Math Appendix for Microeconomics

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October 2020

1 Notation

1.1 Logical

Meaning	*Command*	*Notation*
Not	`\neg`	$\neg$
There exists	`\exists`	$\exists$
For all	`\forall`	$\forall$
Implies	`\implies`	$\Rightarrow$
Equivalent	`\iff`	$\iff$
And	`\land`	$\land$
Or	`\lor`	$\lor$
Defined as	`:=`	$:=$
Logical equivalence	`\equiv`	$\equiv$
Therefore	`\therefore`	$\therefore$
Because	`\because`	$\because$

1.2 Greek letters

Command	*Notation*	*Command*	*Notation*
`\alpha`	α	`\tau`	τ
`\beta`	β	`\theta`	θ
`\chi`	χ	`\upsilon`	υ
`\delta`	δ	`\xi`	ξ
`\epsilon`	ϵ	`\zeta`	ζ
`\varepsilon`	ε	`\Delta`	Δ
`\eta`	η	`\Gamma`	Γ
`\gamma`	γ	`\Lambda`	Λ
`\iota`	ι	`\Omega`	Ω
`\kappa`	κ	`\Phi`	Φ
`\lambda`	λ	`\Pi`	Π
`\mu`	μ	`\Psi`	Ψ
`\nu`	ν	`\Sigma`	Σ
`\omega`	ω	`\Theta`	Θ
`\phi`	ϕ	`\Upsilon`	Υ
`\varphi`	φ	`\Xi`	Ξ
`\pi`	π	`\aleph`	$\aleph$
`\psi`	ψ	`\beth`	$\beth$
`\rho`	ρ	`\daleth`	$\daleth$
`\sigma`	σ	`\gimel`	$\gimel$

1.3 General

- $\mathbb{R}^n := \big\{x = (x_1, \ldots, x_i \ldots, x_n) : x^i \in \mathbb{R}, \quad \forall i = 1, \ldots, n\big\}$

*These notes are intended to summarize the main concepts, definitions and results covered in the first year of micro sequence for the Economics PhD of the University of Minnesota. The material is not my own. Please let me know of any errors that persist in the document. E-mail: pawel042@umn.edu .

For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ we denote
$$x \geq y \iff x_i \geq y_i, \quad \forall i = 1, \ldots, n$$
$$x > y \iff x \geq y \quad and \quad x \neq y$$
$$x \gg y \iff x_i > y_i, \forall i = 1, \ldots, n$$

- $x \cdot y$ or $< x, y >$ denotes the scalar product of x and y so $x \cdot y = \sum_{i=1}^{n} x_i y_i$
- A is a matrix with m rows and n columns and B is a matrix with n rows and l columns, AB denotes the matrix product of A and B.
- H is a $n \times n$ matrix, tr (H) denotes the **trace** of H and $det(H)$ denotes the **determinant** of H.
- $x \in \mathbb{R}^n$ is treated as a row matrix so $1 \times n$.
- x^T denotes the transpose of $x \in \mathbb{R}^n, x^T$ is treated as a column matrix so $n \times 1$.
- $f : X \to \mathbb{R}$ so f is a function from open set $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$

2 Binary relations

Definition 1 *Assumptioons on binary relations ($R :\succeq, P :\succ, I :\sim$)*

a ***reflexive*** *:* $\forall_a \quad aRa$

b ***irreflexive****:* $\forall_a \quad \neg(aRa)$

c ***symmetric****:* $\forall_{a,b} \quad aRb \iff bRa$

d ***asymmetric****:* $\forall_{a,b} \quad aRb \iff \neg(bRa)$

e ***antisymmetric****:* $\forall_{a,b} \quad aRb \wedge bRa \Rightarrow a = b$

f ***complete****:* $\forall_{a,b} \quad aRb \vee bRa$

g ***transitive*** $\forall_{a,b,c} \quad aRb \wedge bRc \Rightarrow aRc$

h ***negative transitive*** $\forall_{a,b,c} \quad \neg(aRb) \wedge \neg(bRc) \Rightarrow \neg(aRc)$

Definition 2 *Main cathegories of binary relations*

a ***(Weak) Preorder aka Preference Relation****- Reflexive, Transitive*

b ***Equivalence Relation****- Reflexive, Symmetric, Transitive*

c ***Strict partial order*** *-Asymmetric, Transitive*

d ***Partial Order****- Reflexive, Antisymmetric, Transitive*

e ***Total (or Linear) Order****- Antisymmetric, Complete, Transitive*

2.1 Monotonicity and Nonsatiation

Assume that relation $R :=\succeq$ is a preorder.

Definition 3 $\succeq$ *is* ***weakly monotone*** *on a set X if $\forall x, y \in X$,*

$$x \geq y \Rightarrow x \succeq y$$

Definition 4 $\succeq$ *is* ***monotone*** *on a set X if $\forall x, y \in X$,*

$$x \gg y \Rightarrow x \succ y$$

.

Definition 5 $\succeq$ *is* ***strongly monotone*** *on a set* X *if* $\forall x, y \in X$,

$$(x \geq y \wedge x \neq y) \Rightarrow x \succ y$$

.

Definition 6 $\succeq$ *is* ***locally nonsatiated*** *on a set* X *if*

$$\forall x \in X \text{ and } \forall \epsilon > 0 \quad \Rightarrow \quad \exists y \in X \ni \| x - y \| < \epsilon \text{ and } y \succ x$$

2.2 Convexity

Definition 7 $\succeq$ *is* ***weakly convex*** *on a set* X *if* $\forall x, y \in X, \forall \lambda \in (0,1)$,

$$x \succeq y \Rightarrow \lambda x + (1-\lambda) y \succeq y$$

Definition 8 $\succeq$ *is* ***convex*** *on a set* X *if* $\forall x, y \in X, \forall \lambda \in (0,1)$,

$$x \succ y \Rightarrow \lambda x + (1-\lambda) y \succ y$$

Definition 9 $\succeq$ *is* ***strongly/strictly convex*** *on a set* X *if* $\forall x, y \in X, \forall \lambda \in (0,1)$,

$$x \sim y \wedge x \neq y \Rightarrow \lambda x + (1-\lambda) y \succ y$$

2.3 Continuity

Definition 10 ***(Sequential definition/ weak continuity)*** *A preorder* $\succeq$ *is continuous on a set* X *if* $\forall \{x_n\}, \{y_n\} \subseteq X$,

$$\forall n \in \mathbb{N}, (x_n \succeq y_n) \wedge (x_n \to x) \wedge (y_n \to y) \Rightarrow x \succeq y$$

Definition 11 ***(Set definition/ strong continuity)*** *A preorder* $\succeq$ *is continuous on a set* X *if* $\forall x \in X$, *the upper contour set* $U(x) = \{y \in X : y \succeq x\}$ *and the lower contour set* $L(x) = \{y \in X : x \succeq y\}$ *are closed in* X.

3 Real analysis

3.1 Topology on $\mathbb{R}^n$

Definition 12 ***(Norm)*** $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ *such that*

1. $\|x\| \geq 0 \wedge \|x\| = 0$ *iff* $x = 0$
2. $\forall_{\alpha \in \mathbb{R}} \forall_{x \in \mathbb{R}^n} \quad \|\alpha x\| = |\alpha| \|x\|$
3. $\triangle$ *-inequality* $\quad \|x + y\| \leq \|x\| + \|y\|$

Examples

- Euclidean norm: $\|x\| = \langle x, x \rangle^{1/2} = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$
- 1-norm: $\|x\|_1 = \sum_{i=1}^n |x_i|$
- L^1 norm: $\|f\|_{L^1} = \int |f(x)| dx$
- sup norm: $\|x\|_\infty = \sup_{i \in \{1, \ldots, n\}} |x_i|$

Theorem 1 ***(Cauchy-Schwarz inequality)***

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Definition 13 ***(Open ball)***. $B_\epsilon(a) = \{x \in \mathbb{R}^n : \|x - a\| < \epsilon\}$.

Definition 14 ***(Convergence)***. $x_n \to x \Longleftrightarrow \forall_{\epsilon > 0} \exists_N \forall_{k \geq N} \quad x_k \in B_\epsilon(a)$

Definition 15 ***(Cauchy sequence)***. $\forall_{\epsilon>0}\exists_N\forall_{m,n>N}\,|x_n - x_m| < \epsilon$

Definition 16 ***(Open set)***. *$V \subseteq \mathbb{R}^n$ is open iff $\forall_{x\in V}\exists_{\epsilon>0}B_\epsilon(x) \subseteq V$*

Definition 17 ***(Closed set)***. *$E \subseteq \mathbb{R}^n$ is closed if $E^c = \mathbb{R}^n \backslash E$ is open.*

Definition 18 ***(Topological space (X, τ))***.
1. $\emptyset, X \in \tau$
2. if $\{v_\alpha\} \in \tau \Rightarrow \bigcup_{\alpha\in I} v_\alpha \in \tau$ any collections
3. if $\{v_\alpha\} \in \tau \Rightarrow \bigcap_{\alpha\in I} v_\alpha \in \tau$ finite

Definition 19 ***(Connected set X)***. $\#_{U,V\in\tau}U \cup V = X \wedge U \cap V = \emptyset$

Definition 20 ***(Interior of a set)***. $E^\circ = \bigcup\{V : V \subseteq E \wedge V \in \tau\,(\mathbb{R}^n)\}$

