Recitation 2

[Definitions used today]

- (conditional) factor demand, cost function, Shephard's lemma, Hotelling's lemma
- Δ-monotone, homogeneous, positive definite matrix, correspondence, upper hemicontinuity (UHC)

Question 1 [Properties of π^* and s^*] 33 [I.1 Fall 2006 majors]

Suppose that production set Y is closed. Let $s^*(p)$ denote supply at price level p and by $\pi^*(p)$ corresponding profit level. Then the following properties hold:

1. π^* is homogeneous of deg. 1 in prices p
2. π^* is a convex function in prices p
3. **correspondence** s^* is homogeneous of deg. 0
4. s^* is Δ-monotone, that is:
$$[s^*(p) - s^*(p')] \cdot [p - p'] \geq 0 \quad \forall p, p'$$
5. **Hotelling's Lemma**: If π^* is differentiable at p (this holds iff s is single-valued at p), then
$$D\pi^*(p) = s^*(p)$$
6. Assuming that π^*,s^* are differentiable at $p \in \mathbb{R}^n$ prove comparative statics **law of supply**:
$$\frac{\partial s_i}{\partial p_i}(p) \geq 0$$
7. If Y is compact, then π^* is a continuous function and s^* is an upper hemicontinuous (UHC) correspondence.

Question 2 [Zero profit CRS]

If Y exhibits CRTS, then $\pi^*(p) = 0$ whenever it is well-defined.

Question 3 [Properties of C and x]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a production function that is strictly increasing (continuous) and satisfies $f(0) = 0$. Let $C^*(w, z)$ be the (minimum) cost function, where $w \in \mathbb{R}^n$ is a vector of input prices and $z > 0$ is an output level. Let $x^*(w, z)$ be the optimizer of cost minimization problem. Prove following properties:

1. $C^*(w, z)$ is homogeneous of degree 1 in factor prices w
2. $C^*(w, z)$is a concave function of w
3. $x^*(w, z)$ is homogeneous of degree zero in w.
4. $x^*(w, z)$ is Δ-monotone for fixed z, in following way:
$$[x^*(w, z) - x^*(w', z)] \cdot [w - w'] \leq 0 \quad \forall w, w' \gg 0$$
5. **Shephard's Lemma** If $C^*(w, z)$ is differentiable at w (this holds $\iff$ $x^*(w, z)$ is single-valued) then
$$D_w C^*(w, z) = x^*(w, z)$$
6. Assuming that C^*, x^* are differentiable at $w \in \mathbb{R}^n$ prove comparative statics property of factor demand:
$$\frac{\partial x_i}{\partial w_i}(w, z) \leq 0$$

7. Show that cost function C is a non-decreasing function of output level z, for every $w \gg 0$.

8. If production function f is concave, then cost function C is a convex function of output level z, for every $w \gg 0$

Question 4 [Aggregation]

Consider two closed production sets $Y_1, Y_2 \subseteq \mathbb{R}^L$ such that $0 \in Y_1$ and $0 \in Y_2$. Let π_1^* and π_2^* denote the profit functions associated with Y_1 and Y_2. Let π^* be the profit functions associated with Y.

1. Let $Y = Y_1 + Y_2$ be the (algebraic) sum of the two production sets. Prove that $\pi_1(p) + \pi_2(p) = \pi(p)$ for every $p \in \mathbb{R}^L$

2. Prove that $Y_1 \subseteq Y_2$ if and only if $\pi_1(p) \leq \pi_2(p)$

3. Let $Y = \text{co}\{Y_1, Y_2\}$ be the convex hull of the two production sets (that is, the set of all convex combinations of elements of Y_1 and Y_2). Prove that $\pi(p) = \max\{\pi_1(p), \pi_2(p)\}$for every for every $p \in \mathbb{R}^L$

Question 5 [Midterm 2006]

Consider the following supply function of a firm

$$s(p_1, p_2) = \left(-\frac{2p_2}{p_1}, \frac{p_2}{p_1} \right)$$

Show that this supply function can not result from profit maximization on any production set.