Statistical and Mathematical Methods for Data Analysis

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Textbooks

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

- **Elementary Statistics: Picturing the World,** 6th Edition, Ron Larson and Betsy Farber

- **Elementary Statistics**, 13th Edition, Mario F. Triola

Reference books

- **Probability and Statistical Inference, Ninth Edition,** Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman

- **Probability Demystified**, Allan G. Bluman

- **Practical Statistics for Data Scientists: 50 Essential Concepts,** Peter Bruce and Andrew Bruce

- **Schaum's Outline of Probability,** Second Edition, Seymour Lipschutz, Marc Lipson

- **Python for Probability, Statistics, and Machine Learning,** José Unpingco

References

Readings for these lecture notes:

- Probability & Statistics for Engineers & Scientists, Ninth edition, Ronald E. Walpole, Raymond H. Myer

- Elementary Statistics, Tenth Edition, Mario F. Triola

These notes contain material from the above resources.

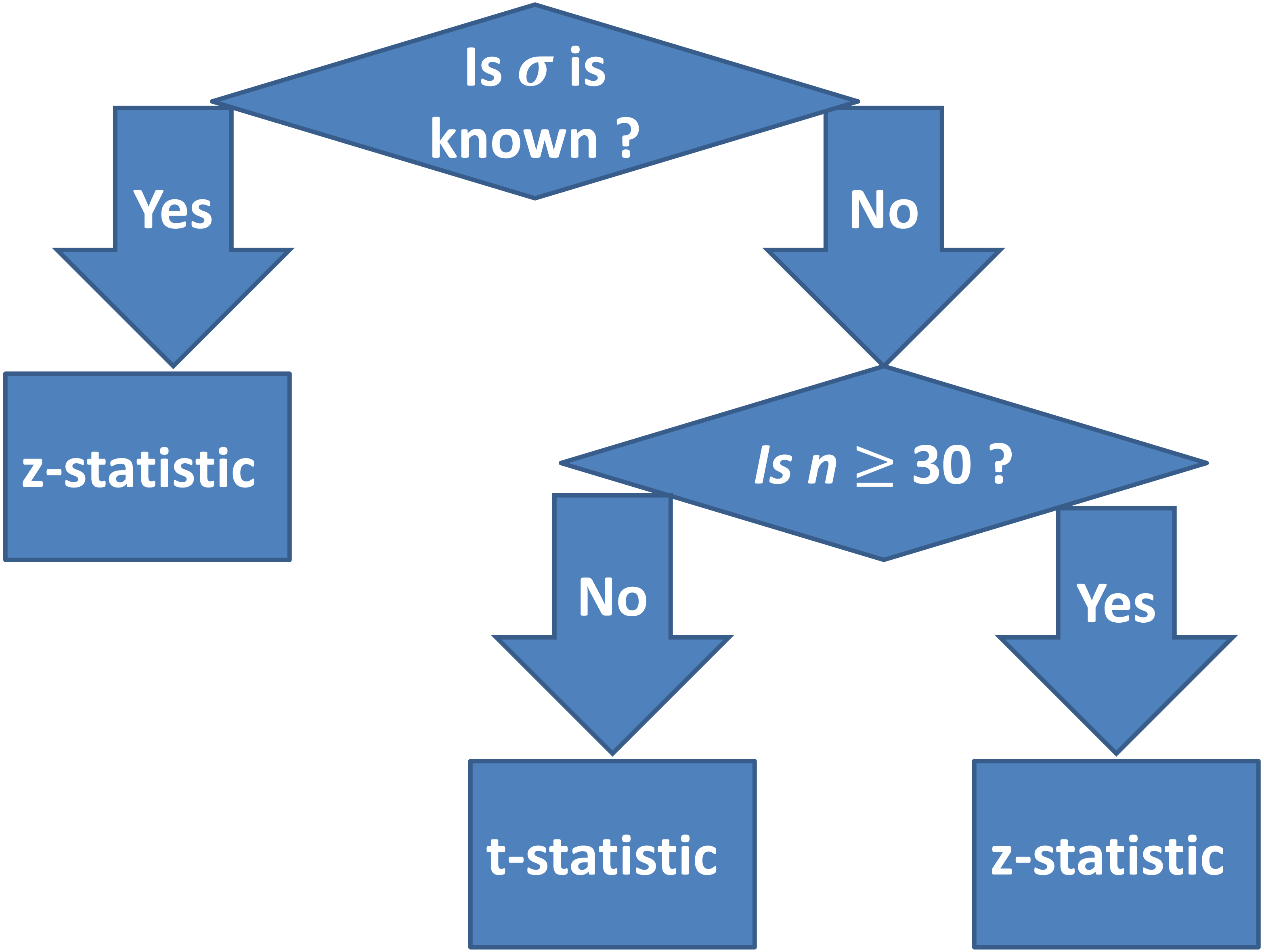
Is σ is known ?

