Recitations 8

[Definitions used today]

- Correspondences: nonempty valued, single valued, closed valued, compact valued, convex valued, closed graph, convex graph, upper hemi-continuity, lower hemi-continuity, continuity.
- Sequential characterization of uhc and lhc, Berge (1963) maximum theorem

This section comes from math appendix chapter 5 - Correspondences Let $\Theta \subseteq \mathbb{R}^n, X \subseteq \mathbb{R}^n$.

Definition 0.1. *A correspondence* $\Gamma : \Theta \rightrightarrows X$ *is a map s.t.* $\Gamma(\Theta) \subseteq X$. $\left(\Gamma : \Theta \to 2^X\right)$

Definition 0.2. ***(Graph of correspondence)***. $Gr(\Gamma) = \{(\theta, x) : \theta \in \Theta, x \in \Gamma(\theta)\}$

Definition 0.3. ***(Properties of correspondences)***.

1. ***not empty valued*** *if* $\Gamma(\theta) \neq \emptyset \quad \forall\theta$
2. ***single valued*** *if* $|\Gamma(\theta)| = 1 \quad \forall\theta$
3. ***closed valued*** *if* $\Gamma(\theta)$ *is closed set* $\forall\theta$
4. ***compact valued*** *if* $\Gamma(\theta)$ *is compact set* $\forall\theta$
5. ***convex valued*** *if* $\Gamma(\theta)$ *is convex set* $\forall\theta$
6. ***closed (graph)*** *if* $\mathrm{Gr}(\Gamma)$ *is closed subset of* $\mathbb{E} \times X$
7. ***convex (graph)*** *if* $Gr(\Gamma)$ *is convex on* $\Theta \times X$

Lemma 0.4. $Gr(\Gamma)$ *is closed graph* $\iff \forall_{\theta:\theta_n \to \theta} \forall_{x_n \to x} : x_n \in \Gamma(\theta_n) \Rightarrow \quad x \in \Gamma(\theta)$

Lemma 0.5. $\mathrm{Gr}(\Gamma)$ *is convex graph* $\iff \forall_\theta, \theta', x \in \Gamma(\theta), x' \in \Gamma(\theta')$ *it holds that* $\lambda x + (1-\lambda)x' \in \Gamma(\theta\lambda + (1-\lambda)\theta') \forall_{x \in [0,1]}$

Lemma 0.6. $\Gamma : \Theta \rightrightarrows X$ *has closed* graph $\Rightarrow$ *it is closed valued. If* X *is compact, than* Γ *is also compact valued.*

Definition 0.7. ***(Upper Hemi-Continuity)*** *Let* $\Gamma : \Theta \rightrightarrows X$ *be a correspondence.*

- Γ *is said to be* ***upper hemi-continuous (uhc)*** *at a point* $\theta \in \Theta$ *if and only if for all open sets* $V \subseteq X$ *such that* $\Gamma(\theta) \subseteq V$, *there exists an open set* $U \subseteq \Theta$ *such that* $\theta \in U$ *and for all* $\theta' \in U$ *it holds that* $\Gamma(\theta') \subseteq V$
- *A compact valued correspondence* $\Gamma : \Theta \rightrightarrows X$ *is u.h.c. at* $\theta \in \Theta$ *if and only if for every* $\{\theta_n\} \subset \Theta$ *such that* $\theta_n \to \theta$ *and every sequence* $\{x_n\} \subset X$ *such that* $x_n \in \Gamma(\theta_n)$ *there exits a convergent subsequence* $\{x_{n_k}\}$ *such that* $x_{n_k} \to x \in \Gamma(\theta)$
$$\forall_{\theta_n \to \theta} \forall_{x_n \in \Gamma(\theta_n)} \exists_{\{x_{n_k}\}} x_{n_k} \to x \in \Gamma(\theta)$$

Definition 0.8. ***(Lower Hemi-Continuity)***. *Let* $\Gamma : \Theta \rightrightarrows X$ *be a correspondence.*

