TESTING THE HOMOGENEITY OF A SET OF VARIANCES

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1. Introduction

When analysing data the experimenter is frequently faced with the necessity of testing the homogeneity in a set of estimated variances. When it is desired to combine a number of variances to obtain an estimate of the common variance it is necessary to apply such a test. Again, if a selected "treatment mean square" is to be compared with an "error mean square", a test for homogeneity has recently been proposed (Wishart, 1938) as a safeguard against the selection of the largest mean square from a set of random ones.

For general use in such cases Neyman & Pearson (1931) have suggested a test: the L_1 test. The statistic L_1 used in this test has been modified by Bartlett (1937) and generalized by Welch (1935, 1936). From recent work (Nair, 1938; Bishop & Nair, 1939; Pitman, 1939) it would appear that Bartlett's statistic μ is the best to use, because it is unbiased in the sense defined by Neyman & Pearson (1936, 1938). Or more precisely, the L_1 test in its original form is biased with regard to the admissible set of alternatives.

Some difficulty has been experienced in obtaining the random sampling distribution of this statistic which is required for a test. Various approximations have been worked out. There are

(*a*) Bartlett's (1937) approximation using the χ^2 distribution.

(*b*) P. P. N. Nayer's (1936) approximation obtained by fitting Pearson-type curves to the distribution in the special case where all mean squares are based on the same number of degrees of freedom.

(*c*) U. S. Nair's (1938) expansion of the exact distribution in the special case mentioned in (*b*).

(*d*) Recently another paper on the subject has appeared by E. J. G. Pitman (1939). In this paper the author transforms the distribution of L_1 into a multiple integral which can be evaluated in special cases (small values of k) by reduction to elliptic integrals.

The accuracy of the approximations (*a*) and (*b*) has recently been tested (Bishop & Nair, 1939) in the special case in which the expansion (*c*) is available. Bartlett's findings were confirmed; it was shown that his approximation is valid only for moderate or large numbers of degrees of freedom ($\geqslant 3$). We shall also show in this paper that even with this restriction for the degrees of freedom the approximation is not very accurate if k, the number of mean squares in the set, is large.

While U. S. Nair's expansion, although it is very complicated, provides a means of working out the exact probability integral in the special case where all mean squares are based on the same number of degrees of freedom, there is still uncertainty in the general case. P. P. N. Nayer has suggested that the test for homogeneity between k mean squares with f_t degrees of freedom ($t = 1, 2, \ldots, k$) is (under certain conditions) identical with testing the homogeneity between k mean squares *all of which have f degrees of freedom*, where f is the arithmetic mean of the f_t. We shall show that, although there is some truth in this statement, the harmonic mean should be used for f rather than the arithmetic mean.

Since Bartlett's approximation does not provide a test of sufficient accuracy in all cases, the main difficulty of dealing with the general case has been the large number of quantities on which the exact distribution depends: if the k mean squares in the set have f_t degrees of freedom respectively ($t = 1, 2, \ldots, k$) the distribution would depend on $k+1$ quantities. We shall now show in this paper that (provided $f_t \geqslant 2$) there is an approximation of sufficient accuracy which depends on *three* quantities only. These three are:

(i) k, the number of mean squares in the set;

(ii) $c_1 = \sum_{t=1}^{k} \frac{1}{f_t} - \frac{1}{F}$, where $F = \sum_{t=1}^{k} f_t$;

(iii) $c_3 = \sum_{t=1}^{k} \frac{1}{f_t^3} - \frac{1}{F^3}$.

This makes the distribution amenable to tabulation, so that the test can be reduced to an inspection of a table of 5 % and 1 % points which can easily be carried out by the experimenter.

In the case where mean squares having one degree of freedom occur in the set, the distribution is of a more complicated character, but our approximation is still fair.