Yes

If either the population is normally distributed or $n \geq 30$, then use the use the standard normal distribution or Z-test

No

If either the population is normally distributed or $n \geq 30$, then use the t-distribution or t-test

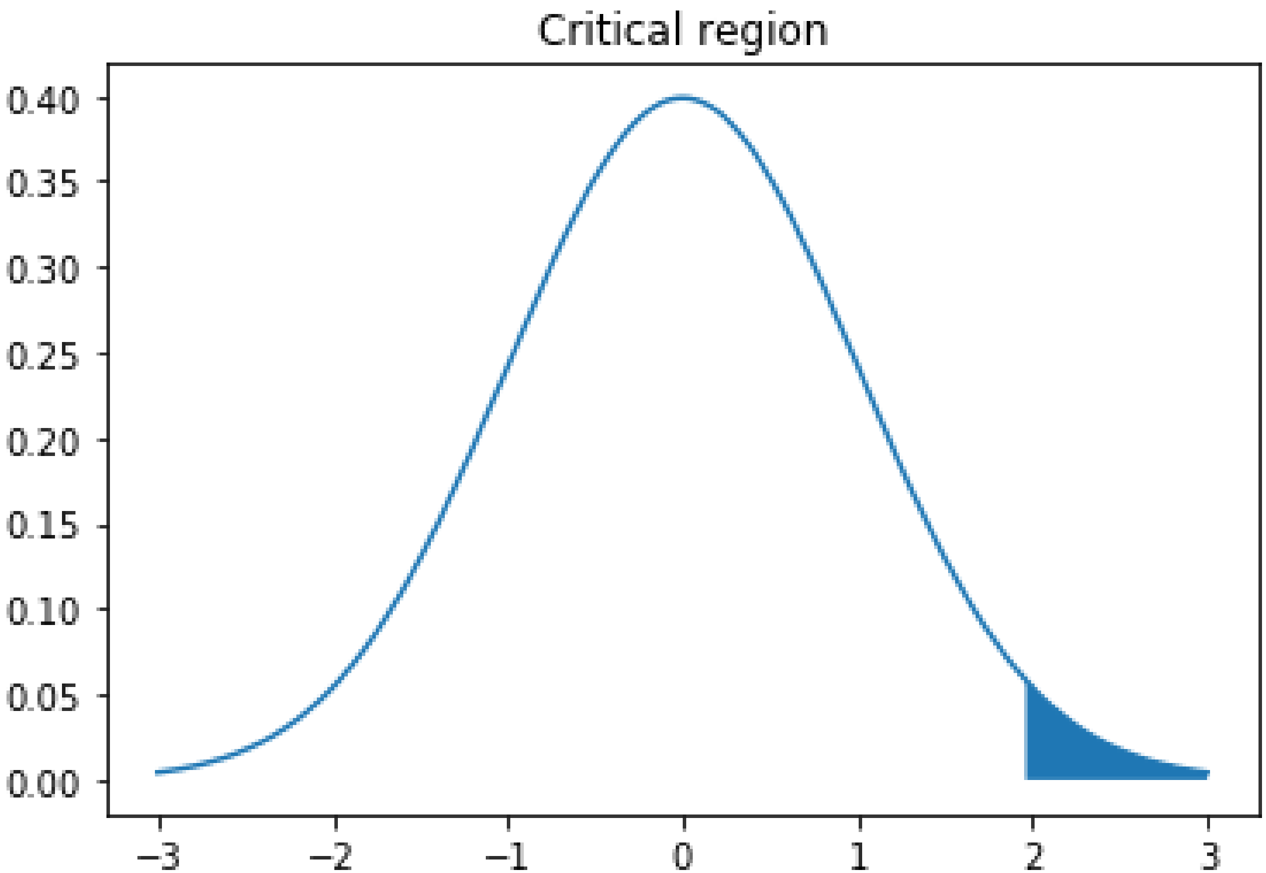

Is σ is known ?
Yes
No
z-statistic
Is $n \geq 30$?
No
Yes
t-statistic
z-statistic

When both ***n* < 30** and the **population is *not* normally distributed**, we *cannot* use the **standard normal distribution** or the ***t*-distribution**.

