$$\int_{2T-b}^{T} f(x)\mathrm{d}x \xlongequal{\text{令 } x=2T-t} \int_{b}^{T} f(2T-t)(-\mathrm{d}t) = -\int_{b}^{T} f[T+(T-t)]\mathrm{d}t$$

$$\xlongequal{f(T+u)=f(T-u)} -\int_{b}^{T} f[T-(T-t)]\mathrm{d}t = -\int_{b}^{T} f(t)\mathrm{d}t = \int_{T}^{b} f(x)\mathrm{d}x,$$

故 $\int_a^b f(x)\mathrm{d}x = 2\int_T^b f(x)\mathrm{d}x + \int_a^{2T-b} f(x)\mathrm{d}x.$

**【例 7】** 设 $f(x)$，$g(x)$在$[a,b]$上连续，且满足

$$\int_a^x f(t)\mathrm{d}t \geqslant \int_a^x g(t)\mathrm{d}t,\ x\in[a,b],\quad \int_a^b f(t)\mathrm{d}t = \int_a^b g(t)\mathrm{d}t,$$

证明：$\int_a^b xf(x)\mathrm{d}x \leqslant \int_a^b xg(x)\mathrm{d}x.$

**证** 令 $F(x)=f(x)-g(x)$，$G(x)=\int_a^x F(t)\mathrm{d}t$，由题设知 $G(x)\geqslant 0$，$x\in[a,b]$，$G(a)=G(b)=0$，$G'(x)=F(x)$. 从而

$$\int_a^b xF(x)\mathrm{d}x = \int_a^b x\mathrm{d}G(x) = xG(x)\Big|_a^b - \int_a^b G(x)\mathrm{d}x = -\int_a^b G(x)\mathrm{d}x.$$

由于 $G(x)\geqslant 0$，$x\in[a,b]$，故有 $-\int_a^b G(x)\mathrm{d}x \leqslant 0$，即 $\int_a^b xF(x)\mathrm{d}x \leqslant 0$.

因此 $\int_a^b xf(x)\mathrm{d}x \leqslant \int_a^b xg(x)\mathrm{d}x.$

### 方法总结

本题为基本证明题. 一般地，证明积分等式或不等式，都应引入变限积分，将其转化为函数等式或不等式.

## 习题 5—3 解答

1. 计算下列定积分：

(1) $\int_{\frac{\pi}{3}}^{\pi} \sin\left(x+\frac{\pi}{3}\right)\mathrm{d}x$； (2) $\int_{-2}^{1} \frac{\mathrm{d}x}{(11+5x)^3}$； (3) $\int_0^{\frac{\pi}{2}} \sin\varphi\cos^3\varphi\mathrm{d}\varphi$；

(4) $\int_0^{\pi} (1-\sin^3\theta)\mathrm{d}\theta$； (5) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 u\mathrm{d}u$； (6) $\int_0^{\sqrt{2}} \sqrt{2-x^2}\,\mathrm{d}x$；

(7) $\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{8-2y^2}\,\mathrm{d}y$； (8) $\int_{\frac{1}{\sqrt{2}}}^{1} \frac{\sqrt{1-x^2}}{x^2}\mathrm{d}x$； (9) $\int_0^a x^2\sqrt{a^2-x^2}\,\mathrm{d}x\ (a>0)$；

(10) $\int_1^{\sqrt{3}} \frac{\mathrm{d}x}{x^2\sqrt{1+x^2}}$； (11) $\int_{-1}^{1} \frac{x\mathrm{d}x}{\sqrt{5-4x}}$； (12) $\int_1^4 \frac{\mathrm{d}x}{1+\sqrt{x}}$；

(13) $\int_{\frac{3}{4}}^{1} \frac{\mathrm{d}x}{\sqrt{1-x}-1}$； (14) $\int_0^{\sqrt{2}a} \frac{x\mathrm{d}x}{\sqrt{3a^2-x^2}}\ (a>0)$； (15) $\int_0^1 te^{-\frac{t^2}{2}}\mathrm{d}t$；

