Recitation 2 Solutions

[Definitions used today]

- (conditional) factor demand, cost function, Shephard's lemma, Hotelling's lemma
- Δ-monotone, homogeneous, positive definite matrix, correspondence, upper hemicontinuity (UHC)

Question 1 [Properties of π^* and s^*] 33 [I.1 Fall 2006 majors]

Suppose that production set Y is closed. Let $s^*(p)$ denote supply at price level p and by $\pi^*(p)$ corresponding profit level. Then the following properties hold:

1. π^* is homogeneous of deg. 1 in prices p
2. π^* is a convex function in prices p
3. **correspondence** s^* is homogeneous of deg. 0
4. s^* is Δ-monotone, that is:
$$[s^*(p) - s^*(p')] \cdot [p - p'] \geq 0 \quad \forall p, p'$$
5. **Hotelling's Lemma**: If π^* is differentiable at p (this holds iff s is single-valued at p), then
$$D\pi^*(p) = s^*(p)$$
6. Assuming that π^*,s^* are differentiable at $p \in \mathbb{R}^n$ prove comparative statics **law of supply**:
$$\frac{\partial s_i}{\partial p_i}(p) \geq 0$$
7. If Y is compact, then π^* is a continuous function and s^* is an upper hemicontinuous (UHC) correspondence.

Solution 1 Let Y satisfy nonemptiness, closedness, and free disposal assumptions.Then $\max_Y$ and $\sup_Y$ of continuous functions are equivalent (Weierstrass Theorem aka Extreme Value Theorem)

Definition 0.1. *Profit maximization at price vector $p \in \mathbb{R}^L$ is represented by the problem:*

$$\sup_{y \in Y} p \cdot y \tag{0.1}$$

Definition 0.2. *The supply of the firm at p is the optimizing vector of the profit maximization problem. We can write*

$$s^*(p) = \arg\max\{p \cdot y : y \in Y\} \tag{0.2}$$
$$s^*(p) = \{y^* \in Y \ : \ p \cdot y^* \geq p \cdot y, \forall y \in Y\} \tag{0.3}$$

Definition 0.3. *Function $g : \mathbb{R}^n \to R$ is **homogeneous of degree k** if $\forall\ \lambda \geq 0$*

$$g(\lambda x) = \lambda^k g(x)$$

Proof. Throughout this proof, I will use "sup" as shorthand for

$$\sup_{y \in Y} \ p \cdot y$$

Step-1 Let $p \in \mathbb{R}^\ell$ and let $\lambda \in \mathbb{R}$. Then

$$\pi^*(\lambda p) = \sup(\lambda p) \cdot y = \sup \lambda(p \cdot y) = \lambda[\sup p \cdot y] = \lambda \pi^*(p).$$

Therefore π^* is homogeneous of degree 1.

Step-2 Let $p, p' \in \mathbb{R}^{\ell}$. Let $\lambda \in [0,1]$ and let $p_\lambda = \lambda p + (1-\lambda)p'$. Then we have

$$\begin{aligned}\pi^*(p_\lambda) &= \sup p_\lambda \cdot y \\ &= \sup(\lambda p + (1-\lambda)p') \cdot y \\ &= \sup[\lambda(p \cdot y) + (1-\lambda)p' \cdot y] \\ &\leq \sup \lambda(p \cdot y) + \sup(1-\lambda)(p' \cdot y) \\ &= \lambda \sup p \cdot y + (1-\lambda) \sup p' \cdot y \\ &= \lambda \pi^*(p) + (1-\lambda)\pi^*(p').\end{aligned}$$

Therefore π^* is convex. **Warning** I didn't consider case when $\pi(p)$ is : empty,$+\infty$, $-\infty$ or single-valued.

