



## Recitations 7 - Additional meeting

### Question 1 247 [I.2 Fall 2017 majors]

There are two assets: a risk-free asset with return  $r_f$  and a risky asset with return  $\tilde{r}$ . Returns are per-dollar returns, that is, total returns for one dollar invested. An agent has expected utility function with von Neumann (or Bernoulli) utility index  $v(x) = -(\alpha - x)^2$ , for  $\alpha > 0$ , and initial wealth  $w > 0$ . Assume that  $\alpha > wr_f$ . Negative investment (i.e., short selling) is permitted for both assets.

- Find the optimal investment in the risky asset as a function of the expected return and the variance of the return on the risky asset, the risk-free return, and the agent's wealth. Consider the following comparative statics exercise. The return  $\tilde{r}$  is changed to a more risky return  $\tilde{r}'$ . The expected return on  $\tilde{r}'$  is the same as on  $\tilde{r}$ , that is,  $E(\tilde{r}') = E(\tilde{r})$ .
- State a definition of more risky return (or more risky random variable).
- Suppose that  $E(\tilde{r}) > r_f$ . Show that the optimal investment in the risky asset with more risky return  $\tilde{r}'$  is smaller than the optimal investment with less risky return  $\tilde{r}$ , everything else being unchanged? Prove your answer.

### Question 2

There are three states with equal probabilities  $\pi_s = \frac{1}{3}$  for  $s \in \{1, 2, 3\}$ . Consider two state contingent consumption plans  $z = (8, 2, 2)$ , and  $y = (3, 3, 6)$

- Does  $y$  FOSD dominate  $z$ ?
- Is  $z$  more risky than  $y$ ?

### Question 3 [Stochastic Dominance and Risk]

Consider two real-valued random variables  $y$  and  $z$  on some finite state space with  $\mathbb{E}[y] = \mathbb{E}[z]$ .

- Prove that if  $y$  is more risky than  $z$ , then  $\mathbb{E}[v(z)] \geq \mathbb{E}[v(y)]$  for every nondecreasing continuous and concave function  $v : \mathbb{R} \rightarrow \mathbb{R}$ . You may assume  $v$  is twice differentiable.
- Give an example of two random variables  $y$  and  $z$  such that  $y \neq z$ ,  $\mathbb{E}[y] = \mathbb{E}[z]$  and neither  $z$  is more risky than  $y$  nor  $y$  is more risky than  $z$ .

### Question 4 [Pratt]

Consider an agent whose preferences over real-valued random variables (or state-contingent consumption plans) are represented by an expected utility function with strictly increasing and twice differentiable vN-M utility  $v : \mathbb{R} \rightarrow \mathbb{R}$ . Let  $\rho(w, \tilde{z})$  denote the risk compensation for random variable  $\tilde{z}$  with  $\mathbb{E}(z) = 0$  at risk-free initial wealth  $w$ . Let  $A(w)$  denote the Arrow-Pratt measure of risk aversion at  $w$ .

- Prove that  $A$  is an increasing function of  $w$  if and only if risk compensation  $\rho$  is an increasing function of  $w$  for every  $\tilde{z}$  with  $\mathbb{E}(\tilde{z}) = 0$  and  $\tilde{z} \neq 0$ .
- Derive an explicit expression for risk compensation for quadratic utility  $v(x) = -(\alpha - x)^2$  where  $\alpha > 0$ . Prove that this quadratic utility is, up to an increasing linear transformation, the only utility function with risk compensation of the form you derived.
- Give an example of two vN-M utility function  $v_1$  and  $v_2$  such that neither  $v_1$  is more risk averse than  $v_2$ , nor  $v_2$  is risk averse than  $v_1$  in the sense of the Theorem of Pratt.

### Question 5

Suppose that  $\tilde{z}$  takes two values  $z_1$  or  $z_2$ , where  $z_1 \leq z_2$ , with probabilities  $\pi$  and  $1 - \pi$ , respectively. Let  $\tilde{y}$  takes two values  $y_1$  or  $y_2$ , where again  $y_1 \leq y_2$ , with probabilities  $\pi$  and  $1 - \pi$ . Assume that  $0 < \pi < 1$

- Under what conditions (necessary and sufficient) on  $z_i$  and  $y_i$  does  $\tilde{z}$  dominate  $\tilde{y}$  in the sense of the First-Order Stochastic Dominance? Justify your answer.

- b Under what conditions does  $\tilde{z}$  dominate  $\tilde{y}$  in the sense of the Second-Order Stochastic Dominance? Under what conditions is  $\tilde{y}$  more risky than  $\tilde{z}$ ?

**Question 6 ~ 223 [I.2 Fall 2016 majors]**

Consider two real-valued random variables  $y$  and  $z$  such that  $\mathbb{E}[y] = \mathbb{E}[z]$ . Suppose that cumulative distribution functions  $F_y$  and  $F_z$  have the following (weak) single-crossing property:

$$\exists t^* \in \mathbb{R} \quad \begin{cases} F_y(t) \geq F_z(t) & \text{for } t \leq t^* \\ F_y(t) \leq F_z(t) & \text{for } t \geq t^* \end{cases}$$

Show that  $y$  is more risky than  $z$ . You may assume that  $y$  and  $z$  are random variables on a state space with  $S$  states.