(16) $\int_1^{e^2} \frac{\mathrm{d}x}{x\sqrt{1+\ln x}}$； (17) $\int_{-2}^{0} \frac{x+2\mathrm{d}x}{x^2+2x+2}$； (18) $\int_0^2 \frac{x\mathrm{d}x}{(x^2-2x+2)^2}$；

1 题视频解析

(19) $\int_{-\pi}^{\pi} x^4 \sin x \mathrm{d}x$;　(20) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^4\theta \mathrm{d}\theta$;　(21) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(\arcsin x)^2}{\sqrt{1-x^2}}\mathrm{d}x$;

(22) $\int_{-5}^{5} \frac{x^3\sin^2 x}{x^4+2x^2+1}\mathrm{d}x$;　(23) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\cos 2x\mathrm{d}x$;　(24) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x-\cos^3 x}\mathrm{d}x$;

(25) $\int_0^{\pi} \sqrt{1+\cos 2x}\mathrm{d}x$;　(26) $\int_0^{2\pi} |\sin(x+1)|\mathrm{d}x$.

解 (1) $\int_{\frac{\pi}{3}}^{\pi} \sin\left(x+\frac{\pi}{3}\right)\mathrm{d}x=\int_{\frac{\pi}{3}}^{\pi} \sin\left(x+\frac{\pi}{3}\right)\mathrm{d}\left(x+\frac{\pi}{3}\right)=\left[-\cos\left(x+\frac{\pi}{3}\right)\right]_{\frac{\pi}{3}}^{\pi}=0.$

(2) $\int_{-2}^{1} \frac{\mathrm{d}x}{(11+5x)^3}=\int_{-2}^{1}\frac{\mathrm{d}(11+5x)}{5(11+5x)^3}=\left[-\frac{1}{10(11+5x)^2}\right]_{-2}^{1}=\frac{51}{512}.$

(3) $\int_0^{\frac{\pi}{2}} \sin\varphi\cos^3\varphi\mathrm{d}\varphi=-\int_0^{\frac{\pi}{2}}\cos^3\varphi\mathrm{d}(\cos\varphi)=\left[-\frac{1}{4}\cos^4\varphi\right]_0^{\frac{\pi}{2}}=\frac{1}{4}.$

(4) $\int_0^{\pi}(1-\sin^3\theta)\mathrm{d}\theta=\pi+\int_0^{\pi}(1-\cos^2\theta)\mathrm{d}(\cos\theta)\xlongequal{u=\cos\theta}\pi+\int_1^{-1}(1-u^2)\mathrm{d}u=\pi-\frac{4}{3}.$

(5) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 u\mathrm{d}u=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1+\cos 2u)\mathrm{d}u=\frac{1}{2}\left[u+\frac{1}{2}\sin 2u\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}=\frac{\pi}{6}-\frac{\sqrt{3}}{8}.$

(6) $\int_0^{\sqrt{2}}\sqrt{2-x^2}\mathrm{d}x\xlongequal{x=\sqrt{2}\sin u}\int_0^{\frac{\pi}{2}}2\cos^2 u\mathrm{d}u=2\times\frac{\pi}{4}=\frac{\pi}{2}.$

(7) $\int_{-\sqrt{2}}^{\sqrt{2}}\sqrt{8-2y^2}\mathrm{d}y\xlongequal{y=2\sin u}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}4\sqrt{2}\cos^2 u\mathrm{d}u=2\sqrt{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1+\cos 2u)\mathrm{d}u$

$$=2\sqrt{2}\left[u+\frac{1}{2}\sin 2u\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=\sqrt{2}(\pi+2).$$

