习题 4—4 解答

求下列不定积分：

1. $\int \frac{x^3}{x+3}\mathrm{d}x$. 2. $\int \frac{2x+3}{x^2+3x-10}\mathrm{d}x$. 3. $\int \frac{x+1}{x^2-2x+5}\mathrm{d}x$.

4. $\int \frac{\mathrm{d}x}{x(x^2+1)}$. 5. $\int \frac{3}{x^3+1}\mathrm{d}x$. 6. $\int \frac{x^2+1}{(x+1)^2(x-1)}\mathrm{d}x$.

7. $\int \frac{x\mathrm{d}x}{(x+1)(x+2)(x+3)}$. 8. $\int \frac{x^5+x^4-8}{x^3-x}\mathrm{d}x$. 9. $\int \frac{\mathrm{d}x}{(x^2+1)(x^2+x)}$.

10. $\int \frac{1}{x^4-1}\mathrm{d}x$. 11. $\int \frac{\mathrm{d}x}{(x^2+1)(x^2+x+1)}$. 12. $\int \frac{(x+1)^2}{(x^2+1)^2}\mathrm{d}x$.

13. $\int \frac{-x^2-2}{(x^2+x+1)^2}\mathrm{d}x$. 14. $\int \frac{\mathrm{d}x}{3+\sin^2 x}$. 15. $\int \frac{\mathrm{d}x}{3+\cos x}$.

16. $\int \frac{\mathrm{d}x}{2+\sin x}$. 17. $\int \frac{\mathrm{d}x}{1+\sin x+\cos x}$. 18. $\int \frac{\mathrm{d}x}{2\sin x-\cos x+5}$.

19. $\int \frac{\mathrm{d}x}{1+\sqrt[3]{x+1}}$. 20. $\int \frac{(\sqrt{x})^3-1}{\sqrt{x}+1}\mathrm{d}x$. 21. $\int \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\mathrm{d}x$.

22. $\int \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[4]{x}}$. 23. $\int \sqrt{\frac{1-x}{1+x}}\,\frac{\mathrm{d}x}{x}$. 24. $\int \frac{\mathrm{d}x}{\sqrt[3]{(x+1)^2(x-1)^4}}$.

1 题视频解析

7 题视频解析

19 题视频解析

21 题视频解析

22 题视频解析

解 1. $\int \frac{x^3}{x+3}\mathrm{d}x=\int \left(x^2-3x+9-\frac{27}{x+3}\right)\mathrm{d}x=\frac{1}{3}x^3-\frac{3}{2}x^2+9x-27\ln|x+3|+C.$

2. $\int \frac{2x+3}{x^2+3x-10}\mathrm{d}x=\int \frac{\mathrm{d}(x^2+3x-10)}{x^2+3x-10}=\ln|x^2+3x-10|+C.$

3. $\int \frac{x+1}{x^2-2x+5}\mathrm{d}x=\int \frac{x-1}{(x-1)^2+4}\mathrm{d}x+\frac{1}{2}\int \frac{1}{\left(\frac{x-1}{2}\right)^2+1}\mathrm{d}x=\frac{1}{2}\ln(x^2-2x+5)+\arctan\frac{x-1}{2}+C.$

4. $\int \frac{\mathrm{d}x}{x(x^2+1)}=\int \left(\frac{1}{x}-\frac{x}{x^2+1}\right)\mathrm{d}x=\ln|x|-\frac{1}{2}\int \frac{\mathrm{d}(x^2+1)}{x^2+1}=\ln|x|-\frac{1}{2}\ln(x^2+1)+C.$

5. $$\begin{aligned}\int \frac{3}{1+x^3}\mathrm{d}x&=\int \frac{3}{(1+x)(x^2-x+1)}\mathrm{d}x=\int \left(\frac{1}{1+x}+\frac{2-x}{x^2-x+1}\right)\mathrm{d}x\\&=\ln|1+x|-\frac{1}{2}\int \frac{\mathrm{d}(x^2-x+1)}{x^2-x+1}+\frac{3}{2}\int \frac{1}{x^2-x+1}\mathrm{d}x\\&=\ln|1+x|-\frac{1}{2}\ln(x^2-x+1)+\sqrt{3}\int \frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}\mathrm{d}\left(\frac{2x-1}{\sqrt{3}}\right)\\&=\ln|1+x|-\frac{1}{2}\ln(x^2-x+1)+\sqrt{3}\arctan\frac{2x-1}{\sqrt{3}}+C.\end{aligned}$$

