Solution to HW2

Question 1

Consider a 2 good, 2 agent world from question 2, where $e_1 = 1$, $e_2 = 1$, and where agent $i = 1$ has utility function $u_1(c_{1,1}; c_{1,2}) = 2\ln(c_{1,1}) + \ln(c_{1,2})$ and agent $i = 2$ has utility function $u_2(c_{2,1}; c_{2,2}) = \ln(c_{2,1}) + 2\ln(c_{2,2})$. Solve the Social Planner's Problem for utility weights $\alpha_1 = \frac{2}{3}$ and $\alpha_2 = \frac{1}{3}$. Use a computer (Matlab or Mathematica) to graph the endowment point and the set $B(c)$ where c is your solution to the social planner's problem. Repeat but with utility weights $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{2}{3}$. For each utility weight specification, what is the separating hyperplane?

Solution 1

Consider an economy with 2 goods and 2 agents, with societal endowments of $e_1 = e_2 = 1$ where agents $i = 1, 2$ have the utility functions:

$$u_1(c_{1,1}, c_{1,2}) = 2\log c_{1,1} + \log c_{1,2}$$
$$u_2(c_{2,1}, c_{2,2}) = \log c_{2,1} + 2\log c_{2,2}$$

1. Let the utility weights be $\alpha_1 = \frac{2}{3}$ and $\alpha_2 = \frac{1}{3}$. The social planner's problem:

$$\max_{c_{1,1},c_{1,2},c_{2,1},c_{2,2}} \alpha_1(2\log c_{1,1} + \log c_{1,2}) + \alpha_2(\log c_{2,1} + 2\log c_{2,2})$$
$$\text{s.t} \sum_{i=1}^{2} c_{i,m} \leq e_m = 1 \quad \forall m = 1, 2$$
$$c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2} \geq 0$$

   Note the utility function is strictly monotonic and satisfies the Inada conditions. Thus the feasibility constraints will bind and none of the non-negativity constraints will bind. Substitute out for agent's 2's consumption using the feasibility constraints and take first-order conditions of the new unconstrained maximization problem:

$$\frac{2\alpha_1}{c_{1,1}} = \frac{\alpha_2}{1-c_{1,1}} \implies 1 - c_{1,1} = \frac{\alpha_2}{2\alpha_1}c_{1,1} \implies 1 = c_{1,1}\left(1 + \frac{\alpha_2}{2\alpha_1}\right)$$
$$\implies c^*_{1,1} = \frac{1}{1+\frac{\alpha_2}{2\alpha_1}} = \frac{1}{\frac{2\alpha_1+\alpha_2}{2\alpha_1}} = \frac{2\alpha_1}{2\alpha_1+\alpha_2} = \frac{4}{5}$$
$$\frac{\alpha_1}{c_{1,2}} = \frac{2\alpha_2}{1-c_{1,2}} \implies 1 - c_{1,2} = \frac{2\alpha_2}{\alpha_1}c_{1,2} \implies 1 = c_{1,2}\left(1 + \frac{2\alpha_2}{\alpha_1}\right)$$
$$\implies c^*_{1,2} = \frac{1}{1+\frac{2\alpha_2}{\alpha_1}} = \frac{1}{\frac{\alpha_1+2\alpha_2}{\alpha_1}} = \frac{\alpha_1}{\alpha_1+2\alpha_2} = \frac{1}{2}$$
$$\implies c^*_{2,1} = \frac{1}{5}, \quad c^*_{2,2} = \frac{1}{2}$$

2. With new utility weights $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{2}{3}$ the new allocations resulting from the social planner's problem will be:

$$c^*_{1,1} = \frac{2\alpha_1}{2\alpha_1+\alpha_2} = \frac{1}{2}$$
$$c^*_{1,2} = \frac{\alpha_1}{\alpha_1+2\alpha_2} = \frac{1}{5}$$
$$\implies c^*_{2,1} = \frac{1}{2}, \quad c^*_{2,2} = \frac{4}{5}$$

3. See figures 1. Note the starred points represent the endowment point and the better-than set $B(c)$ is the area above the dashed curve passing through the endowment point. The solid blue curve represents the social planner's indifference curve over endowments at the initial endowment point. The downward sloping solid red line passing through the endowment point represents the separating hyperplane. Note all curves meet exactly once, at the endowment point.