Definition 21 ***(Closure of a set)*** . *$\bar{E} = cl(E) = \cap\{V : E \subseteq V \wedge V \in \mathcal{F}\,(\mathbb{R}^n)\}$, where $\mathcal{F}\,(\mathbb{R}^n)$ is family of closed sets.*

Lemma 1 *(i) $E^\circ \subseteq E \subseteq \bar{E}$*
(ii) $E^\circ = E$ iff E is open
(iii) $\bar{E} = E$ iff E is closed

Definition 22 ***(Compactness)***. *$E \subseteq \bigcup_{\alpha\in I} v_\alpha v_\alpha \in \tau.E$ is compact if for every open covering of E it has always finite subcover.*

Definition 23 ***(Heine-Borel)***. *$E \subseteq \mathbb{R}^n$ is compact $\iff$ E closed and bounded.*

Lemma 2 *($\exists$ of convergent subsequence). If $E \subseteq \mathbb{R}^n$ is compact, $\{x_n\} \subseteq E \Rightarrow \exists_{x_{n_k}} x_{n_k} \to x$*

Definition 24 ***(Continuity in topological space)***. *$f \in \mathcal{C}^0$ on $X \iff \forall_{V\in\tau(X)}f^{-1}(V) \in \tau \iff \forall_{U\in\mathcal{F}(x)}f^{-1}(U) \in \mathcal{F}$*

Definition 25 ***(Continuity at a point)***. *$f : \Phi \to X$ is continuous at $\Theta \iff \forall_{openV\subseteq X:f(\Theta)\in V}\exists_{openU\subseteq\Phi}\Theta \in U$*

3.2 Continuity and Convergence

Definition 26 ***(Continuous function)*** *f is continuous at $y \in X$ if*

$$\lim_{x\to y} f(x) = f(y)$$

f is continuous on X if f is continuous at every point $y \in X$

Definition 27 ***(Supremum)*** $\sup E \iff \forall_{\epsilon>0}\exists_{a\in E}\sup E - \epsilon < a \leq \sup E$

Definition 28 ***(Convergence)*** $x_n \to x \iff \forall_\mu\exists_N\forall_{n\geq N}\,|x_n - x| < \epsilon$ *divergent sequence:* $(x_n \to \pm\infty) \iff \forall_\mu\exists_N\forall_{n\geq N}x_n > \mu\,(x_n < \mu)$

Lemma 3 *Every convergent sequence is bounded.*

Theorem 2 ***(Bolzano-Weierstrass)***. *Every bounded sequence has convergent subsequence*

Definition 29 ***(Cauchy sequence)***. *$\{x_n\}$ is Cauchy $\iff$*

$$\forall_{\epsilon>0}\exists_N\forall_{n,m\geq N} \mid x_n - x_m \mid < \epsilon$$

Lemma 4 *If $x_n \to x \Rightarrow \{x_n\}$ is Cauchy*

Definition 30 ***(limsup, liminf)***.

$$\limsup x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$$
$$liminf x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$$

Theorem 3 ***(Monotone convergence).***
If x_n is monotone and bounded $\Rightarrow x_n \to x < +\infty$

Lemma 5 $x_n \to x \Longleftrightarrow \limsup x_n = liminf x_n$

Definition 31 ***(Limit of function).***

$$\lim_{x \to a} f(x) = y \Longleftrightarrow \forall_{\epsilon>0} \exists_{\rho>0} |x-a| < \rho \Rightarrow |f(x) - f(y)| < \epsilon$$

Definition 32 ***(Right/left limits).***
Right

$$y = \lim_{x \to a^+} f(x) \Longleftrightarrow \forall_{\epsilon>0} \exists_{\rho>0} a < x < a + \rho \Rightarrow |f(x) - y| < \epsilon$$

Left

$$z = \lim_{x \to a^-} f(x) \Longleftrightarrow \forall_{\epsilon>0} \exists_{\rho>0} a - \rho < x < a \Rightarrow |f(x) - z| < \epsilon$$

Definition 33 ***(Continuity).*** *$f : E \to \mathbb{R}$ is continuous at $a \in E \Longleftrightarrow \forall_{\epsilon>0} \exists_{\rho>0} |x - a| < \rho \Rightarrow |f(x) - f(a)| < \epsilon$*

$$\left(x_n \to x \Rightarrow f\left(x_n\right) \to f(x)\right)$$

Lemma 6 *Equivalently*

- *f is continuous at $y \in X$ if and only if for every open ball J of center $f(y)$ there exists an open ball B of center y such that $f(B \cap X) \subseteq J$*
- *f is continuous at $y \in X$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - y\| < \delta$ and $x \in X \Longrightarrow |f(x) - f(y)| < \varepsilon$*

Lemma 7 ***(Sequentially continuous function)*** *f is continuous at $y \in X$ if and only if f is sequentially continuous at y, that is, for every sequence $\left(x_n\right)_{n \in \mathbb{N}} \subseteq X$ such that $x_n \to y$, we have that*

$$f\left(x_n\right) \to f(y)$$

Theorem 4 ***(Extreme value theorem - Weierstrass).*** *If $H \subseteq \mathbb{R}^n$ compact, $f : H \to, f \in \mathcal{C}^0$ then*

$$\exists y, \underline{x} \quad f(y) = \sup_{x \in H} f(x) \quad \wedge \quad f(\underline{x}) = \inf_{x \in H} f(x)$$

Theorem 5 ***(Intermediate value theorem).*** *$f : I \to \mathbb{R}, a, b \in I, a < b, y_0 \in (f(a), f(b)) \Rightarrow \exists_{x_0 \in (a,b)} f\left(x_0\right) = y_0$*

Definition 34 ***(Uniform continuity).*** *$\forall x_{1,x_2} \forall_{\epsilon>0} \exists_{\rho>0} \left|x_1 - x_2\right| < \rho \Rightarrow \| f\left(x_1\right) - f\left(x_2\right) | < \epsilon$ then $f\left(x_n\right)$ is Cauchy*

Definition 35 ***(Pointwise convergence).***

$$\forall_x f(x) = \lim_{n \to \infty} f_n(x) \Longleftrightarrow \forall_x \forall_{\epsilon>0} \exists_N \forall_{n>N} \left|f_n(x) - f(x)\right| < \epsilon$$

Definition 36 ***(Uniform convergence $\rightrightarrows$).***

$$\forall_{\epsilon>0} \exists_N \forall_{n>N} \forall_{x \in E} \left|f_n(x) - f(x)\right| < \epsilon \Longleftrightarrow \forall_{\epsilon>0} \exists_N \forall_{m,n>N} \forall_{x \in E} \left|f_n(x) - f_m(x)\right| < \epsilon$$

Theorem 6 ***Lebesgue Dominated convergence theorem***

$$\forall_n \left|f_n\right| < M, f_n \rightrightarrows f \Rightarrow f < M$$

Definition 37 *f is **weakly increasing (or non-decreasing)** on X if for all x and y in X*

$$x \leq y \Longrightarrow f(x) \leq f(y)$$

Definition 38 *f is **increasing** on X if for all x and y in X*

$$x \ll y \Longrightarrow f(x) < f(y)$$

Definition 39 *f is **strictly increasing** on X if for all x and y in X,*

$$x < y \Longrightarrow f(x) < f(y)$$

3.3 Differentiability

Definition 40 *$X \subseteq \mathbb{R}^n$ is an open set, f is a function from X to $\mathbb{R}$ and $x \in X$*

$$\nabla f(x) := \left(\frac{\partial f}{\partial x^1}(x), \ldots, \frac{\partial f}{\partial x^h}(x), \ldots, \frac{\partial f}{\partial x^n}(x) \right)$$

*denotes the **gradient** of f at x,*

Definition 41 *$D^2 f(x) = H(x)$ denotes the **Hessian matrix** of f at x*

Definition 42 *$X \subseteq \mathbb{R}^n$ is an open set, $g := (g_1, \ldots, g_j, \ldots, g_m)$ is a mapping from X to $\mathbb{R}^m$ and $x \in X$*

$$\mathrm{J}g(x) := \begin{bmatrix} \frac{\partial g_1}{\partial x^1}(x) & \cdots & \frac{\partial g_1}{\partial x^h}(x) & \cdots & \frac{\partial g_1}{\partial x^n}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_j}{\partial x^1}(x) & \cdots & \frac{\partial g_j}{\partial x^h}(x) & \cdots & \frac{\partial g_j}{\partial x^n}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_m}{\partial x^1}(x) & \cdots & \frac{\partial g_m}{\partial x^h}(x) & \cdots & \frac{\partial g_m}{\partial x^n}(x) \end{bmatrix}_{m \times n} = \begin{bmatrix} \nabla g_1(x) \\ \vdots \\ \nabla g_j(x) \\ \vdots \\ \nabla g_m(x) \end{bmatrix}_{m \times n}$$

*denotes the **Jacobian matrix** of g at x.*

Definition 43 ***(Frechet Differentiable function)*** *f is differentiable at $y \in X$ if*
1. all the partial derivatives of f at y exist,
2. there exists a function E_y defined in some open ball $B(0, \varepsilon) \subseteq \mathbb{R}^n$ such that for every $u \in B(0, \varepsilon)$

$$f(y + u) = f(y) + \nabla f(y) \cdot u + \|u\| E_y(u) \quad \textit{where} \quad \lim_{u \to 0} E_y(u) = 0$$

f is differentiable on X if f is differentiable at every point $y \in X$.

