



## Recitation 2 Solutions

### [Definitions used today]

- (conditional) factor demand, cost function, Shephard's lemma, Hotelling's lemma
- $\Delta$ -monotone, homogeneous, positive definite matrix, correspondence, upper hemicontinuity (UHC)

### Question 1 [Properties of $\pi^*$ and $s^*$ ] 33 [I.1 Fall 2006 majors]

Suppose that production set  $Y$  is closed. Let  $s^*(p)$  denote supply at price level  $p$  and by  $\pi^*(p)$  corresponding profit level. Then the following properties hold:

1.  $\pi^*$  is homogeneous of deg. 1 in prices  $p$
2.  $\pi^*$  is a convex function in prices  $p$
3. **correspondence**  $s^*$  is homogeneous of deg. 0
4.  $s^*$  is  $\Delta$ -monotone, that is:

$$[s^*(p) - s^*(p')] \cdot [p - p'] \geq 0 \quad \forall p, p'$$

5. **Hotelling's Lemma:** If  $\pi^*$  is differentiable at  $p$  (this holds iff  $s$  is single-valued at  $p$ ), then

$$D\pi^*(p) = s^*(p)$$

6. Assuming that  $\pi^*, s^*$  are differentiable at  $p \in \mathbb{R}^n$  prove comparative statics **law of supply**:

$$\frac{\partial s_i}{\partial p_i}(p) \geq 0$$

7. If  $Y$  is compact, then  $\pi^*$  is a continuous function and  $s^*$  is an upper hemicontinuous (UHC) correspondence.

### Solution 1

Let  $Y$  satisfy nonemptiness, closedness, and free disposal assumptions. Then  $\max_Y$  and  $\sup_Y$  of continuous functions are equivalent (Weierstrass Theorem aka Extreme Value Theorem)

**Definition 0.1.** *Profit maximization* at price vector  $p \in \mathbb{R}^L$  is represented by the problem:

$$\sup_{y \in Y} p \cdot y \tag{0.1}$$

**Definition 0.2.** *The supply* of the firm at  $p$  is the optimizing vector of the profit maximization problem. We can write

$$s^*(p) = \arg \max \{p \cdot y : y \in Y\} \tag{0.2}$$

$$s^*(p) = \{y^* \in Y : p \cdot y^* \geq p \cdot y, \forall y \in Y\} \tag{0.3}$$

**Definition 0.3.** Function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is **homogeneous of degree  $k$**  if  $\forall \lambda \geq 0$

$$g(\lambda x) = \lambda^k g(x)$$

*Proof.* Throughout this proof, I will use “sup” as shorthand for

$$\sup_{y \in Y} p \cdot y$$

**Step-1** Let  $p \in \mathbb{R}^L$  and let  $\lambda \in \mathbb{R}$ . Then

$$\pi^*(\lambda p) = \sup(\lambda p) \cdot y = \sup \lambda(p \cdot y) = \lambda[\sup p \cdot y] = \lambda \pi^*(p).$$

Therefore  $\pi^*$  is homogeneous of degree 1.

**Step-2** Let  $p, p' \in \mathbb{R}^\ell$ . Let  $\lambda \in [0, 1]$  and let  $p_\lambda = \lambda p + (1 - \lambda)p'$ . Then we have

$$\begin{aligned}\pi^*(p_\lambda) &= \sup p_\lambda \cdot y \\ &= \sup(\lambda p + (1 - \lambda)p') \cdot y \\ &= \sup[\lambda(p \cdot y) + (1 - \lambda)p' \cdot y] \\ &\leq \sup \lambda(p \cdot y) + \sup(1 - \lambda)(p' \cdot y) \\ &= \lambda \sup p \cdot y + (1 - \lambda) \sup p' \cdot y \\ &= \lambda \pi^*(p) + (1 - \lambda) \pi^*(p').\end{aligned}$$

Therefore  $\pi^*$  is convex. **Warning** I didn't consider case when  $\pi(p)$  is : empty,  $+\infty$ ,  $-\infty$  or single-valued.

