PROBLEMS IN THE ANALYSIS OF GROWTH AND WEAR CURVES

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1. INTRODUCTION

In a number of different fields of application, the problem arises of comparing sets of growth and wear curves. The object of this paper is to describe methods of analysis which have been found useful, to point out what assumptions are being made, and show how these assumptions can themselves be put to the test.

1.1 Examples of wear and growth curves.

Much research is carried on by technologists with the object of improving the abrasion resistance of materials, and various machines have been devised to test this property. In most of these, specimens of materials to be compared are rubbed against a standard abrasive under standard conditions and the loss in weight or decrease in thickness of each of the specimens is noted at suitable intervals. For example, Fig. I shows the weight loss curves of 24 specimens of coated fabrics tested in the Martindale wear tester. The experiment was arranged in the form of a $2 \times 2 \times 3$ factorial design replicated twice, two different fillers F_1 and F_2 being tried in three different proportions Q_1, Q_2, and Q_3 with and without a surface treatment T, and weight losses were recorded after 1000, 2000, and 3000 revolutions of the machine.

Further examples of this type of test arose in road trial assessments of materials for tire treads (see for example Buist et al., 1950). In one of these investigations, for instance, a test vehicle was run with each of its tires constructed in 3 segments: 4 compounds A, B, C, and D were tested on the four tires of the car in a balanced incomplete block design as follows:

Tire 1	Tire 2	Tire 3	Tire 4
$B\ C\ D$	$A\ C\ D$	$A\ B\ D$	$A\ B\ C$

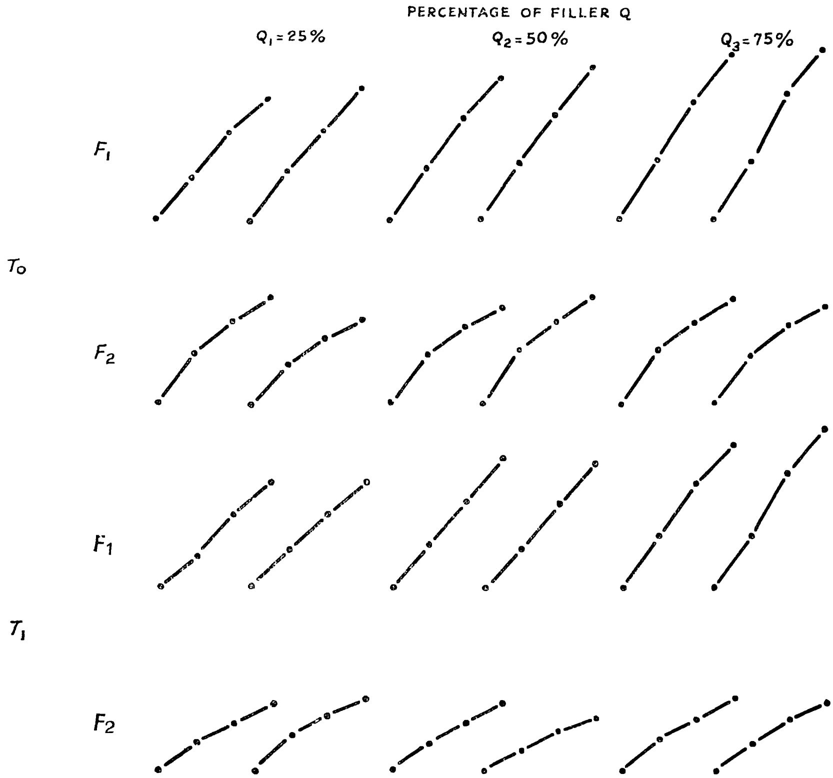

FIGURE I. WEIGHT LOSS CURVES FOR COATED FABRICS

The wear of the compounds was measured by the decrease in tread depth which was observed for each segment of each tire at a number of mileages. Thus for each compound in each tire, there again resulted, not a single result, but a set of results from each of which a wear curve could be plotted.

In biological investigations the growth of an animal or part of an animal is often the subject of study; Fig. II, for example, shows growth curves for 27 rats kept in separate cages. The rats were divided at random into 3 groups containing 10, 7, and 10 rats respectively, (the second group contains fewer rats than the other two, due to an accident at the beginning of the experiment). The first group were kept as a control, the second group had thyroxin, and the third group thiouracil added to their drinking water.

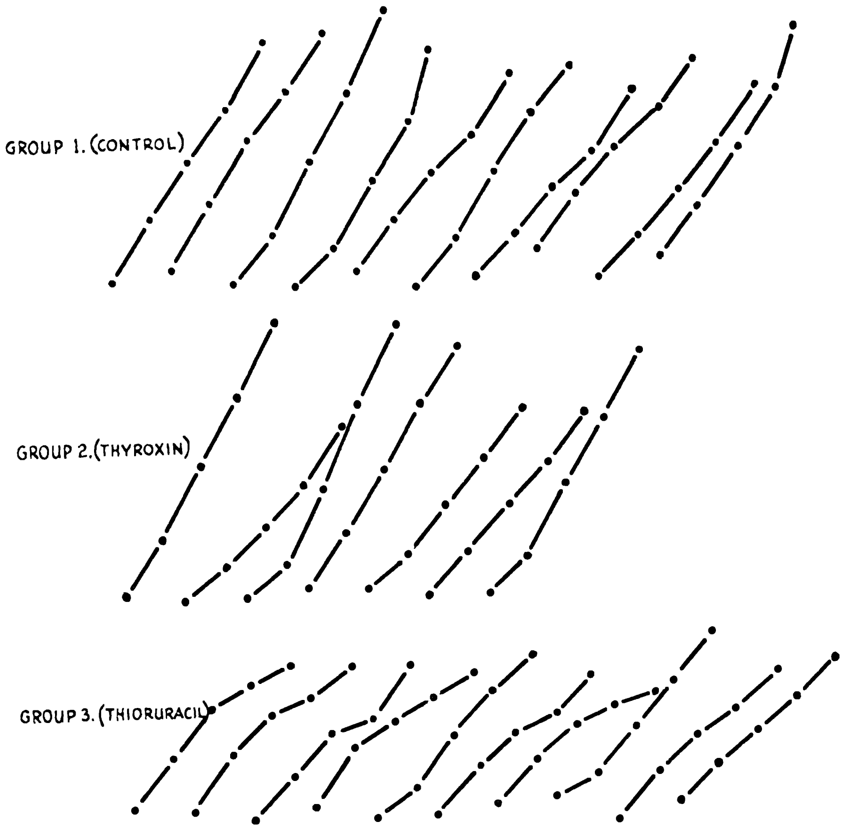

FIGURE II. GROWTH CURVES FOR 27 RATS

In all the examples, it will be seen that we are concerned with experiments which may be of a simple or complex character; each observation, however, consists not of a single value, but of a set of values recorded at intervals of time, from which curves can be plotted.

2. A SIMPLE ANALYSIS

We begin by considering the wear data plotted in Fig. I for coated fabrics prepared in a number of different ways. With data of this kind it has been found to be of value to consider not the wear, after say 1000, 2000, and 3000 revolutions of the machine, but the wear occurring *during* the first thousand, second thousand, and third thousand revolutions, that is to say to consider the first differences of the original data.

