Previous midterms

Question 1 [Midterm 2017]

A normal form game has I players, action sets A^i and utility functions u^i, for $i \in I$. The vector $u^i : i = 1, \ldots, n$ is a utility profile. Fix the set of players and the action sets. The Nash equilibrium correspondence (NEC) associates to every utility profile the set of Nash equilibria of the normal form game with that profile.

a) Prove that one can restrict the domain of the correspondence to a compact sel of ntility profiles.

b) Prove that the NEC is closed valued.

c) Prove that the NEC is upper-hemi-continuous, or find an example to show it is not

Solution 1

Theorem 0.1. *For every NFG the set of Nash equilibria is (1) nonempty and (2) closed.*

Proof. (1) Notice that $NE := \{s : s \in BR(s)\}$, i.e. the set of all fixed points of the correspondence BR. Then since S is a nonempty, compact, and convex subset of $\mathbb{R}^n$ and BR is, by Theorem 2.4, a convex-valued, nonempty, self-correspondence on S that has a closed graph, BR has a fixed point by Kakutani's fixed point theorem. Hence, $NE \neq \emptyset$.

(2) Clearly NE is a subset of the metric space S. So take any $(s_n) \in NE^\infty$ with $s_n \to s \in S$. For all $n, s_n \in NE$ implies $s_n \in BR(s_n)$. Since BR is closed-valued and u.h.c., it also has a closed graph. In particular, BR is closed at s, which implies $s \in BR(s)$. Then s_n converges to a point in NE which implies that NE is closed. $\square$

Theorem 0.2. *Fix I and $\left(S^i\right)_{i\in I}$ and define $NE(u) = NE\left(\left(u^i\right)_{i\in I}\right)$ as the set of N.E. of the mixed extension $\left\langle I, \left(S^i\right)_{i\in I}, \left(u^i\right)_{i\in I}\right\rangle$. Then the correspondence mapping $u \mapsto NE(u)$ is uhc*

Proof. Let $U = \times_{i\in I} U^i$, where U^i is the set of all vNM utility functions for player i. Take any sequences $(u_m) \in U^\infty$ and $(s_m) \in S^\infty$ s.t. $u_m \to u$ and $s_m \in NE(u_m)\,\forall m$. Since $s_m \in NE(u_m)$ we have that for each player $i, s_m^i \in BR^i(s_m \mid u_m)$. Equivalently,

$$u_m^i\left(s_m^i, s_m^{-i}\right) \geq u_m^i\left(r^i, s_m^{-i}\right), \quad \forall r^i \in S^i$$

$$\Longleftrightarrow \sum_{a\in A}\left(\prod_{j\in I} s_m^j\left(a^j\right)\right) u_m^i(a) \geq \sum_{a\in A}\left(r^i\left(a^i\right) \prod_{j\in I\setminus\{i\}} s_m^j\left(a^j\right)\right) u_m^i(a), \quad \forall r^i \in S^i$$

Taking the limit as $m \to \infty$ of this expression we have

$$\lim_{m\to\infty} \sum_{a\in A}\left(\prod_{j\in I} s_m^j\left(a^j\right)\right) u_m^i(a) \geq \lim_{m\to\infty} \sum_{a\in A}\left(r^i\left(a^i\right) \prod_{j\in I\setminus\{i\}} s_m^j\left(a^j\right)\right) u_m^i(a), \quad \forall r^i \in S^i$$

Since the sum and product operators are continuous functions, we have move the limit operator through, yielding

$$\sum_{a\in A}\left(\prod_{j\in I} \lim_{m\to\infty} s_m^j\left(a^j\right)\right) \lim_{m\to\infty} u_m^i(a) \geq \sum_{a\in A}\left(r^i\left(a^i\right) \prod_{j\in I\setminus\{i\}} \lim_{m\to\infty} s_m^j\left(a^j\right)\right) \lim_{m\to\infty} u_m^i(a), \quad \forall r^i \in S^i$$