6. $$\begin{aligned}\int \frac{x^2+1}{(x+1)^2(x-1)}\mathrm{d}x&=\int \left[\frac{1}{2(x-1)}+\frac{1}{2(x+1)}-\frac{1}{(x+1)^2}\right]\mathrm{d}x\\&=\frac{1}{2}\ln|x-1|+\frac{1}{2}\ln|x+1|+\frac{1}{x+1}+C=\frac{1}{2}\ln|x^2-1|+\frac{1}{x+1}+C.\end{aligned}$$

7. $\int \frac{x\mathrm{d}x}{(x+1)(x+2)(x+3)}=\int \left[-\frac{1}{2(x+1)}+\frac{2}{x+2}-\frac{3}{2(x+3)}\right]\mathrm{d}x=-\frac{1}{2}\ln|x+1|+2\ln|x+2|-\frac{3}{2}\ln|x+3|+C.$

8. $\int \frac{x^5+x^4-8}{x^3-x}\mathrm{d}x=\int \left(x^2+x+1+\frac{8}{x}-\frac{3}{x-1}-\frac{4}{x+1}\right)\mathrm{d}x=\frac{x^3}{3}+\frac{x^2}{2}+x+8\ln|x|-3\ln|x-1|-4\ln|x+1|+C.$

9. $$\begin{aligned}\int \frac{\mathrm{d}x}{(x^2+1)(x^2+x)}&=\int \left[\frac{1}{x}-\frac{1}{2(x+1)}-\frac{1+x}{2(x^2+1)}\right]\mathrm{d}x\\&=\ln|x|-\frac{1}{2}\ln|x+1|-\frac{1}{2}\arctan x-\frac{1}{4}\int \frac{\mathrm{d}(x^2+1)}{x^2+1}\end{aligned}$$

$$=\ln|x|-\frac{1}{2}\ln|x+1|-\frac{1}{2}\arctan x-\frac{1}{4}\ln(x^2+1)+C.$$

10. $\displaystyle\int\frac{1}{x^4-1}\mathrm{d}x=\int\frac{1}{(x-1)(x+1)(x^2+1)}\mathrm{d}x=\frac{1}{4}\int\frac{1}{x-1}\mathrm{d}x-\frac{1}{4}\int\frac{1}{x+1}\mathrm{d}x-\frac{1}{2}\int\frac{1}{x^2+1}\mathrm{d}x$

$$=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\arctan x+C.$$

11. $\displaystyle\int\frac{\mathrm{d}x}{(x^2+1)(x^2+x+1)}=\int\left(\frac{-x}{x^2+1}+\frac{x+1}{x^2+x+1}\right)\mathrm{d}x$

$$=-\frac{\ln(x^2+1)}{2}+\frac{1}{2}\int\frac{\mathrm{d}(x^2+x+1)}{x^2+x+1}+\frac{1}{2}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x$$

$$=-\frac{\ln(x^2+1)}{2}+\frac{\ln(x^2+x+1)}{2}+\frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C.$$

12. $\displaystyle\int\frac{(x+1)^2}{(x^2+1)^2}\mathrm{d}x=\int\frac{x^2+1}{(x^2+1)^2}\mathrm{d}x+\int\frac{2x\mathrm{d}x}{(x^2+1)^2}=\arctan x-\frac{1}{x^2+1}+C.$

13. $\displaystyle\int\frac{-x^2-2}{(x^2+x+1)^2}\mathrm{d}x=\int\left[-\frac{1}{x^2+x+1}+\frac{x-1}{(x^2+x+1)^2}\right]\mathrm{d}x$

$$=-\int\frac{1}{x^2+x+1}\mathrm{d}x+\frac{1}{2}\int\frac{\mathrm{d}(x^2+x+1)}{(x^2+x+1)^2}-\frac{3}{2}\int\frac{1}{(x^2+x+1)^2}\mathrm{d}x,$$

令 $u=x+\dfrac{1}{2}$，并记 $a=\dfrac{\sqrt{3}}{2}$，则

$$\int\frac{1}{(x^2+x+1)^2}\mathrm{d}x=\int\frac{1}{(u^2+a^2)^2}\mathrm{d}u\xlongequal{(*)}\frac{1}{2a^2}\left[\frac{u}{u^2+a^2}+\int\frac{1}{u^2+a^2}\mathrm{d}u\right]$$

$$=\frac{u}{2a^2(u^2+a^2)}+\frac{1}{2a^2}\int\frac{1}{u^2+a^2}\mathrm{d}u,$$

由此得 $$\int\frac{1}{x^2+x+1}\mathrm{d}x+\frac{3}{2}\int\frac{1}{(x^2+x+1)^2}\mathrm{d}x$$

$$=\int\frac{1}{u^2+a^2}\mathrm{d}u+\frac{3}{2}\left[\frac{u}{2a^2(u^2+a^2)}+\frac{1}{2a^2}\int\frac{1}{u^2+a^2}\mathrm{d}u\right]$$

$$=\frac{3u}{4a^2(u^2+a^2)}+\left(\frac{3}{4a^2}+1\right)\int\frac{1}{u^2+a^2}\mathrm{d}u$$

$$=\frac{3u}{4a^2(u^2+a^2)}+\frac{1}{a}\left(\frac{3}{4a^2}+1\right)\arctan\frac{u}{a}+C_1$$

$$=\frac{2x+1}{2(x^2+x+1)}+\frac{4}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C_1.$$