- Γ *is said to be* ***lower hemi-continuous (1hc)*** *at a point* $\theta \in \Theta$ *if and only if for all open sets* $V \subseteq X$ *such that* $\Gamma(\theta) \cap V \neq \emptyset$, *there exists an open set* $U \subseteq \Theta$ *such that* $\theta \in U$ *and for all* $\theta' \in U$ *it holds that* $\Gamma(\theta') \cap V \neq \emptyset$
- *A correspondence* $\Gamma : \Theta \rightrightarrows X$ *is l.h.c. at* $\theta \in \Theta$ *if for all* $x \in \Gamma(\theta)$ *and all sequences* $\{\theta_n\} \subset \theta$ *such that* $\theta_n \to \theta$ *there exits a sequence* $\{x_n\} \subset X$ *such that* $x_n \in \Gamma(\theta_n)$ *and* $x_n \to x$
$$\forall_{\theta_n \to \theta} \forall_{x \in \Gamma(\theta)} \exists_{x_n \in \Gamma(\theta_n)} x_n \to x$$

Definition 0.9. ***(Continuity)*** Γ *is said to be continuous at a point* $\theta \in \Theta$ *if it is both UHC an LHC.*

Lemma 0.10. ***(u.h.c and Closed graph)*** *Let* $\Gamma : \Theta \rightrightarrows X$. *If* Γ *is u.h.c, then* Γ *is closed (has a closed graph).*

Lemma 0.11. ***(Closed graph and u.h.c.)*** *Let* $\Gamma : \Theta \rightrightarrows X$. *If* X *is compact and* Γ *is closed (has a closed graph), then* Γ *is u.h.c.*

Theorem 0.12. ***(Berge (1961) of Maximum)*** *Let $\Theta \subseteq \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$, let $f : \Theta \times X \to \mathbb{R}$ be a continuous function and $\Gamma : \Theta \rightrightarrows X$ a nonempty, compact valued, continuous correspondence. Define:*

$$v(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta) \quad G(\theta) = \{x \in \Gamma(\theta) \mid f(x, \theta) = v(\theta)\}$$

Then

- *$v : \Theta \to X$ is continuous*
- *$G : \Theta \rightrightarrows X$ is nonempty and compact valued, and UHC*

Proof. The proof is divided in three parts. First it is proven that G is nonempty and compact valued, then that it is u.h.c. and finally that v is continuous.

