Recitations 19

JAKUB PAWELCZAK

MIN II

SPRING 2021

RECITATION 19

me: JAKUB PAWELCZAK

mail: PAWEL042@UMN.EDU

www: JAKUBPAWELCZAK.COM

office hours: MONDAY 5:30-6:30
ZOOM

Today:

MIDTERM

- SOLUTION
- TOPOLOGY
- PERFECT EQ.

$\bar{x}$ 69
s.d. 19
k 5.4

Non Cooperative Game Theory, 8103
Midterm Examination

Aldo Rustichini
University of Minnesota

February 11, 2021

Problem 1. *Fix the number of players, and for each player fix the cardinality of the set of actions. You have a vector $(I, (S^i)_{i \in I})$ of player set and mixed strategies, each mixed strategy set on a given actions set. Consider all the games with this vector of players and mixed strategies, for all different preferences over consequences that satisfy (A1) (A2) and (A3).*

10 1. *Can you define a selection function choosing for each preference over consequences a unique representative of the utility function for each player?*

10 2. *Give a topology (possibly a metric) to the set of utility functions.*

5+15 3. *Define the correspondence associating to each vector of utility function the set of Nash equilibria in the mixed extensions, and prove that this correspondence is upper hemicontinuous.*

Problem 2. *The outcome of an Iterated Eliminations of Strictly Dominated Strategies (IESDS) in a normal form game is unique. In contrast, the outcome of an Iterated Eliminations of Weakly Dominated Strategies (IEWDS) is not necessarily unique. A crucial lemma in the proof of the statement of uniqueness of outcome of IESDS states, informally, that if an action can be eliminated at an earlier stage in an IESDS, then it can also be eliminated at a later stage.*

10 1. *Give a precise statement of the lemma.*

25 2. *Clearly, the statement of the lemma is false when we consider IEWDS. State clearly where in the proof of the lemma use is made of the fact that dominance is strict in the elimination procedure, and why the step fails in the IEWDS case.*

15 **Problem 3.** 1. *Provide an example of a finite normal form game where the set of Nash equilibria (1) is not finite and (2) for no player the set of mixed strategies used in some Nash equilibrium is the entire set of mixed strategies of the player; explicitly, condition (2) is:*

0,0 1,0
1,3 0,1

$$\nexists i \in I \text{ such that } \cup_{\{s=(s^1,\ldots,s^i,\ldots,s^n)\in NE\}} \{s^i\} = \Delta(A^i)$$

In particular you cannot use a game where all actions profiles give the same utility to one of the players.

10 2. *Can you state and prove a general statement on the cardinality if the set of Nash equilibria in normal form games with two players, each player with a two-actions set?*

~~10~~ 1, 2, 3 , $+\infty$

1.10 Iterated elimination

Definition 48 (Iterated elimination of strictly dominated strategies (IESDS)). *An IESDS is a sequence $C_t = (C_t^1, \ldots, C_t^i, \ldots, C_t^n)$ for $t = 0, \ldots, T$, where:*

1. $\forall i, \quad C_0^i = A^i$
2. $\forall i \forall t, \quad C_{t+1}^i \subseteq C_t^i$
3. $\forall i \forall a^i \forall t, \quad a^i \in C_t^i \backslash C_{t+1}^i$ *if and only if* $\exists s^i \in \Delta\left(C_t^i\right)$ *such that*

$$\forall b^{-i} \in C_t^{-i}, u^i\left(s^i, b^{-i}\right) > u^i\left(a^i, b^{-i}\right)$$

Definition 49 (Iterated elimination of weakly dominated strategies (IEWDS)). *An IEWDS is a sequence $C_t = (C_t^1, \ldots, C_t^i, \ldots, C_t^n)$ for $t = 0, \ldots, T$, where:*

1. $\forall i, \quad C_0^i = A^i$
2. $\forall i \forall t, \quad C_{t+1}^i \subseteq C_t^i$
3. $\forall i \forall a^i \forall t, \quad a^i \in C_t^i \backslash C_{t+1}^i$ *if and only if* $\exists s^i \in \Delta\left(C_t^i\right)$ *such that*

$$\forall b^{-i} \in C_t^{-i}, u^i\left(s^i, b^{-i}\right) \geq u^i\left(a^i, b^{-i}\right)$$

$$\exists c^{-i} \in C_t^{-i}, u^i\left(s^i, c^{-i}\right) > u^i\left(a^i, c^{-i}\right)$$

Example 9. *Find all the solutions obtained by IESDS*

		Mr 2 L	Mr 2 R
Mr 1	T	3,0	0,1
	M	0,0	3,1
	B	1,1	1,0

In this game, $A_1^1 = \{T, M, B\}$ and $A_1^2 = \{L, R\}$. No (pure) strategy dominates any other (pure) strategy for both players. However, the mixed strategy $s_1(T) = s_1(M) = \frac{1}{2}$ and $s_1(B) = 0$ strictly dominates B since $\forall q \in [0, 1]$

$$u_1(s_1, q) = 3q\frac{1}{2} + 3(1-q)\frac{1}{2} = \frac{3}{2} > 1 = u_1(B, q)$$

So B is eliminated from player 1's set of actions. Given that player 2 knows this, $s^2 = (0, 1)DL$. Thus L is eliminated from player 2's action set. Finally, given that player 2 will only play R, M dominates T. Thus player 1 will eliminate T as well. This leads to a final action set $C_T = \{M\} \times \{R\}$.