Since for each $m, u_m^i(\cdot)$ and $s_m^j(\cdot)$ are evaluated only at a particular $a \in A$ over the whole domain of the sequence, in the limit we have

$$\sum_{a\in A}\left(\prod_{j\in I} s^j\left(a^j\right)\right) u^i(a) \geq \sum_{a\in A}\left(r^i\left(a^i\right) \prod_{j\in I\setminus\{i\}} s^j\left(a^j\right)\right) u^i(a), \quad \forall r^i \in S^i$$

But then $u^i\left(s^i, s^{-i}\right) \geq u^i\left(r^i, s^{-i}\right), \forall r^i \in S^i$; that is, $s^i \in BR^i(s \mid u)$. Since this holds for all $i \in I$, $s \in BR(s \mid u)$, which implies $s \in NE(u)$. Hence, $NE(\cdot)$ has a closed graph. Then, since $NE(\cdot)$ is closed-valued by first theorem and has a compact range, $NE(\cdot)$ is upper hemicontinuous. $\square$

Question 2 [Midterm 2017]

Define a Perfect Equilibrium of a normal form game as the limit of sequences of Nash equilibria of a perturbed game.

a) Prove that a perfect equilibrium is a Nash equilibrium.

b) A strategy is fully mixed if it gives positive probability to all the actions in the action set of the player, and a strategy profile is fully mixed if the strategies of all players are fully mixed. Prove that a fully mixed strategy profile is perfect.

c) Give as simple sufficient condition for a Nash equilibrium to be perfect.

Solution 2

Definition 0.3. *A vector of perturbations is* $\varepsilon = \left(\varepsilon^i\left(a^i\right)\right)_{i\in I, a^i\in A^i}$, *with* $\sum_{i\in I}\left|A^i\right|$ *coordinates, such that* $\forall i, a^i \in A^i, \varepsilon^i\left(a^i\right) > 0$ *and* $\forall i, \sum_{a^i\in A^i}\varepsilon^i\left(a^i\right) < 1$

Definition 0.4. *For a NFG* Γ *and a vector of perturbations* ε, *a perturbed game is* $\Gamma_\varepsilon := \left\langle I, \left(S_\varepsilon^i\right)_{i\in I}, \left(u^i\right)_{i\in I}\right\rangle$ *where* $S_\varepsilon^i := \left\{s \in S^i : \forall a^i \in A^i, s^i\left(a^i\right) \geq \varepsilon^i\left(a^i\right)\right\},$. *Defining* $S_\varepsilon \equiv X_{i\in I} S_\varepsilon^i (\text{so } S_\varepsilon \subset S)$, *then for each* $i \in I, u^i : S_\varepsilon \to \mathbb{R}$ *is defined as* $\left.u^i\right|_{S_\varepsilon}$

Definition 0.5. *Player* i *'s best response correspondence in the perturbed game* $BR_\varepsilon^i : S_\varepsilon \rightrightarrows S_\varepsilon^i$ *is defined by* $BR_\varepsilon^i(s) := \left\{r \in S_\varepsilon^i : r = \arg\max_{r'\in S_\varepsilon^i}\sum_{a\in A} r'(a)u^i\left(a, s^{-i}\right)\right\}$

Corollary 0.6. *The correspondence* $BR_\varepsilon^i(\cdot)$ *is nonempty-, closed-and convex-valued and upper hemicontinuous. Hence,* $BR_\varepsilon(\cdot) := \chi_{i\in I} BR_\varepsilon^i(\cdot)$ *is also a nonempty-, closed- and convex-valued upper hemicontinuous correspondence.*

Proof. The argument is identical to that provided in the proof of Theorem 2.4 and Theorem 2.3. □

Definition 0.7. *The set of N.E. of the perturbed game* Γ_ε *is* $NE\left(\Gamma_\varepsilon\right) := \left\{s \in S_\varepsilon : s \in BR_\varepsilon(s)\right\}$.

Corollary 0.8. $NE\left(\Gamma_\varepsilon\right) \neq \emptyset$.

