HW4.

1. In the teleportation protocol, show that the probability distribution for the values of the two qubits that Alice sends to Bob is independent of the state $|\psi\rangle$ of the qubit being transmitted. In other words, an eavesdropper can infer nothing about the value of $|\psi\rangle$ by knowing the values of the two classical bits transmitted.

If we want to teleport the state $\alpha|0\rangle+\beta|1\rangle$ from Alice to Bob, we share an EPR state to Alice and Bob. The total state can be written as $|\psi\rangle = \frac{1}{\sqrt{2}}\left[(\alpha|0\rangle+\beta|1\rangle)(|00\rangle+|11\rangle)\right]$.

First we send the first qubit through a cnot and H gate

$$\Rightarrow |\psi\rangle \xrightarrow{\text{CNOT},\,H} |\psi_1\rangle = \frac{1}{2}\Big[|00\rangle(\alpha|0\rangle+\beta|1\rangle)+|01\rangle(\alpha|1\rangle+\beta|0\rangle)$$
$$+|10\rangle(\alpha|0\rangle-\beta|1\rangle)+|11\rangle(\alpha|1\rangle-\beta|0\rangle)\Big]$$

If we partial trace out Alice system

$$\therefore \sum_{r_A}\langle r_A|\psi_1\rangle\langle\psi|r_A\rangle = \frac{1}{4}\Big(\alpha^2|0\rangle\langle 0|+\beta^2|1\rangle\langle 1|+\alpha^2|1\rangle\langle 1|+\beta^2|0\rangle\langle 0|$$
$$+\alpha^2|0\rangle\langle 0|+\beta^2|1\rangle\langle 1|+\alpha^2|1\rangle\langle 1|+\beta^2|0\rangle\langle 0|\Big)$$
$$=\frac{1}{2}\mathbb{I}$$

This state doesn't depend upon $|\psi\rangle$ being teleported.

2. Show that Alice can teleport two qubits $|\phi\rangle|\psi\rangle$ "through" the gate

$$S = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

so that Bob obtains two qubits in the state $S|\phi\rangle|\psi\rangle$.

More specifically, suppose Alice and Bob share four qubits in the state

$$\frac{1}{2}\big(|0000\rangle + |0101\rangle + |1010\rangle - |1111\rangle\big),$$

with Alice holding the first pair of qubits and Bob holding the second pair. Alice measures $|\phi\rangle$ and her first qubit in the Bell basis, and measures $|\psi\rangle$ and her second qubit in the Bell basis. Alice sends the results of these measurements to Bob over a classical channel, and Bob applies some unitary transformations, which depend on the classical information he receives from Alice, to his two qubits to obtain the state $S|\phi\rangle|\psi\rangle$. Explain how the transformations Bob applies depend on the results of Alice's measurements.

Alice's measurement	Bob's state	expression with $|\phi\rangle$
00	$|00\rangle$	
01	$|01\rangle$	$S(\sigma_b^{(1)} \otimes \sigma_b^{(2)})|\phi\rangle$
10	$|10\rangle$	
11	$-|11\rangle$	

We know $\sigma_{x,y,z}$ commutes with S

For example:

If Bob got $S\sigma_x^{(1)}|\phi\rangle$

from the property of commutation, we get

$S\sigma_x^{(1)}|\phi\rangle = \sigma_z^{(2)}\sigma_x^{(1)}S|\phi\rangle$

We only need to apply $\sigma_x^{(1)} \otimes \sigma_z^{(2)}$,

then we can get $\sigma_x^{(1)}\sigma_z^{(2)}\sigma_z^{(2)}\sigma_x^{(1)}S|\phi\rangle = S|\phi\rangle$

the table for corresponding operator $S(\sigma_b^{(1)}\otimes\sigma_b^{(2)})|\phi\rangle$

$S\,\sigma_b^{(1)}$	$\otimes\,\sigma_b^{(2)}$	$\sigma_b^{(1)}$	$\otimes\,\sigma_b^{(2)}\,S$
I	I	I	I
I	σ_x	σ_z	σ_x
I	σ_y	σ_z	σ_y
I	σ_z	I	σ_z
σ_x	I	σ_x	σ_z
σ_x	σ_x	σ_y	σ_y
σ_x	σ_y	σ_y	σ_x
σ_x	σ_z	σ_x	I
σ_y	I	σ_y	σ_z
σ_y	σ_x	σ_x	σ_y
σ_y	σ_y	σ_x	σ_x
σ_y	σ_z	σ_y	I
σ_z	I	σ_z	I
σ_z	σ_x	I	σ_x
σ_z	σ_y	I	σ_y
σ_z	σ_z	σ_z	σ_z

3. Generalize the superdense coding procedure to three-dimensional quantum states (qutrits). Let $|0\rangle, |1\rangle, |2\rangle$ be an orthonormal basis for the qutrits. Now, suppose Alice and Bob share a pair of qutrits in the state

$$|\mathrm{EPR}_3\rangle = \frac{1}{\sqrt{3}}\big(|00\rangle + |11\rangle + |22\rangle\big)$$

Show that there are 9 unitary operations so that if Alice performs one of these unitary operations on her half of the state $|\mathrm{EPR}_3\rangle$, and sends the resulting qubit to Bob, he can then make a measurement on the two qutrits that he now holds which deterministically tells him which operation Alice performed. This shows that superdense coding can encode $\log_2 9$ bits in one qutrit.

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^2 \end{pmatrix} \quad \text{where } w = e^{\frac{2}{3}\pi i}$$

Alice can make 9 different kind of states through T and R.