Since each player only has one action now, no more actions can be eliminated. This is referred to as a **complete** IESDA. Note that we have need to allow dominance by mixed strategies for this to work; neither T nor M alone strictly dominates B.

Example 10. *Find all the solutions obtained by IESDS*

		Mr 2		
		L	C	R
	T	4,3	5,1	6,2
Mr 1	M	2,1	8,4	3,6
	B	3,0	9,5	2,6

One way to organize our work is put it in table. Observe that

$C_1^0 = \{T, M, B\}$	$C_2^0 = \{L, C, R\}$
$C_1^1 = \{T, M, B\}$	$C_2^1 = \{L, R\}$
$C_1^2 = \{T\}$	$C_2^2 = \{L, R\}$
$C_1^3 = \{T\}$	$C_2^3 = \{L\}$
$\dots$	$\dots$
$C_1^\infty = \{T\}$	$C_2^\infty = \{L\}$

So $\{(T, L)\}$ is our final result of IESDS.

Example 11. *Find all the solutions obtained by IESDS and IEWDS*

		Mr 2		
		L	C	R
	T	1,2	2,3	0,3
Mr 1	M	2,2	2,1	3,2
	B	2,1	0,0	1,0

IESDS: nothing to rule out in pure strategies

IEWDS: For Mr1 M weakly dominates T and B and For Mr2 R weakly dominates C. Consider 3 procedures

- Procedure 1: Mr1 eliminates T
- Procedure 2: Mr1 eliminates B
- Procedure 3: Mr2 eliminates R

For example in Procedure 3 we can have following solution. Mr1 can eliminate T or B. If he eliminates T , Mr2 can eliminate R or B. If we eliminated C, T and R then Mr1 eliminates B and we end up in (M,R).
If we eliminated C, T and B then Mr2 can not eliminate and we end up in (M,(l,r)).

In total we have 4 outcomes

We say IESDS is complete if no elimination is possible in the C_T game

Observe that complete IESDS results in a unique outcome which we prove below. It is not true for IEWDS. Let's ilustrate it with example.

Example 12.

Mr 2

Mr 1		L	R
	T	1,2	2,2
	B	1,2	1,1

- Procedure 1: T weakly dominates B:eliminate B then Mr2 is indifferent between L and R so we get $((1,0) \times (q, 1-q))$
- Procedure 2: L weakly dominates R:eliminate R then Mr1 is indifferent between T and B so we get $((p, 1-p) \times (1,0))$

Furthermore IEWDS can eliminate a NE

Example 13.

Mr 2

Mr 1		L	R
	T	1,1	0,0
	B	0,0	0,0

Observe that $\{(T,L),(B,R))\}$ are pure NE.
Let's do IEWDS for this game: For Mr2 L weakly dominates R so elimiate R. For Mr1 T weakly dominates B so elimiate B so we elimated our NE.

Theorem 12. ***For any normal-form game*** $\left\langle I, (A^i)_{i\in I}, (u^i)_{i\in I} \right\rangle$***, the outcome of a complete IESDS is unique.***

Proof. Let $(C_t^i)^i_{i\in I, t=0,1,\ldots,T}$ be a complete IESDS. We show by induction that, $\forall i \in I$, if a strategy c^i cannot be eliminated at C_T^i, then it cannot be eliminated at C_t^i for any $t \in \{0,1,\ldots,T-1\}$. Fix

$i \in I$ and take any $c^i \in C_T^i$ Basis Step: By definition.

Induction Step: Suppose $c^i \in C_{t+1}^i, t \in \{0, 1, \ldots, T-1\}$, and that it cannot be eliminated at this stage. We want to show that c^i cannot be eliminated earlier. WLOG we can consider $C_t^i = C_{t+1}^i \cup \left\{d^i\right\}$. Suppose for $\lightning$that c^i can be eliminated at C_t^i. Then

$$\exists \hat{s}^i \in \Delta\left(C_t^i\right) \quad \text{s.t.} \quad \forall c^{-i} \in C_t^{-i}, \quad u^i\left(\hat{s}^i, c^{-i}\right) > u^i\left(c^i, c^{-i}\right)$$

Define $s^i \in \Delta\left(C_{t+1}^i\right)$ as follows. If $a^i \in C_{t+1}^i$, then let $s^i\left(a^i\right) = \hat{s}^i\left(a^i\right)$. If $d^i \in C_t^i \backslash C_{t+1}^i$, then d^i must have been eliminated at t. Then $\exists r^i \in \Delta\left(C_t^i\right)$ that strictly dominates d^i. Notice that since $r^i\left(d^i\right) = 0$, we have $\operatorname{supp}\left(r^i\right) \subseteq C_t^i$. Then define

$$s^i\left(a^i\right) \equiv \hat{s}^i\left(a^i\right) + r^i\left(a^i\right)\hat{s}^i\left(d^i\right) \quad \forall a^i \in C_{t+1}^i$$

