Recitation 3

[Definitions used today]

- (weakly/strongly) convex, continuous, monotone preferences, locally non-satiated utility function
- utility maximization, Debreu theorem, lexicographic preferences
- WARP, GWARP, GARP, Topkis theorem, Afriat theorem

Question 1 [Weak vs strong continuity] 182 [Question I.1 Fall 2014 majors]

Let $\succeq$ be a transitive and complete preference relation on (connected) set $X \subseteq \mathbb{R}_+^N$:
Prove that the following statements are equivalent

- $\succeq$ on X is **weakly continuous** if $\forall x \in X$ the preferred-to-x set $U(x) = \{y \in X : y \succeq x\}$ and lower countur set $L(x) = \{y \in X : x \succeq y\}$ are closed.
- $\succeq$ on X is **strongly continuous** if for all sequences $\{x_n\}\{y_n\} \in X$ such that $x_n \to x$, $y_n \to y$, if $\forall n$, $x_n \succeq y_n$, then $x \succeq y$.

Solution 1

TBD

Question 2 [Properties of preferences]

Prove following statements

1. If a preorder $\succeq$ is monotone in $\mathbb{R}^l$, then it is locally nonsatiated.
2. If a preorder $\succeq$ is transitive, weakly monotone, and locally nonsatiated then it is monotone
3. A preorder $\succeq$ is weakly convex $\iff$ the upper contour sets $U(x) = \{y \in X : y \succeq x\}$ are convex for all $x \in X$
4. If a preorder $\succeq$ is continuous and strictly convex then it is convex

Solution 2

The following definitions are following from Debreu(1987).

Definition 0.1. *A* ***preorder*** $\succeq$ *on* X *is a binary relation which is reflexive* ($\forall x \in X, x \succeq x$) *and transitive* ($\forall x, y, z \in X \ni (x \succeq y \wedge y \succeq z) \Rightarrow x \succeq z$)*.A* ***complete preorder*** *is a preorder that is complete* ($\forall x, y \in X, x \succeq y \vee y \succeq x$)*.*

0.1 Monotonicity and Nonsatiation

Definition 0.2. *A preorder* $\succeq$ *is* ***weakly monotone*** *on a set* X *if* $\forall x, y \in X, x \geq y \Rightarrow x \succeq y$*.*
A preorder $\succeq$ *is* ***monotone*** *on a set* X *if* $\forall x, y \in X, x \gg y \Rightarrow x \succ y$*.*
A preorder $\succeq$ *is* ***strongly monotone*** *on a set* X *if* $\forall x, y \in X, (x \geq y \wedge x \neq y) \Rightarrow x \succ y$*.*

Definition 0.3. *A preorder* $\succeq$ *is* ***locally nonsatiated*** *on a set* X *if* $\forall x \in X$ *and* $\forall \epsilon > 0, \exists y \in X \ni \| x - y \| < \epsilon$ *and* $y \succ x$*.*

0.2 Convexity

Definition 0.4. *A preorder* $\succeq$ *is* ***weakly convex*** *on a set* X *if* $\forall x, y \in X, \forall \lambda \in (0, 1)$,

$$x \succeq y \Rightarrow \lambda x + (1 - \lambda) y \succeq y$$

Definition 0.5. *A preorder* $\succeq$ *is* ***convex*** *on a set* X *if* $\forall x, y \in X, \forall \lambda \in (0, 1)$,

$$x \succ y \Rightarrow \lambda x + (1 - \lambda) y \succ y$$

Definition 0.6. *A preorder* $\succeq$ *is* ***strongly/strictly convex*** *on a set* X *if* $\forall x, y \in X, \forall \lambda \in (0, 1)$,

$$x \sim y \wedge x \neq y \Rightarrow \lambda x + (1 - \lambda) y \succ y$$

0.3 Continuity

Definition 0.7. ***(Sequential definition/ weak continuity)*** *A preorder $\succeq$ is continuous on a set X if $\forall \{x_n\}, \{y_n\} \subseteq X$,*

$$\forall n \in \mathbb{N}, (x_n \succeq y_n) \wedge (x_n \to x) \wedge (y_n \to y) \Rightarrow x \succeq y$$

Definition 0.8. ***(Set definition/ strong continuity)*** *A preorder $\succeq$ is continuous on a set X if $\forall x \in X$, the upper contour set $U(x) = \{y \in X : y \succeq x\}$ and the lower contour set $L(x) = \{y \in X : x \succeq y\}$ are closed in X.*

Proposition 0.9. ***Equivalence of Weak and Strong Continuity***

If X is connected(e.g. $X = \mathbb{R}^l_+$), then definitions of weak and strong continuity are equivalent.