(8) $\int_{\frac{1}{\sqrt{2}}}^{1}\frac{\sqrt{1-x^2}}{x^2}\mathrm{d}x\xlongequal{x=\sin u}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos^2 u}{\sin^2 u}\mathrm{d}u=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\csc^2 u-1)\mathrm{d}u=\left[-\cot u-u\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}=1-\frac{\pi}{4}.$

(9) $\int_0^a x^2\sqrt{a^2-x^2}\mathrm{d}x\xlongequal{x=a\sin u}\int_0^{\frac{\pi}{2}}a^4\sin^2 u\cos^2 u\mathrm{d}u=\frac{a^4}{8}\int_0^{\frac{\pi}{2}}(\sin 2u)^2\mathrm{d}(2u)$

$$\xlongequal{t=2u}\frac{a^4}{8}\int_0^{\pi}\sin^2 t\mathrm{d}t=\frac{a^4}{4}\int_0^{\frac{\pi}{2}}\sin^2 t\mathrm{d}t=\frac{a^4}{4}\times\frac{\pi}{4}=\frac{\pi}{16}a^4.$$

(10) $\int_1^{\sqrt{3}}\frac{\mathrm{d}x}{x^2\sqrt{1+x^2}}\xlongequal{x=\frac{1}{u}}\int_1^{\frac{1}{\sqrt{3}}}\frac{-u}{\sqrt{1+u^2}}\mathrm{d}u=\left[-\sqrt{1+u^2}\right]_1^{\frac{1}{\sqrt{3}}}=\sqrt{2}-\frac{2\sqrt{3}}{3}.$

(11)令 $u=\sqrt{5-4x}$,即 $x=\frac{5-u^2}{4}$,得

$$\int_{-1}^{1}\frac{x\mathrm{d}x}{\sqrt{5-4x}}=\int_3^1\frac{u^2-5}{8}\mathrm{d}u=\left[\frac{u^3}{24}-\frac{5}{8}u\right]_3^1=\frac{1}{6}.$$

(12)令 $u=\sqrt{x}$,即 $x=u^2$,得

$$\int_1^4\frac{\mathrm{d}x}{1+\sqrt{x}}=\int_1^2\frac{2u\mathrm{d}u}{1+u}=\left[2u-2\ln(1+u)\right]_1^2=2+2\ln\frac{2}{3}.$$

(13)令 $u=\sqrt{1-x}$,即 $x=1-u^2$,得

$$\int_{\frac{3}{4}}^{1}\frac{\mathrm{d}x}{\sqrt{1-x}-1}=\int_{\frac{1}{2}}^{0}\frac{-2u\mathrm{d}u}{u-1}=-2\left[u+\ln|u-1|\right]_{\frac{1}{2}}^{0}=1-2\ln 2.$$

(14) $\int_0^{\sqrt{2}a}\frac{x\mathrm{d}x}{\sqrt{3a^2-x^2}}=-\frac{1}{2}\int_0^{\sqrt{2}a}\frac{\mathrm{d}(3a^2-x^2)}{\sqrt{3a^2-x^2}}=-\left[\sqrt{3a^2-x^2}\right]_0^{\sqrt{2}a}=(\sqrt{3}-1)a.$

(15) $\int_0^1 te^{-\frac{t^2}{2}}\mathrm{d}t=-\int_0^1 e^{-\frac{t^2}{2}}\mathrm{d}\left(-\frac{t^2}{2}\right)=\left[-e^{-\frac{t^2}{2}}\right]_0^1=1-e^{-\frac{1}{2}}.$

(16) $\int_1^{e^2}\frac{\mathrm{d}x}{x\sqrt{1+\ln x}}\xlongequal{x=e^u}\int_0^2\frac{\mathrm{d}u}{\sqrt{1+u}}=\left[2\sqrt{1+u}\right]_0^2=2\sqrt{3}-2.$

(17) $\int_{-2}^{0}\frac{x+2\mathrm{d}x}{x^2+2x+2}=\int_{-2}^{0}\frac{(x+1)+1}{(x+1)^2+1}\mathrm{d}x=\left[\frac{1}{2}\ln(x^2+2x+2)+\arctan(x+1)\right]_{-2}^{0}=\frac{\pi}{2}.$