Clearly $s^i\left(a^i\right) \geq 0 \forall a^i \in C_{t+1}^i$. Moreover,

$$\begin{aligned}
\sum_{a^i \in C_{t+1}^i} s^i\left(a^i\right) &= \sum_{a^i \in C_{t+1}^i} \hat{s}^i\left(a^i\right) + r^i\left(a^i\right)\hat{s}^i\left(d^i\right) \\
&= \hat{s}^i\left(C_{t+1}^i\right) + \hat{s}^i\left(d^i\right) \sum_{a^i \in C_{t+1}^i} r^i\left(a^i\right) \\
&= \hat{s}^i\left(C_{t+1}^i\right) + \hat{s}^i\left(d^i\right) \\
&= 1 - \hat{s}^i\left(d^i\right) + \hat{s}^i\left(d^i\right) \\
&= 1
\end{aligned}$$

where the second line follows since $\operatorname{supp}\left(r^i\right) \subseteq C_{t+1}^i$. So we have that $s^i \in \Delta\left(C_{t+1}^i\right)$. Now, we have that

$$\begin{aligned}
&\forall c^{-i} \in C_t^{-i}, \quad u^i\left(\hat{s}^i, c^{-i}\right) > u^i\left(c^i, c^{-i}\right) \\
\Longrightarrow &\forall c^{-i} \in C_{t+1}^{-i}, \quad u^i\left(\hat{s}^i, c^{-i}\right) > u^i\left(c^i, c^{-i}\right) \\
\Rightarrow &\forall c^{-i} \in C_{t+1}^{-i}, \quad \sum_{a^i \in C_{t+1}^i} \hat{s}^i\left(a^i\right) u^i\left(a^i, c^{-i}\right) > u^i\left(c^i, c^{-i}\right) \\
\Longrightarrow &\forall c^{-i} \in C_{t+1}^{-i}, \quad \sum_{a^i \in C_{t+1}^i} \left(\hat{s}^i\left(a^i\right) + r^i\left(a^i\right)\hat{s}^i\left(d^i\right)\right) u^i\left(a^i, c^{-i}\right) > u^i\left(c^i, c^{-i}\right) \\
\Longrightarrow &\forall c^{-i} \in C_{t+1}^{-i}, \quad \sum_{a^i \in C_{t+1}^i} s^i\left(a^i\right) u^i\left(a^i, c^{-i}\right) > u^i\left(c^i, c^{-i}\right) \\
\Longrightarrow &\forall c^{-i} \in C_{t+1}^{-i}, \quad u^i\left(s^i, c^{-i}\right) > u^i\left(c^i, c^{-i}\right)
\end{aligned}$$

where the second line holds since $C_{t+1}^i \subseteq C_t^i$ and the third since d^i is strictly dominated. Then s^2 strictly dominates c^i in C_{t+1}^i, which is a $\lightning$. Hence c^i cannot be eliminated at C_t^i. By induction, c^i cannot be eliminated at any $\left(C_t^i\right)_{t=0,1,\ldots,T}$. Since $i \in I$ has been arbitrary, IESDS is unique. $\square$

The non-empty set resulting from IESDS is called the set of *sophisticated equilibria*. Check H. Moulin 'Dominance Solvable Voting Schemes', Econometrica 47, 1979.

Example 14. *This game is called Guess the average*

- *Each player $i \in I$ picks simultaneously an integer x_i between 1 and 999. Hence, $A_i = \{1, \ldots, 999\}$.*

3.1 Perfect Equilibria

Example 17. *Let $I = \{1,2\}$ and consider the game G defined by*

		Mr 2	
		q	$1-q$
		L	R
p	Mr 1 T	1,1	0,0
$1-p$	B	0.0	x.y

where $1 > x > 0$ and $y > 0$. Suppose that $s^1 = (p, 1-p)$ and $s^2 = (q, 1-q)$ so that $s = ((p,1-p),(q,1-q))$. Then we can view each player's best response as a function of the other player's mixed strategy. In particular, if player 2 plays L, his expected utility is p. If he plays R it is $(1-p)y$. So his best response depends on the value of p. Similarly for player 1. Then G has three NE.

$$NE = \left\{((1,0),(1,0)),((0,1),(0,1)),\left(\left(\frac{y}{1+y}, 1-\frac{y}{1+y}\right),\left(\frac{x}{1+x}, 1-\frac{x}{1+x}\right)\right)\right\}$$

