing the parameters according to (6.4). Therefore, strong uniqueness results for $\tilde{X}$ can be easily deduced from Theorem 6.1.1. Since these processes are equivalent up to an affine transformation, one usually works with Wright-Fisher processes on [0, 1], i.e. with $\alpha = 0$ and $\beta = 1$. Another popular choice is to work with $\alpha = -1$ and $\beta = 1$. In this case, the process

$$d\tilde{X}_t = (\tilde{a} - \tilde{k}\tilde{X}_t)dt + \tilde{\sigma}\sqrt{1 - \tilde{X}_t^2}dW_t \tag{6.5}$$

is often named Jacobi process. This is due to its connection with Jacobi polynomials, see Proposition 6.1.4 below. The Jacobi process can be used to model the instantaneous correlation between two Brownian motions. It is well defined as soon as $\tilde{X}_0 \in [-1, 1]$,

$$\tilde{a} + \tilde{k} \geq 0 \text{ and } \tilde{a} - \tilde{k} \leq 0. \tag{6.6}$$

Besides, by Proposition 6.1.2 and (6.4), it never reaches -1 or 1 if, and only if, we assume $\tilde{X}_0 \in (-1, 1)$,

$$\tilde{a} + \tilde{k} \geq \tilde{\sigma}^2 \text{ and } \tilde{k} - \tilde{a} \geq \tilde{\sigma}^2. \tag{6.7}$$

6.1.2 Moments and Density Transition

Wright-Fisher processes, as the affine processes, belong the class of Polynomial processes introduced by Cuchiero et al. [34]. In particular, it is possible to calculate explicitly the moments of X_t^x by using Itô's formula and an induction on the moment order. Namely, we have $\mathbb{E}[X_t^x] = x + \int_0^t (a - k\mathbb{E}[X_s^x])ds$ and we thus get

$$\mathbb{E}[X_t^x] = xe^{-kt} + a\zeta_k(t),\ t \geq 0,$$

as in formula (3.18), with $\zeta_k(t) = \frac{1-e^{-kt}}{k}$ when $k \neq 0$ and $\zeta_0(t) = t$. For $m \geq 2$, we get by Itô's formula

$$d(X_t^x)^m = \left\{ m(X_t^x)^{m-1}(a - k(X_t^x)^{m-1}) + \sigma^2 \frac{m(m-1)}{2}(X_t^x)^{m-2}X_t^x(1 - X_t^x) \right\} dt$$
$$+ \sigma m(X_t^x)^{m-1}\sqrt{X_t^x(1 - X_t^x)}dW_t.$$

Since the process X_t^x is bounded, it is trivial to get that the stochastic integral has a zero expectation. This yields to

$$\frac{d}{dt}\mathbb{E}[(X_t^x)^m] = -m\left(\frac{\sigma^2}{2}(m-1)+k\right)\mathbb{E}[(X_t^x)^m]+m\left(a+\frac{\sigma^2}{2}(m-1)\right)\mathbb{E}[(X_t^x)^{m-1}].$$

By an induction on m, we get that the value of the moments are uniquely determined. In another way, we can write

$$\frac{d}{dt}\begin{bmatrix}\mathbb{E}[X_t^x]\\ \vdots \\ \mathbb{E}[(X_t^x)^m]\end{bmatrix} = A(m)\begin{bmatrix}\mathbb{E}[X_t^x]\\ \vdots \\ \mathbb{E}[(X_t^x)^m]\end{bmatrix},$$

with $A(m) \in \mathcal{M}_m(\mathbb{R})$ defined by $A(m)_{i,i} = -i\left(\frac{\sigma^2}{2}(i-1)+k\right)$ for $1 \le i \le d$, $A(m)_{i,i-1} = i\left(\frac{\sigma^2}{2}(i-1)+a\right)$ for $2 \le i \le d$, and $A(m)_{i,j} = 0$ for $j \notin \{i, i-1\}$. This leads to

$$\begin{bmatrix}\mathbb{E}[X_t^x]\\ \vdots \\ \mathbb{E}[(X_t^x)^m]\end{bmatrix} = \exp(tA(m))\begin{bmatrix}x\\ \vdots \\ x^m\end{bmatrix}.$$

Besides the moments, it is possible to obtain the density transition of Wright-Fisher processes in a explicit way. This is explained in detail by Karlin and Taylor [84], see p. 332 and 335, and we present here a slightly different argument. The density transition is related to the Jacobi polynomials, and it is easier to begin with the Jacobi process $\tilde{X}$ on $[-1, 1]$ defined by (6.5). Let $\gamma, \delta > 0$. We use the standard notation $\binom{\mu}{0} = 1$ and $\binom{\mu}{k} = \frac{\mu(\mu-1)\ldots(\mu-k+1)}{k!}$ for $\mu \in \mathbb{R}$ and $k \in \mathbb{N}^*$. The Jacobi polynomials is a family of orthogonal polynomials, and we recall their main properties that can be found in Abramowitz and Stegun [1]. They are defined by

$$\begin{aligned} P_n^{\gamma,\delta}(x) &= \frac{1}{n!2^n}(1-x)^{1-\gamma}(1+x)^{1-\delta}\frac{d^n}{dx^n}\left[(1-x)^{n+\gamma-1}(1+x)^{n+\delta-1}\right] \\ &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n+\gamma-1}{k}\binom{n+\delta-1}{n-k}(x-1)^{n-k}(x+1)^k, \end{aligned}$$

for $n \ge 1$ and $P_0^{\gamma,\delta}(x) = 1$, and we have $P_n^{\gamma,\delta}(1) = \binom{n+\gamma-1}{n}$. The degree of the polynomial $P_n^{\gamma,\delta}(x)$ is obviously equal to n, and therefore the Jacobi polynomials span all the polynomial functions. Besides, these polynomials are orthogonal in the following sense

$$\int_{-1}^{1} P_m^{\gamma,\delta}(x)P_n^{\gamma,\delta}(x)(1-x)^{\gamma-1}(1+x)^{\delta-1}dx$$
$$= \mathbb{1}_{m=n}\frac{2^{\gamma+\delta-1}(n+\gamma+\delta-1)}{2n+\gamma+\delta-1}\frac{\Gamma(n+\gamma)\Gamma(n+\delta)}{n!\Gamma(n+\gamma+\delta)}, \tag{6.8}$$

and they satisfy the following differential equation:

$$(1-x^2)(P_n^{\gamma,\delta})''(x)+(\delta-\gamma-(\delta+\gamma)x)(P_n^{\gamma,\delta})'(x)+n(n+\gamma+\delta-1)P_n^{\gamma,\delta}(x)=0. \tag{6.9}$$

By applying Itô's formula to the process $\tilde{X}$ defined by (6.5) and taking the expectation, we get

$$\frac{d}{dt}\mathbb{E}[P_n^{\gamma,\delta}(\tilde{X}_t)] = \frac{\tilde{\sigma}^2}{2}\mathbb{E}\left[\frac{2}{\tilde{\sigma}^2}(\tilde{a}-\tilde{k}\tilde{X}_t)(P_n^{\gamma,\delta})'(\tilde{X}_t)+(1-\tilde{X}_t^2)(P_n^{\gamma,\delta})''(\tilde{X}_t)\right].$$

By taking $\delta-\gamma=\frac{2}{\tilde{\sigma}^2}\tilde{a}$ and $\delta+\gamma=\frac{2}{\tilde{\sigma}^2}\tilde{k}$, which is equivalent to

$$\gamma=\frac{1}{\tilde{\sigma}^2}(\tilde{k}-\tilde{a}),\ \delta=\frac{1}{\tilde{\sigma}^2}(\tilde{a}+\tilde{k}), \tag{6.10}$$

we obtain $\frac{d}{dt}\mathbb{E}[P_n^{\gamma,\delta}(\tilde{X}_t)] = -\frac{\tilde{\sigma}^2}{2}n(n+\gamma+\delta-1)\mathbb{E}[P_n^{\gamma,\delta}(\tilde{X}_t)]$, and therefore

$$\mathbb{E}[P_n^{\gamma,\delta}(\tilde{X}_t)] = \mathbb{E}[P_n^{\gamma,\delta}(\tilde{X}_0)]e^{-n(n+\gamma+\delta-1)\tilde{\sigma}^2t/2} = \mathbb{E}[P_n^{\gamma,\delta}(\tilde{X}_0)]e^{-n(n+\frac{2}{\tilde{\sigma}^2}\tilde{k}-1)\tilde{\sigma}^2t/2}.$$

Proposition 6.1.4 *We consider the Jacobi process $\tilde{X}$ defined by (6.5). We assume that $|\tilde{a}|<\tilde{k}$, which amounts to have $\gamma,\delta>0$ with (6.10). Then, $\tilde{X}$ admits the following transition density for $t>0$, $x,y\in(-1,1)$,*

$$\begin{aligned}\tilde{p}(t,x,y) =&(1-y)^{\gamma-1}(1+y)^{\delta-1} \\ &\times\sum_{n=0}^{\infty}\frac{2n+\gamma+\delta-1}{2^{\gamma+\delta-1}(n+\gamma+\delta-1)}\frac{n!\Gamma(n+\gamma+\delta)}{\Gamma(n+\gamma)\Gamma(n+\delta)} \\ &\times e^{-n(n+\gamma+\delta-1)\tilde{\sigma}^2t/2}P_n^{\gamma,\delta}(x)P_n^{\gamma,\delta}(y).\end{aligned} \tag{6.11}$$

Proof Let $m\in\mathbb{N}$. Let $\tilde{p}$ be the function defined by (6.11). We get by (6.8)

$$\int_{-1}^{1}P_m^{\gamma,\delta}(y)\tilde{p}(t,x,y)dy = P_m^{\gamma,\delta}(x)e^{-n(n+\gamma+\delta-1)\tilde{\sigma}^2t/2} = \mathbb{E}[P_m^{\gamma,\delta}(\tilde{X}_t)|\tilde{X}_0=x].$$

We therefore have $\int_{-1}^{1} y^m \tilde{p}(t,x,y)dy = \mathbb{E}[(\tilde{X}_t)^m|\tilde{X}_0 = x]$ for any $m \in \mathbb{N}$ since the Jacobi polynomials span all the polynomial functions. By using the series expansion of the exponential function, we get $\int_{-1}^{1} e^{iuy} \tilde{p}(t,x,y)dy = \mathbb{E}[e^{iu\tilde{X}_t}|\tilde{X}_0 = x]$ for any $u \in \mathbb{R}$, which gives the claim. □

The previous proposition gives in particular the stationary law of the Jacobi process. It is straightforward to obtain that $\tilde{p}(t,x,y)$ converges when $t \to +\infty$ toward

$$\tilde{p}_\infty(y) = \frac{1}{2^{\gamma+\delta-1}} \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)\Gamma(\delta)} (1-y)^{\gamma-1}(1+y)^{\delta-1}.$$

Let us mention here that in the case $\tilde{a} = \tilde{k}$ (resp. $\tilde{a} = -\tilde{k}$), the conditional distribution of $\tilde{X}_t$ given $\tilde{X}_0 = x$ is also known explicitly, but it has some positive mass at 1 (resp. -1). It can be formally obtained as the limit in the sense of distributions of (6.11) when $\tilde{a} \to \tilde{k}$ (resp. $\tilde{a} \to -\tilde{k}$). We leave this as an exercise for the reader.

Now, it is easy to get back the transition density $p(t,x,y)$ of the Wright-Fisher process (6.1). We assume $0 < a < k$. We use the affine transformation $\tilde{X}_t = 2X_t^x - 1$. The process $\tilde{X}$ follows a Jacobi process with parameters $\tilde{a}, \tilde{k}, \tilde{\sigma}$ given by (6.4) with $\alpha = -1$ and $\beta = 1$. For a bounded measurable test function $f : [0,1] \to \mathbb{R}$, we set $\tilde{f}(\tilde{x}) = f((1+\tilde{x})/2)$ for $\tilde{x} \in [-1,1]$ and have

$$\begin{aligned}\int_0^1 f(y)p(t,x,y)dy &= \mathbb{E}[f(X_t^x)] = \mathbb{E}[\tilde{f}(\tilde{X}_t)] \\ &= \int_{-1}^{1} \tilde{f}(\tilde{y})\tilde{p}(t,2x-1,\tilde{y})d\tilde{y} \\ &= 2\int_0^1 f(y)\tilde{p}(t,2x-1,2y-1)dy,\end{aligned}$$

which leads to $p(t,x,y) = 2\tilde{p}(t,2x-1,2y-1)$. Its stationary law is then equal to

$$y \in (0,1),\ p_\infty(y) = 2\tilde{p}_\infty(2y-1) = \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)\Gamma(\delta)}(1-y)^{\gamma-1}y^{\delta-1},$$

which is the Beta distribution $\beta(\delta,\gamma)$. By using (6.4) and (6.10), the expression of the parameters γ and δ in function of the Wright-Fisher parameters is the following:

$$\gamma = \frac{2}{\sigma^2}(k-a),\ \delta = \frac{2}{\sigma^2}a.$$

6.1.3 Connection with the CIR Process

We present now some interesting identities between CIR processes and Wright-Fisher processes. The first one can be seen as a generalization in a dynamic framework of the well known identity between Gamma and Beta distributions: if $Z^1 \sim \Gamma(b_1, \theta)$ and $Z^2 \sim \Gamma(b_2, \theta)$ are independent, $\frac{Z^1}{Z^1+Z^2}$ follows a Beta distribution $\beta(b_1, b_2)$.

Proposition 6.1.5 *Let B^1 and B^2 two independent real Brownian motions. Let $b_1, b_2, z_1, z_2 \geq 0$ and $\sigma > 0$ such that $\sigma^2 \leq 2(b_1 + b_2)$ and $z_1 + z_2 > 0$. We consider the following CIR processes*

$$Z_t^i = z_i + b_i t + \int_0^t \sigma \sqrt{Z_s^i} dB_s^i, \ i = 1, 2.$$

Then, $Y_t = Z_t^1 + Z_t^2$ is a CIR process that never reaches 0, and we define

$$t \geq 0, \ X_t = \frac{Z_t^1}{Y_t}, \ \phi(t) = \int_0^t \frac{1}{Y_s} ds.$$

Then, ϕ is bijective on $\mathbb{R}_+$ and the process $(X_{\phi^{-1}(t)}, t \geq 0)$ is a Wright-Fisher diffusion with parameters $a = b_1$, $k = b_1 + b_2$ and σ that is independent of $(Y_t, t \geq 0)$.

Proof The first assertion is related to the first point of Exercise 1.2.13. Since $\langle \sqrt{Z_t^1} dB_t^1 + \sqrt{Z_t^2} dB_t^2 \rangle = Y_t dt$, we know by Theorem 4.2, p. 170 of [83] that there is a Brownian motion W^2 such that $\sqrt{Z_t^1} dB_t^1 + \sqrt{Z_t^2} dB_t^2 = \sqrt{Y_t} dW_t^2$. Then, we get

$$Y_t = z_1 + z_2 + (b_1 + b_2)t + \int_0^t \sigma \sqrt{Y_s} dW_s^2.$$

This proves that Y is a CIR process. Besides, it never reaches 0 by Proposition 1.2.15 since $\sigma^2 \leq 2(b_1 + b_2)$ and $z_1 + z_2 > 0$. Thus, $\phi(t) = \int_0^t 1/(Y_s) ds$ is well defined and one to one on $\mathbb{R}_+$ by Lemma 6.1.9 and Remark 6.1.11 in Sect. 6.1.4. The process X_t is also well defined, and we have

$$dW_t^2 = \sqrt{X_t} dB_t^1 + \sqrt{1 - X_t} dB_t^2.$$

We now apply Itô's formula and get

$$\begin{aligned} dX_t =& b_1 \frac{dt}{Y_t} + \sigma \frac{1}{Y_t} \sqrt{Z_t^1} dB_t^1 - \frac{X_t}{Y_t} [(b_1 + b_2)dt + \sigma \sqrt{Y_t} (\sqrt{X_t} dB_t^1 + \sqrt{1 - X_t} dB_t^2)] \\ &+ \sigma^2 \frac{X_t}{Y_t} dt - \sigma^2 \frac{X_t}{Y_t} dt \\ =& (b_1 - (b_1 + b_2) X_t) \frac{dt}{Y_t} + \frac{\sigma}{\sqrt{Y_t}} [\sqrt{X_t} (1 - X_t) dB_t^1 - X_t \sqrt{1 - X_t} dB_t^2]. \end{aligned}$$

Since $\langle \sqrt{X_t}(1-X_t)dB^1_t - X_t\sqrt{1-X_t}dB^2_t\rangle = [X_t(1-X_t)^2 + X_t^2(1-X_t)]dt = X_t(1-X_t)dt$ and $\langle \sqrt{X_t}(1-X_t)dB^1_t - X_t\sqrt{1-X_t}dB^2_t, dW^2_t\rangle = 0$, we get again by Theorem 4.2, p. 170 of [83] that there is a Brownian motion W^1 independent from W^2 such that $\sqrt{X_t}(1-X_t)dB^1_t - X_t\sqrt{1-X_t}dB^2_t = \sqrt{X_t(1-X_t)}dW^1_t$. We therefore have

$$dX_t = (b_1 - (b_1+b_2)X_t)\frac{dt}{Y_t} + \sigma\sqrt{X_t(1-X_t)}\frac{dW^1_t}{\sqrt{Y_t}}.$$

Now, let us consider $\tilde{W}^1_t$ the Brownian motion defined by $\tilde{W}^1_{\phi(t)} = \int_0^t \frac{dW^1_s}{\sqrt{Y_s}}$. It is also independent of W^2 since the conditional law of $\tilde{W}^1$ given $(W^2_t, t \geq 0)$ is clearly the law of a Brownian motion and does not depend on $(W^2_t, t \geq 0)$. We eventually obtain that

$$\begin{aligned} dX_{\phi^{-1}(t)} &= (b_1 - (b_1+b_2)X_{\phi^{-1}(t)})\frac{d\phi^{-1}(t)}{Y_{\phi^{-1}(t)}} + \sigma\sqrt{X_{\phi^{-1}(t)}(1-X_{\phi^{-1}(t)})}\frac{dW^1_{\phi^{-1}(t)}}{\sqrt{Y_{\phi^{-1}(t)}}} \\ &= (b_1 - (b_1+b_2)X_{\phi^{-1}(t)})dt + \sigma\sqrt{X_{\phi^{-1}(t)}(1-X_{\phi^{-1}(t)})}d\tilde{W}^1_t, \end{aligned}$$

since $d\phi^{-1}(t) = Y_{\phi^{-1}(t)}dt$. Besides, $X_{\phi^{-1}(t)}$ and Y_t solve autonomous SDEs driven by independent Brownian motions, which gives the independence. □

We now present a similar but different result for Jacobi processes. To make the link between both results, we need the following simple lemma.

Lemma 6.1.6 *Let $\tilde{x} \in [-1,1]$, $\tilde{k} \geq 0$ and $\tilde{\sigma} > 0$. Let $\tilde{X}_t$ be the following Jacobi process*

$$d\tilde{X}_t = -\tilde{k}\tilde{X}_t dt + \tilde{\sigma}\sqrt{1-\tilde{X}_t^2}dW_t.$$

Then, $X_t = \tilde{X}_t^2$ is a Wright-Fisher process starting from $x = \tilde{x}^2$ that solves

$$dX_t = [\tilde{\sigma}^2 - (\tilde{\sigma}^2 + 2\tilde{k})X_t]dt + 2\tilde{\sigma}\sqrt{X_t(1-X_t)}dW'_t,$$

where W' is the Brownian motion defined by $W'_t = \int_0^t(\mathbb{1}_{\tilde{X}_s \geq 0} - \mathbb{1}_{\tilde{X}_s < 0})dW_s$.

Proof We apply Itô's formula and get

$$dX_t = -2\tilde{k}X_t dt + 2\tilde{\sigma}\tilde{X}_t\sqrt{1-X_t}dW_t + \tilde{\sigma}^2(1-X_t)dt.$$

We get the claim by observing that $\tilde{X}_t dW_t = \sqrt{X_t}dW'_t$. □

Proposition 6.1.7 *Let B^1 and B^2 two independent real Brownian motions. Let $b_2, z_2 \geq 0$, $\tilde{z}_1 \in \mathbb{R}$ and $\sigma > 0$ such that $\sigma^2 \leq 4b_2$ and $z_2 + (\tilde{z}_1)^2 > 0$. Let $\tilde{Z}^1_t = \tilde{z}_1 + \frac{\sigma}{2} B^1_t$, $Z^1_t = (\tilde{Z}^1_t)^2$ and Z^2 be the following CIR process*

$$Z^2_t = z_2 + b_2 t + \int_0^t \sigma \sqrt{Z^2_s} dB^2_s.$$

Then, $Y_t = Z^1_t + Z^2_t$ is a CIR process that never reaches 0, and we define

$$t \geq 0,\ \tilde{X}_t = \frac{\tilde{Z}^1_t}{\sqrt{Y_t}},\ \phi(t) = \int_0^t \frac{1}{Y_s} ds.$$

Then, ϕ is bijective on $\mathbb{R}_+$ and the process $(\tilde{X}_{\phi^{-1}(t)}, t \geq 0)$ is a Jacobi diffusion with parameters $\tilde{a} = 0$, $\tilde{k} = b_2/2$ and $\tilde{\sigma} = \sigma/2$ that is independent of $(Y_t, t \geq 0)$.

Proof We use Itô's formula and get

$$dY_t = \left(\frac{\sigma^2}{4} + b_2\right) dt + \sigma(\tilde{Z}^1_t dB^1_t + \sqrt{Z^2_t} dB^2_t).$$

Since $\langle \tilde{Z}^1_t dB^1_t + \sqrt{Z^2_t} dB^2_t \rangle = Y_t dt$, there is a Brownian motion W^2 such that $\tilde{Z}^1_t dB^1_t + \sqrt{Z^2_t} dB^2_t = \sqrt{Y_t} dW^2_t$. This yields to

$$dY_t = \left(\frac{\sigma^2}{4} + b_2\right) dt + \sigma \sqrt{Y_t} dW^2_t.$$

Therefore, Y is a CIR process. It never reaches the origin since it satisfies the Feller condition $\sigma^2 \leq 2(\frac{\sigma^2}{4} + b_2)$ and $z_2 + (\tilde{z}_1)^2 > 0$. Thus, $\tilde{X}_t$ and $\phi(t)$ are well defined, and we have

$$dW^2_t = \tilde{X}_t dB^1_t + \sqrt{1 - \tilde{X}^2_t} dB^2_t.$$

Besides, ϕ is almost surely bijective by Lemma 6.1.9 and Remark 6.1.11.

By Itô's formula, we have

$$\begin{aligned} d\tilde{X}_t &= \frac{\sigma}{2\sqrt{Y_t}} dB^1_t - \frac{\tilde{X}_t}{2Y_t}\left[\left(\frac{\sigma^2}{4} + b_2\right) dt + \sigma \sqrt{Y_t} dW^2_t\right] + \frac{3}{8}\sigma^2 \frac{\tilde{X}_t}{Y_t} dt - \frac{\sigma^2}{4} \frac{\tilde{X}_t}{Y_t} dt \\ &= -\frac{b_2}{2} \tilde{X}_t \frac{dt}{Y_t} + \frac{\sigma}{2\sqrt{Y_t}}\left[(1 - \tilde{X}^2_t) dB^1_t - \tilde{X}_t \sqrt{1 - \tilde{X}^2_t} dB^2_t\right]. \end{aligned} \tag{6.12}$$

We have $\langle (1-\tilde{X}_t^2)dB_t^1 - \tilde{X}_t\sqrt{1-\tilde{X}_t^2}dB_t^2\rangle = [(1-\tilde{X}_t^2)^2 + \tilde{X}_t^2(1-\tilde{X}_t^2)]dt = (1-\tilde{X}_t^2)dt$ and $\langle (1-\tilde{X}_t^2)dB_t^1 - \tilde{X}_t\sqrt{1-\tilde{X}_t^2}dB_t^2, dW_t^2\rangle = 0$. By Theorem 4.2, p. 170 of [83], there is a Brownian motion W^1 independent from W^2 such that $(1-\tilde{X}_t^2)dB_t^1 - \tilde{X}_t\sqrt{1-\tilde{X}_t^2}dB_t^2 = \sqrt{1-\tilde{X}_t^2}dW_t^1$. We eventually get

$$d\tilde{X}_t = -\frac{b_2}{2}\tilde{X}_t\frac{dt}{Y_t} + \frac{\sigma}{2\sqrt{Y_t}}\sqrt{1-\tilde{X}_t^2}dW_t^1.$$

We can now repeat the same argument as in the proof of Proposition 6.1.5 on the time change to conclude that $\tilde{X}_{\phi^{-1}(t)}$ is a Jacobi process independent of Y. □

Remark 6.1.8 Let $X_t = (\tilde{X}_t)^2 = \frac{Z_t^1}{Y_t}$. We know by Proposition 6.1.7 and Lemma 6.1.6 that $X_{\phi^{-1}(t)}$ is a Wright-Fisher process independent of Y with parameters $a = \frac{\sigma^2}{4}$, $k = \frac{\sigma^2}{4} + b_2$ and σ. By using the connection between Ornstein-Uhlenbeck and CIR processes, see Eq. (1.25), this gives back Proposition 6.1.5 in the particular case $b_1 = \frac{\sigma^2}{4}$.

