

## 总习题四解答

1. 填空:

(1)  $\int x^3 e^x dx =$  \_\_\_\_\_.

(2)  $\int \frac{x+5}{x^2-6x+13} dx =$  \_\_\_\_\_.

$$\begin{aligned} \text{解} \quad (1) \int x^3 e^x dx &= \int x^3 d(e^x) = x^3 e^x - 3 \int x^2 de^x \\ &= x^3 e^x - 3 \left( x^2 e^x - \int 2x de^x \right) = x^3 e^x - 3x^2 e^x + 6 \left( x e^x - \int e^x dx \right) \\ &= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C, \end{aligned}$$

因此,应填  $x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$ .

$$\begin{aligned} (2) \int \frac{x+5}{x^2-6x+13} dx &= \frac{1}{2} \int \frac{(x^2-6x+13)'}{x^2-6x+13} dx + \int \frac{8}{x^2-6x+13} dx \\ &= \frac{1}{2} \ln(x^2-6x+13) + \int \frac{8}{(x-3)^2+4} dx \\ &= \frac{1}{2} \ln(x^2-6x+13) + 4 \arctan \frac{x-3}{2} + C. \end{aligned}$$

因此,应填  $\frac{1}{2} \ln(x^2-6x+13) + 4 \arctan \frac{x-3}{2} + C$ .

2. 以下两题中给出了四个结论,从中选出一个正确的结论:

(1) 已知  $f'(x) = \frac{1}{x(1+2\ln x)}$ , 且  $f(1)=1$ , 则  $f(x)$  等于 \_\_\_\_\_.

(A)  $\ln(1+2\ln x) + 1$

(B)  $\frac{1}{2} \ln(1+2\ln x) + 1$

(C)  $\frac{1}{2} \ln(1+2\ln x) + \frac{1}{2}$

(D)  $2\ln(1+2\ln x) + 1$

(2) 在下列等式中,正确的结果是 \_\_\_\_\_.

(A)  $\int f'(x) dx = f(x)$

(B)  $\int d f(x) = f(x)$

(C)  $\frac{d}{dx} \int f(x) dx = f(x)$

(D)  $d \int f(x) dx = f(x)$

解 (1) 由微积分基本定理,有

$$\begin{aligned} f(x) - f(1) &= \int_1^x f'(t) dt = \int_1^x \frac{1}{t(1+2\ln t)} dt = \frac{1}{2} \int_1^x \frac{1}{1+2\ln t} d(1+2\ln t) \\ &= \frac{1}{2} \left[ \ln(1+2\ln t) \right]_1^x = \frac{1}{2} \ln(1+2\ln x), \end{aligned}$$

根据条件  $f(1)=1$ , 得  $f(x) = \frac{1}{2} \ln(1+2\ln x) + 1$ . 故选(B).

(2) 根据微分运算与积分运算的关系,可知

$$\int d f(x) = \int f'(x) dx = f(x) + C, \quad \frac{d}{dx} \int f(x) dx = f(x)$$

$$d \int f(x) dx = \left( \frac{d}{dx} \int f(x) dx \right) dx = f(x) dx,$$

故选(C).



2 题视频解析

3. 已知  $\frac{\sin x}{x}$  是  $f(x)$  的一个原函数, 求  $\int x^3 f'(x) dx$ .

解 根据条件, 有  $\int f(x) dx = \frac{\sin x}{x} + C$ , 即  $f(x) = \left(\frac{\sin x}{x}\right)' = \frac{x \cos x - \sin x}{x^2}$ , 因此

$$\begin{aligned} \int x^3 f'(x) dx &= x^3 f(x) - \int 3x^2 f(x) dx = x(x \cos x - \sin x) - 3 \int x^2 d\left(\frac{\sin x}{x}\right) \\ &= x^2 \cos x - x \sin x - 3 \left( x^2 \cdot \frac{\sin x}{x} - \int \frac{\sin x}{x} \cdot 2x dx \right) \\ &= x^2 \cos x - 4x \sin x - 6 \cos x + C. \end{aligned}$$