1. *G is nonempty valued and compact valued.*
   - *Let $\theta \in \Theta$, by hypothesis $\Gamma(\theta)$ is compact and nonempty. since $f(\cdot, \theta)$ is continuous a maximum is attained on $\Gamma(\theta)$ by the extreme value theorem (Weierstrass). This proves that $G(\theta)$ is nonempty for arbitrary θ.*
   - *Let $\theta \in \Theta$, by hypothesis $\Gamma(\theta)$ is compact and nonempty. since $G(\theta) \subseteq \Gamma(\theta)$ it follows that $G(\theta)$ is bounded, it is left to show closedness to establish compactness. Let $x_n \to x$ and $x_n \in G(\theta)$ for all n. Clearly $x_n \in \Gamma(\theta)$ for all n, since Γ is closed valued it follows that $x \in \Gamma(\theta)$, so its feasible. By definition of G we have $v(\theta) = f(x_n, \theta)$ for all n, since f is continuous we get $v(\theta) = \lim f(x_n, \theta) = f(x, \theta)$, then by definition $x \in G(\theta)$, which proves closedness.*
2. *G is u.h.c. Consider $\theta \in \Theta$, a sequence in Θ such that $\theta_n \to \theta$ and a sequence in X such that $x_n \in G(\theta_n)$ for all n. Note that $x_n \in \Gamma(\theta_n)$. since Γ is u.h.c. there exists a subsequence $x_{n_k} \to x \in \Gamma(\theta)$ Now consider $z \in \Gamma(\theta)$. since Γ is l.h.c. there exists a sequence in X such that $z_n \in \Gamma(\theta_n)$ and $z_n \to z$. In particular the subsequence $\{z_{n_k}\}$ also converges to z since $x_n \in G(\theta_n)$ and $z_n \in \Gamma(\theta_n)$ it follows that $f(x_n, \theta_n) \geq f(z_n, \theta_n)$. since f is continuous in both arguments we get by taking limits: $f(x, \theta) \geq f(z, \theta)$. since the inequality holds for arbitrary $z \in \Gamma(\theta)$ we get the result: $x \in G(\theta)$. This proves u.h.c.*
3. *v is continuous. Let $\theta \in \Theta$ and $\theta_n \to \theta$ an arbitrary sequence converging to θ. Consider an arbitrary sequence in X such that $x_n \in G(\theta_n)$ for all n. Let $\bar{v} = \limsup v(\theta_n)$. By proposition 2.9 there is a subsequence $\{\theta_{n_k}\}$ such that $v(\theta_{n_k}) \to \bar{v}$. since G is u.h.c. there exists a subsequence of $\{x_{n_k}\}$ (call it $\{x_{n_{k_l}}\}$) converging to a point $x \in G(\theta)$. Then*
   $$\bar{v} = \lim v(\theta_{k_l}) = \lim f(x_{k_l}, \theta_{k_l}) = f(x, \theta) = v(\theta)$$
   where the second equality follows from $x_{k_l} \in G(\theta_{k_l})$, the third one from f being continuous and the final one from $x \in G(\theta)$. Let $\underline{v} = \liminf v(\theta_n)$ and by a similar argument we get $v(\theta) = \underline{v}$ since $v(\theta) = \liminf v(\theta_n) = \limsup v(\theta_n)$ we get $v(\theta) = \lim v(\theta_n)$ for arbitrary $\{\theta_n\}$ converging to θ. This proves continuity.

Theorem 0.13. ***(ToM under convexity)*** *Let $\Theta \subseteq \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$, let $f : \Theta \times X \to \mathbb{R}$ be a continuous function and $\Gamma : \Theta \Rightarrow X$ a nonempty, compact valued, continuous correspondence. Define:*

$$v(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta) \quad G(\theta) = \{x \in \Gamma(\theta) \mid f(x, \theta) = v(\theta)\}$$

a *If $f(\cdot, \theta)$ is concave in x for all θ and Γ is convex valued then G is convex valued.*

b *If $f(\cdot, \theta)$ is strictly concave in x for all θ and Γ is convex valued then G is single valued, hence a continuous function.*

c *If f is concave on $\Theta \times X$ and Γ has a convex graph then v is concave and G is convex valued.*

d *If f is strictly concave on $\Theta \times X$ and Γ has a convex graph then v is strictly concave and G is single valued, hence a continuous function.*

Theorem 0.14. ***(ToM under quasi-convexity)****. Let $\Theta \subseteq \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$, let $f : \Theta \times X \to \mathbb{R}$ be a continuous function and $\Gamma : \Theta \rightrightarrows X$ a nonempty, compact valued, continuous correspondence. Define:*

$$v(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta) \quad G(\theta) = \{x \in \Gamma(\theta) \mid f(x, \theta) = v(\theta)\}$$

a *If $f(\cdot, \theta)$ is quasi-concave in x for all θ and Γ is convex valued then G is convex valued.*

b *If $f(\cdot, \theta)$ is strictly quasi-concave in x for all θ and Γ is convex valued then G is single valued, hence a continuous function.*

c *If f is quasi-concave on $\Theta \times X$ and Γ has a convex graph then v is quasi-concave and G is quasi-convex valued.*

d *If f is strictly quasi-concave on $\Theta \times X$ and Γ has a convex graph then v is strictly quasi-concave and G is single valued, hence a continuous function.*

Where are we heading?

