Recitations 20

[Definitions used today]

- SPE, Backward Induction, Behavioral Strategies, Linear Game, Perfect Recall, Dalkey and Kuhn Theorems

Question 1 [84 III.1 Spring 2009 majors]

An extensive form game (EFG) is said to be linear if every information set is crossed at most once by every history.

- Give an example of an EFG which is not linear.
- Give an example of EFG that is linear but not of perfect recall.
- Compare linear games and games with perfect recall. Is one of the two a subset of the other? Prove your answer.

Solution 1

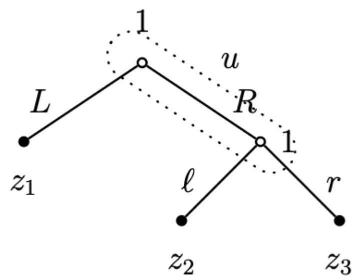

Figure 1: Non linear EFG

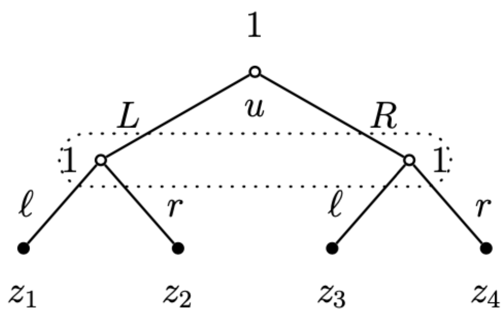

Figure 2: Linear game but not of perfect recall

Theorem 0.1. *Every game of perfect recall is linear.*

Proof. Suppose not, there is some perfect recall EFG that is not linear. Then we know there exists some $i \in I$ and $I_k^i \in \mathcal{I}^i$ such that for some $z \in Z, \# \left\{ P(z) \cap I_k^i \right\} > 1$.

Now take $x, y \in P(z) \cap I_k^i$ such that $x \succeq_c y$ for some action $c_k^i \in C_k^i$, i.e. x follows c_k^i but y does not. Now let $I_l^i = I_k^i$ and $y' = y$.

Then clearly there exists $c_k^i \in C_k^i$ such that $x \succeq_c y'$ but not $y \succeq_c y'$, since a node cannot come after itself. Therefore the game is not of perfect recall, which the hypothesis. $\square$

Question 2 [32 and 45 IV.2 Spring 2006 III.1 Spring 2007 majors]

Consider extensive form games that are finite (that is, that have a finite set of nodes).

a) Give an example to show that in an extensive form game a behavioral strategy may not havean equivalent mixed strategy

b) Define an extensive form linear game.

c) Prove that for any linear game, any player in the game, and any behavioral strategy of the player there is a mixed strategy of the same player that induces the same probability distribution on final nodes for any pure strategy of the other players.

d) Give an example to show that in a linear game for a mixed strategy of the player there may be no behavioral strategy that induces the same distribution on final nodes for some pure strategy of the other players.

Solution 2

a)

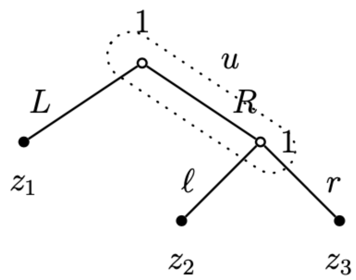

Figure 3: Non linear EFG

Player one forgot he has moved and forget what choice he made

$$C^1_{I_1} = \{L, R\}$$
$$S^1\{L, R\}$$
$$\Sigma^i = \Delta S^1 = (x, 1-x)$$
$$B^i = (p, 1-p)$$
$$Pr_{\sigma^1} = \{x, 0, 1-x\}$$
$$Pr_{\beta^1} = \{p, (1-p)p, (1-p)\cdot(1-p)\}$$

We can not find any mixed strategy to this as long as $p \in (0, 1)$

b) **Intuitively**, every player always knows if they've moved or not.