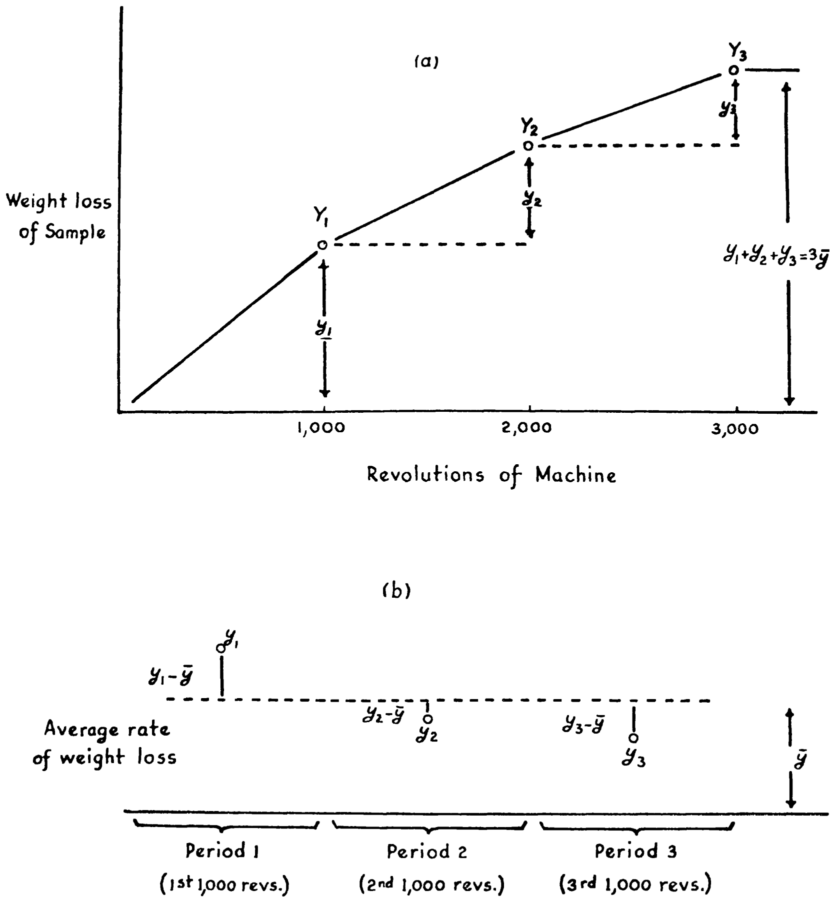

FIGURE III. ANALYSING FIRST DIFFERENCES OF WEAR DATA

In Fig. IIIa, Y_1, Y_2 and Y_3 are the total weight losses for a particular specimen at 1000, 2000, and 3000 revolutions of the machine. We consider $y_1 = Y_1$, $y_2 = Y_2 - Y_1$ and $y_3 = Y_3 - Y_2$; y_1, y_2 and y_3 can be regarded as measuring the average *rates* of wear in milligrams per 1000 revolutions during the three periods, and $\bar{y}$, the mean as measuring the *overall* average rate. Since $3\bar{y}$ is equal to Y_3, $\bar{y}$ is proportional to the total wear during the experiment. The three periods of wear considered can then formally be regarded as a further factor, "periods", and the curve obtained by plotting the differences y_1, y_2, y_3 (Fig. IIIb) indicates the approximate shape of the wear *rate* curve. Now whatever other information is required from the data it will usually be the case that the overall effects of

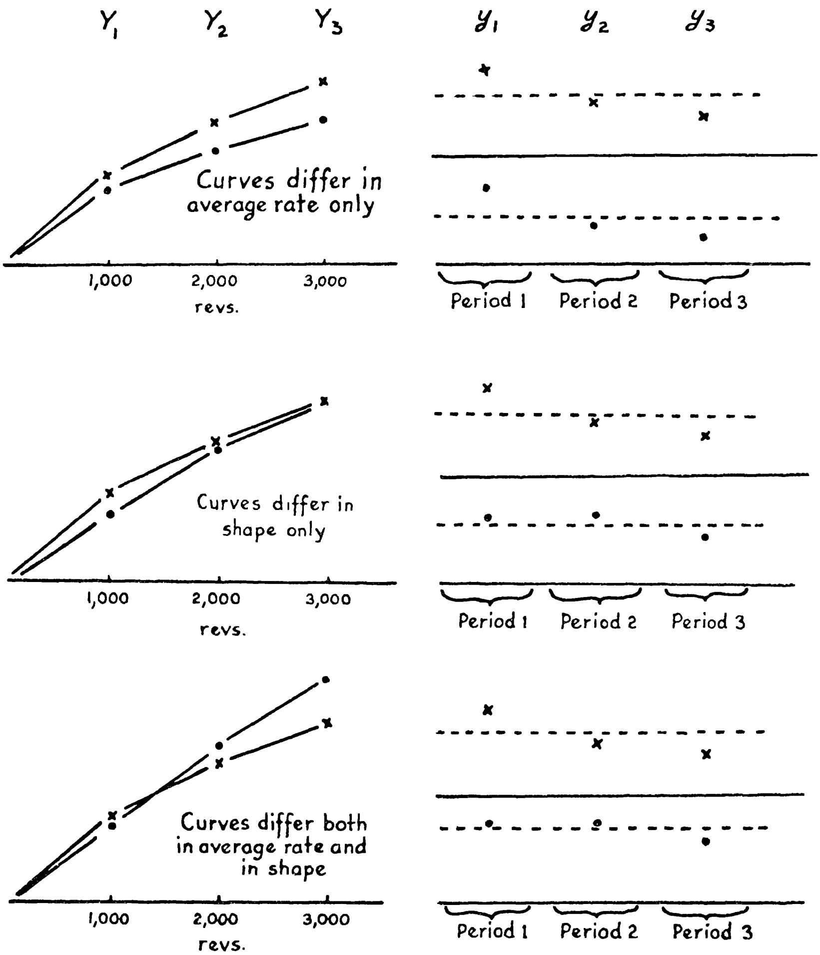

FIGURE IV. COMPARISON OF CURVES USING AVERAGE RATE AND "SHAPE"

treatments will be of major interest and these effects can be elucidated by analyzing the variate $\bar{y}$; it may be, however, that the treatments have effected not only the average rates, but have also influenced the configuration of the individual period rates about their averages, and such effects as this can be regarded as interactions between the factor concerned and "periods".

If (Fig. IV) a factor affects only the average wear rate $\bar{y}$, i.e. alters y_1 , y_2 , and y_3 , equally, (and there is consequently no interaction with "periods"), we shall say that the wear rate curve has been altered only in level, but if the deviations $y_1 - \bar{y}$, $y_2 - \bar{y}$, $y_3 - \bar{y}$ are affected, we shall say that the wear rate curve is also altered in *shape.* This use of the average level $\bar{y}$ and the deviations $y_1 - \bar{y}$, $y_2 - \bar{y}$, $y_3 - \bar{y}$, to describe the curve is analogous to the size and shape analysis used in discrimination problems by Penrose (1947) and Smith (1947). On this definition, then, two curves are said to have different shapes when the wear rates differ by different amounts at different stages of wear.

The data for Fig. I, after differences have been taken, are set out in Table A. The figures in the table corresponding with periods 1, 2, and 3 refer to the wear between 0–1000, 1000–2000, 2000–3000 revolutions of the machine respectively.

TABLE A

WEAR OF COATED FABRICS IN MILLIGRAMS

Surface Treatment	Filler	Proportion of Filler								
		Q_1 (25%)			Q_2 (50%)			Q_3 (75%)		
		Period 1	2	3	Period 1	2	3	Period 1	2	3
T_0	F_1	194	192	141	233	217	171	265	252	207
		208	188	165	241	222	201	269	283	191
	F_2	239	127	90	224	123	79	243	117	100
		187	105	85	243	123	110	226	125	75
T_1	F_1	155	169	151	198	187	176	235	225	166
		173	152	141	177	196	167	229	270	183
	F_2	137	82	77	129	94	78	155	76	91
		160	82	83	98	89	48	132	105	67

2.1 A mathematical model for the experiment.

The error variances for the data are as follows:

Revolutions	Error Variance	Revolutions	Error Variance
0–1000	269	0–1000	269
0–2000	456	1000–2000	200
0–3000	935	2000–3000	222

A characteristic of the cumulative data is seen to be the increase of the error variance as the test proceeds, the variance of the differences however, remains stable from period to period. This is to be expected, for much of the variation probably arises in the operation of the testing machine itself and the variation occurring in equal periods of running might be expected to be the same. Also, in the original data the errors would be expected to be correlated from one observation to the next, since for example abnormally low wear occurring in the first 1000 revolutions would be reflected in subsequent values. So far as the variation of the machine was concerned however, correlation between wear in successive *periods* might be expected to be much smaller and possibly negligible.