6.1.4 Complementary Results on Squared Bessel Processes

This paragraph presents some particular results that are used through Chap. 6. It can be skipped for a first reading.

Lemma 6.1.9 *Let $\beta \geq 2$ and $Z_t = z + \beta t + 2\int_0^t \sqrt{Z_s}dB_s$ be a squared Bessel process of dimension β starting from $z > 0$. Then we have*

$$\mathbb{P}\left(\forall t \geq 0, \int_0^t \frac{ds}{Z_s} < \infty\right) = 1 \textit{ and } \int_0^{+\infty}\frac{ds}{Z_s} = +\infty \ a.s.$$

Proof The first claim is obvious, since the squared Bessel process does never touch zero under the condition of $\beta \geq 2$ by Proposition 1.2.15. Let $\beta' \geq \beta$ and $Z'_t = z + \beta' t + 2\int_0^t \sqrt{Z'_s}dB_s$. By using the comparison result given by Proposition 2.18, p. 293 in Karatzas and Shreve [83], we have

$$\forall t \geq 0, Z_t \leq Z'_t, \ a.s.$$

It is therefore sufficient to prove the second claim for $\beta \in \mathbb{N}$. Thus, we consider a countable family of independent Brownian motions $(W_t^k, t \geq 0)$ indexed by $k \in \mathbb{N}^*$. We know from the calculation made in Eq. (1.25) that $(W_t^1 + \sqrt{z})^2 + \sum_{k=2}^n (W_t^k)^2$ follows a squared Bessel process of dimension n. By the law of the iterated logarithm (Theorem 9.23, p. 112 in [83]), $\limsup_{t\to+\infty} \frac{(W_t^k)^2}{2t\log(\log(t))} = 1$. Therefore, there exists almost surely a time $T > 1$ such that

$$\forall t \geq T,\ (W_t^1 + \sqrt{z})^2 + \sum_{k=2}^{n}(W_t^k)^2 \leq 4nt\log(\log(t)).$$

This gives the desired result since $\int_0^{+\infty} \frac{ds}{(W_s^1+\sqrt{z})^2+\sum_{k=2}^{n}(W_s^k)^2} \geq \int_T^{\infty} \frac{ds}{4ns\log(\log(s))} = +\infty$. □

Lemma 6.1.10 *Let $\beta \geq 6$. Let $Z_t = 1 + \beta t + 2\int_0^t \sqrt{Z_s}dB_s$ be a squared Bessel process of dimension β starting from 1 and $\phi(t) = \int_0^t \frac{1}{Z_s}ds$. Then we have*

$$\mathbb{E}[\phi(t)] = t + \frac{4-\beta}{2}t^2 + O(t^3),\ \mathbb{E}[\phi(t)^2] = t^2 + O(t^3),\ \mathbb{E}[\phi(t)^3] = O(t^3).$$

Proof For a fixed time $t > 0$, we know from Proposition 1.2.11 that the density of Z_t is given by

$$z > 0,\ p(t,z) = \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}}(\frac{1}{2t})^k}{k!} \frac{1}{2t\Gamma(k+\frac{\beta}{2})}(\frac{z}{2t})^{k-1+\frac{\beta}{2}}e^{-\frac{z}{2t}}.$$

Let us consider that $\gamma \in \{1, 2, 3\}$, then all negative moments can be written as

$$\begin{aligned}\mathbb{E}\left[\frac{1}{Z_t^\gamma}\right] &= \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}}(\frac{1}{2t})^{k+\gamma}}{k!} \frac{\Gamma(k+\frac{\beta}{2}-\gamma)}{\Gamma(k+\frac{\beta}{2})} \\ &= \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}}(\frac{1}{2t})^{k+\gamma}}{k!} \frac{1}{(k+\frac{\beta}{2}-1)\times\cdots\times(k+\frac{\beta}{2}-\gamma)}.\end{aligned}$$

We have $\frac{1}{(k+\frac{\beta}{2}-1)} = \frac{1}{k+1} - \frac{\beta-4}{2(k+2)(k+1)} + O(\frac{1}{k^3})$, which yields to the following expansion:

$$\begin{aligned}\mathbb{E}\left[\frac{1}{Z_t}\right] &= \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}}(\frac{1}{2t})^{k+1}}{(k+1)!} - (\beta-4)t\sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}}(\frac{1}{2t})^{k+2}}{(k+2)!} + O\left(\frac{t^2}{2}\sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}}(\frac{1}{2t})^{k+3}}{(k+3)!}\right) \\ &= 1-(\beta-4)t + O(t^2).\end{aligned} \tag{6.13}$$

The first equality is thus obtained. We use the same argument to get:

$$\mathbb{E}\left[\frac{1}{Z_t^2}\right] = \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}}(\frac{1}{2t})^{k+2}}{(k+2)!} + O\left(t \sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}}(\frac{1}{2t})^{k+3}}{(k+3)!}\right) = 1 + O(t)$$

$$\mathbb{E}\left[\frac{1}{Z_t^3}\right] = O\left(\sum_{k=0}^{+\infty} \frac{e^{-\frac{1}{2t}}(\frac{1}{2t})^{k+3}}{(k+3)!}\right) = O(1). \tag{6.14}$$

By Jensen's inequality, one can deduce that $\mathbb{E}\left[\left(\int_0^t \frac{ds}{Z_s}\right)^3\right] \leq t^2 \mathbb{E}\left[\int_0^t \frac{ds}{(Z_s)^3}\right]$. Thanks to the moment expansion in (6.14), we find the third equality. Finally, by Jensen's equality, we obtain that

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t \left[\frac{1}{Z_s} - 1\right] ds\right)^2\right] &\leq t\mathbb{E}\left[\int_0^t \left(\frac{1}{Z_s} - 1\right)^2 ds\right] \\
&= t\mathbb{E}\left[\int_0^t \frac{ds}{(Z_s)^2}\right] - 2t\mathbb{E}\left[\int_0^t \frac{ds}{(Z_s)}\right] + t^2 \\
&= t^2 - 2t^2 + t^2 + O(t^3) = O(t^3).
\end{aligned}$$

It yields that

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t \left[\frac{1}{Z_s}\right] ds\right)^2\right] &= \mathbb{E}\left[\left(\int_0^t \left[\frac{1}{Z_s} - 1\right] ds\right)^2\right] - t^2 + 2t \int_0^t \mathbb{E}\left[\frac{1}{Z_s}\right] ds \\
&= t^2 + O(t^3).
\end{aligned}$$

□

Remark 6.1.11 Let $Z_t = z + at + 2\sigma \int_0^t \sqrt{Z_s} dB_s$ be a CIR process with $z, \sigma > 0$ and $2\sigma^2 \leq a$. Then, we can check easily that $(Z_{\frac{t}{\sigma^2}}, t \geq 0)$ is a squared Bessel process of dimension $a/\sigma^2 \geq 2$. By Lemma 6.1.9, we get

$$\mathbb{P}\left(\forall t \geq 0, \int_0^t \frac{ds}{Z_s} < \infty\right) = 1 \text{ and } \int_0^{+\infty} \frac{ds}{Z_s} = +\infty \ a.s.$$

Let $\phi(t) = \int_0^t \frac{1}{Z_s} ds = \frac{1}{\sigma^2} \int_0^{\sigma^2 t} \frac{1}{Z_{\frac{u}{\sigma^2}}} du$. When $z = 1$ and $a/\sigma^2 \geq 6$, we get from Lemma 6.1.10

$$\mathbb{E}[\phi(t)] = t + \frac{4\sigma^2 - a}{2} t^2 + O(t^3),\ \mathbb{E}[\phi(t)^2] = t^2 + O(t^3),\ \mathbb{E}[\phi(t)^3] = O(t^3).$$

6.1.5 A Second-Order Scheme

We now focus on the simulation of the Wright-Fisher process X^x defined by (6.1). We construct a second order scheme by using again the splitting technique that have been introduced in Chap. 2 and especially Theorem 2.3.8. To apply this theorem, we have to study the Kolmogorov equation related to $\tilde{u}(t,x) = \mathbb{E}[f(X_t^x)]$ when $f : [0,1] \to \mathbb{R}$ is $\mathcal{C}^\infty$. This equation has been studied recently by Epstein and Mazzeo [50] and Chen and Stroock [27]. Theorem 4 of [50] yields to the following result.

Theorem 6.1.12 *Let $f : [0,1] \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function. Then, $\tilde{u}(t,x) = \mathbb{E}[f(X_t^x)]$ is a $\mathcal{C}^\infty$ function on $\mathbb{R}_+ \times [0,1]$ that solves*

$$\partial_t \tilde{u}(t,x) = (a - kx)\partial_x \tilde{u}(t,x) + \frac{\sigma^2}{2} x(1-x)\partial_x^2 \tilde{u}(t,x).$$

In particular, we have for any $T > 0$,

$$\forall l, m \in \mathbb{N},\ \exists C_{l,m} > 0, \forall x \in [0,1], t \in [0,T],\ |\partial_t^l \partial_x^m u(t,x)| \le C_{l,m}.$$

Thanks to this result, the assumption (i) of Theorem 2.3.8 is satisfied by the Wright-Fisher process. We can now focus on constructing a second order scheme.

Since Wright-Fisher and Jacobi processes are the same up to the linear application $x \to 2x - 1$, we will construct in fact a second order scheme for the Jacobi process (6.5). The infinitesimal generator of $\tilde{X}$ is given by

$$\tilde{L} f(x) = (\tilde{a} - \tilde{k}x) f'(x) + \frac{\tilde{\sigma}^2}{2}(1-x^2) f''(x),\ x \in [-1,1],$$

for any $\mathcal{C}^2$ function $f : [-1,1] \to \mathbb{R}$. A first natural idea is to use the Ninomiya and Victoir scheme. It is based on the splitting $\tilde{L} = \tilde{L}' + \tilde{L}_2$ with

$$\tilde{L}' f(x) = \left(\tilde{a} - (\tilde{k} - \frac{\tilde{\sigma}^2}{2})x\right) f'(x),\ \tilde{L}_2 f(x) = -\frac{\tilde{\sigma}^2}{2} x f'(x) + \frac{\tilde{\sigma}^2}{2}(1-x^2) f''(x). \tag{6.15}$$

The SDE associated to $\tilde{L}_2$ can be explicitly solved as follows. For $x \in [-1,1]$, we consider $y \in [-\pi/2, \pi/2]$ such that $\sin(y) = x$. Then, $X_t = \sin(y + \tilde{\sigma} W_t)$ starts from x and satisfies

$$dX_t = \tilde{\sigma} \cos(y + \tilde{\sigma} W_t) dW_t - \frac{\tilde{\sigma}^2}{2} X_t dt = \tilde{\sigma} \sqrt{1 - X_t^2} dW_t' - \frac{\tilde{\sigma}^2}{2} X_t dt,$$

with $dW_t' = [\mathbb{1}_{\cos(y+\tilde{\sigma}W_t)\ge 0} - \mathbb{1}_{\cos(y+\tilde{\sigma}W_t)<0}] dW_t$. The ODE associated to $\tilde{L}'$ is linear and is solved by $\xi(t,x) = x e^{-(\tilde{k} - \frac{\tilde{\sigma}^2}{2})t} + \tilde{a}\zeta_{\tilde{k} - \frac{\tilde{\sigma}^2}{2}}(t)$. This solution stays in $[-1,1]$ if and only if

$$\tilde{a} - (\tilde{k} - \frac{\tilde{\sigma}^2}{2}) \leq 0 \text{ and } \tilde{a} + (\tilde{k} - \frac{\tilde{\sigma}^2}{2}) \geq 0. \tag{6.16}$$

In this case, the Ninomiya and Victoir scheme $\xi(t/2, \sin(\arcsin(\xi(t/2, x)) + \tilde{\sigma} W_t))$ is well defined and has a weak error of order 2 by using Theorems 2.3.17, 6.1.12 and 2.3.8. When condition (6.16) does not hold, one has to find another way to simulate the Jacobi process. We could proceed as for the CIR process, see Sect. 3.3, using the two following recipes.

- First, we use the Ninomiya and Victoir scheme $\xi(t/2, \sin(\arcsin(\xi(t/2, x)) + \tilde{\sigma}\sqrt{t}Y))$ replacing the Gaussian increment by the moment matching variable Y given by (2.27). This scheme is well defined on $[-1 + K(t), 1 - K'(t)]$, with $K(t), K'(t) \geq 0$ that satisfy $K(t) \underset{t \to 0}{=} O(t)$ and $K'(t) \underset{t \to 0}{=} O(t)$.
- Use a scheme valued in $[-1, 1]$ that matches the two first moments of the Jacobi process when the starting point is either in $[-1, -1 + K(t))$ or $(1 - K'(t), 1]$.

This approach leads to a second order scheme that works without any restriction on the parameters, and we leave it as an exercise for the reader.

Here, we consider another approach to construct a second order scheme. In fact, we have seen in Propositions 6.1.5 and 6.1.7 that CIR and Wright-Fisher diffusions are closely related. One may wonder if it would not be possible to reuse the high order schemes that we have developed for the CIR process in order to construct a scheme for Wright-Fisher and Jacobi processes. The answer is positive, and the construction that we present now will be extended later on to construct second order schemes for Wright-Fisher type processes on the set of correlation matrices. Let us start by splitting the infinitesimal generator and write $\tilde{L} = \tilde{L}_1 + \tilde{L}_2$ with

$$\tilde{L}_1 f(x) = \left(\tilde{a} - \tilde{k}x\right) f'(x), \ \tilde{L}_2 f(x) = \frac{\tilde{\sigma}^2}{2}(1 - x^2) f''(x). \tag{6.17}$$

The ODE associated to $\tilde{L}_1$ is linear and is solved by $\xi(t, x) = xe^{-\tilde{k}t} + \tilde{a}\zeta_{\tilde{k}}(t)$. It is important to notice that we have $\xi(t, x) \in [-1, 1]$ for any $x \in [-1, 1]$ and $t \geq 0$ since we have $\tilde{a} + \tilde{k} \geq 0$ and $\tilde{a} - \tilde{k} \leq 0$ by (6.6). Thus, by using Corollary 2.3.14 on the composition of schemes, it is sufficient to construct a second order scheme for $\tilde{L}_2$ in order to get a second order scheme for any Jacobi process.

We now construct a second order scheme for $\tilde{L}_2$ that stays in the domain $\mathbb{D} = [-1, 1]$. This construction is related to the result of Proposition 6.1.7. Let B^1 and B^2 be two independent standard real Brownian motions. Let $z_1 \in \mathbb{R}$ and $z_2 \geq 0$. We define $Z_t^1 = z_1 + \tilde{\sigma} B_t^1$ and

$$Z_t^2 = z_2 + 2\tilde{\sigma} \int_0^t \sqrt{Z_s^2} dB_s^2, \ t \geq 0.$$

The infinitesimal generator of (Z^1, Z^2) is given by

$$Lg = \frac{\tilde{\sigma}^2}{2}\partial^2_{z_1} g + 2\tilde{\sigma}^2 z_2 \partial^2_{z_2} g.$$

We define $p(z_1, z_2) = \frac{z_1}{\sqrt{z_1^2+z_2}} \in [-1, 1]$ that is $\mathcal{C}^\infty$ on $z_1^2 + z_2 > 0$. We want to calculate $L(f \circ p)$ and $L^2(f \circ p)$ for a smooth function $f : [-1, 1] \to \mathbb{R}$. We have

$$\partial_{z_1} p(z_1, z_2) = \frac{1}{\sqrt{z_1^2 + z_2}}(1 - p(z_1, z_2)^2), \quad \partial^2_{z_1} p(z_1, z_2) = -\frac{3p(z_1, z_2)}{z_1^2 + z_2}(1 - p(z_1, z_2)^2),$$

$$\partial_{z_2} p(z_1, z_2) = -\frac{p(z_1, z_2)}{2(z_1^2 + z_2)}, \qquad \partial^2_{z_2} p(z_1, z_2) = \frac{3}{4}\frac{p(z_1, z_2)}{(z_1^2 + z_2)^2}.$$

Using that $\partial_{z_i}(f \circ p) = \partial_{z_i} p \times f' \circ p$ and $\partial^2_{z_i}(f \circ p) = \partial^2_{z_i} p \times f' \circ p + (\partial_{z_i} p)^2 \times f'' \circ p$, we obtain

$$\begin{aligned}
L(f \circ p)(z_1, z_2) = \frac{\tilde{\sigma}^2}{2} & \left\{ -\frac{3p(z_1, z_2)}{z_1^2 + z_2}(1 - p(z_1, z_2)^2) f' \circ p(z_1, z_2) \right. \\
& \left. + \frac{(1 - p(z_1, z_2)^2)^2}{z_1^2 + z_2} f'' \circ p(z_1, z_2) \right\} \\
& + 2\tilde{\sigma}^2 \left\{ \frac{3}{4}\frac{p(z_1, z_2) z_2}{(z_1^2 + z_2)^2} f' \circ p(z_1, z_2) + \frac{p(z_1, z_2)^2 z_2}{4(z_1^2 + z_2)^2} f'' \circ p(z_1, z_2) \right\} \\
= & \frac{\tilde{\sigma}^2}{2(z_1^2 + z_2)}(1 - p(z_1, z_2)^2) f'' \circ p(z_1, z_2),
\end{aligned}$$

since $\frac{z_2}{z_1^2+z_2} = 1 - p(z_1, z_2)^2$. We define $h(x) = \frac{\tilde{\sigma}^2}{2}(1 - x^2) f''(x)$ and have

$$\begin{aligned}
L^2(f \circ p)(z_1, z_2) = & \frac{\tilde{\sigma}^2}{2} \left\{ \frac{6z_1^2 - 2z_2}{(z_1^2 + z_2)^3}(h \circ p)(z_1, z_2) + \partial^2_{z_1}(h \circ p)(z_1, z_2) \right. \\
& \left. - \frac{4z_1}{(z_1^2 + z_2)^2}\frac{1}{\sqrt{z_1^2 + z_2}}(1 - p(z_1, z_2)^2)(h' \circ p)(z_1, z_2) \right\} \\
& + 2\tilde{\sigma}^2 z_2 \left\{ \frac{2}{(z_1^2 + z_2)^3}(h \circ p)(z_1, z_2) + \partial^2_{z_2}(h \circ p)(z_1, z_2) \right. \\
& \left. + \frac{2}{(z_1^2 + z_2)^2}\frac{p(z_1, z_2)}{2(z_1^2 + z_2)}(h' \circ p)(z_1, z_2) \right\} \\
= & \frac{3\tilde{\sigma}^2}{(z_1^2 + z_2)^2}(h \circ p)(z_1, z_2) + L(h \circ p)(z_1, z_2).
\end{aligned}$$

We now consider $x \in [-1, 1]$ and set $z_1 = x$, $z_2 = 1 - x^2$ so that we have $p(z_1, z_2) = x$. Then, we get from the previous formulas that

$$L(f \circ p)(z_1, z_2) = \tilde{L}_2 f(x),\ L^2(f \circ p)(z_1, z_2) = 3\tilde{\sigma}^2 \tilde{L}_2 f(x) + \tilde{L}_2^2 f(x). \quad (6.18)$$

We are now in position to construct the second order scheme. For $x \in [-1, 1]$, we define $\hat{Z}_t^{1,x} = x + \tilde{\sigma}\sqrt{t}Y$, with Y sampled according to (2.27). Let $\hat{Z}_t^{2,x}$ be sampled independently according to the second order scheme of the CIR process given by Proposition 3.3.5 (or Proposition 3.3.8) starting from $1 - x^2$, without drift and with volatility coefficient $2\tilde{\sigma}$. This scheme is obviously a potential second order scheme for L. Since these schemes have discrete values, we can check that

$$\exists t_0 > 0, \forall t \in (0, t_0),\ 1/2 \le (\hat{Z}_t^{1,x})^2 + \hat{Z}_t^{2,x} \le 3/2.$$

Let $f : [-1, 1] \to \mathbb{R}$ be $\mathcal{C}^\infty$. The function $f \circ p$ is $\mathcal{C}^\infty$ on the compact set $\{(z_1, z_2) \in \mathbb{R} \times \mathbb{R}_+,\ 1/2 \le z_1^2 + z_2 \le 3/2\}$ and has thus bounded derivatives on this set. This leads to

$$\mathbb{E}[f(p(\hat{Z}_t^{1,x}, \hat{Z}_t^{2,x}))] = f(x) + t\tilde{L}_2 f(x) + \frac{t^2}{2}(3\tilde{\sigma}^2 \tilde{L}_2 f(x) + \tilde{L}_2^2 f(x)) + Rf(t, x),$$

where $Rf(t, x)$ is a remainder of order 3. Since x belongs to the compact set $[-1, 1]$, we even have $|Rf(t, x)| \le Ct^3$ for some constant $C > 0$ depending on f. Now, we consider

$$\phi(t) = \frac{-1 + \sqrt{1 + 6\tilde{\sigma}^2 t}}{3\tilde{\sigma}^2},$$

so that $\phi(t) \underset{t \to 0}{=} t - 3\tilde{\sigma}^2 \frac{t^2}{2} + O(t^3)$. Then, we have

$$\mathbb{E}[f(p(\hat{Z}_{\phi(t)}^{1,x}, \hat{Z}_{\phi(t)}^{2,x}))] \underset{t \to 0}{=} f(x) + t\tilde{L}_2 f(x) + \frac{t^2}{2}\tilde{L}_2^2 f(x) + O(t^3),$$

which is the desired expansion to get a second order scheme. Such a construction will be generalized later on for multidimensional processes, see Eq. (6.64). In fact, Proposition 6.1.13 is a particular case of Corollary 6.5.1 that has been obtained in Ahdida and Alfonsi [3].

Proposition 6.1.13 *Let* $\xi(t, x) = xe^{-\tilde{k}t} + \tilde{a}\zeta_{\tilde{k}}(t)$ *for* $x \in [-1, 1]$ *and* $t \ge 0$. *The scheme defined by*

$$\hat{X}_t^x = \xi\left(t/2, p(\hat{Z}_{\phi(t)}^{1,\xi(t/2,x)}, \hat{Z}_{\phi(t)}^{2,\xi(t/2,x)})\right)$$

is a second order scheme for the Jacobi process (6.5).

By using the affine transformation, we get that for $x \in [0,1]$, the scheme $(\hat{X}_t^{2x-1}+1)/2$ is a second order scheme for the Wright-Fisher process with coefficients $a = (\tilde{a}+\tilde{k})/2, k = \tilde{k}, \sigma = \tilde{\sigma}$.

