# Convex Project

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***Abstract*—This paper provide a method of convex optimization to optimize the antenna array and the beam pattern. If the main lobe is 6 degree in the center and the side lobe is 80 degree, the maximize difference of the main lobe and the side lobe is 17 dB. At the same time, the power of each antenna is the same.**

***Index Terms*—beamforming, convex optimization, phase-only, semidefinite relaxation**

## I. Introduction

The phase-only beam pattern synthesis in our paper is to optimize the main-lobe beam-pattern by fixing the amplitude of each single antenna. The relative topic is researched in the [1] - [6]. In the six papers, they use the convex optimization to solve the relative problem about our problem. In [1], the author use the convex optimization to find a notch minimization problem, and the paper's basic setting is very similar to our problem. The paper and our paper all set each single antenna have the same power, and fix the phase to make the beam pattern correspond to the shape we want. In [2], the paper use the semi-definite relaxation to solve the optimization problem, also the paper use the concept of convex iteration to solve a non-rank one problem. In [3] and [4], these paper are target to find a particular shape of the beam pattern, and the paper also change the power of each antenna to find a optimized beam pattern. In [5], like [3], this paper optimizes the power and the phase of each antenna to make sure the algorithm they use could give them a specific shape of the beam pattern. Like our formulation, the paper give a region of main lobe and a region of side lobe to optimize the beam pattern. In [6], the paper use the iterative convex optimization to optimize the 2-dimension antenna array. In our problem, we set the power of each single antenna to be the same since the big power antenna is more expensive and harder to design than small power antenna. Therefore, our problem aims to find the better phase, which can make sure the main-lobe beam-pattern and side-lobe beam-pattern have a difference, when given a fixed antenna power by using convex optimization with semidefinite relaxation. Therefore, we design the phase of each antenna by using convex optimization to make sure the main-lobe beam-pattern and side-lobe beam-pattern have a difference. Finally, we make sure the power of the antenna array could be focus on the main-lobe beam-pattern and have the effect as the power-changed antenna array.

## II. System Model

In all the models, we give $\theta \in [0°, 3°]$, $\phi \in [0°, 80°]$, $\alpha \in [1, 2000]$, and $N = 16$ as basic settings, where $\theta$ expresses the region we want to make it higher, $\phi$ expresses the region we want its height could be constrained, $\alpha$ expresses the height magnification of the region of $\theta$ and $\phi$, and $N$ is the number of antenna. We defined $x[k]$ to represent the phase, $\eta_k$, of the $k^{th}$ antenna, and the magnitude of each antenna is 1, which means $x[k] = e^{j\eta_k}$.

### A. Original Problem Before SDR

We want that the middle domain of the beam pattern, which defined by $\theta$, could be $\alpha$ times higher than the side domain, which is define by $\phi$. To show it clearly, we define $\phi' = \frac{\pi}{2} - \phi$. We defined $\omega$ as the phase difference of two adjacent antenna as follows and we set $d = \frac{\lambda}{2}$, and $\mu$ is any angle.

$$\omega = \frac{2\pi d \sin\mu}{\lambda} = \pi \sin\mu \tag{1}$$

We defined

$$\mathfrak{X}(\omega, x) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$
$$= \begin{bmatrix} 1 & e^{-j\omega} & e^{-j2\omega} & \cdots & e^{\cdot -j(N-1)\omega} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \tag{2}$$

where $\omega$ is any angle. Therefore, $|\mathfrak{X}(\omega)|^2$ means the power of the beam at angle $\mu$. We can express the optimization problem as the following equations.

$$\text{maximize } t \tag{3}$$
$$\text{subject to } |\mathfrak{X}(\omega_1, x)|^2 \geq t \tag{3a}$$
$$|\mathfrak{X}(\omega_2, x)|^2 \leq \frac{t}{\alpha} \tag{3b}$$
$$|x[n]| = 1 \tag{3c}$$
$$\forall \omega_1 \in [\pi \sin(-\theta), \pi \sin\theta] \tag{3d}$$
$$\forall \omega_2 \in [\pi \sin(-\frac{\pi}{2}), \pi \sin(-\phi')] \cup [\pi \sin\phi', \pi \sin(\frac{\pi}{2})] \tag{3e}$$
$$\forall n \in \{0, \ldots, N-1\} \tag{3f}$$

### B. Semi-definite Problem Equivalent to The Original Problem

We can rewrite the problem in 2.1 to change it to a semi-definite problem. We apply $\mathbf{X} = \mathbf{x}\mathbf{x}^H$ where $\mathbf{x} = [x[0], x[1], \ldots, x[N-1]]^H$, where $(\cdot)^H$ represent the conjugate transpose. We defined.

$$\mathbf{a}(\omega) = \begin{bmatrix} 1 \\ e^{-j\omega} \\ e^{-j2\omega} \\ \vdots \\ e^{-j(N-1)\omega} \end{bmatrix} \tag{4}$$

And then we defined.

$$\mathbf{Q}(\omega) = \mathbf{a}(\omega)\mathbf{a}(\omega)^H \tag{5}$$

So, we can rewrite (3a) and (3b) as

$$|\mathfrak{X}(\omega)|^2 = \mathbf{x}^H\mathbf{Q}(\omega)\mathbf{x} = \mathbf{tr}(\mathbf{Q}(\omega)\mathbf{X}) \tag{6}$$

