Recitations 20

[Definitions used today]

- SPE, Backward Induction, Behavioral Strategies, Linear Game, Perfect Recall, Dalkey and Kuhn Theorems

Question 1 [84 III.1 Spring 2009 majors]

An extensive form game (EFG) is said to be linear if every information set is crossed at most once by every history.

- Give an example of an EFG which is not linear.
- Give an example of EFG that is linear but not of perfect recall.
- Compare linear games and games with perfect recall. Is one of the two a subset of the other? Prove your answer.

Solution 1

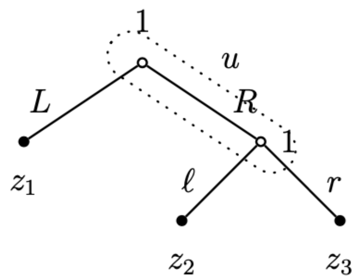

Figure 1: Non linear EFG

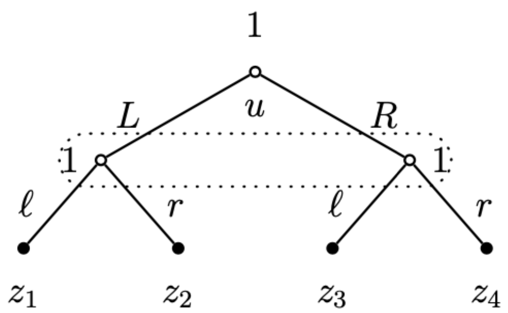

Figure 2: Linear game but not of perfect recall

Theorem 0.1. *Every game of perfect recall is linear.*

Proof. Suppose not, there is some perfect recall EFG that is not linear. Then we know there exists some $i \in I$ and $I_k^i \in \mathcal{I}^i$ such that for some $z \in Z, \# \left\{ P(z) \cap I_k^i \right\} > 1$.

Now take $x, y \in P(z) \cap I_k^i$ such that $x \succeq_c y$ for some action $c_k^i \in C_k^i$, i.e. x follows c_k^i but y does not. Now let $I_l^i = I_k^i$ and $y' = y$.

Then clearly there exists $c_k^i \in C_k^i$ such that $x \succeq_c y'$ but not $y \succeq_c y'$, since a node cannot come after itself. Therefore the game is not of perfect recall, which the hypothesis. $\square$

Question 2 [32 and 45 IV.2 Spring 2006 III.1 Spring 2007 majors]

Consider extensive form games that are finite (that is, that have a finite set of nodes).

a) Give an example to show that in an extensive form game a behavioral strategy may not havean equivalent mixed strategy

b) Define an extensive form linear game.

c) Prove that for any linear game, any player in the game, and any behavioral strategy of the player there is a mixed strategy of the same player that induces the same probability distribution on final nodes for any pure strategy of the other players.

d) Give an example to show that in a linear game for a mixed strategy of the player there may be no behavioral strategy that induces the same distribution on final nodes for some pure strategy of the other players.

Solution 2

a)

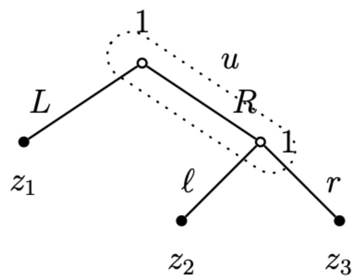

Figure 3: Non linear EFG

Player one forgot he has moved and forget what choice he made

$$C^1_{I_1} = \{L, R\}$$
$$S^1\{L, R\}$$
$$\Sigma^i = \Delta S^1 = (x, 1-x)$$
$$B^i = (p, 1-p)$$
$$Pr_{\sigma^1} = \{x, 0, 1-x\}$$
$$Pr_{\beta^1} = \{p, (1-p)p, (1-p)\cdot(1-p)\}$$

We can not find any mixed strategy to this as long as $p \in (0,1)$

b)

Definition 0.2 (Linear game)**.** *An EFG is linear if no information set intersects a path more than once, i.e.*

$$\forall i \in I, \forall I^i_k \in \mathcal{I}^i, \forall z \in Z, \#\left\{P(z) \cap I^i_k\right\} \leq 1$$

[Note: Intuitively, every player always knows if they've moved or not.

