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## Afriat's Theorem proof by Fostel, Scarf, and Todd (2004)

A version of this proof was provided by Ichiro Obara.

**Definition 0.1.** *We say that the observations satisfy the Generalized Axiom of Revealed Preference* **(GARP)** *if for every ordered subset* $\{i, j, k, \ldots r\} \subset \mathbb{N}$:

$$p_i \cdot x_j \leq p_i \cdot x_i$$

$$p_j \cdot x_k \leq p_j \cdot x_j$$

$$\vdots$$

$$p_r \cdot x_i \leq p_r \cdot x_r$$

*it must be true that each inequality is, in fact, an equality*

Consider a finite data set $D = \left\{ (p^t, x^t) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_+, t = 1, \ldots, T \right\}$. This note proves the following proposition, which is skipped in the class.

**Theorem 0.2.** *Suppose that a finite data set $D$ satisfies GARP. Then there exists $\lambda^t > 0, U^t, t = 1, \ldots, T$ such that*

$$U^j \leq U^i + \lambda^i p^i \cdot \left( x^j - x^i \right) \text{ for all } i, j \in \{1, \ldots, T\}$$

Denote by $M^T$ the set of $T \times T$ matrices where all diagonal elements are 0. We say that $A \in M^T$ satisfies GA if the following condition is satisfied.

$$\text{(GA) For any } \left\{ a_{t(n)t(n+1)} \right\}_{n=1}^N$$
$$\text{if } a_{t(n)t(n+1)} \leq 0 \text{ for } n = 1 \ldots N-1$$
$$\text{then } a_{t(N)t(1)} \geq 0$$

where $a_{N,N+1} = a_{N,1}$. Note that the $T \times T$ matrix where $ij$ entry is given by $a_{ij} = p^i \cdot \left( x^j - x^i \right)$ satisfies GA if $D$ satisfies GARP.

First we show that GA is equivalent to the following condition.

$$\text{(GA*) For any } \left\{ a_{t(n)t(n+1)} \right\}_{n=1}^N$$
$$\text{if } a_{t(n)t(n+1)} \leq 0 \text{ for } n = 1 \ldots N$$
$$\text{then } a_{t(n)t(n+1)} = 0 \text{ for } n = 1 \ldots N$$

**Lemma 0.3.** *GA and GA* *are equivalent.*

*Proof.* Suppose that GA is satisfied. If $a_{t(n)t(n+1)} \leq 0$ for all $n$, then any $a_{t(n)t(n+1)}$ can be regarded as the tail of a cycle (the one starting at $a_{t(n+1)t(n+2)}$) in GA. So $a_{t(n)t(n+1)} \geq 0$, hence $a_{t(n)t(n+1)} = 0$ for all $n$

Conversely, suppose that GA* is satisfied. If $a_{t(n)t(n+1)} \leq 0$ for $n = 1 \ldots N-1$, then $a_{t(N)t(1)} < 0$ cannot be the case because then GA* implies $a_{t(N)t(1)} = 0$, a contradiction. Hence $a_{t(N)t(1)} \geq 0$ $\qquad \square$

Now we prove the above proposition.

*Proof. of* **Theorem (0.2)**.

The proof is based on induction. We show that we can find such $\lambda^t > 0, U^t, t = 1, \ldots, T$ for every $A \in M^T$ that satisfies GA* if we can find them for every $A \in M^{T-1}$ that satisfies GA*. since this is trivially true for $T = 1$ this proves that the same property holds for every $T$. We start with the following lemma.

**Lemma 0.4.** *Suppose that $A \in M^T$ satisfies $GA^*$. Then there exists $t^*$ such that $a_{t^*t} \geq 0$ for $t = 1, \ldots, T$*

*Proof.* Suppose not. Then we can construct a cycle $\{a_{t(n)t(n+1)}, n = 1, \ldots, N\}$ such that $q_{t(n)t(n+1)} < 0$ for all $n$. This contradicts $GA^*$. Suppose that $A \in M^T$ satisfies $GA$ and, without loss of generality, assume that $a_{Tt} \geq 0$ for all $t$. Now define $(T-1) \times (T-1)$ matrix $A'$ as follows.

$$a'_{ij} = \begin{cases} a_{ij} \text{ if } a_{Tj} > 0 \\ \min\{a_{ij}, a_{iT}\} \text{ if } a_{Tj} = 0 \end{cases}$$

(Note that any diagonal element is zero $(a'_{ii} = a_{ii} = 0)$, so $A' \in M^{T-1}$. If not, then $a'_{ii} = a_{iT} < 0$. But $a_{iT} < 0$ and $a_{Ti} = 0$ violates $GA^*$ ). □

**Lemma 0.5.** *If $A \in M^T$ satisfies $GA^*$, then $A' \in M^{T-1}$ satisfies $GA^*$.*

*Proof.* Suppose not. Then we can find a cycle $\left\{a'_{t(n)t(n+1)}, n = 1, \ldots, N\right\}$ such that $a'_{t(n)t(n+1)} \leq 0$ for all $n$ such that at least one inequality is strict. If every $a'_{ij}$ is $a_{ij}$ in this cycle, then this contradicts the assumption that $A$ satisfies GA *. So suppose that there exists $a'_{ij}$ within this cycle such that $a'_{ij} = a_{iT} \leq 0$ and $a_{Tj} = 0$. Then we can replace $a'_{ij}$ with $a_{iT}$ and $a_{Tj}$ in the original cycle. In this way, we can eliminate such $a'_{ij}$ and guarantee that each element of this cycle is from $A$. Thus again we reach a contradiction. Hence $A'$ must satisfy $GA^*$ □

Now we can complete the proof. By the inductive assumption, there exists $\lambda^t > 0, U^t, t = 1, \ldots, T-1$ such that

$$U^j \leq U^i + \lambda^i a'_{ij} \text{ for all } i, j \in \{1, \ldots, T-1\}$$

By definition of $a'_{ij}$, we have

$$U^j \leq U^i + \lambda^i a_{ij} \text{ for all } i, j \in \{1, \ldots, T-1\}$$

Define $U^T$ and $\lambda^T > 0$ as follows.

$$U^T = \min_{i \in \{1, \ldots, T-1\}} \left\{U^i + \lambda^i a_{iT}\right\}$$

$$\lambda^T = \max\left\{1, \max_{j:a_{Tj} \neq 0} \left\{\frac{U^j - U^T}{a_{TJ}}\right\}\right\}$$

We are done if we can show

$$U^T \leq U^i + \lambda^i a_{iT} \text{ for all } i$$
$$U^j \leq U^T + \lambda^T a_{Tj} \text{ for all } j$$

The first inequalities are satisfied by definition. As for the second inequality, it follows from the definition for any $j$ when $a_{Tj} > 0$. If $a_{Tj} = 0$, then

$$U^j \leq U^i + \lambda^i a'_{ij} = U^i + \lambda^i a_{iT} \text{ for any } i \in \{1, \ldots, T-1\}$$

by definition of $a'_{ij}$. Hence we get

$$U^j \leq \min_{i \in \{1, \ldots, T-1\}} \left\{U^i + \lambda^i a_{iT}\right\}$$
$$= U^T$$
$$= U^T + \lambda^T a_{Tj}$$

This proves that we can find such $\lambda^t > 0, U^t, t = 1, \ldots, T$ for every $A \in M^T$ that satisfies $GA^*$. since $GA^*$ is equivalent to GA by the lemma and GA is satisfied when $D$ satisfies GARP, the proposition is proved. □