Then, we can reformulate the original problem into the following equivalent semi-definite form.

$$\begin{aligned}
&\text{maximize } t && (7)\\
&\text{subject to } \mathbf{tr}(\mathbf{Q}(\omega_1)\mathbf{X}) \geq t && (7a)\\
&\mathbf{tr}(\mathbf{Q}(\omega_2)\mathbf{X}) \leq \frac{t}{\alpha} && (7b)\\
&[\mathbf{X}]_{nn} = 1 && (7c)\\
&\mathbf{rank}(\mathbf{X}) = 1 && (7d)\\
&\forall \omega_1 \in [\pi\sin(-\theta), \pi\sin\theta] && (7e)\\
&\forall \omega_2 \in [-\pi, \pi\sin(-\phi')] \cup [\pi\sin\phi', \pi] && (7f)\\
&\forall n \in \{0, \ldots, N-1\} && (7g)
\end{aligned}$$

### C. The Problem After Semi-definite Relaxation

Since the problem (7) is not a convex problem, so we use the semi-definite relaxation to change it to a convex problem as following.

$$\begin{aligned}
&\text{maximize } t && (8)\\
&\text{subject to } \mathbf{tr}(\mathbf{Q}(\omega_1)\mathbf{X}) \geq t && (8a)\\
&\mathbf{tr}(\mathbf{Q}(\omega_2)\mathbf{X}) \leq \frac{t}{\alpha} && (8b)\\
&[\mathbf{X}]_{nn} = 1 && (8c)\\
&\forall \omega_1 \in [\pi\sin(-\theta), \pi\sin\theta] && (8d)\\
&\forall \omega_2 \in [-\pi, \pi\sin(-\phi')] \cup [\pi\sin\phi', \pi] && (8e)\\
&\forall n \in \{0, \ldots, N-1\} && (8f)
\end{aligned}$$

### D. SDR problem with Dattoro's Convex iteration

Unfortunately, the problem (8), although being convex, does not always have an optimal point $\mathbf{X}$ that is rank one. Therefore, we resort to some convex iteration techniques. One of them is the Dattoro's convex iteration described below.

$$\begin{aligned}
&\underset{\mathbf{X}^{(r)}}{\text{maximize}}\ t - \sigma^{(r)} \cdot \mathfrak{Re}[\mathbf{tr}(\mathbf{U'}^{(r-1)}\mathbf{X}^{(r)})] && (9)\\
&\text{subject to } \mathbf{tr}(\mathbf{Q}(\omega_1)\mathbf{X}) \geq t && (9a)\\
&\mathbf{tr}(\mathbf{Q}(\omega_2)\mathbf{X}) \leq \frac{t}{\alpha} && (9b)\\
&[\mathbf{X}]_{nn} = 1 && (9c)\\
&\forall \omega_1 \in [\pi\sin(-\theta), \pi\sin\theta] && (9d)\\
&\forall \omega_2 \in [-\pi, \pi\sin(-\phi')] \cup [\pi\sin\phi', \pi] && (9e)\\
&\forall n \in \{0, ..., N-1\} && (9f)
\end{aligned}$$

where $\sigma^{(0)} = 1000$ and $\sigma^{(r)} = \xi \cdot \sigma^{(r-1)}$. In our algorithm, we choose $\xi = 1.2$ for iteration. Note that $\mathbf{U}'$ is obtained from eigenvalue decomposition of $\mathbf{X}$. Therefore, we can have the definition as below.

$$\mathbf{X} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H = \sum_{k=1}^{N} \lambda_k \mathbf{u}_k \mathbf{u}_k^H \tag{10}$$

where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N \geq 0$, if $\mathbf{X}$ is a rank 1 matrix, $\lambda_2 = 0$. Therefore, $\mathbf{U}'$ has the definition as below.

$$\mathbf{U}' = \sum_{k=2}^{N} \mathbf{u}_k \mathbf{u}_k^H \tag{11}$$

### E. SDR problem with Dattoro's Convex iteration with different direction

Since we want that the main beam could be on the other direction, we append $\delta$ to change the direction of beam. Moreover, note that it is possible for the constrained region to cross to the other side when $|\delta|$ is too large, which is not what we want. To avoid this, we made a minor change in (9e) to ensure that the domain of the set will be correct. We can write the problem as the following.

$$\begin{aligned}
&\underset{\mathbf{X}^{(r)}}{\text{maximize}}\ t - \sigma^{(r)} \cdot \mathfrak{Re}[\mathbf{tr}(\mathbf{U'}^{(r-1)}\mathbf{X}^{(r)})] && (12)\\
&\text{subject to } \mathbf{tr}(\mathbf{Q}(\omega_1)\mathbf{X}) \geq t && (12a)\\
&\mathbf{tr}(\mathbf{Q}(\omega_2)\mathbf{X}) \leq \frac{t}{\alpha} && (12b)\\
&[\mathbf{X}]_{nn} = 1 && (12c)\\
&\forall \omega_1 \in [\pi\sin(-\theta+\delta), \pi\sin(\theta+\delta)] && (12d)\\
&\forall \omega_2 \in [-\pi, \pi\sin(\max\{-\phi'+\delta, -\frac{\pi}{2}\})] && \\
&\cup\ [\pi\sin(\min\{\phi'+\delta, \frac{\pi}{2}\}), \pi] && (12e)\\
&\forall n \in \{0, ..., N-1\} && (12f)
\end{aligned}$$

where $\mathfrak{Re}[\cdot]$ takes the real part of a complex number.