4. 求下列不定积分(其中  $a, b$  为常数):

(1)  $\int \frac{dx}{e^x - e^{-x}}.$

(2)  $\int \frac{x}{(1-x)^3} dx$

(3)  $\int \frac{x^2}{a^6 - x^6} dx \quad (a > 0).$

(4)  $\int \frac{1 + \cos x}{x + \sin x} dx.$

(5)  $\int \frac{\ln \ln x}{x} dx.$

(6)  $\int \frac{\sin x \cos x}{1 + \sin^4 x} dx.$

(7)  $\int \tan^4 x dx.$

(8)  $\int \sin x \sin 2x \sin 3x dx.$

(9)  $\int \frac{dx}{x(x^6 + 4)}.$

(10)  $\int \sqrt{\frac{a+x}{a-x}} dx \quad (a > 0).$

(11)  $\int \frac{dx}{\sqrt{x(1+x)}}.$

(12)  $\int x \cos^2 x dx.$

(13)  $\int e^{ax} \cos bxdx.$

(14)  $\int \frac{dx}{\sqrt{1+e^x}}.$

(15)  $\int \frac{dx}{x^2 \sqrt{x^2 - 1}}.$

(16)  $\int \frac{dx}{(a^2 - x^2)^{\frac{5}{2}}}.$

(17)  $\int \frac{dx}{x^4 \sqrt{1+x^2}}.$

(18)  $\int \sqrt{x} \sin \sqrt{x} dx.$

(19)  $\int \ln(1+x^2) dx.$

(20)  $\int \frac{\sin^2 x}{\cos^3 x} dx.$

(21)  $\int \arctan \sqrt{x} dx.$

(22)  $\int \frac{\sqrt{1+\cos x}}{\sin x} dx.$

(23)  $\int \frac{x^3}{(1+x^8)^2} dx.$

(24)  $\int \frac{x^{11}}{x^8 + 3x^4 + 2} dx.$

(25)  $\int \frac{dx}{16-x^4}.$

(26)  $\int \frac{\sin x}{1+\sin x} dx.$

(27)  $\int \frac{x + \sin x}{1 + \cos x} dx.$

(28)  $\int \frac{e^{\sin x} x \cos^3 x - \sin x}{\cos^2 x} dx.$

(29)  $\int \frac{\sqrt[3]{x}}{x(\sqrt{x} + \sqrt[3]{x})} dx.$

(30)  $\int \frac{dx}{(1+e^x)^2}.$

(31)  $\int \frac{e^{3x} + e^x}{e^{4x} - e^{2x} + 1} dx.$

(32)  $\int \frac{x e^x}{(e^x + 1)^2} dx.$

(33)  $\int \ln^2(x + \sqrt{1+x^2}) dx.$

(34)  $\int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}} dx.$

(35)  $\int \sqrt{1-x^2} \arcsin x dx.$

(36)  $\int \frac{x^3 \arccos x}{\sqrt{1-x^2}} dx.$

(37)  $\int \frac{\cot x}{1 + \sin x} dx.$

(38)  $\int \frac{dx}{\sin^3 x \cos x}.$



3 题视频解析



4 题视频解析



$$(39) \int \frac{dx}{(2+\cos x)\sin x}.$$

$$(40) \int \frac{\sin x \cos x}{\sin x + \cos x} dx.$$

解 (1)  $\int \frac{dx}{e^x - e^{-x}} = \frac{e^x dx}{e^{2x} - 1} = \frac{1}{2} \int \left( \frac{1}{e^x - 1} - \frac{1}{e^x + 1} \right) d(e^x) = \frac{1}{2} \ln \left| \frac{e^x - 1}{e^x + 1} \right| + C.$

$$(2) \int \frac{x}{(1-x)^3} dx \stackrel{u=1-x}{=} \int \left( \frac{1}{u^2} - \frac{1}{u^3} \right) du = -\frac{1}{u} + \frac{1}{2u^2} + C = -\frac{1}{1-x} + \frac{1}{2(1-x)^2} + C.$$