Critical Region: Scenario 1

$H_1: \mu > \mu_0$ (One tailed test)

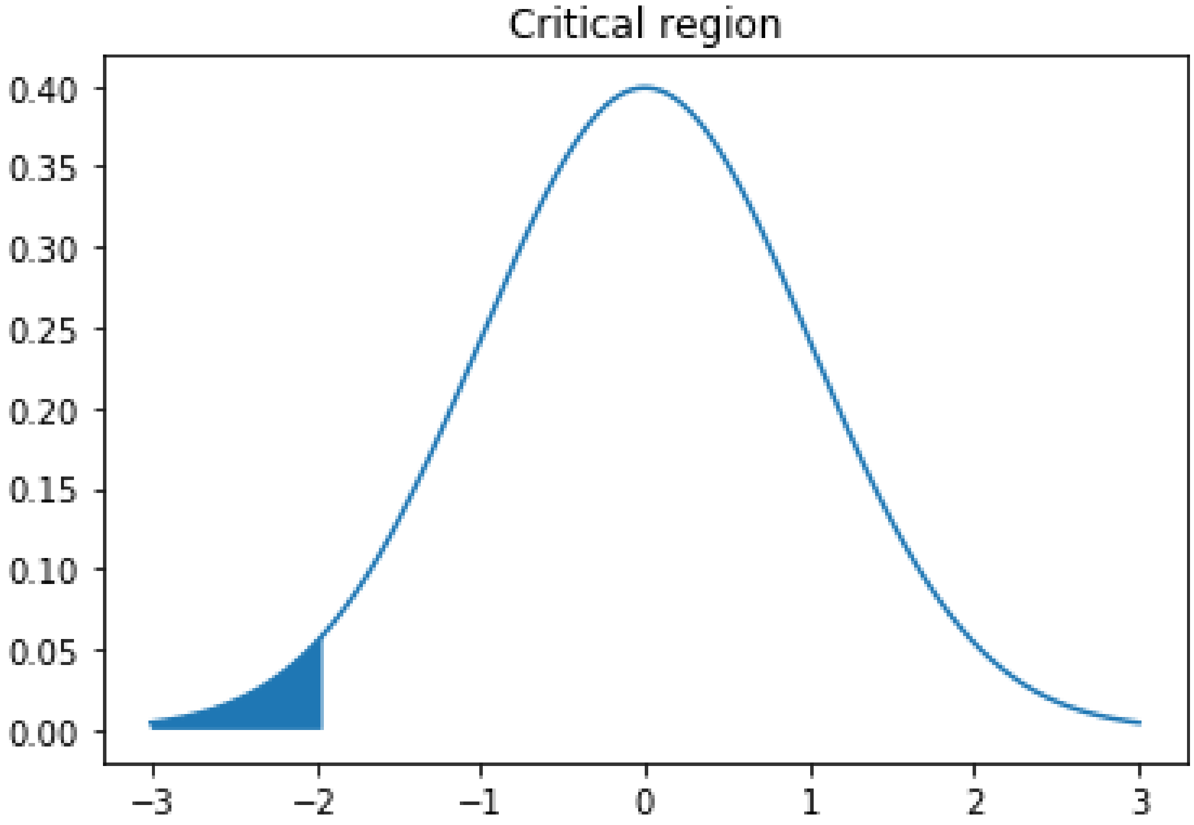
$Z_{cal} > Z_{tab}$, where $Z_{tab} = Z_\alpha$



Critical Region: Scenario 2

H_1: $\mu < \mu_0$ (One tailed test)

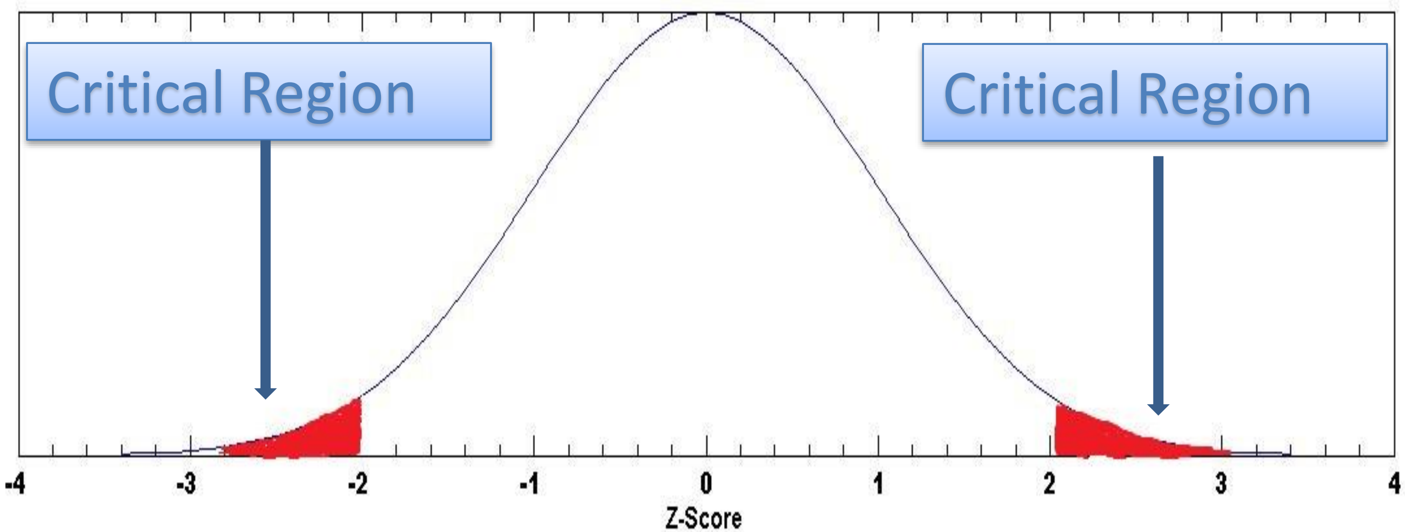
$Z_{cal} < Z_{tab}$, where $Z_{tab} = -Z_{\alpha}$