## III. Algorithm

The sets in problem (12) are infinite sets. To simplify this, we choose the finite subsets, $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$, to replace the sets in (12d) and (12e). We divide the set of $[-\frac{\pi}{2}, \frac{\pi}{2}]$ into 360 slices for sampling. We define

$$\mathcal{B}_1 = \{-\theta+\delta, -\theta+\delta+1^\circ, \dots, \theta+\delta\} \tag{13}$$

$$\mathcal{B}_2 = \{-90^\circ, -89^\circ, \dots, \max\{-\phi'+\delta, -90^\circ\}\} \tag{13a}$$

$$\mathcal{B}_3 = \{\min\{\phi'+\delta, 90^\circ\}, \dots, 89^\circ, 90^\circ\} \tag{13b}$$

Therefore, the reformulated problem is as the following.

$$\underset{\mathbf{X}^{(r)}}{\text{maximize}} \quad t \tag{14}$$

$$\text{subject to } \mathbf{tr}(\mathbf{Q}(\pi\sin\rho)\mathbf{X}) \geq t \tag{14a}$$

$$\mathbf{tr}(\mathbf{Q}(\pi\sin\tau)\mathbf{X}) \leq \frac{t}{\alpha} \tag{14b}$$

$$[\mathbf{X}]_{nn} = 1 \tag{14c}$$

$$\forall\rho \in \mathcal{B}_1 \tag{14d}$$

$$\forall\tau \in \mathcal{B}_2 \cup \mathcal{B}_3 \tag{14e}$$

$$\forall n \in \{0, \dots, N-1\} \tag{14f}$$

and the reformulated problem with Dattoro's convex iteration:

$$\underset{\mathbf{X}^{(r)}}{\text{maximize}} \quad t - \sigma^{(r)} \cdot \mathfrak{Re}[\mathbf{tr}(\mathbf{U}^{'(r-1)}\mathbf{X}^{(r)})] \tag{15}$$

subject to (14a) , (14b) , (14c) , (14d) , (14e) , (14f)

**Algorithm 1** algorithm for solving the semidefinite relaxation problem with Dattoro's convex iteration

**Initialization:** $r = 0$

Solve the solution of problem (14) for a given $\delta$, $\phi$, $\alpha$, $\theta$, and let $\mathbf{X}^{(r)}$ denote the solution.

Set a proper initial $\sigma^{(r)}$

**Iterations:** $r \to r+1$

**1)** Do eigenvalue decomposition of $\mathbf{X}^{(r-1)} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^H = \sum_{k=1}^{N}\lambda_k\mathbf{u}_k\mathbf{u}_k^H$, and compute $\mathbf{U}'^{(r-1)}$ by equation (11).

**2)** Solve (15) with $\{\mathbf{U}'^{(r-1)}, \sigma^{(r)}\}$, and let $\mathbf{X}^{(r)}$ denote the solution.

**3)** If $\mathbf{rank}(\mathbf{X}^{(r)}) = 1$, terminate and go to Step 6.

**4)** Update $\sigma^{(r)} = \xi \cdot \sigma^{(r-1)}$.

**5)** If $r = r_0$ where $r_0$ is the maximum number of iterations, terminate and go to Step. 6.

**6)** Take the principal eigenvector of $\mathbf{X}$ depending on termination. If the solution is not a rank one matrix, then scale every entry to make their amplitude be one.

## IV. Simulation Result

In this section, we provide some example to identify our algorithm.

### A. Performance Comparison of Different Parameters

In Fig. 1, we can discover that when $\theta = 3^\circ$ and $\phi' = 30^\circ$, the maximum value of expected $\alpha$(this will be discussed in Section 4.2) could be 230, but if $\theta = 1^\circ$ and $\phi' = 30^\circ$, the maximum value of expected $\alpha$ could high up to 360. In the other word, when $\theta = 1^\circ$, $\phi' = 30^\circ$ and $\delta = 0^\circ$, the main beam's power will be 25.56dB higher than the other beams'. But at the same situation, the main beam's power of $\theta = 3^\circ$ will only be 23.61dB higher than the other beams'.

From the numerical data, in the situation of $\phi' = 30^\circ$ and $\delta = 0^\circ$, if the $\theta = 1$, the system performance will have 56% higher than the situation of $\theta = 3$. However, in the situation of $\phi' = 10^\circ$ and $\delta = 0^\circ$, if the $\theta = 1$, the system performance will only have 25% higher than the situation of $\theta = 3$. We can also discover that when $\phi'$ is smaller the influences from $\theta$ will not be large. Which means if the constrain from $\phi'$ is too serious, even the $\theta$ is very small, the system cannot find a better rank one solution after convex iterations.