Given the above formulas for consumption at the solution to the social planner's problem (conveniently expressed in terms of α), we can compute the marginal rates of substitution at that point for any given pair of α's. Clearly the separating hyperplane through the endowment point must have this same slope. Given we also know that $e_1 = e_2 = 1$ at this point, we then solve for the equation of the hyperplane.

The social planner's indifference curve can be solved for by first solving for the utility attained at the initial endowment point (call it $\bar{U}$). Then, since we know the social planner will always split the endowments among the two agents in fixed proportions (given the α weights), we can then express the societal utility function in terms of the endowments and equate it to $\bar{U}$. Then, solving for e_2 in terms of e_1 yields:

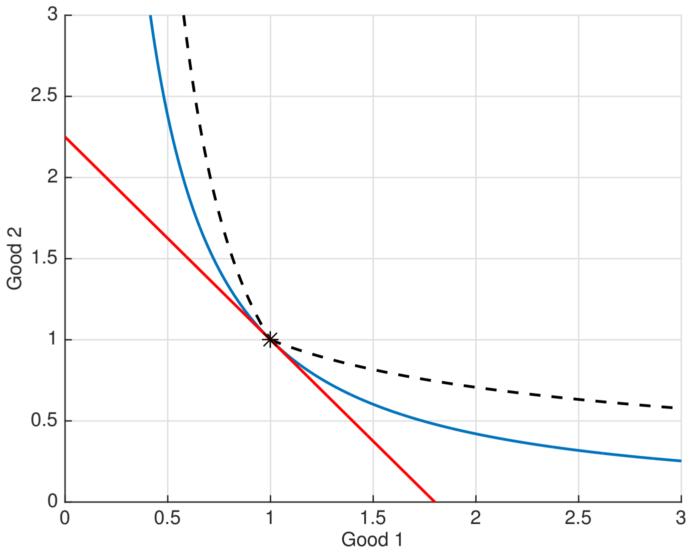
$$e_2 = e_1^{-5/4} \qquad \text{for } \alpha_1 = \frac{2}{3},\ \alpha_2 = \frac{1}{3}$$
$$e_2 = e_1^{-4/5} \qquad \text{for } \alpha_1 = \frac{1}{3},\ \alpha_2 = \frac{2}{3}$$

Similarly, to attain the better-than set, we solve for the better-than sets of each agent (which are independent of α) under the social planner's allocation scheme, which are given by:

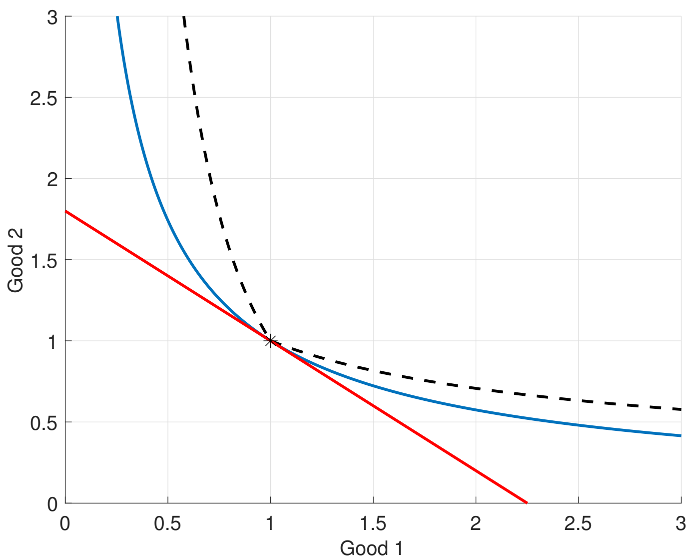
$$e_2 = e_1^{-2} \qquad \text{for agent 1}$$
$$e_2 = e_1^{-\frac{1}{2}} \qquad \text{for agent 2}$$

We then take the maximum of these two functions, which gives us the set where both agents are strictly better off than at the endowment point under the social planner's allocation scheme.