Proof. By Kakutani. □

Definition 0.9. *A mixed strategy* $s \in S$ *is a perfect equilibrium if there exists a sequence of perturbation vectors* $\left(\varepsilon_m\right)$ *with* $\varepsilon_m^i\left(a^i\right) \to 0, \forall i \in I, a^i \in A^i$ *and a corresponding sequence of strategy profiles* $\left(s_{\varepsilon_m}\right) \in S^\infty$ *such that* $s_{\varepsilon_m} \to s$ *such that* $s_{\varepsilon_m} \in NE\left(\Gamma_{\varepsilon_m}\right) \forall m \in \mathbb{N}$

Corollary 0.10. *The requirements of a perfect equilibrium are actually weak since we are able to choose the sequence* $\left(\varepsilon_m\right)$.

Theorem 0.11. *Let* $s \in S$ *be a strategy profile. Then the following are equivalent:*

1. *s is a perfect equilibrium;*

2. *s is a perfect-2 equilibrium;*

3. $\exists\left(s_n\right) \in S^\infty$ *s.t. (i)* s_n *is fully mixed, (ii)* $s_n \to s$ *and (iii)* $\forall i, n \quad s^i \in BR_{s^i}^i\left(s_n^{-i}\right)$.

Theorem 0.12. *If a mixed-strategy profile* $s \in S$ *is a perfect equilibrium of* Γ, *then it is a Nash equilibrium of* Γ, *i.e.* $s \in NE(\Gamma)$.

Proof. Since s is a perfect equilibrium, by Theorem 2.11 there exists a sequence $\left(s_n\right) \in S^\infty$ such that s_n is fully mixed for each $n \in \mathbb{N}, s_n \to s$, and $s^i \in BR^i\left(s^i, s_n^{-i}\right) \forall i \in I$. Fix $i \in I$ arbitrarily and define a sequence $\left(s_m\right) \in S^\infty$ by $s_m^i = s^i$ for all $m \in \mathbb{N}$ and $s_m^{-i} = s_n^i$ for all $m = n$. Then $s_m \to s$ and $s_m^i \in BR^i\left(s_m\right) \forall m \in \mathbb{N}$. Since $BR^i(\cdot)$ is uhc by Theorem 2.3 there exists a strictly increasing sequence $\left(m_k\right) \in \mathbb{N}^\infty$ such that $s_{m_k}^i \to r^i \in BR^i(s)$. But since every subsequence of $\left(s_m^i\right)$ is the stationary sequence $\left(s^i\right)$, we necessarily have $r^i = s^i$. Hence, $s^i \in BR^i(s)$. Since i has been arbitrary, $s \in NE(\Gamma)$ □

Theorem 0.13. *If a strategy profile* $s \in S$ *is a perfect equilibrium then it is undominated.*

Proof. Let $s \in S$ be a perfect equilibrium and suppose s is weakly dominated. Then $\exists i \in I, r^i \in S^i$ such that

$$\begin{array}{ll} \forall a^{-i} \in A^{-i}, & u^i\left(r^i, a^{-i}\right) \geq u^i\left(s^i, a^{-i}\right) \\ \exists b^{-i} \in A^{-i}, & u^i\left(r^i, b^{-i}\right) > u^i\left(s^i, b^{-i}\right) \end{array}$$

Since s is a perfect equilibrium, by Theorem 2.11$\exists\left(s_n\right) \in S^{\infty}$ s.t. $\forall n s_n$ is fully mixed, $s_n \rightarrow s$ and $\forall(i, n) s^i \in BR^i\left(s^i, s_n^{-i}\right)$. Since s^n is fully mixed for each $n \in \mathbb{N}, \operatorname{Pr}_{s_n}\left(a^{-i}\right)>0$ for all $a^{-i} \in A^{-i}$ By multiplying (6) – (7) by $\operatorname{Pr}_{s_n}\left(a^{-i}\right)$ and summing across A^{-i} we have