Critical Region: Scenario 3

H_1: $\mu \neq \mu_0$ **(Two tailed test)**

$|Z_{cal}| > Z_{tab}$, where $Z_{tab} = Z_{\alpha/2}$



Approach to Hypothesis Testing with Fixed Probability of Type I Error

1. State the null and alternative hypotheses.
2. Choose a fixed significance level α.
3. Test statistic to be used it
4. Calculations
5. Critical region
6. Conclusion

Area under the Normal Curve [1]

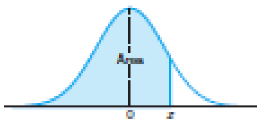


Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Area under the Normal Curve [2]

Table A.3 (continued) Areas under the Normal Curve

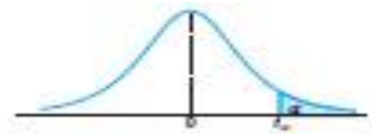
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

H_0	Value of Test Statistic	H_1	Critical Region
$\mu = \mu_0$	$z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$; σ known	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$z < -z_\alpha$ $z > z_\alpha$ $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$
$\mu = \mu_0$	$t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$; $v = n - 1$, σ unknown	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$
$\mu_1 - \mu_2 = d_0$	$z = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$; σ_1 and σ_2 known	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$z < -z_\alpha$ $z > z_\alpha$ $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$
$\mu_1 - \mu_2 = d_0$	$t = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{s_p\sqrt{1/n_1 + 1/n_2}}$; $v = n_1 + n_2 - 2$, $\sigma_1 = \sigma_2$ but unknown, $s_p^2 = \dfrac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$
$\mu_1 - \mu_2 = d_0$	$t' = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$; $v = \dfrac{(s_1^2/n_1 + s_2^2/n_2)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$, $\sigma_1 \neq \sigma_2$ and unknown	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$t' < -t_\alpha$ $t' > t_\alpha$ $t' < -t_{\alpha/2}$ or $t' > t_{\alpha/2}$
$\mu_D = d_0$ paired observations	$t = \dfrac{\bar{d} - d_0}{s_d/\sqrt{n}}$; $v = n - 1$	$\mu_D < d_0$ $\mu_D > d_0$ $\mu_D \neq d_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$

Two Samples: Tests on Two Means

Example: An experiment was performed to compare the abrasive wear of two different laminated materials. **Twelve** pieces of material 1 were tested by exposing each piece to a machine measuring wear. **Ten** pieces of material 2 were similarly tested. In each case, the depth of wear was observed. The **samples** of material 1 gave an **average** (coded) wear of **85** units with a **sample standard deviation** of **4**, while the samples of material **2** gave an **average of 81** with a **sample standard deviation** of **5**. Can we conclude at the **0.05** level of significance that the abrasive wear of material 1 exceeds that of material 2 by more than 2 units? Assume the populations to be approximately normal with **equal variances**.

Table A.4 Critical Values of the t-Distribution



v	α = 0.40	0.30	0.20	0.15	0.10	0.05	0.025
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.538	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

	α						
v	0.02	0.015	0.01	0.0075	0.005	0.0025	0.0005
1	15.894	21.205	31.821	42.433	63.656	127.321	636.578
2	4.849	5.643	6.965	8.073	9.925	14.089	31.600
3	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	2.328	2.491	2.718	2.879	3.106	3.497	4.437
12	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	2.249	2.397	2.602	2.746	2.947	3.286	4.073
16	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	2.197	2.336	2.528	2.661	2.845	3.153	3.850
21	2.189	2.328	2.518	2.649	2.831	3.135	3.819
22	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	2.167	2.301	2.485	2.612	2.787	3.078	3.725
26	2.162	2.296	2.479	2.605	2.779	3.067	3.707
27	2.158	2.291	2.473	2.598	2.771	3.057	3.689
28	2.154	2.286	2.467	2.592	2.763	3.047	3.674
29	2.150	2.282	2.462	2.586	2.756	3.038	3.660
30	2.147	2.278	2.457	2.581	2.750	3.030	3.646
40	2.123	2.250	2.423	2.542	2.704	2.971	3.551
60	2.099	2.223	2.390	2.504	2.660	2.915	3.460
120	2.076	2.196	2.358	2.468	2.617	2.860	3.373
∞	2.054	2.170	2.326	2.432	2.576	2.807	3.290