Definition 0.3 (Games of perfect recall (PR))**.** *An EFG is perfect recall if and only if $\nexists I^i_k, I^i_l \in \mathcal{I}^i, x, y \in I^i_l$ such that x follows some $c^i_k \in C^i_k$ but y does not. [Alternatively, $\nexists I^i_k, I^i_l \in \mathcal{I}^i, x, y \in I^i_l, w \in I^i_k, c^i_k \in C^i_k$, such that $x \succeq_c w$ but not $y \succeq_c w$.*

[Note: In general, for linear games not of perfect recall, $\{\Pr_\sigma \mid \sigma \in \Sigma\} = \Delta(z)$ but $\{\Pr_\beta \mid \beta \in B\} \subset \Delta(z)$ and so $\forall \sigma^i \in \Sigma^i, \nexists \beta^i \in B^i$ such that $\beta^i \sim \sigma^i$.]

Definition 0.4 (Relevant information sets)**.** *The set of pure strategies for player i that lead to $I^i_k \in \mathcal{I}^i$ for some given strategy $s^{-i} \in S^{-i}$ of the other players, i.e. the set of pure strategies relevant for I^i_k, is:*

$$\mathrm{Rel}\left(I^i_k\right) = \left\{s^i \in S^i \mid \exists s^{-i} \in S^{-i}, \Pr_{(s^i, s^{-i})}\left(\left\{z \in Z \mid \Pr(z) \cap I^i_k \neq \emptyset\right\}\right) > 0\right\}$$

Further, the set of pure strategies relevant for I_k^i that play action $c \in C_k^i$ is:

$$\operatorname{Rel}\left(I_k^i, c\right)=\left\{s^i \in \operatorname{Rel}\left(I_k^i\right) \mid s^i\left(I_k^i\right)=c\right\} \subseteq \operatorname{Rel}\left(I_k^i\right)$$

c)

Theorem 0.5 (Dalkey)**.** *In any linear EFG, for any behavioral strategy $\beta^i \in B^i$ there is a mixed strategy $\sigma^i \in \Sigma^i$ such that $\beta^i \sim \sigma^i$.*

Proof. Consider any linear EFG and note that $\forall i, \forall \beta^i \in B^i$ it is possible to construct $\sigma_{\beta^i}^i\left(s^i\right)=\prod_{k=1}^{K^i} \beta_k^i\left(s^i\left(I_k^i\right)\right)$. Further note that, clearly, $\sigma_{\beta^i}^i\left(s^i\right) \in[0,1] \forall s^i \in S^i$. Since the game is linear, we know each path intersects each information set only once, and thus $\sum_{s^i \in S^i} \sigma_{\beta^i}^i\left(s^i\right)=\sum_{s^i \in S^i} \prod_{k=1}^{K^i} \beta_k^i\left(s^i\left(I_k^i\right)\right)=1$, so the constructed $\sigma_{\beta^i}^i$ is a mixed strategy.

Now take any $z \in Z$ and any $\pi^{-i} \in \Gamma^{-i}$. Consider first the cases where z is always reached or z is never reached, regardless of player i 's actions. In these cases, $\operatorname{Pr}(z)=1$ and $\operatorname{Pr}(z)=0$, respectively, for any $\beta^i \in B^i$ and for any $\sigma^i \in \Sigma^i$, so $\beta^i \sim \sigma_{\beta^i}^i$ trivially. Consider now the case where $\operatorname{Pr}(z) \in(0,1)$ and depends on player i 's actions. Define $\tilde{c}_{I_k^i}^i(z)$ as the action of player i at information set I_k^i that leads to final node z and $\tilde{I}^i(z) \equiv \left\{I_k^i \in \mathcal{I}^i \mid P(z) \cap I_k^i \neq \emptyset\right\}$ as the set of player i 's information sets in the path of z. Then the probability on z induced by β^i is $\operatorname{Pr}_{\left(\beta^i, \pi^{-i}\right)}(z)=\prod_{I_k^i \in \tilde{I}^i(z)} \beta_k^i\left(\tilde{c}_{I_k^i}^i(z)\right)$. Now define $\tilde{S}^i(z)$ as the set of player i 's pure strategies that result in z, i.e. $\forall s^i \in \tilde{S}^i(z), \forall I_k^i \in \tilde{I}^i(z), c_k^i=\tilde{c}_k^i(z)$. Then the probability on z induced by $\sigma_{\beta^i}^i$ is:

