

$$\begin{aligned}\int \frac{e^{2x}}{\sqrt{e^x-1}} dx &= \int \frac{e^x}{\sqrt{e^x-1}} de^x = \int \sqrt{e^x-1} de^x + \int \frac{1}{\sqrt{e^x-1}} de^x \\ &= \frac{2}{3}(e^x-1)\sqrt{e^x-1} + 2\sqrt{e^x-1} + C_1,\end{aligned}$$

$$\text{故} \int e^{2x} \arctan \sqrt{e^x-1} dx = \frac{1}{2} e^{2x} \arctan \sqrt{e^x-1} - \frac{1}{6} (e^x+2) \sqrt{e^x-1} + C.$$

方法二 令 $\sqrt{e^x-1} = t, x = \ln(1+t^2)$, 则积分化为 $2 \int (t+t^3) \arctan t dt$.

这是典型的分部积分类型.

习题 4-3 解答

求下列不定积分:

- | | | |
|--|----------------------------------|---------------------------------------|
| 1. $\int x \sin x dx.$ | 2. $\int \ln x dx.$ | 3. $\int \arcsin x dx.$ |
| 4. $\int x e^{-x} dx.$ | 5. $\int x^2 \ln x dx.$ | 6. $\int e^{-x} \cos x dx.$ |
| 7. $\int e^{-2x} \sin \frac{x}{2} dx.$ | 8. $\int x \cos \frac{x}{2} dx.$ | 9. $\int x^2 \arctan x dx.$ |
| 10. $\int x \tan^2 x dx.$ | 11. $\int x^2 \cos x dx.$ | 12. $\int t e^{-2t} dt.$ |
| 13. $\int \ln^2 x dx.$ | 14. $\int x \sin x \cos x dx.$ | 15. $\int x^2 \cos^2 \frac{x}{2} dx.$ |
| 16. $\int x \ln(x-1) dx.$ | 17. $\int (x^2-1) \sin 2x dx.$ | 18. $\int \frac{\ln^3 x}{x^2} dx.$ |
| 19. $\int e^{\sqrt{x}} dx.$ | 20. $\int \cos \ln x dx.$ | 21. $\int (\arcsin x)^2 dx.$ |
| 22. $\int e^x \sin^2 x dx.$ | 23. $\int x \ln^2 x dx.$ | 24. $\int e^{\sqrt{3x+9}} dx.$ |

解 1. $\int x \sin x dx = - \int x d(\cos x) = -x \cos x + \int \cos x dx = -x \cos x + \sin x + C.$

2. $\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C.$

3. $\int \arcsin x dx = x \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}} dx = x \arcsin x + \sqrt{1-x^2} + C.$

4. $\int x e^{-x} dx = - \int x d e^{-x} = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} + C.$

5. $\int x^2 \ln x dx = \frac{1}{3} \int \ln x d(x^3) = \frac{x^3 \ln x}{3} - \frac{1}{3} \int x^3 \cdot \frac{1}{x} dx = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C.$

6. $\begin{aligned} \int e^{-x} \cos x dx &= - \int \cos x d(e^{-x}) = -e^{-x} \cos x + \int e^{-x} (-\sin x) dx \\ &= -e^{-x} \cos x + \int \sin x d(e^{-x}) = -e^{-x} \cos x + e^{-x} \sin x - \int e^{-x} \cos x dx, \end{aligned}$

故有 $\int e^{-x} \cos x dx = \frac{e^{-x} (\sin x - \cos x)}{2} + C.$

7. $\begin{aligned} \int e^{-2x} \sin \frac{x}{2} dx &= -\frac{1}{2} \int \sin \frac{x}{2} d(e^{-2x}) = -\frac{1}{2} e^{-2x} \sin \frac{x}{2} + \frac{1}{2} \int e^{-2x} \cdot \frac{1}{2} \cos \frac{x}{2} dx \\ &= -\frac{1}{2} e^{-2x} \sin \frac{x}{2} - \frac{1}{8} \int \cos \frac{x}{2} d(e^{-2x}) \\ &= -\frac{1}{2} e^{-2x} \sin \frac{x}{2} - \frac{1}{8} e^{-2x} \cos \frac{x}{2} + \frac{1}{8} \int e^{-2x} \cdot \left(-\frac{1}{2} \sin \frac{x}{2}\right) dx \end{aligned}$



