Statistical and Mathematical Methods for Data Analysis

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Textbooks

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

- **Elementary Statistics: Picturing the World,** 6th Edition, Ron Larson and Betsy Farber

- **Elementary Statistics**, 13th Edition, Mario F. Triola

Reference books

- **Probability and Statistical Inference, Ninth Edition,** Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman

- **Probability Demystified**, Allan G. Bluman

- **Practical Statistics for Data Scientists: 50 Essential Concepts,** Peter Bruce and Andrew Bruce

- **Schaum's Outline of Probability,** Second Edition, Seymour Lipschutz, Marc Lipson

- **Python for Probability, Statistics, and Machine Learning,** José Unpingco

References

Readings for these lecture notes:

- **Probability & Statistics for Engineers & Scientists**, Ninth edition, Ronald E. Walpole, Raymond H. Myer

These notes contain material from the above book.

Discrete Probability Distribution

- The set of ordered **pairs (x, $f(x)$)** is a **probability function, probability mass function**, or **probability distribution** of the discrete random variable X if, for each possible outcome x,

1. $f(x) \geq 0$,

2. $\sum_x f(x) = 1$,

3. $P(X = x) = f(x)$.

Example: A shipment of **20 similar laptop computers** to a retail outlet contains **3 that are defective**. If a school makes a random purchase of **2 of these computers**, **find the probability distribution** for the number of **defectives**.

$N = 20$

$n = 2$

$k = 3$

$P(X = x) = h(x; N, n, k) = ({}_kC_x)({}_{N-k}C_{n-x})/({}_NC_n)$, $\max\{0, n-(N-k)\} \leq x \leq \min\{n, k\}$

Let X represent the number of defective computers

$\max\{0, n - (N-k)\} = \max\{0, 2 - (20 - 3)\}$

$= \max(0, -17) = 0$

$\min\{n, k\} = \min(2, 3) = 2$

Probability Distribution	
x	P(X = x)
0	$\frac{68}{95}$
1	$\frac{51}{190}$
2	$\frac{3}{190}$
	$\sum P(X) = 1$

Example : If a car agency **sells 50%** of its inventory of a certain foreign car equipped with side airbags, find a formula for the **probability distribution** of the number of cars with side airbags among the **next 4 cars** sold by the agency.

$$\mathbf{b(x; n, p)} = \binom{\mathbf{n}}{\mathbf{x}} \mathbf{p^x q^{n-x}}, \quad \mathbf{x = 0, 1, 2, ..., n}$$

Here n = 4, p = 0.50, q = 0.50

Let x denotes the number of cars with side airbags

$$\mathbf{b(x; 4, 0.50)} = \binom{\mathbf{4}}{\mathbf{x}} \mathbf{(0.50)^x (0.50)^{4-x}}, \quad \mathbf{x = 0, 1, 2, 3, 4}$$

$$= \binom{4}{x}(0.50)^4, \quad x = 0, 1, 2, 3, 4$$

$$\mathbf{b(x; 4, 0.50)} = \frac{1}{16}\binom{4}{x}, \quad x = 0, 1, 2, 3, 4$$

Cumulative Distribution Function

The **cumulative distribution function *F*(*x*)** of a discrete random variable *X* with probability distribution *f*(*x*) is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \text{ for } -\infty < x < \infty$$

Example A stockroom clerk returns **three safety helmets at random** to three steel mill employees

who had previously checked them. If **Smith, Jones, and Brown**, in that order, receive one of the three hats, list the **sample points for the possible orders of returning the helmets**, and find the value m of the random variable M that represents the number of **correct matches**

If **S, J,** and **B** stand for **Smith's**, **Jones's**, and **Brown's** helmets, respectively, then the possible arrangements in which the helmets may be returned and the number of correct matches are

Sample space	m
SJB	3
SBJ	1
JS**B**	1
B**J**S	1
JBS	0
BSJ	0

For the random variable M, the number of correct matches in the previous example, we have

$$F(2) = P(M \leq 2) = f(0) + f(1) = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$$