(18)令 $x=1+\tan u$,则 $\mathrm{d}x=\sec^2 u\mathrm{d}u$,因此

$$\int_0^2\frac{x\mathrm{d}x}{(x^2-2x+2)^2}=\int_0^2\frac{x\mathrm{d}x}{[(x-1)^2+1]^2}=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{(1+\tan u)\mathrm{d}u}{\sec^2 u}=2\int_0^{\frac{\pi}{4}}\cos^2 u\mathrm{d}u$$

$$=\int_0^{\frac{\pi}{4}}(1+\cos 2u)\mathrm{d}u=\frac{\pi}{4}+\frac{1}{2}.$$

(19)由于被积函数为奇函数,因此 $\int_{-\pi}^{\pi}x^4\sin x\mathrm{d}x=0.$

(20)由于被积函数为偶函数,因此

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4\cos^4\theta\mathrm{d}\theta=2\int_0^{\frac{\pi}{2}}4\cos^4\theta\mathrm{d}\theta=8\times\frac{3}{4}\times\frac{\pi}{4}=\frac{3}{2}\pi.$$

(21)由于被积函数为偶函数,因此

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{(\arcsin x)^2}{\sqrt{1-x^2}}\mathrm{d}x=2\int_0^{\frac{1}{2}}\frac{(\arcsin x)^2}{\sqrt{1-x^2}}\mathrm{d}x=2\int_0^{\frac{1}{2}}(\arcsin x)^2\mathrm{d}(\arcsin x)$$

$$=\frac{2}{3}\left[(\arcsin x)^3\right]_0^{\frac{1}{2}}=\frac{\pi^3}{324}.$$

(22)由于被积函数为奇函数,因此 $\int_{-5}^{5}\frac{x^3\sin^2 x}{x^4+2x^2+1}\mathrm{d}x=0.$

(23) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\cos 2x\mathrm{d}x=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x(1-2\sin^2 x)\mathrm{d}x=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-2\sin^2 x)\mathrm{d}(\sin x)$

$$=\left[\sin x-\frac{2}{3}\sin^3 x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{2}{3}.$$

或者

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\cos 2x\mathrm{d}x=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos 3x+\cos x)\mathrm{d}x$$

$$=\frac{1}{2}\left[\frac{1}{3}\sin 3x+\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{2}{3}.$$

(24) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos x-\cos^3 x}\mathrm{d}x=2\int_0^{\frac{\pi}{2}}\sqrt{\cos x}\sin x\mathrm{d}x\xlongequal{u=\cos x}-2\int_1^0\sqrt{u}\mathrm{d}u=\frac{4}{3}.$

(25) $\int_0^{\pi}\sqrt{1+\cos 2x}\mathrm{d}x=\int_0^{\pi}\sqrt{2\cos^2 x}\mathrm{d}x=\sqrt{2}\int_0^{\pi}|\cos x|\mathrm{d}x$

$$=\sqrt{2}\left(\int_0^{\frac{\pi}{2}}\cos x\mathrm{d}x-\int_{\frac{\pi}{2}}^{\pi}\cos x\mathrm{d}x\right)=\sqrt{2}\left(\sin x\Big|_0^{\frac{\pi}{2}}-\sin x\Big|_{\frac{\pi}{2}}^{\pi}\right)=2\sqrt{2}.$$

(26) $\int_0^{2\pi}|\sin(x+1)|\mathrm{d}x\xlongequal{x=u-1}\int_1^{2\pi+1}|\sin u|\mathrm{d}u,$

由于 $|\sin x|$ 是以 $\pi$ 为周期的周期函数,因此上式 $=2\int_0^{\pi}|\sin u|\mathrm{d}u=4.$

2. 设 $f(x)$ 在 $[a,b]$ 上连续,证明 $\int_a^b f(x)\mathrm{d}x=\int_a^b f(a+b-x)\mathrm{d}x.$