For the moment then, we will assume that the departure of y_{ti} (the loss in weight during the i^{th} period of wear of a sample having the t^{th} factor combination) from its mean value η_{ti} can be represented by two independent random variables ϵ_t and δ_i each being normally distributed about zero, the first with variance σ_1^2 allows for *overall* variation in mean rate in duplicate specimens and the second with variance σ_0^2 allows for variations associated with individual periods.

$$y_{ti} - \eta_{ti} = \epsilon_t + \delta_i \tag{1}$$

If this is true, then the analysis of variance for the data in Table A, will be analogous to that for a split-plot agricultural experiment, the "plots" being the samples of material and the three values for wear in successive periods corresponding to the splitting of the plots. As with split plot experiments, the analysis will consist of two parts, each part having its own error estimate. This analysis for the wear data is set out in Table B. The entries in the table are the mean squares corresponding to the effects; the figures in brackets refer to the degrees of freedom available for the comparisons.

The mean squares in the left-hand column are obtained after averaging over "periods" and they enable hypotheses concerning the average wear rate (or what is equivalent, the overall wear) to be tested; the correction for the mean, denoted by I, is included for completeness. Significance is judged by comparison with the mean square of 312 at the foot of this column; this error term has 12 degrees of freedom and is an estimate of the "between samples" variance $3\sigma_1^2 + \sigma_0^2$ as is each of the mean squares in this column if the treatments are without effect. The asterisks indicate significance at the 5, 1 and 0.1% points respectively.

The mean squares in the right-hand column correspond to interactions with "periods" and significant interactions imply that a factor has

TABLE B
ANALYSIS OF VARIANCE FOR THE DATA OF TABLE A

Source		Averaged over "periods" (level of rate curve)		Interactions with "periods" (Shape of rate curve)	
Mean	(I)	(1)	1,866,956	(2)	30,479***
Main effects					
% Filler	(Q)	(2)	6,785***	(4)	440
Type of Filler	(F)	(1)	107,803***	(2)	9,144***
Surface Treatment	(T)	(1)	24,494***	(2)	4,124***
Interactions					
$Q \times F$		(2)	4,942***	(4)	354
$Q \times T$		(2)	397	(4)	172
$F \times T$		(1)	1,682*	(2)	1,164***
$Q \times F \times T$		(2)	150	(4)	116
Error		(12)	312	(24)	190

affected the quantities $y_1 - \bar{y}$, $y_2 - \bar{y}$, $y_3 - \bar{y}$, that is, that it has altered the "shape" of the rate curve. The interaction of I with the periods factor P is $P \times I = P$ the main effect for periods, which indicates whether the mean *rate* of growth has remained constant from period to period i.e. whether the mean growth curve is a straight line. The error mean square of 190 appropriate for testing these effects has 24 degrees of freedom and is an estimate of σ_0^2 the "within samples" variance, as is each of the mean squares in this column on the hypothesis that the treatments are without effect. The entries in this column can be most easily calculated by first carrying through an analysis of variance for each period separately; and then using the identity;

$$\left\{\begin{array}{l}\text{Sum of squares}\\ \text{for interaction}\\ \text{with periods}\end{array}\right\} = \left\{\begin{array}{l}\text{total of sums}\\ \text{of squares for}\\ \text{individual periods}\end{array}\right\} - \left\{\begin{array}{l}\text{sums of squares}\\ \text{for average}\\ \text{effects.}\end{array}\right\}$$

For example, for the surface treatment T the sums of squares for the three periods of wear were found to be 26,268, 5,017 and 1,457 respectively; the sum of squares for the average effect was 24,494, and the sum of squares for interaction with periods was therefore $26{,}268 + 5{,}017 + 1{,}457 - 24{,}494 = 8{,}248$ and the mean square, 4,124 as shown in the table; all the entries in this column including, of course, the error term, are calculated in this way. This procedure can be followed whether the remaining effects are orthogonal or not. For example, in the balanced incomplete block design used for the assessment of tire treads, the usual (nonorthogonal) analysis for incomplete block designs is first carried through for the means, averaging over periods; this analysis is then repeated for each period separately and the above identity used to determine the interactions with periods.

2.2 Interpretation of the analysis.

Considering first the left-hand column of table B, we see that all the main effects are highly significant and that interactions exist between Q and F, that is, between the proportion of filler and the type of filler used and between F and T, the type of filler and the surface treatment. The appropriate tables of mean values showing the nature of these interactions follow:

% FILLER (Q)

		25	50	75
Type of Filler	F_1	169	199	231
	F_2	121	120	126

SURFACE TREATMENT

		T_0	T_1
Type of Filler	F_1	214	145
	F_2	186	99

From the first table, we see that the average rate of wear is less with the second filler, and this average rate was virtually unaffected by the increase in the percentage of filler, whereas with the first filler, the material wears more, and the wear increases as the percentage is increased. From the second table of means we see that the beneficial effect of the surface treatment is even more pronounced with the second than with the first filler.

So far, we have considered the effect of factors only on the average rate of wear, we now need to consider whether these effects change at different stages of wear; that is to say whether the factors affect the shape of the rate curves. To do this, we consider the mean square for the interaction of each of the effects with periods, shown in the right-hand column of the table; the significance of these items is judged by comparison with the error term at the foot of this column. The % filler (Q) and % filler $\times$ type of filler (QF) effects have no significant interactions with periods, so we can regard our conclusions for these effects as probably true at all stages of wear; however both F (the type of filler) and T (the surface treatment) show strong interactions. The tables of mean values are as follows:

		Period 1	Period 2	Period 3
Type of Filler	F_1	215	213	172
	F_2	181	104	82
		Period 1	Period 2	Period 3
Surface Treatment	T_0	231	173	134
	T_1	165	144	119

From the first table we conclude that not only do the samples having filler (1) wear at a greater rate than those having filler (2), but also the fall off in wear is greater with filler (2). From the second table we conclude that although the surface treatment has a favorable effect at all stages of wear this is (as would be expected) most marked initially. Finally, we see that the interaction $F \times T$, between type of filler and surface treatment, interacts with periods and, on consulting a table of mean values, this is seen to be due to the fact that the interaction between filler and surface treatment found before, is confined to the first period of wear alone.

The conclusions from the experiment can therefore be set out as follows:

1. Filler (2) results in less wear than filler (1), the rate of fall off of wear is greater and the protective effect of surface treatment is more marked with (2) than with (1).
2. Whereas increasing the % of filler (1) results in increased rate of wear, the % of filler (2) can be increased, at least to 75%, without increasing wear.
3. The surface treatment markedly reduces wear especially during the early stages and when filler (2) is used.

It is clear that the two fillers are behaving differently and in a full analysis the data would be split into two, and an analysis made for each filler separately. We shall not however elaborate the analysis further here, since our purpose is merely to show that when the set-up given in equation (1) can be regarded as valid, an exceedingly simple and informative analysis can be made. We now proceed to show how it is possible to test whether significant departure from the simple set-up occurs or not.

