Math Appendix for Microeconomics

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1 Notation

1.1 Logical

Meaning	*Command*	*Notation*
Not	`\neg`	$\neg$
There exists	`\exists`	$\exists$
For all	`\forall`	$\forall$
Implies	`\implies`	$\Rightarrow$
Equivalent	`\iff`	$\iff$
And	`\land`	$\land$
Or	`\lor`	$\lor$
Defined as	`:=`	$:=$
Logical equivalence	`\equiv`	$\equiv$
Therefore	`\therefore`	$\therefore$
Because	`\because`	$\because$

1.2 Greek letters

Command	*Notation*	*Command*	*Notation*
`\alpha`	α	`\tau`	τ
`\beta`	β	`\theta`	θ
`\chi`	χ	`\upsilon`	υ
`\delta`	δ	`\xi`	ξ
`\epsilon`	ϵ	`\zeta`	ζ
`\varepsilon`	ε	`\Delta`	Δ
`\eta`	η	`\Gamma`	Γ
`\gamma`	γ	`\Lambda`	Λ
`\iota`	ι	`\Omega`	Ω
`\kappa`	κ	`\Phi`	Φ
`\lambda`	λ	`\Pi`	Π
`\mu`	μ	`\Psi`	Ψ
`\nu`	ν	`\Sigma`	Σ
`\omega`	ω	`\Theta`	Θ
`\phi`	ϕ	`\Upsilon`	Υ
`\varphi`	φ	`\Xi`	Ξ
`\pi`	π	`\aleph`	$\aleph$
`\psi`	ψ	`\beth`	$\beth$
`\rho`	ρ	`\daleth`	$\daleth$
`\sigma`	σ	`\gimel`	$\gimel$

1.3 General

- $\mathbb{R}^n := \big\{x = (x_1, \ldots, x_i \ldots, x_n) : x^i \in \mathbb{R}, \quad \forall i = 1, \ldots, n\big\}$

$$\begin{gathered}\text{For } x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^n \text{ we denote}\\ x \geq y \Longleftrightarrow x_i \geq y_i, \quad \forall i = 1, \ldots, n\\ x > y \Longleftrightarrow x \geq y \quad and \quad x \neq y\\ x \gg y \Longleftrightarrow x_i > y_i, \forall i = 1, \ldots, n\end{gathered}$$

- $\forall x \in A$ means: For all x in Y
- $x \cdot y$ or $< x, y >$ denotes the scalar product of x and y so $x \cdot y = \sum_{i=1}^{n} x_i y_i$
- A is a matrix with m rows and n columns and B is a matrix with n rows and l columns, AB denotes the matrix product of A and B.
- H is a $n \times n$ matrix, tr (H) denotes the trace of H and $det(H)$ denotes the determinant of H.
- $x \in \mathbb{R}^n$ is treated as a row matrix so $1 \times n$.
- x^T denotes the transpose of $x \in \mathbb{R}^n, x^T$ is treated as a column matrix so $n \times 1$.
- $f : X \to \mathbb{R}$ so f is a function from open set $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$

2 Binary relations

Definition 1 *Assumptioons about binary relations ($R :\succeq, P :\succ, I :\sim$)*

a *reflexive :* $\forall_a \quad aRa$

b *irreflexive:* $\forall_a \quad \neq (aRa)$

c *symmetric:* $\forall_{a,b} \quad aRb \iff bRa$

d *asymmetric:* $\forall_{a,b} \quad aRb \iff \neg(bRa)$

e *antisymmetric:* $\forall_{a,b} \quad aRb \wedge bRa \Rightarrow a = b$

f *complete:* $\forall_{a,b} \quad aRb \vee bRa$

g *transitive* $\forall_{a,b,c} \quad aRb \wedge bRc \Rightarrow aRc$

h *negative transitive* $\forall_{a,b,c} \quad \neg(aRb) \wedge \neg(bRc) \Rightarrow \neg(aRc)$

Definition 2 *Main cathegories of binary relations*

a *(Weak) Preorder aka Preference Relation- Reflexive, Transitive*

b *Equivalence Relation- Reflexive, Symmetric, Transitive*

c *Strict partial order -Asymmetric, Transitive*

d *Partial Order- Reflexive, Antisymmetric, Transitive*

e *Total (or Linear) Order- Antisymmetric, Complete, Transitive*

2.1 Monotonicity and Nonsatiation

Assume that relation $R :=\succeq$ is a preorder.