因此有 $$\int\frac{-x^2-2}{(x^2+x+1)^2}\mathrm{d}x=-\frac{1}{2(x^2+x+1)}-\frac{2x+1}{2(x^2+x+1)}-\frac{4}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C$$

$$=-\frac{x+1}{x^2+x+1}-\frac{4}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C.$$

评注：其中（*）这一步是利用了递推公式：

$$I_{m+1}=\frac{u}{2a^2m(u^2+a^2)^m}+\frac{2m-1}{2a^2m}I_m,$$

取 $m=1$ 即可，且 $I_m=\displaystyle\int\frac{\mathrm{d}u}{(u^2+a^2)^m}$，利用分部积分法可推出上述递推公式.

14. $\displaystyle\int\frac{\mathrm{d}x}{3+\sin^2x}=-\int\frac{\mathrm{d}(\cot x)}{3\csc^2x+1}\xlongequal{u=\cot x}-\int\frac{\mathrm{d}u}{3u^2+4}=-\frac{1}{2\sqrt{3}}\arctan\frac{\sqrt{3}u}{2}+C=-\frac{1}{2\sqrt{3}}\arctan\frac{\sqrt{3}\cot x}{2}+C.$

15. 令 $u=\tan\frac{x}{2}$，则 $\int\frac{\mathrm{d}x}{3+\cos x}=\int\frac{1}{3+\frac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}\mathrm{d}u=\int\frac{1}{2+u^2}\mathrm{d}u=\frac{1}{\sqrt{2}}\arctan\frac{u}{\sqrt{2}}+C$

$$=\frac{1}{\sqrt{2}}\arctan\frac{\tan\frac{x}{2}}{\sqrt{2}}+C.$$

16. 令 $u=\tan\frac{x}{2}$，则 $\int\frac{\mathrm{d}x}{2+\sin x}=\int\frac{1}{2+\frac{2u}{1+u^2}}\cdot\frac{2}{1+u^2}\mathrm{d}u=\int\frac{1}{u^2+u+1}\mathrm{d}u=\int\frac{1}{\left(u+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\mathrm{d}u$

$$=\frac{2}{\sqrt{3}}\arctan\frac{2u+1}{\sqrt{3}}+C=\frac{2}{\sqrt{3}}\arctan\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}+C.$$

17. 令 $u=\tan\frac{x}{2}$，则 $\int\frac{\mathrm{d}x}{1+\sin x+\cos x}=\int\frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}\mathrm{d}u=\int\frac{\mathrm{d}u}{1+u}=\ln|1+u|+C$

$$=\ln\left|1+\tan\frac{x}{2}\right|+C.$$

18. 令 $u=\tan\frac{x}{2}$，则 $\int\frac{\mathrm{d}x}{2\sin x-\cos x+5}=\int\frac{1}{\frac{4u}{1+u^2}-\frac{1-u^2}{1+u^2}+5}\cdot\frac{2}{1+u^2}\mathrm{d}u=\int\frac{1}{3u^2+2u+2}\mathrm{d}u$

$$=\frac{1}{3}\int\frac{1}{\left(u+\frac{1}{3}\right)^2+\left(\frac{\sqrt{5}}{3}\right)^2}\mathrm{d}\left(u+\frac{1}{3}\right)=\frac{1}{\sqrt{5}}\arctan\frac{3u+1}{\sqrt{5}}+C$$

$$=\frac{1}{\sqrt{5}}\arctan\frac{3\tan\frac{x}{2}+1}{\sqrt{5}}+C.$$

19. 令 $u=\sqrt[3]{x+1}$，即 $x=u^3-1$，则

$$\int\frac{\mathrm{d}x}{1+\sqrt[3]{x+1}}=\int\frac{3u^2}{1+u}\mathrm{d}u=\int\left(3u-3+\frac{3}{1+u}\right)\mathrm{d}u=\frac{3}{2}u^2-3u+3\ln|1+u|+C$$