3. A TEST FOR DEPARTURE FROM THE SIMPLE MODEL

Instead of equation (1) let us write

$$y_{ti} - \eta_{ti} = z_{ti} \tag{2}$$

Then if the simple set-up is valid

$$z_{ti} = \epsilon_t + \delta_i \tag{3}$$

and z_{ti} would be distributed in a three-variate multinormal distribution with each variance equal to $\sigma_1^2 + \sigma_0^2$ and each covariance equal to σ_1^2. (For simplicity the test is illustrated for three variates; it can of course be immediately generalized to the p-variate case.) To check the validity of our analysis therefore we must test the hypothesis, that all the variances are equal and all the covariances are equal; that is, that V^*, the matrix of variances and covariances is of the form:

$$V_0 = \begin{bmatrix} a & d & d \\ d & a & d \\ d & d & a \end{bmatrix} \tag{4}$$

against the alternative that the variances are not all the same and the covariances not all the same; that is, that the variance covariance matrix is of the form:

$$V_1 = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \tag{5}$$

where a, b, c are not all equal and d, e, f, are not all equal.

The null hypothesis of (4) is a little more general than that implied by equation (3) for negative values of the covariance d are possible with (4) (although since V_0 must be positive definite d cannot be less than $-a/(p - 1)$), whereas correlation arising from a common component ϵ_t in (1) must clearly be positive. However, an analysis of the type given in Table B is valid for any positive definite matrix of form (4) so that this extension is appropriate.

3.1 The Test Criterion

A criterion for testing a statistical hypothesis of this form has been

*We tacitly assume here that the variance covariance matrix V remains constant from group to group, that is to say is itself unaffected by the treatments; we consider this assumption later in § 7.

obtained by Wilks (1946) using the likelihood ratio method of Neyman and Pearson (1928). Let $c_{ij} = c_{ji}$ denote the error sums of squares and products calculated from the sample for the i^{th} and j^{th} variates, then the criterion is

$$\Lambda = \frac{\Delta_1}{\Delta_0} = \frac{\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}}{\begin{vmatrix} \bar{c}_{ii} & \bar{c}_{ij} & \bar{c}_{ij} \\ \bar{c}_{ij} & \bar{c}_{ii} & \bar{c}_{ij} \\ \bar{c}_{ij} & \bar{c}_{ij} & \bar{c}_{ii} \end{vmatrix}} \tag{6}$$

where $\bar{c}_{ii}$ is the average sum of squares $(c_{11} + c_{22} + c_{33})/3$ or in general $(\sum_i^p c_{ii})/p$ and $\bar{c}_{ij}$ is the average sum of products $(c_{12} + c_{13} + c_{23})/3$, or in general

$$\left\{ \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} c_{ij} \right\} \Big/ \left\{ \frac{1}{2} p(p-1) \right\}$$

and p is the number of variates (3 in this case). The value of the determinant Δ_0 in the denominator can be easily shown to be equal to $[\bar{c}_{ii} + (p-1)\bar{c}_{ij}][\bar{c}_{ii} - \bar{c}_{ij}]^{p-1}$ which simplifies the calculations.

For the wear test example the sums of squares and products for error can be most easily calculated from the differences between duplicate observations in table A. These differences are set out in table C, for example, $14 = 208 - 194$, $-4 = 188 - 192$, etc.

TABLE C

DIFFERENCES BETWEEN DUPLICATES

Period	T_0						T_1					
	F_1			F_2			F_1			F_2		
	Q_1	Q_2	Q_3	Q_1	Q_2	Q_3	Q_1	Q_2	Q_3	Q_1	Q_2	Q_3
1	14	8	4	−52	19	−17	18	−21	−6	23	−31	−23
2	−4	5	31	−22	0	8	−17	9	45	0	−5	29
3	24	30	−16	−5	31	−25	−10	−9	17	6	−30	−24
Total	34	43	19	−79	50	−34	−9	−21	56	29	−66	−18

The sums of squares and products are then obtained as follows:

		Period			
		1	2	3	S
Period	1	3225.0	−80.5	1656.5	4801.0
	2	−80.5	2405.5	−112.0	2213.0
	3	1656.5	−112.0	2662.5	4207.0

$$\text{For example } 3225.0 = [14^2 + 8^2 + 4^2 + \cdots + (-23)^2]/2$$

$$-80.5 = [14 \times (-4) + 8 \times 5 + \cdots + (-23) \times 29]/2.$$

The divisor 2 is employed since the values concerned are differences between two observations. The check column S shows the sum of products between each of the variates and the column totals of table C; this supplies an independent check on the working, for example, for the second row of the matrix

$$-80.5 + 2405.5 - 112.0 = 2213.0.$$

The value of the determinant of this matrix which is the numerator Δ_1 of the criterion is found to be $14{,}026 \times 10^6$. To calculate the denominator Δ_0 of the criterion, we find:

$$\bar{c}_{ii} = 2764.3, \qquad \bar{c}_{ij} = 488.0,$$

$$\bar{c}_{ii} + 2\bar{c}_{ij} = 3740.3 \qquad \bar{c}_{ii} - \bar{c}_{ij} = 2276.3$$

and $\Delta_0 = 3740.3 \times (2276.3)^2 = 19{,}381 \times 10^6$, whence $\Lambda = 0.7237$

3.2 The test of significance

We now wish to test whether or not this value of Λ is exceptionally small. The exact distribution is not known in the general case; however, an expression for the moments of Λ has been given by Wilks (1946) and for these, sufficiently accurate approximations can be calculated. The present author (Box 1949) has given a general distribution theory for a very wide class of what may be called "Λ" statistics, whose moments can be written in the form

$$E(\Lambda^a)^h = \text{constant} \cdot \left[\frac{\prod_{j=1}^{k} (y_j^{y_j})}{\prod_{j=1}^{m} (x_i^{x_i})}\right]^h \frac{\prod_{i=1}^{m} [\Gamma\{x_i(1+h)+\xi_i\}]}{\prod_{j=1}^{k} [\Gamma\{y_j(1+h)+\eta_j\}]} \tag{7}$$

Given the value of k, m, and the x_i, ξ_i, y_i and η_i, general formulae are provided from which taking $M = 2a \log_e \Lambda^{-1}$ as a working statistic, an accurate χ^2 series solution can be obtained, and simple χ^2 and F approximations. For the statistic here considered, Wilks' expression for the moments can be written

$$E(\Lambda^{\nu/2})^h = \text{constant} \cdot (p-1)^{\frac{1}{2}(p-1)\nu h} \frac{\prod_{i=1}^{p-1}\left[\Gamma\left\{\frac{\nu}{2}(1+h) - \frac{i}{2}\right\}\right]}{\Gamma\left\{\frac{\nu}{2}(p-1)(1+h)\right\}} \tag{8}$$

ν being the degrees of freedom of the sums of squares and products tested. This is seen to be of the same form as (7) so the general theory can be applied.

Making the substitutions we find we should take for our working statistic

$$M = \nu \log_e \Lambda^{-1}$$

The more the data depart from the simple set-up, the smaller will be Λ and the greater the value of M. To test whether M is large enough to indicate a "significant" departure from the simple set-up we calculate

$$f_1 = (p^2 + p - 4)/2, \qquad A_1 = \frac{\{p(p+1)^2(2p-3)\}}{\{6\nu(p-1)(p^2+p-4)\}}$$

and refer $(1 - A_1) M$ to tables of χ^2 with f_1 degrees of freedom. An approximation which is rather more precise especially when p is large and/or ν is small is supplied by calculating

$$A_2 = \frac{(p-1)p(p+1)(p+2)}{6\nu^2(p^2+p-4)}, \; f_2 = \frac{f_1 + 2}{A_2 - A_1^2}, \; b = \frac{f_1}{1 - A_1 - f_1/f_2}$$

and referring M/b to tables of the F distribution with f_1 and f_2 degrees of freedom.