Figure 1: **Indifference curve through the endowment point, the better-than set, and the corresponding separating hyperplane**



Note: $\alpha_1 = \frac{2}{3}, \alpha_2 = \frac{1}{3}$.



Note: $\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{2}{3}$.

Question 2

Consider a world with 10 000 pieces of land, in the form of a 100 by 100 grid. On the east side of this grid is a lake. There are 1 000 agents. Each agent is endowed with one one-thousandth of each piece of land and one apple. Each agent must live on exactly one piece of land and a piece of land can have at most one person living on it. An agent who lives adjacent to the lake has utility $\sqrt{c}$ where c is his consumption of apples. An agent who lives one block in from the lake has utility $\sqrt{c} - .1$. Likewise, an agent living n blocks from the lake has utility $\sqrt{c} - .n$. Define a feasible allocation for this environment. Define a competitive equilibrium for this environment. Will a competitive equilibrium in this environment be Pareto efficient? (try to use the proof from class). Find all competitive equilibria. (you will need an equation solver most likely. I did.) (Hint: counting equations and unknowns in this problem is tricky.)

Solution

Denote each row (column) of the grid be indexed by $m = 0, 1, ..., 99$ (n) where $m = 0$ ($n = 0$) represents the northern-most row (eastern-most column) of the grid (i.e. the row (column) closest to the lake. Let $I = 1000$, c_i denote agent i's consumption of apples, m_i (n_i) denote the grid row (column) that agent i lives in. Note each agent is endowed with one apple and an equal claim to each plot of land in the grid. Since there is no difference between two pieces of land on different rows within a single column and each agent has equal right to every piece of land, we will consider dividing up the land by columns instead of individual plots. And from now on agent must live in exactly one column and each column an have at most one hundred agents. Let the utility function of each agent be:

$$u(c, m, n) = u(c, n) = \sqrt{c} - .1n$$

1. Let $C_i = (c_i, n_i, m_i)$ and $\bar{C}_i = \mathbb{R}_+ \times \{0, 1, ..., 99\} \times \{0, 1, ..., 99\}$ denote a consumption bundle and the consumption set of agent i, respectively. Define an allocation as $C = \{C_i\}_{i=1}^{1000}$. Then, in this environment, an allocation C is said to be feasible if:

$$c_i \in \bar{C}_i \quad \forall i \in I$$
$$\sum_{i=1}^{1000} a_i \leq 1000$$
$$\sum_{i=1}^{1000} \mathbf{1}\{n_i = n\} \leq 100 \quad \forall n = 0, 1, \ldots, 99$$

   where $\mathbf{1}$ is the indicator function taking the value of 1 if $n_i = n$ and 0 otherwise.

2. Denote the price of apples and the price of a plot of land in column n as p_a and $p_{l,n}$, respectively.Since we have parcels in 100 columns and in each column value of land endowment is equal then the aggregate value of the land endowment is:

$$X = \sum_{n=0}^{99} 100 p_{l,n}$$

   Recall that a competitive equilibrium in this set up is an allocation c and a set of prices $p = (p_c, p_{l,0}, p_{l,1}, \ldots, p_{l,99})$ such that:

   (a) $p \in \mathbb{R}_+^{101}$
   (b) The allocation is feasible.
   (c) $p_c c_i + p_{l,n_i} \leq p_c + \frac{X}{1000}$ $\forall i \in I$
   (d) $\forall i, \forall \hat{C}_i = (\hat{C}_i, \hat{m}_i, \hat{n}_i) \in C_i$ s.t. $u(\hat{a}_i, \hat{n}_i) > u(a_i, n_i), p_c \hat{c}_i + p_{l,\hat{n}_i} > p_c + \frac{X}{1000}$

3. Recall that the grid row of an agent is irrelevant to their utility; thus all agents within a column will face the same utility maximization problem. For an agent in column n, this optimization problem will be:

$$\max_{c} \quad \sqrt{c} - .1n$$
$$\text{s.t. } p_c c + p_{l,n} \leq p_a + \frac{X}{1000}$$
$$c \geq 0$$

   Note that the utility function is strictly monotonic in apples, and thus the budget constraint will bind with equality since the agent will eat as many apples as possible.

Claim: In equilibrium, $p_c > 0$.

Proof: Suppose not, i.e. in a competitive equilibrium (p, C), $p_c \leq 0$. Since each agent's utility is strictly increasing in apples, the optimal consumption of apples for each agent would be infinity. Since the aggregate endowment of apples is finite, this would be infeasible, contradicting (p, c) being a competitive equilibrium. Therefore $p_c > 0$ in equilibrium.

Claim: Suppose (p, C) is a competitive equilibrium. Then it is Pareto efficient.

Proof: Suppose not, i.e. there exists some other feasible allocation $\hat{C}$ that Pareto dominates C. We immediately know that:

$$\begin{aligned} u(\hat{c}_i, \hat{n}_i) &\geq u(c_i, n_i) \quad \forall i \in I \\ u(\hat{c}_j, \hat{n}_j) &> u(c_j, n_j) \quad \text{for some } j \in I \end{aligned}$$

From the definition of a competitive equilibrium, we also know that:

$$p_c\hat{c}_j + p_{l,\hat{n}_j} > p_c + \frac{X}{1000}$$

It follows that:

$$\begin{aligned} &\sum_{i=1}^{1000} (p_c\hat{c}_i + p_{l,\hat{n}_i}) > \sum_{i=1}^{1000} \left(p_c + \frac{X}{1000}\right) \\ \implies &p_c \left(\sum_{i=1}^{1000} \hat{c}_i - 1000\right) + \sum_{s=0}^{9} p_{l,s}(\#(\hat{n}_i = s) - 100) > 0 \end{aligned}$$

where $\#(\hat{n}_i = s) = \sum_{i=1}^{1000} \mathbf{1}\{\hat{n}_i = s\}$ is the total number of agent's living in column s. Since we know $p_c > 0$ and $p_{l,s} \geq 0$ it must either be that $\sum_{i=1}^{1000} \hat{c}_i > 1000$ or that $p_{l,s} > 0$ and $\#(\hat{n}_i = s) > 100$. Either of these being true would make $\hat{c}$ infeasible, contradicting $\hat{c}$ being a feasible allocation. Therefore there exists no feasible allocation that Pareto dominates c, and therefore c is Pareto efficient.

Characterization of CE.

Since $p_c > 0$we can normalize and get

$$c_i + p_{l,n_i} \leq 1 + \frac{1}{10}\sum_{n=0}^{99} p_{l,n} \quad \forall i \in \{0, \ldots 9\}$$

So agent's consumption is equal:

$$c_i = 1 + \frac{X}{1000} - p_{l,n_i}$$

and utility is equal

$$U_n = \sqrt{1 + \frac{X}{1000} - p_{l,n_i}} - 0.1n$$

The last thing we need to show before computing the equilibrium is that no agent will live further than 10 columns from the lake.

Claim: Suppose (p, c) is a competitive equilibrium. If a parcel in the grid is empty,it's equilibrium price is $p_{l,n_i} = 0$.