Step-3 Let $p \in \mathbb{R}^{\ell}$ and let $y^* \in s^*(p)$. Then $p \cdot y^* \geq p \cdot y,\ \forall y \in Y$. Let $\lambda \in \mathbb{R}$. Then

$$(\lambda p) \cdot y^* = \lambda(p \cdot y^*) \geq \lambda(p \cdot y) = (\lambda p) \cdot y,\ \forall y \in Y,$$

so $y^* \in s^*(\lambda p)$. Therefore s^* is homogeneous of degree 0.

Step-4 We obtain it from **step 6** by applying **Proposition I.1** from Math appendix I-III.

Step-5 Let $f(p,y) = p \cdot y$. Then $\pi^*(p) = \max_{y \in Y}\ f(p,y)$. Let $y^*(p) = s^*(p)$. Then $\pi^*(p) = f(p, y^*(p))$, so

$$D\pi^*(p) = D_p f(p, y^*(p)) = D_p f(p,y)|_{y=y^*(p)} + D_y f(p, y^*(p)) D_p y^*(p).$$

But $D_y f(p, y^*(p)) = 0$ is a FOC of the maximization problem (which we know has a solution since s^* is single-valued and thus nonempty), so we have

$$D\pi^*(p) = D_p f(p,y)|_{y=y^*(p)}.$$

And $D_p f(p,y) = y$, so

$$D\pi^*(p) = y^*(p) = s^*(p).$$

Step-6

Corollary 0.4. *If π^* is twice differentiable, then $D^2\pi^*(p) = Ds^*(p)$. The substitution matrix $Ds^*(p)$ is positive semi-definite and symmetric.*

Every matrix of second partial derivatives is symmetric, and since π^* is convex, $D^2\pi^*(p)$ must be positive semi-definite.

This corollary implies the following **comparative statics** property of supply

$$\frac{\partial s_i^*}{\partial p_i} \geq 0 \tag{0.4}$$

Step-7 Remind me to do it during consumer theory!

□

Question 2 [Zero profit CRS]

If Y exhibits CRTS, then $\pi^*(p) = 0$ whenever it is well-defined.

Solution 2

Proof. The outline of our proof is as follows:
1) Show that $0 \in Y$
2)Show that $\pi_Y^*(p) \geq 0, \forall p$
3)Show that $\pi_Y^*(p) \leq 0, \forall p$such that $\pi_Y^*(p) \neq \infty$.

- Step 1: Let $y \in Y$. $\xrightarrow{(CRTS)}$ $\lambda y \in Y, \forall \lambda \geq 0$. In particular, $\lambda y \in Y$ for $\lambda = 0$. $\Rightarrow \lambda y = 0y = 0 \in Y$.
- Step 2: Since $\pi_Y^*(p) = \sup_{y \in Y} py$, $\pi_Y^*(p) \geq py, \forall y \in Y$. $\xrightarrow{Step1(0 \in Y)}$ $\pi_Y^*(p) \geq p0 = 0$
- Step 3: (By contradiction)

Suppose $\exists p$ s.t. $\pi_Y^*(p) \neq \infty$. and $\pi_Y^*(p) > 0$. Recall $\pi_Y^*(p) = \sup_{y \in Y} py$.
Then $\exists y \in Y$ s.t. $0 < \frac{\pi_Y^*(p)}{2} < py \leq \pi_Y^*(p)$. [1] If we multiply this inequality by three, we get $0 < \frac{3}{2}\pi_Y^*(p) < 3py \leq 3\pi_Y^*(p)$ which implies $\pi_Y^*(p) < \frac{3}{2}\pi_Y^*(p) < 3py \equiv 3 < p, y >=< p, 3y >$.

However, since $y \in Y$ and Y has CRTS, $3y \in Y$. Hence, we have proved that $\exists \hat{y} \in Y$ s.t. $p\hat{y} > \pi_Y^*(p) = \sup_{y \in Y} py$, which is a contradiction. Thus, $\pi_Y^*(p) \leq 0, \forall p$ such that $\pi_Y^*(p) \neq \infty$.

