THE

MESSENGER OF MATHEMATICS.

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CONTENTS.

MESSENGER OF MATHEMATICS.

ON CERTAIN DEFORMABLE FRAMEWORKS.

By *A. C. Dixon, Sc.D.*

§ 1. THE subject of this article was suggested by the theorem—given by Weill in Vol. IV. of the third series of *Liouville's Journal*, p. 269—that the centre of mean position of the points of contact of m successive sides of a polygon inscribed in one fixed circle and circumscribed to another always lies on another fixed circle.*

This theorem can be extended to the case when the successive sides touch different fixed circles, all the given circles being coaxal.

The article consists of three sections: in the first the construction of a class of deformable frameworks is explained; in the second this construction is connected with the theory of coaxal circles; in the third an answer is sought to the question whether there are any deformable frameworks of eight rods with four joints and four fixed pivots other than those thus constructed.

Section I.

§ 2. LEMMA. If the base AB of a pentagon $ABCDE$ is fixed, and the sides are of given lengths, and the angles C and E are equal, then the locus of D is a circle.

Let $\qquad AE=a,\ BC=b,\ CD=c,\ DE=d,$

then $\qquad AD^2=a^2+d^2-2ad\cos AED,$

$$BD^2=b^2+c^2-2bc\cos BCD,$$

* See also a paper in the *Mathematical Gazette* for October, 1897, pp. 121—23.

and thus $$bc.AD^2 - ad.BD^2 = (ab - cd)(ac - bd),$$

if the angles AED, DCB are equal.*

Thus the locus of D is a circle whose centre K divides AB externally in the ratio $ad : bc$.

Now let the parallelogram $CDEF$ be completed; the angles FEA, BCF are equal, and thus the locus of F is a circle whose centre L divides AB externally in the ratio $ac : bd$.

If now the lines KD, AE, LF, BC and the sides of the parallelogram are taken to be rods with joints at C, D, E, F, and fixed pivots at K, A, L, B, it follows that the framework will have one degree of freedom.

Such a framework of eight rods I shall in this article call *complete*.†

§ 3. Let $KD = e$, $LF = f$, then we have

$$(bc - ad)(e^2 - AK.BK) = (ab - cd)(ac - bd),$$

so that

$$e^2 (bc - ad)^2 = (ab - cd)(ac - bd)(bc - ad) + abcd.AB^2.$$

Also $f^2 (bd - ac)^2 =$ the same expression.

Now $$AK = \frac{ad}{ad - bc} AB, \quad AL = \frac{ac}{ac - bd} AB,$$

so that we have KL divided externally by A in the ratio $ed : fc$, and similarly by B in the ratio $ec : fd$, if the signs of e and f are so taken that

$$e(ad - bc) = f(ac - bd).$$

Hence we find

$$cf.KE^2 - de.LE^2 = (ef - cd)(ce - df),$$
$$df.KC^2 - ce.LC^2 = (ef - cd)(de - cf).$$

Thus the figure might have been constructed by taking KL as the fixed base, and $KDCFL$ or $KDEFL$ as the pentagon; and the angles KDC, CFL are equal, as also KDE, EFL, due regard being paid to the signs of e and f.

* In this article the expression, "the angle ABC," will be understood to denote the amount of rotation counter-clockwise that would bring BA into the position BC; and further it will be the angle between the positive directions of measurement along BA, BC, so that if one of the two—BA, BC—is a rod of negative length, "the angle ABC" will be numerically the supplement of what would generally be meant by the phrase.

† If the parallelogram is a rhombus, either A must coincide with B, or K with L; and the framework then consists of a Peaucellier cell with two additional rods.

§ 4. It is possible by means of the above to construct deformable frameworks of more than eight rods in which every movable joint is attached by a rod to some fixed pivot.

Complete the parallelogram $KDCG$.

Let us use the notation (PQ) to denote the end of a line drawn from a fixed point parallel and equal to PQ, and in the same sense. Then $KGCF$ is a variable broken line starting from a fixed point K whose vertices G, C, F move on fixed circles, while the point (GF), that is, E, moves on another. Now take a rod FH of any length and suppose it jointed on at F; then another rod can be taken of suitable length, having one end fixed at some point in the line AB, and the other jointed on at H, and such that the locus of each of the points (GH), (CH) is a circle.

To prove this, complete the parallelograms $KGFE$, $HFEa$. Then a may be taken as the point (GH), so that $HFEa$ is to be a parallelogram belonging to a complete framework of eight rods, and the problem is to construct such a framework, having given, in the original notation, the points A, K, and the lengths e, a, d, c. From these data f, b, and the points B, L may be found;* there are two solutions, real, coincident, or imaginary.

We have to shew that the locus of the point (CH) is also a circle. Let M be the centre of the locus of H. Complete the parallelograms $BCFb$, $HFbc$. Then the angles AEF, FHM are equal from the nature of the complete framework to which $HFEa$ belongs. Also angle $AEF = FCB = BbF$. Thus $FHM = BbF$ in all positions, and $HFbc$ moves as part of a complete framework, so that the locus of c, that is (CH), is a circle.

§ 5. In this way it is possible to joint together n rods, say A_0A_1, A_1A_2, A_2A_3, ..., $A_{n-1}A_n$, and to add such constraints that each joint shall be free to move in a circle, and also that each of the points (A_rA_s) shall describe a circle, A_0 being a fixed point. This has been proved when $n=4$, KG, GC, CF, FH being such a system.

Let C_r denote the pivot round which A_r moves. Complete

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* The problem may sometimes be solved geometrically as follows:—Let the circle with centre K and radius e cut the circles with centre A and radii $a-d$, $a+d$ in P, P' and Q, Q' respectively. With Q as centre and $2c$ as radius describe a circle and find a circle touching this one and passing through P, P'. The centre of this last circle is the required point B, and the length $b = BP + c = BQ - c$, the signs of BP, BQ being so taken that these values are the same. This construction depends on the two limiting forms of the pentagon in which A, E, D are collinear and also B, C, D.

the parallelogram $C_rA_rA_sA_{r,s}$. Then the locus of $A_{r,s}$ is to be a circle, whose centre is $C_{r,s}$, say. The parallelogram $A_{s-1}A_sA_{r,s}A_{r,s-1}$ with the four lines C_sA_s, $C_{s-1}A_{s-1}$, $C_{r,s}A_{r,s}$, $C_{r,s-1}A_{r,s-1}$ shall form a complete framework of eight rods for all values of r and s, such that $r < s-1$.

This has been proved for the values 3 and 4 of n. Suppose it true for the value $n-1$. Then the pivot C_n, and the length C_nA_n, must be so chosen that the parallelogram $A_{n-1}A_nA_{1,n}A_{1,n-1}$ shall move as part of a complete framework. Thus the angles $C_nA_nA_{n-1}$ and $A_{n-1}A_{1,n-1}C_{1,n-1}$ are always equal. But now by hypothesis $A_{n-2}A_{n-1}A_{1,n-1}A_{1,n-2}$ forms part of a complete framework, so that the angles $A_{n-1}A_{1,n-1}C_{1,n-1}$ and $C_{n-2}A_{n-2}A_{n-1}$ are equal, that is, $C_nA_nA_{n-1} = C_{n-2}A_{n-2}A_{n-1}$.

Again, from the complete framework $A_{n-2}A_{n-1}A_{r,n-1}A_{r,n-2}\ldots$,

$$C_{n-2}A_{n-2}A_{n-1} = A_{n-1}A_{r,n-1}C_{r,n-1},$$

that is
$$C_nA_nA_{n-1} = A_{n-1}A_{r,n-1}C_{r,n-1},$$

or $A_{n-1}A_nA_{r,n}A_{r,n-1}$ moves as part of a complete framework with pivots C_{n-1}, C_n, $C_{r,n}$, $C_{r,n-1}$. This holds if $r < n-2$.

If $r = n-2$, $C_{r,n-1}$ is C_{n-2}, and

$$A_{n-1}A_{n-2,n-1}C_{n-2,n-1} = C_{n-2}A_{n-2}A_{n-1},$$

being opposite angles of a parallelogram.

Hence the thing is true for any value of r less than $n-1$, supposing it to be true when $n-1$ is put for n. But it is true when n is 3 or 4, and therefore universally.

§ 6. Let us use ρ_r and ρ_{rs} to denote the lengths of the rods C_rA_r and $C_{rs}A_{rs}$ respectively, with their proper signs. The lengths such as $\rho_{r,r+1}$ are assumed arbitrarily in succession; and the others are determined by them as just shown, except in the case of ρ_{rs}, where $r > s$. But now it is easily seen that $A_{rs}A_{sr}$ and C_sC_r are the diagonals of a parallelogram, and bisect each other, so that $\rho_{rs} = \pm\rho_{sr}$ and $C_{rs}C_{sr}$ is also bisected in the same point, and $C_{rs}A_{rs}$ is parallel to $C_{sr}A_{sr}$.

We have also ($r < n-1$ as before)

$$A_{r,n}A_nC_n = C_{r\ n-1}A_{r,n-1}A_{r,n},$$

and
$$C_{r,n}A_{r,n}A_n = A_nA_{n-1}C_{n-1},$$

and therefore by addition

$$C_{n,r}A_{n,r}A_r = C_{n-1,r}A_{n-1,r}A_r,$$

if we take that
$$\rho_{rs} = (-1)^{r+s+e}\rho_{sr},$$

where e is either 0 or 1, and does not depend on r and s.

Thus it follows that

$$C_{n,r}A_{n,r}A_r = C_{r+1,r}A_{r+1,r}A_r,$$
$$= A_rA_{r+1}C_{r+1}, \text{ if } e = 1,$$
$$= C_{s,r}A_{s,r}A_r \ (s < r)$$

by the former work.

§ 7. Now let r, s, t be any integers, and we see that

$$C_{r,s}A_{r,s}A_s = A_sA_{s-1}C_{s-1},$$
$$= C_{t,s}A_{t,s}A_s,$$
$$= A_sB_{t,s}C_{t,s}$$

if we complete the parallelograms $C_{t,s}$, $A_{t,s}$, A_s, $B_{t,s}$, and A_t, C_t, $C_{t,s}$, $B_{t,s}$. Thus $C_{r,s}A_{r,s}A_sB_{t,s}C_{t,s}$ is a pentagon whose base is fixed, its sides are constant, and its side angles are equal, and a new complete framework may be constructed by forming the parallelogram $A_{r,s}A_sB_{t,s}A_{r,s,t}$, and connecting its vertices respectively with pivots at $C_{r,s}$, C_s, $C_{t,s}$, and a fourth point which may be called $C_{r,s,t}$.

Section II.

§ 8. If $ABCD$ is a quadrilateral with sides of constant length on a given base AB, and if PQR is a triangle inscribed in a given circle whose sides QR, RP, PQ are parallel respectively to BC, CD, DA, then the envelope of each side of PQR is a circle.

Let O be the centre of the circle PQR, and EOF the diameter perpendicular to AB. From E, F draw ES, FT perpendicular to PQ.

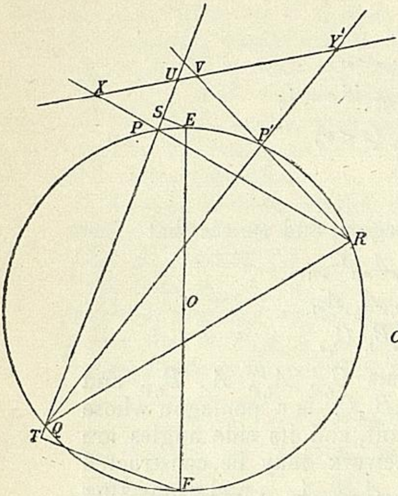
Then
$$ES + FT = EF \cos PRQ,$$
$$= EF \cos BCD,$$
$$ES - FT = EF \cos OES,$$
$$= - EF \cos BAD.$$

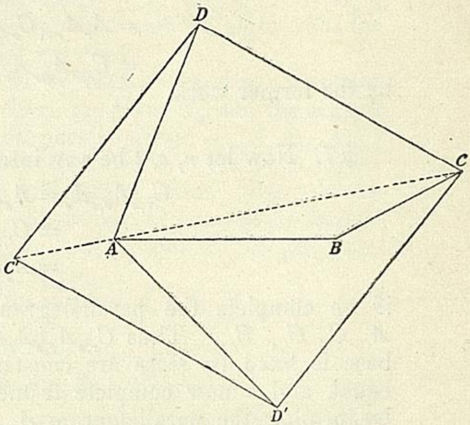
But writing a, b, c, d for AB, BC, CD, DA, we have

$$b^2 + c^2 - 2bc \cos BCD = BD^2 = a^2 + d^2 - 2ad \cos BAD.$$

Hence $(bc + ad)ES + (bc - ad)FT$ is constant, so that PQ always touches a fixed circle whose centre divides EF in the ratio $bc - ad : bc + ad$.

§ 9. Now let D' be the other point such that $CD' = c$, $AD' = d$, and let RP', $P'Q$ be parallel to AD', $D'C$ respectively.





Then RP' is another position of PQ, and touches the same fixed circle, as may be easily verified. Let U, V be the points of contact of PQ, $P'R$. Then UV must be either parallel or perpendicular to AC, and an examination of the signs of ES and FT in the two positions shows that it must be parallel. UV is equally inclined to PR and QP'; suppose it to meet them in X, Y, and let a circle be described to touch PR and QP' in X, Y. Then

$PX : PU :: c : d$, since PXU, DCA are similar;

$P'Y : P'V :: c : d$, „ $P'YV$, $D'CA$ „ „

$QY : QU :: c : d$, „ QUY, DAC' „ „

$RX : RV :: c : d$, „ RVX, $D'AC'$ „ „

C being the fourth vertex of the parallelogram $DCD'C'$. Thus P, P', Q, R are four points on the same circle of a system coaxal with the circles UV and XY; and as the circles PQR and UV are fixed, and the ratio $c : d$ is a constant, the circle XY must also be fixed.

Thus PR envelopes a third fixed circle coaxal with PQR and the envelope of PQ.

QR may be proved to envelope a circle in the same way as PQ was, and the envelopes of QR and RP are circles coaxal with PQR just as those of PQ and RP are. Hence all the four circles are coaxal. The tangents from any point of the circle PQR to the three envelopes are in the ratios $b:c:d$.*

The foregoing proof may be reversed so as to show that if a triangle is inscribed in a circle, and moves in such a way that two sides touch fixed coaxal circles, then the sides are constantly parallel to those of a quadrilateral on a fixed base, and with sides of constant length, while the third side of the triangle touches another circle of the coaxal system.

§ 10. Take now the figure of § 5. In a fixed circle inscribe a triangle $\beta_0\beta_1\beta_2$, such that $\beta_0\beta_1$ is parallel to A_0A_1, $\beta_1\beta_2$ to A_1A_2, $\beta_2\beta_0$ to A_2C_2. From β_2 draw the chord $\beta_2\beta_3$ parallel to A_2A_3, then since $A_0A_1A_2 = C_3A_3A_2$, it follows that $\beta_0\beta_3$ is parallel to C_3A_3; and since $C_2A_2A_3 = A_3A_{13}C_{13}$, $\beta_1\beta_3$ is parallel to $C_{13}A_{13}$.

From β_3 draw the chord $\beta_3\beta_4$ parallel to A_3A_4, then in the same way $\beta_0\beta_4$ is parallel to C_4A_4, $\beta_1\beta_4$ to $C_{14}A_{14}$, and $\beta_2\beta_4$ to $C_{24}A_{24}$. In this way we find a series of points $\beta_0\beta_1\beta_2, \ldots$ on the circle such that $\beta_0\beta_r$ is parallel to C_rA_r, and $\beta_r\beta_s$ to $C_{rs}A_{rs}$. The envelope of any of the chords is a fixed circle belonging to a certain coaxal system, whose radical axis is parallel to the line of pivots.

§ 11. Consider the points of contact of the sides of the polygon $\beta_0\beta_1\ldots\beta_n$ with their envelopes. The radius drawn to any point of contact is of constant length, and is perpendicular to the corresponding piece of the line $A_0A_1\ldots A_nC_n$. Hence, since A_0 and C_n are fixed points, it follows that the centre of mean position of the points of contact for a certain system of multiples is a fixed point. The multiplier for any point of contact is proportional to the radius of its circle inversely, and to the tangent from a fixed point of the circle $\beta_0\beta_1\beta_2\ldots$ to its circle directly.

If we take only m consecutive sides of this polygon, say $\beta_0\beta_1$, $\beta_1\beta_2$, …, $\beta_{m-1}\beta_m$, then since the locus of A_m is a circle, that of the centre of mean position of the m points of contact for the same system of multiples is also a circle. This result includes the theorem of Weill quoted above.

Section III.

§ 12. The question is now suggested whether there are any other cases in which the four corners of a jointed quadrilateral of rods can move on circles.

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* A remarkable property is that the length QU is proportional to the area of the quadrilateral $ABCD$.

Now if two points on two circles are at a constant distance from each other their parameters are connected by a biquadratic relation. The question is thus included in the more general one of finding all cases in which four variables are connected chainwise by four biquadratic equations without being completely determined.

Let the variables be t, x, y, z and the equations

$$(t)^2(x)^2=0,$$

$$(x)^2(y)^2=0,$$

$$(y)^2(z)^2=0,$$

$$(z)^2(t)^2=0.$$

From these it will generally be possible to determine two, say y and z, rationally in terms of the others, and thus the four are all doubly periodic functions of the same argument, with the same periods. It is clear also that each is a function of the second order, that is, has two infinities within the parallelogram of periods. It is known that four such variables would be connected in pairs by biquadratic equations (I.).

§ 13. Now suppose that y and z cannot be expressed rationally in terms of x and t. The expressions for them might be sought as follows. From the equations $(y)^2(z)^2=0$, $(x)^2(y)^2=0$, find y rationally in terms of z and x, and substitute in either so as to eliminate y. The result is of the form

$$(x)^4(z)^4=0,$$

a biquadratic in x and z. By means of the equation $(z)^2(t)^2=0$ reduce the degree in z of this biquartic to one, and thence find z in terms of x and t, and substitute in the expression for y.

This process may fail in two ways: either the equations $(y)^2(z)^2=0$, $(x)^2(y)^2=0$ may, when solved for y, have both roots common, or the equation $(z)^2(t)^2=0$ for z may have both its roots in common with $(x)^4(z)^4=0$.

Let us suppose that

$$(y)^2(z)^2=0, \quad (x)^2(y)^2=0$$

are the same quadratic for y; let these equations, arranged in powers of y, be

$$Z_0y^2+Z_1y+Z_2=0, \quad X_0y^2+X_1y+X_2=0.$$

Thus
$$\frac{Z_0}{X_0}=\frac{Z_1}{X_1}=\frac{Z_2}{X_2}.$$

Hence z and x are connected by the equations

$$Z_0X_1 - Z_1X_0 = 0, \quad Z_0X_2 - Z_2X_0 = 0,$$

biquadratics which must have a common bilinear factor or be identically the same as z and x are variable.

In the latter case there are identities of the form

$$aZ_0 + bZ_1 + cZ_2 = 0,$$

$$aX_0 + bX_1 + cX_2 = 0,$$

a, b, c being constants.

Thus also

$$cy - b : a - cy^2 : by^2 - ay :: Z_0 : Z_1 : Z_2$$
$$:: X_0 : X_1 : X_2.$$

These biquadratics connecting x, y, z are of a special form, and by linear fractional transformations of x, y, z may be reduced to bilinear equations in x^2, y^2, z^2.

Now t can be expressed rationally in terms of x and z unless the equations

$$(t)^2 (x)^2 = 0, \quad (z)^2 (t)^2 = 0$$

are of the same special form, so that by transformation of t they may be made bilinear in t^2 and x^2, t^2 and z^2 respectively. We have then two cases—

(II.) x^2, y^2, z^2, t^2 are connected in pairs by bilinear equations. Three of these may be chosen arbitrarily, say those connecting x^2 and y^2, y^2 and z^2, z^2 and t^2. The consequent relation between t^2 and x^2 will then be bilinear and will complete the set.

(III.) t, x, z are doubly periodic functions of the same argument u, with the same periods, the relation between x and z being bilinear in x^2 and z^2; y^2 is connected with x^2 or z^2 by a bilinear relation. In this case x and z are, save as to a constant factor, the same function of arguments differing by a quarter-period.

§ 14. If the biquadratics $Z_0X_1 - Z_1X_0$, $Z_0X_2 - Z_2X_0$ have a common bilinear factor, it is also a factor in $Z_1X_2 - Z_2X_1$, and the vanishing of this factor may express the relation between z and x. If this is so then z is a linear fractional function of x, and the equations $(y^2)(z)^2 = 0$, $(z)^2(t)^2 = 0$ are immediate consequences of $(x)^2(y)^2 = 0$ and $(t)^2(x)^2 = 0$ respectively by substitution for x of its value in terms of z.

(IV.) If the bilinear factor does not vanish, then the relation between x and z is biquadratic and the case is included either in (I.) or (III.).

§ 15. Let us now take the other alternative and suppose that $(z)^2(t)^2$, as a quadratic in z, is a factor of $(x)^4(z)^4$. If it is a factor rational in x this case is a repetition of (II.) or (III.). Excluding these and (IV.) we have that z can be expressed rationally in terms of y and t, so that its expression in terms of x contains only two simple radicals $P^{\frac{1}{2}}$ and $Q^{\frac{1}{2}}$ where P and Q are quartics. Thus we may write

$$z = \frac{A + BP^{\frac{1}{2}} + CQ^{\frac{1}{2}} + P^{\frac{1}{2}}Q^{\frac{1}{2}}}{E},$$

where A, E, P, Q are quartics in x and B, C quadratics,* E is the coefficient of z^4 in $(x)^4(z)^4$, and $B^2 - Q$, $C^2 - P$, $A - BC$ are constant multiples of E.

Suppose $P^{\frac{1}{2}}$ to be the radical that occurs in the value of y, and $Q^{\frac{1}{2}}$ that which occurs in t. Let

$$A = BC + \lambda E, \quad P = C^2 + \mu E, \quad Q = B^2 + \nu E.$$

Then $(z)^2(t)^2 = 0$ must be the same quadratic for z as

$$(Ez - A - CQ^{\frac{1}{2}})^2 = P(B + Q^{\frac{1}{2}})^2,$$

or $$Ez^2 - 2z(A + CQ^{\frac{1}{2}}) + \lambda(A + BC) - \mu(B^2 + Q) + 2(\lambda C - \mu B)Q^{\frac{1}{2}} = 0.$$

Thus $(A + CQ^{\frac{1}{2}})/E$ must be equal to a fraction whose numerator and denominator are quadratics in t. Constants α and β can now be so chosen that when

$$(A + CQ^{\frac{1}{2}} + \alpha E)/(A + CQ^{\frac{1}{2}} + \beta E)$$

is expressed in terms of t its numerator and denominator are perfect squares, so that we may write

$$\frac{A + \alpha E + CQ^{\frac{1}{2}}}{A + \beta E + CQ^{\frac{1}{2}}} = \left(\frac{F + Q^{\frac{1}{2}}}{G}\right)^2,$$

F and G being quadratics in x.

The denominator of the expression on the left may be made rational. Thus we find

$$\frac{E(\alpha+\lambda)(\beta+\lambda) + (\alpha+\beta+2\lambda)BC - \nu C^2 + (\beta-\alpha)CQ^{\frac{1}{2}}}{E(\beta+\lambda)^2 + 2(\beta+\lambda)BC - \nu C^2} = \frac{(F+Q^{\frac{1}{2}})^2}{G^2}.$$

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* In order to avoid exceptional cases, we suppose z to have undergone an arbitrary linear transformation. The occurrence of a squared factor in P or Q will not affect the argument.

In this identity the denominators and also the numerators can only differ by a constant factor. Hence F must be a constant multiple of C, say $F=\gamma C$.

Also

$$E(\alpha+\lambda)(\beta+\lambda)+(\alpha+\beta+2\lambda)BC-\nu C^2=\frac{\beta-\alpha}{2\gamma}(F^2+Q)$$

$$=\tfrac{1}{2}(\beta-\alpha)\gamma C^2+\frac{\beta-\alpha}{2\gamma}(B^2+\nu E).$$

That is, E is equal to a quadratic in B and C unless the coefficient of E in this equation vanishes, and then B and C differ only by a constant factor. In the latter case a comparison of the denominators shows that E is a quadratic function of C and G. In either case E and consequently A, P, Q are quadratic functions of two quadratics in x. The same is true of the coefficient of each power of z in $(x)^4(z)^4=0$, so that, by a linear transformation of x, this may be written

$$(x^2)^2(z)^4=0.$$

Further, G^2 is now a quadratic in x^2, so that G is either a linear function of x^2, or a mere multiple of x.

Now $F^2-Q=GH$ where H is another quadratic in x.

Since GH is rational in x^2, G and H must both be linear functions of x^2, or both mere multiples of x.

But F, G, H are independent linear combinations of the coefficients of the different powers of t in the relation $(t)^2(x)^2=0$, and F, G, H are not linearly independent on either supposition as to G and H.

Thus the relation $(t)^2(x)^2=0$ is of the special form already noticed. By parity of reasoning each of the other biquadratics must be of the same form, and they may be written

$$X_1/X_2=T_1/T_2,\quad X_3/X_4=Y_3/Y_4,$$
$$Y_1/Y_2=Z_1/Z_2,\quad Z_3/Z_4=T_3/T_4.$$

If the relation $(x)^4(z)^4=0$ is found by eliminating t, the coefficients of different powers of z will be quadratics in X_1 and X_2; on the other hand if the same relation is found by eliminating y they will be quadratics in X_3 and X_4. Hence at least two quadratics in X_3 and X_4 can be expressed also as quadratics in X_1 and X_2. Thus if we exclude the cases (II.) and (III.) above, it must be possible by help of linear transformations to take

X_1 as x^2+1, X_2 as x, X_3 as x^2-1, and X_4 as x.

Thus the relations involving x become

$$x + \frac{1}{x} = T_1 / T_2, \quad x - \frac{1}{x} = Y_3 / Y_4,$$

and those involving z may be reduced in like manner to

$$z + \frac{1}{z} = T_3 / T_4, \quad z - \frac{1}{z} = Y_1 / Y_2.$$

Hence the relation connecting t and y may be written either as

$$(T_1 / T_2)^2 - (Y_3 / Y_4)^2 = 4 \text{ or as } (T_3 / T_4)^2 - (Y_1 / Y_2)^2 = 4.$$

By comparison of the coefficients of powers of y, it follows that T_3^2 and T_4^2 are linear combinations of T_1^2 and T_2^2. Hence by a linear transformation we may write

$$T_1 = t^2 + 1, \quad \varepsilon T_2 = t, \quad \varepsilon \text{ being a constant,}$$

and similarly

$$Y_3 = y^2 + 1, \quad \eta Y_4 = y, \quad \eta \text{ being a constant.}$$

Then we have

$$\varepsilon^2 \left(t + \frac{1}{t}\right)^2 - \eta^2 \left(y + \frac{1}{y}\right)^2 = 4,$$

which may also be written

$$\varepsilon^2 \left(t - \frac{1}{t}\right)^2 - \eta^2 \left(y - \frac{1}{y}\right)^2 = 4\,(1 - \varepsilon^2 + \eta^2).$$

Thus we are to take

$$T_3 = \varepsilon\,(t^2 - 1), \quad T_4 = t\,(1 - \varepsilon^2 + \eta^2)^{\frac{1}{2}},$$

$$Y_1 = \eta\,(y^2 - 1), \quad Y_2 = y\,(1 - \varepsilon^2 + \eta^2)^{\frac{1}{2}};$$

any other possible assumption falls under (I.), (II.), (III.), or (IV.), and we have as case (V.) the following—

$$x + \frac{1}{x} = \varepsilon \left(t + \frac{1}{t}\right), \quad x - \frac{1}{x} = \eta \left(y + \frac{1}{y}\right),$$

$$z + \frac{1}{z} = \zeta \left(t - \frac{1}{t}\right), \quad z - \frac{1}{z} = \theta \left(y - \frac{1}{y}\right),$$

where ε, ζ, η, θ are constants such that

$$\frac{\varepsilon}{\zeta} = \frac{\eta}{\theta} = (1 - \varepsilon^2 + \eta^2)^{\frac{1}{2}}.$$

§ 16. The biquadratic equations to which the problem of the jointed framework leads are of a special form. Let P, Q, R, S be the movable vertices, and A, B, C, D the corresponding pivots. Then if we denote by AP the complex quantity which it represents on Argand's diagram, and by $A'P'$ the conjugate quantity, and use a like notation throughout, we have

$$AP - BQ - AB = QP,$$

so that

$$(AP - BQ - AB)(A'P' - B'Q' - A'B') = |QP|^2, \text{ a constant.}$$

Now
$$A'P' = |AP|^2 \div AP,$$
$$B'Q' = |BQ|^2 \div BQ,$$

and $|AP|$, $|BQ|$ are constants, so that the equation is biquadratic, and AP, BQ are doubly periodic functions of the same variable u of the second order. The equation is unaltered by the substitution for AP and BQ of

$$\frac{AB}{A'B'} A'P' \text{ and } \frac{AB}{A'B'} B'Q'.$$

These substitutions are both homographic, and each is its own reciprocal, so that they must correspond to the addition of a half-period, say ω to u. The form of the equation shows also that AP and BQ have a common infinity and the same residue. Let α, β be the infinities of AP, then $\alpha + \omega$, $\beta + \omega$ are its zeros; and we may put α, γ for the infinities, and $\alpha + \omega$, $\gamma + \omega$ for the zeros of BQ. It follows that β, γ are the infinities, and $\beta + \omega$, $\gamma + \omega$ the zeros of PQ.

§ 17. Let us take the alternative (I.) and suppose CR, DS to be also doubly periodic functions of u with the same periods; then we have the following possible distinct arrangements of the infinities of AP, BQ, CR, DS, since each must have one in common with its neighbour:

AP,	BQ,	CR,	DS,
$\alpha, \beta,$	$\alpha, \gamma,$	$\alpha, \delta,$	$\alpha, \varepsilon,$
$\alpha, \beta,$	$\alpha, \gamma,$	$\delta, \gamma,$	$\delta, \beta,$
$\alpha, \beta,$	$\alpha, \gamma,$	$\alpha, \beta,$	$\delta, \beta.$

The order in which the two infinities are named is not indifferent; the residue for the first in each pair is the same. Let us take the half-period ω to be the same in each case.

§ 18. Take first the case where α is an infinity for each of the functions AP, BQ, CR, DS, and consider the identity

$$PQ+QR+RS+SP=0.$$

The functions $PQ+QR$ and $PS+SR$ are thus seen to be the same. The infinities of each are β and δ, while $\gamma+\omega$ is a zero of $PQ+QR$ and $\varepsilon+\omega$ of $PS+SR$. Thus

$$(\gamma+\omega)+(\varepsilon+\omega)\equiv\beta+\delta \text{ (mod. the periods)},$$

that is, we may say $\gamma+\varepsilon=\beta+\delta$, unless $\gamma=\varepsilon$, in which case $PQ=PS$, and Q, S are the same point.

Again, PQ/PA is a function of the second order with γ, $\alpha+\omega$ for infinities and α, $\gamma+\omega$ for zeros; RQ/RC is another function of the second order with the same infinities and zeros, so that

$$\frac{PQ}{PA}\div\frac{RQ}{RC}$$

is a constant.

By the addition of ω to u this becomes

$$\frac{P'Q'}{P'A'}\div\frac{R'Q'}{R'C'},$$

so that the constant must be real and therefore the angles APQ, CRQ are equal.

In the same way the angles APS, CRS are equal and $PQRS$ is a cyclic quadrilateral.

Consider also the functions PQ/RQ and RS/PS. Each is of the second order. The infinities of the first are β, $\delta+\omega$ and its zeros $\beta+\omega$, δ; it is equal to unity when $u=\gamma$ or $\varepsilon+\omega$. The infinities of the second are $\beta+\omega$, δ; its zeros β, $\delta+\omega$; it is unity when $u=\varepsilon$ or $\gamma+\omega$. Hence the one is the same function of $u+\omega$ that the other is of u, and thus

$$PQ/RQ=R'S'/P'S',$$

so that the triangles PQR, $R'S'P'$ are of the same species and aspect, and since $|PR|=|R'P'|$, it follows that

$$|PQ|=|R'S'|=|RS|,$$

$$|QR|=|S'P'|=|SP|.$$

Hence $PQRS$ is a crossed quadrilateral whose opposite sides are equal, and in every position the angles APQ, CRQ are equal. As at § 2 this ensures the locus of Q being a circle, as also that of S.

§ 19. Now consider the second arrangement in which the infinities of CR are δ, γ, and those of DS δ, β. Here PQ and SR have the same infinities β, γ with the same residues, and also the same zeros $\beta+\omega$, $\gamma+\omega$, so that $PQ = SR$, that is, $PQRS$ is a parallelogram.

Here also AP/PQ and RQ/CR have the same infinities α, $\gamma+\omega$ and the same zeros $\alpha+\omega$, γ, so that they differ by a constant factor, which must again be real, since the addition of ω to u changes the two functions into $A'P'/P'Q'$ and $R'Q'/C'R'$. Hence the angles APQ, QRC are equal. Thus the figure is what has been called a complete framework.

§ 20. In the third arrangement, where the infinities of CR are α, β, it is seen at once that PQ and RQ have the same infinities and residues and the same zeros, so that they are equal, and P coincides with R.

§ 21. So far we have supposed that the half-period ω is the same in all cases, or, what is equivalent, that the fractions $A'B'/AB$, $B'C'/BC$, &c. are equal, and the points A, B, C, D in the same straight line. Let us now assume that $B'C'/BC$ is not equal to $A'B'/AB$. Then by the addition of another half-period, say ω' to u, BQ becomes

$$\frac{BC}{B'C'} \cdot \frac{|BQ|^2}{BQ},$$

and by the addition of $\omega+\omega'$ to u it therefore becomes

$$\frac{AB}{A'B'} \cdot \frac{B'C'}{BC} \cdot BQ.$$

Now $2\omega+2\omega'$ is a period and $\omega+\omega'$ is not, so that

$$\frac{AB}{A'B'} \cdot \frac{B'C'}{BC} = -1,$$

that is, the angle ABC is right. Hence the pivots A, B, C, D form a rectangle, unless A and C or B and D coincide.

None of these suppositions lead to anything if we exclude the case when two opposite vertices of the quadrilateral always coincide, and also that in which AP, BQ, CR, DS are all equal and parallel, so that the quadrilateral $PQRS$ moves without deformation.

§ 22. We have now to consider under what circumstances the biquadratic equation connecting say AP, BQ will be of the special form in which the variables can be separated. To any position of Q there correspond two of P, say P_1, P_2; and now the two positions of Q that correspond to P_1 are the same as the two that correspond to P_2; let them be Q_1, Q_2. Then $P_1Q_1P_2Q_2$ is a rhombus, and P_1P_2, Q_1Q_2 bisect each other at right angles, and therefore pass through B, A respectively. The geometrical characteristic of this case is therefore that AQ, BP are perpendicular, which will be the case in all positions if

$$|AB|^2 + |PQ|^2 = |AP|^2 + |BQ|^2.$$

§ 23. In the alternative (II.), when the four biquadratics are of the separable form, it follows that P_1P_2 passes through both B and D and Q_1Q_2 through both A and C. Hence it is easily found that B and D must coincide, as also A and C. The diagonals PR, QS of the quadrilateral $PQRS$ are perpendicular and pass through B, A respectively.

§ 24. The alternative (III.) is a combination of (I.) and (II.), and is found on examination to lead to the following:

Two opposite pivots—say A, C—coincide; B, D are at infinity, and Q, S describe straight lines cutting at right angles in A; PR is always parallel to one of these lines, say that on which S moves, and $|AP| = |RS|$, $|AR| = |PS|$.

§ 25. If we take alternative (IV.) we have AP and CR, say, homographically related. When P is given there will be two positions of Q, say Q_1 and Q_2, and only one of R. Since $|PQ_1| = |PQ_2|$ and $|RQ_1| = |RQ_2|$, PR bisects Q_1Q_2, and similarly S_1S_2, at right angles. Hence PR passes through B and D. This leads to the following: B and D coincide in a centre of similitude of the loci of P and R, which are inversely corresponding points; $|PQ| = |QR|$ and $|PS| = |SR|$, but these lengths are both arbitrary.

§ 26. Lastly, let us consider alternative (V.). Here for a given position of P there are two of Q, the straight line joining which passes through A, and the same is true all round. In fact the form of the biquadratic equation shows that the two positions of Q are conjugates in an involution and that therefore the chord joining them passes through a fixed point. The pairs of positions of Q corresponding to different positions of R form another involution in which the foci of the former are conjugates. Thus A and C must be conjugate points with respect to the circle on which Q has to lie, and similarly they are conjugate with respect to the locus of S. Also B, D are conjugate with respect to the loci of P and R.

On each circle there are two systems in involution, the common pair being given by the values 0, ∞ of the parameter. By taking one from the common pair on each circle we satisfy the biquadratics, and thus have 16 positions of the quadrilateral. But the common pair on the Q circle lie on AC and those on the P circle lie on BD, and the two pairs form a rhombus. Hence AC, BD are perpendicular, from which it follows that the intersections of the P and R circles lie on BD and those of the Q and S circles on AC.

Let M be an intersection of the P and R circles, and N an intersection of the Q and S circles; then one position of the quadrilateral is $MNMN$, and it is therefore equilateral. In the adjacent positions the quadrilateral must clearly be crossed, which is impossible if the sides are all equal. Thus the alternative (V.) leads to nothing in the present case.

§ 27. Hence there are the following cases in which a jointed quadrilateral of rods can have its four vertices constrained to move on fixed circles and yet have a degree of internal freedom.

(*a*) $PQRS$ is a parallelogram, and the angles APQ, APS are equal to QRC, SRC respectively.

(*b*) $PQRS$ is a crossed quadrilateral whose opposite sides are equal; the angles APQ, APS are equal to CRQ, CRS respectively.

(*c*) $PQRS$ is a symmetrical quadrilateral or kite, so that $PQ = PS$ and $RQ = RS$; Q, S are inverse points with respect to a fixed circle and A, C coincide at the centre of this circle.

(*d*) The diagonals PR, QS are perpendicular in all positions; A, C coincide at a point through which QS always passes and B, D at a point through which PR always passes.

(*e*) As at § 24.

§ 28. To complete the enumeration we must add the cases where the equations considered are not all biquadratic. The equation connecting AP, BQ, for instance, will be bilinear if $APQB$ is a parallelogram or a crossed quadrilateral with opposite sides equal, and it will be linear in AP if the lengths AB, AP, and also the lengths BQ, PQ, are equal. I find the following cases:

(*f*) $APSD$ is an arbitrary quadrilateral; $ABQP$, $PQRS$, $SRCD$ are parallelograms.

(*g*) $ABCD$, $PQRS$ are crossed quadrilaterals such that

$$|AB| = |CD| = |PQ| = |RS|, \ |BC| = |AD| = |QR| = |PS|,$$

$$|AP| = |BQ| = |CR| = |DS|; \ ABQP, \ CDSR, \ BCRQ, \ ADSP$$

are also crossed.

(*h*) A, B, C, D all coincide, so that $PQRS$ rotates about a fixed pivot without deformation.

(*i*) A, B, C, D, are collinear; in one position P, Q, R, S are collinear; if the two straight lines meet in O, then in this position

$$|OQ| = |OA|, \ |OP| = |OB|, \ |OR| = |OD|, \ |OS| = |OC|,$$

so that the range $OABCD$ could be rotated about O into the position $OQPSR$; also

$$\frac{1}{OA} + \frac{1}{OC} = \frac{1}{OB} + \frac{1}{OD}.$$

(*j*) B and D coincide in the middle point of AC, $|DS| = |AB|$, $|AP| = |PS|$, $|CR| = |RS|$, $PQRS$ is a parallelogram.

(*k*) B and D coincide in the middle point of AC, $|DS| = |AB|$, $|AP| = |PS|$, $|CR| = |RS|$, $BQRC$ is a parallelogram.

(*l*) A, B, C, D are collinear; AC is bisected in D;

$$|DA| = |DC| = |DS|,\ |PA| = |PQ| = |PS|,\ |RC| = |RQ| = |RS|;$$

the angles APQ, QRC are supplementary.

(*m*) A, B, C, D are collinear; in one position A, Q, S coincide; in all positions $CRSD$ is a parallelogram, and the angles APQ, CRQ are equal.

(*n*) A, C coincide, as also B, D; in one position $PQRS$ is a rectangle whose circumcircle passes through A, B and in this position AS, BP are parallel, as also AQ, BR.

(*o*) Same figure as (*n*); but P, Q are the fixed pivots, and $ABSR$ is the moving quadrilateral.

(*p*) Same figure as (*n*), A, P being taken as pivots, and $BQRS$ as moving quadrilateral.

(*q*) B and D are at infinity; $PQRS$ is a parallelogram; $|PA| = |PS|$, $|CR| = |RS|$, S moves on AC, Q on the line bisecting AC at right angles.

(*r*) D is at infinity;

$$|PQ| = |PS| = |PA|,\quad |RQ| = |RS| = |RC|,$$

the angles APQ, CRQ are equal; S moves along AC.

(*s*) P, R coincide in all positions.

§ 29. The different positions of four lines of fixed length forming a quadrilateral are given in an explicit form by the following identity

$$\operatorname{sn}(u-\beta)\operatorname{sn}(u-\gamma)\operatorname{sn}(\beta-\gamma) + \operatorname{sn}(u-\gamma)\operatorname{sn}(u-\alpha)\operatorname{sn}(\gamma-\alpha)$$

$$+ \operatorname{sn}(u-\alpha)\operatorname{sn}(u-\beta)\operatorname{sn}(\alpha-\beta) + \operatorname{sn}(\beta-\gamma)\operatorname{sn}(\gamma-\alpha)\operatorname{sn}(\alpha-\beta) = 0.$$

Suppose α, β, γ real constants and u a variable argument whose imaginary part is $\pm \frac{1}{2}iK'$. Then each of the first three terms in this identity is a complex quantity with a constant modulus and variable argument, while the fourth term is constant. The geometrical equivalent is therefore a quadrilateral on a fixed base with sides given in length. The same holds good if α, β, γ are complex quantities such that the imaginary part of each is a multiple of iK'. The identity is still true if the function sn is replaced by qn, where $\operatorname{qn} v = (1 - \operatorname{cn} v)/\operatorname{sn} v$. The modulus of the complex quantity $\operatorname{qn}(v + iK')$ is constant if v is real and $k < 1$. Hence by taking α, β, γ to be real and u to be complex, its imaginary part being always $\pm iK'$, we have another representation of a deformable quadrilateral.

In each case there are three parameters, the modulus k and the differences of α, β, γ; these are to be determined in any case from the ratios of the sides of the quadrilateral. There is no difficulty in proving that one formula or the other is applicable in every case; the second formula will in fact apply to every case* if k is not restricted to fractional, but only to real, values.

§ 30. Explicit forms may be deduced for the different positions of a framework such as has been discussed in this article; thus we may say

$$C_r A_r = l \operatorname{qn} u \operatorname{qn}(u - a_r) \operatorname{qn} a_r,$$

$$C_{r,s} A_{r,s} = l \operatorname{qn}(u - a_r) \operatorname{qn}(u - a_s) \operatorname{qn}(a_s - a_r),$$

$$A_r A_{r+1} = l \operatorname{qn}(u - a_r) \operatorname{qn}(u - a_{r+1}) \operatorname{qn}(a_{r+1} - a_r),$$

$$C_r C_{r+1} = l \operatorname{qn} a_r \operatorname{qn} a_{r+1} \operatorname{qn}(a_{r+1} - a_r),$$

$$C_s C_{r,s} = l \operatorname{qn} a_r \operatorname{qn} a_s \operatorname{qn}(a_r - a_s),$$

l being a multiplier depending on the scale of the figure.

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* If a, b, c are the lengths of the moveable sides and d is that of the base, the equations to be solved are

$$\operatorname{qn}(\gamma - \alpha) \operatorname{qn}(\alpha - \beta) = d/a,$$

$$\operatorname{qn}(\alpha - \beta) \operatorname{qn}(\beta - \gamma) = d/b,$$

$$\operatorname{qn}(\beta - \gamma) \operatorname{qn}(\gamma - \alpha) = d/c,$$

Hence

$$\operatorname{qn}^2(\beta - \gamma) = ad/bc,$$

$$\operatorname{cn}(\beta - \gamma) = (bc - ad)/(bc + ad),$$

$$\operatorname{sn}(\beta - \gamma) = 2(abcd)^{\frac{1}{2}}/(bc + ad), \text{ etc.}$$

From the formulæ for three arguments whose sum vanishes $\operatorname{dn}(\beta - \gamma)$ is found to be $\frac{1}{2}(a^2 - b^2 - c^2 + d^2)/(bc + ad)$. Hence

$$k^2 = (-a + b + c + d)(a - b + c + d)(a + b - c + d)(a + b + c - d)/16abcd.$$

Thus k is real and the differences of α, β, γ only differ from real quantities by multiples of $2iK'$, so that the solution is always admissible.

When k is less than 1, the quadrilateral can be brought from any one position to any other by continuous deformation, but when k is greater than 1 there are two series of positions of which one cannot be changed into the other by continuous deformation in the plane.

§ 31. For the special form of quadrilateral in which the diagonals are perpendicular the sn form may be taken and $\alpha-\gamma$ must be either K or $K+iK'$, the second term of the identity representing the rod with two moveable ends.

§ 32. In the case when $k=1$ the general formula applies; the following identity may also be used, v being a purely imaginary variable and a, b, c real constants:

$$\frac{v-b}{v+b}\cdot\frac{v-c}{v+c}\cdot\frac{b-c}{b+c}+\frac{v-c}{v+c}\cdot\frac{v-a}{v+a}\cdot\frac{c-a}{c+a}$$

$$+\frac{v-a}{v+a}\cdot\frac{v-b}{v+b}\cdot\frac{a-b}{a+b}+\frac{b-c}{b+c}\cdot\frac{c-a}{c+a}\cdot\frac{a-b}{a+b}=0.$$

When $c=0$ or ∞, this becomes

$$\frac{v-a}{v+a}\cdot\frac{v-b}{v+b}\cdot\frac{a-b}{a+b}-\frac{v-a}{v+a}+\frac{v-b}{v+b}=\frac{a-b}{a+b},*$$

which applies to the case of a cyclic or symmetrical quadrilateral with its sides equal in pairs. The following identity is also useful in cases when expressions for such quadrilaterals have to be given in elliptic functions

$$\operatorname{qn}(u+\alpha)\operatorname{qn}(u-\alpha)-\operatorname{qn}(u+\beta)\operatorname{qn}(u-\beta)$$

$$-\operatorname{qn}(u+\alpha)\operatorname{qn}(u-\alpha)\operatorname{qn}(u+\beta)\operatorname{qn}(u-\beta)\operatorname{qn}(\alpha+\beta)\operatorname{qn}(\alpha-\beta)$$

$$+\operatorname{qn}(\alpha+\beta)\operatorname{qn}(\alpha-\beta)=0.$$

Here u is a variable argument whose imaginary part is the constant iK' and α, β are constants, real or purely imaginary.

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* A more general form is

$$\frac{v-a_1}{v+a_2}\cdot\frac{v-b_1}{v+b_2}(a_2-b_2)-\frac{v-a_1}{v+a_2}(a_2+b_1)+\frac{v-b_1}{v+b_2}(a_1+b_2)=a_1-b_1,$$

where a_1, a_2 and b_1, b_2 are pairs of conjugate complex quantities.

NOTE ON A CLASS OF ALGEBRAICAL IDENTITIES.

By Professor *E. J. Nanson.*

Let there be two sets of n letters $x, y, z, \ldots, a, b, c, \ldots$, and let these letters be connected by the $n-h$ equations

$$\Sigma a^r x = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $r = 0, 1, 2, \ldots, n-h-1$. Let

$$A = (a-b)(a-c)(a-d)\ldots$$

and let $B, C, \ldots$ have similar meanings. Then we know that, if $s < n-1$,

$$\Sigma \frac{a^s}{A} = 0;$$

and hence from (1) we have

$$Ax = \lambda_1 + \lambda_2 a + \ldots + \lambda_h a^{h-1} \ldots\ldots\ldots\ldots\ldots(2),$$

and therefore

$$\Sigma A^p a^q x^{p+1} = \Sigma \frac{a^q}{A}(\lambda_1 + \lambda_2 a + \ldots + \lambda_h a^{h-1})^{p+1} \ldots\ldots(3).$$

Hence, if $q + (h-1)p < n-h$, we have

$$\Sigma A^p a^q x^{p+1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If $q + (h-1)p = n-h$, we have

$$\Sigma A^p a^q x^{p+1} = \lambda_h^{p+1},$$

and, taking $p = 0$, this reduces to

$$\Sigma a^{n-h} x = \lambda_h,$$

so that

$$\Sigma A^p a^q x^{p+1} = \{\Sigma a^{n-h} x\}^{p+1} \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

When $h = 2$, the identities (4), (5) reduce to those given by Lachlan, *Messenger*, XVI., p. 21.

To find the corresponding identities when

$$q + (h-1)p > n-h,$$

we expand the second member of (3) by the multinomial theorem, and remembering that

$$\Sigma \frac{a^{n-1+s}}{A} = H_s,$$

where H_s is the sum of the homogeneous products of $a, b, c, \ldots$ of s dimensions, we see that

$$\Sigma A^p a^q x^{p+1} = \Sigma C \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \ldots \lambda_h^{\alpha_h} H_\sigma,$$

where the summation on the second side extends to all zero or positive integral values of $\sigma, \alpha_1, \alpha_2, \ldots, \alpha_h$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_h = p + 1,$$
$$\alpha_1 + 2\alpha_2 + \ldots + h\alpha_h = \sigma + n + p - q,$$

and C is given by

$$C = \frac{(p+1)!}{\alpha_1!\, \alpha_2! \ldots \alpha_h!}.$$

In this result, making $p = 0$ and giving to q the values $n - h$, $n - h + 1$, ..., $n - 1$ in succession, we find

$$\Sigma a^{n-h} x = \lambda_h,$$
$$\Sigma a^{n-h+1} x = H_1 \lambda_h + \lambda_{h-1},$$
$$\Sigma a^{n-h+2} x = H_2 \lambda_h + H_1 \lambda_{h-1} + \lambda_{h-2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Sigma a^{n-1} x = H_{h-1} \lambda_h + H_{h-2} \lambda_{h-1} + \ldots + H_1 \lambda_2 + \lambda_1,$$

whence $\lambda_1, \lambda_2, \ldots, \lambda_h$ can be expressed in terms of $\Sigma a^{n-h} x$, $\Sigma a^{n-h+1} x$, ..., $\Sigma a^{n-1} x$, and hence $\Sigma A^p a^q x^{p+1}$ can be expressed in terms of the same quantities and the H's.

In particular, if $h = 2$, we have

$$\Sigma A^p a^q x^{p+1} \Sigma_t \frac{(p+1)!}{t!\,(p+1-t)!} H_{t+q+1-n} \{\Sigma a^{n-2} x\}^t \{\Sigma a^{n-1} x - H_1 \Sigma a^{n-2} x\}^{p+1-t},$$

the summation with respect to t extending from $t = n - 1 - q$ to $t = p + 1$.

Melbourne,
November 17*th*, 1898.

THE GENERALISATION OF VANDERMONDE'S THEOREM.

By Professor *E. J. Nanson.*

In a paper entitled "Condensed Proof and Generalisation of Vandermonde's Theorem,"* Dr. Hill has given an elementary condensed proof of Vandermonde's Theorem. As stated therein the proof depends on an artifice due to Cayley.† Dr. Hill then applies the same artifice to prove the general extension of Vandermonde's Theorem, viz. that if

$$(x)_n = x(x-1)(x-2)\dots(x-n+1),$$

then

$$(a_1 + a_2 + \dots + a_m)_n = \Sigma \frac{n!}{r_1!\, r_2! \dots r_m!} (a_1)_{r_1} (a_2)_{r_2} \dots (a_m)_{r_m},$$

where Σ denotes summation for all zero or positive integral values of $r_1, r_2, \dots, r_m$, such that

$$r_1 + r_2 + \dots + r_m = n.$$

The object of this note is to indicate a simpler way of proving the extension.

With the aid of the original theorem it is readily shewn, by the method used in a well-known proof of the multinomial theorem for a positive integral exponent, that if the extension be true for m letters $a_1, a_2, \dots, a_m$, then it is true for $m+1$ letters. Hence by induction, the theorem having been proved for two letters, it follows that it is true for any number of letters.

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* *Messenger*, Vol. XXV. 154—6.

† *C. M. P.* 119 and 553.

ON A CLASS OF DEFINITE INTEGRALS CONTAINING HYPERBOLIC FUNCTIONS.

By *G. H. Hardy*, Trinity College, Cambridge.

§ 1. The integrals evaluated in this paper are all of the type

$$\int_{-\infty}^{\infty} \frac{e^{(\alpha+\iota\beta)x}}{ae^x + b + ce^{-x}} R(x)\, dx \;\ldots\ldots\ldots\ldots\ldots (A),$$

where $R(x)$ is a rational function, and the constants are so chosen as to give a finite value for the integral or for its principal value. It is not proposed to consider the most general form of (A); we shall, as a rule, suppose

$$a = \pm c.$$

The integrals thus resulting are the formal analogues, for the hyperbolic functions, of the integrals

$$\int_{-\infty}^{\infty} \frac{\begin{pmatrix}\cos\\ \sin\end{pmatrix}\lambda x}{1 + 2p\cos rx + p^2} R(x)\, dx.$$

The method adopted is that of contour integration, as explained by Briot and Bouquet, or in Forsyth's *Theory of Functions*, pp. 38–42. We consider three main types of integral:

$$\text{(I.)} \qquad \int \frac{e^{\alpha z}}{1 + 2pe^z \pm e^{2z}} \, \frac{dz}{z - b + ia},$$

where a, α are real and positive, $0 < \alpha < 2$, b, p real, and the contour of integration consists of $y = 0$ and the infinite semicircle including the upper half of the plane.

$$\text{(II.)} \qquad \int \frac{e^{\alpha(z - i\pi)}}{1 + 2pe^z \pm e^{2z}} \, \frac{dz}{z - b - i\pi},$$

where the real part a of α satisfies $0 < a < 2$, b, p are real, and the contour is the infinite rectangle whose sides are $y = 0$, $y = 2\pi$, $x = \pm\infty$.

(III.) $$\int \frac{e^{az}}{1+2pe^z \pm e^{2z}}\, dz,$$

where the conditions are the same as for (II.), except that p is not necessarily real. In all cases in which the subject of integration becomes infinite for real values of z the principal value of the resulting definite integral is what is considered.

From these a very large number of definite integrals can be deduced. Those arising from (III.) include those discussed from another point of view in Dr. Glaisher's paper "On the definite integrals connected with the Bernoullian function" (*Messenger*, Vols. XXVI.–XXVII.). Particular cases of the integrals arising from (I.) and (II.) are considered by Schlömilch in connection with the summation of series, and given by Bierens de Haan ("Nouvelles Tables d'Intégrales définies," 1867; "Exposé de la Théorie des Intégrales définies," *Verhandelingen der Koninklijke Akademie van Wetenschappen*, Deel VIII., Amsterdam, 1862). De Haan does not use contour integration; whence the lack of generality of his results.

Expression of the general integral as a series.

§ 2. The function

$$\frac{e^{az}}{1+2pe^z+e^{2z}}\,\frac{1}{z-b+ia}$$

has in the upper half of the plane simple poles at the points

$$(2n+1)\,i\pi \pm \delta,$$

where $p=\cosh\delta$, p being supposed at present positive and >1. If $z=(2n+1)\,i\pi \pm \delta + \varepsilon$, where ε is small, it becomes

$$-\frac{1}{\{(2n+1)\,\pi+a\}\,i \pm \delta - b}\,\frac{e^{(2n+1)ia\pi \pm a\delta}}{\varepsilon e^{\pm\delta}\,(e^{\mp\delta}-e^{\pm\delta})},$$

i.e. $$\frac{1}{\{(2n+1)\,\pi+a\}\,i+\delta-b}\,\frac{e^{(2n+1)ia\pi+(a-1)\delta}}{2\varepsilon\sinh\delta},$$

or $$\frac{1}{\{(2n+1)\,\pi+a\}\,i-\delta-b}\,\frac{e^{(2n+1)ia\pi-(a-1)\delta}}{-2\varepsilon\sinh\delta}.$$

For simplicity we suppose $b=0$; then the sum of the two residues is

$$\frac{e^{(2n+1\; ia\pi}}{2\sinh\delta}\left[\frac{e^{(\alpha-1)\delta}}{\delta+i\{(2n+1)\pi+a\}}+\frac{e^{-(\alpha-1)\delta}}{\delta-i\{(2n+1)\pi+a\}}\right]$$

$$=\frac{e^{(2n+1)ia\pi}}{\sinh\delta}\,\frac{\delta\cosh(\alpha-1)\,\delta-i\{(2n+1)\,\pi+a\}\sinh(\alpha-1)\,\delta}{\delta^2+\{(2n+1)\,\pi+a\}^2}.$$

Thus if we suppose the contour of integration cuts the axis of y between

$$y=(2N-1)\,\pi,\quad y=(2N+1)\,\pi,$$

and that N is then indefinitely increased, we find, putting $\alpha=\mu+1$,

$$\tfrac{1}{2}\int_{-\infty}^{\infty}\frac{e^{\mu x}}{\cosh x+\cosh\delta}\,\frac{dx}{x+ia}$$

$$=-\frac{2\pi i}{\sinh\delta}\sum_{n=0}^{\infty}\frac{e^{(2n+1)i\mu\pi}\,[\delta\cosh\mu\delta-i\{(2n+1)\,\pi+a\}\sinh\mu\delta]}{\delta^2+\{(2n+1)\,\pi+a\}^2}.$$

Equating the real and imaginary parts

$$(1)\quad \tfrac{1}{2}\int_{-\infty}^{\infty}\frac{e^{\mu x}}{\cosh x+\cosh\delta}\,\frac{dx}{x^2+a^2}=\int_0^{\infty}\frac{\cosh\mu x}{\cosh x+\cosh\delta}\,\frac{dx}{x^2+a^2}$$

$$=\frac{2\pi}{a\sinh\delta}$$

$$\times\sum_{n=0}^{\infty}\frac{\delta\cosh\mu\delta\cos(2n+1)\,\mu\pi+\{(2n+1)\pi+a\}\sinh\mu\delta\sin(2n+1)\,\mu\pi}{\delta^2+\{(2n+1)\,\pi+a\}^2}.$$

$$(2)\quad \tfrac{1}{2}\int_{-\infty}^{\infty}\frac{e^{\mu x}}{\cosh x+\cosh\delta}\,\frac{x\,dx}{x^2+a^2}=\int_0^{\infty}\frac{\sinh\mu x}{\cosh x+\cosh\delta}\,\frac{x\,dx}{x^2+a^2}$$

$$=\frac{2\pi}{\sinh\delta}$$

$$\times\sum_{n=0}^{\infty}\frac{\delta\cosh\mu\delta\sin(2n+1)\,\mu\pi-\{(2n+1)\,\pi+a\}\sinh\mu\delta\cos(2n+1)\,\mu\pi}{\delta^2+\{(2n+1)\,\pi+a\}^2}.$$

In both of these formulæ $-1<\mu<+1$. Also the argument is unchanged if $p<1$, in which case we put $p=\cos\delta$ and find formulæ similar to the above.