Definition 3 $\succeq$ *is* ***weakly monotone*** *on a set* X *if* $\forall x, y \in X$,

$$x \geq y \Rightarrow x \succeq y$$

Definition 4 $\succeq$ *is* ***monotone*** *on a set* X *if* $\forall x, y \in X$,

$$x \gg y \Rightarrow x \succ y$$

.

Definition 5 $\succeq$ *is* ***strongly monotone*** *on a set* X *if* $\forall x, y \in X$,

$$(x \geq y \wedge x \neq y) \Rightarrow x \succ y$$

.

Definition 6 $\succeq$ *is* ***locally nonsatiated*** *on a set* X *if*

$$\forall x \in X \text{ and } \forall \epsilon > 0, \exists y \in X \ni \| x - y \| < \epsilon \text{ and } y \succ x$$

2.2 Convexity

Definition 7 $\succeq$ *is* ***weakly convex*** *on a set* X *if* $\forall x, y \in X, \forall \lambda \in (0,1)$,

$$x \succeq y \Rightarrow \lambda x + (1-\lambda) y \succeq y$$

Definition 8 $\succeq$ *is* ***convex*** *on a set* X *if* $\forall x, y \in X, \forall \lambda \in (0,1)$,

$$x \succ y \Rightarrow \lambda x + (1-\lambda) y \succ y$$

Definition 9 $\succeq$ *is* ***strongly/strictly convex*** *on a set* X *if* $\forall x, y \in X, \forall \lambda \in (0,1)$,

$$x \sim y \wedge x \neq y \Rightarrow \lambda x + (1-\lambda) y \succ y$$

2.3 Continuity

Definition 10 ***(Sequential definition/ weak continuity)*** *A preorder* $\succeq$ *is continuous on a set* X *if* $\forall \{x_n\}, \{y_n\} \subseteq X$,

$$\forall n \in \mathbb{N}, (x_n \succeq y_n) \wedge (x_n \to x) \wedge (y_n \to y) \Rightarrow x \succeq y$$

Definition 11 ***(Set definition/ strong continuity)*** *A preorder* $\succeq$ *is continuous on a set* X *if* $\forall x \in X$*, the upper contour set* $U(x) = \{y \in X : y \succeq x\}$ *and the lower contour set* $L(x) = \{y \in X : x \succeq y\}$ *are closed in* X.

3 Real analysis

3.1 Topology on $\mathbb{R}^n$

Definition 12 *(Norm)* $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ *such that*

1. $\|x\| \geq 0 \wedge \|x\| = 0$ *iff* $x = 0$
2. $\forall_{\alpha \in \mathbb{R}} \forall_{x \in \mathbb{R}^n} \|\alpha x\| = |\alpha| \|x\|$
3. $\triangle$ *-inequality* $\quad \|x + y\| \leq \|x\| + \|y\|$

Examples

- Euclidean norm: $\|x\| = \langle x, x \rangle^{1/2} = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$
- 1-norm: $\|x\|_1 = \sum_{i=1}^n |x_i|$
-L^1 norm: $\|f\|_{L^1} = \int |f(x)| dx$
- sup norm: $\|x\|_\infty = \sup_{i \in \{1,\ldots,n\}} |x_i|$

Theorem 1 *(Cauchy-Schwarz inequality).*

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Definition 13 *(Open ball).* $B_\epsilon(a) = \{x \in \mathbb{R}^n : \|x - a\| < \epsilon\}$.

