Homework 1

Do two of the problems 2.29 – 2.32 in Nielsen and Chuang.

2.29: Show that the tensor product of two unitary operators is unitary.

2.30: Show that the tensor product of two Hermitian operators is Hermitian.

2.31: Show that the tensor product of two positive operators is positive.

2.32: Show that the tensor product of two projectors is a projector.

2.29 Suppose that $A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & \cdots & B_{1x} \\ \vdots & \ddots & \vdots \\ B_{y1} & \cdots & B_{xy} \end{bmatrix}$

are unitary which satisfy $A^\dagger A = B^\dagger B = I$

$\because (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$

$\Rightarrow (A \otimes B)(A \otimes B)^\dagger = (A \otimes B)(A^\dagger \otimes B^\dagger) = AA^\dagger \otimes BB^\dagger = I \otimes I = I$

—a

2.30 Suppose that $A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & \cdots & B_{1x} \\ \vdots & \ddots & \vdots \\ B_{y1} & \cdots & B_{xy} \end{bmatrix}$

are Hermitian which satisfy $A = A^\dagger$, $B = B^\dagger$

$\because (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$

$\Rightarrow A \otimes B = A^\dagger \otimes B = A^\dagger \otimes B^\dagger = (A \otimes B)^\dagger$ —a

2.31

Suppose that $A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & \cdots & B_{1x} \\ \vdots & \ddots & \vdots \\ B_{y1} & \cdots & B_{xy} \end{bmatrix}$

are positive which satisfy $A, B > 0$

$\Rightarrow V^T A V > 0$ for all vector V

set $z = x \otimes y$

$(x \otimes y)^T (A \otimes B)(x \otimes y) = x^T A x \otimes y^T B y > 0$

—a

2.32

Suppose that $A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & \cdots & B_{1x} \\ \vdots & \ddots & \vdots \\ B_{y1} & \cdots & B_{xy} \end{bmatrix}$

are projectors which satisfy $A^2 = A$, $B^2 = B$

$$(A \otimes B)^2 = (A \otimes B)(A \otimes B) = A^2 \otimes B^2 = A \otimes B$$

2.59: Suppose we have a qubit in the state $|0\rangle$, and we measure the observable X $[\sigma_x]$. What is the average value of X? What is the standard deviation of X?

We measure $|0\rangle$ along the x axis

σ_x has 2 eigenvectors : $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

with eigenvalue 1 and -1

The average value $= 1 \cdot \frac{1}{2} + -1 \cdot \frac{1}{2} = 0$

SD $= \sqrt{\frac{1}{2}\left((1-0)^2 + (-1-0)^2\right)} = 1$

2.60: Show that $\vec{v} \cdot \vec{\sigma}$ has eigenvalues ± 1, and that the projectors onto the corresponding eigenspaces are given by $P_{\pm} \equiv (I \pm \vec{v} \cdot \vec{\sigma})/2$.

$$\vec{V} \cdot \vec{\sigma} = V_x \sigma_x + V_y \sigma_y + V_z \sigma_z$$

consider $(\vec{V} \cdot \vec{\sigma})^2 = (V_x \sigma_x + V_y \sigma_y + V_z \sigma_z)(V_x \sigma_x + V_y \sigma_y + V_z \sigma_z)$

$$= V_x^2 + V_y^2 + V_z^2 = I$$

$\because$ $\vec{V} \cdot \vec{\sigma}$ is Hermitian and unitary operator

$\Rightarrow$ its eigenvalue is ± 1

So, $\vec{V} \cdot \vec{\sigma}$ can be written in terms of its eigenvectors

$\Rightarrow$ $\vec{V} \cdot \vec{\sigma} = P_+ \cdot 1 + P_- \cdot -1 = P_+ - P_-$

$\Rightarrow$ $\frac{I + \vec{V} \cdot \vec{\sigma}}{2} = P_+$, $\frac{I - \vec{V} \cdot \vec{\sigma}}{2} = P_-$

2.61: Calculate the probability of obtaining the result $+1$ for a measurement of $\vec{v}\cdot\vec{\sigma}$, given that the state prior to measurement is $|0\rangle$. What is the state of the system after the measurement if $+1$ is obtained?