Mr 1

Mr 2

0 $x, y > 0$

$T > B \iff q > x(1-q) \quad q > \frac{x}{1+x}$

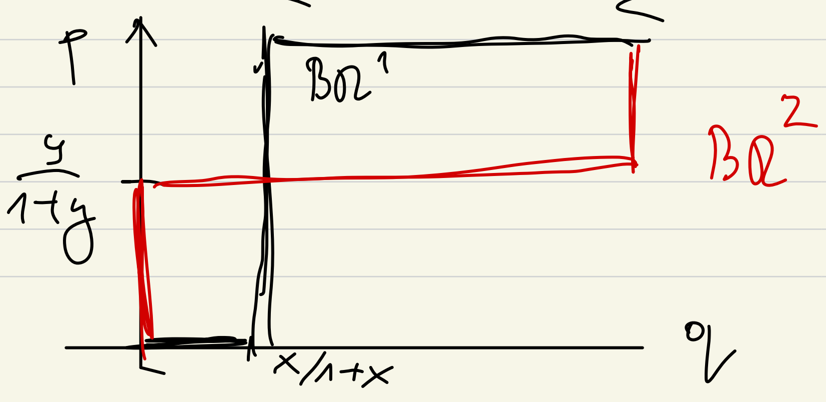
$\sim \qquad = \qquad =$

$< \qquad < \qquad <$

$L > R \iff p > y(1-p) \quad p > \frac{y}{1+y}$

$= \qquad = \qquad =$

$< \qquad < \qquad <$

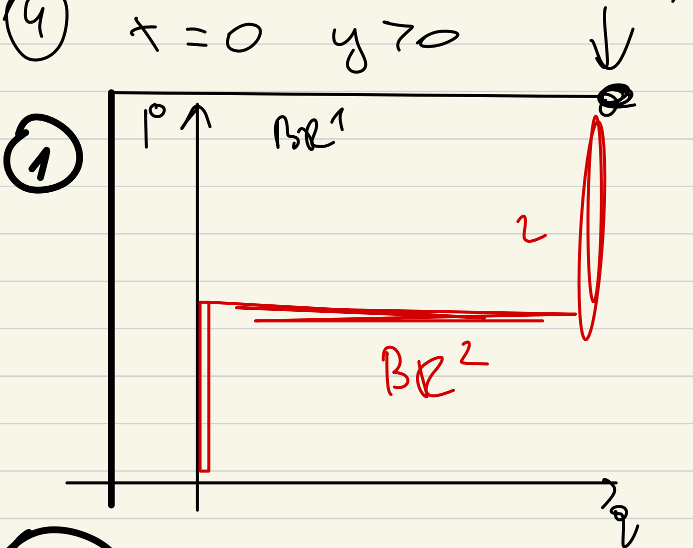


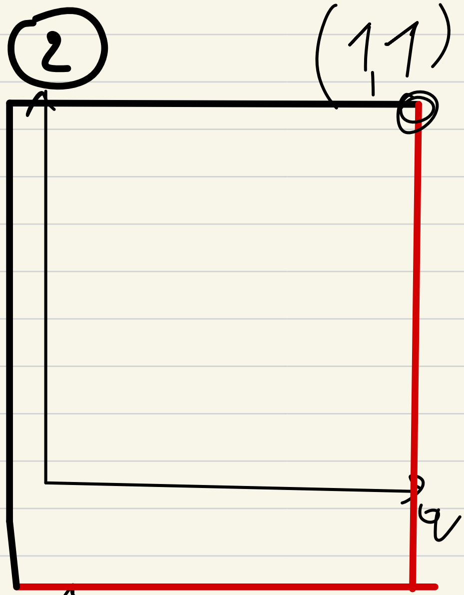
① $x < 0 \quad y > 0$

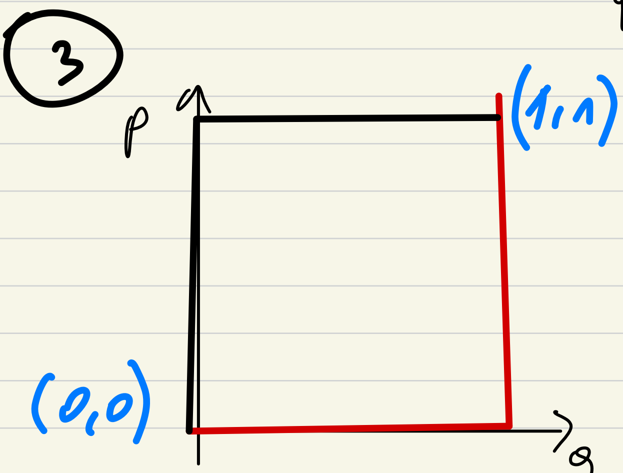
② $x, y < 0$

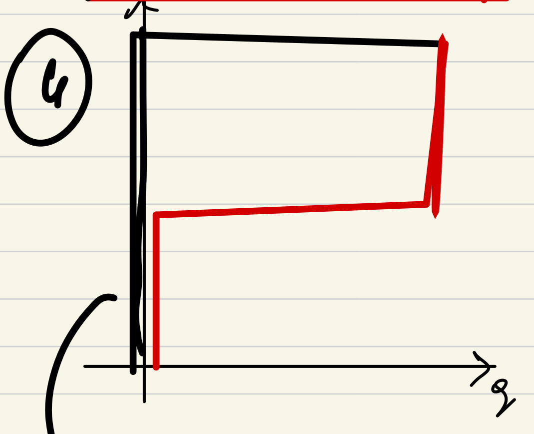
③ $x = y = 0$

④ $x = 0 \quad y > 0$









Q3.1.

- $(1,1)$ is NE always
- $(0,0)$ it is not robust eq.

	L	R
T	1,1	0,0
B	0,0	0,0

$$\underline{\underline{(0,0)}} \longrightarrow \Big(\underset{\substack{\downarrow \\ B}}{(0,1)}, \underset{\substack{\downarrow \\ R}}{(0,1)} \Big)$$

(i) it gives $>$ prob (1!) to WD strategy B

(ii) by perturbating of $(x,y) \to u$
we saw that $(0,1), (0,1)$ disappears from Eq.