The cumulative distribution function of M is

$$\mathbf{F(m)} = \begin{cases} \mathbf{0}, & \text{for } m < 0, \\ \mathbf{\frac{1}{3}}, & \text{for } 0 \leq m < 1, \\ \mathbf{\frac{5}{6}}, & \text{for } 1 \leq m < 3, \\ \mathbf{1}, & \text{for } m \geq 3. \end{cases}$$

Example : Find the **cumulative distribution function** of the random variable X in $f(x) = \frac{1}{16}\binom{4}{x}$, x = 0, 1, 2, 3, 4. Using ***F*(*x*),** verify that ***f*(2) = 3/8**.

$$f(x) = \frac{1}{16}\binom{4}{x}, \quad x = 0, 1, 2, 3, 4$$

$$f(0) = \frac{1}{16}$$

$$f(1) = \frac{4}{16}$$

$$f(2) = \frac{6}{16}$$

$$f(3) = \frac{4}{16}$$

$$f(4) = \frac{1}{16}$$

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \text{ for } -\infty < x < \infty$$

$$F(0) = P(X \leq 0) = f(0) = \frac{1}{16},$$

$$F(1) = P(X \leq 1) = f(0) + f(1) \text{ ---------------------------------(1)}$$

$$= \frac{1}{16} + \frac{4}{16} = \frac{5}{16},$$

$$F(2) = P(X \leq 2) = f(0) + f(1) + f(2) \text{ ---------------------------(2)}$$