Proof. Suppose not. That is there exists an empty plot with a price $p^e > 0$. The value of land be comes

$$X \geq \sum_{n=0}^{999} p_{l,n_i} + p^e$$

Here, we need weak inequality because there may be more empty plots with non-zero prices than we are accounting for. Summing up the budget constraints of the individual agents,we get

$$\sum_{i=0}^{999} c_i + p_{l,n_i} = 1000 + X \Rightarrow \sum_{i=0}^{999} c_i \leq 1000 + p^e > 1000$$

which implies that agents eat more apples than exist. Contradiction.

Claim Suppose (p, c) is a competitive equilibrium and $\#(n = m)$ is the number of agents living in column m. Then $\#(n = m)$= 100 for $m < 9$ and $\#(n = m) = 0$ for $m \geq 9$.
Proof: . If this wasn't the case, then we would have an agent living in one spot located somewhere else and one free spot available at no price. This agent could strictly increase her utility by moving to this place near the lake and this move is feasible. This completes the proof.

Claim Suppose (p, c) is an equilibrium, then$U_0 = U_1 = ... = U9$. Also $U0 \geq 0$.

Proof. Suppose there exists at least one pair $(i, j) \in \{0, ..., 9\}$ such that $U_i > U_j$. Assume, without loss of generality, that $U_i = U_k$ for $k \neq j$. So each agent would prefer to move to column i if not a different column. Since column i is affordable,we know this means there is an allocation which is strictly preferred to the current allocation and is affordable. That contradicts the denition of a competitive equilibrium. Thus $U_0 = U_1 = \cdots = U_9$. Also, suppose that $U_0 < 0$. But then agents would move to a column outside of the rst ten. However,this contradicts the result of previous claim (2 claims ago). So$U_0 \geq 0$. By using

$$U_n = \sqrt{1 + \frac{X}{1000} - p_{l,n_i}} - 0.1n \forall n \in \{0, \dots 9\}$$

by numerical solution (we have 10 variables and 9*8/2 inequalities by picking 10 of them we can find numerical solution):

$$\{\hat{c}_0, \dots \hat{c}_9\} = \{0.2579, 0.3695, 0.5011, 0.6526, 0.8242, 1.0158, 1.2274, 1.4589, 1.7105, 1.9821\}$$

Now again let's normalize $p_c = 1$ and by budget constraint we get:

$$c_i + p_{l,n_i} = 1 + \frac{1}{10} \sum_{n=0}^{9} p_{l,n} \quad \forall i \in \{0, \dots 9\}$$

RHS does not depend on i thus by solving it numerically we obtain following prices

$$P = \{\hat{p}_{l,\hat{n}_0}, \dots \hat{p}_{l,\hat{n}_9}\} = \{1.7265, 1.6149, 1.4833, 1.3317, 1.1602, 0.9686, 0.7570, 0.5254, 0.2739, 0.0023\}$$

We can sum up our results. Let P be set of equilibrium prices

$$P = \{(p_{l,n_i})_{i=0}^{99} : \hat{p}_{l,\hat{n}_0}, \dots \hat{p}_{l,\hat{n}_9} \in P', \hat{p}_{l,n_i} = 0 \quad i \in \{10, \cdots 99\}\}$$

We did all of our analysis assuming that it did not matter where in a column an agent lived. For a specic price p, this gives us unique equilibrium allocations, minus permutations within a column of where agents live,of

$$\bar{C}(p) = \{(\hat{c}_i, \hat{n}_i)_{i=0}^{999} : \{c_i = 1 + \frac{X}{1000} - p_{l,n_i}\}\}$$

are allocations (observe that permutations are separate competitive equilibrium). So the sets of ALL competitive equilibrium are prices $p \in P$ and allocations:

$$C(p) = \{(\hat{c}_i, \hat{m}_i, \hat{n}_i)_{i=0}^{999} : (\hat{c}_i, \hat{n}_i)_{i=0}^{999} \in \hat{C}(p) \quad i, j : i \neq j \quad n_i = n_j \quad m_i \neq m_j\}\}$$

□