Theory of Game Theory

Jakub Pawelczak*

October 2020

Contents

*These notes are intended to summarize the main concepts, definitions and results covered in the first year of micro sequence for the Economics PhD of the University of Minnesota. The material is not my own. Please let me know of any errors that persist in the document. E-mail: pawel042@umn.edu .

1 Normal Form Games

1.1 Games on Consequences

Definition 1 (Games on Consequences). *consists of:*

- $I = \{1, \dots, n\}$ *is the finite set of players.*
- A^i *is the (finite) set of actions for player* i.
- $A \equiv X_{i \in I} A^i$ *is the (finite) set of action profiles*
- C *is finite the set of consequences,* $C = \{c^1, \dots, c^m\}$.
- $\succeq_i$ *preference relation of Mr* i *over* C
- $g : A \to C$ *mapping of actions to consequences*

This will be compactly denoted as a $\left\langle I, \left(A^i\right)_{i \in I}, \left(\succeq^i\right)_{i \in I}, C, g \right\rangle$.

Example 1.

Mr 2

		L	R
Mr 1	T	c^1	c^2
	B	c^3	c^4

Table above induces g

$A^1 = \{T, B\}$, $A^2 = \{L, R\}$

$c^1 = (10, 5)$, $c^1 = (1, 2)$, $c^1 = (3, 2)$, $c^1 = (4, 3)$

Mr 1 : $c^1 \succeq_1 c^3$ and $\succeq_1 c^2 \succeq_1 c^4$

Mr 2 : $c^2 \succeq_2 c^1$ and $\succeq_2 c^4 \succeq_2 c^3$

1.2 Preferences on lotteries

Definition 2 (Simplex).

$$\Delta(C) \equiv \left\{ p = \left(p^1, \dots, p^m\right) \mid \forall i \quad p^i \geq 0 \quad \sum_{i=1}^{m} p^i = 1 \right\}$$

Definition 3 (Lottery). $L \in \Delta(C)$ *is a simple lottery, where*

$$L = \begin{pmatrix} p^1 & \cdots & p^i & \cdots & p^m \\ c^1 & \cdots & c^i & \cdots & c^m \end{pmatrix}$$

Example 2 (Degenerated lottery). $\delta_{c^i} \in \mathcal{L}$

$$\delta_{c^i} = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ c^1 & \cdots & c^i & \cdots & c^m \end{pmatrix}$$

Definition 4. *$\mathcal{L} \equiv \Delta(C)$ is the set of (simple) lotteries.*

Definition 5. *$G = \left(q^1 L^1, \ldots, q^K L^K\right) \in \Delta(\mathcal{L})$ is a compound lottery, where*

$$G = \begin{pmatrix} q^1 & \cdots & q^k \\ L^1 & \cdots & L^k \end{pmatrix}$$

$L^k \in \mathcal{L} \quad \forall k = 1, \ldots, K, q^k \geq 0$ *and* $\sum_{k=1}^{K} q^k = 1$

Definition 6. *$\mathcal{G} \equiv \Delta(\mathcal{L})$ is the set of compound lotteries.*

Note that all simple lotteries can be viewed as compound lotteries with degenerate distributions. For example, the simple lottery $L = \left(p^1, \ldots p^m\right)$ can be viewed as a compound lottery $L = \left(p^1 \delta_{c^1}, \ldots, p^m \delta_{c^m}\right)$, where δ_{c^i} is a degenerate lottery giving fully probability to consequence c^i

Definition 7 (Reduction of a lottery)**.** *. For every $G \in \mathcal{G}, R(G) \in \mathcal{L}$ is the reduction of* G, *and gives probability $\sum_{k=1}^{K} q^k p_k^i$ to consequence c^i*

Definition 8 (Convex combination)**.** *For any F, G and $\alpha \in [0,1]$, denote the convex combination as $F\alpha G \equiv \alpha F + (1-\alpha)G$*

1.3 Assumptions on $\succeq$

We are interested in the binary preference relation $\succeq_i$ on $\mathcal{L}$.

Definition 9 (Complete (C))**.** *$\forall F, G \in \mathcal{G}$ either $F \succeq G$ or $G \succeq F$*

Definition 10 (Reflexive (R))**.** *$\forall F \in \mathcal{G}$ $F \succeq F$*

Definition 11 (Transitive (T))**.** *$\forall F, G, H \in \mathcal{G}$ such that $F \succeq G$, $G \succeq H$ then $F \succeq H$*

Definition 12 (Weak Order (WO)-A1)**.** *$\succeq$ is compete , reflexive, and transitive.*

Definition 13 (Independence (I)-A2)**.** *$\forall F, G, H \in \mathcal{G}$ and $\alpha \in (0,1)$: such that*

$$F \succ G \Rightarrow F\alpha H \succ G\alpha H$$

Definition 14 (Continuity (Cty)-A3)**.** *$\forall F, G, H \in \mathcal{G}$ such that $F \succeq G \succeq H, \forall \alpha \in [0,1]$ such that $\{\alpha | F\alpha H \geq G\}$ and $\{\beta | F\beta H \leq G\}$ are closed sets.*

Alternative definition of Cty

Definition 15 (Continuity (Cty2))**.** *$\forall F, G, H \in \mathcal{G}$ such that $F \succeq G \succeq H, \exists \alpha \in [0,1]$ such that $F\alpha H \sim G$*

Lemma 1. *If $[C, T, Cty]$ holds then $Cty2$ holds too.*

Proof. Suppose $F \succeq G$. Define A=$\{\alpha | F\alpha H \geq G\}$ and B=$\{\beta | F\beta H \leq G\}$. Observe that:

- $A, B \subset [0,1]$
- $1 \in A$, $0 \in B$

- A, B are closed (by Cty)
- $A \cup B = [0,1]$
- $[0,1]$ is a connected set

(1)-(5) implies that $A \cap B \neq \varnothing$. So $\exists \alpha \in A \cap B$ s.t. $F\alpha G \succeq H \succeq F\alpha G$. Thus $F\alpha G \sim H$. $\square$

Lemma 2. *Suppose* [WO, I] *hold then:*

$$\forall_{F\in\mathcal{L}} \quad \delta_{c^1} \succeq F \succeq \delta_{c^m}$$

Proof. Since C is finite then $\exists$ best and worst outcome δ_{c^b} and δ_{c^w}. WTS $\forall L \quad \delta_{c^b} \succeq L \succeq \delta_{c^w}$. I will use (easy to prove) corollary

Corollary 1. *Let $L_0, \dots L_K$ be (1+K) lotteries $\alpha_k \geq 0 : \sum_k \alpha_k = 1$:*

$$\begin{aligned} &\textit{If} \quad \forall k \quad L_k \succeq L_0 \quad \Rightarrow \quad \sum_k \alpha_k L_k \succeq L_0 \\ &\textit{If} \quad \forall k \quad L_0 \succeq L_k \quad \Rightarrow \quad L_0 \succeq \sum_k \alpha_k L_k \end{aligned}$$

Now let lottery L^k yields outcome k with probability 1. Then $\delta_{c^b} \succeq L \succeq \delta_{c^w}$ and any L can be represented as $L = \sum_k p_k L^k$ so by corollary $\delta_{c^b} \succeq L \succeq \delta_{c^w}$ $\square$

Definition 16 (Monotonicity (M)). *$\forall F, G \in \mathcal{G}$ such that $F \succ G$, then for $\alpha, \beta \in (0,1)$:*

$$\alpha > \beta \Leftrightarrow F\alpha G \succ F\beta G$$

Lemma 3. *If I holds and $F \succ G$ $\forall \alpha \in (0,1) \quad \Rightarrow F \succ F\alpha G \succ G$*

Proof.

$$F = \alpha F + (1-\alpha)F \succ^I \alpha F + (1-\alpha)G = F\alpha G = \alpha F + (1-\alpha)G \succ^I \alpha G + (1-\alpha)G = M$$

$\square$

Lemma 4. *Prove that WO, Cty, I imply M.*

Proof. $\Rightarrow$ Suppose $\alpha > \beta$. Observe that

$$F = \alpha F + (1-\alpha)G = \gamma F + (1-\gamma)[\beta F + (1-\beta)G]$$

after rearrangement $\gamma = \frac{\alpha-\beta}{1-\beta} \in (0,1)$ By lemma 3 $F \succ G$: $F \succ F\beta G$

$$F\alpha G = F\gamma(F\beta G) \succ^I (F\beta G)\gamma(F\beta G) = F\beta G$$

Now $\Leftarrow$ part. Suppose $F \succ G$ and $F\alpha G \succ F\beta G$. WTS: $\alpha > \beta$.
Suppose not. So either $\alpha = \beta$ or $\alpha < \beta$. If $\alpha = \beta$ then we have contradiction with $F\alpha G \succ F\beta G$.
If $\alpha < \beta$ by $\Rightarrow$ part $F\beta G \succ F\alpha G$ contradiction. $\square$

Definition 17 (Reduction (R)). $\forall G \in \mathcal{G}, R(G) \sim G$

Definition 18 (Substitution (S):). $\forall G \in \mathcal{G}$, *if* $G = \begin{pmatrix} q^1 & \dots & q^j & \dots & q^K \\ L^1 & \dots & L^j & \dots & L^K \end{pmatrix}$ *is modified by substituting* L^j *for* M^j, *where* $M^j \sim L^j$, *then* $G \sim H$, *where* $H = \begin{pmatrix} q^1 & \dots & q^j & \dots & q^K \\ L^1 & \dots & M^j & \dots & L^K \end{pmatrix}$

1.4 Utility representation

Definition 19 (Utility representation). *The function* $u : \mathcal{G} \to \mathbb{R}$ *is a representation of* $\succeq$ *if and only if:*

$$F \succeq G \Leftrightarrow u(F) \geq u(G)$$

Recall:

$$F \succ G \Leftrightarrow F \succeq G \text{ and not } G \succeq F$$
$$F \sim G \Leftrightarrow F \succeq G \text{ and } G \succeq F$$

Lemma 5. *If* u *represents* $\succeq$ *and* $T : \mathbb{R} \to \mathbb{R}$ *is strictly increasing, then* $T(u(\cdot)) : \mathcal{G} \to \mathbb{R}$ *is a representation of* $\succeq$

Lemma 6 (Recap from MINI 1). *If* $\succeq$ *satisfies WO and C, then* $\succeq$ *has some (continuous) utility representation.*

Definition 20 (Linear utility). *If* u *is linear then* $u(F\alpha G) = u(F)\alpha u(G)$, *where* $\alpha \in [0, 1]]$

Alternative definition of linearity:

Definition 21 (Linear utility). u *is linear if and only if* $u(L) = \sum_{i=1}^{m} p^i u\left(c^i\right)$, *where* $L = \left(p^1, \dots, p^m\right)$

Example 3. *If* u *represents* $\succeq$ *and is linear, then if* $A > 0$ *and* $B \in \mathbb{R}, Au(\cdot) + B$ *also represents* $\succeq$ *and is linear.*

Example 4. $\succeq$ *satisfies* $\mathrm{WO}, \mathrm{Cty}$, *and* M *if and only if* $\forall F \in \mathcal{G} \quad \exists u(F) \in [0,1]$ *such that* $F \sim \delta_{c^1} u(F) \delta_{c^m}$ *and* $u(F)$ *is unique. In particular,* $\forall c^i \in C \quad \exists u\left(c^i\right) \in [0,1]$ *such that* $c^i \sim c^1 u\left(c^i\right) c^m$.

Theorem 1 (von Neumann-Morgenstern (I)). ***1. (existence)*** $\succeq$ ***on*** $\mathcal{L}$ ***satisfies*** $\mathrm{WO}, \mathrm{Cty}, \mathrm{I}$ ***if and only if there exists a linear*** $u : \mathcal{G} \to \mathbb{R}$ ***that represents*** $\succeq$

2. (uniqueness) If u, v ***are linear representations of*** $\succeq$***, then*** $\exists A > 0, B \in \mathbb{R}$ ***such that*** $u(\cdot) = Av(\cdot) + B$

Proof. We will proceed in three steps: 1) (existence): $\Rightarrow$; 2)(existence): $\Leftarrow$; 3)(uniqueness)

- (existence): $\Rightarrow$

By lemma 2: $\exists \delta_{c^1}, \delta_{c^m} : \forall F : \delta_{c^1} \succeq F \succeq \delta_{c^m}$ and $\delta_{c^1} \succ \delta_{c^m}$.
Define $u(F)$:$\delta_{c^1} u(F) \delta_{c^m} \sim F$. By lemma 1 we know that such $u(F)$ is well defined. Our goal is to show for $\alpha = u(F)$ that this is representation, it is unique and linear. We do it with two lemmas. We want to avoid $\alpha \neq \beta$ $\delta_{c^1} \alpha \delta_{c^m} \sim \delta_{c^1} \beta \delta_{c^m}$, we want $\delta_{c^1} \alpha \delta_{c^m} \succ \delta_{c^1} \beta \delta_{c^m} \iff \alpha > \beta$.

Lemma 7. *$u(F) : \delta_{c^1} u(F) \delta_{c^m} \sim F$ is unique*

Proof. Let $\bar{u}(F)$ and $u(F)$ be two different values and WLOG $\bar{u}(F) > u(F)$.

$$\delta_{c^1} u(F) \delta_{c^m} \sim F \sim \delta_{c^1} \bar{u}(F) \delta_{c^m}$$

by applying lemma 4 ($\delta_{c^1} \succ \delta_{c^m}$), $\bar{u}(F) > u(F)$):

$$\delta_{c^1} u(F) \delta_{c^m} \succ \delta_{c^1} \bar{u}(F) \delta_{c^m}$$

contradiction.

□

By last lemma $F \succeq G \iff \delta_{c^1} u(F) \delta_{c^m} \succeq \delta_{c^1} u(G) \delta_{c^m}$ by lemma 4 $\iff u(F) \geq u(G)$. So $u : \mathcal{L} \to \mathbb{R}$ represents $\succeq$.