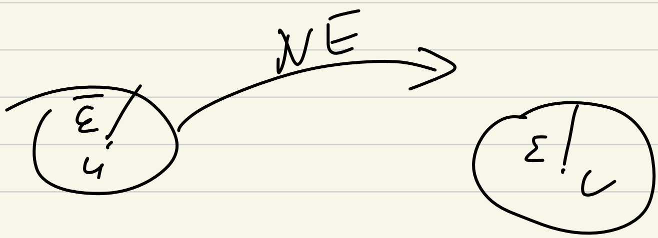
(iii) we force players to make mistakes

Def. Given $s_u \in NE(I, S^i, u^i)$

s_u is u-robust if

$\forall \delta \; \exists \bar{\varepsilon} > 0 \; \forall v \; \|v - u\| < \varepsilon \; \forall \varepsilon < \bar{\varepsilon}$

$\exists \, s_v : \|s_v - s_u\| < \delta$



Ex. $(0,0)$ is it utility robust? -

No. $\forall x, y < 0 \Rightarrow NE$ is $(1,1)$

Def. Perturbations.

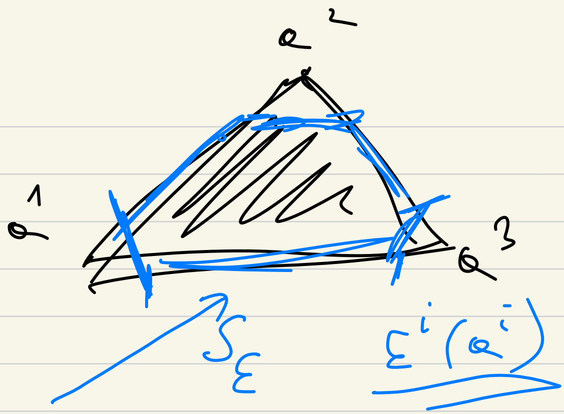
$\varepsilon = (\varepsilon^i)_{i \in I} \qquad \varepsilon^i = (\varepsilon^i(a^i))_{a^i \in A^i}$

$\forall i \; \forall a^i \in A^i \quad \varepsilon^i(a) > 0$

$$\sum \varepsilon^i(a) < 1$$

It is not strategy itself.

$S^i = \Delta A^i$



Def. $S_\varepsilon^i := \{ s^i \in S^i \mid \forall a^i \in A^i \; s^i(a^i) \geq \varepsilon^i(a^i) \}$

Def. NE $\Gamma = \{ I, \{S^i\}_{i \in I}, \{u^i\}_{i \in I} \}$

Def NE of perturbated game

$$\Gamma_\varepsilon = \{ I, \{S_{\underline{\varepsilon}}^i\}_{i \in I}, u^i \}$$

$\forall i \; \forall t^i \in S_\varepsilon^i$:

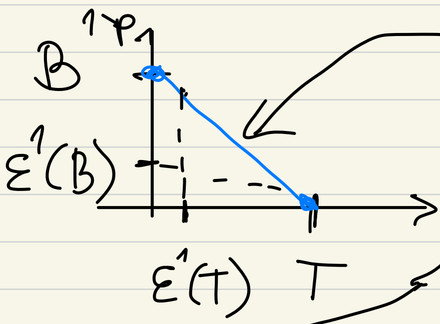
$$u^i(\underline{s}^i, s^{-i}) \geq u^i(t^i, s^{-i})$$

$\Longleftarrow \quad s \in BR(s)$

- Perturbations
- Derive BR Γ_ε
- $NE(\Gamma_\varepsilon)$
- take limit $\varepsilon \to 0$

Ex.
$$\begin{matrix} 1,1 & 0,0 \\ 0,0 & 0,0 \end{matrix}$$

Mr 1



Mr 1 cannot pick (0) or (1)

$p \in [\varepsilon^1(B), \varepsilon^1(T)]$

$$S_{\varepsilon^1} = \{ s^1 \in S^1 : \; s^1(T) \geq \varepsilon^1(T)$$
$$s^1(B) \geq \varepsilon^1(B)$$
$$\varepsilon^1(T), \varepsilon^1(B) > 0 \qquad \varepsilon^1(T) + \varepsilon^1(B) < 1 \}$$

$NE(\Gamma_\varepsilon)$:
- Perturbations
- S_ε:

Remember

$$p \geq \varepsilon^1(T)$$
$$1-p \geq \varepsilon^1(B)$$
$$q \geq \varepsilon^2(L)$$
$$1-q \geq \varepsilon^2(R)$$

		q	$1-q$
		L	R
p	T	1, 1	0, 0
$1-p$	B	0, 0	0, 0

BR^1 B – gives 0 anyways

T $q \cdot 1 + (1-q) \cdot 0 = q$

$q \geq \varepsilon^2(L) > 0$

$T > B$ usually we picked $p = 1$

However $1 - \varepsilon^1(B) \geq p$

So $\underline{p^*} = 1 - \varepsilon^1(B)$

BR^2 R gets 0 always

L gets $1 \cdot p + 0 \cdot (1-p) = p$

$L > R$ Mr 2 wants to $q = 1$

however given restriction $1 - q \geq \varepsilon^2(R)$

$\underline{q^*} = 1 - \varepsilon^2(R)$.

