

所以 $\frac{dy}{dx} \Big|_{(0,1)} = \frac{dy}{dx} \Big|_{t=0} = \frac{1}{2}$, 即曲线在(0,1)处的切线斜率为 $\frac{1}{2}$, 从而知法线斜率为 -2.

法线方程为: $y-1=-2(x-0)$, 即 $y+2x-1=0$.

【例 6】 (2021 数学二, 5 分) 有一圆柱体底面半径与高随时间变化的速率分别为 2cm/s, -3cm/s, 当底面半径为 10cm, 高为 5cm 时, 圆柱体的体积与表面积随时间变化的速率分别为 _____.

- (A) $125\pi \text{cm}^3/\text{s}, 40\pi \text{cm}^2/\text{s}$ (B) $125\pi \text{cm}^3/\text{s}, -40\pi \text{cm}^2/\text{s}$
 (C) $-100\pi \text{cm}^3/\text{s}, 40\pi \text{cm}^2/\text{s}$ (D) $-100\pi \text{cm}^3/\text{s}, -40\pi \text{cm}^2/\text{s}$

解 设圆柱体底面半径为 R , 高为 h , 则 $\frac{dR}{dt}=2, \frac{dh}{dt}=-3$.

体积 $V=\pi R^2 h$, 表面积 $S=2\pi Rh+2\pi R^2$,

$$\text{故 } \frac{dV}{dt}=2\pi R \cdot \frac{dR}{dt}h+R^2\pi \frac{dh}{dt}, \frac{dS}{dt}=2\pi \frac{dR}{dt}h+2\pi R \frac{dh}{dt}+4\pi R \cdot \frac{dR}{dt},$$

$$\text{即 } \frac{dV}{dt} \Big|_{R=10, h=5} = -100\pi, \frac{dS}{dt} \Big|_{R=10, h=5} = 40\pi.$$

故应选(C).

方法总结

求相关变化率问题的步骤:

- (1) 根据题意, 建立相关变量之间的等量关系式;
- (2) 在所得等式两边同时对 t 求导;
- (3) 代入变量在指定时刻的值及变化率, 从而求出未知变化率.

另外, 相关变化率问题大部分是实际问题, 解题的关键在于把实际问题用数学语言表达出来, 即要写出问题的函数表达式.

习题 2—4 解答

1. 求由下列方程所确定的隐函数的导数 $\frac{dy}{dx}$:

- (1) $y^2 - 2xy + 9 = 0$; (2) $x^3 + y^3 - 3axy = 0$;
 (3) $xy = e^{x+y}$; (4) $y = 1 - xe^y$.

解 (1) 在方程两端分别对 x 求导, 得 $2yy' - 2y - 2xy' = 0$, 从而 $y' = \frac{y}{y-x}$.

(2) 在方程两端分别对 x 求导, 得 $3x^2 + 3y^2 y' - 3ay - 3axy' = 0$, 从而 $y' = \frac{ay - x^2}{y^2 - ax}$.

(3) 在方程两端分别对 x 求导, 得 $y + xy' = e^{x+y}(1+y')$, 从而 $y' = \frac{e^{x+y}-y}{x-e^{x+y}}$.

(4) 在方程两端分别对 x 求导, 得 $y' = -e^y - xe^y y'$, 从而 $y' = \frac{-e^y}{1+xe^y}$.

2. 求曲线 $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ 在点 $(\frac{\sqrt{2}}{4}a, \frac{\sqrt{2}}{4}a)$ 处的切线方程和法线方程.

解 由导数的几何意义知, 所求切线的斜率为 $k = y' \Big|_{(\frac{\sqrt{2}}{4}a, \frac{\sqrt{2}}{4}a)}$,

在曲线方程两端分别对 x 求导, 得 $\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}y' = 0$,

从而 $y' = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}$, $y' \Big|_{(\frac{\sqrt{2}}{4}a, \frac{\sqrt{2}}{4}a)} = -1$.