$$u^i\left(r^i, s_n^{-i}\right)=\sum_{a^{-i} \in A^{-i}} u^i\left(r^i, a^{-i}\right) \operatorname{Pr}_{s_n}\left(a^{-i}\right)>\sum_{a^{-i} \in A^{-i}} u^i\left(s^i, a^{-i}\right) \operatorname{Pr}_{s_n}\left(a^{-i}\right)=u^i\left(s^i, s_n^{-i}\right)$$

for each $n \in \mathbb{N}$. But this contradicts s^i being a best response to s_n^{-i} for all $n \in \mathbb{N}$. Hence, s is undominated. □

Theorem 0.14. *In a 2-player, finite NFG, $aN.E. \hat{s} \in S$ is undominated if, and only if, it is a perfect equilibrium.*

Proof. $\Longleftarrow$ Proved above. $\Longrightarrow$ Let $\hat{s}$ be an undominated N.E. By Lemma 2.16$\hat{s}^i$ is a best response to a fully mixed strategy of player $-i$. Let $\tilde{s}^{-i}$ denote this strategy profile for player $-i$; that is, $\hat{s}^i \in BR^i\left(\hat{s}^i, \tilde{s}^{-i}\right)$. Since $\hat{s}$ is a N.E., $\hat{s}^i \in BR^i(\hat{s})$. For any $\varepsilon>0$, the strategy profile $(1-\varepsilon) \hat{s}^{-i}+\varepsilon \tilde{s}^{-i}$ is also fully mixed and $\hat{s}^i \in BR^i\left[(1-\varepsilon) \hat{s}^{-i}+\varepsilon \tilde{s}^{-i}\right]$ since $\hat{s}$ is undominated. Take any $\left(\varepsilon_n\right) \in \mathbb{R}^{\infty}$ with $\varepsilon_n \rightarrow 0$. Then for each $i \in I,\left(1-\varepsilon_n\right) \hat{s}^{-i}+\varepsilon_n \tilde{s}^{-i} \rightarrow \hat{s}^{-i}$ and $\hat{s}^i \in BR^i\left[\left(1-\varepsilon_n\right) \hat{s}^{-i}+\varepsilon_n \tilde{s}^{-i}\right]$. Define $s_n=\left(1-\varepsilon_n\right) \hat{s}+\varepsilon_n \tilde{s}$ where $\hat{s}=\left(\hat{s}^1, \hat{s}^2\right)$ and $\tilde{s}=\left(\tilde{s}^1, \tilde{s}^2\right)$. Then $\left(s_n\right)$ is a sequence of fully mixed strategy profiles with $s_n \rightarrow \hat{s}$ and, $\forall i \in I, n \in \mathbb{N}, \hat{s}^i \in BR^i\left(s_n\right)$. Thus, by Theorem 2.11, $\hat{s}$ is a perfect equilibrium. □

Question 3 [Midterm 2017]

As in question 1, fix the set of players and the action sets.

a) Identify the set of all transformations of the utility function of a player that leave the Best Response correspondence unchanged. Please note you will state and prove statement like "The set of transformations T (put the definition here) of the utility functions of player 1 is such that for all normal formal games with that player set and action sets the Best Response for the utility function is u^1 and for its transformation $T\left(u^1\right)$ are the same."

b) Prove your answer in detail.

Solution 3

We look at positive affine transformations

$$v^i(a^i,a^{-i}) = Au(a^i,a^{-i}) + B$$

Observe that :

$$v^i(s^i,s^{-i}) = \sum_{a^i \in A^i} s^i(a^i) \sum_{a^{-i} \in A^{-i}} PR_{s^{-i}}(a^{-i}) v^i(a^i,a^{-i}) =$$

$$= \sum_{a^i \in A^i} s^i(a^i) \sum_{a^{-i} \in A^{-i}} PR_{s^{-i}}(a^{-i}) Au(a^i,a^{-i}) + B =$$