$$(3) \int \frac{x^2}{a^6 - x^6} dx = \int \frac{d(x^3)}{3(a^6 - x^6)} \stackrel{u=x^3}{=} \int \frac{du}{3(a^6 - u^2)} = \frac{1}{6a^3} \int \left( \frac{1}{a^3 + u} + \frac{1}{a^3 - u} \right) du$$

$$= \frac{1}{6a^3} \ln \left| \frac{a^3 + u}{a^3 - u} \right| + C = \frac{1}{6a^3} \ln \left| \frac{a^3 + x^3}{a^3 - x^3} \right| + C.$$

$$(4) \int \frac{1 + \cos x}{x + \sin x} dx = \int \frac{d(x + \sin x)}{x + \sin x} dx = \ln |x + \sin x| + C.$$

$$(5) \int \frac{\ln \ln x}{x} dx = \int \ln \ln x d(\ln x) = \ln x \ln \ln x - \int \ln x \cdot \frac{1}{x \ln x} dx$$

$$= \ln x (\ln \ln x - 1) + C.$$

$$(6) \int \frac{\sin x \cos x}{1 + \sin^4 x} dx = \frac{1}{2} \int \frac{d(\sin^2 x)}{1 + \sin^4 x} dx = \frac{\arctan(\sin^2 x)}{2} + C.$$

$$(7) \int \tan^4 x dx = \int \tan^2 x (\sec^2 x - 1) dx = \int \tan^2 x d(\tan x) - \int (\sec^2 x - 1) dx$$

$$= \frac{1}{3} \tan^3 x - \tan x + x + C.$$

$$(8) \int \sin x \sin 2x \sin 3x dx = \int \frac{1}{2} (\cos x - \cos 3x) \sin 3x dx = \frac{1}{2} \int \cos x \sin 3x dx - \frac{1}{2} \int \cos 3x \sin 3x dx$$

$$= \frac{1}{4} \int (\sin 2x + \sin 4x) dx - \frac{1}{12} \sin^2 3x$$

$$= -\frac{1}{16} \cos 4x - \frac{1}{8} \cos 2x - \frac{1}{12} \sin^2 3x + C.$$

$$(9) \int \frac{dx}{x(x^6 + 4)} \stackrel{x=\frac{1}{u}}{=} \int \frac{-u^5 du}{1 + 4u^6} = -\frac{1}{24} \int \frac{d(1 + 4u^6)}{1 + 4u^6} = -\frac{1}{24} \ln(1 + 4u^6) + C$$

$$= -\frac{1}{24} \ln \frac{x^6 + 4}{x^6} + C = \frac{1}{4} \ln |x| - \frac{1}{24} \ln(x^6 + 4) + C.$$

(10) 方法一

$$\int \sqrt{\frac{a+x}{a-x}} dx = \int \frac{a+x}{\sqrt{a^2-x^2}} dx = a \int \frac{1}{\sqrt{1-(\frac{x}{a})^2}} d\left(\frac{x}{a}\right) - \frac{1}{2} \int \frac{d(a^2-x^2)}{\sqrt{a^2-x^2}}$$

$$= a \arcsin \frac{x}{a} - \sqrt{a^2-x^2} + C.$$

方法二 令  $u = \sqrt{\frac{a+x}{a-x}}$ , 即  $x = a \frac{u^2-1}{u^2+1}$ , 则

$$\int \sqrt{\frac{a+x}{a-x}} dx = \int u \cdot \frac{4au}{(1+u^2)^2} du = \int (-2au) d\left(\frac{1}{1+u^2}\right) = -\frac{2au}{1+u^2} + \int \frac{2a}{1+u^2} du$$

$$= -\frac{2au}{1+u^2} + 2a \arctan u + C = (x-a) \sqrt{\frac{a+x}{a-x}} + 2a \arctan \sqrt{\frac{a+x}{a-x}} + C$$

$$= -\sqrt{a^2-x^2} + 2a \arctan \sqrt{\frac{a+x}{a-x}} + C.$$



(11) 方法一

$$\begin{aligned}\int \frac{dx}{\sqrt{x(1+x)}} &= \int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} \stackrel{x=-\frac{1}{2}+\frac{1}{2}\sec u}{=} \int \sec u du \\ &= \ln|\sec u + \tan u| + C = \ln|2x+1+2\sqrt{x(1+x)}| + C.\end{aligned}$$