$$\begin{aligned}
\operatorname{Pr}_{\left(\sigma^i, \pi^{-i}\right)}(z) & =\sum_{s^i \in \tilde{S}^i(z)} \sigma_{\beta^i}^i\left(s^i\right)=\sum_{s^i \in \tilde{S}^i(z) I_k^i \in \mathcal{I}^i} \beta_k^i\left(s^i\left(I_k^i\right)\right) \\
& =\sum_{s^i \in \tilde{S}^i(z)} \prod_{I_k^i \in \tilde{I}^i(z)} \beta_k^i\left(\tilde{c}_{I_k^i}^i(z)\right) \prod_{I_k^i \notin \tilde{I}^i(z)} \beta_k^i\left(s^i\left(I_k^i\right)\right) \\
& =\prod_{I_k^i \in \tilde{I}^i(z)} \beta_k^i\left(\tilde{c}_{I_k^i}^i(z)\right) \quad\left(\text { since } \forall s^i \in \tilde{S}^i(z), \forall I_k^i \in \tilde{I}^i(z), s_k^i=\tilde{c}_k^i(z)\right)=\operatorname{Pr}_{\left(\beta^i, \pi^{-i}\right)}(z)
\end{aligned}$$

Thus σ^i and β^i induce the same probability on z, and this is true $\forall z \in Z$. Thus $\beta^i \sim \sigma_{\beta^i}^i$ □

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d)

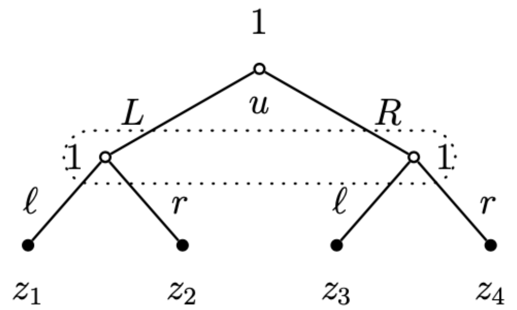

Figure 4: Linear game but not of perfect recall

$$S^1\{L, R\} \times\{l, r\}$$
$$\Sigma^i=\Delta S^1=\left(p_1, p_2, p_3, 1-p_1-p_2-p_3\right)$$
$$B^i=(p, 1-p) \times(q, 1-q)=\{p q, p(1-q),(1-p)(1-q),(1-p) q\}$$

Consider

$$Pr_{\sigma^1}=\left\{\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}\right\}$$
$$\nexists Pr_{\beta^1}=\{p,(1-p) p,(1-p) \cdot(1-p)\}$$

We can not find any behavioral strategy to this mixed strategy.

Question 3

Find all SPE and NE of following games

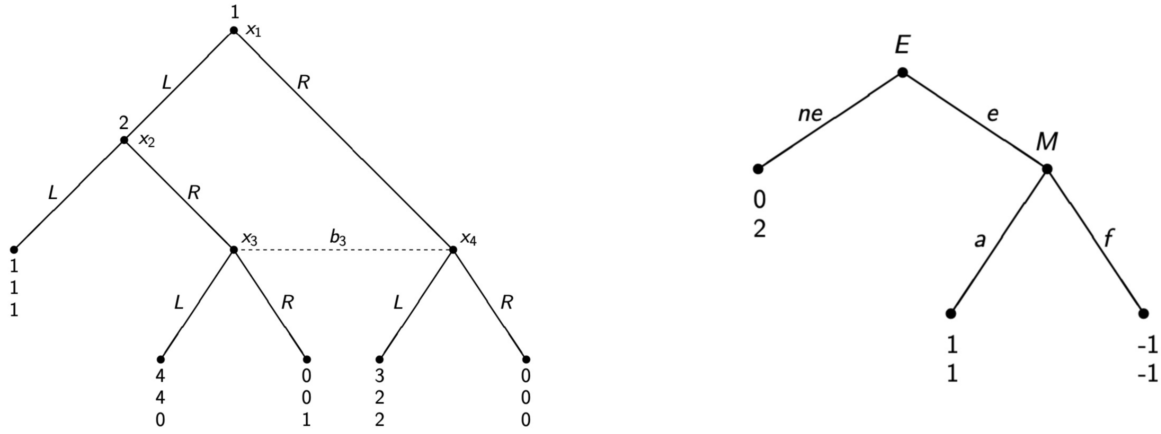

Figure 5

Solution 3

b)