Lemma 8. *$u(\cdot)$ is linear*

Proof. By definition of u

$$F \sim \delta_{c^1} u(F) \delta_{c^m}$$

$$G \sim \delta_{c^1} u(G) \delta_{c^m}$$

by I (and rearrangement) :

$$F \alpha G \sim (\delta_{c^1} u(F) \delta_{c^m}) \alpha G \sim (\delta_{c^1} u(F) \delta_{c^m}) \alpha (\delta_{c^1} u(G) \delta_{c^m}) \sim \delta_{c^1} (u(F) \alpha u(G)) \delta_{c^m}$$

Thus $u(F \alpha G) = u(F) \alpha u(G)$ □

- (existence):$\Leftarrow$

Let's show that $\succeq$ satisfy weak order (WO). Let's start with completeness.

$$\forall F, G \in \mathcal{L} \quad u(F) \geq u(G) \quad \text{or} \quad u(F) \leq u(G) \quad \iff \quad F \succeq G \quad \text{or} \quad G \succeq F$$

since it is order on real line.
Transitivity. WLOG $F \succeq G$ and $G \succeq H$. Observe that since u represents preferences:

$$u(F) \geq u(G) \iff F \succeq G$$

$$u(G) \geq u(H) \iff G \succeq H$$

$$u(F) \geq u(H) \iff F \succeq H$$

we have $u(F) \geq u(G)$,$u(G) \geq u(H) \Rightarrow u(F) \geq u(H)$ comes from linear order on real line. So $F \succeq H$.

Now we show continuity. Consider any sequence $\{\alpha_i\}_{i=1}^{\infty} \rightarrow \alpha$, (where $\forall i, \alpha_i \in [0,1]$) and $\alpha_i F + (1-\alpha_i) G \succsim H, \forall i$ Then,

$$U(\alpha_i F + (1-\alpha_i) G) \geq U(H), \forall i$$

and using the linearity of U

$$\alpha_i U(F) + (1-\alpha_i) U(G) \geq U(H), \forall i$$

which implies (taking limit as $i \rightarrow \infty$)

$$\alpha U(F) + (1-\alpha) U(G) \geq U(H)$$

so that $\alpha F + (1-\alpha) G \succsim H$.

Next, we show independence. Consider $F, G, H \in \mathcal{L}$ and $\alpha \in (0,1)$ Need to show: $F \succsim G \Longleftrightarrow \alpha F + (1-\alpha) H \succsim \alpha G + (1-\alpha) H$ Suppose $F \succsim G$ Then, $U(F) \geq U(G)$ so that

$$\alpha U(F) + (1-\alpha) U(H) \geq \alpha U(G) + (1-\alpha) U(H)$$

which implies

$$\alpha F + (1-\alpha) H \succsim \alpha G + (1-\alpha) H$$

Suppose that $\alpha F + (1-\alpha) H \succsim \alpha G + (1-\alpha) H$ Then,

$$U(\alpha F + (1-\alpha) H) \geq U(\alpha G + (1-\alpha) H)$$

and using linearity of U,

$$\alpha U(F) + (1-\alpha) U(H) \geq \alpha U(G) + (1-\alpha) U(H)$$

which implies that $U(F) \geq U(G)$

- (uniqueness):

Let u, v be linear representations of $\succeq$ and take F such that $F \sim c^1 \alpha c^m$ for some $\alpha \in [0,1]$. Then, by linearity:

$$u(F) = u\left(c^1 \alpha c^m\right) = \alpha u\left(c^1\right) + (1-\alpha) u\left(c^m\right)$$

$$\text{and} \quad v(F) = v\left(c^1 \alpha c^m\right) = \alpha v\left(c^1\right) + (1-\alpha) v\left(c^m\right)$$

$$\alpha = \frac{u(F) - u\left(c^m\right)}{u\left(c^1\right) - u\left(c^m\right)} = \frac{v(F) - v\left(c^m\right)}{v\left(c^1\right) - v\left(c^m\right)} \Longrightarrow u(F) = \frac{u\left(c^1\right) - u\left(c^m\right)}{v\left(c^1\right) - v\left(c^m\right)} v(F) - \frac{u\left(c^1\right) - u\left(c^m\right)}{v\left(c^1\right) - v\left(c^m\right)} v\left(c^m\right) + u\left(c^m\right)$$

$$u(F) = A v(F) + B$$

where $A \equiv \frac{u\left(c^1\right) - u\left(c^m\right)}{v\left(c^1\right) - v\left(c^m\right)}$ and $B \equiv u\left(c^m\right) - \frac{u\left(c^1\right) - u\left(c^m\right)}{v\left(c^1\right) - v\left(c^m\right)} v\left(c^m\right)$

$\square$

Theorem is true under alternative set of axioms. We present proof of it for pedagogical reasons.

Theorem 2 (von Neumann-Morgenstern (M,S,R)). ***1. (existence)*** $\succeq$ ***on*** $\mathcal{L}$ ***satisfies*** $\mathrm{WO}, \mathrm{Cty}, \mathrm{M}, \mathrm{S}, \mathrm{R}$ ***if and only if there exists a linear*** $u : \mathcal{G} \to \mathbb{R}$ ***that represents*** $\succeq$

2. (uniqueness) If u, v ***are linear representations of*** $\succeq$***, then*** $\exists A > 0, B \in \mathbb{R}$ ***such that*** $u(\cdot) = Av(\cdot) + B$

Proof. Below we prove theorem when $\succeq$ on $\mathcal{G}$ satisfies $\mathrm{WO}, \mathrm{Cty}, \mathrm{M}, \mathrm{R} and \mathrm{S}$. We show only (existence)$\Rightarrow$ part. Uniqueness remains the same and $\Leftarrow$ of existence is easy exercise left for a reader.

(existence): $\Rightarrow$

By WO, Cty, and M, we know there exists $u : C \to \mathbb{R}$ and thus $c^i \sim c^1 u\left(c^i\right) c^m$ implies $\bar{u}(L) \equiv \sum_{i=1}^m p^i u\left(c^i\right)$, where $L = \left(p^1, \ldots, p^m\right)$ and $L \sim c^1 u(L) c^m$

Lemma 9. $\bar{u}(L) = u(L)$

Proof: Recall $c^2 \sim c^1 u\left(c^2\right) c^m$ and construct

$$L' = \begin{pmatrix} p^1 & p^2 & \ldots & p^m \\ c^1 & c^1 u\left(c^2\right) c^m & \ldots & c^m \end{pmatrix}$$

where $L' \sim L$ by substitution. Repeat this substitution process for all but c^1 and c^m. Now take the reduction

$$R\left(L'\right) = \begin{pmatrix} p^1 + p^2 u\left(c^2\right) + p^3 u\left(c^3\right) \ldots & 0 & \ldots & 1 - \left(p^1 + \ldots\right) \\ c^1 & c^2 & \ldots & c^m \end{pmatrix}$$

and note $R\left(L'\right) \sim L$ by reduction. Then $u(L) = \sum_{i=1}^m p^i u\left(c^i\right) = \bar{u}(L)$. □

Definition 22 (Sure Thing Principle). *For lotteries* $L, M, N, R \in \mathcal{L}$ *and* $\alpha \in (0,1]$

$$L\alpha M \succ N\alpha M \Leftrightarrow L\alpha R \succ N\alpha R$$

Lemma 10. *If* $\succeq$ *satisfies the vNM axioms, then* $\succeq$ *satisfies the Sure Thing Principle.*

Proof. Since $\succeq$ satisfies the vNM axioms, there exists a linear utility representation $u(\cdot)$. Thus, $\forall \alpha \in (0,1]$:

$$\begin{aligned} L\alpha M \succ N\alpha M &\Leftrightarrow u(L\alpha M) > u(N\alpha M) \\ &\Leftrightarrow \alpha u(L) + (1-\alpha)u(M) > \alpha u(N) + (1-\alpha)u(M) \\ &\Leftrightarrow u(L) > u(N) \\ &\Leftrightarrow \alpha u(L) + (1-\alpha)u(R) > \alpha u(N) + (1-\alpha)u(R) \\ &\Leftrightarrow u(L\alpha R) > u(N\alpha R) \\ &\Leftrightarrow L\alpha R \succ N\alpha R \end{aligned}$$

□

From a game on consequences, we elicit $\succeq_i$ for each player.

We then use the von Neumann-Morgenstern Theorem to construct utility functions $u^i : C \to \mathbb{R}$

Then we construct utility functions $\hat{u}^i : A \to \mathbb{R}$ defined by $\hat{u}^i = u^i(g(a))$.

Thus we transform a game on consequences into a **normal form game**

Definition 23 (Normal Form Game (NFG))**.** *is a tuple* $\left(I, \left(A^i\right)_{i\in I}, \left(u^i\right)_{i\in I}\right)$

1.5 Strategies of Normal Form Games

Definition 24. *A mixed strategy for player i is $s^i \in \Delta\left(A^i\right)$; we denote the mixed strategies of all players $j \neq i$ as $s^{-i} \in \Delta\left(A^{-i}\right)$*

Definition 25. *The set of mixed strategy profiles for player i is $S^i \equiv \Delta\left(A^i\right)$; we denote the set for all players $j \neq i$ as $S^{-i} \equiv \Delta\left(A^{-i}\right)$. Equivalently,*

$$S^i = \left\{ \left\{s^i\left(a^i\right)\right\}_{a^i\in A^i} \mid \sum_{a^i\in A^i} s^i\left(a^i\right) = 1; \forall a^i \in A^i, s^i\left(a^i\right) \geq 0 \right\}$$

[Note: $S^i = \operatorname{co}\left(A^i\right)$, and so S^i is convex. If A^i is finite, then $S^i = \overline{co}\left(A^i\right)$

Definition 26. *A mixed strategy for all players is $s \in S$, where $S \equiv X_{i\in I} S^i$ is the set of all mixed strategy profiles.*

Definition 27. *Fully mixed strategy A mixed strategy $s^i \in \Delta\left(A^i\right)$ is a fully mixed strategy if* $\forall a^i \in A^i, s^i\left(a^i\right) > 0$

1.6 Nash Equilibrium

Definition 28. *A normal form game* (NFG) *is a tuple* $\left(I, \left(A^i, u^i\right)_{i\in I}\right)$, *where* $\forall i u^i : A \to \mathbb{R}$

Definition 29 (Mixed extension of NFG)**.** *For a NFG* $\left(I, \left(A^i, u^i\right)_{i\in I}\right)$, *the mixed extension is* $\left(I, \left(S^i, u^i\right)_{i\in I}\right)$, *where $\forall i s^i \in S^i$ and $u^i : S \to \mathbb{R}$*

We define agent i 's expected utility over mixed strategy profiles as $u^i : S \to \mathbb{R}$, where:

$$\begin{aligned}
u^i(s) &= \sum_{a\in A} \Pr_s(a) u^i(a) \\
&= \sum_{a^i\in A^i} s^i\left(a^i\right) \sum_{a^{-i}\in A^{-i}} \Pr_{s^{-i}}\left(a^{-i}\right) u^i\left(a^i, a^{-i}\right) \\
&= \sum_{a^i\in A^i} s^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) \\
&= u^i\left(s^i, s^{-i}\right)
\end{aligned}$$

We will use this representation extensively.

Definition 30 (Pure action best response correspondence)**.** *The action best response correspon-*

dence of player $i, BR^i_{A^i} : S \rightrightarrows A^i$, *is:*

$$\begin{aligned} BR^i_{A^i}(s) &\equiv \left\{a^i \in A^i \mid \forall b^i \in A^i u^i\left(a^i, s^{-i}\right) \geq u^i\left(b^i, s^{-i}\right)\right\} \\ &= \underset{a^i \in A^i}{\arg\max} u^i\left(a^i, s^{-i}\right) \end{aligned}$$

Definition 31 (Best response correspondence)**.** *The best response correspondence of player* $i, BR^i : S \rightrightarrows S^i$, *is:*

$$\begin{aligned} BR^i\left(s^{-i}\right) = BR^i(s) &\equiv \left\{r^i \in S^i \mid \forall t^i \in S^i u^i\left(r^i, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right)\right\} \\ &= \left\{r^i \in S^i \mid u^i\left(r^i, s^{-i}\right) = \max_{t^i \in S^i} u^i\left(t^i, s^{-i}\right)\right\} \\ &= \underset{s^i \in S^i}{\arg\max} u^i\left(s^i, s^{-i}\right) \end{aligned}$$

The only difference between those two Best responses is on domain of correspondences.

Definition 32 (Best reply correspondence)**.** *The best reply correspondence* $BR : S \rightrightarrows S$ *is defined by:*

$$BR(s) = \prod_{i \in I} BR^i(s)$$

Definition 33 (Nash equilibrium)**.** *If* $\left(I, \left(S^i, u^i\right)_{i \in I}\right)$ *is the mixed extension of a NFG, then* $\hat{s} \in S$ *is a Nash equilibrium if and only if* $\forall i \hat{s}^i \in BR^i(\hat{s})$.

Example 5.