方法二 当  $x>0$  时, 因为  $\frac{1}{\sqrt{x(1+x)}} = \frac{1}{x}\sqrt{\frac{x}{1+x}}$ , 故令  $u=\sqrt{\frac{x}{1+x}}$ , 即  $x=\frac{u^2}{1-u^2}$ , 则

$$\begin{aligned}\int \frac{dx}{\sqrt{x(1+x)}} &= \int \frac{2}{1-u^2} du = \int \left(\frac{1}{1-u} + \frac{1}{1+u}\right) du = \ln\left|\frac{1+u}{1-u}\right| + C \\ &= \ln\left|\frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}-\sqrt{x}}\right| + C = \ln|2x+1+2\sqrt{x(1+x)}| + C,\end{aligned}$$

当  $x<-1$  时, 同样可得  $\int \frac{dx}{\sqrt{x(1+x)}} = \ln|2x+1+2\sqrt{x(1+x)}| + C$ .

$$\begin{aligned}(12) \int x \cos^2 x dx &= \frac{1}{2} \int x(1+\cos 2x) dx = \frac{1}{4} \int x(2x+\sin 2x) \\ &= \frac{x(2x+\sin 2x)}{4} - \frac{1}{4} \int (2x+\sin 2x) dx = \frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C.\end{aligned}$$

(13) 当  $a \neq 0$  时,

$$\begin{aligned}\int e^{ax} \cos bxdx &= \frac{1}{a} \cos bxd(e^{ax}) = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bxdx \\ &= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} \int \sin bxd(e^{ax}) \\ &= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} (e^{ax}) \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bxdx.\end{aligned}$$

因此有  $\int e^{ax} \cos bxdx = \frac{1}{a^2+b^2} e^{ax} (a \cos bx + b \sin bx) + C$ ,

$$\text{当 } a=0 \text{ 时, } \int e^{ax} \cos bxdx = \begin{cases} \frac{\sin bx}{b} + C, & b \neq 0, \\ x + C, & b = 0. \end{cases}$$

(14) 令  $u = \sqrt{1+e^x}$ , 即作换元  $x = \ln(u^2-1)$ , 得

$$\int \frac{dx}{\sqrt{1+e^x}} = \int \frac{2du}{u^2-1} = \ln \left| \frac{u-1}{u+1} \right| + C = \ln \frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1} + C.$$

$$(15) \int \frac{dx}{x^2 \sqrt{x^2-1}} \stackrel{x=\frac{1}{u}}{=} - \int \frac{udu}{\sqrt{1-u^2}} = \sqrt{1-u^2} + C = \frac{\sqrt{x^2-1}}{x} + C,$$

易知当  $x<0$  和  $x>0$  时的结果相同.

(16) 设  $x = a \sin u$  ( $-\frac{\pi}{2} < u < \frac{\pi}{2}$ ), 则  $\sqrt{a^2-x^2} = a \cos u$ ,  $dx = a \cos u du$ , 于是

$$\begin{aligned}\int \frac{dx}{(a^2-x^2)^{\frac{5}{2}}} &= \frac{1}{a^4} \int \sec^4 u du = \frac{1}{a^4} \int (\tan^2 u + 1) d(\tan u) = \frac{\tan^3 u}{3a^4} + \frac{\tan u}{a^4} + C \\ &= \frac{1}{3a^4} \left[ \frac{x^3}{\sqrt{(a^2-x^2)^3}} + \frac{3x}{\sqrt{a^2-x^2}} \right] + C.\end{aligned}$$

$$\begin{aligned}(17) \int \frac{dx}{x^4 \sqrt{1+x^2}} &\stackrel{x=\frac{1}{u}}{=} \int \frac{-u^3 du}{\sqrt{1+u^2}} = - \int \left( u \sqrt{1+u^2} - \frac{u}{\sqrt{1+u^2}} \right) du \\ &= -\frac{1}{3} (1+u^2)^{\frac{3}{2}} + \sqrt{1+u^2} + C = -\frac{1}{3} \frac{\sqrt{(1+x^2)^3}}{x^3} + \frac{\sqrt{1+x^2}}{x} + C,\end{aligned}$$