If we differentiate the series which represents

$$\int_0^\infty \frac{\cosh \mu x}{\cosh x + \cosh \delta} \frac{dx}{a^2 + x^2}$$

with respect to μ, we get the series which represents

$$\int_0^\infty \frac{\sinh \mu x}{\cosh x + \cosh \delta} \frac{x dx}{a^2 + x^2},$$

plus an indeterminate infinite series of the form

$$A \sum_0^\infty \cos (2n + 1) \mu \pi,$$

which shows that the conditions necessary for the differentiation of equation (1) are violated.

Evaluation of the integrals in special cases.

§ 3. For many sets of special values of the constants the series of § 2 (1), (2) can be summed; some of the simplest cases will now be given.

(a) Let $\mu = 0$; then from (1)

$$(1) \quad \int_0^\infty \frac{1}{\cosh x + \cosh \delta} \frac{dx}{a^2 + x^2} = \frac{2\delta\pi}{a \sinh \delta} \sum_0^\infty \frac{1}{\delta^2 + \{(2n+1)\pi + a\}^2},$$

$$(2) \quad \int_0^\infty \frac{1}{\cosh x + \cos \delta} \frac{dx}{a^2 + x^2} = \frac{2\delta\pi}{a \sin \delta} \sum_0^\infty \frac{1}{\{(2n+1)\pi + a\}^2 - \delta^2}, \ (\delta < \pi).$$

In (2) let $a = \pi$. Since

$$\sum_0^\infty \left[\frac{1}{2(n+1)\pi - \delta} - \frac{1}{2(n+1)\pi + \delta}\right]$$

$$= -\tfrac{1}{2} \sum_0^\infty \left[\frac{1}{\frac{1}{2}\delta - (n+1)\pi} + \frac{1}{\frac{1}{2}\delta + (n+1)\pi}\right] = -\tfrac{1}{2}\left(\cot \tfrac{1}{2}\delta - \frac{2}{\delta}\right),$$

$$(3) \quad \int_0^\infty \frac{1}{\cosh x + \cos \delta} \frac{dx}{\pi^2 + x^2} = \frac{1}{\delta \sin \delta} - \frac{1}{4 \sin^2 \frac{1}{2}\delta},$$

$$(4) \quad \int_0^\infty \frac{1}{\cosh x + \cosh \delta} \frac{dx}{\pi^2 + x^2} = \frac{1}{4 \sinh^2 \frac{1}{2}\delta} - \frac{1}{\delta \sinh \delta}.$$

In particular

$$(5)\quad \int_0^\infty \frac{1}{\cosh^2 \frac{1}{2}x} \frac{dx}{\pi^2 + x^2} = \tfrac{1}{6},$$

$$(6)\quad \int_0^\infty \frac{1}{\cosh x} \frac{dx}{\pi^2 + x^2} = \frac{2}{\pi} - \tfrac{1}{2}.$$

The last integral in the form

$$\int_0^\infty \frac{1}{e^{\pi x} + e^{-\pi x}} \frac{dx}{1 + x^2} = 1 - \tfrac{1}{4}\pi$$

is given by De Haan (l. c. Table 97, No. 1).

If we multiply equation (3) by $\sin \delta$ and integrate from $\frac{1}{2}\pi$ to δ, we find

$$(7)\quad \int_0^\infty \log\left(1 + \frac{\cos \delta}{\cosh x}\right) \frac{dx}{\pi^2 + x^2} = \log\left(\frac{\delta \sqrt{2}}{\pi \sin \frac{1}{2}\delta}\right).$$

In the particular case of $\delta = 0$ this may be verified by comparison with De Haan (l. c. 384, 7).

(*b*) Let $\mu = \frac{1}{2}$; then

$$(8)\quad \int_0^\infty \frac{\cosh \frac{1}{2}x}{\cosh x + \cosh \delta} \frac{dx}{a^2 + x^2}$$

$$= \frac{\pi}{a \cosh \frac{1}{2}\delta} \sum_0^\infty (-)^n \frac{(2n+1)\pi + a}{\delta^2 + \{(2n+1)\pi + a\}^2},$$

$$(9)\quad \int_0^\infty \frac{\cosh \frac{1}{2}x}{\cosh x + \cos \delta} \frac{dx}{a^2 + x^2} = \frac{\pi}{a \cos \frac{1}{2}\delta} \sum_0^\infty (-)^n \frac{(2n+1)\pi + a}{\{(2n+1)\pi + a\}^2 - \delta^2}.$$

Let $\delta = 0$; then

$$(10)\quad \int_0^\infty \frac{1}{\cosh \frac{1}{2}x} \frac{dx}{a^2 + x^2} = \frac{2\pi}{a} \sum_0^\infty (-)^n \frac{1}{(2n+1)\pi + a},$$

and if $a = \pi$

$$(11)\quad \int_0^\infty \frac{1}{\cosh \frac{1}{2}x} \frac{dx}{\pi^2 + x^2} = \frac{1}{\pi} \sum_0^\infty (-)^n \frac{1}{n+1} = \frac{1}{\pi} \log 2.$$

This is given by De Haan (l. c. Table 97, 2).

More generally let $a=\dfrac{r\pi}{s}$; then

$$\int_0^\infty \frac{1}{\cosh\frac{1}{2}x}\,\frac{dx}{\left(\dfrac{r\pi}{s}\right)^2+x^2}=\frac{2s^2}{r\pi}\sum_0^\infty(-)^n\frac{1}{2ns+s+r}$$

$$=\frac{2s^2}{r\pi}\int_0^1\frac{x^{r+s-1}}{1+x^{2s}}\,dx,$$

i.e.

$$(12)\qquad \int_0^\infty \frac{1}{\cosh\dfrac{r\pi}{2s}x}\,\frac{dx}{1+x^2}=2s\int_0^1\frac{x^{r+s-1}}{1+x^{2s}}\,dx,$$

and can therefore be evaluated. In the particular case of $r=1$, $s=2$, we find

$$(13)\qquad \int_0^\infty \frac{1}{\cosh\frac{1}{4}\pi x}\,\frac{dx}{1+x^2}=\frac{1}{\sqrt{2}}\{\pi-2\log(\sqrt{2}+1)\}$$

given by De Haan (l. c. 97 (3)).

A simple general value for

$$(a)\qquad \int_0^\infty \frac{\cosh\frac{1}{2}x}{\cosh x+\cos\delta}\,\frac{dx}{\pi^2+x^2},$$

analogous to (3), does not exist. For, while series of the form

$$\sum_0^\infty\left(\frac{1}{x+na}+\frac{1}{x-na}\right)$$

can be summed in circular functions, those of the form

$$\sum_0^\infty\left(\frac{1}{x-na}-\frac{1}{x+na}\right)$$

require logarithms of Gamma-functions. But for any particular case in which δ is commensurable with π, we can reduce the evaluation of (a) to that of

$$\int_0^1 R(x)\,dx,$$

and the same is true of

$$\int_0^\infty \frac{\cosh\frac{1}{2}x}{\cosh x+\cos\dfrac{l\pi}{m}}\,\frac{dx}{\left(\dfrac{r\pi}{s}\right)^2+x^2};$$

thus all such integrals can be found in finite terms. On the other hand, for integrals of the forms (1), (2), finite expressions can be found for particular values of a other than π. For, if $\psi(x)$ denote the function

$$\operatorname*{Limit}_{m=\infty}\left(\log m - \frac{1}{x} - \frac{1}{x+1} - \dots - \frac{1}{x+m}\right),$$

we easily find

$$\int_0^\infty \frac{1}{\cosh x + \cos\delta}\,\frac{dx}{a^2+x^2}$$

$$= \frac{1}{2a\sin\delta}\left\{\psi\left(\frac{a+\delta}{2\pi}+\tfrac{1}{2}\right) - \psi\left(\frac{a-\delta}{2\pi}+\tfrac{1}{2}\right)\right\}.$$

Now $\psi(x) - \psi(1)$ is known to be expressible in finite terms when x is rational; *e.g.*

$$\psi(\tfrac{1}{3}) = \psi(1) - \tfrac{1}{2}\pi\sqrt{\tfrac{1}{3}} - \tfrac{3}{2}\log 3,$$

$$\psi(\tfrac{1}{4}) = \psi(1) - \tfrac{1}{2}\pi - 3\log 2,$$

(Bertrand, *Calcul Intégral*).

Thus the integral in question can be found in finite terms whenever a and δ are commensurable with π.

§ 4. We next consider § 2. (2). Let $\mu = \frac{1}{2}$, then

(1) $$\int_0^\infty \frac{\sinh\frac{1}{2}x}{\cosh x + \cosh\delta}\,\frac{x\,dx}{a^2+x^2} = \frac{\pi\delta}{\sinh\frac{1}{2}\delta}\sum_0^\infty (-)^n \frac{1}{\delta^2+\{(2n+1)\pi+a\}^2},$$

(2) $$\int_0^\infty \frac{\sinh\frac{1}{2}x}{\cosh x + \cos\delta}\,\frac{x\,dx}{a^2+x^2} = \frac{\pi\delta}{\sin\frac{1}{2}\delta}\sum_0^\infty (-)^n \frac{1}{\{(2n+1)\pi+a\}^2-\delta^2}.$$

Let $a = \pi$,

(3) $$\int_0^\infty \frac{\sinh\frac{1}{2}x}{\cosh x + \cos\delta}\,\frac{x\,dx}{\pi^2+x^2} = \frac{\pi\delta}{\sin\frac{1}{2}\delta}\sum_0^\infty \frac{(-)^n}{\{2(n+1)\pi\}^2-\delta^2}$$

$$= \frac{\pi}{4\sin\frac{1}{2}\delta}\sum_0^\infty (-)^{n-1}\left\{\frac{1}{\frac{1}{2}\delta-(n+1)\pi} + \frac{1}{\frac{1}{2}\delta+(n+1)\pi}\right\}$$

$$= \frac{\pi}{4\sin\frac{1}{2}\delta}\left\{\frac{1}{\sin\frac{1}{2}\delta} - \frac{1}{\frac{1}{2}\delta}\right\},$$

(4) $$\int_0^\infty \frac{\sinh\frac{1}{2}x}{\cosh x + \cosh\delta}\,\frac{x\,dx}{\pi^2+x^2} = \frac{\pi}{4\sinh\frac{1}{2}\delta}\left\{\frac{1}{\frac{1}{2}\delta} - \frac{1}{\sinh\frac{1}{2}\delta}\right\}.$$

Another case which naturally presents itself is that of $a=0$. Our formulæ were calculated on the supposition $a>0$; it is, however, easy to see that, if we calculate the principal value of

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+2pe^{x}+e^{2x}} \frac{dx}{x}$$

by the same method, the result only differs from that which we obtain by putting $a=0$ in the general formula by the occurrence of a term

$$\frac{i\pi}{2(1+p)}$$

due to the pole at the origin, and as the formulæ (1), (2) are found by an equation of *real* parts, they will not be affected. Thus we may put $a=0$. That is

$$(5) \int_0^{\infty} \frac{\sinh \frac{1}{2}x}{\cosh x+\cosh \delta} \frac{dx}{x} = \frac{\pi\delta}{\sinh \frac{1}{2}\delta} \sum_0^{\infty} (-)^n \frac{1}{\delta^2+\{(2n+1)\pi\}^2},$$

$$(6) \int_0^{\infty} \frac{\sinh \frac{1}{2}x}{\cosh x+\cos \delta} \frac{dx}{x} = \frac{\pi\delta}{\sin \frac{1}{2}\delta} \sum_0^{\infty} (-)^n \frac{1}{\{(2n+1)\pi\}^2-\delta^2}.$$

These series cannot, in general, be summed; thus if $\delta=0$,

$$\int_0^{\infty} \frac{\sinh \frac{1}{2}x}{2\cosh^2 \frac{1}{2}x} \frac{dx}{x} = \frac{2}{\pi} \sum_0^{\infty} (-)^n \frac{1}{(2n+1)^2},$$

which is not (trigonometrically) expressible in finite terms.

If, however, $\delta=\frac{1}{2}\pi$ in (6),

$$\int_0^{\infty} \frac{\sinh \frac{1}{2}x}{\cosh x} \frac{dx}{x} = \frac{1}{\sqrt{2}} \sum_0^{\infty} (-)^{n-1} \left\{-\frac{1}{2n+\frac{1}{2}}+\frac{1}{2n+\frac{3}{2}}\right\}$$

$$= \sqrt{2}\int_0^1 \frac{1-x^2}{1+x^4}\,dx = \log(\sqrt{2}+1)$$

(this is given by De Haan, (l. c. 95, 1)).

Connection of the integrals with the summation of series.

§ 5. It is clearly impracticable to sum the series of § 2 for general values of μ and δ. In the case, however, in which the denominator reduces to $\cosh x$ they may be greatly simplified; the integrals are then of a class important in the summation of series.

In § 2. (1) put $\delta = \frac{1}{2}i\pi$; the numerator of the general term is then

$$\tfrac{1}{2}\left[\tfrac{1}{2}\pi\left\{\cos(2n+\tfrac{1}{2})\mu\pi + \cos(2n+\tfrac{3}{2})\mu\pi\right\}\right.$$
$$\left. + \left\{(2n+1)\pi + a\right\}\left\{\cos(2n+\tfrac{1}{2})\mu\pi - \cos(2n+\tfrac{3}{2})\mu\pi\right\}\right],$$

and the series becomes

$$\frac{\pi}{a}\sum_0^\infty \left\{\frac{\cos(2n+\frac{1}{2})\mu\pi}{(2n+\frac{1}{2})\pi + a} - \frac{\cos(2n+\frac{3}{2})\mu\pi}{(2n+\frac{3}{2})\pi + a}\right\},$$

so that

$$(1)\quad \int_0^\infty \frac{\cosh\mu x}{\cosh x}\,\frac{dx}{a^2+x^2} = \frac{\pi}{a}\sum_0^\infty \frac{(-)^n\cos(n+\frac{1}{2})\mu\pi}{(n+\frac{1}{2})\pi + a}.$$

This, for the case of $\mu = 0$, is given by De Haan. If we vary the subject of integration or the contour appropriately we can find the principal values of the integrals whose denominators are

$$\cosh x - \cosh\delta, \quad \sinh x - \sinh\delta,$$

e.g. for the latter case we consider

$$\int \frac{e^{az}}{1 + 2pe^z - e^{2z}}\,\frac{dz}{z + ia}.$$

One special case is noticeable, viz. that of

$$\int \frac{e^{az}}{1 - e^{2z}}\,\frac{dz}{z + ia};$$

the poles are $0, \pi i, \ldots, n\pi i, \ldots$; the residue at $n\pi i$ is

$$-\frac{1}{2i}\cdot\frac{e^{na\pi i}}{n\pi + a},$$

and we easily find, if $\alpha = \mu + 1$,

$$\tfrac{1}{2}\int_{-\infty}^\infty \frac{e^{\mu x}}{\sinh x}\,\frac{dx}{x + ia} = \tfrac{1}{2}a\pi + \sum_1^\infty \frac{(-)^n e^{n\mu\pi i}}{n\pi + a}\,\pi.$$

Equating real and imaginary parts, we get two formulæ given by De Haan in the form

$$(2)\quad \int_0^\infty \frac{\sinh(r-p)x}{\sinh rx}\,\frac{dx}{a^2+x^2} = \frac{\pi}{a}\sum_1^\infty \frac{1}{ar + n\pi}\sin\frac{np\pi}{r},$$

$$(3)\quad \int_0^\infty \frac{\cosh(r-p)x}{\sinh rx}\,\frac{xdx}{a^2+x^2} = \frac{\pi}{2ar} + \pi\sum_1^\infty \frac{1}{ar + n\pi}\cos\frac{np\pi}{r}.$$

These are connected with the general formula due to Schlömilch,

$$\sum_0^\infty \phi(n) = \tfrac{1}{2}\phi(0) + \int_{-\infty}^{\infty} \frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}} \frac{\phi(-ix) - \phi(ix)}{2i} dx$$

(Boole, *Finite Differences*, p. 106; Schlömilch, *Crelle*, 42). The integrals (2), (3) can be found in finite terms when $a = 1$, $r = \pi$; thus if $p < \pi$,

$$(4) \int_0^\infty \frac{\sinh px}{\sinh \pi x} \frac{dx}{1 + x^2} = \sum_0^\infty \frac{(-1)^{n-1} \sin up}{n+1}$$

$$= -\tfrac{1}{2}p \cos p + \tfrac{1}{2} \sin p \log \{2(1 + \cos p\},$$

and so on; these are given by De Haan, but not the corresponding formula which we can obtain from equation (1) of this section, viz.

$$(5) \int_0^\infty \frac{\cosh px}{\cosh \pi x} \frac{dx}{1 + x^2} = \sum_0^\infty \frac{(-)^n \cos(n + \frac{1}{2})p}{(n + \frac{3}{2})p}$$

$$= 2 \cos \tfrac{1}{2}p - \tfrac{1}{2}\pi \cos p - \tfrac{1}{2} \sin p \log \frac{1 + \sin \frac{1}{2}p}{1 - \sin \frac{1}{2}p}.$$

In these formulæ p may be complex, provided its real part is less than π; *e.g.*

$$(6) \int_0^\infty \frac{\sin px}{\sinh \pi x} \frac{dx}{1 + x^2} = \tfrac{1}{2}\{e^p \log(1 + e^{-p}) - e^{-p} \log(1 + e^p)\},$$

and many others of similar form. Any of these may be differentiated with respect to p.

All the integrals which have been found are special cases of the general integral (I.). By means of the integrals (II.) we are able in many cases to dispense with the trouble of summation.

Integrals which can be immediately expressed in finite form.

§ 6. For (II.) the contour of integration is the rectangle bounded by $y = 0$, $y = 2\pi$. The first integral is

$$\int \frac{e^{a(z-\pi i)}}{1 + 2pe^z + e^{2z}} \frac{dz}{z - \pi i}, \quad (0 \overline{\overline{<}} a \overline{\overline{<}} 2),$$

the poles within the contour are

$$\pi i, \ \pi i \pm \delta, \ (\text{where } p = \cosh \delta > 1),$$

and the corresponding residues are

$$\frac{1}{2(1-p)},\ \frac{e^{\mu\delta}}{2\delta\sinh\delta},\ \frac{e^{-\mu\delta}}{2\delta\sinh\delta},\ (\mu=\alpha-1).$$

Also along the line $y=2\pi$, $z=x+2\pi i$, and so we find

$$(1)\quad \tfrac{1}{2}\int_{-\infty}^{\infty}\frac{e^{\mu x}}{\cosh x+\cosh\delta}\,\frac{x\sin\mu\pi-\pi\cos\mu\pi}{x^2+\pi^2}\,dx$$
$$=\pi\left\{\frac{\cosh\mu\delta}{\delta\sinh\delta}-\frac{1}{4\sinh^2\frac{1}{2}\delta}\right\},$$

and a corresponding formula for denominator $(\cosh x+\cos\delta)$. If we put $\mu=0$, we get § 3, (3), (4).

In these we may put $\mu=i\lambda$ and so obtain formulæ with circular numerators, but these are not of any interest, as they cannot be implified in particular cases. If $\mu=\frac{1}{2}$, we get § 4, (3), (4).

Equation (1) may be verified by another method; if the left-hand side be denoted by $F(\mu)$, we have

$$F(\mu_2)-F(\mu_1)=\tfrac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{\cosh x+\cosh\delta}\int_{\mu_1}^{\mu_2}e^{\mu x}\sin\mu\pi\,d\mu.$$

Now (§ 10) $\displaystyle\int_{-\infty}^{\infty}\frac{e^{\mu x}}{\cosh x+\cosh\delta}\,dx=\frac{2\pi}{\sinh\delta}\,\frac{\sinh\mu\delta}{\sin\mu\pi},$

thus $$F(\mu_2)-F(\mu_1)=\frac{\pi}{\sinh\delta}\int_{\mu_1}^{\mu_2}\sinh\mu\delta\,d\mu,$$

$$F(\mu)=\pi\frac{\cosh\mu\delta}{\delta\sinh\delta}+C,$$

where C is independent of μ.

§ 7. Let us consider next

$$\int\frac{e^{a(z-\pi i)}}{1+2pe^z-e^{2z}}\,\frac{dz}{z-\pi i},$$

If $p=\sinh\delta$, the poles are

$$\delta,\ \pi i-\delta,\ 2\pi i+\delta,\ \pi i,$$

and we find

$$(1)\quad \int_{-\infty}^{\infty}\frac{e^{\mu x}}{\sinh x-\sinh\delta}\,\frac{x\sin\mu\pi-\pi\cos\mu\pi}{x^2+\pi^2}\,dx$$
$$=\frac{\pi}{\sinh\delta}-\frac{\pi}{\cosh\delta}\left\{\frac{\delta\cos\mu\pi+\pi\sin\mu\pi}{\delta^2+\pi^2}\,e^{\mu\delta}+\frac{1}{\delta}\,e^{-\mu\delta}\right\},$$

where $\mu=\alpha-1$, and on the left the principal value is understood. This can be similarly verified, for if the integral on the left be $F(\mu)$,

$$F(\mu_2)-F(\mu_1)=\int_{-\infty}^{\infty}\frac{dx}{\sinh x-\sinh\delta}\int_{\mu_1}^{\mu_2}e^{\mu x}\sin\mu\pi\,d\mu,$$

and (§ 10)

$$\int_{-\infty}^{\infty}\frac{e^{\mu x}dx}{\sinh x-\sinh\delta}=\frac{\pi}{\cosh\delta}\,\frac{e^{-\mu\delta}-e^{\mu\delta}\cos\mu\pi}{\sin\mu\pi},$$

whence $$F(\mu)\frac{\cosh\delta}{\pi}=C-\frac{1}{\delta}e^{-\mu\delta}-e^{\mu\delta}\frac{\delta\cos\mu\pi+\pi\sin\mu\pi}{\delta^2+\pi^2},$$

where C is independent of μ.*

In (1) put $\mu=0$,

$$(2)\int_{-\infty}^{\infty}\frac{1}{\sinh x-\sinh\delta}\,\frac{dx}{x^2+\pi^2}=\frac{1}{\cosh\delta}\left\{\frac{\delta}{\delta^2+\pi^2}+\frac{1}{\delta}\right\}-\frac{1}{\sinh\delta}.$$

In this case there is not a distinct formula for

$$\int_{-\infty}^{\infty}\frac{1}{\sinh x-\sin\delta}\,\frac{dx}{x^2+\pi^2}.$$

Multiplying by $\cosh\delta$, and integrating from 0 to δ,*

$$(3)\int_{-\infty}^{\infty}\log\text{mod.}\left(1-\frac{\sinh\delta}{\sinh x}\right)\frac{dx}{x^2+\pi^2}=-\log\left\{\frac{\delta\sqrt{(\delta^2+\pi^2)}}{\pi\sinh\delta}\right\}.$$

Among other simple cases may be mentioned

$$(4)\int_{-\infty}^{\infty}\frac{e^{\alpha x}}{1-e^x}\,\frac{\pi\cos\alpha\pi-x\sin\alpha\pi}{x^2+\pi^2}\,dx=\tfrac{1}{2}\pi-\sin\alpha\pi.$$

If $\alpha=0,\ \frac{1}{2}$,

$$(5)\int_{-\infty}^{\infty}\frac{dx}{(1-e^x)(x^2+\pi^2)}=\tfrac{1}{2},\ \text{which is easy to verify,}$$

$$(6)\int_{0}^{\infty}\frac{1}{\sinh\frac{1}{2}x}\,\frac{x\,dx}{x^2+\pi^2}=\tfrac{1}{2}\pi-1,\quad\text{(De Haan 97, (8)).}$$

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* This is an example of the application of the processes of differentiation and integration with respect to a parameter to an integral given only by its principal value. For a general theorem on this subject see the *Educational Times*, Question 14297.

Other more general forms.

§ 8. An exactly similar procedure gives us the corresponding formulæ which come from

$$\int \frac{e^{a(z-\pi i)}}{1+2pe^z+e^{2z}} \frac{dz}{z-\lambda-\pi i},$$

such as

$$\int_{-\infty}^{\infty} \frac{1}{\cosh x + \cos\delta} \frac{dx}{(x-\lambda)^2+\pi^2},$$

but the general formulæ are complicated. Among special cases

$$(1) \int_{-\infty}^{\infty} \frac{1}{1-e^x} \frac{dx}{(x-\lambda)^2+\pi^2} = \frac{1}{1+e^\lambda} + \frac{\lambda}{\lambda^2+\pi^2},$$

$$(2) \int_{-\infty}^{\infty} \frac{1}{\sinh\frac{1}{2}x} \frac{(x-\lambda)\,dx}{(x-\lambda)^2+\pi^2} = \frac{\pi}{\cosh\frac{1}{2}\lambda} - \frac{2\pi^2}{\lambda^2+\pi^2},$$

which include § 7. (5), (6);

$$(3) \int_{-\infty}^{\infty} \frac{1}{1+e^x} \frac{dx}{(x-\lambda)^2+\pi^2} = \frac{1}{\lambda} - \frac{1}{e^\lambda-1},$$

$$(4) \int_{-\infty}^{\infty} \frac{1}{\cosh\frac{1}{2}x} \frac{(x-\lambda)\,dx}{(x-\lambda)^2+\pi^2} = \frac{\pi}{\sinh\frac{1}{2}\lambda} - \frac{2\pi}{\lambda}.$$

§ 9. So far we have always supposed that in (A) (§ 1) $a=\pm c$. Dropping this condition we can obtain integrals of a different form; thus consider

$$\int \frac{e^{a(z-\pi i)}}{e^{2z}+p} \frac{dz}{z-\pi i}, \quad (p>0)$$

the contour being the same as before.

The poles are πi, $\frac{1}{2}\pi i+\gamma$, $\frac{3}{2}\pi i+\gamma$, where $p=e^{2\gamma}$; the residues

$$\frac{1}{1+p}, \quad -\tfrac{1}{2}\frac{e^{(a-2)\gamma-\frac{1}{2}a\pi i}}{\gamma-\frac{1}{2}\pi i}, \quad -\tfrac{1}{2}\frac{e^{(a-2)\gamma+\frac{1}{2}a\pi}}{\gamma+\frac{1}{2}\pi i};$$

whence, if $a-1=\mu$,

$$(1) \int_{-\infty}^{\infty} \frac{e^{\mu x}}{e^x+pe^{-x}} \frac{x\sin\mu\pi - \pi\cos\mu\pi}{x^2+\pi^2}\,dx$$

$$= \pi\left\{\frac{1}{1+p} - e^{(\mu-1)\gamma}\frac{\frac{1}{2}\pi\cos\frac{1}{2}\mu\pi - \gamma\sin\frac{1}{2}\mu\pi}{\gamma^2+\frac{1}{4}\pi^2}\right\}.$$

Put $\mu=0$,

$$(2) \int_{-\infty}^{\infty} \frac{1}{e^x+pe^{-x}} \frac{dx}{x^2+\pi^2} = -\frac{1}{1+p} + \frac{\frac{1}{2}\pi e^{-\gamma}}{\gamma^2+\frac{1}{4}\pi^2}.$$

If $1+p=2a$, $1-p=2b$, $\gamma=\frac{1}{2}\log(a-b)$,

$$(3)\int_{-\infty}^{\infty}\frac{1}{a\cosh x+b\sinh x}\frac{dx}{x^2+\pi^2}$$
$$=\frac{1}{\sqrt{(a-b)}}\frac{4\pi}{\pi^2+\{\log(a-b)\}^2}-\frac{1}{a},$$

where $a>b$, $a+b=1$; if $b=0$ we find the formula § 3, (6) again.

If we put $\mu=\frac{1}{2}$, $(p>0)$,

$$(4)\int_{-\infty}^{\infty}\frac{e^{\frac{1}{2}x}}{e^x+pe^{-x}}\frac{xdx}{x^2+\pi^2}=\pi\left\{\frac{1}{1+p}-p^{-\frac{1}{4}}\sqrt{2}\,\frac{\pi-\log p}{\pi^2+(\log p)^2}\right\}.$$

It is clear from these examples that simple results are only to be expected when $a=\pm c$.

Integrals of the type (III.).

§ 10. Our third type of integral is

$$\int\frac{e^{(\alpha+\beta i)z}}{1+2pe^z+e^{2z}}dz,\quad(0<\alpha<2),$$

the contour being the same as for (II.).

It is not necessary to enter into details of the integration, which is in every way similar to that of §§ 6, 7. We obtain the two following integrals, due to Cauchy*;

$$(1)\int_0^{\infty}\frac{\cosh\rho x\cos\gamma x}{\cosh x+\cos\alpha}dx,\quad(\alpha<\pi,\ \rho<1)$$
$$=\frac{\pi}{\sin\alpha}\,\frac{\cosh(\alpha+\pi)\gamma\cos(\pi-\alpha)\rho-\cosh(\pi-\alpha)\gamma\cos(\alpha+\pi)\rho}{\cosh 2\pi\gamma-\cos 2\pi\rho},$$

$$(2)\int_0^{\infty}\frac{\sinh\rho x\sin\gamma x}{\cosh x+\cos\alpha}dx$$
$$=-\frac{\pi}{\sin\alpha}\,\frac{\sinh(\alpha+\pi)\gamma\sin(\pi-\alpha)\rho-\sinh(\pi-\alpha)\gamma\sin(\alpha+\pi)\rho}{\cosh 2\pi\gamma-\cos 2\pi\rho}.$$

If in these we put $\rho=\frac{1}{2}$, we obtain the integrals given by Dr. Glaisher, *Messenger*, Vol. XXVI.; if we put $\alpha=\frac{1}{2}\pi$ we get Poisson's integrals, quoted in the same paper. If we put $\gamma=0$ or $\rho=0$,

$$(3)\quad\int_0^{\infty}\frac{\cosh\rho x}{\cosh x+\cos\alpha}dx=\frac{\pi}{\sin\alpha}\,\frac{\sin\alpha\rho}{\sin\pi\rho}$$

* *Annales de Math. de Gergonne*, Vol. XVII.; De Haan, T. 267, 2, 6.

and another; the same process which gives these also gives

$$(4)\quad \int_0^\infty \frac{\cosh \rho x}{\cosh x + \cosh \alpha}\, dx = \frac{\pi}{\sinh \alpha}\, \frac{\sinh \alpha\rho}{\sin \pi\rho}$$

and another; and as in these ρ is capable of complex values

$$(5)\quad \int_0^\infty \frac{\cosh \rho x \cos \gamma x}{\cosh x + \cosh \alpha}\, dx$$

$$= \frac{\pi}{\sinh \alpha}\, \frac{\sinh \alpha\rho \sin \pi\rho \cos \alpha\gamma \cosh \pi\gamma + \cosh \alpha\rho \cos \pi\rho \sin \alpha\gamma \sinh \pi\gamma}{\cosh 2\pi\gamma - \cos 2\pi\rho}$$

$$(6)\quad \int_0^\infty \frac{\sinh \rho x \sin \gamma x}{\cosh x + \cosh \alpha}\, dx$$

$$= \frac{\pi}{\sinh \alpha}\, \frac{\cosh \alpha\rho \sin \pi\rho \sin \alpha\gamma \cosh \pi\gamma - \sinh \alpha\rho \cos \pi\rho \cos \alpha\gamma \sinh \pi\gamma}{\cosh 2\pi\gamma - \cos 2\pi\rho}.$$

In any of these we may expand in powers of ρ or γ, and equate coefficients; thus

$$(7)\quad \int_0^\infty \frac{x^2 \cos \gamma x}{\cosh x + \cos \alpha}\, dx = -\frac{2 \sinh \alpha\gamma}{\sin \alpha} \left\{\frac{\pi}{\sinh \pi\gamma}\right\}^3$$

$$+ \frac{\pi}{2 \sin \alpha \sinh^2 \pi\gamma} \{(\alpha+\pi)^2 \cosh (\pi-\alpha)\gamma - (\pi-\alpha)^2 \cosh (\alpha+\pi)\gamma\},$$

and there are three corresponding formulæ for

$$\int_0^\infty \frac{x^2 \cos \gamma x}{\cosh x + \cosh \alpha}\, dx,\ \int_0^\infty \frac{x^2 \cosh \rho x}{\cosh x + \cos \alpha}\, dx,\ \int_0^\infty \frac{x^2 \cosh \rho x}{\cosh x + \cosh \alpha}\, dx.$$

Also

$$(8)\quad \int_0^\infty \frac{x^2 dx}{\cosh x + \cos \alpha} = \tfrac{1}{3}\alpha\, (\pi^2 - \alpha^2) \operatorname{cosec} \alpha,$$

$$(9)\quad \int_0^\infty \frac{x^4 dx}{\cosh x + \cos \alpha} = \tfrac{1}{5}\alpha\, (\pi^2 - \alpha^2)\, (7\pi^2 - 3\alpha^2) \operatorname{cosec} \alpha.$$

These are the integrals of § 9 of Dr. Glaisher's paper; we can easily write down from (3) the general value in Bernoullian numbers of

$$\int_0^\infty \frac{x^{2n} dx}{\cosh x + \cos \alpha};$$

in all of these formulæ we may put $i\alpha$ for α, $\cosh \alpha$ for $\cos \alpha$.

Again, in any of the formulæ (1)—(9), we may multiply up by $\sin\alpha$ or $\sinh\alpha$, and integrate with respect to α between suitable limits; thus from (3) and (4),

$$(10)\quad \int_0^\infty \cos\gamma x \log\left(1+\frac{\cos\alpha}{\cosh x}\right)dx$$
$$= \frac{\pi}{\gamma\sinh\pi\gamma}(\cosh\tfrac{1}{2}\pi\gamma - \cosh\alpha\gamma),$$

$$(11)\quad \int_0^\infty \cosh\rho x \log\left(1+\frac{\cos\alpha}{\cosh x}\right)dx$$
$$= \frac{\pi}{\rho\sin\pi\rho}[\cos\alpha\rho - \cos\tfrac{1}{2}\pi\rho],$$

$$(12)\quad \int_0^\infty \log\left(1+\frac{\cos\alpha}{\cosh x}\right)dx = \tfrac{1}{8}(\pi^2-4\alpha^2),$$

$$(13)\quad \int_0^\infty x^2\log\left(1+\frac{\cos\alpha}{\cosh x}\right)dx = \tfrac{1}{192}(7\pi^2-4\alpha^2)(\pi^2-4\alpha^2),$$

and so on.

To complete the set of formulæ corresponding to (3), (4), we find the principal values of the corresponding integrals whose denominators are

$$\cosh x - \cosh\alpha, \quad \sinh x - \sinh\alpha;$$

$$(14)\quad \int_0^\infty \frac{\cosh\rho x}{\cosh x - \cosh\alpha}dx = -\pi\cot\pi\rho\,\frac{\sinh\alpha\rho}{\sinh\alpha},$$

$$(15)\quad \int_{-\infty}^\infty \frac{\cosh\rho x}{\sinh x - \sinh\alpha}dx = -\pi\cot\tfrac{1}{2}\pi\rho\,\frac{\sinh\alpha\rho}{\cosh\alpha},$$

$$(16)\quad \int_{-\infty}^\infty \frac{\sinh\rho x}{\sinh x - \sinh\alpha}dx = \pi\tan\tfrac{1}{2}\pi\rho\,\frac{\cosh\alpha\rho}{\cosh\alpha}.$$

From any of these follow formulæ analogous to (1), (2), (7), (8), (9); or to (10)–(13).

If we take $\int \frac{e^{(\mu+1)z}}{e^{2z}-2ipe^z-1}dz$, we find, similarly,

$$(17)\quad \int_{-\infty}^\infty \frac{e^{\mu x}}{\sinh x - i\sin\alpha}dx = \frac{2\pi i}{\sin\mu\pi}\cdot\frac{e^{-\frac{1}{2}\mu\pi i}}{\cos\alpha}\sin(\tfrac{1}{2}\pi-\alpha)\mu,$$

where $0<\alpha<\pi$; this equation is obviously untrue if $\alpha=0$ or π.

Hence, multiplying by $-i\cos\alpha$, and integrating from 0 to α, we get, on equating imaginary parts,

$$(18)\ \int_0^\infty \sinh\mu x \tan^{-1}\left(\frac{\sin\alpha}{\sinh x}\right) dx = \frac{\pi \sin\frac{1}{2}\mu\alpha \sin\frac{1}{2}\mu(\pi-\alpha)}{\mu\cos\frac{1}{2}\mu\pi},$$

which happens to be true for $\alpha = 0, \pi$. It may be verified by noticing that, if α is small, it gives

$$\alpha\int_0^\infty \frac{\sinh\mu x}{\sinh x}\,dx = \tfrac{1}{2}\alpha\pi\tan\tfrac{1}{2}\mu\pi,$$

a case of (16). Other integrals of this form may be treated similarly.

§ 11. As in § 9 we may drop the condition $a = \pm c$, and so evaluate integrals whose denominators are of the form

$$l\cosh x + m\sinh x + n,$$

between $-\infty$ and $+\infty$. But the denominator can always be put in the form

$$A\left\{\begin{matrix}\cosh\\ \sinh\end{matrix}(x+a)+p\right\},$$

and, by putting $x + a = y$, the integral is reduced to a sum of types already evaluated. This also shows that the integrals of § 9 can be considered under § 8, or *vice versa.*

§ 12. In conclusion we may mention two simple transformations of some of these integrals.

(1) Consider

$$u = \int_\alpha^\beta F\left\{\log\left(\frac{z-\alpha}{\beta-z}\right)\right\}\frac{dz}{z},$$

where F is an *even* function; let

$$\frac{z-\alpha}{\beta-z} = y,$$

$$u = \int_0^\infty F(\log y)\left(\frac{\beta}{\alpha+\beta y} - \frac{1}{1+y}\right)dy$$

$$= \int_0^1 F(\log y)\left(\frac{\beta}{\alpha+\beta y} - \frac{\alpha}{\beta+\alpha y}\right)dy.$$

Let $y=e^{-x}$, $\beta=\alpha e^{\gamma}$, then

$$u=\sinh\gamma\int_0^\infty F(x)\frac{dx}{\cosh x+\cosh\gamma}.$$

Thus we can find, by the help of § 10,

$$\int_\alpha^\beta \cos\left\{a\log\left(\frac{z-\alpha}{\beta-z}\right)\right\}\frac{dz}{z},\quad \int_\alpha^\beta\left\{\log\left(\frac{z-\alpha}{\beta-z}\right)\right\}^{2n}\frac{dz}{z};$$

the latter are also easily found by integrating round the loop $(\alpha\beta)$ the functions

$$\left\{\log\left(\frac{z-\alpha}{\beta-z}\right)\right\}^{p+1}\frac{1}{z},\ p=0,\ 1,\ 2,\ \ldots$$

(Harkness and Morley, *Th. of Functions*, p. 199).

(2) Integrals of the form $\int_0^\infty \frac{\phi(x)\,dx}{\cosh x+\cos\alpha}$ may be transformed by putting

$$(\cosh x+\cos\alpha)(\cos\theta-\cos\alpha)=\sin^2\alpha;$$

thus from § 10, (8) we find

$$\int_0^a\left\{\log\left(\frac{\tan a+\tan\theta}{\tan a-\tan\theta}\right)\right\}^2 d\theta=\tfrac{1}{3}a\,(\pi^2-4a^2),$$

which, if $a=\frac{1}{4}\pi$, gives

$$\int_0^{\frac{1}{4}\pi}(\log\tan\theta)^2\,d\theta=\tfrac{1}{16}\pi^3,$$

which is easy to verify. Similarly from § 3, (6) we get the value of

$$\int_0^a\frac{d\theta}{\pi^2+\left\{\log\frac{\sin\frac{1}{2}(a+\theta)}{\sin\frac{1}{2}(a-\theta)}\right\}^2},$$

which, in the case of $a=\frac{1}{2}\pi$, gives

$$\int_0^{\frac{1}{4}\pi}\frac{d\theta}{\pi^2+(\log\tan\theta)^2}=\frac{1}{\pi}-\tfrac{1}{4}$$

(De Haan, T. 129, (6))

ON THE REGULAR AND SEMI-REGULAR FIGURES IN SPACE OF n DIMENSIONS.

By *Thorold Gosset.*

THE following is an attempt to give a complete list of all the regular and semi-regular figures in a Euclidean space of n dimensions where n is greater than 3. These results were obtained between two and three years ago, and as far as the regular figures are concerned agree with those given by Mr. Curjel in the April number of this year's *Messenger of Mathematics.** A list of the semi-regular hypersolids has not, it is believed, been previously given. For the sake of brevity a figure in a Euclidean space of n dimensions is called an n-ic figure, and an infinite n-ic fignre an $(n-1)$-ic check.

REGULAR FIGURES.

In space of n dimensions there are four regular figures.

1. n-ic *Pyramid.* Analogous to the tetrahedron. It is bounded by $n+1$ $(n-1)$-ic pyramids, n of which meet in each summit; and by $\dfrac{(n+1)!}{(r+1)!(n-r)!}$ r-ic pyramids, $\dfrac{n!}{r!(n-r)!}$ of which meet in each summit, where r may have any value from $(n-2)$ to 1. It has $n+1$ summits. $\dfrac{(n-t)!}{(r-t)!(n-r)!}$ r-ic pyramids meet in each t-ic pyramid.

The radius of its circumscribing sphere is $\dfrac{\sqrt{\{\frac{1}{2}n(n+1)\}}}{n+1}$ times the length of its edge. It is self-reciprocal.

2. n-ic *Double Pyramid.* Analogous to the octahedron. It is bounded by 2^n $(n-1)$-ic pyramids, 2^{n-1} of which meet in each summit; and by $\dfrac{2^{r+1}n!}{(r+1)!(n-r-1)!}$ r-ic pyramids, $\dfrac{2^r(n-1)!}{r!(n-r-1)!}$ of which meet in each summit, where r may have any value from $(n-2)$ to 1. It has $2n$ summits. $\dfrac{2^{r-t}(n-t-1)!}{(r-t)!(n-r-1)!}$ r-ic pyramids meet in each t-ic pyramid.

The radius of its circumscribing sphere is $\frac{1}{2}\sqrt{2}$ times the length of its edge. Its reciprocal is the n-ic cube.

3. n-ic *Cube.* Analogous to the cube. It is bounded by $2n$ $(n-1)$-ic cubes, n of which meet in each summit; and by

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* Vol. XXVIII., p. 190.

$\frac{2^{n-r}n!}{r!\,(n-r)!}$ r-ic cubes, $\frac{n!}{r!\,(n-r)!}$ of which meet in each summit, where r may have any value from $n-2$ to 1. It has 2^n summits. $\frac{(n-t)!}{(r-t)!\,(n-r)!}$ r-ic cubes meet in each t-ic cube. The radius of its circumscribing sphere is $\frac{1}{2}\sqrt{n}$ times the length of its edge. Its reciprocal is the n-ic double pyramid.

4. $(n-1)$-ic *Check*. Analogous to the pattern of a chess-board. This figure is infinite. It is bounded by ∞ $(n-1)$-ic cubes, 2^{n-1} of which meet in each summit; and by $\frac{(n-1)!}{(n-r-1)!\,r!}\infty$ r-ic cubes, $\frac{2^r(n-1)!}{r!\,(n-r-1)!}$ of which meet in each summit, where r may have any value from $n-2$ to 1. It has ∞ summits, $\frac{2^{r-t}(n-t-1)!}{(r-t)!\,(n-r-1)!}$ r-ic cubes meet in each t-ic cube. It is self-reciprocal.

In space of four dimensions there are three additional regular figures.

1. *Octahedric*. No analogy in three dimensions. It is bounded by 24 octahedra, 6 of which meet in each summit, and 3 of which meet in each edge; 96 triangles, 12 of which meet in each summit and 3 of which meet in each edge; and 96 edges, 8 of which meet in each summit. It has 24 summits. The radius of its circumscribing sphere is equal to the length of its edge. It is self-reciprocal.

2. *Tetrahedric*. Analogous to the icosahedron. It is bounded by 600 tetrahedra, 20 of which meet in each summit and 5 of which meet in each edge; 1200 triangles, 30 of which meet in each summit and 5 of which meet in each edge; and 720 edges, 12 of which meet in each summit. It has 120 summits. The radius of its circumscribing sphere is $\frac{1}{2}(\sqrt{5}+1)$ times the length of its edge. Its reciprocal is the dodecahedric.

3. *Dodecahedric*. Analogous to the dodecahedron. It is bounded by 120 dodecahedra, 4 of which meet in each summit and 3 of which meet in each edge; 720 pentagons, 6 of which meet in each summit and 3 of which meet in each edge; and 1200 edges, 4 of which meet in each summit. It has 600 summits. The radius of its circumscribing sphere is $\frac{1}{2}\sqrt{2}\,(3+\sqrt{5})$ times the length of its edge. Its reciprocal is the tetrahedric.

In space of five dimensions there are two additional regular figures, both of which are infinite.

1. *Octahedric Check.* It is bounded by ∞ octahedrics, 8 of which meet in each summit, 4 of which meet in each edge, and 3 of which meet in each triangle; 12 ∞ octahedra, 24 of which meet in each summit, 6 of which meet in each edge, and 3 of which meet in each triangle; 32 ∞ triangles, 32 of which meet in each summit, and 4 of which meet in each edge; and 24 ∞ edges, 16 of which meet in each summit. It has 3 ∞ summits. Its reciprocal is the double pyramidal check.

2. *Double Pyramidal Check.* It is bounded by 3 ∞ 4-ic double pyramids, 24 of which meet in each summit, 6 of which meet in each edge, and 3 of which meet in each triangle; 24 ∞ tetrahedra, 96 of which meet in each summit, 12 of which meet in each edge, and 3 of which meet in each triangle; 32 ∞ triangles, 96 of which meet in each summit, and 8 of which meet in each edge; and 12 ∞ edges, 24 of which meet in each summit. It has ∞ summits. Its reciprocal is the octahedric check.

Semi-regular Figures.

In space of n dimensions there is one infinite semi-regular figure.

$(n-1)$-ic *Semi-check.* Analogous to the pattern of a single file of a chess board. It is bounded by 2 $(n-2)$-ic semi-checks and ∞ $(n-1)$-ic cubes, one of the former meeting 2^{n-2} of the latter in each summit. It is also bounded by $\frac{(2n-r-2)(n-2)!}{r!(n-r-1)!}\infty$ r-ic cubes, of which $\frac{2^{r-1}(2n-r-2)(n-2)!}{r!(n-r-1)!}$ meet in each summit, where r may have any value from $(r-2)$ to 1. It has 2 ∞ summits. Either

$$\frac{2^{r-t-1}(2n-r-t-2)(n-t-2)!}{(r-t)!(n-r-1)!} \text{ or } \frac{2^{r-t}(n-t-1)!}{(r-t)!(n-r-1)!}$$

r-ic cubes meet in each t-ic cube according to whether the t-ic cube is or is not a boundary of an $(n-2)$-ic semi-check.

In space of four dimensions there are six additional semi-regular figures.

1. *Tetroctahedric.* It is bounded by 5 octahedra and 5 tetrahedra, 3 of the former meeting 2 of the latter in each summit, and 2 of the former meeting 1 of the latter in each edge. It is also bounded by 30 triangles, 9 of which meet in each summit, and 3 of which meet in each edge; and by

30 edges, 6 of which meet in each summit. It has 10 summits. The radius of its circumscribing sphere is $\frac{1}{5}\sqrt{(15)}$ times the length of its edge.

2. *Tetricosahedric.* It is bounded by 24 icosahedra and 120 tetrahedra, 3 of the former meeting 5 of the latter in each summit; 480 triangles, 15 of which meet in each summit; and 432 edges, 9 of which meet in each summit. 2 icosahedra, 1 tetrahedron and 3 triangles meet in 6 out of the 9 edges meeting in each summit; and 1 icosahedron, 3 tetrahedra and 4 triangles meet in the remaining 3 out of the 9 edges meeting in each summit. It has 96 summits. The radius of its circumscribing sphere is $\frac{1}{2}(\sqrt{5}+1)$ times the length of its edge.

3. *Octicosahedric.* It is bounded by 120 icosahedra and 600 octahedra, 2 of the former meeting 5 of the latter in each summit, and 1 of the former meeting 2 of the latter in each edge. It is also bounded by 3600 triangles 15 of which meet n each summit, and 3 of which meet in each edge; and by 3600 edges, 10 of which meet in each summit. It has 720 summits. The radius of its circumscribing sphere is $\sqrt{(5+2\sqrt{5})}$ times the length of its edge.

4. *Simple Tetroctahedric Check.* This figure is infinite. It is bounded by ∞ octahedra and $2\,\infty$ tetrahedra, 6 of the former and 8 of the latter meeting in each summit; and 2 of the former and 2 of the latter meeting in each edge, in such a way that the 2 octahedra, and consequently the 2 tetrahedra are not adjacent. It is also bounded by $8\,\infty$ triangles, 24 of which meet in each summit, and 4 of which meet in each edge; and $6\,\infty$ edges, 12 of which meet in each summit. It has ∞ summits.

5. *Complex Tetroctahedric Check.* This figure is infinite. It is bounded by ∞ octahedra and $2\,\infty$ tetrahedra, 6 of the former and 8 of the latter meeting in each summit; and 2 of the former and 2 of the latter meeting in each edge. It is also bounded by $8\,\infty$ triangles, 24 of which meet in each summit and 4 of which meet in each edge; and $6\,\infty$ edges, 12 of which meet in each summit. It has ∞ summits. This figure differs from the simple tetroctahedric check in having the 2 octahedra and 2 tetrahedra, which meet in 6 out of the 12 edges meeting in each summit, placed so that the 2 octahedra and consequently the 2 tetrahedra are adjacent. The 2 octahedra and 2 tetrahedra, which meet in the remaining 6 out of the 12 edges meeting in each summit, are placed so that the 2 octahedra and consequently the 2 tetrahedra are not adjacent.

6. *Tetroctahedric Semi-check.* This figure is infinite. It is bounded by ∞ octahedra, 2 ∞ tetrahedra, and 2 triangular checks (that is to say infinite 3-ic figures bounded by equilateral triangles meeting 6 at a point); 3 octahedra, 4 tetrahedra, and 1 triangular check meeting in each summit. It is also bounded by 10 ∞ triangles, 15 of which meet in each summit; and 9 ∞ edges, 9 of which meet in each summit. 1 octahedron, 1 tetrahedron, 1 triangular check, and 3 triangles meet in 6 out of the 9 edges meeting in each summit; and 2 octahedra, 2 tetrahedra, and 4 triangles meet in the remaining 3 out of the 9 edges meeting in each summit, in such a way that the 2 octahedra and consequently the 2 tetrahedra are not adjacent.

In space of five dimensions there is one additional semi-regular figure.

5-ic *Semi-regular.* It is bounded by 10 4-ic double pyramids, 16 4-ic pyramids, 120 tetrahedra, 160 triangles, 80 edges and 16 summits. 5 4-ic double pyramids, 5 4-ic pyramids, 30 tetrahedra, 30 triangles, and 10 edges meet in each summit; 3 4-ic double pyramids, 2 4-ic pyramids, 9 tetrahedra, and 6 triangles meet in each edge; 2 4-ic double pyramids, 1 4-ic pyramid, and 3 tetrahedra meet in each triangle. The radius of its circumscribing sphere is $\frac{1}{4}\sqrt{(10)}$ times the length of its edge.

In space of six dimensions there is one additional semi-regular figure.

6-ic *Semi-regular.* It is bounded by 27 5-ic double pyramids, 72 5-ic pyramids, 648 4-ic pyramids, 1080 tetrahedra, 720 triangles, 216 edges and 27 summits. 10 5-ic double pyramids, 16 5-ic pyramids, 120 4-ic pyramids, 160 tetrahedra, 80 triangles and 16 edges meet in each summit. As many r-ic boundaries meet in each t-ic boundary as there are $(r-1)$-ic boundaries meeting in each $(t-1)$-ic boundary in the 5-ic semi-regular. The radius of its circumscribing sphere is $\frac{1}{3}\sqrt{6}$ times the length of its edge.

In space of seven dimensions there is one additional semi-regular figure.

7-ic *Semi-regular.* It is bounded by 126 6-ic double pyramids, 576 6-ic pyramids, 6048 5-ic pyramids, 12096 4-ic pyramids, 10080 tetrahedra, 4032 triangles, 756 edges and 56 summits. 27 6-ic double pyramids, 72 6-ic pyramids, 648 5-ic pyramids, 1080 4-ic pyramids, 720 tetrahedra, 216 triangles, and 27 edges meet in each summit. As many r-ic boundaries meet in each t-ic boundary as there are $(r-1)$-ic boundaries meeting in each $(t-1)$-ic boundary in the 6-ic semi-regular, or $(r-2)$-ic boundaries meeting in each $(t-2)$-ic boundary in the 5-ic semi-regular. The radius of its circumscribing sphere is $\frac{1}{2}\sqrt{3}$ times the length of its edge.

In space of eight dimensions there is one additional semi-regular figure.

8-ic *Semi-regular.* It is bounded by 2160 7-ic double pyramids, 17280 7-ic pyramids, 207360 6-ic pyramids, 483840 5-ic pyramids, 4838404-ic pyramids, 241920 tetrahedra, 60480 triangles, 6720 edges and 240 summits. It has as many r-ic boundaries meeting in each summit as there are $(r-1)$-ic boundaries to the 7-ic semi-regular. As many r-ic boundaries meet in each t-ic boundary as there are $(r-1)$-ic boundaries meeting in each $(t-1)$-ic boundary in the 7-ic semi-regular, or $(r-2)$-ic boundaries meeting in each $(t-2)$-ic boundary in the 6-ic semi-regular, or $(r-3)$-ic boundaries meeting in each $(t-3)$-ic boundary in the 5-ic semi-regular. The radius of its circumscribing sphere is equal to the length of its edge.

In space of nine dimensions there is one additional semi-regular figure, which is infinite.

9-ic *Semi-regular.* It is bounded by 135 ∞ 8-ic double pyramids, 1920 ∞ 8-ic pyramids, 25920 ∞ 7-ic pyramids, 61920 ∞ 6-ic pyramids, 80640 ∞ 5-ic pyramids, 48384 ∞ 4-ic pyramids, 15120 ∞ tetrahedra, 2240 ∞ triangles, 120 ∞ edges, and ∞ summits. It has as many r-ic boundaries meeting in each summit as there are $(r-1)$-ic boundaries to the 8-ic semi-regular; as many r-ic boundaries meeting in each edge as there are $(r-2)$-ic boundaries to the 7-ic semi-regular. As many r-ic boundaries meet in each t-ic boundary as there are $(r-1)$-ic boundaries meeting in each $(t-1)$-ic boundary in the 8-ic semi-regular, or $(r-2)$-ic boundaries meeting in each $(t-2)$-ic boundary in the 7-ic semi-regular, or $(r-3)$-ic boundaries meeting in each $(t-3)$-ic boundary in the 6-ic semi-regular, or $(r-4)$-ic boundaries meeting in each $(t-4)$-ic boundary in the 5-ic semi-regular.

[The absolute (as distinguished from the relative) magnitude of the coefficients of ∞, which occur in giving the number of r-ic boundaries of the infinite figures are of course arbitrary, and are merely chosen so as to avoid fractions, and keep the integral coefficients as small as possible.]

FUNDAMENTAL THEOREMS RELATING TO THE BERNOULLIAN NUMBERS.

By *J. W. L. Glaisher.*

I.

Staudt's Theorem.

§ 1. STAUDT'S well-known theorem may be enunciated as follows: If $1, 2, a, a', \ldots, 2n$ be all the divisors of $2n$, and if 1 be added to each of these divisors so as to form the series $2, 3, a+1, a'+1, \ldots, 2n+1$, and if from this series only the prime numbers $2, 3, p, p', \ldots$ be selected, then the fractional part of the nth Bernoullian number B_n is equal to

$$(-1)^n\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{p}+\frac{1}{p'}+\ldots\right).$$

Thus the denominator of B_n consists of a simple product $2, 3, p, p', \ldots$ of prime factors; and a prime p is one of these factors if $p-1$ is a divisor of $2n$.

It is convenient to call a prime factor of the denominator of B_n a Staudt factor of B_n, so that for a prime p to be a Staudt factor of B_n it is sufficient that $p-1$ should be a divisor of $2n$.

The theorems of Sylvester and Adams.

§ 2. In the *Comptes Rendus** for 1861 Sylvester enunciated the theorem that if p^i is a divisor of n, and $p-1$ is not, then the numerator of B_n will be divisible by p^i; and he pointed out as a corollary that if n itself is prime, the numerator of B_n is divisible by n.

Subsequently Adams observed this theorem† to be true in the case $i=1$ for the Bernoullian numbers up to B_{62}, and

* Vol. LII., p. 162. Sylvester's exact words are worth quoting. After referring to Staudt's theorem he proceeds: "Mais on paraît ne pas avoir fait la remarque importante que le numérateur de B_n contiendra tous les facteurs de n qui ne sont pas puissances des facteurs du dénominateur, de telle sorte que, si n contient p^i, mais ne contient pas $p-1$, le numérateur de B_n contiendra p^i; comme corollaire, on peut remarquer que, p étant un nombre premier quelconque, le numérateur de B_p contiendra toujours p."

† Appendix I. to Vol. XXIII. of the *Cambridge Observations*, p. iii (*Collected Works*, Vol. I., p. 430). Adams's values of the Bernoullian numbers first appeared in the *British Association Report* for 1877, and in *Crelle's Journal*, Vol. LXXXV. p. 269; in both places the sentences quoted in the text also occur.

proved that n, if prime, was a divisor of the numerator of B_n. His own words, in concluding the account of his calculation of the thirty-one Bernoullian numbers from B_{32} to B_{62} inclusive, are: "I have found that if n be a prime number other than 2 and 3, then the numerator of the nth number of Bernoulli will be divisible by n. This forms an excellent test of the correctness of the work.

"I have also observed that if q be a prime factor of n, which is not likewise a factor of the denominator of B_n, then the numerator of B_n will be divisible by q. I have not succeeded, however, in obtaining a general proof of this proposition, though I have no doubt of its truth."

So far as I know no proof has been published of Sylvester's general theorem, or of the result observed by Adams, which is included in it. The object of this paper is to give a proof of Sylvester's theorem.

