Convex Project

Ming Hao, Hsu

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Abstract

1 Introduction

2 Model

2.1 Original Problem Before SDR

$$\text{maximize } t \tag{1.a}$$
$$\text{subject to } |X(e^{j\omega_1})|^2 \geq t \tag{1.b}$$
$$|X(e^{j\omega_2})|^2 \leq \frac{t}{\alpha} \tag{1.c}$$
$$x[n]| = 1 \tag{1.d}$$
$$\forall \omega_1 \in [-\pi \sin\theta, \pi \sin\theta] \tag{1.e}$$
$$\forall \omega_2 \in [-\pi, -\pi \sin\phi] \cup [\pi \sin\phi, \pi] \tag{1.f}$$
$$\forall n \in \{0, ..., N-1\} \tag{1.g}$$

where, $\theta \in \{1^\circ, 2^\circ, 3^\circ, 4^\circ, 5^\circ\}$, $\phi \in \{20^\circ, 30^\circ, ..., 80^\circ\}$, $\alpha \in \{100, 110, ..., 1000\}$, $N = 16$

2.2 Semidefinite Problem Equivalent to The Original Problem

We apply $\mathbf{X} = \mathbf{x}\mathbf{x}^H$ where $\mathbf{x} = [x[0], x[1], ..., x[N-1]]^T$, and reformulate the original problem into the following equivalent semidefinite form.

$$\text{maximize } t \tag{2.a}$$
$$\text{subject to } \mathbf{x}^T\mathbf{Q}(\omega_1)\mathbf{x} = \text{trace}(\mathbf{Q}(\omega_1)\mathbf{X}) \geq t \tag{2.b}$$
$$\mathbf{x}^T\mathbf{Q}(\omega_2)\mathbf{x} = \text{trace}(\mathbf{Q}(\omega_2)\mathbf{X}) \leq \frac{t}{\alpha} \tag{2.c}$$
$$[\mathbf{x}\mathbf{x}^H]_{nn} = [\mathbf{X}]_{nn} = 1 \tag{2.d}$$
$$\text{rank } \mathbf{X} = 1 \tag{2.e}$$
$$\forall \omega_1 \in [-\pi \sin\theta, \pi \sin\theta] \tag{2.f}$$
$$\forall \omega_2 \in [-\pi, -\pi \sin\phi] \cup [\pi \sin\phi, \pi] \tag{2.g}$$
$$\forall n \in \{0, ..., N-1\} \tag{2.h}$$

where $\mathbf{Q}(\omega)$ can be expressed as

$$\mathbf{Q}(\omega) = \mathbf{a}(\omega)\mathbf{a}(\omega)^H \tag{3}$$

with

$$\mathbf{a}(\omega) = \begin{bmatrix} 1 \\ e^{j\omega} \\ e^{j2\omega} \\ \vdots \\ e^{j(N-1)\omega} \end{bmatrix} \tag{4}$$

Also note that

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=0}^{N-1} x[n]e^{-j\omega n} \\ &= \begin{bmatrix} 1 & e^{-j\omega} & e^{-j2\omega} & \cdots \\ e^{-j(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \\ &= \mathbf{a}(\omega)^H \mathbf{x} \end{aligned} \tag{5}$$

Also note that the third constraint (2.d) is equivalent to

$$\mathrm{diag}(\mathbf{x}\mathbf{x}^H) = \mathbf{1}_N \tag{6}$$

2.3 The Problem After Semidefinite Relaxation

$$\text{maximize } t \tag{7.a}$$

$$\text{subject to } \mathbf{x}^T\mathbf{Q}(\omega_1)\mathbf{x} = \mathrm{trace}(\mathbf{Q}(\omega_1)\mathbf{X}) \geq t \tag{7.b}$$

$$\mathbf{x}^T\mathbf{Q}(\omega_2)\mathbf{x} = \mathrm{trace}(\mathbf{Q}(\omega_2)\mathbf{X}) \leq \frac{t}{\alpha} \tag{7.c}$$

$$[\mathbf{x}\mathbf{x}^H]_{nn} = [\mathbf{X}]_{nn} = 1 \tag{7.d}$$

$$\forall \omega_1 \in [-\pi \sin\theta, \pi \sin\theta] \tag{7.e}$$

$$\forall \omega_2 \in [-\pi, -\pi \sin\phi] \cup [\pi \sin\phi, \pi] \tag{7.f}$$

$$\forall n \in \{0, ..., N-1\} \tag{7.g}$$

where $\phi = \mathbf{90}°$, $w^{(0)} = 1000$, $w^{(r)} = 1.2w^{(r-1)}$. Unfortunately, the problem, although being convex, does not always have an optimal point $\mathbf{X}$ that is rank one. So we resort to some convex iteration techniques. One of them is the Dattoro's Convex interation described below.