于是所求的切线方程为 $y - \frac{\sqrt{2}}{4}a = -1 \left(x - \frac{\sqrt{2}}{4}a \right)$, 即 $x + y = \frac{\sqrt{2}}{2}a$.

法线方程为 $y - \frac{\sqrt{2}}{4}a = 1 \cdot \left(x - \frac{\sqrt{2}}{4}a \right)$, 即 $x - y = 0$.

3. 设曲线 C 的方程为 $x^2y - xy^2 = 2$, 试找出 C 上有水平切线和铅直切线的点.

解 在 $x^2y - xy^2 = 2$ 两边对 x 求导, 得 $2xy + x^2y' - y^2 - 2xyy' = 0$, $y' = \frac{2xy - y^2}{2xy - x^2}$.

(1) 令 $y' = 0$, 得 $2xy - y^2 = 0$, 所以 $y = 2x$ ($y = 0$ 舍去), 代入 $x^2y - xy^2 = 2$ 可得 $x = -1, y = -2$. 即 C 上有水平切线的点为 $(-1, -2)$.

(2) 令 $2xy - x^2 = 0$, 得 $x = 2y$ ($x = 0$ 舍去), 代入 $x^2y - xy^2 = 2$ 可得 $x = 2, y = 1$, 即 C 上有铅直切线的点为 $(2, 1)$.

4. 求由下列方程所确定的隐函数的二阶导数 $\frac{d^2y}{dx^2}$:

(1) $b^2x^2 + a^2y^2 = a^2b^2$; (2) $y = \tan(x+y)$; (3) $y = 1 + xe^y$; (4) $y - 2x = (x-y)\ln(x-y)$.

解 (1) 应用隐函数的求导方法, 得 $2xb^2 + 2a^2yy' = 0$,

$$\text{于是 } y' = -\frac{b^2x}{a^2y}, y'' = -\frac{b^2}{a^2} \cdot \frac{y - xy'}{y^2} = -\frac{b^4}{a^2y^3}.$$

(2) 应用隐函数的求导方法, 得

$$y' = \sec^2(x+y)(1+y') = [1 + \tan^2(x+y)](1+y') = (1+y^2)(1+y'),$$

$$\text{于是 } y' = \frac{1+y^2}{1-(1+y^2)} = -\frac{1}{y^2} - 1, y'' = \frac{2y'}{y^3} = -\frac{2(1+y^2)}{y^5} = -2\csc^2(x+y)\cot^3(x+y).$$

(3) 应用隐函数的求导方法, 得 $y' = e^y + xe^y y'$, 于是

$$\begin{aligned} y' &= \frac{e^y}{1-xe^y}, \\ y'' &= \frac{e^y \cdot y'(1-xe^y) - e^y(-e^y - xe^y y')}{(1-xe^y)^2} = \frac{e^y y' + e^{2y}}{(1-xe^y)^2} = \frac{e^{2y}(2-xe^y)}{(1-xe^y)^3} = \frac{e^{2y}(3-y)}{(2-y)^3}. \end{aligned}$$

(4) 在方程 $y - 2x = (x-y)\ln(x-y)$ 两边对 x 求导, 得

$$y' - 2 = (1-y')\ln(x-y) + (1-y') \quad \text{①}, \quad y' = \frac{3 + \ln(x-y)}{2 + \ln(x-y)},$$

在①式两边再对 x 求导, 得

$$y'' = -y'' \ln(x-y) + \frac{(1-y')^2}{x-y} - y'',$$

$$y'' = \frac{(1-y')^2}{[2 + \ln(x-y)](x-y)} = \frac{1}{[2 + \ln(x-y)]^3(x-y)}.$$

5. 用对数求导法求下列函数的导数:

$$(1) y = \left(\frac{x}{1+x} \right)^x; \quad (2) y = \sqrt[5]{\frac{x-5}{\sqrt[5]{x^2+2}}};$$

$$(3) y = \frac{\sqrt{x+2}(3-x)^4}{(x+1)^5}; \quad (4) y = \sqrt{x \sin x \sqrt{1-e^x}}.$$

解 (1) 在 $y = \left(\frac{x}{1+x}\right)^x$ 两端取对数, 得 $\ln y = x[\ln x - \ln(1+x)]$.