I have also, in a second Part (p. 60), enunciated another theorem relating to the Bernoullian numbers.

Proof of Sylvester's theorem.

§ 3. The proof consists in showing that if a is any integer, then $\dfrac{(a^{2n}-1)B_n}{n}$ is always of the form $\dfrac{\text{integer}}{a^{\mu}2^{\nu}}$. From this result it can easily be deduced (§ 8) that if p^i is a divisor of n and p is not a Staudt factor, then the numerator of B_n must be divisible by p^i.

§ 4. The Bernoullian numbers may be regarded as defined by the equation

$$\frac{1}{e^x-1}=\frac{1}{x}-\frac{1}{2}+\frac{B_1}{2!}x-\frac{B_2}{4!}x^3+\frac{B_3}{6!}x^5-\&c.$$

Putting ax for x, we have

$$\frac{a}{e^{ax}-1}=\frac{1}{x}-\frac{a}{2}+\frac{B_1}{2!}a^2x-\frac{B_2}{4!}a^4x^3+\frac{B_3}{6!}a^6x^5-\&c.;$$

whence, by subtraction,

$$\frac{1}{e^x-1}-\frac{a}{e^{ax}-1}=\frac{a-1}{2}-\frac{(a^2-1)B_1}{2!}x+\frac{(a^4-1)B_2}{4!}x^3-\&c.$$

Now let

$$\psi_n(a)=\frac{(a^{2n}-1)B_n}{2n},\ n>0,$$

and for convenience put

$$\psi_0(a) = \frac{a-1}{2}.$$

Thus we have

$$\frac{1}{e^x - 1} - \frac{a}{e^{ax} - 1} = \psi_0(a) - \frac{\psi_1(a)}{1\,!}\,x + \frac{\psi_2(a)}{3\,!}\,x^3 - \frac{\psi_3(a)}{5\,!}\,x^5 + \&c.$$

§ 5. Now, supposing a to be a positive integer,

$$\frac{1}{e^x - 1} - \frac{a}{e^{ax} - 1} = \frac{e^{(a-1)x} + e^{(a-2)x} + \ldots + e^x + 1 - a}{e^{ax} - 1},$$

and the numerator

$$= (e^x - 1)\,\{e^{(a-2)x} + 2e^{(a-3)x} + 3e^{(a-4)x} + \ldots + (a-2)\,e^x + a - 1\},$$

so that

$$\frac{1}{e^x - 1} - \frac{a}{e^{ax} - 1} = \frac{e^{(a-2)x} + 2e^{(a-3)x} + \ldots + (a-2)\,e^x + a - 1}{e^{(a-1)x} + e^{(a-2)x} + \ldots + e^x + 1}.$$

§ 6. Thus we have

$$e^{(a-2)x} + 2e^{(a-3)x} + 3e^{(a-4)x} + \ldots + (a-2)\,e^x + a - 1$$
$$= \{e^{(a-1)x} + e^{(a-2)x} + e^{(a-3)x} + \ldots + e^x + 1\}$$
$$\times \left\{\psi_0(a) - \frac{\psi_1(a)}{1\,!}\,x + \frac{\psi_2(a)}{3\,!}\,x^3 - \frac{\psi_3(a)}{5\,!}\,x^5 + \&c.\right\}.$$

Now let

$$S_r(a) = 1^r + 2^r + 3^r + \ldots + (a-1)^r;$$

then the right-hand side

$$= \left\{a + S_1(a)\,x + \frac{S_2(a)}{2\,!}\,x^2 + \frac{S_3(a)}{3\,!}\,x^3 + \&c.\right\}$$
$$\times \left\{\psi_0(a) - \frac{\psi_1(a)}{1\,!}\,x + \frac{\psi_2(a)}{3\,!}\,x^3 - \frac{\psi_3(a)}{5\,!}\,x^5 + \&c.\right\}.$$

Equating the coefficients of x^{2n-1}, we find

$$a\psi_n(a) - (2n-1)_2\,S_2(a)\,\psi_{n-1}(a) + (2n-1)_4\,S_4(a)\,\psi_{n-2}(a) - \ldots$$
$$+ (-1)^{n-1}\,(2n-1)_{2n-2}\,S_{2n-2}(a)\,\psi_1(a) + (-1)^n\,S_{2n-1}(a)\,\psi_0(a)$$
$$= (-1)^n\,\{(a-2)^{2n-1} + 2\,(a-3)^{2n-1} + 3\,(a-4)^{2n-1} + \ldots\,(a-2)\,1^{2n-1}\},$$

that is,

$$a\psi_n(a)-(2n-1)_2\,S_2(a)\,\psi_{n-1}(a)+(2n-1)_4\,S_4(a)\,\psi_{n-2}(a)-\dots$$
$$+(-1)^{n-1}(2n-1)_{2n-2}\,S_{2n-2}(a)\,\psi_1(a)$$
$$=(-1)^n\left[(a-2)^{2n-1}+2(a-3)^{2n-1}+3(a-4)^{2n-1}+\dots+(a-2)\,1^{2n-1}\right.$$
$$\left.-\tfrac{1}{2}(a-1)\{1^{2n-1}+2^{2n-1}+3^{2n-1}+\dots+(a-1)^{2n-1}\}\right],$$

where $(n)_r$ denotes the number of combinations of n things taken r together.

§ 7. In this recurring equation the coefficients $(2n-1)_2$, $(2n-1)_4$, ... are all integers, as also are $S_2(a)$, $S_4(a)$, ... which are sums of powers. The right-hand side also is an integer if a is uneven, and may have the denominator 2 if a is even.

Calculating $\psi_1(a)$, $\psi_2(a)$, $\psi_3(a)$, ... from this recurring equation by putting $n=1, 2, 3, \dots$, we see that $\psi_n(a)$ can contain only powers of a in the denominator besides the factor 2 which may occur when a is even, viz. $\psi_1(a)$, $\psi_2(a)$, ... are given by the equations

$$a\psi_1(a)=-\{a-2+2(a-3)+\dots+(a-2)\,1\}$$
$$+\tfrac{1}{2}(a-1)\{1+2+3+\dots+a-1\},$$
$$a\psi_2(a)-3S_2(a)\,\psi_1(a)=(a-2)^3+2(a-1)^3+\dots+(a-2)\,1^3$$
$$-\tfrac{1}{2}(a-1)\{1^3+2^3+3^3+\dots+(a-1)^3\},$$

and so on.

Thus $\psi_n(a)$ must be of the form $\dfrac{\text{integer}}{a^\mu 2^\nu}$, and we see further that μ cannot be greater than n (since only one additional a can enter into the denominator with each new equation), and $\nu=0$ if a is uneven, and may $=1$ if a is even.

§ 8. Now suppose that p is a prime factor of n, and take a to be a primitive root of p, *i.e.* so that a^{p-1} is the lowest power of a for which the residue, mod. p, is 1. We know that $a^n\times 2\times\dfrac{(a^{2n}-1)\,B_n}{2n}$ must be an integer, and therefore p must be either a divisor of $a^{2n}-1$ or a divisor of the numerator of B_n. Now, if p is not a Staudt factor of B_n, *i.e.* if $p-1$ is not a divisor of $2n$, then a^{2n} cannot be $\equiv 1$, mod. p, *i.e.* p

cannot be a divisor of $a^{2n}-1$; and therefore the numerator of B_n must be divisible by p. This is the result which Adams observed to be true.

If p is a Staudt factor, then $2n = lp(p-1)$, l being an integer, and therefore (by the generalised Fermat's theorem) $a^{2n}-1$ is divisible by p^2, as it should be.

Exactly the same reasoning holds good when p^i is a factor of n; for, if p is not a Staudt factor of B_n, $a^{2n}-1$ cannot be divisible by p, and therefore the numerator of B_n must be divisible by p^i. This is Sylvester's theorem.

If p is a Staudt factor, then $2n = lp^i(p-1)$, and therefore $a^{2n}-1$ is divisible by p^{i+1} as it should be.

The cases $a=2$ and $a=3$.

§ 9. The proof of Sylvester's theorem is very easily completed when it has been shown that $a^n \times \dfrac{(a^{2n}-1)B_n}{n}$ is integral, a being an arbitrary integer. For having a at our disposal we can take it to be a primitive root of p, and so secure that $a^{2n}-1$ cannot be divisible by p unless p is a Staudt factor of B_n. The integral character of $a^n \dfrac{(a^{2n}-1)B_n}{n}$ was established by means of a recurring formula (§ 6) in which the coefficient of $\psi_n(a)$, the highest ψ, was independent of n.

Now it can be shown by considerations of an entirely different kind that $\dfrac{(2^{2n}-1)B_n}{n}$ and $\dfrac{(3^{2n}-1)B_n}{n}$ are integers, except for powers of 2 and 3 respectively, and it is therefore of some interest to examine for what values of p, and for how many consecutive values of n beginning with $n=1$, either of these particular results affords a demonstration of the theorem.

§ 10. The formula

$$(2n-1)_1 E_{n-1} - (2n-1)_3 E_{n-2} + \ldots + (-1)^{n-1} E_0 = 2^{2n-1}(2^{2n}-1)\frac{B_n}{n},$$

in which $E_0 = 1$ and E_1, E_2, ... are the Eulerian numbers and therefore integers, shows that $2^{2n-1}(2^{2n}-1)\dfrac{B_n}{n}$ must be integral.

The fact that $\dfrac{(2^{2n}-1)B_n}{n}$ is an integer (except for powers

* *Quarterly Journal*, Vol. XXIX., p. 144.

of 2) affords a proof of the theorem that if n is prime, the numerator of B_n is divisible by n; for if n is prime, $2^{2n-2} \equiv 1$, mod. n; therefore $2^{2n}-1$ cannot be divisible by n, and therefore the numerator of B_n must be so divisible. It also, of course, proves the general theorem for all values of p of which 2 is a primitive root, and, further, for values of p such that 2 belongs to the exponent $\frac{p-1}{2}$ if $\frac{p-1}{2}$ is uneven. For in this latter case if $a^{2n}-1$ is divisible by p, $2n$ must be a multiple of $\frac{p^i(p-1)}{2}$, $=\frac{lp^i(p-1)}{2}$, and when $\frac{p-1}{2}$ is uneven, l must be even $=2m$; thus $2n = mp^i(p-1)$, and p is a Staudt factor. If therefore p is not a Staudt factor, $a^{2n}-1$ cannot be divisible by p, and therefore the numerator of B_n must be divisible by p^i. Thus, if $2^{\frac{p-1}{k}}$ is the lowest power of 2 which $\equiv 1$, mod. p, the values of p for which the integral character of $\frac{(2^{2n}-1)B_n}{n}$ fails to prove the theorem are (i) $k=2$, $p-1$ even, (ii) $k>2$.

The lowest powers of 2 which $\equiv 1$, mod. p, for values of p up to 31 are:

$2^2 \equiv 1$, mod. 3,	$2^8 \equiv 1$, mod. 17,
$2^4 \equiv 1$, mod. 5,	$2^{18} \equiv 1$, mod. 19,
$2^3 \equiv 1$, mod. 7,	$2^{11} \equiv 1$, mod. 23,
$2^{10} \equiv 1$, mod. 11,	$2^{28} \equiv 1$, mod. 29,
$2^{12} \equiv 1$, mod. 13,	$2^{10} \equiv 1$, mod. 31.

Thus the first case of failure is for $p=17$, *i.e.* we know that $\frac{2(2^{136}-1)B_{68}}{17}$ is integral, but since $2^{136}-1$ is divisible by 17,* we have no proof that the numerator of B_{68} is divisible by 17, though 17 is not a Staudt factor. The next failure is

* It is divisible by 17^2; in fact, generally, if $a^{2n}-1$ is divisible by a prime p, which is also a divisor of n, it is divisible by p^2; for, if a belongs to the exponent $\frac{p-1}{k}$, then $2n = l\frac{p(p-1)}{k}$, and, since $a^{\frac{p-1}{k}} = 1 + \alpha p$, therefore $a^{lp\frac{p-1}{k}} = 1+\beta p^2$, *i.e.*, $a^{2n}-1 \equiv 0$, mod. p^2. Similarly we see that if $a^{2n}-1$ is divisible by p, and p^i is a divisor of n, then $a^{2n}-1$ is divisible by p^{i+1}

for $p=31$, viz. the fact that $\frac{2(2^{310}-1)B_{155}}{31}$ is integral does not show that the numerator of B_{155} is divisible by 31 for $2^{310}-1$ is so divisible.

§ 11. The formula

$$(2n-1)_1 I_{n-1}-(2n-1)_3 I_{n-2}+\ldots+(-1)^n(2n-1)_{2n-3} I_1$$
$$+(-1)^{n-1} I_0=\tfrac{1}{8}(3^{2n+1}-3)\frac{B_n}{n}*$$

shows that $\frac{(3^{2n}-1)B_n}{n}$ is an integer, except for powers of 3, for it can be proved that the I's can only have powers of 3 as denominators.

From this result we may prove the theorem as before (i) for n itself, when prime, (ii) for any prime p of which 3 is a primitive root, and (iii) for values of p for which 3 belongs to the exponent $\frac{p-1}{2}$ if $\frac{p-1}{2}$ is uneven.

The lowest powers of 3 which $\equiv 1$, mod. p, for values of p up to $p=31$ are:

$3^4 \equiv 1$, mod. 5 ,	$3^{18} \equiv 1$, mod. 19,
$3^6 \equiv 1$, mod. 7 ,	$3^{11} \equiv 1$, mod. 23,
$3^5 \equiv 1$, mod. 11,	$3^{28} \equiv 1$, mod. 29,
$3^3 \equiv 1$, mod. 13,	$3^{30} \equiv 1$, mod. 31,
$3^{16} \equiv 1$, mod. 17.	

Thus with 3 as base the first case of failure occurs as early as $p=13$, for the fact that $\frac{(3^{78}-1)B_{39}}{13}$ is integral does not show that B_{39} is divisible by 13 (which is not a Staudt factor), because $3^{78}-1$ is so divisible.

It is noticeable that with the base 2 the first case of failure occurs beyond the range of Adams's Bernoullian numbers. Up to $p=31$ there is no case of failure common to both the bases 2 and 3.

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* *Quarterly Journal*, Vol. XXIX., p. 56. It was shown that the I's are either integers or of the form $\frac{\text{integer}}{\text{power of 3}}$ in a paper 'On a set of coefficients analogous to the Eulerian numbers,' which was read before the *London Mathematical Society* on June 8, 1899.

Recurring relations for $\psi_n(a)$.

§ 12. The importance of the recurring series in § 6 arises from the fact that the coefficient of $\psi_n(a)$ is a constant.

The most natural way of obtaining a recurring series for $\psi_n(a)$ is by equating coefficients in the formula

$$\frac{e^{(a-1)x}+e^{(a-2)x}+\ldots+e^{x}+1}{e^{ax}-1}=\psi_0(a)-\frac{\psi_1(a)}{1\,!}x+\frac{\psi_2(a)}{3\,!}x^3-\ldots$$

after multiplying up by $e^{ax}-1$.

By equating the coefficients of x^{2n} we find

$$\begin{aligned}\text{(i)}\ \ &(2n)_1\,a\psi_n(a)-(2n)_3\,a^3\psi_{n-1}(a)+\ldots\\&\quad+(-1)^{n-1}(2n)_{2n-1}\,a^{2n-1}\psi_1(a)+(-1)^n a^{2n}\psi_0(a)\\&\quad=(-1)^n\{1^{2n}+2^{2n}+\ldots+(a-1)^{2n}\},\end{aligned}$$

and, by equating the coefficients of x^{2n+1},

$$\begin{aligned}\text{(ii)}\ \ &(2n+1)_2\,a^2\psi_n(a)-(2n+1)_4\,a^4\psi_{n-1}(a)+\ldots\\&\quad+(-1)^{n-1}(2n+1)_{2n}\,a^{2n}\psi_1(a)+(-1)^n a^{2n+1}\psi_0(a)\\&\quad=(-1)^n\{1^{2n+1}+2^{2n+1}+\ldots+(a-1)^{2n+1}\}.\end{aligned}$$

Neither of these recurring relations throws any light upon the integral character of $\psi_n(a)$; but, as shown in § 6, if we notice that both numerator and denominator of the left-hand side of the original equation contain e^x-1 as a factor, and if we divide out of this factor, and then equate the coefficients of x^{2n-1}, we obtain the recurring equation of § 6 in which the coefficient of $\psi_n(a)$ is a.

§ 13. We do not obtain another recurring equation of the same kind as that in § 6 (*i.e.*, in which the coefficient of $\psi_n(a)$ is a constant) by equating the coefficients of x^{2n}, instead of x^{2n-1}, in § 6, for the result so obtained is

$$\begin{aligned}&(2n)_1\,S_1(a)\,\psi_n(a)-(2n)_3\,S_3(a)\,\psi_{n-1}(a)+(2n)_5\,S_5(a)\,\psi_{n-2}(a)-\ldots\\&+(-1)^{n-1}(2n)_{2n-1}\,S_{2n-1}(a)\,\psi_1(a)=(-1)^n[(a-2)^{2n}+2(a-3)^{2n}+\ldots\\&\qquad+(a-2)\,1^{2n}-\tfrac{1}{2}(a-1)\{1^{2n}+2^{2n}+\ldots+(a-1)^{2n}\}],\end{aligned}$$

in which the coefficients of $\psi_n(a)$ is $na(a-1)$.

§ 14. It is interesting to connect the recurring formula in § 6 with a recurring formula for $\phi_n(a) = (a^{2n} - 1) B_n$, which was given in the last volume of the *Messenger*.*

This formula is

$$q^{2n}\phi_n(a) - (2n)_2 q^{2n-2}p^2\phi_{n-1}(a) + (2n)_4 q^{2n-4}p^4\phi_{n-2}(a) - \dots$$
$$+ (-1)^{n-1}(2n)_{2n-2}p^{2n-2}q^2\phi_1(a)$$
$$= (-1)^{n-1}\left\{n(a-1)p^{2n-1}q - q^{2n}V_{2n}\left(\frac{p}{q}\right) + (aq)^{2n}V_{2n}\left(\frac{p}{aq}\right)\right\}\dagger,$$

where a, p, and q are any real quantities.

Putting
$$\psi_n(a) = \frac{\phi_n(a)}{2n},$$

and $q = 1$, and replacing p by r, the formula becomes, after dividing out by $2n$,

$$\psi_n(a) - (2n-1)_2 r^2\psi_{n-1}(a) + (2n-1)_4 r^4\psi_{n-2}(a) - \dots$$
$$+ (-1)^{n-1}(2n-1)_{2n-2} r^{2n-2}\psi_1(a)$$
$$= (-1)^{n-1}\left\{\tfrac{1}{2}(a-1)r^{2n-1} - A_{2n}(r) + a^{2n}A_{2n}\left(\frac{r}{a}\right)\right\}.$$

Let a be a positive integer, and put $r = 1, 2, \dots, a-1$, and add the results: we thus find

$$(a-1)\psi_n(a) - (2n-1)_2 S_2(a)\psi_{n-1}(a) + \dots$$
$$+ (-1)^{n-1}(2n-1)_{2n-2} S_{2n-2}(a)\psi_1(a)$$
$$= (-1)^{n-1}\left[\tfrac{1}{2}(a-1)S_{2n-1}(a) - A_{2n}(1) - A_{2n}(2) - \dots - A_{2n}(a-1)\right.$$
$$\left. + a^{2n}\left\{A_{2n}\left(\frac{1}{a}\right) + A_{2n}\left(\frac{2}{a}\right) + \dots + A_{2n}\left(\frac{a-1}{a}\right)\right\}\right].$$

Now

$$a^{2n}\left\{A_{2n}\left(\frac{1}{a}\right) + A_{2n}\left(\frac{2}{a}\right) + \dots + A_{2n}\left(\frac{a-1}{a}\right)\right\} = (-1)^n(a^{2n} - a)\frac{B_n}{2n}.$$

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* Vol. XXVIII., p. 65.

† The function $V_n(x) = nA_n(x)$; $A_n(x)$ and $V_n(x)$ are defined in the *Quarterly Journal*, Vol. XXIX., pp. 19 and 115.

and $A_{2n}(r) = 1^{2n-1} + 2^{2n-1} + \ldots + (r-1)^{2n-1} + (-1)^{n-1} \dfrac{B_n}{2n}.$*

so that $\qquad A_{2n}(1) + A_{2n}(2) + \ldots + A_{2n}(a-1)$

$$= (-1)^{n-1} (a-1) \frac{B_n}{2n} + (a-2)\, 1^{2n-1} + (a-3)\, 2^{2n-1} + \ldots$$
$$+ 2\,(a-3)^{2n-1} + (a-2)^{2n-1}.$$

Thus the right-hand side of the recurring equation

$$= (-1)^{n-1} \{\tfrac{1}{2}(a-1)\, S_{2n-1}(a) + (-1)^n \frac{(a^{2n}-1) B_n}{2n}$$
$$- (a-2)^{2n-1} - 2\,(a-3)^{2n-1} - \ldots - (a-2)\, 1^{2n-1}\},$$

and, since $\qquad \dfrac{(a^{2n}-1) B_n}{2n} = \psi_n(a),$

the recurring formula becomes

$$a\psi_n(a) - (2n-1)_2\, S_2(a)\, \psi_{n-1}(a) + \ldots$$
$$+ (-1)^{n-1} (2n-1)_{2n-2} S_{2n-2}(a)\, \psi_1(a)$$
$$= (-1)^{n-1} \{\tfrac{1}{2}(a-1)\, S_{2n-1}(a) - (a-2)^{2n-1} - 2\,(a-3)^{2n-1} - \ldots$$
$$- (a-2)\, 1^{2n-1}\},$$

which is the formula obtained in § 6.

§ 15. Another recurring formula for $\phi_n(a)$ was given on the same page of the *Messenger* (Vol. XXVIII., p. 65), viz.,

$$(2n+1)_1 pq^{2n} \phi_n(a) - (2n+1)_3 p^3 q^{2n-2} \phi_{n-1}(a) + \ldots$$
$$+ (-1)^{n-1} (2n+1)_{2n-1} p^{2n-1} q^2 \phi_1(a)$$
$$= (-1)^{n-1} \left\{ \tfrac{1}{2}(2n+1)(a-1)\, p^{2n} q - q^{2n+1} V_{2n+1}\left(\frac{p}{q}\right)\right.$$
$$\left. + (aq)^{2n+1} V_{2n+1}\left(\frac{p}{aq}\right)\right\},$$

a, p, q being unrestricted.

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* *Quarterly Journal*, Vol. XXIX., pp. 20, 21.

This gives the ψ-formula

$$(2n)_1 r\psi_n(a) - (2n)_3 r^3\psi_{n-1}(a) + \ldots + (-1)^{n-1}(2n)_{2n-1} r^{2n-1}\psi_1(a)$$
$$= (-1)^{n-1}\left\{\tfrac{1}{2}(a-1)\,r^{2n} - A_{2n+1}(r) + a^{2n+1}A_{2n+1}\left(\frac{r}{a}\right)\right\},$$

a and r being unrestricted.

Taking a to be a positive integer, and putting $r = 1, 2, \ldots, a-1$, and adding as in the last section, we find

$$(2n)_1 S_1(a)\,\psi_n(a) - (2n)_3 S_3(a)\,\psi_{n-1}(a) + \ldots$$
$$+ (-1)^{n-1}(2n)_{2n-1} S_{2n-1}(a)\,\psi_1(a)$$
$$= (-1)^{n-1}\left[\tfrac{1}{2}(a-1)\,S_{2n}(a) - A_{2n+1}(1) - A_{2n+1}(2) - \ldots - A_{2n+1}(a-1)\right.$$
$$\left. + a^{2n+1}\left\{A_{2n+1}\left(\frac{1}{a}\right) + A_{2n+1}\left(\frac{2}{a}\right) + \ldots + A_{2n+1}\left(\frac{a-1}{a}\right)\right\}\right].$$

The last line is zero, so that the right-hand side

$$= (-1)^{n-1}\left\{\tfrac{1}{2}(a-1)\,S_{2n}(a) - (a-2)\,1^{2n} - (a-3)\,2^{2n} - \ldots - (a-2)^{2n}\right\}.$$

The equation is therefore the same as that obtained in § 13.

§ 16. If we put $r = a$ in the preceding section, we find

$$(2n)_1 a\psi_n(a) - (2n)_3 a^3\psi_{n-1}(a) + \ldots + (-1)^{n-1}(2n)_{2n-1} a^{2n-1}\psi_1(a)$$
$$= (-1)^{n-1}\left\{\tfrac{1}{2}(a-1)\,a^{2n} - A_{2n+1}(a) + a^{2n+1}A_{2n+1}(1)\right\}$$
$$= (-1)^{n-1}\left\{\tfrac{1}{2}(a-1)\,a^{2n} - 1^{2n} - 2^{2n} - \ldots - (a-1)^{2n}\right\},$$

since $A_{2n+1}(1) = 0$.

This is equation (i) of § 12.

§ 17. Putting $r = a$ in the corresponding formula in § 14, we find

$$\psi_n(a) - (2n-1)_2 a^2\psi_{n-1}(a) + \ldots + (-1)^{n-1}(2n-1)_{2n-2} a^{2n-2}\psi_1(a)$$
$$= (-1)^{n-1}\left\{\tfrac{1}{2}(a-1)\,a^{2n-1} - A_{2n}(a) + a^{2n}A_{m}(1)\right\}.$$

Now

$$A_{2n}(a) = 1^{2n-1} + 2^{2n-1} + \ldots + (a-1)^{2n-1} + (-1)^{n-1}\frac{B_n}{2n},$$

and
$$A_{2n}(1) = (-1)^{n-1}\frac{B_n}{2n},$$

so that the right-hand side

$$= (-1)^{n-1}\{\tfrac{1}{2}(a-1)a^{2n-1} - 1^{2n-1} - 2^{2n-1} - \ldots - (a-1)^{2n-1}\} + \frac{(a^{2n}-1)B_n}{2n}.$$

The second term $= \psi_n(a)$, so that, changing the sign throughout, the equation becomes

$$(2n-1)_2 a^2\psi_{n-1}(a) - (2n-1)_4 a^4\psi_{n-2}(a) + \ldots + (-1)^n(2n-1)_{2n-2}a^{2n-2}\psi_1(a)$$
$$= (-1)^n\{\tfrac{1}{2}(a-1)a^{2n-1} - 1^{2n-1} - 2^{2n-1} - \ldots - (a-1)^{2n-1}\},$$

which is (ii) of § 12 with $n-1$ written for n.

II.

§ 18. By means of the recurring equation in § 6, I have proved a much more general theorem relating to Bernoullian numbers which, so far as I know, has not been enunciated, viz., that if p is any uneven prime, then, putting $j = \frac{p-1}{2}$,

$$\frac{B_n}{n} \equiv (-1)^{tj}\frac{B_{n-tj}}{n-tj}, \quad \text{mod. } p,$$

t being any integer, such that $n-tj$ is positive.

This includes Sylvester's theorem for the case of a simple prime; for, if p is a divisor of n, but not a Staudt factor of B_n, *i.e.*, if n is a multiple of p, but not of $\frac{p-1}{2}$, then p is not a Staudt factor of B_{n-tj}, nor is p a divisor of $n-tj$, and therefore we must have $B_n \equiv 0$, mod. p.

The theorem enables us to assign the residue of the numerator of B_n for any modulus p, if the first $\frac{p-1}{2}$ Bernoullian numbers are known; for t can always be so chosen that $n-t\frac{p-1}{2}$ is one of the numbers $1, 2, 3, \ldots, \frac{p-1}{2}$, and the denominator of B_n is known by Staudt's theorem.

It may be remarked that the theorem shows that

$$\frac{B_p}{p} \equiv 1, \quad \text{mod. } p;$$

for, putting $n=p$, $t=2$, we have

$$\frac{B_p}{p} \equiv B_1 \equiv \tfrac{1}{6}, \quad \text{mod. } p.$$

Now the denominator of B_p is 2.3. $(2p+1)$ or 2.3, according as $2p+1$ is, or is not, a prime, so that in either case this denominator $\equiv 6$, mod. p. Thus, if B_p' be the numerator of B_p, we have

$$B_p' \equiv 1, \quad \text{mod. } p,$$

i.e., when the numerator of B_p has been divided by p, the quotient $\equiv 1$, mod. p.

§ 19. I do not here consider the theorem in more detail, or its connection with Staudt's theorem, or the results obtained by assigning particular values to p, as I propose to devote a separate paper to the proof and applications of the theorem. I may, however, briefly indicate the nature of the proof, as part of the reasoning depends upon the principles which have been used in proving Sylvester's theorem in this paper.

In a paper recently communicated to the London Mathematical Society* it was shown (among other results of a similar kind) that if $X_1, X_3, \ldots$ are any quantities connected by the recurring equation

$$(\lambda+1)X_{2n+1} + (2n+1)_2 b_2 X_{2n-1} + (2n+1) b_4 X_{2n-3} + \ldots$$
$$+ (2n+1)_{2n-2} b_{2n-2} X_3 + (2n+1)_{2n} b_{2n} X_1 = c_{2n+1},$$

where λ is a constant and b_{2r} and c_{2r+1} satisfy the congruences

$$b_{2r} \equiv b_{2r-t(p-1)}, \quad \text{mod. } p,$$

$$c_{2r+1} \equiv c_{2r+1-t(p-1)}, \quad \text{mod. } p,$$

* "On a congruence theorem having reference to an extensive class of coefficients." Read June 8th, 1899.

then X_{2r+1} satisfies the congruence

$$X_{2r+1} \equiv X_{2r+1-t(p-1)}, \text{ mod. } p.$$

The modulus p may be any uneven prime which is not a divisor of $\lambda+1$ or of the denominator of any of the b's, c's, or X's.

Putting $n-1$ for n in this recurring relation and comparing it with the recurring relation for $\psi_n(a)$ in § 6, we have

$$\lambda+1=a,$$

$$X_{2r-1}=(-1)^r\psi_r(a),$$

$$b_{2r}=S_{2r}(a),$$

$$c_{2r-1}=P_{2r-1}-\tfrac{1}{2}(a-1)\,S_{2r-1}(a),$$

where

$$S_r(a)=1^r+2^r+3^r+\ldots+(a-1)^r,$$

$$P_r=(a-2)^r+2\,(a-3)^r+3\,(a-4)^r+\ldots+(a-2)\,1^r.$$

Since, by Fermat's theorem, any power m^r satisfies the congruence

$$m^r \equiv m^{r-t(p-1)}, \text{ mod. } p,$$

we see that $S_r(a)$ and P_r, being sums of powers, satisfy the congruence

$$u_r \equiv u_{r-t(p-1)}, \text{ mod. } p,$$

and therefore b_{2r} and c_{2r-1} satisfy their required congruences. Thus all the conditions are fulfilled, and, putting $j=\dfrac{p-1}{2}$ as before, the theorem gives

$$(-1)^n\psi_n(a)\equiv(-1)^{n-tj}\psi_{n-tj}(a), \text{ mod. } p,$$

that is

$$\psi_n(a)\equiv(-1)^{tj}\psi_{n-tj}(a), \text{ mod. } p,$$

a being any integer.

Since $\lambda+1=a$, and $\psi_r(a)$ has only powers of a in the denominator (except 2, which we do not consider), and since all the b's and c's are integers the only values of p which are excluded are the prime divisors of a.

§ 20. Now $a^{2n}-1\equiv a^{2n-t(p-1)}-1, \text{ mod. } p,$

and if a is a primitive root of p, neither side of the congruence is divisible by p. Taking therefore a to be a primitive root of p, and dividing out by these factors, neither of which can be $\equiv 0$, mod. p, we obtain the theorem

$$\frac{B_n}{n} \equiv (-1)^{tj} \frac{B_{n-tj}}{n-tj}, \text{ mod. } p,$$

There is no restriction with respect to the values of p, as p cannot be a divisor of a when a is a primitive root of p.

§ 21. This theorem is more difficult to prove than the corresponding theorem

$$E_n \equiv (-1)^{tj} E_{n-tj}, \text{ mod. } p^*$$

relating to the Eulerian numbers, on account of the denominators which occur in the Bernoullian numbers. In the proof which has just been described we consider not $\frac{B_n}{n}$ but $\frac{(a^{2n}-1)B_n}{n}$, which has been shown to be an integer except for powers of a, so that we are enabled to establish the theorem

$$\psi_n(a) = (-1)^{tj} \psi_{n-tj}(a), \text{ mod. } p$$

without the exclusion of any values of p except the divisors of a; then, having a at our disposal, we can, as in § 8, so choose a as to ensure that $a^{2n}-1$ and $a^{2n-1(p-1)}-1$ are not divisible by p, which enables us to divide out by these congruent factors for every value of p.

§ 22. It was shown by Stern in Vol. LXXXVIII., p. 96, of *Crelle's Journal*† that if

$$T_{2n-1} = 2^{2n-1}(2^{2n}-1)\frac{B_n}{n},$$

then
$$T_{2n-1+p-1} \equiv (-1)^{\frac{1}{2}(p-1)} T_{2n-1}, \text{ mod. } p.$$

This is equivalent to the ψ-congruence in the case $a=2$. It affords a proof of the theorem for values of p for which $2^{2n}-1$ is not divisible by p, *i.e.* for the values considered in § 10.

* Other theorems of this kind are contained in the paper referred to in § 19.

† "Zur Theorie der Bernoullischen Zahlen," pp. 85–95.

THE THEORY OF THE GAMMA FUNCTION.

By *E. W. Barnes, B.A.*, Fellow of Trinity College, Cambridge.

§ 1. The Gamma Function occupies in many ways an important place in analysis.

It is the simplest function to satisfy a transformation relation; it enters into analysis historically as a definite integral which appears in numerous investigations; it satisfies the simplest difference equation which has not constant coefficients; it has a curious connection with the algebraic polynomials known as Bernoullian functions; and finally, it is the simplest known function which does not satisfy a differential equation with algebraic or periodic coefficients.

The present paper attemps to develop in an elementary manner a complete theory of the function.

In Part I. the logarithmic differential of the function is defined by an infinite series, and its elementary properties are deduced much in the same way as Weierstrass has developed the theory of *pu*.

In Part II., such a brief account of the Bernouillian function is given as is necessary for the subsequent theory.

Part III. deals with contour integrals connected with the Gamma function. The latter is represented by a contour integral, and by means of a natural extension of Riemann's ζ-function the Gamma and Bernouillian functions are shown to be particular cases of the same integral; and certain asymptotic formulæ, necessary in Part IV., are obtained without the intervention of the Maclaurin-sum-formula.

In Part IV. the complete asymptotic expansion of the Gamma function near infinity is deduced for real as well as complex values of the variable.

And finally, in Part V. it is shown that the function cannot be obtained as a solution of a differential equation whose coefficients are not functions of the same nature.

No references are attempted in Parts I. or II. as this would, to be satisfactory, necessitate bibliographies which have already been given. In the remainder of the paper references are given which will serve to differentiate proofs and results which appear for the first time.

Part I.

The Gamma Function and its elementary properties.

§ 2. We denote the simple Gamma function with parameter ω by

$$\Gamma_1(z/\omega),$$

and we derive the series of functions

$$\psi_1^{(1)}(z/\omega) = \frac{d}{dz}\log\Gamma_1(z/\omega),$$

$$\psi_1^{(2)}(z/\omega) = \frac{d^2}{dz^2}\log\Gamma_1(z/\omega),$$

the notation having a close analogy with that introduced by Gauss.

For the fundamental definition, we take

$$\psi_1^{(2)}(z/\omega) = \sum_0^\infty \frac{1}{(z+n\omega)^2},\text{*}$$

the series being absolutely convergent, and admitting poles only at the points

$$z = -n\omega, \quad n = 0, 1, 2, \ldots, \infty.$$

It is at once evident that

$$\psi_1^{(2)}(z+\omega/\omega) = -\frac{1}{z^2} + \psi_1^{(2)}(z/\omega).$$

We determine $\Gamma_1(z/\omega)$ by two successive integrations coupled with the conditions, that

$$\Gamma_1(z+\omega/\omega) = z\Gamma_1(z/\omega),$$

and

$$\Gamma_1(\omega/\omega) = 1.$$

On integrating the fundamental series (1), we have

$$\psi_1^{(1)}(z/\omega) = \gamma_{11} - \frac{1}{z} - \sum_{n=1}^\infty \left(\frac{1}{z+n\omega} - \frac{1}{n\omega}\right),$$

where γ_{11} is a constant.

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* Vide a suggestion by Méray, *L'Analyse Infinitesimale*, 2[me] Partie, 1895, concluding pages.

Integrating again, we find

$$\Gamma_1^{-1}(z/\omega) = Ce^{-\gamma_{11}z}z \prod_1^\infty \left[\left(1+\frac{z}{n\omega}\right)e^{-\frac{z}{n\omega}}\right],$$

But by definition, we have the relation

$$\Gamma_1(z+\omega/\omega) = z\Gamma_1(z/\omega).$$

Hence

$$e^{-\gamma_{11}\omega}.\frac{z+\omega}{z}\prod_1^\infty \left\{\frac{\left(1+\frac{z+\omega}{n\omega}\right)e^{-\frac{z+\omega}{n\omega}}}{\left(1+\frac{z}{n\omega}\right)e^{-\frac{z}{n\omega}}}\right\} = \frac{1}{z}.$$

Thus

$$\operatorname*{Lt.}_{p=\infty} e^{-\gamma_{11}\omega}\left[\frac{z+(p+1)}{z}\omega\,.\,e^{-\sum_1^p \frac{1}{n}}\right] = \frac{1}{z},$$

which at once gives us,* γ being Euler's constant,

$$e^{-(\gamma+\gamma_{11}\omega)}\,\omega = 1\,;$$

whence $$\gamma_{11} = -\frac{\gamma}{\omega} + \frac{\log\omega}{\omega}.$$

We thus have

$$\Gamma_1^{-1}(z/\omega) = C\omega^{-\frac{z}{\omega}}e^{\frac{\gamma z}{\omega}}.z\prod_1^\infty\left[\left(1+\frac{z}{n\omega}\right)e^{-\frac{z}{n\omega}}\right].$$

We now put $z=\omega$, and assign the condition

$$\Gamma_1^{-1}(\omega/\omega) = 1.$$

We find

$$1 = Ce^{\gamma}\operatorname*{Lt.}_{p=\infty}\prod_1^p\left[\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}}\right],$$

or $$C = 1.$$

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* Chrystal's *Algebra*, p. 81, Vol. II.

Thus finally

$$\Gamma_1^{-1}(z/\omega) = \omega^{-\frac{z}{\omega}} \cdot e^{\frac{\gamma z}{\omega}} \cdot z \cdot \prod_1^\infty \left[\left(1 + \frac{n\omega}{z}\right) e^{-\frac{z}{n\omega}}\right].$$

§ 3. We may now write down the connection between $\Gamma_1(z)\omega$, and the function $\Gamma(z)$, as usually defined. The usual definition of the latter is*

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_1^\infty \left[\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}\right].$$

We have therefore

$$\Gamma^{-1}\left(\frac{z}{\omega}\right) = e^{\gamma \frac{z}{\omega}} \cdot \frac{z}{\omega} \cdot \prod_1^\infty \left[\left(1 + \frac{z}{n\omega}\right) e^{-\frac{z}{n\omega}}\right],$$

We have just obtained

$$\Gamma_1^{-1}(z/\omega) = \omega^{-\frac{z}{\omega}} \cdot e^{\gamma \frac{z}{\omega}} \cdot z \prod_1^\infty \left[\left(1 + \frac{z}{n\omega}\right) e^{-\frac{z}{n\omega}}\right].$$

Hence $$\Gamma_1^{-1}(z/\omega) = \omega^{1-\frac{z}{\omega}} \Gamma^{-1}\left(\frac{z}{\omega}\right),$$

or $$\Gamma_1(z/\omega) = \omega^{\frac{z}{\omega}-1} \Gamma\left(\frac{z}{\omega}\right).$$

The notation adopted in this paper introduces analogies of great importance in connection with higher allied transcendents, and can, by the formula just found, be at once changed when necessary into the stereotyped form.

Notice as a Corollary that

$$\left[\frac{\Gamma_1^{-1}(z/\omega)}{z}\right]_{z=0} = 1.$$

§ 4. We proceed now to investigate the multiplication of the simple gamma function.

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* Forsyth, *Theory of Functions*, p. 84; Baker, "Note on the Gamma Function," *Messenger of Mathematics*, Vol. xxv., p. 125.

We have by definition

$$\psi_1^{(2)}(z/\omega) = \sum_{m=0}^{\infty} \frac{1}{(z+m\omega)^2}.$$

Hence, if n be an integer,

$$n^2\psi_1^{(2)}(nz/\omega) = \sum_{m=0}^{\infty} \frac{1}{\left(z+\frac{m\omega}{n}\right)^2} = \sum_{q=0}^{n-1} \psi_1^{(2)}\left(z+\frac{q\omega}{n}/\omega\right),$$

whence, on integration,

$$n\psi_1^{(1)}(nz/\omega) = r + \sum_{q=0}^{n-1} \psi_0^{(1)}\left(z+\frac{q\omega}{n}/\omega\right),$$

where r is some constant.

Now $\psi_1^{(1)}(z/\omega) = \gamma_{11} - \frac{1}{z} - \underset{p=\infty}{\mathrm{Lt.}} \sum_1^p \left(\frac{1}{z+m\omega} - \frac{1}{m\omega}\right)$,

where, (§ 2), $\gamma_{11} = -\frac{\gamma}{\omega} + \frac{\log\omega}{\omega}$.

We have then

$$n \underset{p=\infty}{\mathrm{Lt.}} \sum_{m=1}^{np+n-1} \left(\frac{1}{m\omega}\right) = r + n\sum_1^p \left(\frac{1}{m\omega}\right),$$

the number of terms on each side being determined by the manner in which z is involved.

We thus have

$$r = \frac{n}{\omega} \underset{p=\infty}{\mathrm{Lt.}} \left[\log(np+n-1) - \log p\right],$$

so that
$$r = \frac{n}{\omega}\log n.$$

Integrating again we have

$$\Gamma_1(nz/\omega) = e^{rz+s} \prod_{q=0}^{n-1} \Gamma_1\left(z+\frac{q\omega}{n}/\omega\right),$$

s being the new constant of integration.

Now

$$\Gamma_1^{-1}(z/\omega) = e^{-\gamma_{11}z} \,.\, z \,.\, \prod_{m=1}^{\infty}\left[\left(1+\frac{z}{m\omega}\right)e^{-\frac{z}{m\omega}}\right].$$

Hence, making $z=0$ in the identity which results on substitution, we find

$$n = e^{-s}\frac{1.2\ldots(n-1)}{n^{n-1}}\,\underset{=\infty}{\text{Lt.}}\;\prod_{q=0}^{n-1}\left\{\prod_{m=1}^{p}\left[\left(1+\frac{q}{mn}\right)e^{-\frac{q}{mn}}\right]\right\}\times e^{-\gamma_{11}\frac{n-1}{2}\omega},$$

so that

$$n = e^{-s-\gamma_{11}\frac{1}{2}(n-1)\omega}\,.\,\omega^{n-1}\,.\,\underset{p=\infty}{\text{Lt.}}\;\frac{\prod\limits_{m=1}^{pn+n-1}\left[\dfrac{m}{n}\right]\prod\limits_{m=1}^{p}\left[e^{-\frac{n-1}{2m}}\right]}{\prod\limits_{m=1}^{p}[m]^n},$$

whence, by Stirling's theorem,*

$$n = e^{-s}e^{\gamma\frac{1}{2}(n-1)}\omega^{+\frac{1}{2}(n-1)}$$

$$\times\,\underset{p=\infty}{\text{Lt.}}\left[\frac{(np+n-1)^{np+n-1+\frac{1}{2}}e^{-np+n-1}}{n^{np+n-1}(2\pi)^{\frac{1}{2}(n-1)}p^{np+\frac{1}{2}n}e^{-np}}\,.\,e^{-\frac{1}{2}(n-1)(\gamma+\log p)}\right].$$

And thus $\qquad n = e^{-s}\omega^{+\frac{1}{2}(n+1)}\dfrac{n^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}(n-)1}},$

so that $\qquad e^{s} = \omega^{+\frac{1}{2}(n-1)}n^{-\frac{1}{2}}(2\pi)^{-\frac{1}{2}(n-1)}.$

And finally

$$\Gamma_1(nz/\omega) = \frac{n^{\frac{nz}{\omega}-\frac{1}{2}}}{(2\pi/\omega)^{\frac{1}{2}(n-1)}}\prod_{q=0}^{n-1}\Gamma_1\left(z+q\frac{\omega}{n}/\omega\right),$$

which agrees with the result of writing Gauss' Theorem in the notation adopted in this paper.

As a *Corollary* we have Euler's Theorem, on putting $z=0$,

$$\prod_{q=1}^{n-1}\Gamma_1\left(q\frac{\omega}{n}/\omega\right) = n^{-\frac{1}{2}}\left(\frac{2\pi}{\omega}\right)^{\frac{1}{2}(n-1)}.$$

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* Chrystal's *Algebra*, Vol. II., p. 344.

§ 5. The multiplication theorem for $n=2$ is important, in that it at once furnishes us with the value of

$$\Gamma_1\left(\frac{\omega}{2}\,/\,\omega\right).$$

We have, on making $n=2$,

$$\Gamma_1(2z/\omega)=\frac{\omega^{\frac{1}{2}}}{(4\pi)^{\frac{1}{2}}}\,2^{\frac{2z}{\omega}}\,\Gamma_1(z/\omega)\,\Gamma_1\left(z+\frac{\omega}{2}\,/\,\omega\right).$$

Make $z=0$ and we find

$$\Gamma_1\left(\frac{\omega}{2}\,/\,\omega\right)=\left(\frac{4\pi}{\omega}\right)^{\frac{1}{2}}\underset{z=0}{\text{Lt.}}\ \frac{\Gamma_1(2z/\omega)}{\Gamma_1(z/\omega)},$$

or

$$\Gamma_1\left(\frac{\omega}{2}\,/\,\omega\right)=\left(\frac{\pi}{\omega}\right)^{\frac{1}{2}}.$$

This is the theorem equivalent to

$$\Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}.$$

It will be noted that this value is obtained without the intervention of the formula

$$\Gamma(z)\,\Gamma(1-z)=\frac{\pi}{\sin z\pi};$$

in fact, the value of $\Gamma_1\left(\frac{\omega}{z}\,/\,\omega\right)$ is shown to have a most intimate connection with the constant arising from Stirling's Theorem.

§ 6. We may now express the sine function in terms of products of simple Gamma Functions.

We have, by definition,

$$\psi_1^{(2)}(z/\omega)=\sum_{n=0}^{\infty}\frac{1}{(z+n\omega)^2},$$

and

$$-\frac{d^2}{dz^2}\log\sin\frac{z\pi}{\omega}=\sum_{n=-\infty}^{\infty}\frac{1}{(z+n\omega)^2}.$$

Hence

$$-\frac{d^2}{dz^2}\log\sin\frac{z\pi}{\omega}=\psi_1^{(2)}(z/\omega)+\psi_1^{(2)}(\omega-z/\omega),$$

whence, integrating,

$$-\frac{d}{dz}\log\sin\frac{z\pi}{\omega}=\psi_1^{(1)}(z/\omega)-\psi_1^{(1)}(\omega-z/\omega),$$

the constant of integration vanishing on making $z=0$.

Integrating again, we find

$$\Gamma_1(z/\omega)\,\Gamma_1(\omega-z/\omega)=\frac{C}{\sin\frac{\pi z}{\omega}}.$$

Again, comparing the values of both sides when $z=0$, we find $C=\frac{\pi}{\omega}$, so that finally

$$\sin\frac{\pi z}{\omega}=\frac{\pi}{\omega\Gamma_1(z/\omega)\,\Gamma_1(\omega-z/\omega)}.$$

§ 7. The transformation of simple Gamma functions.

The theory of the parametric transformation of simple Gamma functions is practically identical with that of multiplication of the argument.

For the sake of some important analogies it is useful to express the formulæ from the different standpoint.

We have seen that

$$\Gamma_1^{-1}(z/\omega)=e^{\gamma\frac{z}{\omega}}\,.\,\omega^{-\frac{z}{\omega}}\,.\,z\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n\omega}\right)e^{-\frac{z}{n\omega}}\right],$$

whence

$$\Gamma_1^{-1}\left(z/\frac{\omega}{n}\right)=\left(\frac{1}{n}\right)^{-\frac{nz}{\omega}+1}\Gamma_1^{-1}(nz/\omega)$$

$$=\left(\frac{2\pi}{\omega}\right)^{\frac{1}{2}(n-1)}n^{-\frac{1}{2}}\prod_{q=0}^{n-1}\Gamma_1^{-1}\left(z+\frac{q\omega}{n}/\omega\right)\text{ by § 4.}$$

Thus the required formula of transformation may be written

$$\Gamma_1\left(z/\frac{\omega}{n}\right)=\left(\frac{2\pi}{\omega}\right)^{-\frac{1}{2}(n-1)}n^{+\frac{1}{2}}\prod_{q=0}^{n-1}\Gamma_1\left(z+\frac{q\omega}{n}/\omega\right).$$

It will be noted that this formula is somewhat simpler than the multiplication formula.

Note also that we have

$$\psi_1^{(1)}\left(z \,/\, \frac{\omega}{n}\right) = \sum_{q=0}^{n-1} \psi_1^{(1)}\left(z + \frac{q\omega}{n} \,/\, \omega\right),$$

and similar relations among the higher derived functions.

§ 8. We proceed now to evaluate

$$\int_0^\omega \log \Gamma_1 (z + a/\omega)\, dz.$$

Let $$\phi(a/\omega) = \int_0^\omega \log \Gamma_1 (z + a/\omega)\, dz.$$

Then we have

$$\frac{\phi(a/\omega)}{a} = \int_0^\omega \frac{\partial}{\partial a} \log \Gamma_1 (z + a/\omega)\, dz$$

$$= \int_0^\omega \frac{\Gamma_1'(z + a/\omega)}{\Gamma_1(z + z/\omega)}\, dz = \log \frac{\Gamma_1(a + \omega/\omega)}{\Gamma_1(a/\omega)}$$

$$= \log a \text{ by definition,}$$

whence $$\phi(a/\omega) = a \log a - a + C,$$

where C is independent of a.

To determine C we make use of the multiplication formula for $n = 2$. We have

$$\int_0^{2\omega} \log \Gamma_1 (z + a/\omega)\, dz$$

$$= \int_0^\omega \log \Gamma_1 (z + a/\omega)\, dz + \int_0^\omega \log \Gamma_1 (z + a + \omega/\omega)\, dz$$

$$= 2\int_0^\omega \log \Gamma_1 (z + a/\omega) + \int_0^\omega \log (z + a)\, dz$$

on using the fundamental formula

$$\Gamma_1 (z + \omega/\omega) = z\Gamma_1 (z/\omega).$$

Hence we have

$$\int_0^{2\omega} \log \Gamma_1 (z + a/\omega)\, dz = 2C - 2a + a \log a - \omega$$

$$+ (a + \omega) \log (a + \omega).$$

Now, by the multiplication theorem when $n = 2$, (§ 5),

$$\int_0^{\omega} \log \Gamma_1 (2z/\omega)\, dz = -\tfrac{1}{2} \int_0^{\omega} \log \left(\frac{4\pi}{\omega}\right) dz + \int_0^{\omega} \frac{2z}{\omega} \log 2 dz$$

$$+ \int_0^{\omega} \log \Gamma_1 (z/\omega)\, dz + \int_0^{\omega} \log \Gamma_1 \left(z + \frac{\omega}{2} / \omega\right) dz.$$

And $\displaystyle\int_0^{\omega} \log \Gamma_1 (2z/\omega)\, dz = \tfrac{1}{2} \int_0^{2\omega} \log \Gamma_1 (z/\omega)\, dz.$

We thus have

$$\tfrac{1}{2} \{2C - \omega + \omega \log \omega\} = -\frac{\omega}{2} \log \frac{4\pi}{\omega} + \omega \log 2 + C$$

$$+ C + \frac{\omega}{2} \log \frac{\omega}{2} - \frac{\omega}{2},$$

so that $$C = \frac{\omega}{2} \log 2 + \frac{\omega}{2} \log \pi\omega,$$

$$= \frac{\omega}{2} \log \frac{2\pi}{\omega},$$

and, finally,

$$\int_0^{\omega} \log \Gamma_1 (z + a/\omega)\, dz = a \log a - a + \frac{\omega}{2} \log (2\pi\omega),$$

which is equivalent to Raabe's* formula in the stereotyped notation.

Cor. Evidently the method just used, combined with the multiplication theory, will furnish immediately the value of

$$\int_0^{n\omega} \log \Gamma_1 (z + a/\omega)\, dz,$$

n being a positive integer.

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* Raabe, *Crelle*, Bd. 28, p. 12.

§ 9. In the foregoing paragraphs we have deduced most of the theorems usually proved in the text-books.

Before any further progress can be made, it is necessary to consider briefly the theory of Bernoullian functions.

Part II.

The Bernoullian Function.

§ 10. It is unfortunate that so many varying notations exist for simple Bernoullian numbers and functions. We have adopted M. Hermite's notation,* $S_n(x)$, for the simple Bernoullian function, which is the algebraic solution of the differential equation

$$f(x+1) - f(x) = x^n,$$

such that $f(0) = 0$.

Dr. Glaisher, in his recent exhaustive memoir,† "On products and series involving prime numbers only," denotes this function by $B_n(x)$. The function $V_n(x)$ which he subsequently finds‡ it convenient to use is identical with Hermite's $S_n'(x)$—the prime denoting differentiation with regard to x. We have generalized Hermite's notation, writing $S_n(x/\omega)$ for the algebraic polynomial, satisfying the difference equation

$$f(x+\omega) - f(x) = x^n$$

with the condition $f(0) = 0$. Thus $S_n(x/1) = S_n(x)$. Where there is no doubt as to the presence of the parameter, the generalised function is written $S_n(x)$.

A complete bibliography of Bernoullian numbers and functions has been given by Worpitzky,§ and Dr. Glaisher‖ has noticed several supplementary memoirs.

The present method of investigation closely resembles one recently suggested by M. Sonin.¶

§ 11. It is first necessary to prove that there exists one and only one rational integral algebraic solution of the difference equation

$$S_n(a+\omega/\omega) - S_n(a/\omega) = a^n,$$

which is such that $S_n(0/\omega) = 0$, n being a positive integer.

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* Hermite, *Crelle*, Bd. 115, pp. 201 et seq

† Glaisher, *Quarterly Journal of Mathematics*, Vol. XXVIII., §§ 373 and 384.

‡ Glaisher, *Ibid*, Vol. XXIX., § 217 "On the Bernoullian Function."

§ Worpitzky, *Crelle*, Bd. XCIV., pp. 203 et seq.

‖ Loc. cit., §§ 312 et seq.

¶ Sonin, *Crelle*, Bd. CXVI, pp. 133 et seq.

In the first place, if an algebraic solution exist at all it is unique; for the difference of any two solutions is a solution of the equation

$$f(a+\omega)-f(a)=0,$$

and, as is well known, there exists no algebraic simply periodic function except a constant.

In the second place an algebraic solution does exist; for

$$n\,!\,\frac{a}{\omega}+\lambda_1$$

is evidently a solution of

$$S_n^{(n)}(a+\omega)-S_n^{(n)}(a)=n\,!,$$

where
$$S_n^{(n)}(a)=\frac{d^n}{da^n}\,S_n(a).$$

On integrating with respect to a it is evident that

$$n\,!\,\frac{a^2}{2\omega}+\lambda_1 a+\lambda_2$$

is a solution of

$$S_n^{(n-1)}(a+\omega)-S_n^{(n-1)}(a)=\frac{n\,!}{1\,!}\,a+\mu_1,$$

and we may determine λ_1 so that μ_1 vanishes.

Integrating again, we see that

$$\frac{n\,!}{3\,!}\,\frac{a^3}{\omega}+\frac{\lambda_1}{2\,!}\,a^2+\lambda_2 a+\lambda_3$$

is a solution of

$$S_n^{(n-2)}(a+\omega)-S_n^{(n-2)}(a)=\frac{n\,!}{2\,!}\,a^2+\mu_2,$$

and again we can determine λ_2, so that μ_2 vanishes.

Proceeding in this way we see finally that an algebraic solution of degree $(n+1)$ exists, which is the unique solution of this kind of the equation

$$S_n(a+\omega)-S_n(a)=a^n$$

with the assigned condition $S_n(0)=0$.

This solution we define as the nth simple Bernoullian function. We note at once that $S_n(\omega/\omega) = 0$.

Cor. No algebraic solution of the difference equation

$$f(a+\omega) - f(a) = a^n$$

exists when n is a negative integer.

§ 12. It can now be proved that, when $n > 0$,

$$S_n(a/\omega) = (-)^{n-1} S_n(\omega - a/\omega).$$

By definition we have

$$S_n(a+\omega) - S_n(a) = a^n,$$

so that $$S_n(-a) - S_n(-a+\omega) = (-)^{n-1} a^n.$$

Now $$S_n(\omega/\omega) = 0,$$

and hence $$(-)^{n-1} S_n\{-(a-\omega)\}$$

is the unique algebraic solution of

$$f(a+\omega) - f(a) = a^n,$$

with the condition $f(0) = 0$.

Hence, by § 11,

$$S_n(a) = (-)^{n-1} S_n(\omega - a),$$

the theorem enunciated.

§ 13. It is now at once evident that, when $n > 0$,

$$\int_0^\omega S_{2n}(a)\, da = 0;$$

for

$$\int_0^\omega S_{2n}(a)\, da = \int_0^\omega S_{2n}(\omega - a)\, da = -\int_0^\omega S_{2n}(a)\, da \text{ by § 12.}$$

We thus have $$\int_0^\omega S_{2n}(a)\, da = 0,$$

and we define the nth simple Bernoullian number B_n by the relation

$$\int_0^\omega S_{2n-1}(a/\omega)\, da = \frac{(-)^n \omega^{2n} B_n}{2n} \quad (n > 0).$$

When $n=0$ we have

$$S_n(a/\omega) = \frac{a}{\omega},$$

and
$$\int_0^\omega S_0(a/\omega)\,da = \tfrac{1}{2}\omega.$$

§ 14. We proceed now to express

$$S_n^{(k)}(0/\omega)$$

in terms of Bernoullian numbers.

We will first show that when $n-k \mathrel{\overline{>}} 0$, both n and k being positive integers,

$$S_n^{(k)}(0/\omega) = -\frac{n\,!}{\omega\,.\,(n-k)\,!}\int_0^\omega S_{n-k}(x)\,dx.$$

From the fundamental difference equation we have

$$S_n^{(k)}(x+\omega) - S_n^{(k)}(x) = n\,.\,(n-1)\ldots(n-k+1)\,x^{n-k}$$

$$= \frac{n\,!}{(n-k)\,!}\left[S_{n-k}(x+\omega) - S_{n-k}(x)\right].$$

Hence, if

$$f(x) = S_n^{(k)}(x) - \frac{n\,!}{(n-k)\,!}S_{n-k}(x),$$

we have
$$f(x+\omega) - f(x) = 0.$$

Hence $f(x)$, being an algebraical polynomial in x, must be a constant, whose value is at once given by putting $x=0$.