Remark 6.1.14 It is possible to deduce (6.18) from Proposition 6.1.7. Let $x \in [-1,1]$, $\tilde{Z}_t^1 = x + \sigma B_t^1$ and $Z_t^2 = 1 - x^2 + 2kt + \int_0^t 2\sigma \sqrt{Z_s^2} dB_s^2$ with $2k \geq \sigma^2$. We consider $\hat{L}f(x) = -kxf'(x) + \frac{\sigma^2}{2}(1-x^2)f''(x) = -kxf'(x) + \tilde{L}_2 f(x)$, for $f \in \mathcal{C}^\infty([-1,1],\mathbb{R})$. By Proposition 6.1.7, we get that

$$\mathbb{E}[f(p(\tilde{Z}_t^1, Z_t^2))|\phi(t)] = f(x) + \phi(t)\hat{L}f(x) + \frac{\phi(t)^2}{2}\hat{L}^2 f(x) + O(\phi(t)^3).$$

From Lemma 6.1.10 and Remark 6.1.11, we get that the following expectations are finite when $2k \geq 5\sigma^2$ and have the following expansions $\mathbb{E}[\phi(t)] = t + (\frac{3\sigma^2}{2} - k)t^2 + O(t^3)$, $\mathbb{E}[\phi(t)^2] = t^2 + O(t^3)$ and $\mathbb{E}[\phi(t)^3] = O(t^3)$, which gives

$$[(L - kz_1\partial_{z_1})(f \circ p)](x, 1-x^2) = \hat{L}f(x),$$
$$[(L - kz_1\partial_{z_1})^2(f \circ p)](x, 1-x^2) = \hat{L}^2 f(x) + (3\sigma^2 - 2k)\hat{L}f(x).$$

These operators are polynomials of order 2 with respect to k and coincide for $2k \geq 5\sigma^2$. Therefore, the constant terms ($k=0$) are equal, which precisely gives (6.18).

Algorithm 6.1: Second order scheme for the Jacobi process (6.5) starting from x with time step $t > 0$.

Input: $x \in [-1,1]$, $\tilde{a}, \tilde{k} \in \mathbb{R}$ satisfying (6.6), $\tilde{\sigma} > 0$ and $t > 0$.
Output: X.
$X = xe^{-\tilde{k}t/2} + \tilde{a}\zeta_{\tilde{k}}(t/2)$,
$Z_1 = X + \tilde{\sigma}\sqrt{\phi(t)}\tilde{N}$, with $N \sim \mathcal{N}(0,1)$.
Sample independently Z_2 by using Algorithm 3.1 (or 3.3) with time step $\phi(t)$, starting point $1 - X^2$ with CIR parameters $a = 0$, $k = 0$ and $\sigma = 2\tilde{\sigma}$.
$X = \frac{Z_1}{\sqrt{(Z_1)^2 + Z_2}}$,
$X = Xe^{-\tilde{k}t/2} + \tilde{a}\zeta_{\tilde{k}}(t/2)$.

Algorithm 6.2: Second order scheme for the Wright-Fisher process (6.1) starting from x with time step $t > 0$.

Input: $x \in [0,1]$, $a \geq 0$, k satisfying $a - k \leq 0$, $\sigma > 0$ and $t > 0$.
Output: X.
Sample X with Algorithm 6.1 starting from $2x - 1$ and parameters
$\tilde{a} = 2a - k, \tilde{k} = k, \tilde{\sigma} = \sigma$.
$X = (X+1)/2$.

6.2 A Mean-Reverting Process on Correlation Matrices: Definition and First Properties

We now present a mean-reverting process on correlation matrices of Wright-Fisher type that has been recently introduced by Ahdida and Alfonsi [3]. The motivation for considering such a process is to model the dependence dynamics between different SDEs through their driving Brownian motion, by assuming that their instantaneous quadratic covariation is described by the correlation process. This point of view is particularly relevant in financial modelling. In fact, financial models for one dimensional assets are now rather well developed, and they correctly fit market data. Instead, modelling their dependence is still an open issue. Besides, one would like to have a model for a basket of assets that is consistent with the model chosen for each asset. Working with the instantaneous correlation between assets is a simple way to achieve this task. In fact, one can chose for each asset one's favourite model and then plug between the driving Brownian motions the instantaneous correlation given by some process valued in the set of the correlation matrices.

Let $d \geq 2$. We consider $(W_t, t \geq 0)$, a d-by-d square matrix process whose elements are independent real standard Brownian motions, and focus on the following SDE on the correlation matrices $\mathfrak{C}_d(\mathbb{R}) := \{x \in \mathcal{S}_d^+(\mathbb{R}), \forall 1 \leq i \leq d, x_{i,i} = 1\}$:

$$X_t = x + \int_0^t (\kappa(c - X_s) + (c - X_s)\kappa)\, ds \tag{6.19}$$
$$+ \sum_{n=1}^{d} a_n \int_0^t \left(\sqrt{X_s - X_s e_d^n X_s} dW_s e_d^n + e_d^n dW_s^\top \sqrt{X_s - X_s e_d^n X_s}\right),$$

where $x, c \in \mathfrak{C}_d(\mathbb{R})$ and $\kappa = diag(\kappa_1, \dots, \kappa_d)$ and $a = diag(a_1, \dots, a_d)$ are nonnegative diagonal matrices such that

$$\kappa c + c\kappa - (d-2)a^2 \in \mathcal{S}_d^+(\mathbb{R}) \text{ or } d = 2. \tag{6.20}$$

We recall that $(e_d^n)_{i,j} = \mathbb{1}_{i=j=n}$ and invite the reader to give a look at Appendix A.3 to observe that $\sqrt{x - x e_d^n x}$ is well defined for $x \in \mathfrak{C}_d(\mathbb{R})$. When $a = 0$ and κ has positive diagonal elements, X is an ordinary differential equation that reverts to the correlation matrix c, while staying in the set of correlation matrices. When $a \neq 0$, this mean reversion is perturbed by some noise. Under the assumptions above, we will show in Sect. 6.4 that this SDE has a unique weak solution which is well-defined on correlation matrices, i.e. $\forall t \geq 0, X_t \in \mathfrak{C}_d(\mathbb{R})$. We will also show that strong uniqueness holds if we assume moreover that $x \in \mathfrak{C}_d^*(\mathbb{R})$ and

$$\kappa c + c\kappa - da^2 \in \mathcal{S}_d^+(\mathbb{R}). \tag{6.21}$$

Looking at the diagonal coefficients, conditions (6.20) and (6.21) imply respectively $\kappa_i \geq (d-2)a_i^2/2$ and $\kappa_i \geq d a_i^2/2$. This heuristically means that the speed of the mean-reversion has to be high enough with respect to the noise in order to stay in $\mathfrak{C}_d(\mathbb{R})$.

Definition 6.2.1 We will denote $MRC_d(x,\kappa,c,a)$ the law of the process $(X_t)_{t\geq 0}$ and $MRC_d(x,\kappa,c,a;t)$ the law of X_t. When using these notations, we implicitly assume that (6.20) holds.

The acronym *MRC* stands for Mean-Reverting Correlation process.

In dimension $d=2$, the only non trivial component is $(X_t)_{1,2}$. We can show easily that there is a real Brownian motion $(B_t, t\geq 0)$ such that

$$d(X_t)_{1,2} = (\kappa_1+\kappa_2)(c_{1,2}-(X_t)_{1,2})dt + \sqrt{a_1^2+a_2^2}\sqrt{1-(X_t)_{1,2}^2}dB_t.$$

Thus, the process (6.19) is simply the Jacobi process considered in Eq. (6.5). Our parametrization is however redundant in dimension 2, and we can assume without loss of generality that $\kappa_1=\kappa_2$ and $a_1=a_2$. Then, the condition $\kappa c + c\kappa \in \mathcal{S}_d^+(\mathbb{R})$ is always satisfied, while assumption (6.21) is equivalent to $\kappa_1 \geq a_1^2$ and $(\kappa_1 c_{1,2})^2 \leq (\kappa_1 - a_1^2)^2$, which is also equivalent to $\kappa_1 c_{1,2} \leq \kappa_1 - a_1^2$ and $\kappa_1 c_{1,2} \geq a_1^2 - \kappa_1$. This is precisely the condition (6.7) that ensures $\forall t\geq 0, (X_t)_{1,2}\in(-1,1)$, i.e. that -1 and 1 are never reached. In larger dimensions $d\geq 3$, we can also show that each non-diagonal element of (6.19) follows a Jacobi process, see Proposition 6.2.3 below.

6.2.1 The Infinitesimal Generator

Lemma 6.2.2 *Let $(\mathcal{F}_t)_{t\geq 0}$ denote the filtration generated by $(W_t, t\geq 0)$. We consider a process $(Y_t)_{t\geq 0}$ valued in $\mathcal{S}_d(\mathbb{R})$ such that*

$$dY_t = B_t dt + \sum_{n=1}^{d} (A_t^n dW_t e_d^n + e_d^n dW_t^\top (A_t^n)^\top),$$

where $(A_t^n)_{t\geq 0}$, $(B_t)_{t\geq 0}$ are continuous $(\mathcal{F}_t)$-adapted processes respectively valued in $\mathcal{M}_d(\mathbb{R})$, and $\mathcal{S}_d(\mathbb{R})$. Then, we have for $1\leq i,j,k,l\leq d$:

$$\begin{aligned}\langle d(Y_t)_{i,j}, d(Y_t)_{k,l}\rangle = \Big[&\mathbb{1}_{i=k}(A_t^i(A_t^i)^\top)_{j,l} + \mathbb{1}_{i=l}(A_t^i(A_t^i)^\top)_{j,k} \\ &+\mathbb{1}_{j=k}(A_t^j(A_t^j)^\top)_{i,l} + \mathbb{1}_{j=l}(A_t^j(A_t^j)^\top)_{i,k}\Big]dt.\end{aligned} \tag{6.22}$$

Proof Since $(A_t^n dW_t e_d^n)_{i,j} = \mathbb{1}_{j=n}(A_t^j dW_t)_{i,j}$ and $(e_d^n dW_t^\top (A_t^n)^\top)_{i,j} = \mathbb{1}_{i=n}(A_t^i dW_t)_{j,i}$, we get:

$$d(Y_t)_{i,j} = (B_t)_{i,j}dt + \sum_{n=1}^{d}(A_t^j)_{i,n}(dW_t)_{n,j} + (A_t^i)_{j,n}(dW_t)_{n,i}.$$

This yields to

$$\begin{aligned}\langle d(Y_t)_{i,j}, d(Y_t)_{k,l}\rangle_t = \Bigg[&\mathbb{1}_{j=l}\sum_{n=1}^{d}(A_t^j)_{i,n}(A_t^j)_{k,n} + \mathbb{1}_{j=k}\sum_{n=1}^{d}(A_t^j)_{i,n}(A_t^j)_{l,n} \\ &+\mathbb{1}_{i=l}\sum_{n=1}^{d}(A_t^i)_{j,n}(A_t^i)_{k,n} + \mathbb{1}_{i=k}\sum_{n=1}^{d}(A_t^i)_{j,n}(A_t^i)_{l,n}\Bigg]dt,\end{aligned}$$

which precisely gives (6.22). □

By Lemma 6.2.2, we obtain:

$$\begin{aligned}\langle d(X_t)_{i,j}, d(X_t)_{k,l}\rangle &= \Big[a_i^2(\mathbb{1}_{i=k}(X_t - X_t e_d^i X_t)_{j,l} + \mathbb{1}_{i=l}(X_t - X_t e_d^i X_t)_{j,k}) \\ &\quad + a_j^2(\mathbb{1}_{j=k}(X_t - X_t e_d^j X_t)_{i,l} + \mathbb{1}_{j=l}(X_t - X_t e_d^j X_t)_{i,k})\Big]dt \\ &= \Big[a_i^2(\mathbb{1}_{i=k}((X_t)_{j,l} - (X_t)_{i,j}(X_t)_{i,l}) \qquad (6.23) \\ &\quad + \mathbb{1}_{i=l}((X_t)_{j,k} - (X_t)_{i,j}(X_t)_{i,k})) + a_j^2(\mathbb{1}_{j=k}((X_t)_{i,l} \\ &\quad - (X_t)_{j,i}(X_t)_{j,l}) + \mathbb{1}_{j=l}((X_t)_{i,k} - (X_t)_{j,i}(X_t)_{j,k}))\Big]dt.\end{aligned}$$

We remark in particular that $\langle d(X_t)_{i,j}, d(X_t)_{k,l}\rangle = 0$ when i, j, k, l are distinct.

We are now in position to calculate the infinitesimal generator of $MRC_d(x, \kappa, c, a)$. As for Wishart processes in Sect. 5.1.2, we calculate the infinitesimal generator on $\mathcal{M}_d(\mathbb{R})$ and on $\mathcal{S}_d(\mathbb{R})$ and use the same notations. By straightforward calculations, we get from (6.23) that the generator on $\mathcal{M}_d(\mathbb{R})$ is given by

$$\begin{aligned}L^{\mathcal{M}} = &\sum_{\substack{1\le i,j\le d \\ j\neq i}} (\kappa_i + \kappa_j)(c_{i,j} - x_{i,j})\partial_{i,j} \\ &+\frac{1}{2}\sum_{\substack{1\le i,j,k\le d \\ j\neq i,k\neq i}} a_i^2(x_{j,k} - x_{i,j}x_{i,k})[\partial_{i,j}\partial_{i,k} + \partial_{i,j}\partial_{k,i} + \partial_{j,i}\partial_{i,k} + \partial_{j,i}\partial_{k,i}].\end{aligned}$$

We can then easily deduce the generator on $\mathcal{S}_d(\mathbb{R})$ that we denote by L:

$$L = \sum_{i=1}^{d} \left(\sum_{\substack{1 \le j \le d \\ j \ne i}} \kappa_i (c_{\{i,j\}} - x_{\{i,j\}}) \partial_{\{i,j\}} + \frac{1}{2} \sum_{\substack{1 \le j,k \le d \\ j \ne i, k \ne i}} a_i^2 (x_{\{j,k\}} - x_{\{i,j\}} x_{\{i,k\}}) \partial_{\{i,j\}} \partial_{\{i,k\}} \right). \tag{6.24}$$

Then, we will say that a process $(X_t, t \ge 0)$ valued in $\mathfrak{C}_d(\mathbb{R})$ solves the martingale problem of $MRC_d(x, \kappa, c, a)$ if for any $n \in \mathbb{N}^*$, $0 \le t_1 \le \cdots \le t_n \le s \le t$, $g_1, \ldots, g_n \in \mathcal{C}(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$, $f \in \mathcal{C}^2(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$ we have:

$$\mathbb{E}\left[\prod_{i=1}^{n} g_i(X_{t_i}) \left(f(X_t) - f(X_s) - \int_s^t Lf(X_u) du \right) \right] = 0, \text{ and } X_0 = x. \tag{6.25}$$

Now, we state simple but interesting properties of mean-reverting correlation processes. Each non-diagonal coefficient follows a Wright-Fisher type diffusion and any principal submatrix is also a mean-reverting correlation process.

Proposition 6.2.3 *Let $(X_t)_{t \ge 0} \sim MRC_d(x, \kappa, c, a)$. For $1 \le i \ne j \le d$, there is Brownian motion $(\beta_t^{i,j}, t \ge 0)$ such that*

$$d(X_t)_{i,j} = (\kappa_i + \kappa_j)(c_{i,j} - (X_t)_{i,j}) dt + \sqrt{a_i^2 + a_j^2} \sqrt{1 - (X_t)_{i,j}^2} \, d\beta_t^{i,j}. \tag{6.26}$$

Let $I = \{k_1 < \cdots < k_{d'}\} \subset \{1, \ldots, d\}$ such that $1 < d' < d$. For $x \in \mathcal{M}_d(\mathbb{R})$, we define $x^I \in \mathcal{M}_{d'}(\mathbb{R})$ by $(x^I)_{i,j} = x_{k_i, k_j}$ for $1 \le i, j \le d'$. We have:

$$(X_t^I)_{t \ge 0} \overset{law}{=} MRC_{d'}(x^I, \kappa^I, c^I, a^I).$$

Proof Without loss of generality, we consider the case $I = \{1, \ldots, d'\}$ and have $(x^I)_{i,j} = x_{i,j}$ for $1 \le i, j \le d'$. Let $f \in \mathcal{C}^2(\mathcal{S}_{d'}(\mathbb{R}), \mathbb{R})$. It can be naturally extended to a function $f \in \mathcal{C}^2(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$ by setting $f(x) = f(x^I)$ for $x \in \mathcal{S}_d(\mathbb{R})$, and the generator of X^I is given by $\lim_{t \to 0} \frac{\mathbb{E}[f(X_t^I)] - f(x^I)}{t} = Lf(x)$. Since $\partial_{\{i,j\}} f(x) = 0$ if $i > d'$ or $j > d'$, we obtain $Lf(x) =$

$$\sum_{i=1}^{d'} \left(\sum_{\substack{1 \le j \le d' \\ j \ne i}} \kappa_i (c_{\{i,j\}} - x_{\{i,j\}}) \partial_{\{i,j\}} f(x) \right.$$

$$\left. + \frac{1}{2} \sum_{\substack{1 \le j,k \le d' \\ j \ne i, k \ne i}} a_i^2 (x_{\{j,k\}} - x_{\{i,j\}} x_{\{i,k\}}) \partial_{\{i,j\}} \partial_{\{i,k\}} f(x) \right).$$

This is the infinitesimal generator of $MRC_{d'}(x^I, \kappa^I, c^I, a^I)$. We conclude by using the weak uniqueness obtained in Corollary 6.2.5. $\square$

6.2.2 Calculation of Moments and the Ergodic Law

We first introduce some notations that have been already used in the proof of Proposition 5.5.2, and are useful to characterise the general form for moments. For every $x \in \mathcal{S}_d(\mathbb{R}), m \in \mathcal{S}_d(\mathbb{N})$, we set:

$$x^m = \prod_{1 \le i \le j \le d} x_{\{i,j\}}^{m_{\{i,j\}}} \text{ and } |m| = \sum_{1 \le i \le j \le d} m_{\{i,j\}}.$$

A function $f : \mathcal{S}_d(\mathbb{R}) \to \mathbb{R}$ is a polynomial function of degree smaller than $n \in \mathbb{N}$ if there are real numbers a_m such that $f(x) = \sum_{|m| \le n} a_m x^m$, and we define the norm of f by $\|f\|_{\mathbb{P}} = \sum_{|m| \le n} |a_m|$.

We want to calculate the moments $\mathbb{E}[X_t^m]$ of $(X_t, t \ge 0) \sim MRC_d(x, \kappa, c, a)$. Since the diagonal elements are equal to 1, we will take $m_{\{i,i\}} = 0$. Let us also remark that for $i \ne j$ such that $\kappa_i = \kappa_j = 0$, we have from (6.20) that $a_i = a_j = 0$. Therefore we get $(X_t)_{i,j} = x_{i,j}$ by (6.26).

Proposition 6.2.4 *Let $m \in \mathcal{S}_d(\mathbb{N})$ such that $m_{i,i} = 0$ for $1 \le i \le d$. Let $(X_t)_{t \ge 0} \sim MRC_d(x, \kappa, c, a)$. We have $Lx^m = -K_m x^m + f_m(x)$, with*

$$K_m = \sum_{i=1}^d \sum_{j=1}^d \kappa_i m_{\{i,j\}} + \frac{1}{2} \sum_{i=1}^d a_i^2 \sum_{j,k=1}^d m_{\{i,j\}}(m_{\{i,k\}} - \mathbf{1}_{j=k})$$

and

$$f_m(x) = \sum_{i=1}^d \sum_{j=1}^d \kappa_i c_{\{i,j\}} m_{\{i,j\}} x^{m - e_d^{\{i,j\}}}$$
$$+ \frac{1}{2} \sum_{i=1}^d a_i^2 \sum_{j,k=1}^d m_{\{i,j\}}(m_{\{i,k\}} - \mathbf{1}_{j=k}) x^{m - e_d^{\{i,j\}} - e_d^{\{i,k\}} + e_d^{\{j,k\}}}$$

is a polynomial function of degree smaller than $|m| - 1$. We have

$$\mathbb{E}\left[X_t^m\right] = x^m \exp(-tK_m) + \exp(-tK_m) \int_0^t \exp(sK_m) \mathbb{E}[f_m(X_s)] ds. \tag{6.27}$$

Proof The calculation of Lx^m is straightforward from (6.24). By using Itô's formula, we get easily that $\frac{d\mathbb{E}[X_t^m]}{dt} = -K_m \mathbb{E}[X_t^m] + \mathbb{E}[f_m(X_t)]$, which gives (6.27). $\square$

Equation (6.27) allows us to calculate explicitly any moment by induction on $|m|$, which shows in particular that X is a polynomial process in the sense of Cuchiero et al. [34]. Here are the formula for moments of order 1 and 2:

$$\forall 1 \le i \ne j \le d,\ \mathbb{E}\left[(X_t)_{i,j}\right] = x_{i,j} e^{-t\left(\kappa_i+\kappa_j\right)} + c_{i,j}(1 - e^{-t(\kappa_i+\kappa_j)}),$$

and for given $1 \le i \ne j \le d$ and $1 \le k \ne l \le d$ such that $\kappa_i + \kappa_j > 0$ and $\kappa_k + \kappa_l > 0$,

$$\begin{aligned}\mathbb{E}\left[(X_t)_{i,j}(X_t)_{k,l}\right] &= x_{i,j} x_{k,l} e^{-tK_{i,j,k,l}} + (\kappa_i + \kappa_j) c_{i,j} \gamma_{k,l}(t) + (\kappa_k + \kappa_l) c_{k,l} \gamma_{i,j}(t) \\ &\quad + a_i^2 \left(\mathbb{1}_{i=k} \gamma_{j,l}(t) + \mathbb{1}_{i=l} \gamma_{j,k}(t)\right) + a_j^2 \left(\mathbb{1}_{j=k} \gamma_{i,l}(t) + \mathbb{1}_{j=l} \gamma_{i,k}(t)\right),\end{aligned}$$

where $K_{i,j,k,l} = \kappa_i + \kappa_j + \kappa_k + \kappa_l + a_i^2 \left(\mathbb{1}_{i=k} + \mathbb{1}_{i=l}\right) + a_j^2 \left(\mathbb{1}_{j=k} + \mathbb{1}_{j=l}\right)$ and

$$\forall m, n \in \{i, j, k, l\},\ \gamma_{m,n}(t) = c_{m,n} \frac{1 - e^{-tK_{i,j,k,l}}}{K_{i,j,k,l}} + (x_{m,n} - c_{m,n}) \frac{e^{-t(\kappa_m+\kappa_n)} - e^{tK_{i,j,k,l}}}{K_{i,j,k,l} - \kappa_m - \kappa_n}.$$

Let f be a polynomial function of degree smaller than $n \in \mathbb{N}$. From Proposition 6.2.4, L is a linear mapping on the polynomial functions of degree smaller than n, and there is a constant $C_n > 0$ such that $\|Lf\|_{\mathbb{P}} \le C_n \|f\|_{\mathbb{P}}$. On the other hand, we have by Itô's formula $\mathbb{E}[f(X_t)] = f(x) + \int_0^t \mathbb{E}[Lf(X_s)]ds$, and by iterating $\mathbb{E}[f(X_t)] = \sum_{i=0}^k \frac{t^i}{i!} L^i f(x) + \int_0^t \frac{(t-s)^k}{k!} \mathbb{E}[L^{k+1} f(X_s)]ds$. Since $\|L^i f\|_{\mathbb{P}} \le C_n^i \|f\|_{\mathbb{P}}$, the series converges and we have

$$\mathbb{E}[f(X_t)] = \sum_{i=0}^{\infty} \frac{t^i}{i!} L^i f(x) \tag{6.28}$$

for any polynomial function f. We also remark that the same iterated Itô's formula gives

$$\begin{gathered}\forall f \in \mathcal{C}^\infty(\mathcal{S}_d(\mathbb{R}), \mathbb{R}), \forall k \in \mathbb{N}^*, \exists C > 0, \forall x \in \mathfrak{C}_d(\mathbb{R}), \\ \left|\mathbb{E}[f(X_t)] - \sum_{i=0}^{k} \frac{t^i}{i!} L^i f(x)\right| \le Ct^{k+1},\end{gathered} \tag{6.29}$$

since $L^{k+1} f$ is a bounded function on $\mathfrak{C}_d(\mathbb{R})$.

Let us discuss some interesting consequences of Proposition 6.2.4. Obviously, we can calculate explicitly in the same manner $\mathbb{E}[X_{t_1}^{m_1} \ldots X_{t_n}^{m_n}]$ for $0 \le t_1 \le \cdots \le t_n$ and $m_1, \ldots, m_n \in \mathcal{S}_d(\mathbb{N})$. Therefore, the law of $(X_{t_1}, \ldots, X_{t_n})$ is entirely determined and we get the weak uniqueness for the SDE (6.19).