在上式两端分别对 x 求导, 并注意到 $y = y(x)$, 得

$$\frac{y'}{y} = [\ln x - \ln(1+x)] + x\left(\frac{1}{x} - \frac{1}{1+x}\right) = \ln \frac{x}{1+x} + \frac{1}{1+x},$$

于是

$$y' = y \left(\ln \frac{x}{1+x} + \frac{1}{1+x} \right) = \left(\frac{x}{1+x} \right)^x \left(\ln \frac{x}{1+x} + \frac{1}{1+x} \right).$$

(2) 在 $y = \sqrt[5]{\frac{x-5}{\sqrt[5]{x^2+2}}}$ 两端取对数, 得

$$\ln y = \frac{1}{5} \left[\ln(x-5) - \frac{1}{5} \ln(x^2+2) \right] = \frac{1}{5} \ln(x-5) - \frac{1}{25} \ln(x^2+2).$$

在上式两端分别对 x 求导, 并注意到 $y = y(x)$, 得

$$\frac{y'}{y} = \frac{1}{5} \cdot \frac{1}{x-5} - \frac{1}{25} \cdot \frac{2x}{x^2+2},$$

$$\text{于是 } y' = y \left[\frac{1}{5(x-5)} - \frac{2x}{25(x^2+2)} \right] = \sqrt[5]{\frac{x-5}{\sqrt[5]{x^2+2}}} \left[\frac{1}{5(x-5)} - \frac{2x}{25(x^2+2)} \right].$$

(3) 在 $y = \frac{\sqrt{x+2}(3-x)^4}{(x+1)^5}$ 两端取对数, 得

$$\ln y = \frac{1}{2} \ln(x+2) + 4 \ln(3-x) - 5 \ln(1+x).$$

在上式两端分别对 x 求导, 并注意到 $y = y(x)$, 得

$$\frac{y'}{y} = \frac{1}{2} \cdot \frac{1}{x+2} + 4 \cdot \frac{(-1)}{3-x} - 5 \cdot \frac{1}{1+x}$$

$$\text{于是 } y' = y \left[\frac{1}{2(x+2)} - \frac{4}{3-x} - \frac{5}{1+x} \right] = \frac{\sqrt{x+2}(3-x)^4}{(x+1)^5} \left[\frac{1}{2(x+2)} - \frac{4}{3-x} - \frac{5}{1+x} \right].$$

(4) 在 $y = \sqrt{x \sin x \sqrt{1-e^x}}$ 两端取对数, 得

$$\ln y = \frac{1}{2} \left[\ln x + \ln \sin x + \frac{1}{2} \ln(1-e^x) \right].$$

在上式两端分别对 x 求导, 并注意到 $y = y(x)$, 得

$$\frac{y'}{y} = \frac{1}{2} \left[\frac{1}{x} + \frac{\cos x}{\sin x} + \frac{1}{2} \cdot \frac{(-e^x)}{1-e^x} \right],$$

$$\text{于是 } y' = y \left[\frac{1}{2x} + \frac{\cos x}{2\sin x} - \frac{e^x}{4(1-e^x)} \right] = \frac{1}{2} \sqrt{x \sin x \sqrt{1-e^x}} \left[\frac{1}{x} + \cot x - \frac{e^x}{2(1-e^x)} \right].$$

6. 求下列参数方程所确定的函数的导数 $\frac{dy}{dx}$:

$$(1) \begin{cases} x = at^2, \\ y = bt^3; \end{cases} \quad (2) \begin{cases} x = \theta(1-\sin \theta), \\ y = \theta \cos \theta. \end{cases}$$

$$\text{解 (1)} \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3bt^2}{2at} = \frac{3b}{2a} t.$$

$$(2) \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos \theta - \theta \sin \theta}{1 - \sin \theta + \theta(-\cos \theta)} = \frac{\cos \theta - \theta \sin \theta}{1 - \sin \theta - \theta \cos \theta}.$$