We thus have, when $n-k \mathrel{\overline{>}} 0$,

$$S_n^{(k)}(x) - \frac{n\,!}{(n-k)\,!}S_{n-k}(x) = S_n^{(k)}(0).$$

And, in particular, since $S_{n-k}(\omega) = 0$,

$$S_n^{(k)}(\omega) = S_n^{(k)}(0).$$

Integrate now the result just obtained between 0 and ω, and we have, when $n-k \mathrel{\overline{>}} 0$,

$$\omega S_n^{(k)}(0) = S_n^{(k-1)}(\omega) - S_n^{(k-1)}(0) - \frac{n\,!}{(n-k)\,!}\int_0^\omega S_{n-k}(x/\omega)\,dx,$$

or
$$S_n^{(k)}(0) = -\frac{n\,!}{\omega\,.\,(n-k)\,!}\int_0^\omega S_{n-k}(x/\omega)\,dx.$$

And now, by § 13, when $n-k$ is even and >0,

$$S_n^{(k)}(0/\omega)=0;$$

when $n-k=0$, $\quad S_n^{(k)}(0/\omega)=-\tfrac{1}{2}(n\,!)$;

when $n-k$ is odd and >0,

$$S_n^{(k)}(0/\omega)=(-)^{\frac{1}{2}(n-k-1)}\frac{n\,!}{(n-k+1)\,!}\,\omega^{n-k}B_{\frac{1}{2}(n-k+1)}.$$

When $n-k=-1$, we have

$$S_n^{(k)}(a+\omega)-S_n^{(k)}(a/\omega)=0,$$

an equation of which the only solution is a constant, whose value we shall shew in § 16 to be $\dfrac{n\,!}{\omega}$.

In consequence $S_n^{(k)}(0/\omega)=0$ when $n-k<-1$.

It is convenient to tabulate these results for references;—

$S_n^{(k)}(0/\omega)=0$, $n-k$ even and >0,

$$S_n^{(k)}(0/\omega)=(-)^{\frac{1}{2}(n-k-1)}\omega^{n-k}\frac{n!}{(n-k+1)!}B_{\frac{1}{2}(n-k+1)},$$

$n-k$ odd and >0,

$$S_n^{(k)}(0/\omega)=-\tfrac{1}{2}n!,\qquad n-k=0,$$

$$S_n^{(k)}(0/\omega)=\frac{n!}{\omega},\qquad n-k=-1,$$

$$S_n^{(k)}(0/\omega)=0,\qquad n-k<-1.$$

§ 15. The constants $S_n'(0/\omega)$ are of great importance.

We have, provided $n-k>0$ and $k>0$,

$$\int_0^\omega S_{n-k}(x)\,dx=-\omega\,\frac{(n-k)!}{n!}\,S_n^{(k)}(0).$$

Whence, writing n for $n-k$,

$$\int_0^\omega S_n(x)\,dx=-\frac{\omega n!}{(n+k)!}\,S^k{}_{n+k}(0).$$

Now $\int_0^\omega S_n(x)\,dx$ is independent of k; hence

$$\int_0^\omega S_n(x)\,dx=-\frac{\omega}{n+1}\,S'{}_{n+1}(0).$$

And $$S^{(k)}{}_{n+k}(0/\omega) = \frac{(n+k)!}{(n+1)!} S^{(1)}{}_{n+1}(0/\omega),$$

or $$S_n^{(k)}(0/\omega) = \frac{n!}{(n-k+1)!} S'_{n-k+1}(0).$$

In accordance with a more general notation, we shall sometimes write

$$\frac{S_n'(0)}{n} = {}_1B_n(\omega).$$

The connection of this modified number with the usual B_n is at once given by the table

$S_n'(0/\omega) = (-)^{\frac{1}{2}n-1}\omega^{n-1}B_{\frac{1}{2}n}$, when n is even,

$S_n'(0/\omega) = 0$, when n is odd and > 1,

$S_n'(0/\omega) = -\frac{1}{2}$, when $n = 1$.

Cor. By § 14, provided $n > k$,

$$S_n^{(k)}(x/\omega) = \frac{n!}{(n-k)!}\left\{S_{n-k}(x) + \frac{S'_{n-k+1}(0)}{n-k+1}\right\},$$

or as it may be written

$$= \frac{n!}{(n-k)!}\{S_{n-k}(x) + {}_1B_{n-k+1}(\omega)\}.$$

§ 16. We may now obtain the formal expression of $S_n(x)$. We have by Maclaurin's theorem,

$$S_n(x/\omega) = S_n(0) + xS_n'(0) + \ldots + \frac{x^k}{k!} S_n^{(k)}(0) + \ldots.$$

By § 14,

$S_n^{(k)}(0) = 0$, when $k > n+1$,

$S_n^{(k)}(0/\omega) = 0$, when $n-k$ is even and > 0,

$S_n^{(k)}(0/\omega) = -\frac{1}{2}(n!)$, when $n-k=0$,

$$S_n^{(k)}(0/\omega) = \frac{n!}{(n-k+1)!}(-)^{\frac{1}{2}(n-k-1)}\omega^{n-k}B_{\frac{1}{2}(n-k+1)},$$

when $n-k$ is odd and > 0.

Thus

$$S_n(x/\omega) = \frac{S_n^{(n+1)}(0)}{(n+1)!} x^{n+1} - \tfrac{1}{2}x^n + \frac{x^{n-1}}{n-1!} \frac{n!}{2!} \omega B_1$$
$$- \frac{x^{n-3}}{(n-3)!} \frac{n!}{4!} \omega^3 B_2 + \dots .$$

Substituting in the equation

$$S_n(x+\omega) - S_n(x) = x^n,$$

we find $$S_n^{(n+1)}(0/\omega) = \frac{n!}{\omega},$$

the result needed to complete the set in § 14.

And finally

$$S_n(x/\omega) = \frac{x^{n+1}}{(n+1)\,\omega} - \tfrac{1}{2}x^n + \tfrac{1}{2}B_1 \binom{n}{1} \omega x^{n-1} - \tfrac{1}{3}B_2\,\omega^3 \binom{n}{3} x^{n-3} + \dots$$

Cor. 1. We note that

$$S_n\left(\frac{x}{\omega}\right) = \frac{1}{\omega^n} S_n(x/\omega).$$

Cor. 2. We also note that

$$S_n(x/-\omega) = -\left[\frac{x^{n+1}}{(n+1)\,\omega} + \tfrac{1}{2}x^n + \tfrac{1}{2}B_1\binom{n}{1} x^{n-1}\omega + \dots\right]$$
$$= -\left[x_n + S_n(x/\omega)\right].$$

Thus $$S_n(x/-\omega) = -S_n(x+\omega/\omega).$$

§ 17. We will next show that

$$\int_0^x S_n(x)\,dx = \frac{x}{\omega}\int_0^\omega S_n(x)\,dx + \frac{S_{n+1}(x)}{n+1}.$$

We have $$S_n(x+\omega) - S_n(x) = x^n.$$

Hence $$\int_0^x S_n(x+\omega)\,dx - \int_0^x S_n(x)\,dx = \frac{x^{n+1}}{n+1},$$

or $$\int_\omega^{x+\omega} S_n(x)\,dx - \int_0^x S_n(x)\,dx = \frac{x^{n+1}}{n+1},$$

so that if $$f(x) = \int_0^x S_n(x)\,dx,$$

$$f(x+\omega) - f(x) = \int_0^\omega S_n(x)\,dx + \frac{x^{n+1}}{n+1}.$$

But this difference equation is also satisfied by

$$\frac{x}{\omega}\int_0^\omega S_n(x)\,dx + \frac{1}{n+1}\,S_{n+1}(x).$$

Now both solutions are rational integral algebraic functions of x, and hence they can only differ by a constant which vanishes on making $x=0$, whence the theorem.

By § 15 the formula which we have just obtained may evidently be written

$$\int_0^x S_n(x)\,dx = -\frac{x}{n+1}\,S'_{n+1}(0) + \frac{S_{n+1}(x)}{n+1}$$

$$= \frac{S_{n+1}(x)}{n+1} - {}_1B_{n+1}(\omega)\,.\,x.$$

§ 18. We now proceed to show that

$$S_n\left(x \middle/ \frac{\omega}{m}\right) = \sum_{k=0}^{m-1} S_n\left(x + \frac{k\omega}{m} \middle/ \omega\right) + (-)^{\frac{1}{2}(n-1)} B_{\frac{1}{2}(n+1)}\left(\frac{\omega}{m}\right)^n \frac{m^{n+1}-1}{n+1},$$

when n is odd and > 0, and

$$= \sum_{k=0}^{m-1} S_n\left(x + \frac{k\omega}{m} \middle/ \omega\right),$$

when n is even and > 0, this being the expression of the transformation theory of Bernoullian functions.

Let $$f(x) = \sum_{k=0}^{m-1} S_n\left(\frac{x}{m} + \frac{k\omega}{m} \middle/ \omega\right).$$

Then

$$f(x+\omega) - f(x) = S_n\left(\frac{x}{m} + \omega \middle/ \omega\right) - S_n\left(\frac{x}{m} \middle/ \omega\right) = \left(\frac{x}{m}\right)^n.$$

But $\dfrac{S_n(x/\omega)}{m^n}$ is also an algebraic solution of this difference equation, so that, as two algebraic solutions can only differ by a constant

$$S_n(x/\omega) = m^n \sum_{k=0}^{m-1} \left(\frac{x}{m} + \frac{k\omega}{m} \Big/ \omega\right) - A_n.$$

Integrate now between 0 and ω, and we have

$$\int_0^\omega S_n(x/\omega)\,dx = -A_n\omega + m^{n+1}\int_0^\omega S_n(x/\omega)\,dx,$$

since
$$\sum_{k=0}^{m-1} \int_0^\omega S_n\left(\frac{x}{m} + \frac{k\omega}{m} / \omega\right) dx$$

$$= m\sum_{k=0}^{m-1} \int_{\frac{k\omega}{m}}^{\frac{(k+1)\omega}{m}} S_n(x/\omega)\,dx = m\int_0^\omega S_n(x/\omega)\,dx.$$

Thus $A_n = (1 - m^{n+1})\,{}_1B_{n+1}(\omega)$, and therefore

$$A_n = (m^{n+1} - 1)\,\frac{(-)^{\frac{n+1}{2}}\,\omega^n B_{\frac{n+1}{2}}}{n+1}$$

when n is odd;

$= 0$ when n is even.

Since
$$S_n(x/\omega) = m^n S_n\left(\frac{x}{m}\Big/\frac{\omega}{m}\right),$$

we may obviously write the theorem in the form enunciated.

When $n = 0$ we find $A_0 = \frac{1}{2}(m-1)$.

Cor. Making $x = 0$ in the results just obtained, we find

$$\sum_{k=0}^{m-1} S_n\left(\frac{k\omega}{m} / \omega\right) = 0 \text{ when } n \text{ is even and } > 0,$$

$$= (-)^{\frac{m+1}{2}} \left(\frac{\omega}{m}\right)^n \frac{m^{n+1} - 1}{n+1} B_{\frac{n+1}{2}} \text{ when } n \text{ is odd},$$

and therefore for all positive integral values of n the value of the series is $\dfrac{1 - m^{n+1}}{m^n}\,{}_1B_{n+1}(\omega)$.

§ 19. We next proceed to prove that

$$\frac{e^{ax}-1}{e^{\omega x}-1}=\frac{a}{\omega}+\frac{S_1(a/\omega)}{1!}x+\ldots+\frac{S_n(a/\omega)}{n!}x^n+\ldots$$

for such values of x as make the series convergent.

Consider the function

$$f(x, a, \omega)=\frac{a}{\omega}+\frac{S_1(a/\omega)}{1!}x+\ldots+\frac{S_n(a/\omega)}{n!}x^n+\ldots,$$

which exists within that circle round the origin in the x plane for which the series is convergent.

We shall have

$$f(x, a+\omega, \omega)-f(x, a, \omega)$$

$$=1+\frac{S_1(a+\omega)-S_1(a)}{1!}x+\ldots+\frac{S_n(a+\omega)-S_n(a)}{n!}x^n+\ldots$$

$$=1+\frac{ax}{1!}+\ldots+\frac{(ax)^n}{n!}+\ldots=e^{ax}.$$

We thus have a functional equation of which the most general solution may be written

$$f(x, a, \omega)=\frac{e^{ax}-1}{e^{\omega x}-1}+\sum_{\lambda=\infty}^{\infty}p_\lambda(a/\omega)\,x^\lambda,$$

where $p_\lambda(a/\omega)$ must be an algebraic simply periodic function of a of period ω, whose coefficients are functions of x.

Now the coefficients of the various powers of x in the expansion defining $f(x, a, \omega)$ are algebraic functions of a. Hence the functions $p_\lambda(a/\omega)$ must be an algebraic simply periodic function of a, and is therefore independent of a.

They must therefore vanish, since $f(x, a, \omega)$ and $\dfrac{e^{ax}-1}{e^{\omega x}-1}$ both vanish when $a=0$. And we have finally

$$\frac{e^{ax}-1}{e^{\omega x}-1}=\frac{a}{\omega}+\frac{S_1(a/\omega)}{1!}x+\ldots+\frac{S_n(a/\omega)}{n!}x^n+\ldots.$$

This expansion is valid provided $|x|$ is less than $\left|\dfrac{2\pi i}{\omega}\right|$. For, within a circle whose radius is less than this quantity, the function $\dfrac{e^{ax}-1}{e^{\omega x}-1}$ has no poles and is therefore expansible in a Taylor's series in powers of x.

§ 20. Differentiate the preceding relation with regard to a. We obtain

$$\frac{xe^{ax}}{e^{\omega x}-1}=\frac{1}{\omega}+\frac{S_1'(a/\omega)}{1!}x+\ldots+\frac{S_n'(a/\omega)}{n!}x^n+\ldots,$$

an important result which it is often convenient to write in the equivalent form

$$\frac{-xe^{-ax}}{1-e^{-\omega x}}=-S_0'(a/\omega)+\frac{S_1'(a/\omega)}{1!}x+\ldots+(-)^{n-1}\frac{S_n'(a/\omega)}{n!}z^n+\ldots.$$

Make now $a=0$, and employ the results of 15.

We deduce the expansion

$$\frac{x}{e^{\omega x}-1}=\frac{1}{\omega}-\tfrac{1}{2}x+\ldots+\frac{(-)^{n-1}B_n\omega^{2n-1}x^{2n}}{2n!}+\ldots,$$

which, when $\omega=1$, is the well-known formula

$$\frac{x}{a^x-1}=1-\tfrac{1}{2}x+\frac{B_1}{2!}x^2-\frac{B_2}{4!}x^4+\ldots.$$

The brief sketch which has now been given of simple Bernoullian functions will serve for the formulæ which it is necessary to employ in connection with the asymptotic approximations which follow.

Part III.

Contour integrals connected with the Gamma function.

§ 21. We proceed now to express $\Gamma(z)$ as a definite integral.

We have seen that

$$\Gamma(z)=e^{-\gamma z}\frac{1}{z}\prod_{m=1}^{\infty}\left[\frac{m}{z+m}e^{\frac{z}{m}}\right].$$

Consider the expression

$$\frac{1}{z}\prod_{m=1}^{n}\left[\frac{m}{z+m}\right].$$

It may evidently be put into the form

$$\sum_{m=0}^{n}\frac{A_m}{z+m}.$$

If we determine the constants A by the usual method of partial fractions, we find

$$A_m = \frac{n!}{-m.(1-m).(2-m)...-1.1.2...(n-m)}$$

$$= (-)^m \frac{n!}{m!(n-m)!}.$$

And hence

$$\frac{1}{z}\prod_{m=1}^{n}\left[\frac{m}{z+m}e^{\frac{z}{m}}\right] = e^{\sum_{m=1}^{n}\frac{z}{m}} . \sum_{m=0}^{n}(-)^m\binom{n}{m}\frac{1}{z+m}.$$

Now for all values of α let us define $(-x)^{\alpha-1}$ by the relation

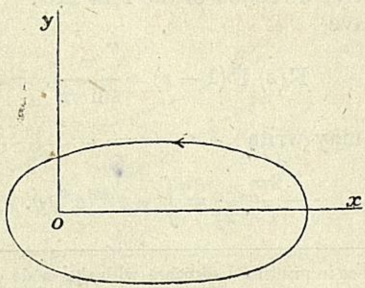
$$(-x)^{(\alpha-1)} = e^{(\alpha-1)\log(-x)},$$

where the logarithm is to be rendered unifrom by the dissection of the plane from 0 to $+\infty$ and that value is to be taken which is real for negative values of x.

Thus $$\int(-x)^{\alpha-1}dx = -\frac{e^{\pi i\alpha}}{\alpha} + \frac{e^{-\pi i\alpha}}{\alpha}$$

$$= -\frac{2i\sin\pi\alpha}{\alpha},$$

the integral being taken round the contour enclosing the origin and cutting the axis in the point 1 as indicated in the figure.



Hence, the integral being taken round the same contour,*

$$\frac{1}{z}\prod_{m=1}^{n}\left[\frac{m}{z+m}e^{\frac{z}{m}}\right]=\frac{i}{2\sin\pi z}e^{\sum_{m=1}^{n}\frac{z}{m}}\sum_{m=0}^{n}\int(-)^{2m}\binom{n}{m}(-x)^{z+m-1}dx,$$

so that

$$e^{-\gamma z}\cdot\frac{1}{z}\cdot\prod_{m=1}^{n}\left[\frac{m}{z+m}e^{\frac{z}{m}}\right]$$

$$=\frac{i}{2\sin\pi z}e^{z\left(\log n+\frac{1}{2n}-\frac{1}{12n^2}+\dots\right)}\int(-x)^{z-1}(1-x)^n dx.$$

Thus, putting $y=nx$,

$$e^{-\gamma z}\cdot\frac{1}{z}\cdot\prod_{m=1}^{n}\left[\frac{m}{z+m}e^{\frac{z}{m}}\right]$$

$$=\frac{i}{2\sin\pi z}e^{z\left(\frac{1}{2n}-\frac{1}{12n^2}+\dots\right)}\int(-y)^{z-1}\left(1-\frac{y}{n}\right)^n dy,$$

the new integral being taken round a contour enclosing the point 0 and cutting the real axis in the point n.

Make now n increase without limit, and we have

$$\Gamma(z)=\frac{i}{2\sin\pi z}\int(-y)^{z-1}e^{-y}dy,$$

the integral extending from $+\infty$ round the origin to $+\infty$ again, and in it

$$(-y)^{z-1}=e^{(z-1)\log(-y)},$$

where the real value of $\log(-y)$ is to be taken when y is negative, and the logarithm is rendered uniform by a cut along the positive direction of the real axis.

Now we have

$$\Gamma(z)\,\Gamma(1-z)=\frac{\pi}{\sin\pi z}.$$

Hence we may write

$$\frac{2\pi}{i\Gamma(1-z)}=\int(-y)^{z-1}e^{-y}dy.$$

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* For convenience in printing I dispense with the mode of contour-indication adopted by Klein, *Hypergeometrische Functionen*, pp. 136–140.

The relation

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int (-y)^{-z} e^{-y} dy$$

gives an expression for $\dfrac{1}{\Gamma(z)}$ as a contour integral which is valid for all values of z.

When z is real and positive it is easy to see that we have the usual expression

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx.$$

§ 22. It is convenient to write down the contour integral which expresses $\Gamma_1(z/\omega)$.

We have (§ 3)

$$\Gamma_1(z/\omega) = \omega^{\frac{z}{\omega}-1} \Gamma\left(\frac{z}{\omega}\right),$$

and hence

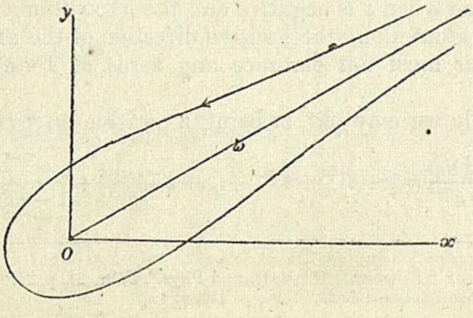
$$\frac{1}{\Gamma_1(z/\omega)} = \frac{i}{2\pi} \omega^{1-\frac{z}{\omega}} \int (-y)^{-\frac{z}{\omega}} e^{-y} dy$$

$$= \frac{i\omega}{2\pi} \int (-\omega y)^{-\frac{z}{\omega}} e^{-y} dy;$$

or, writing ωy for y,

$$\frac{1}{\Gamma_1(z/\omega)} = \frac{i}{2\pi} \int (-y)^{-\frac{z}{\omega}} e^{-\frac{y}{\omega}} dy,$$

and the integral is to be taken from $+\infty$ round the origin to $+\infty$ along the axis of ω.



It is also convenient to state the identity for all values, real or complex of n, except $n=0$,

$$\frac{i\Gamma(1-s)}{2\pi}\int(-z)^{s-1}e^{-nz}dz=\frac{1}{n^s},$$

which results at once on changing the variable of integration, the integral being taken along the axis of $\frac{1}{n}$.

If $\mathfrak{R}(n)$ is positive, the value of the integral along the axis of $\frac{1}{n}$ is equal to its values along the axis of real quantities; for their difference, the integral along an arc of the great circle at infinity, will vanish.

§ 23. We now proceed to the consideration of a more general contour integral, particular cases of which include expressions for the Bernoullian functions and the logarithm of the Gamma function and its derivatives.

From the consideration of this integral it is possible to deduce several asymptotic approximations of considerable utility.

The integral in question we call the extended Riemann ζ function. For all values of s, a, ω, such that the real part of $\frac{a}{\omega}$ is positive, we define it by*

$$\zeta(s, a, \omega)=\frac{i\Gamma(1-s)}{2\pi}\int\frac{e^{-az}}{1-e^{-\omega z}}(-z)^{s-1}dz,$$

the integral being taken round a contour extending along the axis of ω^{-1} round the origin from $+\infty$ to $+\infty$, and $(-z)^{s-1}$ being equal to $e^{(s-1)\log(-z)}$, where the real value of the logarithm is to be taken when z is negative and the logarithm is rendered uniform by a cut along the positive direction of the axis of ω^{-1}. The contour must not embrace any zeros of $1-e^{-\omega z}$ except the origin.

Evidently we may put, m being a positive integer,

$$\frac{1}{1-e^{-\omega z}}=1+e^{-\omega z}+e^{-2\omega z}+\ldots+e^{-(m-1)\omega z}+\frac{e^{-m\omega z}}{1-e^{-\omega z}};$$

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* See Hurwitz, "Zeitschrift für Matt. und Phys.", Jahr. 27, p. 27. Bachmann, "Die Analytische Zahlen-theorie," 1894, p. 340, et seq.

and hence, since

$$\zeta(s, a, \omega) = \frac{i\Gamma(1-s)}{2\pi} \int \frac{e^{-az}}{1-e^{-\omega z}} (-z)^{s-1} dz,$$

we have

$$\zeta(s, a, \omega) = \frac{i\Gamma(1-s)}{2\pi} \int \sum_{n=0}^{m-1} e^{-(n+a\omega)z} (-z)^{s-1} dz$$

$$+ \frac{i\Gamma(1-s)}{2\pi} \int \frac{e^{-(a+n\omega)z}}{1-e^{-\omega z}} (-z)^{s-1} dz.$$

Now, with the restrictions which have been assigned, we have (§ 22),

$$\frac{i\Gamma(1-s)}{2\pi} \int \sum_{n=0}^{m-1} e^{-(n\omega+a)z} (-z)^{s-1} dz = \sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^s}.$$

Again, with the same restrictions, we may obtain an asymptotic approximation for

$$\frac{i\Gamma(1-s)}{2\pi} \int \frac{e^{-(m\omega+a)z}}{1-e^{-\omega z}} (-z)^{s-1} dz,$$

when m is large.

This expression may be written

$$\frac{i\Gamma(1-s)}{2\pi} \int (-z)^{v-z} e^{-m\omega z} \left(\frac{-ze^{-az}}{1-e^{-\omega z}} \right) dz.$$

Now (§ 20) we have obtained the expansion

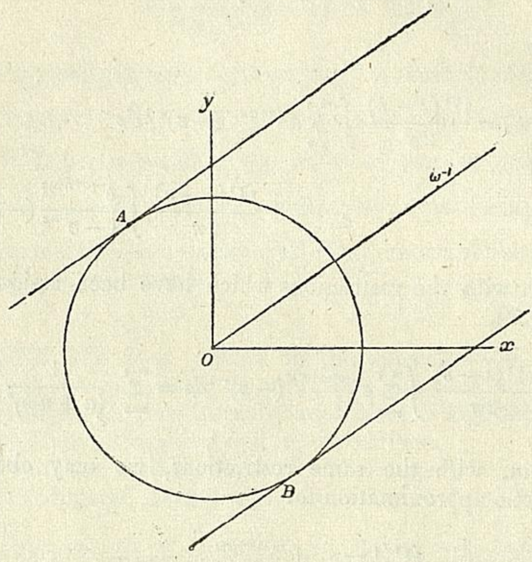
$$\frac{-ze^{-az}}{1-e^{-\omega z}} = -\frac{1}{\omega} + \frac{S_1'(a/\omega)}{1!} z + \ldots + (-1)^{n-1} \frac{S_n'(a/\omega)}{n!} z^n + \ldots$$

valid within a circle of radius $\left| \frac{2\pi i}{\omega} \right|$.

The expansion ceases to converge outside of this circle, for the function

$$\frac{-ze^{-az}}{1-e^{-\omega z}}$$

has upon the circle the two poles $z = \pm \dfrac{2\pi i}{\omega}$ marked A and B in the figure.



The other poles of the function are given by $z = \pm \dfrac{2n\pi i}{\omega}$ and therefore lie on the line joining the origin to the two poles A and B.

But it may readily be proved by a procedure analogous to that introduced by Borel,* that the series

$$-\frac{1}{\omega} + \frac{S_1'(a/\omega)}{1!} z + \dots\dots + (-)^{n-1} \frac{S_n'(a/\omega)}{n!} z^n + \dots$$

is "*summable*" within the region included by the tangents at A and B to the circle.

That is to say, the series is divergent within the region

---

* Borel, "Theorie des séries divergentes sommables." *Liouville*, 5 Sér., T. 2, pp. 103 et seq. I have omitted details of the proof as the statement in question is merely an example of a subsequent theorem due to M. Borel, and recently published in his "Mémoire sur les séries divergentes," pp. 62–67. *Ann., de l'Ecole Normale Supérieure*, Ser. 3, T. 6 (Feb. 1899).

between these tangents outside the circle of radius $\left|\frac{2\pi i}{\omega}\right|$, but the numerical value of the function

$$\frac{-ze^{-az}}{1-e^{-\omega z}}$$

can be calculated for values of z within this space from the numerical values of terms of the series by a procedure only depending on these numerical values.

Now in the contour integral

$$\frac{i\Gamma(1-s)}{2\pi}\int(-z)^{s-2}e^{-m\omega z}\left\{\frac{-ze^{-az}}{1-e^{-\omega z}}\right\}dz$$

the variable z may be supposed to lie entirely within the domain included by the tangents at A and B for the contour extends to infinity along the axis of ω^{-1}. We may then employ the expansion for

$$\frac{-ze^{-az}}{1-e^{-\omega z}},$$

and we shall expect to obtain a divergent expansion for the contour integral, which, however, is such that from it can be calculated with as much exactness as we please the value of the integral. The series proves in fact to belong to a class known as asymtotic, such that for large values of m the difference between the integral and the first n terms of the series is a quantity of order $\frac{1}{m^{n+s-1}}$.

For at once

$$\frac{i\Gamma(1-s)}{2\pi}\int\frac{e^{-(m\omega+a)z}}{1-e^{-\omega z}}(-z)^{s-1}dz$$

$$=-\frac{i\Gamma(1-s)}{2\pi}\int(-z)^{s-2}e^{-m\omega z}\left\{\frac{1}{\omega}+\sum_{n=1}^{\infty}\frac{(-)^n S_n'(a/\omega)}{n!}z^n\right\}dz$$

$$=-\frac{\Gamma(1-s)}{\Gamma(2-s)}\frac{1}{\omega}\frac{1}{(m\omega)^{s-1}}-\sum_{n=1}^{\infty}\frac{1}{n!}\frac{\Gamma(1-s)}{\Gamma(2-n-s)}\frac{S_n'(a/\omega)}{(m\omega)^{n+s-1}}$$

by § 22, provided m does not vanish.

And therefore

$$\zeta(s, a, \omega)-\sum_{n=0}^{m-1}\frac{1}{(a+n\omega)^s}$$

$$=-\frac{1}{1-s}\cdot\frac{1}{\omega^s m^{s-1}}+\sum_{n=1}^{\infty}\binom{-s}{n}\frac{S_n'(a/\omega)}{(s+n-1)(m\omega)^{n+s-1}}.$$

This general asymptotic expansion is the first which has arisen in this paper.

The general theory of such expansions has been dealt with by Poincaré.*

Briefly the identity

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^s} = \zeta(s, a, \omega) + \frac{1}{1-s} \cdot \frac{1}{\omega^s m^{s-1}}$$
$$- \sum_{n=1}^{\infty} \binom{-s}{n} \frac{S_n'(a/\omega)}{(s+n-1)(m\omega)^{s+n-1}} \ldots (1)$$

is such that the second series represents, as successive terms are added, with more and more truth the value of the difference between

$$\sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^s} \text{ and } \zeta(s, a, \omega) - \frac{1}{1-s} \cdot \frac{1}{\omega^s m^{s-1}}.$$

So far we have only defined the extended Riemann function $\zeta(s, a, \omega)$ by means of a contour integral valid when a is positive with respect to ω.

We now use the asymptotic identity (1) to define the function for all values of a, negative or positive, with respect to ω.

We see that this definition coincides with the former definition by means of a contour integral when $\mathfrak{R}\left(\frac{a}{\omega}\right)$ is positive, and gives the natural continuation when $\mathfrak{R}\left(\frac{a}{\omega}\right)$ is negative.

§ 24. When $a = \omega$ our extended function is practically the same as that used by Rieman.

In this case the integral completely represents the function and we have evidently

$$\zeta(s, \omega, \omega) = \frac{i}{2\pi} \int \Gamma(1-s) \frac{(-z)^{s-1}}{e^{\omega z} - 1} dz$$
$$= \frac{i\Gamma(1-s)}{2\pi\omega^s} \int \frac{(-z)^{s-1}}{e^z - 1} dz, \text{ when } \mathfrak{R}(\omega) \text{ is positive,}$$
$$= \frac{\zeta(s)}{\omega^s}, \text{ where } \zeta(s) \text{ is Riemann's } \zeta \text{ function.}\dagger$$

---

* Poincaré, *Acta Mathematica*, T. 8, pp. 295–344, and *Mécanique Celeste*, T. 2, pp. 12–14.

† Riemann, *Ges. Werke*, pp. 136–144.

The formula of § 23 now becomes

$$\frac{1}{\omega^s}\zeta(s) = \sum_{n=1}^{m}\frac{1}{n^s\omega^s} - \frac{1}{(1-s)\,\omega^s}\cdot\frac{1}{m^{s-1}}$$
$$+\sum_{n=1}^{\infty}\binom{-s}{n}\frac{S_n'(0/\omega)}{(s+n-1)(m\omega)^{n+s-1}}.$$

Now (§ 15)

$$\begin{aligned} S_n'(0/\omega) &= -\tfrac{1}{2}, \text{ when } n=1, \\ &= 0, \text{ when } n \text{ is odd and} > 1, \\ &= (-)^{\frac{1}{2}n-1}\omega^{n-1}B_{\frac{1}{2}n}, \text{ when } n \text{ is even and} > 0. \end{aligned}$$

Thus, when m is large,

$$\zeta(s) = \sum_{n=1}^{m}\frac{1}{n^s} - \frac{1}{1-s}\cdot\frac{1}{m^{s-1}} - \tfrac{1}{2}m^s + \sum_{t=1}^{\infty}\binom{-s}{2t}\frac{(-)^{t-1}B_t}{(s+2t-1)\,m^{s+2t-1}}.$$

We thus obtain the asymptotic expression for

$$\sum_{n=1}^{m}\frac{1}{n^s},$$

which is usually found in text-books on the calculus of finite differences by means of the Maclaurin-sum-formula. We have, however, obtained an expansion true for all values, real or complex, of s, and we have shewn that the constant which enters is the Riemann ζ function $\zeta(s)$. When s is complex, we take the principal value of the many valued function n^s, as usually defined in text books on Trigonometry.

From the above formula, when $s=1$, we have

$$\underset{\substack{s=1\\ m=\infty}}{\text{Lt.}}\left[\zeta(s) + \frac{m^{1-s}}{1-s} - \log m\right]$$
$$= \underset{m=\infty}{\text{Lt.}}\left[\sum_{n=1}^{m-1}\frac{1}{n} - \log m\right]$$
$$= \gamma.$$

When $s=0$, we have

$$\zeta(0) = \underset{m=\infty}{\text{Lt.}}\left[m - 1 - m + \tfrac{1}{2}\right] = -\tfrac{1}{2}$$

agreeing with von Mangoldt's correction of the value given by Riemann.*

§ 25. In a few special cases it is possible to express $\zeta(s)$ by the preceding formulæ in terms of Bernoullian numbers.

Thus, when s is a positive even integer, we have, on making m infinite,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{2^{s-1}\pi^s}{s!} B_{s+1}.$$

When $s = 1$ we have from the integral

$$\zeta(1) = \infty.$$

When s is a negative integer we may resolve the contour integral into a small circle round the origin and two line integrals which destroy one another.

Thus

$$\frac{i\Gamma(1-s)}{2\pi} \int \frac{(-)^{s-1}}{e^z - 1} dz$$

$$= -\Gamma(1-s) \left[\text{Residue of } \frac{(-z)^{s-1}}{e^z - 1} \text{ at the origin} \right].$$

Thus when s is a negative even integer

$$\zeta(s) = 0.$$

When s is a negative odd integer $= -(2t+1)$, the residue of $\dfrac{(-z)^{s-1}}{e^z - 1}$ at the origin

$$= \text{residue of } \frac{1}{z^{2t+3}} \left[1 - \frac{z}{2} + \frac{B_1}{2!} z^2 + \ldots + \frac{(-)^t B_{t+1} x^{2t+2}}{(2t+2)!} + \ldots \right]$$

$$= \frac{(-)^t B_{t+1}}{(2t+2)!}.$$

And hence

$$\zeta\{-(2t+1)\} = -(2t+1)! \frac{(-)^t B_{t+1}}{(2t+2)!}$$

$$= \frac{(-)^{t-1} B_{t+1}}{2t+1}.$$

---

* Riemann, *Ges. Werke*, 2te Auflage, § 155.

§ 26. Suppose that in § 23 we had employed the formula

$$\frac{1}{1-e^{-\omega z}} = 1 + e^{-\omega z} + \ldots + e^{-m\omega z} + \frac{1-e^{-\omega z}}{e^{-(m+1)\omega z}},$$

we should evidently have obtained, when m is large,

$$\zeta(s, a, \omega)$$

$$= \sum_{n=0}^{m} \frac{1}{(a+n\omega)^s} - \frac{1}{1-s} \cdot \frac{1}{\omega^s m^{s-1}} + \sum_{n=1}^{m} \binom{-s}{n} \frac{S_n'(\omega + a/\omega)}{(s+n-1)(m\omega)^{n+s-1}}$$

By § 15, *Cor.*, when $n > 1$,

$$S_n'(a+\omega/\omega) = n\left\{S_{n-1}(a+\omega/\omega) + \frac{S_n'(0/\omega)}{n}\right\},$$

and when $n = 1$,

$$S_1'(a+\omega/\omega) = \frac{d}{da}\left\{\frac{(a+\omega)^2}{2\omega} - \frac{a+\omega}{2}\right\} = \frac{a}{\omega} + \tfrac{1}{2}.$$

Hence we have, when m is a large integer, the asymptotic equality

$$\sum_{n=0}^{m} \frac{1}{(a+n\omega)^s} = \zeta(s, a, \omega) + \frac{1}{1-s} \cdot \frac{1}{\omega^s m^{s-1}} + \frac{\frac{a}{\omega} + \frac{1}{2}}{m^s \omega^s}$$

$$+ \frac{1}{m^s \omega^s} \sum_{r=1}^{\infty} \frac{(-)^r}{(m\omega)^r} \binom{s+r-1}{r} \{S_r(a+\omega/\omega) + {}_1B_{r+1}(\omega)\},$$

where, as in § 15,

$${}_1B_n(\omega) = \frac{S_n'(0/\omega)}{n}.$$

§ 27. We can now see the intimate connection between the Gamma-function and its derivatives and the Bernoullian functions.

We will show that

$$\zeta(s, a, \omega) = \frac{(-)^s}{(s-1)!} \frac{d^s}{da^s} \log \Gamma_1(a),$$

when s is a positive integer > 1;

$$\underset{s=1}{\text{Lt.}} \left[\zeta(s, a, \omega) + \frac{1}{(1-s)\,\omega}\right] = -\frac{d}{da} \log \Gamma_1(a),$$

$$\zeta(s, a, \omega) = -\left(\frac{a}{\omega} - \tfrac{1}{2}\right), \text{ when } s = 0;$$

and $$\zeta(s, a, \omega) = -[S_{-s}(a/\omega) + {}_1B_{1-s}(\omega)],$$

when s is a negative integer.

We know that, when s is a positive integer > 1,

$$\frac{(-)^s}{(s-1)!}\frac{d^s}{da^s}\log\Gamma_1(a) = \sum_{n=0}^{\infty}\frac{1}{(a+n\omega)^s}.$$

Hence, if we make m infinite in the result of the preceding section, we have the first result.

When $s = 1$ the result of § 25 becomes

$$\sum_{n=0}^{m}\frac{1}{a+n\omega} = \underset{s=1}{\text{Lt.}}\left[\zeta(s, a, \omega) + \frac{1}{(1-s)\,\omega}\right] + \frac{\log m\omega}{\omega} + \frac{\frac{a}{\omega}+\frac{1}{2}}{m\omega}$$
$$+ \frac{1}{m\omega}\sum_{r=1}^{\infty}\frac{(-)^r}{(m\omega)^r}\{S_r(a+\omega/\omega) + {}_1B_{r+1}(\omega)\}.$$

Now

$$-\frac{d}{da}\log\Gamma_1(a) = -\frac{\gamma}{\omega} - \frac{\log\omega}{\omega} + \frac{1}{a} + \sum_{n=1}^{\infty}\left[\frac{1}{a+n\omega} - \frac{1}{n\omega}\right].$$

Comparing these two results and making m infinite, we have*

$$\underset{s=1}{\text{Lt.}}\left[\zeta(s, a, \omega) + \frac{1}{(1-s)\,\omega}\right] = -\frac{d}{da}\log\Gamma_1(a).$$

Nextly make $s = 0 + \epsilon$ in the result of § 26, we find, retaining only the first term in the expansion in powers of ϵ,

$$m + -\epsilon\log\prod_{n=0}^{m}(a+n\omega)$$
$$= \zeta(\epsilon, a, \omega) + \left[m + \frac{a}{\omega} + \frac{1}{2}\right][1 + \epsilon\log\omega m] + \epsilon m$$
$$+ \sum_{r=1}^{\infty}\frac{(-)^r\,\epsilon}{r\,(m\omega)^r}\cdot\{S_r(a+\omega/\omega) + {}_1B_{r+1}(\omega)\}.$$

Equating the absolute term, we find, when m is infinite,

$$\zeta(0, a, \omega) = -\left(\frac{a}{\omega} - \frac{1}{2}\right),$$

---

* Hurwitz (loc. cit.) has previously shown that $\zeta(s, a, \omega)$ is infinite at $s = 1$ like $\frac{1}{s-1\,.\,\omega}$.

which agrees with the value of $\zeta(0)$ previously obtained. On equating coefficients of ϵ, we should ultimately get a contour integral for $\log \Gamma_1(a)$.

When s is a negative integer, we may take the contour integral for $\zeta(s, a, \omega)$* and resolve it into a small circle round the origin and two line integrals which destroy one another.

We thus get

$$\zeta(s, a, \omega) = \frac{i\Gamma(1-s)}{2\pi} \int \frac{e^{-az}}{1-e^{-\omega z}} (-z)^{s-1}\, dz$$

$$= -\Gamma(1-s) \left[\text{residue of } (-z)^{s-1} \frac{e^{-az}}{1-e^{-\omega z}} \text{ at the origin}\right]$$

$$= -\Gamma(1-s) \frac{\frac{d}{da} S_{1-s}(a/\omega)}{(1-s)!},$$

as may be readily seen by aid of the expansion of § 19.

Thus by §§ 12 and 15, *Cor.*

$$\zeta(s, a, \omega) = -[S_{-s}(a/\omega) + {}_1B_{1-s}(\omega)],$$

and we have proved the group of formulæ enunciated.

As a *Corollary* note that we have proved incidentally that

$$\sum_{n=0}^{m} \frac{1}{a+n\omega} = -\frac{d}{da} \log \Gamma_1(a) + \frac{1}{\omega} \log m + \frac{1}{m\omega}\left(\frac{a}{\omega} + \tfrac{1}{2}\right)$$

$$+ \sum_{r=1}^{\infty} \frac{(-)^r}{m^{r+1}\omega^{r+1}} \{S_r(a+\omega/\omega) + {}_1B_r(\omega)\}.$$

But, from the product expression of $\Gamma_1(a)$,

$$\underset{a=0}{\text{Lt.}} \left[-\frac{d}{da} \log \Gamma_1(a) - \frac{1}{a}\right] = \frac{\gamma - \log \omega}{\omega},$$

Hence, making $a = 0$, $\omega = 1$, we have, when m is large,

$$\sum_{n=1}^{m} \frac{1}{n} = \log m + \gamma + \frac{1}{2m} + \sum_{r=1}^{\infty} \frac{(-)^r}{m^{r+1}}\, {}_1B_r(1).$$

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* I wish at this point to take the first opportunity of correcting an omission which occurred in that part of the paper which appeared in the last number of this journal.

In § 23 it should have been been stated that we limit ourselves to those values of ω for which $R(\omega)$ is positive, in which case alone it is possible with the given definition of $\zeta(s, a, \omega)$ to apply the fundamental theorem stated at the end of § 22 so as always to obtain many-valued functions with s as index whose principal value is to be taken.

§ 28. Before it is possible to deduce from § 27 the usual formula which expresses $\log \Gamma_1(z)$ as a definite integral, it is necessary to express Euler's constant γ in that manner and to show that when ϵ is small and real

$$\int_\epsilon^\infty \frac{e^{-z}}{z}\, dz = -\log \epsilon - \gamma + \text{terms vanishing with } \epsilon.$$

We proceed to show first that the limit when n is very large of

$$1 + \tfrac{1}{2} + \ldots + \frac{1}{n} - \log n$$

is a finite quantity

$$\gamma = \int_0^1 \frac{1 - e^{-z} - e^{-\frac{1}{z}}}{z}\, dz.$$

Since $$\frac{1-z^n}{1-z} = 1 + z + z^2 + \ldots + z^{n-1},$$

we evidently have

$$\int_0^1 \frac{1-z^n}{1-z}\, dz = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n}.$$

Hence, writing $1 - z$ for z,

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} = \int_0^1 \frac{1 - (1-z)^n}{z}\, dz,$$

so that

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} - \log n = \int_0^n \frac{1 - \left(1 - \dfrac{z}{n}\right)^n}{z}\, dz - \int_1^n \frac{dz}{z}$$

$$= \int_0^1 \frac{1 - \left(1 - \dfrac{z}{n}\right)^n}{z}\, dz - \int_1^n \frac{\left(1 - \dfrac{z}{n}\right)^n}{z}\, dz.$$

Now $\underset{n=\infty}{\text{Lt.}} \left(1 - \dfrac{z}{n}\right)^n = e^{-z}$, and hence

$$\underset{n=\infty}{\text{Lt.}} \left[1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} - \log n\right] = \int_0^1 \frac{1 - e^{-z}}{z}\, dz - \int_1^\infty \frac{e^{-z}}{z}\, dz,$$

and this is equal to the finite quantity

$$\gamma = \int_0^1 \frac{1 - e^{-z} - e^{-\frac{1}{z}}}{z}\, dz.$$

The second part of the theorem follows at once.

Corollary. When $\mathfrak{R}(\omega)$ is positive and ϵ lies on the axis of ω^{-1}, the line integral along that axis

$$\int_\epsilon^\infty \frac{e^{-z}}{z}\, dz = -\log \epsilon - \gamma + \text{terms vanishing with } |\epsilon|.$$

For an integral with this subject of integration will vanish when taken round a contour formed by those arcs of circles of radii $|\epsilon|$ and ∞ cut off between the axes of 1 and ω^{-1}, and including the intercepted parts of these axes.

§ 29. It is now possible to obtain a contour integral for

$$\log \Gamma_1 (a\omega).$$

We have seen in § 27 that

$$-\frac{d}{da} \log \Gamma_1 (a) = \underset{s=1}{\text{Lt.}} \left[\zeta(s, a, \omega) + \frac{1}{(1-s)\,\omega} \right];$$

or, putting $1 - \epsilon = s$, and substituting in the integral which represents $\zeta(s, a, \omega)$, we have, when $\mathfrak{R}\left(\dfrac{a}{\omega}\right)$ is positive,

$$-\frac{d}{da} \log \Gamma_1 (a) = \underset{\epsilon=0}{\text{Lt.}} \left[\frac{i\Gamma(\epsilon)}{2\pi} \int \frac{e^{-az}}{1-e^{-\omega z}} (-z)^{-\epsilon}\, dz + \frac{1}{\epsilon\omega} \right]$$

$$= \underset{\epsilon=0}{\text{Lt.}} \left[\frac{i}{2\pi\epsilon} \int \frac{e^{-az}}{1-e^{-\omega z}} \{1 - \epsilon \log(-z) + \ldots\} \{1 - \gamma\epsilon + \ldots\}\, dz + \frac{1}{\epsilon\omega} \right].$$

This expression just written must be finite, and hence

$$\frac{i}{2\pi} \int \frac{e^{-az}}{1-e^{-\omega z}}\, dz = -\frac{1}{\omega},$$

as is evidently true; and also, equating coefficients of ϵ,

$$\frac{d}{da} \log \Gamma_1 (a) = \frac{i}{2\pi} \int \frac{e^{-az}}{1-e^{-\omega z}} \{\log(-z) + \gamma\}\, dz.$$

Integrate the result with respect to a, and we get

$$\log \Gamma_1 (a) = \frac{i}{2\pi} \int \frac{e^{-az}(-z)^{-1}}{1-e^{-\omega z}} \{\log(-z) + \gamma\}\, dz + \mu,$$

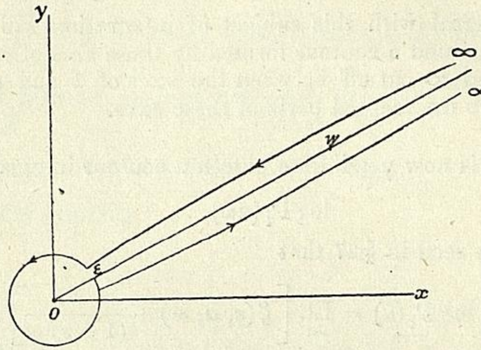
where μ is constant with respect to a.

To determine μ integrate this relation with respect to a between 0 and ω. Then, by Raabe's formula, we find

$$\frac{\omega}{2} \log \frac{2\pi}{\omega} = \mu\omega - \frac{i}{2\pi} \int \frac{dz}{z^2} \{\log(-z) + \gamma\}.$$

We can readily see that the last integral vanishes.

Reduce the contour to a straight line from $+\infty$ to ϵ, a small circle of radius $|\epsilon|$ round the origin, and a straight line back from $+\epsilon$ to $+\infty$, as in the figure.



The axis of ω^{-1} is a cross cut serving to render $\log(-z)$ uniform; hence the value of $\log(-z)$ at any point along the first of these lines falls short of its value at the same point along the second by $2\pi i$.

Hence
$$\frac{-i}{2\pi}\int\frac{dz}{z^2}\{\log(-z)+\gamma\}$$
$$=\int_\epsilon^\infty\frac{dz}{z^2}-\frac{i}{2\pi}\int\frac{dz}{z^2}\{\log(-z)+\gamma\},$$
the second integral being taken round a circle of radius $|\epsilon|$.

Writing $z=\epsilon e^{i\theta}$, the latter integral becomes
$$\frac{i}{2\pi}\int_0^{2\pi}\frac{d\theta}{\epsilon}e^{-i\theta}\{\log\epsilon+i(\theta-\pi)+\gamma\}=-\frac{1}{\epsilon}.$$

Hence
$$\frac{-i}{2\pi}\int\frac{dz}{z^2}\{\log(-z)+\gamma\}=0,$$

and therefore $\mu=\frac{1}{2}\log\frac{2\pi}{\omega}$.

Finally then, provided a is positive with respect to ω, we may express $\log\Gamma_1(a/\omega)$ as a definite integral by the formula
$$\log\Gamma_1(a/\omega)=\frac{i}{2\pi}\int\frac{e^{-az}(-z)^{-1}}{1-e^{-\omega z}}\{\log(-z)+\gamma\}\,dz+\tfrac{1}{2}\log\frac{2\pi}{\omega},$$
the contour being drawn along the axis of ω^{-1}. So far this expression has been proved to hold when $\mathfrak{R}(\omega)$ is positive.

We shall afterwards see that it is true for all values of ω, provided a different specification is given to the value of the logarithm in the subject of integration.

§ 30. In the preceding paragraph we have integrated with respect to a under the sign of contour integration to find the integral which expresses $\log \Gamma_1(a/\omega)$ from that for

$$\frac{d}{da} \log \Gamma_1 (a/\omega);$$

and we have not investigated the validity of this process. But this is not necessary; for the integral which has been obtained would at once result from the identity of § 27

$$\log \prod_{n=0}^{m} (a+n\omega) = -\underset{\epsilon=0}{\text{Lt.}} \left(\frac{\zeta(\epsilon, a, \omega) + \frac{a}{\omega} - \frac{1}{2}}{\epsilon}\right) + \left(m + \frac{a}{\omega} + \tfrac{1}{2}\right) \log \omega m$$

$$-m + \sum_{v=1}^{\infty} \frac{(-)^{r-1}}{r(m\omega)^r} [S_r(a+\omega/\omega) + {}_1B_{r+1}(\omega)]$$

on substituting for $\log \Gamma_1(a/\omega)$ its value

$$\frac{a}{\omega} \log \omega - \frac{\gamma a}{\omega} - \underset{m=\infty}{\text{Lt.}} \log a \prod_{n=1}^{m} \left[\left(1 + \frac{a}{n\omega}\right) e^{-\frac{a}{n\omega}}\right]$$

$$= \frac{a}{\omega} \log \omega - \frac{\gamma a}{\omega} - \underset{m=\infty}{\text{Lt.}} \left[\log \prod_{n=0}^{m} (a+n\omega) - m \log \omega - \log m! - \sum_{n=1}^{m} \frac{a}{n\omega}\right]$$

$$= -\underset{m=\infty}{\text{Lt.}} \left[\log \prod_{n=0}^{m} (a+n\omega) + m - \left(m + \frac{a}{\omega} + \tfrac{1}{2}\right) \log m\omega\right] + \tfrac{1}{2} \log \frac{2\pi}{\omega}.$$

We thus have, on making m infinite,

$$\log \Gamma_1 (a/\omega) - \tfrac{1}{2} \log \frac{2\pi}{\omega} = + \underset{\epsilon=0}{\text{Lt.}} \left(\frac{\zeta(\epsilon, a, \omega) + \frac{a}{\omega} - \frac{1}{2}}{\epsilon}\right).$$

And when a is positive with respect to ω, the limit just written may be expressed by the integral

$$\frac{i}{2\pi} \int \frac{e^{-az}(-z)^{-1}}{1 - e^{-\omega z}} \{\log(-z) + \gamma\} dz.$$

In a similar manner it can be shewn from the formula

$$\zeta(s, a, \omega) = \frac{(-)^s}{(s-1)!} \frac{d^s}{da^s} \log \Gamma_1 (a)$$

that

$$\frac{d^s}{da^s}\log\Gamma_1(a/\omega)=\frac{i}{2\pi}\int\frac{e^{-az}(-z)^{s-1}}{1-e^{-\omega z}}\{\log(-z)+\gamma\}\,dz,$$

when s has any integral value greater than unity.

Note that the present paragraph solves the important problem of the asymptotic approximation to the factorial

$$a.(a+\omega)(a+2\omega)\ldots(a+m\omega).$$

In fact we have proved that, when m is large,

$$\log\prod_{n=0}^{m}(a+n\omega)=-\log\Gamma_1(a/\omega)+\tfrac{1}{2}\log\frac{2\pi}{\omega}+\left(m+\frac{a}{\omega}+\tfrac{1}{2}\right)\log m\omega$$

$$-m+\sum_{r=1}^{\infty}\frac{(-)^{r-1}}{r(m\omega)^r}\{S_r(a+\omega/\omega)+{}_1B_{r+1}(\omega)\}.$$

Thus we have as the most general form of Stirling's theorem

$$\begin{aligned}\log\prod_{n=0}^{pm}(a+n\omega)&=p\{m\log m-m\}\\&+m\{S_1^{(2)}(a+\omega)\,p\omega\log p\omega\}+[1+S_1'(a)]\log m\\&-\log\Gamma_1(a/\omega)+\log\sqrt{\left(\frac{2\pi}{\omega}\right)}+S_1'(a+\omega)\log p\omega\\&+\sum_{r=1}^{\infty}\frac{(-)^{r-1}}{rm^r}\,\frac{S_r(a+\omega)+{}_1B_{r+1}(\omega)}{(p\omega)^r}.\end{aligned}$$

This formula will be afterwards extended to any number of parameters.

§ 31. Suppose that we write $\rho_1(\omega)=\left(\frac{2\pi}{\omega}\right)^{\frac{1}{2}}$, in accordance with a more general notation which will be subsequently introduced.

Then we have, from § 29,

$$\log\frac{\Gamma_1(a/\omega)}{\rho_1(\omega)}=-\frac{i}{2\pi}\int\frac{e^{-az}}{1-e^{-\omega z}}\,\frac{\log(-z)+\gamma}{z}\,dz,$$

and, since
$$\frac{i}{2\pi}\int\frac{e^{-az}}{1-e^{-\omega z}}\,\frac{dz}{z}=S_1'(a/\omega),$$

we may write this result in the form

$$\log\frac{\Gamma_1(a)}{\rho_1(\omega)}=-\frac{i}{2\pi}\int\frac{e^{-az}}{1-e^{-\omega z}}\,\frac{\log(-z)}{z}\,dz-\gamma S_1'(a/\omega).$$

Suppose now that we had integrated the expression of § 29 for $\frac{d}{da}\log\Gamma_1(a)$ with respect to a between ω and a. We should have obtained

$$\log\Gamma_1(a/\omega) = \frac{-i}{2\pi}\int\frac{e^{-az}-e^{-\omega z}}{1-e^{-\omega z}}\cdot\frac{\log(-z)+\gamma}{z}\,.\,dz,$$

since $\Gamma_1(\omega/\omega)=1$.

Hence we see that

$$\log\rho_1(\omega) = \frac{i}{2\pi}\int\frac{e^{-\omega z}}{1-e^{-\omega z}}\cdot\frac{\log(-z)+\gamma}{z}\,.\,dz,$$

an expression as a contour integral which might easily be proved independently.

Under the previous limitation that a is positive with respect to ω, coupled with the further condition that $\mathfrak{R}(\omega)$ is positive, we can readily change the contour integral for $\log\Gamma_1(a/\omega)$ into a line integral. For this purpose reduce the contour as in § 29. Then by the previous argument

$$\frac{i}{2\pi}\int\frac{e^{-az}(-z)^{-1}}{1-e^{-\omega z}}\{\log(-z)+\gamma\}\,dz$$

$$=\int_\epsilon^\infty\frac{e^{-az}}{1-e^{-\omega z}}\cdot\frac{dz}{z}+\frac{i}{2\pi}\int\frac{e^{-az}(-z)^{-1}}{1-e^{-\omega z}}\{\log(-z)+\gamma\}\,dz,$$

the second integral being taken round the circle of radius $|\epsilon|$.

The value of this second integral is

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-a\epsilon e^{i\theta}}\,d\theta}{1-e^{-\omega\epsilon e^{i\theta}}}\{\log\epsilon+i(\theta-\pi)+\gamma\},$$

which, on employing the expansion of § 20, is evidently equal to

$$-\{\log\epsilon+\gamma\}\,S_1'(a/\omega)$$
$$+\frac{i}{2\pi}\int_0^{2\pi}\frac{(\theta-\pi)}{\omega\epsilon}e^{-i\theta}\left\{1-\frac{\omega S_1'(a/\omega)}{1!}\epsilon e^{i\theta}+\ldots\right\}d\theta$$

$$=-\{\log\epsilon+\gamma\}\,S_1'(a/\omega)-\frac{1}{\epsilon\omega}+\text{terms which vanish with }|\epsilon|.$$

Hence

$$\log\frac{\Gamma_1(a/\omega)}{\rho_1(\omega)}=\int_\epsilon^\infty\frac{e^{-az}}{1-e^{-\omega z}}\cdot\frac{dz}{z}-\{\log\epsilon+\gamma\}\,S_1'(a/\omega)-\frac{1}{\omega\epsilon}$$
$$+\text{terms which vanish with }|\epsilon|.$$

But, by § 28, when $|\varepsilon|$ is small,

$$\int_\epsilon^\infty \frac{e^{-z}\,dz}{z} = -\log \varepsilon - \gamma + \text{terms which vanish with } |\varepsilon|.$$

Hence $$\int_\epsilon^\infty \frac{e^{-az}\,dz}{z} = -\log \varepsilon - \log a - \gamma + \text{similar terms},$$

and obviously $$\int_\epsilon^\infty \frac{dz}{z^2} = \frac{1}{\epsilon}.$$

We thus have

$$\log \frac{\Gamma_1(a/\omega)}{\rho_1(\omega)} = \int_\epsilon^\infty \frac{dz}{z} \left\{ \frac{e^{-az}}{1-e^{-\omega z}} + e^{-z} S_1'(a/\omega) - z\, S_1^{(2)}(a/\omega) \right\}$$
$$+ \text{terms which vanish with } |\epsilon|.$$

Make now the point ϵ coincide with the origin, then the integral remains finite and we have finally, when $\mathfrak{R}(\omega)$ is positive,

$$\log \frac{\Gamma_1(a/\omega)}{\rho_1(\omega)} = \int_0^\infty \frac{dz}{z} \left\{ \frac{e^{-az}}{1-e^{-\omega}} - z\, S_1^{(2)}(a/\omega) + e^{-z} S_1'(a/\omega) \right\},$$

an expression valid so long as a is positive with respect to ω.