Lemma 8 *If f is differentiable at y, then f is continuous at y*

Definition 44 ***(Gateaux differentiable- Directional derivative)*** *Let $v \in \mathbb{R}^n, v \neq 0$. The directional derivative $D_v f(y)$ of f at $y \in X$ in the direction v is defined as*

$$\lim_{t \to 0^+} \frac{f(y + tv) - f(y)}{t}$$

if this limit exists and it is finite.

Lemma 9 *(Differentiable function/Directional derivative) If f is differentiable at $y \in X$, then for every $v \in \mathbb{R}^n$ with $v \neq 0$*

$$D_v f(y) = \nabla f(y) \cdot v$$

3.4 Compactness

Definition 45 ***(Compact set/Subsequences)*** *X is compact if and only if for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of the sequence $(x_n)_{n \in \mathbb{N}}$ such that $(x_{n_k})_{k \in \mathbb{N}}$ converges to some point $y \in X$.*[1]

Lemma 10 ***(Compact set)*** *$X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded*

Definition 46 ***(Closed set)*** *X is closed if its complement $\mathcal{C}(X) := \mathbb{R}^n \backslash X$ is open*

Lemma 11 ***(Sequentially closed)*** *X is closed if and only if it is sequentially closed, that is, for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ such that $x_n \to y$, we havey $\in X$*

Definition 47 ***(Bounded set)*** *X is bounded if it is included in some ball, that is, there exists $\varepsilon > 0$ such that for all $x \in X, \|x\| < \varepsilon$.*

3.5 Concavity and quasi-concavity

In this section, we assume that C is a convex subset of $\mathbb{R}^n$ and f is a function from C to $\mathbb{R}$.

Definition 48 ***(Convex set).*** *A set C is convex* $\iff \forall_{x,y\in \mathrm{C}}\forall_{\lambda\in[0,1]}\lambda x + (1-\ \lambda)y \in \mathrm{C}$

Definition 49 ***(Convex combination).*** *Let* $\{x_i\}_{i=1}^m \subseteq \mathbb{R}^n, \{\lambda_i\}_{i=1}^m \subseteq \mathbb{R}_+, \sum \lambda_i = 1$. *The vector* $\sum \lambda_i x_i = 1$ *is called a convex combination of* $\{x_i\}$ *.*

Lemma 12 C *is convex* $\iff$ C *contains all convex combinations of its elements.*

Definition 50 ***(Hyperplane).*** $H \subseteq \mathbb{R}^n$ *is hyperplane* $\iff \exists_{\beta\in\mathbb{R},b\in\mathbb{R}^n} H = \{x \in \mathbb{R}^n : x \cdot b = \beta\}$

Lemma 13 *(hyperplane generates two halfspaces)* $\{x \in \mathbb{R}^n : x \cdot b \leq \beta\}$ *and* $\{x \in \mathbb{R}^n : x \cdot b \geq \beta\}$

Definition 51 ***(Convex hull) .*** *Let* $Co = \cap\{C : E \subseteq C, C$ *convex* $\}$.
Note $Co = \left\{x \in \mathbb{R}^n : \exists_{\{x_i\}\subseteq E}\exists_{\{\lambda_i\}\subseteq\mathbb{R}:\sum\lambda_i=1} x = \sum \lambda_i x_i\right\}$

Definition 52 ***(Simplex).*** *A set* $S \subseteq \mathbb{R}^n$ *is* m *-dimensional simplex* $\iff S = \{(b_0, \dots, b_m) \in \mathbb{R}^m : b_i$ *affinely independent* $\}$

Theorem 7 *If* $\forall_i C_i$ *convex following sets are convex*

- $\mathrm{C} = \bigcap C_i$
- $C_1 + a$
- $C_1 + C_2$ *is convex*
- $\mathrm{C} = \{x \in \mathbb{R}^n : x \cdot b \leq \beta\}$
- *if f is quasi concave function, then* $\mathrm{C} = \{x \in \mathbb{R}^n : f(x) \leq \beta\}$ *is convex*

Theorem 8 ***(Separating hyperplane theorem).*** *Let* $C_1 \subseteq \mathbb{R}^n, C_2 \subseteq \mathbb{R}^n$. $H = \{x \in \mathbb{R}^n : x \cdot b = \beta\}$ *is separating hyperplane of* $C_1\&C_2 \iff$

$$\forall_{x\in C_1} x \cdot b \leq \beta \qquad \forall_{y\in C_2} y \cdot b \geq \beta$$

Separation is strong if at least one is $<$, $>$)
" $\iff$ " part of theorem is true when

- $ri\,(C_1) \cap ri\,(C_2) = \emptyset$
- $ri(A) = \{x \in A : B(x, \epsilon) \cap aff(A) \subseteq A\}$
- $aff(A) = \{\sum \alpha_i x_i : x_i \in A, \sum \alpha_i = 1\}$

Conditions for "$\Rightarrow$ " separating hyperplane theorem: for all C_1, C_2 non empty, convex, $x \in \mathbb{R}^n$

- $x \notin C_1 \Rightarrow H(b, \beta)$ separates strongly $x\&C_1$
- $C_1 \cap C_2 = \emptyset \Rightarrow H(b, \beta)$ separates $C_1\&C_2$
- C_1 open $\Rightarrow H(b, \beta)$ separates strongly $C_1\&C_2$
- C_1, C_2 closed, C_1 compact$\Rightarrow H(b, \beta)$ separates strongly $C_1\&C_2$

Definition 53 ***(Support).*** *The support function* $S(\cdot \mid C)$ *of convex set* $C \subseteq \mathbb{R}^n$ *is defined as:*

$$S(x, y) = \sup_{y\in \mathrm{C}} x \cdot y$$

Definition 54 ***(Concave function)*** *f is concave if for all $t \in [0, 1]$ and for all x and y in C,*

$$f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y)$$

Lemma 14 *f is concave if and only if the set*

$$\{(x, \alpha) \in C \times \mathbb{R} : f(x) \geq \alpha\}$$

*is a convex subset of $\mathbb{R}^{n+1}$. The set above is called **hypograph** of f*

Lemma 15 ***(Jensen's Inequality)*** *f is concave if and only if $f(\lambda_1 x_1 + \ldots + \lambda_k x_k) \geq \lambda_1 f(x_1) + \ldots + \lambda_k f(x_k)$ for $x_1, \ldots, x_k \in \Gamma$ and $\lambda_i \geq 0$ and $\sum \lambda_1 = 1$*

Lemma 16 *C is open and f is differentiable on $C.f$ is concave if and only if for all x and y in C,*

$$f(x) \leq f(y) + \nabla f(y) \cdot (x - y)$$

Lemma 17 C *is open and f is twice continuously differentiableon C . f is concave if and only if for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative semidefinite, that is, for all $x \in C$*

$$v\mathrm{H}f(x)v^T \leq 0, \forall v \in \mathbb{R}^n$$

Definition 55 ***(Strictly concave function)*** *f is strictly concave if for all $t \in]0, 1[$ and for all x and y in C with $x \neq y$*

$$f(tx + (1 - t)y) > tf(x) + (1 - t)f(y)$$

Lemma 18 *C is open and f is differentiable on $C.f$ is strictly concave if and only if for all x and y in C with $x \neq y$,*

$$f(x) < f(y) + \nabla f(y) \cdot (x - y)$$

Lemma 19 *C is open and f is twice continuously differentiable on C. If for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative definite, that is, for all $x \in C$*

$$v\mathrm{H}f(x)v^T < 0, \forall v \in \mathbb{R}^n, v \neq 0$$

then f is strictly concave

Lemma 20 ***Monotone transformation*** *If f quasi convex, g monotone, nondecreasing, then $g \circ f$ is quasi-convex.*

Definition 56 ***(Quasi-concave function)*** *f is quasi-concave if and only if for all $\alpha \in \mathbb{R}$ the set*

$$\{x \in C : f(x) \geq \alpha\}$$

is a convex subset of $\mathbb{R}^n$. The set above is called upper contour set of f at α.

Lemma 21 *f is quasi-concave if and only if for all $t \in [0, 1]$ and for all x and y in C,*

$$f(tx + (1 - t)y) \geq \min\{f(x), f(y)\}$$

Lemma 22 *C is open and f is differentiable on $C.f$ is quasiconcave if and only if for all x and y in C,*

$$f(x) \geq f(y) \Longrightarrow \nabla f(y) \cdot (x - y) \geq 0$$

Lemma 23 *C is open and f is differentiable on C. If f is quasiconcave and $\nabla f(x) \neq 0$ for all $x \in C$, then for all x and y in C with $x \neq y$,*

$$f(x) > f(y) \Longrightarrow \nabla f(y) \cdot (x - y) > 0$$

Definition 57 ***(Kernel)*** *$Ker_{g(x)} := \{v \in \mathbb{R}^n, v \neq 0 \text{ and } \nabla g(x) \cdot v = 0\}$*

Lemma 24 *C is open and f is twice continuously differentiable on C. If f is quasi-concave, then for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative semidefinite on $Ker\nabla f(x)$, that is, for all $x \in C$*

$$v \in \mathbb{R}^n \quad \text{and} \quad \nabla f(x) \cdot v = 0 \Longrightarrow v\mathrm{H}f(x)v^T \leq 0$$

Definition 58 ***(Strictly quasi-concave function)*** *f is strictly quasi-concave if and only if for all $t \in]0,1[$ and for all x and y in C with $x \neq y$,*

$$f(tx + (1-t)y) > \min\{f(x), f(y)\}$$

Is concave function differentiable? almost everywhere. Moreover derivative is continuous a.s.