Definition 0.2 (Linear game)**.** *An EFG is linear if no information set intersects a path more than once, i.e.*

$$\forall i \in I, \forall I^i_k \in \mathcal{I}^i, \forall z \in Z, \#\left\{P(z) \cap I^i_k\right\} \leq 1$$

Informally: Effectively, the assumption is one that players never forget information once it is acquired]

Definition 0.3 (Games of perfect recall (PR))**.** *An EFG is perfect recall if and only if $\nexists I^i_k, I^i_l \in \mathcal{I}^i, x, y \in I^i_l$ such that x follows some $c^i_k \in C^i_k$ but y does not. [Alternatively, $\nexists I^i_k, I^i_l \in \mathcal{I}^i, x, y \in I^i_l, w \in I^i_k, c^i_k \in C^i_k$, such that $x \succeq_c w$ but not $y \succeq_c w$.*

In general, for linear games not of perfect recall, $\{\Pr_\sigma \mid \sigma \in \Sigma\} = \Delta(z)$ but $\{\Pr_\beta \mid \beta \in B\} \subset \Delta(z)$ and so $\forall \sigma^i \in \Sigma^i, \nexists \beta^i \in B^i$ such that $\beta^i \sim \sigma^i$.]

Definition 0.4 (Relevant information sets)**.** *The set of pure strategies for player i that lead to $I^i_k \in \mathcal{I}^i$ for some given strategy $s^{-i} \in S^{-i}$ of the other players, i.e. the set of pure strategies relevant for I^i_k, is:*

$$\text{Rel}\left(I^i_k\right) = \left\{s^i \in S^i \mid \exists s^{-i} \in S^{-i}, \Pr_{(s^i, s^{-i})}\left(\left\{z \in Z \mid \Pr(z) \cap I^i_k \neq \emptyset\right\}\right) > 0\right\}$$

Further, the set of pure strategies relevant for I_k^i that play action $c \in C_k^i$ is:

$$\operatorname{Rel}\left(I_k^i, c\right)=\left\{s^i \in \operatorname{Rel}\left(I_k^i\right) \mid s^i\left(I_k^i\right)=c\right\} \subseteq \operatorname{Rel}\left(I_k^i\right)$$

c)

Theorem 0.5 (Dalkey)**.** *In any linear EFG, for any behavioral strategy $\beta^i \in B^i$ there is a mixed strategy $\sigma^i \in \Sigma^i$ such that $\beta^i \sim \sigma^i$.*

Proof. Consider any linear EFG and note that $\forall i, \forall \beta^i \in B^i$ it is possible to construct $\sigma_{\beta^i}^i\left(s^i\right)=\prod_{k=1}^{K^i} \beta_k^i\left(s^i\left(I_k^i\right)\right)$. Further note that, clearly, $\sigma_{\beta^i}^i\left(s^i\right) \in[0,1] \forall s^i \in S^i$. Since the game is linear, we know each path intersects each information set only once, and thus $\sum_{s^i \in S^i} \sigma_{\beta^i}^i\left(s^i\right)=\sum_{s^i \in S^i} \prod_{k=1}^{K^i} \beta_k^i\left(s^i\left(I_k^i\right)\right)=1$, so the constructed $\sigma_{\beta^i}^i$ is a mixed strategy.

Now take any $z \in Z$ and any $\pi^{-i} \in \Gamma^{-i}$. Consider first the cases where z is always reached or z is never reached, regardless of player i 's actions. In these cases, $\operatorname{Pr}(z)=1$ and $\operatorname{Pr}(z)=0$, respectively, for any $\beta^i \in B^i$ and for any $\sigma^i \in \Sigma^i$, so $\beta^i \sim \sigma_{\beta^i}^i$ trivially. Consider now the case where $\operatorname{Pr}(z) \in(0,1)$ and depends on player i 's actions. Define $\tilde{c}_{I_k^i}^i(z)$ as the action of player i at information set I_k^i that leads to final node z and $\tilde{I}^i(z) \equiv \left\{I_k^i \in \mathcal{I}^i \mid P(z) \cap I_k^i \neq \emptyset\right\}$ as the set of player i 's information sets in the path of z. Then the probability on z induced by β^i is $\operatorname{Pr}_{\left(\beta^i, \pi^{-i}\right)}(z)=\prod_{I_k^i \in \tilde{I}^i(z)} \beta_k^i\left(\tilde{c}_{I_k^i}^i(z)\right)$. Now define $\tilde{S}^i(z)$ as the set of player i 's pure strategies that result in z, i.e. $\forall s^i \in \tilde{S}^i(z), \forall I_k^i \in \tilde{I}^i(z), c_k^i=\tilde{c}_k^i(z)$. Then the probability on z induced by $\sigma_{\beta^i}^i$ is:

$$\begin{aligned}
\operatorname{Pr}_{\left(\sigma^i, \pi^{-i}\right)}(z) & =\sum_{s^i \in \tilde{S}^i(z)} \sigma_{\beta^i}^i\left(s^i\right)=\sum_{s^i \in \tilde{S}^i(z) I_k^i \in \mathcal{I}^i} \beta_k^i\left(s^i\left(I_k^i\right)\right) \\
& =\sum_{s^i \in \tilde{S}^i(z)} \prod_{I_k^i \in \tilde{I}^i(z)} \beta_k^i\left(\tilde{c}_{I_k^i}^i(z)\right) \prod_{I_k^i \notin \tilde{I}^i(z)} \beta_k^i\left(s^i\left(I_k^i\right)\right) \\
& =\prod_{I_k^i \in \tilde{I}^i(z)} \beta_k^i\left(\tilde{c}_{I_k^i}^i(z)\right) \quad\left(\text{since } \forall s^i \in \tilde{S}^i(z), \forall I_k^i \in \tilde{I}^i(z), s_k^i=\tilde{c}_k^i(z)\right)=\operatorname{Pr}_{\left(\beta^i, \pi^{-i}\right)}(z)
\end{aligned}$$

Thus σ^i and β^i induce the same probability on z, and this is true $\forall z \in Z$. Thus $\beta^i \sim \sigma_{\beta^i}^i$ □

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d)

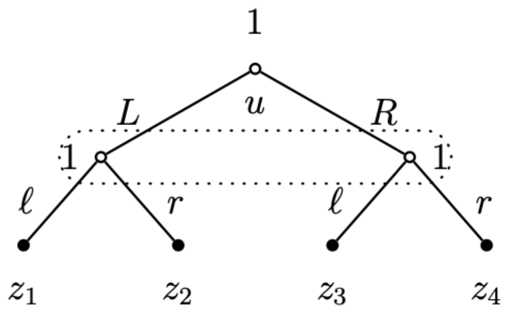


Figure 4: Linear game but not of perfect recall

$$S^1\{L, R\} \times \{l, r\}$$
$$\Sigma^i = \Delta S^1 = (p_1, p_2, p_3, 1 - p_1 - p_2 - p_3)$$
$$B^i = (p, 1-p) \times (q, 1-q) = \{pq, p(1-q), (1-p)(1-q), (1-p)q\}$$

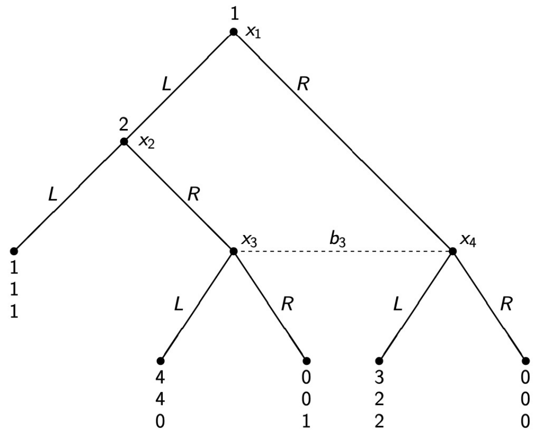
Consider

$$Pr_{\sigma^1} = \{\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}\}$$
$$\nexists Pr_{\beta^1} = \{p, (1-p)p, (1-p) \cdot (1-p)\}$$

We can not find any behavioral strategy to this mixed strategy.