Comparing the Fig. 1 to Fig. 5, we can discover that when the $\delta$ is bigger, the maximum value of $\alpha$ will be smaller. In Fig. 2, we can discover that is the main beam has a $15^\circ$ shift, the $\alpha$ will be very close to the the $\alpha$ if no shift, which means if the shift is only $15^\circ$, the system can still provide us a solution similar to the situation of no shift. However, in Fig. 3 to Fig. 5, we can discover that if the shift is bigger than $30^\circ$, the performance of the system will be not so good as the small shift situation. When the shift is higher to $60^\circ$, the $\alpha$ only be 120 at most, 66% lower than the small shift situation.

### B. Comparison of Ideal and Expected Situation

$$\alpha_{result} = \frac{min[\mathbf{tr}(\mathbf{Q}(\pi\sin\rho)\mathbf{X})]}{max[\mathbf{tr}(\mathbf{Q}(\pi\sin\tau)\mathbf{X})]} \tag{16}$$

$$\forall\rho \in \mathcal{B}_1 \tag{16a}$$

$$\forall\tau \in \mathcal{B}_2 \cup \mathcal{B}_3 \tag{16b}$$

(13), (13a), (13b)

In Fig. 5, 7, 11, 13, 17, 19, the expected $\alpha$ means the value we use in the algorithm. However, since in the last step of our algorithm, we select the eigenvector with the largest eigenvalue to represent the 16 antennas, and set their amplitude to one, but which may make the $\alpha$ not fit the the constrain in (14b) if the solution is not a rank one matrix. Therefore, we substitute this eigenvector in the original problem and calculate its $\alpha$ as result $\alpha$.

We can discover that in a specific region, the result $\alpha$ and the predict $\alpha$ will be the same, which means that the system could give us a rank one solution, so the maximum eigenvector can represent the 16 antennas as the original problem. However, after the specific region, we can discover that the predict $\alpha$ will be better than the result $\alpha$, which means the solution is not a rank one matrix in the situation. Therefore, if we use the maximum eigenvector to be the solution of our original problem, the $\alpha$ will have a difference between the realistic and ideal situation.

### C. Comparison the Result with and without Dattoro's Convex Iteration

In Fig. 5 and Fig. 17, the two figures compares the result with and without Dattoro's convex iteration. In Fig. 5, we can see that with the iterative algorithm, the performance will be better than the result without the iteration at the point of maximum useful data, which is about 75% higher than the result without iteration. Also, from the figure, we can discover that if we do not use the method of iteration and use the maximum eigenvector as the antenna array, the result $\alpha$ will not correspond to the expect $\alpha$. In Fig. 17, we can see that at the point of maximum usable data ($\alpha = 120$), the result with iterative algorithm will be 50% higher than the result without the iterative algorithm. Therefore, though the result $\alpha$ with smaller $\phi'$ is not high as the result $\alpha$ with larger $\phi'$, but the system can get a larger promotion in the situation of smaller $\phi'$ with the iterative algorithm than the situation with larger $\phi'$.

Compare the Fig. 5 and Fig. 7, the maximum usable value in Fig. 7 with the iterative algorithm is 69% higher than the value without the iteration. Therefore, even the $\delta$ is different in the two situation, the performance will not have a large difference.

## V. Conclusion

This paper use the convex optimization to optimize the magnitude of the antenna array when fixing the power of each antenna, and we also make the power-fixed antenna array could have a performance like other antenna arrays which has different powers. In a nutshell, we make the low-power antennas could have a performance like the expensive and hard-designed antennas.

## VI. Figure

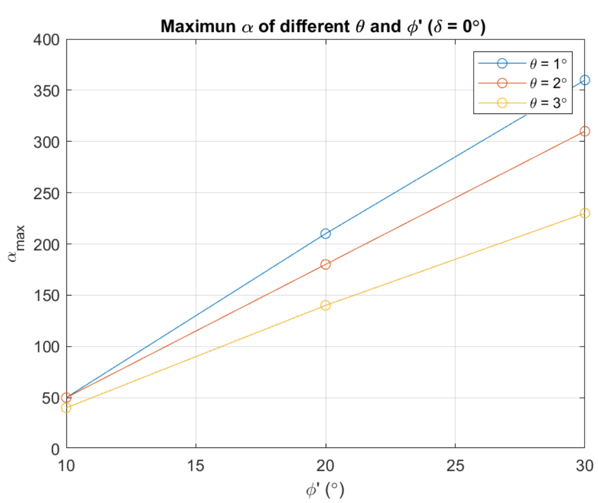


Figure 1. Maximum $\alpha$ of different $\theta$ and $\phi'$ ($\delta = 0°$)

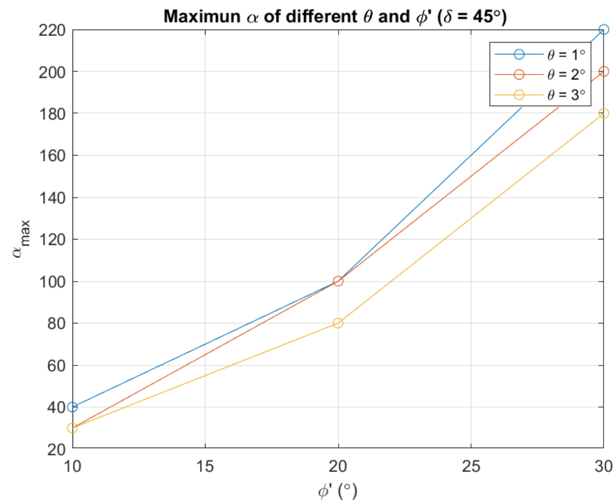


Figure 2. Maximum $\alpha$ of different $\theta$ and $\phi'$ ($\delta = 45°$)