**Question 7**

Consider two random variables  $y$  and  $z$  with the same expectations  $\mathbb{E}[y] = \mathbb{E}[z]$ .

- Show that if  $y$  is more risky than  $z$  then  $\text{Var}[y] \geq \text{Var}[z]$ .
- Suppose both  $y$  and  $z$  are normal. Show that  $y$  is more risky than  $z$  if and only if  $\text{Var}[y] \geq \text{Var}[z]$ .
- Show that if  $z$  is more risky than  $y$  and  $y$  is more risky than  $z$  then  $y$  and  $z$  have the same distribution, i.e.  $F_y(t) = F_z(t)$  for all  $t$ . You may assume  $y$  and  $z$  take only finitely many values.

**Question 8**

Consider two random variables  $y$  and  $z$  on a probability space. You may think about  $y$  and  $z$  as statecontingent consumption plans on a finite state space. Let  $\{y_1, y_2, \dots, y_k\}$  be the values  $y$  can take. Show that if  $\mathbb{E}[z | y] = 0$  ( $= \mathbb{E}[z | y = y_i]$  for all  $i \in \{1, 2, \dots, k\}$ ) then  $z + y$  is more risky than  $y$ .

**Question 9 271 [I.2 Spring 2019 majors]**

Consider two real-valued random variables  $\tilde{y}$  and  $\tilde{z}$  on some state space (i.e. probability space). Let  $F_{\tilde{y}}$  and  $F_{\tilde{z}}$  be their cumulative distribution functions, and  $E(\tilde{z})$  and  $E(\tilde{y})$  their expected values. You may assume that  $\tilde{y}$  and  $\tilde{z}$  take values in a finite interval  $[a, b]$ .

- State a definition of  $\tilde{z}$  first-order stochastically dominating (FSD)  $\tilde{y}$ . State a definition of second-order stochastic dominance (SSD). Your definitions should be stated in terms of cumulative distribution functions  $F_{\tilde{y}}$  and  $F_{\tilde{z}}$ .
- State a definition of  $\tilde{y}$  being more risky than  $\tilde{z}$ . Give a brief justification for why it is a sensible definition of more risky.
- Suppose that  $\tilde{z}$  has uniform distribution on an interval  $[\underline{z}, \bar{z}]$  while  $\tilde{y}$  has uniform distribution on  $[y, \bar{y}]$ . Under what conditions on the bounds  $\underline{z}, \bar{z}, y, \bar{y}$  does  $\tilde{z}$  FSD  $\tilde{y}$ ? Prove your statement.
- Suppose again that  $\tilde{z}$  and  $\tilde{y}$  have uniform distributions as in (c). Show that if  $\underline{y} \leq \underline{z}$  and  $\bar{z} \leq \bar{y}$  and  $E(\tilde{z}) = E(\tilde{y})$ , then  $\tilde{y}$  is more risky than  $\tilde{z}$ . You may assume some specific (distinct) numerical values of  $\underline{z}, \bar{z}, y, \bar{y}$  in your proof, if you find it convenient.

**Question 10 255 [I.2 Spring 2018 majors]**

Consider two real-valued random variables  $\tilde{y}$  and  $\tilde{z}$  on some state space (i.e. probability space). Let  $F_{\tilde{y}}$  and  $F_{\tilde{z}}$  be their cumulative distribution functions, and  $E(\tilde{z})$  and  $E(\tilde{y})$  their expected values. You may assume that  $\tilde{y}$  and  $\tilde{z}$  take values in a finite interval  $[a, b]$ .

- State a definition of  $\tilde{z}$  second-order stochastically dominating (SSD)  $\tilde{y}$ . Your definition should be stated in terms of cumulative distribution functions  $F_{\tilde{y}}$  and  $F_{\tilde{z}}$ .
- State a definition of  $\tilde{y}$  being more risky than  $\tilde{z}$ . Provide a justification for why it is a sensible definition of more risky.
- Extend the definition of more risky to random variables that may have different expected values as follows:  $\tilde{y}$  is more risky than  $\tilde{z}$  if and only if  $\tilde{y} - E(\tilde{y})$  is more risky than  $\tilde{z} - E(\tilde{z})$ . Show that if  $\tilde{y}$  is more risky than  $\tilde{z}$  and  $E(\tilde{z}) \geq E(\tilde{y})$ , then  $\tilde{z}$  second-order stochastically dominates  $\tilde{y}$ .
- Show that  $\lambda \tilde{z}$  is more risky than  $\tilde{z}$  for every  $\lambda \geq 1$  and every  $\tilde{z}$ . Note that  $E(\tilde{z})$  may be different from 0.