0.4 Relationships Among Preorder Properties

Proposition 0.10. *If a preorder $\succeq$ is strictly monotone, then it is monotone.*

Proof. Let $x, y \in X$ such at $y \gg x$. Then $y \geq x$ and $y \neq x$. Since $\succeq$ is strictly monotone, this implies that $y \succ x$. Thus, $\succeq$ is monotone. □

Proposition 0.11. *If a preorder $\succeq$ is monotone in $\mathbb{R}^l$, then it is locally nonsatiated.*

Proof. Let $x \in X$ and fix $\epsilon > 0$. Let e denote the unit vector in $\mathbb{R}^l$. Pick $y = x + \frac{\epsilon}{2\sqrt{l}}e$. Note that $\parallel x - y \parallel = \epsilon/2 < \epsilon$, and $y \gg x$. Since $\succeq$ is monotone, $y \succ x$. Thus, $\succeq$ is l.n.s. □

Proposition 0.12. *If a preorder $\succeq$ is transitive, weakly monotone, and locally nonsatiated then it is monotone.*

Proof. Let $x, y \in X$ such that $y \gg x$. Let $\epsilon = \min\{y_1 - x_1, \ldots, y_\ell - x_\ell\}$. Then for all $z \in X$ such that $\|x - z\| < \epsilon$, $y \gg z$. By local nonsatiation, $\exists z' \in X$ such that $\|x - z'\| < \epsilon$ and $z' \succ x$. Since $y \gg z'$, weak monotoncity implies that $y \succeq z'$. By transitivity, $y \succ x$. Thus $\succeq$ is monotone. □

Proposition 0.13. *A preorder $\succeq$ is weakly convex $\iff$ the upper contour sets $U(x) = \{y \in X : \ y \succeq x\}$ are convex for all $x \in X$.*

Proof. Suppose $\succeq$ is weakly convex. Let $x \in X$, let $y, y' \in U(x)$ such that $y' \succeq y$, and let $\lambda \in [0, 1]$. By weak convexity, $\lambda y + (1 - \lambda)y' \succeq y$. By definition of $U(x)$, $y \succeq x$. By transitivity, $\lambda y + (1 - \lambda)y' \succeq x$, so $\lambda y + (1 - \lambda)y' \in U(x)$. Thus $U(x)$ is convex.

Now suppose that $U(x)$ are convex for all $x \in X$. Let $x, y \in X$ such that $y \succeq x$ and let $\lambda \in [0, 1]$. By reflexivity, $x \succeq x$, so $x, y \in U(x)$. Since $U(x)$ is convex, $\lambda x + (1 - \lambda)y \in U(x)$. By definition of $U(x)$, $\lambda x + (1 - \lambda)y \succeq x$. Thus $\succeq$ is weakly convex. Therefore $\succeq$ is convex $\iff$ $U(x)$ is convex for all $x \in X$. □

Proposition 0.14. *If a preorder $\succeq$ is continuous and strictly convex then it is convex.*

Proof. Let $x, y \in X$ such that $y \succ x$ and $x \neq y$, and let $\lambda \in (0, 1)$. Then $y \succeq x$ and $x \neq y$, so strict convexity implies that $\lambda y + (1 - \lambda)x \succ x$. Thus we have $y \succ x \Rightarrow \lambda y + (1 - \lambda)x \succ x$, so $\succeq$ is convex. □

Question 4 Give an example of preferences/utility function such that :

1. satisfy non-satiation, but not weak monotonicity
2. satisfy non-satiation, but not local non-satiation
3. satisfy local non-satiation, strict monotonicity, but not quasi-concave
4. does not satisfy continuous but it is representable by a utility function

Solution 4

TBD

Question 3 Consider the following preference relations on $\mathbb{R}^2_+$

1. $x \succeq y \iff \min\{x_1, x_2\} \geq \min\{y_1, y_2\}$
2. $x \succeq y \iff \max\{x_1, x_2\} \geq \max\{y_1, y_2\}$

are they convex? Are they strictly convex?

Solution 3

a) It is convex but not strictly convex

$$(2,1) \succeq (1,1) \quad (3,1) \succeq (1,1)$$

but

$$\frac{1}{2}(2,1) + \frac{1}{2}(3,1) \succeq (2.5,1) \not\succ (1,1)$$

b) it is not even convex. Take

$$(5,3) \succeq (4.5,4.5) \quad (3,5) \succeq (4.5,4.5)$$

but

$$\frac{1}{2}(5,3) + \frac{1}{2}(3,5) \succeq (4,4) \not\succ (4.5,4.5)$$

Question 5 [Utility representation] 157 [I.1 Fall 2013 majors]

Consider preference relation $\succeq$ on the consumption set $\mathbb{R}^L_+$. Suppose that $\succeq$ is reflexive and complete.