§ 32. Suppose that the same process is carried out with regard to the relations

$$\log \rho_1(\omega) = \frac{i}{2\pi} \int \frac{e^{-\omega z}}{1-e^{-\omega z}} \frac{\log(-z)+\gamma}{z}\, dz.$$

We evidently obtain

$$\log \rho_1(\omega) = -\int_0^\infty \frac{dz}{z} \left\{ \frac{e^{-\omega z}}{1-e^{-\omega z}} - \frac{S_1^{(2)}(\omega/\omega)}{z} + e^{-z} S_1'(\omega/\omega) \right\},$$
$$= -\int_0^\infty \frac{dz}{z} \left\{ \frac{1}{e^{\omega z}-1} - \frac{1}{\omega z} + \frac{e^{-z}}{2} \right\}.$$

Hence we have, when $\mathfrak{R}(a)$ and $\mathfrak{R}\left(\dfrac{a}{\omega}\right)$ are both positive,

$$\log \Gamma_1(a) = \int_0^\infty \frac{dz}{z} \left[\frac{e^{-az} - e^{-\omega z}}{1-e^{-\omega z}} + e^{-z} \{S_1'(a) - S_1'(\omega)\} \right]$$
$$= \int_0^\infty \frac{dz}{z} \left\{ \frac{e^{-az} - e^{-\omega z}}{1-e^{-\omega z}} + \frac{a-\omega}{\omega} e^{-z} \right\}.$$

When $\omega = 1$ this reduces to the form usually given in the text books.*

Carrying out a similar process we find

$$\frac{d}{da}\log\Gamma_1(a) = -\int_0^\infty dz\left\{\frac{e^{-az}}{1-e^{-\omega z}} + e^{-z}S_1^{(2)}(a/\omega)\right\},$$

and, when s is a positive integer greater than unity,

$$\frac{d^s}{da^s}\log\Gamma_1(a) = -\int_0^\infty dz\left\{\frac{(-z)^{s-1}e^{-az}}{1-e^{-\omega z}}\right\},$$

the line integral being taken along the axis of ω^{-1}, and a being positive with respect to ω.

When $\mathfrak{R}(\omega)$ is positive, all the contour and line integrals which have been taken along the axis of ω^{-1} in the preceding paragraphs may be replaced by integrals having for axis the positive half of the real axis. This is an immediate deduction from the theorem stated at the end of § 22 (p. 88).

§ 33. In order to extend the previous integrals and approximations to the case when $\mathfrak{R}(\omega)$ is not positive, it is advisable to give *in extenso* the proof of the fundamental theorem stated at the end of § 22. This theorem is as follows: The integral

$$\int(-z)^{s-1}e^{-z}dz,$$

where $(-z)^{s-1} = e^{(s-1)\log(-z)}$ and that value of $\log(-z)$ is to be taken, which is made uniform by a cross-cut along the axis of the contour of the integral, and which is such that $\log(-z)$ is real when z is real and negative, has the same value when the contour embraces any axis in the positive half of the z-plane.

The contour may be regarded as one which encloses the points 0 and ∞, which are joined by an axis along which there is a cross-cut which serves to render uniform the many-valued subject of integration. Along that part of the contour which is at infinity the integral vanishes when $\mathfrak{R}(z)$ is positive. Hence we may, without altering the value of the integral, make the contour expand so as to include any two lines OA and OB passing from the origin O to two points A and B at infinity in the positive half of the z-plane.

Now the subject of integration is a one-valued function of

* Jordan, *Cours d'Analyse*, Vol II., p. 180.

position on the infinite sheeted Neumann sphere with a cross cut from 0 to ∞ on which $\log(-z)$ is uniform. Hence the value of the integral will not be altered if the cross-cut is deformed in any way inside the contour, provided such deformation does not cause $\log(-z)$ to have a different value along the axis of -1. The latter phenomenon only takes place when the cross-cut in its deformation passes over the axis of -1, and will therefore not occur when we deal with cross-cuts which are axes in the positive half of the z-plane.

If then we start with the integral along the axis OA, we expand its contour till it also includes the axis OB; we then deform the cross-cut till it comes into the position OB, and finally compress the contour till it has OB for axis. By all these changes the integral is unaltered in value. If then we denote the integral taken along the contour which embraces the axis of n^{-1} by

$$\int_{\frac{1}{n}} (-z)^{s-1} e^{-z} dz,$$

where that value of $(-z)^{s-1} = e^{(s-1)\log(-z)}$ is to be taken, which is such that $\log(-z)$ is rendered uniform by a cross-cut along the axis of n^{-1}, while that value of $\log(-z)$ is taken which is real when z is real and negative; we see that, when $\mathfrak{R}(n)$ is positive,

$$\int_{\frac{1}{n}} (-z)^{s-1} e^{-z} dz = \int_{1} (-z)^{s-1} e^{-z} dz.$$

Corollary. Evidently under the same limitations we have the theorem stated at the end of § 22,

$$\int_{\frac{1}{n}} (-z)^{s-1} e^{-nz} dz = \int_{1} (-z)^{s-1} e^{-nz} dz.$$

§ 34. It is now necessary to define the many-valued functions in the subject of integration of our integrals in a different manner. When $\mathfrak{R}(\omega)$ is positive this definition is equivalent to that hitherto adopted; its importance lies in the fact that it allows us to extend the previous contour-integral expressions for the Gamma and extended Riemann ζ-functions, so that they are valid for all values of ω.

In any integral of the same type as those which have arisen

$$\int_{\frac{1}{\omega}} e^{(s-1)\log(-z)} f(z)\, dz,$$

where $f(z)$ is a uniform function, and the contour of the integral embraces the axis of ω^{-1}, we take this axis as a cross-cut for $\log(-z)$, but that value of the logarithm is to be taken which is such that the imaginary part of the initial value of $\log(-\omega^{-1})$ lies between 0 and $-2\pi i$. In other words, that value is to be taken which is such that the imaginary part of $\log(\omega^{-1})$ lies between $\pm\pi i$.

Now we may prove an extension of the previous theorem which was implicitly assumed in § 23. With the definition of the integrals just given, the theorem states that when $\mathfrak{R}(\omega)$ and $\mathfrak{R}\left(\frac{n}{\omega}\right)$ are both positive,

$$\int_{\frac{1}{\omega}} (-z)^{s-1} e^{-nz} dz = \int_{\frac{1}{n}} (-z)^{s-1} e^{-nz} dz.$$

The two integrals are at once seen to be equal to one another by the procedure of § 33; for, since $\mathfrak{R}(n\omega^{-1})$ is positive, the axis of n^{-1} lies within a range of a right angle on either side of the axis of ω^{-1}, and, since $\mathfrak{R}(\omega)$ is positive, the smaller angle between these axes cannot embrace the negative part of the real axis. Hence, when $\mathfrak{R}(n\omega^{-1})$ and $\mathfrak{R}(\omega)$ are both positive,

$$\frac{i\Gamma(1-s)}{2\pi} \int_{\frac{1}{\omega}} (-z)^{s-1} e^{-nz} dz = \frac{i\Gamma(1-s)}{2\pi n} \int_{1} e^{(s-1)\log\left(-\frac{y}{n}\right)-y} dy,$$

and in the last integral the logarithm has a cross-cut along the axis of n^{-1}, and that value is assigned to it for which the imaginary part of $\log(n^{-1})$ lies between $\pm\pi i$. The logarithm is therefore equal to $\log_1(-y) + \log n^{-1}$, where the second logarithm has its principal value; for, both $\log(-yn^{-1})$ and $\log_1(-y)$ increase by $2\pi i$ when y passes in the positive direction over the axis of 1, and the initial value of $\log(-yn^{-1})$ is such that its imaginary part lies between 0 and $-2\pi i$, while the initial value of $\log_1(-1)$ is $-\pi i$.

And now, by § 22,

$$\frac{i\Gamma(1-s)}{2\pi} \int_{\frac{1}{\omega}} (-z)^{s-1} e^{-nz} dz = \frac{1}{n^s},$$

where the principal value of the latter function must be taken.

§ 35. We have now been able to see more in detail how, when $\mathfrak{R}(\omega)$ and $\mathfrak{R}(n\omega^{-1})$ are both positive, the fundamental asymptotic expansion (1) § 23 is such that the principal value of all the many-valued functions with s as index is to be taken. Put now in this expression $\omega = 1$, and write $a\omega^{-1}$ for ω. Multiply the resulting expansion throughout by ω^{-s}, and transform the contour integral. We obtain the expansion

$$\sum_{n=0}^{m-1} \frac{1}{\omega^s\left(\frac{a}{\omega}+n\right)^s} = \frac{i\Gamma(1-s)}{2\pi}\int_{\frac{1}{\omega}} \frac{e^{-ay}}{1-e^{-\omega y}}(-y)^{s-1}dy + \frac{1}{1-s}\cdot\frac{1}{m^{s-1}\omega^s}$$

$$- \sum_{n=1}^{\infty}\binom{-s}{n}\frac{S_n'(a/\omega)}{(s+n-1)(n+\omega)^{s+n-1}}$$

valid, when $\mathfrak{R}(a\omega^{-1})$ is positive, for all values of ω whose amplitudes lie between $\pm\pi$. The logarithm is defined as at the commencement of § 34, and in all other terms the principal values of the many-valued functions must be taken.

Now it may be readily seen that

$$\omega^{-s}\left(\frac{a}{\omega}+n\right)^{-s} = e^{-s\,[\log(a+n\omega)+2k\pi i]},$$

where $k=0$ unless the points $a+n\omega$ and ω embrace the axis of -1, in which case $k=\pm 1$, the upper or lower sign being taken as ω lies above or below the real axis. Assuming then that each term of the series $\sum_{n=0}^{m-1}\frac{1}{(a+n\omega)^s}$ has the value given by this expression, we have, as the general definition of $\zeta(s, a, \omega)$ for all values of s, a, ω, the asymptotic equality

$$\sum_{n=0}^{m-1}\frac{1}{(a+n\omega)^s} = \zeta(s, a, \omega) + \frac{1}{(1-s)\,\omega^s m^{s-1}}$$

$$- \sum_{n=1}^{\infty}\binom{-s}{n}\frac{S_n'(a/\omega)}{(s+n-1)(m\omega)^{s+n-1}},$$

The value of $\zeta(s, a, \omega)$ here taken is represented by the contour integral considered above when $\mathfrak{R}(a\omega^{-1})$ is positive, and accords with the value taken in § 23 when $\mathfrak{R}(\omega)$ is positive. It is evident that, in this general asymptotic expansion defining $\zeta(s, a, \omega)$, the many-valued functions have their principal values with respect to a cross-cut along the axis of $-\omega$.* This line is thus both a cross-cut to render the functions uniform, and also the line on which the possible poles of $\zeta(s, a, \omega)$ lie. It is the *barrier-line* of the function.

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* This prescription was omitted in the extension of the definition to the case when $\mathfrak{R}\left(\frac{a}{\omega}\right)$ is negative at the end of § 23.

§ 36. The value of the typical term in the series just considered, from whose asymptotic expansion $\zeta(s, a, \omega)$ arises, may be be obtained *à priori* when $\mathfrak{R}(a\omega^{-1})$ is positive from the consideration of the integral for $\zeta(s, a, \omega)$. In fact, the procedure of § 23 applied to the integral of § 35 leads to the integral

$$\frac{i\Gamma(1-s)}{2\pi}\int_{\frac{1}{\omega}} e^{-(a+n\omega)z}(-z)^{s-1}dz$$

when $\mathfrak{R}(a\omega^{-1})$ is positive. This integral will, by the theory of the deformation of cross-cuts given in § 33, be equal to the same integral along a contour embracing the axis of $(a+n\omega)^{-1}$, provided the axes of ω^{-1} and $(a+n\omega)^{-1}$ do not form an angle less than two right angles embracing the axis of -1. In the latter case the prescriptions of the logarithms which enter into the expressions of the subjects of integration of the respective integrals will be different, and the integral along the axis of ω^{-1} will be equal to that along the axis of $(a+n\omega)^{-1}$ multiplied by $e^{\mp 2\pi i s}$, the upper or lower sign being taken as ω lies above, by, or below the real axis. We thus get the same expansion as before.

§ 37. We now see that with the new prescription of the integrals we may obtain for all values of ω such formulæ as

$$\log\Gamma_1(a/\omega) = \frac{i}{2\pi}\int_{\frac{1}{\omega}} \frac{e^{-az}(-z)^{-1}}{1-e^{-\omega z}}\{\log(-z)+\gamma\}\,dz + \tfrac{1}{2}\log\frac{2\pi}{\omega}.$$

In fact the whole theory of §§ 24–32, except that relating to expressions as line-integrals, is valid with this modification for all values of ω. It will be noted that in the method of deriving the formula for $\log\Gamma_1(a/\omega)$ in § 30 we should, when $\mathfrak{R}(\omega)$ is negative, assume that such a value of $\log(a+n\omega)$, derived from the term $(a+n\omega)^{-s}$, is to be taken as is equal to

$$\log n\omega + \log\left(1+\frac{a}{n\omega}\right),$$

which is precisely the prescription shown to be necessary in § 36. The extended formulæ may also be proved by putting $\omega = 1$, writing $a\omega^{-1}$ for a, and transforming the integrals in the manner indicated in § 36.

§ 38. Let us consider finally the nature of the asymptotic expansions which have been obtained. In the first place it is evident that they form absolutely divergent series, and yet the first few terms of the series would for large values of m give a very near estimate of the function which the series has been formed to represent. As stated in § 23 a theory of asymptotic approximations has been developed by Poincaré which attempts to place such expansions on a satisfactory basis.

If we have the asymptotic expansion

$$J(m) = a_0 + \frac{a_1}{m} + \ldots + \frac{a_n}{m^n} + \ldots,$$

then in Poincaré's theory the series, though divergent for all values of m, will for large values of the modulus of that quantity represent $J(m)$ if the expression

$$m^n [J(m) - s_n],$$

where s_n represents the sum of the first $(n+1)$ terms of the series, tends to zero when m tends to infinity. This hypothesis, however, necessitates the investigation of a superior limit to the remainder after the first $(n+1)$ terms of the series, and it is precisely this tedious investigation which has not been undertaken. Poincaré, entrenching himself behind physical applications of such series, makes no attempt thus to deal with the remainders of the asymptotic approximations which occur in celestial dynamics. From the point of view, however, of pure mathematics, either some such investigation must be undertaken or some means must be devised by which such a series may be regarded as arithmetically defining some function or functions.

The recent investigations of M. Borel open up a line of thought which indicates a consistent theory of such series as have arisen in §§ 23–37 of this paper. In investigating series of convergency zero, Borel* takes such a series to be

$$a_0 + a_1 z + \ldots + a_n z^n + \ldots,$$

and by means of the associated function

$$F(u) = a_0 + \frac{a_1 u}{1!} + \ldots + a_n \frac{u^n}{n!} - \ldots$$

* *Ann. de l'Ec. Norm. Sup.*, 1899, loc. cit., Chap. 3, § 3.

he establishes that we may regard

$$f(z) = \int_0^\infty F(u)\, e^{-\frac{u}{z}}\, du$$

as the result of the problem of interpolation which the summation of such a series proposes, whenever $F(u)$ is an integral function and the expression for $f(z)$ is finite.

Now it may be readily seen that the argument is equally valid if we consider as associated function

$$G(u) = a_0 c_0 + a_1 c_1 u + \ldots + a_n c_n u^n + \ldots,$$

where $c_0 + c_1 z + \ldots + c_n z^n + \ldots$ is the expansion of an integral function $\chi(z)$. And the sum of the series may be defind as

$$f(z) = \int_0^\infty G(u) \frac{1}{\chi\left(\frac{u}{z}\right)} \frac{du}{z},$$

provided $G(u)$ is an integral function of u and the expression for $f(z)$ is finite.

All the asymptotic series which have so far been obtained in this paper are series of powers of a real variable of zero line of convergency from the point $m = \infty$; so that, if we put $z = \frac{1}{m}$, they are series whose length of convergence in the z plane vanishes.

Again, the general term of all the series was substantially

$$a_n = \binom{-s}{n} \frac{S_n'(a)}{(s+n-1)\,\omega^n}.$$

Now, since the expansion

$$-S_0'(a/\omega) + \frac{S_1'(a/\omega)}{1!} x + \ldots + (-)^{n-1} \frac{S_n'(a/\omega)}{n!} x^n + \ldots$$

is convergent within, and has two poles on, the circle of radius $\left|\frac{2\pi i}{\omega}\right|$, we have

$$\underset{n=\infty}{\text{Lt.}} \left| \left\{ \frac{S_n'(a/\omega)}{n!} \right\}^{\frac{1}{n}} \right| \overline{\lessdot} \frac{|\omega|}{2\pi}.$$

[Actually this limit vanishes with a when n is odd, and when n is even the inequality can be replaced by an equality if $|a|$ be finite.]

Hence
$$\underset{n=\infty}{\text{Lt.}}\ |\sqrt[n]{(a_n c_n)}|$$
$$=\underset{n=\infty}{\text{Lt.}}\left|\sqrt[n]{\left(\frac{\binom{-s}{n} S_n'(a)}{\omega^n (s+n^{-1})} c_n\right)}\right|$$
$$\overline{\overline{<}}\ \frac{n}{2\pi}\sqrt[n]{(c_n)}\,\mu,$$

where μ is some finite quantity.

Suppose now that the quantities c are the coefficients of some integral function $\chi(z)$ of "order"* $\frac{1}{\rho}$, where ρ is any number greater than unity. Then it may be proved† that c_n is of the same order of infinity when n is large, as $\frac{1}{n!^\rho}$.

Hence
$$\underset{n=\infty}{\text{Lt.}}\ |\sqrt[n]{(a_n c_n)}| = \frac{\mu'}{n^{\rho-1}},$$

and therefore the series
$$G(u) = a_0c_0 + a_1c_1u + \ldots + a_nc_nu^n + \ldots$$
is an integral function of u.

Such a function as that indicated is given by
$$\chi(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n^\rho}\right).$$

If then we take any one of such auxiliary functions, and they are infinite in number, we shall obtain the "sum" of the divergent series in question. This "sum" will, as Borel's researches show, be one of the functions which is approximately given by such an expansion.

The process which has just been sketched of obtaining a rigid conception of these series is in many ways an extended reversal of the process by which the series were originally obtained. It would of course be interesting to obtain an

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* According to M. Borel's definition, *Act. Math.*, Tome XX., pp. 357 et seq.

† An extension of a theorem due to Hadamard (Liouville (4), Tome IX., pp. 171 et seq) which I hope to publish in connection with certain investigations of integral functions.

expression for the remainder after a finite number of terms of such a series. The evaluation of this remainder can actually be carried out by a process similar to that which is employed subsequently in § 41. This may be called the arithmetic, as distinct from the function-theoretic, point of view, inasmuch as, given a large numerical value for m, we can by it see how many terms of the divergent series will give the nearest approximation to the value of the function. From the point of view of the theory of functions we are satisfied when we have proved that there are processes by which the dependence between the series and the function which it approximately represents is established.

Part IV.

The asymptotic expansion of $\Gamma_1(z)$ near infinity.

§ 39. From the asymptotic approximation to the factorial

$$a(a+\omega)\dots(a+n\omega)$$

given in § 30, it is possible to deduce an asymptotic approximation for $\log\Gamma_1(a+n\omega)$, where n is a large positive integer.

For we can at once see that the factorial is equal to

$$\log\Gamma_1\{a+(n+1)\omega\}-\log\Gamma_1(a),$$

and therefore we have the asymptotic equality

$$\log\Gamma_1\{a+(n+1)\omega\}=\tfrac{1}{2}\log\frac{2\pi}{\omega}+\left(n+\frac{a}{\omega}+\tfrac{1}{2}\right)\log n\omega-n$$
$$+\sum_{r=1}^{\infty}\frac{(-)^{r-1}}{(n\omega)^r}\{S_r(a+\omega/\omega)+{}_1B_{r+1}(\omega)\},$$

so that, writing a for $a+\omega$,

$$\log\Gamma_1(a+n\omega)=\left(n+\frac{a}{\omega}-\tfrac{1}{2}\right)\log n\omega-n+\tfrac{1}{2}\log\frac{2\pi}{\omega}$$
$$+\sum_{r=1}^{\infty}\frac{(-)^{r-1}}{(n\omega)^r}\{S_r(a/\omega)+{}_1B_{r+1}(\omega)\}.$$

Thus, when n is a large positive integer, and a any finite quantity, real or complex, we have the asymptotic approximation

$$\log\Gamma(n+a)=(n+a-\tfrac{1}{2})\log n-n+\tfrac{1}{2}\log 2\pi$$
$$+\sum_{r=1}^{\infty}\frac{(-)^{r-1}S_r(a)}{rn^r}+\sum_{r=1}^{\infty}\frac{(-)^{r-1}B_r}{2r.(2r-1)n^{2r-1}},$$

a formula recently proved under precisely similar limitations by Hermite and Sonin.*

Make $a=0$ and we get the asymptotic approximation

$$\log\Gamma(n) = (n-\tfrac{1}{2})\log n - n + \tfrac{1}{2}\log 2\pi$$
$$+\frac{B_1}{2n} - \frac{B_2}{3.4n^3} + \ldots + \frac{(-)^{r-1}B_r}{2r.2r-1\,n^{2r-1}} + \ldots.$$

Note also the compactness of the asymptotic expansion

$$\log\frac{\Gamma(n+a)}{\Gamma(n)} = a\log n + \sum_{r=1}^{\infty}\frac{(-)^{r-1}S_r(a)}{rn^r}.$$

As a verification of the results, this formula may be integrated with regard to a between 0 and 1, and we shall find the expansion for $\log\Gamma(n)$.

§ 40. We now enter on the investigation which attempts to obtain an asymptotic expansion for $\log\Gamma_1(z+a)$ when z is any complex quantity, such that $|z|$ is very large. And first let us remark that such an extension is not by any means obvious, nor perhaps *à priori* probable; for it was the fact that

$$\Gamma_1(z+\omega) = z\Gamma_1(z),$$

which at bottom enabled us to find the asymptotic expansion for $\log\Gamma_1(z+n\omega)$—n being a large positive integer. But there is no formula which expresses $\Gamma_1(z+\omega')$ in terms of $\Gamma_1(z)$ when the ratio of ω' to ω is not real, and an attempt to give asymptotic expansions when z/ω is not real must base itself on methods entirely different.

Previous to attacking the main problem, it proves necessary to establish the following Lemma of fundamental importance in the integrals which will be employed.

LEMMA. If $\zeta(s, a, \omega)$ be the extended Riemann ζ function as defined in § 35, the quantity

$$|\,\omega^s.\zeta(s, a, \omega)\,|$$

is finite for all values of $|s|$, such that $\mathfrak{R}(s)$ is greater than some finite negative quantity.

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* Hermite, *Crelle*, Bd. 115, pp. 201 et seq; Sonin, *Annales de l'Université de Varsovie*, 1889; Hermite and Sonin, *Crelle*, Bd. 116, pp. 133 et seq.

For all values of s, a, ω, $\zeta(s, a, \omega)$ has been defined by the asymptotic equality,

$$\omega^s \zeta(s, a, \omega) = \sum_{n=0}^{m} \frac{1}{\left(\frac{a}{\omega} + n\right)^s} - \frac{1}{(1-s)\, m^{s-1}} - \frac{\frac{a}{\omega} + \frac{1}{2}}{m^s}$$

$$- \frac{1}{(m)^s} \sum_{r=0}^{\infty} \frac{(-)^r}{(m\omega)^r} \binom{s+r-1}{r} \{S_r(a + \omega / \omega) + {}_1B_{r+1}(\omega)\},$$

where, on the right-hand side, m tends to grow without limit, and their principal values are assigned to the various terms. We may, by the remarks previously made on asymptotic series, ultimately neglect those terms in the second series for which $r + s$ is positive.

Thus, when $\mathfrak{R}(s) > 1$, we see that

$$|\omega^s \zeta(s, a, \omega)| \overline{\underset{<}{=}} \sum_{n=0}^{\infty} \frac{1}{\left|\left(\frac{a}{\omega} + n\right)^s\right|},$$

and therefore $|\omega^s \zeta(s, a, \omega)|$ is finite for all values of $|s|$ such that $\mathfrak{R}(s) > 1$.

To prove the theorem when $\mathfrak{R}(s)$ is a finite quantity < 1, we employ a modification of the process by which Mittag-Leffler's theorem in uniform functions is established.

Evidently

$$\frac{1}{\left(\frac{a}{\omega} + n\right)^s} = \frac{1}{n^s} \left\{1 - s \frac{a}{n\omega} + \frac{s \,.\, s+1}{2!} \left(\frac{a}{n\omega}\right)^2 \ldots \right.$$

$$\left. + (-)^r \frac{(s+r-2)!}{(r-1)} \left(\frac{a}{n\omega}\right)^{r-1} + R_r \right\};$$

and, provided $|a| < |n\omega|$, by a theorem due to Cauchy

$$|R_r| \leqq \left| \left(\frac{a}{n\omega}\right)^r \right| \frac{\mu}{(r-1)!},$$

where μ is a finite real quantity.

Thus, provided $|a| < |n\omega|$, the modulus of

$$\frac{1}{\left(\frac{a}{\omega} + n\right)^s \omega^s} - \frac{1}{(n\omega)^s} + s \frac{a}{(n\omega)^{s+1}} - \frac{s \,.\, s+1}{2!} \frac{a^2}{(n\omega)^{s+2}} - \ldots$$

$$- (-)^r \frac{(s+r-2)!}{(r-1)!} \frac{a^{r-1}}{(n\omega)^{s+r-1}}$$

is equal to

$$\left|\frac{R_r}{(n\omega)^s}\right| \leqq \left|\frac{a^r}{(n\omega)^{r+s}}\right| \frac{\mu}{(r-1)!}.$$

Now we have the asymptotic equality

$$\sum_{n=1}^{m} \frac{1}{n^s} - \frac{1}{(1-s)\,m^{s-1}} - \frac{1}{2m^s}$$

$$-\frac{1}{m^s}\sum_{r=1}^{\infty} \frac{(-)^r}{(m\omega)^r}\binom{s+r-1}{r}\{S_r(\omega/\omega) + {}_1B_{r+1}(\omega)\} = \omega^s\zeta(s, 0, \omega),$$

and a series of derived equalities obtained by differentiating the fundamental one successively with regard to a, and making $a = 0$.

Introduce these expressions into the fundamental equality, then ignoring a finite number of terms for which $|a| \overline{\gtrless} |n\omega|$, we see that if $\mathfrak{R}(r+s-1)$ be positive, $|\omega^s\zeta(s, a, \omega)|$ is less than or equal to the modulus of the expression

$$M + \sum_{n=k}^{m}\left[\frac{1}{\left(\frac{a}{\omega}+n\right)^s} - \frac{1}{n^s} + \ldots + (-)^r\frac{(s+r-2)!}{(r-1)!}\frac{\left(\frac{a}{\omega}\right)^{r-1}}{n^{s+r-1}}\right]$$

$$+\,\omega^s\zeta(s, 0, \omega) - a\omega^{s+1}\zeta'(s, 0, \omega) + \frac{a^2}{2!}\omega^{s+2}\zeta^{(2)}(s, 0, \omega)\ldots$$

$$+\frac{(-a)^{r-1}}{(r-1)!}\omega^{s+r-1}\zeta^{(r-1)}(s, 0, \omega),$$

plus an asymptotic series, each term of which involves $\frac{1}{m^{r+s-1}}$ or some higher inverse power. It is thus less than or equal to

$$|M| + \frac{\mu}{(r-1)!}\sum_{n=k}^{\infty}\left|\frac{a^r\omega^s}{(n\omega)^{r+s}}\right| + |\omega^s\zeta(s, 0, \omega)| + \ldots$$

$$+\left|\frac{a^{r-1}}{(r-1)!}\omega^{s+r-1}\zeta^{(r-1)}(s, 0, \omega)\right|,$$

where $|M|$ is a finite quantity, as also is μ.

Now $\omega^s\zeta(s, 0, \omega)$ is completely defined by the contour-integral

$$\frac{i\Gamma(1-s)}{2\pi}\int_1 \frac{(-z)^{s-1}}{1-e^{-z}}\,dz,$$

and hence, when $\mathfrak{R}(s) < 1$, $|\omega^s\zeta(s, 0, \omega)|$ is finite when s tends to infinity either way parallel to the imaginary axis.

Hence, when $0 < \mathfrak{R}(s) < 1$, we see, on putting $r=1$ in the inequality written above, that

$$|\omega^s \zeta(s, a, \omega)|$$

is finite.

Again, we evidently have

$$\omega^{s+1}\zeta'(s, 0, \omega) = \frac{i\Gamma(1-s)}{2\pi}\int_1 \frac{(-z)^s}{1-e^{-z}}\,dz,$$

so that $$|\omega^{s+1}\zeta'(s, 0, \omega)|$$

is finite if $\mathfrak{R}(s) < 0$. Hence, writing $r=2$ in our fundamental inequality, we see that when $-1 < \mathfrak{R}(s) < 0$,

$$|\omega^s \zeta(s, a, \omega)|$$

is finite.

And so on in general, since

$$\omega^{s+k}\zeta^{(k)}(s, 0, \omega) = \frac{i\Gamma(1-s)}{2\pi}\int_1 \frac{(-z)^{s+k-1}}{1-e^{-z}}\,dz,$$

we see that when $\mathfrak{R}(s) < -(k-1)$,

$$|\omega^{s+k}\zeta^{(k)}(s, 0, \omega)|$$

is finite when s tends to infinity either way. Put now $r=k+1$ in the fundamental inequality, and we see that, when $-k < \mathfrak{R}(s) < -(k-1)$, the quantity $|\omega^s\zeta(s, a, \omega)|$ is finite.

The Lemma is thus completely established.

§ 41. We may now prove that, for all large values of $|z|$ which are such that z is not in the vicinity of the negative part of the axis of ω,

$$\log\Gamma_1(z+a) = \left(\frac{z+a}{\omega} - \tfrac{1}{2}\right)\log z - \frac{z}{\omega} + \tfrac{1}{2}\log\frac{2\pi}{\omega}$$
$$+ \sum_{r=1}^{n} \frac{(-)^{r-1}}{rz^r}\{S_r(a/\omega) + {}_1B_{r+1}(\omega)\} + J_n(z, a, \omega),$$

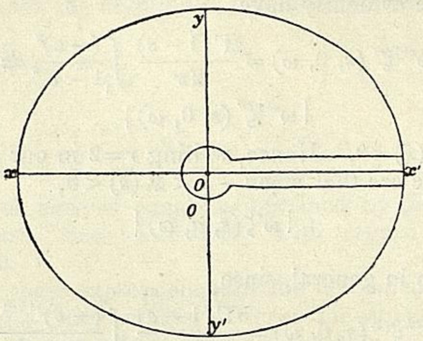
where $|J_n(z, a, \omega)|$ is a quantity of the same order of infinity as $\frac{1}{|z^{n+1}|}$, and $\log z$ has the axis of $-\omega$ as a cross-cut, and is real when z is real and positive.

When a vanishes we have the expansion

$$\log\Gamma_1(z) = \left(\frac{z}{\omega} - \tfrac{1}{2}\right)\log z - \frac{z}{\omega} + \tfrac{1}{2}\log\frac{2\pi}{\omega}$$
$$+ \sum_{r=1}^{n} \frac{(-)^{r-1}}{rz^r}\,{}_1B_{r+1}(\omega) + J_n(z, 0, \omega),$$

and this expansion is valid for all points in the domain of $z = \infty$, except those in the vicinity of the negative part of the axis of ω.

When $\omega = 1$ this region is indicated in the figure.

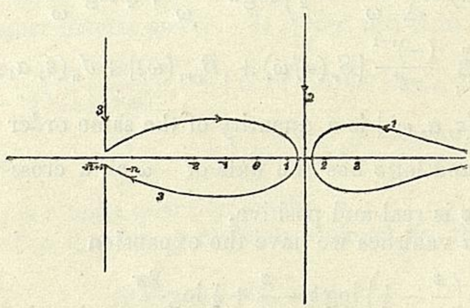


Consider the contour integrals

$$\frac{1}{2\pi i}\int \frac{\pi}{s \sin \pi s}\,\zeta(s, a, \omega)\, z^s ds,$$

where $z^s = e^{s \log z}$ and the logarithm is defined as in the enunciation. We suppose that s moves on the s-plane of the complex variable; though an easier geometrical conception of the alteration of the contours of the integrals to which we proceed is perhaps given by the Neumann sphere.

In the first place suppose that the integral is taken along a contour cutting the axis of real quantities in some point between $s = 1$ and $s = 2$, and that the contour otherwise is the same as that considered in § 21. The contour is numbered 1 in the figure.



Then the integral will be finite provided $|z|$ be sufficiently small, for $\mathfrak{R}(s)$ is > 1, and therefore

$$|\omega^s \zeta(s, a, \omega)|$$

is finite. But the integral is equal to the sum of the residues of the subject of integration at the points within the contour, *i.e.* at the points 2, 3, ..., ∞.

Now in § 27 we have seen that, when n is a positive integer greater than unity,

$$\zeta(n, a, \omega) = \frac{(-)^n}{(n-1)!} \frac{d^n}{da^n} \log \Gamma_1(a/\omega).$$

Hence the integral is equal to

$$\sum_{n=2}^{\infty} \frac{z^n}{n!} \frac{d^n}{da^n} \log \Gamma_1(a/\omega),$$

under the proviso that $|z|$ is sufficiently small.

But provided we take $|z|$ so small as to be within the circle of convergency for $\log \dfrac{\Gamma_1(z+a)}{\Gamma_1(a)}$, the series last written is equal to

$$\log \frac{\Gamma_1(z+a)}{\Gamma_1(a)} - z \frac{d}{da} \log \Gamma_1(a).$$

Suppose now that the contour is made to expand so that it still cuts the real axis in the same point, but ultimately becomes perpendicular to that axis, as the contour numbered 2 in the figure. [The contours are slightly separated to give greater distinctness to the figure.]

Since $\mathfrak{R}(s)$ is greater than unity the contour integral remains finite; and since the contour has passed over no poles of the subject of integration, this finite value will be equal to the former for sufficiently small values of $|z|$. But the new integral is finite for all finite values of $|z|$, for, since

$$1 < \mathfrak{R}(s) < 2,$$

we see that

$$\left| \frac{\zeta(s, a, \omega) z^s}{s \sin \pi s} \right| \leqq \sum_{n=0}^{\infty} \left| \frac{1}{\left(\frac{a}{\omega} + n\right)^s} \right| \frac{\left| \left(\frac{z}{\omega}\right)^s \right|}{|s| e^{\pi |s|}};$$

and, since $\left(\dfrac{z}{\omega}\right)^s = e^{s \log \frac{z}{\omega}}$, where the principal value of the logarithm is taken, this expression decreases when the absolute value of $\mathfrak{R}(is)$ increases, $\mathfrak{R}(s)$ remaining finite.

Hence the integral along the contour 2 must, for all values of $|z|$, be equal to the value which we saw that it had when $|z|$ was sufficiently small; that is, it must be equal to

$$\log \frac{\Gamma_1(z+a)}{\Gamma_1(a)} - z \frac{d}{da} \log \Gamma_1(a),$$

except when z is such that $\Gamma_1(z+a)$ is infinite, in which case the Taylor's series expansion would cease to hold good.

Suppose now that the contour of the integral is bent still further round until it becomes a contour enclosing the points $1, 0, -1, \ldots, -n$, and after the n^{th} point again going off to infinity perpendicularly to the real axis, as in the contour numbered 3 in the figure.

So far as the value of the integral is concerned, this contour will differ from that numbered 2, only by two strips at infinity of length less than $(n+2)$ parallel to the real axis, and since

$$|\omega^s \zeta(s, a, \omega)|$$

is finite if $\mathfrak{R}(s)$ is greater than some finite negative quantity the integral along these strips will vanish.

By Cauchy's Theorem the value of the integral along this third contour will be equal to minus the sum of its residues at the points $1, 0, -1, \ldots, -n$ together with the integral along the perpendicular line cutting the real axis between the points $-n$ and $-(n+1)$.

At the point $s=1$, we have, putting $s=1+\varepsilon$,

$$\frac{\pi\zeta(s, a, \omega)}{s \sin \pi s} z^s$$

$$= -\frac{z}{\omega\varepsilon^2}\left\{1 - \varepsilon\omega \frac{d}{da} \log \Gamma_1(a) + \ldots\right\}(1 - \varepsilon + \ldots)[1 + \varepsilon \log z + \ldots],$$

and therefore the residue is

$$-\frac{z}{\omega}\left\{\log z - \omega \frac{d}{da} \log \Gamma_1(a) - 1\right\}.$$

At the point $s=0$, we have, if $s=\varepsilon$, an expression for the subject of integration which may be written

$$\frac{1}{\varepsilon^2}\{1 + \varepsilon \log z + \ldots\}\left\{-\frac{a}{\omega} + \tfrac{1}{2} + \varepsilon\zeta'(0, a, \omega) + \ldots\right\}.$$

Hence, by § 30, the residue at the origin in the s plane is

$$\left(-\frac{a}{\omega} + \tfrac{1}{2}\right)\log z + \log \Gamma_1(a/\omega) - \tfrac{1}{2}\log\frac{2\pi}{\omega}.$$

The residue at the point $s=-r$, where r is a positive integer, is

$$\frac{(-)^r}{rz^r}\{S_r(a/\omega) + {}_1R_{r+1}(\omega)\}.$$

The original integral which was proved equal to

$$\log \frac{\Gamma_1(z+a)}{\Gamma_1(a)} - z\frac{d}{da} \log \Gamma_1(a)$$

is therefore equal to

$$\frac{z}{\omega}\left\{\log z - \omega \frac{d}{da}\log\Gamma_1(a) - 1\right\} - \log\Gamma_1(a) + \tfrac{1}{2}\log\frac{2\pi}{\omega}$$
$$+ \left(\frac{a}{\omega} - \tfrac{1}{2}\right)\log z + \sum_{r=1}^{n} \frac{(-)^{r-1}}{rz^r}\left[S_r(a/\omega) + {}_1B_{r+1}(\omega)\right],$$

together with the integral along the perpendicular contour which cuts the real axis between the points $-n$ and $-(n+1)$.

We thus have, denoting this integral by $J_n(z, a, \omega)$,

$$\log\Gamma_1(z+a) = \left(\frac{z+a}{\omega} - \tfrac{1}{2}\right)\log z - \frac{z}{\omega} + \tfrac{1}{2}\log\frac{2\pi}{\omega}$$
$$+ \sum_{r=1}^{n} \frac{(-)^{r-1}}{rz^r}\left[S_r(a/\omega) + {}_1B_{r+1}(\omega)\right] + J_n(z, a, \omega),$$

where $\log z$ is real, when z is real and positive, and has the axis of $-\omega$ for a cross-cut.

Now $$|J_n(z, a, \omega)| = \left|\frac{1}{2\pi i}\int \frac{\pi}{s \sin \pi s}\zeta(s, a, \omega)\, z^s ds\right|,$$

when the integral is taken along the perpendicular contour, for which

$$-(n+1) < \Re(s) < -n.$$

By the Lemma of § 40 the integral is evidently finite for such values of s as occur in its line of integration, and therefore when $|z|$ is large,

$$|J_n(z, a, \omega)|$$

is of an order of greatness less than or equal to that of $\mu|z^{n+1}|$, when μ is a finite quantity independent of z. Our theorem is thus completely proved.*

It is worth careful consideration to notice how the contour integrals along the lines numbered 1 and 3 in the figure give in the one case the convergent or summable divergent series (according as $|z|$ is $>$ or $<$ the radius of convergency of the Taylor series for $\log\dfrac{\Gamma_1(z+a)}{\Gamma_1(a)}$), which is the Taylor expansion for

$$\log\frac{\Gamma_1(z+a)}{\Gamma_1(a)} - z\frac{d}{da}\log\Gamma_1(a),$$

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* The theorem was first established in all generality by Stieltjes, *Liouville's Journal* (4), v., pp. 425–444, 1889.

and in the other case the asymptotic expansion proceeding by powers of $\frac{1}{z}$ which is of zero radius of convergency.

The mode of connection thus established admits of extension to the theory of multiple Gamma functions.

Part V.

The impossibility of $\Gamma(z)$ being given as the solution of a differential equation with algebraic coefficients.

§ 42. Hitherto we have considered $\Gamma(z)$ either as an infinite product or a definite integral or the solution of the difference equation

$$f(z+1) = zf(z).$$

Nowhere at any point of the investigation has there seemed any possibility of $\Gamma(z)$ satisfying a simple differential equation. It is of course evident that it will satisfy any number of equations whose coefficients are formed from $\Gamma(z)$ and its derivatives. But the question of interest is the possibility that $\Gamma(z)$ may be given as the solution of an equation whose coefficients are known simple functions of z, such as rational algebraic functions or simply or doubly periodic functions.

We proceed to show that $\Gamma(z)$ does not satisfy such an equation. The result is a negative one; but it is important as showing that $\Gamma(z)$ is a function differing entirely in character from functions which possess the properties common to solutions of differential equations. It indicates that there is little probability of there being many elementary properties of $\Gamma(z)$ hitherto uninvestigated, and marks the fact that it is the first and most elementary example of groups of functions having properties differing widely from most of the functions at present generally considered in elementary analysis.

Two proofs of the main theorem here considered have been given by Hölder* and Moore† respectively. They each differ from that which follows.

§ 43. We shall find it convenient to write

$$\psi(x) = \frac{d}{dx} \log \Gamma(x),$$

* Hölder, *Math. Ann.*, Bd. 28, pp. 1—13.
† Moore, *Math. Ann.*, Bd. 48, pp. 49 et seq.

so that $\psi(x)$ satisfies the difference equation

$$\psi(x+1)-\psi(x)=\frac{1}{x},$$

We will first show that $\psi(x)$ cannot satisfy a differential equation with algebraic coefficients

$$f(x, y, y', \ldots, y^{(n)})=0$$

of finite order n.

Suppose that

$$R(x)\, y^{m_1}(y')^{m_2}\ldots(y^{(n)})^{m_{n+1}}$$

is a term of the differential equation

$$f(x, y, y', \ldots, y^{(n)})=0,$$

written so that no terms appear in the denominator. Then this term is defined to be of class s, where

$$s=m_1+2m_2+\ldots+nm_{n+1}.$$

Suppose that the highest terms of the differential equation are of class s, and let them be written symbolically

$$R_0(x)\, Q_s^0(y),\ R_1(x)\, Q_s^1(y),\ \ldots,\ R_k(x)\, Q_s^k(y),$$

the terms of class $(s-1)$ being

$$S_0(x)\, Q_{s-1}^0(y),\ S_1(x)\, Q_{s-1}^1(y),\ \ldots,\ S_l(x)\, Q_{s-1}^l(y).$$

Supposing that $\psi(x)$ satisfies this differential equation, $\psi(x)+\frac{1}{x}$ will satisfy the differential equations in which $(x+1)$ is written for x.

Subtract the equation as previously written from the one so formed, having first reduced each by dividing by $R_0(x)$ and $R_0(x+1)$ respectively.

We get the equation

$$\frac{R_1(x+1)}{R_0(x+1)}\, Q_s'\left[\psi(x)+\frac{1}{x}\right]+\ldots+\frac{R_k(x+1)}{R_0(x+1)}\, Q_s^k\left[\psi(x)+\frac{1}{x}\right]$$

$$-\left\{\frac{R_1(x)}{R_0(x)}\, Q_s'\left[\psi(x)\right]+\ldots+\frac{R_k(x)}{R_0(x)}\, Q_s^k\left[\psi(x)\right]\right\}$$

$$+\, Q_s^0\left[\psi(x)+\frac{1}{x}\right]-Q_s^0\left[\psi(x)\right]$$

$+$ terms of lower class $=0$.

Now $$Q_s^0\left[\psi(x)+\frac{1}{x}\right]-Q_s^0[\psi(x)]$$

gives rise solely to terms of lower class than s.

We thus obtain for $\psi(x)$ the derived equation

$$\left[\frac{R_1(x+1)}{R_0(x+1)}-\frac{R_1(x)}{R_0(x)}\right]Q_s'[\psi(x)]+\ldots$$
$$+\left[\frac{R_k(x+1)}{R_0(x+1)}-\frac{R_k(x)}{R_0(x)}\right]Q_s^k[\psi(x)]$$

+ terms of lower class = 0.

Thus we have obtained for $\psi(x)$ an equation with only k instead of $(k+1)$ terms of class s, and it is not possible in general for this equation to vanish identically. For the terms $Q_s'(y), \ldots, Q_s^{(k)}(y)$ are by hypothesis all of different type; hence such a possibility would demand that we should have

$$\left.\begin{array}{c}\dfrac{R_1(x+1)}{R_0(x+1)}-\dfrac{R_1(x)}{R_0(x)}=0\\ \vdots\\ \dfrac{R_k'(x+1)}{R_0(x+1)}-\dfrac{R_k(x)}{R_0(x)}=0\end{array}\right\},$$

that is to say, the ratios

$$\frac{R_1(x)}{R_0(x)},\ \ldots,\ \frac{R_k(x)}{R_0(x)}$$

would have to be simply periodic functions, which, since the functions $R(x)$ are by hypothesis algebraic, is impossible unless these ratios are all constants. Thus, unless the terms of class s of the differential equation are the set

$$R_0(x)\left[\alpha_0 Q_s^0(y)+\alpha_1 Q_s'(y)+\ldots+\alpha_k Q_s^k(y)\right],$$

where the α's are mere constants, it is possible to reduce the number of these terms.

We thus have for $\psi(x)$ the differential equation

$$R_0(x)\left[\alpha_0 Q_s^0(y)+\alpha_1 Q_s'(y)+\ldots+\alpha_k Q_s^{(k)}(y)\right]$$
$$+S_0(x)\,Q_{s-1}^0(y)+S_1(x)\,Q_{s-1}'(y)+\ldots+S_l(x)\,Q_{s-1}^l(y)$$

+ terms of lower class = 0 ....(1).

Divide this equation by $R_0(x)$, form a new equation by changing x into $(x+1)$, and subtract the one just written down from this. We obtain for $\psi(x)$ the equation

$$\alpha_0\left[Q_s^0\left(y+\frac{1}{x}\right)-Q_s^0(y)\right]+\ldots+\alpha_k\left[Q_s^{(k)}\left(y+\frac{1}{x}\right)-Q_s^{(k)}(y)\right]$$
$$+\frac{S_0(x+1)}{R_0(x+1)}Q^0_{s-1}\left(y+\frac{1}{x}\right)-\frac{S_0(x)}{R_0(x)}Q^0_{s-1}(y)+\ldots$$
$$\ldots+\frac{S_l(x+1)}{R_0(x+1)}Q^l_{s-1}\left(y+\frac{1}{x}\right)-\frac{S_l(x)}{R_0(x)}Q^l_{s-1}(y)$$

+ terms of lower class $=0$.

Now $Q_s^0\left(y+\dfrac{1}{x}\right)-Q_s^0(y)=Q^{1,0}_{s-1}(y)P_0\left(\dfrac{1}{x}\right)$

+ terms of lower class, where $P_0\left(\dfrac{1}{x}\right)$ is a rational integral algebraic function of $\dfrac{1}{x}$, and $Q^{1,0}_{s-1}(y)$ denotes a term or terms of class $(s-1)$.

We have thus obtained for $\psi(x)$ an equation of class $(s-1)$ which cannot vanish identically unless the functions

$$\frac{S_0(x)}{R_0(x)},\ \ldots,\ \frac{S_l(x)}{R_l(x)}$$

satisfy equations of the type

$$\frac{S_m(x+1)}{R_0(x+1)}-\frac{S_m(x)}{R_0(x)}=P_m\left(\frac{1}{x}\right).$$

—equations which admit no rational algebraic function (§ 11, *Cor.*) unless all the functions $P(x)$ vanish, in which case

the ratios $$\frac{S_0(x)}{R_0(x)},\ \ldots\ldots,\ \frac{S_l(x)}{R_l(x)}$$

must be all constants.

We have thus obtained for $\psi(x)$ an equation of class $(s-1)$ unless the reduced equation (1) can be written

$$R_0(x)\left[\alpha_0 Q_s^0(y)+\ldots+\alpha_k Q_s^{(k)}(y)+\beta_0 Q^0_{s-1}(y)+\ldots+\beta_l Q^l_{s-1}(y)\right]$$
$$+T_0(x)\,Q^0_{s-2}(y)+\ldots+T_m(x)\,Q^{(m)}_{s-2}(y)$$

+ terms of lower class $=0$, where the quantities α and β are constants.

In this equation, after division by $R_0(x)$, change x into $(x+1)$, and subtract from the equation so formed the one just written.

We shall obtain an equation of class $(s-2)$ unless the quantities

$$\frac{T_0(x)}{R_0(x)}, \ldots, \frac{T_m(x)}{R_0(x)}$$

satisfy difference equations of the form

$$f(x+1) - f(x) = Q\left(\frac{1}{x}\right),$$

where $Q\left(\frac{1}{x}\right)$ is a rational integral algebraic function of $\frac{1}{x}$.

Repeating the argument we see that either the equation is of the form

$$R_0(x)\left[f_1(y', y^{(2)}, \ldots, y^{(n)})\right] + k_0(x) f_2(y) = 0,$$

where the functions f_1 and f_2 are algebraic functions, and f_1 such that when $y + \frac{1}{x}$ is written for y, it reproduces itself, save for an additional integral algebraic function of y and $\frac{1}{x}$; *or* the equation is capable of reduction to one of class $(s-1)$.

In the first case, changing x into $(x+1)$ after division by $R_0(x)$ and subtracting, we get an algebraic equation for $\psi(x)$, which is a manifest impossibility.

If, however, the equation can be reduced from class s to class $s-1$, it can be further reduced by employing the same process until we finally obtain a differential equation of class 1.

But such an equation is

$$f_1(x)\,y + f_0(x) = 0,$$

which is absurd, since $f_0(x)$ and $f_1(x)$ must be algebraic functions of x.

We thus see finally that $\psi(x)$ cannot satisfy a differential equation with algebraic coefficients.

§ 44. Suppose now that we had made the assumption that $\psi(x)$ satisfied a differential equation with simply periodic coefficients, or coefficients which were rational algebraic functions of x and of simply periodic functions of x.

We could repeat the argument of § 43 with few alterations

We should obtain instead of the equation (1) the equation

$$\begin{aligned} R_0\{x, \phi(x)\}\left[f_0(x)\, Q_s^0(y) + f_1(x)\, Q_s'(y) + \ldots + f_k(k)\, Q_s^k(y)\right] \\ + S_0\{x, \phi(x)\}\, Q''_{s-1}(y) + \ldots + S_l(x, \phi(x)\, Q^l_{s-1}(y) \\ + \text{terms of lower class} = 0, \end{aligned}$$

where $R\{x, \phi(x)\}$ denotes a rational integral algebraic function of x and of simply periodic functions of x, and the quantities $f(x)$ are simply periodic functions of x of period unity.

And we should be able to reduce this to an equation of class $s-1$ unless the functions

$$\frac{S_0\{x, \phi(x)\}}{R_0\{x, \phi(x)\}}, \ldots, \frac{S_l\{x, \phi(x)\}}{R_0\{x, \phi(x)\}}$$

satisfied equations of the type

$$f(x+1) - f(x) = P\left(\frac{1}{x}\right).$$

But the only solution of such an equation is evidently an algebraic combination of a finite number of differentials of $\psi(x)$ multiplied by an arbitrary simply periodic function of x of period unity.

So that we are again faced by an impossibility unless the quantities P all vanish.

The argument now proceeds exactly as before, and we finally see that $\psi(x)$ cannot satisfy a differential equation whose coefficients are finite combinations of

(*a*) rational algebraic functions of x,

(*b*) simply (and therefore *à fortiori* doubly) periodic functions of x.

§ 45. We can now deduce that $\Gamma(x)$ cannot satisfy any equation of the type just mentioned.

Let $$y = \psi(x), \quad z = \Gamma(x),$$

so that $$y = \frac{z'}{z}.$$

And suppose that z satisfies the differential equation

$$f[z, z', \ldots, z^{(n)}] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Then, since $$z' = zy,$$

we have $$z'' = zy' + z'y = zy' + zy^2,$$

and so on.

And hence, from (1), we find

$$\phi[z, y, y', \ldots, y^{(n-1)}] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Differentiate this equation with respect to x and substitute zy for z', and we obtain

$$\psi(z, y, y', \ldots, y^{(n-1)}, y^{(n)}) = 0 \quad \ldots\ldots\ldots\ldots(3).$$

The z-eliminant of equations (2) and (3) evidently cannot be an identity and is therefore an equation for y.

Thus if $\Gamma(x)$ satisfies an equation of the given type, $\psi(x)$ satisfies one of that type. This has been shown to be impossible, and therefore $\Gamma(x)$ cannot satisfy an equation of the type mentioned.

The same theorem is evidently true for the more general function $\Gamma_1(x/\omega)$.

In a subsequent paper I hope to show that a similar theorem can be proved for extended classes of functions.

NOTE. While the foregoing paper has been passing through the press, I have received from Herr Mellin a paper* in which he has shortly preceded me by using an extended ζ function to establish results in the theory of the Gamma function. Whereas I start from the point of view of a contour integral, he uses a series, substantially equivalent to the asymptotic identity (1) of § 23, as his fundamental definition. His theory proceeds on different lines to my own, and is unfortunately expressed in a different notation. Herr Mellin in his paper obtains a series of results for a class of functions which are degenerate cases (when the parameters are all equal to one another) of what I propose to call multiple Gamma functions, whose theory I have already developed and hope to publish in a series of subsequent papers. This theory, to which the present paper serves as an introduction, formed the substance of an essay which obtained the first Smith's Prize in the University of Cambridge in the year 1898. It has been re-written and amplified in consequence of some kind criticism of Professor Forsyth, to whom I would express my warmest thanks.

ERRATA AND ADDENDA.

p. 67, l. 2, *for* $\frac{n\omega}{z}$ *read* $\frac{z}{n\omega}$.

p. 72, l. 10, *for* $\frac{\phi(a/\omega)}{a}$ *read* $\frac{d\phi(a/\omega)}{da}$.

p. 73, l. 11, *for* $\log \pi\omega$ *read* $\log \frac{\pi}{\omega}$.

p. 73, l. 14 *for* $\log 2\pi\omega$ *read* $\log \frac{2\pi}{\omega}$.

p. 83, l. 19, *for* functions *read* function.

p. 88, l. 9, *for* values *read* value

p. 88, ll. 18, 19, *for* part of $\frac{a}{\omega}$ is *read* parts of $\frac{a}{\omega}$ and ω are.

p. 92, l. 17, *add*: The specification to be given to the many-valued function will be determined later (§ 35).

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* Mellin, "Ueber eine Verallgemeinerung der Riemannschen Function $\zeta(s)$," *Acta Societatis Fennicae*, T. 24, No. 10, 1899.

FUNDAMENTAL THEOREMS RELATING TO THE BERNOULLIAN NUMBERS.

(Second paper.)

By *J. W. L. Glaisher.*

§ 1. In my paper, with the above title, in the August number (pp. 49–63 of the present volume) there is a slight slip on pp. 62, 63, where it is stated, in reference to the theorem

$$\frac{B_n}{n} \equiv (-1)^{tj} \frac{B_{n-tj}}{n-tj}, \text{ mod. } p,$$

that 'there is no restriction with respect to the values of p.' The statement should have been that 'there is no restriction with respect to the values of p, except that p must not be a Staudt factor of B_n.' This is evident from the mode of proof, for if p is a Staudt factor of B_n, then $a^{2n}-1$ and $a^{2n-t(p-1)}-1$ are divisible by p, and we cannot divide out by these factors.* For all other values of p the reasoning in §§ 20, 21 of the paper holds good.

§ 2. I take this opportunity of giving the formulæ which are derivable from the general theorem in the case of the first few primes, and also of deducing Staudt's theorem. It is convenient first to re-state the general theorem in a form which is more suitable for these applications.

The general theorem.

§ 3. Let p be any uneven prime and let $j = \frac{1}{2}(p-1)$. Every number n is of one of the forms

$$tj,\ tj+1,\ tj+2,\ \ldots,\ tj+j-1;$$

---

* In the last two lines of p. 62 after 'neither side of the congruence is divisible by p' the words 'unless $p-1$ is a divisor of $2n$' should be added. The 'dividing out' in this case is considered in § 17 of the present paper.

There is also an error in the statement of Sylvester's theorem on p. 49, viz., the theorem is that "if p^i is a divisor of $2n$ [not n as on p. 49], and $p-1$ is not, then the numerator of B_n will be divisible by p^i." The error of n for $2n$ occurs both in the text and in Sylvester's own statement quoted in the note. The theorem is given correctly on p. 50 (§ 3), for p is a Staudt factor of B_n when $p-1$ is a divisor of $2n$ (p. 49).

and if $n = tj + r$, where r has any one of the values 1, 2, 3, .. $j-1$ (excluding zero), then the theorem asserts that

$$\frac{B_n}{n} \equiv (-1)^{tj} \frac{B_r}{r}, \text{ mod. } p,$$

i.e.

$$\frac{B_n}{n} \equiv (-1)^{n-r} \frac{B_r}{r}, \text{ mod. } p.$$

If we denote the numerator of B_n by B_n' and its denominator (that is, the product of the Staudt factors of B_n) by F_n, so that $B_n = \frac{B_n'}{F_n}$, we may enunciate the theorem as follows: if $n \equiv r$, mod. j, r not being zero, then

$$\text{(i)} \quad B_n' \equiv (-1)^{n-1} \frac{B_r}{r} nF_n, \text{ mod. } p.$$

§ 4. The case $r = 0$ corresponds to Staudt's theorem, which may be enunciated in this form*:—if $n \equiv 0$, mod. j, then p is a Staudt factor of B_n, and

$$\text{(ii)} \quad B_n' \equiv (-1)^n F_n', \text{ mod. } p,$$

where F_n' denotes the product of all the Staudt factors of B_n except p, *i.e.* $F_n' = \frac{F_n}{p}$.

The case $p = 3$.

§ 5. When $p = 3$, $j = 1$, and therefore, as is well known, 3 is a Staudt factor of all the Bernoullian numbers. The theorem (i) does not apply, and from (ii) we have

$$B_n' \equiv (-1)^{n-1} \times \left.\begin{array}{l}\text{product of all the Staudt factors} \\ \text{of } B_n \text{ except 2 and 3,}\end{array}\right\}, \text{ mod. } 3.$$

The case $p = 5$.