Lemma 25 *If f concave, $|f(x)| \leq M$ on open neighborhood of convex X, then f continuous.*

Lemma 26 *C is open and f is differentiable on C.*

1. *If for all x and y in C with $x \neq y$,*

$$f(x) \geq f(y) \Longrightarrow \nabla f(y) \cdot (x-y) > 0$$

then f is strictly quasi-concave

2. *If f is strictly quasi-concave and $\nabla f(x) \neq 0$ for all $x \in C$, then for all $x, y \in C$, $x \neq y$*

$$f(x) \geq f(y) \Longrightarrow \nabla f(y) \cdot (x-y) > 0$$

Lemma 27 *C is open and f is twice continuously differentiable on C. If for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative definite on $Ker\nabla f(x)$, that is, for all $x \in C$*

$$v \in Ker\nabla f(x), \quad \Longrightarrow v\mathrm{H}f(x)v^T < 0$$

then f is strictly quasi-concave

Lemma 28 *We remark that*

$$\begin{array}{ccccc} f \text{ linear or affine} & \Rightarrow & f \text{ concave} & \Leftarrow & f \text{ strictly concave} \\ & & \downarrow & & \\ & & f \text{ quasi-concave} & \Leftarrow & f \text{ strictly quasi-concave} \end{array}$$

We remind the definitions and some properties of negative definite/semidefinite matrices. Let H be a $n \times n$ symmetric matrix.

Definition 59 ***(nsd, nd matrix)***

- H *is* ***negative semidefinite (nsd)*** *if $v\mathrm{H}v^T \leq 0$ for all $v \in \mathbb{R}^n$*
- *H is* ***negative definite (nd)*** *if $v\mathrm{H}v^T < 0$ for all $v \in \mathbb{R}^n$ with $v \neq 0$*

Theorem 9 ***(Eigen values and definitness)***

1. *H has n real eigenvalues. We denote $\lambda_1, \ldots, \lambda_n$ the eigenvalues of H.*
2. *H is negative semidefinite if and only $\lambda_i \leq 0$ for every $i = 1, ..., n$*
3. *H is negative definite if and only $\lambda_i < 0$ for every $i = 1, ..., n$*

Theorem 10 ***($n = 2$ and definitness of matrix)***

1. *If H is negative semidefinite, then tr$(H) \leq 0$ and det$(H) \geq 0$ if n is even, det$(H) \leq 0$ if n is odd*
2. *If H is negative definite, then tr$(H) < 0$ and det$(H) > 0$ if n is even, det$(H) < 0$ if n is odd*

We remark that if n = 2, then the conditions stated in the proposition above also are sufficient conditions, that is

1. *H is negative semidefinite if and only if tr(H) $\leq$ 0 and det(H) $\geq$ 0.*
2. *H is negative definite if and only if tr$(H) < 0$ and det$(H) > 0$.*

4 Optimization

4.1 Karush-Kuhn-Tucker Conditions

In this section, we assume that $C \subseteq \mathbb{R}^n$ is convex and open
- the following functions f and g_j with $j = 1, \ldots, m$ are differentiable on C

$$f : x \in C \subseteq \mathbb{R}^n \longrightarrow f(x) \in \mathbb{R} \text{ and}$$
$$g_j : x \in C \subseteq \mathbb{R}^n \longrightarrow g_j(x) \in \mathbb{R}, \forall j = 1, \ldots, m$$

Maximization problem

$$\max_{x \in C} f(x)$$
$$\text{subject to } g_j(x) \geq 0, \forall j = 1, \ldots, m$$

where f is the objective function, and g_j with $j = 1, \ldots, m$ are the constraint functions.

The Karush-Kuhn-Tucker conditions associated with problem are given below

$$\begin{cases} \nabla f(x) + \sum_{j=1}^{m} \lambda_j \nabla g_j(x) = 0 \\ \lambda_j g_j(x) = 0, \forall j = 1, \ldots, m \\ g_j(x) \geq 0, \forall j = 1, \ldots, m \\ \lambda_j \geq 0, \forall j = 1, \ldots, m \end{cases}$$

where for every $j = 1, \ldots, m, \lambda_j \in \mathbb{R}$ is called Lagrange multiplier associated with the inequality constraint g_j

Definition 60 *Let $x^* \in C$, we say that the constraint j is binding at x^* if f $g_j(x^*) = 0$. We denote*

1. *$B(x^*)$ the set of all binding constraints at x^*, that is*

$$B(x^*) := \{j = 1, \ldots, m : g_j(x^*) = 0\}$$

2. *$m^* \leq m$ the number of elements of $B(x^*)$ and*

3. *$g^* := (g_j)_{j \in B(x^*)}$ the following mapping*

$$g^* : x \in C \subseteq \mathbb{R}^n \longrightarrow g^*(x) = (g_j(x))_{j \in B(x^*)} \in \mathbb{R}^{m^*}$$

Theorem 11 *(Karush-Kuhn-Tucker **necessary** conditions) Let x^* be a solution to problem above. Assume that one of the following conditions is satisfied.*

1. *For all $j = 1, \ldots, m, g_j$ is a **linear** or **affine function**.*

2. *Slater's Condition :*

*for all $j = 1, \ldots, m, g_j$ is a **concave** function or g_j is a **quasiconcave** function with $\nabla g_j(x) \neq 0$ for all $x \in C$, and there exists $y \in C$ such that $g_j(y) > 0$ for all $j = 1, \ldots, m$*

3. ***Rank Condition** : rank $\mathrm{J}g^*(x^*) = m^* \leq n$ Then, there exists $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*) \in \mathbb{R}_+^m$ such that (x^*, λ^*) satisfies the Karush-Kuhn-Tucker Conditions .*

Theorem 12 *(Karush-Kuhn-Tucker **sufficient** conditions) Suppose that there exists $\lambda^* = \left(\lambda_1^*, \ldots, \lambda_j^*, \ldots, \lambda_m^*\right) \in \mathbb{R}_+^m$ such that $(x^*, \lambda^*) \in C \times \mathbb{R}_+^m$*

satisfies the Karush-Kuhn-Tucker Conditions (2). Assume that

1. *f is a **concave** function or f is a **quasi-concave** function with $\nabla f(x) \neq 0$ for all $x \in C$, and*

2. *g_j is a **quasi-concave** function for all $j = 1, \ldots, m$*

Then, x^* is a solution to problem.

5 Correspondences

Let $\Theta \subseteq \mathbb{R}^n, X \subseteq \mathbb{R}^n$.

Definition 61 *A correspondence* $\Gamma : \Theta \rightrightarrows X$ *is a map s.t.* $\Gamma(\Theta) \subseteq X$. $\left(\Gamma : \Theta \to 2^X\right)$

Definition 62 ***(Graph of correspondence).*** $Gr(\Gamma) = \{(\theta, x) : \theta \in \Theta, x \in \Gamma(\theta)\}$

Definition 63 ***(Properties of correspondences).***

1. ***not empty valued*** *if* $\Gamma(\theta) \neq \emptyset \quad \forall \theta$
2. ***single valued*** *if* $|\Gamma(\theta)| = 1 \quad \forall \theta$
3. ***closed valued*** *if* $\Gamma(\theta)$ *is closed set* $\forall \theta$
4. ***compact valued*** *if* $\Gamma(\theta)$ *is compact set* $\forall \theta$
5. ***convex valued*** *if* $\Gamma(\theta)$ *is convex set* $\forall \theta$
6. ***closed (graph)*** *if* $Gr(\Gamma)$ *is closed subset of* $\mathbb{E} \times X$
7. ***convex (graph)*** *if* $Gr(\Gamma)$ *is convex on* $\Theta \times X$

Lemma 29 $Gr(\Gamma)$ *is closed graph* $\iff \forall_{\theta : \theta_n \to \theta} \forall_{x_n \to x} : x_n \in \Gamma(\theta_n) \Rightarrow \quad x \in \Gamma(\theta)$

Lemma 30 $Gr(\Gamma)$ *is convex graph* $\iff \forall_\theta, \theta', x \in \Gamma(\theta), x' \in \Gamma(\theta')$ *it holds that* $\lambda x + (1-\lambda)x' \in \Gamma(\theta\lambda + (1-\lambda)\theta') \, \forall_{x \in [0,1]}$

Lemma 31 $\Gamma : \Theta \rightrightarrows X$ *has closed graph* $\Rightarrow$ *it is closed valued. If* X *is compact, than* Γ *is also compact valued.*

Definition 64 ***(Upper Hemi-Continuity)*** *Let* $\Gamma : \Theta \rightrightarrows X$ *be a correspondence.*

- Γ *is said to be* ***upper hemi-continuous (uhc)*** *at a point* $\theta \in \Theta$ *if and only if for all open sets* $V \subseteq X$ *such that* $\Gamma(\theta) \subseteq V$, *there exists an open set* $U \subseteq \Theta$ *such that* $\theta \in U$ *and for all* $\theta' \in U$ *it holds that* $\Gamma(\theta') \subseteq V$
- *A compact valued correspondence* $\Gamma : \Theta \rightrightarrows X$ *is u.h.c. at* $\theta \in \Theta$ *if and only if for every* $\{\theta_n\} \subset \Theta$ *such that* $\theta_n \to \theta$ *and every sequence* $\{x_n\} \subset X$ *such that* $x_n \in \Gamma(\theta_n)$ *there exits a convergent subsequence* $\{x_{n_k}\}$ *such that* $x_{n_k} \to x \in \Gamma(\theta)$

$$\forall_{\theta_n \to \theta} \forall_{x_n \in \Gamma(\theta_n)} \exists_{\{x_{n_k}\}} x_{n_k} \to x \in \Gamma(\theta)$$