Corollary 6.2.5 *Every solution $(X_t, t \ge 0)$ to the martingale problem (6.25) have the same law.*

Proposition 6.2.4 allows us to compute the limit $\lim_{t\to+\infty}\mathbb{E}[X_t^m]$ that we note $\mathbb{E}[X_\infty^m]$ by a slight abuse of notation. Let us observe that $K_m > 0$ if and only if there is i, j such that $\kappa_i + \kappa_j > 0$ and $m_{i,j} > 0$. We have

$$\mathbb{E}[X_\infty^m] = x^m \text{ if } m \in \mathcal{S}_d(\mathbb{N}) \text{ is such that } m_{\{i,j\}} > 0 \iff \kappa_i = \kappa_j = 0, \tag{6.30}$$
$$\mathbb{E}[X_\infty^m] = \mathbb{E}[f_m(X_\infty)]/K_m \text{ otherwise.}$$

Thus, X_t converges in law when $t \to +\infty$, and the moments $\mathbb{E}[X_\infty^m]$ are uniquely determined by (6.30) with an induction on $|m|$. In addition, if $\kappa_i + \kappa_j > 0$ for any $1 \le i, j \le d$ (which means that at most only one coefficient of κ is equal to 0), the law of X_∞ does not depend on the initial condition and is the unique invariant law. In this case the ergodic moments of order 1 and 2 are given by:

$$\mathbb{E}\left[(X_\infty)_{i,j}\right] = c_{i,j}, \tag{6.31}$$
$$\mathbb{E}\left[(X_\infty)_{i,j}(X_\infty)_{k,l}\right] = \frac{1}{K_{i,j,k,l}}\Big[(\kappa_i + \kappa_j + \kappa_k + \kappa_l)c_{i,j}c_{k,l} + a_i^2(\mathbb{1}_{i=k}c_{j,l} + \mathbb{1}_{i=l}c_{j,k}) + a_j^2(\mathbb{1}_{j=k}c_{i,l} + \mathbb{1}_{j=l}c_{i,k})\Big].$$

6.3 MRC and Wishart Processes

6.3.1 The Connection Between Elementary Processes

Proposition 6.1.7 illustrates a connection between Jacobi and CIR processes in dimension one. Here, we present somehow an extension of this result between some elementary Wishart processes and elementary MRC processes.

Wishart processes have been widely studied in Chap. 5. We consider the following one:

$$Y_t^y = y + \int_0^t \left((\alpha+1)a^\top a + bY_s^y + Y_s^y b^\top\right) ds + \int_0^t \left(\sqrt{Y_s^y}\,dW_s a + a^\top dW_s^\top \sqrt{Y_s^y}\right), \tag{6.32}$$

where $a, b \in \mathcal{M}_d(\mathbb{R})$ and $y \in \mathcal{S}_d^+(\mathbb{R})$. We recall that from Theorem 5.1.1, strong uniqueness holds when $\alpha \ge d$ and $y \in \mathcal{S}_d^{+,*}(\mathbb{R})$; weak existence and uniqueness holds when $\alpha \ge d - 2$. This is in fact very similar to the results that we obtain for mean-reverting correlation processes, see conditions (6.21) and (6.20).

Once we have a positive semidefinite matrix $y \in \mathcal{S}_d^+(\mathbb{R})$ such that $y_{i,i} > 0$ for $1 \le i \le d$, a trivial way to construct a correlation matrix is to consider $\mathbf{p}(y)$, where $\mathbf{p}$ is defined by

$$(\mathbf{p}(y))_{i,j} = \frac{y_{i,j}}{\sqrt{y_{i,i}y_{j,j}}},\ 1 \le i, j \le d. \tag{6.33}$$

Thus, it is somehow natural then to look at the dynamics of $\mathbf{p}(Y_t^y)$, provided that the diagonal elements of the Wishart process do not vanish. In general, this does not lead to an autonomous SDE. However, the particular case where the Wishart parameters are $a = e_d^1$ and $b = 0$ is interesting since it leads to the SDE satisfied by the mean-reverting correlation processes, up to a change of time. Obviously, we have a similar property for $a = e_d^i$ and $b = 0$ by a permutation of the ith and the first coordinates.

Proposition 6.3.1 *Let $\alpha \geq \max(1, d-2)$ and $y \in \mathcal{S}_d^+(\mathbb{R})$ such that $y_{i,i} > 0$ for $1 \leq i \leq d$. Let $(Y_t^y)_{t\geq 0} \sim WIS_d(y, \alpha+1, 0, e_d^1)$. Then, $(Y_t^y)_{i,i} = y_{i,i}$ for $2 \leq i \leq d$ and $(Y_t^y)_{1,1}$ follows a squared Bessel process of dimension $\alpha + 1$ and a.s. never vanishes. We set*

$$X_t = \mathbf{p}(Y_t^y),\ \phi(t) = \int_0^t \frac{1}{(Y_s^y)_{1,1}} ds.$$

The function ϕ is a.s. one-to-one on $\mathbb{R}_+$ and defines a time-change such that:

$$(X_{\phi^{-1}(t)}, t \geq 0) \overset{law}{=} MRC_d(\mathbf{p}(y), \frac{\alpha}{2} e_d^1, I_d, e_d^1).$$

In particular, there is a weak solution to $MRC_d(\mathbf{p}(y), \frac{\alpha}{2} e_d^1, I_d, e_d^1)$. Besides, the processes $(X_{\phi^{-1}(t)}, t \geq 0)$ and $((Y_t^y)_{1,1}, t \geq 0)$ are independent.

Proof From (6.32), $a = e_d^1$ and $b = 0$, we get $d(Y_t^y)_{i,j} = 0$ for $2 \leq i, j \leq d$ and

$$d(Y_t^y)_{1,1} = (\alpha+1)dt + 2\sum_{k=1}^d (\sqrt{Y_t^y})_{1,k} (dW_t)_{k,1},\ d(Y_t^y)_{1,i} = \sum_{k=1}^d (\sqrt{Y_t^y})_{i,k} (dW_t)_{k,1}. \tag{6.34}$$

In particular, $\langle d(Y_t^y)_{1,1}\rangle = 4(Y_t^y)_{1,1} dt$ and $(Y_t^y)_{1,1}$ is a squared Bessel process of dimension $\alpha + 1$. Since $\alpha + 1 \geq 2$, it almost surely never vanishes and $(X_t, t \geq 0)$ is well defined. Since $(X_t)_{1,i} = \frac{(Y_t^y)_{1,i}}{\sqrt{(Y_t^y)_{1,1} y_{i,i}}}$, we get from Itô's formula

$$d(X_t)_{1,i} = -\frac{\alpha}{2}(X_t)_{1,i} \frac{dt}{(Y_t^y)_{1,1}} + \sum_{k=1}^d \left(\frac{(\sqrt{Y_t^y})_{i,k}}{\sqrt{(Y_t^y)_{1,1} y_{i,i}}} - (X_t)_{1,i} \frac{(\sqrt{Y_t^y})_{1,k}}{(Y_t^y)_{1,1}} \right) (dW_t)_{k,1}. \tag{6.35}$$

By Lemma 6.1.9, $\phi(t)$ is a.s. one-to-one on $\mathbb{R}_+$, and we consider the Brownian motion $(\tilde{W}_t, t \geq 0)$ defined by $(\tilde{W}_{\phi(t)})_{i,j} = \int_0^t \frac{(dW_s)_{i,j}}{\sqrt{(Y_s^y)_{1,1}}}$. We have by straightforward calculus

$$\begin{aligned} d(X_{\phi^{-1}(t)})_{1,i} &= -\frac{\alpha}{2}(X_{\phi^{-1}(t)})_{1,i}dt \\ &+ \sum_{k=1}^{d}\left(\frac{(\sqrt{Y^y_{\phi^{-1}(t)}})_{i,k}}{\sqrt{y_{i,i}}} - (X_{\phi^{-1}(t)})_{1,i}\frac{(\sqrt{Y^y_{\phi^{-1}(t)}})_{1,k}}{\sqrt{(Y^y_{\phi^{-1}(t)})_{1,1}}}\right)(d\tilde{W}_t)_{k,1}, \qquad (6.36) \\ \langle d(X_{\phi^{-1}(t)})_{1,i}, d(X_{\phi^{-1}(t)})_{1,j}\rangle &= [(X_{\phi^{-1}(t)})_{i,j} - (X_{\phi^{-1}(t)})_{1,i}(X_{\phi^{-1}(t)})_{1,j}]dt, \end{aligned}$$

which shows by uniqueness of the solution of the martingale problem (Corollary 6.2.5) that $(X_{\phi^{-1}(t)}, t \geq 0) \overset{law}{=} MRC_d(\mathbf{p}(y), \frac{\alpha}{2}e_d^1, I_d, e_d^1)$.

Let us now show the independence. We can check easily that

$$\langle d(X_t)_{1,i}, d(X_t)_{1,j}\rangle = \frac{1}{(Y_t^y)_{1,1}}[(X_t)_{i,j} - (X_t)_{1,i}(X_t)_{1,j}]dt \text{ and } \langle d(X_t)_{1,i}, d(Y_t^y)_{1,1}\rangle = 0. \qquad (6.37)$$

We define $\Psi(y) \in \mathcal{S}_d(\mathbb{R})$ for $y \in \mathcal{S}_d^+(\mathbb{R})$ such that $y_{i,i} > 0$ by $\Psi(y)_{1,i} = \Psi(y)_{i,1} = y_{1,i}/\sqrt{y_{1,1}y_{i,i}}$ and $\Psi(y)_{i,j} = y_{i,j}$ otherwise. By (6.34) and (6.35), $(\Psi(Y_t), t \geq 0)$ solves an SDE on $\mathcal{S}_d(\mathbb{R})$. This SDE has a unique weak solution. Indeed, we can check that for any solution $(\tilde{Y}_t, t \geq 0)$ starting from $\Psi(y)$, $(\Psi^{-1}(\tilde{Y}_t), t \geq 0) \sim WIS_d(y, \alpha+1, 0, e_d^1)$, which gives our claim since Ψ is one-to-one and weak uniqueness holds for $WIS_d(y, \alpha+1, 0, e_d^1)$ by Theorem 5.1.1. Let $(B_t, t \geq 0)$ denote a real Brownian motion independent of $(W_t, t \geq 0)$. We consider a weak solution to the SDE

$$\begin{aligned} d(\bar{Y}_t)_{1,1} &= (\alpha+1)dt + 2\sqrt{(\bar{Y}_t)_{1,1}}dB_t,\ d(\bar{Y}_t)_{i,j} = 0 \text{ for } 2 \leq i, j \leq d, \\ d(\bar{Y}_t)_{1,i} &= -\frac{\alpha}{2}(\bar{Y}_t)_{1,i}\frac{dt}{(\bar{Y}_t)_{1,1}} \\ &\quad + \sum_{k=1}^{d}\left(\frac{(\sqrt{\bar{Y}_t})_{i,k}}{\sqrt{(\bar{Y}_t)_{1,1}y_{i,i}}} - (\bar{Y}_t)_{1,i}\frac{(\sqrt{\bar{Y}_t})_{1,k}}{(\bar{Y}_t)_{1,1}}\right)(dW_t)_{k,1},\ i = 2, \ldots, d \end{aligned}$$

that starts from $\bar{Y}_0 = \Psi(y)$. It solves the same martingale problem as $\Psi(Y_t)$, and therefore $(\Psi(Y_t), t \geq 0) \overset{law}{=} (\bar{Y}_t, t \geq 0)$. We set $\bar{\phi}(t) = \int_0^t \frac{1}{(\bar{Y}_s)_{1,1}}ds$. As above, $((\bar{Y}_{\bar{\phi}^{-1}(t)})_{1,i}, i = 2\ldots, d)$ solves an SDE driven by $(W_t, t \geq 0)$ and is therefore independent of $((\bar{Y}_t)_{1,1}, t \geq 0)$, which gives the desired independence. □

6.3.2 A Remarkable Splitting of the Infinitesimal Generator

In this section, we present a remarkable splitting for the mean-reverting correlation matrices. We have already obtained in Theorem 5.5.1 a very similar properties for Wishart processes. Of course, these properties are related through Proposition 6.3.1, which is illustrated in the proof below. As for Wishart processes, this remarkable splitting will be very convenient to construct simulation schemes.

Theorem 6.3.2 *Let $\alpha \geq d-2$. Let L be the generator associated to the $MRC_d(x, \frac{\alpha}{2}a^2, I_d, a)$ on $\mathfrak{C}_d(\mathbb{R})$ and L_i be the generator associated to $MRC_d(x, \frac{\alpha}{2}e^i_d, I_d, e^i_d)$, for $i \in \{1,\ldots,d\}$. Then, we have*

$$L = \sum_{i=1}^{d} a_i^2 L_i \text{ and } \quad \forall i,j \in \{1,\ldots,d\},\ L_i L_j = L_j L_i. \tag{6.38}$$

Proof The formula $L = \sum_{i=1}^d a_i^2 L_i$ is obvious from (6.24). The commutativity property can be obtained directly by a tedious but simple calculus, as we have made for Theorem 5.5.1. This is left as a calculation exercise. Here, we give another proof that uses the link between Wishart and Mean-Reverting Correlation processes given by Proposition 6.3.1.

Let L_i^W denotes the generator of $WIS_d(x, \alpha+1, 0, e^i_d)$. From Theorem 5.5.1, we have $L_i^W L_j^W = L_j^W L_i^W$ for $1 \leq i,j \leq d$. Let us consider $\alpha \geq \max(5, d-2)$ and $x \in \mathfrak{C}_d(\mathbb{R})$. We set for $i=1,2$ $(Y_t^{i,x}, t \geq 0) \sim WIS_d(x, \alpha+1, 0, e^i_d)$, and we assume that the Brownian motions of their associated SDEs are independent. Since $L_1^W L_2^W = L_2^W L_1^W$, we know from Proposition 5.5.2 that $Y_t^{1,Y_t^{2,x}} \overset{law}{=} Y_t^{2,Y_t^{1,x}}$ and thus

$$\mathbb{E}[f(\mathbf{p}(Y_t^{1,Y_t^{2,x}}))] = \mathbb{E}[f(\mathbf{p}(Y_t^{2,Y_t^{1,x}}))],$$

for any polynomial function f. By Proposition 6.3.1, $\mathbf{p}(Y_t^{1,Y_t^{2,x}}) \overset{law}{=} X^{1,\mathbf{p}(Y_t^{2,x})}_{(\phi^1)^{-1}(\phi^1(t))}$, where $(X^{1,\mathbf{p}(Y_t^{2,x})}_{(\phi^1)^{-1}(u)}, u \geq 0)$ is a mean-reverting correlation process independent of $\phi^1(t) = \int_0^t \frac{1}{(Y_s^{1,Y_t^{2,x}})_{1,1}} ds$. Since $(Y_t^{2,x})_{1,1} = 1$, $(Y_s^{1,Y_t^{2,x}})_{1,1}$ follows a squared Bessel of dimension $\alpha+1$ starting from 1. Using the independence, we get by (6.29)

$$\begin{aligned}\mathbb{E}[f(\mathbf{p}(Y_t^{1,Y_t^{2,x}}))|Y_t^{2,x}, \phi^1(t)] = f(\mathbf{p}(Y_t^{2,x})) + \phi^1(t) L_1 f(\mathbf{p}(Y_t^{2,x}))\\ + \frac{\phi^1(t)^2}{2} L_1^2 f(\mathbf{p}(Y_t^{2,x})) + O(\phi^1(t)^3).\end{aligned}$$

By Lemma 6.1.10, we have $\mathbb{E}[\phi^1(t)] = t + \frac{3-\alpha}{2}t^2 + O(t^3)$, $\mathbb{E}[\phi^1(t)^2] = t + O(t^3)$, $\mathbb{E}[\phi^1(t)^3] = O(t^3)$. Thus, we get:

$$\begin{aligned}\mathbb{E}[f(\mathbf{p}(Y_t^{2,Y_t^{1,x}}))|Y_t^{2,x}] =& f(\mathbf{p}(Y_t^{2,x})) + tL_1 f(\mathbf{p}(Y_t^{2,x})) \\ &+ \frac{t^2}{2}[L_1^2 f(\mathbf{p}(Y_t^{2,x})) + (3-\alpha)L_1 f(\mathbf{p}(Y_t^{2,x}))] + O(t^3).\end{aligned}$$

Once again, we use Proposition 6.3.1 and (6.29) to get similarly that $\mathbb{E}[f(\mathbf{p}(Y_t^{2,x}))] = f(x) + tL_2 f(x) + \frac{t^2}{2}[L_2^2 f(x) + (3-\alpha)L_2 f(x)] + O(t^3)$ for any polynomial function f. We finally get:

$$\begin{aligned}\mathbb{E}[f(\mathbf{p}(Y_t^{1,Y_t^{2,x}}))] =& f(x) + t(L_1 + L_2)f(x) \\ &+ \frac{t^2}{2}[L_1^2 f(x) + 2L_2L_1 f(x) + L_2^2 f(x) \\ &+ (3-\alpha)(L_1+L_2)f(x)] + O(t^3).\end{aligned}$$

Similarly, we also have

$$\begin{aligned}\mathbb{E}[f(\mathbf{p}(Y_t^{2,Y_t^{1,x}}))] =& f(x) + t(L_1 + L_2)f(x) \\ &+ \frac{t^2}{2}[L_1^2 f(x) + 2L_1L_2 f(x) + L_2^2 f(x) \\ &+ (3-\alpha)(L_1+L_2)f(x)] + O(t^3),\end{aligned} \tag{6.39}$$

and since both expectations are equal, we get $L_1L_2 f(x) = L_2L_1 f(x)$ for any $\alpha \geq \max(5, d-2)$. However, we can write $L_i = \frac{1}{2}(\alpha L_i^D + L_i^M)$, with

$$L_i^D = \sum_{\substack{1\leq j\leq d \\ j\neq i}} x_{\{i,j\}}\partial_{\{i,j\}} \text{ and } L_i^M = \sum_{\substack{1\leq j,k\leq d \\ j\neq i, k\neq i}} (x_{\{j,k\}} - x_{\{i,j\}}x_{\{i,k\}})\partial_{\{i,j\}}\partial_{\{i,k\}}.$$

Thus, we have $\alpha^2 L_1^D L_2^D + \alpha(L_1^D L_2^M + L_1^M L_2^D) + L_1^M L_2^M = \alpha^2 L_2^D L_1^D + \alpha(L_2^D L_1^M + L_2^M L_1^D) + L_2^M L_1^M$ for any $\alpha \geq \max(5, d-2)$. This gives $L_1^D L_2^D = L_2^D L_1^D$, $L_1^D L_2^M + L_1^M L_2^D = L_2^D L_1^M + L_2^M L_1^D$, $L_1^M L_2^M = L_2^M L_1^M$, and therefore $L_1L_2 = L_2L_1$ holds without restriction on α. □

Remark 6.3.3 Let $x \in \mathfrak{C}_d(\mathbb{R})$, $(Y_t^{1,x}, t \geq 0) \sim WIS_d(x, \alpha+1, 0, e_d^1)$ and L_1^W its infinitesimal generator. Equation (6.39) and the formula $E[f(\mathbf{p}(Y_t^{1,x}))] = f(x) + tL_1 f(x) + \frac{t^2}{2}[L_1^2 f(x) + (3-\alpha)L_1 f(x)] + O(t^3)$ used in the proof above lead formally to the following identities for $x \in \mathfrak{C}_d(\mathbb{R})$ and $f \in \mathcal{C}^\infty(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$,

$$L_1^W(f \circ \mathbf{p})(x) = L_1 f(x),\ (L_1^W)^2(f \circ \mathbf{p})(x) = L_1^2 f(x) + (3-\alpha)L_1 f(x),$$
$$L_1^W L_2^W(f \circ \mathbf{p})(x) = L_1 L_2 f(x),$$

that can be checked by basic calculations.

The property given by Theorem 6.3.2 will help us to prove the weak existence of mean-reverting correlation processes. It plays also a key role to construct discretization scheme for these diffusions. It gives a simple way to generate the law $MRC_d(x, \frac{\alpha}{2}a^2, I_d, a; t)$. Let $x \in \mathfrak{C}_d(\mathbb{R})$. We construct iteratively:

- $X_t^{1,x} \sim MRC_d(x, \frac{\alpha}{2}a_1^2 e_d^1, I_d, a_1 e_d^1; t)$.
- For $2 \le i \le d$, conditionally to $X_t^{i-1,\dots X_t^{1,x}}$, $X_t^{i,\dots X_t^{1,x}} \sim MRC_d(X_t^{i-1,\dots X_t^{1,x}}, \frac{\alpha}{2}a_i^2 e_d^i, I_d, a_i e_d^i; t)$ is sampled independently according to the distribution of a mean-reverting correlation process at time t with parameters $(\frac{\alpha}{2}a_i^2 e_d^i, I_d, a_i e_d^i)$ starting from $X_t^{i-1,\dots X_t^{1,x}}$.

Proposition 6.3.4 *Let $X_t^{d,\dots X_t^{1,x}}$ be defined as above. Then, $X_t^{d,\dots X_t^{1,x}} \sim MRC_d(x, \frac{\alpha}{2}a^2, I_d, a; t)$.*

Let us notice that

$$MRC_d(x, \frac{\alpha}{2}a_i^2 e_d^i, I_d, a_i e_d^i; t) \overset{law}{=} MRC_d(x, \frac{\alpha}{2}e_d^i, I_d, e_d^i; a_i^2 t)$$

and that $MRC_d(x, \frac{\alpha}{2}e_d^i, I_d, e_d^i; t)$ and $MRC_d(x, \frac{\alpha}{2}e_d^1, I_d, e_d^1; t)$ are the same law up to the permutation of the first and the i-th coordinate. Thus, for simulation purposes, it is sufficient to be able to focus on the approximation of the law $MRC_d(x, \frac{\alpha}{2}e_d^1, I_d, e_d^1; t)$.

Proof The proof is the same as the one of Proposition 5.5.2 for Wishart processes. Let f be a polynomial function and $X_t^x \sim MRC_d(x, \frac{\alpha}{2}a^2, I_d, a; t)$. By (6.28), $\mathbb{E}[f(X_t^x)] = \sum_{j=0}^\infty \frac{t^j}{j!} L^j f(x)$. Using once again (6.28),

$$\mathbb{E}[f(X_t^{d,\dots X_t^{1,x}})] = \mathbb{E}[\mathbb{E}[f(X_t^{d,\dots X_t^{1,x}})|X_t^{d-1,\dots X_t^{1,x}}]] = \sum_{j=0}^\infty \frac{t^j}{j!}\mathbb{E}[L_d^j f(X_t^{d-1,\dots X_t^{1,x}})],$$

and we finally obtain by iterating

$$\mathbb{E}[f(X_t^{d,\dots X_t^{1,x}})] = \sum_{j_1,\dots,j_d=0}^{\infty} \frac{t^{j_1+\dots+j_d}}{j_1!\dots j_d!} L_1^{j_1} \dots L_d^{j_d} f(x)$$
$$= \sum_{j=0}^{d} \frac{t^j}{j!}(L_1 + \dots + L_d)^j f(x) = \mathbb{E}[f(X_t^x)],$$

since the operators commute by Theorem 6.3.2. □

We can also extend Proposition 6.3.4 to the limit laws. More precisely, let us denote by $MRC_d(x, \kappa, c, a; \infty)$ the law characterized by (6.30). We define similarly for $x \in \mathfrak{C}_d(\mathbb{R})$, $X_\infty^{1,x} \sim MRC_d(x, \frac{\alpha}{2}a_1^2 e_d^1, I_d, a_1 e_d^1; \infty)$ and, conditionally to $X_\infty^{i-1,\dots X_\infty^{1,x}}$, $X_\infty^{i,\dots X_\infty^{1,x}} \sim MRC_d(X_\infty^{i-1,\dots X_\infty^{1,x}}, \frac{\alpha}{2}a_i^2 e_d^i, I_d, a_i e_d^i; \infty)$ for $2 \le i \le d$. We have:

$$X_\infty^{d,\dots X_\infty^{1,x}} \sim MRC_d(x, \frac{\alpha}{2}a^2, I_d, a; \infty). \tag{6.40}$$

To check this we consider $(X_t, t \ge 0) \sim MRC_d(x, \frac{\alpha}{2}a^2, I_d, a)$ and $m \in \mathcal{S}_d(\mathbb{N})$ such that $m_{i,i} = 0$. By Proposition 6.2.4, $\mathbb{E}[X_t^m]$ is a polynomial function of x that we write $\mathbb{E}[X_t^m] = \sum_{m' \in \mathcal{S}_d(\mathbb{N}), |m'| \le |m|} \gamma_{m,m'}(t) x^{m'}$. From the convergence in law (6.30), we get that the coefficients $\gamma_{m,m'}(t)$ go to a limit $\gamma_{m,m'}(\infty)$ when $t \to +\infty$, and $\mathbb{E}[X_\infty^m] = \sum_{|m'| \le |m|} \gamma_{m,m'}(\infty) x^{m'}$. Similarly, the moment m of $MRC_d(x, \frac{\alpha}{2}a_i^2 e_d^i, I_d, a_i e_d^i; t)$ can be written as $\sum_{|m'| \le |m|} \gamma_{m,m'}^i(t) x^{m'}$. We get from Proposition 6.3.4:

$$\mathbb{E}[X_t^m] = \sum_{|m_1| \le \dots \le |m_d| \le |m|} \gamma_{m,m_d}^d(t) \gamma_{m_d,m_{d-1}}^{d-1}(t) \dots \gamma_{m_2,m_1}^1(t) x^{m_1},$$

which gives (6.40) by letting $t \to +\infty$.