Solution:

$n_1 = 12$ (Sample size of material 1)

$\overline{x_1} = 85$ (Sample mean of material 1)

$s_1 = 4$ (sample standard deviation of material 1)

$n_2 = 10$ (Sample size of material 2)

$\overline{x_2} = 81$ (Sample mean of material 2)

$s_2 = 5$ (sample standard deviation of material 1)

$\alpha = 0.05$ (Level of significance

Given assumption the populations to be approximately normal with **equal variances**

$\Rightarrow$ $\boldsymbol{\sigma_1^2 = \sigma_1^2}$ **but unknown**

1. **We state our hypothesis as**:

$H_0: \mu_1 - \mu_2 = 2$

$H_1: \mu_1 - \mu_2 > 2$

(One tailed test)

2. **The level of significance is set** $\alpha = 0.05$.

3. **Test statistic to be used is**

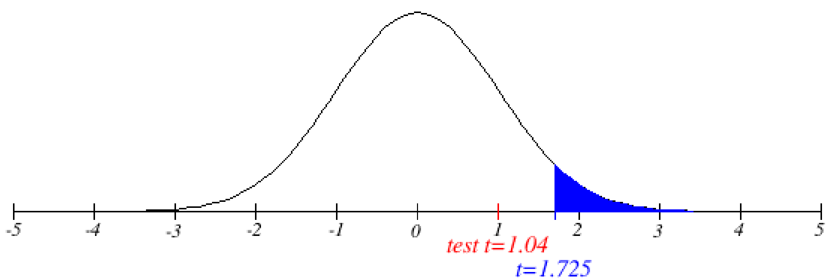
$$t_{cal} = \frac{(\overline{x}_1 - \overline{x}_2) - (\mu_1 - \mu_2)}{s_p \sqrt{1/n_1 + 1/n_2}}$$

$$\text{where, } sp = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

$$s_1^2 = \frac{\sum(x_1 - \overline{x}_1)^2}{n_1 - 1} \quad \text{and} \quad s_2^2 = \frac{\sum(x_2 - \overline{x}_2)^2}{n_2 - 1}$$

4. **Calculations:** $s_p = \sqrt{\dfrac{(12-1)4^2 + (10-1)5^2}{12+10-2}} = 4.478$

$$t_{cal} = \frac{(85-81) - 2}{4.478\sqrt{1/12 + 1/10}} = 1.04$$



5. **Critical region:**

$t_{cal} > t_{tab}$

Where $t_{tab} = t_{(\alpha,\, n1 + n2 - 2)} = t_{(0.05,\, 20)} = 1.725$

6. **Conclusion:** Since calculated value of t_{cal} is less than the tabulate value of t, so we accept H_O

Two Samples: Tests on Two Means

Example: An experiment was performed to compare the abrasive wear of two different laminated materials. **Twelve** pieces of material 1 were tested by exposing each piece to a machine measuring wear. **Ten** pieces of material 2 were similarly tested. In each case, the depth of wear was observed. The **samples** of material 1 gave an **average** (coded) wear of **85** units with a **sample standard deviation** of **4**, while the samples of material **2** gave an **average of 81** with a **sample standard** deviation of **5**.

Cont.

Can we conclude at the **0.05** level of significance that the abrasive wear of material 1 exceeds that of material 2 by more than 2 units? Assume the populations to be approximately normal with **unequal variances**.

Solution:

$n_1 = 12$ (Sample size of material 1)

$\overline{x_1} = 85$ (Sample mean of material 1)

$s_1 = 4$ (sample standard deviation of material 1)

$n_2 = 10$ (Sample size of material 2)

$\overline{x_2} = 81$ (Sample mean of material 2)

$s_2 = 5$ (sample standard deviation of material 1)

$\alpha = 0.05$ (Level of significance

Given assumption the populations to be approximately normal with **unequal variances**

$\Rightarrow$ $\boldsymbol{\sigma_1^2 \neq \sigma_1^2}$ **but unknown**

1. **We state our hypothesis as**:

$H_0: \mu_1 - \mu_2 = 2$

$H_1: \mu_1 - \mu_2 > 2$

(One tailed test)

2. **The level of significance is set** $\alpha = 0.05$.

