A Two-stage Signal Decomposition into Jump, Oscillation and Trend using ADMM

Martin Huska[1], Antonio Cicone[2], Sung Ha Kang[3], Serena Morigi[1]

IPOL/MLBriefs, December 12, 2023

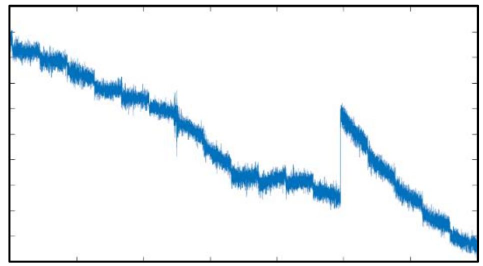
[1] Department of Mathematics, University of Bologna, Italy
[2] DISIM, University of L'Aquila, Italy
[3] School of Mathematics, Georgia University of Technology, Atlanta

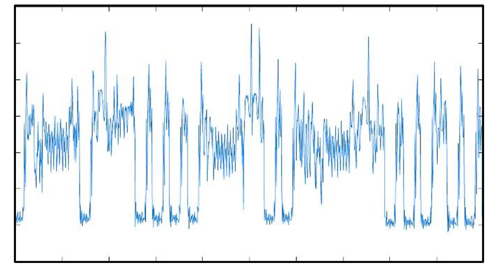
Outline

- Motivation and Problem Statement
- 1D Signal JOT Decomposition Framework
  - Stage 1: Variational Decomposition Model $f = v + w + n$
    - Choice of regularizers
    - Numerical Solution
  - Stage 2: Residual-aided refinement
- Examples/Applications

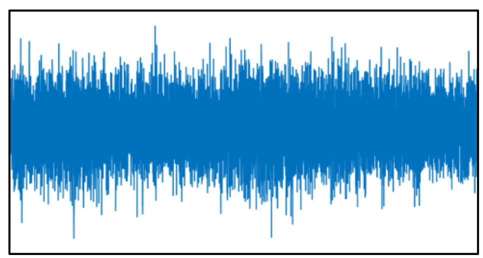
Motivation

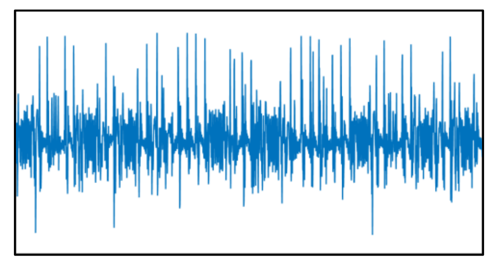
- Data acquisition generates various artifacts in the signal
  - Noise, unwanted trend, even jump(-discontinuity) artifacts
- Artifact sources: environment, vibrations, operational break for data packaging (satellites)





- Studying raw data in time-frequency analysis – jump discontinuities can be present/active in every frequency
- Goal: Improvement of the analysis for signals containing piece-wise constant (jump) artifacts
- Ideally, data should be without trend and artifacts





[1] Cicone A., H. M., Kang S.-H., Morigi S., JOT: A Variational Signal Decomposition into Jump, Oscillation and Trend, IEEE Transactions on Signal Processing, 70, pp. 772 - 784, 2022

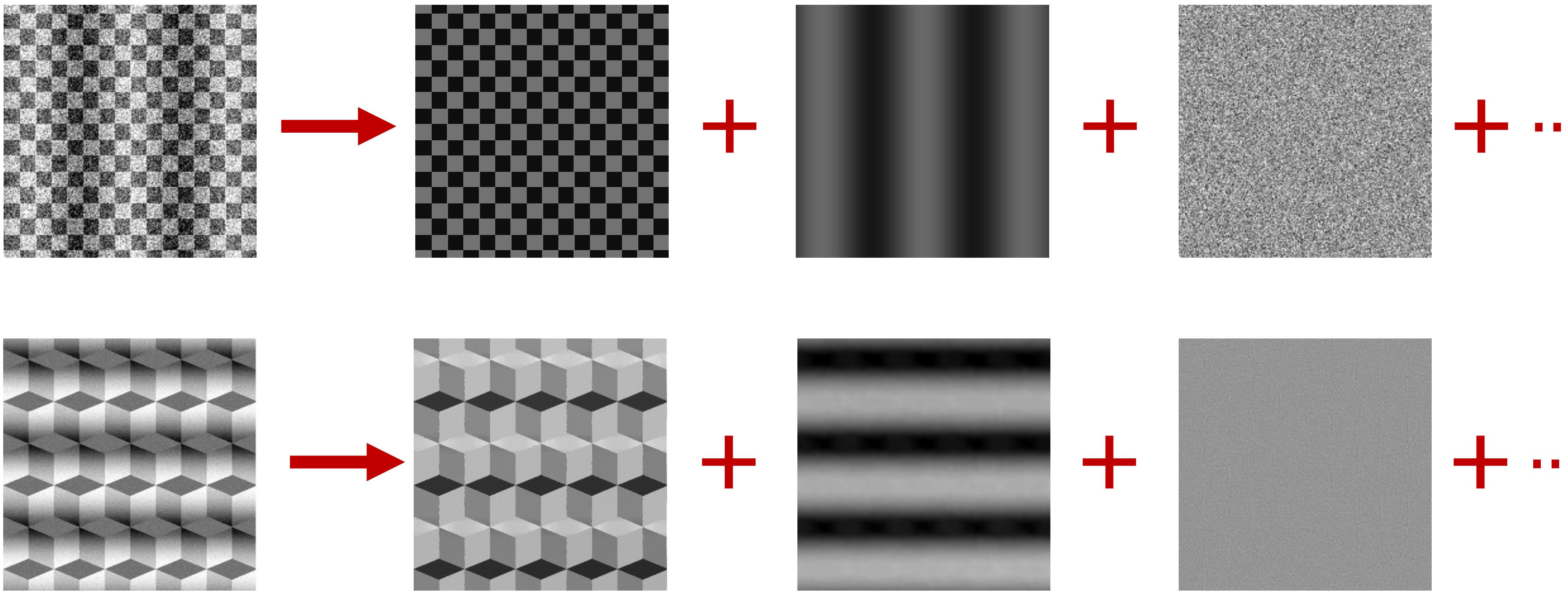
Problem Statement

- Additive Decomposition – separate the observed signal/image/function $f \in \mathbb{R}^d, d = \{1,2\}$ into meaningful components, such that $f = v + w + n + \cdots$

- The choice of components depends on the application:
  - Structure retrieval
  - Denoising
  - Detrending
  - Artifacts removal