§ 6. This case is particularly interesting as it gives a rule for assigning the last figure of the numerator of any Bernoullian number.

---

* The theorem is expressed in this form in the *Quarterly Journal*, Vol. XXX., p. 367.

When $p=5, j=2$, so that the theorem (i) applies to uneven values of n and the theorem (ii) to even values.

Taking first the case of n uneven, the theorem (i) shows that, n being uneven,

$$B_n' \equiv nF_n, \text{ mod. } 5.$$

Now the numerator of a Bernoullian number is always uneven and the denominator is always even. It follows therefore from this congruence that, if we multiply the denominator of B_n by n and add 5 to the last figure, we obtain the last figure of the numerator.

As examples, (1) let $n=27$; the last figure of the denominator of B_{27} is 8 and the last figure of the product of 27 by 8 (*i.e.* of 7×8) is 6; and adding 5 we obtain 1 as the last figure of the numerator of B_{27}.

(2) Let $n=57$; the last figure of the denominator of B_{57} is 2, and 7×2 gives 4, so that the numerator of B_{57} ends in 9.

(3) Taking an example beyond the range of Adams's table (which extends to B_{62}), let $n=81$. The Staudt factors of B_{81} are 2, 3, 7, 19, 163, and the last figure of their product multiplied by 81 is 4, so that the numerator of B_{81} ends in 9.

§ 7. The rule shows that when n ends in 5 the numerator of B_n must also end in 5; for the denominator being even, its product by 5 ends in 0, and therefore the numerator ends in 5.

It also affords a still simpler rule for the last figure of the numerator when n is prime. For when n is prime the Staudt factors of B_n are 2 and 3, if $2n+1$ is not prime, and are 2, 3, and $2n+1$, if $2n+1$ is prime.

By separating these cases and supposing successively that n ends in 1, 3, 7, 9, we find

(i) if n is prime, and $2n+1$ is not prime, then the numerator of B_n has the same last figure as n.

(ii) if n is prime and $2n+1$ is also prime, then the numerator of B_n ends in 3 if n ends in 1, and ends in 1 if n ends in 3 or 9.

If n ends in 7, $2n+1$ cannot be prime, so that whenever n is a prime ending in 7, the numerator of B_n always ends in 7.

§ 8. When n is even 5 is always a Staudt factor of B_n and the last figure of the numerator can be assigned by means of Staudt's theorem (ii) of § 4. For this theorem shows that the numerator is congruent, mod. 5, to the denominator with the factor 5 omitted. Now the denominator, when n is even, always ends in 0, and the last figure of the quotient when this denominator is divided by 5 may be obtained by doubling the figure immediately preceding the final 0. The rule for the last figure of the numerator may therefore be stated:—double the figure immediately preceding the final 0 in the denominator and add 5.

As examples, (1) let $n=8$; the last two figures of the denominator of B_8 are 10; therefore the last figure of the numerator is given by $2\times 1+5$ and is 7.

(2) Let $n=52$; the last two figures of the denominator of B_{52} are 90; therefore the last figure of the numerator is given by $2\times 9+5$ and is 3.

(3) Taking an example beyond the range of Adams's table, let $n=78$. The Staudt factors of B_{78} are 2, 3, 5, 7, 13, 53, 79, 157. Leaving out 5, the product of the other Staudt factors ends in 4, and therefore the last figure of the numerator is given by $4+5$ and is 9.

§ 9. The rules for the last figure of B_n may therefore be briefly stated as follows:

(i) n uneven; the last figure of the product of the Staudt factors multiplied by the last figure of n with 5 added.

(ii) n even; the last figure of the product of the Staudt factors, except 5, with 5 added.

The case $p=7$.

§ 10. From the general theorem (§ 3) we have

$$\frac{B_{3t+1}}{3t+1}\equiv(-1)^{3t}\frac{B_1}{1}\equiv(-1)^{3t}\tfrac{1}{6}\equiv(-1)^{3t-1},\ \text{mod. } 7,$$

$$\frac{B_{3t+2}}{3t+2}\equiv(-1)^{3t}\frac{B_2}{2}\equiv(-1)^{3t}\tfrac{1}{60}\equiv(-1)^{3t}\,2,\ \text{mod. } 7,$$

and from Staudt's theorem (§ 4)

$$7B_{3t}\equiv(-1)^{3t},\ \text{mod. } 7;$$

therefore, putting as before $B_n=\dfrac{B_n'}{F_n}$, where B_n' is the numerator of B_n, and F_n is the denominator, or product of the Staudt factors, we have

$$\begin{array}{llll} \text{if} & n \equiv 1, \text{ mod. } 3, & B_n' \equiv (-1)^n nF_n, & \text{mod. } 7, \\ ,, & n \equiv 2, \quad ,, \;, & B_n' \equiv (-1)^n 2nF_n, & ,, \\ ,, & n \equiv 0, \quad ,, \;, & B_n' \equiv (-1)^n F_n', & ,, \end{array}$$

where F_n' is the product of all the Staudt factors of B_n except 7.

Taking as examples $n = 4$, 5, 6, these congruences give

$$\begin{array}{lll} n = 4, & 1 \equiv 4 \times 30, & \text{mod. } 7, \\ n = 5, & 5 \equiv -10 \times 66, & ,, \\ n = 6, & 691 \equiv 390, & ,, \;. \end{array}$$

The case $p = 11$.

§ 11. When $p = 11, j = 5$, and the general theorem gives

$$\frac{B_{5t+1}}{5t+1} \equiv (-1)^{5t} \tfrac{1}{6} \equiv (-1)^{5t} 2, \text{ mod. } 11,$$

$$\frac{B_{5t+2}}{5t+2} \equiv (-1)^{5t} \tfrac{1}{60} \equiv (-1)^{5t} 9, \quad ,,$$

$$\frac{B_{5t+3}}{5t+3} \equiv (-1)^{5t} \tfrac{1}{126} \equiv (-1)^{5t} 9, \quad ,,$$

$$\frac{B_{5t+4}}{5t+4} \equiv (-1)^{5t} \tfrac{1}{120} \equiv (-1)^{5t} 10, \quad ,,$$

and therefore

$$\begin{array}{llll} \text{if} & n \equiv 1, \text{ mod. } 5, & B_n' \equiv (-1)^{n-1} 2nF_n, & \text{mod. } 11, \\ ,, & n \equiv 2, \quad ,, \;, & B_n' \equiv (-1)^n \; 9nF_n, & ,, \\ ,, & n \equiv 3, \quad ,, \;, & B_n' \equiv (-1)^{n-1} 9nF_n, & ,, \\ ,, & n \equiv 4, \quad ,, \;, & B_n' \equiv (-1)^n \; 10nF_n, & ,, \\ ,, & n \equiv 0, \quad ,, \;, & B_n' \equiv (-1)^n F_n', & ,, \end{array}$$

where in the last congruence (which is derived from Staudt's theorem) F_n' is the product of all the Staudt factors of B_n except 11.

As examples we have

$$
\begin{array}{lrll}
n=6, & 691 \equiv -12 \times 2730, & \text{mod. } 11, \\
n=7, & 7 \equiv -63 \times 6, & \quad ,, \\
n=8, & 3617 \equiv -72 \times 510, & \quad ,, \\
n=9, & 43867 \equiv -90 \times 798, & \quad ,, \\
n=10, & 174611 \equiv 30, & \quad ,, & .
\end{array}
$$

It will be noticed that the second and third congruences show that, if n is any number $\equiv 2$, mod. 5, then

$$\frac{B_n}{n} \equiv \frac{B_{n+1}}{n+1}, \text{ mod. } 11.$$

The case $p=13$.

§ 12. In this case the formulæ are:

$$
\begin{array}{llll}
\text{if} & n \equiv 1, \text{ mod. } 6, & B_n' \equiv 11nF_n, & \text{mod. } 13, \\
,, & n \equiv 2, \quad ,, & B_n' \equiv 5nF_n, & \quad ,, \\
,, & n \equiv 3, \quad ,, & B_n' \equiv 3nF_n, & \quad ,, \\
,, & n \equiv 4, \quad ,, & B_n' \equiv 9nF_n, & \quad ,, \\
,, & n \equiv 5, \quad ,, & B_n' \equiv nF_n, & \quad ,, \\
,, & n \equiv 0, \quad ,, & B_n' \equiv F_n', & \quad ,,
\end{array}
$$

where F_n' is the product of all the Staudt factors of B_n except 13.

As examples we have

$$
\begin{array}{lrll}
n=7, & 7 \equiv 77 \times 6, & \text{mod. } 13, \\
n=8, & 3617 \equiv 40 \times 510, & \quad ,, \\
n=9, & 43867 \equiv 27 \times 798, & \quad ,, \\
n=10, & 174611 \equiv 90 \times 330, & \quad ,, \\
n=11, & 854513 \equiv 11 \times 138, & \quad ,, \\
n=12, & 236364091 \equiv 210, & \quad ,, & .
\end{array}
$$

The case $p=17$.

§ 13. The formulæ are:

$$\begin{array}{llll}
\text{if} & n\equiv 1, \text{ mod. } 8, & B_n'\equiv 3nF_n, & \text{mod. } 17,\\
,, & n\equiv 2, \quad ,, & B_n'\equiv 2nF_n, & ,,\\
,, & n\equiv 3, \quad ,, & B_n'\equiv 5nF_n, & ,,\\
,, & n\equiv 4, \quad ,, & B_n'\equiv nF_n, & ,,\\
,, & n\equiv 5, \quad ,, & B_n'\equiv 8nF_n, & ,,\\
,, & n\equiv 6, \quad ,, & B_n'\equiv 5nF_n, & ,,\\
,, & n\equiv 7, \quad ,, & B_n'\equiv 3nF_n, & ,,\\
,, & n\equiv 0, \quad ,, & B_n'\equiv F_n', & ,,
\end{array}$$

where F_n' is the product of all the Staudt points of B_n except 17.

As examples we have

$$\begin{array}{lrl}
n=9, & 43867\equiv 27\times 798, & \text{mod. } 17,\\
n=10, & 174611\equiv 20\times 330, & ,,\\
n=11, & 854513\equiv 55\times 138, & ,,\\
n=12, & 236364091\equiv 12\times 2730, & ,,\\
n=13, & 8553103\equiv 104\times 6, & ,,\\
n=14, & 23749461029\equiv 70\times 870, & ,,\\
n=15, & 8615841276005\equiv 45\times 14322, & ,,\\
n=16, & 7709321041217\equiv 30, & ,, .
\end{array}$$

It is unnecessary to consider higher values of p, as the results given by the theorem will evidently be of the same kind as those already obtained.

General remarks.

§ 14. Staudt's theorem not only gives the denominator of a Bernoullian number, but also assigns the residue of the numerator with respect to any of the primes which occur in the denominator: and the general formula (i) of § 3 completes this result by assigning the residue of the numerator with respect to all the primes which do not occur in the denominator.

We can, therefore, always assign the residue of the numerator of B_n with respect to any prime p, subject only to the two following conditions:

(i) We must be able to resolve n into its factors (in order to determine the Staudt factors of B_n).

(ii) When n is divided by $\frac{p-1}{2}$ the remainder must not not exceed 62 (the extent of Adams's table).

Thus we can certainly assign the residue of the numerator of B_n with respect to p if n does not exceed 9,000,000 and if p does not exceed 127. In other cases the actual determination of the residue depends upon the possibility of resolving n into its prime factors, and upon the remainder, when n is divided by $\frac{p-1}{2}$, being not greater than 62.

§ 15. It will be noticed that in the case of $p = 17$ (§ 13), if $n \equiv 1$, mod. 8,

$$\frac{B_n}{n} \equiv \frac{B_{n+6}}{n+6}, \text{ mod. } 17.$$

A similar result holds good for all primes above 13, for in general we have, j being $\frac{p-1}{2}$ as before,

$$\begin{aligned}
\frac{B_{tj+1}}{tj+1} &\equiv (-1)^{tj}\tfrac{1}{6}, && \text{mod. } p, \\
\frac{B_{tj+2}}{tj+2} &\equiv (-1)^{tj}\tfrac{1}{60}, && \text{,,} \\
\frac{B_{tj+3}}{tj+3} &\equiv (-1)^{tj}\tfrac{1}{126}, && \text{,,} \\
\frac{B_{tj+4}}{tj+4} &\equiv (-1)^{tj}\tfrac{1}{120}, && \text{,,} \\
\frac{B_{tj+5}}{tj+5} &\equiv (-1)^{tj}\tfrac{1}{66}, && \text{,,} \\
\frac{B_{tj+6}}{tj+6} &\equiv (-1)^{tj}\tfrac{691}{16380}, && \text{,,} \\
\frac{B_{tj+7}}{tj+7} &\equiv (-1)^{tj}\tfrac{1}{6}, && \text{,,} \\
&\text{\&c.} && \text{\&c.}
\end{aligned}$$

Thus in general if $j>7$, *i.e.* if $p>13$, we have if $n\equiv 1$, mod. j, and $t\equiv 7$, mod. j,

$$\frac{B_n}{n}\equiv\frac{B_t}{t}, \text{ mod. } p.$$

There are obviously other relations of the same kind which are less simple, for example j being >4 (*i.e.* $p>7$), we have if $n\equiv 2$, mod. j and $t\equiv 4$, mod. j,

$$\frac{B_n}{n}\equiv 2\frac{B_t}{t}, \text{ mod. } p.$$

Relation of Staudt's theorem to the general theorem.

§ 16. I now proceed to deduce Staudt's theorem from the formula

$$\psi_n(a)\equiv(-1)^{tj}\psi_{n-tj}(a), \text{ mod. } p$$

given in § 19 of the previous paper (p. 62).

It is convenient to prove first that, p being prime and $j=\frac{p-1}{2}$ as before, B_j has p as a factor of its denominator, and that $pB_j\equiv(-1)^j$, mod. p.

These results may be derived from the recurring formula

$$(2n+1)_2B_1-(2n+1)_4B_2+\ldots+(-1)^{n-1}(2n+1)_{2n}B_n=n-\tfrac{1}{2},$$

in which $(n)_r$ denotes the number of combinations of n things taken r together.

By putting $n=1, 2, \ldots, j-1$ (so that $2n+1=3, 5, \ldots, p-2$), we may calculate $B_1, B_2, \ldots, B_{j-1}$ from this recurring formula; and, since p cannot be a factor of $2n+1$, the coefficient of B_n (the highest B), we see that p cannot occur as a factor in the denominator of any of these B's.

Putting $n=j$; that is $2n+1=p$, we have

$$(p)_2B_1-(p)_4B_2+\ldots+(-1)^{j-2}(p)_{p-2}B_{j-1}+(-1)^{j-1}(p)_{p-1}B_j=\tfrac{1}{2}p-1.$$

This formula shows that B_j must have p as a factor in the denominator and that $pB_j\equiv(-1)^j$, mod. p. For on the left-hand side the coefficients of all the terms except the last are $\equiv 0$, mod. p, and none of these B's has p in the denominator, so that the formula gives the congruence

$$(-1)^{j-1}pB_j\equiv -1, \text{ mod. } p,$$

which shows that the denominator of B_j must contain p as a factor and that

$$pB_j \equiv (-1)^j, \text{ mod. } p.$$

If we use $[B_j]$ to denote the value of B_j when the factor p has been removed from the denominator, *i.e.* so that $[B_j] = pB_j$, this formula becomes

$$[B_j] \equiv (-1)^j, \text{ mod. } p.$$

§ 17. The theorem

$$\psi_n(a) \equiv (-1)^{tj} \psi_{n-tj}(a), \text{ mod. } p,$$

written at length is

$$\frac{(a^{2n}-1)\,B_n}{n} \equiv (-1)^{tj} \frac{(a^{2n-2tj}-1)\,B_{n-tj}}{n-tj}, \text{ mod. } p.$$

Each side of this congruence is an integer except for powers of a and 2. Now let $n = (t+1)j$; the congruence then becomes

$$\frac{(a^{2(t+1)j}-1)\,B_{(t+1)j}}{(t+1)j} \equiv (-1)^{tj} \frac{(a^{2j}-1)\,B_j}{j}, \text{ mod. } p,$$

that is, putting $t-1$ for t and dividing out by j,

$$\frac{(a^{2tj}-1)\,B_{tj}}{t} \equiv (-1)^{(t-1)j} (a^{2j}-1)\,B_j, \text{ mod. } p.$$

Now, a being any number prime to p, $a^{2j}-1$ contains p as a factor, and we have

$$(a^{2j}-1)\,B_j = \frac{a^{p-1}-1}{p} \times [B_j],$$

in which $\dfrac{a^{p-1}-1}{p}$ is an integer, and $[B_j]$ is a fraction not containing p as a factor either in the numerator or the denominator. Now it is possible for $\dfrac{a^{p-1}-1}{p}$ to be divisible by p, but we can always choose a so that this is not the case (*e.g.* by taking $a = p \pm 1$), and we shall suppose that a is so chosen.

We have then $a^{p-1}=1+\lambda p$, where λ is not divisible by p, whence

$$a^{2tj}=(1+\lambda p)^t=1+t\lambda p+(t)_2\lambda^2p^2+\ldots+(t)_{t-1}\lambda^{t-1}p^{t-1}+\lambda^t p^t,$$

and therefore

$$\frac{a^{2tj}-1}{tp}\div\frac{a^{p-1}-1}{p}=1+\frac{t-1}{2!}\lambda p+\ldots+\lambda^{t-2}p^{t-2}+\frac{\lambda^{t-1}p^{t-1}}{t}.$$

The quantity on the right-hand side is congruent to 1, mod. p, whether t be prime to p or contain p^i as a factor. Dividing, therefore, both sides of the congruence by $\dfrac{a^{p-1}-1}{p}$ we find

$$pB_{tj}\equiv(-1)^{(t-1)j}[B_j], \text{ mod. } p.$$

This congruence shows that B_{tj} has p as a factor of its denominator, and if we denote by $[B_{tj}]$ the fraction obtained by omitting p from the denominator of B_{tj}, *i.e.* so that $[B_{tj}]=pB_{tj}$, we have

$$[B_{tj}]\equiv(-1)^{(t-1)j}[B_j], \text{ mod. } p.$$

In § 16 it was shown that

$$[B_j]\equiv(-1)^j, \text{ mod. } p,$$

and therefore $\quad [B_{tj}]\equiv(-1)^{tj}, \text{ mod. } p,$

which is Staudt's theorem (§ 4).

Thus the congruence

$$\psi_n(a)\equiv(-1)^{tj}\psi_{n-tj}(a), \text{ mod. } p$$

gives rise to the general theorem (i) of § 3 when n is not a multiple of j; and, when n is a multiple of j, it gives rise to Staudt's theorem, the two theorems conjointly serving to assign the residue of the numerator of any Bernoullian number with respect to any prime modulus.

The case when p is a divisor of n.

§ 18. When p is prime the numerator of B_p is divisible by p, and it was shown in § 18 of the previous paper (p. 61) that, when the numerator of B_p is divided by p, the quotient $\equiv 1$, mod. p.* In general if n is a multiple of p, but not of

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* In line 14 of p. 61 p is omitted from the denominator of the congruence which should be

$$\frac{B'_p}{p}\equiv 1, \text{ mod. } p;$$

but the congruence is correctly stated in words.

$j = \frac{p-1}{2}$ (so that p is not a Staudt factor of B_n), the numerator of B_n is divisible by p (Adams's result p. 50). If B_n'' denotes the quotient when the numerator of B_n is divided by p, and if n' denotes $\frac{n}{p}$, and if $n \equiv r$, mod. j, then the general theorem (§ 3) shows that

$$B_n'' \equiv (-1)^{n-r} \frac{B_r}{r} n' F_n, \text{ mod. } p,$$

where F_n is the product of the Staudt factors of B_n. We thus obtain the residue of the quotient B_n'' with respect to p.

§ 19. The result for $p = 5$ is interesting. In this case n is any number ending in 5, the numerator of n ends in 5, and $r = 1$, so that the congruence becomes

$$B_n'' \equiv n' F_n'', \text{ mod. } 5,$$

where F_n'' is the product of all the Staudt factors of B_n except 2 and 3.

Now the last figure in B_n'' and in n' may be obtained by doubling the last two figures in the numerator of B_n and in n, and throwing off the final 0 in each case; so that we obtain the rule:—if we double the last two figures of the numerator of B_n and of n, and throw off the final 0 in each case, then the last figure of the former is the same as the figure obtained by multiplying the last figure of the latter by all the Staudt factors of B_n except 2 and 3.

This rule does not actually assign the penultimate figure of B_n, but it limits the possible figures to two.

As examples, (1) let $n = 15$; the last two figures of the numerator of B_n are 05, and the Staudt factors of B_{15} are 2, 3, 7, 11, 31. Doubling 05 and throwing off the final 0 we obtain 1, which is also the last figure in the product of 3 by $7 \times 1 \times 1$.

(2) Let $n = 45$, the last two figures of the numerator of B_n are 35, and the Staudt factors are 2, 3, 7, 11, 19, 31. Doubling 35 and throwing off the 0 we obtain 7, which is also the last figure in the product of 9 by 7, 1, 9, 1.

§ 20. From the general theorem of § 3, we have, k being any number,

$$\frac{B_{kp}}{kp} \equiv \frac{B_k}{k}, \text{ mod. } p.$$

If k is not a multiple of $\frac{p-1}{2}$, so that p is not a Staudt factor of B_{kp} or B_k, the numerator of B_{kp} is divisible by p, and we have

$$\frac{B_{kp}'}{p} \equiv B'_k \times \frac{F_{kp}}{F_k}, \text{ mod. } p,$$

Now $F_{kp} \equiv F_k$, mod. p; for, let $1, 2, a, a', \ldots, 2k$ be all the divisors of kp, then the Staudt factors of B_k are those among the numbers $2, 3, a+1, a'+1, \ldots, 2k+1$, which are primes. The Staudt factors of B_k are the Staudt factors of B_k, and also those among the numbers $p+1, 2p+1, ap+1, \ldots, 2kp+1$, which are primes; and all of these numbers are $\equiv 1$, mod. p; so that $F_{kp} \equiv F_k$, mod. p. We therefore have

$$\frac{B_{kp}'}{p} \equiv B_k', \text{ mod. } p.$$

§ 21. For $k=1$, this becomes the theorem given in § 18 (p. 61) of the previous paper, referred to in § 18 of the present paper; and since the numerator of B_k is also unity for $k=2, 3, 4$, we have also

$$\frac{B_{2p}'}{p} \equiv 1, \quad \text{mod. } p,$$

$$\frac{B_{3p}'}{p} \equiv 1, \quad \text{mod. } p,$$

$$\frac{B_{4p}'}{p} \equiv 1, \quad \text{mod. } p.$$

From the first and third of these formulæ $p=5$ is excluded, and from the second $p=7$. All other values of $p>3$ are admissible.

As an example, let $p=7$ in the first formula; B'_{14} is 23749461029 which divided by 7 gives 3392780147, which $\equiv 1$, mod. 7.

§ 22. When $k=5$ and 7, the numerator of B_k is k itself, and in both cases B_{kp}' is divisible by k, so that we have

$$\frac{B_{5p}'}{5p} \equiv 1, \quad \text{mod. } p,$$

$$\frac{B_{7p}'}{7p} \equiv 1, \quad \text{mod. } p.$$

§ 23. The formulæ in the last three sections may be generalised by the substitution of p^i for p. For let $n = kp^i$ where k is not a multiple of p* or of $\frac{p-1}{2}$, then since $kp^i - k$ is a multiple of $p - 1$, we have, n being $= kp^i$,

$$\frac{B_n}{n} \equiv \frac{B_k}{k}, \quad \text{mod. } p.$$

Now it can be shown by the same reasoning as in § 20 that $F_n \equiv F_k$, mod. p; for, as before, all the Staudt factors of B_n which are not Staudt factors of B_k are $\equiv 1$, mod. p: and therefore we have, if $n = kp^i$,

$$\frac{B_n'}{p^i} \equiv B_k', \quad \text{mod. } p.$$

§ 24. Giving to k the same values as in §§ 21 and 22, we have for $n = p^i$, $2p^i$, $3p^i$, and $4p^i$,

$$\frac{B_n'}{p^i} \equiv 1, \quad \text{mod. } p,$$

and for $n = 5p^i$ and $7p^i$,

$$\frac{B_n'}{n} \equiv 1, \quad \text{mod. } p.$$

Since this paper has been in type I have published, in a complete form, the investigation† of the theorem

$$\frac{B_n}{n} \equiv (-1)^{tj} \frac{B_{n-tj}}{n-tj}, \quad \text{mod. } p,$$

of which the mode of proof was merely indicated in pp. 60–63.

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* In § 20 k may be a multiple of p, but the result in that case given by the congruence is of no value as both sides are divisible by p.

† "A congruence theorem relating to the Bernoullian numbers," *Quarterly Journal*, Vol. XXXI., pp. 253—263.

The paper referred to in the note on p. 61 is published in *Proc. Lond. Math. Soc.*, Vol. XXXI., pp. 193—215.

ON LINEAR TRANSFORMATION BY INVERSIONS.

By *G. G. Morrice, M.D.*

It is known that any linear transformation of the complex variable of the normal type,

$$\lambda' = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta}, \quad \alpha\delta - \beta\gamma = 1 \text{ ....} (1)$$

can be effected by inverting consecutively on two circles in the plane of the variable.

It is also known that the same linear transformation may be effected by considering the figure in the plane as the stereographic projection of the figure on a sphere, and rotating the sphere round a certain axis through an angle ϕ, the formulæ being

$$a = \rho\xi \sin\tfrac{1}{2}\phi, \quad b = \rho\eta \sin\tfrac{1}{2}\phi, \quad c = \rho\zeta \sin\tfrac{1}{2}\phi, \quad d = \rho \cos\tfrac{1}{2}\phi,$$

$$\rho = \sqrt{(a^2 + b^2 + c^2 + d^2)},$$

$$\lambda' = \frac{(d + ic)\lambda - (b - ia)}{(b + ia)\lambda + (d - ic)}.$$

And it can be shown that the angle of rotation of the sphere, ϕ, is twice the angle of intersection of the two circles of inversion.

Now let us take λ to be one of the parameters of a generator of the quadric surface,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \text{ ....} (2),$$

Klein has shown (*Mathematische Annalen*, Bd. XXXVII., p. 549) that to a linear transformation of λ,

$$\lambda' = \frac{(d + ic)\lambda - (b - ia)}{(b + ia)\lambda + (d - ic)},$$

corresponds a linear transformation of the x's,

$$\left.\begin{aligned} x_1' &= dx_1 - cx_2 + bx_3 + ax_4 \\ x_2' &= cx_1 + dx_2 - ax_3 + bx_4 \\ x_3' &= -bx_1 + ax_2 + dx_3 + cx_4 \\ x_4' &= -ax_1 - bx_2 - cx_3 + dx_4 \end{aligned}\right\} \text{ ....} (3),$$

and the non-euclidean distance of the two points x, x' is

$$E = \text{arc cos} \frac{x_1x_1' + x_2x_2' + x_3x_3' + x_4x_4'}{\sqrt{(x_1^2 + x_2^2 + x_3^2 + x_4^2)}\sqrt{(x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2)}} \cdots (4),$$

i.e. the point x has been moved through a distance $\frac{1}{2}\phi$ along the non-euclidean parallel.

But now give x_1, x_2, x_3, x_4 another interpretation and regard them as the power-coordinates of a circle in the original plane of the complex variable; then I say that:

The system of such circles corresponding to the modular group of transformations (1) *and* (2) *forms a polygon whose angles are equal to those of the polygon formed by the system of circles of inversion by which the said transformations may be effected.*

For Clifford has shown that, choosing four fundamental circles with radii r_1, r_2, r_3, r_4, the radius of any other circle in the plane whose powers with respect to the fundamental circles are y_1, y_2, y_3, y_4 is given by

$$r^2 = \tfrac{1}{4}\left(\frac{y_1^2}{r_1^2} + \frac{y_2^2}{r_2^2} + \frac{y_3^2}{r_3^2} + \frac{y_4^2}{r_4^2}\right),$$

and the power of two circles y, y'

$$= -\tfrac{1}{2}\left(\frac{y_1y_1'}{r_1^2} + \frac{y_2y_2'}{r_2^2} + \frac{y_3y_3'}{r_3^2} + \frac{y_4y_4'}{r_4^2}\right),$$

or simplifying this still further by changing the coordinates to

$$x_1 = \frac{y_1}{r_1}, \quad x_2 = \frac{y_2}{r_2}, \quad x_3 = \frac{y_3}{r_3}, \quad x_4 = \frac{y_4}{r_4},$$

we have $$r = \tfrac{1}{2}\sqrt{(x_1^2 + x_2^2 + x_3^2 + x_4^2)},$$

$$\text{power} = -\tfrac{1}{2}(x_1x_1' + x_2x_2' + x_3x_3' + x_4x_4').$$

Substituting in (4), we find:

Power of the circles x, $x' = -2rr' \cos E$,

i.e. the angle between the circles is equal to the non-euclidean distance of the points x, $x' = \frac{1}{2}\phi$. This corresponds to a rotation of the sphere round an axis through an angle ϕ, which is double the angle between the inversion circles corresponding to the linear transformation which proves the theorem.

The following papers have been consulted:—(1) Burnside, *Messenger of Mathematics*, Vol. XX. "On a property of linear substitutions." (2) Clifford, *Mathematical Papers.* "On Power-coordinates in general." (3) Klein, *Mathematische Annalen*, Vol. XXXVII. "Zur Nicht-Euklidischen Geometrie."

PERIOD-LENGTHS OF CIRCULATES.

By *Lt.-Col. A. Cunningham, R.E.*, Fellow of King's College, London.

[The author's acknowledgments are due to Mr. C. E. Bickmore for help in preparing and revising this Paper].

1. It may be interesting to supplement recent* Papers on the subject of Pure Circulates by regarding the question of their period-lengths from another point of view. Let l, r, N† be three quantities defined in any *one* of the following ways:

$$\left.\begin{array}{l} l = \textit{least}\text{ exponent which gives } r^l \equiv +1, \text{ (mod. } N); \\ l = \textit{least}\text{ power which gives } (r^l - 1) \text{ divisible by } N; \end{array}\right\} \begin{array}{c} r \text{ is called the} \\ \textit{root}\text{ or }\textit{base} \end{array} \left.\vphantom{\begin{array}{c}a\\b\end{array}}\right\} \;..(1);$$

l = *period-length* of the *repetend* fraction $1 \div N$ in the scale of radix r.

[Here r, N are supposed whole numbers, and r *prime* to N, so that $1 \div N$ is necessarily a *circulator*, and l also a whole number].

Thus any *one* of the three definitions involves the other two, so that any question expressed in any one of the ways may be interpreted in each of these ways, and any problem expressed in any one of the ways may be solved in the phraseology of another. The relations of l to r and N have been considered in the Papers referred to almost wholly in the view of repetend fractions, but there is great advantage in considering them in the other two ways, partly because they have been much more studied in those ways, and partly because the use of the Gaussian symbol ($\equiv$) gives great help in the study, inasmuch as the resulting "congruences" can be treated to a great extent like ordinary equations.

2. *Period-Lengths.* Thus a great deal has been written (in the first two modes of expression) on the *general* Law‡ for the period-length (l) of $1 \div N$, when N is prime, viz.

$l = (N-1)$, or *some*‡ submultiple of $(N-1)$, when N is *prime* ..(2),

so that this Law may be now much more closely restricted, as the *particular submultiple*, say $l = (N-1) \div \nu$ can now be

* See three Papers by Mr. B. Reynolds in the *Messenger of Mathematics*, Vol. XXVII., p. 177; Vol. XXVIII., pp. 38 and 88.

† The notation of Mr. Reynolds's Papers has been to a great extent preserved.

‡ Given as Law 2 in Mr. Reynolds's Paper in Vol. XXVII., p. 178, of the *Messenger* (but without defining the submultiple).

to some extent foretold (by rules which are simple of application) for small values* of ν. The Rules when $\nu = 2, 3, 4, 8$ have been worked out and stated in a general manner applicable to all radices or bases (r) by Legendre, Jacobi, Gauss, and Goldscheider respectively. They have been collected together and reduced to a convenient form for the small radices or bases $r=2, 3, 5, 6, 7, 10, 11, 12$ in two Papers by† Mr. C. E. Bickmore; these rules are easily modified so as to apply to the complementary bases $N-2$, $N-3$, &c. (this application is given in several instances in the Papers† quoted): the particular case of $\nu=16$ when $r=2$ (discovered by the present author) is also given; considerable progress has also been made in the case of $\nu=5$, as there† shown.

The method of congruences shows also that the Rules for casting out mutually prime factors ν_1, ν_2, &c., from the exponent $(N-1)$ are independent of one another, and *can therefore be used in combination.*

These Rules suffice to give the *least exponent* or period-length (l) in many cases when $N = 2^{\alpha}.3^{\beta}.q + 1$, especially when $\alpha \not> 3$, $\beta \not> 1$, and q is a *prime.*

Ex. $N = 5197 = 4.3.433 + 1 = 29^2 + 66^2 = 65^2 + 3.18^2$; the period-length ($l$) in various scales of radix r is at once given by the above rules as

$r =$	2,	3,	5,	6,	7,	10,	11,	12,
$l =$	1732,	866,	1732,	1732,	5196,	433,	1299,	433,
$\nu =$	3,	6,	3,	3,	1,	12,	4,	12.

There are many extensive Tables extant‡ giving the least exponents (l) of the smaller bases (r), or period-lengths (l) in scales of small radix (r).

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* This divisor ν is often called the *Residue-index.*

† See two papers *On the Numerical Factors of* $(a^n - 1)$ in the *Messenger*, Vol. XXV., p. 1, and Vol. XXVI., p. 1, *passim.*

‡ (1) Burckhardt's *Table des Diviseurs pour tous les nombres du premier million*, Paris, 1817, gives (p. 114) a Table of l (in scale of 10) for all primes < 2545 and a few larger ones: (it contains several Errata).

(2) Dr. Glaisher *On Circulating Decimals* (in *Proc. Cam. Phil. Soc.*, Vol. III. Part 5) gives a Table (p. 204) of l (in scale of 10) for all numbers < 1024 and prime to 10.

(3) Dr. Salmon and Mr. Shanks in *Proc. Royal Soc.*, Vol. XXII., pp. 200 and 384 give Tables of l (in scale of 10) for all primes < 30000: these have been extended to 60000 by Mr. Shanks, and to 81,150 by Dr. Salmon, but their results are unpublished: (the printed Tables contain a few errors).

(4) Prof. Reuschle's *Neue zahlentheoretische Tabellen*, Stuttgart, 1856, gives (p. 42) a Table of l and ν in scales of 2, 3, 5, 6, 7, 10 for all primes < 1000; another Table (p. 47) of l and ν in scales of 2 and 10 for all primes from 1000 to 5000; and another Table (p. 53) of l and ν, in scale of 10 only, for all primes from 5000 to 15000: (these Tables contain many errors).

3. Much help in finding period-lengths can often be had by the use of congruences. Thus if the least exponent or period-length (l_1) of $1 \div N$ for one radix (r_1) be known, and any congruence such as $r^y \equiv r_1^x$ be also given, the least exponent or period-length (l) for the radix r may be at once found, (or at any rate approximated to).

Thus given $r \equiv r_1^x$, and $r_1^{l_1} \equiv 1$, (mod. N) ....(3),

and suppose

$$x = \xi G, \; l_1 = \lambda G, \text{ where } G = \text{G.C.M. of } x, l_1 \ldots (3a).$$

Then $r^\lambda \equiv (r_1^x)^\lambda \equiv r_1^{\xi G \lambda} \equiv (r_1^{l_1})^\xi \equiv 1$, (mod. N) ...(4),

and $l = \lambda = l_1 \div G$, the period-length (for radix r) required ....(4a).

But if the data be $r^y \equiv r_1^x$, and $r_1^{l_1} \equiv 1$, (mod. N)...(3′),

and $x = \xi G$, $l_1 = \lambda_1 G$, where G = G.C.M. of x_1, l_1 ...(3′a)

and g = G.C.M. of $y\lambda$, $(N-1)$; (here g is a mult. of λ) (3′b),

then $(r^y)^\lambda \equiv (r_1^x)^\lambda \equiv (r_1^{l_1})^\xi \equiv 1$ ....(4′);

$$\left.\begin{array}{l} \text{whence } l = y\lambda, \text{ or } \textit{some} \text{ submult. thereof.} \\ \text{Thus } l = \lambda, \text{ if } y \text{ is prime to } (N-1), \\ \text{otherwise } l = g, \text{ or } \textit{some} \text{ submult. thereof.} \end{array}\right\} \ldots\ldots\ldots\ldots (4'a).$$

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(5) Herr H. Bork's *Periodische Dezimalbrüche*, Berlin, 1895, contains (p. 36) a Table (computed by Dr. Kessler) of the value of the residue-index (ν) in the scale of 10 for all primes $< 100{,}000$, except those for which $\nu = 1$ or 2 only, which are omitted to save space; these cases are easily recognised by rules given on p. 36: the period-length (l) is easily obtained as $l = (N-1) \div \nu$.

(6) Reuschle's *Tafeln complexer Primzahlen*, Berlin, 1875, contain numerous Tables which may be used for the inverse process of finding the bases or radices (r), which have a *given* least exponent or period-length (l) for a *given* prime (N) < 1000: there is a separate Table for each value of $l < 100$.

(7) Desmarest's *Théorie des Nombres*, Paris, 1852, contain (p. 288) a Table giving one primitive root (r) for every prime $< 10{,}000$; for each such number therefore $l = N - 1$, the max. value: (this Table has several Errata).

(8) The present author has prepared an extension of Reuschle's Tables, No. (4) above (which it is hoped to publish shortly), giving the values of l and ν for the base 2 for all primes $< 10{,}000$; a similar extension for the bases 3, 5, 6 is also far advanced to completion.

(9) The present author is preparing an extension of part of Reuschle's Tables, No. (6) above, giving the values of the bases or radices (r), which can have the least exponent or period-length $l = 5$, 10, 15, 7, 9, 11, for all primes and powers of primes $< 10{,}000$: (these Tables are nearly complete).

4. One of the problems considered in the Papers referred to is the finding* the period-length (l) of the fraction $1 \div N$ in *all* scales of radix $r < N - 1$, where N is *prime*; the above problem gives a *direct*† solution of this.

Let r_1 be a *primitive root*‡ of a *prime* N, *i.e.* such that the period-length (l_1) of $1 \div N$ in the scale of radix r is the *maximum* $l_1 = N - 1$. Let a Table be formed giving the *Residues* (r) of every successive power r_1^x, ($x = 1, 2, 3, \ldots$ to $N-1$) left after division of r_1^x by N, so that, as (3) of Art. 3, for all values of x,

$$r \equiv r_1^x, \text{ and } r_1^{N-1} \equiv 1, \text{ (mod. } N) \quad \ldots\ldots\ldots(3).$$

Such a Table, once formed, may be conveniently used for the above purpose: unfortunately it is very laborious to form when N is not a *small* prime. But Jacobi's *Canon Arithmeticus*, Berlin, 1839, contains such Tables *ready formed* for all primes§ $N < 1000$; so that for all such primes the most laborious part of the work is ready to hand: the rest of the work is comparatively slight. Thus, taking the example in the Papers quoted ($N = 31$), form the following Table:

$r_1^x \equiv r$ =	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,	14,	15,	&c.,
when x =	30,	12,	13,	24,	20,	25,	4,	6,	26,	2,	29,	7,	23,	16,	3,	&c.,
G =	30,	6,	1,	6,	10,	5,	2,	6,	2,	2,	1,	1,	1,	2,	3,	&c.,
l =	1,	5,	30,	5,	3,	6,	15,	5,	15,	15,	30,	30,	30,	15,	10,	&c,

The two lines of r, x are copied *direct* out of Jacobi's Table; the third line contains G = G.C.M. of x, $(N-1)$, *i.e.* of x and 30; and the last line contains $l = (N-1) \div G = 30 \div G$, as in ($4a$) of Art. (3).

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* *Messenger of Math.*, Vol. XXVIII., p. 89.

† practically the *same* as in the papers quoted; but here stated in other terms, chiefly to show the help that can be had from *existing* Tables.

‡ The finding of a primitive root (as hereby required) is often a most laborious matter. The process is detailed in Mathews's *Theory of Numbers*, 1892, Pt. I. Art. 20. There are Tables extant, each giving *one* primitive root of every prime as below:

(1) *Théorie des Nombres*, by E. Desmarest, Paris, 1852, p. 298, for every prime < 10,000; (this Table has several Errata).

(2) *Neue zahlentheoretische Tabellen*, by Prof. Reuschle, Stuttgart, 1856, p. 42, for every prime < 5,000; (this Table has several Errata).

(3) *Acta Mathematica*, Vol. XX.; 2, by C. Wertheim, 1876, p. 153, for primes > 3,000 and < 5,000. This gives the *smallest* primitive root of each prime.

§ The *Canon Arithmeticus* contains also similar Tables for all *powers of primes* $N^x < 1000$. These Tables can be used in a quite similar way.

Thus finally, l is the *required** period-length of $1 \div 31$ in each of the scales of radix r above, or least exponent giving $r^l \equiv 1$, (mod. 31).

5. Much help in determining the least exponent or period-length (L) of $1 \div N$ in a scale whose radix (R) is the *product* or *quotient* of two radices (r_1, r_2), or of the powers thereof may be gained by the use of congruences.

i. Given $r_1^l \equiv 1$ and $r_2^l \equiv 1$, (l being the *least* exponent in each case) ....(5),

and given $R = r_1 r_2$, or $r_1 \div r_2$, or $r_1^\alpha . r_2^\beta$, or $r_1^\alpha \div r_2^\beta$, (where α, β are both *prime*† to l) ....(5a).

Then $R^L \equiv 1$, where $L = l$, or *some* submult. of l...(5b).

Also, if $l = (N-1) \div \nu$, and $L = (N-1) \div n$; so that ν, n are the max. Residue-indices ....(6).

Then $n = \nu$, or a multiple of ν ....(6a).

ii. Again, given $r_1^{l_1} \equiv 1$, $r_2^{l_2} \equiv 1$; (l_1, l_2 being the *least* exponents in each case ....(7),

and $l_1 = \lambda_1 G$, $l_2 = \lambda_2 G$, where G is the G. C. M. of l_1, l_2...(7a).

Given also, $R = r_1 r_2$, or $r_1 \div r_2$, or $r_1^\alpha . r_1^\beta$, or $r_1^\alpha \div r_1^\beta$, (where α, β are *prime*‡ to l_1, l_2 respectively.........(7b).

Then $R^L \equiv 1$, where $L = \lambda_1 \lambda_2 G = l_1 l_2 \div G$, or *some* sub-multiple thereof ....(7c).

Also, if $l_1 = (N-1) \div \nu_1$, $l_2 = (N-1) \div \nu_2$, and $L = (N-1) \div n$, so that ν_1, ν_2, n are the max. Residue-indices ......(8),

and, if $\nu_1 = \gamma . \nu_1'$, $\nu_2 = \gamma . \nu_2'$, where $\gamma =$ G. C. M. of ν_1', ν_2', so that ν_1' is prime to ν_2' ....(8a).

Then $n = \gamma$, or a multiple of γ, but contains neither ν_1', ν_2' ....(8b).

When l in (i) is *even*, and also when l_1, l_2 in (ii) are *both even* (provided that the ratio $l_1 : l_2$ can be reduced to $\omega_1 : \omega_2$, where ω_1, ω_2 are *both odd*), the previous approximation to L can be *halved*. Thus

iii. Given $r_1^l \equiv 1$, $r_2^l \equiv 1$, (l being the *least* exponent, and *even*) ....(9),

and, given $R = r_1 r_2$, or $r_1 \div r_2$, or $r_1^\alpha . r_2^\beta$, or $r_1^\alpha \div r_2^\beta$, ($\alpha$, β *both prime*† to l) ....(9a).

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* A *number* of Tables of this kind have been prepared by Mr. H. J. Woodall of Stockport.

† If α, β have any factors common with l, Cases i, iii may be brought under Cases ii, iv respectively.

‡ If α, β have any factors oommon with l_1, l_2, a further reduction can generally be made in L, with a corresponding increase in n.

Then $R^L \equiv 1$, where $L = \frac{1}{2}l$, or *some* submult. of $\frac{1}{2}l$...(10),

and $n = 2\nu$, or a mult. of 2ν, $[\nu=(N-1)\div l,\ n=(N-1)\div L]$ (10*a*).

iv. Again, given $r_1^{l_1} \equiv 1$, $r_2^{l_2} \equiv 1$, (l_1, l_2 being the *least* exponents in each case and *both even*).........(11),

and $l_1 = \lambda_1 G$, $l_2 = \lambda_2 G$, where G = G.C.M. of l_1, l_2....(11*a*),

and, given also λ_1, λ_2 *both odd* ....(11*b*).

And, if $R = r_1 r_2$ or $r_1 \div r_2$, &c., (as in Case iii), then $R^L \equiv 1$, where $L = \frac{1}{2}\lambda_1\lambda_2 G = \frac{1}{2}l_1 l_2 \div G$, or *some* submult. of L, (λ_1, λ_2 being *both odd* and G *even*)..................(12).

Also, with the notation of Case ii, $n = 2\gamma$, or a multiple of 2γ, but contains neither ν_1', ν_2' ....(12*a*).

Ex. $N = 2857 = 2^3.3.7.17 + 1 = \frac{1}{4}(41^2 + 27.19^2)$. Given also* $10^{102} \equiv 1$, $2^{408} \equiv 1$, to find L† when $r = 5$.

Here $r_1 = 10$, $l_1 = 102$, $\nu_1 = 28$; $r_2 = 2$, $l_2 = 408$, $\nu_2 = 7$.

Hence $G = 102$, giving $\lambda_1 = 1$, $\lambda_2 = 4$, (so that this does not fall under Case iv). Hence (Case ii), $L = 1.4.102 = 8.3.17 = 408$, or *some* submultiple thereof.

Now the Rules quoted in Art. 2 show that n cannot contain 2 or 3, so that L must contain 8.3; also L must be > 8.3 because the factors of $(5^{12} + 1)$ are already known not to include N. Therefore $L = 408$.

Ex. $N = 4999 = 2.3.7^2.17 + 1 = \frac{1}{4}(139^2 + 27.5^2)$. Given also* $10^{357} \equiv 1$, $2^{357} \equiv 1$; to find† L when $r = 5$.

Here $r_1 = 10$, $r_2 = 2$, $l_1 = l_2 = 357 = l$.

Hence (Case i), $L = l = 357 = 3.7.17$, or some submultiple thereof. Now the Rules quoted in Art. 2 show that n contains 3, which must therefore be removed from L. Also the factors of $(5^7 - 1)$ and $(5^{17} - 1)$ are already known not to include N. Therefore $L = 7.17 = 119$.

6. *Congruence* $r^l \equiv 1$ (mod. N^t). The property that in particular scales of radix r_1, r_2, &c., the fractions $1 \div N$, $1 \div N^2$, $1 \div N^3$, ..., $1 \div N^t$ may all have the‡ *same period-length* (l) when N is an *odd prime*, and the *radix* $r < N^t$ (t being the highest power of N in question), and in some few cases even when $r < N^{t-1}$, may be conveniently studied by the aid of congruences. The expression of the property is, in fact,

$r^l \equiv 1$, (mod. N, N^2, N^3, ..., N^t, and $r < N^t$, or $< N^{t-1}$)...(13).

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* See Reuschle's Tables No. (4) quoted in footnote‡ of Art. 2.

† Not in Reuschle's Table quoted.

‡ A good many examples of this are given in the papers referred to; but no general rules are given showing how the results were or could be arrived at.

6*a*. This property (13) is, however, somewhat *exceptional*: in fact when $r^l \equiv 1$, (mod. N), l being the *least* exponent, then the *general* rule as to the exponents for the moduli N^2, N^3, &c., with the *same radix* r, appears to be

$$r^{lN} \equiv 1, \text{ (mod. } N^2), \quad r^{lN^2} \equiv 1, \text{ (mod. } N^3), \text{ and so on} \ldots (14).$$

But lN, lN^2, &c. are here not necesssarily the *least* exponents for the moduli N^2, N^3, &c. These exponents *must*, however, contain the factor l, whilst the factor N, N^2, &c., may in certain (quite exceptional) cases be divided out.

Ex. 1. $2^3 \equiv 1$ (mod. 7), $2^{21} \equiv 1$ (mod. 7^2), $2^{147} \equiv 1$ (mod. 7^3), &c.

Ex. 2. $3^5 \equiv 1$, (mod. 11 and 11^2); $10^{486} \equiv 1$, (mod. 487 and 487^2) are instances (rare with such *low* bases as 3, 10) of the removal of the prime N from the exponent (usually $= lN$) proper to the squared modulus (N^2).

[Hence, when r and N are *given*, the least exponent of r for the moduli N^2, N^3, &c. may be found by this Rule by actual trial of the several exponents l, lN, lN^2, &c., l being the *least* exponent giving $r^l \equiv 1$, (mod. N)].

7. *General Rules for* $r^l \equiv 1$, (mod. N^t). The principal Rules for solving (13) are summarised* below for the case of $N=$† prime.

i. N must be of the form $N = \nu l + 1$, and conversely l must $= (N-1) \div \nu$, *i.e.* $l = N - 1$ or *some* submultiple thereof. [This is independent of t].

ii. The *number* of (incongruous) roots (r_t), each $< N^t$, of the congruence $r^l \equiv 1$, (mod. N^t) is the *same for all values* of t, and therefore the same as for $r^l \equiv$ (mod. N), and is therefore $= \phi(l)$, the *Totient* of l.

Thus if $l =$ (a prime p), $\phi(l) = p - 1$; if $l = p^\alpha$, where $p =$ *prime*, then $\phi(l) = (p-1) . p^{\alpha-1}$. If $l = p_1^{\alpha}, p_2^{\beta}, p_3^{\gamma} \ldots$, where p_1, p_2, p_3, &c. are primes; then

$$\phi(l) = \{(p_1 - 1)(p_2 - 1)(p_3 - 1) \ldots\ldots\} . l \div (p_1 p_2 p_3 \ldots).$$

Ex. If $l = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$, &c., then $\phi(l) = 2, 2, 4, 2, 6, 4, 6, 4, 10, 4$, &c.

iii. If $\rho_t < N^t$ be a root of $r^l \equiv 1$ (mod. N^t), then $r_t = m.N^t + \rho_t$ also satisfies the same congruence for *all* (integer) values of m: hence if $r_t > N^t$ be a given root, the *least* root ρ_t congruous to r_t, and therefore $< N^t$, can be found by subtracting a sufficient multiple of N^t. Such roots r_t are called *congruous roots* (*i.e.* congruous to ρ_t). Hence it suffices to discover the system of $\phi(l)$ *incongruous* roots each $< N^t$.

iv. If ρ_1 be a root of $r^l \equiv 1$, (mod. N), then *one root of each* of the other congruences in (13) is contained in the general expression $\rho_t = m_t N + \rho_t$ (the value of m_t being in general different for, and peculiar to, each value of t).

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* Many of these rules are well known, but no systematic statement of them is known to the author.

† When N is *not prime* the rules are not nearly so simple.

v. Any root ρ_t of the congruence $r^l \equiv 1$ (mod N^t) is a root of each of the congruences (13) for all lesser values of t, by Rule iii, although not in general $< N^{t-1}$; it may, however, be made to yield one root of each congruence $<$ the modulus (N, N^2, N^3, &c. respectively) thereof by subtracting a sufficient number of multiples of N. These roots, one from each congruence of (13), are styled *congruous roots*, being all congruous to ρ_1 a root of $r^l \equiv 1$, (mod. N).

vi. The complete set of $\phi(l)$ incongruous roots, each $< N^t$, of the congruence $r^l \equiv 1$, (mod. N^t) is also a *complete set of incongruous roots* of each of the congruences (13), though, not of course, in general, $< N^{t-1}$: the least values proper to each congruence, (*i.e.* $< N, N^2, N^3, \ldots, N^{t-1}$ respectively) may be obtained by subtracting a sufficient multiple of N from each.

vii. If ρ_t be any root of $r^l \equiv 1$ (mod N^t), then the powers of ρ (*i. e.* $\rho, \rho^\alpha, \rho^\beta, \ldots$, to ρ^{l-1}), taking as exponents only the $\phi(l)$ numbers $1, \alpha, \beta, \gamma, \ldots, l-\beta, l-\alpha, l-1$, which are $< l$ and also *prime* to l, are a *complete set* of the incongruous roots thereof: these may be reduced, if $> N^t$, to their least values, each $< N^t$ by taking their Least Residues to modulus N^t. (Hence, if *one* root be known, all the rest can be found by this Rule).

Cor. If l be *even*, the indices $1, \alpha, \beta, \gamma$, &c. to $(l-1)$ must be all *odd*. Again, if l be *prime* the indices $1, 2, 3, \ldots, (l-1)$ will suffice. *Ex.* When $l = 12$, $\phi(l) = 4$ and the four indices required are $1, 5, 7, 11$, (being the four numbers prime to l and $< l$).

viii. If r, r' be the least Residues of $\rho_t^\alpha, \rho_t^{l-\alpha}$, (mod. N^t), when ρ_t is *any* root of $r^l \equiv 1$, (mod. N^t), then r, r' are roots of the same. And, since $r \cdot r' \equiv \rho_t^\alpha \cdot \rho_t^{l-\alpha} \equiv \rho_t^l \equiv 1$, (mod. N^t), either root can be found from the other, by solution of the simple congruence $rr' \equiv 1$, (mod. N^t). Such roots are called *reciprocal*: clearly every root has its reciprocal. (Thus if $\frac{1}{2}\phi(l)$ roots be found by Rule vii, taking α, β, &c. $< \frac{1}{2}(l-1)$, the remaining $\frac{1}{2}\phi(l)$ roots can be found by this Rule).

ix. If ρ_1, ρ_2, ρ_3 &c. .. be a set of *congruous roots* (Rule v), of one of each of the congruences (13), so that

$$\rho_1^l \equiv 1, \text{ (mod. } N), \quad \rho_2^l \equiv 1, \text{ (mod. } N^2), \quad \rho_3^l \equiv 1, \text{ (mod. } N^3), \text{ \&c. } \ldots (15);$$

then, by (14),

$$\rho_2 \equiv \rho_1^N \text{(mod. } N^2), \quad \rho_3 \equiv \rho_1^{N^2} \equiv \rho_2^N \text{ (mod. } N^3), \quad \ldots, \quad \rho_t \equiv \rho_{t-1}^N, \text{ (mod. } N^t) \quad \ldots\ldots\ldots (16).$$

[This Rule enables ρ_2, ρ_3, &c. to be found in succession from ρ_1, a *given* solution of $r^l \equiv 1$, (mod. N). It has the inconvenience of requiring the formation of a Residue of an N^{th} power in every case, whereas the process detailed below only proceeds to the l^{th} powers (l *odd*) or $\frac{1}{2}l^{\text{th}}$ powers (l *even*)].

x. If l be *even*, the proper roots of the two congruences $r^l \equiv +1$, (mod. N^t), $r^{\frac{1}{2}l} \equiv -1$, (mod. N^t) are *the same*, for each value of t.

xi. If $\frac{1}{2}l$ be *odd*, each root ρ_t of $\rho^{\frac{1}{2}l} \equiv 1$, (mod. N^t) is *complementary* to one root r_t of $r^{\frac{1}{2}l} \equiv -1$, and of $r^l \equiv +1$, (mod. N^t), [the latter pair being the same roots by Rule x] for each value of t; *i.e.* $\rho_t + r_t = N^t$; so that each root of either congruence gives a root of the other, and either set gives the other (by simple subtraction from N^t).

xii. If $\frac{1}{2}l$ be *even*, each root (r_t) of $r^{\frac{1}{2}l} \equiv -1$, and $r^l \equiv +1$, (mod. N^t) is *complementary* to some other root (r_t') of the same for each value of t; *i.e.* $r_t + r_t' = N^t$, so that every known root gives another root (by simple subtraction from N^t).

7*a*. *Case of* $l=4$, 3, or 6. A few general applications will now be given showing how the computation of *some* of the $\phi(l)$ incongruous roots may be greatly simplified, especially for *small values* of l, when one or more roots have been found.

Application (1). Let $l=4$. By Rule x, the roots of $r^4 \equiv +1$ and $r^2 \equiv -1$, (mod. N^t) are the *same* for each value of t. By Rule ii, there are only $\phi(4)=2$ incongruous roots, say r_t, r_t', each $< N^t$. By Rule xii, $r_t + r_t' = N^t$, so that either root gives the other by simple subtraction from N^t, and only *one root need be specially computed* for each value of t.

[Many examples will be found in the Table ($l=4$) on p. 166].

Application (2). Let $l=3$ or 6. By Rule ii, $\phi(l)=2$, so that there are only two incongruous roots, each $< N^t$, of each congruence, for each value of t, say

$$\rho_t, \rho_t' \text{ of } \rho^3 \equiv +1\,;\; r_t, r_t' \text{ of } r^3 \equiv -1 \text{ and of } r^6 \equiv +1, \text{ (mod. } N^t).$$

Now $\quad \rho^3 - 1 \equiv 0$ gives $(\rho-1)(\rho^2+\rho+1) \equiv 0$, (mod. N^t),

$$r^3 + 1 \equiv 0 \text{ gives } (r+1)(r^2 - r + 1) \equiv 0, \text{ (mod. } N^t).$$

But $\rho - 1$ and $r+1$ cannot $\equiv 0$, (mod. N^t), because 3, 6 are the *least* exponents giving $\rho^l \equiv 1$ and $r^{\frac{1}{2}l} \equiv -1$, (mod. N^t).

Therefore $\quad \rho_t^2 + \rho_t + 1 \equiv 0$ and $r_t^2 - r_t + 1 \equiv 0$, (mod. N^t)......(17).

But, by Rule vii (l being $=3$, or 6), $\rho_t' \equiv \rho_t^2$, and $r_t' \equiv r_t^5$ (mod. N^t)

Therefore $\quad \rho_t' + \rho_t + 1 \equiv 0$, (mod. N^t) ....(17*a*).

Hence, since ρ_t', ρ_t are both $< N^t$, $\quad \rho_t' + \rho_t = N^t - 1$ ....(18).

Again, Rule xi gives $\rho_t + r_t = N^t$, and $\rho_t' + r_t' = N^t$..........(18*a*).

These last three results give also

$$r_t + r_t' = N^t + 1\,;\quad r_t - \rho_t' = 1\,;\quad r_t' - \rho_t = 1 \text{ ......(18}b).$$

By the last six results any one of the four roots ρ_t, ρ_t', r_t, r_t' gives the other three by simple subtractions, &c., so that *only one of the four needs special computation.*

[Many examples will be found in the Tables ($l=3$ and 6) on p. 167].

