Recitation 4

[Definitions used today]

- Topkis theorem, Supermodularity, Increasing Differences

Question 1

Suppose that a firm with production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ such that $f(0) = 0$ chooses its production plan $(x; z)$ at prices $w \in \mathbb{R}^n_{++}$ of inputs and $q \in \mathbb{R}_{++}$ of the output in such a way that minimizes the cost of producing z at prices w, and the marginal cost $\frac{\partial C^*}{\partial z}(w; z)$ equals the output price q:

a Under what conditions on f is the firm maximizing its production? Be as general as you can. Prove you answer.

b Suppose that cost function C^* is strictly concave in$z$. Show that the firm makes a loss (strictly negative profit) when following the marginal cost rule whenever the output is non-zero.

Solution 1

a) f concave $\to$ Y is convex so $\pi(p) \in \partial Y$ or f concave $\to$ C convex in z so $\pi(q, w) = \sup_{z\geq 0} qz - C(w, z)$ is concave and this representation holds (envelope)
b) strict concavity means stric convexity of profit. It implies minimization of profit in FOCs gives profit below zero. Or from concavity of C in z and envelope for profit

$$0 \leq C(w, 0) \leq C(w, z) - z \cdot \frac{\partial C^*}{\partial z}(w; z) \qquad \pi(p) \leq 0$$

Question 2 [Topkis theorem]

If S is a lattice, f is supermodular in x, and f has nondecreasing differences in $(x; t)$, then $\varphi^*(t) = \arg\max_{x\in S} f(x, t)$ is monotone nondecreasing in t.

Question 3 [Midterm 2017] or $\sim$ 82,89 [II.1 Spring 2009 majors]

Consider a profit maximizing firm with single output and n inputs, with production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ assumed strictly increasing, continuous (but possibly nondifferentiable), and $f(0) = 0$. Let $q \in \mathbb{R}_{++}$ be the price of output and $w \in \mathbb{R}^n_{++}$ be the vector of prices of inputs. The firm's profit maximization problem is

$$\max_{x\geq 0}[qf(x) - wx]$$

a Show that if the production function f is supermodular, then the firm's input demand x is monotone non-increasing in input prices, that is if $w \leq w'$ for $w, w \in \mathbb{R}^N_{++}$ then $x(w, q) \geq x(w, q)$. You may assume that input demand x is single valued. Production function is strictly increasing but need not be differentiable.

b Under what conditions on f is the solution $x(w, q)$ unique? Be as general as you can and prove your answer

c Give an example of strictly increasing function that is not supermodular.

Solution 3

Function f is assumed strictly increasing. If f is nondecreasing, then the objective function $F(x, q) = qf(x) - wx$ has nondecreasing differences in $(x; q)$. If f is supermodular, then $F(x, q)$ is supermodular in x. Theorem **??** implies that input demand $x^*(q)$ is monotone nondecreasing in output price q.

Question 4

Consider a $C \subset \mathbb{R}^L$, $T \subset \mathbb{R}$. Define function F in following way:

$$F : \mathbb{R}^L \times T \to \mathbb{R} \quad F(x, t) = \bar{F}(x) + f(x, t)$$

where $f : \mathbb{R} \times T \to \mathbb{R}$ is supermodular and $\bar{F} : \mathbb{R}^L \to \mathbb{R}$. Assume that:

$$\forall \quad t'' > t' \quad x'' \in \operatorname*{argmax}_{x\in C} F(x, t'') \quad x' \in \operatorname*{argmax}_{x\in C} F(x, t')$$

Show that if $x_i' > x_i''$ then

$$\forall \quad t'' > t' \quad x'' \in \operatorname*{argmax}_{x\in C} F(x,t') \quad x' \in \operatorname*{argmax}_{x\in C} F(x,t'')$$