**证** 令 $x=a+b-u$,则 $\int_a^b f(x)\mathrm{d}x=-\int_b^a f(a+b-u)\mathrm{d}u=\int_a^b f(a+b-u)\mathrm{d}u=\int_a^b f(a+b-x)\mathrm{d}x.$

3. 证明:$\int_x^1\frac{\mathrm{d}t}{1+t^2}=\int_1^{\frac{1}{x}}\frac{\mathrm{d}t}{1+t^2}\quad(x>0).$

**证** $\int_x^1\frac{\mathrm{d}t}{1+t^2}\xlongequal{t=\frac{1}{u}}-\int_{\frac{1}{x}}^1\frac{\mathrm{d}u}{1+u^2}=\int_1^{\frac{1}{x}}\frac{\mathrm{d}u}{1+u^2}=\int_1^{\frac{1}{x}}\frac{\mathrm{d}t}{1+t^2}.$

4. 证明:$\int_0^1 x^m(1-x)^n\mathrm{d}x=\int_0^1 x^n(1-x)^m\mathrm{d}x\quad(m,n\in\mathbf{N}).$

证　令 $x=1-u$，则 $\int_0^1 x^m(1-x)^n\mathrm{d}x=\int_1^0 -(1-u)^m u^n\mathrm{d}u=\int_0^1 x^n(1-x)^m\mathrm{d}x$.

5. 设 $f(x)$ 在 $[0,1]$ 上连续，$n\in\mathbf{Z}$，证明：

$$\int_{\frac{n}{2}\pi}^{\frac{n+1}{2}\pi} f(|\sin x|)\mathrm{d}x=\int_{\frac{n}{2}\pi}^{\frac{n+1}{2}\pi} f(|\cos x|)\mathrm{d}x=\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x.$$

证　令 $x=u+\frac{n}{2}\pi$，则 $\mathrm{d}x=\mathrm{d}u$，因此

$$\int_{\frac{n}{2}\pi}^{\frac{n+1}{2}\pi} f(|\sin x|)\mathrm{d}x=\int_0^{\frac{\pi}{2}} f\left(\left|\sin\left(u+\frac{n}{2}\pi\right)\right|\right)\mathrm{d}u=\begin{cases}\int_0^{\frac{\pi}{2}} f(\sin u)\mathrm{d}u，n\text{ 为偶数}，\\ \int_0^{\frac{\pi}{2}} f(\cos u)\mathrm{d}u，n\text{ 为奇数}.\end{cases}$$

$$\int_{\frac{n}{2}\pi}^{\frac{n+1}{2}\pi} f(|\cos x|)\mathrm{d}x=\int_0^{\frac{\pi}{2}} f\left(\left|\cos\left(u+\frac{n}{2}\pi\right)\right|\right)\mathrm{d}u=\begin{cases}\int_0^{\frac{\pi}{2}} f(\cos u)\mathrm{d}u，n\text{ 为偶数}，\\ \int_0^{\frac{\pi}{2}} f(\sin u)\mathrm{d}u，n\text{ 为奇数}.\end{cases}$$

由于 $\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x=\int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x$，因此结论成立.

6. 若 $f(x)$ 是连续的奇函数，证明 $\int_0^x f(t)\mathrm{d}t$ 是偶函数；若 $f(x)$ 是连续的偶函数，证明 $\int_0^x f(t)\mathrm{d}t$ 是奇函数.

6 题视频解析

证　记 $F(x)=\int_0^x f(t)\mathrm{d}t$，则有 $F(-x)=\int_0^{-x} f(t)\mathrm{d}t\xlongequal{t=-u}-\int_0^x f(-u)\mathrm{d}u$，

当 $f(x)$ 为奇函数时，$F(-x)=\int_0^x f(u)\mathrm{d}u=F(x)$，故 $\int_0^x f(t)\mathrm{d}t$ 是偶函数.