**Step-3** Let  $p \in \mathbb{R}^\ell$  and let  $y^* \in s^*(p)$ . Then  $p \cdot y^* \geq p \cdot y$ ,  $\forall y \in Y$ . Let  $\lambda \in \mathbb{R}$ . Then

$$(\lambda p) \cdot y^* = \lambda(p \cdot y^*) \geq \lambda(p \cdot y) = (\lambda p) \cdot y, \forall y \in Y,$$

so  $y^* \in s^*(\lambda p)$ . Therefore  $s^*$  is homogeneous of degree 0.

**Step-4** We obtain it from **step 6** by applying **Proposition I.1** from Math appendix I-III.

**Step-5** Let  $f(p, y) = p \cdot y$ . Then  $\pi^*(p) = \max_{y \in Y} f(p, y)$ . Let  $y^*(p) = s^*(p)$ . Then  $\pi^*(p) = f(p, y^*(p))$ , so

$$D\pi^*(p) = D_p f(p, y^*(p)) = D_p f(p, y)|_{y=y^*(p)} + D_y f(p, y^*(p)) D_p y^*(p).$$

But  $D_y f(p, y^*(p)) = 0$  is a FOC of the maximization problem (which we know has a solution since  $s^*$  is single-valued and thus nonempty), so we have

$$D\pi^*(p) = D_p f(p, y)|_{y=y^*(p)}.$$

And  $D_p f(p, y) = y$ , so

$$D\pi^*(p) = y^*(p) = s^*(p).$$

**Step-6**

**Corollary 0.4.** If  $\pi^*$  is twice differentiable, then  $D^2\pi^*(p) = Ds^*(p)$ . The substitution matrix  $Ds^*(p)$  is positive semi-definite and symmetric.

Every matrix of second partial derivatives is symmetric, and since  $\pi^*$  is convex,  $D^2\pi^*(p)$  must be positive semi-definite.

This corollary implies the following **comparative statics** property of supply

$$\frac{\partial s_i^*}{\partial p_i} \geq 0 \quad (0.4)$$

**Step-7** Remind me to do it during consumer theory!

□

### Question 2 [Zero profit CRS]

If  $Y$  exhibits CRTS, then  $\pi^*(p) = 0$  whenever it is well-defined.

### Solution 2

*Proof.* The outline of our proof is as follows:

- 1) Show that  $0 \in Y$
- 2) Show that  $\pi_Y^*(p) \geq 0, \forall p$
- 3) Show that  $\pi_Y^*(p) \leq 0, \forall p$  such that  $\pi_Y^*(p) \neq \infty$ .

- Step 1: Let  $y \in Y$ .  $\xrightarrow{(CRTS)} \lambda y \in Y, \forall \lambda \geq 0$ . In particular,  $\lambda y \in Y$  for  $\lambda = 0 \Rightarrow \lambda y = 0y = 0 \in Y$ .
- Step 2: Since  $\pi_Y^*(p) = \sup_{y \in Y} py$ ,  $\pi_Y^*(p) \geq py, \forall y \in Y$ .  $\xrightarrow{Step 1(0 \in Y)} \pi_Y^*(p) \geq p0 = 0$
- Step 3: (By contradiction)

Suppose  $\exists p$  s.t.  $\pi_Y^*(p) \neq \infty$ . and  $\pi_Y^*(p) > 0$ . Recall  $\pi_Y^*(p) = \sup_{y \in Y} py$ .

Then  $\exists y \in Y$  s.t.  $0 < \frac{\pi_Y^*(p)}{2} < py \leq \pi_Y^*(p)$ .<sup>1</sup> If we multiply this inequality by three, we get  $0 < \frac{3}{2}\pi_Y^*(p) < 3py \leq 3\pi_Y^*(p)$  which implies  $\pi_Y^*(p) < \frac{3}{2}\pi_Y^*(p) < 3py \equiv 3 < p, y > < p, 3y >$ .

However, since  $y \in Y$  and  $Y$  has CRTS,  $3y \in Y$ . Hence, we have proved that  $\exists \hat{y} \in Y$  s.t.  $p\hat{y} > \pi_Y^*(p) = \sup_{y \in Y} py$ , which is a contradiction. Thus,  $\pi_Y^*(p) \leq 0, \forall p$  such that  $\pi_Y^*(p) \neq \infty$ .