E/M	a	f
ne	0,2	0,2
e	1,1	-1,-1

Two Nash pure equilibria: (ne, f), (e, a). And we have mixed too

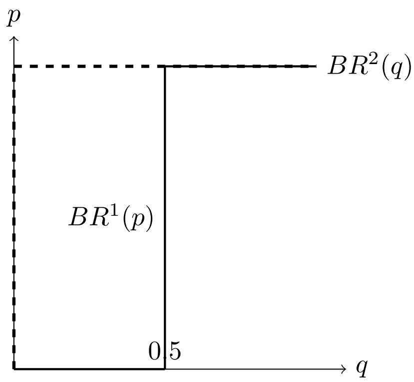

Figure 6: Best Responses

yield the set of Nash equilibria

$$\text{NE} = \left\{ \big((0,1),(0.1)\big), \big((1,0),(q,1-q)\big), \forall 0 \leq q \geq \frac{1}{2} \right\}.$$

- Consider the Nash equilibrium (ne, f).

Entrant plays not to enter (ne) and the telephone rings: Monopolist, it is your turn!!! M's information set is out of equilibrium path. To play f was part of the optimal behavior because this information set was not reached. Strategy gives plan for moves in all information sets, even though some of them won't be reached. The threat of playing f is what makes optimal for the entrant to play ne.

OK but, f is a non-credible threat, so E should not believe that M will play f (it is not rational for him) if he plays e. **Subgame Perfect Equilibrium** requires rational behavior even in information sets that are not reached in equilibrium (equilibrium should not be based on incredible threats).

- Obtain (e,a) as the unique Subgame Perfect Equilibrium (which coincides with the one obtained by backwards induction).

(e,a) could be obtained also in the normal form as applying the principal ìn ever a dominated strategy since f is dominated by a.
But this is not always true.

(e,a) is the unique SPE of Γ and (ne,f) is not a SPE since f is not a Nash equilibrium of the subgame starting at the unique node that belongs to M.

In general, SPE requires two different things:

- SPE gives a solution everywhere (in all subgames), even in subgames where the solution says that they will not be reached (information sets with zero probability).
- SPE imposes rational behavior everywhere, even in the subgames of the game that SPE says that cannot be reached. In out-of-equilibrium subgames, the solution is disapproved, yet players evaluate their actions taking as given the behavior of the other players, that have been demonstrated incorrect since we are in an out-of-equilibrium path.

a) This game is called **Selten Horse example 1975** Players are not absolutely perfect, there exists a possibility (that it may be very small) that they make mistakes when computing the optimal behavior, when implementing their strategies, etc. Rationality should be understood as the limit of a process where mistakes tend to disappear

Remark: Since there is only one subgame (the game itself), all Nash equilibria are SPE.
Critical point: behavior out-of-equilibrium path.
Let p_i be the probability that player i chooses L.
There are two types of equilibria:

- Type1: $p_1 = 1$,$p_2 = 1$ and $p_3 \in [0, \frac{1}{4}]$.
- Type2: $p_1 = 0$,$p_2 \in [\frac{1}{3}, 1]$ and $p_3 = 1$.

Consider first the particular equilibrium of type 2: $(p_1, p_2, p_3) = (0, 1, 1)$.The same argument will work for all other equilibria of type 2, but it will be less transparent.
Given $p_1 = 0$ and $p_3 = 1$, is it reasonable to think that player 2 will play $p_2 = 1$?
NO. Or is (0,1,1) an stable agreement? Suppose they agree on playing (0,1,1)
- Player 2 arrives home (he does not have to play) but suddenly, the telephone rings and says: "It is your turn, decide between L and R''.
- He knows that he is at x_2 (player 1 did a mistake), but given $p_3 = 1$, player 2 cannot play $p_2 = 1$ but rather he has to play R.

Type 2 equilibria are not sensible since they disappear as soon as there is a probability that players make mistakes when implementing their strategies. Consider now the type 1 equilibrium $(p_1, p_2, p_3) = (1, 1, 0)$ Now, suppose player 3 is called to play (an out-of-equilibrium play).