7. 已知 $\begin{cases} x = e^t \sin t, \\ y = e^t \cos t, \end{cases}$ 求当 $t = \frac{\pi}{3}$ 时 $\frac{dy}{dx}$ 的值.

$$\text{解 } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t \cos t - e^t \sin t}{e^t \sin t + e^t \cos t} = \frac{\cos t - \sin t}{\sin t + \cos t}.$$

$$\text{于是 } \left. \frac{dy}{dx} \right|_{t=\frac{\pi}{3}} = \frac{\frac{1}{2} - \frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2} + \frac{1}{2}} = \sqrt{3} - 2.$$

8. 写出下列曲线在所给参数值相应的点处的切线方程和法线方程:

$$(1) \begin{cases} x = \sin t, \\ y = \cos 2t, \end{cases} \text{ 在 } t = \frac{\pi}{4} \text{ 处;}$$

$$(2) \begin{cases} x = \frac{3at}{1+t^2}, \\ y = \frac{3at^2}{1+t^2}, \end{cases} \text{ 在 } t = 2 \text{ 处.}$$



8 题视频解析

$$\text{解 } (1) \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin 2t}{\cos t} = -4\sin t, \left. \frac{dy}{dx} \right|_{t=\frac{\pi}{4}} = -2\sqrt{2}.$$

$t = \frac{\pi}{4}$ 对应点 $(\frac{\sqrt{2}}{2}, 0)$, 曲线在点 $(\frac{\sqrt{2}}{2}, 0)$ 处的切线方程为

$$y - 0 = -2\sqrt{2} \left(x - \frac{\sqrt{2}}{2} \right), \quad \text{即 } 2\sqrt{2}x + y - 2 = 0.$$

法线方程为

$$y - 0 = \frac{1}{2\sqrt{2}} \left(x - \frac{\sqrt{2}}{2} \right), \quad \text{即 } \sqrt{2}x - 4y - 1 = 0.$$

$$(2) \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\left(\frac{3at^2}{1+t^2} \right)'}{\left(\frac{3at}{1+t^2} \right)'} = \frac{\frac{3a[2t(1+t^2) - t^2 \cdot 2t]}{(1+t^2)^2}}{\frac{3a[(1+t^2) - t \cdot 2t]}{(1+t^2)^2}} = \frac{2t}{1-t^2}, \left. \frac{dy}{dx} \right|_{t=2} = -\frac{4}{3},$$

$t = 2$ 对应点 $(\frac{6}{5}a, \frac{12}{5}a)$, 曲线在点 $(\frac{6}{5}a, \frac{12}{5}a)$ 处的切线方程为

$$y - \frac{12}{5}a = -\frac{4}{3} \left(x - \frac{6}{5}a \right), \quad \text{即 } 4x + 3y - 12a = 0.$$

法线方程为

$$y - \frac{12}{5}a = \frac{3}{4} \left(x - \frac{6}{5}a \right), \quad \text{即 } 3x - 4y + 6a = 0.$$

9. 求下列参数方程所确定的函数的二阶导数 $\frac{d^2y}{dx^2}$:

$$(1) \begin{cases} x = \frac{t^2}{2}, \\ y = 1 - t, \end{cases}$$

$$(2) \begin{cases} x = a \cos t, \\ y = b \sin t; \end{cases}$$

$$(3) \begin{cases} x = 3e^{-t}, \\ y = 2e^t; \end{cases}$$

$$(4) \begin{cases} x = f'(t), \\ y = tf'(t) - f(t), \end{cases} \text{ 设 } f''(t) \text{ 存在且不为零.}$$

$$\text{解 } (1) \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-1}{t}, \quad \frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{1}{t^2} = \frac{1}{t^3}.$$