Problem Statement

Examples from image decomposition:



$$f \quad = \quad v \quad + \quad w \quad + \quad n \quad + \quad \cdots$$

Data/input — Piece-wise constant — Smooth — Noise

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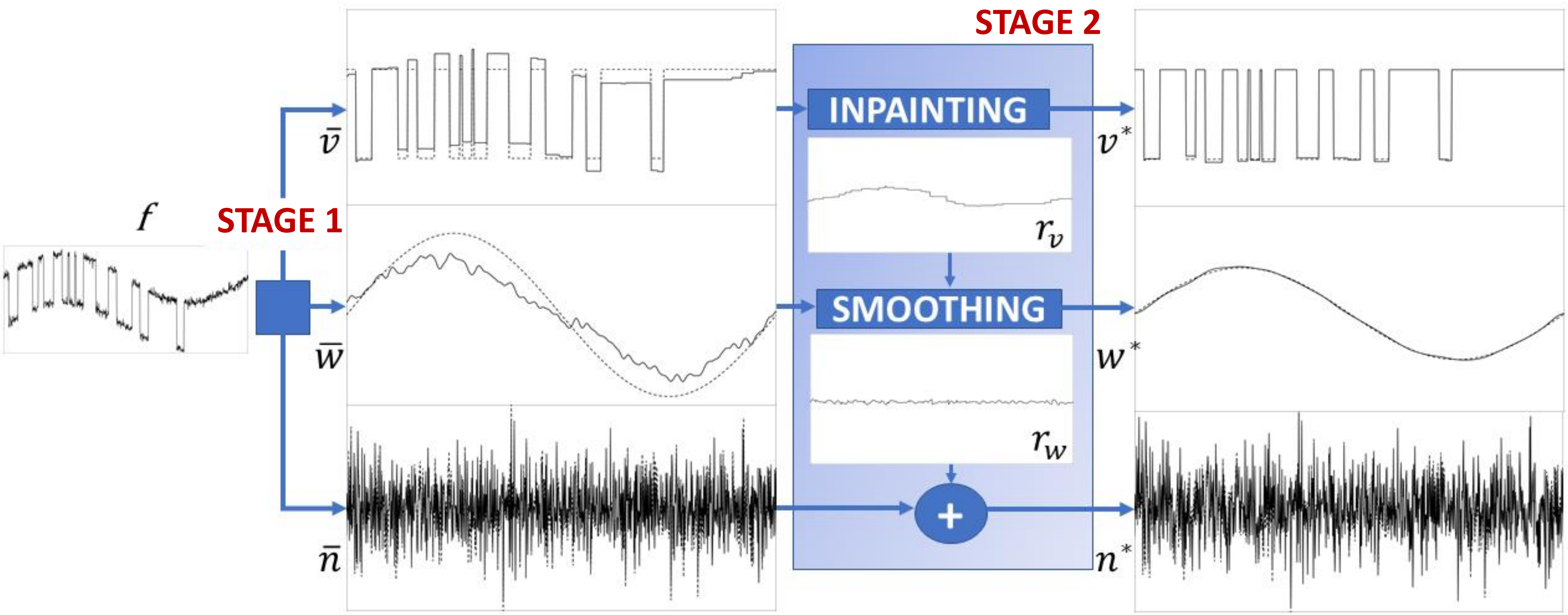
JOT Framework

- Two-stage framework:

  Decomposition + Residual Aided Refinement



Decomposition $f \rightarrow v + w + n$

- Goal: Signal decomposition into piece-wise constant, smooth and oscillatory component
- Variational model ingredients

$$\{v^*, w^*, n^*\} \leftarrow \underset{v,w,n \in \mathbb{R}^m}{\operatorname{argmin}} \mathcal{J}(v, w, n; \lambda, \eta, a)$$

$$\mathcal{J}(v, w, n; \gamma_1, \gamma_2, \gamma_3, a) := F(v, w, n) + \gamma_1 R_v(v; a) + \gamma_2 R_w(w) + \gamma_3 R_n(n)$$

- Data term imposing $f = v + w + n$

- v – regularization
- Piece-wise constant reconstruction
- Penalize (sparsify) first-order derivatives

- w – regularization
- Smooth reconstructions
- Penalize second-order derivatives

- n – regularization
- Oscillatory reconstructions
- Penalize smooth and low frequency signals

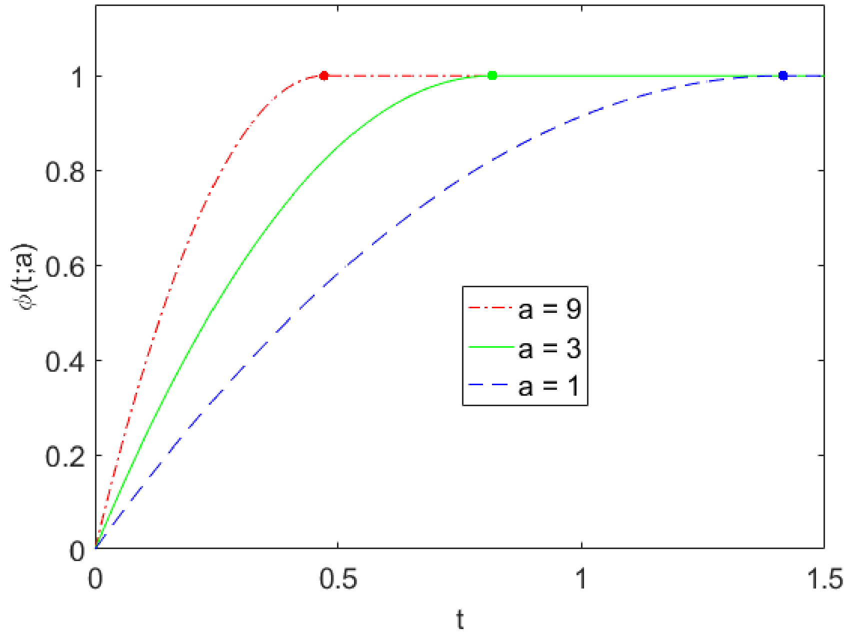
Choice of Regularizers – Jumps v

- Falls in family of separable, parametrized non-convex penalty functions
- Asymptotic behavior overperforms other penalty functions in the family
- We use the rescaled and reparametrized version of the MiniMax penalty
- For any $a > 0$, $\phi(\cdot\,; a): \mathbb{R}_+ \to \mathbb{R}_+$

$$\phi(t;a) = \begin{cases} -\frac{a}{2}t^2 - \sqrt{2a}t & for\ t \in [0, \sqrt{2/a}) \\ 1 & for\ t \in [\sqrt{2/a}, +\infty) \end{cases}$$

$$\phi(t;a) \in C^1([0,+\infty)) \cap C^\infty([0,+\infty)\backslash\{\sqrt{2/a}\})$$

$$\phi''(t;a) = \begin{cases} -a & for\ t \in [0, \sqrt{2/a}) \\ 0 & for\ t \in [\sqrt{2/a}, +\infty) \end{cases}$$



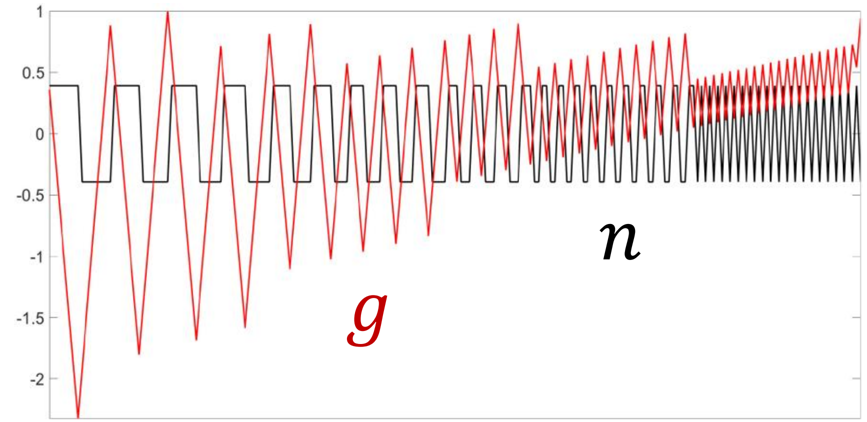
- $a > 0$ represents the degree of concavity