$p_3 = 0$ is still rational since he can be either at x_3 or at x_4 (the mistake may come from either player 1 or player 2). Even with a probability of mistakes. (1,1,0) is still rational.

Remeber last class:

- There are games for which the set of perfect equilibria is an strict subset of the set of subgame perfect equilibria.
- Type 2 equilibria of the horse game are subgame perfect but not perfect.
- Type 1 equilibria of the horse game are perfect equilibria.

Theorem 0.6 (Selten, 1975)**.** *Let $\hat{\sigma}$ be a perfect equilibrium of Γ. Then, $\hat{\sigma}$ is a subgame perfect equilibrium of Γ.*

Proof. (idea):- Along the sequence, players behave rationally except for the fact that all choices have to receive strictly positive probability. - Since payoff functions are continuous, in the limit also rational behavior is required, even in information sets that are out-of-equilibrium play.

To prove theorem I need three lemmas:

Lemma 0.7. *Every equilibrium $\hat{\sigma} \in \hat{\Sigma}(\varepsilon)$ of $\Gamma(\varepsilon)$ is a subgame perfect equilibrium of $\Gamma(\varepsilon)$*

Lemma 0.8. *Let $\hat{\sigma} \in \hat{\Sigma}$ be a perfect equilibrium of Γ. Then, in each subgame $\Gamma_x, \hat{\sigma}^x$ is a perfect equilibrium of Γ_x.*

Lemma 0.9. *Let $\hat{\sigma}^* \in \hat{\Sigma}$ be a perfect equilibrium of Γ. Then, $\hat{\sigma}^*$ is an equilibrium of Γ.*

Proof of the Theorem
Let $\hat{\sigma}$ be a perfect equilibrium of Γ.
By Lemma 0.9, for all subgames $\Gamma_x, \hat{\sigma}^x$ is a perfect equilibrium of Γ_x.
By Lemma 0.8 $\hat{\sigma}^x$ is an equilibrium of Γ_x.
By the definition of subgame perfection, $\hat{\sigma}$ is a subgame perfect equilibrium of Γ. □

We just showed

$$PE \subseteq SPE.$$

$$PE \neq \varnothing$$

There exists a game Γ (Selten's horse game) such that

$$PE \subsetneq SPE$$

Be careful above I used Perfect equilibrium for extensive form game (as behavioral strategy). In general PE in extensive form game and in NFG indused by the same EFG maight be different.
The reason is that in the normal form trembles are correlated while in the extensive form trembles in different information sets are uncorrelated

Question 4 [Final 2019]

- Prove that for any finite EFG of perfect information, there is a last move node, that is a move node x such that $IS(x) \subseteq Z$.
- Prove, or disprove by showing a counter-example to the statement: In any finite EFG of perfect recall, there is a last information set I^i for some player i, that is, an information set such that for any node $x \in I^i, IS(x) \subseteq Z$.

Solution 4

1) Suppose not. Then $\exists$ node $y_1 \in IS(x), y_1 \notin Z$ and
Then $\exists$ node $y_2 \in IS(y_1), y_1 \notin Z$
...
$\forall n \quad \exists$ node $y_n \in IS(y_1), y_{n-1} \notin Z$ violates EFG beign finite.

2) Suppose not . Thn every info set I_F^i for each player i has a moving node
a) all these info sets belong to different player $\rightarrow$ infinite number of players violates finite EFG
or b) $\exists$ some player i his info set repeated shows up on some path, rename the info set as $I^i_{k_1}, \dots I^i_{k_n}, \dots$ by order of the path : $\exists y_1 \in I^i_{k_1}, c_1 \in C_{I^i_{k_1}}, y_2 \in I^i_{k_2}, c_2 \in C_{I^i_{k_2}}$ and $y_2 \succeq y_1$. By perfect recall (eliminate the possibility of having redundant structure): $\forall x \in I^i_{k_2}\ x \succeq y_1$ i.e. all the nodes of $I^i_{k_2}$ should come after info set $I^i_{k_1}$.
Some $I^i_{k_3}, \dots I^i_{k_n}, \dots$ means that i has infinite number of info sets which violates finite EFG.