In the present example $\nu = 12$ and $p = 3$. Whence $f_1 = 4$, $A_1 = 0.125$, $A_2 = 0.0174$, $M = 12 \log_e (1/0.7237) = 3.880$. $(1 - A_1) M = 3.395$ and this quantity is to be referred to tables of χ^2 with four degrees of freedom, from which we can conclude at once that on this data there is no reason to question the simple set-up. The actual probability given by this approximation for the chance occurrence of a value of M as great or greater than this is slightly less than 0.5. Using the F approximation a very similar result would have been obtained, we find

$$f_1 = 4, \qquad f_2 = 3456, \qquad b = 4.578.$$

Thus $M/b = 0.8476$ is to be referred to tables of F with 4 and 3456 degrees of freedom, and consulting the table of Thompson and Merrington (1943) we again find a value for the probability of chance occurrence slightly below 0.5. In this particular case both approximations are quite accurate, and the values they give are almost identical. When p is larger and ν is smaller the χ^2 approximation is less accurate and the F approximation differs more markedly from it, f_2 being no longer very large as it is in this example.

4. AN ALTERNATIVE SET-UP

We have seen how, by the device of taking differences, the problem of interpreting wear data was facilitated and a simple analysis was possible if it was reasonable to assume a particular set-up. We have shown that for a particular example the hypothesis that the set-up was of this simple form was not contradicted by the data.

The simple set-up would be expected to represent wear data satisfactorily if most of the variation arose from the operation of the machine itself or if the variation within replicate specimens of material were mainly confined to changes in *average* abrasion resistance and not to changes in "shape" of the wear curves which might give rise to serial correlation between differences. Observational errors would also tend to cause departures from the set-up, for an apparent increase in wear rate during one period would be compensated by an apparent decrease in the next, and this would lead to negative correlation between successive differences. In some cases therefore we would not expect the variances and covariances, even after differencing of the data, to be capable of adequate representation by the simple pattern of equation (4) and the more general set-up typified by equation (5) would have to be adopted.

When a research program is being carried out in which the results will appear in the form of wear curves or growth curves, it is worthwhile paying particular attention, in the preliminary experiments, to the form of the variance co-variance matrices, so that decisions may be reached concerning a set-up, and method of analysis, suitable for use in this *particular* investigation; that is to say with this particular type of material and interval between observations. Consider now the rat growth data shown in detail in Table D at end of article, which was plotted in Figure II. For the reasons given above we should not expect the weight gains for these animals, even after differencing, to be uncorrelated from one period to another. In fact, if we denote the five variates recording initial weight and weight after one, two, three, and four weeks by Y_0, Y_1, Y_2, Y_3, Y_4 and the differences, $Y_1 - Y_0$, $Y_2 - Y_1$, $Y_3 - Y_2$, $Y_4 - Y_3$ by y_1, y_2, y_3, y_4, the matrix of sums of squares and products for the 24 error degrees of freedom for y_1, y_2, y_3 and y_4, is found to be

$$\begin{bmatrix} 582.3 & 42.5 & -55.5 & -74.6 \\ 42.5 & 609.0 & 626.5 & 344.5 \\ -55.5 & 626.5 & 1046.7 & 459.0 \\ -74.6 & 344.5 & 459.0 & 853.0 \end{bmatrix} \tag{9}$$

From which proceeding as before we find

$$\Lambda = 0.3534, \qquad M = 25.0, \qquad 1 - A_1 = 0.9277, \qquad f = 8.$$

Referring $(1 - A_1)M = 23.2$ to tables of χ^2 with 8 degrees of freedom we conclude that the probability of a chance deviation from the simple set-up as great as this is less than 0.01; a similar result is given by the F approximation and we are therefore led to reject this set-up.

Although differencing is not completely successful in transforming the data into a form in which the variances are equal and the covariances are equal, it is worth noting that the differences do show a great deal more uniformity in this respect than the data before differencing. The corresponding error matrix for sums of squares and products for the overall weight gains $Y_1 - Y_0$, $Y_2 - Y_0$, $Y_3 - Y_0$, and $Y_4 - Y_0$ is

$$\begin{bmatrix} 582.3 & 624.8 & 569.3 & 494.7 \\ 624.8 & 1276.3 & 1847.3 & 2117.2 \\ 569.3 & 1847.3 & 3465.0 & 4193.9 \\ 494.7 & 2117.2 & 4193.9 & 5775.8 \end{bmatrix} \tag{10}$$

and the value of χ^2 obtained on applying the test to these values is 109.1; the transformation has thus gone a good deal of the way towards bringing the data to the simpler form.

In making tests of an actual set-up, it should be borne in mind that the important consideration is how far departures from the assumptions made will affect the *tests based on these assumptions*. This problem has been investigated in a number of cases by the present author and it is hoped to publish the results elsewhere in the near future. It appears that minor departures of the data from independence and homoscedasticity of the type considered here will not seriously influence subsequent tests of significance; a similar conclusion was reached by Daniels (1938). Thus, in the wear curve example we found no significant departure from the simple set-up, and although this did not mean that no such departure could have occurred, but possibly, merely that the data were not sufficiently extensive for it to be detected, it is probable that little error was made by adopting the simple set-up in this case. In the growth curve example however a marked departure is apparent and it would be safer to adopt the less restrictive assumptions of the alternative hypothesis of equation (5).

We shall assume that the variates Y_0, Y_1, Y_2, Y_3 and Y_4 are distributed about their mean values in a 5-variate normal distribution, the same for each rat, and this of course implies that any set of variates derived from these, by linear transformation, will also be distributed multinormally; in particular the differences will be distributed in that form.

4.1 The Multivariate Test

Now if a single variate only, say the overall increase in weight, were being analyzed, the hypothesis concerning the significance of treatment differences could be tested by means of the analysis of variance, that is to say the criterion

$$\frac{\text{mean square for treatments}}{\text{mean square for error}}$$

would be referred to tables of the F distribution with the appropriate numbers of degrees of freedom. Alternatively (see for example Kolodziejczyk 1935) the criterion*

$$\Lambda = \frac{\text{sum of squares for error}}{\text{sum of squares for error} + \text{sum of squares for treatment}}$$

could be employed, and the test carried out by referring to tables of the incomplete B-function (Karl Pearson; 1934, Thomson; 1941), the result of course would be precisely the same.

For our present purpose the latter criterion is of more interest, because it can be directly generalized (Wilks; 1932, Pearson and Wilks; 1933, Bartlett; 1934, 1938) to the case where the observations are not single variates but multivariates, whereas the former criterion cannot. Thus the hypothesis that the mean value for each of the variates is the same from group to group; (for example, that the gains in weight during the first week are all equal, *and* the gains during the second week are all equal, etc.) can be tested by calculating the criterion;

*Λ is used in the paper to denote a criterion of the form associated with the likelihood ratio method of Neyman and Pearson. M is used to denote a logarithmic statistic derived from Λ. The likelihood statistic Λ and the derived quantity M referred to in this section are of course different from the criterion discussed in § 3.

$$\Lambda = \frac{|\text{ sums of squares and products for error }|}{\left|\begin{array}{c}\text{sums of squares and products for error}\\ + \text{ sums of squares and products for treatment}\end{array}\right|}$$

where *determinants* whose elements are sums of squares and products replace the single sums of squares of the univariate criterion.

