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MIN II

SPRING 2021

RECITATION

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Today:
- INTRO
- NOTES
- DECISION THEORY

Recitations 15

[Definitions used today]

- players, actions, action profiles, consequences
- game on consequences, game in normal form
- lotteries: simple and compound
- vNM axioms: weak order, continuity, monotonicity, reduction, substitution

Question 1

Suppose [WO, I] hold. Let $\mathcal{L} \equiv \Delta(C)$ and $C = \{c_1, \ldots, c_m\}$. Show that:

$$\forall_{F \in \mathcal{L}} \quad \delta_{c^1} \succeq F \succeq \delta_{c^m}$$

where δ_{c^i} gives probability 1 to the consequence i

Question 2 [von Neumann-Morgenstern]

1. (existence) $\succeq$ on $\mathcal{L}$ satisfies WO, Cty, I if and only if there exists a linear $u : \mathcal{G} \to \mathbb{R}$ that represents $\succeq$
2. (uniqueness) If u, v are linear representations of $\succeq$, then $\exists A > 0, B \in \mathbb{R}$ such that $u(\cdot) = Av(\cdot) + B$

Show $\Rightarrow$ part of existence .

Question 3 [234 III.1 Fall 2016 majors]

Consider a preference order $\succeq$, and assume that it satisfies the von Neumann-Morgenstern (vNM) axioms. Let, for any two lotteries L and M, and any $\alpha \in [0, 1]$, (L, α, M) be the compound lottery that gives the lottery L with probability α and the lottery M with probability $1 - \alpha$

a) State what a vNM representation is, and then state the vNM axioms in the form you prefer: the axioms you state must characterize preferences with the vNM representation.

b) Prove that $\succeq$ satisfies the Sure Thing Principle (STP), namely that for any lotteries L, M, N and R and any $\alpha \in [0, 1]$

$$(L, \alpha, M) \succ (N, \alpha, M) \quad \text{if and only if} \quad (L, \alpha, R) \succ (N, \alpha, R)$$

If you assume the STP among your axioms, then prove that your axioms imply that the preference order has a vNM representation.

c) Suppose that $L \succ M$; prove that for any $\alpha \in (0, 1]$

$$(L, \alpha, M) \succ M$$

d) Prove that if u and v are two linear utility functions representing $\succeq$, then u is a positive affine transformation of v

Question 4 [Marschak Machina Triangle]

Consider a set C of three consequences 1,2,3 and a set of lotteries over C.

- Draw a 2D diagram that represents a three dimensional simplex.
- Draw two simple lotteries L_1 and L_2. Consider a compound lottery $L_3 = (L_1, p; L_2, 1 - p)$. How to represent it on the diagram?
- Suppose preferences are given by a Bernoulli function $u : C \to \mathbb{R}$. Write an equation for an indifference curve. Show that indifference curves are parallel. Draw some indifference curves on the diagram.

Question 5 The weighted utility model represents preferences $\succsim$ over lotteries in the MM triangle given above as follows:

$$(p_1, p_3) \succ (p_1', p_3') \iff \sum_{i=1}^{3} \frac{v_i p_i}{\sum_{j=1}^{3} v_j p_j} u_i > \sum_{i=1}^{3} \frac{v_i p_i}{\sum_{j=1}^{3} v_j p_j} u_i$$

where $u_i = u(i), u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing utility function, v_i are strictly positive weights and $p_2 = 1 - p_1 - p_3$

a) Write down the formula for an indifference curve implied by these preferences, i.e. the set of lotteries indifferent to each other (an equivalence class)

b) Show that any equivalence class for these preferences is convex, i.e. if E denotes some equivalence class of these preference, then if $P, Q \in E$, then $\alpha P + (1 - \alpha)Q \in E$, where $\alpha \in (0, 1)$. Another word for this property is betweenness.

c) Show that in general these preferences may not satisfy independence: $P \succ Q \Longrightarrow \alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R$, for $\alpha \in (0, 1)$

d) Show that betweenness is implied by independence.

e) Suppose that (0.2,0.8)$\prec$ $(0, 0)$ and $(0.8, 0.2) \succ (0.75, 0)$. Can this model accommodate such a pattern? If yes, specify values of u_i and v_i that may do the job.