$$= A \sum_{a^i \in A^i} s^i(a^i) \sum_{a^{-i} \in A^{-i}} PR_{s^{-i}}(a^{-i}) u(a^i,a^{-i}) + B \sum_{a^i \in A^i} s^i(a^i) \sum_{a^{-i} \in A^{-i}} PR_{s^{-i}}(a^{-i}) =$$

$$= Au^i(s^i,s^{-i}) + B$$

Next, BR set remains the same

$$u^i(s^i,s^{-i}) \geq u^i(t^i,s^{-i}) \quad \forall s^i \in BR^i(s) \forall t^i \in S^i$$

$$A \cdot u^i(s^i,s^{-i}) + B \geq A \cdot u^i(t^i,s^{-i}) + B \quad \forall s^i \in BR^i(s) \forall t^i \in S^i$$

$$v^i(s^i,s^{-i}) \geq v^i(t^i,s^{-i}) \quad \forall s^i \in BR^i(s) \forall t^i \in S^i$$

We can be more general than above (but this is for extra credit):

Consider two games $G = \left(I, (S_i)_{i \in I}, (u_i)_{i \in I}\right)$ and $\hat{G} = \left(I, (S_i)_{i \in I}, (\hat{u}_i)_{i \in I}\right)$ so that the set of players I is the same and the set of pure strategies S_i is also the same for every $i \in I$. For every i, suppose that $\hat{u}_i\left(s_i, s_{-1}\right) =$

$a_i u_i(s_i, s_{-i}) + b_i(s_{-i})$ for every s_{-i} where $a_i > 0$ and $b_i : S_{-i} \to \mathbb{R}$. For the rest of argument, consider a specific $i \in I$. Observe that, for any belief $\mu_{-i} \in \Delta S_{-i}$,

$$\begin{aligned}
\hat{u}_i(s_i, \mu_{-i}) &= \sum_{s_{-i} \in S_{-i}} \hat{u}_i(s_i, s_{-i}) \mu_{-i}(s_{-i}) \\
&= \sum_{s_{-i} \in S_{-i}} [a_i u_i(s_i, s_{-i}) + b_i(s_{-i})] \mu_{-i}(s_{-i}) \\
&= a_i \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \mu_{-i}(s_{-i}) + \sum_{s_{-i} \in S_{-i}} b_i(s_{-i}) \mu_{-i}(s_{-i}) \\
&= a_i \sum_{s_{-i} \in S_{-i}} u_i(s_i, \mu_{-i}) + \sum_{s_{-i} \in S_{-i}} b_i(s_{-i}) \mu_{-i}(s_{-i})
\end{aligned}$$

Note that the second term in above only depends on μ_{-i}, and therefore for any $s_i, s_i' \in S_i$, from above we know that

$$\hat{u}_i(s_i, \mu_{-i}) - \hat{u}_i(s_i', \mu_{-i}) = a_i [u_i(s_i, \mu_{-i}) - u_i(s_i', \mu_{-i})]$$

Since $a_i > 0$, from above we know

$$\hat{u}_i(s_i, \mu_{-i}) \geq \hat{u}_i(s_i', \mu_{-i}) \Leftrightarrow u_i(s_i, \mu_{-i}) \geq u_i(s_i', \mu_{-i})$$

Note that here I take into account correlated beliefs, but this proof is just as fine if we only consider independent beliefs that we uses to define best response in class. By last equation, we now that the best response in pure strategy, $BR_i(\mu_{-i})$, are the same for the two game G and $\hat{G}$. But recall that,

$$BR_i^*(\mu_{-i}) = \text{co}\{\delta_{s_i} : s_i \in BR_i(\mu_{-i})\}$$

where $BR_i^*(\mu_{-i})$ is the best response in mixed strategies. So we must also have $BR_i^*(\mu_{-i})$ the same.

Question 4 [Midterm 2017]

a) Consider the utility function of player 1 :

	l	c	r
T	7	5	1
M	1	4	3
B	4	1	7

What are the correlated strategies for which player 1 will want to follow the recommended action for each of his actions?

b) What are the set of correlated equilibria of a zero-sum game?