$$(2) \frac{dy}{dx} = \frac{b \cos t}{-a \sin t} = -\frac{b}{a} \cot t, \quad \frac{d^2 y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{-\frac{b}{a} (-\csc^2 t)}{-a \sin t} = \frac{-b}{a^2 \sin^3 t}.$$

$$(3) \frac{dy}{dx} = \frac{2e^t}{-3e^{-t}} = -\frac{2}{3} e^{2t}, \quad \frac{d^2 y}{dx^2} = \frac{-\frac{4}{3} e^{2t}}{-3e^{-t}} = \frac{4}{9} e^{3t}.$$

$$(4) \frac{dy}{dx} = \frac{f'(t) + t f''(t) - f'(t)}{f''(t)} = t, \quad \frac{d^2 y}{dx^2} = \frac{1}{f''(t)}.$$

10. 求下列参数方程所确定的函数的三阶导数 $\frac{d^3 y}{dx^3}$:

$$(1) \begin{cases} x = 1 - t^2, \\ y = t - t^3; \end{cases}$$

$$(2) \begin{cases} x = \ln(1+t^2), \\ y = t - \arctan t. \end{cases}$$

$$\text{解 } (1) \frac{dy}{dx} = \frac{1-3t^2}{-2t} = -\frac{1}{2t} + \frac{3}{2}t, \quad \frac{d^2 y}{dx^2} = \frac{\frac{1}{2t^2} + \frac{3}{2}}{-2t} = -\frac{1}{4} \left(\frac{1}{t^3} + \frac{3}{t} \right),$$

$$\frac{d^3 y}{dx^3} = \frac{-\frac{1}{4} \left(-\frac{3}{t^4} - \frac{3}{t^2} \right)}{-2t} = -\frac{3}{8t^5}(1+t^2).$$

$$(2) \frac{dy}{dx} = \frac{1 - \frac{1}{1+t^2}}{\frac{2t}{1+t^2}} = \frac{t}{2}, \quad \frac{d^2 y}{dx^2} = \frac{\frac{1}{2}}{\frac{2t}{1+t^2}} = \frac{1+t^2}{4t} = \frac{1}{4} \left(\frac{1}{t} + t \right),$$

$$\frac{d^3 y}{dx^3} = \frac{\frac{1}{4} \left(-\frac{1}{t^2} + 1 \right)}{\frac{2t}{1+t^2}} = \frac{t^4 - 1}{8t^3}.$$



10 题视频解析

11. 落在平静水面上的石头,产生同心波纹.若最外一圈波纹半径的增大速率总是 6m/s ,问在 2s 末扰动水面面积的增大的速率为多少?

解 设最外一圈波纹的半径为 $r=r(t)$,圆的面积 $S=S(t)$. 在 $S=\pi r^2$ 两端分别对 t 求导,得

$$\frac{dS}{dt} = 2\pi r \frac{dr}{dt} \text{ 当 } t=2 \text{ 时, } r=6 \times 2=12, \frac{dr}{dt}=6 \text{ 代入上式得}$$

$$\left. \frac{dS}{dt} \right|_{t=2} = 2\pi \times 12 \times 6 = 144\pi (\text{m}^2/\text{s}).$$

12. 注水入深 8m 、上顶直径 8m 的正圆锥形容器中,其速率为 $4\text{m}^3/\text{min}$. 当水深为 5m 时,其表面上升的速率为多少?

解 如图 2-1 所示,设在时刻 t 容器中的水深为 $h(t)$,水的容积为 $V(t)$,

$$\frac{r}{4} = \frac{h}{8}, \text{ 即 } r = \frac{h}{2}, V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3.$$

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}, \quad \text{即} \quad \frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}.$$

$$\text{故} \left. \frac{dh}{dt} \right|_{h=5} = \frac{4}{25\pi} \times 4 = \frac{16}{25\pi} \approx 0.204 (\text{m/min}).$$

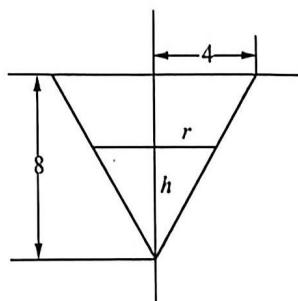


图 2-1

13. 溶液自深 18cm 、顶直径 12cm 的正圆锥形漏斗中漏入一直径为 10cm 的圆柱形筒中. 开始时漏斗中盛满了溶液. 已知当溶液在漏斗中深为 12cm 时,其表面下降的速率为 1cm/min . 问此时圆柱形筒中溶液表面上升的速率为多少?