From steps 2 and 3,  $\pi_Y^*(p) \geq 0$  and  $\pi_Y^*(p) \leq 0 \Rightarrow \pi_Y^*(p) = 0, \forall p$  such that  $\pi_Y^*(p) \neq \infty$ . □

### Question 3 [Properties of $C$ and $x$ ]

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a production function that is strictly increasing (continuous) and satisfies  $f(0) = 0$ . Let  $C^*(w, z)$  be the (minimum) cost function, where  $w \in \mathbb{R}^n$  is a vector of input prices and  $z > 0$  is an output level. Let  $x^*(w, z)$  be the optimizer of cost minimization problem. Prove following properties:

1.  $C^*$  is homogeneous of degree 1 in factor prices  $w$
2.  $C^*$  is a concave function of  $w$
3.  $x^*(w, z)$  is homogeneous of degree zero in  $w$ .
4.  $x$  is  $\Delta$ -monotone for fixed  $z$ , in following way:

$$[x^*(w, z) - x^*(w', z)] \cdot [w - w'] \leq 0 \quad \forall w, w' \gg 0$$

5. **Shephard's Lemma** If  $C^*$  is differentiable at  $p$  (this holds  $\iff x^*$  is single-valued) then

$$D_w C^*(w, z) = x^*(w, z)$$

6. Assuming that  $C^*, x^*$  are differentiable at  $w \in \mathbb{R}^n$  prove comparative statics property of factor demand:

$$\frac{\partial x_i}{\partial w_i}(w, z) \leq 0$$

7. Show that cost function  $C$  is a non-decreasing function of output level  $w$ , for every  $z$ .
8. If production function  $f$  is concave, then cost function  $C$  is a convex function of output level  $z$ , for every  $w \gg 0$

### Solution 3

**Definition 0.5.** The problem of **cost minimization** for a producer with production function  $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  is

$$\begin{aligned} C^*(w, z) = & \inf w \cdot x \\ \text{s.t.} & f(x) \geq z \\ & x \geq 0 \end{aligned}$$

where

- $w = (w_1, \dots, w_n) \gg 0$  is a vector of input prices.
- $x = (x_1, \dots, x_n) \geq 0$  is vector of inputs.
- $z \geq 0$  is the single-output produced.
- $Y^f$  satisfies free disposal, closed.

**Definition 0.6.** (**conditional factor demand correspondence** (or function, if single-valued) of the firm at  $(w, z)$ , denoted by

$$x^*(w, z) = \operatorname{argmin}_x \{w \cdot x : f(x) \geq z\} \tag{0.5}$$

is the optimizing vector (minimizer) of the cost minimization problem.

---

<sup>1</sup>If  $\alpha = \sup A$  then  $\forall \varepsilon > 0, \exists \beta$  s.t.  $\alpha - \varepsilon < \beta \leq \alpha$ .

*Proof.*

□

**Step-3**  $x^*(\lambda w, z) = \{x^* \in \mathbb{R}_+^n : \lambda w x^* \leq \lambda w x, \forall x \in \mathbb{R}_+^n\} = \{x^* \in \mathbb{R}_+^n : w x^* \leq w x, \forall x \in \mathbb{R}_+^n\} = x^*(w, z)$   
This is non-empty because  $Y^f$  is closed.

**Step-1**

$$c^*(\lambda w, z) = \lambda w \cdot x^*(\lambda w, z) \xrightarrow{\text{by 3}} \lambda w \cdot x^*(w, z) = \lambda c^*(w, z)$$

**Step-2** From support functions.  $C^*(\cdot, z) = \inf_{x \in V(z)} w x$  is the support function of  $V(z)$ . Since it is the inf (and not the sup as in the profit maximization problem), the support function is concave (and not convex, like the profit function).