Solution We didn't cover correlated equilibria

Question 1 [Midterm 2018]

You are told that the following game: has a unique equilibrium in mixed strategies. What are

	L	R
T	(a, b)	(c, d)
B	(e, f)	(g, h)

the conditions on the utilities $\{a, b, \ldots\}$ that are necessary and sufficient for this statement to be true?

Solution It was a homework question

Question 2 [Midterm 2018]

a) State and prove precisely the relation between the two best response correspondences of player i, $\text{BR}^i_{A^i}$ and $\text{BR}^i \equiv \text{BR}^i_{\Delta(A^i)}$

b) Prove that the two correspondences are closed valued.

c) Consider the case when the action sets of all players A^i are countable. Can you identify conditions on the utility functions of the players so that the best response correspondence $\mathrm{BR}^i_{\Delta(A^i)}$ is non-empty valued, convex and upper-hemicontinuous?

Solution We did it in class

Question 3 [Midterm 2018]

a) Prove that the set of Nash equilibria of a normal form game with finite actions is closed.

b) Show that the set of Nash equilibria of a normal form game with finitely many players may be empty if one of the players has a countably infinite set of actions.

Solution Check Q1 2017

Question 4 [Midterm 2018]

a) Define a Perfect Equilibrium of a normal form game as the limit of sequences of Nash equilibria of a perturbed game.

b) Prove that a perfect equilibrium is a Nash equilibrium.

c) A strategy is fully mixed if it gives positive probability to all the actions in the action set of the player, and a strategy profile is fully mixed if the strategies of all players are fully mixed. Prove that a fully mixed strategy profile is perfect.

d) Give as simple sufficient condition for a Nash equilibrium to be perfect.

Solution Check Q2 2017

Question 1 [Midterm 2019]

The outcome of an Iterated Eliminations of Strictly Dominated Strategies (IESDS) in a normal form game is unique. In contrast, the outcome of an Iterated Eliminations of Weakly Dominated Strategies (IEWDS) is not necessarily unique, and depends on the consequence of eliminations. A crucial lemma in the proof of the statement on IESDS states, informally, that if an action can be eliminated at an earlier stage in an IESDS, then it can also be eliminated at a later stage.

a) Give a precise statement of the lemma.

b) Clearly, the statement of the lemma is false when we consider IEWDS. State clearly where in the proof of the lemma use is made of the fact that dominance is strict in the elimination procedure, and why the step fails in the IEWDS case.

Question 2 [Midterm 2019]

The following result is used in the proof of the existence of Nash equilibria. Let for $i = 1, \ldots, n$ $F^i : S \rightrightarrows S^i$ be a correspondence from S to S^i, where $S^i \equiv \Delta\left(A^i\right)$ as usual, and let F be the product of the correspondences.

a) Prove that F is uhc if each F^i is;

b) Prove that F is closed valued if each F^i is;

c) Prove that F is convex valued if each F^i is;

d) Is the result true when the action set is countable?

Solution 2

Theorem 0.15 (Existence of Nash Equilibrium 1950). *The correspondence $BR : S \rightrightarrows S$ defined by $BR(s) = X_{i\in I} BR^i(s)$ is*
(1) nonempty-valued
(2) closed-valued
(3) convex-valued
(4) upper hemicontinous. Thus by Kakutani fixed point theorem it has fixed point $s \in BR(s)$.

Proof. Fix $s = \left(s^1, s^2, \ldots, s^n\right) \in S$ arbitrarily.
(1) BR maps s into the set $BR^1(s) \times BR^2(s) \times \cdots \times BR^n(s)$. Since each $BR^i(s), i \in I$, is nonempty and I is finite, we can choose an element $r^i \in BR^i(s)$ for each $i \in I$. Then $\left(r^1, r^2, \ldots, r^n\right) \in BR^1(s) \times \cdots \times BR^n(s) = BR(s)$. Then, since s has been arbitrary, $BR(s)$ is nonempty for all $s \in S$. Hence, BR is nonempty-valued.