1/2	L	R
T	3,1	0,0
B	0,0	1,3

- Define: pure actions, mixed actions, best correspondences
- Find all Nash Equilibria

pure strategies: $A^1 = \{T, B\}$, $A^2 = \{L, R\}$,$A = A^1 A^2$
mixed strategies:

$$S = S^1 \times S^2 = \Delta(A^1) \times \Delta(A^2) = \{((p, 1-p), (q, 1-q)) \,|\, p, q \in [0, 1]\}$$

We can solve for the best responses as follows: Mr 1 best response:

$$BR^1\left((q, 1-q)\right) : \left\{ \begin{array}{cc} T & B \\ 3(q) + 0(1-q) & 0(q) + 1(1-q) \end{array} \right\}$$

Equality only holds when $q = \frac{1}{4}$. $T > B \iff p > \frac{1}{4}$. $T < B \iff p < \frac{1}{4}$ Therefore, player 1 sets $p = 1$ if $q > frac14$ and sets $p = 0$. She picks $p \in [0, 1]$ where is indifferent between T and B.

otherwise.

$$BR^1\left((q,1-q)\right)=\begin{cases}0 & \text{if } p<\frac{1}{4}\\ [0,1] & \text{if } p=\frac{1}{4}\\ 1 & \text{if } p>\frac{1}{4}\end{cases}$$

Mr 2 best response:

$$BR^2\left((p,1-p)\right):\left\{\begin{array}{cc} L & R\\ p+0\left(1-p\right) & 0\left(p\right)+3\left(1-p\right)\end{array}\right\}$$

Equality only holds when $p=\frac{3}{4}$. $L>R \iff p>\frac{3}{4}$,$L<R \iff p<\frac{3}{4}$ Similarly, player 2 sets $q=1$ if $p>\frac{3}{4}$ and sets $q=0$ otherwise.

$$BR^2\left((p,1-p)\right)=\begin{cases}0 & \text{if } p<\frac{3}{4}\\ [0,1] & \text{if } p=\frac{3}{4}\\ 1 & \text{if } p>\frac{3}{4}\end{cases}$$

These best responses can be graphed :

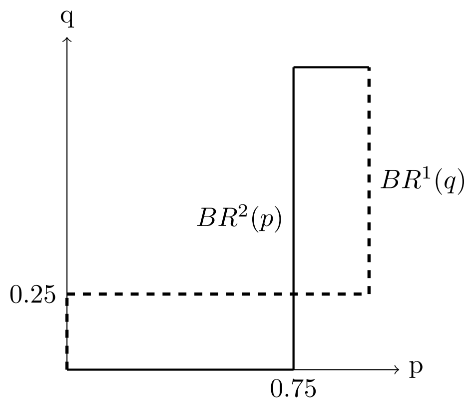


Figure 1: Best Responses

The points of interesection

$$\left(\frac{3}{4},\frac{1}{4}\right),(1,1),(0,0)$$

yield the set of Nash equilibria

$$\mathrm{NE}=\Big\{\big((1,0),(1,0)\big),\big((0,1),(0,1)\big),\big((\frac{3}{4},\frac{1}{4}),(\frac{1}{4},\frac{3}{4})\big)\Big\}.$$

Corollary 2. *A NE exists if and only if the best response correspondence $BR:S\rightrightarrows S$ has a fixed point (i.e. $s\in BR(s)$)*

Lemma 11. *Show that* $\mathrm{BR}_i(s)=\mathrm{co}\left(\left\{\delta_{b^i}:b^i\in\mathrm{BR}^i_{A^i}(s)\right\}\right)$

Proof. • $\mathrm{BR}_i(s) \subset \mathrm{co}\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right)$

We present here small but important result: if strategy is not best response in pure best response, corresponding probability in best response in mixed strategies is zero.
Let $s^i \in \mathrm{BR}^i(s)$.

Lemma 12.

$$\forall b^i \notin BR_{A^i}(s), b^i \in A^i \Rightarrow s^i(b^i) = 0$$

Proof. Suppose not. if the strategy $s^i \in \mathrm{BR}^i(s)$ uses some pure action $b^i \in A^i$ which $\notin \mathrm{BR}_{A^i}(s)$, i.e. $s^i(b^i) > 0$ then

$$\forall c^i \in \mathrm{BR}_{A^i}(s): \quad u^i(c^i, s^{-i}) > u^i(b^i, s^{-i})$$

Consider another mixed strategy r^i, defined as follows:

$$\begin{cases} r^i(a^i) = s^i(a^i) & \forall a^i \in A^i / \{b^i, c^i\} \\ r^i(b^i) = 0 \\ r^i(c^i) = s^i(b^i) + s^i(c^i) \end{cases}$$

then

$$u^i(r^i, s) = \sum_{a^i \in A^i} r^i(a^i) u(a^i, s^{-i}) + r^i(b^i) u^i(b^i, s^{-i}) + r^i(c^i) u^i(c^i, s^{-i}) =$$

$$= \sum_{a^i \in A^i} s^i(a^i) u^i(a^i, s) + [s^i(b^i) + s^i(c^i)] u^i(c^i, s^{-i}) >$$

$$\sum_{a^i \in A^i} s^i(a^i) u^i(a^i, s^{-i}) + s^i(b^i) u^i(b^i, s^{-i}) + s^i(c^i) u^i(c^i, {}^{-i}s) = u^i(s^i, s^{-i})$$

contradiction with $s^i \in \mathrm{BR}^i(s)$. □

$\mathrm{BR}_i(s) \subset \mathrm{co}\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right)$ comes straight from lemma (our mixed best response has zeros when it is not in pure best response).

- $\mathrm{BR}_i(s) \supset \mathrm{co}\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right)$

BR is convex valued. We need to show that $\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right) \subset \mathrm{BR}^i(s)$
Suppose not Let $b^i \in \mathrm{BR}^i(s)$ and suppose $\delta_{b^i} \notin \mathrm{BR}^i(s)$ then

$$\exists s^i \in \Delta(A^i) \quad u^i(s^i, s^{-i}) > u^i(b^i, s^{-i})$$

$$\sum_{a^i \in A^i} s^i(a) u^i(a^i, s^{-i}) > u^i(b^i, s^{-i}) = \sum_{a^i \in A^i} s^i(a^i) u^i(b^i, s^{-i})$$

for at least one a^i $u^i(a^i, s^{-i}) > u^i(b^i, s^{-i})$ contradicts $b^i \in \mathrm{BR}^i_{A^i}(s)$ □

Lemma 13. $\forall i \quad \forall s^{-i} \quad u^i\left(\cdot, s^{-i}\right) : s^i \to u^i\left(s^i, s^{-i}\right)$ *is linear, and thus it is continuous.*

Lemma 14. $\forall i \quad u^i : S \to \mathbb{R}$ *is continuous and linear in each argument, fixing other arguments.*

Lemma 15. *If A^i is finite then S^i is closed.*

Proof. Let A^i be finite. Take any $\{s_n^i\}_{n\in\mathbb{N}} \in S^{i^{\mathbb{N}}}$ such that $s_n^i \to s^i$. Then $\forall n \sum_{a^i\in A^i} s_n^i\left(a^i\right) = 1$. Taking limits:

$$\begin{aligned} &\lim_{n\to\infty} \sum_{a^i\in A^i} s_n^i\left(a^i\right) = \lim_{n\to\infty} 1 \\ \Longrightarrow &\sum_{a^i\in A^i} \lim_{n\to\infty} s_n^i\left(a^i\right) = 1 \\ \Longrightarrow &\sum_{a^i\in A^i} s^i\left(a^i\right) = 1 \end{aligned}$$

Also $\forall n \forall a^i \in A^i s_n^i\left(a^i\right) \geq 0$. Taking limits again, clearly $s^i\left(a^i\right) \geq 0$. Thus S^i is closed. $\square$

1.7 Correspondences

Let $\Theta \subseteq \mathbb{R}^n, X \subseteq \mathbb{R}^n$.

Definition 34. *A correspondence* $\Gamma : \Theta \rightrightarrows X$ *is a map s.t.* $\Gamma(\Theta) \subseteq X$. $\left(\Gamma : \Theta \to 2^X\right)$

Definition 35. ***(Graph of correspondence)****.* $Gr(\Gamma) = \{(\theta, x) : \theta \in \Theta, x \in \Gamma(\theta)\}$

Definition 36. ***(Properties of correspondences)****.*

1. ***not empty valued*** *if* $\Gamma(\theta) \neq \emptyset \quad \forall \theta$
2. ***single valued*** *if* $|\Gamma(\theta)| = 1 \quad \forall \theta$
3. ***closed valued*** *if* $\Gamma(\theta)$ *is closed set* $\forall \theta$
4. ***compact valued*** *if* $\Gamma(\theta)$ *is compact set* $\forall \theta$
5. ***convex valued*** *if* $\Gamma(\theta)$ *is convex set* $\forall \theta$
6. ***closed (graph)*** *if* $\mathrm{Gr}(\Gamma)$ *is closed subset of* $\mathbb{E} \times X$
7. ***convex (graph)*** *if* $Gr(\Gamma)$ *is convex on* $\Theta \times X$

Lemma 16. *$Gr(\Gamma)$ is closed graph* $\Longleftrightarrow \forall_{\theta:\theta_n\to\theta} \forall_{x_n\to x} : x_n \in \Gamma\left(\theta_n\right) \Rightarrow \quad x \in \Gamma\left(\theta\right)$

Lemma 17. $\mathrm{Gr}(\Gamma)$ *is convex graph* $\Longleftrightarrow \forall_\theta, \theta', x \in \Gamma(\theta), x' \in \Gamma\left(\theta'\right)$ *it holds that* $\lambda x + (1-\lambda)x' \in \Gamma\left(\theta\lambda + (1-\lambda)\theta'\right) \forall_{x\in[0,1]}$

Lemma 18. $\Gamma : \Theta \rightrightarrows X$ *has closed* graph $\Rightarrow$ *it is closed valued. If X is compact, than Γ is also compact valued.*

Definition 37. ***(Upper Hemi-Continuity)*** *Let* $\Gamma : \Theta \rightrightarrows X$ *be a correspondence.*

- Γ *is said to be* ***upper hemi-continuous (uhc)*** *at a point* $\theta \in \Theta$ *if and only if for all open sets* $V \subseteq X$ *such that* $\Gamma(\theta) \subseteq V$*, there exists an open set* $U \subseteq \Theta$ *such that* $\theta \in U$ *and for all* $\theta' \in U$ *it holds that* $\Gamma\left(\theta'\right) \subseteq V$

- *A compact valued correspondence $\Gamma : \Theta \rightrightarrows X$ is u.h.c. at $\theta \in \Theta$ if and only if for every $\{\theta_n\} \subset \Theta$ such that $\theta_n \to \theta$ and every sequence $\{x_n\} \subset X$ such that $x_n \in \Gamma(\theta_n)$ there exits a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x \in \Gamma(\theta)$*

$$\forall_{\theta_n \to \theta} \forall_{x_n \in \Gamma(\theta_n)} \exists_{\{x_{n_k}\}} x_{n_k} \to x \in \Gamma(\theta)$$

Definition 38. ***(Lower Hemi-Continuity)****. Let $\Gamma : \Theta \rightrightarrows X$ be a correspondence.*

- *Γ is said to be* ***lower hemi-continuous (lhc)*** *at a point $\theta \in \Theta$ if and only if for all open sets $V \subseteq X$ such that $\Gamma(\theta) \cap V \neq \emptyset$, there exists an open set $U \subseteq \Theta$ such that $\theta \in U$ and for all $\theta' \in U$ it holds that $\Gamma(\theta') \cap V \neq \emptyset$*

- *A correspondence $\Gamma : \Theta \rightrightarrows X$ is l.h.c. at $\theta \in \Theta$ if for all $x \in \Gamma(\theta)$ and all sequences $\{\theta_n\} \subset \theta$ such that $\theta_n \to \theta$ there exits a sequence $\{x_n\} \subset X$ such that $x_n \in \Gamma(\theta_n)$ and $x_n \to x$*

$$\forall_{\theta_n \to \theta} \forall_{x \in \Gamma(\theta)} \exists_{x_n \in \Gamma(\theta_n)} x_n \to x$$

Definition 39. ***(Continuity)*** *Γ is said to be continuous at a point $\theta \in \Theta$ if it is both UHC an LHC.*

Lemma 19. ***(u.h.c and Closed graph)*** *Let $\Gamma : \Theta \rightrightarrows X$. If Γ is u.h.c, then Γ is closed (has a closed graph).*

Lemma 20. ***(Closed graph and u.h.c.)*** *Let $\Gamma : \Theta \rightrightarrows X$. If X is compact and Γ is closed (has a closed graph), then Γ is u.h.c.*

Theorem 3. ***(Berge (1961) of Maximum) Let $\Theta \subseteq \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$, let $f : \Theta \times X \to \mathbb{R}$ be a continuous function and $\Gamma : \Theta \rightrightarrows X$ a nonempty, compact valued, continuous correspondence. Define:***

$$v(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta) \quad G(\theta) = \{x \in \Gamma(\theta) \mid f(x, \theta) = v(\theta)\}$$

Then

- ***$v : \Theta \to X$ is continuous***

- ***$G : \Theta \rightrightarrows X$ is nonempty and compact valued, and UHC***

Proof. The proof is divided in three parts. First it is proven that G is nonempty and compact valued, then that it is u.h.c. and finally that v is continuous.

1. G is nonempty valued and compact valued.

    - Let $\theta \in \Theta$, by hypothesis $\Gamma(\theta)$ is compact and nonempty. since $f(\cdot, \theta)$ is continuous a maximum is attained on $\Gamma(\theta)$ by the extreme value theorem (Weierstrass). This proves that $G(\theta)$ is nonempty for arbitrary θ.

- Let $\theta \in \Theta$, by hypothesis $\Gamma(\theta)$ is compact and nonempty. since $G(\theta) \subseteq \Gamma(\theta)$ it follows that $G(\theta)$ is bounded, it is left to show closedness to establish compactness. Let $x_n \to x$ and $x_n \in G(\theta)$ for all n. Clearly $x_n \in \Gamma(\theta)$ for all n, since Γ is closed valued it follows that $x \in \Gamma(\theta)$, so its feasible. By definition of G we have $v(\theta) = f(x_n, \theta)$ for all n, since f is continuous we get $v(\theta) = \lim f(x_n, \theta) = f(x, \theta)$, then by definition $x \in G(\theta)$, which proves closedness.