$$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} = \frac{11}{16},$$

$$F(3) = P(X \leq 3) = f(0) + f(1) + f(2) + f(3)$$

$$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16}$$

$$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} = \frac{15}{16},$$

$$F(4) = P(X \le 4) = f(0) + f(1) + f(2) + f(3) + f(4)$$

$$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16}$$

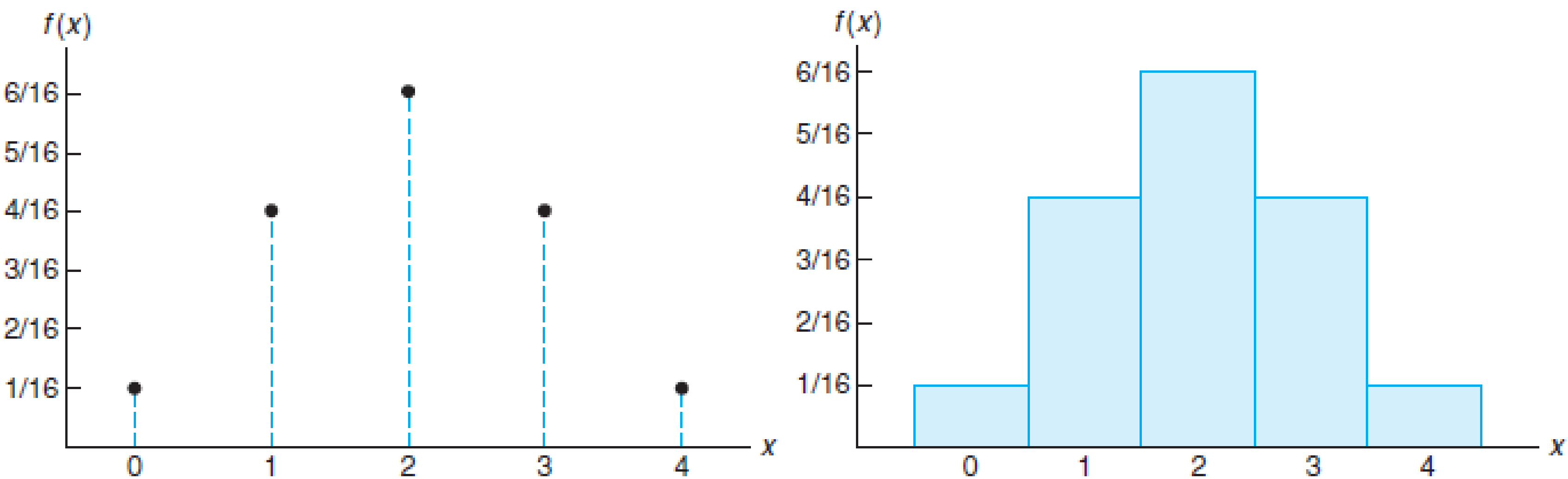
$$= \frac{16}{16} = 1$$

$$\therefore F(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{1}{16}, & \text{for } 0 \le x < 1, \\ \frac{5}{16}, & \text{for } 1 \le x < 2, \\ \frac{11}{16}, & \text{for } 2 \le x < 3, \\ \frac{15}{16}, & \text{for } 3 \le x < 4, \\ 1, & \text{for } x \ge 4. \end{cases}$$

(2) –(1):

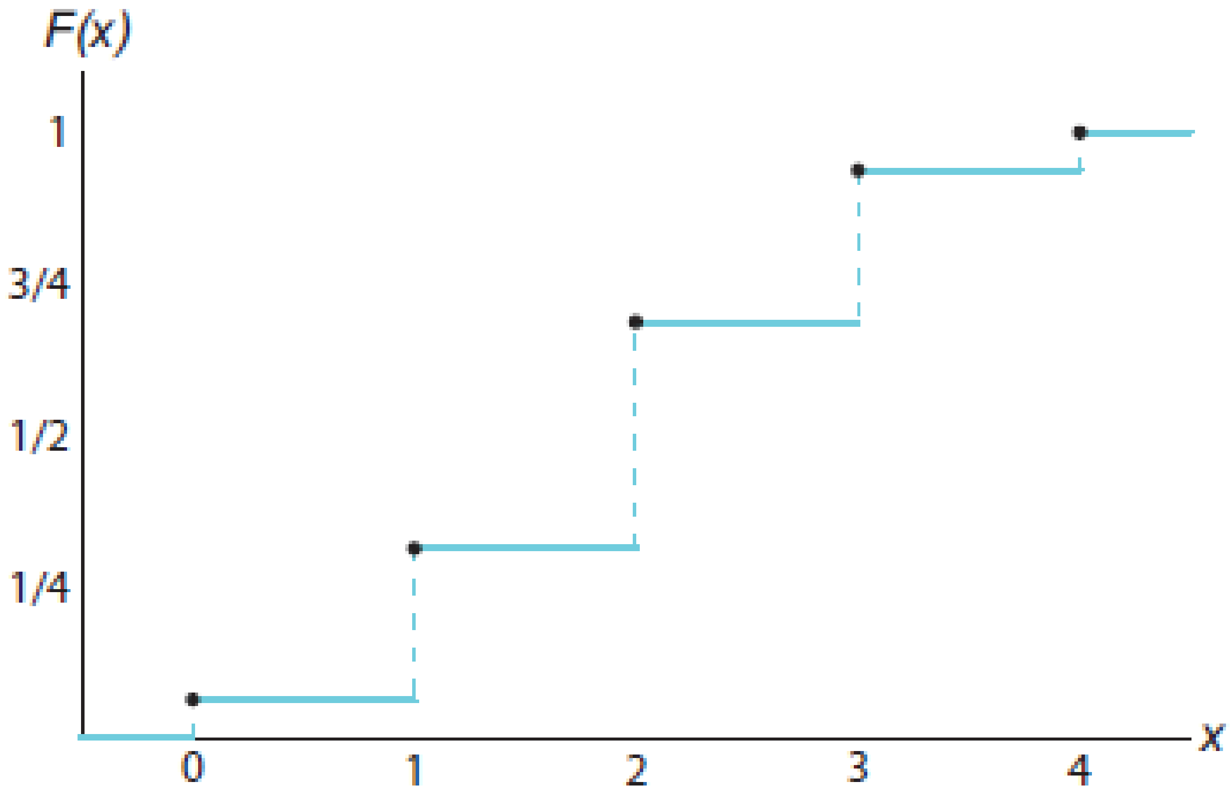
$f(2) = F(2) – F(1) = \frac{11}{16} - \frac{5}{16} = \frac{6}{16} = \frac{\mathbf{3}}{\mathbf{8}}$

Probability mass function plot vs. Probability histogram



Probability mass function plot vs. Probability histogram

Discrete cumulative distribution function



Discrete cumulative distribution function

Continuous Probability Distributions

- A **continuous random variable** has a probability of **0** of assuming *exactly* any of its values.