解 如图 2-2,设在时刻 t 漏斗中的水深为 $H=H(t)$,圆柱形筒中水深为 $h=h(t)$,建立 h 与 H 之间的关系:



$$\frac{1}{3}\pi \times 6^2 \cdot 18 - \frac{1}{3}\pi r^2 H = \pi \times 5^2 h.$$

又 $\frac{r}{6} = \frac{H}{18}$, 即 $r = \frac{H}{3}$. 故

$$\frac{1}{3}\pi \times 6^2 \times 18 - \frac{1}{3}\pi \left(\frac{H}{3}\right)^2 H = \pi \times 5^2 h,$$

即 $216 - \frac{1}{27}H^3 = 25h$.

上式两端分别对 t 求导, 得 $-\frac{3}{27}H^2 \frac{dH}{dt} = 25 \frac{dh}{dt}$.

当 $H=12$ 时, $\frac{dH}{dt}=-1$, 此时 $\frac{dh}{dt} = \frac{1}{25} \left(-\frac{3}{27}H^2 \frac{dH}{dt}\right) \Big|_{\substack{H=12 \\ \frac{dH}{dt}=-1}} = \frac{16}{25} = 0.64 (\text{cm/min})$.

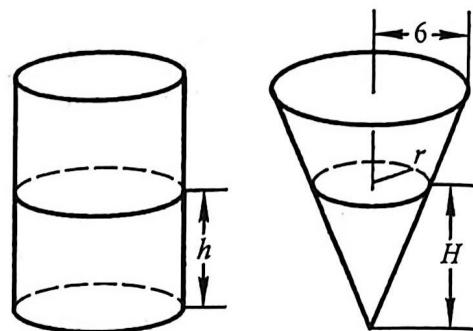


图 2-2

第五节 函数的微分

一、主要内容归纳

1. 微分定义 若函数 $f(x)$ 在 x 点的增量 $\Delta y = f(x+\Delta x) - f(x)$, 可表示为 $\Delta y = Ax + o(\Delta x)$, 其中: A 是与 Δx 无关的量; 当 $\Delta x \rightarrow 0$ 时, $o(\Delta x)$ 是比 Δx 高阶的无穷小. 则称 $y=f(x)$ 在 x 点可微, 而线性主部 $A\Delta x$ 称为 $y=f(x)$ 在 x 点的微分, 记为 dy 或 $df(x)$, 即 $dy = df(x) = A \cdot \Delta x$.

当函数 $f(x)$ 可微时, 微分中 Δx 的系数 $A = f'(x)$, 记 $dx = \Delta x$, 称之为自变量的微分, 微表达式通常写为对称形式 $dy = f'(x)dx$, 而导数就是函数微分与自变量微分之商(微商)

$$f'(x) = \frac{dy}{dx}.$$

2. 基本初等函数的微分公式

- | | |
|--|---|
| (1) $d(C) = 0$; | (2) $d(x^\mu) = \mu x^{\mu-1} dx$ (μ 为实数); |
| (3) $d(\sin x) = \cos x dx$; | (4) $d(\cos x) = -\sin x dx$; |
| (5) $d(\tan x) = \sec^2 x dx$; | (6) $d(\cot x) = -\csc^2 x dx$; |
| (7) $d(\sec x) = \sec x \tan x dx$; | (8) $d(\csc x) = -\csc x \cot x dx$; |
| (9) $d(a^x) = a^x \ln a dx$ ($a > 0, a \neq 1$); | (10) $d(e^x) = e^x$ |