Solution 4

Let's take $x_i' \geq x_i^{''}$, $t^{''} \geq t'$ and consider $z' = (x_i', t')$ and $z^{''} = (x_i'', t'')$ thus $z'' \wedge z' = (x_i^{''}, t')$, $z'' \vee z' = (x_i', t'')$. From Supermodularity of $f(x_i, t)$:

$$f(z' \vee z'') + f(z \wedge z'') \geq f(z') + f(z'')$$

$$f(x_i', t'') + f(x_i^{''}, t') \geq f(x_i'', t'') + f(x_i', t')$$

and add to both sides $\bar{F}(x'') + \bar{F}(x')$

$$F(x'', t') + F(x', t'') \geq F(x', t') + F(x'' + t'')$$

$$F(x'', t') - F(x', t') \geq F(x'', t'') - F(x', t'')$$

$x' \in \operatorname{argmax} F(x,, t')$ so $F(x', t') \geq F(x'', t')$ $x'' \in \operatorname{argmax} F(x,, t')$ so $F(x'', t'') \geq F(x', t'')$

$$0 \geq F(x'', t') - F(x', t') \geq F(x'', t'') - F(x', t'') \geq 0$$

$$0 = F(x'', t') - F(x', t') = F(x'', t'') - F(x', t'') = 0$$

$$F(x'', t') = F(x', t') = F(x'', t'') = F(x', t'')$$

so $x'' \in \operatorname{argmax} F(x,, t')$ and $x' \in \operatorname{argmax} F(x,, t'')$

Question 5

Let $\{f(s,t)\}$ $t \in T$ be a family of density functions on $S \subset R$. T is a poset (partially ordered set). Consider

$$v(x,t) = \int_S u(x,s) f(s,t) ds$$

Prove the following statement. Suppose u has increasing differences and that $\{f(\cdot,t)\}$ $t \in T$ are ordered with t by first order stochastic dominance. Then v has increasing differences in (x,t).

Solution 5

For $x' > x$ and $t' > t$ we define $\gamma(s) := u(x', s) - u(x, s)$. It is incresing function and look at difference of v (we have to prove that is increasing differences):

$$v(x', t') - v(x, t') = \int_S [u(x', s) - u(x, s)] f(s, t') ds = \int_S \gamma(s) f(s, t')$$

$f(\cdot, t)$ is FOSD in t and γ is increasing so the value $v(x', t') - v(x, t')$ itself is increasing in t , i.e. $v(x', t') - v(x, t') \geq v(x', t) - v(x, t)$.

Question 7 Suppose that utility function $u : \mathbb{R}_+^\ell \to \mathbb{R}$ is supermodular, strictly concave, and locally non-satiated. Then the Walrasian demand function $x^*(\cdot)$ is a nondecreasing function of income, i.e.,

$$x^*(p, w') \geq x^*(p, w), \ \forall w' \geq w \geq 0, \ \forall p \gg 0.$$

In other words, the demand for every good is normal.

Solution 7

If $w = w'$, the proof is trivial. Let $p \gg 0$, let $w > w'$, let $x = x^*(p, w)$, and let $y = x^*(p, w')$. Since u is locally non-satiated, we have $p \cdot x = w$ and $p \cdot y = w'$ (by lemma **??**). Clearly, $p \cdot [x \wedge y] \leq w$. Since $p \cdot y = w' > w$, $\exists \lambda \in [0, 1)$ such that

$$p \cdot (\lambda[x \wedge y] + (1 - \lambda)y) = w.$$

Let $\underline{z}_\lambda = \lambda[x \wedge y] + (1 - \lambda)x$ and let $\bar{z}_\lambda = \lambda[x \vee y] + (1 - \lambda)y$. Note that

$$\underline{z}_\lambda + \bar{z}_\lambda = x + y$$

by the fact that $x \wedge y + x \vee y = x + y$. Then we have

$$p \cdot \underline{z}_\lambda = w$$

and

$$p \cdot \bar{z}_\lambda = w'.$$

Since x is the unique maximizer at w and $\underline{z}_\lambda$ is affordable at w, it must be that $u(x) \geq u(\underline{z}_\lambda)$. Then by lemma **??**, $u(\bar{z}_\lambda) \geq u(y)$. But since y is the unique maximizer at w' and $\bar{z}_\lambda$ is affordable at w', then it must be that $u(y) \geq u(\bar{z}_\lambda)$. Then we have $u(y) = u(\bar{z}_\lambda)$ so $y = \bar{z}_\lambda$. Since $\underline{z}_\lambda + \bar{z}_\lambda = x + y$, this means that we also have $x = \underline{z}_\lambda$.