易知当  $x < 0$  和  $x > 0$  时结果相同.

$$\begin{aligned}
 (18) \int \sqrt{x} \sin \sqrt{x} dx &\stackrel{x=u^2}{=} \int 2u^2 \sin u du = - \int 2u^2 d(\cos u) = -2u^2 \cos u + \int 4u \cos u du \\
 &= -2u^2 \cos u + \int 4u d(\sin u) = -2u^2 \cos u + 4u \sin u - \int 4 \sin u du \\
 &= -2u^2 \cos u + 4u \sin u + 4 \cos u + C \\
 &= -2x \cos \sqrt{x} + 4\sqrt{x} \sin \sqrt{x} + 4 \cos \sqrt{x} + C.
 \end{aligned}$$

$$(19) \int \ln(1+x^2) dx = x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} dx = x \ln(1+x^2) - 2x + 2 \arctan x + C.$$

$$\begin{aligned}
 (20) \int \frac{\sin^2 x}{\cos^3 x} dx &= \int \tan^2 x \sec x dx = \int \sec^3 x dx - \int \sec x dx \\
 &= \left( \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x dx \right) - \int \sec x dx \\
 &= \frac{1}{2} \sec x \tan x - \frac{1}{2} \int \sec x dx = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C.
 \end{aligned}$$

$$\begin{aligned}
 (21) \int \arctan \sqrt{x} dx &= \int \arctan \sqrt{x} d(1+x) = (1+x) \arctan \sqrt{x} - \int \frac{1}{2\sqrt{x}} dx \\
 &= (1+x) \arctan \sqrt{x} - \sqrt{x} + C.
 \end{aligned}$$

$$(22) \int \frac{\sqrt{1+\cos x}}{\sin x} dx = \int \frac{\sqrt{2} \left| \cos \frac{x}{2} \right|}{2 \sin \frac{x}{2} \cos \frac{x}{2}} dx = \pm \sqrt{2} \int \csc \frac{x}{2} d\left(\frac{x}{2}\right) = \pm \sqrt{2} \ln \left| \csc \frac{x}{2} - \cot \frac{x}{2} \right| + C.$$

上式当  $\cos \frac{x}{2} > 0$  时取正, 当  $\cos \frac{x}{2} < 0$  时取负.

$$\text{当 } \cos \frac{x}{2} > 0 \text{ 时, } \ln \left| \csc \frac{x}{2} - \cot \frac{x}{2} \right| = \ln \frac{1 - \cos \frac{x}{2}}{\left| \sin \frac{x}{2} \right|} = \ln \left( \left| \csc \frac{x}{2} \right| - \left| \cot \frac{x}{2} \right| \right),$$

$$\begin{aligned}
 \text{当 } \cos \frac{x}{2} < 0 \text{ 时, } \ln \left| \csc \frac{x}{2} - \cot \frac{x}{2} \right| &= \ln \frac{1 - \cos \frac{x}{2}}{\left| \sin \frac{x}{2} \right|} = \ln \left( \left| \csc \frac{x}{2} \right| + \left| \cot \frac{x}{2} \right| \right) \\
 &= -\ln \left( \left| \csc \frac{x}{2} \right| - \left| \cot \frac{x}{2} \right| \right),
 \end{aligned}$$

$$\text{因此有 } \int \frac{\sqrt{1+\cos x}}{\sin x} dx = \sqrt{2} \ln \left( \left| \csc \frac{x}{2} \right| - \left| \cot \frac{x}{2} \right| \right) + C.$$

$$(23) \int \frac{x^3}{(1+x^8)^2} dx = \frac{1}{4} \int \frac{1}{(1+x^8)^2} d(x^4) \stackrel{u=x^4}{=} \frac{1}{4} \int \frac{1}{(1+u^2)^2} du.$$

