Recitation 4

[Definitions used today]

- Topkis theorem, Supermodularity, Increasing Differences

Question 1

Suppose that a firm with production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ such that $f(0) = 0$ chooses its production plan $(x; z)$ at prices $w \in \mathbb{R}^n_{++}$ of inputs and $q \in \mathbb{R}_{++}$ of the output in such a way that minimizes the cost of producing z at prices w, and the marginal cost $\frac{\partial C^*}{\partial z}(w; z)$ equals the output price q:

a Under what conditions on f is the firm maximizing its production? Be as general as you can. Prove you answer.

b Suppose that cost function C^* is strictly concave in$z$. Show that the firm makes a loss (strictly negative profit) when following the marginal cost rule whenever the output is non-zero.

Solution 1

a) f concave $\to Y$ is convex so $\pi(p) \in \partial Y$ or f concave $\to C$ concave in z so $\pi(q, w) = \sup_{z \geq 0} qz - C(w, z)$ is convex (envelope)
b) strict concavity means stric convexity of profit. It implies minimization of profit in FOCs gives profit below zero. Or from concavity of C in z and envelope for profit

$$0 \leq C(w, 0) \leq C(w, z) - z \cdot \frac{\partial C^*}{\partial z}(w; z) \qquad \pi(p) \leq 0$$

Question 2 [Topkis theorem]

If S is a lattice, f is supermodular in x, and f has nondecreasing differences in $(x; t)$, then $\varphi^*(t) = \arg\max_{x \in S} f(x, t)$ is monotone nondecreasing in t.

Solution 2 Monotone comparative statics is based on mathematical theories of super-modularity and vector lattices developed by D.M. Topkis and others (see the book by Topkis (1998), Topkis (1978) or Milgrom Shannon (19).

0.1 Definitions

Definition 0.1. *For two vectors $x, y \in \mathbb{R}^n$, the **lattice operations** are the **supremum**:*

$$x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$$

*and the **infimum**:*

$$x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}).$$

Note: $x + y = x \vee y + x \wedge y$.

Definition 0.2. *A set $S \subseteq \mathbb{R}^n$ is said to be a **lattice** if $x \vee y \in S$ and $x \wedge y \in S$ for all $x, y \in S$.*

Interval $[a, b] \subset \mathbb{R}^n$ and $\mathbb{R}^n_+$ is a lattice.

Definition 0.3. *Let $X \subset \mathbb{R}^n$ be a lattice. A function $f : X \to \mathbb{R}$ is **supermodular** on X if*

$$f(x \vee y) - f(x) \geq f(y) - f(x \wedge y), \forall x, y \in X \tag{0.1}$$

Note: *An equivalent formulation:*

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y), \forall x, y \in X$$

TO DO: Draw Figure

Figure **insert reference** illustrates this definition for $n = 2$. For a production function or a utility function f, supermodularity is a form of complementarity among goods.

0.2 Nondecreasing maximizers and the Theorem of Topkis

Let X be a subset of $\mathbb{R}^n$. We will generally assume that either $X = \mathbb{R}^n$ or $X = \mathbb{R}^n_+$. Let $T \subseteq \mathbb{R}^m$. For a function $f : X \times T \to \mathbb{R}^n$ and a set $S \subseteq X$, consider the problem

$$\max_{x \in S} f(x,t)$$

Let the correspondence $\varphi^*(t)$ denote the set of solutions for a given t, i.e.,

$$\varphi^*(t) = \arg\max_{x \in S} f(x,t)$$

Definition 0.4. *$\leq_{sso}$ is the **strong set order** if for every $x \in \varphi^*(t)$ and $x' \in \varphi^*(t')$, $x \wedge x' \in \varphi^*(t)$ and $x \vee x' \in \varphi^*(t')$. Note that if $\varphi^*(t)$ and $\varphi^*(t')$ are singletons, the strong set order is the same as the usual order on vectors.*

*The correspondence φ^* is **monotone nondecreasing in** t if*

$$\varphi^*(t) \leq_{sso} \varphi^*(t'),\ \forall t \leq t'$$

Definition 0.5. *A function $f : X \times T \to \mathbb{R}$ has **nondecreasing differences in** $(x;t)$ if the difference $f(x',t) - f(x,t)$ is monotone nondecreasing in t for every $x' \geq x$, i.e.,*

$$f(x',t') - f(x,t') \geq f(x',t) - f(x,t),\ \forall x' \geq x,\ \forall t' \geq t.$$

Theorem 0.6. Topkis Theorem

If S is a lattice, f is supermodular in x, and f has nondecreasing differences in $(x;t)$, then $\varphi^*(t) = \arg\max_{x \in S} f(x,t)$ is monotone nondecreasing in t.