2. G is u.h.c. Consider $\theta \in \Theta$, a sequence in Θ such that $\theta_n \to \theta$ and a sequence in X such that $x_n \in G(\theta_n)$ for all n. Note that $x_n \in \Gamma(\theta_n)$. since Γ is u.h.c. there exists a subsequence $x_{n_k} \to x \in \Gamma(\theta)$ Now consider $z \in \Gamma(\theta)$. since Γ is l.h.c. there exists a sequence in X such that $z_n \in \Gamma(\theta_n)$ and $z_n \to z$. In particular the subsequence $\{z_{n_k}\}$ also converges to z since $x_n \in G(\theta_n)$ and $z_n \in \Gamma(\theta_n)$ it follows that $f(x_n, \theta_n) \geq f(z_n, \theta_n)$. since f is continuous in both arguments we get by taking limits: $f(x, \theta) \geq f(z, \theta)$. since the inequality holds for arbitrary $z \in \Gamma(\theta)$ we get the result: $x \in G(\theta)$. This proves u.h.c.

3. v is continuous. Let $\theta \in \Theta$ and $\theta_n \to \theta$ an arbitrary sequence converging to θ. Consider an arbitrary sequence in X such that $x_n \in G(\theta_n)$ for all n. Let $\bar{v} = \limsup v(\theta_n)$. By proposition 2.9 there is a subsequence $\{\theta_{n_k}\}$ such that $v(\theta_{n_k}) \to \bar{v}$. since G is u.h.c. there exists a subsequence of $\{x_{n_k}\}$ (call it $\{x_{n_{kl}}\}$) converging to a point $x \in G(\theta)$. Then

$$\bar{v} = \lim v(\theta_{k_l}) = \lim f(x_{k_l}, \theta_{k_l}) = f(x, \theta) = v(\theta)$$

where the second equality follows from $x_{k_l} \in G(\theta_{k_l})$, the third one from f being continuous and the final one from $x \in G(\theta)$. Let $\underline{v} = \liminf v(\theta_n)$ and by a similar argument we get $v(\theta) = \underline{v}$ since $v(\theta) = \liminf v(\theta_n) = \limsup v(\theta_n)$ we get $v(\theta) = \lim v(\theta_n)$ for arbitrary $\{\theta_n\}$ converging to θ. This proves continuity.

$\square$

Theorem 4. ***(ToM under convexity) Let $\Theta \subseteq \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$, let $f : \Theta \times X \to \mathbb{R}$ be a continuous function and $\Gamma : \Theta \rightrightarrows X$ a nonempty, compact valued, continuous correspondence. Define:***

$$v(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta) \qquad G(\theta) = \{x \in \Gamma(\theta) \mid f(x, \theta) = v(\theta)\}$$

- ***a If $f(\cdot, \theta)$ is concave in x for all θ and Γ is convex valued then G is convex valued.***
- ***b If $f(\cdot, \theta)$ is strictly concave in x for all θ and Γ is convex valued then G is single valued, hence a continuous function.***
- ***c If f is concave on $\Theta \times X$ and Γ has a convex graph then v is concave and G is convex valued.***
- ***d If f is strictly concave on $\Theta \times X$ and Γ has a convex graph then v is strictly concave and G is single valued, hence a continuous function.***

Theorem 5. ***Kakutani's Fixed Point Theorem – u.h.c. correspondence***

Let $S \subset \mathbb{R}^n$ be nonempty, compact, and convex, and $\Gamma : S \rightrightarrows S$ be a nonempty, convex-valued, and u.h.c. correspondence.
Then Γ has a fixed point in S, i.e. $\exists x^* \in S : x^* \in \Gamma(x^*)$

Since S is compact, u.h.c. is equivalent to Γ having a closed graph.

Example 6. *Under standard assumptions, prove the following properties of $\mathrm{BR}^i_{A^i}(s)$:*
i) non-empty valued,
ii) compact valued,
iii) upper hemi continuous.
iv) Is it convex-valued?

Example 7. *Under standard assumptions, prove the following properties of $\mathrm{BR}_i(s)$:*
i) non-empty valued,
ii) compact valued,
iii) upper hemi continuous.
iv) Is it convex-valued?

Proof. (i)Take any $s \in S$. Then $BR^i(s) = \arg\max_{r^i \in S^i} u^i\left(r^i, s^{-i}\right)$. Since $u^i\left(\cdot, s^{-i}\right)$ is continuous and $S^i = \Delta\left(A^i\right)$ is compact, by the Weierstrass Theorem u^i achieves a maximum on S^i. Hence, $BR^i(s)$ is nonempty. Since s has been arbitrary, $BR^i(\cdot)$ is nonempty-valued.

(ii) Fix $s \in S$ arbitrarily and take any sequence $\left(r^i_m\right) \in BR^i(s)^\infty$ that converges in S^i, i.e. $r^i_m \rightarrow r^i \in S^i$. By definition we have $u^i\left(r^i_m, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right) \forall t^i \in S^i, m \in \mathbb{N}$. Then since $u^i\left(\cdot, s^{-i}\right)$ is continuous,

$$u^i\left(r^i, s^{-i}\right) = u^i\left(\lim_{m\to\infty} r^i_m, s^{-i}\right) = \lim_{m\to\infty} u^i\left(r^i_m, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right) \quad \forall t^i \in S^i$$

Hence, $r^i \in BR^i(s)$. Since s has been arbitrary, $BR^i(\cdot)$ is closed-valued.

(iii) Since S^i (the range of $BR^i(\cdot)$) is compact, it is sufficient to establish that $BR^i(\cdot)$ has a closed graph. Fix $s \in S$ arbitrarily and take any sequences $(s_m) \in S^\infty$ and $\left(r^i_m\right) \in S^{i\infty}$ with $s_m \rightarrow s \in S, r^i_m \rightarrow r^i \in S^i$ and $r^i_m \in BR^i\left(s_m\right) \forall m \in \mathbb{N}$. Then $u^i\left(r^i_m, s^{-i}_m\right) \geq u^i\left(t^i, s^{-i}_m\right), \forall t^i \in S^i$. Since $u^i(\cdot,\cdot)$ is continuous it follows that $\forall t^i \in S^i$

$$\begin{aligned} u^i\left(r^i, s^{-i}\right) = u^i\left(\lim_{m\to\infty} r^i_m, \lim_{m\to\infty} s^{-i}_m\right) &= \lim_{m\to\infty} u^i\left(r^i_m, s^{-i}_m\right) \\ &\geq \lim_{m\to\infty} u^i\left(t^i, s^{-i}_m\right) \\ &= u^i\left(t^i, \lim_{m\to\infty} s^{-i}_m\right) \\ &= u^i\left(t^i, s^{-i}\right) \end{aligned}$$

Hence, $r^i \in BR^i(s)$ and $BR^i(\cdot)$ is closed at s. Since s has been arbitrary, $BR^i(\cdot)$ has a closed graph.

(iv) Fix $s \in S$ arbitrarily and take any $r_a^i, r_b^i \in BR^i(s)$ and $\lambda \in (0,1)$. Then it must be that $u^i\left(r_a^i, s^{-i}\right) = u^i\left(r_b^i, s^{-i}\right) \geq u^i\left(r^i, s^{-i}\right) \forall r^i \in S^i$. Or, equivalently,

$$\sum_{a^i \in A^i} r_a^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) = \sum_{a^i \in A^i} r_b^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) \geq \sum_{a^i \in A^i} r^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) \quad \forall r^i \in S^i$$

Now consider the mixed strategy $\lambda r_a^i + (1-\lambda) r_b^i$. The utility of this strategy profile is

$$\begin{aligned} u^i\left[\lambda r_a^i + (1-\lambda) r_b^i, s^{-i}\right] &= \sum_{a^i \in A^i}\left[\lambda r_a^i\left(a^i\right) + (1-\lambda) r_b^i\left(a^i\right)\right] u^i\left(a^i, s^{-i}\right) \\ &= \lambda \sum_{a^i \in A^i} r_a^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) + (1-\lambda) \sum_{a^i \in A^i} r_b^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) \\ &= \sum_{a^i \in A^i} r_a^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) \\ &\geq u^i\left(r^i, s^{-i}\right) \quad \forall r^i \in S^i, \end{aligned}$$

where the third line follows from (2) and the inequality holds since $r_a^i \in BR^i(s)$. Hence, $\lambda r_a^i + (1-\lambda) r_b^i \in BR^i(s)$ and, since s has been arbitrary, $BR^i(\cdot)$ is convex-valued. $\square$

Lemma 21 (Properties of Best Response Correspondence). *$BR^i : S \rightrightarrows S^i$ is nonempty-valued, compact-valued, convex-valued, and upper hemicontinuous.*

Proof. Assume A^i is nonempty and finite. Then recall BR^i is the argmax of the problem (for a given s^{-i})

$$\max_{s^i \in S^i} u^i\left(s^i, s^{-i}\right)$$

then by Berge theorem we have that $BR^i : S \rightrightarrows S^i$ is nonempty-valued, compact-valued, convex-valued, and upper hemicontinuous. $\square$

Theorem 6 (Existence of Nash Equilibrium 1950). ***The correspondence** $BR : S \rightrightarrows S$ **defined by** $BR(s) = X_{i \in I} BR^i(s)$ **is***

(1) nonempty-valued

(2) closed-valued

(3) convex-valued

(4) upper hemicontinous.

Proof. Fix $s = \left(s^1, s^2, \ldots, s^n\right) \in S$ arbitrarily.
(1) BR maps s into the set $BR^1(s) \times BR^2(s) \times \cdots \times BR^n(s)$. Since each $BR^i(s), i \in I$, is nonempty and I is finite, we can choose an element $r^i \in BR^i(s)$ for each $i \in I$. Then $\left(r^1, r^2, \ldots, r^n\right) \in BR^1(s) \times \cdots \times BR^n(s) = BR(s)$. Then, since s has been arbitrary, $BR(s)$ is nonempty for all $s \in S$. Hence, BR is nonempty-valued.

(2) Take any $r_a, r_b \in BR(s)$ and $\lambda \in (0,1)$. Then

$$\lambda r_a + (1-\lambda) r_b = \left(\lambda r_a^1 + (1-\lambda) r_b^1, \ldots, \lambda r_a^n + (1-\lambda) r_b^n\right)$$

Since for each $i \in I$ the set $BR^i(s)$ is convex, $\lambda r_a^i + (1-\lambda) r_b^i \in BR^i(s) \forall i \in I$. Then $\lambda r_a + (1-\lambda) r_b \in BR(s)$ and, hence, $BR(s)$ is a convex set for all $s \in S$, i.e., BR is convex- valued.

(3) Take any point $v = (v^1, \ldots, v^n) \notin BR(s)$. Then for some $i \in I, v^i \notin BR^i(s)$. Since $BR^i(S)$ is closed in S^i, v^i is not a limit point of $BR^i(s)$. That is, there exists an open set ${}^1U^i \subset S^i$ containing v^i that contains no more than a finite number of points of $BR^i(s)$. Now, $\forall j \neq i$, choose any $U^j \subset S^j$. Then the neighborhood $U = X_{i \in I} U^i$ of v contains no more than a finite number of points of $BR(s)$, i.e. v is not a limit point of $BR(s)$. Since v has been arbitrary, for all $v \notin BR(s) v$ is not a limit point of $BR(s)$, which implies that $BR(s)$ contains all of its limit points and is, hence, closed in S.
Since $S^i \subset \mathbb{R}_+^{m_i}, \forall i \in I$, where m_i is the cardinality of A^i, I consider each S^i as a metric subspace of $\mathbb{R}^{m_i}$ with the Euclidean metric. Then $S = \mathrm{X}_{i \in I} S^i$ is considered as a metric subspace with the usual product metric.

(4) Take any sequences $(s_m), (r_m) \in S^\infty$ such that $s_m \rightarrow s$ and $r_m \in BR(s_m) \forall m$.[2] Then for all $i \in I, (s_m^i), (r_m^i) \in S^{i\infty}, s_m^i \rightarrow s^i$, and $r_m^i \in BR^i(s_m) \forall m$. Since BR^i is u.h.c., this implies that there exists a subsequence $r_{m_k}^i \rightarrow r^i \in BR^i(s)$. Then the sequence $r_{m_k} = (r_{m_k}^1, \ldots, r_{m_k}^n)$ of r_m converges to $r = (r^1, \ldots, r^n) \in BR^1(s) \times \cdots \times BR^n(s)$. Hence, BR is upper hemicontinuous

□

1.8 Zero sum games

Definition 40. *A two players finite action normal form game is zero sum if the sum of the utilities of the two players is equal to 0 for any action profile, so $u^1 = -u^2$.*

Theorem 7 (Minimax- von Neumann 1928)**.** ***For any 2-player zero-sum game,***

$$\min_{\alpha^2 \in \Delta(A^2)} \max_{\alpha^1 \in \Delta(A^1)} u(\alpha^1, \alpha^2) = \max_{\alpha^1 \in \Delta(A^1)} \min_{\alpha^2 \in \Delta(A^2)} u(\alpha^1, \alpha^2) \equiv v$$

Proof. We will do it in two steps: First we will prove that $\geq$ holds. Secondly that $\leq$ holds.