From steps 2 and 3, $\pi_Y^*(p) \geq 0$ and $\pi_Y^*(p) \leq 0 \Rightarrow \pi_Y^*(p) = 0, \forall p$ such that $\pi_Y^*(p) \neq \infty$. □

Question 3 [Properties of C and x]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a production function that is strictly increasing (continuous) and satisfies $f(0) = 0$. Let $C^*(w, z)$ be the (minimum) cost function, where $w \in \mathbb{R}^n$ is a vector of input prices and $z > 0$ is an output level. Let $x^*(w, z)$ be the optimizer of cost minimization problem. Prove following properties:

1. C^* is homogeneous of degree 1 in factor prices w
2. $C^*$is a concave function of w
3. $x^*(w, z)$ is homogeneous of degree zero in w.
4. x is Δ-monotone for fixed z, in following way:
$$[x^*(w, z) - x^*(w', z)] \cdot [w - w'] \leq 0 \quad \forall w, w' \gg 0$$
5. **Shephard's Lemma** If C^* is differentiable at p (this holds $\iff$ x^* is single-valued) then
$$D_w C^*(w, z) = x^*(w, z)$$
6. Assuming that C^*, x^* are differentiable at $w \in \mathbb{R}^n$ prove comparative statics property of factor demand:
$$\frac{\partial x_i}{\partial w_i}(w, z) \leq 0$$
7. Show that cost function C is a non-decreasing function of output level w, for every z.
8. If production function f is concave, then cost function C is a convex function of output level z, for every $w \gg 0$

Solution 3

Definition 0.5. *The problem of* ***cost minimization*** *for a producer with production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ is*

$$\begin{aligned} C^*(w, z) = \quad & \inf w \cdot x \\ s.t. \quad & f(x) \geq z \\ & x \geq 0 \end{aligned}$$

where

- $w = (w_1, \ldots, w_n) \gg 0$ is a vector of input prices.
- $x = (x_1, \ldots, x_n) \geq 0$ is vector of inputs.
- $z \geq 0$ is the single-output produced.
- Y^f satisfies free disposal, closed.

Definition 0.6. ***(conditional) factor demand correspondence*** *(or function, if single-valued) of the firm at (w, z), denoted by*

$$x^*(w, z) = argmin_x\{w \cdot x : \ f(x) \geq z\} \tag{0.5}$$

is the optimizing vector (minimizer) of the cost minimization problem.

[1] If $\alpha = \sup A$ then $\forall \varepsilon > 0$, $\exists \beta$ s.t. $\alpha - \varepsilon < \beta \leq \alpha$.

Proof. □

Step-3 $x^*(\lambda w, z) = \{x^* \in \mathbb{R}^n_+ : \lambda w x^* \leq \lambda w x,\ \forall x \in \mathbb{R}^n_+\} = \{x^* \in \mathbb{R}^n_+ : w x^* \leq w x,\ \forall x \in \mathbb{R}^n_+\} = x^*(w, z)$
This is non-empty because Y^f is closed.

Step-1

$$c^*(\lambda w, z) = \lambda w \cdot x^*(\lambda w, z) \xrightarrow{\text{by } 3} = \lambda w \cdot x^*(w, z) = \lambda c^*(w, z)$$

Step-2 From support functions. $C * (\centerdot, z) = \inf_{x \in V(z)} wz$ is the support function of $V(z)$. Since it is the inf (and not the sup as in the profit mazimization problem), the support function is concave (and not convex, like the profit function).

Step-4 We obtain it from **step 6** by applying **Proposition I.1** from Math appendix I-III. Other proof:

$$x(w, z) - x(w', z)[w - w'] = [wx(w, z) - wx(w', z)] + [w'x(w', z) - w'x(w, z)] \leq 0 + 0 \leq 0$$