1. State a definition of $\succeq$ having a utility representation. Is utility representation, if it exists, unique?
2. State a theorem providing sufficient conditions on $\succeq$ to have a utility representation. Be as general as you can and clearly define any extra properties of $\succeq$ that you use
3. [**Debreu Theorem**] Let $\succeq$ be a complete, transitive and continuous, strictly increasing (i.e. strongly monotone) preference relation on $\mathbb{R}^L_+$, show that it has a continuous utility representation

Solution 5

Check out Debreu's proof by Ariel Rubinstein on my webpage!
Existence theorems: Rader RES 1964, Hildebrand , Mathematical Economics:Debreu 1981,

Definition 0.15. *A function $u : X \to \mathbb{R}$ is a **utility representation** of a preorder $\succeq$ if for all $x, y \in X$,*

$$u(y) \geq u(x) \iff y \succeq x.$$

Theorem 0.16. ***(Existence of Utility Representation)***
If a preorder $\succeq$ is complete, continuous and monotone on $X = \mathbb{R}^l_+$, then it has a continuous utility representation.

Proof. **Step-1** Constructing $\alpha(x)$
Let $\succeq$ be a complete, continuous and monotone preorder on $\mathbb{R}^l_+$, and let $x \in \mathbb{R}^l_+$. Let $e = (1, 1, \ldots, 1) \in \mathbb{R}_{++}$.
Then $\alpha e \in \mathbb{R}^l_+ \ \forall \alpha \in \mathbb{R}_+$. Choose $\bar{\alpha}$ such that $\bar{\alpha} e \geq x$. Since $\succeq$ is monotone, $\bar{\alpha} e \succeq x$. Also note that $0e \leq x$, so $x \succeq 0e$.
Define A^+ and A^- as

$$A^+ = \{\alpha \in \mathbb{R}_+ : \alpha e \succeq x\} \qquad A^- = \{\alpha \in \mathbb{R}_+ : x \succeq \alpha e\}$$

Since $\succeq$ is continuous, A^+ and A^- are closed. $\bar{\alpha} \in A^+$ and $0 \in A^-$, so both sets are also nonempty. By completeness of $\succeq$, $\mathbb{R}_+ = A^+ \cup A^-$.
Since $\mathbb{R}_+$ is connected, $A^+ \cap A^- \neq \emptyset$. Let $\alpha(x) \in A^+ \cap A^-$. Then $\alpha(x)e \sim x$.

Step-2 Uniqueness

Suppose that $\exists \alpha' \in A^+ \cap A^-$ such that $\alpha' \neq \alpha(x)$. Then either $\alpha' > \alpha(x)$ or $\alpha(x) > \alpha'$. If the former case is true, then since $\succeq$ is monotone, $\alpha' e \succ \alpha(x)e \sim x$. If the latter, then $\alpha' e \prec \alpha(x)e \sim x$. In either case, it cannot be true that $\alpha' \in A^+ \cap A^-$. Therefore $\alpha(x)$ is unique.

Step-3 Utility Representation

($\Rightarrow$) Let $x, y \in \mathbb{R}^l_+$such that $y \succeq x$. Then by definition of $\alpha(x)$, we have $\alpha(y)e \sim y \succeq x \sim \alpha(x)e$.
By transitivity, $\alpha(y)e \succeq \alpha(x)e$, and so by monotonicity, $\alpha(y) \geq \alpha(x)$.
($\Leftarrow$) Suppose that $\alpha(y) \geq \alpha(x)$. Then by monotonicity of $\succeq$, $\alpha(y)e \succeq \alpha(x)e$.
By definition of $\alpha(x)$, we have $y \sim \alpha(y)e \succeq \alpha(x)e \sim x$. By transitivity, $y \succeq x$
Thus we have $y \succeq x \iff \alpha(y) \geq \alpha(x)$, so $u(x) = \alpha(x)$ is a utility representation of $\succeq$

Step-4 Continuity

Note that $u : \mathbb{R}^l_+ \to \mathbb{R}_+$. For any $x \in \mathbb{R}^l_+$, we can write $u^{-1}([0, \alpha(x)])$ and $u^{-1}([\alpha(x), \infty))$ as

$$u^{-1}([0, \alpha(x)]) = \{y \in \mathbb{R}^l_+ : \alpha(x) \geq u(y)\} = \{y \in \mathbb{R}^l_+ : \alpha(x)e \succeq y\}$$
$$u^{-1}([\alpha(x), \infty)) = \{y \in \mathbb{R}^l_+ : u(y) \geq \alpha(x)\} = \{y \in \mathbb{R}^l_+ : y \succeq \alpha(x)e\}$$

