

## Problem Set 2

**Both theory and programming questions** are due **Tuesday, September 27 at 11:59PM**. Please submit your solutions on <http://alg.csail.mit.edu>. The site will prompt you for your answers to the questions, so you do not need to create a solution template. The site will be open for PS2 submissions by Thursday, September 22. You don't need to wait until then to work on your solutions, and we encourage you to start early.

Remember, your goal is to communicate. Full credit will be given only to a correct solution which is described clearly. Convolved and obtuse descriptions might receive low marks, even when they are correct. Also, aim for concise solutions, as it will save you time spent on write-ups, and also help you conceptualize the key idea of the problem.

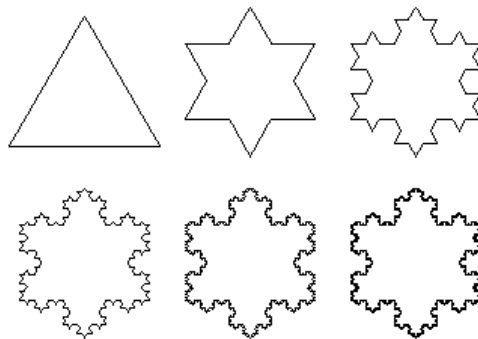
We will provide the solutions to the problem set 10 hours after the problem set is due, which you will use to find any errors in the proof that you submitted. You will need to submit a critique of your solutions by **Thursday, September 29th, 11:59PM**. Your grade will be based on both your solutions and your critique of the solutions.

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**Collaborators:** None.

### Problem 2-1. [40 points] Fractal Rendering

You landed a consulting gig with Gopple, who is about to introduce a new line of mobile phones with Retina HD displays, which are based on unicorn e-ink and thus have infinite resolution. The high-level executives heard that fractals have infinite levels of detail, and decreed that the new phones' background will be the *Koch snowflake* (Figure 1).



**Figure 1:** The Koch snowflake fractal, rendered at Level of Detail (LoD) 0 through 5.

Unfortunately, the phone's processor (CPU) and the graphics chip (GPU) powering the display do not have infinite processing power, so the Koch fractal cannot be rendered in infinite detail. Gopple engineers will stop the recursion at a fixed depth  $n$  in order to cap the processing requirement. For

example, at  $n = 0$ , the fractal is just a triangle. Because higher depths result in more detailed drawing, this depth is usually called the *Level of Detail (LoD)*.

The Koch snowflake at LoD  $n$  can be drawn using an algorithm following the sketch below:

SNOWFLAKE( $n$ )

- 1  $e_1, e_2, e_3 =$  edges of an equilateral triangle with side length 1
- 2 SNOWFLAKE-EDGE( $e_1, n$ )
- 3 SNOWFLAKE-EDGE( $e_2, n$ )
- 4 SNOWFLAKE-EDGE( $e_3, n$ )

SNOWFLAKE-EDGE( $edge, n$ )

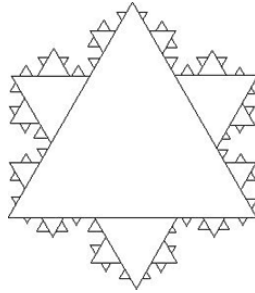
- 1 **if**  $n == 0$
- 2      $edge$  is an edge on the snowflake
- 3 **else**
- 4      $e_1, e_2, e_3 =$  split  $edge$  in 3 equal parts
- 5     SNOWFLAKE-EDGE( $e_1, n - 1$ )
- 6      $f_2, g_2 =$  edges of an equilateral triangle whose 3rd edge is  $e_2$ , pointing outside the snowflake
- 7      $\Delta(f_2, g_2, e_2)$  is a triangle on the snowflake's surface
- 8     SNOWFLAKE-EDGE( $f_2, n - 1$ )
- 9     SNOWFLAKE-EDGE( $g_2, n - 1$ )
- 10    SNOWFLAKE-EDGE( $e_3, n - 1$ )

The sketch above should be sufficient for solving this problem. If you are curious about the missing details, you may download and unpack the problem set's .zip archive, and read the CoffeeScript implementation in `fractal/src/fractal.coffee`.

In this problem, you will explore the computational requirements of four different methods for rendering the fractal, as a function of the LoD  $n$ . For the purpose of the analysis, consider the recursive calls to SNOWFLAKE-EDGE; do not count the main call to SNOWFLAKE as part of the recursion tree. (You can think of it as a super-root node at a special level -1, but it behaves differently from all other levels, so we do not include it in the tree.) Thus, the recursion tree is actually a forest of trees, though we still refer to the entire forest as the “recursion tree”. The root calls to SNOWFLAKE-EDGE are all at level 0.