Definition 14 *(Convergence).* $x_n \to x \iff \forall_{\epsilon > 0} \exists_N \forall_{k \geq N} x_k \in B_\epsilon(a)$

Definition 15 *(Cauchy sequence). iff* $\forall_{\epsilon > 0} \exists_N \forall_{m,n > N} |x_n - x_m| < \epsilon$

Definition 16 *(Open set).* $V \subseteq \mathbb{R}^n$ *is open iff* $\forall_{x\in V}\exists_{\epsilon>0} B_\epsilon(x) \subseteq V$

Definition 17 *(Closed set).* $E \subseteq \mathbb{R}^n$ *is closed if* $E^c = \mathbb{R}^n \backslash E$ *is open.*

Definition 18 *(Topological space* (X, τ)*).*
1. if $\{v_\alpha\} \in \tau \Rightarrow \bigcup_{\alpha\in I} v_\alpha \in \tau$ *any collections*
2. if $\{v_\alpha\} \in \tau \Rightarrow \bigcap_{\alpha\in I} v_\alpha \in \tau$ *finite*
3. $\emptyset, X \in \tau$

Definition 19 *(Connected set* X *).* $\#_{U,V\in\tau} U \cup V = X \wedge U \cap V = \emptyset$

Definition 20 *(Interior of a set).* $E^\circ = \bigcup \{V : V \subseteq E \wedge V \in \tau(\mathbb{R}^n)\}$

Definition 21 *(Closure of a set).*$\bar{E} = cl(E) = \cap\{V : E \subseteq V \wedge V \in \mathcal{F}(\mathbb{R}^n)\}$ *, where* $\mathcal{F}(\mathbb{R}^n)$ *is family of closed sets.*

Lemma 1 *(i)* $E^\circ \subseteq E \subseteq \bar{E}$
(i i) $E^\circ = E$ *iff* E *is open*
(iii) $\bar{E} = E$ *iff* E *is closed*

Definition 22 *(Compactness).* $E \subseteq \bigcup_{\alpha\in I} v_\alpha v_\alpha \in \tau.E$ *is compact if for every open covering of* E *it has always finite subcover.*

Definition 23 *(Heine-Borel).* $E \subseteq \mathbb{R}^n$ *is compact* $\Longleftrightarrow E$ *closed and bounded.*

Corollary (existence of convergent subsequence). If $E \subseteq \mathbb{R}^n$ is compact, $\{x_n\} \subseteq E \Rightarrow \exists_{x_{n_k}} x_{n_k} \to x$

Definition 24 *(Continuity in topological space).* $f \in \mathcal{C}^0$ *on* $X \Longleftrightarrow \forall_{V\in\tau(X)} f^{-1}(V) \in \tau \Longleftrightarrow \forall_{U\in\mathcal{F}(x)} f^{-1}(U) \in \mathcal{F}$

Definition 25 *Continuity at a point).* $f : \Phi \to X$ *is continuous at* $\Theta \Longleftrightarrow \forall_{openV\subseteq X: f(\Theta)\in V} \exists_{openU\subseteq\Phi} \Theta \in U$

Theorem 2 *(Extreme value theorem -* ***Weierstrass****). If* $H \subseteq \mathbb{R}^n$ *compact,* $f : H \to, f \in \mathcal{C}^0$ *then*

$$\exists \bar{x}, \underline{x} \quad f(\bar{x}) = \sup_{x\in H} f(x) \quad \wedge \quad f(\underline{x}) = \inf_{x\in H} f(x)$$

Definition 26

Theorem 3 *(Intermediate value theorem).* $f : I \to \mathbb{R}, a, b \in I, a < b, y_0 \in (f(a), f(b)) \Rightarrow \exists_{x_0\in(a,b)} f(x_0) = y_0$