Choice of Regularizers – Oscillations n

- Key idea: Modelling of textures in images via dual space to TV (G-space) [Meyer; Ajoul, Chambolle]
  - Assuming $n = D^T g$ and selecting appropriate space for g leading to G-norm

$$\|n\|_G = \inf\{\|g\|_\infty \mid n = \mathrm{D}^\mathrm{T} g, \qquad g \in \mathbb{R}^N\}$$

  - n – oscillating, zero mean component

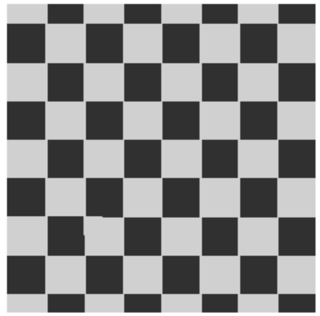
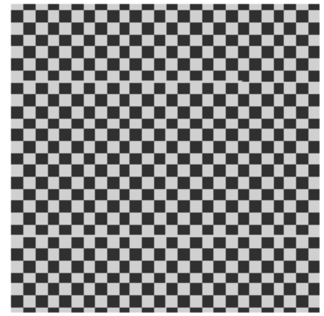
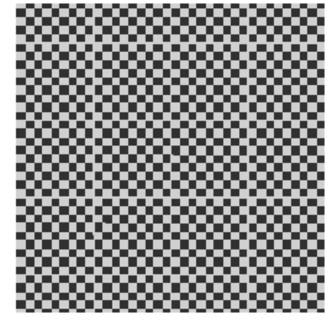


- Using negative Sobolev space H^{-1} (dual of H_0^1) with seminorm

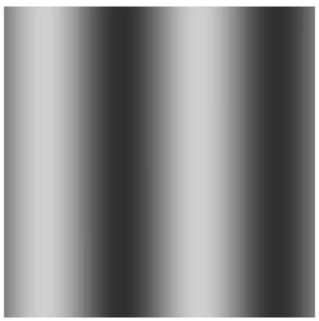
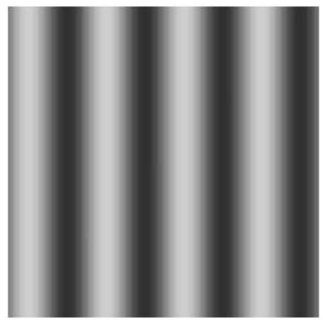
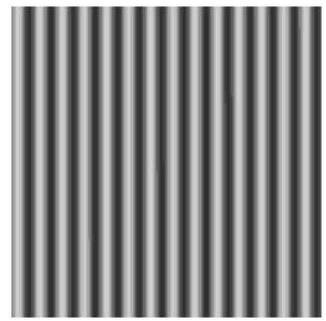
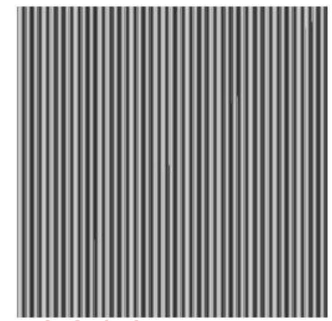
$$\|n\|_{H^{-1}} = inf\left\{\sqrt{\sum_i |g_i|^2} \mid n = \mathrm{D}^\mathrm{T} g\right\} \approx ||g||_2$$

Choice of Regularizers – Oscillations n

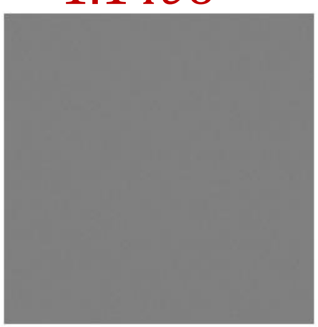
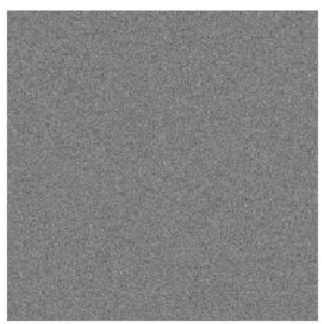
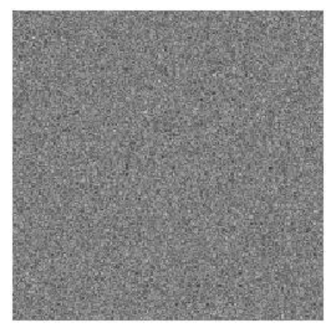
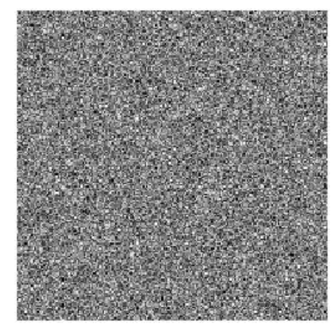
H^{-1} norm for piece-wise constant, smooth and noise images of dimension 512×512 pixels:

$\|\cdot\|_{H^{-1}} \approx 2.39e6$ | 527582 | 63056 | 31843

$\|\cdot\|_{H^{-1}} \approx 1.14e6$ | 296022 | 148488 | 38027

$\|\cdot\|_{H^{-1}} \approx 499$ | 5164 | 10581 | 33071

Variational Model $\boldsymbol{f = v + w + n}$

- Model of $f = v + w + n$
- Goal: Signal decomposition into piece-wise constant, smooth and oscillatory component

$$\mathcal{J}(v, w, n; \gamma_1, \gamma_2, \gamma_3, a) = \frac{1}{2}\|v + w + n - f\|_2^2 + \gamma_1 \sum_{j=1}^{N-1} \left[\phi\left(\left\|(Dv)_j\right\|_2; a\right)\right] + \frac{\gamma_2}{2}\|Hw\|_2^2 + \frac{\gamma_3}{2}\|n\|_{H^{-1}}^4$$

- First- and second-order difference operators:

$$D = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix} \in \mathbb{R}^{(N-1)\times N}, \qquad \mathrm{H} = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix} \in \mathbb{R}^{N\times N}$$

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Numerical Solution - ADMM

$$\mathcal{J}(v,w,n;\gamma_1,\gamma_2,\gamma_3,a) = \frac{1}{2}\|v+w+n-f\|_2^2 + \gamma_1 \sum_{j=1}^{N-1}\left[\phi\left(\left\|(Dv)_j\right\|_2;a\right)\right] + \frac{\gamma_2}{2}\|Hw\|_2^2 + \frac{\gamma_3}{2}\|n\|_{H^{-1}}^4$$

$$\mathcal{J}(v,w,g;\gamma_1,\gamma_2,\gamma_3,a) = \frac{1}{2}\|v+w+D^T g-f\|_2^2 + \gamma_1 \sum_{j=1}^{N-1}\left[\phi\left(\left\|(Dv)_j\right\|_2;a\right)\right] + \frac{\gamma_2}{2}\|Hw\|_2^2 + \frac{\gamma_3}{2}\|g\|_2^4$$