(2) Take any $r_a, r_b \in BR(s)$ and $\lambda \in (0,1)$. Then

$$\lambda r_a + (1-\lambda) r_b = \left(\lambda r_a^1 + (1-\lambda) r_b^1, \ldots, \lambda r_a^n + (1-\lambda) r_b^n\right)$$

Since for each $i \in I$ the set $BR^i(s)$ is convex, $\lambda r_a^i + (1-\lambda) r_b^i \in BR^i(s) \forall i \in I$. Then $\lambda r_a + (1-\lambda) r_b \in BR(s)$ and, hence, $BR(s)$ is a convex set for all $s \in S$, i.e., BR is convex- valued.

(3) Take any point $v = \left(v^1, \ldots, v^n\right) \notin BR(s)$. Then for some $i \in I, v^i \notin BR^i(s)$. Since $BR^i(S)$ is closed in S^i, v^i is not a limit point of $BR^i(s)$. That is, there exists an open set ${}^1U^i \subset S^i$ containing v^i that contains no more than a finite number of points of $BR^i(s)$. Now, $\forall j \neq i$, choose any $U^j \subset S^j$. Then the neighborhood $U = X_{i\in I} U^i$ of v contains no more than a finite number of points of $BR(s)$, i.e. v is not a limit point of $BR(s)$. Since v has been arbitrary, for all $v \notin BR(s) v$ is not a limit point of $BR(s)$, which implies that $BR(s)$ contains all of its limit points and is, hence, closed in S.
Since $S^i \subset \mathbb{R}_+^{m_i}, \forall i \in I$, where m_i is the cardinality of A^i, I consider each S^i as a metric subspace of $\mathbb{R}^{m_i}$ with the Euclidean metric. Then $S = \mathrm{X}_{i\in I} S^i$ is considered as a metric subspace with the usual product metric.

(4) Take any sequences $\left(s_m\right), \left(r_m\right) \in S^\infty$ such that $s_m \rightarrow s$ and $r_m \in BR\left(s_m\right) \forall m$.[2] Then for all $i \in I, \left(s_m^i\right), \left(r_m^i\right) \in S^{i\infty}, s_m^i \rightarrow s^i$, and $r_m^i \in BR^i\left(s_m\right) \forall m$. Since BR^i is u.h.c., this implies that there exists a subsequence $r_{m_k}^i \rightarrow r^i \in BR^i(s)$. Then the sequence $r_{m_k} = \left(r_{m_k}^1, \ldots, r_{m_k}^n\right)$ of r_m converges to $r = \left(r^1, \ldots, r^n\right) \in BR^1(s) \times \cdots \times BR^n(s)$. Hence, BR is upper hemicontinuous ☐

Question 3 [Midterm 2019]

Provide an example of a finite normal form game where the set of Nash equilibria

a) (1) is not finite and (2) for no player the set of mixed strategies used in some Nash equilibrium is the entire set of mixed strategies of the player. The condition (2) is important so we write it explicitly as:

b) if and only if $\nexists i \in I$ such that $\cup_{s=(s^i,\ldots,s^i,\ldots,s^n)\in NE} s^i = \delta\left(A^i\right)$ In particular you cannot use the game where all actions profiles give the same utility.

Solution 3

	L	R
T	0,0	1,0
B	1,3	0,1

can solve for the best responses as follows: Mr 1 best response:

$$BR^1\left((q, 1-q)\right) : \left\{ \begin{array}{cc} T & B \\ 0\,(q) + 1\,(1-q) & 1\,(q) + 0\,(1-q) \end{array} \right\}$$

Equality only holds when $q = \frac{1}{2}$. $T > B \iff q < \frac{1}{2}$. $T < B \iff q > \frac{1}{2}$ Therefore, player 1 sets $p = 1$ if $q < \frac{1}{2}$ and sets $p = 0$. She picks $p \in [0,1]$ where is indifferent between T and B. otherwise.