Gopple's engineers have prepared a prototype of the Koch fractal drawing software, which you can use to gain a better understanding of the problem. To use the prototype, download and unpack the problem set's .zip archive, and use Google Chrome to open `fractal/bin/fractal.html`.

First, in 3D hardware-accelerated rendering (e.g., iPhone), surfaces are broken down into triangles (Figure 2). The CPU compiles a list of coordinates for the triangles' vertices, and the GPU is responsible for producing the final image. So, from the CPU's perspective, rendering a triangle costs the same, no matter what its surface area is, and the time for rendering the snowflake fractal is proportional to the number of triangles in its decomposition.



**Figure 2:** Koch snowflake drawn with triangles.

(a) [1 point] What is the depth of the recursion tree for rendering a snowflake of LoD  $n$ ?

1.  $\log n$
2.  $n$
3.  $3n$
4.  $4n$

**Answer: 2.**

For each recursion, the level of details would changes from  $n$  to  $n-1$ . The level of details would becomes 0 at the bottom of the tree. Therefore, the depth of the recursion tree for rendering a snowflake of LoD is  $n$ .

(b) [2 points] How many nodes are there in the recursion tree at level  $i$ , for  $1 \leq i \leq n$ ?

1.  $3^i$
2.  $4^i$
3.  $4^{i+1}$
4.  $3 \cdot 4^i$

**Answer: 4.**

At  $i = 0$ , there are only three nodes represented three edges. When  $0 < i < n$ , the function SNOWFLAKE-EDGE generates 4 nodes in a round. Therefore, there are  $3 \cdot 4^i$  nodes.

(c) [1 point] What is the asymptotic rendering time (triangle count) for a node in the recursion tree at level  $i$ , for  $0 \leq i < n$ ?

1. 0
2.  $\Theta(1)$
3.  $\Theta(\frac{1}{9}^i)$
4.  $\Theta(\frac{1}{3}^i)$

**Answer: 2.**

For each node, the function divided a side into 4 line segment, which can make a triangle. Therefore, the rendering time for a node is  $\Theta(1)$ .

- (d) [1 point] What is the asymptotic rendering time (triangle count) at each level  $i$  of the recursion tree, for  $0 \leq i < n$ ?

1. 0
2.  $\Theta(\frac{4^i}{9})$
3.  $\Theta(3^i)$
4.  $\Theta(4^i)$

**Answer: 4.**

From Problem (b) we know that there are  $3 \cdot 4^i$  nodes at level  $i$ . Therefore, the rendering time at level  $i$  is  $3 \cdot 4^i \times \Theta(1) = \Theta(4^i)$ .

- (e) [2 points] What is the total asymptotic cost for the CPU, when rendering a snowflake with LoD  $n$  using 3D hardware-accelerated rendering?

1.  $\Theta(1)$
2.  $\Theta(n)$
3.  $\Theta(\frac{4^n}{3})$
4.  $\Theta(4^n)$

**Answer: 4.**

Suppose that the cost for the CPU is  $T(n)$ . The total cost is the summation of the costs for all nodes. So  $T(n) = \sum_{i=0}^{n-1} \Theta(4^i) = \Theta(4) + \Theta(4^1) + \dots + \Theta(4^n) = \Theta(4^n)$ .

Second, when using 2D hardware-accelerated rendering, the surfaces' outlines are broken down into open or closed paths (list of connected line segments). For example, our snowflake is one closed path composed of straight lines. The CPU compiles the list of coordinates in each path to be drawn, and sends it to the GPU, which renders the final image. This approach is also used for talking to high-end toys such as laser cutters and plotters.

- (f) [1 point] What is the depth of the recursion tree for rendering a snowflake of LoD  $n$  using 2D hardware-accelerated rendering?

1.  $\log n$
2.  $n$
3.  $3n$
4.  $4n$

**Answer: 2.**

Same as Problem (a). For each recursion, the level of details would changes from  $n$  to  $n-1$ . The level of details would becomes 0 at the bottom of the tree. Therefore, the depth of the recursion tree for rendering a snowflake of LoD is  $n$ .

(g) [1 point] How many nodes are there in the recursion tree at level  $i$ , for  $1 \leq i \leq n$ ?