2.4 SDR problem with Dattoro's Convex iteration

$$\underset{\mathbf{X}^{(r)}}{\text{maximize}} \;\; t - w[\mathcal{R}](\mathrm{trace}\{\mathbf{U}^{(r-1)}\mathbf{X}^{(r)}\})] - w[\mathcal{I}\Updownarrow(\mathrm{trace}\{\mathbf{U}^{(r-1)}\mathbf{X}^{(r)}\})] \tag{8.a}$$

$$\text{subject to } \mathbf{x}^H\mathbf{Q}(\omega_1)\mathbf{x} = \mathrm{trace}(\mathbf{Q}(\omega)\mathbf{X}) \geq t \tag{8.b}$$

$$\mathbf{x}^H\mathbf{Q}(\omega_2)\mathbf{x} = \mathrm{trace}(\mathbf{Q}(\omega)\mathbf{X}) \leq \frac{t}{\alpha} \tag{8.c}$$

$$[\mathbf{x}\mathbf{x}^H]_{nn} = [\mathbf{X}]_{nn} = 1 \tag{8.d}$$

$$\forall \omega_1 \in [-\pi \sin\theta], \pi \sin\theta] \tag{8.e}$$

$$\forall \omega_2 \in [-\pi, -\pi \sin\phi] \cup [\pi \sin\phi, \pi] \tag{8.f}$$

$$\forall n \in \{0, ..., N-1\} \tag{8.g}$$

2.5 SDR problem with Dattoro's Convex iteration with different direction

$$\underset{\mathbf{X}^{(r)}}{\text{maximize}} \;\; t - w[\mathcal{R}](\text{trace}\{\mathbf{U}^{(r-1)}\mathbf{X}^{(r)}\})] - w[\mathcal{I}\Updownarrow(\text{trace}\{\mathbf{U}^{(r-1)}\mathbf{X}^{(r)}\})] \tag{9.a}$$

$$\text{subject to } \mathbf{x}^H\mathbf{Q}(\omega_1)\mathbf{x} = \text{trace}(\mathbf{Q}(\omega)\mathbf{X}) \geq t \tag{9.b}$$

$$\mathbf{x}^H\mathbf{Q}(\omega_2)\mathbf{x} = \text{trace}(\mathbf{Q}(\omega)\mathbf{X}) \leq \frac{t}{\alpha} \tag{9.c}$$

$$[\mathbf{x}\mathbf{x}^H]_{nn} = [\mathbf{X}]_{nn} = 1 \tag{9.d}$$

$$\forall\omega_1 \in [\pi\sin(-\theta+\delta), \pi\sin(\theta+\delta)] \tag{9.e}$$

$$\forall\omega_2 \in [\pi\sin(-\frac{\pi}{2}), \pi\sin(-\frac{\pi}{2}+\phi+\delta)] \cup [\pi\sin(\frac{\pi}{2}-\phi+\delta), \pi\sin\frac{\pi}{2}] \tag{9.f}$$

$$\forall n \in \{0, ..., N-1\} \tag{9.g}$$

2.6 SDR problem with Dattoro's Convex iteration with different direction algorithm

$$\underset{\mathbf{X}^{(r)}}{\text{maximize}} \;\; t - w[\mathcal{R}](\text{trace}\{\mathbf{U}^{(r-1)}\mathbf{X}^{(r)}\})] - w[\mathcal{I}\Updownarrow(\text{trace}\{\mathbf{U}^{(r-1)}\mathbf{X}^{(r)}\})] \tag{10.a}$$

$$\text{subject to } \mathbf{x}^H\mathbf{Q}(\omega_1)\mathbf{x} = \text{trace}(\mathbf{Q}(\omega)\mathbf{X}) \geq t \tag{10.b}$$