$\geq$. Note that for any $\bar{s}^1 \in \Delta(A^1)$ and $\bar{s}^2 \in \Delta(A^2)$ it holds that:

$$u(\bar{s}^1, \bar{s}^2) \geq \min_{s^2 \in \Delta(A^2)} u(\bar{s}^1, s^2)$$

Then by taking maximum at both sides with respect to s^1 :

$$\max_{s^1 \in \Delta(A^1)} u(s^1, \bar{s}^2) \geq \max_{s^1 \in \Delta(A^1) s^2 \in \Delta(A^2)} \min_x u(s^1, s^2)$$

Note that the RHS is now constant, and a lower bound to the LHS across s^2, then:

$$\min_{s^2 \in \Delta(A^2)} \max_{s^1 \in \Delta(A^1)} u(s^1, s^2) \geq \max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} u(s^1, s^2) \quad (1)$$

$\leq$. Note that for any $\bar{s}^1 \in \Delta\left(A^1\right)$ it holds that:

$$\max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} u\left(s^1, s^2\right) \geq \min_{s^2 \in \Delta(A^2)} u\left(\bar{s}^1, s^2\right)$$

In particular for $\hat{s}^1$ a NE of the game the inequality must hold. We assume that such NE exists in mixed strategies. Note that if $\left(\hat{s}^1, \hat{s}^2\right)$ it is defined as an strategy profile such that:

$$u\left(\hat{s}^1, \hat{s}^2\right) = \max_{s^1 \in \Delta(A^1)} u\left(s^1, \hat{s}^2\right) \quad - u\left(\hat{s}^1, \hat{s}^2\right) = \max_{s^2 \in \Delta(A^2)} -u\left(\hat{s}^1, s^2\right)$$

The second condition implies:

$$u\left(\hat{s}^1, \hat{s}^2\right) = \min_{s^2 \in \Delta(A^2)} u\left(\hat{s}^1, s^2\right) = \max_{s^1 \in \Delta(A^1)} u\left(s^1, \hat{s}^2\right)$$

thus

$$\begin{aligned} \min_{s^2 \in \Delta(A^2)} u^1\left(\hat{s}^1, s^2\right) &= u^1\left(\hat{s}^1, \operatorname*{argmin}_{s^2 \in \Delta(A^2)} u^1\left(\hat{s}^1, s^2\right)\right) \\ &= u^1\left(\hat{s}^1, \operatorname*{argmax}_{s^2 \in \Delta(A^2)} u^2\left(\hat{s}^1, s^2\right)\right) \\ &= u^1\left(\hat{s}^1, \hat{s}^2\right) \\ &= \max_{s^1 \in \Delta(A^1)} u^1\left(s^1, \hat{s}^2\right) \\ &\geq \min_{s^2 \in \Delta(A^2)} \max_{s^1 \Delta(A^1)} u^1\left(s^1, s^2\right) \end{aligned}$$

Then by taking max over $\Delta\left(A^1\right)$:

$$\max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} u\left(s^1, s^2\right) \geq \min_{s^1 \in \Delta(A^1)} u\left(s^1, \hat{s}^2\right) \geq \min_{s^2 \in \Delta(A^2)} \max_{s^1 \in \Delta(A^1)} u\left(s^1, s^2\right) \tag{2}$$

Inequalities (1) and (2) gives us thesis of minimax theorem. □

Definition 41. *For a zero sum game of two players define the value of the game as* $V : \mathbb{R}^{nm} \to \mathbb{R}$ *(where* $n = \#A^1$ *and* $m = \#A^2$*) :*

$$V(u) = \max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u\right)$$

where for a given strategy profile $s^1 = (p_1, \ldots, p_n), s^2 = (q_1, \ldots, q_n)$ *and payoffs* $u \in \mathbb{R}^{nm}$ *we define*

$$U\left(s^1, s^2 \mid u\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j u_{ij}$$

Lemma 22. *Show that* ***The value of a game*** *is*

a) continuous

b) non-decreasing

c) homogenous of degree one in payoffs.

- Consider the problem:

$$v\left(s^1, u\right) = \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u\right)$$

note that U is continuous in s_1, s_2 and u and that the minimum is being taken over s^2 in a compact set that does not vary with s^1 or u. By the theorem of the maximum the value of this problem, as a function of s^1 and u is a continuous function. Now consider:

$$V(u) = \max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u\right) = \max_{s^1 \in \Delta(A^1)} v\left(s^1, u\right)$$

again since v is continuous and s^1 varies in a compact set independent of u by the theorem of the maximum V is a continuous function of u.

- Let $u_1 \leq u_2$. Clearly for all s^1, s^2:

$$U\left(s^1, s^2 \mid u_1\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j u_{ij}^1 \leq U\left(s^1, s^2 \mid u_1\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j u_{ij}^2$$

so $U\left(s^1, s^2 \mid u_1\right) \leq U\left(s^1, s^2 \mid u_2\right)$. Then:

$$\min\nolimits_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u_1\right) \leq \min\nolimits_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u_2\right)$$
$$V\left(u_1\right) = \max\nolimits_{s^1 \in \Delta(A^1)} \min\nolimits_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u_1\right) \leq \max\nolimits_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u_2\right) = V\left(u_2\right)$$

- Let $\lambda \in \mathbb{R}$, note that $U\left(s^1, s^2 \mid \lambda u\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j \lambda u_{ij} = \lambda U\left(s^1, s^2 \mid u\right)$ and $\max_x \lambda f(x) = \lambda \max_x f(x)$. Thus $V(\lambda u) = \lambda V(u)$

1.9 Perfect Equilibria

Example 8. *Let $I = \{1, 2\}$ and consider the game G defined by*

	L	R
T	*1,1*	*0,0*
B	*0,0*	x, y

where $1 > x > 0$ and $y > 0$. Suppose that $s^1 = (p, 1-p)$ and $s^2 = (q, 1-q)$ so that $s = ((p, 1-p), (q, 1-q))$. Then we can view each player's best response as a function of the other player's mixed strategy. In particular, if player 2 plays L, his expected utility is p. If he plays R it is $(1-p)y$. So his best response depends on the value of p. Similarly for player 1. Then G has three NE.

$$NE = \left\{((1,0),(1,0)),((0,1),(0,1)),\left(\left(\frac{y}{1+y}, 1-\frac{y}{1+y}\right),\left(\frac{x}{1+x}, 1-\frac{x}{1+x}\right)\right)\right\}$$

Typically $|NE|$ is odd. However, not in general. For instance, in G let $x = y = 0$. Compute equilibrium. Show it is strange in that it gives positive probability to a weakly dominated strategy. Motivate perfect equilibria/perterbations by show that if player 2 plays L with some small but positive probability, this strange equilibrium goes away.

Definition 42 (Utility robust NE). *Given a* NEs_u *of* (I, S^i, u^i) , s_u *for* u *is utility robust if* $\forall\delta\exists\bar{\epsilon} > 0$ *such that* $\forall v$ *such that* $\|v - u\| < \epsilon$ *where* $\epsilon < \bar{\epsilon}, \exists s_v$ *such that* $\|s_v - s_u\| < \delta$

Definition 43 (Perturbation). *A perturbation is* $\epsilon = (\epsilon^i)_{i\in I}$, *where* $\forall i \in I \epsilon^i = (e^i(a^i))_{a^i\in A^i}$, *such that:*

$$\forall i \in I \forall a^i \in A^i, \epsilon^i(a^i) > 0 \quad \wedge \quad \forall i \in I, \sum_{a^i\in A^i} \epsilon^i(a^i) < 1$$

Perturbation is not a mixed strategy.

Definition 44 (Perturbed strategy set). *The perturbed strategy set for player i is*

$$S^i_{\epsilon^i} \equiv \{s^i \in S^i \mid \forall a^i \in A^i, s^i(a^i) \geq \epsilon^i(a^i)\}$$

The perturbed strategy set for all players is $S_\epsilon \equiv X_{i\in I} S^i_\epsilon$

Definition 45. *NE of ϵ -perturbed game $s \in S_\epsilon$ is a* NE *of the ϵ -perturbed game if* $\forall i \in I, \forall t^i \in S^i_{\epsilon i}, u^i(s^i, s^{-i}) \geq u^i(t^i, s^{-i})$

[A NE of the ϵ -perturbed game is $\hat{s} \in S_\epsilon$ such that $\forall i \in I \hat{s}^i \in BR^i_{S^i_\epsilon}(\hat{s})$.

Definition 46. *Perfect equilibrium Let* $\left(I, (S^i, u^i)_{i\in I}\right)$ *be a NFG. Then* $s \in S$ *is a* PE *if* $\exists \{\epsilon_m\}_{m\in\mathbb{N}}, \{s_m\}_{m\in\mathbb{N}}$ *such that* $\epsilon_m \to 0, s_m \to s$, *and* s_m *is a* NE *of the* ϵ_m *-perturbed game (for each m)* [$s \in S$ is PE *if it is the limit of a sequence of* NE *of some* ϵ *-perturbed game, where* $\epsilon \to 0$.

Theorem 8. ***The set of* PE *is nonempty***

Proof. As proved in Theorem 2.2, for any finite game the set of NE is nonempty. It follows immediately that for any ϵ -perturbation of a finite game, the set of NE is nonempty. Then, for any sequence of perturbations $\epsilon_n \to 0$, there exists $s_n \in S_{\epsilon_n}$ such that s_n is a NE of the ϵ_n

-perturbed game. Then s_n is a sequence in S, and since S is compact, there exists a convergent subsequence $s_{n_k} \to s \in S$. Then s is a perfect equilibrium by definition, and thus the set of PE is nonempty. $\square$

Theorem 9. ***If $s \in S$ is a PE, then it is also a NE.***

Proof. Let $s \in S$ be a PE. Then $\exists \{\epsilon_m\}_{m\in\mathbb{N}}, \{s_m\}_{m\in\mathbb{N}}$ such that $\epsilon_m \to 0, s_m \to s$, and $\forall m \in \mathbb{N}, s_m$ is a NE of the ϵ_m -perturbed game. Take any $i \in I$ and any $t^i \in S^i$. Since $\epsilon_m \to 0$, it follows that $\epsilon_m^i \to 0$, and thus there exists a sequence $t_m^i \in S^i_{\epsilon_i^i}$ such that $t_m^i \to t^i$. Take such a sequence. Then, since s_m is a NE of the ϵ_m -perturbed game, it follows that

$$u^i\left(s_m^i, s_m^{-i}\right) \geq u^i\left(t_m^i, s_m^{-i}\right) \quad \forall m \in \mathbb{N}$$

Since $u^i(\cdot)$ is continuous $\forall i \in I$, then

$$\begin{aligned}
&\lim u^i\left(s_m^i, s_m^{-i}\right) \geq \lim u^i\left(t_m^i, s_m^{-i}\right) \\
&\Longrightarrow u^i\left(s^i, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right)
\end{aligned}$$

Since $t^i \in S^i$ was taken arbitrarily, $s^i \in BR^i\left(s^{-i}\right)$. Since $i \in I$ was taken arbitrarily, $s \in BR(s)$, so s is a NE $\square$

Theorem 10. ***If $s \in S$ is a fully mixed* NE, *then it is also a* PE.**

Proof. Let $s \in S$ be a fully mixed NE for some finite NFG, i.e. $\forall i \in I, \forall a^i \in A^i$, $s^i\left(a^i\right) > 0$. From this, note there exists

$$\bar{s}^i \equiv \min_{a^i \in A^i} s^i\left(a^i\right) \forall i \in I \text{ and } \bar{s} \equiv \min_{i\in I} \bar{s}^i$$

and that $\bar{s} > 0$. It follows that, for any sequence of perturbations $\{\epsilon_n\}_{n\in\mathbb{N}}$ such that $\epsilon_n \to 0, \exists N \in \mathbb{N}$ such that, $\forall m \geq N$

$$\forall i \in I \forall a^i \in A^i, \quad e_m^i\left(a^i\right) < \bar{s}$$

so $\forall m \geq N, s \in S_{\epsilon_m}$. Now recall that since s is a NE of the original game,

$$\forall i \in I, \forall t^i \in S^i, \quad u^i\left(s^i, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right)$$

Note that $S^i_{\epsilon_m^i} \subseteq S^i$, so since $\forall m \geq N, s \in S_{\epsilon_m}$, we know that $\forall m \geq N, s$ is a NE of the ϵ_m -perturbed game. Now take a sequence $\{s_m\}$ such that $s_m = s \forall m \in \mathbb{N}$ and construct a new sequence of perturbations $\{\hat{\epsilon}_m\} = \{\epsilon_m\}_{m\geq N}$. Then s is a PE by definition. $\square$

1.10 Iterated Elimination and Rationalizability

Definition 47 (Weak dominance)**.** *An action $a^i \in A^i$ is weakly dominated if $\exists s^i \in \Delta\left(A^i\right)$ such that:*

$$\forall b^{-i} \in A^{-i}, \quad u^i\left(s^i, b^{-i}\right) \geq u^i\left(a^i, b^{-i}\right)$$

for at least one $c^{-i} \in A^{-i}$, $\quad u^i\left(s^i, c^{-i}\right) > u^i\left(a^i, c^{-i}\right)$

Definition 48 (Strict dominance)**.** *An action $a^i \in A^i$ is strictly dominated if $\exists s^i \in \Delta\left(A^i\right)$ such that:*

$$\forall b^{-i} \in A^{-i}, u^i\left(s^i, b^{-i}\right) > u^i\left(a^i, b^{-i}\right)$$

Definition 49 (Weakly undominated)**.** *A strategy profile $s \in S$ is weakly undominated if and only if $\forall i \in I, s^i$ isn't weakly dominated.*

Definition 50 (Strictly undominated)**.** *A strategy profile $s \in S$ is strictly undominated if and only if $\forall i \in I, s^i$ isn't strictly dominated*

Definition 51 (Belief)**.** *We call μ^{-i} player i 's belief if and only if $\mu^{-i} \in \Delta\left(A^{-i}\right)$. [Note: $u^i\left(a^i, \mu\right) = \sum_{a^{-i} \in A^{-i}} \mu\left(a^{-i}\right) u^i\left(a^i, a^{-i}\right)$]*

Definition 52 (Never a best response)**.** *An action $a^i \in A^i$ is never a best response if $\nexists \mu \in \Delta\left(A^{-i}\right)$ such that $a^i \in BR^i_{\Delta i}(\mu)$.*

Theorem 11. ***The following three statements are equivalent:***

$$\begin{gathered} u^i\left(s^i, a^{-i}\right) > u^i\left(a^i, a^{-i}\right) \forall a^{-i} \in A^{-i} \\ u^i\left(s^i, s^{-i}\right) > u^i\left(a^i, s^{-i}\right) \forall s^{-i} \in S^{-i} \\ u^i\left(s^i, \mu^{-i}\right) > u^i\left(a^i, \mu^{-i}\right) \forall \mu^{-i} \in \Delta\left(A^{-i}\right) \end{gathered}$$

Proof. $(1) \Longrightarrow (3) : u^i\left(s^i, \mu^{-i}\right) - u^i\left(a^i, \mu^{-i}\right) = \sum_{a^{-i} \in A^{-i}} \mu\left(a^{-i}\right)\left[u^i\left(s^i, a^{-i}\right) - u^i\left(a^i, a^{-i}\right)\right]$, and the first term is greater than or equal to zero and the second is strictly greater than 0 by hypothesis. Thus the difference is strictly greater than 0 .