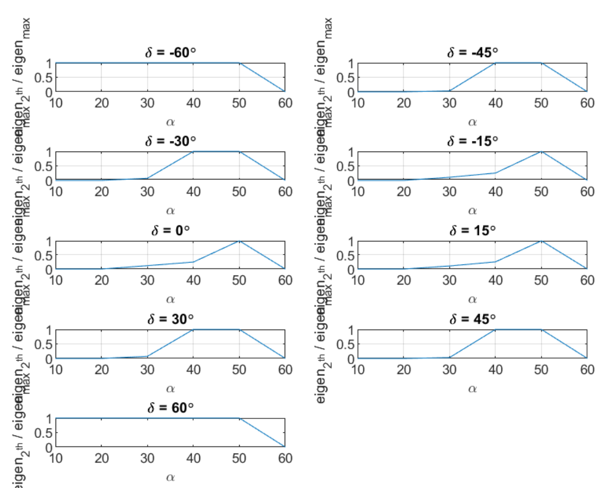


Figure 3. 2nd largest eigenvalue/1st largest eigenvalue in different situation($\phi' = 10°$, $\theta = 3°$)

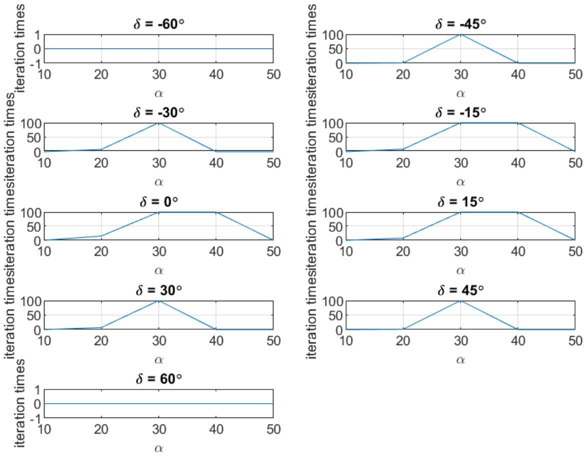


Figure 4. iteration time in different situation($\phi' = 10°$, $\theta = 3°$)

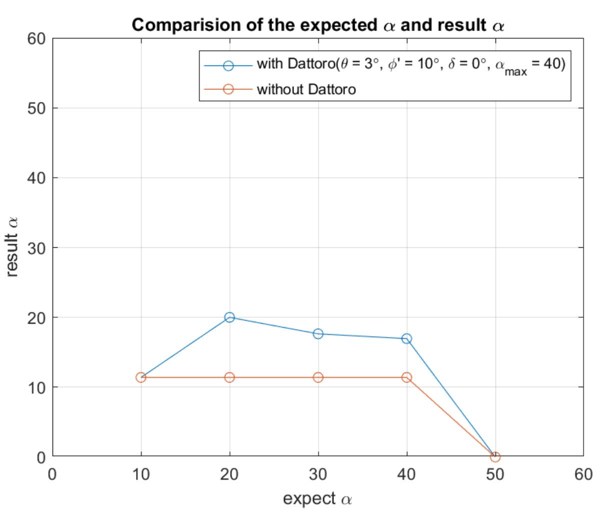


Figure 5. Comparison of result $\alpha$ with or without Datorro's convex iteration($\phi' = 10°, \delta = 0°$)