- Solved via two-block ADMM for $x = (v^T, w^T, g^T)^T \in \mathbb{R}^{3N}$ and $t = Dv \in \mathbb{R}^{N-1}$
- Augmented Lagrangian:

$$\mathcal{L}(x,t;\rho) = \frac{1}{2}\|v+w+D^T g-f\|_2^2 + \gamma_1 \sum_{j=1}^{N-1}\left[\phi\left(\left\|(Dv)_j\right\|_2;a\right)\right] + \frac{\gamma_2}{2}\|Hw\|_2^2 + \frac{\gamma_3}{2}\|g\|_2^4 - \langle\rho, t-Dv\rangle + \frac{\beta}{2}\|t-Dv\|_2^2$$

- Considering the saddle point problem for x, t, ρ; we seek the solution by set of iterations

$$x^{(k+1)} \leftarrow \arg\min_{x\in\mathbb{R}^{3N}} \mathcal{L}\left(x, t^{(k)}, \rho^{(k)}; \lambda, \eta, a\right)$$
$$t^{(k+1)} \leftarrow \arg\min_{t\in\mathbb{R}^{N-1}} \mathcal{L}\left(x^{(k+1)}, t, \rho^{(k)}; \lambda, \eta, a\right)$$
$$\rho^{(k+1)} \leftarrow \rho^{(k)} - \beta\left(t^{(k+1)} - D\, v^{(k+1)}\right)$$

ADMM: Subproblem for x

- The optimality conditions read

$$\begin{cases} \left(v^{(k+1)} + w^{(k+1)} + D^T g^{(k+1)} - f\right) + D^T \rho^{(k)} - \beta D^T\left(t^{(k)} - Dv^{(k)}\right) & = 0 \\ \left(v^{(k+1)} + w^{(k+1)} + D^T g^{(k+1)} - f\right) + \gamma_2 H^T H w^{(k+1)} & = 0 \\ D^T\left(v^{(k+1)} + w^{(k+1)} + D^T g^{(k+1)} - f\right) + 2\gamma_3 ||g^{(k+1)}||_2^2 g^{(k+1)} & = 0 \end{cases}$$

- Replacing $g^{(k+1)}$ with $g^{(k)}$, we obtain a linear system $Lx^{(k+1)} = y$, with

$$L = \begin{pmatrix} I + \beta D^T D & I & D^T \\ I & I + \gamma_2 H^T H & D^T \\ D & D & DD^T + 2\gamma_3 ||g^{(k)}||_2^2 I \end{pmatrix}, \quad y = \begin{pmatrix} f + \beta D^T\left(t^{(k)} - \frac{1}{\beta}\rho^{(k)}\right) \\ f \\ Df \end{pmatrix}$$

- Block with $H^T H$ slightly worsens the conditioning, therefore a regularized system with small $\kappa > 0$ is solved – CG or sparse Cholesky solver

$$(L + \kappa I)x^{(k+1)} = y$$

ADMM: Subproblem for $\boldsymbol{t}$

- Separability property of $\phi(\cdot\,; a)$ allows to solve $(N-1)$ 1-dimensional problems of form

$$t_j^{(k+1)} \leftarrow \underset{t\in\mathbb{R}}{\operatorname{argmin}} \left\{ \frac{1}{2}\left\| t - q_j \right\|_2^2 + \frac{1}{\lambda}\phi(|t|; a\,) \right\}, \qquad j = 1, \ldots, N,$$

with $\lambda = \frac{\beta}{\gamma_1}$ and $q_j = \left(Dv^{(k)}\right)_j + \frac{\rho_j^{(k)}}{\beta}$

- Strong convexity of the subproblems can be imposed

$$a < \lambda \implies \beta > a\gamma_1 \implies \beta = \tau a \gamma_1, \quad \text{for } \tau \in \mathbb{R}, \tau > 1$$

- Then, unique solutions can be obtained in closed form as

$$t_j^{(k+1)} = \min\left(\max\left(\nu - \frac{\zeta}{|q_j|}, 0\right), 1\right) q_j$$

where $\nu = \frac{\lambda}{\lambda - a}$ and $\zeta = \frac{\sqrt{2a}}{\lambda - a}$

Stage 1 Algorithm overview

Algorithm 1: Stage 1 Decomposition

input : f
output : $\bar{v}$, $\bar{w}$, $\bar{n}$ components
parameters: $\bar{a}$, γ_1, γ_2, γ_3, τ
Generate discrete operators D in (3) and H in (4).
$a = 2/\bar{a}^2$, $\beta = \tau a \gamma_1$
while $k < iter$ **and** $r > th$ **do**
  $x^{(k+1)} \leftarrow$ solve $(L + \kappa I)x^{(k+1)} = y$, using L, y in (15) *subproblem for* $x = (v, w, g)$
  $t^{(k+1)} \leftarrow$ compute (20) *subproblem for* t
  $\rho^{(k+1)} = \rho^{(k)} - \beta(t^{(k+1)} - Dv^{(k+1)})$ *subproblem for* ρ
  Update L in (15)
  $r = \|x^{(k+1)} - x^{(k)}\|/\|x^{(k)}\|$
  $k = k + 1$
Get $\{\bar{v}, \bar{w}, g\}$ from x
$\bar{n} = D^T g$

- The output components $\bar{v}$, $\bar{w}$, $\bar{n}$ may be slightly mixed due to the morphology of the signal
- Modular Stage 2 – Residual-aided refinement has been proposed