Definition 27 *(Uniform continuity).* $\forall x_{1,x_2} \forall_{\epsilon>0} \exists_{\rho>0} |x_1 - x_2| < \rho \Rightarrow \| f(x_1) - f(x_2) | < \epsilon$ *then* $f(x_n)$ *is Cauchy*

3.2 Convergence of functions

Definition 28 *(Pointwise convergence).*

$$\forall_x f(x) = \lim_{n\to\infty} f_n(x) \Longleftrightarrow \forall_x \forall_{\epsilon>0} \exists_N \forall_{n>N} |f_n(x) - f(x)| < \epsilon$$

Definition 29 *(Uniform convergence* $\rightrightarrows$ *).*

$$\forall_{\epsilon>0} \exists_N \forall_{n>N} \forall_{x\in E} |f_n(x) - f(x)| < \epsilon \Longleftrightarrow \forall_{\epsilon>0} \exists_N \forall_{m,n>N} \forall_{x\in E} |f_n(x) - f_m(x)| < \epsilon$$

Theorem 4

$$\forall_n |f_n| < M, f_n \rightrightarrows f \Rightarrow f < M$$

3.3 Continuity

Definition 30 *(Continuous function) f is continuous at $\bar{x} \in X$ if*

$$\lim_{x \to \bar{x}} f(x) = f(\bar{x})$$

f is continuous on X if f is continuous at every point $\bar{x} \in X$

Definition 31 $\sup E \iff \forall_{\epsilon>0} \exists_{a \in E} \sup E - \epsilon < a \leq \sup E$

Definition 32 *Convergence $x_n \to x \iff \forall_{\mu} \exists_N \forall_{n \geq N} |x_n - x| < \epsilon$ divergent sequence: $(x_n \to \pm\infty) \iff \forall_{\mu} \exists_N \forall_{n \geq N} x_n > \mu \, (x_n < \mu)$*

Lemma 2 *Every convergent sequence is bounded.*

Theorem 5 *(Bolzano-Weierstrass). Every bounded sequence has convergent subsequence*

Definition 33 *(Cauchy sequence). $\{x_n\}$ is Cauchy $\iff$*

$$\forall_{\epsilon>0} \exists_N \forall_{n,m \geq N} \mid x_n - x_m \mid < \epsilon$$

Lemma 3 *If $x_n \to x \Rightarrow \{x_n\}$ is Cauchy*

Definition 34 *(limsup, liminf).*

$$\begin{array}{l} \limsup x_n = \lim_{n \to \infty} \left(\sup_{k \geq n} x_k \right) \\ liminf x_n = \lim_{n \to \infty} \left(\inf_{k \geq n} x_k \right) \end{array}$$

Theorem 6 *(Monotone convergence).*
If x_n is monotone and bounded $\Rightarrow x_n \to x < +\infty$

Lemma 4 $x_n \to x \iff \limsup x_n = liminf x_n$

Definition 35 *(Limit of function).*

$$\lim_{x \to a} f(x) = y \iff \forall_{\epsilon>0} \exists_{\rho>0} |x - a| < \rho \Rightarrow |f(x) - f(y)| < \epsilon$$

Definition 36 *(Right/left limits).*
Right

$$y = \lim_{x \to a^+} f(x) \iff \forall_{\epsilon>0} \exists_{\rho>0} a < x < a + \rho \Rightarrow |f(x) - y| < \epsilon$$

Left

$$z = \lim_{x \to a^-} f(x) \iff \forall_{\epsilon>0} \exists_{\rho>0} a - \rho < x < a \Rightarrow |f(x) - z| < \epsilon$$

Definition 37 *(Continuity). $f : E \to \mathbb{R}$ is continuous at $a \in E \iff \forall_{\epsilon>0} \exists_{\rho>0} |x - a| < \rho \Rightarrow |f(x) - f(a)| < \epsilon$*

$$(x_n \to x \Rightarrow f(x_n) \to f(x))$$

Equivalently 1. f is continuous at $\bar{x} \in X$ if and only if for every open ball J of center $f(\bar{x})$ there exists an open ball B of center $\bar{x}$ such that $f(B \cap X) \subseteq J$

