Recitation 5

[Definitions used today]

- Topkis theorem, Supermodularity, Increasing Differences
- Recursive and dynamically consistent family of utility function, time separablity, ICC axiom

Question 1

Consider the following utility functions

- a $u(c_1, c_2, c_3) = \min\{2c_1 + c_2 + c_3, c_1 + c_2 + 2c_3\}$
- b $u(c_1, c_2, c_3) = c_1 + \sqrt{c_2} + \sqrt{c_3}$
- 1. Show that (a) does not have state-separable representation
- 2. Show that (b) does not have expected utility representation
- 3. Find $\pi \in \Delta \subset \mathbb{R}^3$ s.t. $u(c_1, c_2, c_3)$ is risk averse with respect to π
- 4. Show that there is no $\pi \in \Delta \subset \mathbb{R}^3$ s.t. $u(c_1, c_2, c_3)$ is strictly risk averse with respect to π

Question 2 [Properties of state separable u]

- a Prove that every recursive family of utility functions $\{U_t\}$ is dynamically consistent if the aggregator function $G(\cdot,\cdot)$ is strictly increasing in continuation
- b $S \geq 3$ and \succeq increasing and continuous. Prove that \succeq has state-separable representation then ICC holds.
- c For S=2 all increasing functions obey ICC. Show that for $u(c_1,c_2)=c_1\sqrt{c_2}+c_1+c_2$ it does not have state separable utility function

Question 3 [Topkis theorem]

If S is a lattice, f is supermodular in x for fixed t, and f has nondecreasing differences in (x;t), then $\varphi^*(t) = \arg\max_{x \in S} f(x,t)$ is monotone nondecreasing in t.

Question 4 254 [I.1 Spring 2018 majors]

Consider the problem of finding a Pareto optimal allocation of aggregate resources $\omega \in \mathbb{R}^n_+$ in an economy with two agents:

$$\max_{x} \mu_1 u_1(x) + \mu_2 u_2(\omega - x)$$

subject to $0 \le x \le \omega$

where $u_i : \mathbb{R}^n_+ \to \mathbb{R}$ are agents' utility functions (assumed continuous) and $\mu_i > 0$ are welfare weights for i = 1, 2. Let $x^* (\mu_1, \mu_2)$ be the set of solutions.

- a State a definition of utility function u_i being supermodular. Show that if u_i is supermodular, then $u_i(\omega x)$ is supermodular in x
- b Show that, if u_1 and u_2 are strictly increasing and supermodular in x then $x^*(\mu_1, \mu_2)$ is non-decreasing in μ_1 . You may assume that $x^*(\mu)$ is single-valued. Is $x^*(\mu_1, \mu_2)$ non-increasing in μ_2 ? Justify your answer. If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.
- c Under what conditions on u_1 and u_2 is the solution $x^*(\mu_1, \mu_2)$ unique. Justify your answer.