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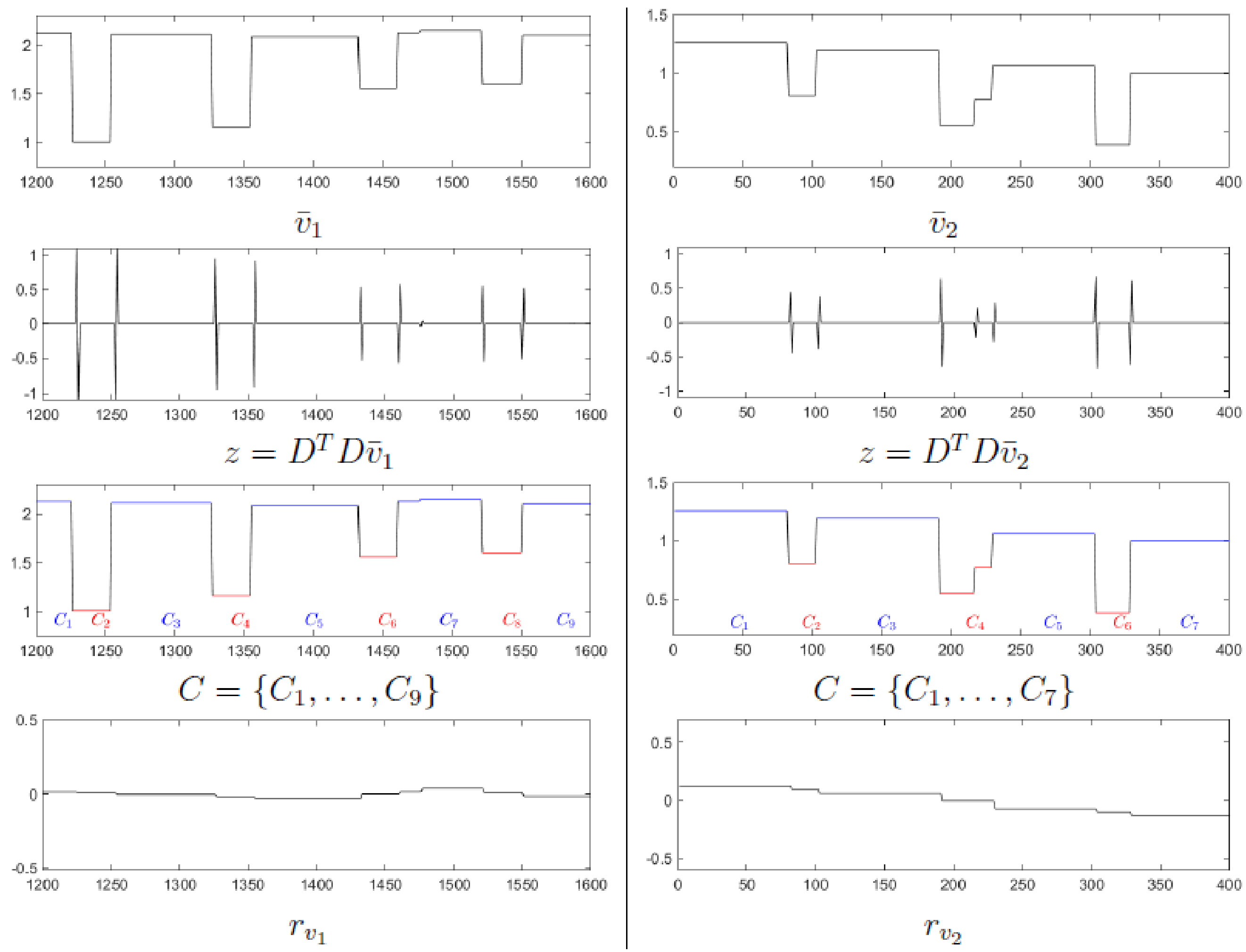
Stage 2 – Refinement of $\bar{v}$

- Useful to use when $\bar{v}$ should capture a reoccurring jumps from a constant signal, while containing spurious trend



- Idea: Localize jumps and 'inpaint' the data
- v-values similarity threshold th_v

- Residual r_v is obtained
- Resulting component $v^* = \bar{v} - r_v$

Stage 2 – Refinement of $\bar{\boldsymbol{w}}, \bar{\boldsymbol{n}}$

- Spurious trend r_v should be added to w^*, while residual oscillations may be present in $\bar{w}$
- The refinement of $\bar{w}$ is a smoothing problem

$$w^* = \underset{w \in \mathbb{R}^N}{\operatorname{argmin}} \frac{1}{2} \|w - (\bar{w} + r_v)\|_2^2 + \alpha \|Dw\|_2^2$$

with $\alpha > 0$ controlling the level of smoothness of w^*

- First optimality conditions lead to the solution of

$$(I_N + \alpha D^T D)w = \bar{w} + r_v$$

with sparse, symmetric positive definite coefficient matrix

- Refinement of $\bar{n}$ from updates it by adding to it the highly oscillatory residual from w^*, r_w, and the model residual $\bar{r}$

$$n^* = \bar{n} + r_w + \bar{r}$$

where $\bar{r} = f - \bar{v} - \bar{w} - \bar{n}$, and $r_w = (\bar{w} + r_v) - w^*$

Stage 2 Algorithm overview

Algorithm 2: Stage 2: Refinement

input : $\bar{v}$, $\bar{w}$, $\bar{n}$ components, $\bar{r} = f - \bar{v} - \bar{w} - \bar{n}$ residual
output : v^*, w^*, n^* components
parameters: th_v, α
Generate discrete operator D in (3)
Solve inpainting problem for v^*
- $z = D^T D\,\bar{v}$
- Compute the set $\{\bar{C}_i\}$ from z
- Generate the set $\{C_i\}$ from $\{\bar{C}_i\}$ and $\bar{v}$
- set r_v according to (21)
- $v^* = \bar{v} - r_v$

Solve smoothing problem for w^*
- Solve (25) for w^*
- $r_w = \bar{w} + r_v - w^*$

Refinement for n^*
- $n^* = \bar{n} + r_w + \bar{r}$

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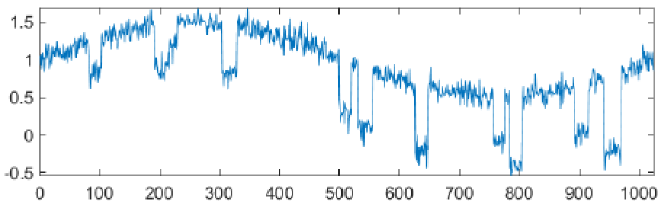
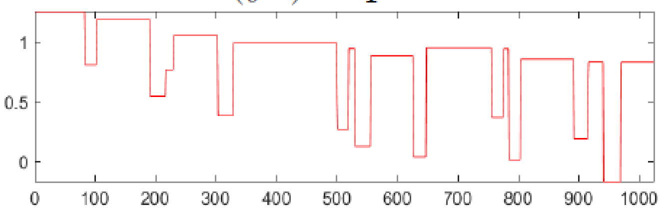
Example 1