Definition 65 ***(Lower Hemi-Continuity).*** *Let* $\Gamma : \Theta \rightrightarrows X$ *be a correspondence.*

- Γ *is said to be* ***lower hemi-continuous (1hc)*** *at a point* $\theta \in \Theta$ *if and only if for all open sets* $V \subseteq X$ *such that* $\Gamma(\theta) \cap V \neq \emptyset$, *there exists an open set* $U \subseteq \Theta$ *such that* $\theta \in U$ *and for all* $\theta' \in U$ *it holds that* $\Gamma(\theta') \cap V \neq \emptyset$
- *A correspondence* $\Gamma : \Theta \rightrightarrows X$ *is l.h.c. at* $\theta \in \Theta$ *if for all* $x \in \Gamma(\theta)$ *and all sequences* $\{\theta_n\} \subset \theta$ *such that* $\theta_n \to \theta$ *there exits a sequence* $\{x_n\} \subset X$ *such that* $x_n \in \Gamma(\theta_n)$ *and* $x_n \to x$

$$\forall_{\theta_n \to \theta} \forall_{x \in \Gamma(\theta)} \exists_{x_n \in \Gamma(\theta_n)} x_n \to x$$

Definition 66 ***(Continuity)*** Γ *is said to be continuous at a point* $\theta \in \Theta$ *if it is both UHC an LHC.*

Lemma 32 ***(u.h.c and Closed graph)*** *Let* $\Gamma : \Theta \rightrightarrows X$. *If* Γ *is u.h.c, then* Γ *is closed (has a closed graph).*

Lemma 33 ***(Closed graph and u.h.c.)*** *Let* $\Gamma : \Theta \rightrightarrows X$. *If* X *is compact and* Γ *is closed (has a closed graph), then* Γ *is u.h.c.*

Theorem 13 ***(Berge (1961) of Maximum)*** *Let $\Theta \subseteq \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$, let $f : \Theta \times X \to \mathbb{R}$ be a continuous function and $\Gamma : \Theta \rightrightarrows X$ a nonempty, compact valued, continuous correspondence. Define:*

$$v(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta) \quad G(\theta) = \{x \in \Gamma(\theta) \mid f(x, \theta) = v(\theta)\}$$

Then

- *$v : \Theta \to X$ is continuous*
- *$G : \Theta \rightrightarrows X$ is nonempty and compact valued, and UHC*

Proof. *The proof is divided in three parts. First it is proven that G is nonempty and compact valued, then that it is u.h.c. and finally that v is continuous.*

1. *G is nonempty valued and compact valued.*
    - *Let $\theta \in \Theta$, by hypothesis $\Gamma(\theta)$ is compact and nonempty. since $f(\cdot, \theta)$ is continuous a maximum is attained on $\Gamma(\theta)$ by the extreme value theorem (Weierstrass). This proves that $G(\theta)$ is nonempty for arbitrary θ.*
    - *Let $\theta \in \Theta$, by hypothesis $\Gamma(\theta)$ is compact and nonempty. since $G(\theta) \subseteq \Gamma(\theta)$ it follows that $G(\theta)$ is bounded, it is left to show closedness to establish compactness. Let $x_n \to x$ and $x_n \in G(\theta)$ for all n. Clearly $x_n \in \Gamma(\theta)$ for all n, since Γ is closed valued it follows that $x \in \Gamma(\theta)$, so its feasible. By definition of G we have $v(\theta) = f(x_n, \theta)$ for all n, since f is continuous we get $v(\theta) = \lim f(x_n, \theta) = f(x, \theta)$, then by definition $x \in G(\theta)$, which proves closedness.*

2. *G is u.h.c. Consider $\theta \in \Theta$, a sequence in Θ such that $\theta_n \to \theta$ and a sequence in X such that $x_n \in G(\theta_n)$ for all n. Note that $x_n \in \Gamma(\theta_n)$. since Γ is u.h.c. there exists a subsequence $x_{n_k} \to x \in \Gamma(\theta)$ Now consider $z \in \Gamma(\theta)$. since Γ is l.h.c. there exists a sequence in X such that $z_n \in \Gamma(\theta_n)$ and $z_n \to z$. In particular the subsequence $\{z_{n_k}\}$ also converges to z since $x_n \in G(\theta_n)$ and $z_n \in \Gamma(\theta_n)$ it follows that $f(x_n, \theta_n) \geq f(z_n, \theta_n)$. since f is continuous in both arguments we get by taking limits: $f(x, \theta) \geq f(z, \theta)$. since the inequality holds for arbitrary $z \in \Gamma(\theta)$ we get the result: $x \in G(\theta)$. This proves u.h.c.*

3. *v is continuous. Let $\theta \in \Theta$ and $\theta_n \to \theta$ an arbitrary sequence converging to θ. Consider an arbitrary sequence in X such that $x_n \in G(\theta_n)$ for all n. Let $\bar{v} = \limsup v(\theta_n)$. By proposition 2.9 there is a subsequence $\{\theta_{n_k}\}$ such that $v(\theta_{n_k}) \to \bar{v}$. since G is u.h.c. there exists a subsequence of $\{x_{n_k}\}$ (call it $\{x_{n_{k_l}}\}$) converging to a point $x \in G(\theta)$. Then*

$$\bar{v} = \lim v(\theta_{k_l}) = \lim f(x_{k_l}, \theta_{k_l}) = f(x, \theta) = v(\theta)$$

where the second equality follows from $x_{k_l} \in G(\theta_{k_l})$, the third one from f being continuous and the final one from $x \in G(\theta)$. Let $\underline{v} = \liminf v(\theta_n)$ and by a similar argument we get $v(\theta) = \underline{v}$ since $v(\theta) = \liminf v(\theta_n) = \limsup v(\theta_n)$ we get $v(\theta) = \lim v(\theta_n)$ for arbitrary $\{\theta_n\}$ converging to θ. This proves continuity.

Theorem 14 ***(ToM under convexity)*** *Let $\Theta \subseteq \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$, let $f : \Theta \times X \to \mathbb{R}$ be a continuous function and $\Gamma : \Theta \Rightarrow X$ a nonempty, compact valued, continuous correspondence. Define:*

$$v(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta) \quad G(\theta) = \{x \in \Gamma(\theta) \mid f(x, \theta) = v(\theta)\}$$

a *If $f(\cdot, \theta)$ is concave in x for all θ and Γ is convex valued then G is convex valued.*

b *If $f(\cdot, \theta)$ is strictly concave in x for all θ and Γ is convex valued then G is single valued, hence a continuous function.*

c *If f is concave on $\Theta \times X$ and Γ has a convex graph then v is concave and G is convex valued.*

d *If f is strictly concave on $\Theta \times X$ and Γ has a convex graph then v is strictly concave and G is single valued, hence a continuous function.*

Theorem 15 ***(ToM under quasi-convexity).*** *Let $\Theta \subseteq \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$, let $f : \Theta\times\ X \to \mathbb{R}$ be a continuous function and $\Gamma : \Theta \rightrightarrows X$ a nonempty, compact valued, continuous correspondence. Define:*

$$v(\theta) = \max_{x\in\Gamma(\theta)} f(x,\theta) \quad G(\theta) = \{x \in \Gamma(\theta) \mid f(x,\theta) = v(\theta)\}$$

a *If $f(\cdot,\theta)$ is quasi-concave in x for all θ and Γ is convex valued then G is convex valued.*

b *If $f(\cdot,\theta)$ is strictly quasi-concave in x for all θ and Γ is convex valued then G is single valued, hence a continuous function.*

c *If f is quasi-concave on $\Theta \times X$ and Γ has a convex graph then v is quasi-concave and G is quasi-convex valued.*

d *If f is strictly quasi-concave on $\Theta \times X$ and Γ has a convex graph then v is strictly quasi-concave and G is single valued, hence a continuous function.*

5.1 Berge theorem applied to micro

One, useful, application of the material covered above is to determine properties of the budget correspondence, that indicates the feasible consumption bundles for a consumer given a price vector p and an endowment vector e. Suppose there are l goods, and that the agent has a fixed endowment of each good given by the vector $e \in \mathbb{R}^l_{++}$, the price of the goods is a vector $p \in \Delta$, where Δ is the n -dimensional open simplex. Define the budget set correspondence $B(\cdot,e) : \Delta \rightrightarrows \mathbb{R}^l_+$by

$$B(p,e) = \left\{x \in \mathbb{R}^l_+ \mid p\cdot x \leq p \cdot e\right\}$$

Theorem 16 *$B(\cdot,e)$ is continuous on prices.*

Proof. The claim is proved establishing u.h.c. and l.h.c. of B.

1. $B(\cdot,e)$ is upper hemi-continuous on prices. Let $p\in\Delta$, $\{p_n\} \subset \Delta$ with $p_n \to p$ and $\{x_n\} \subset \mathbb{R}^l_+$ a sequence such that $x_n \in B(p_n,e)$ since $p_n \to p \in \Delta$ there exists a closed ball, C, around p such that $C \subset \Delta$ and for n large enough $p_n \in C$. Let $\xi_i = \max_{p\in C} \frac{p\cdot e}{p_i}$ for $i = 1,\ldots,l$. ξ_i is the maximum amount of x_i that can be bought in the neighborhood of p. Define $\xi = \max\{\xi_i\}+1$, it is clear that for n large enough $x_n \in B_\xi(0)$, then $\{x_n\}$ is a bounded sequence, hence it admits a convergent subsequence $x_{n_k} \to x$. since $x_{n_k} \in B(p_{n_k},e)$ we have: $p_{n_k}\cdot x_{n_k} \leq p_{n_k}\cdot e$, since dot product is a continuous function taking limits we have $p\cdot x \leq p\cdot e$, which is $x \in B(p,e)$, proving u.h.c. of B.