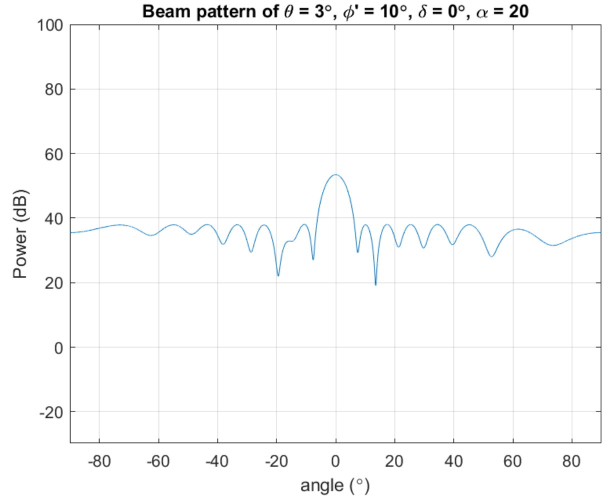


Figure 6. Beam pattern of maximum $\alpha$ of $\phi' = 10°, \delta = 0°$

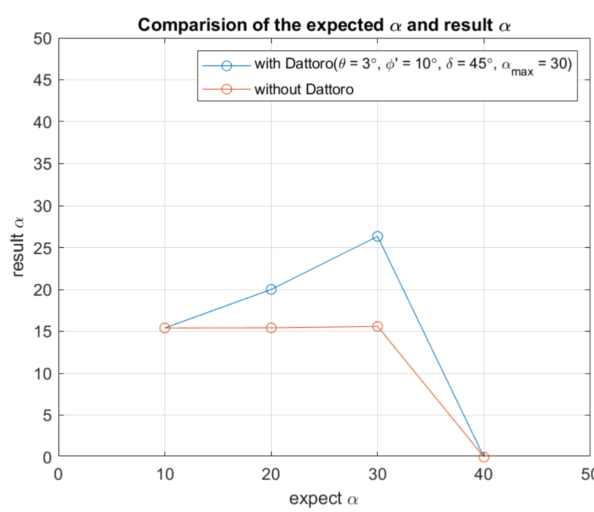


Figure 7. Comparison of result $\alpha$ with or without Datorro's convex iteration($\phi' = 10°, \delta = 45°$)

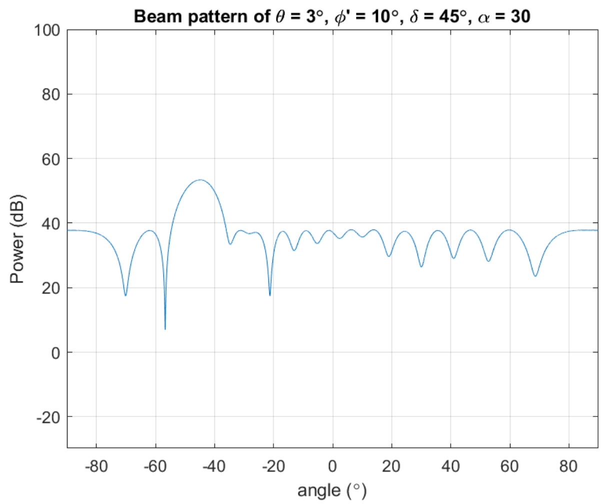


Figure 8. Beam pattern of maximum $\alpha$ of $\phi' = 10°, \delta = 45°$

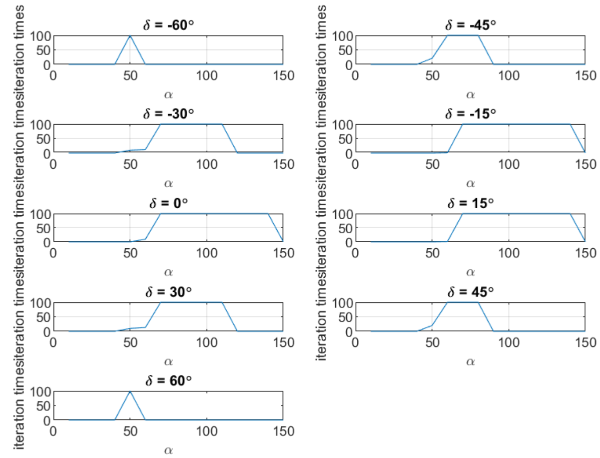


Figure 10. iteration time in different situation($\phi' = 20°$, $\theta = 3°$)

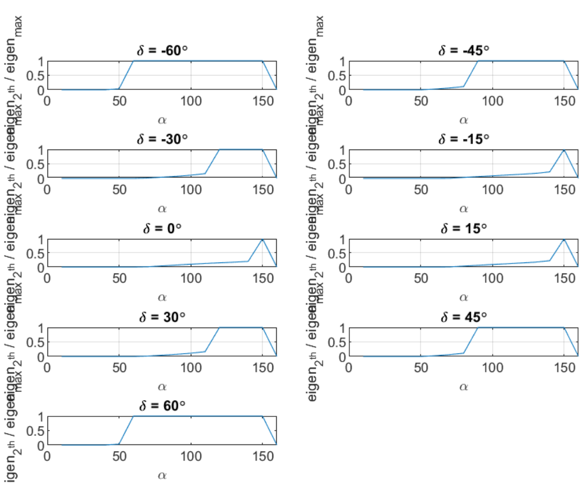


Figure 9. 2nd largest eigenvalue/1st largest eigenvalue in different situation($\phi' = 20°$, $\theta = 3°$)

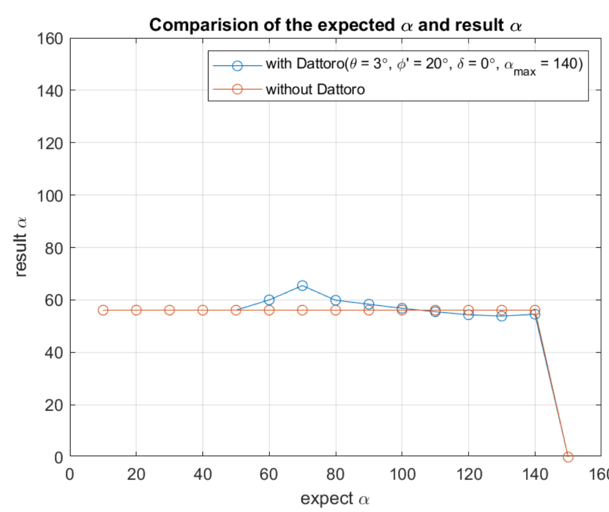


Figure 11. Comparison of result $\alpha$ with or without Datorro's convex iteration($\phi' = 20°, \delta = 0°$)