Thus, had we desired to use the more general set-up in the wear test example, an *overall* test for each of the main effects and interactions could have been applied by calculating the 3×3 matrix of sums of squares and products first for error and then for the particular main effect or interaction concerned, and hence calculating the criterion Λ. This test would not have distinguished between changes in average and changes in shape but would have been an overall test including both.

The exact distribution of Λ is known only for certain special cases, however, this is another of the general class of statistics whose moments can be written in the form of equation (7), and simple approximations which are perfectly general and which are usually sufficient for most practical purposes can be obtained. To preserve generality, even when the exact distribution is available, these approximations will be used in all the tests that follow. If n is the number of degrees of freedom for *treatments plus error*, q the number of degrees of freedom for treatments and p the number of variates then Bartlett's (1938) χ^2 approximation is obtained by calculating

$$M = n \log_e \Lambda^{-1} \qquad A_1 = (p + q + 1)/2n \qquad f_1 = pq$$

and referring $(1 - A_1)M$ to tables of χ^2 with f_1 degrees of freedom. This approximation tends to break down if n is small or p and q are large and in these cases it is worthwhile calculating the more accurate F approximation (Box 1949). For this we calculate in addition

$$f_2 = \frac{12n^2(pq + 2)}{p^2 + q^2 - 5}, \qquad b = \frac{pq}{1 - A_1 - f_1/f_2},$$

and refer M/b to tables of F with f_1 and f_2 degrees of freedom.

In the growth curve example if we consider the variates y_1, y_2, y_3, y_4 (that is, the first differences of the weight gains) the matrix for sums of squares and products for treatments is found to be

$$\begin{bmatrix} 81.7 & 37.2 & 11.5 & 112.9 \\ 37.2 & 476.9 & 782.7 & 787.4 \\ 11.5 & 782.7 & 1315.9 & 1260.1 \\ 112.9 & 787.4 & 1260.1 & 1334.0 \end{bmatrix} \tag{11}$$

The corresponding matrix for sums of squares and products for error has already been given (9). The error plus treatment matrix is obtained by adding each element in the error matrix to the corresponding element in the treatment matrix. The ratio Λ of the error determinant to the error plus treatment determinant is then found to be 0.2661, and $n = 26$, $p = 4$, $q = 2$. For such large values of n and comparatively small values of p and q Bartlett's χ^2 approximation will be adequate and we find

$$M = 34.4, \qquad A_1 = 7/52, \qquad f_1 = 8$$

and referring $(1 - A_1)M = 29.8$ to tables of χ^2 with 8 degrees of freedom we conclude that the mean values for y_1, y_2, y_3, and y_4 representing the growth during the first, second, third, and fourth weeks differ very significantly from group to group ($P < 0.001$).

4.2 Special Properties of the Criterion.

Before proceeding further we note certain important properties of this criterion used in the multivariate extension of the analysis of variance.

1. The criterion is invariant under non-singular linear transformation of the variates. Thus, in the example above, if we had analyzed the total gains in weight instead of the differences, or had applied any other linear transformation of this sort to the data, the value of the overall criterion would have been unchanged.

2. The sums of squares and products matrix for any *new* set of variates obtained by linear transformation can be found directly by applying the transformation to the rows and columns of the matrix of sums of squares and products of the *old* set of variates. For example if the matrix (10) for total weight gains were known, (9) the corresponding matrix after differencing the data, could be obtained by applying the differencing process to the rows and columns of (10) itself; it is not necessary to make the transformation to the original data and recalculate.

3. In the calculation of determinants, the method of pivotal condensation (see for example Aitken, 1948) provides a rapid practical procedure; *this device also provides a useful method for the elimination of variables.* As an example consider a determinant Δ of sums of squares and products for, say, three variates y_1, y_2, y_3,

$$\Delta = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} \qquad \text{where } c_{ij} = c_{ji}$$

dividing through the first row by c_{11} we obtain a new first row

$$1 \quad \frac{c_{12}}{c_{11}} \quad \frac{c_{13}}{c_{11}}$$

if this is subtracted c_{12} times from the second row and c_{13} times from the third, we have

$$\Delta = c_{11} \begin{vmatrix} 1 & \frac{c_{12}}{c_{11}} & \frac{c_{13}}{c_{11}} \\ 0 & c_{22} - \frac{c_{12}^2}{c_{11}} & c_{23} - \frac{c_{11}c_{13}}{c_{11}} \\ 0 & c_{23} - \frac{c_{12}c_{13}}{c_{11}} & c_{33} - \frac{c_{13}^2}{c_{11}} \end{vmatrix}$$

and writing $c_{22} - c_{12}^2/c_{11}$ as $c_{22.1}$, $c_{23} - c_{12}c_{13}/c_{11}$, as $c_{23.1}$, etc. and expanding the determinant along the first column, we have

$$\Delta = c_{11} \begin{vmatrix} c_{22.1} & c_{23.1} \\ c_{23.1} & c_{33.1} \end{vmatrix}$$

that is

$$\Delta = \Delta_{123} = c_{11}\, \Delta_{23.1} \;.$$

As is well known, to compute the value of any $p \times p$ determinant, the process may be repeated $p - 1$ times till the determinant is reduced to the product of p known quantities, and this process is the basis of the Gauss-Doolittle method for the solution of the linear equations, and can be still further simplified (see for example Dwyer; 1942). What is interesting for our purpose is the fact that the elements of $\Delta_{23.1}$ are the sums of squares and products for y_2 and y_3 after eliminating the variable y_1 , that is to say they are the sums of squares and products of deviations from the regressions of y_2 and y_3 on y_1 . Now if condensation of this sort is applied simultaneously to numerator and denominator of Λ we have

$$\Lambda = \frac{c_{11}\text{ (error)}}{c_{11}\text{ (error + treatments)}} \quad \frac{\Delta_{23.1}\text{ (error)}}{\Delta_{23.1}\text{ (error + treatments)}}$$

$$= \Lambda_1\, \Lambda_{23.1}$$

In the above equation Λ_1 corresponds to a univariate analysis of variance for the variable y_1 and the second component to a multivariate analysis of covariance for the remaining variables with the first variate y_1 eliminated. Any number of variables can be eliminated in this way, the number of degrees of freedom for error being correspondingly reduced after each elimination. A successive elimination of variates of this sort was applied by Bartlett to the linear quadratic and cubic components fitted to the growth curves of pigs by Wishart (1939).

4.3 Further Analysis of the Data.

Now even though the taking of differences does not result in a simplification of the set-up, it allows the changes in the wear curves to be more easily appreciated and may still be employed in the interpretation of the overall criterion. We shall therefore again consider the mean growth rate $\bar{y}$ and the deviations from the mean $y_1 - \bar{y}$, etc. Only three of the four deviations from the mean are linearly independent and all the information concerning departures from the mean is contained in any three of them, we therefore consider the variates $\bar{y}$, $y_1 - \bar{y}$, $y_2 - \bar{y}$ and $y_3 - \bar{y}$; $y_4 - \bar{y}$ is omitted from the analysis, (exactly the same result will be obtained whichever of the deviations is omitted). Using the second property noted above, the entries for sums of squares and products for the new variates are obtained by direct transformation. For example we obtain the error matrix for the new variates from the error matrix (9) for y_1, y_2, y_3, and y_4 as follows; first applying the transformation to the rows, corresponding to the operation of taking the mean $\bar{y}$ we replace the elements of the first row of (9) by their column means, the second row is then obtained by subtracting these values from the elements of the first row of (9) corresponding to the operation of taking $y_1 - \bar{y}$; the third and fourth rows are found similarly, and the whole set of operations is then carried out on the columns. A partial check is supplied by the symmetry of the final transformed matrix and a complete check can be made by calculating the sums for each row and column of the final matrix and confirming that these totals agree with the values found by operating on the sums of rows and columns of the original matrix. From the error matrix (9) we obtain