**Step-4** We obtain it from **step 6** by applying **Proposition I.1** from Math appendix I-III. Other proof:

$$x(w, z) - x(w', z)[w - w'] = [w x(w, z) - w x(w', z)] + [w' x(w', z) - w' x(w, z)] \leq 0 + 0 \leq 0$$

**Step-5 Shephard's Lemma** - we will see it again in consumer theory (derivative of expenditure function is Hicksian (compensated) demand over prices. The cost function is **nondecreasing** in factor prices Differentiate for  $f(w, y) = w \cdot y$ :

$$C^*(w, z) \equiv w \cdot x^*(w, z) \quad (0.6)$$

$$D_w C^*(w, z) = D_w f(w, x^*(w, z)) = D_w f(w, y)|_{y=x^*(w, z)} + D_z f(w, y^*(w, z)) D_w x^*(w, z).$$

And  $D_w f(w, y) = y$ , so

$$D_w C^*(w, z) = y^*(w, z) = x^*(w, z).$$

**Step-6**

**Corollary 0.7.** If  $C^*$  is twice-differentiable with respect to prices, then  $D_w^2 C^*(w, z) = D_w x^*(w, z)$ . The matrix  $D_w x^*$  is negative semi-definite and symmetric.

Corollary 0.7 implies the following comparative statics property of factor demand:

$$\frac{\partial x_i^*}{\partial w_i} \leq 0 \quad (0.7)$$

The matrix  $D_w x^*$  is singular. This is so because  $D_w x^*(w, z) w = 0$  as follows from Theorem 1.7.1 part 3 and Euler's Theorem (see MWG, Appendix).

**Step-7** To show that  $C(w, z)$  is non decreasing in  $w$  take any  $w_1, w_2$   $w_1 \geq w_2$  and  $x$  s.t.  $f(x) \geq z$ , we have  $w x_1 \geq w x_2$ . Now following inequalities holds:

$$w_1 \cdot x \geq w_1 x(w_1, z) \geq w_2 x(w_1, z) \geq w_2 x(w_2, z)$$

where first is optimality of  $x(w_1, z)$ , second comes from  $w x_1 \geq w x_2$ , third again from optimality of  $x(w_2, z)$   
WARNING: look longer at last inequality

**Step-8** Take  $z_1 \neq z_2$

$$\forall x_1 \geq 0 \quad f(x_1) \geq z_1$$

$$\forall x_2 \geq 0 \quad f(x_2) \geq z_2$$

$f$  is concave so  $\forall \lambda \in [0, 1]$

$$f(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda f(x_1) + (1 - \lambda)f(x_2) \geq \lambda z_1 + (1 - \lambda)z_2$$

$$C^*(w, \lambda x_1 + (1 - \lambda)x_2) \leq w \cdot (\lambda x_1 + (1 - \lambda)x_2) = \lambda(w x_1) + (1 - \lambda)(w x_2)$$

Since  $x_1, x_2$  are taken arbitrary in particular take  $x_1 = x^*(w, z_1)$  and  $x_2 = x^*(w, z_2)$  and we can take min on RHS and we hence we obtain

$$C^*(w, \lambda x_1 + (1 - \lambda)x_2) \leq \lambda C(w, x_1) + (1 - \lambda)C(w, x_2)$$

#### Question 4 [Aggregation]

Consider two closed production sets  $Y_1, Y_2 \subseteq \mathbb{R}^L$  such that  $0 \in Y_1$  and  $0 \in Y_2$ . Let  $\pi_1^*$  and  $\pi_2^*$  denote the profit functions associated with  $Y_1$  and  $Y_2$ . Let  $\pi^*$  be the profit functions associated with  $Y$ .

1. Let  $Y = Y_1 + Y_2$  be the (algebraic) sum of the two production sets. Prove that  $\pi_1(p) + \pi_2(p) = \pi(p)$  for every  $p \in \mathbb{R}^L$
2. Prove that  $Y_1 \subseteq Y_2$  if and only if  $\pi_1(p) \leq \pi_2(p)$
3. Let  $Y = \text{co}\{Y_1, Y_2\}$  be the convex hull of the two production sets (that is, the set of all convex combinations of elements of  $Y_1$  and  $Y_2$ ). Prove that  $\pi(p) = \max\{\pi_1(p), \pi_2(p)\}$  for every  $p \in \mathbb{R}^L$

#### Solution 4

a) Let denote for convenience  $Y_1$  by  $Y$ ,  $Y_2$  by  $Y'$  and  $Y$  by  $Y''$  respectively. Take  $y'' \in Y''$ . From the definition of  $Y''$ , it exists  $(y, y') \in Y \times Y'$  such that  $y'' = y + y'$ .