- Consequently, its **probability distribution cannot** be given in **tabular form**.

Continuous Probability Distributions

- We shall concern ourselves with **computing probabilities for various intervals** of continuous random variables such as $P(a < X < b)$, $P(W \geq c)$, and so forth.

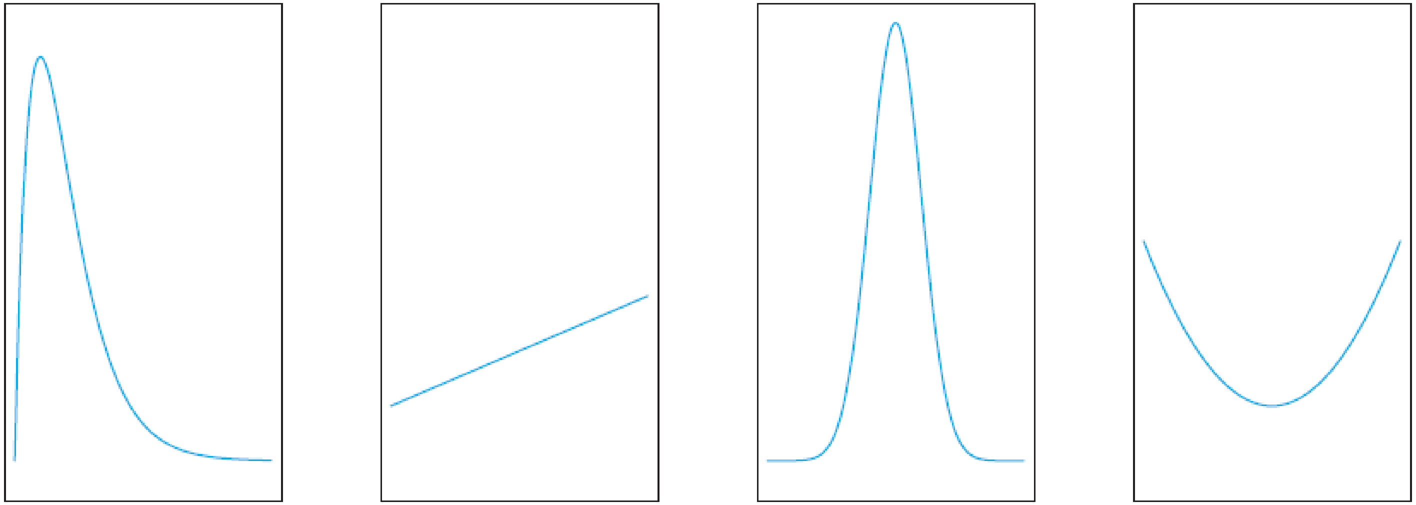
- Note that when X is continuous,

$$P(a < X \leq b) = P(a < X < b) + P(X = b) = P(a < X < b).$$

- That is, it does not matter whether we include an endpoint of the interval or not.

- This is not true, though, when X is discrete.

- Because **areas** will be used to represent probabilities and probabilities are **positive numerical values**, the **density function** must lie entirely **above the *x* axis**.