Theorem 0.15. ***Brouwer's Fixed Point Theorem – continuous function***
Let $S \subset \mathbb{R}$ be nonempty, compact, and convex, and $f : S \to S$ be a continuous function. Then f has (at least) a fixed point in S, i.e. $\exists x^* \in S : x^* = f(x^*)$

Theorem 0.16. ***Tarsky's Fixed Point Theorem – weakly increasing functions***
Let $f : [0,1]^n \to [0,1]^n$, where $[0,1]^n = [0,1] \times ... \times [0,1]$, an n-dimensional cube. If f is nondecreasing, then f has a fixed point in $[0,1]^n$.

Theorem 0.17. ***Kakutani's Fixed Point Theorem – u.h.c. correspondence***
Let $S \subset \mathbb{R}$ be nonempty, compact, and convex, and $\Gamma : S \rightrightarrows S$ be a nonempty, convex-valued, and u.h.c. correspondence. Then Γ has a fixed point in S, i.e. $\exists x^* \in S : x^* \in \Gamma(x^*)$

Since S is compact, u.h.c. is equivalent to Γ having a closed graph.

Theorem 0.18. ***Fixed Point Theorem – l.h.c. correspondence***
Let $S \subset \mathbb{R}$ be nonempty, compact, and convex, and $\Gamma : S \rightrightarrows S$ be a nonempty, convex-valued, closed-valued, and l.h.c. correspondence. Then Γ has a fixed point in S.

Definition 0.19. ***Aggregate excess demand*** *$Z : \bar{\Delta} \to \mathbb{R}^\ell$ is defined as*

$$Z(p) = \sum_{i \in I} x_i(p, e_i) - \sum_{i \in I} e_i.$$

Theorem 0.20. ***Easy Existence Theorem***
Let $Z : \Delta \to \mathbb{R}^l$ be a continuous function that is bouded from below, satisfying Walras' Law and the boundary condition: $p_n \to p \in \partial\Delta \Rightarrow \|Z(p_n)\| \to \infty$. Then $\exists p^* \in \Delta$ such that $Z(p^*) = 0$.

Outline:

- Z is defined on Δ not $\bar{\Delta}$
- Define $\mu : \bar{\Delta} \rightrightarrows \bar{\Delta}$ that is nonempty, convex-valued, u.h.c. Use Kakutani's to find a fixed point in $\bar{\Delta}$
- Argue that the fixed point is in Δ and it is CE.

Define $\mu : \bar{\Delta} \rightrightarrows \bar{\Delta}$ by

$$\mu(p) = \begin{cases} \{\bar{q} \in \bar{\Delta} | \bar{q} \in argmax_{q \in \bar{\Delta}} q \cdot Z(p)\}, & \text{if } p \in \Delta \\ \{\bar{q} \in \bar{\Delta} | \bar{q} \cdot p = 0\}, & \text{if } p \in \partial\Delta \end{cases}$$

Question 1

Let $\Gamma : \Theta \rightrightarrows X$ be a correspondence.

1. Show that if a correspondence Γ has a closed graph then it is closed valued.
2. If Γ is compact valued and u.h.c then Γ has a closed graph.
3. If X is compact and Γ has a closed graph then Γ is u.h.c.