2. $B(\cdot,e)$ is lower hemi-continuous on prices. Let $p\in\Delta$, $\{p_n\}\subset\Delta$ with $p_n \to p$ and $x \in B(p,e)$. Define $\eta^i_n = \max\left\{0, \frac{p_n\cdot x - p_n\cdot e}{lp^i_n}\right\}$ and let $x_n = x - \eta_n$ Clearly $x_n \in B(p_n,e)$ since either $x \in B(p_n,e)$ or

$$p_n\cdot x_n = p_n\cdot x - \sum p^i_n \left(\frac{p_n\cdot x - p_n\cdot e}{lp^i_n}\right) = p_n\cdot x - (p_n\cdot x - p_n\cdot e) = p_n \cdot e$$

   then $p_n\cdot x_n \leq p_n\cdot e$ Moreover $x_n \to x$, since $x \in B(p,e)$ and $p_n \to p$ it follows that $p_n\cdot x - p_n\cdot e \to p\cdot x - p\cdot e \leq 0$, then $\eta_n = \max\{0, p_n\cdot x - p_n\cdot e\} \to 0$ which is $x_n \to x$. Then B is l.h.c.

3. Note that it wasn't checked if $x_n \geq 0$ for all n. This is not guaranteed by the construction above. With extra notation it can be guaranteed that $x^i_n \geq 0$.

The consumer problem is often laid out without explicit endowments of the goods, instead the parameters are prices $p \in \mathbb{R}^l_{++}$ and a nominal income level $I \in \mathbb{R}_+$. The set of parameters is $\Theta = \mathbb{R}^l_{++}\times\mathbb{R}$. The **indirect utility function** and the **Marshalian demand correspondence** are:

$$v(p,I) = \max_{x\in B(p,I)} u(x) \quad G(p,I) = \{x \in B(p,I) \mid u(x) = v(p,I)\}$$

where the budget set is given by the correspondence:

$$B(p,I)=\left\{x\in\mathbb{R}_{+}^{l}\mid p\cdot x\leq I\right\}$$

I take as given that B is a nonempty, convex valued and continuous correspondence, and that u is a continuous function.

Theorem 17 *v and G have the following properties on Θ.*

a v is a continuous function on Θ and G is a nonempty, compact valued, u.h.c. correspondence.

b v is nondecreasing in I for fixed p and non-increasing in p for fixed I.

c v is jointly quasi-convex on (p,I).

d If u is (quasi) concave then v is (quasi) concave in I for fixed p.

e If u is (quasi) concave then G is a convex valued correspondence.

f If u is strictly (quasi) concave then G is a continuous function.

5.2 Nash equilibrium in normal form games

Definition 67 *A normal form game is formed by:*

a A finite set of agents $I=\{1,\ldots,N\}$. A generic player is denoted i and the set of other players $-i$.

b For each player a finite action set A_i. Note $A=\times A_i$.

c For each player a payoff function $u^i:A\to\mathbb{R}$.

From the set of pure strategies of a player one can define the set of mixed strategies. $S_i=\Delta(A_i)$, a mixed strategy is a probability distribution over the set of possible actions A_i. Formally:

$$S_i=\Delta(A_i)=\left\{s_i:A_i\to[0,1]\mid\sum_{a_i\in A_i}s_i(a_i)=1\right\}$$

Note that S_i is convex and compact. In fact S_i is the convex hull of A_i. If players play mixed strategies they rank alternative strategies according to their expected payoffs, the expected payoffs are given by function $v^i:S_i\times S_{-i}\to\mathbb{R}$ which is:

$$v^i(t,s_{-i})=\sum_{a_i\in A_i}t(a_i)\left(\sum_{a_{-i}\in A_{-i}}\prod_{j\neq i}s_j(a_j)\,u^i(a_i,a_{-i})\right)=\sum_{a\in A}\left(\left(t(a_i)\prod_{j\neq i}s_j(a_j)\right)u^i(a)\right)$$

In a game where players play simultaneously in a noncooperative manner they have to answer optimally to a given strategy profile of the other players.

Definition 68 *The best response of a player to s_{-i} is given by:*

$$\mathrm{BR}_i(s_i,s_{-i})=\mathrm{BR}_i(s)=\left\{t\in S_i\mid\forall_{r\in S_i}u^i(t,s_{-i})\geq u^i(r,s_{-i})\right\}=t\in S_i argmax v^i(t,s_{-i})$$

Note that BR^i is the solution to the problem $V(s)=\max_{t\in S_i}v^i(t,s_{-i})$ since S_i is a fixed set it is also a constant correspondence with argument s, a strategy profile. It is then continuous as well as nonempty, compact and convex valued. Moreover v^i is continuous in s_{-i} and constant in s_i by construction, then it is continuous in $s.v$ is also linear in t holding s_{-i} constant, then it is concave. It follows that the ToM under convexity applies, then the BR is a nonempty, compact and convex valued and u.h.c. correspondence for each player.

Definition 69 *A Nash Equilibrium is defined as a strategy profile $s^\star\in S$ such that $s_i^\star\in\mathrm{BR}_i(s^\star)$ for all i. A way to think about it is to form a correspondence with the cartesian product of the individual BR correspondences, this is $\mathrm{BR}:S\to S$ defined as:*

$$\mathrm{BR}(s)=\times\mathrm{BR}_i(s)$$

Note that BR is by construction a nonempty, compact and convex valued and u.h.c. correspondence.

A NE is then a fixed point of the correspondence BR. The following theorem will establish the existence of such fixed point.

6 Powerful theorems of analysis

6.1 Fixed Point Theorems

Theorem 18 ***Brouwer's Fixed Point Theorem – continuous function***
Let $S \subset \mathbb{R}$ be nonempty, compact, and convex, and $f : S \to S$ be a continuous function. Then f has (at least) a fixed point in S, i.e. $\exists x^* \in S : x^* = f(x^*)$

Theorem 19 ***Tarsky's Fixed Point Theorem – weakly increasing functions***
Let $f : [0,1]^n \to [0,1]^n$, where $[0,1]^n = [0,1] \times ... \times [0,1]$, an n-dimensional cube. If f is nondecreasing, then f has a fixed point in $[0,1]^n$.

Theorem 20 ***Kakutani's Fixed Point Theorem – u.h.c. correspondence***
Let $S \subset \mathbb{R}$ be nonempty, compact, and convex, and $\Gamma : S \rightrightarrows S$ be a nonempty, convex-valued, and u.h.c. correspondence. Then Γ has a fixed point in S, i.e. $\exists x^* \in S : x^* \in \Gamma(x^*)$

Since S is compact, u.h.c. is equivalent to Γ having a closed graph.

Theorem 21 ***Fixed Point Theorem – l.h.c. correspondence***
Let $S \subset \mathbb{R}$ be nonempty, compact, and convex, and $\Gamma : S \rightrightarrows S$ be a nonempty, convex-valued, closed-valued, and l.h.c. correspondence. Then Γ has a fixed point in S.

6.2 Other Powerful theorems of analysis

Theorem 22 *(Inverse Function Theorem) Let V be open in $\mathbb{R}^n$ and $f : V \to \mathbb{R}^n$ be $\mathcal{C}^1$ on V. If $\Delta_f(a) \neq 0$ for some $a \in V$, then there exists an open set W containing a such that*

- *f is $1-1$ on W*
- *f^{-1} is $\mathcal{C}^1$ on $f(W)$, and*
- *for each $y \in f(W)$*
$$D\left(f^{-1}\right)(y) = \left[Df\left(f^{-1}(y)\right)\right]^{-1}$$

Notation: $[\cdot]^{-1}$ represents matrix inversion, $\Delta_f(a) = det(Df(a))$ (the Jacobian of f at a)

Theorem 23 *(Mean Value Theorem on $\mathbb{R}^n$) Let $V \subset \mathbb{R}^n$ be open and convex, and let $f : V \to \mathbb{R}$ be a function that is differentiable everywhere on V. Then, for any $a, b \in V$, there is $\lambda \in (0,1)$ such that*
$$f(b) - f(a) = Df((1-\lambda)a + \lambda b) \cdot (b-a)$$

Notation: $L(a,b) := \{(1-t)\mathbf{a} + tb : t \in [0,1)\}$ is called line segment

Theorem 24 *(Taylor Theorem on $\mathbb{R}^n$) Let $p \in \mathbb{N}$, let V be open in $\mathbb{R}^n$, let $x, a \in V$, and suppose that $f : V \to \mathbb{R}$. If the pth total differential of f exists on V and $L(x;a) \subseteq V$, then there is a point $c \in L(x,a)$, $h := x - a$ such that*
$$f(x) = f(a) + \sum_{k=1}^{p-1} \frac{1}{k!} D^{(k)} f(a;h) + \frac{1}{p!} D^{(p)} f(c,h)$$

Theorem 25 *(Implicit Function Theorem) Let $F : S \subset \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ function, where S is open. Let (x^*, y^*) be a point in S such that $DF_y(x^*, y^*)$ is invertible, and let $F(x^*, y^*) = c$. Then, there is a neighborhood $U \subset \mathbb{R}^m$ of x^* and a $\mathcal{C}^1$ function $g : U \to \mathbb{R}^n$ such that*