Proof. **Step 1:** Show that $x \vee x' \in \varphi^*(t')$

Let $t \leq t'$, let $x \in \varphi^*(t)$, and let $x' \in \varphi^*(t')$. First, we will show that $x \vee x' \in \varphi^*(t')$. Supermodularity in x implies that

$$f(x \vee x', t') \geq f(x',t') + f(x,t') - f(x \wedge x', t'). \tag{0.2}$$

Nondecreasing differences in $(x;t)$ implies

$$f(x,t') - f(x \wedge x', t') \geq f(x,t) - f(x \wedge x', t). \tag{0.3}$$

Since S is a lattice, $x \wedge x' \in S$. Using this and the fact that $x \in \varphi^*(t)$, we have

$$f(x,t) - f(x \wedge x', t) \geq 0 \tag{0.4}$$

Combining (0.2), (0.3), and (0.4), we get

$$f(x \vee x', t') - f(x',t') \geq 0$$

and so

$$f(x \vee x', t') \geq f(x',t') \tag{0.5}$$

Again, since S is a lattice, $x \vee x' \in S$. Since $x' \in \varphi^*(t')$, (0.5) implies that $x \vee x' \in \varphi^*(t')$.

Step 2: Show that $x \wedge x' \in \varphi^*(t)$.

Supermodularity in x implies that

$$f(x \wedge x', t) \geq f(x,t) + f(x',t) - f(x \vee x', t). \tag{0.6}$$

Nondecreasing differences in $(x;t)$ implies

$$f(x \vee x', t') - f(x',t') \geq f(x \vee x', t) - f(x',t). \tag{0.7}$$

We can rearrange this as

$$f(x',t) - f(x \vee x', t) \geq f(x',t') - f(x \vee x', t'). \tag{0.8}$$

Since S is a lattice, $x \vee x' \in S$. Using this and the fact that $x' \in \varphi^*(t')$, we have

$$f(x',t') - f(x \vee x', t') \geq 0$$

$$f(x',t') - f(x \vee x', t') \geq 0 \tag{0.9}$$

Combining (0.6) and (0.9), we get

$$f(x \wedge x', t) \geq f(x,t). \tag{0.10}$$

Again, since S is a lattice, $x \wedge x' \in S$. Since $x \in \varphi^*(t)$, (0.10) implies that $x \wedge x' \in \varphi^*(t)$. □

0.3 Characterization of supermodularity and non-decreasing differences

Supermodularity and nondecreasing differences in $(x;t)$ can be characterized using second-order derivatives.

Proposition 0.7. *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be twice-differentiable on an interval $(a,b) \subset \mathbb{R}^n \times \mathbb{R}^m$. Then*

[**(i)**] *f has nondecreasing differences in $(x;t)$ if and only if*

$$\frac{\partial^2 f}{\partial x_i \partial t_k}(x,t) \geq 0,\ \forall i,k,\ \forall (x,t) \in (a,b).$$

[**(ii)**] *f is supermodular in x if and only if*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x,t) \geq 0,\ \forall i \neq j,\ \forall (x,t) \in (a,b).$$

1 Examples

1.1 Normal demand

If u is concave and supermodular, then

$$u(\lambda[x \vee y] + (1-\lambda)y) - u(y) \geq u(x) - u(\lambda[x \wedge y] + (1-\lambda)x),\ \forall x,y \in \mathbb{R}^\ell_+,\ \forall \lambda \in [0,1]. \tag{1.1}$$

Proof. Let $x, y \in \mathbb{R}^\ell_+$ and let $\lambda \in [0,1]$. By concavity of u,

$$u(\lambda[x \vee y] + (1-\lambda)y) \geq \lambda u(x \vee y) + (1-\lambda)u(y), \tag{1.2}$$
$$u(\lambda[x \wedge y] + (1-\lambda)x) \geq \lambda u(x \wedge y) + (1-\lambda)u(x). \tag{1.3}$$