9 题视频解析



13 题视频解析



21 题视频解析



22 题视频解析



24 题视频解析



$$= -\frac{1}{8} \left(4\sin \frac{x}{2} + \cos \frac{x}{2} \right) e^{-2x} - \frac{1}{16} \int e^{-2x} \sin \frac{x}{2} dx,$$

$$\text{故 } \int e^{-2x} \sin \frac{x}{2} dx = -\frac{2}{17} \left(4\sin \frac{x}{2} + \cos \frac{x}{2} \right) e^{-2x} + C.$$

$$8. \int x \cos \frac{x}{2} dx = 2 \int x d\left(\sin \frac{x}{2}\right) = 2x \sin \frac{x}{2} - 2 \int \sin \frac{x}{2} dx = 2x \sin \frac{x}{2} + 4 \cos \frac{x}{2} + C.$$

$$\begin{aligned} 9. \int x^2 \arctan x dx &= \frac{1}{3} \int \arctan x d(x^3) = \frac{1}{3} x^3 \arctan x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx \\ &= \frac{1}{3} x^3 \arctan x - \frac{1}{3} \int \left(x - \frac{x}{1+x^2} \right) dx = \frac{1}{3} x^3 \arctan x - \frac{1}{6} x^2 + \frac{1}{6} \ln(1+x^2) + C. \end{aligned}$$

$$10. \int x \tan^2 x dx = \int x (\sec^2 x - 1) dx = \int x d(\tan x) - \frac{x^2}{2} = x \tan x + \ln |\cos x| - \frac{x^2}{2} + C.$$

$$\begin{aligned} 11. \int x^2 \cos x dx &= \int x^2 d(\sin x) = x^2 \sin x - 2 \int x \sin x dx = x^2 \sin x + \int 2x d(\cos x) \\ &= x^2 \sin x + 2x \cos x - \int 2 \cos x dx = x^2 \sin x + 2x \cos x - 2 \sin x + C. \end{aligned}$$

$$12. \int t e^{-2t} dt = -\frac{1}{2} \int t d(e^{-2t}) = -\frac{1}{2} t e^{-2t} + \frac{1}{2} \int e^{-2t} dt = -\frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t} + C.$$

$$13. \int \ln^2 x dx = x \ln^2 x - \int 2 \ln x dx = x \ln^2 x - 2x \ln x + \int 2 dx = x \ln^2 x - 2x \ln x + 2x + C.$$

$$14. \int x \sin x \cos x dx = \int -\frac{x}{4} d(\cos 2x) = -\frac{x \cos 2x}{4} + \frac{1}{4} \int \cos 2x dx = -\frac{x \cos 2x}{4} + \frac{\sin 2x}{8} + C.$$

$$\begin{aligned} 15. \int x^2 \cos^2 \frac{x}{2} dx &= \frac{1}{2} \int x^2 (1 + \cos x) dx = \frac{1}{6} x^3 + \frac{1}{2} \int x^2 d(\sin x) = \frac{1}{6} x^3 + \frac{1}{2} x^2 \sin x - \int x \sin x dx \\ &= \frac{1}{6} x^3 + \frac{1}{2} x^2 \sin x + \int x d(\cos x) = \frac{1}{6} x^3 + \frac{1}{2} x^2 \sin x + x \cos x - \int \cos x dx \\ &= \frac{1}{6} x^3 + \frac{1}{2} x^2 \sin x + x \cos x - \sin x + C. \end{aligned}$$

$$\begin{aligned} 16. \int x \ln(x-1) dx &= \frac{1}{2} \int \ln(x-1) d(x^2-1) = \frac{1}{2} (x^2-1) \ln(x-1) - \frac{1}{2} \int (x+1) dx \\ &= \frac{1}{2} (x^2-1) \ln(x-1) - \frac{1}{4} x^2 - \frac{1}{2} x + C. \end{aligned}$$

$$\begin{aligned} 17. \int (x^2-1) \sin 2x dx &= -\frac{1}{2} \int (x^2-1) d(\cos 2x) = -\frac{1}{2} (x^2-1) \cos 2x + \int x \cos 2x dx \\ &= -\frac{1}{2} (x^2-1) \cos 2x + \frac{1}{2} \int x d(\sin 2x) \\ &= -\frac{1}{2} (x^2-1) \cos 2x + \frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x dx \\ &= -\frac{1}{2} \left(x^2 - \frac{3}{2} \right) \cos 2x + \frac{1}{2} x \sin 2x + C. \end{aligned}$$

$$\begin{aligned} 18. \int \frac{\ln^3 x}{x^2} dx &= \int -\ln^3 x d\left(\frac{1}{x}\right) = -\frac{\ln^3 x}{x} - 3 \int \ln^2 x d\left(\frac{1}{x}\right) \\ &= -\frac{\ln^3 x}{x} - 3 \left[\frac{\ln^2 x}{x} + 2 \int \ln x d\left(\frac{1}{x}\right) \right] = -\frac{\ln^3 x + 3 \ln^2 x + 6 \ln x + 6}{x} + C. \end{aligned}$$