当 $f(x)$ 为偶函数时，$F(-x)=-\int_0^x f(u)\mathrm{d}u=-F(x)$，故 $\int_0^x f(t)\mathrm{d}t$ 是奇函数.

7. 设 $x=\varphi(y)$ 是单调函数 $y=x\mathrm{e}^{x^2}$ 的反函数，求 $\int_0^{\mathrm{e}}\varphi(y)\mathrm{d}y$.

7 题视频解析

解　当 $y=0$ 时，$x=0$，当 $y=\mathrm{e}$ 时，$x=1$.

$$\int_0^{\mathrm{e}}\varphi(y)\mathrm{d}y=\int_0^1 x\mathrm{d}(x\mathrm{e}^{x^2})=x^2\mathrm{e}^{x^2}\Big|_0^1-\int_0^1 x\mathrm{e}^{x^2}\mathrm{d}x=\mathrm{e}-\frac{1}{2}\mathrm{e}^{x^2}\Big|_0^1=\mathrm{e}-\frac{1}{2}(\mathrm{e}-1)$$

$$=\frac{\mathrm{e}+1}{2}.$$

8. 计算下列定积分：

8 题视频解析

(1) $\int_0^1 x\mathrm{e}^{-x}\mathrm{d}x$；　(2) $\int_1^{\mathrm{e}} x\ln x\mathrm{d}x$；

(3) $\int_0^{\frac{2\pi}{\omega}} t\sin\omega t\mathrm{d}t$（$\omega$ 为常数）；　(4) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{x}{\sin^2 x}\mathrm{d}x$；

(5) $\int_1^4\frac{\ln x}{\sqrt{x}}\mathrm{d}x$；　(6) $\int_0^1 x\arctan x\mathrm{d}x$；

(7) $\int_0^{\frac{\pi}{2}}\mathrm{e}^{2x}\cos x\mathrm{d}x$；　(8) $\int_1^2 x\log_2 x\mathrm{d}x$；

(9) $\int_0^{\pi}(x\sin x)^2\mathrm{d}x$；　(10) $\int_1^{\mathrm{e}}\sin(\ln x)\mathrm{d}x$；

(11) $\int_{\frac{1}{\mathrm{e}}}^{\mathrm{e}}|\ln x|\mathrm{d}x$；　(12) $\int_0^1(1-x^2)^{\frac{m}{2}}\mathrm{d}x(m\in\mathbf{N}_+)$；

(13) $J_m=\int_0^{\pi}x\sin^m x\mathrm{d}x(m\in\mathbf{N}_+)$.

**解** (1) $\int_0^1 xe^{-x}dx=-\int_0^1 xd(e^{-x})=-\left[xe^{-x}\right]_0^1+\int_0^1 e^{-x}dx=-e^{-1}+\left[-e^{-x}\right]_0^1=1-\frac{2}{e}$.

(2) $\int_1^e x\ln xdx=\int_1^e \frac{\ln x}{2}d(x^2)=\left[\frac{1}{2}x^2\ln x\right]_1^e-\int_1^e \frac{x}{2}dx=\frac{e^2+1}{4}$.

(3) $\int_0^{\frac{2\pi}{\omega}} t\sin \omega tdt=-\frac{1}{\omega}\int_0^{\frac{2\pi}{\omega}} td(\cos \omega t)=-\frac{1}{\omega}\left[t\cos \omega t\right]_0^{\frac{2\pi}{\omega}}+\frac{1}{\omega}\int_0^{\frac{2\pi}{\omega}}\cos \omega tdt$

$=-\frac{2\pi}{\omega^2}+\frac{1}{\omega^2}\left[\sin \omega t\right]_0^{\frac{2\pi}{\omega}}=-\frac{2\pi}{\omega^2}$.