	$\bar{y}$	$y_1 - \bar{y}$	$y_2 - \bar{y}$	$y_3 - \bar{y}$
$\bar{y}$	361.0	−237.3	44.6	158.2
$y_1 - \bar{y}$	−237.3	695.9	−125.8	−337.4
$y_2 - \bar{y}$	44.6	−125.8	158.7	62.7
$y_3 - \bar{y}$	158.2	−337.4	62.7	369.3

and applying the same procedure to the error plus treatment matrix we have

	$\bar{y}$	$y_1 - \bar{y}$	$y_2 - \bar{y}$	$y_3 - \bar{y}$
$\bar{y}$	935.5	−751.0	−8.8	426.2
$y_1 - \bar{y}$	−751.0	1230.5	−96.0	−654.7
$y_2 - \bar{y}$	−5.8	−96.0	168.0	56.3
$y_3 - \bar{y}$	426.2	−654.7	56.3	574.6

The Λ criterion for means alone is therefore:

$$\Delta\,(\bar{y}) = \frac{360.9}{935.4} = 0.3859$$

and for reasons already given we shall employ Bartlett's approximation to make the test of significance. We find $(1 - A_1)M = 22.9$, should be referred to χ^2 tables with two degrees of freedom whence we deduce that the mean growth rates differ very significantly ($P < .001$) from group to group. To test the deviations from the means, that is to test whether the "shape" of the growth curve varies from group to group we calculate the ratio of the 3 x 3 determinant for error to that for error + treatments for the three variates $y_1 - \bar{y}$, $y_2 - \bar{y}$, $y_3 - \bar{y}$. We find

$$\Lambda(y_1 - \bar{y},\, y_2 - \bar{y},\, y_3 - \bar{y}) = 0.4366$$

and $(1 - A_1)M = 19.0$ is referred to tables of χ^2 with 6 degrees of freedom. This value is significant ($P < .01$) and we therefore conclude that, not only the mean level, but also the shape of the curve is changing from group to group. A table of mean values indicates the nature of the differences.

MEAN GAINS IN WEIGHT (GRAMS)

Period	Group		
	1	2	3
1st week	24.5	20.3	21.6
2nd week	27.5	29.0	19.5
3rd week	24.1	29.3	12.4
4th week	30.5	30.1	15.8
Mean rate (grams/week)	26.7	27.2	17.3

Further tests show that no significant differences occur between groups 1 and 2, i.e., that the treatment of group 2 is without significant effect, however group 3 differs from the other groups both in average level and in shape. In the first two groups we find a fairly steady rate, which if anything, is tending to increase, whereas a fall in growth rate is found in the third group.

So far the average effects and "shape" effects have been treated separately, but it may be relevant to inquire whether the effects found in the two parts of the analysis can be regarded as separate entities, or whether they are really manifestations of the same thing. In Fig. II it is noticeable that the growth curves of group 3 not only show a low average rate, but also tend to be convex upwards whereas the growth curves of groups 1 and 2 have a higher average and are if anything concave. Now there may also be a tendency *within* the groups, for these curves with low average rates of growth to be also those which are most convex; we may therefore wish to test whether *given the change in mean growth rate*, any differences in "shape" occur, other than would be expected from the internal evidence of the groups concerning the relation between "shape" and mean value. To make the test we use the third property of the Λ criterion mentioned above; the criterion for variables $y_1 - \bar{y}$, $y_2 - \bar{y}$, $y_3 - \bar{y}$ *given* $\bar{y}$ is calculated by dividing the overall criterion for the 4 variables by that for the single variable $\bar{y}$.

$$\begin{aligned}\Lambda(y_1 - \bar{y}, y_2 - \bar{y}, y_3 - \bar{y} : \bar{y}) &= \Lambda(y_1 - \bar{y}, y_2 - \bar{y}, y_3 - \bar{y}, \bar{y})/\Lambda(\bar{y}) \\ &= \Lambda(y_1\,, y_2\,, y_3\,, y_4)/\Lambda(\bar{y}) \\ &= \frac{0.2661}{0.3859} = 0.6896\end{aligned}$$

Since one variable $\bar{y}$ has been eliminated we have $n = 25$, $q = 2$, $p = 3$, and $(1 - A_1)M = 8.18$ is referred to tables of χ^2 with 6 degrees of freedom. The probability of chance occurrence of such a value is about 0.25. We see, therefore, that there is no evidence of differences in shape other than would be expected from the *internal* relation between average and shape.

5. REDUCTION OF THE DATA

In some experiments, weighings are made at very short intervals of time, and the number of points p' for each growth curve is large. Usually however the salient features of the curves will be described by employing fewer than p' constants. Thus to reduce his data Wishart (1938) fitted orthogonal polynomials up to the third degree to the overall weight gain for each of a number of pigs receiving different rations, and analyzed the regression constants. The essential idea is to reduce the data without sacrificing the extra precision given by the larger number of points available. When the method of analysis given here is to be used, this can be done by first applying some process of graduation, to produce a smooth curve through each set of points corresponding to each animal, and analyzing the smoothed values read off from the curves, at a number of equal intervals sufficient to give an adequate description of the curves. This graduation of the data can sometimes be accomplished quite satisfactorily by fitting the curves by eye, but some may prefer a method which is more objective. Since any polynomial of degree p is uniquely determined by specifying $p + 1$ points through which it passes, the division of the smoothed curve into p periods, specified by $p + 1$ points, is equivalent to the description of the curve by a polynomial of degree p. In the growth experiment described here, two weighings were made per week, and the values actually plotted in Fig. II and analyzed, are the means of these pairs of values.

6. ELIMINATION OF THE INITIAL WEIGHT

In the analysis of growth curves the increases in weight of the animals may be correlated with their initial weights. In this case greater precision may be obtained if the analysis is made after elimination of the initial weight by covariance analysis (see for example Fisher 1941). The elimination of y_0, the initial weight, can be accomplished in a precisely similar manner to that used for the elimination of $\bar{y}$ from the criterion for "shape" analysis of 4.3, that is to say we have for the *overall* criterion after the elimination of y_0

$$\Lambda_{1234.0} = \Lambda_{01234}/\Lambda_0$$

This criterion serves to assess the differences in growth rates during the four periods, after the regression of each of these variates on the initial weight has been allowed for, and its significance is assessed as before, the degrees of freedom for error being one less than for the corresponding criterion Λ_{1234}. We shall normally wish to analyze $\Lambda_{1234.0}$ further, however, and to do this we shall need the corresponding matrices of sums of squares and products. These can be found in the manner described in §4.2. The matrices for error and for error plus treatments for the five variates y_0, y_1, y_2, y_3 and y_4 are reduced by a single pivotal condensation based on the element corresponding to the sum of squares for y_0. This gives the desired matrices for the numerator and denominator of $\Lambda_{1234.0}$. Further analysis of the data into differences in mean growth rate and differences in "shape" can be accomplished by operating on the determinants of $\Lambda_{1234.0}$ in precisely the same manner as has already been described for Λ_{1234}.

The two components $\Lambda(\bar{y}:y_0)$ and $\Lambda(y_1 - \bar{y}, y_2 - \bar{y}, y_3 - \bar{y}:y_0)$ assess respectively, the change in overall growth rate when change in initial weight is allowed for, and the change in shape when change in initial weight is allowed for; again apart from the loss of a degree of freedom the tests are the same.

Finally $\Lambda(y_1 - \bar{y}, y_2 - \bar{y}, y_3 - \bar{y}:y_0, \bar{y})$ can be calculated by eliminating $\bar{y}$ in a precisely similar way as before and this criterion will assess whether group differences occur other than can be explained by the relation within the groups between the "shape" and the initial weight and average growth rate.