Question 1

Suppose [WO, I] hold. Let $\mathcal{L} \equiv \Delta(C)$ and $C = \{c_1, \dots, c_m\}$. Show that:

$$\forall_{F\in\mathcal{L}} \quad \delta_{c^1} \succeq F \succeq \delta_{c^m}$$

where δ_{c^i} gives probability 1 to the consequence i

$\exists \; \delta_{c^1} \; \delta_{c^m} \; : \quad \delta_{c^i}$

$\boxed{\text{WTS: } \forall L \in \mathcal{L} \quad \delta_{c^b} \succsim L \succsim \delta_{c^w}}$

$\delta_{c^b} \succsim \delta_{c^i} \succsim \delta_{c^j} \succsim \dots \succsim \delta_{c^w}$

Lemma. $L_0, L_1, \dots L_K$ $(K+1)$ lotteries $L_i \in \mathcal{L}$

If $L_k \succsim L_0 \;\; \forall k \Rightarrow \sum_{k=1}^{K} \alpha_k L_k \succsim L_0$

If $L_0 \succsim L_k \;\; \forall k \Rightarrow \sum \alpha_k L_k \lesssim L_0$

$\alpha_k : \quad \sum_{k=1}^{K} \alpha_k = 1 \quad \alpha_k \geq 0.$

Proof: Induction. $K=1$ easy

Let $K \geq 2$. Assume true $K-1$. By def of $\Delta(\mathcal{L})$

$$\sum_{k=1}^{K} \alpha_k L_k = (1-\alpha_K) \sum_{k=1}^{K-1} \frac{\alpha_k}{1-\alpha_K} \cdot L_k + $$

$$+ \alpha_K L_K$$

WLOG $L_k \succsim L_0 \;\; \forall k$. By $(K-1)$-step

$$M = \sum_{k=1}^{K-1} \frac{\alpha_k}{1-\alpha_K} L_k \succsim L_0.$$

By I $\quad (1-\alpha_k)M + \alpha_k L_k \geq (1-\alpha_k)L_0 + \alpha_k L_k \quad (*)$

Again by I ($L_k \geq L_0$)

$$(1-\alpha_k)L_0 + \alpha_k L_k \geq (1-\alpha_k)L_0 + \alpha_k L_0 = L_0 \quad (**)$$

T (*), (**)

$$\sum_{k=1}^{K} \alpha_k L_k \geq L_0 \qquad \boxtimes$$

Case $L_0 \leq L_k$ $\forall k$ similar

Now. L^k – c^k w.p. 1

$$L = \begin{pmatrix} p^1 & \dots & p^m \\ c^1 & \dots & c^m \end{pmatrix} \qquad L = \sum_{k=1}^{m} L^k p_k$$

By lemma $\delta_{c^b} \geq \delta_{c^k}$ $\forall k \in \{1,\dots,m\}$

$$\delta_{c^b} \geq \sum_{k=1}^{m} p_k \delta_{c^k} = L$$

$$\underline{\delta_{c^b}} \geq L \geq \underline{\delta_{c^w}} \quad . \; L \in \Delta(C)$$

Question 2 [von Neumann-Morgenstern]

1. (existence) $\succeq$ on $\mathcal{L}$ satisfies WO, Cty, I if and only if there exists a linear $u : \mathcal{G} \to \mathbb{R}$ that represents $\succeq$
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Show $\Rightarrow$ part of existence .

lemma ① (WO, I, Cty), $L \succeq N \succeq M$

$\Rightarrow \exists\, \alpha \in [0,1] \quad L\alpha M \sim N$

Proof: $L \succeq M$. Define $A = \{\alpha \mid L\alpha M \succeq N\}$

$B = \{\beta \mid L\beta M \preceq N\}$

$\boxed{L\alpha M = \alpha L + (1-\alpha) M}$

Observe that

① $A, B \subseteq [0,1]$

② $0 \in B$, $1 \in A$

$L0M = 0 \cdot L + (1-0) \cdot M = M$

$M \preceq N$

$0 \in B$

③ By Cty A, B are closed

④ $A \cup B = [0,1]$

(1)-(4), $[0,1]$ connected set $\Rightarrow$

$\Rightarrow \exists\, \delta \in A \cap B \qquad L\delta M \succeq N \qquad L\delta M \preceq N$

$\Rightarrow$ by T $\quad L\delta M \succeq N \succeq L\delta M \Rightarrow N \sim L\delta M$

The rest is HW 1

Lemma 1. $L > M$ $\forall \alpha \in (0,1)$ $L > L\alpha M > M$

$L, M \in \mathcal{L}$

Proof: by ex 1. $\exists \delta_c^b, \delta_c^w$ $\delta_c^1 \geq L \geq \delta_c^m$

$\delta_c^b > \delta_c^w$.