Step-5 **Shephard's Lemma** - we will see it again in consumer theory (derivative of expenditure function is Hicksian (compensated)[comes from min expenditures] demand over prices. The cost function is **nondecreasing** in factor prices Differentiate for $f(w, y) = w \cdot y$:

$$C^*(w, z) \equiv w \cdot x^*(w, z) \tag{0.6}$$

$$D_w C^*(w, z) = D_w f(w, x^*(w, z)) = D_w f(w, y)|_{y = x^*(w,z)} + D_z f(w, y^*(w, z)) D_w x^*(w, z).$$

And $D_w f(w, y) = y$, so

$$D_w C^*(w, z) = y^*(w, z) = x^*(w, z).$$

Step-6

Corollary 0.7. *If C^* is twice-differentiable with respect to prices, then $D^2_w C^*(w, z) = D_w x^*(w, z)$. The matrix $D_w x^*$ is negative semi-definite and symmetric.*

Corollary 0.7 implies the following comparative statics property of factor demand:

$$\frac{\partial x_i^*}{\partial w_i} \leq 0 \tag{0.7}$$

The matrix $D_w x^*$ is singular. This is so because $D_w x^*(w, z)\, w = 0$ as follows from Theorem 1.7.1 part 3 and Euler's Theorem (see MWG, Appendix).

Step-7 To show that $C(w, z)$ is non decreasing in w take any w_1, w_2 $w_1 \geq w_2$ and x s.t. $f(x) \geq z$, we have $wx_1 \geq w_{x2}$. Now following inequalities holds:

$$w_1 \cdot x \geq w_1 x(w_1, z) \geq w_2 x(w_1, z) \geq w_2 x(w_2, z)$$

where first is optimality of $x(w_1, z)$, second comes from $wx_1 \geq w_{x2}$, third again from optimality of $x(w_2, z)$ WARNING: look longer at last inequality

Step-8 Take $z_1 \neq z_2$

$$\forall x_1 \geq 0 \quad f(x_1) \geq z_1$$

$$\forall x_2 \geq 0 \quad f(x_2) \geq z_2$$

f is concave so $\forall \lambda \in [0, 1]$

$$f(\lambda x_1 + (1 - \lambda) x_2) \geq \lambda f(x_1) + (1 - \lambda) f(x_2) \geq \lambda z_1 + (1 - \lambda) z_2$$

$$C^*(w, \lambda x_1 + (1 - \lambda) x_2) \leq w \cdot (\lambda x_1 + (1 - \lambda) x_2) = \lambda (w x_1) + (1 - \lambda)(w x_2)$$

Since x_1, x_2 are taken arbitrary in particular take $x_1 = x^*(w, z_1)$ and $x_2 = x^*(w, z_2)$ and we can take min on RHS and we hence we obtain

$$C^*(w, \lambda x_1 + (1 - \lambda) x_2) \leq \lambda C(w, x_1) + (1 - \lambda) C(w, x_2)$$

Question 4 [Aggregation]

Consider two closed production sets $Y_1, Y_2 \subseteq \mathbb{R}^L$ such that $0 \in Y_1$ and $0 \in Y_2$. Let π_1^* and π_2^* denote the profit functions associated with Y_1 and Y_2. Let π^* be the profit functions associated with Y.

1. Let $Y = Y_1 + Y_2$ be the (algebraic) sum of the two production sets. Prove that $\pi_1(p) + \pi_2(p) = \pi(p)$ for every $p \in \mathbb{R}^L$
2. Prove that $Y_1 \subseteq Y_2$ if and only if $\pi_1(p) \leq \pi_2(p)$
3. Let $Y = \text{co}\{Y_1, Y_2\}$ be the convex hull of the two production sets (that is, the set of all convex combinations of elements of Y_1 and Y_2). Prove that $\pi(p) = \max\{\pi_1(p), \pi_2(p)\}$for every for every $p \in \mathbb{R}^L$

Solution 4

a) Let denote for convenience Y_1 by Y, Y_2 by $Y^{'}$ and Y by $Y^{''}$ respectively. Take $y'' \in Y''$. From the definition of Y'', it exists $(y, y') \in Y \times Y'$ such that $y'' = y + y'$.