Observe that we here $\exists! \; NE(\Gamma_\varepsilon) = \sigma^*$

$(p^*, 1-p^*)\,(q^*, 1-q^*) =$

$= (1-\varepsilon^1(B), \varepsilon^1(B)), (1-\varepsilon^2(R), \varepsilon^2(R))$

Define $\varepsilon = (\varepsilon^1(T), \varepsilon^1(B), \varepsilon^2(L), \varepsilon^2(R))$

$\varepsilon^i(a^i) > 0 \qquad \sum_{a^i} \varepsilon^i(a^i) < 1$

take $\varepsilon \to 0$. In particular

$\varepsilon^1(B), \varepsilon^2(R) \to 0$

$\overset{\shortparallel}{1/n} \qquad 1/n \qquad \frac{1}{n} \xrightarrow{n\to\infty} 0$

$NE(\Gamma_\varepsilon) \xrightarrow{\varepsilon \to 0}$?

$1-\varepsilon^1(B) \xrightarrow{\varepsilon\to 0} \underline{1}, \qquad \varepsilon^1(B) \xrightarrow{\varepsilon\to 0} 0$

$1-\varepsilon^2(R) \xrightarrow{\varepsilon\to 0} 1, \qquad \varepsilon^2(R) \xrightarrow{\varepsilon\to 0} 0$

So

$NE\,\Gamma_\varepsilon \xrightarrow{\varepsilon\to 0} \underline{((1,0),(1,0))} \in NE(\Gamma)$

PERFECT!

(1) I found sequence of ε | $\varepsilon \to 0$

(2) $NE(\Gamma_\varepsilon) \to \underbrace{NE\,\Gamma}$

(3) such $NE\,\Gamma$ I call

PERFECT EQUILIBRIUM

- $(1,0)\ (1,0)$ is a PE
- $(0,1)\ (0,1)$ is it PE ?

NO ! Bc I did not put any constraints on ε

Def. (PE) $s \in S$ is Perfect Equilibrium

$\Leftrightarrow$ $\exists\, \varepsilon_m, s_m$

$\varepsilon_m \to 0$ $\quad s_m \in NE(\Gamma_{\varepsilon_m})$

$s_m \to s$ $\quad s \in NE(\Gamma)$

Lemma. $s \in S$ is PE $\Rightarrow$ $s \in S$ is NE.

Thm. If $s \in NE$ & fully mixed $\Rightarrow$ PE

Thm. $|I| = 2$. If $s \in NE$ & fully mixed $\Longleftrightarrow$ s is PE.

3 Equilibrium Refinements

3.1 Perfect Equilibria

Example 17. *Let $I = \{1,2\}$ and consider the game G defined by*

		Mr 2 L	Mr 2 R
Mr 1	T	1,1	0,0
	B	0.0	x.y

where $1 > x > 0$ and $y > 0$. Suppose that $s^1 = (p, 1-p)$ and $s^2 = (q, 1-q)$ so that $s = ((p,1-p),(q,1-q))$. Then we can view each player's best response as a function of the other player's mixed strategy. In particular, if player 2 plays L, his expected utility is p. If he plays R it is $(1-p)y$. So his best response depends on the value of p. Similarly for player 1. Then G has three NE.

$$NE = \left\{((1,0),(1,0)),((0,1),(0,1)),\left(\left(\frac{y}{1+y},1-\frac{y}{1+y}\right),\left(\frac{x}{1+x},1-\frac{x}{1+x}\right)\right)\right\}$$

Typically $|NE|$ is odd. However, not in general. For instance, in G let $x = y = 0$. Compute equilibrium. Show it is strange in that it gives positive probability to a weakly dominated strategy. Motivate perfect equilibria/perterbations by show that if player 2 plays L with some small but positive probability, this strange equilibrium goes away.

Definition 80 (Utility robust NE). *Given a* NEs_u *of* $\left(I, S^i, u^i\right)$, s_u *for* u *is utility robust if* $\forall\delta\exists\bar{\epsilon} > 0$ *such that* $\forall v$ *such that* $\|v-u\| < \epsilon$ *where* $\epsilon < \bar{\epsilon}$, $\exists s_v$ *such that* $\|s_v - s_u\| < \delta$

Definition 81 (Perturbation). *A perturbation is* $\epsilon = \left(\epsilon^i\right)_{i\in I}$, *where* $\forall i \in I \epsilon^i = \left(\epsilon^i\left(a^i\right)\right)_{a^i\in A^i}$, *such that:*

$$\forall i \in I \quad \forall a^i \in A^i, \quad \epsilon^i\left(a^i\right) > 0 \quad \wedge \quad \sum_{a^i\in A^i} \epsilon^i\left(a^i\right) < 1$$

Perturbation is not a mixed strategy.