6.3.3 A Link with the Multi-allele Wright-Fisher Model

Theorem 6.3.2 and Proposition 6.3.4 show that any law $MRC_d(x, \frac{\alpha}{2}a^2, I_d, a; t)$ can be obtained by composition with the elementary law $MRC_d(x, \frac{\alpha}{2}, I_d, e_d^1; t)$. Here, we give an identity in law which enables us to focus on the the distribution of $MRC_d(x, \frac{\alpha}{2}, I_d, e_d^1; t)$ when $(x_{i,j})_{2 \le i,j \le d} = I_{d-1}$.

Proposition 6.3.5 *Let $x \in \mathfrak{C}_d(\mathbb{R})$. Let $u \in \mathcal{M}_{d-1}(\mathbb{R})$ and $\check{x} \in \mathfrak{C}_d(\mathbb{R})$ such that $x = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \check{x} \begin{pmatrix} 1 & 0 \\ 0 & u^\top \end{pmatrix}$ and $(\check{x})_{2 \le i,j \le d} = I_{d-1}$ (Lemma A.3.3 gives a construction* *of such matrices). Then, for* $\alpha \geq 2$*,*

$$MRC_d(x, \frac{\alpha}{2}e_d^1, I_d, e_d^1) \overset{law}{=} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} MRC_d(\check{x}, \frac{\alpha}{2}e_d^1, I_d, e_d^1) \begin{pmatrix} 1 & 0 \\ 0 & u^\top \end{pmatrix}.$$

Proof Let $(\check{X}_t, t \geq 0) \sim MRC_d(\check{x}, \frac{\alpha}{2}e_d^1, I_d, e_d^1)$. We set $X_t = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \check{X}_t \begin{pmatrix} 1 & 0 \\ 0 & u^\top \end{pmatrix}$. Clearly, $((\check{X}_t)_{i,j})_{2 \leq i,j \leq d} = I_{d-1}$ and the matrix $((X_t)_{i,j})_{2 \leq i,j \leq d}$ is constant and equal to $uu^\top = (x_{i,j})_{2 \leq i,j \leq d}$. We have for $2 \leq i \leq d$, $(X_t)_{1,i} = \sum_{k=2}^d u_{i-1,k-1}(\check{X}_t)_{1,k}$. By (6.23), we get $\langle d(\check{X}_t)_{1,k}, d(\check{X}_t)_{1,l} \rangle = [\mathbb{1}_{k=l} - (\check{X}_t)_{1,k}(\check{X}_t)_{1,l}]dt$. Therefore, the quadratic variations

$$\begin{aligned} \langle d(X_t)_{1,i}, d(X_t)_{1,j} \rangle &= \left(\sum_{k=2}^d u_{i-1,k-1} u_{j-1,k-1} \right. \\ &\qquad \left. - \sum_{k,l=2}^d u_{i-1,k-1}(\check{X}_t)_{1,k} u_{i-1,l-1}(\check{X}_t)_{1,l} \right) dt \\ &= \big((X_t)_{i,j} - (X_t)_{1,i}(X_t)_{1,j} \big)\, dt, \end{aligned}$$

are by (6.23) the one of $MRC_d(x, \frac{\alpha}{2}e_d^1, I_d, e_d^1)$. This gives the claim by using the weak uniqueness (Corollary 6.2.5). □

For $x \in \mathcal{S}_d(\mathbb{R})$ such that $(x_{i,j})_{2 \leq i,j \leq d} = I_{d-1}$ and $x_{1,1} = 1$, we have $\det(x) = 1 - \sum_{i=2}^d x_{1,i}^2$ and therefore

$$x \in \mathfrak{C}_d(\mathbb{R}) \iff \sum_{i=2}^d x_{1,i}^2 \leq 1. \tag{6.41}$$

The process $(X_t)_{t \geq 0} \sim MRC_d(x, \frac{\alpha}{2}, I_d, e_d^1; t)$ is such that $((X_t)_{i,j})_{2 \leq i,j \leq d} = I_{d-1}$. In this case, the only non constant elements are on the first row (or column). More precisely, $((X_t)_{1,i})_{i=2,\ldots,d}$ is a vector process on the unit ball in dimension $d-1$ such that

$$\langle d(X_t)_{1,i}, d(X_t)_{1,j} \rangle = (\mathbb{1}_{i=j} - (X_t)_{1,i}(X_t)_{1,j})dt.$$

For $i = 1, \ldots, d-1$, we set $\zeta_t^i = (X_t)_{1,i+1}^2$. We have $\langle d\zeta_t^i, d\zeta_t^j \rangle = 4\zeta_t^i(\mathbb{1}_{i=j} - \zeta_t^j)dt$ and the drift of ζ_t^i is $(1-(1+2\alpha)\zeta_t^i)dt$. Thus, $(\zeta_t^i)_{1 \leq i \leq d-1}$ satisfies $\sum_{i=1}^{d-1} \zeta_t^i \leq 1$ and has the following infinitesimal generator

$$\sum_{i=1}^{d-1}[1 - (1 + 2\alpha)z_i]\partial_{z_i} + 2 \sum_{1 \leq i,j \leq d-1} z_i(\mathbb{1}_{i=j} - z_j)\partial_{z_i}\partial_{z_j}.$$

This is a particular case of the multi-allele Wright-Fisher diffusion (see for example the lecture notes of Etheridge [51], Chap. 4), where $(\zeta_t^1, \dots, \zeta_t^{d-1}, 1 - \sum_{i=1}^{d-1} \zeta_t^i)$ describes population ratios along the time. Similar diffusions have also been considered by Gourieroux and Jasiak [68] in a different context. Roughly speaking, $((X_t)_{1,i})_{2 \le i \le d}$ can be seen as a square-root of a multi-allele Wright-Fisher diffusion that is such that its drift coefficient remains linear.

Also, the identity in law given by Proposition 6.3.5 allows us to compute more explicitly the ergodic limit law. Let $x \in \mathfrak{C}_d(\mathbb{R})$ such that $(x_{i,j})_{2 \le i,j \le d} = I_{d-1}$, $(X_t^x)_{t \ge 0} \sim MRC_d(x, \frac{\alpha}{2} e_d^1, I_d, e_d^1)$ and $(Y_t^x)_{t \ge 0} \sim WIS_d(x, \alpha + 1, 0, e_d^1)$. We know by Theorem 5.5.3 that $((Y_t^x)_{i,j})_{1 \le i,j \le d} = I_{d-1}$ and

$$((Y_t^x)_{1,i})_{1 \le i \le d} \overset{law}{=} (Z_t^{x_{1,1}} + \sum_{i=2}^{d} (x_{1,i} + \sqrt{t} N_i)^2, x_{1,2} + \sqrt{t} N_2, \dots, x_{1,d} + \sqrt{t} N_d),$$

where $N_i \sim \mathcal{N}(0,1)$ are independent standard Gaussian variables and $Z_t^{x_{1,1}} = x_{1,1} + (\alpha + 2 - d)t + 2 \int_0^t \sqrt{Z_u^{x_{1,1}}} d\beta_u$ is a Bessel process independent of the Gaussian variables starting from $x_{1,1}$. By a time scaling, we have $Z_t^{x_{1,1}} \overset{law}{=} t Z_1^{x_{1,1}/t}$, and thus:

$$(\mathbf{p}(Y_t^x)_{1,i})_{2 \le i \le d} \overset{law}{=} \frac{\left(\frac{x_{1,2}}{\sqrt{t}} + N_2, \dots, \frac{x_{1,d}}{\sqrt{t}} + N_d \right)}{\sqrt{Z_1^{x_{1,1}/t} + \sum_{i=2}^{d} (\frac{x_{1,i}}{\sqrt{t}} + N_i)^2}} \underset{t \to +\infty}{\to} \frac{(N_2, \dots, N_d)}{\sqrt{Z_1^0 + \sum_{i=2}^{d} N_i^2}}.$$

On the other hand, we know that X_t^x converges in law when $t \to +\infty$, and Proposition 6.3.1 immediately gives, with the help of Lemma 6.1.9 that $((X_\infty^x)_{1,i})_{2 \le i \le d} \overset{law}{=} \frac{(N_2, \dots, N_d)}{\sqrt{Z_1^0 + \sum_{i=2}^{d} N_i^2}}$. By simple calculations, we get that $((X_\infty^x)_{1,i})_{2 \le i \le d}$ has the following density:

$$\mathbb{1}_{\sum_{i=2}^{d} z_i^2 \le 1} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{(\sqrt{\pi})^{d-1} \Gamma\left(\frac{\alpha+2-d}{2}\right)} \left(1 - \sum_{i=2}^{d} z_i^2\right). \tag{6.42}$$

In particular, we can check that $((X_\infty^x)_{1,i}^2)_{2 \le i \le d}$ follows a Dirichlet law, which is known as the ergodic limit of multi-allele Wright-Fisher models. Last, let us mention that we can get an explicit but cumbersome expression of the density of the law $MRC_d(x, \frac{\alpha}{2} a^2, I_d, a; \infty)$ by combining (6.40), Proposition 6.3.5 and (6.42).

6.4 Existence and Uniqueness Results for MRC Processes

In this section we show weak and strong existence results for the SDE (6.19), respectively under assumptions (6.20) and (6.21). These assumptions are of the same nature as the one known for Wishart processes. To prove the strong existence and uniqueness, we make assumptions on the coefficients that ensures that X_t remains in the set of the invertible correlation matrices where the coefficients are locally Lipschitz. Then, we prove the weak existence by introducing a sequence of processes defined on $\mathfrak{C}_d(\mathbb{R})$, which is tight such that any subsequence limit solves the martingale problem (6.25). Next, we extend our existence results when the parameters are no longer constant.

6.4.1 Strong Existence and Uniqueness

Theorem 6.4.1 *Let $x \in \mathfrak{C}^*_d(\mathbb{R})$. We assume that (6.21) holds. Then, there is a unique strong solution of the SDE (6.19) that is such that $\forall t \geq 0$, $X_t \in \mathfrak{C}^*_d(\mathbb{R})$.*

Proof For $x \in \mathcal{S}_d(\mathbb{R})$ and $1 \leq n \leq d$, we denote by $x^{[n]}$ the symmetric matrix obtained from x by removing the n^{th} row and column and x^n the vector obtained from the n^{th} column of x by removing its n^{th} element, see the notations page xi.

By Lemma A.3.1, we have $\left(\sqrt{x - xe^n_d x}\right)^{[n]} = \sqrt{x^{[n]} - x^n(x^n)^\top}$ and $x^{[n]} - x^n(x^n)^\top \in \mathcal{S}^{+,*}_{d-1}(\mathbb{R})$ when $x \in \mathfrak{C}^*_d(\mathbb{R})$. For $x \in \mathcal{S}^{+,*}_d(\mathbb{R})$ such that $x^{[n]} - x^n(x^n)^\top \in \mathcal{S}^{+,*}_{d-1}(\mathbb{R})$, we define $f^n(x) \in \mathcal{S}^+_d(\mathbb{R})$ by $(f^n(x))_{n,j} = 0$ for $1 \leq j \leq d$ and $(f^n(x))^{[n]} = \sqrt{x^{[n]} - x^n(x^n)^\top}$. The function f^n is well defined on an open set of $\mathcal{S}_d(\mathbb{R})$ that includes $\mathfrak{C}^*_d(\mathbb{R})$, and is such that $f^n(x) = \sqrt{x - xe^n_d x}$ for $x \in \mathfrak{C}^*_d(\mathbb{R})$. Since the square-root of a positive semi-definite matrix is locally Lipschitz on the positive definite matrix set, we get that the SDE

$$X_t = x + \int_0^t (\kappa(c - X_s) + (c - X_s)\kappa)\, ds + \sum_{n=1}^d a_n \int_0^t \left(f^n(X_s) dW_s e^n_d \right.$$
$$\left. + e^n_d dW_s^\top f^n(X_s)\right),$$

has a unique strong solution for $0 \leq t < \tau$, where

$$\tau = \inf\{t \geq 0, X_t \notin \mathcal{S}^{+,*}_d(\mathbb{R}) \text{ or } \exists i \in \{1, \dots, d\},$$
$$X_t^{[i]} - X^i_t (X^i_t)^\top \notin \mathcal{S}^{+,*}_{d-1}(\mathbb{R})\}, \ \inf \emptyset = +\infty.$$

For $1 \leq i \leq d$, we have $(f^n(X_s) dW_s e^n_d)_{i,i} = \mathbb{1}_{i=n} \sum_{j=1}^d f^n(X_s)_{n,j} (dW_s)_{j,n} = 0$ and then:

$$d(X_t)_{i,i} = 2\kappa_{i,i}(1 - (X_t)_{i,i})dt,$$

which immediately gives $(X_t)_{i,i} = 1$ for $0 \le t < \tau$. Thus, $X_t \in \mathfrak{C}_d^*(\mathbb{R})$ for $0 \le t < \tau$ and $\tau = \inf\{t \ge 0, X_t \notin \mathfrak{C}_d^*(\mathbb{R})\}$ by Lemma A.3.1, and the process X_t is solution of (6.19) up to time τ. From (5.13), we have by Itô's Formula gives for $t < \tau$:

$$\frac{d(\det(X_t))}{\det(X_t)} = \sum_{1 \le i,j \le d} (X_t^{-1})_{i,j} d(X_t)_{i,j}$$
$$+ \frac{1}{2} \sum_{\substack{1 \le i,j \le d \\ 1 \le k,l \le d}} \left((X_t^{-1})_{i,j}(X_t^{-1})_{k,l} - (X_t^{-1})_{i,k}(X_t^{-1})_{j,l}\right) \langle d(X_t)_{i,j}, d(X_t)_{k,l}\rangle.$$

On the one hand we have

$$\sum_{1 \le i,j \le d} (X_t^{-1})_{i,j} d(X_t)_{i,j} = \mathrm{Tr}[X_t^{-1}(\kappa c + c\kappa)]dt - \mathrm{Tr}(2\kappa)dt$$
$$+ 2\sum_{i=1}^{d} a_i \mathrm{Tr}\left[X_t^{-1} e_d^i dW_s^\top \sqrt{X_t - X_t e_d^i X_t}\right].$$

On the other hand we get by (6.23):

$$\sum_{\substack{1 \le i,j \le d \\ 1 \le k,l \le d}} \left((X_t^{-1})_{i,j}(X_t^{-1})_{k,l} - (X_t^{-1})_{i,k}(X_t^{-1})_{j,l}\right) \langle d(X_t)_{i,j}, d(X_t)_{k,l}\rangle$$
$$= \sum_{\substack{1 \le i,j \le d \\ 1 \le k,l \le d}} \left((X_t^{-1})_{i,j}(X_t^{-1})_{k,l} - (X_t^{-1})_{i,k}(X_t^{-1})_{j,l}\right) \times \Big\{a_j^2 \mathbb{1}_{j=k}(X_t - X_t e_d^j X_t)_{i,l}$$
$$+ a_j^2 \mathbb{1}_{j=l}(X_t - X_t e_d^j X_t)_{i,k} + a_i^2 \mathbb{1}_{i=l}(X_t - X_t e_d^i X_t)_{j,k}$$
$$+ a_i^2 \mathbb{1}_{i=k}(X_t - X_t e_d^i X_t)_{j,l}\Big\}$$
$$= \sum_{j=1}^{d}\left(\sum_{1 \le i,k \le d} a_j^2 (X_t - X_t e_d^j X_t)_{i,k}\left((X_t^{-1})_{i,j}(X_t^{-1})_{k,j} - (X_t^{-1})_{i,k}(X_t^{-1})_{j,j}\right)\right)$$
$$+ \sum_{i=1}^{d}\left(\sum_{1 \le j,l \le d} a_i^2 (X_t - X_t e_d^i X_t)_{j,l}\left((X_t^{-1})_{i,j}(X_t^{-1})_{i,l} - (X_t^{-1})_{i,i}(X_t^{-1})_{j,l}\right)\right)$$
$$= 2\sum_{i=1}^{d} a_i^2 \left(\mathrm{Tr}\left[(X_t - X_t e_d^i X_t) X_t^{-1} e_d^i X_t^{-1}\right] - (X_t^{-1})_{i,i} \mathrm{Tr}\left[(X_t - X_t e_d^i X_t) X_t^{-1}\right]\right).$$

Since $X_t \in \mathfrak{C}_d^*(\mathbb{R})$, we obtain that $\mathrm{Tr}\left[(X_t - X_t e_d^i X_t) X_t^{-1} e_d^i X_t^{-1}\right] = (X_t^{-1})_{i,i} - 1$ and $\mathrm{Tr}\left[X_t^{-1}(X_t - X_t e_d^i X_t)\right] = d - (X_t)_{i,i} = d - 1$. We finally get:

$$\frac{d(\det(X_t))}{\det(X_t)} = \mathrm{Tr}[X_t^{-1}(\kappa c + c\kappa - (d-2)a^2)]dt - \mathrm{Tr}(2\kappa + a^2)dt \tag{6.43}$$
$$+ 2\sum_{i=1}^d a_i \mathrm{Tr}\left[X_t^{-1} e_d^i dW_s^\top \sqrt{X_t - X_t e_d^i X_t}\right].$$

Now, we compute the quadratic variation of $\det(X_t)$ by using (6.23):

$$\begin{aligned}
\frac{\langle d \det(X_t)\rangle}{\det(X_t)^2} &= \sum_{\substack{1 \le i,j \le d \\ 1 \le k,l \le d}} (X_t^{-1})_{i,j}(X_t^{-1})_{k,l} \\
&\quad \times \Big\{ a_j^2 \mathbb{1}_{j=k}(X_t - X_t e_d^j X_t)_{i,l} + a_j^2 \mathbb{1}_{j=l}(X_t - X_t e_d^j X_t)_{i,k} \\
&\quad + a_i^2 \mathbb{1}_{i=l}(X_t - X_t e_d^i X_t)_{j,k} + a_i^2 \mathbb{1}_{i=k}(X_t - X_t e_d^i X_t)_{j,l} \Big\} dt \\
&= 4\sum_{i=1}^d a_i^2 \mathrm{Tr}\left[X_t^{-1} e_d^i X_t^{-1}(X_t - X_t e_d^i X_t)\right] dt \\
&= 4\sum_{i=1}^d a_i^2((X_t^{-1})_{i,i} - 1)dt = 4[\mathrm{Tr}(a^2 X_t^{-1}) - \mathrm{Tr}(a^2)]dt.
\end{aligned}$$

It is indeed nonnegative: we can show by diagonalizing and using the convexity of $z \mapsto 1/z$ for $z > 0$ that $x_{i,i}^{-1} \ge 1/x_{i,i} = 1$ for $x \in \mathfrak{C}_d^*(\mathbb{R})$. Then, there is a Brownian motion $(\beta_t, t \ge 0)$ such that

$$d\log(\det(X_t)) = \mathrm{Tr}[X_t^{-1}(\kappa c + c\kappa - da^2)]dt - \mathrm{Tr}(2\kappa - a^2)dt$$
$$+ 2\sqrt{\mathrm{Tr}\left[a^2(X_t^{-1} - I_d)\right]}d\beta_t. \tag{6.44}$$

We now define $Y_t = \log(\det(X_t)) + \mathrm{Tr}(2\kappa - a^2)t$, and obtain from (6.44)

$$\begin{aligned}
Y_t &= Y_0 + \int_0^t \mathrm{Tr}[X_s^{-1}(\kappa c + c\kappa - da^2)]ds + 2\int_0^t \sqrt{\mathrm{Tr}[a^2(X_t^{-1} - I_d)]}d\beta_s \\
&\ge Y_0 + 2\int_0^t \sqrt{\mathrm{Tr}[a^2(X_t^{-1} - I_d)]}d\beta_s,
\end{aligned}$$

since $\kappa c + c\kappa - da^2 \in \mathcal{S}_d^+(\mathbb{R})$ by Assumption (6.21). Now, we use the McKean argument as for Wishart processes: on $\{\tau < \infty\}$, $Y_t \underset{t \to \tau}{\to} -\infty$, which implies that the local martingale $\int_0^t \sqrt{\mathrm{Tr}[a^2(X_t^{-1} - I_d)]} d\beta_s$ also goes to $-\infty$ when $t \to \tau$. We deduce that $\tau = +\infty$, a.s. □

6.4.2 Linear ODEs on Correlation Matrices

To prepare the proof of the weak existence of MRC processes, we first need to characterize linear ordinary differential equations that naturally stay in the set of correlation matrices. Let $b \in \mathcal{S}_d(\mathbb{R})$ and $\kappa \in \mathcal{M}_d(\mathbb{R})$. In this section, we consider the following linear ODE

$$x'(t) = b - (\kappa x(t) + x(t)\kappa^\top),\ x(0) = x \in \mathfrak{C}_d(\mathbb{R}), \tag{6.45}$$

and we are interested in necessary and sufficient conditions on κ and b such that

$$\forall x \in \mathfrak{C}_d(\mathbb{R}), \forall t \geq 0, x(t) \in \mathfrak{C}_d(\mathbb{R}). \tag{6.46}$$

Let us first look at necessary conditions. We have for $1 \leq i, j \leq d$:

$$x'_{i,j}(t) = b_{i,j} - \sum_{k=1}^{d} \kappa_{i,k} x_{k,j}(t) + x_{i,k}(t)\kappa_{j,k}.$$

In particular, we necessarily have $x'_{i,i}(t) = 0$. This gives for $t = 0$, $l \neq i$ and $x(0) = I_d + \rho(e_d^{i,l} + e_d^{l,i})$ that $b_{i,i} - 2\kappa_{i,i} - 2\rho\kappa_{i,l} = 0$ for any $\rho \in [-1, 1]$. It comes out that:

$$\kappa_{i,l} = 0 \text{ if } l \neq i,\ b_{i,i} = 2\kappa_{i,i}.$$

Thus, the matrix κ is diagonal and we denote $\kappa_i = \kappa_{i,i}$. We get $x'_{i,j}(t) = b_{i,j} - (\kappa_i + \kappa_j)x_{i,j}(t)$ for $i \neq j$. If $\kappa_i + \kappa_j = 0$, we have $x_{i,j}(t) = x_{i,j} + b_{i,j}t$, which implies that $b_{i,j} = 0$. Otherwise, $\kappa_i + \kappa_j \neq 0$ and we get:

$$x_{i,j}(t) = x_{i,j} \exp\big(-(\kappa_i + \kappa_j)t\big) + \frac{b_{i,j}}{\kappa_i + \kappa_j}\big[1 - \exp\big(-(\kappa_i + \kappa_j)t\big)\big].$$

Once again, this implies that $\kappa_i + \kappa_j > 0$ since the initial value $x \in \mathfrak{C}_d(\mathbb{R})$ is arbitrary. We set for $1 \leq i, j \leq d$,

$$c_{i,i} = 1, \text{ and for } i \neq j,\ c_{i,j} = \begin{cases} \frac{b_{i,j}}{\kappa_i + \kappa_j} \text{ if } \kappa_i + \kappa_j > 0 \\ 0 \text{ if } \kappa_i + \kappa_j = 0. \end{cases} \tag{6.47}$$

We have $b = \kappa c + c\kappa$ and for $x = I_d$, $c = \lim_{t\to+\infty} x(t) \in \mathfrak{C}_d(\mathbb{R})$, and deduce the following result.