1.  $3^i$
2.  $4^i$
3.  $4^{i+1}$
4.  $3 \cdot 4^i$

**Answer: 4.**

Same as Problem (b). At  $i = 0$ , there are only three nodes represented three edges. When  $0 < i < n$ , the function SNOWFLAKE-EDGE generates 4 nodes in a round. Therefore, there are  $3 \cdot 4^i$  nodes.

(h) [1 point] What is the asymptotic rendering time (line segment count) for a node in the recursion tree at level  $i$ , for  $0 \leq i < n$ ?

1. 0
2.  $\Theta(1)$
3.  $\Theta(\frac{1}{9}^i)$
4.  $\Theta(\frac{1}{3}^i)$

**Answer: 1.**

Different to Problem (c). Line segments are only rendered at the bottom of the tree, so the cost of each node is 0 at level  $0 \leq i < n$ .

(i) [1 point] What is the asymptotic rendering time (line segment count) for a node in the last level  $n$  of the recursion tree?

1. 0
2.  $\Theta(1)$
3.  $\Theta(\frac{1}{9}^n)$
4.  $\Theta(\frac{1}{3}^n)$

**Answer: 2.**

At the last level, each node represents a line in the pattern, so the cost is  $\Theta(1)$  for each node in the last level.

(j) [1 point] What is the asymptotic rendering time (line segment count) at each level  $i$  of the recursion tree, for  $0 \leq i < n$ ?

1. 0
2.  $\Theta(\frac{4}{9}^i)$
3.  $\Theta(3^i)$
4.  $\Theta(4^i)$

**Answer: 1.**

Based on the results from Problem (h), the cost at each level  $i$  for  $0 \leq i < n$  is 0.

(k) [1 point] What is the asymptotic rendering time (line segment count) at the last level  $n$  in the recursion tree?

1.  $\Theta(1)$
2.  $\Theta(n)$
3.  $\Theta(\frac{4^n}{3})$
4.  $\Theta(4^n)$

**Answer: 4.**

Based on the results from Problem (i), the cost at the last level  $n$  is  $3 \cdot 4^n \times \Theta(1) = \Theta(4^n)$ .

(l) [1 point] What is the total asymptotic cost for the CPU, when rendering a snowflake with LoD  $n$  using 2D hardware-accelerated rendering?

1.  $\Theta(1)$
2.  $\Theta(n)$
3.  $\Theta(\frac{4^n}{3})$
4.  $\Theta(4^n)$

**Answer: 4.**

The total cost is equivalent to the cost at the last level  $n$ :  $T(n) = 3 \cdot 4^n \times \Theta(1) = \Theta(4^n)$ .

Third, in 2D rendering without a hardware accelerator (also called software rendering), the CPU compiles a list of line segments for each path like in the previous part, but then it is also responsible for “rasterizing” each line segment. Rasterizing takes the coordinates of the segment’s endpoints and computes the coordinates of all the pixels that lie on the line segment. Changing the colors of these pixels effectively draws the line segment on the display. We know an algorithm to rasterize a line segment in time proportional to the length of the segment. It is easy to see that this algorithm is optimal, because the number of pixels on the segment is proportional to the segment’s length. Throughout this problem, assume that all line segments have length at least one pixel, so that the cost of rasterizing is greater than the cost of compiling the line segments.

It might be interesting to note that the cost of 2D software rendering is proportional to the total length of the path, which is also the power required to cut the path with a laser cutter, or the amount of ink needed to print the path on paper.

(m) [1 point] What is the depth of the recursion tree for rendering a snowflake of LoD  $n$ ?

1.  $\log n$
2.  $n$
3.  $3n$
4.  $4n$

**Answer: 2.**

Same as Problem (a). For each recursion, the level of details would change from  $n$  to  $n-1$ . The level of details would become 0 at the bottom of the tree. Therefore, the depth of the recursion tree for rendering a snowflake of LoD is  $n$ .

(n) [1 point] How many nodes are there in the recursion tree at level  $i$ , for  $1 \leq i \leq n$ ?

1.  $3^i$
2.  $4^i$
3.  $4^{i+1}$
4.  $3 \cdot 4^i$

**Answer: 4.**

Same as Problem (b). At  $i = 0$ , there are only three nodes represented three edges. When  $0 < i < n$ , the function SNOWFLAKE-EDGE generates 4 nodes in a round. Therefore, there are  $3 \cdot 4^i$  nodes.

(o) [1 point] What is the asymptotic rendering time (line segment length) for a node in the recursion tree at level  $i$ , for  $0 \leq i < n$ ? Assume that the sides of the initial triangle have length 1.