Definition 82 (Perturbed strategy set). *The perturbed strategy set for player i is*

$$S^i_{\epsilon^i} \equiv \left\{s^i \in S^i \mid \forall a^i \in A^i, s^i\left(a^i\right) \geq \epsilon^i\left(a^i\right)\right\}$$

The perturbed strategy set for all players is $S_\epsilon \equiv \prod_{i\in I} S^i_\epsilon$

Definition 83. *NE of ϵ -perturbed game $s \in S_\epsilon$ is a* NE *of the ϵ -perturbed game if*

$$\forall i \in I, \forall t^i \in S^i_{\epsilon i}, \quad u^i\left(s^i, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right)$$

[A NE of the ϵ -perturbed game is $\hat{s} \in S_\epsilon$ such that $\forall i \in I\hat{s}^i \in BR^i_{S^i_\epsilon}(\hat{s})$ or $NE(\Gamma_\varepsilon) := \{s \in S_\varepsilon : s \in BR_\varepsilon(s)\}$

Definition 84. *Perfect equilibrium Let* $\left(I, \left(S^i, u^i\right)_{i\in I}\right)$ *be a NFG. Then* $s \in S$ *is a* PE *if* $\exists \{\epsilon_m\}_{m\in\mathbb{N}}, \{s_m\}_{m\in\mathbb{N}}$ *such that* $\epsilon_m \to 0, s_m \to s$, *and* s_m *is a* NE *of the* ϵ_m *-perturbed game (for each m)*

[$s \in S$ is PE if it is the limit of a sequence of NE of some ϵ -perturbed game, where $\epsilon \to 0$.

Theorem 28. ***The set of*** PE ***is nonempty***

Proof. As proved in Theorem 2.2, for any finite game the set of NE is nonempty. It follows immediately that for any ϵ -perturbation of a finite game, the set of NE is nonempty. Then, for any sequence of perturbations $\epsilon_n \to 0$, there exists $s_n \in S_{\epsilon_n}$ such that s_n is a NE of the ϵ_n -perturbed game. Then s_n is a sequence in S, and since S is compact, there exists a convergent subsequence $s_{n_k} \to s \in S$. Then s is a perfect equilibrium by definition, and thus the set of PE is nonempty. □

Theorem 29. ***If*** $s \in S$ ***is a PE, then it is also a NE.***

Proof. Let $s \in S$ be a PE. Then $\exists \{\epsilon_m\}_{m \in \mathbb{N}}, \{s_m\}_{m \in \mathbb{N}}$ such that $\epsilon_m \to 0, s_m \to s$, and $\forall m \in \mathbb{N}, s_m$ is a NE of the ϵ_m -perturbed game. Take any $i \in I$ and any $t^i \in S^i$. Since $\epsilon_m \to 0$, it follows that $\epsilon^i_m \to 0$, and thus there exists a sequence $t^i_m \in S^i_{\epsilon^i_i}$ such that $t^i_m \to t^i$. Take such a sequence. Then, since s_m is a NE of the ϵ_m -perturbed game, it follows that

$$u^i\left(s^i_m, s^{-i}_m\right) \geq u^i\left(t^i_m, s^{-i}_m\right) \quad \forall m \in \mathbb{N}$$

Since $u^i(\cdot)$ is continuous $\forall i \in I$, then

$$\begin{aligned} &\lim u^i\left(s^i_m, s^{-i}_m\right) \geq \lim u^i\left(t^i_m, s^{-i}_m\right) \\ &\Longrightarrow u^i\left(s^i, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right) \end{aligned}$$

Since $t^i \in S^i$ was taken arbitrarily, $s^i \in BR^i\left(s^{-i}\right)$. Since $i \in I$ was taken arbitrarily, $s \in BR(s)$, so s is a NE □

Theorem 30. ***If*** $s \in S$ ***is a fully mixed*** NE, ***then it is also a*** PE.

Proof. Let $s \in S$ be a fully mixed NE for some finite NFG, i.e. $\forall i \in I, \forall a^i \in A^i, s^i\left(a^i\right) > 0$. From this, note there exists

$$\bar{s}^i \equiv \min_{a^i \in A^i} s^i\left(a^i\right) \forall i \in I \text{ and } \bar{s} \equiv \min_{i \in I} \bar{s}^i$$

and that $\bar{s} > 0$. It follows that, for any sequence of perturbations $\{\epsilon_n\}_{n \in \mathbb{N}}$ such that $\epsilon_n \to 0, \exists N \in \mathbb{N}$ such that, $\forall m \geq N$

$$\forall i \in I \forall a^i \in A^i, \quad \epsilon^i_m\left(a^i\right) < \bar{s}$$

so $\forall m \geq N, s \in S_{\epsilon_m}$. Now recall that since s is a NE of the original game,

$$\forall i \in I, \forall t^i \in S^i, \quad u^i\left(s^i, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right)$$

Note that $S^i_{\epsilon^i_m} \subseteq S^i$, so since $\forall m \geq N, s \in S_{\epsilon_m}$, we know that $\forall m \geq N, s$ is a NE of the ϵ_m -perturbed game. Now take a sequence $\{s_m\}$ such that $s_m = s \forall m \in \mathbb{N}$ and construct a new sequence of perturbations $\{\hat{\epsilon}_m\} = \{\epsilon_m\}_{m \geq N}$. Then s is a PE by definition. □