By supermodularity of u,

$$u(x \vee y) + u(x \wedge y) \geq u(x) + u(y). \tag{1.4}$$

Multiply the last inequality by λ:

$$\lambda u(x \vee y) + \lambda u(x \wedge y) \geq \lambda u(x) + \lambda u(y). \tag{1.5}$$

Summing (1.2), (1.3), and (1.5), we get

$$u(\lambda[x \vee y] + (1-\lambda)y) + u(\lambda[x \wedge y] + (1-\lambda)x) \geq u(x) + u(y). \tag{1.6}$$

Rerranging, we get

$$u(\lambda[x \vee y] + (1-\lambda)y) - u(y) \geq u(x) - u(\lambda[x \wedge y] + (1-\lambda)x). \tag{1.7}$$

□

Question 3 254 [I.1 Spring 2018 majors]

Consider the problem of finding a Pareto optimal allocation of aggregate resources $\omega \in \mathbb{R}^n_+$ in an economy with two agents:

$$\max_x \mu_1 u_1(x) + \mu_2 u_2(\omega - x)$$
$$\text{subject to } x \leq \omega, x \geq 0$$

where $u_i : \mathbb{R}^n_+ \to \mathbb{R}$ are agents' utility functions (assumed continuous) and $\mu_i > 0$ are welfare weights for $i = 1, 2$. Let $x^*(\mu_1, \mu_2)$ be the set of solutions.

a State a definition of utility function u_i being supermodular. Show that if u_i is supermodular, then $u_i(\omega - x)$ is supermodular in x

b Show that, if u_1 and u_2 are strictly increasing and supermodular in x then $x^*(\mu_1, \mu_2)$ is non-decreasing in μ_1. You may assume that $x^*(\mu)$ is single-valued. Is $x^*(\mu_1, \mu_2)$ non-increasing in μ_2? Justify your answer. If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.

c Under what conditions on u_1 and u_2 is the solution $x^*(\mu_1, \mu_2)$ unique. Justify your answer.

Question 4 [Midterm 2017] or ∼ 82,89 [II.1 Spring 2009 majors]

Consider a profit maximizing firm with single output and n inputs, with production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ assumed strictly increasing, continuous (but possibly nondifferentiable), and $f(0) = 0$. Let $q \in \mathbb{R}_{++}$ be the price of output and $w \in \mathbb{R}^n_{++}$ be the vector of prices of inputs. The firm's profit maximization problem is

$$\max_{x \geq 0}[qf(x) - wx]$$

a Show that if the production function f is supermodular, then the firm's input demand x is monotone non-increasing in input prices, that is if $w \leq w'$ for $w, w \in \mathbb{R}^N_{++}$ then $x(w, q) \geq x(w, q)$. You may assume that input demand x is single valued. Production function is strictly increasing but need not be differentiable.

b Under what conditions on f is the solution $x(w, q)$ unique? Be as general as you can and prove your answer

c Give an example of strictly increasing function that is not supermodular.

Solution 4

Function f is assumed strictly increasing. If f is nondecreasing, then the objective function $F(x, q) = qf(x) - wx$ has nondecreasing differences in $(x; q)$. If f is supermodular, then $F(x, q)$ is supermodular in x. Theorem 0.6 implies that input demand $x^*(q)$ is monotone nondecreasing in output price q.

Question 5

Consider a $C \subset \mathbb{R}^L$, $T \subset \mathbb{R}$. Define function F in following way:

$$F : \mathbb{R}^L \times T \to \mathbb{R} \quad F(x, t) = \bar{F}(x) + f(x, t)$$

where $f : \mathbb{R} \times T \to \mathbb{R}$ is supermodular and $\bar{F} : \mathbb{R}^L \to \mathbb{R}$. Assume that:

$$\forall \quad t'' > t' \quad x'' \in \operatorname*{argmax}_{x \in C} F(x, t'') \quad x' \in \operatorname*{argmax}_{x \in C} F(x, t')$$

Show that if $x'_i > x''_i$ then

$$\forall \quad t'' > t' \quad x'' \in \operatorname*{argmax}_{x \in C} F(x, t') \quad x' \in \operatorname*{argmax}_{x \in C} F(x, t'')$$