3. **Test statistic to be used is**

$$t_{cal} = \frac{(\overline{x}_1 - \overline{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1{}^2}{n_1} + \frac{s_2{}^2}{n_2}}}$$

where $s_1^2 = \frac{\sum(x_1 - \overline{x}_1)^2}{n_1 - 1}$ and $s_2^2 = \frac{\sum(x_2 - \overline{x}_2)^2}{n_2 - 1}$

4. **Calculations:** $t_{cal} = \frac{(85 - 81) - 2}{\sqrt{4^2/12 + 5^2/10}} = 2/1.8708 = 1.0691$

5. **Critical region:**

$t_{cal} > t_{tab}$

where $t_{tab} = t_{(\alpha, v)}$

$$v = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{[(s_1^2/n_1)^2/(n_1-1) + (s_2^2/n_2)^2/(n_2-1)]}$$

$$v = \frac{(4^2/12 + 5^2/10)^2}{[(4^2/12)^2/11 + (5^2/10)^2/9]}$$

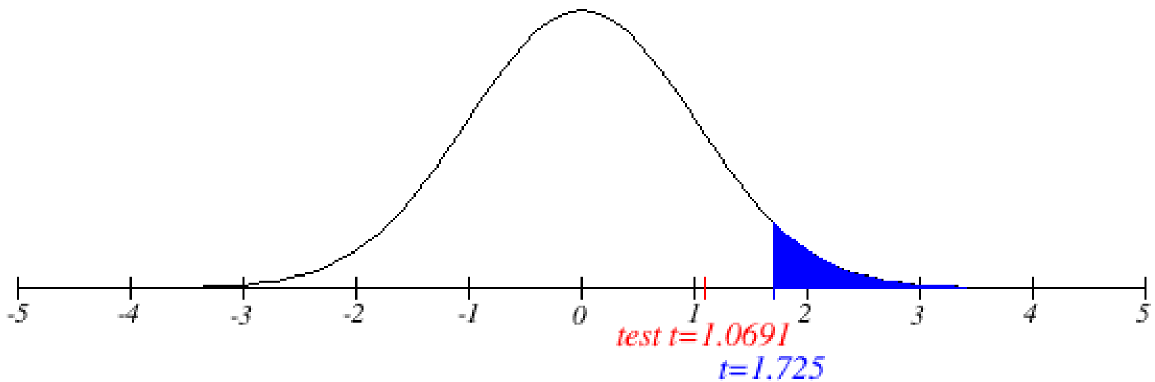
$v = 16.6944/(0.1616+0.6944) = 19.5028 = 20$

$\mathbf{t_{(\alpha, v)} = t_{(0.05, 20)} = 1.725}$

$\mathbf{t_{cal} = 1.0691}$

$t_{cal} < t_{tab}$

$1.0691 < 1.725$



t_{cal} = 1.0691

$t_{cal} < t_{tab}$

$1.0691 < 1.725$

6. **Conclusion:** Since calculated value of t_{cal} is less than the tabulate value of t, so we accept H_O

1. When $\alpha = 0.05$ or $\alpha = 5\%$

$z_{\alpha/2} = z_{0.0250} = 1.96$ $\because 1 - \alpha/2 = 1 - 0.250 = 0.9750$

2. When $\alpha = 0.01$ or $\alpha = 1\%$

$z_{\alpha/2} = z_{0.005} = 2.575$ $\because 1 - \alpha/2 = 1 - 0.005 = 0.9950$

3. When $\alpha = 0.10$ or $\alpha = 10\%$

$z_{\alpha/2} = z_{0.05} = 1.645$ $\because 1 - \alpha/2 = 1 - 0.05 = 0.9500$

Testing a Proportion

1. H_0: $p = p_0$.

One of the alternatives H_1: $p < p_0$, $p > p_0$, or $p \neq p_0$.

2. Choose a level of significance equal to α.

3. Test statistic: Binomial variable X with $p = p_0$.

$$Z_{cal} = \frac{x - np_0}{\sqrt{np_0q_0}} \quad \text{or } Z_{cal} = \frac{\hat{p} - p_0}{\sqrt{p_0q_{0\,/\,n}}}$$

Where **p_0** is the **population proportion** and **$\hat{p}$** is the **sample proportion**

4. Computations: Find x, the number of successes, and compute the appropriate P-value.

5. Decision: Draw appropriate conclusions based on the *P*-value.

Example: A **builder claims** that heat pumps are installed in **70%** of all homes being constructed today in the city of Richmond, Virginia. Would you **agree with this claim** if a **random survey** of new homes in this city showed that **8 out of 15** had heat pumps installed? Use a **0.10 level of significance**.

Solution:

p or $p_0 = 0.70$ **(population proportion)**

$\hat{p} = x/n = 8/15 = 0.53$ **(sample proportion)**

$\hat{q} = 1 - 0.53 = 0.47$

$n=15$

1. **We state our hypothesis as**

H_0: p = 0.7.