Thus  $p \cdot y'' = p \cdot y + p \cdot y' \leq \pi(p) + \pi'(p)$ .

Taken the supremum on  $Y''$  in the right side of the equality, one gets the inequality  $\pi''(p) \leq \pi(p) + \pi'(p)$ .

Conversely if  $\pi(p) = +\infty$ , then there exists a sequence  $(y'')_{\nu \in \mathbb{N}}$  of  $Y$  such that the sequence  $(p \cdot y'')$  converges to  $+\infty$ . Let  $y'$  be any element of  $Y'$ .

Then we have  $\lim_{\nu} p \cdot (y'' + y') = +\infty$  and since  $y'' + y' \in Y''$  one deduces that  $\pi''(p) = +\infty$ . A symmetric argument shows that the result is identical if  $\pi'(p) = +\infty$ . If  $\pi(p)$  and  $\pi'(p)$  are finite, for all  $\varepsilon > 0$ , it exists  $(y, y') \in Y \times Y'$  such that  $p \cdot y \geq \pi(p) - \varepsilon$  and  $p \cdot y' \geq \pi'(p) - \varepsilon$

Hence  $p \cdot (y + y') \geq \pi(p) + \pi'(p) - 2\varepsilon$ . since  $y + y' \in Y''$ , one deduces that  $\pi''(p) \geq \pi(p) + \pi'(p) - 2\varepsilon$ . since the inequality holds true for every  $\varepsilon > 0$  one can conclude that  $\pi''(p) \geq \pi(p) + \pi'(p)$

Let  $(y, y') \in s(p) \times s'(p)$ . Hence  $p \cdot y = \pi(p)$  and  $p \cdot y' = \pi'(p)$ . So  $p \cdot (y + y') = \pi(p) + \pi'(p) = \pi''(p)$ . since  $y + y' \in Y''$ , this implies that  $y + y' \in s''(p)$ . Conversely let  $y'' \in s''(p)$  and let  $(y, y') \in Y \times Y'$  such that  $(y, y') = y + y'$ .

Then,  $p \cdot (y + y') = \pi''(p) = \pi(p) + \pi'(p)$ . since  $p \cdot y \leq \pi(p)$  and  $p \cdot y' \leq \pi'(p)$ , this implies that  $p \cdot y = \pi(p)$  and  $p \cdot y' = \pi'(p)$ . Consequently  $y \in s(p)$  and  $y' \in s'(p)$

b)  $\Rightarrow$

$$\max_{x \in Y_2} px = \max_{x \in Y_1 \cup (Y_2/Y_1)} px = \max\{\max_{x \in Y_1} px, \max_{x \in (Y_2/Y_1)} px\} \geq \max_{x \in Y_1} px$$

Maximum of the function on bigger (in sense of inclusion) set is higher.

$\Leftarrow$  Suppose not. There exist  $x \in Y_1$  and  $x \notin Y_2$ . Sets  $Y_2$  and  $\{x\}$  are convex, closed and disjoint so we can apply strict separating hyperplane theorem, i.e. there exists  $q, b \in \mathbb{R}^L$ :

$$q \cdot x > b \quad x \in Y_1 \quad \text{and} \quad b > q \cdot y \quad \forall y \in Y_2$$

this means that by taking  $\max : \pi_1(q) > \pi_2(q)$  which contradicts our notion. □

#### Question 5 [Midterm 2006]

Consider the following supply function of a firm

$$s(p_1, p_2) = \left( -\frac{2p_2}{p_1}, \frac{p_2}{p_1} \right)$$

Show that this supply function can not result from profit maximization on any production set.

#### Solution 5

Consider following choice of prices which gives us formula for supply (2nd good is produced with 1st good as input)

$$p = (1, 1) \quad p' = \left( \frac{1}{3}, \frac{1}{6} \right) \quad s(p) = (-2, 1) \quad s(p') = \left( -1, \frac{1}{2} \right)$$

$$[s^*(p) - s^*(p')] \cdot [p - p'] = \left( -1, \frac{1}{2} \right) \cdot \left( \frac{2}{3}, \frac{5}{6} \right) = -\frac{2}{3} + \frac{5}{12} = -\frac{3}{12} < 0$$

□