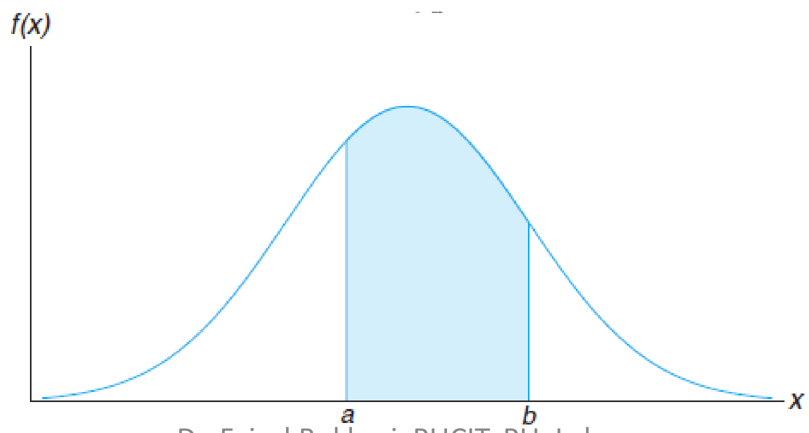


Typical density functions.

Probability Density Function

The function ***f*(*x*)** is a **probability density function (pdf)** for the continuous random variable *X*, defined over the set of real numbers, if

1. $f(x) \geq 0$, for all $x \in R$.
2. $\int_{-\infty}^{+\infty} \mathbf{f(x)\ dx} = 1.$
3. $P(a < X < b) = \int_{a}^{b} \mathbf{f(x)\ dx}$



Example: Suppose that the error in the reaction temperature, in ◦C, for a controlled laboratory experiment is a continuous random variable *X* having the probability density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere} \end{cases}$$

(a) Verify that ***f*(*x*)** is a **density function**.

(b) Find $P(0 < X \leq 1)$.

- $f(x) \geq 0.$
- $\int_{-\infty}^{+\infty} f(x)\, dx = 1.$

$$\text{LHS} = \int_{-1}^{2} \frac{x^2}{3} dx$$

$$= [\frac{x^3}{9}]_{-1}^{2}$$

$$= \frac{[(2)^3 - (-1)^3]}{9}$$

$$= 1$$

LHS = RHS

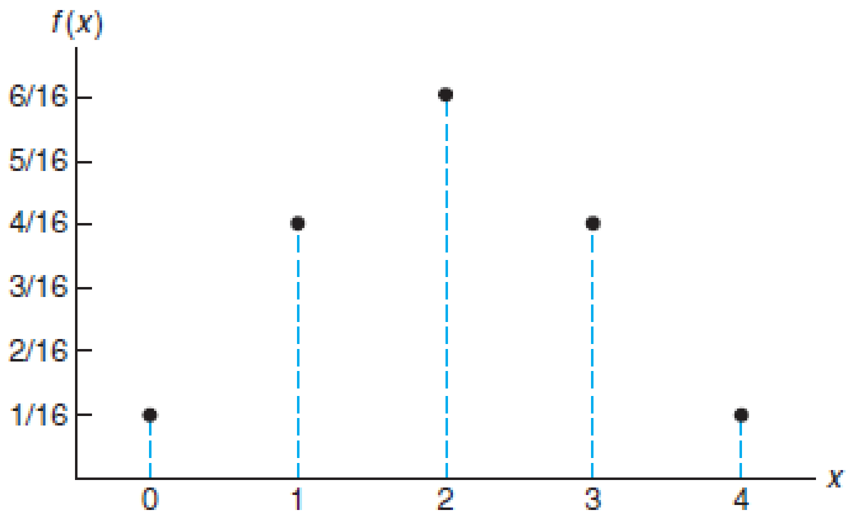
$$P(0 < X \leq 1) = \int_0^1 \frac{x^2}{3}\,dx$$