(4) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\sin^2 x}dx=-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} xd(\cot x)=\left[-x\cot x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}+\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\cot xdx=-\frac{\pi}{3\sqrt{3}}+\frac{\pi}{4}+\left[\ln\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$=\left(\frac{1}{4}-\frac{\sqrt{3}}{9}\right)\pi+\frac{1}{2}\ln\frac{3}{2}$.

(5) $\int_1^4 \frac{\ln x}{\sqrt{x}}dx=\int_1^4 2\ln xd\sqrt{x}=\left[2\sqrt{x}\ln x\right]_1^4-\int_1^4\frac{2}{\sqrt{x}}dx=8\ln 2-\left[4\sqrt{x}\right]_1^4=4(2\ln 2-1)$.

(6) $\int_0^1 x\arctan xdx=\frac{1}{2}\int_0^1 \arctan xd(x^2)=\left[\frac{1}{2}x^2\arctan x\right]_0^1-\frac{1}{2}\int_0^1\frac{x^2}{1+x^2}dx$

$=\frac{\pi}{8}-\frac{1}{2}\left[x-\arctan x\right]_0^1=\frac{\pi}{4}-\frac{1}{2}$.

(7) $\int_0^{\frac{\pi}{2}} e^{2x}\cos xdx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos xd(e^{2x})=\frac{1}{2}\left[e^{2x}\cos x\right]_0^{\frac{\pi}{2}}+\frac{1}{2}\int_0^{\frac{\pi}{2}} e^{2x}\sin xdx$

$=-\frac{1}{2}+\frac{1}{4}\int_0^{\frac{\pi}{2}}\sin xd(e^{2x})=-\frac{1}{2}+\frac{1}{4}\left[e^{2x}\sin x\right]_0^{\frac{\pi}{2}}-\frac{1}{4}\int_0^{\frac{\pi}{2}} e^{2x}\cos xdx$,

因此有$\int_0^{\frac{\pi}{2}} e^{2x}\cos xdx=\frac{1}{5}(e^{\pi}-2)$.

(8) $\int_1^2 x\log_2 xdx=\frac{1}{2}\int_1^2 \log_2 xd(x^2)=\frac{1}{2}\left[x^2\log_2 x\right]_1^2-\frac{1}{2}\int_1^2\frac{x}{\ln 2}dx$

$=2-\frac{1}{4\ln 2}\left[x^2\right]_1^2=2-\frac{3}{4\ln 2}$.

(9) $\int_0^{\pi}(x\sin x)^2dx=\frac{1}{2}\int_0^{\pi}x^2(1-\cos 2x)dx=\frac{\pi^3}{6}-\frac{1}{4}\int_0^{\pi}x^2d(\sin 2x)$

$=\frac{\pi^3}{6}-\frac{1}{4}\left[x^2\sin 2x\right]_0^{\pi}+\frac{1}{2}\int_0^{\pi}x\sin 2xdx=\frac{\pi^3}{6}-\frac{1}{4}\int_0^{\pi}xd(\cos 2x)$

$=\frac{\pi^3}{6}-\frac{1}{4}\left[x\cos 2x\right]_0^{\pi}+\frac{1}{4}\int_0^{\pi}\cos 2xdx=\frac{\pi^3}{6}-\frac{\pi}{4}$.

(10) $\int_1^e \sin(\ln x)dx\xlongequal{x=e^u}\int_0^1 e^u\sin udu=\left[e^u\sin u\right]_0^1-\int_0^1 e^u\cos udu$

$=e\sin1-\left[e^u\cos u\right]_0^1-\int_0^1 e^u\sin udu=e(\sin 1-\cos1)+1-\int_0^1 e^u\sin udu$

所以$\int_1^e\sin(\ln x)dx=\frac{e}{2}(\sin 1-\cos1)+\frac{1}{2}$.