$$\mathbf{x}^H\mathbf{Q}(\omega_2)\mathbf{x} = \text{trace}(\mathbf{Q}(\omega)\mathbf{X}) \leq \frac{t}{\alpha} \tag{10.c}$$

$$[\mathbf{x}\mathbf{x}^H]_{nn} = [\mathbf{X}]_{nn} = 1 \tag{10.d}$$

$$\forall\omega_1 \in \{[180^\circ \times \sin(-\theta+\delta)], ..., [180^\circ \times \sin(\theta+\delta)]\} \tag{10.e}$$

$$\forall\omega_2 \in \{180^\circ \times \sin(-90^\circ), ..., [180^\circ \times \sin(-90^\circ+\phi+\delta)]\} \cup \{[180^\circ \times \sin(90^\circ-\phi+\delta)], ..., 180^\circ \times \sin 90^\circ\} \tag{10.f}$$

$$\forall n \in \{0, ..., N-1\} \tag{10.g}$$

where, $\theta \in \{1^\circ, 2^\circ, 3^\circ\}$, $\phi \in \{20^\circ, 30^\circ, ..., 80^\circ\}$, $\alpha \in \{1000, 1100, ..., 2000\}$, $N = 16$.

3 Simulation Result

In Fig. 1, we can discover that there is a dramatic gap of maximum α between $\phi = 40^\circ$ and $\phi = 50^\circ$. Fig. 2 is the side view of Fig. 1, from this figure, we can know that when θ is smaller, the result will be better.

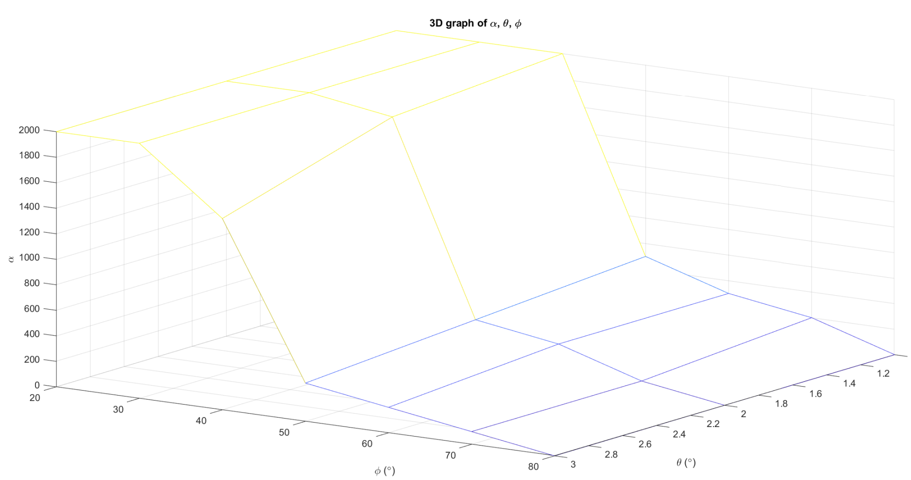
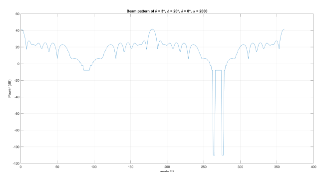
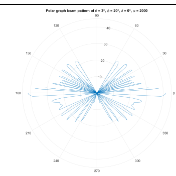
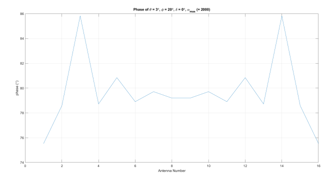
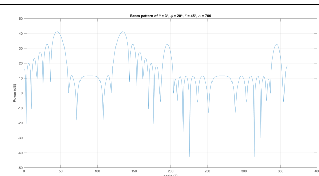
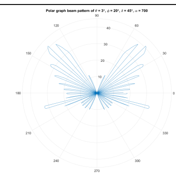
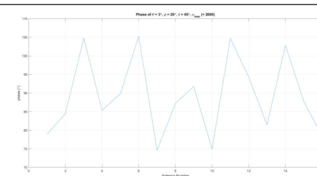
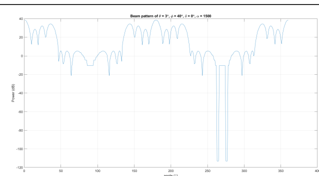
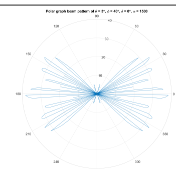
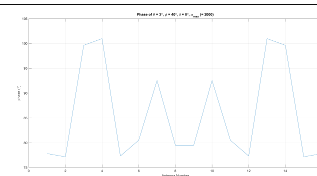
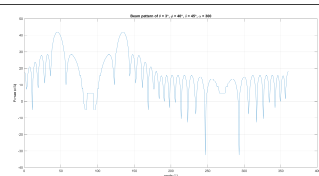
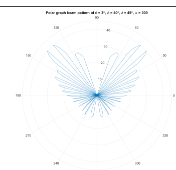
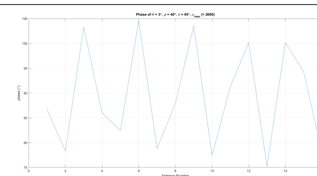
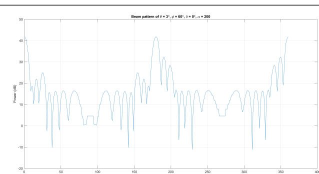
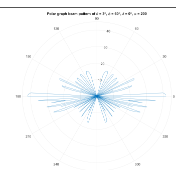
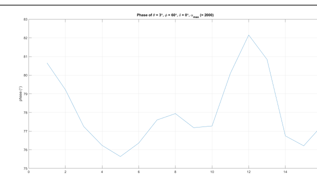
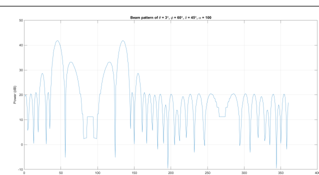
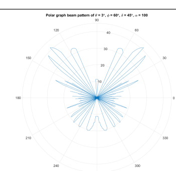
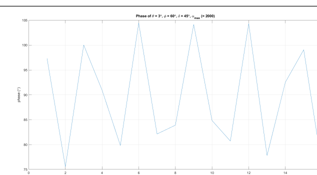