Solution 5

Let's take $x'_i \geq x''_i$, $t'' \geq t'$ and consider $z' = (x'_i, t')$ and $z'' = (x''_i, t'')$ thus $z'' \wedge z' = (x''_i, t')$, $z'' \vee z' = (x'_i, t'')$. From Supermodularity of $f(x_i, t)$:

$$f(z' \vee z'') + f(z \wedge z'') \geq f(z') + f(z'')$$

$$f(x'_i, t'') + f(x''_i, t') \geq f(x''_i, t'') + f(x'_i, t')$$

and add to both sides $\bar{F}(x'') + \bar{F}(x')$

$$F(x'', t') + F(x', t'') \geq F(x', t') + F(x'' + t'')$$

$$F(x'', t') - F(x', t') \geq F(x'', t'') - F(x', t'')$$

$x' \in \operatorname{argmax} F(x, , t')$ so $F(x', t') \geq F(x'', t')$ $x'' \in \operatorname{argmax} F(x, , t')$ so $F(x'', t'') \geq F(x', t'')$

$$0 \geq F(x'', t') - F(x', t') \geq F(x'', t'') - F(x', t'') \geq 0$$

$$0 = F(x'', t') - F(x', t') = F(x'', t'') - F(x', t'') = 0$$

$$F(x'', t') = F(x', t') = F(x'', t'') = F(x', t'')$$

so $x'' \in \operatorname{argmax} F(x, , t')$ and $x' \in \operatorname{argmax} F(x, , t'')$

Question 6

Let $\{f(s, t)\}$ $t \in T$ be a family of density functions on $S \subset R$. T is a poset (partially ordered set). Consider

$$v(x, t) = \int_S u(x, s) f(s, t) ds$$

Prove the following statement. Suppose u has increasing differences and that $\{f(\cdot, t)\}$ $t \in T$ are ordered with t by first order stochastic dominance. Then v has increasing differences in (x, t).

Solution 6

For $x' > x$ and $t' > t$ we define $\gamma(s) := u(x', s) - u(x, s)$. It is incresing function and look at difference of v (we have to prove that is increasing differences):

$$v(x', t') - v(x, t') = \int_S [u(x', s) - u(x, s)]f(s, t')ds = \int_S \gamma(s)f(s, t')$$

$f(\cdot, t)$ is FOSD in t and γ is increasing so the value $v(x', t') - v(x, t')$ itself is increasing in t , i.e. $v(x', t') - v(x, t') \geq v(x', t) - v(x, t)$.

Question 7 Suppose that utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ is supermodular, strictly concave, and locally non-satiated. Then the Walrasian demand function $x^*(\cdot)$ is a nondecreasing function of income, i.e.,

$$x^*(p, w') \geq x^*(p, w), \ \forall w' \geq w \geq 0, \ \forall p \gg 0.$$

In other words, the demand for every good is normal.

Solution 7

If $w = w'$, the proof is trivial. Let $p \gg 0$, let $w > w'$, let $x = x^*(p, w)$, and let $y = x^*(p, w')$. Since u is locally non-satiated, we have $p \cdot x = w$ and $p \cdot y = w'$ (by lemma 1.1). Clearly, $p \cdot [x \wedge y] \leq w$. Since $p \cdot y = w' > w$, $\exists \lambda \in [0, 1)$ such that

$$p \cdot (\lambda[x \wedge y] + (1 - \lambda)y) = w.$$

Let $\underline{z}_\lambda = \lambda[x \wedge y] + (1 - \lambda)x$ and let $\bar{z}_\lambda = \lambda[x \vee y] + (1 - \lambda)y$. Note that

$$\underline{z}_\lambda + \bar{z}_\lambda = x + y$$

by the fact that $x \wedge y + x \vee y = x + y$. Then we have

$$p \cdot \underline{z}_\lambda = w$$

and

$$p \cdot \bar{z}_\lambda = w'.$$

Since x is the unique maximizer at w and $\underline{z}_\lambda$ is affordable at w, it must be that $u(x) \geq u(\underline{z}_\lambda)$. Then by lemma 1.1, $u(\bar{z}_\lambda) \geq u(y)$. But since y is the unique maximizer at w' and $\bar{z}_\lambda$ is affordable at w', then it must be that $u(y) \geq u(\bar{z}_\lambda)$. Then we have $u(y) = u(\bar{z}_\lambda)$ so $y = \bar{z}_\lambda$. Since $\underline{z}_\lambda + \bar{z}_\lambda = x + y$, this means that we also have $x = \underline{z}_\lambda$. Seems like there's an error in Werner's notes in the defs of $\bar{z}_\lambda$ and $\underline{z}_\lambda$. Finish this!