Recitation 5

[Definitions used today]

- Topkis theorem, Supermodularity, Increasing Differences
- Recursive and dynamically consistent family of utility function, time separablity, ICC axiom

Question 1

Consider the following utility functions

a $u(c_1, c_2, c_3) = \min\{2c_1 + c_2 + c_3, c_1 + c_2 + 2c_3\}$

b $u(c_1, c_2, c_3) = c_1 + \sqrt{c_2} + \sqrt{c_3}$

1. Show that (a) does not have state-separable representation
2. Show that (b) does not have expected utility representation
3. Find $\pi \in \Delta \subset \mathbb{R}^3$ s.t. $u(c_1, c_2, c_3)$ is strictly risk averse with respect to π
4. Show that there is no $\pi\Delta \subset \mathbb{R}^3$ s.t. $u(c_1, c_2, c_3)$ is strictly risk averse with respect to π

Question 2 [Properties of state separable u]

a Prove that every recursive family of utility functions $\{U_t\}$ is dynamically consistent if the aggregator function $G(\cdot, \cdot)$ is strictly increasing in continuation

b $S \geq 3$ and $\succeq$ increasing and continuous. Prove that $\succeq$ has state-separable representation then ICC holds.

c For $S = 2$ all increasing functions obey ICC. Show that for $u(c_1, c_2) = c_1\sqrt{c_2} + c_1 + c_2$ it does not have state separable utility function

Question 3 [Topkis theorem]

If S is a lattice, f is supermodular in x for fixed t, and f has nondecreasing differences in $(x; t)$, then $\varphi^*(t) = \arg\max_{x \in S} f(x, t)$ is monotone nondecreasing in t.

Question 4 254 [I.1 Spring 2018 majors]

Consider the problem of finding a Pareto optimal allocation of aggregate resources $\omega \in \mathbb{R}^n_+$ in an economy with two agents:

$$\max_x \mu_1 u_1(x) + \mu_2 u_2(\omega - x)$$
$$\text{subject to } 0 \leq x \leq \omega$$

where $u_i : \mathbb{R}^n_+ \to \mathbb{R}$ are agents' utility functions (assumed continuous) and $\mu_i > 0$ are welfare weights for $i = 1, 2$. Let $x^*(\mu_1, \mu_2)$ be the set of solutions.

a State a definition of utility function u_i being supermodular. Show that if u_i is supermodular, then $u_i(\omega - x)$ is supermodular in x

b Show that, if u_1 and u_2 are strictly increasing and supermodular in x then $x^*(\mu_1, \mu_2)$ is non-decreasing in μ_1. You may assume that $x^*(\mu)$ is single-valued. Is $x^*(\mu_1, \mu_2)$ non-increasing in μ_2? Justify your answer. If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.

c Under what conditions on u_1 and u_2 is the solution $x^*(\mu_1, \mu_2)$ unique. Justify your answer.