Figure 1: 3D graph of α, θ and ϕ

$\phi(\circ)$ / $\delta(\circ)$	beam pattern	polar beam pattern	phase
20/0			
20/45			
40/0			
40/45			
60/0			
60/45			

Note: Bold categories are biologically correct growth models.

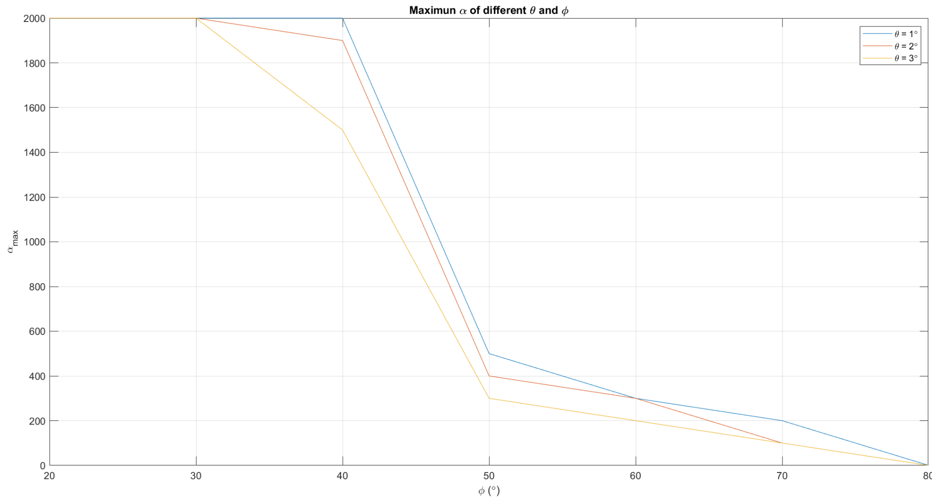


Figure 2: α versus ϕ of different θ

4 Conclusion

After fixing the problem, wediscover that if ϕ is smaller than $40°$, then the α could be very high (higher than 5000).