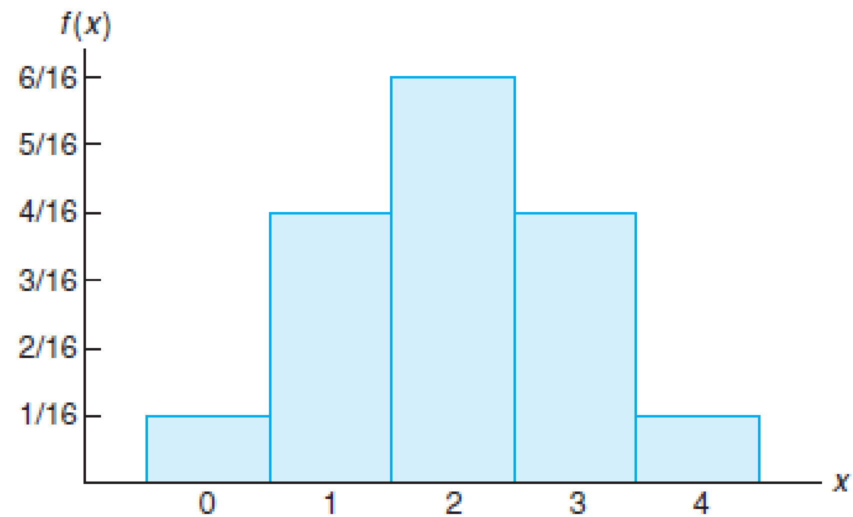
$$= \left[\frac{x^3}{9}\right]_0^1$$

$$= \frac{[(1)^3 - (0)^3]}{9}$$

$$= \frac{1}{9}$$

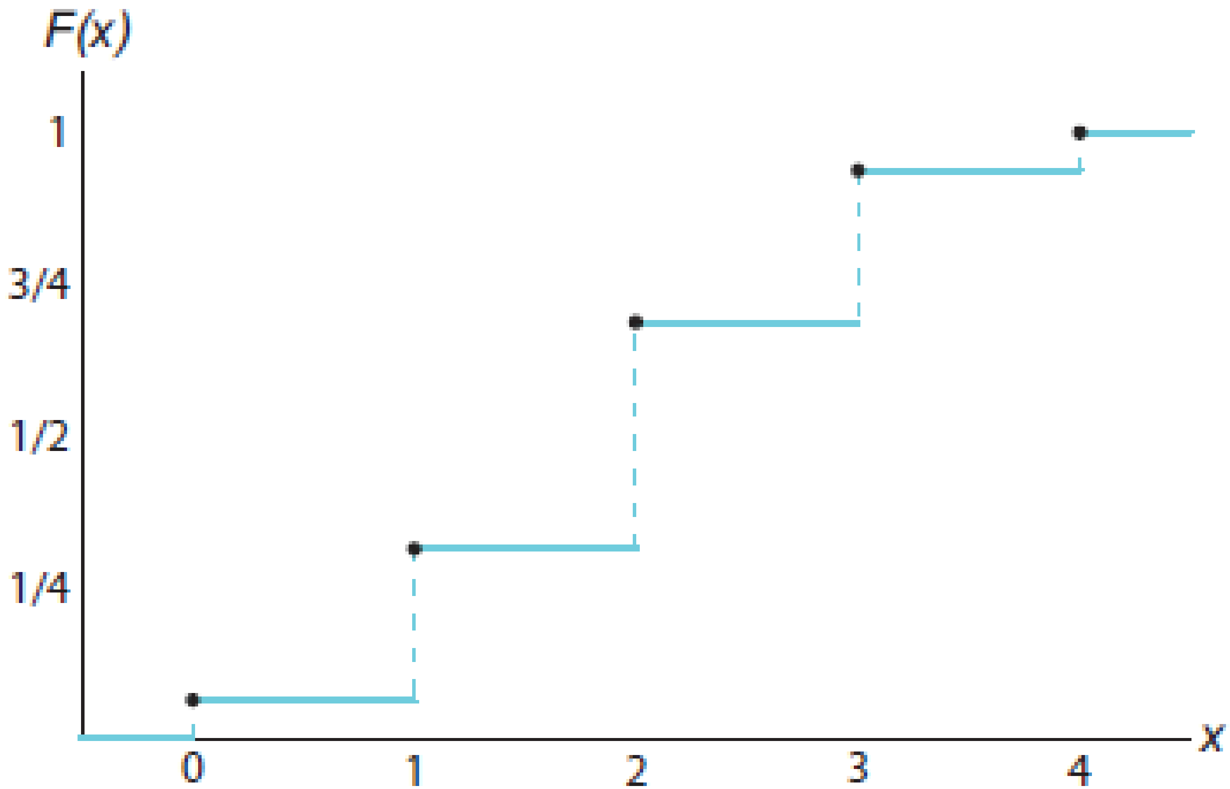
Probability mass function plot vs. Probability histogram