设  $u = \tan t$  ( $-\frac{\pi}{2} < t < \frac{\pi}{2}$ ), 则  $1+u^2 = \sec^2 t$ ,  $du = \sec^2 t dt$ , 于是

$$\text{原式} = \frac{1}{4} \int \cos^2 t dt = \frac{2t + \sin 2t}{16} + C = \frac{\arctan x^4}{8} + \frac{x^4}{8(1+x^8)} + C.$$

$$\begin{aligned}
 (24) \int \frac{x^{11}}{x^8 + 3x^4 + 2} dx &\stackrel{u=x^4}{=} \frac{1}{4} \int \frac{u^2}{u^2 + 3u + 2} du = \frac{1}{4} \int \left( 1 + \frac{1}{u+1} - \frac{4}{u+2} \right) du \\
 &= \frac{1}{4} u + \frac{1}{4} \ln |1+u| - \ln |2+u| + C = \frac{x^4}{4} + \ln \frac{\sqrt{1+x^4}}{2+x^4} + C.
 \end{aligned}$$

$$\begin{aligned}
 (25) \int \frac{dx}{16-x^4} &= \int \frac{1}{(2-x)(2+x)(4+x^2)} dx = \int \left[ \frac{1}{32(2-x)} + \frac{1}{32(2+x)} + \frac{1}{8(4+x^2)} \right] dx \\
 &= \frac{1}{32} \ln \left| \frac{2+x}{2-x} \right| + \frac{1}{16} \arctan \frac{x}{2} + C.
 \end{aligned}$$



(26) 方法一 令  $u = \tan \frac{x}{2}$ , 则

$$\begin{aligned}\int \frac{\sin x}{1 + \sin x} dx &= \int \frac{4u}{(1+u)^2(1+u^2)} du = \int \left[ \frac{-2}{(1+u)^2} + \frac{2}{1+u^2} \right] du \\ &= \frac{2}{1+u} + 2\arctan u + C = \frac{2}{1 + \tan \frac{x}{2}} + x + C.\end{aligned}$$

方法二  $\int \frac{\sin x}{1 + \sin x} dx = \int \frac{\sin x(1 - \sin x)}{\cos^2 x} dx = -\int \frac{1}{\cos^2 x} d(\cos x) - \int (\sec^2 x - 1) dx$   
 $= \sec x - \tan x + x + C.$

(27)  $\int \frac{x + \sin x}{1 + \cos x} dx = \int \frac{x}{2} \sec^2 \frac{x}{2} dx + \int \tan \frac{x}{2} dx = \int x d\left(\tan \frac{x}{2}\right) + \int \tan \frac{x}{2} dx = x \tan \frac{x}{2} + C.$

(28)  $\int e^{\sin x} \frac{x \cos^3 x - \sin x}{\cos^2 x} dx = \int x e^{\sin x} \cos x dx - \int e^{\sin x} \tan x \sec x dx = \int x d(e^{\sin x}) - \int e^{\sin x} d(\sec x)$   
 $= x e^{\sin x} - \int e^{\sin x} dx - (\sec x e^{\sin x} - \int e^{\sin x} dx) = (x - \sec x) e^{\sin x} + C.$

(29)  $\int \frac{\sqrt[3]{x}}{x(\sqrt{x} + \sqrt[3]{x})} dx \stackrel{x=u^6}{=} \int \frac{6}{u(u+1)} du = 6 \int \left( \frac{1}{u} - \frac{1}{u+1} \right) du$   
 $= 6 \ln \left| \frac{u}{1+u} \right| + C = \ln \frac{x}{(\sqrt[6]{x} + 1)^6} + C.$

(30)  $\int \frac{dx}{(1+e^x)^2} \stackrel{x=\ln u}{=} \int \frac{du}{u(1+u)^2} = \int \left[ \frac{1}{u} - \frac{1}{1+u} - \frac{1}{(1+u)^2} \right] du$   
 $= \ln u - \ln(1+u) + \frac{1}{1+u} + C = x - \ln(1+e^x) + \frac{1}{1+e^x} + C.$