2. f is continuous at $\bar{x} \in X$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - \bar{x}\| < \delta$ and $x \in X \Longrightarrow |f(x) - f(\bar{x})| < \varepsilon$

Lemma 5 *(Sequentially continuous function) f is continuous at $\bar{x} \in X$ if and only if f is sequentially continuous at $\bar{x}$, that is, for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ such that $x_n \to \bar{x}$, we have that*

$$f(x_n) \to f(\bar{x})$$

Definition 38 *f is weakly increasing (or non-decreasing) on X if for all x and $\bar{x}$ in X*

$$x \leq \bar{x} \Longrightarrow f(x) \leq f(\bar{x})$$

Definition 39 *f is increasing on X if for all x and $\bar{x}$ in X*

$$x \ll \bar{x} \Longrightarrow f(x) < f(\bar{x})$$

Definition 40 *f is strictly increasing on X if for all x and $\bar{x}$ in X,*

$$x < \bar{x} \Longrightarrow f(x) < f(\bar{x})$$

Definition 41 *f strictly increasing on $X \Longrightarrow f$ increasing on X*

Definition 42 *f strictly increasing on $X \Longrightarrow f$ weakly increasing (or non-decreasing) on X*

Definition 43 *$X \subseteq \mathbb{R}^n$ is an open set, f is a function from X to $\mathbb{R}$ and $x \in X$*

$$\nabla f(x) := \left(\frac{\partial f}{\partial x^1}(x), \ldots, \frac{\partial f}{\partial x^h}(x), \ldots, \frac{\partial f}{\partial x^n}(x) \right)$$

denotes the gradient of f at x,

Definition 44 *$D^2 f(x) = H(x)$ denotes the Hessian matrix of f at x*

Definition 45 *$X \subseteq \mathbb{R}^n$ is an open set, $g := (g_1, \ldots, g_j, \ldots, g_m)$ is a mapping from X to $\mathbb{R}^m$ and $x \in X$*

$$\mathrm{J}g(x) := \begin{bmatrix} \frac{\partial g_1}{\partial x^1}(x) & \cdots & \frac{\partial g_1}{\partial x^h}(x) & \cdots & \frac{\partial g_1}{\partial x^n}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_j}{\partial x^1}(x) & \cdots & \frac{\partial g_j}{\partial x^h}(x) & \cdots & \frac{\partial g_j}{\partial x^n}(x) \\ \vdots & \vdots & & & \vdots \\ \frac{\partial g_m}{\partial x^1}(x) & \cdots & \frac{\partial g_m}{\partial x^h}(x) & \cdots & \frac{\partial g_m}{\partial x^n}(x) \end{bmatrix}_{m\times n} = \begin{bmatrix} \nabla g_1(x) \\ \vdots \\ \nabla g_j(x) \\ \vdots \\ \nabla g_m(x) \end{bmatrix}_{m\times n}$$

denotes the Jacobian matrix of g at x.

3.4 Differentiability

Definition 46

Definition 4 (Differentiable function) f is differentiable at $\bar{x} \in X$ if
1. all the partial derivatives of f at $\bar{x}$ exist,
2. there exists a function $E_{\bar{x}}$ defined in some open ball $B(0, \varepsilon) \subseteq \mathbb{R}^n$ such that for every $u \in B(0, \varepsilon)$

$$f(\bar{x} + u) = f(\bar{x}) + \nabla f(\bar{x}) \cdot u + \|u\| E_{\bar{x}}(u) where \lim_{u \to 0} E_{\bar{x}}(u) = 0$$

f is differentiable on X if f is differentiable at every point $\bar{x} \in X$.