$(3) \Longrightarrow (2)$ Since $S^{-i} \equiv \Delta\left(A^{-i}\right)$, the result is immediate.

$(2) \Longrightarrow (1)$ Since $S^{-i} \equiv \Delta\left(A^{-i}\right), A^{-i} \subseteq S^{-i}$, and thus the result follows immediately.

□

Lemma 23. *If $s \in S$ is a NE and $a^i \in A^i$ is strictly dominated, then $s^i\left(a^i\right) = 0$*

Proof. Since a^i is strictly dominated, it is never a best response. Then it must be that $s^i\left(a^i\right) = 0$ □

Theorem 12 (Theorem of the Alternative)**.** *.*

$$\exists x \;\textbf{\textit{s.t.}} \left\{\begin{array}{c} Ax \gg a \\ Bx \geq b \\ Cx = c \end{array}\right\} \Longleftrightarrow \nexists \mu \geq 0, \lambda \geq 0, \nu \;\textbf{\textit{s.t.}} \left\{\begin{array}{c} \mu A + \lambda B + \nu c = 0 \\ \mu a + \lambda b + \nu c \geq 0 \\ \mu(a+c) + \lambda b + \nu c > 0 \end{array}\right\}$$

Theorem 13. ***A strategy $b^i \in A^i$ is strictly dominated if, and only if, it is never a best response.***

Proof. Define

$$U := \begin{bmatrix} u^i\left(a_1^i, a_1^{-i}\right) & \cdots & u^i\left(a_{\#A^i}^i, a_1^{-i}\right) \\ \vdots & \ddots & \vdots \\ u^i\left(a_1^i, a_{\#A^{-i}}^{-i}\right) & \cdots & u^i\left(a_{\#A^i}^i, a_{\#A^{-i}}^{-i}\right) \end{bmatrix}$$

Take any $b^i \in A^i$ and define

$$u := \begin{bmatrix} u^i\left(b^i, a_1^{-i}\right) \\ \vdots \\ u^i\left(b^i, a_{\#A^{-i}}^{-i}\right) \end{bmatrix}$$

So b^i is never a best response if $\nexists \mu = \left[\mu_1, \ldots, \mu_{\#A^{-i}}\right]^T \in \Delta\left(A^{-i}\right)$ such that $\mu^T U \leq \mu^T u e^T$, i.e.

$$\begin{bmatrix} u^i\left(a^i, \mu\right) \\ \vdots \\ u^i\left(a_{\#A^i}^i, \mu\right) \end{bmatrix} \leq \begin{bmatrix} u^i\left(b^i, \mu\right) \\ \vdots \\ u^i\left(b^i, \mu\right) \end{bmatrix}$$

Moreover, b^i is strictly dominated if $\exists s^i = \left[s^i\left(a_1^i\right), \ldots, s^i\left(a_{\#A^i}^i\right)\right]^T$ such that $Us^i \gg u, Is^i \geq 0$, and $e^T s^i = 1$, where I is the $\#A^i$ dimensional identity matrix. The first condition gives dominance while the second two ensure that s^i is a mixed strategy. Now, suppose b^i is never a best response but is not dominated. Then $\nexists s^i$ such that

$$\left\{ \begin{array}{c} Us^i \gg u \\ Is^i \geq 0 \\ e^T s^i = 1 \end{array} \right\}$$

Then by the Theorem of the Alternative, $\exists \mu \geq 0, \lambda \geq 0, \nu$ such that

$$\left\{ \begin{array}{c} \mu^T U + \lambda I + \nu e^T = 0 \\ \mu^T u + \lambda \cdot 0 + \nu \cdot 1 \geq 0 \\ \mu^T (u+1) + \lambda \cdot 0 + \nu \cdot 1 > 0 \end{array} \right\}$$

Notice that if $\mu = 0$ then $\lambda I + \nu e^T = 0$ and $\nu > 0$, which contradicts $\lambda \geq 0$. So $\mu \geq 0, \mu \neq 0$. Now, normalize μ, λ and ν so that $\mu \in \Delta\left(A^{-i}\right.$. Then (8) reduces to

$$\mu^T U + \nu e^T \leq 0 \quad \text{and} \quad \mu^T u + \nu \geq 0$$

which implies

$$\mu^T U \leq \mu^T u e^T$$

which contradicts b^i as a never best response. □

Definition 53 (Iterated elimination of strictly dominated strategies (IESDS)). *An IESDS is a sequence $C_t = \left(C_t^1, \ldots, C_t^i, \ldots, C_t^n\right)$ for $t = 0, \ldots, T$, where:*

1. $\forall i, C_0^i = A^i$
2. $\forall i \forall t, C_{t+1}^i \subseteq C_t^i$

3. $\forall i \forall a^i \forall t, a^i \in C_t^i \backslash C_{t+1}^i$ *if and only if* $\exists s^i \in \Delta\left(C_t^i\right)$ *such that* $\forall b^{-i} \in C_t^{-i}, u^i\left(s^i, b^{-i}\right) > u^i\left(a^i, b^{-i}\right)$

We say IESDS is complete if no elimination is possible in the C_T game

[Note: Complete IESDS results in a unique outcome.

Definition 54 (Rationalizable sets)**.** *A tuple* $R = \left(R^1, \ldots, R^n\right)$ *where* $\forall i R^i \subseteq A^i$*, is rationalizable if and only if* $\forall i, \forall a^i \in R^i, \exists \mu \in \Delta\left(R^{-i}\right)$ *such that* $\forall b^i \in A^i, u^i\left(\bar{a}^i, \mu\right) \geq u^i\left(b^i, \mu\right)$

Lemma 24. *If* R *and* S *are two rationalizable sets, then* $R \cup S = \left(R^1 \cup S^1, \ldots, R^n \cup S^n\right)$ *is rationalizable as well.*

Proof. or any $i \in I$ and any $a^i \in R^i$, since R^i is rationalizable we know $\exists \mu \in \Delta\left(R^{-i}\right)$ such that $\forall b^i \in A^i, u^i\left(a^i, \mu\right) \geq u^i\left(b^i, \mu\right)$. Therefore $\exists \mu \in \Delta\left((R \cup S)^{-i}\right)$ such that $\forall b^i \in A^i, u^i\left(a^i, \mu\right) \geq u^i\left(b^i, \mu\right)$, and thus $R \cup S$ is rationalizable. □

Lemma 25. *There is a unique maximal rationalizable set* R*, i.e.* $\nexists S \supset R$ *where* S *is rationalizable.*

Proof. Suppose not, i.e. both sets R and S are rationalizable, maximal, and $R \neq S$. Then, by the above lemma, $R \cup S$ is rationalizable as well and $R \cup S \supset R$, which contradicts R being maximal. □

Theorem 14. ***Let*** C_T ***be the outcome of a complete IESDS and let*** R ***be the unique maximal rationalizable set. Then*** $R \subseteq C_T$***.***

Proof. We proceed by induction on the elimination stages of IESDS. Note in $t = 0, \forall i R^i \subseteq C_0^i \equiv A^i$. From this, assume $\forall i R^i \subseteq C_t^i$. Then $\forall i, \forall a^i \in R^i$ it must be that:

$$\exists \mu \in \Delta\left(R^{-i}\right) \text{ such that } \forall b^i \in A^i, u^i\left(a^i, \mu\right) \geq u^i\left(b^i, \mu\right) \qquad \text{(by definition)}$$

$$\implies \exists \mu \in \Delta\left(C_t^{-i}\right) \text{ such that } \forall b^i \in A^i, u^i\left(a^i, \mu\right) \geq u^i\left(b^i, \mu\right) \qquad \text{(by hypothesis)}$$

$$\left.\begin{array}{l} \implies \exists \mu \in \Delta\left(C_t^{-i}\right) \text{ such that } \forall b^i \in C_t^i, u^i\left(a^i, \mu\right) \geq u^i\left(b^i, \mu\right) \\ \implies \nexists s^i \in \Delta\left(C_t^i\right) \text{ such that } \forall b^{-i} \in C_t^{-i}, u^i\left(s^i, b^{-i}\right) > u^i\left(a^i, b^{-i}\right) \end{array}\right. \quad \left(\text{since } C_t^i \subseteq A^i\right)$$

Thus $\forall i, \forall a^i \in R^i, a^i \in C_{t+1}^i$, so $R^i \subseteq C_{t+1}^i$. Then, by induction, $R \subseteq C_T$.

□

Theorem 15. ***Let*** C_T ***be the outcome of a complete IESDS and let*** R ***be the unique maximal rationalizable set. Then*** $C_T = R$

Proof. Since C_T is the outcome of a complete IESDS, $\forall i, \forall a^i \in C_T^i$ it must be that:

$$\nexists s^i \in \Delta\left(C_T^i\right) \text{ such that } \forall b^{-i} \in C_T^{-i}, u^i\left(s^i, b^{-i}\right) > u^i\left(a^i, b^{-i}\right)$$

$$\implies \nexists s^i \in \Delta\left(A^i\right) \text{ such that } \forall b^{-i} \in C_T^{-i}, u^i\left(s^i, b^{-i}\right) > u^i\left(a^i, b^{-i}\right)$$

$$\implies \exists \mu \in \Delta\left(C_T^{-i}\right) \text{ such that } \forall b^i \in A^i, u^i\left(a^i, \mu\right) \geq u^i\left(b^i, \mu\right)$$

with the first implication following from the fact that $\forall a^i \in A^i \backslash C_T^i, a^i$ is strictly dominated.

Since i and a^i were arbitrarily taken, it follows that C_T is rationalizable, and recall by the previous theorem $R \subseteq C_T$. Further, since R is the unique maximal rationalizable set, by the above lemma, it must be that $C_T = R$. $\square$

Theorem 16. ***If a strategy profile $s \in S$ is a perfect equilibrium then it is undominated.***

Proof. Let $s \in S$ be a perfect equilibrium and suppose s is weakly dominated. Then $\exists i \in I, r^i \in S^i$ such that

$$\begin{aligned} &\forall a^{-i} \in A^{-i}, \quad u^i\left(r^i, a^{-i}\right) \geq u^i\left(s^i, a^{-i}\right) \\ &\exists b^{-i} \in A^{-i}, \quad u^i\left(r^i, b^{-i}\right) > u^i\left(s^i, b^{-i}\right) \end{aligned}$$

Since s is a perfect equilibrium, by Theorem 2.11$\exists \left(s_n\right) \in S^\infty$ s.t. $\forall n s_n$ is fully mixed, $s_n \to s$ and $\forall (i,n) s^i \in BR^i\left(s^i, s_n^{-i}\right)$. Since s^n is fully mixed for each $n \in \mathbb{N}, \operatorname{Pr}_{s_n}\left(a^{-i}\right) > 0$ for all $a^{-i} \in A^{-i}$ By multiplying (6) – (7) by $\operatorname{Pr}_{s_n}\left(a^{-i}\right)$ and summing across A^{-i} we have

$$u^i\left(r^i, s_n^{-i}\right) = \sum_{a^{-i} \in A^{-i}} u^i\left(r^i, a^{-i}\right) \operatorname{Pr}_{s_n}\left(a^{-i}\right) > \sum_{a^{-i} \in A^{-i}} u^i\left(s^i, a^{-i}\right) \operatorname{Pr}_{s_n}\left(a^{-i}\right) = u^i\left(s^i, s_n^{-i}\right)$$

for each $n \in \mathbb{N}$. But this contradicts s^i being a best response to s_n^{-i} for all $n \in \mathbb{N}$. Hence, s is undominated. $\square$

2 Extensive Form Games

Definition 55 (Extensive Form Game)**.** *consists of*

- *The set of all nodes is* $\mathcal{X}$.
- *The set of all final nodes is* $Z = \left(z^1, z^2, \ldots\right)$, *where* z *are the consequences of the EFG.*
- *The initial node is* α. *Nature, sometimes denoted player 0 , chooses* α *with* $p \in \Delta(IS(\alpha))$
- *The set of move nodes for player* i *is* X^i; *also called player* i *'s partition. Note* $\forall i \neq j$, $X^i \cap X^j = \emptyset$ *and* $\cup_{i \in I} X^i \equiv X = \mathcal{X} \backslash \{\alpha, Z\}$
- *Let* $\succeq$ *be an asymmetric partial order over* $\mathcal{X}$, *where for* $x, y \in \mathcal{X}, x \succeq y$ *means* x *comes after* y. *Note that* $\forall x \in \mathcal{X}, x \succeq \alpha$
- $\forall x, y \in \mathcal{X}$, *let* $x \succeq_c y$ *mean* x *follows action* c *played at* y.
- $\forall x \in \mathcal{X}$, *the set of predecessor nodes is* $P(x) \equiv \{y \in \mathcal{X} \mid x \neq y, x \succeq y\}$.
- $\forall x \in \mathcal{X}$, *the set of immediate predecessor nodes is*

$$IP(x) \equiv \{z \in P(x) \mid \nexists y \neq z, y \neq x, z \preceq y \preceq x\}$$

- $\forall x \in \mathcal{X}$, *the set of successor nodes is* $S(x) \equiv \{y \in \mathcal{X} \mid x \neq y, x \preceq y\}$.
- $\forall x \in \mathcal{X}$, *the set of immediate successor nodes is*

$$IS(x) \equiv \{z \in S(x) \mid \nexists y \neq z, y \neq x, z \succeq y \succeq x\}$$

  Observe that $Z = \{x \in \mathcal{X} \mid S(x) = \emptyset\}$.
- $\forall i, u^i : Z \rightarrow \mathbb{R}$ *is a vNM utility function.*
- *An information set for player* i *is* I_k^i, *where* $k = 1, \ldots, K^i$. *Note* $\forall k \neq j, I_k^i \cap I_j^i = \emptyset$ *and* $\cup_{k=1}^{K^i} I_k^i = X^i$. *Player* i *'s set of information sets is* $\mathcal{I}^i \equiv X_{k=1}^{K^i} I_k^i$
- *For each* $I_k^i \in \mathcal{I}^i$, *an action for player* i *is* $c_{I_k^i}^i$, *or equivalently* c_k^i. *The set of actions for player* i *is* $C_{I_i^i}^i$, *or equivalently* C_k^i.