8. *Successive Congruence Solutions* (*Simple Method*). Another method of computing ρ_2, ρ_3, &c. in succession from ρ_1 (simpler in application than that of Rule ix) will now be developed.

By Rule iv, $\quad \rho_2 = m_2 N + \rho_1$ ....(19),
therefore

$$\rho_2^{\,l} = (m_2 N + \rho_1)^l = \text{multiples of } N^2 + l.m_2 N.\rho_1^{\,l-1} + \rho_1^{\,l},$$

therefore $\quad lN\rho_1^{\,l-1}.m_2 + \rho_1^{\,l} \equiv \rho_2^{\,l} \equiv 1$, (mod. N^2).

Transpose, and multiply by $\nu\rho_1$,

$$-\nu l.N.\rho_1^{\,l}.m_2 \equiv \nu\rho_1.(\rho_1^{\,l} - 1), \text{ (mod. } N^2) \text{ ......(20}').$$

Dividing by N gives $\quad -\nu l.\rho_1^{\,l}.m_2 \equiv \nu\rho_1.\dfrac{\rho_1^{\,l}-1}{N}$, (mod. N).

But $N = l\nu + 1$; therefore $\nu l \equiv -1$ and $\rho_1^{\,l} \equiv 1$, (mod. N), and $(\rho^l - 1) \div N$ is an *integer*.

Therefore $\quad m_2 \equiv \nu\rho_1.\dfrac{\rho_1^{\,l}-1}{N}$, (mod. N) ....(20).

Hence m_2, and therefore also ρ_2 [by (19)] can always be found by this simple Rule by proceeding to the l^{th} power of ρ_1 only, (whereas Rule ix involves proceeding to the N^{th} power). The labor of forming the actual quotient of $(\rho_1^{\,l} - 1) \div N$ is *considerable* when $\rho_1^{\,l}$ is large: but this also may be avoided, for the Residue, say $M_1'N$ of $(\rho_1^{\,l} - 1)$ to the modulus N^2 may evidently be substituted for the quantity $(\rho_1^{\,l} - 1)$ in the congruence (20′): thus, suppose

$$\rho_1^{\,l} = M_1.N + 1 = (M_2N + M_1').N + 1.$$

Then $\quad \rho_1^{\,l} - 1 \equiv M_1'.N$, (mod. N^2);

or $\quad M_1'N =$ Residue of $(\rho_1^{\,l} - 1)$ to mod. N^2......(20*b*);

and, finally, $\quad m_2 \equiv \nu\rho_1.\dfrac{(M_1'N)}{N} \equiv \nu\rho_1.M_1'$, (mod. N)...(20*a*),

whereby the actual division of $(\rho_1^{\,l} - 1)$ by N is avoided.

Again, to find ρ_3;—

By (an extension of) Rule iv, $\quad \rho_3 = m_3N^2 + \rho_2$ ....(21);

and, by a precisely similar process to that by which m_2 was found, it may be shown that

$m_3 \equiv \nu\rho_2.\dfrac{\rho_2^{\,l}-1}{N^2}$, (mod. N). Here $\dfrac{\rho_2^{\,l}-1}{N^2} =$ integer...(22),

and $\quad m_3 \equiv \nu\rho_2.\dfrac{(M_2'N^2)}{N^2} \equiv \nu\rho_2.M_2'$, (mod. N) ....(22*a*),

where $\quad M_2'N^2 =$ Residue of $(\rho_2^{\,l} - 1)$ to mod. N^3.......(22*b*).

Similarly, $\quad \rho_t = m_tN^{t-1} + \rho_{t-1}$ ....(23), where $m_\iota \equiv \nu\rho_{\iota-1} \cdot \dfrac{\rho^l_{\iota-1} - 1}{N^{\iota-1}}$, (mod. N) ....(24)

$$\equiv \nu\rho_{\iota-1} \cdot \frac{(M'_{\iota-1}.N^{\iota-1})}{N^{\iota-1}} \equiv \nu\rho_{\iota-1}.M'_{\iota-1}, \text{ (mod. } N) \text{..(24}a\text{)},$$

where $M'_{\iota-1}.N^{\iota-1}$ = Residue of $(\rho^l_{\iota-1} - 1)$ to mod. N^ι ...(24*b*).

8*a*. *Use of half-index* ($\frac{1}{2}l$). When, however, l is *even*, the congruence $r^{\frac{1}{2}l} = -1$, (mod. N^ι) may be substituted for the congruence $\rho^l = +1$ (mod. N^ι), which was used above, because by Rule x the two congruences have the *same* roots; so that it will hereby be necessary to proceed only to the $\frac{1}{2}l^{\text{th}}$ powers of r_1, r_2, &c. Using the same process as before, and making the slight necessary changes, viz.

Writing now $\quad \frac{1}{2}l, \quad r^{\frac{1}{2}l} \equiv -1, \quad r^{\frac{1}{2}l} + 1,$

instead of the previous $\quad l, \quad \rho^l \equiv +1, \quad \rho^l - 1,$

r_2, r_3, &c.; m_2, m_3, &c. are now given by

$$r_2 = m_2 N + r_1 \text{ ....(25)},$$

$$m_2 \equiv -2\nu r_1 \cdot \frac{r_1^{\frac{1}{2}l} + 1}{N}, \text{ (mod. } N) \text{ ....(26)}$$

$$\equiv -2\nu r_1 \cdot \frac{(M_1''N)}{N} = -2\nu r_1.M_1'', \text{ (mod. } N) \text{...(26}a\text{)},$$

$$r_3 = m_3 N^2 + r_2 \text{ ....(27)},$$

$$m_3 \equiv -2\nu r_2 \cdot \frac{r_2^{\frac{1}{2}l} + 1}{N^2}, \text{ (mod. } N) \text{ ....(28)}$$

$$\equiv -2\nu r_2 \cdot \frac{(M_2''N^2)}{N^2} \equiv -2\nu r_2.M_2'', \text{ (mod. } N) \text{...(28}a\text{)},$$

$$r_\iota = m_\iota N^{\iota-1} + r_{\iota-1} \text{ ....(29)},$$

$$m_\iota \equiv -2\nu r_{\iota-1} \cdot \frac{\rho^{\frac{1}{2}l}_{\iota-1} + 1}{N^{\iota-1}}, \text{ (mod. } N) \text{ ....(30)}$$

$$\equiv -2\nu r_{\iota-1} \cdot \frac{(M''_{\iota-1}N^{\iota-1})}{N^{\iota-1}} \equiv -2\nu r_{\iota-1}.M''_{\iota-1}, \text{ (mod. } N) \text{...(30}a\text{)},$$

where $(M_1''N)$ = Residue of $(r_1^{\frac{1}{2}l} + 1)$ to mod. N^2 ...(26*b*),

$(M_2''N^2)$ = Residue of $(r_2^{\frac{1}{2}l} + 1)$ to mod. N^3 ...(28*b*),

$(M''_{\iota-1}N^{\iota-1})$ = Residue of $(r_{\iota-1}^{\frac{1}{2}l} + 1)$ to mod. N^ι ...(30*b*).

8*b*. The quantities M_2', M_2'' required for forming m_3 by (22*a*), (28*a*) may be reduced in another way. For, by (19),

$$\rho_2^l - 1 \equiv (m_2 N + \rho_1)^l - 1$$

$$= \text{multiples of } N^3 + \tfrac{1}{2}l(l-1)\,m_2^2 N^2 \rho_1^{l-2} + l m_2 N \rho_1^{l-1} + \rho_1^l - 1,$$

$$\therefore (M_2' N^2) \equiv \tfrac{1}{2}l(l-1)\,m_2^2 N^2 \rho_1^{l-2} + l m_2 N \rho_1^{l-1} + \rho_1^l - 1, \text{ (mod. } N^3),$$

$$\therefore M_2' \equiv \tfrac{1}{2}l(l-1)\,m_2^2 \rho_1^{l-2} + \frac{1}{N}\left(l m_2 \rho_1^{l-1} + \frac{\rho_1^l - 1}{N}\right), \text{ mod. } N \text{ ...(32).}$$

Similarly, it may be shown that

$$M_2'' \equiv \tfrac{1}{2}\cdot\tfrac{1}{2}l\,(\tfrac{1}{2}l - 1)\,m_2^2 r^{\frac{1}{2}l-2} + \frac{1}{N}\left(\tfrac{1}{2}l m_1 r_1^{\frac{1}{2}l-1} + \frac{r_1^{\frac{1}{2}l} + 1}{N}\right), \text{ mod. } N \text{ ....(33).}$$

Thus, by this process M_2', M_2'' are found by taking residues of ρ_1^{l-2}, ρ_1^{l-1}, ρ_1^l, or of $r_1^{\frac{1}{2}l-2}$, $r_1^{\frac{1}{2}l-1}$, $r_1^{\frac{1}{2}l}$ instead of residues of ρ_2^l, $r_2^{\frac{1}{2}l}$.

Similar formulæ (increasing in complexity) may be formed for M_3', M_3'', &c.

[This process is advantageous when (as often occurs) ρ_2, r_2, &c. are very much greater than ρ_1, r_1].

8*c*. *Method of Art.* 8, 8*a*, *b compared with Rule* ix, *&c.* It will be seen that, by the process of Art. 8, 8*a*, 8*b*,

only the *quotients* $(\rho_1^l - 1) \div N$, $(\rho_2^l - 1) \div N^2$, &c.,

or the *residues* of $(\rho_1^l - 1)$ to mod. N^2, $(\rho_2^l - 1)$ to mod. N^3, &c.,

or the *quotients* $(r_1^{\frac{1}{2}l} + 1) \div N_1$, $(r_2^{\frac{1}{2}l} + 1) \div N^2$, &c.,

or the *residues* of $(r_1^{\frac{1}{2}l} + 1)$ to mod. N^2, $(r_2^{\frac{1}{2}l} + 1)$ to mod. N^3, &c.,

have to be formed, so that it is only necessary to proceed to the lth or $\frac{1}{2}l$th power of ρ_t or r_t, (instead of to the Nth power as required by Rule ix). When $(\rho_t^l - 1)$ or $(r_t^{\frac{1}{2}l} + 1)$ can be factorised, it is often convenient to factorise them first, and then divide the large residual "algebraic prime factor" by N^t.

8*d*. *Summary of Art.* 6–8*c*. Thus it has been shown that all the congruences of the system (13) are *always possible* (for all values of t) when N is a *prime* of form $N = \nu l + 1$; and a *general* method has been given of completely solving them, *i.e.* of finding the complete system of $\phi(l)$ *incongruous* roots (r_t), each $< N^t$, which shall satisfy the whole system of congruences (13), when l is a *given* exponent, and N a *given* (prime) modulus $= \nu l + 1$.

9. *Expression of roots* (r) *in scale of* N. Since

$$\rho_2 = m_2 N + \rho_1,\ \rho_3 = m_3 N^2 + \rho_2,\ \ldots,\ \rho_t = m_t N^{t-1} + \rho_{t-1};$$

therefore

$$\rho_t = m_t . N^{t-1} + m_{t-1} N^{t-2} + \ldots + m_3 N^2 + m_2 N + \rho_1 \ldots (34).$$

This result shows that ρ_1 is the *units digit*, and m_2, m_3, &c. are the *successive digits* (reckoning from the right) of ρ_t *expressed* in the scale whose radix is N, *i.e.*

$\rho_t = m_t,\ m_{t-1},\ \ldots,\ m_3, m_2, \rho_1$ (reading *arithmetically* in scale of N) ....(34*a*).

When expressed in this manner any given root ρ_t of $r^l \equiv 1$, (mod. N^t), gives *visibly* one root of every congruence $r^l \equiv 1$, (mod. N^τ), for *all* values of $\tau < t$, as it suffices to *efface the higher digits* m_t, m_{t-1}, &c. down to $m_{\tau+1}$; thus

ρ_1; $\rho_2 = m_2, \rho_1$; $\rho_3 = m_3, m_2, \rho_1$; $\rho_4 = m_4, m_3, m_2, \rho_1$, &c. (reading *arithmetically* in the scale of N) ....(34*b*)

are the set of *congruous* roots (Rule v) of the congruences (13) for the successive moduli N, N^2, N^3, N^4, &c., all congruous to the initial root ρ_1.

When expressed in this way the root ρ_t' (or r_t) complementary to ρ_t under Rules xi and xii may be written

$\rho_t' = m_t'\ m'_{t-1},\ \ldots,\ m_3', m_2', \rho_1'$ (reading arithmetically in scale of N).........(35),

where $$\rho_t' + \rho_t = N^t \ \ldots\ldots\ldots\ldots\ldots\ldots(36a),$$

$$m_2' + m_2 = m_3' + m_3 = \ldots = m_t' + m_t = N - 1 \ldots (36b).$$

Example. The following are the roots r_t, r_t' of the congruences $r^4 \equiv 1$, (mod. 5^t) *expressed in the scale of radix* = 5, and will sufficiently illustrate the above results:

t =	1,	2,	3,	4,	5,	6,	7,	8,
mod. =	5,	5^2,	5^3,	5^4,	5^5,	5^6,	5^7,	5^8,
r_t =	2;	12;	212;	1212;	31212;	431212;	2431212;	32431212;
r_t' =	3;	33;	233;	3233;	13233;	13233;	2013233;	12013233.

10. *Case of* $r_t < N^{t-1}$. In the solutions above given every root r_t is $< N^t$, but not usually $< N^{t-1}$. The cases in which $r_t < N^{t-1}$ are in fact *few*, but of special interest. The author is not aware of any mode of determining them *a priori*; *i.e.*

other than by computing the whole* set of $\phi(l)$ incongruous roots $(r_t) < N^t$, and picking out those roots which are $< N^{t-1}$.

The following Table shows 27† cases of this kind $r_t^l \equiv 1$, (mod. N^t), and $r_t < N^{t-1}$:

l	$r^l \equiv 1$	mod. N^t	$r_t <$	l	$r^l \equiv 1$	mod. N^t	$r <$
3	18^3	7^3	7^2	6	19^6	7^3	7^2
3	1353^3	7^5	7^4	6	1354^6	7^5	7^4
3	82681^3	7^7	7^6	6	82682^6	7^7	7^6
3	2819^3	19^4	19^3	6	2820^6	19^4	19^3
4	1068^4	5^6	5^5	28	14^{28}	29^2	29
4	239^4	13^4	13^3	29	53^{29}	59^2	59
5	3^5	11^2	11	30	69^{30}	631^2	631
5	9^5	11^2	11	35	60^{35}	71^2	71
5	58^5	131^2	131	36	18^{36}	37^2	37
13	44^{13}	53^2	53	39	31^{39}	79^2	79
14	26^{14}	71	71	42	19^{42}	43^2	43
16	158^{16}	17^3	17^2	48	53^{48}	97^2	97
18	333^{18}	19^3	19^2	70	11^{70}	71^2	71
				486	‡10^{486}	487^2	487

the Table shows (in the right-hand column of each line) the power of the modulus (N^t) next > the root (r_t); this will be seen to be N^{t-1} in every case: no case has yet been discovered in which§ $r_t < N^{t-2}$.

10*a*. *Case of* $l = 3$ *or* 6. The cases when $l = 3$ and $l = 6$ have been printed side by side to illustrate an interesting connexion between the two. Thus ρ_t, ρ_t' being the roots of $\rho^3 \equiv 1$, (mod. N^t) and r_t, r_t' of $r^6 \equiv 1$, (mod. N^t), it has been shown (18*b*) that

$$r_t = \rho_t' + 1 \text{ and } r_t' = \rho_t + 1.$$

Hence if either root ρ_t, ρ_t' be $< N^{t-1}$, the corresponding root r_t', r_t will *also* be $< N^{t-1}$. The Table shows *four* examples of this.

The prime modulus 7 has the further peculiarity that, when $l = 3$ or 6, the above relations are satisfied for a *succession of three* of its powers, viz.

$18^3 \equiv 1$, (mod. 7^2 & 7^3), $1353^3 \equiv 1$, (mod. 7^4 & 7^5), $82681^3 \equiv 1$, (mod. 7^6 & 7^7),
$19^6 \equiv 1$, (mod. 7^2 & 7^3), $1354^6 \equiv 1$, (mod. 7^4 & 7^5), $82682^6 \equiv 1$, (mod. 7^6 & 7^7),

but this singular property does not‖ extend to the next higher powers (7^8 and 7^9).

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* This is, of course, very laborious.

† Some of these were known before, see Bickmore's 2nd paper quoted above, Art. 2; but the greater part are believed to be *new*.

‡ This case (the most interesting of all yet known) is due to Desmarest.

§ If any such exist, they are probably much rarer than those in which r is $< N^{t-1}$ only,

‖ See the detailed Tables (Case of $l = 3$ and 6), to mod. $N = 7$ following Art. 14.

10*b*. *Limits of possibility of* $r_t < N^{t-\tau}$. Although the problem of finding cases wherein $r_t < N^{t-1}, N^{t-2}, \dots, N^{t-\tau}$ respectively does not seem directly solvable, certain limits may be found outside which the solution is *impossible*. The cases of l odd or even are considered separately.

i. *l odd*, ($l \nless 3$, $t \nless 2$). The conditions are

$$\rho^l > N^t \text{ and } \rho < N^{t-\tau}, \text{ giving } \rho > N^{t \div l}, \text{ but } < N^{t-\tau},$$

where $$\frac{t}{l} < t - \tau, \text{ or } l + \frac{t}{\tau} < l \cdot \frac{t}{\tau} \quad \dots\dots\dots\dots\dots (37a).$$

Ex. 1. $\tau = 1$; then $l + t < lt$, ($l \nless 3$, $t \nless 2$), which is always possible.

Ex. 2. $\tau = 2$; then $l + \frac{1}{2}t < l \cdot \frac{1}{2}t$, ($l \nless 3$, $t \nless 3$), which is always possible, except when $l = 3 = t$.

ii. *l even* ($l \nless 4$, $t \nless 2$). The conditions are

$$r^{\frac{1}{2}l} \nless N^t + 1, \text{ and } r < N^{t-\tau}.$$

These usually involve $r \nless N^{2t \div l}$ and $< N^{t-\tau}$; whence usually,

$$\frac{2t}{l} < t - \tau, \text{ or } \tfrac{1}{2}l + \frac{t}{\tau} < \tfrac{1}{2}l \cdot \frac{t}{\tau} \quad \dots\dots\dots\dots\dots (37b).$$

Ex. 1. $\tau = 1$; then $\frac{1}{2}l + t < \frac{1}{2}l \cdot t$, ($\frac{1}{2}l \nless 2$, $t \nless 2$), which is always possible except when $\frac{1}{2}l = 2 = t$.

Ex. 2. $\tau = 2$; then $\frac{1}{2}l + \frac{1}{2}t < \frac{1}{2}l \cdot \frac{1}{2}t$, or $l + t < l \cdot t$, ($\frac{1}{2}l \nless 2$, $t \nless 3$).

When $l = 4, 6, 8$, this is possible only when $t \nless 5, 4, 3$ respectively.

Summary. When l is *odd* ($l \nless 3$), $r_t < N^{t-1}$ has no inherent impossibility; $r_t < N^{t-2}$ is impossible when $l = 3 = t$. When l is *even* ($l \nless 4$) $r_t < N^{t-1}$ is impossible when $\frac{1}{2}l = 2 = t$; $r_t < N^{t-2}$ is possible only when $t \nless 5, 4, 3$ respectively.

11. *Examples.* Here follow several examples on Art. 6–8*b*.

Ex. 1. Find the radices (r) of scales of period-length $l = 4$ for the powers of the fraction $\frac{1}{5}$: in other words, find the incongruous roots (r) of the congruences $r^4 \equiv 1$, (mod. $5, 5^2, 5^3$, &c.), each $r_t < 5^t$.

By Art. 7*a* there are *only two* incongruous roots $r_t, r_t' < N^t$ to each congruence, and $r_t + r_t' = N^t$ in each case (so that either root gives the other by simple subtraction).

Since $2^2 + 1 = 5$; therefore

$$2^4 \equiv +1, \text{ (mod. 5); hence } r_1 = 2, \qquad r_1' = 5 - 2 = 3.$$

Next, by Rule ix,

$$r_2 \equiv 2^5 \equiv 7, \text{ (mod. 25); hence } r_2 = 7, \qquad r_2' = 25 - 7 = 18.$$

Similarly,

$$r_3 \equiv 7^5 \equiv 57, \text{ (mod. 125); hence } r_3 = 57, \quad r_3' = 125 - 57 = 68,$$
$$r_4 \equiv 57^5 \equiv 182, \text{ (mod. 625); hence } r_4 = 182, \quad r_4' = 625 - 182 = 443,$$
$$r_5 \equiv 182^5 \equiv 2057, \text{ (mod. 3125); hence } r_5 = 2057, \quad r_5' = 1068.$$

Beyond this point the labor of using Rule ix, which involves the use of *fifth* powers, becomes considerable, and it becomes better to use the method of Art. 8*a*: since $l = 4$ is *even*, it suffices (Art. 8*a*) to proceed to the power $\frac{1}{2}l = 2$ only in each case.

Noting that $l = 4$, $\nu = (N-1) \div l = 1$, and using formulæ (30 & 30*a*, *b*) and taking $r_5' = 1068$ (the lesser of r_5, r_5' found above) to find m_6',

$$m_6' \equiv -2\nu r_5'.\frac{r_5'^2+1}{N^5} = -2.1.1068.\frac{1068^2+1}{3125} \equiv -2136.365 \equiv 0, \quad (\text{mod. } 5);$$

therefore, by (29), $r_6' = m_6'.N^5 + r_5' = r_5' = 1068$, and $r_6 = 5^6 - r_6' = 14557$.

Next

$$m_7' \equiv -2\nu.r_6'.\frac{r_6'^2+1}{N^6} = -2.1.1068.\frac{1068^2+1}{15625} \equiv -2136.73 \equiv -1.3 \equiv 2, (\text{mod. } 5).$$

Therefore

$$r_7' = m_7'.N^6 + r_6' = 2.15625 + 1068 = 32318, \text{ and } r_7 = 5^7 - r_7' = 45807.$$

Ex. 2. Find the radices (ρ) of scales of period-length $l = 5$ for the powers of the fraction $\frac{1}{11}$; in other words find the incongruous roots (ρ) of the congruences $r^5 \equiv 1$, (mod. 11, 11^2, 11^3, &c.), each ρ_t to be $< 11^t$.

By Rule ii the number of incongruous roots ρ_t, each $< 11^t$, is $\phi(l) = \phi(5) = 4$ in each case, (say $\rho_t, \rho_t', \rho_t'', \rho_t'''$). Then, by Rule vii any one root (ρ_t) gives the rest as

$$\rho_t' \equiv \rho_t^2, \quad \rho_t'' \equiv \rho_t^3, \quad \rho_t''' \equiv \rho_t^4, \quad (\text{mod. } N^t).$$

Since $3^5 - 1 = 2.11^2$; therefore $3^5 \equiv 1$, (mod. 11 and 11^2). Hence $\rho_1 = 3$, and $\rho_2 = 3$; and, taking the residues of the powers of ρ_1 and ρ_2 to moduli 11 and 11^2,

$$\rho_1 = 3, \ \rho_1' = 9, \ \rho_1'' = 5, \ \rho_1''' = 4, \ (\text{mod. } 11),$$

$$\rho_2 = 3, \ \rho_2' = 9, \ \rho_2'' = 27, \ \rho_2''' = 81, \ (\text{mod. } 11^2).$$

Next, by Rule ix, $\rho_3 \equiv \rho_2^N \equiv 3^{11} \equiv 124$, (mod. 11^3).

Hence $\rho_3' \equiv 124^2 \equiv 735$, $\rho_3'' \equiv 124^3 \equiv 632$, $\rho_3''' \equiv 124^4 \equiv 1170$, (mod. 11^3).

Beyond this point the use of Rule ix, which involves proceeding to the 11th powers, becomes very laborious, and it is better to use the method of Art. 8.

Here $l = 5$, $\nu = (N-1) \div l = 2$. Hence using formulæ (24, 24*a*, *b*), and taking $\rho_3 = 124$, (the least of $\rho_3, \rho_3', \rho_3'', \rho_3'''$) to find m_4,

$$m_4 \equiv \nu.\rho_3.\frac{\rho_3^l - 1}{N^3} = 2.124.\frac{124^5-1}{1331} \equiv 2.124.22025733 \equiv 2.3.4 \equiv 2, \ (\text{mod. } 11),$$

therefore, by (23),

$$\rho_4 = m_4 N^3 + \rho_3 = 2.1331 + 124 = 2786 \ ; \ \rho_4' \equiv 2786^2 \equiv 2066, \ (\text{mod. } 11^4),$$

$$\rho_4'' \equiv 2786^3 \equiv 1963 \ ; \ \rho_4''' \equiv 2786^4 \equiv 2066^2 \equiv 7825, \ (\text{mod. } 11^4).$$

Ex. 2*a*. Find the radices (r) of scales of period-length $l = 10$ for the powers of the fraction $\frac{1}{11}$; in other words find the incongruous roots (r) of the congruences $r^{10} \equiv 1$, (mod. 11, 11^2, 11^3, &c.), each r_t to be $< 11^t$.

By Rule xi, the rooots ρ_t of the congruences $\rho^5 \equiv 1$, (mod. 11^t) are *complementary* to those now required, so that the solutions given in Ex. 2 suffice to give these also (by simple subtraction $r_t = N^t - \rho_t$). The roots of the present case can also be found independently by a process precisely to that of Ex. 2. Note that since $l = 10$ is here *even*, the method of Art. 8*a* should now be used (which only requires proceeding to the 5th $= \frac{1}{2}l$th power) to find m_4, &c.

Ex. 3. Find the radices (ρ_2 and r_2) of scales giving period-length $l = 25$ and 50 for the fraction $\frac{1}{101^2}$; in other words find a root (ρ_2, r_2) of each of the congruences $\rho^{25} \equiv 1$, $r^{50} \equiv 1$, (mod. 101^2).

Jacobi's Canon Arithmeticus gives $5^{25} \equiv 1$, (mod. 101). Thus, when $l = 25$, $\nu = (N - 1) \div l = 4$, $\rho_1 = 5$. Hence, by (20),

$$m_2 \equiv \nu\rho_1 . \frac{\rho_1^{25} - 1}{101} \equiv 4.5.\frac{5^{25} - 1}{101}, \text{ (mod. 101)}.$$

One way of reducing this is to compute, see (20*b*), the *Residue* ($M_1'N$) of ($5^{25} - 1$) to the modulus 101^2, and substitute it (as explained in Art. 8) for the large quantity ($5^{25} - 1$). Thus, by the ordinary rules,

$$5^6 \equiv 5424,\ 5^{12} \equiv 92,\ 5^{24} \equiv 8464,\ 5^{25} \equiv 1515, \text{ (mod. } 101^2),$$

so that $M_1'N =$ Residue of ($5^{25} - 1$) = 1515 to modulus 101^2,

and $\qquad m_2 \equiv 4.5. \frac{1515}{101} = 20.15 = 300 \equiv 98$, (mod. 101);

therefore $\qquad \rho_2 = m_2.N + \rho_1 = 98.101 + 5 = 9903.$

And, by Rule xi, when $l = 50$, $r_2 = 101^2 - \rho_2 = 298$.

Another way of reducing m_2 is by partially factorising ($5^{25} - 1$), thus

$$m_2 \equiv 4.5.(5^5 - 1).\frac{5^{25} - 1}{101.(5^5 - 1)} = 20.3124.(251.401).9384251^*$$

$$\equiv 20.94.(49.98).38 \equiv 62.55.38 = 129580 \equiv 98, \text{ (mod. 101), as before.}$$

12. *Simple cases of* ρ_2, r_2. Here follow a number of *simple* cases of $\rho_2^{\,l} \equiv +1$, $r_2^{\frac{1}{2}l} \equiv -1$, (mod. N^2), N being a *prime*, such that the auxiliary quantity m_2 required for finding ρ_2, r_2 by Art. 8, 8*a* reduces to a *very simple* form: whilst ρ_2, r_2 are given by (19), (25) in every Case.

12*a*. Index l or $\frac{1}{2}l$ = *odd prime*. [$\nu l = N - 1 \equiv -1$, (mod. N)].

i. $N = \dfrac{\rho_1^{\,l} - 1}{\rho_1 - 1}$, where l = *odd prime*. Here $\rho_1^{\,l} \equiv +1$, (mod. N).

By (20), $m_2 \equiv \nu\rho_1(\rho_1 - 1) \equiv -\dfrac{\rho_1(\rho_1 - 1)}{l}$, (mod. N) ....(38).

i*a*. *Mersenne's Primes*. Defined by $\rho_1 = 2$; then $N = 2^l - 1$,

$$m_2 = 2\nu, \text{ (mod. } N), \quad \rho_2 = 2(\nu N + 1) \quad \text{..........(38}a).$$

ii. $N = \dfrac{1}{l}\cdot\dfrac{\rho_1^{\,l} - 1}{\rho_1 - 1}$, where l = *odd prime*. Here $\rho_1^{\,l} \equiv +1$, (mod. N).

By (20), $\qquad m_2 \equiv \nu l\rho_1(\rho_1 - 1) \equiv -\rho_1(\rho_1 - 1)$, (mod. N)............(39).

iii. $N = \dfrac{r_1^{\frac{1}{2}l} + 1}{r_1 + 1}$, where $\frac{1}{2}l$ = *odd prime*. Here $r_1^{\frac{1}{2}l} \equiv -1$, (mod. N)

By (26), $\qquad m_2 \equiv -2\nu r_1(r_1 + 1) \equiv \dfrac{2r_1(r_1 + 1)}{l}$, (mod. N)..........(40).

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* The factorisation of ($5^{25} - 1$) is taken from the Table on p. 43 of Mr. Bickmore's first paper quoted.

iv. $N = \frac{1}{\frac{1}{2}l} \cdot \frac{r_1^{\frac{1}{2}l}+1}{r_1+1}$, where $\frac{1}{2}l$ = *odd prime*. Here $r_1^{\frac{1}{2}l} \equiv -1$, (mod. N).

By (26), $\quad m_2 \equiv -\nu l r_1 (r_1+1) \equiv r_1 (r_1+1)$, (mod. N) ....(41).

12*b*. *Index l or $\frac{1}{2}l$* = 3. Here m_2 can be further reduced; thus

$l = 3$ gives i. $\quad m_2 \equiv \frac{1}{3}(2\rho_1+1)$; and ii. $\quad m_2 \equiv 2\rho_1+1$, (mod, N) ..(42),

$\frac{1}{2}l = 3$ gives iii. $\quad m_2 \equiv \frac{1}{3}(2r_1-1)$; and iv. $\quad m_2 \equiv 2r_1-1$, (mod. N)..(42*a*).

Also, taking the pair of roots ρ_1, r_1' related as as in (18*b*) of Art. 7*a*, *i.e.* so that r_1' (of iii) = $\rho_1 + 1$ (of i), and r_1' (of iv) = $\rho_1 + 1$ (of ii), it follows that

m_2 (of i) = m_2' (of iii); and m_2 (of ii) = m_2' (of iv) ......(43).

Nearly every prime N of form $(3n+1)$ falls under both i, iii, or under both ii, iv. The Tables of $l = 3$ and 6 at the end of this Paper contains the Results for all primes of this form < 101. Two larger examples are subjoined.

Ex. of ii. $\quad l = 3$: $\rho_1 = 1000$ gives $N = 333667$, whence $m_2 = 20001$.

Ex. of iv. $\quad \frac{1}{2}l = 3$: $r_1 = 5000$ gives $N = 8331667$, whence $m_2 = 9999$.

12*c*. *Examples* of i, ii, iii, iv. Here follow solutions of $\rho_2^l \equiv +1$, $r_2^{\frac{1}{2}l} \equiv -1$, (mod. N^2) when l or $\frac{1}{2}l = 5, 7, 11$, &c., including *all* (prime) values of N < 10,000. To save space the larger values of ρ_2, r_2 are not evaluated (they can be worked out by (19), (25)). The primes N marked* are *Mersenne*'s *primes* (see i*a* above).

i. $N = (\rho_1^l - 1) \div (\rho_1 - 1)$.

	$l = 5$		$l = 7$		$l = 13$	$l = 17$
$\rho_1 =$	2,	7;	2,	3;	2;	2;
$N =$	31*,	2801;	127*,	1093;	8191*;	131071*;
$m_2 =$	12,	1112;	36,	916;	1260;	1542;
$\rho_2 =$	374,	ρ_2;	4574,	ρ_2;	ρ_2;	ρ_2;

iii. $N = (r_1^{\frac{1}{2}l}+1) \div (r_1+1)$.

	$\frac{1}{2}l = 5$				$\frac{1}{2}l = 7$		$\frac{1}{2}l = 11$	$\frac{1}{2}l = 13$
$r_1 =$	2,	3,	5,	10;	2,	3;	2;	2;
$N =$	11,	61,	521,	9091;	43,	547;	683;	2731;
$m_2 =$	10,	39,	6,	22;	7,	158;	311;	1470;
$r_2 =$	112,	2382,	3131,	200012;	303,	102229;	212415;	r_2;

ii. $N = (\rho_1^l - 1) \div l(\rho_1 - 1)$.

	$l = 5$	
$\rho_1 =$	6,	11;
$N =$	311,	3221;
$m_2 =$	281,	3111;
$\rho_2 =$	87397,	ρ_2;

iv. $N = (r_1^{\frac{1}{2}l}+1) \div l(r_1+1)$.

	$\frac{1}{2}l = 5$	
$r_1 =$	4,	9;
$N =$	41,	1181;
$m_2 =$	20,	90;
$r_2 =$	824,	10629;

12*d*—v. *Index* $\frac{1}{2}l = Q = 2^q$. Primes (N) of the two forms

$$N = (r_1^Q + 1), \ (r_1 \textit{ even}); \ N = \tfrac{1}{2}(r_1^Q+1), \ (r_1 \textit{ odd}),$$

each of which give $\frac{1}{2}l = 2^q$, $r_1^{\frac{1}{2}l} \equiv -1$, (mod. N),

$$\left.\begin{aligned} m_2 &\equiv -2\nu r_1 \equiv \frac{2r_1}{l}, \text{ (mod. } N\text{), when } r_1 \text{ is even} \\ &\equiv \tfrac{1}{2}r_1, \tfrac{1}{4}r_1, \tfrac{1}{8}r_1, \tfrac{1}{16}r_1, \text{ \&c., (mod. } N\text{), when } q = 1, 2, 3, 4,\dagger \text{ \&c.} \end{aligned}\right\} \ldots(44),$$

† In practice $q \not> 4$, otherwise N runs too high.

$$\left.\begin{aligned} m_2 &\equiv -4\nu r_1 \equiv \frac{4r_1}{l}, \text{ (mod. } N\text{), when } r_1 \text{ is } odd \\ &\equiv r_1, \tfrac{1}{2}r_1, \tfrac{1}{4}r_1, \tfrac{1}{8}r_1, \text{ \&c., (mod. } N\text{), when } q = 1, 2, 3, 4,\dagger \text{ \&c.} \end{aligned}\right\} \ldots(44a).$$

va. Fermat's Primes. These are $N = 2^K + 1$, where $K = 2^k$; then $r_1 = 2^{K'}$, $r_1^Q \equiv -1$, (mod. N), where $K' = 2^{k-q}$, $Q = 2^q$;

$$\left.\begin{aligned} m_2 &\equiv \frac{r_1}{\frac{1}{2}l} \equiv 2^{Q'}, \text{ (mod. } N\text{), where } Q' = 2^{k-q} - q \\ &\equiv 2^{Q'}, \text{ or } N - 2^{Q''}, \text{ according as } 2^{k-q} >< q; \text{ here } Q'' = 2^{k-q} + 2^k - q \end{aligned}\right\} \ldots\ldots\ldots(45).$$

These primes are marked thus* in the examples which follow: the values of m_2 are all expressible in simple powers of 2, by (45); thus Taking $k = 4$, $N = 2^{16} + 1 = 65537$, (the highest Fermat's prime known)

$$m_2 \equiv 2^7, 2^2, -2^{15}, -2^{13}, \text{ (mod. } N\text{), when } q = 1, 2, 3, 4.$$

12*e*. *Examples* of v. (r_1 *even*). Here follow solutions of $r_2^{\frac{1}{2}l} \equiv -1$, (mod. N^2), when r_1 is *even*.

I. $q = 1$, $\frac{1}{2}l = 2$, $N = r_1^2 + 1$, $m_2 = \frac{1}{2}r_1$, $r_2^2 \equiv -1$, (mod. N^2),

$r_1 =$	2,	4,	6,	10,	14,	16,	20,	24,	26,	36,	40,	.. 256,
$N =$	5*,	17*,	37,	101,	197,	257*,	401,	577,	677,	1297,	1601,	..65537*,
$r_2 =$	7,	38,	117,	515,	1393,	2072,	4030,	6948,	8827,	23382,	32060,	..2^7N+r_1

II. $q=2$, $\frac{1}{2}l=4$, $N=r_1^4+1$, $m_2=\frac{1}{4}r_1$ or $\frac{1}{4}(2N+r_1)$, $r_2^4 \equiv -1$, (mod. N^2),

$r_1 =$	2,	4,	6,	16,	20,	24,	28,
$N =$	17*,	257,*	1297,	65537*,	160001,	331777,	614657,
$m_2 =$	9,	1,	650,	4,	5,	6,	7,
$r_2 =$	155,	261,	843056,	262164,	800025,	1990686,	4302627,

III. $q = 3$, $\frac{1}{2}l=8$, $N=r_1^8+1$, $m_2 \equiv \frac{1}{8}r_1$, (mod. N), $r_2^8 \equiv -1$, (mod. N^2),

$r_1 = 2$ gives $N = 257$*, $m_2 = 193$, $r_2 = 49603$;
$r_1 = 4$ gives $N = 65537$*, $m_2 = 32769$, $r_2 = m_2N + r_1$.

IV. $q = 4$, $\frac{1}{2}l=16$, $N=r_1^{16}+1$, $m_2 \equiv \frac{1}{16}r_1$, (mod. N), $r_2^{16} \equiv -1$, (mod. N^2)

$r_1 = 2$ gives $N = 65537$*, $m_2 = 57345$, $r_2 = m_2N + r_1$.

12*f*. *Examples* of v. (r_1 *odd*). Here follow solutions of $r_2^{\frac{1}{2}l} \equiv -1$, (mod. N^2), when r_1 is *odd*.

I. $q=1$, $\frac{1}{2}l=2$, $N=\frac{1}{2}(r_1^2+1)$, $m_2=r_1$, $r_2=r_1(N+1)$, $r_2^2 \equiv -1$, (mod. N^2),

$r_1 =$	3,	5,	7,	9,	11,	15,	19,	25,	29,	35,	39,	45,	49,
$N =$	5,	13,	25,	41,	61,	113,	181,	313,	421,	613,	761,	1013,	1201,
$r_2 =$	18,	70,	182,	378,	682,	1710,	3458,	7850,	12238,	21490,	29718,	45630,	58898,

II. $q=2$, $\frac{1}{2}l=4$, $N=\frac{1}{2}(r_1^4+1)$, $m_2=\frac{1}{2}(N+r_1)$, $r_2^4 \equiv -1$, (mod. N^2),

$r_1 =$	3,	5,	7,	11,	13,	17,	21,
$N =$	41,	313,	1201,	7321,	14281,	41761,	97241,
$m_2 =$	22,	159,	604,	3666,	7147,	20881,	48631,
$r_2 =$	905,	49752,	725411,	$r_2 = (m_2N + r_1)$.			

III. $q=3$, $\frac{1}{2}l=8$, $N=\frac{1}{2}(r_1^8+1)$ has no *prime* values < 417 million.

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* The primes marked thus * are Fermat's primes.

† In practice $q \ngtr 4$, otherwise N runs too high.

IV. $q=4$, $\frac{1}{2}l=16$, $r_1=3$ gives $N=\frac{1}{2}(r_1^{16}+1) = 21523361$,

$$m_2 = \tfrac{1}{8}(5N+3) = 13452101.$$

12*g*. *Examples* of v—III. *bis*. ($q = 3$, $\frac{1}{2}l = 8$). All the lower values of $\frac{1}{2}(r_1^8+1)$, and also all those of (r_1^8+1), except when $r_1 = 2$, or 4, are divisible by 17, and the quotients are all *prime*; so, taking

$$N = \tfrac{1}{17}(r_1^8+1) \text{ gives } m_2 \equiv -34\nu r_1 \equiv \tfrac{17}{8}r_1, \text{ (mod. } N\text{), when } r_1 \text{ is } even \ldots (46a),$$

$$N = \tfrac{1}{17}\cdot\frac{r_1^8+1}{2} \text{ gives } m_2 \equiv -68\nu r_1 \equiv \tfrac{17}{4}r_1, \text{ (mod. } N\text{), when } r_1 \text{ is } odd \ldots (46b);$$

When $r_1 =$ 6 , 10 ; 3 , 5 , 7 , 11 ;

$N =$ 98801, 5852353 ; 193, 11489, 169553, 6304673 ;

and m_2 is now given by (46*a*, *b*) and r_2 by (25).

13. *Simple cases of* ρ_t, r_t ($t > 2$). The facility of solution of $\rho_2^{\,l} \equiv +1$, $r_2^{\frac{1}{2}l} \equiv -1$, (mod. N^2) due to the simple forms obtained for m_2 (in Art. 12) extends, to some extent only, to the higher orders ρ_t, r_t ($t > 2$).

13*a*. *Index* l *or* $\frac{1}{2}l$ = *odd prime*. Conditions similar to those of Art. 11*a*, viz.

$$\text{i, ii.} \quad \frac{\rho_{t-1}^{l} - 1}{\rho - 1} = N^{t-1}, \text{ or } lN^{t-1}, \quad (t > 2),$$

$$\text{iii, iv.} \quad \frac{r_{t-1}^{\frac{1}{2}l} + 1}{r_{t-1} + 1} = N^{t-1}, \text{ or } \tfrac{1}{2}lN^{t-1}, \quad (t > 2),$$

would give *easy* solutions of $\rho_t^{\,l} \equiv 1$, $r_t^{\frac{1}{2}l} \equiv -1$, (mod. N^t) for they reduce the auxiliary m_t of (24), (30) to

i. $m_t \equiv \nu\rho_{t-1}(\rho_{t-1} - 1)$; ii. $m_t \equiv -\rho_{t-1}(\rho_{t-1} - 1)$, (mod. N)..(47);

iii. $m_t \equiv -2\nu r_{t-1}(r_{t-1} + 1)$; iv. $m_t \equiv r_{t-1}(r_{t-1} + 1)$, (mod. N) ..(48).

13*b*. *Index* l *or* $\frac{1}{2}l = 3$. These reduce m_t, as in Art. 12*b*, to

i. $m_t \equiv \frac{1}{3}(2\rho_{t-1} + 1)$; ii. $m_t \equiv (2\rho_{t-1} + 1)$, (mod. N) ..(49);

iii. $m_t \equiv \frac{1}{3}(2r_{t-1} - 1)$; iv. $m_t \equiv (2r_{t-1} - 1)$, (mod. N) ... (50).

Also taking the pair of roots ρ_{t-1}, r'_{t-1} of i and iii, or of ii and iv, related as in (18*b*) of Art. 7*a*. *i.e.* so that r'_{t-1} (of iii) $= \rho_{t-1} + 1$ (of i), and r'_{t-1} (of iv) $= \rho_{t-1} + 1$ of (ii), it follows that

$$m_{t-1} \text{ (of i)} = m'_{t-1} \text{ (of iii), and } m_{t-1} \text{ (of ii)} = m'_{t-1} \text{ of (iv)} \ldots (51).$$

13*c*. *Order* $t = 3$. *Index* l or $\frac{1}{2}l = 3$. Cases ii, iv here admit of *general* solution: for, denoting either ρ_2 or r_2 by r, the conditions may be written together

$$\frac{r^3 \mp 1}{r \mp 1} = r^2 \pm r + 1 = 3N^2, \text{ whence } (2N)^2 - 3\left(\frac{2r \pm 1}{3}\right)^2 = 1.$$

The solutions (N, r) of this Diophantine, all derivable from the fundamental $2^2 - 3.1^2 = 1$, are given below, (including only *prime* values of N, and) substituting ρ_2, r_2 for r; the values of ρ_1, r_1 have been derived from those of ρ_2, r_2 by (19) (25); those of m_3 are given by (49) (50), and those of ρ_3, r_3 may then be found by (21), (27). [Note that since $r_2 = \rho_2 + 1$, therefore m_3 (of ii) $= m_3$ (of iv), by (51)].

$N = 13,\ 181,\ 2521,\ 489061$, &c.; $N = 13,\ 181,\ 2521,\ 489061$, &c.
$\rho_1 = 9,\ 132,\ 1845,\ 358017$, &c.; $r_1 = 10,\ 133,\ 1846,\ 358018$, &c.;
$\rho_2 = 22,\ 313,\ 4366,\ 847078$, &c.; $r_2 = 23,\ 314,\ 4367,\ 847079$, &c.;
$m_3 = 6,\ 84,\ 1170,\ 226974$, &c.; $m_3 = 6,\ 84,\ 1170,\ 226974$, &c.!

13*d*. *Other Cases, Examples.* ($t > 2$, l or $\frac{1}{2}l$ = odd prime). Here follow examples of *other** cases.

Ex. of i. $l=5$; $\rho_2=3$ gives $\dfrac{3^5-1}{3-1}=11^2$; here $N=11$, $t=3$; $m_3=1$; $\rho_3=124$.

Ex. of ii. $l=3$; $\rho_3=18$ gives $\dfrac{18^3-1}{18-1}=7^3$; here $N=7$, $t=4$; $m_4=3$; $\rho_4=1047$.

Ex. of iii. $\frac{1}{2}l=3$; $r_3=19$ gives $\dfrac{19^3+1}{19+1}=7^3$; here $N=7$, $t=4$; $m_4=3$; $r_4=1048$.

13*e*. v. *Order* $t=3$. *Index* $\frac{1}{2}l=2$. The condition

$$\tfrac{1}{2}(r_2^2+1)=N^2, \text{ or } 2N^2-r_2^2=1$$

gives an *easy* solution of $r_3^2 \equiv -1$, (mod. N^3); for it reduces m_3 of (28) to $m_3 \equiv -4\nu r_2 \equiv r_2 \equiv r_1$, (mod. N), whence $m_3=r_1$, $r_3=r_1N^2+r_2$.. (52).

The solutions (N, r_2) of the above Diophantine, all derivable from the fundamental $2.1^2-1^2=1$, are given below, (including only *prime* values of N); the lower root r_1 has been found from (25).

$N =$	5 ;	29 ;	5741 ;	33461 ,
$r_1, r_2 =$	2, 7 ;	12, 41 ;	2378, 8119 ;	13860, 47321,
$r_3 =$	57 ;	10133 ;		$(r_1N^2+r_2)$.

v*a*. *Order* $t=5$. *Index* $\frac{1}{2}l=2$. The condition $\frac{1}{2}(r_4^2+1)=N^4$ gives an *easy* solution of $r_5^2\equiv -1$, (mod. N^5); for it reduces m_5 to $m_5=r_1$ (as above), whence $r_5=r_1N^4+r_4$.

Ex. $r_4=239$ gives $\frac{1}{2}(239^2+1)=13^4$; here $N=13$, $t=5$; whence $r_1=5$, $r_2=70$, $r_3=239=r_4$, $r_5=143044$.

14. *Tables of Congruence Solutions.* The Tables subjoined† give the *complete* set of $\phi(l)$ incongruous roots (r, r', r'', &c. ...) of the congruences

$$r_1^{\,l}\equiv 1,\ (\text{mod. } N)\ [r_1<N];\quad r_2^{\,l}\equiv 1,\ (\text{mod. } N^2)\ [r_2<N^2];$$

for *every prime* $N>3$ up to $N=101$, and for every value of $l>2$ possible to each prime, *i.e.* for every value of l of form $l=(N-1)\div\nu$; and in the case of the smaller primes, (5, 7, 11, &c.), the solutions are also given of

$$r_t^{\,l}\equiv 1,\ (\text{mod. } N^t),\ [r_t<N^t] \text{ when } t=3,\ 4,\ 5,\ \&c.$$

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* No other instances are known to the author; but it seems probable that an extended Table of factorization of $(x^3\mp 1)\div(x\mp 1)$ would give others for the index l or $\frac{1}{2}l=3$.

† Specially computed by the author for this paper: they may be looked on as a supplement to Reuschle's Tables, No. (6) in footnote of Art. 2, which give the roots (r) only for *prime* moduli (N). Considerable help has been derived from Jacobi's *Canon Arithmeticus* for the cases when $N^t<1000$.

There is a separate Table for each value of l; and the complete set of $\phi(l)$ *incongruous* roots ρ, ρ', ρ'', &c. or r, r', r'', &c. of any one congruence ($r_\iota^l \equiv 1$, mod. N^ι) are ranged in the *same horizontal row* as the modulus (N^ι); and the top line of the Table shows the particular set of powers r, r^α, r^β, &c., used under Rule vii for the set of roots.

The set of *congruous* roots ($r_1, r_2, r_3, \ldots, r_\iota$), one of each congruence of (13), all congruous to a particular root (ρ_1 or r_1) of the base-congruence $r_1^l = 1$, (mod. N) [see Rule iv] are ranged in the *same vertical column* under the particular root (ρ_1 or r_1) in question.

Also, when $l = \omega$ an *odd* number, the two Tables for $l = \omega$ and $l = 2\omega$ have been placed close together, (*i.e.* alongside when space permitted, or the latter under the former), so as to bring out prominently the property of *complementary* roots ($\rho_\iota + r_\iota = N^\iota$) explained in Rule xi.

[☞ For Index of Tables, see page 179.]

Roots (r) *of* $r^l \equiv 1$ (mod. N^ι), [$r < N^\iota$, N prime].

N^ι	$r^2 \equiv -1$ & $r^4 \equiv +1$: r	r^3
5	2	3
5^2	7	18
5^3	57	68
5^4	182	443
5^5	2057	1068
5^6	14557	1068
5^7	45807	32318
5^8	280182	110443
13	5	8
13^2	70	99
13^3	239	1958
13^4	239	28329
13^5	143044	228249
17	4	13
17^2	38	251
17^3	2928	1985
17^4	27493	56028
29	12	17
29^2	41	800
29^3	10133	14256
37	6	31
37^2	117	1252
37^3	41187	9466

N^ι	$r^2 \equiv -1$ & $r^4 \equiv +1$: r	r^3
41	9	32
41^2	378	1303
41^3	57532	11389
53	23	30
53^2	500	2309
53^3	22972	125905
61	11	50
61^2	682	3039
61^3	175569	51412
73	27	46
73^2	4553	776
73^3	153765	235252
89	34	55
89^2	3861	4060
97	22	75
97^2	5357	4052
101	10	91
101^2	515	9686
109	33	76
109^2	6137	5744
113	15	98
113^2	1710	11059

N^ι	$r^4 \equiv -1$ & $r^8 \equiv +1$: r	r^3	r^5	r^7
17	2,	8	15,	9
17^2	155,	110	134,	179
17^3	1022,	399	3891,	4514
17^4	59978,	20051	23553,	63470
41	3,	27	38,	14
41^2	905,	847	776,	834
73	10,	51	63,	22
73^2	1032,	3847	4297,	1482
89	12,	37	77,	52
89^2	6776,	927	1145,	6994
97	33,	47	64,	50
97^2	3234,	2569	6175,	6840
113	18,	69	95,	44
113^2	6346,	1990	6423,	10779
137	10,	41	127,	96
137^2	15628,	12919	3141,	5850
193	9,	150	184,	43
193^2	33398,	31223	3851,	6026
199	92,	106	107,	93
199^2	26161,	13439	13440,	26162

N^t	$\rho^3 \equiv +1$		$r^3 \equiv -1$ & $r^6 \equiv 1$	
	$\rho \equiv r'^2$	$\rho' \equiv \rho^2$	$r \equiv -\rho$	$r' \equiv r^5$
7	2,	4	5,	3
7^2	30,	18	19,	31
7^3	324,	18	19,	325
7^4	1353,	1047	1048,	1354
7^5	1353,	15453	15454,	1354
7^6	34967,	82681	82682,	34968
7^7	740861,	82681	82682,	740862
7^8	2387947,	3376853	3376854,	2387948
7^9	25447151,	14906455	14906456,	25447152
13	3,	9	10,	4
13^2	146,	22	23,	147
13^3	1160,	1036	1037,	1161
13^4	20933,	7627	7628,	20934
13^5	220860,	150432	150433,	220861
19	7,	11	12,	8
19^2	292,	68	69,	293
19^3	2819,	4039	4040,	2820
19^4	2819,	127501	127502,	2820
19^5	133140,	2342958	2342959,	133141
31	5,	25	26,	6
31^2	439,	521	522,	440
31^3	23503,	6287	6288,	23504
37	10,	26	27,	11
37^2	787,	581	582,	788
37^3	36381,	14271	14272,	36382
43	6,	36	37,	7
43^2	1425,	423	424,	1426
43^3	34707,	44799	44800,	34708
61	13,	47	48,	14
61^2	1660,	2060	2061,	1661
67	29,	37	38,	30
67^2	699,	3789	3790,	700
73	8,	64	65,	9
73^2	2198,	3130	3131,	2199
79	23,	55	56,	24
79^2	4526,	1714	1715,	4527
97	35,	61	62,	36
97^2	4788,	4620	4621,	4789

N^t	$r^6 \equiv -1$ and $r^{12} \equiv +1$			
	r	r^5	r^7	r^{11}
13	2,	6	11,	7
13^2	80,	19	89,	150
13^3	418,	1540	1779,	657
37	8,	23	29,	14
37^2	896,	356	473,	1013
61	21,	29	40,	32
61^2	936,	2103	2785,	1618
73	3,	24	70,	49
73^2	368,	4185	4961,	1144
97	6,	16	91,	81
97^2	9027,	5739	382,	3670

N^t	$r^8 \equiv -1$ and $r^{16} - +1$							
	r	r^3	r^5	r^7	r^9	r^{11}	r^{13}	r^{15}
17	3,	10,	5,	11	14,	7,	12,	6
17^2	224,	214,	158,	249	65,	75,	131,	40
17^3	802,	4260,	158,	827	4111,	653,	4755,	4086
97	8,	27,	79,	12	89,	70,	18,	85
97^2	978,	7981,	7742,	9130	8431,	1428,	1667,	279

N^t	$r^{12} \equiv -1$ and $r^{24} \equiv +1$							
	r	r^5	r^7	r^{11}	r^{13}	r^{17}	r^{19}	r^{23}
73	7,	17,	30,	52	66,	56,	43,	21
73^2	3511,	4543,	1417,	2899	1818,	786,	3912,	2430
97	4,	54,	88,	24	93,	43,	9,	73
97^2	6600,	4031,	6587,	412	2809,	5378,	2822,	8997

N^t	$r^{16} \equiv -1$ and $r^{32} \equiv +1$															
	r	r^3	r^5	r^7	r^9	r^{11}	r^{13}	r^{15}	r^{17}	r^{19}	r^{21}	r^{23}	r^{25}	r^{27}	r^{29}	r^{31}
97	19,	69,	77,	55,	67,	34,	52,	51	78,	28,	20,	42,	30,	63,	45,	46
97^2	5742,	4337,	2114,	7912,	7536,	6921,	3738,	2961	3667,	5072,	7295,	1497,	1873,	2488,	5671,	6448

N^t	$r^{24} \equiv -1$ and $r^{48} \equiv +1$															
	r	r^5	r^7	r^{11}	r^{13}	r^{17}	r^{19}	r^{23}	r^{25}	r^{29}	r^{31}	r^{35}	r^{37}	r^{41}	r^{43}	r^{47}
97	2,	32,	31,	11,	44,	25,	3,	48	95,	65,	66,	86,	53,	72,	94,	49
97^2	1651,	226,	4978,	3018,	9356,	6330,	2040,	2764	7758,	9183,	4431,	6391,	53,	3079,	7369,	6645

N^t	$r^{48} \equiv -1$ and $r^{96} \equiv +1$															
	r	r^5	r^7	r^{11}	r^{13}	r^{17}	r^{19}	r^{23}	r^{25}	r^{29}	r^{31}	r^{35}	r^{37}	r^{41}	r^{43}	r^{47}
97	5,	21,	40,	71,	29,	83,	38,	82,	13,	74,	7,	10,	56,	80,	60,	58
97^2	5340,	2252,	525,	750,	5364,	9007,	5179,	2022,	3020,	1626,	8543,	107,	9271,	1147,	6171,	2095
N^t	r^{49}	r^{53}	r^{55}	r^{59}	r^{61}	r^{65}	r^{67}	r^{71}	r^{73}	r^{77}	r^{79}	r^{83}	r^{85}	r^{89}	r^{91}	r^{95}
97	92,	76,	57,	26,	68,	14,	59,	15,	84,	23,	90,	87,	41,	17,	37,	39
97^2	4069,	7157,	8884,	8659,	4045,	402,	4230,	7387,	6389,	7783,	866,	9302,	138,	8262,	3238,	7314

N^t	$\rho^9\equiv+1$: $\rho\equiv(r''')^2$	ρ^2	ρ^4	ρ^5	ρ^7	ρ^8	$r^9\equiv-1$ and $r^{18}\equiv+1$: $r=-\rho$,	$r'\equiv-r^2$,	$r''\equiv-r^4$,	$r'''\equiv+r^5$,	$r^{iv}\equiv+r^7$,	$r^v\equiv-r^8$
19	4,	16,	9,	17,	6,	5	15,	3,	10,	2,	13,	14
19^2	99,	54,	28,	245,	234,	62	262,	307,	333,	116,	127,	299
19^3	6236,	4025,	6526,	1689,	956,	1145	623,	2834,	333,	5170,	5903,	5714
37	7,	12,	33,	9,	34,	16	30,	25,	4,	28,	3,	21
37^2	932,	678,	1069,	1045,	737,	1015	437,	691,	300,	324,	632,	354
73	2,	4,	16,	32,	55,	37	71,	69,	57,	41,	18,	36
73^2	2849,	734,	527,	3974,	1953,	621	2480,	4595,	4802,	1355,	3376,	4708

$r^{18}\equiv-1$ and $r^{36}\equiv+1$

N^t	r	r^5	r^7	r^{11}	r^{13}	r^{17}	r^{19}	r^{23}	r^{25}	r^{29}	r^{31}	r^{35}
37	2,	32,	17,	13,	15,	18	35,	5,	20,	24,	22,	19
37^2	76,	476,	424,	494,	348,	18	1293,	893,	945,	875,	1021,	1351
73	6,	38,	54,	50,	48,	12	67,	35,	19,	23,	25,	61
73^2	3948,	4710,	711,	3043,	2092,	3662	1381,	619,	4618,	2286,	3237,	1667