(31)  $\int \frac{e^{3x} + e^x}{e^{4x} - e^{2x} + 1} dx = \int \frac{e^x + e^{-x}}{e^{2x} - 1 + e^{-2x}} dx = \int \frac{d(e^x - e^{-x})}{(e^x - e^{-x})^2 + 1} = \arctan(e^x - e^{-x}) + C.$

(32)  $\int \frac{x e^x}{(e^x + 1)^2} dx = -\int x d\left(\frac{1}{e^x + 1}\right) = -\frac{x}{e^x + 1} + \int \frac{dx}{e^x + 1} = -\frac{x}{e^x + 1} + \int \frac{e^{-x} dx}{1 + e^{-x}}$   
 $= -\frac{x}{e^x + 1} - \ln(1 + e^{-x}) + C.$

(33)  $\int \ln^2(x + \sqrt{1+x^2}) dx = x \ln^2(x + \sqrt{1+x^2}) - \int \frac{2x \ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx$   
 $= x \ln^2(x + \sqrt{1+x^2}) - \int 2 \ln(x + \sqrt{1+x^2}) d(\sqrt{1+x^2})$   
 $= x \ln^2(x + \sqrt{1+x^2}) - 2 \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) + 2x + C.$

(34)  $\int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}} dx \stackrel{x=\frac{1}{u}}{=} \int \frac{u \ln u}{(1+u^2)^{\frac{3}{2}}} du$   
 $= -\int \ln u d((1+u^2)^{-\frac{1}{2}}) = -\frac{\ln u}{\sqrt{1+u^2}} + \int \frac{du}{u \sqrt{1+u^2}}$   
 $= \frac{x \ln x}{\sqrt{1+x^2}} - \int \frac{dx}{\sqrt{1+x^2}} = \frac{x \ln x}{\sqrt{1+x^2}} - \ln(x + \sqrt{1+x^2}) + C.$

(35) 设  $x = \sin u$  ( $-\frac{\pi}{2} < u < \frac{\pi}{2}$ ), 则  $\sqrt{1-x^2} = \cos u$ ,  $dx = \cos u du$ , 于是

$$\begin{aligned}\int \sqrt{1-x^2} \arcsin x dx &= \int u \cos^2 u du = \frac{1}{2} \int u(1 + \cos 2u) du = \frac{1}{4} \int u d(2u + \sin 2u) \\ &= \frac{u(2u + \sin 2u)}{4} - \frac{1}{4} \int (2u + \sin 2u) du = \frac{u^2}{4} + \frac{u}{4} \sin 2u - \frac{\sin^2 u}{4} + C\end{aligned}$$



$$= \frac{(\arcsin x)^2}{4} + \frac{x}{2} \sqrt{1-x^2} \arcsin x - \frac{x^2}{4} + C.$$

(36) 设  $x = \cos u (0 < u < \pi)$ , 则  $\sqrt{1-x^2} = \sin u$ ,  $dx = -\sin u du$ , 于是

$$\begin{aligned} \int \frac{x^3 \arccos x}{\sqrt{1-x^2}} dx &= - \int u \cos^3 u du = - \int u d\left(\sin u - \frac{1}{3} \sin^3 u\right) \\ &= -u\left(\sin u - \frac{1}{3} \sin^3 u\right) + \int \left(\sin u - \frac{1}{3} \sin^3 u\right) du \\ &= -u\left(\sin u - \frac{1}{3} \sin^3 u\right) - \frac{1}{3} \int (2 + \cos^2 u) d(\cos u) \\ &= -u\left(\sin u - \frac{1}{3} \sin^3 u\right) - \frac{2}{3} \cos u - \frac{1}{9} \cos^3 u + C \\ &= -\frac{1}{3} \sqrt{1-x^2} (2+x^2) \arccos x - \frac{1}{9} x(6+x^2) + C. \end{aligned}$$