Theorem 31. $s_\varepsilon \in NE(\Gamma_\varepsilon)$ ***if, and only if,***

$$\forall i \in I, a, b \in A^i, \quad u^i\left(a, s_\varepsilon^{-i}\right) < u^i\left(b, s_\varepsilon^{-i}\right) \Longrightarrow s_\varepsilon^i(a) = \varepsilon^i(a)$$

Proof. The argument is identical to that provided in the proof of theorem 7 □

Definition 85. *Let $\eta > 0$. A mixed-strategy profile $s \in S$ is $\eta-$ perfect if, and only if,*

1. *it is fully mixed, i.e.* $\forall i \in I, a \in A^i \quad s^i(a) > 0$,
2. $\forall i \in I, a^i, b^i \in A^i \quad u^i\left(a^i, s^{-i}\right) < u^i\left(b^i, s^{-i}\right) \Longrightarrow s^i\left(a^i\right) \leq \eta$

Definition 86. *Let $\eta > 0$. A mixed-strategy profile $s \in S$ is $\eta-$ proper if, and only if,*

1. *it is fully mixed, and*
2. $\forall i \in I, a^i, b^i \in A^i \quad u^i\left(a^i, s^{-i}\right) < u^i\left(b^i, s^{-i}\right) \Longrightarrow s^i\left(a^i\right) \leq \eta s^i\left(b^i\right)$

Theorem 32. ***If a strategy profile $s \in S$ is η - proper, then s is η - perfect.***

Proof. Let $s \in S$ be η -proper, $\eta > 0$. Then s is fully mixed and $\forall i \in I, a^i, b^i \in A^i$

$$u^i\left(a^i, s^{-i}\right) < u^i\left(b^i, s^{-i}\right) \Longrightarrow s^i\left(a^i\right) \leq \eta s^i\left(b^i\right) \Longrightarrow s^i\left(a^i\right) \leq \eta$$

since $s^i\left(b^i\right) \in (0,1)$. Hence, s is η -perfect. □

Example 18. *Consider the following game:*

		Mr 2 L	Mr 2 R
Mr 1	T	1,10	0,0
	M	1,2	1,2
	B	3,-10	0,0

- *Find all the Nash equilibria*
- *Find all the perfect equilibria*
- *Take a perfect equilibrium s of any normal form finite game, and remove the action of a player i which is not a best response to the strategy of the others, s^i. Is the restriction of the strategy profile s to the new game a perfect equilibrium of the new game?*

Definition 87. *A strategy profile $s \in S$ is a perfect-2 equilibrium if, and only if, $\exists(\eta_n) \in \mathbb{R}_{++}^\infty$ with $\eta_n \to 0$ and a corresponding sequence of mixed strategy profiles $\left(s_{\eta_n}\right) \in S^\infty$ such that such that s_{η_n} is η_n -perfect for all $n \in \mathbb{N}$ and $s_{\eta_n} \to s$*

Definition 88. *A strategy profile $s \in S$ is a proper equilibrium if, and only if, $\exists(\eta_n) \in \mathbb{R}_{++}^\infty$ with $\eta_n \to 0$ and a corresponding sequence of mixed strategy profiles $\left(s_{\eta_n}\right) \in S^\infty$ such that s_{η_n} is η_n -proper for all $n \in \mathbb{N}$ and $s_{\eta_n} \to s$*

Theorem 33. ***Let*** $s \in S$ ***be a strategy profile. Then the following are equivalent:***

1. s ***is a perfect equilibrium;***

2. s ***is a perfect-2 equilibrium;***

3. $\exists\, (s_n) \in S^\infty$ ***s.t. (i)*** s_n ***is fully mixed, (ii)*** $s_n \to s$ ***and (iii)*** $\forall i, n \quad s^i \in BR^i_{s^i}\left(s_n^{-i}\right)$

Theorem 34. ***If a mixed-strategy profile*** $s \in S$ ***is a perfect equilibrium of*** Γ***, then it is a Nash equilibrium of*** Γ***, i.e.*** $s \in NE(\Gamma)$***.***

Proof. Since s is a perfect equilibrium, by previous theorem there exists a sequence $(s_n) \in S^\infty$ such that s_n is fully mixed for each $n \in \mathbb{N}, s_n \to s$, and $s^i \in BR^i\left(s^i, s_n^{-i}\right) \forall i \in I$.
Fix $i \in I$ arbitrarily and define a sequence $(s_m) \in S^\infty$ by $s^i_m = s^i$ for all $m \in \mathbb{N}$ and $s_m^{-i} = s_n^i$ for all $m = n$. Then $s_m \to s$ and $s^i_m \in BR^i\,(s_m)\, \forall m \in \mathbb{N}$.
Since $BR^i(\cdot)$ is uhc by Theorem 6 there exists a strictly increasing sequence $(m_k) \in \mathbb{N}^\infty$ such that $s^i_{m_k} \to r^i \in BR^i(s)$. But since every subsequence of $\left(s^i_m\right)$ is the stationary sequence $\left(s^i\right)$, we necessarily have $r^i = s^i$. Hence, $s^i \in BR^i(s)$. Since i has been arbitrary, $s \in NE(\Gamma)$. □