Definition 56 (A pure strategy for player i)**.** *is* $s^i = \cup_{k=1}^{K^i} c_k^i$. *A pure strategy can also be viewed as a* map $s^i : \mathcal{I}^i \rightarrow \cup_{k=1}^{K^i} c_k^i$ *such that* $\forall k s^i\left(I_k^i\right) \in C_k^i$. *The set of pure strategies for player* i *is* $S^i = X_{k=1}^{K^i} C_k^i$. *A pure strategy profile for all players is* $s = \left(s^1, \ldots, s^n\right)$

Definition 57 (A mixed strategy for player i)**.** *is* $\sigma^i \in \Sigma^i = \Delta\left(S^i\right)$, *where* Σ^i *is the set of mixed strategies for player* i. *A mixed strategy profile for all players is* $\sigma = \left(\sigma^1, \ldots, \sigma^n\right)$.

Definition 58. *The probability of reaching final node* z *under pure strategy profile* s *is* $\Pr_s(z) \in \Delta(Z)$ *The probability of reaching final node* z *under pure strategy profile* s *is* $\Pr_s(z) \in \Delta(Z)$.

Definition 59. *The probability of the pure strategy profile s being played under the mixed strategy profile σ is* $\Pr_\sigma^s(s) = \prod_{i \in I} \sigma^i\left(s^i\right)$

Definition 60. *The probability of reaching final node z under mixed strategy profile σ is* $\Pr_\sigma(z) = \sum_{s \in S} \Pr_\sigma^s(s) \Pr_s(z)$

Definition 61 (Player i 's expected utility)**.** *from playing pure strategy s is* $E_{\Pr_s}\left[u^i\right] = \sum_{z \in Z} \Pr_s(z) u^i(z)$.

Definition 62 (Extended form game (EFG))**.** *An EFG is* $G = \left(I, \mathcal{X}, \succeq, p, \left(X^i, u^i, (I_k^i, C_k^i)_{k=1,\ldots,K^i}\right)_{i \in I}\right)$

Definition 63 (Associated NFG)**.** *The pure strategy NFG associated with an EFG is* $\left(I, \left(S^i, E_{Pr.}\left[u^i\right]\right)_{i \in I}\right)$

Definition 64 (Associated mixed extension NFG)**.** *The mixed extension NFG associated with an EFG is* $\left(I, \left(\Sigma^i, E_{Prr}\left[u^i\right]\right)_{i \in I}\right)$.

Definition 65 (Nash equilibrium of EFG)**.** *A NE of an EFG is a NE of the associated mixed extension NFG, and vice-versa.*

Definition 66 (Normal form perfect equilibrium)**.** *A normal form PE of an EFG is a PE of the associated mixed extension NFG.*

Theorem 17. ***In any finite EFG,*** $\{x \in X \mid IS(x) \subseteq Z\} \neq \emptyset$***.***

Proof. Suppose not, i.e. $\forall x \in X, \exists y \in X$ such that $y \in IS(x) \backslash Z$. Then, since $y \in X$, we know $\exists w \in X$ such that $w \in IS(y) \backslash Z$. By induction, for any move node there will always be a following move node that itself has a following move node, and thus there will be an infinite amount of move nodes, contradicting the EFG being finite. □

Theorem 18. ***For any finite EFG, the set of NE is nonempty.***

Proof. Immediate by Nash's Existence Theorem □

2.1 Strategies of Extensive Form Games

Definition 67 (Behavioral strategy)**.** *A behavioral strategy for player i is* $\beta^i = \left(\beta_{I_1^i}^i, \ldots, \beta_{I_{K^i}^i}^i\right)$, *or equivalently* $\beta^i = \left(\beta_1^i, \ldots, \beta_{K^i}^i\right)$ *where* $\beta_{I_k^i}^i \in \Delta\left(C_{I_k^i}^i\right) \forall k$. *The set of behavioral strategies for player i is* $B^i \equiv X_{k=1}^{K^i} \Delta\left(C_{I_k^i}^i\right)$

[Note: A player using mixed strategy σ^i randomizes once over the set of all pure strategies. A player using behavioral strategy β^i randomizes at each information set over only the available choices at that information set.]

Definition 68 (General strategy)**.** *A general strategy for player i is π^i.*
The set of general strategies for player i is $\Gamma^i = \Delta\left(B^i\right)$

2.2 Kuhn and Dalkey Theorems

Definition 69 (Equivalence)**.** *A behavioral strategy $\beta^i \in B^i$ and a mixed strategy $\sigma^i \in \Sigma^i$ are equivalent, denoted as $\beta^i \sim \sigma^i$, if and only if they induce the same probability on final nodes Z for any given $\pi^{-i} \in \Gamma^{-i}$, i.e.:*

$$\forall \pi^{-i} \in \Gamma^{-i}, \forall z \in Z, \Pr_{(\beta^i, \pi^{-i})}(z) = \Pr_{(\sigma^i, \pi^{-i})}(z)$$

Definition 70 (Linear game)**.** *An EFG is linear if no information set intersects a path more than once, i.e.*

$$\forall i \in I, \forall I_k^i \in \mathcal{I}^i, \forall z \in Z, \# \left\{ P(z) \cap I_k^i \right\} \leq 1$$

[Note: Intuitively, every player always knows if they've moved or not.

Definition 71 (Games of perfect recall (PR))**.** *An EFG is perfect recall if and only if $\nexists I_k^i, I_l^i \in \mathcal{I}^i, x, y \in I_l^i$ such that x follows some $c_k^i \in C_k^i$ but y does not. [Alternatively, $\nexists I_k^i, I_l^i \in \mathcal{I}^i, x, y \in I_l^i, w \in I_k^i, c_k^i \in C_k^i$, such that $x \succeq_c w$ but not $y \succeq_c w$.*

[Note: In general, for linear games not of perfect recall, $\{\Pr_\sigma \mid \sigma \in \Sigma\} = \Delta(z)$ but $\{\Pr_\beta \mid \beta \in B\} \subset \Delta(z)$ and so $\forall \sigma^i \in \Sigma^i, \nexists \beta^i \in B^i$ such that $\beta^i \sim \sigma^i$.]

Definition 72 (Relevant information sets)**.** *The set of pure strategies for player i that lead to $I_k^i \in \mathcal{I}^i$ for some given strategy $s^{-i} \in S^{-i}$ of the other players, i.e. the set of pure strategies relevant for I_k^i, is:*

$$\operatorname{Rel}\left(I_k^i\right) = \left\{ s^i \in S^i \mid \exists s^{-i} \in S^{-i}, \Pr_{(s^i, s^{-i})}\left(\left\{ z \in Z \mid \Pr(z) \cap I_k^i \neq \emptyset \right\}\right) > 0 \right\}$$

Further, the set of pure strategies relevant for I_k^i that play action $c \in C_k^i$ is:

$$\operatorname{Rel}\left(I_k^i, c\right) = \left\{ s^i \in \operatorname{Rel}\left(I_k^i\right) \mid s^i\left(I_k^i\right) = c \right\} \subseteq \operatorname{Rel}\left(I_k^i\right)$$

Theorem 19. ***Every game of perfect recall is linear.***

Proof. Suppose not, i.e. there is some perfect recall EFG that is not linear. Then we know there exists some $i \in I$ and $I_k^i \in \mathcal{I}^i$ such that for some $z \in Z, \# \left\{ P(z) \cap I_k^i \right\} > 1$. Now take $x, y \in P(z) \cap I_k^i$ such that $x \succeq_c y$ for some action $c_k^i \in C_k^i$, i.e. x follows c_k^i but y does not. Now let $I_l^i = I_k^i$ and $y' = y$. Then clearly there exists $c_k^i \in C_k^i$ such that $x \succeq_c y'$ but not $y \succeq_c y'$, since a node cannot come after itself. Therefore the game is not of perfect recall, which contradicts the hypothesis. □

Theorem 20 (Dalkey)**.** ***In any linear EFG, for any behavioral strategy $\beta^i \in B^i$ there is a mixed strategy $\sigma^i \in \Sigma^i$ such that $\beta^i \sim \sigma^i$.***

Proof. Consider any linear EFG and note that $\forall i, \forall \beta^i \in B^i$ it is possible to construct $\sigma^i_{\beta^i}\left(s^i\right) = \prod_{k=1}^{K^i} \beta^i_k\left(s^i\left(I^i_k\right)\right)$. Further note that, clearly, $\sigma^i_{\beta^i}\left(s^i\right) \in [0,1] \forall s^i \in S^i$. Since the game is linear, we know each path intersects each information set only once, and thus $\sum_{s^i \in S^i} \sigma^i_{\beta^i}\left(s^i\right) = \sum_{s^i \in S^i} \prod_{k=1}^{K^i} \beta^i_k\left(s^i\left(I^i_k\right)\right) = 1$, so the constructed $\sigma^i_{\beta^i}$ is a mixed strategy.

Now take any $z \in Z$ and any $\pi^{-i} \in \Gamma^{-i}$. Consider first the cases where z is always reached or z is never reached, regardless of player i 's actions. In these cases, $\Pr(z) = 1$ and $\Pr(z) = 0$, respectively, for any $\beta^i \in B^i$ and for any $\sigma^i \in \Sigma^i$, so $\beta^i \sim \sigma^i_{\beta^i}$ trivially. Consider now the case where $\Pr(z) \in (0,1)$ and depends on player i 's actions. Define $\tilde{c}^i_{I^i_k}(z)$ as the action of player i at information set I^i_k that leads to final node z and $\tilde{I}^i(z) \equiv \left\{I^i_k \in \mathcal{I}^i \mid P(z) \cap I^i_k \neq \emptyset\right\}$ as the set of player i 's information sets in the path of z. Then the probability on z induced by β^i is $\Pr_{(\beta^i, \pi^{-i})}(z) = \prod_{I^i_k \in \tilde{I}^i(z)} \beta^i_k\left(\tilde{c}^i_{I^i_k}(z)\right)$. Now define $\tilde{S}^i(z)$ as the set of player i 's pure strategies that result in z, i.e. $\forall s^i \in \tilde{S}^i(z), \forall I^i_k \in \tilde{I}^i(z), c^i_k = \tilde{c}^i_k(z)$. Then the probability on z induced by $\sigma^i_{\beta^i}$ is:

$$\begin{aligned}
\Pr_{(\sigma^i, \pi^{-i})}(z) &= \sum_{s^i \in \tilde{S}^i(z)} \sigma^i_{\beta^i}\left(s^i\right) \\
&= \sum_{s^i \in \tilde{S}^i(z) I^i_k \in \mathcal{I}^i} \beta^i_k\left(s^i\left(I^i_k\right)\right) \\
&= \sum_{s^i \in \tilde{S}^i(z)} \prod_{I^i_k \in \tilde{I}^i(z)} \beta^i_k\left(\tilde{c}^i_{I^i_k}(z)\right) \prod_{I^i_k \notin \tilde{I}^i(z)} \beta^i_k\left(s^i\left(I^i_k\right)\right) \\
&= \prod_{I^i_k \in \tilde{I}^i(z)} \beta^i_k\left(\tilde{c}^i_{I^i_k}(z)\right) \quad \left(\text{since } \forall s^i \in \tilde{S}^i(z), \forall I^i_k \in \tilde{I}^i(z), s^i_k = \tilde{c}^i_k(z)\right) \\
&= \Pr_{(\beta^i, \pi^{-i})}(z)
\end{aligned}$$

Thus σ^i and β^i induce the same probability on z, and this is true $\forall z \in Z$. Thus $\beta^i \sim \sigma^i_{\beta^i}$ □

Theorem 21 (Kuhn). ***For games of perfect recall,*** $\forall i, \forall \sigma^i \in \Sigma^i, \exists \beta^i \in B^i$ ***such that*** $\sigma^i \sim \beta^i$.

Proof. dd □

2.3 Backward Induction and Subgame Perfect Equilibrium

Definition 73. *An extensive form game is a game of perfect information if* u^i *is a singleton* $\forall u^i \in U^i, i \in I$ *The following procedure is useful for finding pure strategy NE*

Definition 74 (The backward induction procedure). *in a game with perfect information is as follows:*

1. *For each node* $x \in X$ *such that* $IS(x) \subseteq Z$, *choose an action* $s^i(x) \in C^i_x$ *of the player* $i \in I$ *with* $x \in P^i$ *such that* $s^i(x)$ *leads to a node* $z \in IS(x)$ *with* $u^i(z) \geq u^i\left(z'\right) \forall z' \in IS(x)$
2. *Replace* x *with a final node with utilities* u_x, *where* u_x *is the vector of utilities resulting from* $s^i(x)$
3. *Repeat* $(1)-(2)$ *until an action* $s^i(x)$ *has been assigned to every* $x \in P^i$ *for all* $i \in I$.

The resulting pure strategy profile $\left(s^i(x)\right)_{x\in P^i, i\in I}$ is called a solution to the backwards induction procedure.