H_1: p $\neq$ 0.7 (two tailed test)

2. **The level of significance is** set at α = 0.10.

3. **Test statistic to be used is**:

$$Z_{cal} = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0 / n}}$$

4. **Calculations:**

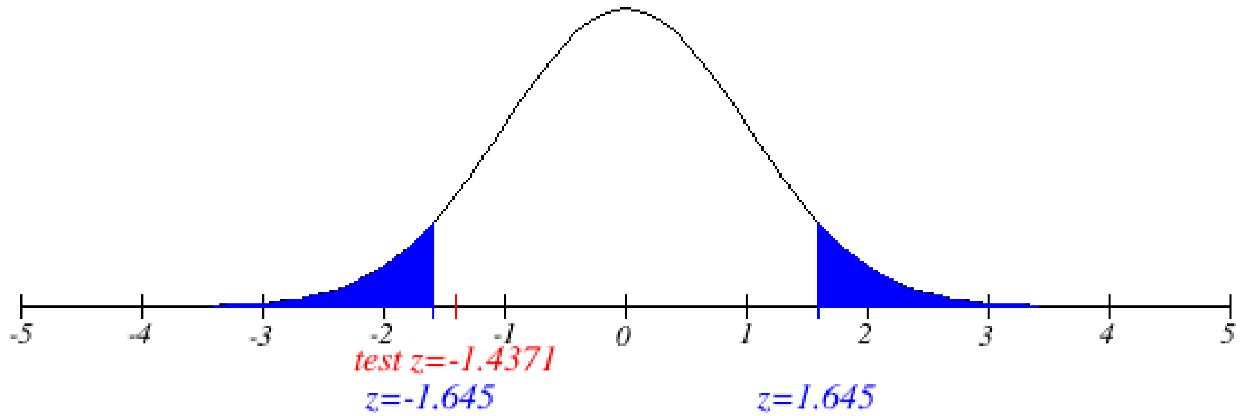
$$Z_{cal} = \frac{0.53 - 0.70}{\sqrt{0.70(0.30)/15}}$$

$$= -0.17/0.1183 = -1.4371$$

5. **Critical region:**

$|Z_{cal}| > Z_{tab}$

Where $Z_{tab} = Z_{\alpha/2} = Z_{0.05} = 1.645$



6. **Conclusion:** Since calculated value of Z is less than the tabulated value of Z, so we accept H_o and conclude that there is insufficient reason to doubt the builder's claim.

Example: A commonly prescribed drug for relieving nervous tension is believed to be only **60% effective**. Experimental results with a new drug administered to a random sample of **100 adults** who were suffering from nervous tension show that **70 received relief**. Is this sufficient evidence to conclude that the new drug is **superior** to the one **commonly prescribed**? Use a 0.05 level of significance.

Solution:

p or $p_0 = 0.60$ **(population proportion)**

$\hat{p} = x/n = 70/100 = 0.70$ **(sample proportion)**

$\hat{q} = 1 - 0.70 = 0.30$

$n = 100$ **(sample size)**

1. We state our hypothesis as

H_0: p = 0.60

H_1: p > 0.60 (one tailed test)

2. **The level of significance is set at α = 0.05**.

3. **Test statistic to be used is**:

$$Z_{cal} = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0 / n}}$$

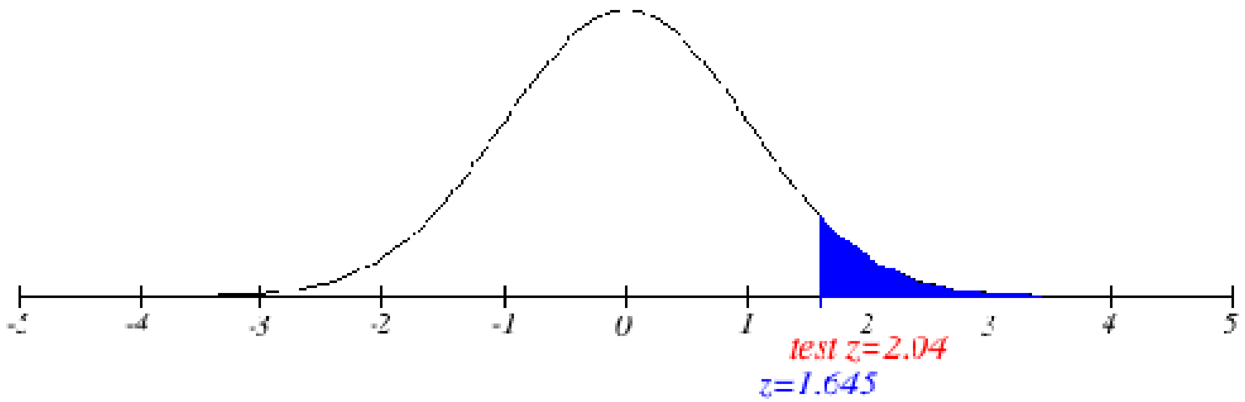
4. **Calculations:**

$$Z_{cal} = \frac{0.70 - 0.60}{\sqrt{(0.60)(0.40)/10}} = 2.04$$

5. **Critical region:**

$Z_{cal} > Z_{tab}$

Where $\mathbf{Z_{tab} = Z_{\alpha} = Z_{0.05} = 1.645}$



6. **Conclusion:** Since calculated value of Z is greater than the tabulated value of Z, so we reject H_o.

