

高等数学



4.2 积分的换元法

基础部数学教研室

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● 基本公式——导数公式与积分公式

$$\left(\frac{x^{a+1}}{a+1}\right)' = x^a$$



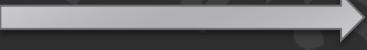
$$\int x^a \, dx = \frac{x^{a+1}}{a+1} + C \quad (a \neq -1)$$

$$(\ln|x|)' = \frac{1}{x}$$



$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\left(\frac{a^x}{\ln a}\right)' = a^x$$



$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$(\sin x)' = \cos x$$



$$\int \cos x \, dx = \sin x + C$$

基本公式——导数公式与积分公式

$$(\tan x)' = \sec^2 x$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$(\sec x)' = \sec x \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$$

$$\int \frac{1}{\sqrt{1 - x^2}} \, dx = \arcsin x + C$$

$$(\arctan x)' = \frac{1}{1 + x^2}$$

$$\int \frac{1}{1 + x^2} \, dx = \arctan x + C$$

不定积分的第一类换元法

不定积分的第二类换元法



定理1 设 $f(u)$ 具有原函数 $F(u)$, $u = \varphi(x)$ 可导, 则

$$\int f[\varphi(x)]\varphi'(x)dx \stackrel{u=\varphi(x)}{=} \left[\int f(u)du \right]_{u=\varphi(x)} = F(\varphi(x)) + C$$

称为**不定积分的第一类换元积分公式** (也称 **凑微分法**).

例1 求以下不定积分

$$\int \frac{1}{5 + 7x} dx$$

$$\int \frac{x^2}{(x + 2)^3} dx$$

$$\int 2xe^{x^2} dx$$

$$\int x\sqrt{1 - x^2} dx$$

$$\int \sin^3 x dx$$

$$\int \tan x dx$$

例2 求不定积分

$$\int \sec^6 x dx$$

$$\int \cos 3x \cos 2x dx$$

$$\int \frac{1+x^4}{1+x^6} dx$$

例3 求 $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C$$

$$\int \sqrt{a^2 + x^2} dx \quad \text{留待分部积分解决}$$

例4 求

$$\int \frac{dx}{x^2 + 4x + 5}$$

$$\int \frac{dx}{x^2 + 4x + 6}$$

$$\int \frac{10^{2\arccos x}}{\sqrt{1-x^2}} dx$$

$$\int \frac{\arctan \sqrt{x}}{\sqrt{x}(1+x)} dx$$

例5 求 $\int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

$$\int \cos^2 x dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

$$\int \csc x dx = \ln|\csc x - \cot x| + C$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C$$

例6 求不定积分

$$\int \frac{dx}{\cos x \sqrt{\sin x}}$$

$$\int \frac{\ln(\tan x)}{\sin 2x} dx$$

$$\int \frac{dx}{\sqrt{x(4-x)}}$$

$$\int \frac{1+x}{x(1+xe^x)} dx$$

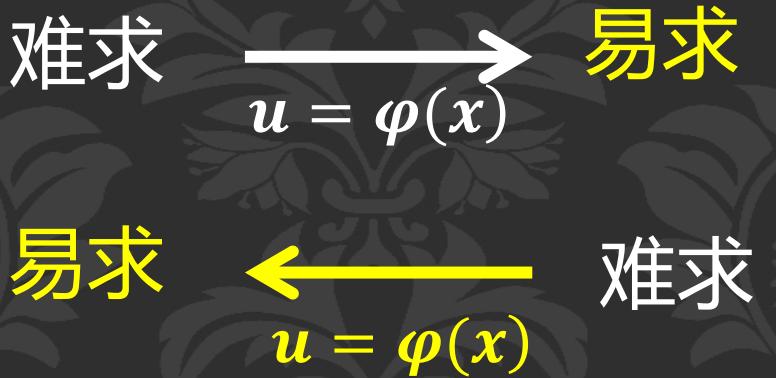
例7 求

$$\int \frac{xdx}{x^2 - 2x\cos\alpha + 1}$$

其中 $\alpha \neq k\pi$

不定积分第一类换元法

$$\int f [\varphi(x)] \varphi'(x) dx = \int f(u) du \Big|_{u=\varphi(x)}$$



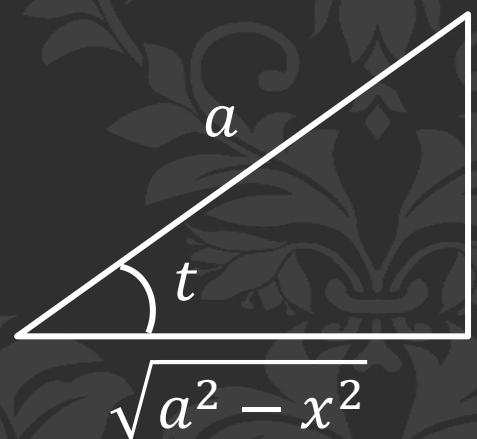
不定积分第二类换元法

定理2 设 $x = \varphi(t)$ 是单调的、可导的函数，并且 $\varphi'(t) \neq 0$ ，
又设 $f[\varphi(t)]\varphi'(t)$ 具有原函数 $\Phi(t)$ ，则有换元公式

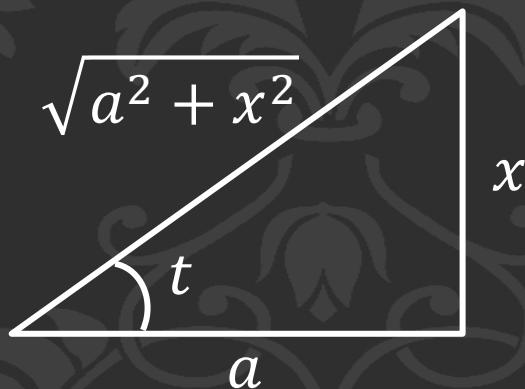
$$\int f(x)dx = \left[\int f[\varphi(t)]\varphi'(t)dt \right]_{t=\varphi^{-1}(x)}.$$

不定积分的第二类换元积分公式

例8 求 $\int \sqrt{a^2 - x^2} dx$ ($a > 0$).



$$\int \frac{1}{\sqrt{x^2 + a^2}} dx \quad (a > 0).$$

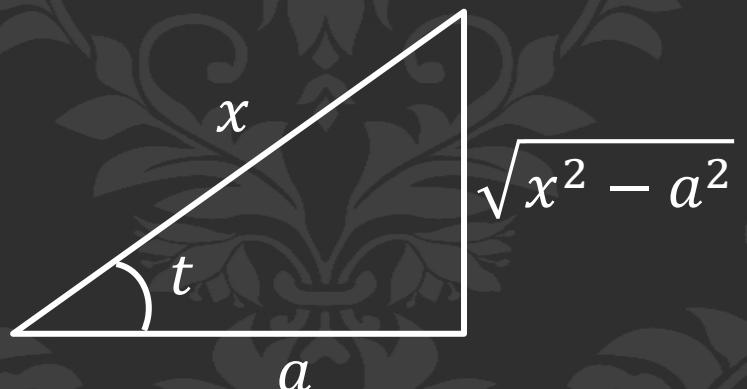


$$\int \frac{1}{\sqrt{x^2 - a^2}} dx \quad (a > 0).$$

($x = a \sin t$)

($x = a \tan t$)

($x = a \sec t$)



例9 求 $\int \sqrt{4 - 3x - x^2} dx$

例10 求 $\int \frac{1}{\sqrt{e^x - 1}} dx$

例11 求 $\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$

$$\int \frac{x^5 - x}{x^8 + 1} dx$$

$$\int \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}}$$