- *$(x, g(x)) \in S, \forall x \in U$*
- *$g(x^*) = y^*$*
- *$F(x, g(x)) \equiv c, \forall x \in U$*
- *$Dg(x) = (DF_y(x,y))^{-1} \cdot DF_x(x,y)$*

7 Comparative statics ala Topkis

Definition 70 *(**Meet and Joint**) Given $x, y \in \mathbb{R}^n$, **the meet** of x and y, denoted $x \wedge y$, is*

$$x \wedge y = (\min\{x_1, y_1\}, \cdots, \min\{x_n, y_n\})$$

***The joint** of x and y, denoted $x \vee y$, is*

$$x \vee y = (\max\{x_1, y_2\}, \cdots, \max\{x_n, y_n\})$$

Definition 71 *(Lattice) $X \subset \mathbb{R}^n$ is a lattice of $\mathbb{R}^n$ if $\forall x, y \in X, x \wedge y \in X$ and $x \vee y \in X$*

Remark: A budget set is generally not a lattice of $\mathbb{R}^n$. More for lattice: we can define compact, sup/inf on it:

Definition 72 *(compact lattice) $X \subset \mathbb{R}^n$ is a compact lattice if X is a lattice and X is compact under the Euclidean metric.*

Definition 73 *$x^* \in X$ is a greatest element of lattice X if $x^* \geq x, \forall x \in X, \hat{x} \in X$ is a least element of lattice X if $\hat{x} \leq x, \forall x \in X$*

Definition 74 *(Uniqueness of greatest and least element) Suppose $X \subset \mathbb{R}^n$ is a non-empty, compact lattice. Then, X has a greatest element and a least element.*

Definition 75 *(Supermodular) $f : S \times \Theta \to \mathbb{R}$ is supermodular in (x, θ) if $\forall z = (x, \theta)$ and*

$$z' = (x', \theta')\, inS \times \Theta f(z) + f(z') \leq f(z \vee z') + f(z \wedge z')$$

Theorem 26 *(Supermodularity) $f : S \times \Theta \to \mathbb{R}$ is supermodular in (x, θ), then for any fixed θ, f is supermodular inx, i.e.*

$$f(x, \theta) + f(x', \theta) \leq f(x \vee x', \theta) + f(x \wedge x', \theta))$$

Proposition 15.1

Definition 76 *(Increasing Differences) $f : S \times \Theta \to \mathbb{R}$ satisfies increasing differences in (x, θ) if $\forall (x, \theta), (x', \theta') \in S \times \Theta$ such that $x \geq x'$ and $\theta \geq \theta'$*

$$f(x, \theta) - f(x', \theta) \geq f(x, \theta') - f(x', \theta')$$

If the inequality is strict whenever $x > x'$ and $\theta < \theta'$, then f satisfies strictly increasing differences $in(x, \theta)$

Theorem 27 *(Supermodularity vs. Increasing Differences) $f : Z \subseteq \mathbb{R}^n \to \mathbb{R}$ is supermodular in z iff f has increasing return in z*

Theorem 28 *(Topkis' Characterization Theorem) Let Z be an open lattice of $\mathbb{R}^n$. A $\mathcal{C}^2$ function $h : Z \to \mathbb{R}$ is supermodular on Z iff $\forall z \in Z$*

$$\frac{\partial^2 h}{\partial z_i \partial z_j}(z) \geq 0, \forall i \neq j$$

7.1 Parametric Monotonicity

Now let's consider the optimization problem:

$$\max_{x \in S} f(x; \theta)$$

with

$$f^*(\theta) = \max\{f(x; \theta) \mid x \in S\}, \quad D^*(\theta) = argmax\{f(x; \theta) \mid x \in S\}$$

A correspondence $D^*(\theta)$ is nondecreasing in θ if for every $\theta \leq \theta'$

$$D^*(\theta) \leq D^*(\theta')$$

Above inequality between sets means the strong set order: for every $x \in D^*(\theta)$ and $x' \in D^*(\theta')$, it holds $x \vee x' \in D^*(\theta'), x \wedge x' \in D^*(\theta)$

Theorem 29 *(Topkis' Monotonicity Theorem) Let S be compact lattice of $\mathbb{R}^n, \Theta$ be a lattice of $\mathbb{R}^l$, and $f : S \times \Theta \to \mathbb{R}$ be a continuous function on S, for each fixed θ. Suppose f satisfies increasing differences in (x, θ), and is supermodular in x for each fixed θ. Then D^* is nondecreasing in θ.*

8 Stochastic analysis

Definition 77 *(σ algebra) Let S be a set and let $\mathcal{F}$ be a family of subsets of S.$\mathcal{F}$ is called a σ -algebra if*

- $\emptyset, S \in \mathcal{F}$
- $(A \in \mathcal{S}) \Rightarrow (A^c = S \backslash A \in \mathcal{F})$ *(close under complements)*
- $(A_n \in \mathcal{S}, n = 1, 2, \cdots) \Rightarrow (\cup_{n=1}^{\infty} A_n \in \mathcal{F})$ *(close under countable unions / intersections)*

Definition 78 *A pair $(S, \mathcal{F})$, where S is a set and $\mathcal{F}$ is a σ -algebra of its subsets is called a measurable space. Any set $A \in \mathcal{F}$ is called an $\mathcal{F}$ -measurable set.*

Definition 79 *Given a set S and a collection $\mathcal{A}$ of subsets of S, the intersection of all σ -algebras (which is also a σ -algebras) containing $\mathcal{A}$ is called the σ -algebra generated by $\mathcal{A}$.*

Example:

- The power set of S;
- The family $\{\varnothing, S\}$ (trivial σ -algebra)
- For the set $S = \{1, 2, 3, 4\}, \mathcal{F} = \{\{1, 3\}, \{2, 4\}, \varnothing, S\}$ is also a σ -algebra
- Let $\mathcal{B}$ be the open ball in $\mathbb{R}^l, \sigma$ -algebra generated by $\mathcal{B}$ is called Borel-algebra generated by $\mathcal{B}$. Similarly, it can also be defined by open rectangles (or closed intervals, half-open intervals if in $\mathbb{R}$

Definition 80 *(Measure) Let ($S, \mathcal{F}$) be a measurable space. A measure is an extended real-valued function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ such that*

a $\mu(\emptyset) = 0$

b $\mu(A) \geq 0$ *for all* $A \in \mathcal{F}$

c μ *is countably additive: if* $\{A_n\}_{n=1}^{\infty}$ *is a countable, disjoint sequence in* $\mathcal{A}$, *then* $\mu\left(\cup A_n\right) = \sum \mu\left(A_n\right)$

Definition 81 *If furthermore $\mu(S) < \infty$, then μ is said to be a finite measure and if $\mu(S) = 1$ then μ is said to be a probability measure.*

Definition 82 *A triple $(S, \mathcal{F}, \mu)$ where S is a set, $\mathcal{F}$ is a σ -algebra of its subsets and μ is a measure on $\mathcal{F}$ is called a measure space. The triple is called a probability space if μ is a probability measure*

Definition 83 *(Measurable Functions) Given a measurable space $(S, \mathcal{F})$, a real-valued function $f : S \to \mathbb{R}$ is measurable with respect to $\mathcal{F}$ (or $\mathcal{F}$ -measurable) if*

$$\{s \in S \mid f(s) \leq a\} \in \mathcal{F}, \forall a \in \mathbb{R}$$

If the space is a probability space, then f is called a random variable.

Definition 84 *(Simple Function) Let $(S, \mathcal{F})$ be a measurable space, a function $\phi : S \to \mathbb{R}$ is called a simple function if it is of the form*

$$\phi(s) = \sum_{i=1}^{n} a_i \chi_{A_i}(s)$$

where $a_1, \cdots, a_n$ are distinct real numbers, $\{A_i\}$ is a partition of S, and χ_{A_i} are indicator functions. A simple function is measurable if and only if $A_i \in \mathcal{F}$.

Theorem 30 *(Pointwise convergence preserves measurability) Let $(S, \mathcal{F})$ be a measurable space, and let $\{f_n\}$ be a sequence of $\mathcal{F}$ -measurable functions converging pointwise to f. Then f is also measurable.*

Theorem 31 *(Approximation of measurable functions by simple functions) Let $(S, \mathcal{F})$ be measurable space. If $f : S \to \mathbb{R}$ is $\mathcal{F}$ -measurable, then there is a sequence of measurable simple functions $\{\phi_n\}$, such that $\phi_n \to f$ pointwise. If $0 \leq f$, then the sequence can be chosen so that*

$$0 \leq \phi_n \leq \phi_{n+1} \leq f, \forall n$$

If f is bounded, then the sequence can be chosen so that $\phi_n \to f$ uniformly.