2. The formal solution

Consider k normal populations with variances σ_t^2 ($t = 1, 2, \ldots, k$). Let s_t^2 be an unbiased estimate of σ_t^2 based on f_t degrees of freedom, and let us denote by F the total number of degrees of freedom,

$$F = \sum_{t=1}^{k} f_t. \qquad \ldots\ldots(1)$$

Bartlett's statistic μ is then given by

$$-2\log\mu = F\log\{\sum_t (f_t s_t^2)/F\} - \sum_t f_t \log s_t^2. \qquad \ldots\ldots(2)$$

The equivalence to a special case of the generalized L_1 statistic (Welch, 1935, 1936) is expressed by the relation

$$F\log L_1' = 2\log\mu, \qquad \ldots\ldots(3)$$

where $$L_1' = \prod_{t=1}^{k}\left(\frac{F}{f_t}\right)^{f_t/F}\prod_{t=1}^{k}\left\{\frac{f_t s_t^2}{\sum_t f_t s_t^2}\right\}^{f_t/F}. \qquad \ldots\ldots(4)$$

For our test we require the random sampling distribution $\phi(L_1')$ of the statistic L_1' under the null hypothesis
$$\sigma_t^2 = \sigma^2, \quad t = 1, 2, \ldots, k.$$

Under these conditions it has recently been shown (Welch, 1936) that the $(q-1)$th sampling moment of L_1' is given by

$$M_{q-1} = \int_0^1 \phi(L_1')\, L_1'^{\,q-1}\, dL_1'$$
$$= \prod_{t=1}^{k}\left(\frac{F}{f_t}\right)^{\{(q-1)f_t\}/F}\frac{\Gamma(\frac{1}{2}F)}{\Gamma(\frac{1}{2}F+q-1)}\prod_{t=1}^{k}\left\{\frac{\Gamma(\frac{1}{2}f_t+\{(q-1)f_t\}/F)}{\Gamma(\frac{1}{2}f_t)}\right\}. \qquad \ldots\ldots(5)$$

From general principles it may now be inferred that equation (5) is valid for all complex q with
$$\text{Real Part of } q > 1.$$

Further, by Mellin's inversion formula, we obtain from (5)

$$\phi(L_1') = \Gamma(\tfrac{1}{2}F)\prod_{t=1}^{k}\Gamma(\tfrac{1}{2}f_t)^{-1}$$
$$\times\frac{1}{2\pi i}\int_{Q-i\infty}^{Q+i\infty}\prod_{t=1}^{k}\left[\left(\frac{F}{f_t}\right)^{\{f_t(q-1)\}/F}\Gamma\left[f_t\left(\frac{1}{2}+\frac{q-1}{F}\right)\right]\right]\frac{L_1'^{-q}\,dq}{\Gamma(\frac{1}{2}F+q-1)}, \qquad \ldots\ldots(6)$$

where Q (>1) is an arbitrary positive quantity.

Introducing as a new statistic
$$x = -F\log L_1' = -2\log\mu, \qquad \ldots\ldots(7)$$

and as a new variable of integration
$$\lambda = \tfrac{1}{2} + (q-1)/F,$$

we obtain for the distribution function of x ($\psi(x)$ say)

$$\psi(x) = \Gamma(\tfrac{1}{2}F)\prod_{t=1}^{k}\left(\frac{F}{f_t}\right)^{-\frac{1}{2}f_t}\Gamma(\tfrac{1}{2}f_t)^{-1}e^{-\frac{1}{2}x}$$
$$\times\frac{1}{2\pi i}\int_{\Lambda-i\infty}^{\Lambda+i\infty}\prod_{t=1}^{k}\left\{\left(\frac{F}{f_t}\right)^{\lambda f_t}\Gamma(\lambda f_t)\right\}\frac{e^{x\lambda}}{\Gamma(F\lambda)}d\lambda, \qquad \ldots\ldots(9)$$

where Λ is an arbitrary positive quantity.

Using now Binet's integral representation of $\log\Gamma$ (Whittaker & Watson, 1927, p. 249), we may write equation (9) in the form

$$\psi(x) = (\tfrac{1}{2})^{\frac{1}{2}(k-1)}e^{-E(\frac{1}{2})}e^{-\frac{1}{2}x}\times\frac{1}{2\pi i}\int_{\Lambda-i\infty}^{\Lambda+i\infty}\lambda^{-\frac{1}{2}(k-1)}e^{x\lambda}e^{E(\lambda)}d\lambda, \qquad \ldots\ldots(10)$$

where $$E(\lambda) = \int_0^\infty\left(\frac{1}{2}-\frac{1}{\tau}+\frac{1}{e^\tau-1}\right)\frac{1}{\tau}\left[\sum_{t=1}^{k}e^{-\tau f_t\lambda}-e^{-\tau F\lambda}\right]d\tau. \qquad \ldots\ldots(11)$$