Theorem 22. ***The backward induction procedure produces N.E. profiles in pure strategies.***

Definition 75 (Subgaem)**.** *Let $G = \left(I, \mathcal{X}, \succeq, p, \left(X^i, u^i, \mathcal{I}^i, (C^i_k)_{k=1,\ldots,K^i}\right)_{i\in I}\right)$ be a finite EFG and $x \in X$ such that:*

1. *For the player i such that $x \in X^i$, the I^i_k containing x is a singleton.*

2. *$\forall i \in I, \forall I^i_k \in \mathcal{I}^i$, either $I^i_k \subset S(x) \cup \{x\}$ or $I^i_k \cap (S(x) \cup \{x\}) = \emptyset$.*

Then $G_x = \left(I, \mathcal{X}_x, \succeq_x, \left(X^i_x, u^i_x, \mathcal{I}^i_x, \left(C^i_{I^i_k}\right)_{I^i_k \in \mathcal{I}^i_x}\right)_{i\in I}\right)$ is the subgame following x where:

$$\begin{aligned}
&\mathcal{X}_x = S(x) \cup \{x\} \\
&\succeq_x = \succeq \mid \mathcal{X}_x \\
&X^i_x = X^i \cap \mathcal{X}_x \\
&Z_x = Z \cap S(x) \\
&u^i_x : Z_x \to \mathbb{R} \\
&\mathcal{I}^i_x = \left\{ I^i_k \in \mathcal{I}^i \mid I^i_k \subseteq S(x) \right\}
\end{aligned}$$

[Note: For any finite EFG, the full game is a subgame of itself. All others (if there are any) are called proper subgames.

Definition 76 (Minimal subgame)**.** *A minimal subgame is a subgame with no subgames other than itself.*

Definition 77 (Games of perfect information)**.** *An EFG is perfect information if $\forall i, \forall I^i_k \in \mathcal{I}^i, I^i_k$ is a singleton.*

[Note: In an EFG of perfect information, every $x \in \mathcal{X} \backslash Z$ induces a subgame.

Definition 78 (Subgame perfect equilibrium (SPE))**.** *For a finite EFG, a behavioral strategy profile $\beta \in B$ is a subgame perfect equilibrium if and only if $\forall x \in \mathcal{X}$ for which G_x is a well defined EFG the restriction of β to G_x is a* NE *of G_x. Intuitively, a SPE is a NE that is also a NE in every subgame.*

[Note: SPE make sure behavior on and off the equilibrium path is rational, in contrast to NE which only make sure behavior on the equilibrium path is rational.

Theorem 23. ***For any finite EFG of perfect information, the BIP produces a set of*** NE ***in pure strategies, but this set does not necessarily contain all NE in pure strategies.***

Theorem 24 (Zermelo)**.** ***Every finite EFG of perfect information has at least one pure strategy SPE.***

Proof. Let G be any finite EFG of perfect information. Apply the BIP, first considering each $x \in P(z)$ and breaking ties arbitrarily. By the BIP, each node $x \in P(z)$ will then be reduced to a single action. Repeat this until only a single node x is left. Then the record generated by the BIP will constitute an action for i at each $x \in X^i \forall i \in I$, i.e. $s = \left(s^1, \ldots, s^n\right)$. Since s is a NE of all subgames by construction, including the full game, it is by definition a pure strategy SPE. $\square$

Theorem 25. ***For any finite EFG, the set of SPE is nonempty.***

Proof. Let G be any finite EFG. If G is of perfect information, then we know by Zermelo's theorem that at least one SPE exists. If G is of imperfect information and has no proper subgames, then every NE is also a SPE, and thus by Nash's Existence Theorem, the set of SPE is nonempty. $\square$

Consider now the case where G is of imperfect information and has proper subgames. [TODO]

2.4 Sequential Equilibrium

Definition 79 (System of beliefs)**.** *A system of beliefs is* $\mu = \left(\mu^i_{I^i_k}\right)_{i \in I, I^i_k \in \mathcal{I}^i}$ *where* $\mu^i_{I^i_k} \in \Delta\left(I^i_k\right) \forall i, \forall k$. *Alternatively, a system of beliefs is a map* $\mu : X \to [0,1]$ *such that* $\forall i, \forall I^i_k \in \mathcal{I}^i, \sum_{x \in I^i_k} \mu(x) = 1$

Definition 80 (Consistency)**.** *A system of beliefs* μ *is consistent with behavioral strategy profile* β *if* $\exists \{(\beta_n, \mu_n)\}_{n \in \mathbb{N}}$ *such that:*

1. $\forall n \in \mathbb{N}, \beta_n$ *is fully mixed*

2. $\lim_{n \to \infty} \beta_n = \beta$

3. $\forall n \in \mathbb{N}, \mu_n$ *is induced by* β_n *according to Bayes' rule*

4. $\lim_{n \to \infty} \mu_n = \mu$

Definition 81 (Optimality)**.** *A behavioral strategy* $\beta^i \in B^i$ *for player* i *is optimal with respect to* μ *at* $I^i_k \in \mathcal{I}^i$, *given* β^{-i}, *if:*

$$\beta^i \in \underset{b^i \in B^i}{\arg\max} V^i\left(I^i_k, \left(b^i, \beta^{-i}\right), \mu\right) = \underset{b^i \in B^i}{\arg\max} \sum_{x \in I^i_k} \mu^i_{I^i_k}(x) \sum_{z \in Z} \Pr_{(x,(b^i, \beta^{-i}))}(z) u^i(z)$$

Definition 82 (Sequential rationality)**.** *A behavioral strategy profile* $\beta \in B$ *is sequentially rational with respect to* μ *if* $\forall i \in I, \forall I^i_k \in \mathcal{I}^i, \beta^i$ *is optimal with respect to* μ *given* β^{-i}.

Definition 83 (Sequential equilibrium (SE))**.** *A sequential equilibrium is a pair* (β, μ), *where* $\beta \in B$ *and* μ *is a system of beliefs, such that:*

1. μ *is consistent with* β

2. β is sequentially rational with respect to μ

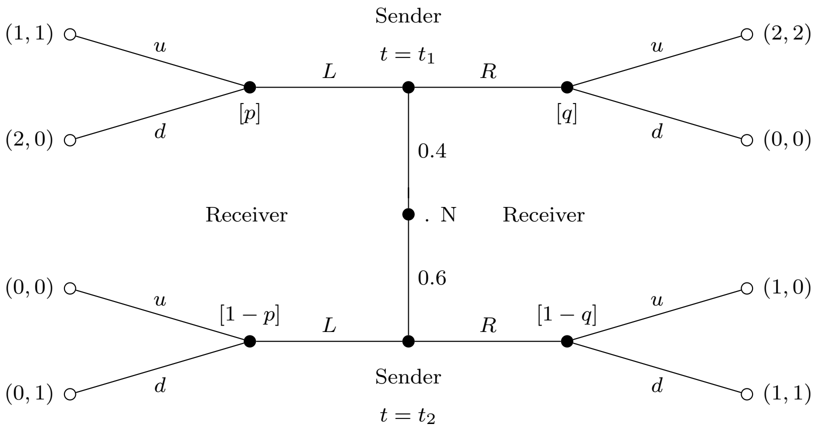
Note that a sequential equilibrium is a pair, not just a strategy profile. Hence, in order to identify a sequential equilibrium, one must identify a strategy profile β which describes what a player does at every information set, and a belief assessment μ, which describes what a player believes at every information set. In order to check that that (β, μ) is a sequential equilibrium, one must check that

1. (Sequential Rationality) s is a best response to belief $\mu(\cdot \mid I)$ and the belief that the other players will follow s in the continuation games in every information set$I$, and

2. (Consistency) there exist trembling probabilities that go to zero such that the conditional probabilities derived from Bayes rule under the trembles approach $\mu(\cdot \mid I)$ at every information set I. If all the information sets are reached under strategy β, we just need to use the Bayes rule in order to check consistency. If not do trembling hand.

[Note: SE make sure no strictly dominated strategies are played; weakly dominated strategies may still be played under certain beliefs, e.g. when some nodes in information sets are reached with zero probability.

Figure 2: Signaling Game



Example 9.

Theorem 26. ***Every SE is a SPE.***

Proof. Suppose not, i.e. there exists some $\beta \in B$ that's a part of an SE of some finite game G but that's not a SPE for G. Then we know there exists some subgame G_x such that β_x, the restriction of β to G_x, is not a NE. Thus there exists some $b_x^i \in B_x^i$ for some player i such that:

$$\sum_{z \in Z_x} u^i(z) \prod_{x \in \operatorname{Path}(z) \cap \mathcal{I}_x^i} b_x^i\left(x, c_z\right) \prod_{x \in \operatorname{Path}(z) \backslash \mathcal{I}_x^i} \beta_x^{-i}\left(x, c_z\right) > \sum_{z \in Z_x} u^i(z) \prod_{x \in \operatorname{Path}(z)} \beta_x\left(x, c_z\right)$$

Note that the l.h.s. of the above inequality can be expressed as some system of beliefs μ induced by β_x. This implies b_x^i is optimal with respect to μ given β_x^{-i}, but since β is a SE, β_x^i is optimal with respect to μ given β_x^{-i}. But this is a contradiction of the above inequality. So it must be that for $\beta \in B$ that's a part of an SE it is also a SPE. □

Theorem 27. ***Suppose that Γ^e is an extensive-form game with perfect recall and the behavioralstrategy profile $b \in B$ is a perfect equilibrium of Γ^e. Then there exists a system of beliefs μ such that (b, μ) is a sequential equilibrium of Γ^e***

Proof. Since b is a perfect equilibrium, by Theorem 2.11$\exists\left(b_n\right) \in B^{\infty}$ such that (i) b_n is fully mixed $\forall n \in \mathbb{N}$, (ii)$b_n \rightarrow b$, and (iii)$b^i \in \operatorname{argmax}_{d^i \in B^i} u^i\left(d^i, b_n^{-i}\right) \forall n \in \mathbb{N}, i \in I . \forall n \in \mathbb{N}, v \in U^i, y \in u$ define

$$\mu_n(y)=\frac{\operatorname{Pr}\left(y \mid b_n, \theta\right)}{\sum_{y' \in v} \operatorname{Pr}\left(y' \mid b_n, \theta\right)}$$

Notice that since b_n is fully mixed, $\operatorname{Pr}\left(y \mid b_n, \theta\right)>0$ for every $y \in v$, so $\sum_{y' \in v} \operatorname{Pr}\left(y' \mid b_n, \theta\right)>0$. Let

$$\mu(y)=\lim_{n \rightarrow \infty} \mu_n(y)$$

Then by (i) and (ii), μ is a system of beliefs fully consistent with b. Let $V^{(i, v)}(\cdot)$ denote the utility function of agent (i, v). When this agent uses a randomized strategy $d^{(i, v)} \in \Delta\left(C_v^i\right)$, his expected payoff is

$$V^{(i,v)}\left(b_n^{-(i,v)},d^{(i,v)}\right)=\sum_{x\in v}\Pr\left(x\mid b_n^{-(i,v)},d^{(i,v)},\theta\right)U^{(i,v)}\left(b_n^{-(i,v)},d^{(i,v)}\mid x\right)+\sum_{z\notin S(v)}\Pr\left(z\mid b_n^{-(i,v)},d^{(i,v)},\theta\right)u^{(i,v)}(z)$$

where $b_n^{-(i,v)}=b_n\backslash\left\{b_n^{(i,v)}\right\}, U^{(i,v)}(b\mid x)=\sum_{z\in Z}\Pr(z\mid b,x)u^{(i,v)}(z)$ and $S(v)=\cup_{x\in v}S(x)$. Note that for any $x\in v, \Pr\left(x\mid b_n^{-(i,v)},d^{(i,v)},\theta\right)=\Pr\left(x\mid b_n,\theta\right)$ since the probability of the node x occurring depends only on the strategies of the agents who move before v occurs. Then

$$\begin{aligned}V^{(i,v)}\left(b_n^{-(i,v)},d^{(i,v)}\right)&=\sum_{x\in v}\Pr\left(x\mid b_n,\theta\right)U^{(i,v)}\left(b_n^{-(i,v)},d^{(i,v)}\mid x\right)+\sum_{z\notin S(v)}\Pr\left(z\mid b_n,\theta\right)u^{(i,v)}(z)\\&=\left(\sum_{x\in v}\mu_n(x)U^{(i,v)}\left(b_n^{-(i,v)},d^{(i,v)}\mid x\right)\right)\left(\sum_{x\in v}\Pr\left(x\mid b_n,\theta\right)\right)+\sum_{z\notin S(v)}\Pr\left(z\mid b_n,\theta\right)u^{(i,v)}(z)\end{aligned}$$

Since (b_n) supports b as a perfect equilibrium of the multi-agent representation of Γ^e we have that

$$b^{(i,v)}\in\underset{d^{(i,v)}\in\Delta(C_v^i)}{\operatorname{argmax}}\ V^{(i,v)}\left(b_n^{-(i,v)},d^{(i,v)}\right)$$

which implies that

$$b^{(i,v)}\in\underset{d^{(i,v)}\in\Delta(C_v^i)}{\operatorname{argmax}}\sum_{x\in v}\mu_n(x)U^{(i,v)}\left(b_n^{-(i,v)},d^{(i,v)}\mid x\right)$$

because these two objectives differ only by a strictly increasing affine transformation whose coeefi-cients are independent of $d^{(i,v)}$. Then by the upper-hemicontinuity of the best-response correspondence we have

$$b^{(i,v)}\in\underset{d^{(i,v)}\in\Delta(C_v^i)}{\operatorname{argmax}}\sum_{x\in v}\mu(x)U^{(i,v)}\left(b^{-(i,v)},d^{(i,v)}\mid x\right)$$

This implies that $\forall v\in U^i$ (the information partition),

$$b^i\in\underset{d^i\in B^i}{\operatorname{argmax}}V^i\left[v,\mu,\left(d^i,b^{-i}\right)\right]$$

Since $i\in I$ has been arbitrary b is sequentially rational given the system of beliefs μ. Hence, (b,μ) is a sequential equilibrium.

$\square$