I need to demonstrate the orthogonality of these 9 states.

they are $R^{a^{(1)}}T^{b^{(1)}}|EPR_3\rangle$, where $0 \le a, b \le 2$

$$\langle EPR_3| T^{\dagger b^{(1)}} R^{\dagger a^{(1)}} R^{a^{(1)}} T^{b^{(1)}} |EPR_3\rangle$$

$$T^{\dagger b}R^{\dagger a}R^{a}T^{b} = T^{\dagger b}\begin{bmatrix} 1 & & \\ & (w^*w)^a & \\ & & (w^{2*}w^2)^a \end{bmatrix} T^b = T^{\dagger b}T^{b} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^b$$

$$= \mathbb{I}$$

$\because R^3 = \mathbb{I}$, $T^2 = \mathbb{I}$, $T^3 = T$

$$T^{\dagger b'}R^{\dagger a'}R^{a}T^{b} = T^{\dagger b'}\begin{bmatrix} 1 & & \\ & w^{*a'}w^{a} & \\ & & w^{2a'*}w^{2a} \end{bmatrix} T^b$$

4 situations

$b' = 1,\ b = 1$ $= \begin{bmatrix} 1 & & \\ & w^{(a-a')} & \\ & & w^{2(a-a')} \end{bmatrix}$

$b' = 1,\ b = 0$ $= \begin{bmatrix} 0 & 0 & w^{2(a-a')} \\ 1 & 0 & 0 \\ 0 & w^{(a-a')} & 0 \end{bmatrix}$

$b' = 0,\ b = 1$ $= \begin{bmatrix} 0 & w^{(a-a')} & 0 \\ 0 & 0 & w^{2(a-a')} \\ 1 & 0 & 0 \end{bmatrix}$

$b' = 0,\ b = 0$ $= \begin{bmatrix} 1 & & \\ & w^{(a-a')} & \\ & & w^{2(a-a')} \end{bmatrix}$

$$\boxed{b'=1,\ b=1} = \boxed{b'=0,\ b=0} = \langle EPR | R^{\dagger a'} R^{a} | EPR \rangle$$

$$= \begin{cases} 1, & \text{if } a'=a \\ 0, & \text{if } a' \neq a \end{cases}$$

$$\boxed{b'=0,\ b=1} = \langle EPR | \begin{pmatrix} 0 & \omega^{(a-a')} & 0 \\ 0 & 0 & \omega^{2(a-a')} \\ 1 & 0 & 0 \end{pmatrix} | EPR \rangle$$

$$= \boxed{b'=1,\ b=0}$$

$$= \begin{cases} 1, & \text{if } a'=a \\ 0, & \text{if } a' \neq a \end{cases}$$

$$\Rightarrow \langle EPR | T^{\dagger b'} R^{\dagger a'} R^{a} T^{b} | EPR \rangle$$

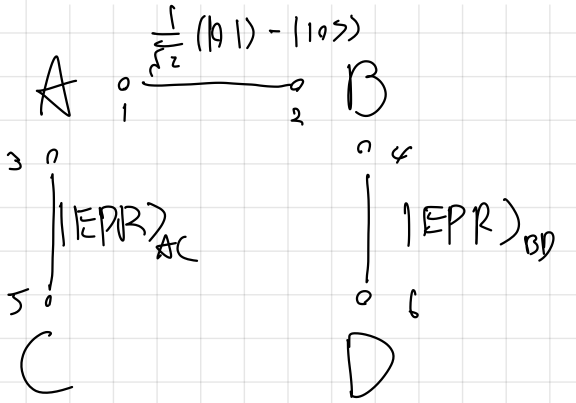
$$= \delta_{a-a'} \delta_{b-b'}$$

These 9 states have orthogonality.

4. Suppose that we have four parties, Alice, Bob, Cathy, and David. Alice and Cathy share a pair of qubits which are in one of the four Bell basis states,

$$\frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), \qquad \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle),$$

but they don't know which state it's in. Bob and David share a pair of qubits in the same Bell state. Suppose further that Alice and Bob share a pair of qubits in the state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Show that there is a protocol that lets Cathy and David end up sharing a pair of qubits in the state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.
Hint: Let Alice teleport her half of the Alice-Bob pair to Cathy, and Bob teleport his half of this pair to David. Show that even though Cathy and David don't know which Bell state they had, they can still apply Pauli transformations to their teleported qubits to end up with the state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.



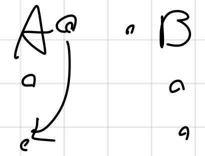
$|EPR\rangle_{BD}$ and $|EPR\rangle_{AC}$ can be written as

$I^{(B)} \otimes \sigma_1^{(C)} |EPR\rangle_{AC}$ and $I^{(B)} \otimes \sigma_1^{(D)} |EPR\rangle_{BD}$

where σ_i is unknown

① Alice teleport qubit 1 to qubit 5

$\Rightarrow$

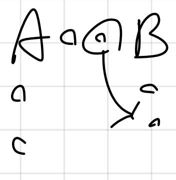


C and B hold $\sigma_1^{(C)} \sigma_2^{(C)} |EPR\rangle_{CB}$

$\sigma_1^{(C)} \sigma_2^{(C)} |EPR\rangle_{CB}$

② Bob teleport qubit 2 to qubit 6

$\Rightarrow$



C and D got $\sigma_1^{(C)} \sigma_2^{(C)} \otimes \sigma_1^{(D)} \sigma_3^{(D)} |EPR\rangle_{CD}$

$= \pm \sigma_2^{(C)} \sigma_1^{(C)} \otimes \sigma_3^{(D)} \sigma_1^{(D)} |EPR\rangle_{CD}$

Because they know what σ_2, σ_3 are, they can eliminate them

$\Rightarrow \pm \sigma_1^{(C)} \otimes \sigma_1^{(D)} |EPR\rangle_{CD}$

phase doesn't change the state.

The same basis transformation applied to EPR can't change its state.