Definition 85 *Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be measurable spaces. Then the function $f : S \to T$ is measurable if the inverse image of every measurable set is measurable, i.e. if $\{s \in \mathcal{S} : f(s) \in A\} \in \mathcal{T}$ for all $A \in \mathcal{T}$*

Definition 86 *(Measurable Selection from a Correspondence) Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be measurable spaces, and let Γ be a correspondence of S into T. Then the function $h : S \to T$ is a measurable selection from Γ if h is measurable and $h(s) \in \Gamma(s), \forall s \in S$.*

Theorem 32 *Measurable Selection Theorem)) Let $S \subset \mathbb{R}^l$ and $T \subset \mathbb{R}^m$ be Borel sets, with their Borel subsets $\mathcal{S}$ and $\mathcal{T}$. Let $\Gamma : S \to T$ be a (nonempty) compact-valued and uhc correspondence. Then there exists a measurable selection from Γ*

Some notation: $M(S, \mathcal{S})$: space of measurable, extended real-valued functions on S $M^+(S, \mathcal{S})$: space of measurable, extended real-valued, non-negative functions on S

Definition 87 *Let $\phi \in M^+(S, \mathcal{S})$ be a measurable simple function, with the standard representation $\phi(s) = \sum_{i=1}^n a_i \chi_{A_i}(s)$. Then the integral of ϕ with respect to μ is*

$$\int_S \phi(s)\mu(ds) = \sum_{i=1}^n a_i \mu\left(A_i\right)$$

Definition 88 *For $f \in M^+(S, \mathcal{S})$, the integral of f with respect to μ is*

$$\int_S f(s)\mu(ds) = \sup \int_S \phi(s)\mu(ds)$$

where the supremum is taken over all simple functions ϕ in $M^+(S, \mathcal{S})$ with $0 \leq \phi \leq f$. If $A \in \mathcal{S}$, then the integral of f over A with respect to μ is

$$\int_A f(s)\mu(ds) = \int_S f(s)\chi_A(s)\mu(ds)$$

Every $f \in M^+(S, \mathcal{S})$ can be written as the limit of an increasing sequence $\{\phi_n\}$ of simple functions. The next theorem tells us that the integral $\int f d\mu$ is also the unique limit t of $\int \phi_n d\mu$, i.e. it does not depend on the particular sequence $\{\phi_n\}$ chosen.

Theorem 33 *(Monotone Convergence Theorem) If $\{f_n\}$ is a monotone increasing sequence of functions in $M^+(S, \mathcal{S})$ converging pointwise to f then*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

Now we can think about generate the definition to functions that take negative value. Define the positive parts and negative parts as below: let $f : S \to \mathbb{R}$ be an arbitrary function. We denote

$$f^+(s) = \begin{cases} f(s) & \text{if} \quad f(s) \geq 0 \\ 0 & \text{if} \quad f(s) < 0 \end{cases}$$

and

$$f^-(s) = \begin{cases} -f(s) & \text{if} \quad f(s) \leq 0 \\ 0 & \text{if} \quad f(s) > 0 \end{cases}$$

Definition 89 *Let $(S, \mathcal{S}, \mu)$ be a measure space, and let f be a measurable, real-valued function on S. If f^+ and f^- both have finite integrals with respect to μ, then f is integrable and the integral of f with respect to μ is*

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

Definition 90 *If $(S, \mathcal{S}, \mu)$ is a probability space and f is integrable, then call $\int f d\mu$ the expected value of f*

8.1 Transition Functions

Definition 91 *(Transition Function) Let* $(Z, \mathcal{Z})$ *be a measurable space. A transition function is a function* $Q : Z \times \mathcal{Z} \to [0,1]$ *such that*

- $\forall z \in Z, Q(z, \cdot)$ *is a probability measure on* $(Z, \mathcal{Z})$*, and*
- $\forall A \in \mathcal{Z}, Q(\cdot, A)$ *is a* $\mathcal{Z}$ *-measurable function.*

Interpretation: $\forall a \in Z, A \in \mathcal{Z}$

$$Q(a, A) = Pr\left\{z_{t+1} \in A \mid z_t = a\right\}$$

A Markov process can be completely described by this transition function, and the most important property of Markov process is the same transition function can be used in all periods, making each period's problem symmetric. Define T to be the operator from $M^{+}(Z, \mathcal{Z})$

$$(Tf)(z) = \int f\left(z'\right) Q\left(z, dz'\right), \forall z \in Z$$

Interpretation: expected value of f next period if the current state is z, called the Markov operator associated with Q. Define $T^{*}\lambda$ to be the operator from the set of probability measure on $(Z, \mathcal{Z})$:

$$\left(T^{*}\lambda\right)(A) = \int Q(z, A)\lambda(dz), \forall A \in \mathcal{Z}$$

Interpretation: probability that the state will be in A next period, given that current values of the state are drawn according to the probability measure λ. Theorem 20.1 (1) T maps the space of bounded $\mathcal{Z}$ -measurable functions, $B(Z; \mathcal{Z})$ into itself. (2) T^{*} maps the space of probability measures on $(Z; \mathcal{Z})$, that is $\Lambda(Z, \mathcal{Z})$ into itself. (3)

$$\int (Tf)(z)\lambda(dz) = \int f\left(z'\right)\left(T^{*}\lambda\right)\left(dz'\right)$$

There are other two properties a transition function may have:

Definition 92 *(Feller property) A transition function* Q *on* $(Z, \mathcal{Z})$ *has the Feller property if the associated operator* T *maps the space of bounded continuous functions on* Z *into itself; that is if* $T : C(Z) \to C(Z)$

Definition 93 *(Monotone) A transition function* Q *on* $(Z, \mathcal{Z})$ *is monotone if the associated operator* T *has the property that for every nondecreasing function* $f : Z \to \mathbb{R}$*, the function* Tf *is also nondecreasing.*

8.2 Probability Measures on Space of Sequences

Given a transition function Q on $(Z, \mathcal{Z})$, we want to look at partial (finite) histories of shocks and complete (infinite) histories generated by this transition function:

$$z^{t} = \left(z_1, \cdots, z_t\right), t = 1, 2, \cdots \quad z^{\infty} = \left(z_1, z_2, \cdots\right)$$

Let $(Z, \mathcal{Z})$ be a measurable space, and for any finite $t = 1, 2, \cdots$, let

$$\left(Z^{t}, \mathcal{Z}^{t}\right) = (Z \times \cdots \times Z, \mathcal{Z} \times \cdots \times \mathcal{Z})$$

denote the product space. We can define a measure on $\left(Z^{t}, \mathcal{Z}^{t}\right)$

$$\mu^{t}\left(z_0, \cdot\right) = \mathcal{Z}^{t} \to [0,1], \quad t = 1, 2, \ldots$$

as follow: $\forall B = A_1 \times \cdots \times A_t \in \mathcal{Z}^{t}$

$$\mu^{t}\left(z_0, B\right) = \int_{A_1} \cdots \int_{A_{t-1}} \int_{A_t} 1 Q\left(z_{t-1}, dz_t\right) Q\left(z_{t-2}, dz_{t-1}\right) \cdots Q\left(z_0, dz_1\right)$$

This approach can also be used to define probability over infinite sequences $z^{\infty}=(z_1, z_2, \ldots)$ (So we will work with infinite product space $Z^{\infty}=Z \times Z \times \cdots$.) Define a finite measurable rectangle $B \subset Z^{\infty}$:

$$B=A_1 \times A_2 \times \cdots \times A_T \times Z \times Z \times \cdots$$

where $A_t \in \mathcal{Z}, t=0,1,2, \cdots, T<\infty$. Let $\mathcal{C}$ be the family of all finite measurable rectangles, and $\mathcal{A}^{\infty}$ the family of all finite unions of sets in $\mathcal{C}$. Then we can show that $\mathcal{A}^{\infty}$ is an algebra. Let $\mathcal{Z}^{\infty}$ be the σ -algebra generated by $\mathcal{A}^{\infty}$. Define the measure similar as before;

$$\mu^{\infty}\left(z_0, B\right)=\int_{A_1} \cdots \int_{A_{t-1}} \int_{A_t} Q\left(z_{t-1}, d z_t\right) Q\left(z_{t-2}, d z_{t-1}\right) \cdots Q\left(z_0, d z_1\right)$$

We can check that this measure will satisfy the three conditions imposed on measures on an algebra. By the Caratheodory and Hahn Extension Theorem, exists a unique extension of μ^{∞} to $\mathcal{Z}^{\infty}$. Therefore, $\left(Z^{\infty}, \mathcal{Z}^{\infty}, \mu^{\infty}\right)$ is a probability space.

Definition 94 *(Stochastic Process) A stochastic process on* $(\Omega, \mathcal{F}, P)$ *is an increasing sequence of* σ *-algebra* $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots \subset \mathcal{F}$; *a measurable space* $(Z, \mathcal{Z})$; *and a sequence of functions* $\sigma: \Omega \rightarrow Z, t=1,2, \cdots$ *such that each* σ_t *is* $\mathcal{F}_t$ *measurable.*

Definition 95 *A stochastic process is called stationary if* $P_{t+1, \cdots, t+n}$ *is independent of* t., *i.e*

$$F_{t_1+k, t_2+k, \cdots, t_s+k}\left(b_1, b_2, \cdots, b_s\right)=F_{t_1, t_2, \cdots, t_s}\left(b_1, b_2, \cdots, b_s\right)$$

for any finite set of indices $\left\{t_1, t_2, \cdots, t_s\right\} \subset \mathbb{Z}$ *with* $s \in \mathbb{Z}^{+}$, *and any* $k \in \mathbb{Z}$

Definition 96 *A stochastic process is called a (first-order) Markov process if*

$$P_{t+1, \cdots, t+n}\left(C \mid a_{t-s}, \cdots, a_{t-1}, a_t\right)=P_{t+1, \cdots, t+n}\left(C \mid a_t\right)$$

9 Acknowledgment

- Elena del Mercato *Mathematical Appendix for Economics, 2015*
- Simeng Zeng *Math refresher notes, 2020*