Proposition 6.4.2 *Let $b \in \mathcal{S}_d(\mathbb{R})$ and $\kappa \in \mathcal{M}_d(\mathbb{R})$. If the linear ODE (6.45) satisfies (6.46), then we have necessarily:*

$$\exists c \in \mathfrak{C}_d(\mathbb{R}), \exists \kappa_1, \dots, \kappa_d \in \mathbb{R}, \forall i \neq j, \kappa_i + \kappa_j \geq 0,$$
$$\kappa = diag(\kappa_1, \dots, \kappa_d) \text{ and } b = \kappa c + c\kappa. \tag{6.48}$$

Conversely, let us assume that (6.48) holds and $b \in \mathcal{S}_d^+(\mathbb{R})$. We get that $\kappa_i = b_{i,i}/2 \geq 0$ and the solution of (6.45) satisfies for $t \geq 0$,

$$\exp(\kappa t)x(t)\exp(\kappa t) = x + \int_0^t \exp(\kappa s)b\exp(\kappa s)ds.$$

It is then clearly positive semidefinite. Besides, we have $x'_{i,i}(t) = b_{i,i} - 2\kappa_i x_{i,i}(t)$ with $x_{i,i}(0) = 1$, which gives $x_{i,i}(t) = 1$. Therefore, (6.46) holds. We get the following result.

Proposition 6.4.3 *Let $\kappa_1, \dots, \kappa_d \geq 0$, $\kappa = diag(\kappa_1, \dots, \kappa_d)$ and $c \in \mathfrak{C}_d(\mathbb{R})$. If $\kappa c + c\kappa \in \mathcal{S}_d^+(\mathbb{R})$ or $d = 2$, the ODE*

$$x'(t) = \kappa(c - x) + (c - x)\kappa, \ x(0) = x \in \mathfrak{C}_d(\mathbb{R}) \tag{6.49}$$

satisfies (6.46).

Let us note here that the parametrization of the ODE (6.49) is redundant when $d = 2$, and we can assume without loss of generality that $\kappa_1 = \kappa_2$ for which $\kappa c + c\kappa \in \mathcal{S}_d^+(\mathbb{R})$ is clearly satisfied.

Remark 6.4.4 The condition given by Proposition 6.4.2 is necessary but not sufficient, and the condition given by Proposition 6.4.3 is sufficient but not necessary. Let $d = 3$ and $c = I_3$. We can check that for $\kappa = (1, \frac{1}{2}, -\frac{1}{2})$, (6.48) holds but (6.46) is not true. Also, we can check that for $\kappa = (1, 1, -\frac{1}{2})$, (6.46) holds.

Lemma 6.4.5 *Let κ^1, κ^2 be diagonal matrices and $c^1, c^2 \in \mathfrak{C}_d(\mathbb{R})$ such that $\kappa^1 c^1 + c^1\kappa^1 + \kappa^2 c^2 + c^2\kappa^2 \in \mathcal{S}_d^+(\mathbb{R})$. Then, the ODE*

$$x' = \kappa^1(c^1 - x) + (c^1 - x)\kappa^1 + \kappa^2(c^2 - x) + (c^2 - x)\kappa^2$$

satisfies (6.46). Besides, $x' = \kappa(c - x) + (c - x)\kappa$ with $\kappa = \kappa^1 + \kappa^2 \in \mathcal{S}_d^+(\mathbb{R})$ and $c \in \mathfrak{C}_d(\mathbb{R})$ defined by:

$$c_{i,i} = 1, \ \text{and for } i \neq j, \ c_{i,j} = \begin{cases} \frac{(\kappa_i^1 + \kappa_j^1)c_{i,j}^1 + (\kappa_i^2 + \kappa_j^2)c_{i,j}^2}{\kappa_i + \kappa_j} & \text{if } \kappa_i + \kappa_j > 0 \\ 0 \text{ if } \kappa_i + \kappa_j = 0. \end{cases}$$

Proof Since $b = \kappa^1 c^1 + c^1\kappa^1 + \kappa^2 c^2 + c^2\kappa^2 \in \mathcal{S}_d^+(\mathbb{R})$, (6.46) holds for $x' = b - \kappa x + x\kappa$. Then, we know by (6.47) that c is a correlation matrix. □

6.4.3 Weak Existence and Uniqueness

The weak uniqueness has already been obtained in Proposition 6.2.4, and we provide in this section a constructive proof of a weak solution to the SDE (6.19). In the case $d = 2$, this result is already well-known. In fact, by Proposition 6.2.3, the associated martingale problem is the one of a one-dimensional Jacobi process. For this SDE, strong (and therefore weak) existence and uniqueness holds since the diffusion coefficient is $1/2$-Hölderian as it has been shown in Theorem 6.1.1.

Thus, we can assume without loss of generality that $d \geq 3$. The first step is to focus on the existence when $a = diag(a_1, \dots, a_d) \in \mathcal{S}_d^+(\mathbb{R})$, $\alpha \geq d-2$, $\kappa = \frac{\alpha}{2}a^2$ and $c = I_d$. By Proposition 6.3.1, we know that weak existence holds for $MRC_d(x, \frac{\alpha}{2}e_d^1, I_d, e_d^1)$, and thus for $MRC_d(x, \frac{\alpha}{2}a_i^2 e_d^i, I_d, a_i e_d^i)$ for $i = 1, \dots, d$ and $a_i \geq 0$, by using a permutation of the coordinates and a linear time-scaling. Therefore, by using Proposition 6.3.4, the distribution $MRC_d(x, \frac{\alpha}{2}a^2, I_d, a; t)$ is also well-defined on $\mathfrak{C}_d(\mathbb{R})$ for any $t \geq 0$. Let $T > 0$ be a time-horizon, $N \in \mathbb{N}^*$, and $t_i^N = iT/N$. We define $(\hat{X}_t^N, t \in [0, T])$ as follows.

- We set $\hat{X}_0^N = x$.
- For $i = 0, \dots, N-1$, $\hat{X}_{t_{i+1}^N}^N$ is sampled according to the law $MRC_d(\hat{X}_{t_i^N}^N, \frac{\alpha}{2}a^2, I_d, a; T/N)$, conditionally to $\hat{X}_{t_i^N}^N$.
- For $t \in [t_i^N, t_{i+1}^N]$, $\hat{X}_t^N = \frac{t-t_i^N}{T/N}\hat{X}_{t_i^N}^N + \frac{t_{i+1}^N - t}{T/N}\hat{X}_{t_{i+1}^N}^N = \hat{X}_{t_i^N}^N + \frac{t-t_i^N}{T/N}(\hat{X}_{t_{i+1}^N}^N - \hat{X}_{t_i^N}^N)$.

The process $(\hat{X}_t^N, t \in [0, T])$ is continuous and such that almost surely, $\forall t \in [0, T]$, $\hat{X}_t^N \in \mathfrak{C}_d(\mathbb{R})$. We endow the set of matrices with the norm $\|x\| = \left(\sum_{i,j=1}^d x_{i,j}^4\right)^{1/4}$. The sequence of processes $(\hat{X}_t^N, t \in [0, T])_{N \geq 1}$ satisfies the following Kolmogorov tightness criterion.

Lemma 6.4.6 *Under the assumptions above, there is a constant $K > 0$ such that:*

$$\forall 0 \leq s \leq t \leq T,\ \mathbb{E}[\|\hat{X}_t^N - \hat{X}_s^N\|^4] \leq K(t-s)^2. \tag{6.50}$$

To prove this Kolmogorov criterion, we will need the following lemma.

Lemma 6.4.7 *Let $k, \theta, \eta \geq 0$. For a given $x \in [-1.1]$, let us consider a process $(X_t^x)_{t \geq 0}$, starting from x, and defined as the solution of the following SDE*

$$dX_t^x = k(\theta - X_t^x)dt + \eta\sqrt{1-(X_t^x)^2}dB_t, \tag{6.51}$$

where $(B_t)_{t\geq 0}$ is a real Brownian motion. Then there exists a positive constant $K > 0$, such that

$$\forall t \geq 0, \forall x \in [-1,1],\ \mathbb{E}\left[(X_t^x - x)^4\right] \leq Kt^2.$$

Proof For a given $x \in [-1,1]$, we set $f^x(y) = (y-x)^4$. We denote $Lf(x) = k(\theta - x)f'(x) + \frac{1}{2}\eta^2(1-x^2)f''(x)$ the infinitesimal generator. We notice that $f^x(x) = Lf^x(x) = 0$. Besides, $(x,y) \in [-1,1]^2 \mapsto L^2 f^x(y)$ is continuous and therefore bounded:

$$\exists K > 0, \forall x, y \in [-1,1], |L^2 f^x(y)| \leq 2K. \tag{6.52}$$

Since the process $(X_t^x)_{t\geq 0}$ is defined on $[-1,1]$, we get by applying twice Itô's formula:

$$\mathbb{E}\left[f^x(X_t^x)\right] = \int_0^t \int_0^s \mathbb{E}\left[L^2 f^x(X_u^x)\right] du ds.$$

From (6.52), one can deduce that $\left|\int_0^t \int_0^s \mathbb{E}\left[L^2 f^x(X_u^x)\right] du ds\right| \leq Kt^2$, and obtain the final result. □

Proof of Lemma 6.4.6 We first consider the case $s = t_k^N$ and $t = t_l^N$ for some $0 \leq k \leq l \leq N$. Then, by Proposition 6.3.4, we know that conditionally on $\hat{X}_{t_k^N}^N$, $\hat{X}_{t_l^N}^N$ follows the law of $MRC_d(\hat{X}_{t_k^N}^N, \frac{\alpha}{2}a^2, I_d, a)$. In particular, each element $(\hat{X}_{t_l^N}^N)_{i,j}$ follows the marginal law of a one-dimensional Jacobi process with parameters given by Eq. (6.26). Thus, by Lemma 6.4.7 there is a constant still denoted by $K > 0$ such that for any $1 \leq i, j \leq d$, $\mathbb{E}[((\hat{X}_{t_l^N}^N)_{i,j} - (\hat{X}_{t_k^N}^N)_{i,j})^4] \leq K(t_l^N - t_k^N)^2$, and therefore

$$\mathbb{E}[\|\hat{X}_{t_l^N}^N - \hat{X}_{t_k^N}^N\|^4] \leq Kd^2(t_l^N - t_k^N)^2.$$

Let us consider now $0 \leq s \leq t \leq T$. If there exists $0 \leq k \leq N-1$, such that $s, t \in [t_k^N, t_{k+1}^N]$, then $\mathbb{E}[\|\hat{X}_t^N - \hat{X}_s^N\|^4] = \left(\frac{s-t}{T/N}\right)^4 \mathbb{E}[\|\hat{X}_{t_{k+1}^N}^N - \hat{X}_{t_k^N}^N\|^4] \leq Kd^2(s-t)^2$. Otherwise, there are $k \leq l$ such that $t_k^N - T/N < s \leq t_k^N \leq t_l^N \leq t < t_l^N + T/N$, and $\mathbb{E}[\|\hat{X}_t^N - \hat{X}_s^N\|^4] \leq Kd^2[(t_k^N - s)^2 + (t - t_l^N)^2 + (t_l^N - t_k^N)^2] \leq 3Kd^2(t-s)^2$. □

The sequence $(\hat{X}_t^N, t \in [0,T])_{N\geq 1}$ is tight by using the Kolmogorov criterion, see Problem 4.11, p. 64 in [83]. We will show that any limit of subsequence solves the martingale problem (6.25). More precisely, we will show that for any $n \in \mathbb{N}^*$, $0 \leq t_1 \leq \cdots \leq t_n \leq s \leq t \leq T$, $g_1, \ldots, g_n \in \mathcal{C}(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$, $f \in \mathcal{C}^\infty(\mathcal{S}_d(\mathbb{R}), \mathbb{R})$ we have:

$$\lim_{N\to+\infty} \mathbb{E}\left[\prod_{i=1}^n g_i(\hat{X}_{t_i}^N)\left(f(\hat{X}_t^N) - f(\hat{X}_s^N) - \int_s^t Lf(\hat{X}_u^N) du\right)\right] = 0. \tag{6.53}$$

We set $k^N(s)$ and $l^N(t)$ the indices such that $t^N_{k^N(s)} - T/N < s \leq t^N_{k^N(s)}$ and $t^N_{l^N(t)} \leq t < t^N_{l^N(t)} + T/N$. Clearly, f is Lipschitz and Lf is bounded on $\mathfrak{C}_d(\mathbb{R})$. It is therefore sufficient to show that

$$\lim_{N\to+\infty} \mathbb{E}\left[\prod_{i=1}^n g_i(\hat{X}^N_{t_i})\left(f(\hat{X}^N_{t^N_{l^N(t)}}) - f(\hat{X}^N_{t^N_{k^N(s)}}) - \int_{t^N_{k^N(s)}}^{t^N_{l^N(t)}} Lf(\hat{X}^N_u)du\right)\right] = 0. \tag{6.54}$$

We decompose the expectation as the sum of

$$\mathbb{E}\left[\prod_{i=1}^n g_i(\hat{X}^N_{t_i}) \int_{t^N_{k^N(s)}}^{t^N_{l^N(t)}} (Lf(\hat{X}^N_{t^N_{l^N(u)}}) - Lf(\hat{X}^N_u))du\right] \tag{6.55}$$
$$+\,\mathbb{E}\left[\prod_{i=1}^n g_i(\hat{X}^N_{t_i})\left(\sum_{j=k^N(s)}^{l^N(t)-1} f(\hat{X}^N_{t^N_{j+1}}) - f(\hat{X}^N_{t^N_j}) - \frac{T}{N}Lf(\hat{X}^N_{t^N_j})\right)\right].$$

To get that the first expectation goes to 0, we claim that:

$$\mathbb{E}\left[\int_{t^N_{k^N(s)}}^{t^N_{l^N(t)}} |\beta(u, \hat{X}^N_u) - \beta(t^N_{l^N(u)}, \hat{X}^N_{t^N_{l^N(u)}})|du\right] \to 0 \tag{6.56}$$

when $\beta : (t, x) \in [0, T] \times \mathfrak{C}_d(\mathbb{R}) \to \mathbb{R}$ is continuous. This formulation will be reused later on. By Lemma 6.4.6, (6.56) holds when β is Lipschitz with respect to (t, x). If β is not Lipschitz, we can still approximate it uniformly on the compact set $[0, T] \times \mathfrak{C}_d(\mathbb{R})$ by using for example the Stone-Weierstrass theorem, which gives (6.56).

On the other hand, we know by (6.29) that the second expectation of (6.55) goes to 0. To be precise, (6.29) has been obtained by using Itô's formula while we do not know yet at this stage that the process $MRC_d(x, \frac{\alpha}{2}a^2, I_d, a)$ exists. It is nevertheless true: (6.29) holds for $MRC_d(x, \frac{\alpha}{2}a_i^2 e^i_d, I_d, e^i_d)$ since this process is already known to be well defined, and we get by using Proposition 6.3.4 and Exercise 2.3.16 that

$$\exists K > 0, |f(\hat{X}^N_{t^N_{j+1}}) - f(\hat{X}^N_{t^N_j}) - (T/N)Lf(\hat{X}^N_{t^N_j})| \leq K/N^2.$$

Thus, $(\hat{X}^N_t, t \in [0, T])_{N\geq 1}$ converges in law to a solution of the martingale problem (6.25). This concludes the existence of $MRC_d(x, \frac{\alpha}{2}a^2, I_d, a)$.

Now, we are in position to show the existence of $MRC_d(x, \kappa, c, a)$ under Assumption (6.20). We denote by $\xi(t, x)$ the solution to the linear ODE:

$$\xi'(t, x) = \kappa(c - x) + (c - x)\kappa - \frac{d-2}{2}[a^2(I_d - x) + (I_d - x)a^2],\ \xi(0, x) = x \in \mathfrak{C}_d(\mathbb{R}). \tag{6.57}$$

By Lemma 6.4.5 and (6.20), we know that $\forall t \geq 0, \xi'(t,x) \in \mathfrak{C}_d(\mathbb{R})$. It is also easy to check that:

$$\exists K > 0, \forall x \in \mathfrak{C}_d(\mathbb{R}),\ \|\xi(t,x) - x\| \leq Kt.$$

Now, we define $(\hat{X}_t^N, t \in [0,T])$ as follows.

- We set $\hat{X}_0^N = x \in \mathfrak{C}_d(\mathbb{R})$.
- For $i = 0, \ldots, N-1$, $\hat{X}^N_{t^N_{i+1}}$ is sampled according to $MRC_d(\xi(T/N, \hat{X}^N_{t^N_i}), \frac{d-2}{2}a^2, I_d, a; T/N)$, conditionally to $\hat{X}^N_{t^N_i}$. More precisely, we denote by $(\bar{X}^N_t, t \in [t^N_i, t^N_{i+1}])$ a solution to

$$\bar{X}^N_t = \xi(T/N, \hat{X}^N_{t^N_i}) + \frac{d-2}{2}\int_{t^N_i}^t \left[a^2(I_d - \bar{X}^N_u) + (I_d - \bar{X}^N_u)a^2\right] du$$
$$+ \sum_{n=1}^d a_n \int_{t^N_i}^t \left(\sqrt{\bar{X}^N_u - \bar{X}^N_u e^n_d \bar{X}^N_u}\, dW_u e^n_d + e^n_d dW_u^\top \sqrt{\bar{X}^N_u - \bar{X}^N_u e^n_d \bar{X}^N_u}\right),$$

  and we set $\hat{X}^N_{t^N_{i+1}} = \bar{X}^N_{t^N_{i+1}}$.
- For $t \in [t^N_i, t^N_{i+1}]$, $\hat{X}^N_t = \hat{X}^N_{t^N_i} + \frac{t - t^N_i}{T/N}(\hat{X}^N_{t^N_{i+1}} - \hat{X}^N_{t^N_i})$.

We proceed similarly and show that the Kolmogorov criterion (6.50) holds for $(\hat{X}^N_t, t \in [0,T])_{N\geq 1}$. As already shown in Lemma 6.4.6, it is sufficient to check that this criterion holds for $s = t^N_k \leq t = t^N_l$. We have

$$\|\hat{X}^N_{t^N_l} - \hat{X}^N_{t^N_k}\|^4 = \|\sum_{j=k}^{l-1} \hat{X}^N_{t^N_{j+1}} - \xi(T/N, \hat{X}^N_{t^N_j}) + \xi(T/N, \hat{X}^N_{t^N_j}) - \hat{X}^N_{t^N_j}\|^4$$
$$\leq 2^3 \left(\|\sum_{j=k}^{l-1} \bar{X}^N_{t^N_{j+1}} - \bar{X}^N_{t^N_j}\|^4 + (l-k)^4 \left(\frac{KT}{N}\right)^4 \right).$$

Since $(\bar{X}^N_t, t \in [0,T])$ is valued in the compact set $\mathfrak{C}_d(\mathbb{R})$, we get easily by using Burkholder-Davis-Gundy inequality that $\mathbb{E}[\|\sum_{j=k}^{l-1} \bar{X}^N_{t^N_{j+1}} - \bar{X}^N_{t^N_j}\|^4] \leq K(t_l - t_k)^2$ and then $\mathbb{E}[\|\hat{X}^N_{t^N_l} - \hat{X}^N_{t^N_k}\|^4] \leq K(t_l - t_k)^2$ for some constant $K > 0$ that does not depend on N.

Thus, $(\hat{X}^N_t, t \in [0,T])_{N\geq 1}$ satisfies the Kolmogorov criterion and is tight. We refer again to Problem 4.11, p. 64 in [83] for this tightness criterion. It remains to show that any subsequence converges in law to the solution of the martingale problem (6.25). We proceed as before and reuse the same notations. From (6.55), it is sufficient to show that

$$\exists K>0, |f(\hat{X}^N_{t^N_{j+1}}) - f(\hat{X}^N_{t^N_j}) - (T/N)Lf(\hat{X}^N_{t^N_j})| \leq K/N^2.$$

Once again, we cannot directly use (6.29) since we do not know at this stage that the process $MRC_d(x,\kappa,c,a)$ exists. We have $L = L^\xi + \tilde{L}$, where L^ξ is the operator associated to $\xi(t,x)$ and $\tilde{L}$ is the infinitesimal generator of $MRC_d(x, \frac{d-2}{2}a^2, I_d, a)$. Using the compactness of $\mathfrak{C}_d(\mathbb{R})$, we have

$$\exists K>0, \forall x \in \mathfrak{C}_d(\mathbb{R}), |f(\xi(t,x)) - f(x) - tL^\xi f(x)| \leq Kt^2,$$

and (6.29) holds for $\tilde{L}$. By using Exercise 2.3.16, we get: $\exists K > 0, \forall x \in \mathfrak{C}_d(\mathbb{R}), |f(\xi(t,x)) - f(x) - tf(x)| \leq Kt^2$, which gives (6.53) and concludes the proof of the weak existence.

Theorem 6.4.8 *Under assumption* (6.20)*, there is a unique weak solution* $(X_t, t\geq 0)$ *to SDE* (6.19) *such that* $\mathbb{P}(\forall t \geq 0, X_t \in \mathfrak{C}_d(\mathbb{R})) = 1$.

Remark 6.4.9 Assumption (6.20) has only be used in the proof of Theorem 6.4.8 to ensure that ξ defined by (6.57) satisfies

$$\forall t \geq 0, x \in \mathfrak{C}_d(\mathbb{R}),\ \xi(t,x) \in \mathfrak{C}_d(\mathbb{R}). \tag{6.58}$$

As pointed by Remark 6.4.4, this is a sufficient but not necessary condition. In fact, a weak solution of (6.19) exists under (6.58), which is a more general but less tractable condition than (6.20).

Before concluding this section, we have to mention that it is rather easy to consider an extension of the MRC process with coefficients κ, c and a that are non longer constant and depend on time and space. This issue is considered in Ahdida and Alfonsi [3]. However, by introducing this space dependence, we lose nice features such as the explicit calculation of the moments.

6.5 Second Order Discretization Schemes for MRC Processes

Through this section, we consider a regular time grid $t^N_i = iT/N$, $i = 0, \ldots, N$ for a given time horizon $T > 0$. First, let us mention that the Euler-Maruyama scheme is not defined for (6.19) as well as for other square-root diffusions.

It is given by

$$\hat{X}^N_{t^N_{i+1}} = \hat{X}^N_{t^N_i} + \left(\kappa(c - \hat{X}^N_{t^N_i}) + (c - \hat{X}^N_{t^N_i})\kappa\right)\frac{T}{N} \tag{6.59}$$
$$+ \sum_{n=1}^{d} a_n \left(\sqrt{\hat{X}^N_{t^N_i} - \hat{X}^N_{t^N_i} e^n_d \hat{X}^N_{t^N_i}}(W_{t^N_{i+1}} - W_{t^N_i})e^n_d\right.$$
$$\left. + e^n_d (W_{t^N_{i+1}} - W_{t^N_i})^\top \sqrt{\hat{X}^N_{t^N_i} - \hat{X}^N_{t^N_i} e^n_d \hat{X}^N_{t^N_i}}\right).$$

Thus, even if $\hat{X}^N_{t^N_i} \in \mathfrak{C}_d(\mathbb{R})$, $\hat{X}^N_{t^N_{i+1}}$ can no longer be in $\mathfrak{C}_d(\mathbb{R})$ and the matrix square-root can no longer be defined at the next time-step. A possible correction is to consider the following modification of the Euler scheme:

$$\hat{X}^N_{t^N_{i+1}} = \mathbf{p}((\tilde{X}_{t^N_{i+1}})^+), \tag{6.60}$$

where $\tilde{X}^N_{t^N_{i+1}}$ denotes the right hand side of (6.59). Here, $x^+ \in \mathcal{S}^+_d(\mathbb{R})$ is defined for $x \in \mathcal{S}_d(\mathbb{R})$ as the unique symmetric semidefinite matrix that shares the same eigenvectors as x, but the eigenvalues are the positive part of the one of x. Namely, $x^+ = odiag(\lambda^+_1, \dots, \lambda^+_d)o$ for $x \in \mathcal{S}_d(\mathbb{R})$ such that $x = odiag(\lambda_1, \dots, \lambda_d)o$ where o is an orthogonal matrix. Let us check that this scheme is well defined if we start from $\hat{X}^N_{t^N_0} \in \mathfrak{C}_d(\mathbb{R})$. By Lemma A.3.1, the square-roots are well defined, we have $(\tilde{X}_{t^N_1})_{i,i} = 1$ and thus $(\tilde{X}_{t^N_1})^+_{i,i} \geq 1$ and $\mathbf{p}((\tilde{X}_{t^N_1})^+)$ is well defined. By induction, this modified Euler scheme is always defined and takes values in the set of correlation matrices. However, as we will see in the numerical experiments, it is time-consuming and converges rather slowly.