7. TESTING AN ASSUMPTION IN THE MULTIVARIATE ANALYSIS

Just as in a single-variate analysis of variance the assumption is usually made that the observations are normally distributed about their population mean values with *constant variance* so an analogous assumption that the variates are multinormally distributed about their mean values with constant variance-covariance matrix is made in the multivariate analysis of variance of §4.1. If the variance-covariance pattern changes markedly from group to group, this test may be invalidated. Also in the test of §3.1 concerning the form of the variance-covariance matrix, the sums of squares and products are pooled on the tacit assumption that the variances and covariances do not change from one treatment group to the next. If this were not so an averaging effect might occur so that even though individual groups showed departure from the simple set-up, the overall criterion computed from the pooled error sums of squares and products for all the groups, might show no such departure.

To test the assumption that the matrix of variances and covariances remains constant from one treatment group to the next, the present author (Box 1949) has employed the multivariate analogue of Bartlett's (1937) criterion which is used to test for constancy of variance in the univariate case.

We take as our criterion

$$M = N \log_e | s_{ij} | - \sum_l (\nu_l \log_e | s_{ijl} |)$$

where s_{ijl} is the usual unbiased estimate of variance or covariance between the i-th and j-th variable in the l-th sample based on sums of squares and products having ν_l degrees of freedom and there are k such samples, s_{ij} is the *average* variance or covariance $(\sum_l \nu_l s_{ijl})/N$ and $N = \sum_l \nu_l$ the total of the degrees of freedom. It will be noted that, as usual, the *determinants* of variances and covariances replace the single variances of the univariate criterion. It is perhaps worth noting that again this criterion is invariant under linear transformation of the data, that is to say data which show lack of constancy in variance and covariance cannot be made more homogeneous by linear transformation.

The test will be illustrated for the growth data of rats set out in Table D for the variates y_1, y_2, y_3 and y_4, recording increases in weight in four successive weekly periods.

The individual matrices for sums of squares and products are as follows:

GROUP I

$$\begin{bmatrix} 210.5 & 13.5 & -7.5 & -13.5 \\ 13.5 & 202.5 & 224.5 & 110.5 \\ -7.5 & 224.5 & 310.9 & 117.5 \\ -13.5 & 110.5 & 117.5 & 258.5 \end{bmatrix}$$

$$\nu_1 = 9$$

GROUP II

$$\begin{bmatrix} 111.4 & 83.0 & 78.4 & 39.7 \\ 83.0 & 246.0 & 292.0 & 157.0 \\ 78.4 & 292.0 & 473.4 & 264.7 \\ 39.7 & 157.0 & 264.7 & 174.9 \end{bmatrix}$$

$$\nu_2 = 6$$

GROUP III

$$\begin{bmatrix} 260.4 & -54.0 & -126.4 & -100.8 \\ -54.0 & 160.5 & 110.0 & 77.0 \\ -126.4 & 110.0 & 262.4 & 76.8 \\ -100.8 & 77.0 & 76.8 & 419.6 \end{bmatrix}$$

$$\nu_3 = 9$$

Since $| s_{ijl} | = | c_{ijl} |/\nu_l^p$ the determinants of the variance-covariance matrices can be obtained directly from the determinants for the sums of squares and products and we find

$$\log_e | s_{ij1} | = 11.2700, \log_e | s_{ij2} | = 10.8357, \log_e | s_{ij3} | = 12.7008$$

the determinant for the average variances and covariances is found in a similar way from the total sums of squares and products matrix (9);

$$\log_e | s_{ij} | = 12.4473$$

and $M = 24 \times 12.4473 - 9 \times 11.2700 - 6 \times 10.8357 - 9 \times 12.7008 = 17.9838$. This logarithmic statistic M is a further example of the class discussed in §3.2, and approximations have been derived using the general theory referred to before.

For the χ^2 approximation the following quantities are calculated

$$A_1 = \frac{2p^2 + 3p - 1}{6(p + 1)(k - 1)} \left(\sum_l \frac{1}{\nu_l} - \frac{1}{N} \right) \qquad f_1 = \frac{1}{2}(k - 1)p(p + 1)$$

and $(1 - A_1)M$ is distributed as χ^2 with f_1 degrees of freedom. As before, a more precise approximation, which is useful when some of the degrees of freedom ν_l are small or p and/or k are large can be obtained using tables of F. We calculate

$$A_2 = \frac{(p - 1)(p + 2)}{6(k - 1)} \left(\sum_l \frac{1}{\nu_l^2} - \frac{1}{N^2} \right)$$

and refer M/b to tables of F with f_1 and f_2 degrees of freedom where

$$f_2 = \frac{f_1 + 2}{A_2 - A_1^2} \qquad \text{and} \qquad b = \frac{f_1}{1 - A_1 - f_1/f_2}$$

In this particular example $A_1 = 0.2408, f_1 = 20$, $(1 - A_1)M = 13.5$ is therefore referred to tables of χ^2 with 20 degrees of freedom. The probability for the occurrence of a value as great or greater than this, when the variances and covariances are in fact constant from one group to the next, is thus about 0.85, and there is therefore no reason to doubt the homogeneity of the data in this respect.

This paper originated partly as the result of a note published by O. L. Davies (1947) criticising a method of analysis for growth curves proposed by W. S. Weil (1947).

I am indebted to Dr. Davies for proposing this problem, and to those of my colleagues who were responsible for the investigations which are mentioned. In conclusion I wish to warmly acknowledge the help and guidance I have received from Dr. H. O. Hartley in this work.

SUMMARY

In the analysis of growth and wear curves, the effects can often be simply interpreted by differencing the original data; these differences correspond to the average growth rates during successive periods. If the successive periods are treated as the level of a further factor "periods", the effect of treatments on mean rate is measured by the variation in the period averages and on the "shape" of the rate curve by the interaction of these treatments with "periods". The taking of differences sometimes results, at least approximately, in a very simple covariance pattern for the errors, and the analysis can then be made by a simple application of the technique of the analysis of variance. A test is given which makes it possible to decide whether this simple set-up is contradicted by the data. When the simple set-up is not appropriate, a multivariate extension of

the analysis of variance is used to make the tests. Certain simple properties of the criterion are discussed which facilitate the analysis and the elimination of variables such as initial weight. Finally, it is shown how an important assumption made in the multivariate analysis may be tested.

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TABLE D

INITIAL WEIGHT AND WEEKLY GAINS IN WEIGHT FOR 27 RATS

Group 1. Control						Group 2. Thyroxin						Group 3. Thiouracil					
Rat	y_0	y_1	y_2	y_3	y_4	Rat	y_0	y_1	y_2	y_3	y_4	Rat	y_0	y_1	y_2	y_3	y_4
1	57	29	28	25	33	11	59	26	36	35	35	18	61	25	23	11	9
2	60	33	30	23	31	12	54	17	19	20	28	19	59	21	21	10	11
3	52	25	34	33	41	13	56	19	33	43	38	20	53	26	21	6	27
4	49	18	33	29	35	14	59	26	31	32	29	21	59	29	12	11	11
5	56	25	23	17	30	15	57	15	25	23	24	22	51	24	26	22	17
6	46	24	32	29	22	16	52	21	24	19	24	23	51	24	17	8	19
7	51	20	23	16	31	17	52	18	35	33	33	24	56	22	17	8	5
8	63	28	21	18	24							25	58	11	24	21	24
9	49	18	23	22	28							26	46	15	17	12	17
10	57	25	28	29	30							27	53	19	17	15	18

y_0 represents initial weight of rat
y_1 gain in 1st week
y_2 gain in 2nd week
y_3 gain in 3rd week
y_4 gain in 4th week