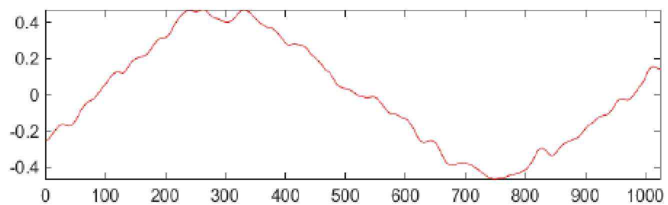
$f_1 \in \mathbb{R}^{1024}$



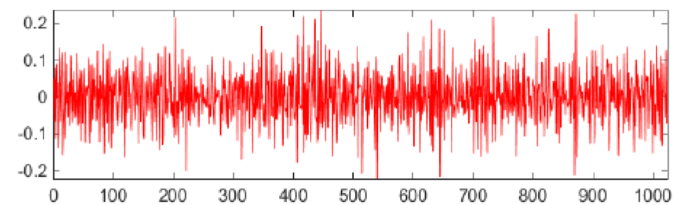
(f_1) input



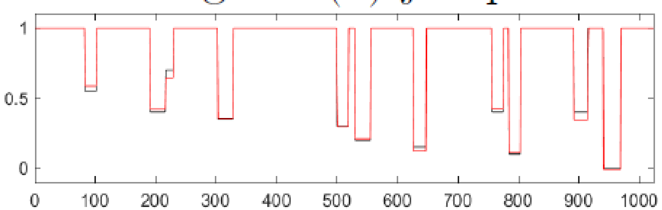
Stage 1: $(\bar{v})$ jumps



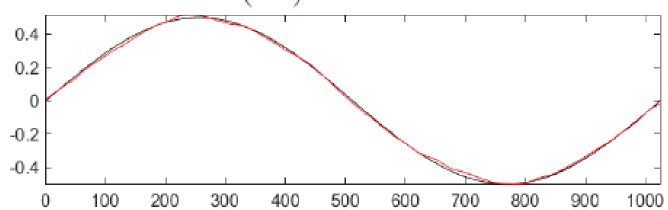
$(\bar{w})$ trend



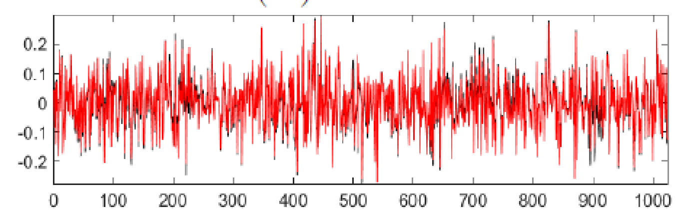
$(\bar{n})$ noise



Stage 2: (v^*) jumps



(w^*) trend

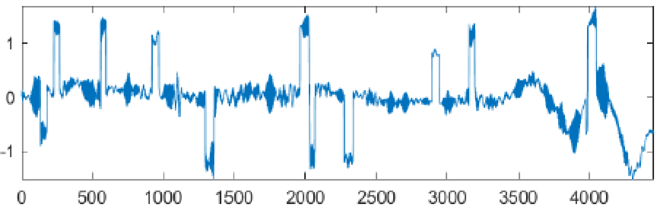


(n^*) noise

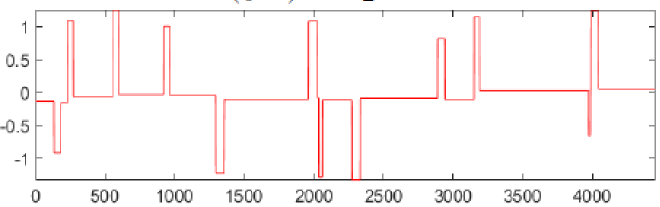
	$SNR(\bar{v}, v)$	$SNR(\bar{w}, w)$	$SNR(\bar{n}, n)$	$SNR(v^*, v)$	$SNR(w^*, w)$	$SNR(n^*, n)$
f_1	8.57	6.66	5.63	25.19	30.06	14.01

Example 2

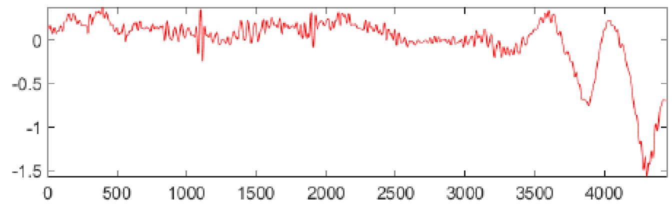
$f_2 \in \mathbb{R}^{4451}$



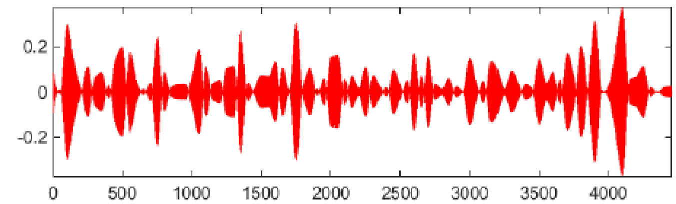
(f_2) input



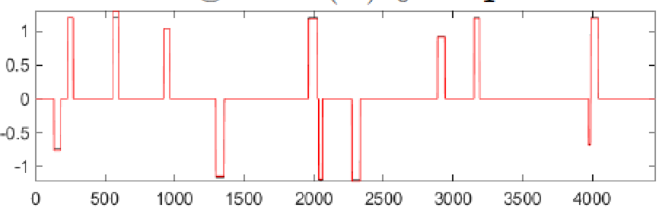
Stage 1: $(\bar{v})$ jumps



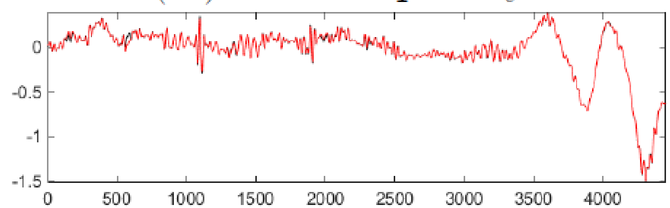
$(\bar{w})$ low frequency



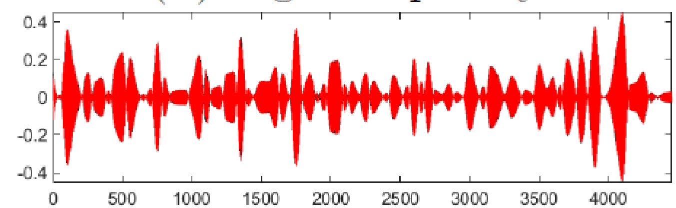
$(\bar{n})$ high frequency



Stage 2: (v^*) jumps



(w^*) low frequency

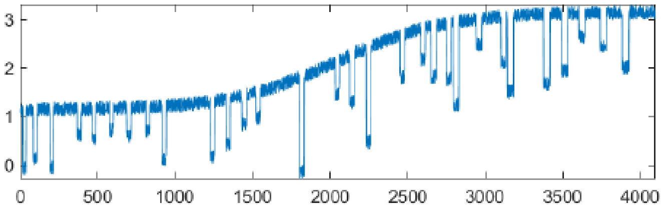


(n^*) high frequency

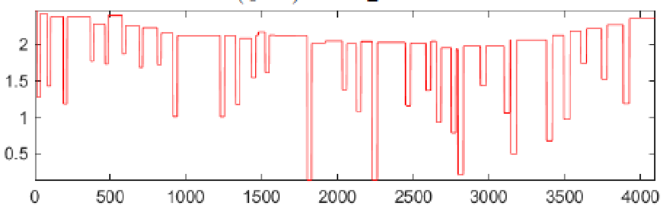
	$SNR(\bar{v}, v)$	$SNR(\bar{w}, w)$	$SNR(\bar{n}, n)$	$SNR(v^*, v)$	$SNR(w^*, w)$	$SNR(n^*, n)$
f_2	15.45	13.61	11.25	29.37	27.78	21.44