Lemma 6 *If f is differentiable at $\bar{x}$, then f is continuous at $\bar{x}$*

Definition 47 *(Directional derivative) Let $v \in \mathbb{R}^n, v \neq 0$. The directional derivative $D_v f(\bar{x})$ of f at $\bar{x} \in X$ in the direction v is defined as*

$$\lim_{t \to 0^+} \frac{f(\bar{x} + tv) - f(\bar{x})}{t}$$

if this limit exists and it is finite.

Lemma 7 *(Differentiable function/Directional derivative) If f is differentiable at $\bar{x} \in X$, then for every $v \in \mathbb{R}^n$ with $v \neq 0$*

$$D_v f(\bar{x}) = \nabla f(\bar{x}) \cdot v$$

3.5 Compactness

Lemma 8 *(Compact set/Subsequences) X is compact if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of the sequence $(x_n)_{n\in\mathbb{N}}$ such that $(x_{n_k})_{k\in\mathbb{N}}$ converges to some point $\bar{x} \in X$.*[1]

Lemma 9 *(Compact set) $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded*

Definition 48 *(Closed set) X is closed if its complement $\mathcal{C}(X) := \mathbb{R}^n \backslash X$ is open*

Lemma 10 *(Sequentially closed) X is closed if and only if it is sequentially closed, that is, for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $x_n \to \bar{x}$, we have*

$$\bar{x} \in X$$

Definition 49 *(Bounded set) X is bounded if it is included in some ball, that is, there exists $\varepsilon > 0$ such that for all $x \in X, \|x\| < \varepsilon$.*

3.6 Concavity and quasi-concavity

In this section, we assume that C is a convex subset of $\mathbb{R}^n$ and f is a function from C to $\mathbb{R}$.

Definition 50 *(Convex set). A set C is convex $\iff \forall_{x,y\in \mathrm{C}} \forall_{\lambda\in[0,1]} \lambda x + (1-\lambda)y \in \mathrm{C}$*

Definition 51 *(Convex combination). Let $\{x_i\}_{i=1}^m \subseteq \mathbb{R}^n, \{\lambda_i\}_{i=1}^m \subseteq \mathbb{R}_+$, $\sum \lambda_i = 1$. The vector $\sum \lambda_i x_i = 1$ is called a convex combination of $\{x_i\}$.*

Lemma 11 C *is convex* $\iff$ C *contains all convex combinations of its elements.*

Definition 52 *(Hyperplane). $H \subseteq \mathbb{R}^n$ is hyperplane $\iff \exists_{\beta\in\mathbb{R}, b\in\mathbb{R}^n} H = \{x \in \mathbb{R}^n : x \cdot b = \beta\}$*

Lemma 12 *(hyperplane generates two halfspaces)* $\{x \in \mathbb{R}^n : x \cdot b \leq \beta\}$

and $\{x \in \mathbb{R}^n : x \cdot b \geq \beta\}$

Definition 53 *(Convex hull) . Let $Co = \cap\{C : E \subseteq C, C$ convex $\}$. Note $Co = \{x \in \mathbb{R}^n : \exists_{\{x_i\}\subseteq E} \exists_{\{\lambda_i\}\subseteq\mathbb{R}:\sum\lambda_i=1} x = \sum$*

Definition 54 *(Simplex). A set $S \subseteq \mathbb{R}^n$ is m -dimensional simplex $\iff S = \{(b_0, \ldots, b_m) \in \mathbb{R}^m : b_i$ affinely independent $\}$*

Theorem 7 $\forall_i C_i$ *convex* $\Rightarrow$

$$\mathrm{C} = \bigcap C_i is convex$$

$C_1 + C_2$ is convex $C_1 + a$ is convex $\mathrm{C} = \{x \in \mathbb{R}^n : x \cdot b \leq \beta\}$ *is convex if f is quasi concave function, then* $\mathrm{C} = \{x \in \mathbb{R}^n : f(x) \leq \beta\}$ *is convex*