Introducing
$$g(\tau) = \left(\frac{1}{2} - \frac{1}{\tau} + \frac{1}{e^\tau - 1}\right)\frac{1}{\tau}, \qquad \ldots\ldots(12)$$

we see that $g(\tau)$ has continuous derivatives of any order for $0 \leqslant \tau \leqslant \infty$. We may therefore transform the integral (11) by integration by parts, differentiating $g(\tau)$ and integrating the exponential functions. We obtain

$$E(\lambda) = \frac{1}{12\lambda}\left[\sum_{t=1}^{k}\left(\frac{1}{f_t}\right) - \frac{1}{F}\right] - \frac{1}{360\lambda^3}\left[\sum_{t=1}^{k}\left(\frac{1}{f_t^3}\right) - \frac{1}{F^3}\right] + \frac{1}{\lambda^3}\int_0^\infty \frac{d^3 g(\tau)}{d\tau^3}\left[\sum_{t=1}^{k}\left(\frac{e^{-\tau\lambda f_t}}{f_t^3}\right) - \frac{e^{-\tau\lambda F}}{F^3}\right]d\tau. \qquad \ldots\ldots(13)$$

We now approximate to $E(\lambda)$, and therefore to $\psi(x)$, by ignoring the last summand in equation (13), and write

$$E(\lambda) \cong \frac{c_1}{12\lambda} - \frac{c_3}{360\lambda^3}, \qquad \ldots\ldots(14)$$

where
$$c_1 = \sum_{t=1}^{k}\left(\frac{1}{f_t}\right) - \frac{1}{F} \quad \text{and} \quad c_3 = \sum_{t=1}^{k}\left(\frac{1}{f_t^3}\right) - \frac{1}{F^3}. \qquad \ldots\ldots(15)$$

It can be shown that this approximation is sufficient for all practical purposes provided

$$f_t \geqslant 2, \quad t = 1, 2, \ldots, k.$$

Substituting (14) in (10), expanding $e^{E(\lambda)}$ and integrating the single terms we obtain*

$$\psi(x) \cong 2^{-\frac{1}{2}(k-1)} e^{-\frac{1}{6}c_1 + \frac{1}{45}c_3} \times \sum_{i=0}^{\infty} \alpha_i 2^{-i} \Gamma\left(\frac{k-1}{2} + i\right)^{-1} x^{\frac{1}{2}(k-3)+i} e^{-\frac{1}{2}x}, \qquad \ldots\ldots(16)$$

where the α_i are the coefficients of the expansion

$$e^{\frac{1}{6}c_1 t - \frac{1}{45}c_3 t^3} \qquad \ldots\ldots(17)$$

in ascending powers of t.

From (16) it is obvious that (to the degree of accuracy considered) the distribution of x is a weighted sum of χ^2 distributions with degrees of freedom ranging between $k-1$ and ∞. We now denote by $P_j(X)$ the probability integral of χ^2 based on j degrees of freedom, i.e. we introduce

$$P_j(X) = \Gamma\left(\frac{j}{2}\right)^{-1} 2^{-\frac{1}{2}j} \int_X^\infty x^{\frac{1}{2}(j-2)} e^{-\frac{1}{2}x}\,dx. \qquad \ldots\ldots(18)$$

We further denote by $P(X)$ the probability integral of our variate x defined in (7), viz.

$$P(X) = \int_X^\infty \psi(x)\,dx. \qquad \ldots\ldots(19)$$

* We make use of the well-known integral representation of $1/\Gamma(z)$, viz.

$$\{\Gamma(z)\}^{-1} = \frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} e^\rho \rho^{-z}\,d\rho.$$

From equation (16) we obtain by integration

$$P(X) = \sum_{i=0}^{\infty} \alpha_i P_{k-1+2i}(X) \times \left(\sum_{i=0}^{\infty} \alpha_i\right)^{-1}, \qquad \ldots\ldots(20)$$

where the α_i are the coefficients of the expansion of

$$e^{\frac{1}{2}c_1 t - \frac{1}{12}c_3 t^3} \qquad \ldots\ldots(21)$$

in ascending powers of t.