$$P = \langle 0|\vec{v}\cdot\vec{\sigma}|0\rangle = V_x \underbrace{\cancel{\langle 0|\sigma_x|0\rangle}}_{0} + V_y \underbrace{\cancel{\langle 0|\sigma_y|0\rangle}}_{0} + V_z \underbrace{\cancel{\langle 0|\sigma_z|0\rangle}}_{1}$$

$$= V_z = P(+1) - P(-1)$$

$\because P(+1) + P(-1) = 1$

$$\Rightarrow P(+1) = \frac{1+V_z}{2}, \quad \text{stay at } |0\rangle$$

1: Consider the state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Suppose all three qubits of this state are measured in the $\{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$ basis. What are the possible joint outcomes of these measurements? With what probabilities do they occur? Suppose the first qubit is measured in the $\{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$ basis and the second and third qubits in the $\{\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\}$ basis. Again, what are the possible joint outcomes of these measurements? With what probabilities do they occur? What is the expected value of the observable $\sigma_x(1)\otimes\sigma_x(2)\otimes\sigma_x(3)$? Of the observable $\sigma_x(1)\otimes\sigma_y(2)\otimes\sigma_y(3)$?

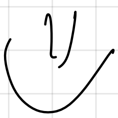
$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle, \quad \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$

$$\Rightarrow \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = |0\rangle, \quad \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) = |1\rangle$$

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = \frac{1}{4}\Big((|+\rangle+|-\rangle)\otimes(|+\rangle+|-\rangle)\otimes(|+\rangle+|-\rangle) + (|+\rangle-|-\rangle)\otimes(|+\rangle-|-\rangle)\otimes(|+\rangle-|-\rangle)\Big)$$

$$= \frac{1}{2}(|+++\rangle + |+--\rangle + |-+-\rangle + |--+\rangle)$$

You can only get $|+++\rangle$, $|+--\rangle$, $|-+-\rangle$, $|--+\rangle$ with probability $\frac{1}{4}$ for each outcomes.



$$\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = |+i\rangle, \quad \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) = |-i\rangle$$

$$\Rightarrow \frac{1}{\sqrt{2}}(|+i\rangle + |-i\rangle) = |0\rangle, \quad \frac{-i}{\sqrt{2}}(|+i\rangle - |-i\rangle) = |1\rangle$$

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = \frac{1}{4}\Big[(|+\rangle + |-\rangle) \otimes (|+i\rangle + |-i\rangle) \otimes (|+i\rangle + |-i\rangle)$$

$$- (|+\rangle - |-\rangle) \otimes (|+i\rangle - |-i\rangle) \otimes (|+i\rangle - |-i\rangle)\Big]$$

$$= \frac{1}{2}\big(|+,+i,-i\rangle + |+,-i,+i\rangle + |-,+i,+i\rangle + |-,-i,-i\rangle\big)$$

You can only get $|+,+i,-i\rangle + |+,-i,+i\rangle + |-,+i,+i\rangle + |-,-i,-i\rangle$ with probability $\frac{1}{4}$ for each outcomes.

expected value of $\sigma_x \otimes \sigma_x \otimes \sigma_x$: $1 \cdot \frac{1}{4} \cdot 4 = 1$

" $\sigma_x \otimes \sigma_y \otimes \sigma_y = -1 \cdot \frac{1}{4} \cdot 4 = -1$

2: (Note: This problem by itself has no quantum mechanics in it.) Let f_1, f_2, f_3 be functions mapping the set $\{x, y\}$ to the set $\{1, -1\}$. Define

$$A_1 = f_1(x)f_2(x)f_3(x), \quad A_2 = f_1(y)f_2(y)f_3(x),$$

$$A_3 = f_1(y)f_2(x)f_3(y), \quad A_4 = f_1(x)f_2(y)f_3(y).$$

Show that either $A_1 = -1$ or one of A_2, A_3, A_4 equals $+1$.

$$A_1A_2A_3A_4 = f_1(x)^2 f_1(y)^2 f_2(x)^2 f_2(y)^2 f_3(x)^2 f_3(y)^2$$

$$\because f_i^2 = 1 \Rightarrow A_1A_2A_3A_4 = 1$$

Equation holds either $A_1 = -1$ or one of $A_2, A_3, A_4 = 1$

3: Use the answers to problems 1 and 2 to show that quantum mechanics is not a local realistic theory.

We can take advantages of Problem 2. back to Problem 1

If δ_1 is the observable $\sigma_x \otimes \sigma_x \otimes \sigma_x$, $\delta_2 = \sigma_y \otimes \sigma_y \otimes \sigma_x$,

$\delta_3 = \sigma_y \otimes \sigma_x \otimes \sigma_y$, $\delta_4 = \sigma_x \otimes \sigma_y \otimes \sigma_y$

δ_1 has expectation value 1, and $\delta_2, \delta_3, \delta_4$ have expectation value -1.

It shows that the setup in Problem 2 violate the rule in Problem 2, which violate a local realistic theory, the measurement chosen to apply to one qubit can't affect the outcome obtained on another qubits.