Two Samples: Tests on Two Proportions

$$Z_{cal} = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{P_c q_c \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where $p_c = \frac{x_1 + x_2}{n_1 + n_2}$ and $q_c = 1 - p_c$

Where $\mathbf{p_1}$ and $\mathbf{p_2}$ are the **population proportion from population 1 and population 2** and $\widehat{\mathbf{p_1}}$ and $\widehat{\mathbf{p_2}}$ are the corresponding respective **sample proportions.**

Example: A vote is to be taken among the residents of a town and the surrounding county to determine whether a proposed chemical plant should be constructed. The construction site is within the town limits, and for this reason many voters in the county believe that the proposal will pass because of the large proportion of town voters who favor the construction. To determine if there is a significant difference in the proportions of town voters and county voters favoring the proposal, a poll is taken. If **120 of 200** town voters favor the proposal and **240 of 500** county residents favor it, would you agree **that the proportion of town voters favoring the proposal is higher than the proportion of county voters**? Use an $\alpha = 0.05$ level of significance.

Solution:

$\widehat{p_1}$ = x_1/n_2 = 120/200 = 0.60 **(sample proportion of town voters, favoring it)**

$\widehat{q_1}$ = 1 – 0.60 = 0.40

n_1 = 100 (sample size of town voters)

$\widehat{p_2}$ = x_2/n_2 = 240/500 = 0.60 **(sample proportion of county residents, favoring it)**

$\widehat{q_2}$ = 1 – 0.60 = 0.40

n_2 = 100 (sample size of county residents)

$p_c = \frac{x_1 + x2}{n_1 + n2}$ = (120 + 240)/(200 + 500) = 0.51

and q_c = 1 - p_c = 0.49

1. We state our hypothesis as

H_0: $p_1 = p_2$

H_1: $p_1 > p_2$ (one tailed test)

2. The level of significance is set at $\alpha = 0.05$.

3. Test statistic to be used is:

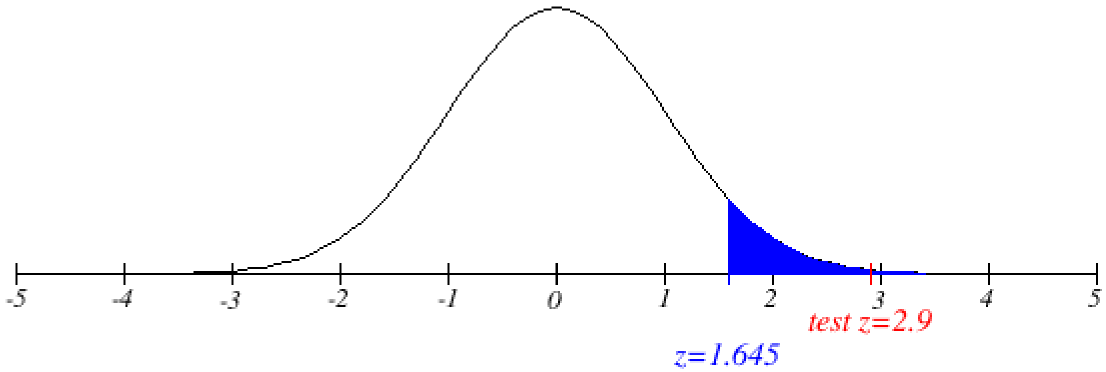
$$Z_{cal} = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{P_c q_c \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$ where $p_c = \frac{x_1+x_2}{n_1+n_2}$, and $q_c = 1 - p_c$

4. **Calculations:**

$$Z_{cal} = \frac{(0.60 - 0.48) - 0}{\sqrt{(0.51)(0.49)\left(\frac{1}{200} + \frac{1}{500}\right)}} = 2.9$$

5. **Critical region:**

$Z_{cal} > Z_{tab}$,where $Z_{tab} = Z_{\alpha} = Z_{0.05} = 1.645$



6. **Conclusion:** Since calculated value of Z is greater than the tabulated value of Z, so we reject H_o agree that the proportion of town voters favoring the proposal is higher than the proportion of county voters.