$$L = \alpha L + (1-\alpha) L = \underbrace{L\alpha L \overset{I}{>} L\alpha M}_{} = \alpha L + (1-\alpha) M$$

$$\alpha L + (1-\alpha) M = \underbrace{L\alpha M \overset{I}{>} \alpha M + (1-\alpha) M = M}_{}$$

[In (I) we have $\geq$ (I^* with $>$)]

Def. (I^*) $\forall L, M, N \in \mathcal{L}$

$L > M \iff \forall \alpha \in (0,1)$ $L\alpha N > M\alpha N$

$I \subseteq I^*$, $I^* \subseteq I$

(I) $\Leftarrow$ existence

Question 4 [Marschak Machina Triangle]

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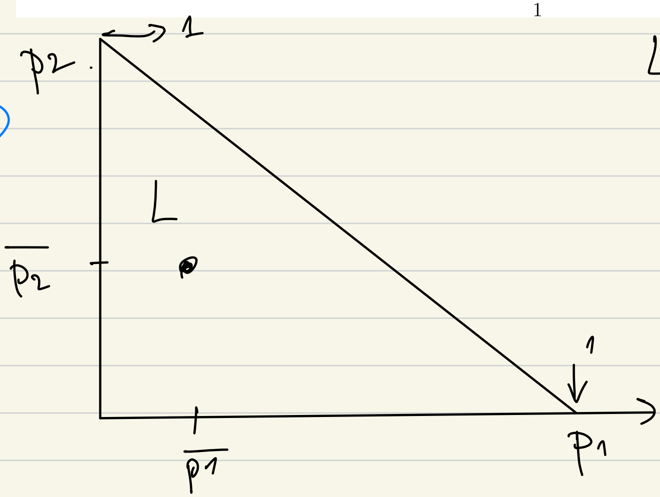
a



$$L = \begin{pmatrix} 1 & 2 & 3 \\ \bar{p}_1 & \bar{p}_2 & \bar{p}_3 \end{pmatrix}$$

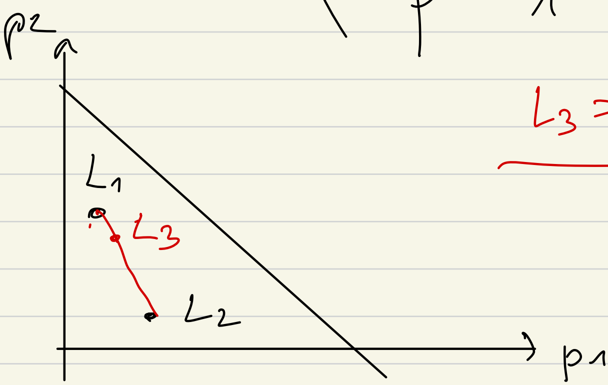
$$\bar{p}_3 = 1 - \bar{p}_1 - \bar{p}_2$$

$$L_1 = \begin{pmatrix} 1 & 2 & 3 \\ \bar{\bar{p}}_1 & \bar{\bar{p}}_2 & \bar{\bar{p}}_3 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 1 & 2 & 3 \\ \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_3 \end{pmatrix}$$

b

$$L_3 = \begin{pmatrix} L_1 & L_2 \\ p & 1-p \end{pmatrix}$$



$$\underbrace{L_3 = L_1 \, p \, L_2}_{\in \mathcal{L}} \in \mathcal{G}$$

(c) $\bar{u}$:

$$p_1 u(1) + p_2 u(2) + (1 - p_1 - p_2)\, u(3) = \bar{u}$$

- $u(3) > u(2) > u(1)$

$$p_2 \left(u(3) - u(2)\right) = p_1 \left(u(1) - u(3)\right) + u(3) - \bar{u}$$

$$p_2 = p_1 \cdot \frac{u(1) - u(3)}{u(3) - u(2)} + \frac{u(3) - \bar{u}}{u(3) - u(2)}$$

fixed nr, < 0 — does not depend on $\bar{u}$

fixed for fixed $\bar{u}$

$$p_2 = p_1\, a(\bar{u}) + b(\bar{u})$$

(1) Fix $\bar{u}$, $u(i)$. $\Rightarrow$ IC linear

(2) BC $a(\bar{u})$ does not depend on $\bar{u}$

IC are parallel

Ex. 5 .. IC $\underline{\underline{a(\bar{u})}}$

How does Indifference curves looks like?

IC are convex parallel curves.

We can depict them in Marschak-
-Machina triangle.