Thus $p \cdot y'' = p \cdot y + p \cdot y' \leq \pi(p) + \pi'(p)$.
Taken the supremum on Y'' in the right side of the equality, one gets the inequality $\pi''(p) \leq \pi(p) + \pi'(p)$.
Conversely if $\pi(p) = +\infty$, then there exists a sequence $(y^\nu)_{\nu \in \mathbb{N}}$ of Y such that the sequence $(p \cdot y^\nu)$ converges to $+\infty$. Let y' be any element of Y'.

Then we have $\lim_\nu p \cdot (y^\nu + y') = +\infty$ and since $y^\nu + y' \in Y''$ one deduces that $\pi''(p) = +\infty$. A symmetric argument shows that the result is identical if $\pi'(p) = +\infty$ If $\pi(p)$ and $\pi'(p)$ are finite, for all $\varepsilon > 0$, it exists $(y, y') \in Y \times Y'$ such that $p \cdot y \geq \pi(p) - \varepsilon$ and $p \cdot y' \geq \pi'(p) - \varepsilon$

Hence $p \cdot (y + y') \geq \pi(p) + \pi'(p) - 2\varepsilon$. since $y + y' \in Y''$, one deduces that $\pi''(p) \geq \pi(p) + \pi'(p) - 2\varepsilon$. since the inequality holds true for every $\varepsilon > 0$ one can conclude that $\pi''(p) \geq \pi(p) + \pi'(p)$

Let $(y, y') \in s(p) \times s'(p)$. Hence $p \cdot y = \pi(p)$ and $p \cdot y' = \pi'(p)$. So $p \cdot (y + y') = \pi(p) + \pi'(p) = \pi''(p)$. since $y + y' \in Y''$, this implies that $y + y' \in s''(p)$. Conversely let $y'' \in s''(p)$ and let $(y, y') \in Y \times Y'$ such that $(y, y') = y + y'$.

Then, $p \cdot (y + y') = \pi''(p) = \pi(p) + \pi'(p)$. since $p \cdot y \leq \pi(p)$ and $p \cdot y' \leq \pi'(p)$, this implies that $p \cdot y = \pi(p)$ and $p \cdot y' = \pi'(p)$. Consequently $y \in s(p)$ and $y' \in s\,(p')$

b) $\Rightarrow$

$$\max_{x \in Y_2} px = \max_{x \in Y_1 \cup (Y_2/Y_1)} px = \max\{\max_{x \in Y_1} px, \max_{x \in (Y_2/Y_1)} px\} \geq \max_{x \in Y_1} px$$

Maximum of the function on bigger (in sense of inclusion) set is higher.

$\Leftarrow$ Suppose not. There exist $x \in Y_1$ and $x \notin Y_2$. Sets Y_2 and $\{x\}$ are convex, closed and disjoint so we can apply strict separating hyperplane theorem, i.e. there exists $q, b \in \mathbf{R}^L$:

$$q \cdot x > b \quad x \in Y_1 \quad \text{and} \quad b > q \cdot y \quad \forall y \in Y_2$$

this means that by taking max : $\pi_1(q) > \pi_2(q)$ which contradicts our notion. $\square$

Question 5 [Midterm 2006]

Consider the following supply function of a firm

$$s(p_1, p_2) = \Big(-\frac{2p_2}{p_1}, \frac{p_2}{p_1}\Big)$$

Show that this supply function can not result from profit maximization on any production set.

Solution 5

Consider following choice of prices which gives us formula for supply (2nd good is produced with 1st good as input)

$$p = (1, 1) \quad p' = (\frac{1}{3}, \frac{1}{6}) \qquad s(p) = (-2, 1) \quad s(p') = (-1, \frac{1}{2})$$

$$[s^*(p) - s^*(p')] \cdot [p - p'] = (-1, \frac{1}{2}) \cdot (\frac{2}{3}, \frac{5}{6}) = -\frac{2}{3} + \frac{5}{12} = -\frac{3}{12} < 0$$

$\square$