In this section, we present a second order discretization schemes that is obtained by composition. It relies on the remarkable splitting of the infinitesimal generator given by Theorem 6.3.2. Thanks to this splitting, it is basically sufficient to focus on the approximation of the process $MRC_d(x, \frac{d-2}{2}e^1_d, I_d, e^1_d)$. It is possible to construct directly a potential second order scheme for this process. This is made in Ahdida and Alfonsi [3]. Here, we present the other construction given in [3]. It relies on Proposition 6.3.1 and reuses the second order scheme for Wishart processes that we have obtained in Chap. 5. This approach generalizes the second order scheme for Jacobi processes given by Proposition 6.1.13.

6.5.1 A Second-Order Scheme for MRC Processes

First, we split the infinitesimal generator of $MRC_d(x, \kappa, c, a)$ as the sum

$$L = L^\xi + \tilde{L},$$

where $\tilde{L}$ is the infinitesimal generator of $MRC_d(x, \frac{d-2}{2}a^2, I_d, a)$ and L^ξ is the operator associated to $\xi(t,x)$ given by (6.57). Obviously, the ODE (6.57) can be solved explicitly and we have to focus on the sampling of $MRC_d(x, \frac{d-2}{2}a^2, I_d, a)$. We use now Theorem 6.3.2 and consider the splitting

$$\tilde{L} = \sum_{i=1}^{d} a_i^2 \tilde{L}_i,$$

where $\tilde{L}_i$ is the infinitesimal generator of $MRC_d(x, \frac{d-2}{2}e_d^i, I_d, e_d^i)$. We claim now that it is sufficient to have a potential second order scheme for $MRC_d(x, \frac{d-2}{2}e_d^1, I_d, e_d^1)$ in order to get a potential second order scheme for $MRC_d(x, \kappa, c, a)$. Indeed, if we have such a scheme, we also get by a permutation of the coordinates a potential second order scheme $\hat{X}_t^{i,x}$ for $MRC_d(x, \frac{d-2}{2}e_d^i, I_d, e_d^i)$. Then, by time-scaling, $\hat{X}_{a_i^2 t}^{i,x}$ is a potential second order scheme for $MRC_d(x, \frac{d-2}{2}a_i^2 e_d^i, I_d, a_i e_d^i)$. Thanks to the commutativity, we get by Corollary 2.3.13 that $\hat{X}_{a_d^2 t}^{d, \dots \hat{X}_{a_1^2 t}^{1,x}}$ is a potential second order scheme for $\tilde{L}$. Last, we obtain by using Corollary 2.3.14 that

$$\xi(t/2, \hat{X}_{a_d^2 t}^{d, \dots \hat{X}_{a_1^2 t}^{1,\xi(t/2,x)}}) \text{ is a potential second order scheme for } MRC_d(x,\kappa,c,a). \tag{6.61}$$

Now, we focus on getting a second order scheme for $MRC_d(x, \frac{d-2}{2}e_d^1, I_d, e_d^1)$. It is possible to construct such a scheme by using an ad-hoc splitting of the infinitesimal generator. This is made in [3]. Here, we achieve this task by using the connection between Wishart and MRC processes and the existing scheme for Wishart processes. The combination of Theorem 5.6.3 and Proposition 3.3.5 gives a potential second order scheme $\hat{Y}_t^{1,x}$ for $WIS_d(x, d-1, 0, e_d^1)$, see Algorithm 5.4. Besides, this scheme is constructed with discrete random variables, and we can check that there is a constant $K > 0$ such that for any $1 \le i \le d$, $|(\hat{Y}_t^{1,x})_{i,i} - 1| \le K\sqrt{t}$ holds almost surely for $x \in \mathfrak{C}_d(\mathbb{R})$ (we even have $(\hat{Y}_t^{1,x})_{i,i} = 1$ for $2 \le i \le d$). Therefore, we have $1/2 \le (\hat{Y}_t^{1,x})_{i,i} \le 3/2$ for $t \le 1/(4K^2)$. Let $f \in \mathcal{C}^\infty(\mathfrak{C}_d(\mathbb{R}))$. Then $f(\mathbf{p}(y))$ is $\mathcal{C}^\infty$ with bounded derivatives on $\{y \in \mathcal{S}_d^+(\mathbb{R}) \ s.t.\ \forall 1 \le i \le d,\ 1/2 \le y_{i,i} \le 3/2\}$. Since $\hat{Y}_t^{1,x}$ is a potential second order scheme, it comes that there are constants $C, \eta > 0$ that only depend on a good sequence of f such that

$$\forall t \in (0,\eta), \left| \mathbb{E}[f(\mathbf{p}(\hat{Y}_t^{1,x}))] - f(x) - t\tilde{L}_1^W(f \circ \mathbf{p})(x) - \frac{t^2}{2}(\tilde{L}_1^W)^2(f \circ \mathbf{p})(x) \right| \le Ct^3, \tag{6.62}$$

where $\tilde{L}_1^W$ is the generator of $WIS_d(x, d-1, 0, e_d^1)$. Thanks to Remark 6.3.3, we get that there are constants C, η depending only on a good sequence of f such that

$$\forall t \in (0,\eta), \left|\mathbb{E}[f(\mathbf{p}(\hat{Y}_t^{1,x}))] - f(x) - \left(t + (5-d)\frac{t^2}{2}\right)\tilde{L}_1 f(x) - \frac{t^2}{2}(\tilde{L}_1)^2 f(x)\right| \le Ct^3. \tag{6.63}$$

In particular, $\mathbf{p}(\hat{Y}_t^{1,x})$ is a potential first order scheme for L_1 and even a second order scheme when $d = 5$. We can improve this by taking a simple time-change. We set:

$$\phi(t) = \begin{cases} t - (5-d)\frac{t^2}{2} \text{ if } d \ge 5 \\ \frac{-1+\sqrt{1+2(5-d)t}}{5-d} \text{ otherwise,} \end{cases}$$

so that in both cases, $\phi(t) = t - (5-d)\frac{t^2}{2} + O(t^3)$. Then, we have that there are constants C, η still depending only on a good sequence of f such that

$$\forall t \in (0,\eta), \left|\mathbb{E}[f(\mathbf{p}(\hat{Y}_{\phi(t)}^{1,x}))] - f(x) - t\tilde{L}_1 f(x) - \frac{t^2}{2}(\tilde{L}_1)^2 f(x)\right| \le Ct^3, \tag{6.64}$$

and therefore

$$\mathbf{p}(\hat{Y}_{\phi(t)}^{1,x}) \text{ is a potential second order scheme for } MRC_d(x, \frac{d-2}{2}e_d^1, I_d, e_d^1). \tag{6.65}$$

This result generalizes Proposition 6.1.13. The study of the fundamental equation that has been made for one dimensional Wright-Fisher processes is still an open question for MRC processes. However, we can still get a weaker result by using the compactness of $\mathfrak{C}_d(\mathbb{R})$ and the explicit calculation of the moments. In fact, since we have an explicit formula for the moments (6.27), we obtain easily that for any polynomial function f, the second point of Theorem 2.3.8 is satisfied. By the Stone-Weierstrass theorem, we can approximate for the supremum norm any continuous function by a polynomial function and get the following interesting corollary.

Corollary 6.5.1 *Let $\hat{X}_t^x$ be the potential weak second order scheme for MRC_d (x, κ, c, a) given by (6.61) and (6.65). Let f be a continuous function on $\mathfrak{C}_d(\mathbb{R})$. Then,*

$$\forall \varepsilon > 0, \exists K > 0, \ |\mathbb{E}[f(\hat{X}_{t_N^N})] - \mathbb{E}[f(X_T^x)]| \le \varepsilon + K/N^2.$$

Algorithm 6.3: Potential second order scheme for $MRC_d(x, \frac{d-2}{2}e_d^1, I_d, e_d^1; t)$.

Input: $x \in \mathfrak{C}_d(\mathbb{R})$ and $t > 0$.
Output: X.
Sample X by using Algorithm 5.4 (second order is enough) with parameter $\alpha = d - 1$, starting point x and time step $\phi(t)$.
$X = \mathbf{p}(X)$.

Algorithm 6.4: Potential second order scheme of $MRC_d(x, \kappa, c, a; t)$.

Input: $x \in \mathfrak{C}_d(\mathbb{R})$, $\kappa, a, c \in \mathfrak{C}_d(\mathbb{R})$ and $t > 0$.
Output: X.
Function $B(x)$: **return** $(\frac{d-2}{2}a^2 - \kappa)x + x(\frac{d-2}{2}a^2 - \kappa)$;
$m = \kappa c + c\kappa - (d-2)a^2$,
$y = \sum_{k=0}^{\infty} \frac{(t/2)^k}{k!} B^{(k)}(x) + \frac{(t/2)^{k+1}}{(k+1)!} B^{(k)}(m)$.
for $k = 1$ **to** d **do**
  Set $p_{k,1} = p_{1,k} = p_{i,i} = 1$ for $i \notin \{1, k\}$, and $p_{i,j} = 0$ otherwise (permutation of the first and k^{th} coordinates).
  $y = pYp$ where Y is sampled according to the potential second order scheme of $MRC_d(pyp, \frac{d-2}{2}e_d^1, I_d, e_d^1; a_k^2 t)$ by using Algorithm 6.3.
end
$X = \sum_{k=0}^{\infty} \frac{(t/2)^k}{k!} B^{(k)}(y) + \frac{(t/2)^{k+1}}{(k+1)!} B^{(k)}(m)$.

Remark 6.5.2 This is the same comment as Remark 5.6.8 for Wishart processes. Unless $B(x) = (\frac{d-2}{2}a^2 - \kappa)x + x(\frac{d-2}{2}a^2 - \kappa)$ is proportional to the identity on $\mathcal{S}_d(\mathbb{R})$, the calculation of the series in Algorithm 6.4 may be time consuming. However,

$$\hat{\xi}(t, x) = \mathbf{p}\left(\left(\sum_{k=0}^{2} \frac{t^k}{k!} B^{(k)}(x) + \sum_{k=0}^{1} \frac{t^{k+1}}{(k+1)!} B^{(k)}(\kappa c + c\kappa - (d-2)a^2)\right)^+\right)$$

is also second order scheme for L^ξ for $x \in \mathfrak{C}_d(\mathbb{R})$ by using the same argument as in Remark 5.6.8 and that $\mathbf{p}$ is Lipschitz on the compact set $\{y \in \mathcal{S}_d^+(\mathbb{R})\ s.t.\ \forall 1 \leq i \leq d,\ 1/2 \leq y_{i,i} \leq 3/2\}$. Therefore, $\hat{\xi}(t/2, \hat{X}_{a_d^2 t}^{d,\dots \hat{X}_{a_1^2 t}^{1,\hat{\xi}(t/2,x)}})$ is also a potential second order scheme for $MRC_d(x, \kappa, c, a)$.

6.5.2 A Faster Second-Order Scheme for MRC Processes Under Assumption (6.66)

We now focus on the time complexity of the scheme given by (6.61) and (6.65) with respect to the dimension d. The second order scheme for $WIS_d(x, d-1, 0, e_d^1)$ requires $O(d^3)$ operations as it has been discussed in Sect. 5.6.4. Since it is used d times in (6.61) to generate a sample, the overall complexity is in $O(d^4)$. However, similarly to what is presented in Sect. 5.6.4 for Wishart processes, it is possible to get a faster second order scheme with complexity $O(d^3)$ if we make the following assumption:

$$a_1 = \cdots = a_d \ (i.e.\ a = a_1 I_d) \text{ and } \kappa c + c\kappa - (d-1)a^2 \in \mathcal{S}_d^+(\mathbb{R}). \tag{6.66}$$

This latter assumption is stronger than (6.20) but weaker than (6.21), which respectively ensures weak and strong solutions to the SDE. Under (6.66), we can check by Lemma 6.4.5 that

$$\zeta'(t,x) = \kappa(c-x)+(c-x)\kappa-\frac{d-1}{2}[a^2(I_d-x)+(I_d-x)a^2],\ \zeta(0,x) = x \in \mathfrak{C}_d(\mathbb{R}) \tag{6.67}$$

takes values in $\mathfrak{C}_d(\mathbb{R})$. Then, we split the infinitesimal generator of $MRC_d(x, \kappa, c, a)$ as the sum

$$L = L^{\zeta} + a_1^2 \bar{L},$$

where L^{ζ} is the operator associated to the ODE ζ, and $\bar{L}$ is the infinitesimal generator of $MRC_d(x, \frac{d-1}{2} I_d, I_d, I_d)$. Proposition 5.6.9 gives a second order scheme $\hat{Y}_t^x$ for $WIS_d(x, d, 0, I_d)$ that has a time-complexity in $O(d^3)$. We then consider $f \in \mathcal{C}^{\infty}(\mathfrak{C}_d(\mathbb{R}))$ and denote by $\bar{L}^W$ the infinitesimal generator of $WIS_d(x, d, 0, I_d)$. By using the same arguments that we used to get (6.62), we obtain that there are constants $C, \eta > 0$ depending only on a good sequence of f such that

$$\forall t \in (0,\eta), \left|\mathbb{E}[f(\mathbf{p}(\hat{Y}_t^x))] - f(x) - t\bar{L}^W(f \circ \mathbf{p})(x) - \frac{t^2}{2}(\bar{L}^W)^2(f \circ \mathbf{p})(x)\right| \le Ct^3.$$

Thanks to Remark 6.3.3, we get that

$$\forall t \in (0,\eta), \left|\mathbb{E}[f(\mathbf{p}(\hat{Y}_t^x))] - f(x) - \left(t + (4-d)\frac{t^2}{2}\right)\bar{L}f(x) - \frac{t^2}{2}\bar{L}^2 f(x)\right| \le Ct^3.$$

In particular, $\mathbf{p}(\hat{Y}_t^x)$ is a first order scheme for $MRC_d(x, \frac{d-1}{2}I_d, I_d, I_d)$. By using Exercise 2.3.16, we obtain that

$$\zeta(t, \mathbf{p}(\hat{Y}^x_{a_1^2 t})) \text{ is a potential first order scheme for } MRC_d(x, \kappa, c, a). \tag{6.68}$$

As before, we can improve this by using the following time-change: $\psi(t) = t - (4-d)\frac{t^2}{2}$ if $d \geq 4$ and $\psi(t) = \frac{-1+\sqrt{1+2(4-d)t}}{4-d}$ otherwise, so that

$$\psi(t) = t - (4-d)\frac{t^2}{2} + O(t^3)$$

in both cases. We get that $\mathbf{p}(\hat{Y}^x_{\psi(t)})$ is a potential second order scheme for $MRC_d(x, \frac{d-1}{2}I_d, I_d, I_d)$. Then, we obtain that

$$\zeta(t/2, \mathbf{p}(\hat{Y}^{\zeta(x,t/2)}_{a_1^2\psi(t)})) \text{ is a potential second order scheme for } MRC_d(x, \kappa, c, a) \tag{6.69}$$

by using Corollary 2.3.14. Its time complexity is in $O(d^3)$.

Algorithm 6.5: Fast potential second order scheme of $MRC_d(x, \kappa, c, a; t)$ under assumption (6.66).

Input: $x \in \mathfrak{C}_d(\mathbb{R})$, κ, $a_1 > 0$, $c \in \mathfrak{C}_d(\mathbb{R})$ and $t > 0$.
Output: X.
Function $B(x)$: **return** $(\frac{d-1}{2}a^2 - \kappa)x + x(\frac{d-1}{2}a^2 - \kappa)$;
$m = \kappa c + c\kappa - (d-1)a_1^2 I_d$,
$X = \sum_{k=0}^{\infty} \frac{(t/2)^k}{k!} B^{(k)}(x) + \frac{(t/2)^{k+1}}{(k+1)!} B^{(k)}(m)$.
Calculate c the Cholesky decomposition of X.
Sample $\hat{G}$, a d-by-d matrix with independent elements following the law (2.27), and set
$X = \mathbf{p}\left((c + a_1\sqrt{\psi(t)}\hat{G})^\top (c + a_1\sqrt{\psi(t)}\hat{G})\right)$.
$X = \sum_{k=0}^{\infty} \frac{(t/2)^k}{k!} B^{(k)}(X) + \frac{(t/2)^{k+1}}{(k+1)!} B^{(k)}(m)$.

As for Algorithm 6.4, the calculation of the matrix series may be avoided by using Remark 6.5.2.

6.5.3 Numerical Experiments on the Discretization Schemes

In this paragraph, we present briefly some numerical results obtained with the discretization schemes for MRC processes. We first focus on the computation time of the different algorithms. In Table 6.1, we have indicated the time required to sample 10^6 scenarios for different time-grids in dimension $d = 3$ and $d = 10$. These times have been obtained with a 2.50 GHz CPU computer. As expected, the modified Euler scheme given by (6.60) is the most time consuming. This is mainly due to the computation of the matrix square-roots that require several diagonalizations. Between the second order schemes that are defined for any parameters satisfying (6.20), the second order scheme given by (6.61) and (6.65) is rather faster than the "direct" one presented in [3]. However, it has a larger bias on our example in Fig. 6.1, and their overall efficiency is similar. Nonetheless, both are as expected overtaken by the fast second order scheme (6.69). Let us recall that it is only defined under Assumption (6.66) which is satisfied by our set of parameters. Also, the fast first order scheme given by (6.68) requires roughly the same computation time.

Let us switch now to Fig. 6.1 that illustrates the weak convergence of the different schemes. To be more precise, we have plotted the following combinations the moments of order 3 and 1 (i.e. respectively

$$\mathbb{E}\left[\sum_{\substack{1\le i\neq j\le 3\\ 1\le k\neq l\le 3}}\left[(\hat{X}_T^N)_{i,j}(\hat{X}_T^N)^2_{k,l}\right]+(\hat{X}_T^N)_{1,2}(\hat{X}_T^N)_{2,3}(\hat{X}_T^N)_{1,3}\right], \tag{6.70}$$

Table 6.1 Computation time in seconds to generate 10^6 paths up to $T = 1$ with $N = 10$ time-steps of the following *MRC* process: $\kappa = 1.25 I_d$, $c = I_d$, $a = I_d$, and $x_{i,j} = 0.7$ for $i \neq j$

	$d = 3$	$d = 10$
Second order "fast"	19	224
Second order	65	1,677
Second order "direct"	90	3,105
First order "fast"	19	224
Corrected Euler	400	14,322

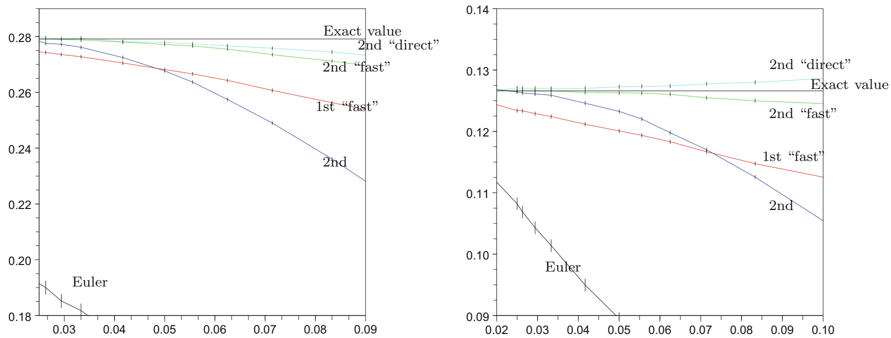

Fig. 6.1 $d = 3$, same parameters as for Table 6.1. In the *left* (resp. *right*) side is plotted (6.70) (resp. $\mathbb{E}\left[\sum_{1\le i\neq j\le d}(\hat{X}_T^N)_{i,j}\right]$) in function of the time step $1/N$. The width of each point represents the 95 % confidence interval (10^7 scenarios for the modified Euler scheme and 10^8 for the others)

and $\mathbb{E}\left[\sum_{1\le i\neq j\le d}(\hat{X}_T^N)_{i,j}\right]$) in function of the time-step T/N. These expectations can be calculated exactly for the MRC process thanks to Proposition 6.2.4, and the exact value is reported in both graphics. As expected, we observe a quadratic convergence for the second order schemes, and a linear convergence for the first order scheme. In particular, this demonstrates numerically the gain that we get by considering the simple change of time ψ between the schemes (6.68) and (6.69). Last, the modified Euler scheme shows a roughly linear convergence. It has however a much larger bias and is clearly not competitive.

Appendix A
Some Results on Matrices

A.1 Some Basic Results

We first recall that the usual scalar product on $\mathcal{M}_d(\mathbb{R})$ is given by

$$(x, y) \mapsto \mathrm{Tr}(x^\top y) = \sum_{i,j=1}^{d} x_{i,j} y_{i,j}.$$

The Frobenius norm is given by $\|x\| = \sqrt{\mathrm{Tr}(x^\top x)}$ and is the sum of the eigenvalues of the positive semidefinite matrix $x^\top x$. The Cauchy-Schwarz inequality immediately gives

$$|\mathrm{Tr}(x^\top y)| \leq \sqrt{\mathrm{Tr}(x^\top x)}\sqrt{\mathrm{Tr}(y^\top y)}. \tag{A.1}$$

The restriction of this scalar product on $\mathcal{S}_d(\mathbb{R})$ is simply given by $(x, y) \mapsto \mathrm{Tr}(xy)$. If $B : \mathcal{S}_d(\mathbb{R}) \to \mathcal{S}_d(\mathbb{R})$ is a linear mapping, it is continuous and there is a constant $C > 0$ such that $\|B(x)\| \leq C\|x\|$ for any $x \in \mathcal{S}_d(\mathbb{R})$. Together with the Cauchy-Schwarz inequality, we get

$$\forall x \in \mathcal{S}_d(\mathbb{R}),\ |\mathrm{Tr}(xB(x))| \leq C\,\mathrm{Tr}(x^2). \tag{A.2}$$

Lemma A.1.1 *Let $b, c \in \mathcal{S}_d(\mathbb{R})$. If either $b \in \mathcal{S}_d^+(\mathbb{R})$ or $c \in \mathcal{S}_d^+(\mathbb{R})$, then $I_d + ibc$ is invertible. In particular, if $b \in \mathcal{S}_d^{+,*}(\mathbb{R})$, $b + ic$ is invertible.*

Proof We start with the first assertion. Since $(I_d + ibc)^\top = I_d + icb$, it is sufficient to check the case where $c \in \mathcal{S}_d^+(\mathbb{R})$. By a way of contradiction, let us assume that there is $x \in \mathbb{C}^d \setminus \{0\}$ such that $x + ibcx = 0$. We respectively denote by $x_R \in \mathbb{R}^d$ and $x_I \in \mathbb{R}^d$ the real and imaginary part of x. One gets easily that

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$x_R = bcx_I$ and $x_I = -bcx_R$. Since $x \neq 0$, we have necessarily $x_R \neq 0$, $cx_R \neq 0$, $bcx_R \neq 0$ and $cbcx_R \neq 0$. Since c is nonnegative, we get by decomposing on an orthonormal basis that $cx_R.x_R > 0$ and $cbcx_R.bcx_R > 0$. However, we also have $cx_R.x_R = -cx_R.bcbcx_R$, which leads to a contradiction. The second assertion is now obvious since $b + ic = b(I_d + ib^{-1}c)$. □

Lemma A.1.2 *If $x, y \in \mathcal{S}_d^+(\mathbb{R})$, then* $\mathrm{Tr}(xy) \geq 0$.