$$(37) \int \frac{\cot x}{1+\sin x} dx = \int \frac{\cos x}{\sin x(1+\sin x)} dx = \int \left(\frac{1}{\sin x} - \frac{1}{1+\sin x}\right) d(\sin x) = \ln \left| \frac{\sin x}{1+\sin x} \right| + C.$$

$$\begin{aligned} (38) \int \frac{dx}{\sin^3 x \cos x} &= - \int \cot x \sec^2 x d(\cot x) \stackrel{u=\cot x}{=} - \int u \left(1 + \frac{1}{u^2}\right) du \\ &= -\frac{u^2}{2} - \ln|u| + C = -\frac{\cot^2 x}{2} - \ln|\cot x| + C \end{aligned}$$

$$\begin{aligned} (39) \int \frac{dx}{(2+\cos x)\sin x} &= \int \frac{d(\cos x)}{(2+\cos x)(\cos^2 x - 1)} \stackrel{u=\cos x}{=} \int \frac{du}{(2+u)(u^2-1)} \\ &= \int \left[ \frac{1}{6(u-1)} - \frac{1}{2(u+1)} + \frac{1}{3(u+2)} \right] du \\ &= \frac{1}{6} \ln|u-1| - \frac{1}{2} \ln|u+1| + \frac{1}{3} \ln|u+2| + C \\ &= \frac{1}{6} \ln(1-\cos x) - \frac{1}{2} \ln(1+\cos x) + \frac{1}{3} \ln(2+\cos x) + C. \end{aligned}$$

(40) 方法一

$$\begin{aligned} \int \frac{\sin x \cos x}{\sin x + \cos x} dx &= \int \frac{\frac{1}{2}(\sin x + \cos x)^2 - \frac{1}{2}}{\sin x + \cos x} dx = \frac{1}{2} \int (\sin x + \cos x) dx - \frac{1}{2} \int \frac{1}{\sin x + \cos x} dx \\ &= \frac{1}{2} (-\cos x + \sin x) - \frac{1}{2} \int \frac{1}{\sin x + \cos x} dx, \end{aligned}$$

令  $u = \tan \frac{x}{2}$ , 则  $\sin x = \frac{2u}{1+u^2}$ ,  $\cos x = \frac{1-u^2}{1+u^2}$ ,  $dx = \frac{2}{1+u^2} du$ , 故有

$$\begin{aligned} \int \frac{1}{\sin x + \cos x} dx &= \int \frac{2}{2u+1-u^2} du = - \int \frac{2}{(u-1)^2 - (\sqrt{2})^2} du \\ &= -\frac{1}{\sqrt{2}} \int \frac{1}{u-1-\sqrt{2}} du + \frac{1}{\sqrt{2}} \int \frac{1}{u-1+\sqrt{2}} du = \frac{1}{\sqrt{2}} \ln \left| \frac{u-1+\sqrt{2}}{u-1-\sqrt{2}} \right| + C', \end{aligned}$$

$$\text{因此有 } \int \frac{\sin x \cos x}{\sin x + \cos x} dx = \frac{1}{2} (\sin x - \cos x) - \frac{1}{2\sqrt{2}} \ln \left| \frac{\tan \frac{x}{2} - 1 + \sqrt{2}}{\tan \frac{x}{2} - 1 - \sqrt{2}} \right| + C.$$

$$\begin{aligned} \text{方法二} \quad \int \frac{\sin x \cos x}{\sin x + \cos x} dx &= \int \frac{\sin x \cos x}{\sqrt{2} \sin\left(x + \frac{\pi}{4}\right)} dx \stackrel{u=x+\frac{\pi}{4}}{=} \int \frac{2\sin^2 u - 1}{2\sqrt{2} \sin u} du \\ &= \frac{1}{\sqrt{2}} \int \sin u du - \frac{1}{2\sqrt{2}} \int \csc u du = -\frac{\cos\left(x + \frac{\pi}{4}\right)}{\sqrt{2}} - \frac{1}{2\sqrt{2}} \ln \left| \csc\left(x + \frac{\pi}{4}\right) - \cot\left(x + \frac{\pi}{4}\right) \right| + C. \end{aligned}$$