1. Convexity of IC

$L \sim M$ $\quad \lambda L + (1-\lambda) M \sim \lambda L + (1-\lambda) L$ $\quad$ ($L \sim M$)

$\underline{L \sim \lambda L + (1-\lambda) M}$ $\rightarrow$ convex IC

2. IC are parallel

Let $L \sim M$ consider $N \in \mathcal{L}$ $(N \neq L \sim M)$

$S \stackrel{\text{def}}{=} N + (M - L) \in \Delta$

I claim that $N \sim S$ (same IC)

Proof: ~~As~~ $N > S$. By I $\alpha = \frac{1}{2}$

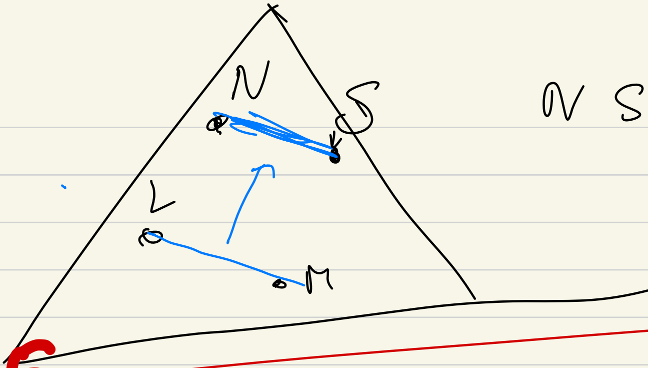
$\underline{\frac{1}{2} N + \frac{1}{2} M} > \frac{1}{2} S + \frac{1}{2} M \sim \frac{1}{2} S + \frac{1}{2} L$ (by $L \sim M$) (*)

but $\overline{S + L = N + M}$ so $\underline{\frac{1}{2} S + \frac{1}{2} L \sim \frac{1}{2} N + \frac{1}{2} M}$ (**)

It is a $\lightning$ with (*) (**)

It can not be $N > S$. similar not $S > N$.

So $N \sim S$



$N\ S\ (1c) \,\|\, L\ M\ (1c)$

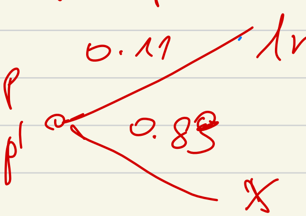
Ex 6

Violation of I Kahneman & Tversky 1979

$P = (1mln, 1)$ $Q = \begin{pmatrix} 5mln, & 0.1, \\ 1mln, & 0.89 \\ 0mln & 0.01 \end{pmatrix}$

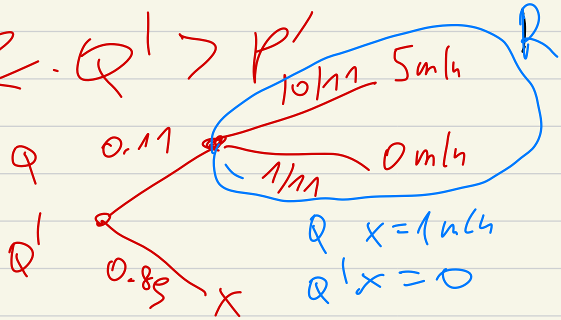
$P' = \begin{pmatrix} 1mln, & 0.11 \\ 0mln, & 0.89 \end{pmatrix}$ $Q' = \begin{pmatrix} 5mln, & 0.1 \\ 0mln & 0.9 \end{pmatrix}$

People $P > Q$ & $Q' > P'$



$P \quad x = 1mln$

$P' \quad x = 0mln$



$Q \quad x = 1mln$

$Q' \quad x = 0$

People pick $P > Q$ & $Q' > P'$.

People pick $P \succ Q$ & $Q' \succ P$

Observe that: $\quad (L \alpha M = \alpha L + (1-\alpha) M)$

$P = 1\text{mln} \ 0.11 \ 1\text{mln}$

$P' = 1\text{mln} \ 0.11 \ 0\text{mln}$

$R = 5\text{mln} \ \frac{10}{11} \ 0\text{mln}$

$Q = R \ 0.11 \ 1\text{mln}$

$Q = R \ 0.11 \ 0\text{mln}$

Let $1\text{mln} \succ R$. By $\underline{I}$

$P = 1\text{mln} \ 0.11 \ 1\text{mln} \succ R \ 0.11 \ 1\text{mln} = Q$

by $\underline{I}$

$P' = 1\text{mln} \ 0.11 \ 0\text{mln} \succ R \ 0.11 \ 0\text{mln} = Q'$

Hence both $P \succ Q$ & $P' \succ Q'$

contradicts what Kahneman & Tversky found in 1979.