By continuity of $\succeq$, both of these sets are closed. So we have for every $x \in \mathbb{R}^l_+$, $[0, \alpha(x)]$ and $u^{-1}([0, \alpha(x)])$ are closed, and $[\alpha(x), \infty)$ and $u^{-1}([\alpha(x), \infty))$ are closed.
Also, for $[\alpha(x), \alpha(y)]$ which is closed, we have

$$u^{-1}([\alpha(x), \alpha(y)]) = u^{-1}([\alpha(x), \infty)) \cap u^{-1}([0, \alpha(y)])$$

which is closed.
Thus by the topological definition of continuity, $u(x) = \alpha(x)$ is continuous.

□

Proposition 0.17. *If $\succeq$ is a lexicographic preference then $\succeq$ has no utility representation.*

Proof. Let $\succeq$ be a lexicographic preference relation on $\mathbb{R}^2_+$ and suppose for contradiction that $\exists u : X \to \mathbb{R}$ such that u is a utility representation of $\succeq$. For each $x \in \mathbb{R}_+$, we can find a rational number $r(x)$ such that $u(x, 2) > r(x) > u(x, 1)$. Let $x, x' \in \mathbb{R}_+$ such that $x' > x$. Then

$$u(x', 2) > r(x') > u(x', 1) > u(x, 2) > r(x) > u(x, 1).$$

Thus $r(x') > r(x)$, so we have a one-to-one map $r : \mathbb{R}_+ \to \mathbb{Q}$. This is a contradiction since $\mathbb{R}_+$ is uncountable and $\mathbb{Q}$ is countable. Therefore $\succeq$ has no utility representation. □

Question 6 [Lexicographic preference]

Consider the following lexicographic preferences on the consumption set $\mathbb{R}^2_+$: the value $x_1 + x_2$ has the first priority, the value of x_2 has the second priority.

1. Is this preference relation continuous? Prove of give a counter example.
2. Does this preference relation have the utility representation? Prove of give a counter example.
3. Consider the lexicographic preferences on $\mathbb{R}^N_{++}$ such that the first priority is described by an increasing and continuous utility function $u_1(x)$ and the second priority is described by another increasing and continuous utility function $u_2(x)$. Show that, if u_1 is strictly concave, then the Walrasian demand of the lexicographic preference coincides with the Walrasian demand of u_1 for every $p \in \mathbb{R}^N_+$, $p \neq 0$ and $w > 0$.

Question 7 [Midterm 2018]

Consider a list of observations $\{(p_1, x_1), \ldots, (p_T, x_T)\}$ where $p_t \in \mathbb{R}^N_+$ and $x_t \in \mathbb{R}^N_+$ are price vector and a corresponding consumption plan of a consumer respectively, for every $t \in \{1, \ldots, T\}$.

1. State the Generalized Weak Axiom of Revealed Preference (GWARP) and Generalized (strong) Axiom of Revealed Preference (GARP) for these observations.
2. Show that if a locally non-satiated utility function rationalized observations then GARP holds.
3. Suppose that the observations are generated by a demand function $d(p, w)$ that is $x_t = d(p_t, w_t)$ for every t. Function d is given as
$$d(p,w) = \begin{cases} (\frac{w}{p_1}, 0) & \text{if } p_1 \geq p_2 \\ (\frac{w}{p_1+p_2}, \frac{w}{p_1+p_2}) & \text{if } p_2 > p_1 \end{cases}$$
Does GWARP hold for arbitrary observations generated by d? Can demand d be rationalized by a locally non-satiated utility function?
4. Show that if a locally non-satiated utility function rationalized observations then GWARP holds.
5. Show that the assumption of local non-satiation in the previous point cannot be dispensed with - i.e. give an example of a utility function that rationalizes a set of pairs of prices and consumption bundles that violates GWARP

Question 8 [Properties of Walrasian Demand]

Prove following claims

1. **[Walras Law]** Show that if a preference relation $\succeq$ is continuous and locally non-satiated then $p \cdot x^*(p, w) = w$, for all $x^*(p, w)$ that belong to the Walrasian Demand correspondence.
2. **[GWARP]** Show that if a preference relation $\succeq$ is continuous and locally non-satiated then for all $w > 0$
$$w' > 0, p >> 0 \text{ and } p' >> 0: \quad p \cdot x^*(p', w') \leq w \Rightarrow p' \cdot x^*(p, w) \geq w'$$