3. Tabulation of percentage points

Equation (20) provides a means of calculating tables of the probability integral $P(X)$ (or its 5 % and 1 % points). For the quantities $P_j(X)$ are given by Elderton's tables of the probability integral of χ^2, while the coefficients α_i are readily obtained from the expansion of (21).

For practical purposes tables of the 5 % and 1 % points could be prepared. These percentage points would depend on three quantities, viz.

$$k, \quad c_1 = \sum_{t=1}^{k}\left(\frac{1}{f_t}\right) - \frac{1}{F} \quad \text{and} \quad c_3 = \sum_{t=1}^{k}\left(\frac{1}{f_t^3}\right) - \frac{1}{F^3}.$$

The effect of c_3 is small, and it would be convenient to make k and c_1 the respective row and column headings of two-way tables of percentage points, and to prepare such tables for two or three selected values of c_3. It is hoped to prepare such tables shortly.

4. Comparison with U. S. Nair's expansion

It would lead us too far afield if we gave here a complete mathematical proof of the accuracy of the approximation (20). It is, however, of interest to check the accuracy in a few cases numerically. U. S. Nair's expansion mentioned above will be used for this check. The most stringent test of the accuracy of equation (20) is established by choosing the f_t small and k large. Numerical results have been obtained from U. S. Nair's expansion (2) in the case $f_t = f = 2$; $k = 10$. The result of the test is given below.

Lower percentage points of $L_1' = e^{-X/F}$

	5% point	1% point
(a) Bartlett's approximation	0·367	0·277
(c) U. S. Nair's expansion	0·375	0·288
(d) Equation (20)	0·378	0·291

The agreement between U. S. Nair's expansion (c) and equation (20), (d) is satisfactory in this case where the approximation would be expected to be worst. For comparison, Bartlett's approximation (a) is also shown.

5. The relation between the special case $f_t = f$, $t = 1, 2, \ldots, k$ and the general case

P. P. N. Nayer has considered the general case, and has provided some evidence for believing that this case can be reduced to the special case provided that the f_t are not too small and not too dissimilar in value. He has suggested using the mean of the f_t as a substitute for the common value f. It is easy to see from the approximation (20) that there is some truth in Nayer's conjecture. However, it is not correct to use the arithmetic mean. The correct value is given by

$$f = \left(k - \frac{1}{k}\right)\left[\sum_{t=1}^{k}\left(\frac{1}{f_t}\right) - \frac{1}{F}\right]^{-1}, \qquad \ldots\ldots(22)$$

and is approximately equal to the harmonic mean of the f_t. If all $f_t \geqslant 4$, the general case of unequal f_t can *always* be reduced to the special case $f_t = f$, no matter how dissimilar the f_t. For if in equation (21) we replace c_3 by $\frac{1}{16}c_1$, and consider the function

$$e^{\frac{1}{6}c_1(t - \frac{1}{16}t^2)},$$

we find that the coefficients of this function when expanded in ascending powers of t will be approximations of sufficient accuracy to the coefficients α_t in (20). The probability integral of X is therefore determined by the quantities k and c_1, so that the identity of the general case and the special case is obvious, provided f is defined by (22).

6. Some remarks on Bartlett's approximation

Bartlett (1937) has given an approximation to the distribution of

$$-2\log\mu = x.$$

He suggests as an approximate test that we enter the table of χ^2 for $k-1$ degrees of freedom with the statistic

$$3x(k-1)/c_1,$$

where c_1 is given by (15).

It can be shown that this approximation is equivalent to equation (20) provided $\frac{1}{6}c_1$ is small, so that higher order terms in the expansion (20) may be ignored. For large or moderate values of $\frac{1}{6}c_1$, however, discrepancies may occur even if all mean squares are based on moderate or large numbers of degrees of freedom. We shall confine ourselves here to demonstrating this with the help of a single example, viz. $f = 5$ and $k = 30$. While for values of k of this order U. S. Nair's expansion is very complicated, equation (20) yields results which are accurate to 3 figures. Below are given the probabilities of exceeding Bartlett's 5 % and 1 % values; they are

	5 % level	1 % level
True $P(X)$	0·047	0·0081

Thus Bartlett's approximation has an error of 6 % and 19 % respectively.

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