Example 3

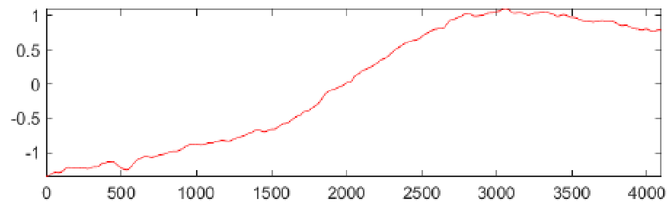
$f_3 \in \mathbb{R}^{4096}$



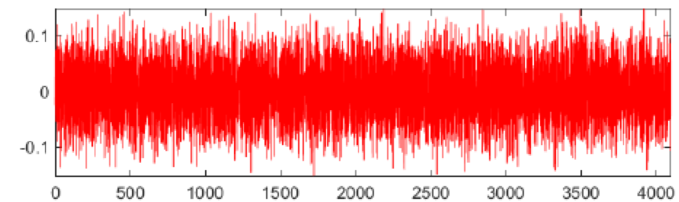
(f_3) input



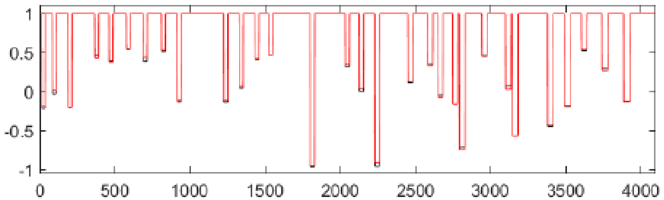
Stage 1: $(\bar{v})$ jumps



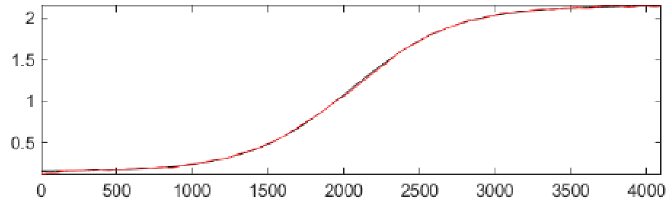
$(\bar{w})$ trend



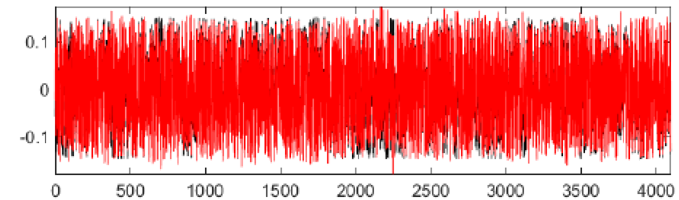
$(\bar{n})$ high frequency



Stage 2: (v^*) jumps



(w^*) trend

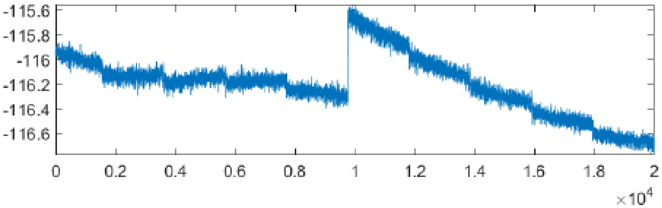


(n^*) high frequency

	$SNR(\bar{v}, v)$	$SNR(\bar{w}, w)$	$SNR(\bar{n}, n)$	$SNR(v^*, v)$	$SNR(w^*, w)$	$SNR(n^*, n)$
f_3	9.54	14.77	2.58	30.67	39.23	16.75

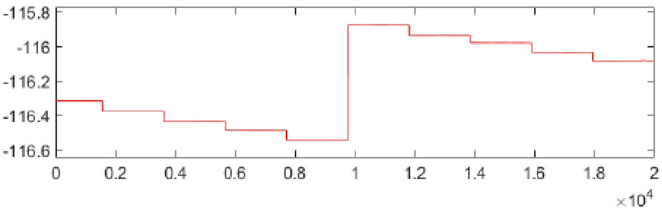
Example 4

$f_4 \in \mathbb{R}^{20000}$

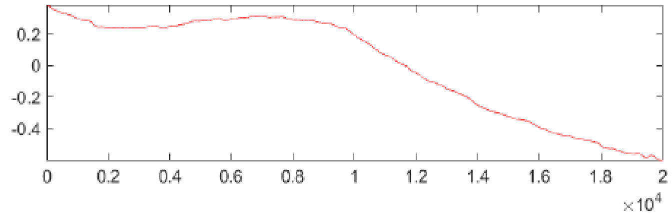


(f_4) input

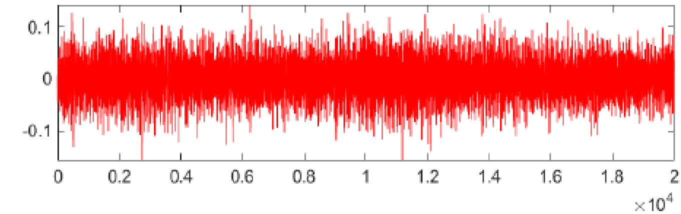
- Earth's magnetic field data corrupted due data packaging pauses during acquisition



Stage 1: $(\bar{v})$ jumps



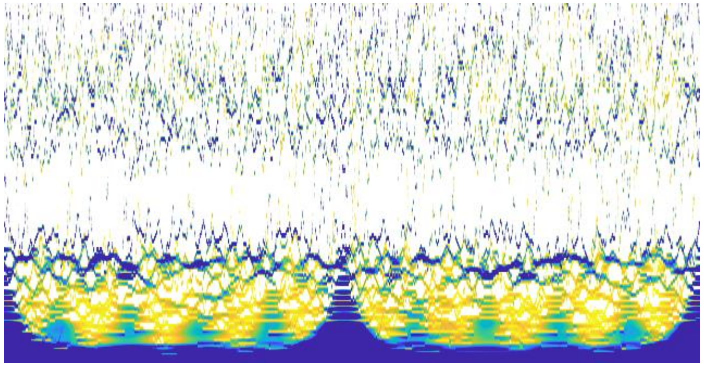
$(\bar{w})$ trend



$(\bar{n})$ high frequency

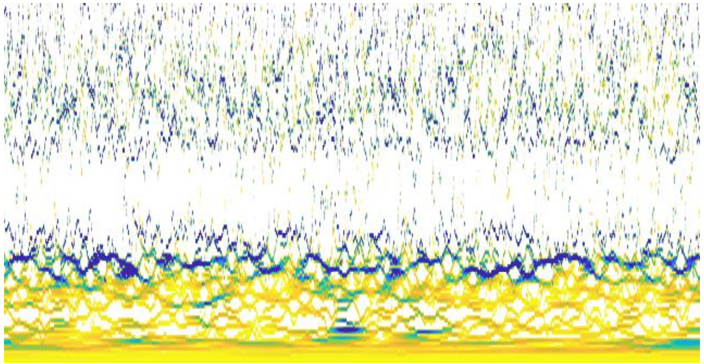
Frequency analysis (IMFogram)

of f



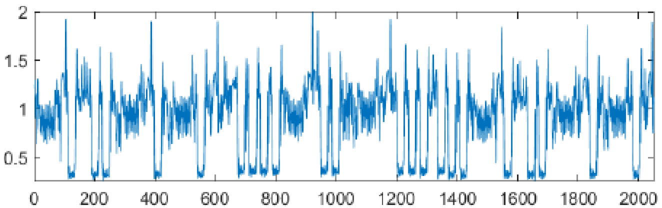
Active frequencies of v eliminated

of $\bar{n}$



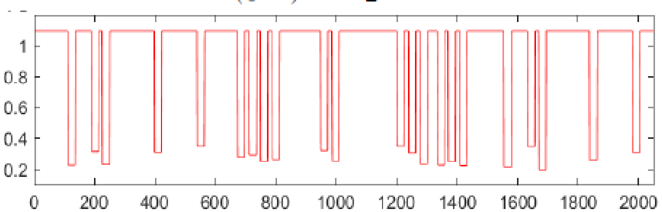
Example 5

$f_5 \in \mathbb{R}^{2\times10^4}$

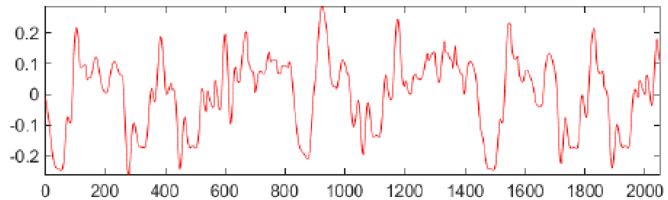


(f_5) input

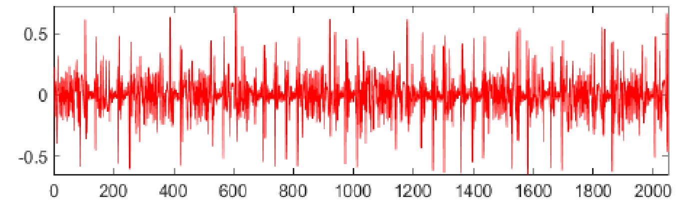
- CPU Emanated EM signal with artifact created by various process loads