$$\begin{aligned} 19. \int e^{\sqrt[3]{x}} dx &\stackrel{x=u^3}{=} \int 3u^2 e^u du = \int 3u^2 d(e^u) = 3u^2 e^u - \int 6u d(e^u) = (3u^2 - 6u + 6) e^u + C \\ &= 3e^{\sqrt[3]{x}} \left(x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 2 \right) + C. \end{aligned}$$

$$20. \int \cos \ln x dx \stackrel{x=e^u}{=} \int e^u \cos u du,$$

$$\text{而 } \int e^u \cos u du = \int \cos u d(e^u) = e^u \cos u + \int e^u \sin u du = e^u \cos u + \int \sin u d(e^u)$$



$$= e^u \cos u + e^u \sin u - \int e^u \cos u du,$$

$$\text{因此 } \int e^u \cos u du = \frac{e^u (\cos u + \sin u)}{2} + C, \text{ 故有 } \int \cos \ln x dx = \frac{x (\cos \ln x + \sin \ln x)}{2} + C.$$

$$\begin{aligned} 21. \int (\arcsin x)^2 dx &= x(\arcsin x)^2 - \int \frac{2x \arcsin x}{\sqrt{1-x^2}} dx = x(\arcsin x)^2 + \int 2 \arcsin x d(\sqrt{1-x^2}) \\ &= x(\arcsin x)^2 + 2 \sqrt{1-x^2} \arcsin x - 2x + C. \end{aligned}$$

$$\begin{aligned} 22. \int e^x \sin^2 x dx &= \frac{1}{2} \int e^x (1 - \cos 2x) dx = \frac{1}{2} e^x - \frac{1}{2} \int e^x \cos 2x dx, \\ \int e^x \cos 2x dx &= \int \cos 2x d(e^x) = e^x \cos 2x + 2 \int e^x \sin 2x dx = e^x \cos 2x + 2 \int \sin 2x d(e^x) \\ &= e^x \cos 2x + 2e^x \sin 2x - 4 \int e^x \cos 2x dx, \\ \text{得 } \int e^x \cos 2x dx &= \frac{e^x \cos 2x + 2e^x \sin 2x}{5} + C, \text{ 因此有} \end{aligned}$$

$$\int e^x \sin^2 x dx = \frac{1}{2} e^x - \frac{1}{5} e^x \sin 2x - \frac{1}{10} e^x \cos 2x + C.$$

$$\begin{aligned} 23. \int x \ln^2 x dx &= \int \ln^2 x d\left(\frac{x^2}{2}\right) = \frac{x^2}{2} \ln^2 x - \int x \ln x dx = \frac{x^2}{2} \ln^2 x - \int \ln x d\left(\frac{x^2}{2}\right) \\ &= \frac{x^2}{2} \ln^2 x - \frac{x^2}{2} \ln x + \int \frac{x}{2} dx = \frac{x^2}{4} (2 \ln^2 x - 2 \ln x + 1) + C. \end{aligned}$$

$$\begin{aligned} 24. \text{ 设 } \sqrt{3x+9} &= u, \text{ 即 } x = \frac{1}{3}(u^2-9), dx = \frac{2}{3}u du, \text{ 则} \\ \int e^{\sqrt{3x+9}} dx &= \int \frac{2}{3} u e^u du = \int \frac{2}{3} u d(e^u) = \frac{2}{3} u e^u - \int \frac{2}{3} e^u du \\ &= \frac{2}{3} u e^u - \frac{2}{3} e^u + C = \frac{2}{3} e^{\sqrt{3x+9}} (\sqrt{3x+9} - 1) + C. \end{aligned}$$

第四节 有理函数的积分

一、主要内容归纳

1. 有理函数的积分 一般要经过两个步骤:

(1) 如果被积函数是假分式, 则需先将其化为多项式与真分式之和; 对于真分式 $\frac{P(x)}{Q(x)}$ 在实数范围内将 $Q(x)$ 分解成一次因式与二次质因式的乘积, 分解结果只含 $(x-a)^k$ 和 $(x^2+px+q)^L$ ($p^2-4q < 0$) 两种类型的因式.

(2) 根据 $Q(x)$ 的分解结果将真分式化为部分分式, 用待定系数法确定真分式的分子中的常数, 最后得到以下 4 个基本类型的积分:

$$\textcircled{1} \int \frac{A}{x-a} dx, \quad \textcircled{2} \int \frac{A}{(x-a)^2} dx, \quad \textcircled{3} \int \frac{A}{(x-a)^3} dx, \quad \textcircled{4} \int \frac{A}{(x-a)^4} dx, \quad \dots$$