$r^{36}\equiv-1$ and $r^{72}\equiv+1$

N^t	r	r^5	r^7	r^{11}	r^{13}	r^{17}	r^{19}	r^{23}	r^{25}	r^{29}	r^{31}	r^{35}
73	5,	59,	15,	31,	45,	20,	62,	60,	40,	34,	47,	29
73^2	1392,	5023,	672,	4630,	2162,	3889,	1595,	2980,	770,	4195,	923,	3679

$r^{36}\equiv-1$ and $r^{72}\equiv+1$

N^t	r^{37}	r^{41}	r^{43}	r^{47}	r^{49}	r^{53}	r^{55}	r^{59}	r^{61}	r^{65}	r^{67}	r^{71}
73	68,	14,	58,	42,	28,	53,	11,	13,	33,	39,	26,	44
73^2	3937,	306,	4657,	699,	3167,	1440,	3734,	2349,	4559,	1134,	4406,	1650

N^t	$\rho^5 \equiv +1$				$r^5 \equiv -1$ and $r^{10} \equiv +1$			
	$\rho \equiv r''^2$,	$\rho' \equiv \rho^2$,	$\rho'' \equiv \rho^3$,	$\rho''' \equiv \rho^4$	$r \equiv -\rho$,	$r' \equiv -r^2$,	$r'' \equiv +r^3$,	$r''' \equiv -r^4$
11	3,	9,	5,	4	8,	2,	6,	7
11^2	3,	9,	27,	81	118,	112,	94,	40
11^3	124,	735,	632,	1170	1207,	596,	699,	161
11^4	2786,	2066,	1963,	7825	11855,	12575,	12678,	6816
31	2,	4,	8,	16	29,	27,	23,	15
31^2	374,	531,	628,	388	587,	430,	333,	573
31^3	10945,	3414,	8316,	7115	18846,	26377,	21475,	22676
41	10,	18,	16,	37	31,	23,	25,	4
41^2	51,	920,	1533,	857	1630,	761,	148,	824
41^3	55524,	9325,	26748,	46244	13397,	59596,	42173,	22677
61	9,	20,	58,	34	52,	41,	3,	27
61^2	3120,	264,	1339,	2718	601,	3457,	2382,	1003
71	5,	25,	54,	57	66,	46,	17,	14
71^2	1922,	4072,	2752,	1335	3119,	969,	2289,	3706
101	36,	84,	95,	87	65,	17,	6,	14
101^2	1854,	9780,	4943,	3824	8347,	421,	5258,	6377

N^t	$r^{10} \equiv -1$ and $r^{20} \equiv +1$							
	r	r^3	r^7	r^9	r^{11}	r^{13}	r^{17}	r^{19}
41	2,	8,	5,	20	39,	33,	36,	21
41^2	207,	787,	1194,	471	1474,	894,	487,	1210
61	8,	24,	33,	38	53,	37,	28,	23
61^2	618,	2281,	2168,	3149	3103,	1440,	1553,	572
101	32,	44,	39,	41	69,	57,	62,	60
101^2	7607,	4084,	9634,	5596	2594,	6117,	567,	4605

N^t	$r^{20} \equiv -1$ and $r^{40} \equiv +1$							
	r	r^3	r^7	r^9	r^{11}	r^{13}	r^{17}	r^{19}
41	6,	11,	29,	19,	28,	24,	26,	34
41^2	744,	913,	644,	962,	1176,	1172,	313,	1141
	r^{21}	r^{23}	r^{27}	r^{29}	r^{31}	r^{33}	r^{37}	r^{39}
41	35,	30,	12,	22,	13,	17,	15,	7
41^2	937,	768,	1037,	719,	505,	509,	1368,	540

N^t	$\rho^{15} \equiv +1$: $\rho \equiv (r'^{r})^2$,	ρ^2,	ρ^4,	ρ^7,	ρ^8,	ρ^{11},	ρ^{13},	ρ^{14}	$r^{15} \equiv -1$ and $r^{30} \equiv +1$: $r \equiv -\rho$,	$-r^2$,	$-r^4$,	$+r^7$,	$-r^8$,	$+r^{11}$,	$+r^{13}$,	$-r^{14}$
31	7,	18,	14,	28,	10,	20,	19,	9	24,	13,	17,	3,	21,	11,	12,	22
31^2	844,	235,	448,	338,	816,	547,	732,	846	117,	726,	513,	623,	145,	414,	229,	115
61	12,	22,	57,	42,	16,	15,	25,	56	49,	39,	4,	19,	45,	46,	36,	5
61^2	2696,	1303,	1033,	1079,	2883,	2028,	574,	3289	1025,	2418,	2688,	2642,	838,	1693,	3147,	432

$r^{30} \equiv -1$ and $r^{60} \equiv +1$

N^t	r	r^7	r^{11}	r^{13}	r^{17}	r^{19}	r^{23}	r^{29}	r^{31}	r^{37}	r^{41}	r^{43}	r^{47}	r^{49}	r^{53}	r^{59}
61	2,	6,	35,	18,	44,	54,	10,	30	59,	55,	26,	43,	17,	7,	51,	31
61^2	673,	3056,	2841,	1116,	2484,	1518,	498,	2958	3048,	665,	880,	2605,	1237,	2203,	3223,	763

$\rho^{25} \equiv +1$

N^t	$\rho \equiv (r^x)^2$	ρ^2	ρ^3	ρ^4	ρ^6	ρ^7	ρ^8	ρ^9	ρ^{11}	ρ^{12}	ρ^{13}	ρ^{14}	ρ^{16}	ρ^{17}	ρ^{18}	ρ^{19}	ρ^{21}	ρ^{22}	ρ^{23}	ρ^{24}
101	5,	25,	24,	19,	71,	52,	58,	88,	79,	92,	56,	78,	31,	54,	68,	37,	16,	80,	97,	81
101^2	9903,	7196,	8003,	2140,	6131,	9142,	9552,	9784,	8563,	8677,	5308,	9572,	2960,	5407,	472,	2158,	3046,	181,	7268,	6949

$r^{25} \equiv -1$ and $r^{50} \equiv +1$

N^t	$r \equiv -\rho$	$-r^2$	r^3	$-r^4$	$-r^6$	r^7	$-r^8$	r^9	r^{11}	$-r^{12}$	r^{13}	$-r^{14}$	$-r^{16}$	r^{17}	$-r^{18}$	r^{19}	r^{21}	$-r^{22}$	r^{23}	$-r^{24}$
101	96,	76,	77,	82,	30,	49,	43,	13,	22,	9,	45,	23,	70,	47,	33,	64,	85,	21,	4,	20
101^2	298,	3005,	2198,	8061,	4070,	1059,	649,	417,	1638,	1524,	4893,	629,	7241,	4794,	9729,	8243,	7155,	10020,	2933,	3252

N^t	$r^{50} \equiv -1$ and $r^{100} \equiv +1$									
	r	r^3	r^7	r^9	r^{11}	r^{13}	r^{17}	r^{19}	r^{21}	r^{23}
101	2,	8,	27,	7,	28,	11,	75,	98,	89,	53
101^2	8385,	8795,	1744,	4451,	7704,	617,	2398,	4845,	392,	7224

N^t	$r^{50} \equiv -1$ and $r^{100} \equiv +1$									
	r^{51}	r^{53}	r^{57}	r^{59}	r^{61}	r^{63}	r^{67}	r^{69}	r^{71}	r^{73}
101	99,	93,	74,	94,	73,	90,	26,	3,	12,	48
101^2	1816,	1406,	8457,	5750,	2497,	9584,	7803,	5356,	9809,	2977

$r^{50} \equiv -1$ and $r^{100} \equiv +1$										N^t
r^{27}	r^{29}	r^{31}	r^{33}	r^{37}	r^{39}	r^{41}	r^{43}	r^{47}	r^{49}	
40,	59,	34,	35,	55,	18,	72,	86,	63,	50	101
747,	7937,	539,	9933,	9953,	7088,	9667,	4732,	9860,	9746	101^2

$r^{50} \equiv -1$ and $r^{100} \equiv +1$										N^t
r^{77}	r^{79}	r^{81}	r^{83}	r^{87}	r^{89}	r^{91}	r^{93}	r^{97}	r^{99}	
61,	42,	67,	66,	46,	83,	29,	15,	38,	51	101
9454,	2264,	9662,	268,	248,	3113,	534,	5469,	341,	455	101^2

N^t	$\rho^7 \equiv +1$						$r^7 \equiv -1$ and $r^{14} \equiv +1$					
	$\rho \equiv (r''')^2$	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	$r \equiv -\rho$	$-r^2$	$+r^3$	$-r^4$	$+r^5$	$-r^6$
29	7,	20,	24,	23,	16,	25	22,	9,	5,	6,	13,	4
29^2	645,	571,	778,	574,	190,	605	196,	270,	63,	267,	651,	236
43	4,	16,	21,	41,	35,	11	39,	27,	22,	2,	8,	32
43^3	1208,	403,	537,	1546,	78,	1774	641,	1446,	1312,	303,	1771,	75
71	20,	45,	48,	37,	30,	32	51,	26,	23,	34,	41,	39
71^2	3286,	5015,	261,	676,	3296,	2588	1755,	26,	4780,	4365,	1745,	2453

N^t	$r^{14} \equiv -1$ and $r^{28} \equiv +1$											
	r	r^3	r^5	r^9	r^{11}	r^{13}	r^{15}	r^{17}	r^{19}	r^{23}	r^{25}	r^{27}
29	2,	8,	3,	19,	18,	14	27,	21,	26,	10,	11,	15
29^2	60,	704,	467,	425,	221,	14	781,	137,	374,	416,	620,	827

N^t	$\rho^{21} \equiv +1$											
	$\rho \equiv (r^{vi})^2$	ρ^2	ρ^4	ρ^5	ρ^8	ρ^{10}	ρ^{11}	ρ^{13}	ρ^{16}	ρ^{17}	ρ^{19}	ρ^{20}
43	10,	14,	24,	25,	17,	23,	15,	38,	31,	9,	40,	13
43^2	1085,	1261,	1830,	1573,	361,	367,	660,	210,	891,	1557,	1588,	1561

N^t	$r^{21} \equiv -1$ and $r^{42} \equiv +1$											
	$r \equiv -\rho$	$-r^2$	$-r^4$	$+r^5$	$-r^8$	$-r^{10}$	$+r^{11}$	$+r^{13}$	$-r^{16}$	$+r^{17}$	$+r^{19}$	$-r^{20}$
43	33,	29,	19,	18,	26,	20,	28,	5,	12,	34,	3,	30
43	764,	588,	19,	276,	1488,	1482,	1189,	1639,	958,	292,	261,	288

						$\rho^{35}\equiv+1$						
N^t	$\rho\equiv(r^{\times\prime\prime})^2$	ρ^2	ρ^3	ρ^4	ρ^6	ρ^8	ρ^9	ρ^{11}	ρ^{12}	ρ^{13}	ρ^{16}	ρ^{17}
71	2,	4,	8,	16,	64,	43,	15,	60,	49,	27,	3,	6
71^2	286,	1140,	3416,	4063,	4182,	3735,	4559,	60,	1895,	2583,	1778,	4408

						$r^{35}\equiv-1$ and $r^{70}\equiv+1$						
N^t	$r=-\rho$	$-r^2$	$+r^3$	$-r^4$	$-r^6$	$-r^8$	$+r^9$	$+r^{11}$	$-r^{12}$	$+r^{13}$	$-r^{16}$	$+r^{17}$
71	69,	67,	63,	55,	7,	28,	56,	11,	22,	44,	68,	65
71^2	4755,	3901,	1625,	978,	859,	1306,	482,	11,	3146,	2458,	3263,	633

					$\rho^{35}\equiv+1$							
ρ^{18}	ρ^{19}	ρ^{22}	ρ^{23}	ρ^{24}	ρ^{26}	ρ^{27}	ρ^{29}	ρ^{31}	ρ^{32}	ρ^{33}	ρ^{34}	N^t
12,	24,	50,	29,	58,	19,	38,	10,	40,	9,	18,	36	71
438,	4284,	121,	4360,	1833,	2646,	606,	223,	2170,	577,	3710,	2450	71^2

					$r^{35}\equiv-1$ and $r^{70}\equiv+1$							
$-r^{18}$	$+r^{19}$	$-r^{22}$	$+r^{23}$	$-r^{24}$	$-r^{26}$	$+r^{27}$	$+r^{29}$	$+r^{31}$	$-r^{32}$	$+r^{33}$	$-r^{34}$	N^t
59,	47,	21,	42,	13,	52,	33,	61,	31,	62,	53,	35	71
4603,	757,	4920,	681,	3208,	2395,	4435,	4818,	2871,	4464,	1331,	2591	71^2

N^t	$\rho^{11} \equiv +1$										$r^{11} \equiv -1$ and $r^{22} \equiv +1$									
	$\rho \equiv (r^{\nu})^2$,	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	ρ^7	ρ^8	ρ^9	ρ^{10}	$r \equiv -\rho$	$-r^2$	$+r^3$	$-r^4$	$+r^5$	$-r^6$	$+r^7$	$-r^8$	$+r^9$	$-r^{10}$
23	2,	4,	8,	16,	9,	18,	13,	3,	6,	12	21,	19,	15,	7,	14,	5,	10,	20,	17,	11
23^2	255,	487,	399,	177,	170,	501,	266,	118,	466,	334	274,	42,	130,	352,	359,	28,	263,	411,	63,	195
67	9,	14,	59,	62,	22,	64,	40,	25,	24,	15	58,	53,	8,	5,	45,	3,	27,	42,	43,	52
67^2	143,	2493,	1868,	2273,	1831,	1471,	3859,	4179,	560,	3767	4346,	1996,	2621,	2216,	2658,	3018,	630,	310,	3929,	722
89	2,	4,	8,	16,	32,	64,	39,	78,	67,	45	87,	85,	81,	73,	57,	25,	50,	11,	22,	44
89^2	1070,	4276,	4903,	2508,	6262,	7095,	3332,	790,	5674,	3694	6851,	3645,	3018,	5413,	1659,	826,	4589,	7131,	2247,	4227

N^t	$r^{22} \equiv -1$ and $r^{44} \equiv +1$																			
	r	r^3	r^5	r^7	r^9	r^{13}	r^{15}	r^{17}	r^{19}	r^{21}	r^{23}	r^{25}	r^{27}	r^{29}	r^{31}	r^{35}	r^{37}	r^{39}	r^{41}	r^{43}
89	5,	36,	10,	72,	20,	40,	21,	80,	42,	71	84,	53,	79,	17,	69,	49,	68,	9,	47,	18
89^2	7214,	2172,	3926,	3187,	2690,	2977,	3492,	1148,	5649,	605	707,	5749,	3995,	4744,	5231,	4944,	4429,	6773,	2272,	7316

N^t	$r^{44} \equiv -1$ and $r^{88} \equiv +1$																			
	r	r^3	r^5	r^7	r^9	r^{13}	r^{15}	r^{17}	r^{19}	r^{21}	r^{23}	r^{25}	r^{27}	r^{29}	r^{31}	r^{35}	r^{37}	r^{39}	r^{41}	r^{43}
89	3,	27,	65,	51,	14,	66,	60,	6,	54,	41,	13,	28,	74,	43,	31,	19,	82,	26,	56,	59
89^2	6411,	6702,	5316,	4323,	3663,	5139,	1573,	184,	2635,	842,	7667,	6436,	1765,	1556,	3858,	7495,	5867,	5455,	3171,	3352
N^t	r^{45}	r^{47}	r^{49}	r^{51}	r^{53}	r^{57}	r^{59}	r^{61}	r^{63}	r^{65}	r^{67}	r^{69}	r^{71}	r^{73}	r^{74}	r^{79}	r^{81}	r^{83}	r^{85}	r^{87}
89	86,	62,	24,	38,	75,	23,	29,	83,	35,	48,	76,	61,	15,	46,	58,	70,	7,	63,	33,	30
89^2	1510,	1219,	2605,	3598,	4258,	2782,	6348,	7737,	5286,	7079,	254,	1485,	6156,	6365,	4063,	426,	2054,	2466,	4750,	4569

$\rho^{33} \equiv +1$

N^t	$\rho\equiv(r^x)^2$	ρ^2	ρ^4	ρ^5	ρ^7	ρ^8	ρ^{10}	ρ^{13}	ρ^{14}	ρ^{16}	ρ^{17}	ρ^{19}	ρ^{20}	ρ^{23}	ρ^{25}	ρ^{26}	ρ^{28}	ρ^{29}	ρ^{31}	ρ^{32}
67	4,	16,	55,	19,	36,	10,	26,	56,	23,	33,	65,	35,	6,	49,	47,	54,	60,	39,	21,	17
67^2	875,	2495,	3271,	2632,	3922,	2154,	897,	4210,	2770,	2579,	3147,	504,	1078,	1121,	248,	1528,	1199,	3188,	4041,	3032

$r^{33} \equiv -1$ and $r^{66} \equiv +1$

N^t	$r\equiv-\rho$	$-r^2$	$-r^4$	$+r^5$	$+r^7$	$-r^8$	$-r^{10}$	$+r^{13}$	$-r^{14}$	$-r^{16}$	$+r^{17}$	$+r^{19}$	$-r^{20}$	$+r^{23}$	$+r^{25}$	$-r^{26}$	$-r^{28}$	$+r^{29}$	$+r^{31}$	$-r^{32}$
67	63,	51,	12,	48,	31,	57,	41,	11,	44,	34,	2,	32,	61,	18,	20,	13,	7,	28,	46,	50
67^2	3614,	1994,	1218,	1857,	567,	2335,	3592,	279,	1719,	1910,	1342,	3985,	3411,	3368,	4241,	2961,	3290,	1301,	448,	1457

$\rho^{41} \equiv +1$

N^t	$\rho\equiv(r^{xx})^2$	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	ρ^7	ρ^8	ρ^9	ρ^{10}	ρ^{11}	ρ^{12}	ρ^{13}	ρ^{14}	ρ^{15}	ρ^{16}	ρ^{17}	ρ^{18}	ρ^{19}	ρ^{20}
83	3,	9,	27,	81,	77,	65,	29,	4,	12,	36,	25,	75,	59,	11,	33,	16,	48,	61,	17,	51
83^2	2161,	6068,	3181,	5808,	6219,	5709,	5839,	4320,	925,	1115,	5254,	822,	5869,	260,	3851,	90,	380,	1389,	4914,	3205

$r^{41} \equiv -1$ and $r^{82} \equiv +1$

N^t	$r\equiv-\rho$	$-r^2$	$+r^3$	$-r^4$	$+r^5$	$-r^6$	$+r^7$	$-r^8$	$+r^9$	$-r^{10}$	$+r^{11}$	$-r^{12}$	$+r^{13}$	$-r^{14}$	$+r^{15}$	$-r^{16}$	$+r^{17}$	$-r^{18}$	$+r^{19}$	$-r^{20}$
83	80,	74,	56,	2,	6,	18,	54,	79,	71,	47,	58,	8,	24,	72,	50,	67,	35,	22,	66,	32
83^2	4728,	821,	3708,	1081,	670,	1180,	1050,	2569,	5964,	5774,	1635,	6067,	1020,	6629,	3038,	6790,	6509,	5500,	1975,	3684

$\rho^{41} \equiv +1$

N^t	ρ^{21}	ρ^{22}	ρ^{23}	ρ^{24}	ρ^{25}	ρ^{26}	ρ^{27}	ρ^{28}	ρ^{29}	ρ^{30}	ρ^{31}	ρ^{32}	ρ^{33}	ρ^{34}	ρ^{35}	ρ^{36}	ρ^{37}	ρ^{38}	ρ^{39}	ρ^{40}
83	70,	44,	49,	64,	26,	78,	68,	38,	31,	10,	30,	7,	21,	63,	23,	69,	41,	40,	37,	28
83^2	2560,	293,	6274,	562,	2018,	161,	3471,	5599,	2355,	5073,	2354,	2912,	3175,	6620,	4256,	401,	5436,	1451,	1116,	526

$r^{41} \equiv -1$ and $r^{82} \equiv +1$

N^t	$+r^{21}$	$-r^{22}$	$+r^{23}$	$-r^{24}$	$+r^{25}$	$-r^{26}$	$+r^{27}$	$-r^{28}$	$+r^{29}$	$-r^{30}$	$+r^{31}$	$-r^{32}$	$+r^{33}$	$-r^{34}$	$+r^{35}$	$-r^{36}$	$+r^{37}$	$-r^{38}$	$+r^{39}$	$-r^{40}$
83	13,	39,	34,	19,	57,	5,	15,	45,	52,	73,	53,	76,	62,	20,	60,	14,	42,	43,	46,	55
83_2	4329,	6596,	615,	6327,	4871,	6728,	3418,	1290,	4534,	1816,	4535,	3977,	3714,	269,	2633,	6488,	1453,	5438,	5773,	6363

N^t	$\rho^{13} \equiv +1$											
	$\rho \equiv (r^{\nu'})^2$	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	ρ^7	ρ^8	ρ^9	ρ^{10}	ρ^{11}	ρ^{12}
53	10,	47,	46,	36,	42,	49,	13,	24,	28,	15,	44,	16
53^2	2077,	2114,	311,	2686,	148,	1215,	1073,	1084,	1459,	2241,	44,	1500
79	8,	64,	38,	67,	62,	22,	18,	65,	46,	52,	21,	10
79^2	4827,	2276,	2092,	146,	5750,	1523,	5864,	2593,	3206,	3923,	1127,	4118

N^t	$r^{13} \equiv -1$ and $r^{26} \equiv +1$											
	$r \equiv -\rho$	$-r^2$	$+r^3$	$-r^4$	$+r^5$	$-r^6$	$+r^7$	$-r^8$	$+r^9$	$-r^{10}$	$+r^{11}$	$-r^{12}$
53	43,	6,	7,	17,	11,	4,	40,	29,	25,	38,	9,	37
53^2	732,	695,	2498,	123,	2661,	1594,	1736,	1725,	1350,	568,	2365,	1309
79	71,	15,	41,	12,	17,	57,	61,	14,	33,	27,	58,	69
79^2	1414,	3965,	4149,	6095,	491,	4718,	377,	3648,	3035,	2318,	5114,	2123

N^t	$r^{26} \equiv -1$ and $r^{52} \equiv +1$											
	r	r^3	r^5	r^7	r^9	r^{11}	r^{15}	r^{17}	r^{19}	r^{21}	r^{23}	r^{25}
53	2,	8,	32,	22,	35,	34,	14,	3,	12,	48,	33,	26
53^2	1009,	856,	1940,	1824,	406,	2154,	862,	851,	1443,	737,	2471,	2358

N^t	$r^{26} \equiv -1$ and $r^{52} \equiv +1$											
	r^{27}	r^{29}	r^{31}	r^{33}	r^{35}	r^{37}	r^{41}	r^{43}	r^{45}	r^{47}	r^{49}	r^{51}
53	51,	45,	21,	31,	18,	19,	39,	50,	41,	5,	20,	27
53^2	1800,	1953,	869,	985,	2403,	655,	1947,	1958,	1366,	2072,	338,	451

						$\rho^{39}\equiv+1$						
N^t	$\rho\equiv(r^{x''})^2$	ρ^2	ρ^4	ρ^5	ρ^7	ρ^8	ρ^{10}	ρ^{11}	ρ^{14}	ρ^{16}	ρ^{17}	ρ^{19}
79	2,	4,	16,	32,	49,	19,	76,	73,	31,	45,	11,	44,
79^2	3004,	5771,	2465,	3034,	3209,	3732,	5922,	2838,	31,	4153,	6094,	439,

						$r^{39}\equiv-1$ and $r^{78}\equiv+1$						
N^t	$r\equiv-\rho$	$-r^2$	$-r^4$	$+r^5$	$+r^7$	$-r^8$	$-r^{10}$	$+r^{11}$	$-r^{14}$	$-r^{16}$	$+r^{17}$	$+r^{19}$
79	77,	75,	63,	47,	30,	60,	3,	6,	48,	34,	68,	35,
79^2	3237,	470,	3776,	3207,	3032,	2509,	319,	3403,	6210,	2088,	147,	5802

					$\rho^{39}\equiv+1$							
ρ^{20}	ρ^{22}	ρ^{23}	ρ^{25}	ρ^{28}	ρ^{29}	ρ^{31}	ρ^{32}	ρ^{34}	ρ^{35}	ρ^{37}	ρ^{38}	N^t
9,	36,	72,	51,	13,	26,	25,	50,	42,	5,	20,	40	79
1905,	3354,	2442,	604,	961,	3502,	1684,	3526,	2886,	795,	810,	5491	79^2

					$r^{39}\equiv-1$ and $r^{78}\equiv+1$							
$-r^{20}$	$-r^{22}$	r^{23}	r^{25}	$-r^{28}$	r^{29}	r^{31}	$-r^{32}$	$-r^{34}$	r^{35}	r^{37}	$-r^{38}$	N^t
70,	43,	7,	28,	66,	53,	54,	29,	37,	74,	59,	39	79
4336,	2887,	3799,	5637,	5280,	2739,	4557,	2715,	3355,	5446,	5431,	750	79^2

					$\rho^{23}\equiv+1$						
N^t	$\rho\equiv(r^{x'})^2$	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	ρ^7	ρ^8	ρ^9	ρ^{10}	ρ^{11}
47	2,	4,	8,	16,	32,	17,	34,	21,	42,	37,	27
47^2	1929,	1085,	1042,	2037,	1771,	1145,	1914,	867,	230,	1870,	2142

					$r^{23}\equiv-1$ and $r^{46}\equiv+1$						
N^t	$r\equiv-\rho$	$-r^2$	r^3	$-r^4$	r^5	$-r^6$	r^7	$-r^8$	r^9	$-r^{10}$	r^{11}
47	45,	43,	39,	31,	15,	30,	13,	26,	5,	10,	20
47^2	280,	1124,	1167,	172,	438,	1064,	295,	1342,	1979,	339,	67

					$\rho^{23}\equiv+1$						
ρ^{12}	ρ^{13}	ρ^{14}	ρ^{15}	ρ^{16}	ρ^{17}	ρ^{18}	ρ^{19}	ρ^{20}	ρ^{21}	ρ^{22}	N^t
7,	14,	28,	9,	18,	36,	25,	3,	6,	12,	24	47
1088,	202,	874,	479,	629,	600,	2093,	1554,	53,	623,	71	47^2

				$\rho^{23}\equiv-1$ and $r^{46}\equiv+1$							
$-r^{12}$	r^{13}	$-r^{14}$	r^{15}	$-r^{16}$	r^{17}	$-r^{18}$	r^{19}	$-r^{20}$	r^{21}	$-r^{22}$	N^t
40,	33,	19,	38,	29,	11,	22,	44,	41,	35,	23	47
1121,	2007,	1335,	1730,	1580,	1609,	116,	655,	2156,	1586,	2138	47^2

$\rho^{29} \equiv +1$

N^t	$\rho \equiv (r^{x/\nu})^2$	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	ρ^7	ρ^8	ρ^9	ρ^{10}	ρ^{11}	ρ^{12}	ρ^{13}	ρ^{14}
59	3,	9,	27,	22,	7,	21,	4,	12,	36,	49,	29,	28,	25,	16
59^2	298,	1779,	1030,	612,	1364,	2676,	299,	2077,	2809,	1642,	1976,	559,	2975,	2376

$r^{29} \equiv -1$ and $r^{58} \equiv +1$

N^t	$r \equiv -\rho$	$-r^2$	$+r^3$	$-r^4$	$+r^5$	$-r^6$	$+r^7$	$-r^8$	$+r^9$	$-r^{10}$	$+r^{11}$	$-r^{12}$	$+r^{13}$	$-r^{14}$
59	56,	50,	32,	37,	52,	38,	55,	47,	23,	10,	30,	31,	34,	43
59^2	3183,	1702,	2451,	2869,	2117,	805,	3182,	1404,	672,	1839,	1505,	2922,	506,	1105

$\rho^{29} \equiv +1$

ρ^{15}	ρ^{16}	ρ^{17}	ρ^{18}	ρ^{19}	ρ^{20}	ρ^{21}	ρ^{22}	ρ^{23}	ρ^{24}	ρ^{25}	ρ^{26}	ρ^{27}	ρ^{28}	N^t
48,	26,	19,	57,	53,	41,	5,	15,	45,	17,	51,	35,	46,	20	59
1405,	970,	137,	2535,	53,	1870,	300,	2375,	1107,	2672,	2588,	1923,	2170,	2675	59^2

$r^{29} \equiv -1$ and $r^{58} \equiv +1$

$+r^{15}$,	$-r^{16}$	$+r^{17}$	$-r^{18}$	$+r^{19}$	$-r^{20}$	$+r^{21}$	$-r^{22}$	$+r^{23}$	$-r^{24}$	$+r^{25}$	$-r^{26}$	$+r^{27}$	$-r^{28}$	N^t
11,	33,	40,	2,	6,	18,	54,	44,	14,	42,	8,	24,	13,	39	59
2076,	2511,	3344,	946,	3428,	1611,	3181,	1106,	2374,	809,	893,	1558,	1311,	806	59^2

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PROOF OF A FUNDAMENTAL FACT AS TO FUNCTIONS OF DIFFERENCES.

By Prof. E. B. ELLIOTT.

1. GIVEN a function of p quantities $\alpha_1, \alpha_2, \ldots, \alpha_p$, it is of course a function of their differences if, and only if, it is annihilated by $\frac{\partial}{\partial\alpha_1} + \frac{\partial}{\partial\alpha_2} + \ldots + \frac{\partial}{\partial\alpha_p}$.

Given a rational integral function of dimensions w throughout—say of weight w—in $\alpha_1, \alpha_2, \ldots, \alpha_p$, which is a function of their differences, it can quite readily be expressed as a rational integral function of the differences. Such an expression is, for instance, effected by diminishing all the quantities by one of them, *e.g.* by writing $0, \alpha_2 - \alpha_1, \alpha_3 - \alpha_1, \ldots, \alpha_p - \alpha_1$ in place of $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_p$. Thus, for example, the function of differences of three things

$$\beta^2 - \gamma^2 - 2\alpha\beta + 2\alpha\gamma$$

may be written

$$(\beta - \alpha)^2 - (\gamma - \alpha)^2.$$

In fact the excesses of any $p-1$ of the p quantities over the p^{th} constitute a complete irreducible system of functions of differences.

But an expression thus written down is generally a linear function of products of differences, into which products, taken singly, a particular root enters to a higher degree than in the expanded function. Thus, in the example taken, α occurs only to the first degree in the function, but to the second degree in the parts $(\beta - \alpha)^2$, $(\gamma - \alpha)^2$.

It is, however, a fact of fundamental importance, too often assumed tacitly,* that the function can always be rationally integrally expressed in terms of the whole system of $\frac{1}{2}p(p-1)$ differences of $\alpha_1, \alpha_2, \ldots, \alpha_p$, without introducing terms which cancel against one another in the expanded sum and are of higher degree in any one of $\alpha_1, \alpha_2, \ldots, \alpha_p$ than occur in the sum itself. For instance, the function in the example may be written

$$(\beta - \gamma)(\beta - \alpha) + (\beta - \gamma)(\gamma - \alpha).$$

* As for instance for the case of symmetric functions in my "Algebra of Quartics," § 75. One way to remedy the defect there is afforded by the present note. Another, and perhaps more natural, one is to introduce at that stage in the work a proof, *e.g.* Clebsch's, of the fact that the determinants of a number of linear forms constitute the irreducible invariant system of those forms, which I have only demonstrated much later in the book (§256).

The precise theorem to be proved is as follows:—"A rational integral function of weight w, and of partial degrees $i_1, i_2, \ldots, i_p$ respectively in $\alpha_1, \alpha_2, \ldots, \alpha_p$, which is a function of the differences of these letters, can be expressed as a sum of numerical multiples of products of w differences in such a way that, in every product, α_1 occurs in not more than i_1 factors, α_2 in not more than $i_2, \ldots$, and α_p in not more than i_p."

2. We need the lemma that $2w$ cannot exceed

$$i_1 + i_2 + \ldots + i_p.$$

If the function under consideration at any stage have $\alpha_1 - \alpha_2$ for a factor, we may remove that factor as often as it occurs, before commencing the argument as to that function. For to remove $(\alpha_1 - \alpha_2)^\lambda$ diminishes each of w, i_1, i_2 by λ without altering i_3, i_4, &c., and so leaves unaltered the difference

$$i_1 + i_2 + \ldots + i_p - 2w.$$

We may suppose then that we are dealing with a function which does not vanish when we put α_2 for α_1.

Make this substitution in the function of characteristics $w;\ i_1, i_2, \ldots, i_p$. A function of the differences of $\alpha_2, \alpha_3, \ldots, \alpha_p$ results. It is rational and integral; its weight is w; its partial degrees in $\alpha_2, \alpha_3, \ldots, \alpha_p$ are at most $i_1 + i_2, i_3, \ldots, i_p$ respectively. Now suppose that, if possible,

$$2w > i_1 + i_2 + i_3 + \ldots + i_p,$$

i.e.
$$> (i_1 + i_2) + i_3 + \ldots + i_p.$$

It would follow that for the function of differences of $p - 1$ letters which we have derived

$$2w > \text{sum of partial degrees.}$$

By repetition of the argument the same conclusion could in succession be drawn as to certain derived functions of differences of $p - 2, p - 3, \ldots 2$ letters.

Now two things, α_{p-1}, α_p, have only one difference, an isobaric function of which, rational and integral in α_{p-1}, α_p, is necessarily of the form $k(\alpha_{p-1} - \alpha_p)^w$, for which

$$2w = \text{sum of partial degrees.}$$

Thus we have supposed an impossibility; and we have always

$$2w \leqq i_1 + i_2 + \ldots + i_p.$$

3. Now let a function of the differences of characteristics $w;\ i_1, i_2, \ldots, i_p$ be arranged by powers of α_1 and written

$$\alpha_1^{i_1}P + \alpha_1^{i_1-1}Q + \ldots + Z.$$

(We no longer have to pay special attention to cases where a difference is a factor.) Here P is a function of the differences of $\alpha_2, \alpha_3, \ldots, \alpha_p$. For, as $\frac{\partial}{\partial\alpha_1} + \frac{\partial}{\partial\alpha_2} + \ldots + \frac{\partial}{\partial\alpha_p}$ annihilates the sum, the fact that the coefficient of $\alpha_1^{i_1}$ in the result of operating must vanish tells us that $\frac{\partial}{\partial\alpha_2} + \ldots + \frac{\partial}{\partial\alpha_p}$ annihilates P. Moreover P is rational and integral, is of weight $w - i_1$, and of degrees which do not exceed $i_2, i_3, \ldots, i_p$ respectively.

We proceed to prove that if the theorem stated at the end of § 1 holds for $p-1$ letters it must hold for p letters. If it hold for $p-1$ letters, P can be arranged as a sum of parts like

$$\lambda\,(\alpha_2 - \alpha_3)^{a_{23}}\,(\alpha_2 - \alpha_4)^{a_{24}} \ldots (\alpha_{p-1} - \alpha_p)^{a_{p-1,p}},$$

where $\qquad a_{23} + a_{24} + \ldots + a_{2p} \leqq i_2' = i_2 - n_2$, say,

$$a_{23} + a_{34} + \ldots + a_{3p} \leqq i_3' = i_3 - n_3, \text{ say},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$a_{2p} + a_{3p} + \ldots + a_{p-1,p} \leqq i_p' = i_p - n_p, \text{ say}.$$

By addition of these we obtain

$$2\,(w - i_1) = i_2' + i_3' + \ldots + i_p' - (n_2 + n_3 + \ldots + n_p);$$

so that $n_2 + n_3 + \ldots + n_p = i_1 + i_2 + i_3 + \ldots + i_p - 2w + i_1,$

$$\geqq i_1,$$

by the lemma.

Hence there are numbers (or some zero) $\nu_2, \nu_3, \ldots, \nu_p$, not greater than $n_2, n_3, \ldots, n_p$ respectively, such that

$$\nu_2 + \nu_3 + \ldots + \nu_p = i_1.$$

Write down then

$$(\alpha_1 - \alpha_2)^{\nu_2}\,(\alpha_1 - \alpha_3)^{\nu_3} \ldots (\alpha_1 - \alpha_p)^{\nu_p}.$$
$$\times\,\lambda\,(\alpha_2 - \alpha_3)^{a_{23}}\,(\alpha_2 - \alpha_4)^{a_{24}} \ldots (\alpha_{p-1} - \alpha_p)^{a_{p-1,p}}.$$

This is λ times a product of w differences into which α_1 enters in i_1 factors, and $\alpha_2, \alpha_3, \ldots, \alpha_p$ in numbers of factors not exceeding $i_2, i_3, \ldots, i_p$ respectively.

Form in like manner products with the same property from all the other parts of which $\alpha_1^{i_1}P$ consists when P is replaced according to our assumption, and add. The result is a linear function of products of the kind described in our enunciation, and the highest terms in α_1 which it contains are $\alpha_1^{i_1}P$. By subtraction we have

Given function—(sum of products as described)

$$= \alpha_1^{i_1-1}P' + \alpha_1^{i_1-2}Q' + \ldots + Z',$$

where it may happen that some or all of P', Q', ..., Z' vanish. If they do not it is a function of the differences, is rational and integral, of weight w, and of degree in α_1 at least one less than the given function.

Repeat the argument with this lower degree in place of i_1. After at most i_1 applications of it we obtain

$$\text{Given function} = \text{sum of products as described} + Z_r,$$

where Z_r, if it do not vanish, is rational, integral, of weight w, and a function of differences of α_2, α_3, ..., α_p only, and is consequently, by present assumption, itself a sum of products as described. Accordingly on this assumption that the theorem holds for $p-1$ letters, it holds for p.

Now it holds for two letters α_{p-1}, α_p, as $(\alpha_{p-1}-\alpha_p)^w$, the only possible rational integral function of weight w of this one difference, is of the form described.

The mathematical induction is then at once completed; and the certainty that we can in all cases express a function of differences, as given, in the form of a linear function of products of differences as described, is established.

4. When in the given function $i_1 = i_2 = \ldots = i_p \,(= i \text{ say})$, the function can be expressed as a linear function of products of differences in none of which any α occurs in more than i factors. If it is a symmetric function we can, by addition of all the $p!$ results of permuting α_1, α_2, ..., α_p in the linear function of products, and dividing by $p!$, express it as a linear function of symmetric sums of such products. This represents the facts as to seminvariants of weight w and degree i, divided by a_0^i.

When $$i_1 = i_2 = \ldots = i_p \,(= i) = \frac{2w}{p},$$

the whole number of factors in each product of differences is $\frac{1}{2}ip$. Let, now, in any one of the products, i_1', i_2', ..., i_p' be the numbers of factors in which occur α_1, α_2, ..., α_p respectively.

None of i_1', i_2', ..., i_p' exceeds i. Also, as each factor involves two a's, the whole number of factors is

$$\tfrac{1}{2}(i_1' + i_2' + \ldots + i_p').$$

This has to be equal to $\frac{1}{2}ip$. Hence the p numbers i_1', i_2', ..., i_p', none greater than i, must all be equal to i. The function then can be written as a linear function of products, in each of which every a occurs in exactly i factors. If symmetric it can, as above, be further written as a linear function of symmetric sums of such products. This represents the facts as to invariants of degree i, divided by a_0^i.

AN ALGEBRAIC IDENTITY WITH TWO GEOMETRICAL APPLICATIONS.

By *T. J. I'a. Bromwich*, St. John's College, Cambridge.

THE algebraical theorem contained in this note is very simple, but does not appear to occur very frequently in elementary mathematics; and I have not succeeded in finding any form of it in the current text-books. It seems probable, however, that the theorem may be contained in some of Prof. Cayley's papers which I have not had leisure to examine. I have attempted to deduce more general forms of it, but hitherto without success.

I was originally led to the theorem in the course of solving the two problems subjoined; I think they are not without interest on their own account, and they serve to illustrate the application of the algebraical identity.

1. *An algebraic identity.*

Suppose that two sets of variables (xyz), $(x_0y_0z_0)$ are related by the identities

$$S \equiv (abcfgh\between xyz)^2 = 0,$$

$$T \equiv (abcfgh\between xyz\between x_0y_0z_0) = 0,$$

to prove that

$$\frac{ax+hy+gz}{yz_0-y_0z} = \frac{hx+by+fz}{zx_0-z_0x} = \frac{gx+fy+cz}{xy_0-x_0y} = \left(-\frac{\Delta}{S_0}\right)^{\frac{1}{2}},$$

where Δ is the discriminant of S, or is

$$abc + 2fgh - af^2 - bg^2 - ch^2,$$

and S_0 is the value of S when (xyz) are replaced by $(x_0y_0z_0)$.

Write for brevity,

$$X = ax + hy + gz,$$
$$Y = hx + by + fz,$$
$$Z = gx + fy + cz,$$

then $T = 0$ gives us

$$Xx_0 + Yy_0 + Zz_0 = 0,$$

also

$$Xx + Yy + Zz = S.$$

Solve first on the hypothesis that S is not zero, then

$$\frac{X}{yz_0 - y_0 z} = \frac{Y}{zx_0 - z_0 x} = \frac{Z - S/z}{xy_0 - x_0 y} = \theta \text{ say.}$$

But we have the well-known identity,

$$SS_0 - T^2 = (ABCFGH \between yz_0 - y_0 z,\ zx_0 - z_0 x,\ xy_0 - x_0 y)^2,$$

where we use the usual notation

$$(ABCFGH) = (bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch).$$

Substituting for the values of $yz_0 - y_0 z$, $zx_0 - z_0 x$, $xy_0 - x_0 y$, we now have, since $T = 0$,

$$\theta^2 SS_0 = (ABCFGH \between X,\ Y,\ Z - S/z)^2$$
$$= (ABCFGH \between XYZ)^2 - 2(S/z)(GX + FY + CZ) + C(S/z)^2,$$

and

$$(ABCFGH \between XYZ)^2 = S\Delta,$$
$$GX + FY + CZ = z\Delta.$$

Thus

$$\theta^2 S_0 = -\Delta + CS/z^2.$$

Now let us take $S = 0$, and we have, under the assumed conditions,

$$\frac{X}{yz_0 - y_0 z} = \frac{Y}{zx_0 - z_0 y} = \frac{Z}{xy_0 - x_0 y} = \left(-\frac{\Delta}{S_0}\right)^{\frac{1}{2}}.$$

An alternative proof is the following:—Put

$$x_0 = \beta Z - \gamma Y,$$
$$y_0 = \gamma X - \alpha Z,$$
$$z_0 = \alpha Y - \beta X,$$

which we may do, since $Xx_0 + Yy_0 + Zz_0 = 0$.

Then $S_0 = (abcfgh \between \beta Z - \gamma Y, \gamma X - \alpha Z, \alpha Y - \beta X)^2$,

and, using the same general identity as before,

$$(ABCFGH \between \alpha\beta\gamma)^2 . (ABCFGH \between XYZ)^2 - [(ABCFGH \between \alpha\beta\gamma \between XYZ)]^2$$
$$= \Delta\,(abcfgh \between \beta Z - \gamma Y,\ \gamma X - \alpha Z,\ \alpha Y - \beta X)^2,$$

for $BC - F^2 = a\Delta$, $GH - AF = f\Delta$, and so forth, by the properties of adjoint determinants.

Here $\qquad (ABCFGH \between XYZ)^2 = S\Delta = 0,$

and $(ABCFGH \between \alpha\beta\gamma \between XYZ) = \alpha\,(AX + HY + GZ)$
$$+ \beta\,(HX + BY + FZ) + \gamma\,(GX + FY + CZ)$$
$$= (\alpha x + \beta y + \gamma z)\,\Delta.$$

Hence $\qquad S_0 = -\,\Delta\,(\alpha x + \beta y + \gamma z)^2,$

and $\quad yz_0 - y_0 z = \alpha\,(xX + yY + zZ) - X\,(\alpha x + \beta y + \gamma z)$
$$= -X(\alpha x + \beta y + \gamma z), \text{ since } S = 0;$$

thus $$\frac{X}{yz_0 - y_0 z} = \frac{Y}{zx_0 - z_0 x} = \frac{Z}{xy_0 - x_0 y} = \left(-\frac{\Delta}{S_0}\right)^{\frac{1}{2}}.$$

2. *Application to find the radius of curvature in homogeneous coordinates, the curve being given by a tangential equation.*

We consider the curve as the envelope of the line $lx + my + nz = 0$, where $f(lmn) = 0$. It appears simplest to find first the curvature of the envelope, for the case when l, m, n are regarded as known functions of an independent variable t.

The coordinates x, y, z are considered as constant multiples of areal coordinates defined by a fundamental triangle, the sides and angles of which will be denoted by a, b, c, A, B, C, the area by ∇. The arbitrary constants contained in x, y, z will be known when the equation to the line at infinity is known: I take this equation to be

$$I \equiv px + qy + rz = 0,$$

so that the ratios of the areal coordinates are those of

$$px : qy : rz.$$

For brevity, write

$$\Omega\,(lmn) = a^2 l^2/p^2 + \ldots - 2bc \cos A\ mn/qr - \ldots,$$
$$\Phi\,(lmnl'm'n') = a^2 ll'/p^2 + \ldots - bc \cos A\ (mn' + m'n)/qr - \ldots,$$

and these will be often still further abbreviated to Ω, Φ. Then $\Omega=0$ is the condition that the line $lx+my+nz=0$ should pass through one of the circular points at infinity, and $\Phi=0$ is the condition that the lines

$$lx+my+nz=0, \quad l'x+m'y+n'z=0$$

should be at right angles.

It is known that θ is the angle between the lines (lmn), $(l'm'n')$, where

$$\frac{pqr\Phi}{2\nabla}\tan\theta = \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ p, & q, & r \end{vmatrix}.$$

Further δ, the distance between (xyz), $(x'y'z')$, is given by

$$\delta^2=(pqr/II')^2\,\Omega\,(yz'-y'z,\ \ zx'-z'x,\ \ xy'-x'y).$$

Results from which these can be at once deduced are to be found in a paper by W. K. Clifford (*Quarterly Journal of Mathematics*, Vol. VII., pp. 54–67, "*Analytical Metrics*," *cf. Coll. Works*, XI. and XVI.).

Having stated these results for convenience of reference I proceed to discuss the problem in hand. The point of contact of the line (lmn) with its envelope will be given by

$$\frac{x}{m\dot{n}-\dot{m}n}=\frac{y}{n\dot{l}-\dot{n}l}=\frac{z}{l\dot{m}-\dot{l}m}=k,$$

where dots indicate differentiation with respect to t. Further, we may take k to be unity without loss of generality.

Then the distance between two points on the envelope is δ, where

$$\delta^2=(pqr/I^2)\,(dt)^2\,\Omega\,(y\dot{z}-\dot{y}z,\ \ z\dot{x}-\dot{z}x,\ \ x\dot{y}-\dot{x}y).$$

We have put $I'=I$ in the denominator, to avoid retaining higher powers of dt. Now we have, on substitution,

$$y\dot{z}-\dot{y}z=lD,\quad z\dot{x}-\dot{z}x=mD,\quad x\dot{y}-\dot{x}y=nD,$$

where

$$D\equiv\begin{vmatrix} l, & m, & n \\ \dot{l}, & \dot{m}, & \dot{n} \\ \ddot{l}, & \ddot{m}, & \ddot{n} \end{vmatrix}.$$

Thus $$\delta=dt\,(pqr/I^2)\,D\Omega^{\frac{1}{2}}.$$

Similarly the angle between the corresponding tangents is given by

$$\theta = \frac{2\nabla . dt}{pqr\Omega} \begin{vmatrix} l, & m, & n \\ \dot{l}, & \dot{m}, & \dot{n} \\ p, & q, & r \end{vmatrix}$$

$$= 2\nabla . dt\,(I/pqr\Omega),$$

for $I = px + qy + rz = p(m\dot{n} - \dot{m}n) + q(n\dot{l} - \dot{n}l) + r(l\dot{m} - \dot{l}m)$.

Hence $\quad \rho = \delta/\theta = (p^2q^2r^2/I^3)\,\Omega^{\frac{3}{2}}\,D/2\nabla,$

or written out in full

$$\rho = \frac{p^2q^2r^2}{2\nabla}\left(\frac{a^2l^2}{p^2} + \ldots - 2bc\cos A\frac{mn}{qr} - \ldots\right)^{\frac{3}{2}} \begin{vmatrix} l, & m, & n \\ \dot{l}, & \dot{m}, & \dot{n} \\ \ddot{l}, & \ddot{m}, & \ddot{n} \end{vmatrix} \Bigg/ \begin{vmatrix} l, & m, & n \\ \dot{l}, & \dot{m}, & \dot{n} \\ p, & q, & r \end{vmatrix}.$$

If we put $p = 1/b$, $q = 1/a$, $r = 1$, $C = \omega$, and then make both a, b infinite, we shall have the corresponding result in Cartesians,

$$\rho = (\dot{l}^2 - 2\dot{l}\dot{m}\cos\omega + \dot{m}^2)^{\frac{3}{2}}\,D/(l\dot{m} - \dot{l}m)^3\sin\omega.$$

Passing now to the case when l, m, n are connected by the functional relation $f(lmn) = 0$, we find an application of the identity proved in paragraph (1).

Differentiating the relation $f = 0$, we have

$$\dot{l}f_1 + \dot{m}f_2 + \dot{n}f_3 = 0,$$

where, for brevity,

$$f_1 = \frac{\partial f}{\partial l},\quad f_2 = \frac{\partial f}{\partial m},\quad f_3 = \frac{\partial f}{\partial n}.$$

By Euler's theorem on homogeneous functions,

$$lf_1 + mf_2 + nf_3 = 0.$$

Hence $\quad \dfrac{f_1}{m\dot{n} - \dot{m}n} = \dfrac{f_2}{n\dot{l} - \dot{n}l} = \dfrac{f_3}{l\dot{m} - \dot{l}m} = \dfrac{pf_1 + qf_2 + rf_3}{I}.$

Differentiating again with respect to t,

$$\ddot{l}f_1 + \ddot{m}f_2 + \ddot{n}f_3 + (\dot{l}^2f_{11} + \ldots + 2\dot{m}\dot{n}f_{23} + \ldots) = 0,$$

where $\quad f_{11} = \dfrac{\partial^2 f}{\partial l^2},\quad f_{23} = \dfrac{\partial^2 f}{\partial m\partial n},$ and so forth.

Substituting for f_1, f_2, f_3 in the last equation, we shall have

$$(pf_1 + qf_2 + rf_3)\, D/I = -(\dot{l}^2 f_{11} + \ldots + 2\dot{m}\dot{n} f_{23} + \ldots).$$

Suppose now that the degree of $f(lmn)$ is s, then f_1, f_2, f_3 are each of degree $s-1$; hence

$$(s-1)\,f_1 = lf_{11} + mf_{12} + nf_{13},$$
$$(s-1)\,f_2 = lf_{12} + mf_{22} + nf_{23},$$
$$(s-1)\,f_3 = lf_{13} + mf_{23} + nf_{33}.$$

Thus the two relations

$$lf_1 + mf_2 + nf_3 = 0,$$
$$\dot{l}f_1 + \dot{m}f_2 + \dot{n}f_3 = 0$$

will become

$$(f_{11}, f_{22}, f_{33}, f_{23}, f_{31}, f_{12} \between lmn)^2 = 0,$$
$$(f_{11}, f_{22}, f_{33}, f_{23}, f_{31}, f_{12} \between lmn \between \dot{l}\dot{m}\dot{n}) = 0.$$

Applying the theorem of the first paragraph, we now find

$$\frac{lf_{11} + mf_{12} + nf_{13}}{m\dot{n} - \dot{m}n} = \frac{lf_{12} + mf_{22} + nf_{23}}{n\dot{l} - \dot{n}l} = \frac{lf_{13} + mf_{23} + nf_{33}}{l\dot{m} - \dot{l}m} = \left(-\frac{\Delta}{S_0}\right)^{\frac{1}{2}},$$

where $\qquad S_0 = (f_{11}, f_{22}, f_{33}, f_{23}, f_{31}, f_{12} \between \dot{l}\dot{m}\dot{n})^2,$

or $\qquad S_0 = -(pf_1 + qf_2 + rf_3)\, D/I.$

From the equalities just proved and the values of f_1, f_2, f_3,

$$(-\Delta/S_0)^{\frac{1}{2}} = (s-1)\,(pf_1 + qf_2 + rf_3)/I,$$

hence

$$(s-1)^2\,(pf_1 + qf_2 + rf_3)^3\, D = I^3 \Delta.$$

Thus the expression for ρ gives

$$\rho = (p^2q^2r^2)\,\Omega^{\frac{3}{2}}\,\Delta/2\,(s-1)^2\,(pf_1 + qf_2 + rf_3)^3\,\nabla.$$

Written in full, this is

$$\rho = \frac{p^2q^2r^2}{2\nabla\,(s-1)^2}\left(\frac{a^2l^2}{p^2} + \ldots - 2bc\cos A\,\frac{mn}{qr} - \ldots\right)^{\frac{3}{2}}$$
$$\times \begin{vmatrix} f_{11}, & f_{12}, & f_{13} \\ f_{12}, & f_{22}, & f_{23} \\ f_{13}, & f_{23}, & f_{33} \end{vmatrix} \Big/ (pf_1 + qf_2 + rf_3)^3,$$

and in Cartesians this will reduce to

$$\rho = \frac{(l^2 - 2lm\cos\omega + m^2)^{\frac{3}{2}}}{(s-1)^2 f_3^{\,3}\sin\omega}\begin{vmatrix} f_{11}, & f_{12}, & f_{13} \\ f_{12}, & f_{22}, & f_{23} \\ f_{13}, & f_{23}, & f_{33} \end{vmatrix}.$$

3. *Conditions that a line may be a generator of the general central quadric.*

The quadric is taken to be $(abcfgh\between xyz)^2 + d = 0$, and the line is supposed given by its coordinates $(lmnl'm'n')$. If $(x_0y_0z_0)$ be any point on the line, the conditions are found by expressing that the point $(x_0 + lr,\ y_0 + mr,\ z_0 + nr)$ is on the surface for all values of r. The conditions are seen to be

$$(abcfgh\between lmn)^2 = 0,$$

$$(abcfgh\between lmn\between x_0y_0z_0) = 0,$$

$$(abcfgh\between x_0y_0z_0)^2 + d = 0.$$

Again applying the theorem of the first paragraph,

$$\frac{al+hm+gn}{ny_0 - mz_0} = \frac{hl+bm+fn}{lz_0 - nx_0} = \frac{gl+fm+cn}{mx_0 - ly_0} = \left(-\frac{\Delta}{S_0}\right)^{\frac{1}{2}},$$

here $\qquad S_0 = -d$ and $l' = ny_0 - mz_0,$

$$m' = lz_0 - nx_0,$$

$$n' = mx_0 - ly_0.$$

Thus our conditions are

$$al + hm + gn = l'(\Delta/d)^{\frac{1}{2}},$$

$$hl + bm + fn = m'(\Delta/d)^{\frac{1}{2}},$$

$$gl + fm + cn = n'(\Delta/d)^{\frac{1}{2}}.$$

These equations were given by me in a College problem paper in 1897; and similar conditions for the quadric referred to any origin will be found in the *Math. Trip. Part I.* papers for 1898. The latter set can be deduced from those above by transferring the origin to the centre and substituting the modified values of $(l'm'n')$. But the best form of the conditions is one due to Mr. J. H. Grace (of Peterhouse); as follows:—Let A, B be any two lines conjugate with respect to the quadric and let G be a generator; then the ratio of the mutual moments (A, G) and (B, G) is the square root of the complete discriminant of the quadric.

This statement includes all the conditions above by taking three positions of A through the centre parallel to the coordinate axes; the corresponding positions of B will be the lines at infinity in the planes

$$ax + hy + gz = 0, \quad hx + by + fz = 0, \quad gx + fy + cz = 0.$$

NOTE ON RECIPROCATION.

By *R W. H. T. Hudson*, St. John's College, Cambridge.

LET the line joining the points (x, y, z, t), (x', y', z', t') have coordinates

$$l = xt' - x't, \quad m = yt' - y't, \quad n = zt' - z't,$$
$$l' = yz' - y'z, \quad m' = zx' - z'x, \quad n' = xy' - x'y.$$

Make the linear transformation

$$\begin{aligned} x_1 &= p't - qz + ry, \\ y_1 &= q't - rx + pz, \\ z_1 &= r't - py + qx, \\ t_1 &= -p'x - q'y - r'z. \end{aligned}$$

Then

$$\begin{aligned} x_1t_1' - x_1't_1 &= p'(p'l + q'm + r'n + pl' + qm' + rn') \\ &\qquad - l'(pp' + qq' + rr') \\ y_1z_1' - y_1'z_1 &= p(p'l + q'm + r'n + pl' + qm' + rn') \\ &\qquad - l(pp' + qq' + rr'), \end{aligned}$$

&c.

Putting $\phi = p'l + q'm + r'n + pl' + qm' + rn'$,

$\theta = -pp' - qq' - rr'$,

we may write these formulæ as follows:

$$\begin{aligned} l_1 &= l'\theta + p'\phi, \\ m_1 &= m'\theta + q'\phi, \\ n_1 &= n'\theta + r'\phi, \\ l_1' &= l\theta + p\phi, \\ m_1' &= m\theta + q\phi, \\ n_1' &= n\theta + r\phi. \end{aligned}$$

Hence, if the line $(lmnl'm'n')$ belongs to the linear complex $\phi = 0$, the substitution

$$\begin{pmatrix} \cdot\,, & r\,, & -q, & p' \\ -r, & \cdot\,, & p\,, & q' \\ q\,, & -p, & \cdot\,, & r' \\ p'\,, & q'\,, & r'\,, & \cdot \end{pmatrix}$$

on point coordinates gives the result of reciprocation.

It is important to notice that this process fails when $pp' + qq' + rr' = 0$, *i.e.*, when the complex is special.

A more instantaneous proof may be given as follows:

We have

$$\phi \equiv x_1'x + y_1'y + z_1'z + t_1't = 0,$$

$$-\phi \equiv x_1x' + y_1y' + z_1z' + t_1t' = 0,$$

also identically $$x_1x + y_1y + z_1z + t_1t = 0,$$

$$x_1'x' + y_1'y' + z_1'z' + t_1't' = 0,$$

showing that x_1, y_1, z_1, t_1, and x_1', y_1', z_1', t_1' are the coordinates of two planes containing the line. Hence, taking x_1, y_1, z_1, t_1 for new point coordinates, the coordinates of the reciprocal line are obtained.

Applying this we have the theorem that a scroll whose generators are lines of a linear complex can be linearly transformed into its reciprocal.

Thus, if $f(x, y, z, t) = 0$ be the surface generated by a line belonging to $\phi = 0$, and satisfying any two other conditions, the reciprocal surface is $f(x_1, y_1, z_1, t_1) = 0$ [see Cayley, *Coll. Works*, VI., pp. 316–321].

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