Solution 1 i) Suppose Γ has closed graph let $\theta_n \subset \Theta$ be such that $\theta_n = \theta$. Let x_n be s.t. $x_n \in \Gamma(\theta_n)$ and $x_n \to x$. WTS: $x \in \Gamma(\theta)$.
$(\theta_n, x_n) \to (\theta, x)$ θ_n by construction $x_n by assumption$. Moreover $(\theta_n, x_n) \in \text{Gr}(\Gamma)$. Therefore since graph is close then $(\theta, x) \in \text{Gr}(\Gamma)$. It means that $x\Gamma(\theta)$ so $\Gamma(\theta)$ is a closet set. ii) Let $x \in X$. Consider any $\{x_n\} \in X$ s.t. $x_n \to x$ and $\{y_n\} \in Y$ s.t. $y_n \in \Gamma(x)$ and $y_n \to y$. Since Γ is u.h.c., there exists a convergent subsequence $\{y_{n_k}\}$ such that $\lim_{n_k \to \infty} y_{n_k} \in \Gamma(x)$. Then $y = \lim_{n \to \infty} y_n = \lim_{n_k \to \infty} y_{n_k} \in \Gamma(x)$. Thus, Γ is closed.
iii)
Let Y be compact and Γ be closed. Since closed graph implies closed-valued, Γ is compact-valued. Let $x \in X, x_n \to x$ and $\{y_n\} \subset Y$ such that $\forall n, y_n \in \Gamma(x_n)$. Since Y is compact, there exists convergent subsequence $y_{n_k} \to y$. Since Γ is closed, $y \in \Gamma(x)$. Thus Γ is u.h.c.

Question 2

Let consumer budget set at a price $p \in \Delta^{\ell}(p >> 0)$ and endowment e_i be

$$B\left(p, e_i\right)=\left\{x \in X_i: p \cdot x \leqslant p \cdot e_i\right\}$$

i) Show that $B\left(p, e_i\right)$ is homogenous of degree zero in prices, non-empty valued and compact valued.
ii) Show that $B\left(p, e_i\right)$ is continuous.

Solution 2

One, useful, application of the material covered above is to determine properties of the budget correspondence, that indicates the feasible consumption bundles for a consumer given a price vector p and an endowment vector e. Suppose there are l goods, and that the agent has a fixed endowment of each good given by the vector $e \in \mathbb{R}_{++}^l$, the price of the goods is a vector $p \in \Delta$, where Δ is the n -dimensional open simplex. Define the budget set correspondence $B(\cdot, e): \Delta \rightrightarrows \mathbb{R}_{+}^l$by

$$B(p, e)=\left\{x \in \mathbb{R}_{+}^l \mid p \cdot x \leq p \cdot e\right\}$$

Theorem 0.21. *$B(\cdot, e)$ is continuous on prices.*

Proof. The claim is proved establishing u.h.c. and l.h.c. of B.

1. $B(\cdot, e)$ is upper hemi-continuous on prices. Let $p \in \Delta,\left\{p_n\right\} \subset \Delta$ with $p_n \rightarrow p$ and $\left\{x_n\right\} \subset \mathbb{R}_{+}^l$ a sequence such that $x_n \in B\left(p_n, e\right)$ since $p_n \rightarrow p \in \Delta$ there exists a closed ball, C, around p such that $C \subset \Delta$ and for n large enough $p_n \in C$. Let $\xi_i=\max_{p \in C} \frac{p \cdot e}{p_i}$ for $i=1, \ldots, l$. ξ_i is the maximum amount of x_i that can be bought in the neighborhood of p. Define $\xi=\max \left\{\xi_i\right\}+1$, it is clear that for n large enough $x_n \in B_{\xi}(0)$, then $\left\{x_n\right\}$ is a bounded sequence, hence it admits a convergent subsequence $x_{n_k} \rightarrow x$. since $x_{n_k} \in B\left(p_{n_k}, e\right)$ we have: $p_{n_k} \cdot x_{n_k} \leq p_{n_k} \cdot e$, since dot product is a continuous function taking limits we have $p \cdot x \leq p \cdot e$, which is $x \in B(p, e)$, proving u.h.c. of B. [Suppose, by contradiction, that $\bar{x} \notin B(\bar{p}, e_i)$. Then $\bar{p} \cdot \bar{x}>\bar{p} \cdot e_i$.
Fix $\epsilon>0$. Then by continuity of dot product, $\forall x^{\prime}$ such that $\left\|x^{\prime}-\bar{x}\right\|<\epsilon, \bar{p} \cdot x^{\prime}>\bar{p} \cdot e_i$.
We know that $x_{n_k} \in B(p_{n_k}, e_i)$, $\forall n_k$. Then $x_{n_k} \rightarrow \bar{x}$ implies that $\exists \bar{k}$ such that $\forall k>\bar{k},\left\|x_{n_k}-\bar{x}\right\|<\epsilon$ and $p_{n_k} \cdot x_{n_k} \leq p_{n_k} \cdot e_i$, $\forall n_k$. By continuity of the dot product, $\bar{p} \cdot \bar{x} \leq \bar{p} \cdot e_i$. Then $\bar{x} \in B(\bar{p}, e_i)$. Therefore $B(\cdot, e_i)$ is u.h.c]