Probability mass function plot vs. Probability histogram

Discrete cumulative distribution function



Discrete cumulative distribution function

Cumulative Distribution Function

The **cumulative distribution function** ***F*(*x*)** of a continuous random variable *X* with density function *f*(*x*) is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt, \text{ for } -\infty < x < \infty$$

$P(a < X < b) = F(b) - F(a)$ and $f(x) = \frac{dF(x)}{dx}$, if the derivative exists.

Example: For the density function

$$f(x) = \begin{cases} \frac{x^2}{3}, -1 < x < 2, \\ 0, \text{ elsewhere} \end{cases}$$

, **find *F*(*x*),** and use it to evaluate ***P*(0 < *X* ≤ 1).**

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt, \text{ for } -\infty < x < \infty$$

For $-1 < x < 2$,

$$F(x) = \int_{-1}^{x} \frac{t^2}{3}\,dt$$

$$= \left[\frac{t^3}{9}\right]_{-1}^{x}$$

$$= \frac{[(x)^3 - (-1)^3]}{9}$$

$$= \frac{x^3 + 1}{9}$$

$$F(x) = \begin{cases} 0, & \text{for } x < -1, \\ \dfrac{x^3+1}{9}, & \text{for } -1 \leq x < 2, \\ 1, & \text{for } x \geq 2 \end{cases}$$

$$P(0 < X \leq 1) = \mathrm{F}(1) - \mathrm{F}(0)$$

$$\mathrm{F}(1) = \frac{1^3 + 1}{9} = \frac{2}{9}$$

$$\mathrm{F}(0) = \frac{0^3 + 1}{9} = \frac{1}{9}$$

$$P(0 < X \leq 1) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

Example: The **Department of Energy (DOE)** puts projects out on bid and generally estimates what a reasonable bid should be. Call the **estimate *b***. The DOE has determined that the **density function** of the **winning (low) bid** is

$$f(x) = \begin{cases} \frac{5}{8b}, & \frac{2}{5}b \le y \le 2b, \\ 0, & \text{elsewhere} \end{cases}$$

Find ***F*(*y*)** and use it to **determine the probability** that the **winning bid is less than** the DOE's preliminary **estimate *b***.

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt, \text{ for } -\infty < x < \infty$$

$$\frac{2}{5}b \leq y \leq 2b$$

$$F(y) = \int_{\frac{2}{5}b}^{y} \frac{5}{8b}\, dy$$

$$= \left[\frac{5}{8b} y\right]_{\frac{2}{5}b}^{y}$$

$$= \frac{5}{8b} y - \frac{5}{8b}\left(\frac{2}{5} b\right)$$

$$= \frac{5}{8b} y - \frac{1}{4}$$

$$F(y) = \begin{cases} \mathbf{0}, & y < \frac{2}{5}b, \\ \frac{\mathbf{5}}{\mathbf{8b}}\boldsymbol{y} - \frac{\mathbf{1}}{\mathbf{4}}, & \frac{2}{5}b \le y \le 2b \\ \mathbf{1}, & y \ge 2b. \end{cases}$$

To determine the probability that the **winning bid** is less than the **preliminary bid estimate *b***, we have

$$F(y) = \frac{5}{8b}\mathbf{y} - \frac{1}{4}$$

$$\Rightarrow F(b) = \frac{5}{8b}\mathbf{b} - \frac{1}{4}$$

$$\Rightarrow F(b) = \frac{5}{8} - \frac{1}{4}$$

$$\therefore \boldsymbol{P(Y \le b) = F(b) = \frac{5}{8} - \frac{1}{4} = \frac{3}{8}}$$