Question 3

Find all SPE and NE of following games



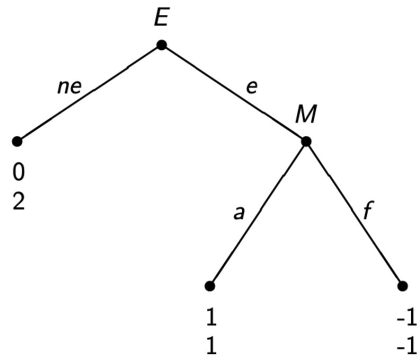


Figure 5

Solution 3

b)

E/M	a	f
ne	0,2	0,2
e	1,1	-1,-1

Two Nash pure equilibria: (ne, f), (e, a). And we have mixed too

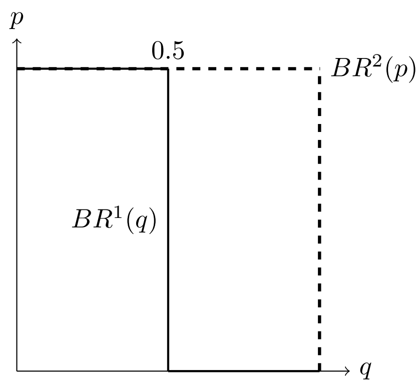


Figure 6: Best Responses

yield the set of Nash equilibria

$$\text{NE} = \left\{ \big((0,1),(1,0)\big), \big((1,0),(q,1-q)\big), \forall 0 \geq q \geq \frac{1}{2} \right\}.$$

Let's focus on two NE.

- Consider the Nash equilibrium (ne, f).

Entrant plays not to enter (ne) and the telephone rings: Monopolist, it is your turn!!!
M's information set is out of equilibrium path. To play f was part of the optimal behavior because this information set was not reached. Strategy gives plan for moves in all information sets, even though some of them won't be reached. The threat of playing f is what makes optimal for the entrant to play ne.

OK but, f is a non-credible threat, so E should not believe that M will play f (it is not rational for him) if he plays e. **Subgame Perfect Equilibrium** requires rational behavior even in information sets that are not reached in equilibrium (equilibrium should not be based on incredible threats).

- Obtain (e,a) as the unique Subgame Perfect Equilibrium (which coincides with the one obtained by backwards induction).

(e,a) could be obtained also in the normal form as applying the principal ǹ ever a dominated strategy since f is dominated by a.
But this is not always true.

(e,a) is the unique SPE of Γ and (ne,f) is not a SPE since f is not a Nash equilibrium of the subgame starting at the unique node that belongs to M.

In general, SPE requires two different things:

- SPE gives a solution everywhere (in all subgames), even in subgames where the solution says that they will not be reached (information sets with zero probability).
- SPE imposes rational behavior everywhere, even in the subgames of the game that SPE says that cannot be reached. In out-of-equilibrium subgames, the solution is disapproved, yet players evaluate their actions taking as given the behavior of the other players, that have been demonstrated incorrect since we are in an out-of-equilibrium path.

a) skipped for next class

Question 4 [Final 2019]

- Prove that for any finite EFG of perfect information, there is a last move node, that is a move node x such that $IS(x) \subseteq Z$.
- Prove, or disprove by showing a counter-example to the statement: In any finite EFG of perfect recall, there is a last information set I^i for some player i, that is, an information set such that for any node $x \in I^i, IS(x) \subseteq Z$.

Solution 4

1) Suppose not.
Then $\exists$ node $y_1 \in IS(x), y_1 \notin Z$ and
Then $\exists$ node $y_2 \in IS(y_1), y_1 \notin Z$
...
$\forall n \quad \exists$ node $y_n \in IS(y_1), y_{n-1} \notin Z$ violates EFG beign finite.

2) Suppose not . Thn every info set I^i_F for each player i has a moving node

- all these info sets belong to different player $\rightarrow$ infinite number of players violates finite EFG
- $\exists$ some player i his info set repeated shows up on some path, rename the info set as $I^i_{k_1}, \ldots I^i_{k_n}, \ldots$ by order of the path : $\exists y_1 \in I^i_{k_1}, c_1 \in C_{I^i_{k_1}}$, $y_2 \in I^i_{k_2}, c_2 \in C_{I^i_{k_2}}$ and $y_2 \succeq y_1$. By perfect recall (eliminate the possibility of having redundant structure): $\forall x \in I^i_{k_2}\ x \succeq y_1$ i.e. all the nodes of $I^i_{k_2}$ should come after info set $I^i_{k_1}$.
Some $I^i_{k_3}, \ldots I^i_{k_n}, \ldots$ means that i has infinite number of info sets which violates finite EFG.