2. $B(\cdot, e)$ is lower hemi-continuous on prices. Let $p \in \Delta,\left\{p_n\right\} \subset \Delta$ with $p_n \rightarrow p$ and $x \in B(p, e)$. Define $\eta_n^i=\max \left\{0, \frac{p_n \cdot x-p_n \cdot e}{l p_n^i}\right\}$ and let $x_n=x-\eta_n$ Clearly $x_n \in B\left(p_n, e\right)$ since either $x \in B\left(p_n, e\right)$ or

$$p_n \cdot x_n=p_n \cdot x-\sum_i \max \{p_n^i\left(\frac{p_n \cdot x-p_n \cdot e}{l p_n^i}\right)\} \leq p_n \cdot x-\sum_i p_n^i\left(\frac{p_n \cdot x-p_n \cdot e}{l p_n^i}\right)=p_n \cdot x-\left(p_n \cdot x-p_n \cdot e\right)=p_n \cdot e$$

then $p_n \cdot x_n \leq p_n \cdot e$ Moreover $x_n \rightarrow x$, since $x \in B(p, e)$ and $p_n \rightarrow p$ it follows that $p_n \cdot x-p_n \cdot e \rightarrow p \cdot x-p \cdot e \leq 0$, then $\eta_n=\max \left\{0, p_n \cdot x-p_n \cdot e\right\} \rightarrow 0$ which is $x_n \rightarrow x$. Then B is l.h.c.

3. Note that it wasn't checked if $x_n \geq 0$ for all n. This is not guaranteed by the construction above. With extra notation it can be guaranteed that $x_n^i \geq 0$.

Question 3

Let consumer i demand correspondence at a price p and endowment e_i be

$$x_i\left(p, e_i\right)=\left\{x \in B\left(p, e_i\right): x_i \succeq_i y \quad \forall_{y \in B\left(p, e_i\right)}\right\}$$

i) Show that if $B\left(p, e_i\right)$ is compact and $\succeq_i$ is complete and transitive preorder with upper contour sets $U_i(x)= \left\{y \in X_i: y \succeq_i x\right\}$ that are closed for all $x \in X t_i$ then the demand is non-empty.
ii) Give an example illustrating that compactness is indeed a necessary condition.

Solution 3

Proof. Since $B(p, e_i)$ is compact and $U_i(x)$ are closed for all $x \in X_i$, $U_i(x) \cup B(p, e_i)$ is also compact for all $x \in X_i$. By completeness and transitivity of $\succeq_i$, given any subset $\{x_1, \ldots, x_n\} \subset B(p, e_i)$, we can rearrange the elements so that $x_1 \preceq_i x_2 \preceq_i, \ldots, \preceq_i x_n$. Then the upper contour sets of these allocations satisfy $U_i(x_1) \supseteq U_i(x_2) \supseteq, \ldots, \supseteq U_i(x_n)$. Thus

$$U_i(x_1) \cup B(p, e_i) \supseteq U_i(x_2) \cup B(p, e_i) \supseteq, \ldots, \supseteq U_i(x_n) \cup B(p, e_i).$$