1. 0
2.  $\Theta(1)$
3.  $\Theta(\frac{1}{9}^i)$
4.  $\Theta(\frac{1}{3}^i)$

**Answer: 1.**

Same as Problem (h). Line segments are only rendered at the bottom of the tree, so the cost of each node is 0 at level  $0 \leq i < n$ .

(p) [1 point] What is the asymptotic rendering time (line segment length) for a node in the last level  $n$  of the recursion tree?

1. 0
2.  $\Theta(1)$
3.  $\Theta(\frac{1}{9}^n)$
4.  $\Theta(\frac{1}{3}^n)$

**Answer: 4.**

In the beginning, we suppose that the length of each side in the triangle is 1. After we do SNOWFLAKE-EDGE for a side, the length of the side is divided into 4 parts, and the length of each fraction is  $\frac{1}{3}$ . At level  $i$  in  $0 \leq i < n$ , the length of each fraction is  $\frac{1}{3}^i$ . Therefore, the cost for a node at the last level is  $\Theta(\frac{1}{3}^n)$ .

(q) [1 point] What is the asymptotic rendering time (line segment length) at each level  $i$  of the recursion tree, for  $0 \leq i < n$ ?

1. 0
2.  $\Theta(\frac{4^i}{9})$
3.  $\Theta(3^i)$
4.  $\Theta(4^i)$

**Answer: 1.**

Based on the result from Problem (o). Line segments are only rendered at the bottom of the tree, so the total cost is 0 at the level  $i$  such that  $0 \leq i < n$ .

(r) [1 point] What is the asymptotic rendering time (line segment length) at the last level  $n$  in the recursion tree?

1.  $\Theta(1)$
2.  $\Theta(n)$
3.  $\Theta(\frac{4^n}{3})$
4.  $\Theta(4^n)$

**Answer: 3.**

Based on the result from Problem (n) and (p). The cost at the last level is  $3 \cdot 4^n \times \Theta(\frac{1}{3}) = \Theta(\frac{4^n}{3})$

(s) [1 point] What is the total asymptotic cost for the CPU, when rendering a snowflake with LoD  $n$  using 2D software (not hardware-accelerated) rendering?

1.  $\Theta(1)$
2.  $\Theta(n)$
3.  $\Theta(\frac{4^n}{3})$
4.  $\Theta(4^n)$

**Answer: 3.**

The total cost is equivalent to the cost at the last level is  $T(n) = 3 \cdot 4^n \times \Theta(\frac{1}{3}) = \Theta(\frac{4^n}{3})$

The fourth and last case we consider is 3D rendering without hardware acceleration. In this case, the CPU compiles a list of triangles, and then rasterizes each triangle. We know an algorithm to rasterize a triangle that runs in time proportional to the triangle's surface area. This algorithm is optimal, because the number of pixels inside a triangle is proportional to the triangle's area. For the purpose of this problem, you can assume that the area of a triangle with side length  $l$  is  $\Theta(l^2)$ . We also assume that the cost of rasterizing is greater than the cost of compiling the line segments.

(t) [4 points] What is the total asymptotic cost of rendering a snowflake with LoD  $n$ ? Assume that initial triangle's side length is 1.

1.  $\Theta(1)$
2.  $\Theta(n)$



3.  $\Theta(\frac{4^n}{3})$
4.  $\Theta(4^n)$

**Answer: 1.**

The illustration is down below.

(u) [15 points] Write a succinct proof for your answer using the recursion-tree method.

**Answer:**

According to Problem (b), we know that there are  $3 \cdot 4^i$  nodes in level  $i$  such that  $0 \leq i < n$ . We assume that the area of a triangle with side length  $l$  is  $\Theta(l^2)$ . So, the cost for each node is  $\Theta(\frac{1}{9})$  at  $i$  level. The cost at level  $i$  can be written as  $3 \cdot 4^i \times \Theta(\frac{1}{9})$ . The total cost is the summation of the cost in all levels:

$$T(n) = \sum_{i=0}^n 3 \cdot 4^i \times \Theta(\frac{1}{9}) \approx \Theta(1). \quad (1)$$

$T(n)$  is a geometric series, which would be asymptotic to a constant.

### Problem 2-2. [60 points] Digital Circuit Simulation

Your 6.006 skills landed you a nice internship at the chip manufacturer AMDtel. Their hardware verification team has been complaining that their circuit simulator is slow, and your manager decided that your algorithmic chops make you the perfect candidate for optimizing the simulator.