Question 9 230 [I.1 Fall 2016 minors]

Let d : be a demand function of prices and income satisfying budget equation $pd(p, w) = w$ for every p and w

1. Show that if d is a Walrasian demand function of a consumer with strictly increasing utility function, then the Generalized Weak Axiom of Revealed Preference (GWARP) holds for every T -tuple of price-quantity pairs $\{p^t, x^t\}_{t=1}^T$, where $x^t = d(p^t, w^t)$ $p^t \in \mathbb{R}^L_{++}$ and $w^t \in \mathcal{R}_+$ for every $t = 1, \ldots, T$. State GWARP
2. onsider the following demand function for $L = 2$ and show that GWARP does not hold for $\hat{d}$.:
$$\hat{d}(p,w) = \begin{cases} \left(\frac{w}{p_1}, 0\right) & \text{if } p_1 \geq p_2 \\ \left(0, \frac{w}{p_2}\right) & \text{if } p_2 > p_1 \end{cases}$$
3. State the Afriat's Theorem. The proof is not required
4. Prove the necessity of an axiom for rationalizability

Solution 9

Problem 4.7 [F17] Consider a list of observations $(p_1, x_1), \ldots, (p_T, x_T)$ where $p_t \in \mathbb{R}^N_+$ and $x_t \in R^N_+$ are price vector and a corresponding consumption plan of a consumer respectively, for every $t \in \{1, \ldots, T\}$.

i) State the Generalized Weak Axiom of Revealed Preference (GWARP) and Generalized (strong) Axiom of Revealed Preference (GARP) for these observations.

ii) State the Afriat theorem. Prove the necessity of an axiom for rationalizability.

def. utility function u on $\mathbb{R}^L_+$ rationalizes observations $\{(p_1, x_1), \ldots, (p_T, x_T)\}$ if :

$$\forall_{t \in \{1,\ldots,T\}} \ \forall_{x \in \mathbb{R}^L_+} \quad p^t x \leq p^t x^t \Rightarrow u(x^t) \geq u(x)$$

def (GWARP) A set of observations satisfies GWARP if:

$$\forall_{s,t} \quad p^t x^s \leq p^t x^t \Rightarrow p^s x^t \geq p^s x^s$$

or

$$x^t R x^s \Rightarrow \sim (x^s P x^t) \quad \text{for } x^t \neq x^s$$

def (GARP) A set of observations satisfies GARP if:

For every subset of observations $\{(p^{t_1}, x^{t_1}), \ldots, (p^{t_n}, x^{t_n})\}$

$$x^{t_1} R x^{t_2}, \ x^{t_2} R x^{t_3}, \ldots, \ x^{t_{n-1}} R x^{t_n} \Rightarrow \sim (x^{t_n} P x^{t_1}) \quad \text{for } x^{t_n} \neq x^{t_1}$$

OR

$$\begin{cases} p^{t_1} x^{t_1} \geq p^{t_1} x^{t_2} \\ \vdots \\ p^{t_{n-1}} x^{t_{n-1}} \geq p^{t_{n-1}} x^{t_n} \end{cases} \Rightarrow p^{t_n} x^{t_n} \leq p^{t_n} x^{t_1}$$

Theorem (Afriat)

Observations $\{(p^1, x^1), \ldots, (p^T, x^T)\}$ satisfy GARP iff there exists a lns utility function u that rationalizes them.

ii) WTS: if there exists a lns u that rationalizes the observations then GARP holds.

Suppose for a contradiction that there exists a lns utility function u that rationalizes the observations but GARP doesn't hold. That means :

$$\exists \ (p^{t_1}, x^{t_1}), \ldots, (p^{t_n}, x^{t_n}) \ \text{s.t.} \begin{cases} p^{t_1} x^{t_1} \geq p^{t_1} x^{t_2} \\ \vdots \\ p^{t_{n-1}} x^{t_{n-1}} \geq p^{t_{n-1}} x^{t_n} \end{cases} \text{and} \quad p^{t_n} x^{t_n} > p^{t_n} x^{t_1}$$

Since utility function is lns :

$$p^t x^t \geq p^t x^s \Rightarrow u(x^t) \geq u(x^s) \quad \forall_{t,s}$$
$$p^t x^t > p^t x^s \Rightarrow u(x^t) > u(x^s). \quad \forall_{t,s}$$

Then :

$$u(x_1) \geq u(x_2) \geq u(x_3) \geq \ldots \geq u(x^{n-1}) \geq u(x^n) > u(x_1) \ \lightning$$