$$BR^1\left((q, 1-q)\right) = \begin{cases} 0 & \text{if } q > \frac{1}{2} \\ [0,1] & \text{if } q = \frac{1}{2} \\ 1 & \text{if } q < \frac{1}{2} \end{cases}$$

Mr 2 best response:

$$BR^2\left((p,1-p)\right):\left\{\begin{array}{cc} L & R \\ 0\cdot p+3\cdot(1-p) & 0\,(p)+1\,(1-p)\end{array}\right\}$$

Equality only holds when $p=1$. $L>R \iff p<1$,$L\sim R \iff p=1$ Similarly, player 2 sets $q=1$ if $p<1$ and sets $q=0$ otherwise.

$$BR^2\left((p,1-p)\right)=\begin{cases}1 & \text{if } p<1\\ [0,1] & \text{if } p=1\end{cases}$$

These best responses can be graphed :

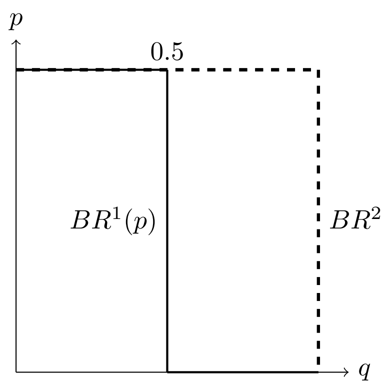

Figure 1: Best Responses

The points of interesection

$$(q,1)\,,\forall 0\leq q\leq\frac{1}{2},\quad (1,0)$$

yield the set of Nash equilibria

$$\text{NE}=\left\{\big((0,1),(1,0)\big),\big((1,0),(q,1-q)\big),\forall 0\leq q\leq\frac{1}{2}\right\}.$$

which is uncountable and no player uses the entire mixed strategy set

Question 4 [Midterm 2019]

In the following, fix the set of players I and the action set A^i of each player $i\in I$ for all normal form games considered.

a) Identify the set of all transformations of the utility function of a player that leave the Best Response correspondence in mixed strategies (this is the correspondence that we have denote $BR^i_{S^i}$ for each player unchanged.

b) Prove your answer in detail.

Solution Check Q3 2017

Question 1 [Midterm 2020]

Consider a normal form game with finitely many players and actions. Denote $\mathcal{U}$the set of all vNM utility representations (taking A to be the set of consequences) for player i and $\mathcal{U}\equiv\times_{i=1}^{n}\mathcal{U}^i$, the Cartesian product over the set of players

a) Prove that the correspondence from $\mathcal{U}$ to the set of Nash equilibria is well defined

b) Prove the correspondence is upper-hemi-continuous.

Solution Check Q1 2017

Question 2 [Midterm 2020]

Find all the solutions obtained by iterated elimination of weakly dominated strategies in the game:

	L	C	R
T	$1,2$	$2,3$	$0,3$
M	$2,2$	$2,1$	$3,2$
B	$2,1$	$0,0$	$1,0$

Question 3 [Midterm 2020]

The Beauty Contest is the following normal form game:

1. A set I of n players; the set $A^i \equiv \{0, 1, \dots, 100\}$ (non-negative integers) for every $i \in I$;
2. The payoff is as follows.
   (a) For any $a \in A$, define $M(a)$ as the average value:
   $$M(a) \equiv \sum_{i=1}^{n} \frac{a^i}{n}$$
   and let $\theta \in (0, 1)$ be fixed (for example, $\theta = 2/3$);
   (b) The player whose a^i is the unique closest to $M(a)\theta$ (note: the average times the θ factor). wins a desirable prize; the others get nothing.
   (c) If a tie at the closest position occurs, no one gets anything.
   (d) Does the value of θ matter, and if so how?

In this game:

1. Find the set of Nash equilibria.
2. Apply iterated elimination of strictly dominated strategies. What do you get?