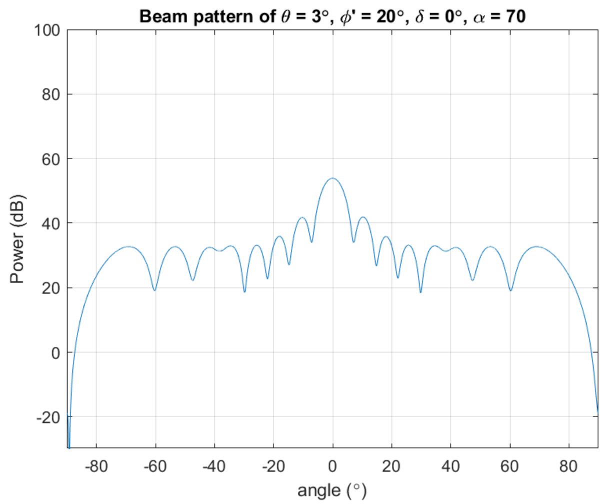


Figure 12. Beam pattern of maximum $\alpha$ of $\phi' = 20°, \delta = 0°$

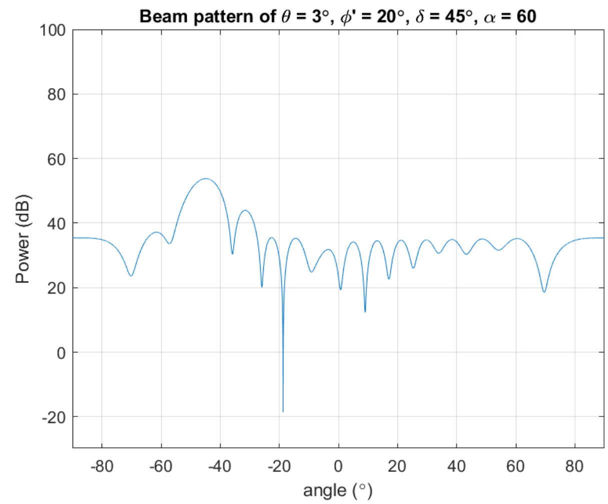


Figure 14. Beam pattern of maximum $\alpha$ of $\phi' = 20°, \delta = 45°$

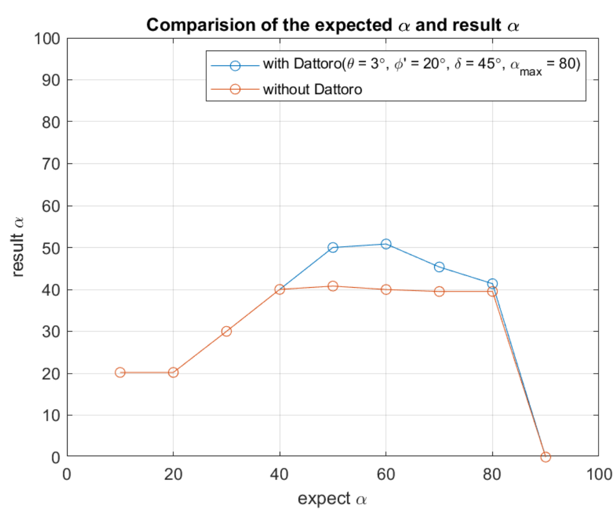


Figure 13. Comparison of result $\alpha$ with or without Datorro's convex iteration($\phi' = 20°, \delta = 45°$)

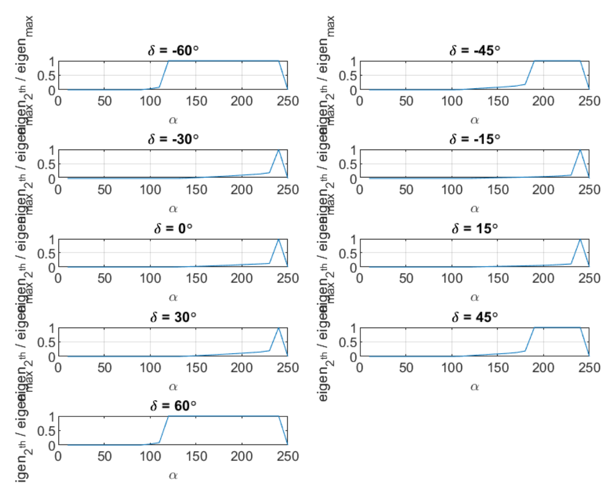


Figure 15. 2nd largest eigenvalue/1st largest eigenvalue in different situation($\phi' = 30°, \theta = 3°$)

