HW 4

1. What is the density matrix obtained if you have a qubit which is in state

$$\begin{array}{ll} |0\rangle & \text{with probability } \frac{1}{3}, \\ -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle & \text{with probability } \frac{1}{3}, \\ -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2u}|1\rangle & \text{with probability } \frac{1}{3}. \end{array}$$

$$\rho = \frac{1}{3}|0\rangle\langle 0| + \frac{1}{3}\left(-\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right)\left(-\frac{1}{2}\langle 0| + \frac{\sqrt{3}}{2}\langle 1|\right)$$

$$+ \frac{1}{3}\left(-\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right)\left(-\frac{1}{2}\langle 0| - \frac{\sqrt{3}}{2}\langle 1|\right)$$

$$= \frac{1}{3}|0\rangle\langle 0| + \frac{1}{3}\left(\frac{1}{4}|0\rangle\langle 0| - \cancel{\frac{\sqrt{3}}{4}|0\rangle\langle 1|} - \cancel{\frac{\sqrt{3}}{4}|1\rangle\langle 0|} + \frac{3}{4}|1\rangle\langle 1|\right)$$

$$+ \frac{1}{3}\left(\frac{1}{4}|0\rangle\langle 0| + \cancel{\frac{\sqrt{3}}{4}|0\rangle\langle 1|} + \cancel{\frac{\sqrt{3}}{4}|1\rangle\langle 0|} + \frac{3}{4}|1\rangle\langle 1|\right)$$

$$= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

2. What is the density matrix obtained if you take the partial trace over the second qubit of the following state; i.e., what is

$$\mathrm{Tr}_2 \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle).$$

$$\rho_1 = tr_2 \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$$

$$= \frac{1}{3}\sum_{\mu_2} \langle \mu_2|\left(|00\rangle + |01\rangle + |10\rangle\right)\left(\langle 00| + \langle 01| + \langle 10|\right)|\mu_2\rangle$$

$$= \frac{1}{3}\left(|0\rangle\langle 0| + |0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 1| + |1\rangle\langle 0|\right)$$

$$= \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| + \frac{1}{3}|0\rangle\langle 1| + \frac{1}{3}|1\rangle\langle 0|$$

3. One way to obtain a noisy quantum operation is to have the input quantum state interact with another "environment" quantum system, and then take a partial trace that removes the "environment" system.

Suppose we start with a qubit in state $|\psi\rangle$, and an "environment qubit" $|e\rangle$ in state $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$. We then apply the quantum gate controlled σ_z

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

to the state $|\psi\rangle \otimes |e\rangle$, and take the partial trace to remove $|e\rangle$. Express the resulting quantum operation in operator sum notation,

$$\rho \to \sum_i A_i \rho A_i^\dagger.$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes \sigma_z$$

$$|\Psi\rangle = |\psi\rangle \otimes \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right)$$

after control σ_z

$$\Rightarrow |\Psi\rangle_{after} = \frac{\sqrt{3}}{2}|\psi\rangle \otimes |0\rangle + \frac{1}{2}\sigma_z|\psi\rangle \otimes |1\rangle$$

$$\mathrm{tr}_e \rho_{after} = \sum_{\mu} \langle \mu_e| \left(\frac{\sqrt{3}}{2}|\psi\rangle\otimes|0\rangle + \frac{1}{2}\sigma_z|\psi\rangle\otimes|1\rangle\right)\left(\frac{\sqrt{3}}{2}\langle\psi|\otimes\langle 0| + \frac{1}{2}\langle\psi|\sigma_z^\dagger \otimes \langle 1|\right)|\mu_e\rangle$$

$$= \frac{3}{4}|\psi\rangle\langle\psi| + \frac{1}{4}\sigma_z|\psi\rangle\langle\psi|\sigma_z^\dagger$$

Operator sum notation

$$|\psi\rangle\langle\psi| \to \sum_{i=1}^{2} A_i |\psi\rangle\langle\psi| A_i^\dagger, \quad A_1 = \frac{\sqrt{3}}{2}\mathbb{I}, \quad A_2 = \frac{1}{2}\sigma_z$$