A **combinational circuit** is made up of **gates**, which are devices that take Boolean (True / 1 and False / 0) input signals, and output a signal that is a function of the input signals. Gates take some time to compute their functions, so a gate's output at time  $\tau$  reflects the gate's inputs at time  $\tau - \delta$ , where  $\delta$  is the gate's delay. For the purposes of this simulator, a gate's output transitions between 0 and 1 instantly. Gates' output terminals are connected to other gates' inputs terminals by **wires** that propagate the signal instantly without altering it.

For example, a 2-input XOR gate with inputs A and B (Figure 3) with a 2 nanosecond (ns) delay works as follows:

Time (ns)	Input A	Input B	Output O	Explanation
0	0	0		Reflects inputs at time -2
1	0	1		Reflects inputs at time -1
2	1	0	0	0 XOR 0, given at time 0
3	1	1	1	0 XOR 1, given at time 1
4			1	1 XOR 0, given at time 2
5			0	1 XOR 1, given at time 3

The circuit simulator takes an input file that describes a circuit layout, including gates' delays, probes (indicating the gates that we want to monitor the output), and external inputs. It then



**Figure 3:** 2-input XOR gate; A and B supply the inputs, and O receives the output.

simulates the transitions at the output terminals of all the gates as time progresses. It also outputs transitions at the probed gates in the order of the timing of those transitions.

This problem will walk you through the best known approach for fixing performance issues in a system. You will profile the code, find the performance bottleneck, understand the reason behind it, and remove the bottleneck by optimizing the code.

To start working with AMDtel's circuit simulation source code, download and unpack the problem set's .zip archive, and go to the `circuit/` directory.

The circuit simulator is in `circuit.py`. The AMDtel engineers pointed out that the simulation input in `tests/5devadas13.in` takes too long to run. We have also provided an automated test suite at `test-circuit.py`, together with other simulation inputs. You can ignore these files until you get to the last part of the problem set.

- (a) [8 points] Run the code under the python profiler with the command below, and identify the method that takes up most of the CPU time. If two methods have similar CPU usage times, ignore the simpler one.

```
python -m cProfile -s time circuit.py < tests/5devadas13.in
```

*Warning:* the command above can take 15-30 minutes to complete, and bring the CPU usage to 100% on one of your cores. Plan accordingly.

What is the name of the method with the highest CPU usage?

**Answer:** `method_name`

- (b) [6 points] How many times is the method called?

**Answer:** 0

- (c) [8 points] The class containing the troublesome method is implementing a familiar data structure. What is the tightest asymptotic bound for the worst-case running time of the method that contains the bottleneck? Express your answer in terms of  $n$ , the number of elements in the data structure.

1.  $O(1)$ .
2.  $O(\log n)$ .
3.  $O(n)$ .
4.  $O(n \log n)$ .
5.  $O(n \log^2 n)$ .
6.  $O(n^2)$ .

**Answer: 0**

- (d) [8 points] If the data structure were implemented using the most efficient method we learned in class, what would be the tightest asymptotic bound for the worst-case running time of the method discussed in the questions above?

1.  $O(1)$ .
2.  $O(\log n)$ .
3.  $O(n)$ .
4.  $O(n \log n)$ .
5.  $O(n \log^2 n)$ .
6.  $O(n^2)$ .

**Answer: 0**

- (e) [30 points] Rewrite the data structure class using the most efficient method we learned in class. **Please note that you are not allowed to import any additional Python libraries and our test will check this.**

We have provided a few tests to help you check your code's correctness and speed. The test cases are in the `tests/` directory. `tests/README.txt` explains the syntax of the simulator input files. You can use the following command to run all the tests.

```
python circuit_test.py
```

To work on a single test case, run the simulator on the test case with the following command.

```
python circuit.py < tests/1gate.in > out
```

Then compare your output with the correct output for the test case.

```
diff out tests/1gate.gold
```

For Windows, use `fc` to compare files.

```
fc out tests/1gate.gold
```

We have implemented a visualizer for your output, to help you debug your code. To use the visualizer, first produce a simulation trace.

```
TRACE=jsonp python circuit.py < tests/1gate.in > circuit.jsonp
```

On Windows, use the following command instead.

```
circuit_jsonp.bat < tests/1gate.in > circuit.jsonp
```

Then use Google Chrome to open `visualizer/bin/visualizer.html`

We recommend using the small test cases numbered 1 through 4 to check your implementation's correctness, and then use test case 5 to check the code's speed.

When your code passes all tests, and runs reasonably fast (the tests should complete in less than 30 seconds on any reasonably recent computer), upload your modified `circuit.py` to the course submission site.

**Answer:** Please upload `circuit.py` with your modifications separately.