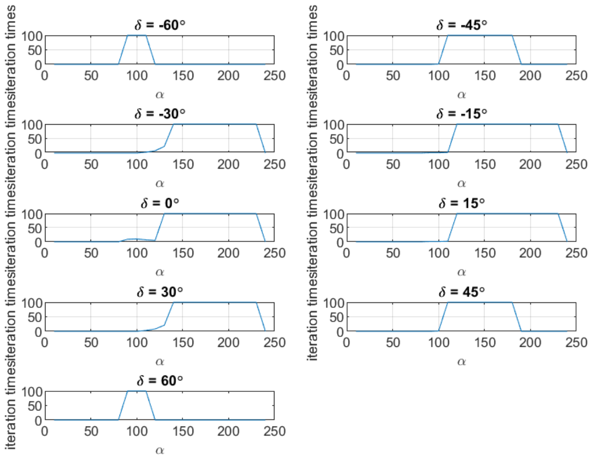


Figure 16. iteration time in different situation($\phi' = 30°$, $\theta = 3°$)

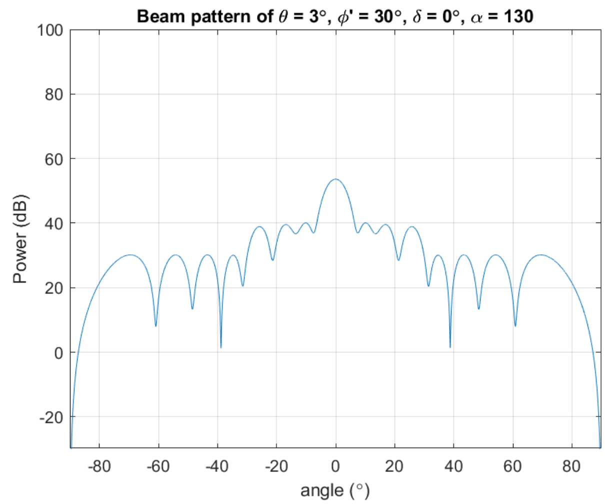


Figure 18. Beam pattern of maximum $\alpha$ of $\phi' = 30°, \delta = 0°$

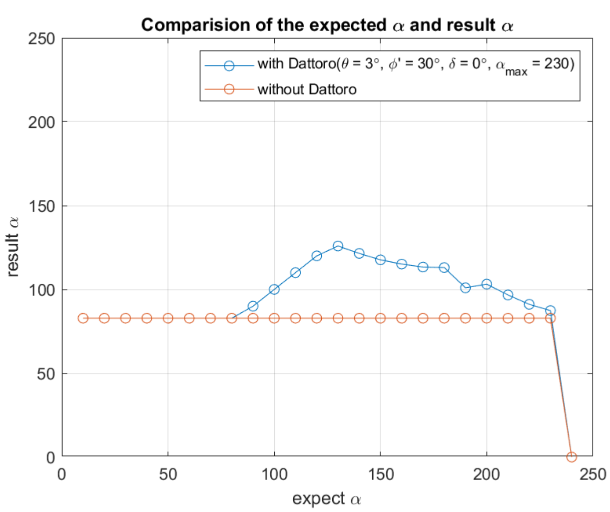


Figure 17. Comparison of result $\alpha$ with or without Datorro's convex iteration($\phi' = 30°, \delta = 0°$)

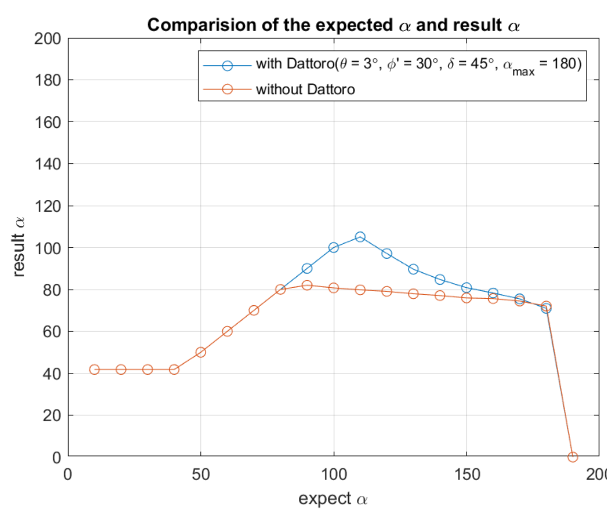


Figure 19. Comparison of result $\alpha$ with or without Datorro's convex iteration($\phi' = 30°, \delta = 45°$)

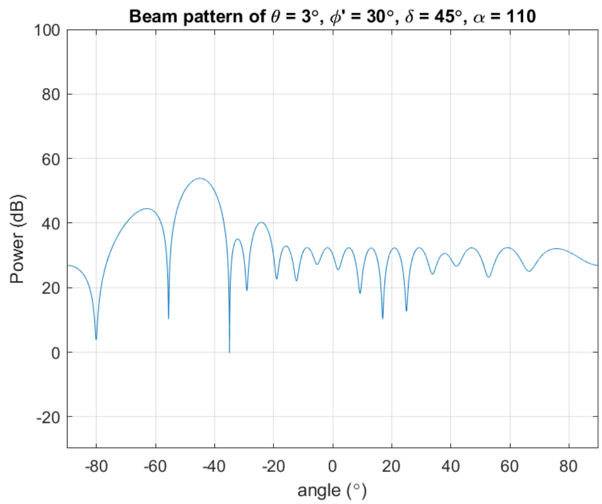


Figure 20. Beam pattern of maximum $\alpha$ of $\phi' = 30°, \delta = 45°$

## VII. References


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