$$=\frac{3}{2}\sqrt[3]{(x+1)^2}-3\sqrt[3]{x+1}+3\ln|1+\sqrt[3]{x+1}|+C.$$

20. $\int\frac{(\sqrt{x})^3-1}{\sqrt{x}+1}\mathrm{d}x=\int\left(x-\sqrt{x}+1-\frac{2}{\sqrt{x}+1}\right)\mathrm{d}x=\frac{x^2}{2}-\frac{2}{3}x\sqrt{x}+x-\int\frac{4t}{t+1}\mathrm{d}t$（其中 $t=\sqrt{x}$）

$$=\frac{x^2}{2}-\frac{2}{3}x\sqrt{x}+x-4\int\left(1-\frac{1}{t+1}\right)\mathrm{d}t$$

$$=\frac{x^2}{2}-\frac{2}{3}x\sqrt{x}+x-4\sqrt{x}+4\ln(\sqrt{x}+1)+C.$$

21. 令 $u=\sqrt{x+1}$，即 $x=u^2-1$，则

$$\int\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\mathrm{d}x=\int\frac{u-1}{u+1}\cdot 2u\mathrm{d}u=2\int\left(u-2+\frac{2}{u+1}\right)\mathrm{d}u=u^2-4u+4\ln|u+1|+C$$

$$=x-4\sqrt{x+1}+4\ln(\sqrt{x+1}+1)+C.$$

22. 令 $u=\sqrt[4]{x}$，即 $x=u^4$，则

$$\int\frac{\mathrm{d}x}{\sqrt{x}+\sqrt[4]{x}}=\int\frac{1}{u^2+u}\cdot 4u^3\mathrm{d}u=4\int\left(u-1+\frac{1}{u+1}\right)\mathrm{d}u=2u^2-4u+4\ln|u+1|+C$$

$$=2\sqrt{x}-4\sqrt[4]{x}+4\ln(\sqrt[4]{x}+1)+C.$$

23. **方法一**

令 $u=\sqrt{\frac{1-x}{1+x}}$，即 $x=\frac{1-u^2}{1+u^2}$，则

$$\int\sqrt{\frac{1-x}{1+x}}\cdot\frac{\mathrm{d}x}{x}=\int u\cdot\frac{1+u^2}{1-u^2}\cdot\frac{-4u}{(1+u^2)^2}\mathrm{d}u=\int\frac{-4u^2}{(1-u^2)(1+u^2)}\mathrm{d}u$$

$$=\int\left(\frac{2}{1+u^2}-\frac{1}{1-u}-\frac{1}{1+u}\right)\mathrm{d}u$$

$$=2\arctan u+\ln|1-u|-\ln|1+u|+C$$

$$=2\arctan\sqrt{\frac{1-x}{1+x}}+\ln\left|\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right|+C.$$

方法二

$$\int\sqrt{\frac{1-x}{1+x}}\,\frac{\mathrm{d}x}{x}=\int\frac{1-x}{x\sqrt{1-x^2}}\mathrm{d}x\xlongequal{x=\sin u}\int\frac{1-\sin u}{\sin u}\mathrm{d}u=\int\csc u\mathrm{d}u-\int\mathrm{d}u$$

$$=\ln|\csc u-\cot u|-u+C=\ln\frac{1-\sqrt{1-x^2}}{|x|}-\arcsin x+C.$$

24. $\displaystyle\int\frac{\mathrm{d}x}{\sqrt[3]{(x+1)^2(x-1)^4}}=\int\frac{1}{x^2-1}\sqrt[3]{\frac{x+1}{x-1}}\mathrm{d}x$,

令 $u=\sqrt[3]{\dfrac{x+1}{x-1}}$，即 $x=\dfrac{u^3+1}{u^3-1}$，得到

$$\int\frac{\mathrm{d}x}{\sqrt[3]{(x+1)^2(x-1)^4}}=\int\frac{u}{\left(\frac{u^3+1}{u^3-1}\right)^2-1}\cdot\frac{-6u^2}{(u^3-1)^2}\mathrm{d}u=-\frac{3}{2}\int\mathrm{d}u$$

$$=-\frac{3}{2}u+C=-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}+C.$$

第五节　积分表的使用

（略）

习题 4—5 解答

利用积分表计算下列不定积分：

1. $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{4x^2-9}}$.　　2. $\displaystyle\int\frac{1}{x^2+2x+5}\mathrm{d}x$.　　3. $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{5-4x+x^2}}$.

4. $\displaystyle\int\sqrt{2x^2+9}\,\mathrm{d}x$.　　5. $\displaystyle\int\sqrt{3x^2-2}\,\mathrm{d}x$.　　6. $\displaystyle\int e$

7. $\displaystyle\int x\arcsin\frac{x}{2}\mathrm{d}x$.　　8. $\displaystyle\int\frac{\mathrm{d}x}{(x^2+9)^2}$.

10. $\displaystyle\int e^{-2x}$