By construction, $x_k \in U_i(x_k) \cap B(p, e_i)$ for all $k \in \{1, \dots, n\}$. So $U_i(x_k) \cap B(p, e_i)$ is nonempty and compact for all $k \in \{1, \dots, n\}$. Since the intersections are nested, this implies that $\cap_{k=1}^{n}[U_i(x_k) \cap B(p, e_i)] \neq \emptyset$ for every finite subset $\{x_1, \dots, x_n\} \subseteq B(p, e_i)$. By the finite intersection property of compact sets,

$$x_i(p, e_i) = \cap_{x \in B(p,e_i)}[U_i(x) \cap B(p, e_i)] \neq \emptyset.$$

□

If the budget set is not compact, demand is often not well-defined. Let $\ell = 2$, let $e_i = (1, 1)$ and let consumer i's preferences be represented by the strictly increasing utility function

$$u_i(x, y) = \log x + \log y.$$

Let $p = (0, 1)$. Then $B(p, e_i) = \{(x, y) \in \mathbb{R}_+^\ell : \ x \in [0, \infty), \ y \leq 1\}$. This set is obviously not bounded so $B(p, e_i)$ is not compact.

Suppose that $x_i(p, e_i) \neq \emptyset$, i.e., $\exists (x, y) \in x_i(p, e_i)$. Then $u_i(x, y) \geq u_i(x', y')$ for all $(x', y') \in B(p, e_i)$. But $(x + 1, y) \in B(p, e_i)$ and $u_i(x + 1, y) > u_i(x, y)$. This is a contradiction, so $(x, y) \notin x_i(p, e_i)$. Therefore $x_i(p, e_i) = \emptyset$.

Question 4

The consumer problem is often laid out without explicit endowments of the goods, instead the parameters are prices $p \in \mathbb{R}_{++}^l$ and a nominal income level $e \in \mathbb{R}_+$. The set of parameters is $\Theta = \mathbb{R}_{++}^l \times \mathbb{R}$. The **indirect utility function** and the **Marshalian demand correspondence** are:

$$v(p, e) = \max_{x \in B(p,e)} u(x) \quad x(p, e) = \{x \in B(p, e) \mid u(x) = v(p, e)\}$$

I take as given that B is a nonempty, convex valued and continuous correspondence, and that u is a continuous function. Show for v and x the following properties on Θ.

a v is a continuous function on Θ and x is a nonempty, compact valued, u.h.c. correspondence.

b v is nondecreasing in r for fixed p and non-increasing in p for fixed xe.

c v is jointly quasi-convex on (p, e).

d If u is (quasi) concave then v is (quasi) concave in e for fixed p.

e If u is (quasi) concave then x is a convex valued correspondence.

f If u is strictly (quasi) concave then x is a continuous function.

Solution 4

a Thm of Max

b For $v(\cdot, e)$ non-increasing for fixed e:

$$\forall p > p' \quad B(p, e)(p', e)$$

Hence $v(p', e) \geq v(p, e)$
Similar for $v(p, \cdot)$ non-decreasing for fix p

c If u concave, take any $p, p' \in \mathbb{R}_{++}^l$, $\lambda \in [0, 1]$

$$v(\lambda p + (1\lambda)p', e) = \max\{u(x) \text{s.t.} \quad [\lambda p + (1\lambda)p'] \cdot xe$$

The budget constraint can be rewritten as

$$\lambda(e - px) + (1\lambda)p'(e - p'x) \geq 0$$

Hence either$p \cdot x \leq e$ or $p' \cdot x \leq e$ is true (or both). This means every affordable package when facing $\lambda p + (1\lambda)p'$ is affordable either when facing p or p', i.e.

$$B(\lambda p + (1\lambda)p', e) \subset B(p, e) \cap B(p', e)$$

$$v(\lambda p + (1\lambda)p') \leq \max\{v(p, e), v(p', e)\}$$

which means v is quasi-convex in p holding e fixed. Similar for I when fixed p. x($\cdot$) part comes from Thm of Max under Convexity.

d Thm of Max under Convexity