Proof We have $\mathrm{Tr}(xy) = \mathrm{Tr}(\sqrt{x}y\sqrt{x})$. Besides, we clearly have $\sqrt{x}y\sqrt{x} \in \mathcal{S}_d^+(\mathbb{R})$ since for $z \in \mathbb{R}^d$, $z^\top \sqrt{x}y\sqrt{x}z = (\sqrt{x}z)^\top y\sqrt{x}z \geq 0$. □

Lemma A.1.3 *We recall that for $x \in \mathcal{S}_d(\mathbb{R})$, $x^+ \in \mathcal{S}_d^+(\mathbb{R})$ is defined by (5.11). We have*

$$\forall x, y \in \mathcal{S}_d(\mathbb{R}), \|y^+ - x^+\| \leq \|y - x\|.$$

Proof For $x \in \mathcal{S}_d(\mathbb{R})$, we define $x^- = (-x)^+ \in \mathcal{S}_d^+(\mathbb{R})$ so that $x = x^+ - x^-$ and $\mathrm{Tr}(x^+x^-) = 0$. Then, we have

$$\begin{aligned}
\mathrm{Tr}((y-x)^2) &= \mathrm{Tr}((y^+ - x^+)^2) - 2\mathrm{Tr}((y^+ - x^+)(y^- - x^-)) + \mathrm{Tr}((y^- - x^-)^2) \\
&= \mathrm{Tr}((y^+ - x^+)^2) + 2\mathrm{Tr}(y^+x^-) + 2\mathrm{Tr}(y^-x^+) + \mathrm{Tr}((y^- - x^-)^2) \\
&\geq \mathrm{Tr}((y^+ - x^+)^2),
\end{aligned}$$

by Lemma A.1.2. □

A.2 The Extended Cholesky Decomposition

Lemma A.2.1 *Let $q \in \mathcal{S}_d^+(\mathbb{R})$ be a matrix with rank r. Then there is a permutation matrix p, an invertible lower triangular matrix $c_r \in \mathcal{G}_r(\mathbb{R})$ and $k_r \in \mathcal{M}_{d-r\times r}(\mathbb{R})$ such that:*

$$pqp^\top = cc^\top, \ c = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix}.$$

The triplet (c_r, k_r, p) is called an extended Cholesky decomposition of q. Besides, $\tilde{c} = \begin{pmatrix} c_r & 0 \\ k_r & I_{d-r} \end{pmatrix} \in \mathcal{G}_d(\mathbb{R})$, and we have:

$$q = (\tilde{c}^\top p)^\top I_d^r \tilde{c}^\top p.$$

The proof and a numerical procedure to get such a decomposition can be found in Golub and Van Loan [67, Algorithm 4.2.4]. When $r = d$, we can take $p = I_d$, and c_r is the usual Cholesky decomposition.

Lemma A.2.2 *Let $y \in \mathcal{S}_d^+(\mathbb{R})$. We set $r = \mathrm{Rk}((y_{i,j})_{2\le i,j\le d})$, $y_1^r = (y_{1,i+1})_{1\le i\le r}$ and $y_1^{r,d} = (y_{1,i+1})_{r+1\le i\le d}$. We assume that there are an invertible matrix c_r and a matrix k_r defined on $\mathcal{M}_{d-r-1\times r}(\mathbb{R})$, such that*

$$(y_{i,j})_{2\le i,j\le d} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix} \begin{pmatrix} c_r^\top & k_r^\top \\ 0 & 0 \end{pmatrix}.$$

Then, we have $y_1^{r,d} = k_r c_r^{-1} y_1^r$.

Proof We set $p = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{array}\right)$ and have $p^{-1} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & c_r^{-1} & 0 \\ 0 & -k_r c_r^{-1} & I_{d-r-1} \end{array}\right)$. Since the matrix

$$p^{-1} y (p^{-1})^\top = \left(\begin{array}{c|cc} y_{1,1} & (c_r^{-1} y_1^r)^\top & (y_1^{r,d} - k_r c_r^{-1} y_1^r)^\top \\ \hline c_r^{-1} y_1^r & I_r & 0 \\ y_1^{r,d} - k_r c_r^{-1} y_1^r & 0 & 0 \end{array}\right)$$

is positive semidefinite, we necessarily have $y_1^{r,d} - k_r c_r^{-1} y_1^r = 0$. □

A.3 Some Algebraic Results on Correlation Matrices

We use the notations of page xi.

Lemma A.3.1 *Let $c \in \mathfrak{C}_d(\mathbb{R})$ and $1 \le i \le d$. Then we have: $c - ce_d^i c \in \mathcal{S}_d^+(\mathbb{R})$, $(c - ce_d^i c)_{i,j} = 0$ for $1 \le j \le d$, $\left(c - ce_d^i c\right)^{[i]} = c^{[i]} - c^i (c^i)^\top$ and:*

$$\left(\sqrt{c - ce_d^i c}\right)^{[i]} = \sqrt{c^{[i]} - c^i (c^i)^\top} \text{ and } \left(\sqrt{c - ce_d^i c}\right)_{i,j} = 0.$$

Besides, if $c \in \mathfrak{C}_d^*(\mathbb{R})$, $c^{[i]} - c^i (c^i)^\top \in \mathcal{S}_{d-1}^{+,*}(\mathbb{R})$.

Proof Up to a permutation, it is sufficient to prove the result for $i = 1$. We have

$$c - ce_d^1 c = \begin{pmatrix} 0 & 0_{d-1}^\top \\ 0_{d-1} & c^{[1]} - c^1 (c^1)^\top \end{pmatrix} = aca^\top, \text{ with } a = \begin{pmatrix} 0 & 0_{d-1} \\ -c^1 & I_{d-1} \end{pmatrix} \in \mathcal{S}_d^+(\mathbb{R}).$$

Besides, we have $\mathrm{Rk}(aca^\top) = \mathrm{Rk}(a\sqrt{c}) = d - 1$ when $c \in \mathfrak{C}_d^*(\mathbb{R})$, which gives $c^{[i]} - c^i (c^i)^\top \in \mathcal{S}_{d-1}^{+,*}(\mathbb{R})$. □

Lemma A.3.2 *Let $c \in \mathfrak{C}_d(\mathbb{R})$ and $1 \leq n \leq d$. Then $I_d - \sqrt{c}e_d^n\sqrt{c} \in \mathcal{S}_d^+(\mathbb{R})$ and is such that*

$$\sqrt{I_d - \sqrt{c}e_d^n\sqrt{c}} = I_d - \sqrt{c}e_d^n\sqrt{c}.$$

Proof The matrix $(\sqrt{c}e_d^n\sqrt{c})_{i,j} = (\sqrt{c})_{i,n}(\sqrt{c})_{j,n}$ is of rank 1 and $\sum_{j=1}^d(\sqrt{c}e_d^n \sqrt{c})_{i,j}(\sqrt{c})_{j,n} = (\sqrt{c})_{i,n}$ since $\sum_{j=1}^d(\sqrt{c})_{j,n}^2 = c_{j,j} = 1$. Therefore $((\sqrt{c})_{i,n})_{1\leq i\leq d}$ is an eigenvector, and the eigenvalues of $I_d - \sqrt{c}e_d^n\sqrt{c}$ are 0 and 1 (with multiplicity $d-1$). □

Lemma A.3.3 *Let $c \in \mathfrak{C}_d(\mathbb{R})$, $r = \mathrm{Rk}((c_{i,j})_{2\leq i,j\leq d})$ and $(m_r, k_r, \tilde{p})$ an extended Cholesky decomposition of $(c_{i,j})_{2\leq i,j\leq d}$. We set $p = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{p}^\top \end{pmatrix}$, $m = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & m_r & 0 \\ 0 & k_r & 0 \end{array}\right)$ and $\check{c} = \left(\begin{array}{c|cc} 1 & (m_r^{-1}c_1^r)^\top & 0 \\ \hline m_r^{-1}c_1^r & I_r & 0 \\ 0 & 0 & I_{d-r-1} \end{array}\right)$, where $c_1^r \in \mathbb{R}^r$, with $(c_1^r)_i = (p^\top cp)_{1,i+1}$ for $1 \leq i \leq r$. We have:*

$$c = pm\check{c}m^\top p^\top \text{ and } \check{c} \in \mathfrak{C}_d(\mathbb{R}).$$

Proof By straightforward block-matrix calculations, on has to check that the vector $c_1^{r,d} \in \mathbb{R}^{d-(r+1)}$ defined by $(c_1^{r,d})_i = (p^\top cp)_{1,i}$ for $r+1 \leq i \leq d$ is equal to $k_r m_r^{-1}c_1^r$. To get this, we introduce the matrix $q = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & m_r & 0 \\ 0 & k_r & I_{d-r-1} \end{array}\right)$ and have $q^{-1} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & m_r^{-1} & 0 \\ 0 & -k_r m_r^{-1} & I_{d-r-1} \end{array}\right)$. Since the matrix

$$q^{-1}p^\top cp(q^{-1})^\top = \left(\begin{array}{c|cc} 1 & (m_r^{-1}c_1^r)^\top & (c_1^{r,d} - k_r m_r^{-1}c_1^r)^\top \\ \hline m_r^{-1}c_1^r & I_r & 0 \\ c_1^{r,d} - k_r m_r^{-1}c_1^r & 0 & 0 \end{array}\right)$$

is positive semidefinite, we have $c_1^{r,d} = k_r m_r^{-1}c_1^r$, $\begin{pmatrix} 1 & (m_r^{-1}c_1^r)^\top \\ m_r^{-1}c_1^r & I_r \end{pmatrix} \in \mathcal{S}_{r+1}^+(\mathbb{R})$ and thus $\check{c} \in \mathfrak{C}_d(\mathbb{R})$. □

A.4 Matrix Riccati Differential Equations

Riccati differential equations play a key role to determine the characteristic functions of affine diffusions. The goal of this appendix is to present standard results on matrix Riccati differential equations that are given in the paper by Levin [96].

For $d_1, d_2 \in \mathbb{N}^*$, we denote by $\mathcal{M}_{d_1 \times d_2}(\mathbb{R})$ the set of real matrices with d_1 rows and d_2 columns. When $d_1 = d_2$, we simply use the notation $\mathcal{M}_{d_1}(\mathbb{R})$. We consider the following differential equation:

$$X'(t) = M_2 + M_1 X(t) + X(t) M_4 + X(t) M_3 X(t),\ X(0) = X_0 \in \mathcal{M}_{d_1 \times d_2}(\mathbb{R}), \tag{A.3}$$

where $M_2 \in \mathcal{M}_{d_1 \times d_2}(\mathbb{R})$, $M_1 \in \mathcal{M}_{d_1}(\mathbb{R})$, $M_4 \in \mathcal{M}_{d_2}(\mathbb{R})$ and $M_3 \in \mathcal{M}_{d_2 \times d_1}(\mathbb{R})$. We note that $X \in \mathcal{M}_{d_1 \times d_2}(\mathbb{R}) \mapsto M_2 + M_1 X + X M_4 + X M_3 X$ is locally Lipschitz, which gives by the Cauchy-Lipschitz theorem that there exists a unique solution of (A.3) on the maximal interval $(\underline{t}, \overline{t}) \ni 0$, where $\overline{t} = \inf\{t \geq 0, \|X(t)\| = +\infty\}$ and $\underline{t} = -\inf\{t \geq 0, \|X(-t)\| = +\infty\}$, with $\inf \emptyset = +\infty$.

To solve the matrix Riccati equation (A.3), we consider a related linear equation on $\mathcal{M}_{d_1+d_2}(\mathbb{R})$

$$Z'(t) = \begin{bmatrix} M_1 & M_2 \\ -M_3 & -M_4 \end{bmatrix} Z(t),\ Z(0) = I_{d_1+d_2}. \tag{A.4}$$

We know that the solution of this linear equation is well defined for $t \in \mathbb{R}$, and is given by

$$Z(t) = \exp\left(t \begin{bmatrix} M_1 & M_2 \\ -M_3 & -M_4 \end{bmatrix} \right) =: \begin{bmatrix} Z_1(t) & Z_2(t) \\ Z_3(t) & Z_4(t) \end{bmatrix}.$$

We recall that the matrix exponential is given by $\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ for $A \in \mathcal{M}_d(\mathbb{R})$. When $Z_3(t)X_0 + Z_4(t)$ is invertible, we set

$$\tilde{X}(t) = [Z_1(t)X_0 + Z_2(t)][Z_3(t)X_0 + Z_4(t)]^{-1}.$$

From (A.4), we have

$$Z_1'(t) = M_1 Z_1(t) + M_2 Z_3(t), \qquad Z_2'(t) = M_1 Z_2(t) + M_2 Z_4(t),$$
$$Z_3'(t) = -(M_3 Z_1(t) + M_4 Z_3(t)), \qquad Z_4'(t) = -(M_3 Z_2(t) + M_4 Z_4(t)).$$

Therefore, we get

$$\begin{aligned}\tilde{X}'(t) =& [Z_1'(t)X_0 + Z_2'(t)][Z_3(t)X_0 + Z_4(t)]^{-1} - \tilde{X}(t)[Z_3'(t)X_0 \\ &+ Z_4'(t)][Z_3(t)X_0 + Z_4(t)]^{-1}\end{aligned}$$

$$
\begin{aligned}
=&[M_1 Z_1(t)X_0 + M_2 Z_3(t)X_0 + M_1 Z_2(t) + M_2 Z_4(t)][Z_3(t)X_0 + Z_4(t)]^{-1}\\
&+ \tilde{X}(t)[M_3 Z_1(t)X_0 + M_4 Z_3(t)X_0 + M_3 Z_2(t) + M_4 Z_4(t)][Z_3(t)X_0\\
&+ Z_4(t)]^{-1} = M_1\tilde{X}(t) + M_2 + \tilde{X}(t)M_3\tilde{X}(t) + \tilde{X}(t)M_4.
\end{aligned}
$$

Since $\tilde{X}(0) = X_0$, we get that $\tilde{X}(t) = X(t)$ on a neighbourhood of 0. We now claim that

$$\bar{t} = \inf\{t \geq 0, \det[Z_3(t)X_0 + Z_4(t)] = 0\}, \tag{A.5}$$

$$\underline{t} = -\inf\{t \geq 0, \det[Z_3(-t)X_0 + Z_4(-t)] = 0\}. \tag{A.6}$$

We prove this for $\bar{t}$. Clearly, we have $\bar{t} \geq \inf\{t \geq 0, \det[Z_3(t)X_0 + Z_4(t)] = 0\}$. Otherwise, we could define the solution of (A.3) on a time interval that is strictly larger than $(\underline{t}, \bar{t})$. This is in contradiction with the fact that $(\underline{t}, \bar{t})$ is a maximal interval. Let us assume now that $\bar{t} > t' = \inf\{t \geq 0, \det[Z_3(t)X_0 + Z_4(t)] = 0\}$. Let $\mu \in \mathbb{R}^{d_2} \setminus \{0\}$ such that $[Z_3(t')X_0 + Z_4(t')]\mu = 0$. We have for $t \in [0, t')$,

$$X(t)[Z_3(t)X_0 + Z_4(t)]\mu = [Z_1(t)X_0 + Z_2(t)]\mu,$$

and by letting $t \to t'$, we get $[Z_1(t')X_0 + Z_2(t')]\mu = 0$. Therefore, we get $Z(t')\begin{bmatrix} X_0\mu \\ \mu \end{bmatrix} = 0$, which is absurd because a matrix exponential is always invertible.

Theorem A.4.1 *The maximal solution of the matrix Riccati differential equation* (A.3) *is given by*

$$X(t) = [Z_1(t)X_0 + Z_2(t)][Z_3(t)X_0 + Z_4(t)]^{-1},\ t \in (\underline{t}, \bar{t}),$$

where $\begin{bmatrix} Z_1(t) & Z_2(t) \\ Z_3(t) & Z_4(t) \end{bmatrix} = \exp\left(t \begin{bmatrix} M_1 & M_2 \\ -M_3 & -M_4 \end{bmatrix}\right)$, $\bar{t} = \inf\{t \geq 0, \det[Z_3(t)X_0 + Z_4(t)] = 0\}$ *and* $\underline{t} = -\inf\{t \geq 0, \det[Z_3(-t)X_0 + Z_4(-t)] = 0\}$.

We note in particular that Theorem A.4.1 gives the solution of the scalar Riccati equation when $d_1 = d_2 = 1$. In this case, we can assume without loss of generality that $M_4 = 0$ since the scalar product is commutative. We can then calculate the two eigenvalues of M and then diagonalize M (or triangularize when they are equal). Thus, we can compute explicitly Z and then X. Another way to proceed is to calculate a root $\bar{X} \in \mathbb{C}$ of the second-degree polynomial $M_2 + M_1 X + M_3 X^2$. Then, $X - \bar{X}$ solves a scalar Riccati equation with $M_2 = 0$. Dividing the equation by X^2, we get a linear differential equation on $1/X$ that can be solved explicitly.

Appendix B
Simulation of a Gamma Random Variable

The methods that we present now to sample the Gamma distribution (3.2) are based on the rejection method. We recall the following classical result.

Proposition B.0.1 *Let $(X_i)_{i\geq 1}$ be a sequence of independent and identically distributed random variables on $\mathbb{R}^d$. Let $A \subset \mathbb{R}^d$ a Borel set such that $p := \mathbb{P}(X_1 \in A) > 0$ and*

$$T = \inf\{i \geq 1, X_i \in A\}.$$

Then, T follows a geometric distribution with probability of success p and is independent of X_T, which follows the conditional law of X_1 given that $X_1 \in A$. To be more explicit, this means that $\mathbb{P}(X_T \in B) = \mathbb{P}(X_1 \in B \cap A)/p$ for any Borel set $B \subset \mathbb{R}^d$.

One easily check that $Z \sim \Gamma(\alpha, \beta)$ if, and only if $\beta Z \sim \Gamma(\alpha, 1)$. Therefore, if we are able to sample a random variable distributed according to $\Gamma(\alpha, \beta(\alpha))$ for any $\alpha > 0$ and some $\beta(\alpha) > 0$, then we are able to sample any random variable distributed according to $\Gamma(\alpha, \beta)$ by using a simple scaling. We present now a method to sample $\Gamma(\alpha, \alpha - 1)$ when $\alpha > 1$ and $\Gamma(\alpha, 1)$ when $\alpha \leq 1$.

We consider the case $\alpha > 1$ and present the method of Cheng and Feast [28]. We take for X_i the uniform law on $[0, 1]^2$ and

$$A = \left\{(x_1, x_2) \in (0, 1)^2,\ \frac{2}{\alpha - 1}\log(x_1) - \log\left(\frac{\lambda x_2}{x_1}\right) + \frac{\lambda x_2}{x_1} - 1 \leq 0\right\},$$

for $\lambda > 0$. We claim that $\frac{\lambda (X_T)_2}{(X_T)_1} \sim \Gamma(\alpha, \alpha - 1)$ when $\lambda > 0$ is suitably chosen, where $(X_T)_i$ denotes for $i \in \{1, 2\}$ the ith coordinate of X_T. In fact, we have

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by Proposition B.0.1

$$\mathbb{E}\left[f\left(\frac{\lambda(X_T)_2}{(X_T)_1}\right)\right] = \frac{1}{p}\int_0^1\int_0^1 f\left(\frac{\lambda x_2}{x_1}\right)\mathbb{1}_{\frac{2}{\alpha-1}\log(x_1)-\log\left(\frac{\lambda x_2}{x_1}\right)+\frac{\lambda x_2}{x_1}-1\leq 0}dx_1dx_2$$
$$= \frac{1}{p}\int_0^\infty f(v)\left(\int_0^1 \mathbb{1}_{\frac{2}{\alpha-1}\log(x_1)-\log(v)+v-1\leq 0}\mathbb{1}_{x_1<\frac{\lambda}{v}}\frac{x_1}{\lambda}dx_1\right)dv,$$

where f is a bounded measurable test function. We observe that $\frac{2}{\alpha-1}\log(x_1) - \log(v) + v - 1 \leq 0 \iff x_1 \leq \exp\left(\frac{\alpha-1}{2}(1-v)\right)v^{\frac{\alpha-1}{2}}$. We check easily that the maximum of the function $v \in \mathbb{R}_+ \mapsto \exp\left(\frac{\alpha-1}{2}(1-v)\right)v^{\frac{\alpha-1}{2}}$ is reached for $v = 1$ and is equal to 1. Also, the maximum of the function $v \in \mathbb{R}_+ \mapsto \exp\left(\frac{\alpha-1}{2}(1-v)\right)v^{\frac{\alpha+1}{2}}$ is reached for $v = 1 + \frac{2}{\alpha-1}$ and is equal to $e^{-1}\left(\frac{\alpha+1}{\alpha-1}\right)^{\frac{\alpha+1}{2}}$. Thus, for any $\lambda \geq e^{-1}\left(\frac{\alpha+1}{\alpha-1}\right)^{\frac{\alpha+1}{2}}$, we have

$$\mathbb{E}\left[f\left(\frac{\lambda(X_T)_2}{(X_T)_1}\right)\right] = \frac{1}{2\lambda p}\int_0^\infty f(v)\exp\left((\alpha-1)(1-v)\right)v^{\alpha-1}dv$$
$$= \int_0^\infty \frac{(\alpha-1)^\alpha}{\Gamma(\alpha)}\exp\left(-(\alpha-1)v\right)v^{\alpha-1}dv.$$

Then, we get also that $(\alpha-1)\frac{\lambda(X_T)_2}{(X_T)_1} \sim \Gamma(\alpha, 1)$. In practice, Cheng and Feast suggest to use $\lambda = \frac{\alpha-1/(6\alpha)}{\alpha-1}$, which avoids to calculate $\left(\frac{\alpha+1}{\alpha-1}\right)^{\frac{\alpha+1}{2}}$. Also, they suggest to check first the condition $X_i \in B$ with

$$B = \left\{(x_1, x_2) \in (0,1)^2,\ \frac{2}{\alpha-1}(x_1-1) + \frac{x_1}{\lambda x_2} + \frac{\lambda x_2}{x_1} - 2 \leq 0\right\}.$$

We have $B \subset A$ since $\log(x) \leq x - 1$ for $x > 0$, and this avoids to calculate a logarithm each time that X_i falls into B.

We now consider the case $\alpha \leq 1$ and present the method of Ahrens and Dieter [5]. We set $\lambda = \frac{\alpha+e}{e}$, we take for X_i the uniform law on $[0,1]^2$ and the acceptance set is

$$A = \Big\{(x_1, x_2) \in (0,1)^2,$$
$$x_2 \leq \mathbb{1}_{\lambda x_1 \leq 1}\exp(-(\lambda x_1)^{\frac{1}{\alpha}}) + \mathbb{1}_{\lambda x_1 > 1}\left(-\log(\lambda(1-x_1)/\alpha)\right)^{\alpha-1}\Big\}.$$

Then, we claim that $Z = \mathbb{1}_{\lambda(X_T)_1 \leq 1}(\lambda(X_T)_1)^{\frac{1}{\alpha}} - \mathbb{1}_{\lambda(X_T)_1>1}\log(\lambda(1-(X_T)_1)/\alpha)$ follows the distribution $\Gamma(\alpha, 1)$. In fact, we have from Proposition B.0.1

$$\begin{aligned}
p\mathbb{E}[f(Z)] &= \int_0^1 \int_0^1 f((\lambda x_1)^{\frac{1}{\alpha}})\mathbb{1}_{\lambda x_1 \leq 1}\mathbb{1}_{x_2 \leq \exp(-(\lambda x_1)^{\frac{1}{\alpha}})} dx_2 dx_1 \\
&+ \int_0^1 \int_0^1 f(-\log(\lambda(1-x_1)/\alpha))\mathbb{1}_{\lambda x_1 > 1}\mathbb{1}_{x_2 \leq (-\log(\lambda(1-x_1)/\alpha))^{\alpha-1}} dx_2 dx_1 \\
&= \int_0^{1/\lambda} f((\lambda x_1)^{\frac{1}{\alpha}}) \exp(-(\lambda x_1)^{\frac{1}{\alpha}}) dx_1 \\
&+ \int_{1/\lambda}^1 f(-\log(\lambda(1-x_1)/\alpha))\,(-\log(\lambda(1-x_1)/\alpha))^{\alpha-1}\, dx_1,
\end{aligned}$$

since we have $\frac{\lambda}{\alpha}(1-x_1) \in (0, \frac{1}{e})$ and thus $(-\log(\lambda(1-x_1)/\alpha))^{\alpha-1} \leq 1$ for $x_1 \in (1/\lambda, 1)$. By a change of variable, we easily get that

$$\mathbb{E}[f(Z)] = \frac{\alpha}{p\lambda}\left(\int_0^1 f(y)y^{\alpha-1}e^{-y}dy + \int_1^\infty f(y)y^{\alpha-1}e^{-y}dy\right),$$

which gives the claim.

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