(11) $\int_{\frac{1}{e}}^{e}|\ln x|dx=-\int_{\frac{1}{e}}^{1}\ln xdx+\int_1^e\ln xdx=-\left[x\ln x\right]_{\frac{1}{e}}^{1}+\int_{\frac{1}{e}}^{1}dx+\left[x\ln x\right]_1^e-\int_1^e dx$

$=2-\frac{2}{e}$.

(12) $\int_0^1(1-x^2)^{\frac{m}{2}}dx\xlongequal{x=\sin u}\int_0^{\frac{\pi}{2}}\cos^{m+1}xdx$

$$=\begin{cases}\frac{m}{m+1}\times\frac{m-2}{m-1}\times\cdots\times\frac{1}{2}\times\frac{\pi}{2}, & m\text{ 为奇数},\\ \frac{m}{m+1}\times\frac{m-2}{m-1}\times\cdots\times\frac{2}{3}, & m\text{ 为偶数}.\end{cases}$$

(13)由教材本节的例 6，可得 $J_m=\int_0^{\pi} x\sin^m x\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi}\sin^m x\mathrm{d}x.$

而 $\int_0^{\pi}\sin^m x\mathrm{d}x \xlongequal{x=\frac{\pi}{2}+t} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^m t\mathrm{d}t=2\int_0^{\frac{\pi}{2}}\cos^m t\mathrm{d}t=2\int_0^{\frac{\pi}{2}}\sin^m x\mathrm{d}x,$

故 $J_m=\pi\int_0^{\frac{\pi}{2}}\sin^m x\mathrm{d}x.$

从而有
$$J_m=\begin{cases}\dfrac{2\times4\times6\times\cdots\times(m-1)}{1\times3\times5\times\cdots\times m}\times\pi, & m\text{ 为大于 1 的奇数},\\ \dfrac{1\times3\times5\times\cdots\times(m-1)}{2\times4\times6\times\cdots\times m}\times\dfrac{\pi^3}{2}, & m\text{ 为偶数},\end{cases}$$
$J_1=\pi.$

# 第四节 反常积分

## 一、主要内容归纳

**1. 无穷区间上的反常积分** 设函数 $f(x)$ 在区间 $[a,+\infty)$ 上有定义，在 $[a,b]$ 上可积，若极限 $\lim\limits_{b\to+\infty}\int_a^b f(x)\mathrm{d}x$ 存在，则定义 $\int_a^{+\infty}f(x)\mathrm{d}x=\lim\limits_{b\to+\infty}\int_a^b f(x)\mathrm{d}x$，并称 $\int_a^{+\infty}f(x)\mathrm{d}x$ 为 $f(x)$ 在 $[a,+\infty)$ 上的反常积分，这时也称反常积分 $\int_a^{+\infty}f(x)\mathrm{d}x$ 存在或收敛；若上述极限不存在，则称反常积分 $\int_a^{+\infty}f(x)\mathrm{d}x$ 不存在或发散.

类似地，定义 $\int_{-\infty}^{b}f(x)\mathrm{d}x=\lim\limits_{a\to-\infty}\int_a^b f(x)\mathrm{d}x$，

$$\int_{-\infty}^{+\infty}f(x)\mathrm{d}x=\int_{-\infty}^{c}f(x)\mathrm{d}x+\int_{c}^{+\infty}f(x)\mathrm{d}x=\lim_{a\to-\infty}\int_a^c f(x)\mathrm{d}x+\lim_{b\to+\infty}\int_c^b f(x)\mathrm{d}x.$$

**2. 无界函数的反常积分(瑕积分)** 设函数 $f(x)$ 在 $[a,b)$ 上连续，而且 $\lim\limits_{x\to b^-}f(x)=\infty$，若极限 $\lim\limits_{\varepsilon\to0^+}\int_a^{b-\varepsilon}f(x)\mathrm{d}x$ 存在，则定义

$$\int^{b}f(x)\mathrm{d}x=\lim\int^{b-\varepsilon}f(x)\mathrm{d}x.$$