Stage 2: (v^*) jumps



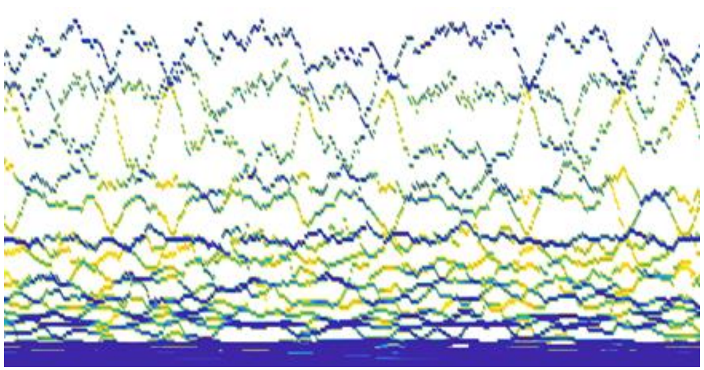
(w^*) low frequency



(n^*) high frequency

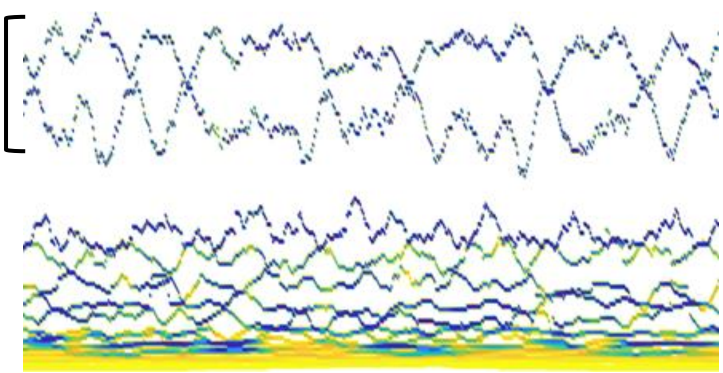
Frequency analysis (IMFogram)

of f



$[3,4]MHz$

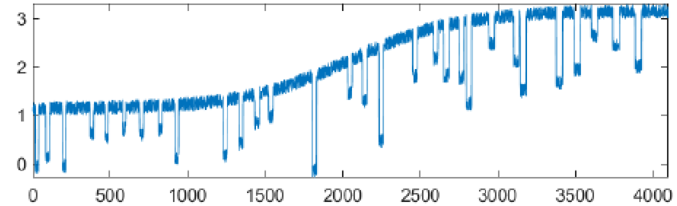
of n^*

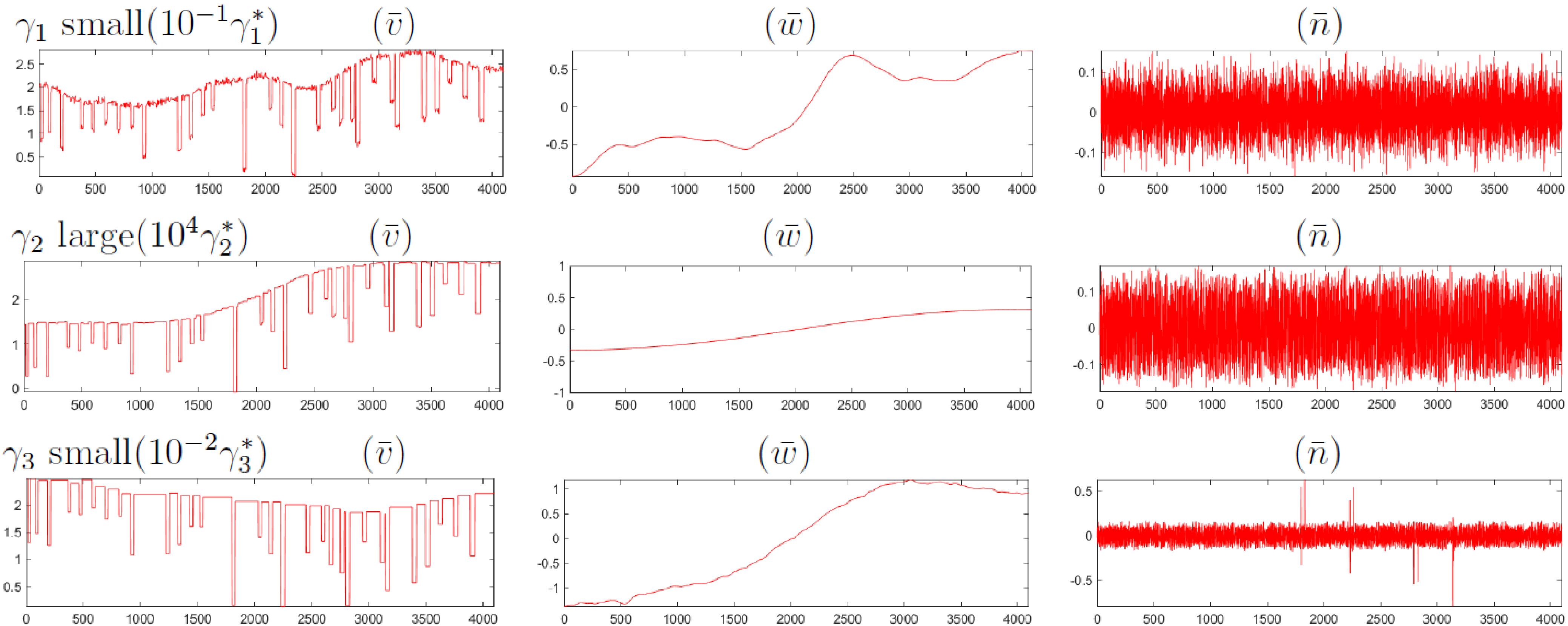


Example 6

Effect of regularization parameters for f_3

- Stage 1 – variational model depending on $\gamma_1, \gamma_2, \gamma_3$





Conclusions

- Enlargement of the class of variational decomposition models with non-convex penalty
- Useful pre-processing tool in 1D signal frequency analysis
- Simple and effective alternating minimization algorithm

- Future directions
  - Parameter estimation
  - Numerical solution for large data
  - Handling different artifacts, e.g. spike decomposition
  - Considering additional information in the model, e.g. periodicity

References

[1] Cicone A., H. M., Kang S.-H., Morigi S., JOT: A Variational Signal Decomposition into Jump, Oscillation and Trend, *IEEE Transactions on Signal Processing*, **70**, pp. 772 – 784, 2022

[2] H. M., Cicone A., Kang S.-H., Morigi S., A two-stage signal decomposition into Jump, Oscillation and Trend using ADMM, Image Processing On Line, **13**, pp. 153–166, 2023

Thank you for your attention!