4. Suppose we start with a qubit and first apply the dephasing operation

$$\rho \to (1-p)\rho + p\sigma_z \rho \sigma_z^\dagger$$

and then apply the amplitude damping operation

$$\rho \to \sum_{i=1}^{2} A_i \rho A_i^\dagger$$

where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-q} \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & \sqrt{q} \\ 0 & 0 \end{pmatrix}.$$

Show the resulting transformation can be expressed in the operator-sum notation with just three matrices A_i:

$$\rho \to \sum_{i=1}^{3} A_i \rho A_i^\dagger.$$

$$\rho \to (1-p)\rho + p\sigma_z \rho \sigma_z^\dagger = \sum_{i=1}^{2} B_i \rho B_i^\dagger, \quad B_1 = \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, \quad B_2 = \sqrt{p}\,\sigma_z$$

$$\rho \to (1-p)\rho + p\sigma_z \rho \sigma_z^\dagger = \rho' \to \sum_{i=1}^{2} A_i \rho' A_i^\dagger$$

$$\Rightarrow \rho \to \sum_{i=1}^{2} \sum_{j=1}^{2} A_i B_j \rho B_j^\dagger A_i^\dagger$$

$$A_1 B_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-q} \end{pmatrix} \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{(1-q)(1-p)} \end{pmatrix} = \sqrt{1-p}\,A_1$$

$$A_1 B_2 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-q} \end{pmatrix} \begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{pmatrix} = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p(1-q)} \end{pmatrix}$$

combine

$$\left\{ \begin{aligned} A_2 B_1 &= \begin{pmatrix} 0 & \sqrt{q} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{q(1-p)} \\ 0 & 0 \end{pmatrix} \\ A_2 B_2 &= \begin{pmatrix} 0 & \sqrt{q} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{qp} \\ 0 & 0 \end{pmatrix} \end{aligned} \right. + = A_3$$

$$\rho \to \sum_{i=1}^{3} A_i \rho A_i^\dagger, \quad A_1 = \sqrt{1-p} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}, \quad A_2 = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{q-1} \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & \sqrt{q} \\ 0 & 0 \end{pmatrix}$$

5. Consider the depolarizing quantum operation $\mathcal{D}$:

$$\mathcal{D}(\rho) = (1-p)\rho + \frac{p}{3}\sum_{a=x,y,z} \sigma_a \rho \sigma_a^\dagger,$$

with $p < 3/4$. Suppose we apply $\mathcal{D}$ to a density matrix ρ_{in} to obtain $\rho_{\text{out}} = \mathcal{D}(\rho_{\text{in}})$. Show that the minimum possible eigenvalue of a density matrix output from this operation is $2p/3$.

Hint: use the identity

$$\frac{1}{4}\rho + \frac{1}{4}\sum_{a=x,y,z} \sigma_a \rho \sigma_a^\dagger = \frac{I}{2},$$

$$D(\rho) = (1-p)\rho + \frac{p}{3}\sum_{a=x,y,z} \sigma_a \rho \sigma_a^\dagger$$

$$\left(\because \frac{1}{4}\sum_{a=x,y,z} \sigma_a \rho \sigma_a^\dagger = \left(\frac{I}{2} - \frac{1}{4}\rho\right) \times 4 \right.$$

$$\Rightarrow D(\rho) = (1-p)\rho + \frac{p}{3}(2I - \rho) = \left(1 - \frac{4}{3}p\right)\rho + \frac{2}{3}p\, I$$

Suggest that the eigenvalues of ρ are a and b,

the eigenvalue of $D(\rho)$ are $\left(1-\frac{4}{3}p\right)a + \frac{2}{3}p$, $\left(1-\frac{4}{3}p\right)b + \frac{2}{3}p$

$\because p < \frac{3}{4}$, $a, b \geq 0$

$\therefore$ the minimum possible eigenvalue of $D(\rho)$ is $\frac{2}{3}p$