Theorem 8 *(Separating hyperplane theorem). Let $C_1 \subseteq \mathbb{R}^n, C_2 \subseteq \mathbb{R}^m$. $H = \{x \in \mathbb{R}^n : x \cdot b = \beta\}$ is separating hyperplane of $C_1 \& C_2 \iff$*

$$\begin{array}{ll} \forall_{x\in C_1} & x \cdot b \leq \beta \\ \forall_{y\in C_2} & y \cdot b \geq \beta \end{array}$$

Separation is strong if at least one is ”¡” (”¿”) • ” $\iff$ ” part of theorem is true when

$$ri\,(C_1) \cap ri\,(C_2) = \emptyset$$
$$ri(A) = \{x \in A : B(x, \epsilon) \cap aff(A) \subseteq A\}$$
$$aff(A) = \{\sum \alpha_i x_i : x_i \in A, \sum \alpha_i = 1\}$$

Conditions for ”$\Rightarrow$ ” separating hyperplane theorem:

•C_1, C_2 non empty, convex, $x \in \mathbb{R}^n$

(i) $x \notin C_1 \Rightarrow H(b, \beta)$ separates strongly $x \& C_1$

(ii) $C_1 \cap C_2 = \emptyset \Rightarrow H(b, \beta)$ separates $C_1 \& C_2$

(a) C_1 open $\Rightarrow H(b, \beta)$ separates strongly $C_1 \& C_2$

(b) C_1, C_2 closed, C_1 compact$\Rightarrow H(b, \beta)$ separates strongly $C_1 \& C_2$

Lemma 13 $\bullet f$ *convex* $\Longleftrightarrow$ *epi*$(f) = \{(x, y) \in X \times \mathbb{R} : y \geq f(x)\}$ *convex.*

$\bullet f$ *quasi convex* $\Longleftrightarrow Lf(a) = \{a \in X : f(x) \leq a\}$ *convex.*

Definition 55 *(Support). The support function $S(\cdot \mid C)$ of convex set $C \subseteq \mathbb{R}^n$ is defined as:*

$$S(x, y) = \sup_{y \in \mathrm{C}} x \cdot y$$

It is well defined for compact C

Lemma 14 *If f concave, $|f(x)| \leq M$ on open neighborhood of convex X, then f continuous.*

Definition 56 *(Directional derivative). Directional derivative of g at x in the direction of y is:*

$$Df(x, y) = \lim_{t \to 0^+} \frac{g(x + ty) - g(x)}{t}$$

derivatives are well defined (may be ∞).

Is concave function differentiable? almost everywhere. Moreover derivative is continuous a.s.

Lemma 15 $\bullet$ *concave and* $\mathcal{C}^1 \Longleftrightarrow \forall_{x,y \in X} Df(x)(y - x) \geq f(y) - f(x)$ - f *convex and* $\mathcal{C}^1 \Longleftrightarrow \forall_{x,y \in X} Df(x)(y - x) \leq f(y) - f(x)$

Lemma 16 *X convex, $f : X \to \mathbb{R}, f\mathcal{C}^2$. 1. f concave $\Longleftrightarrow D^2 f(x)$ negative semi definite. 2. $D^2 f(x)$ negative definite $\Rightarrow f$ strictly concave. Proposition* 1.4.10. *If f quasi convex, g monotone, nondecreasing, then $g \circ f$ is quasi-convex.*

Definition 57 *(Concave function) f is concave if for all $t \in [0, 1]$ and for all x and $\bar{x}$ in C,*

$$f(tx + (1 - t)\bar{x}) \geq tf(x) + (1 - t)f(\bar{x})$$

Lemma 17 *f is concave if and only if the set*

$$\{(x, \alpha) \in C \times \mathbb{R} : f(x) \geq \alpha\}$$

is a convex subset of $\mathbb{R}^{n+1}$. The set above is called hypograph of f

Lemma 18 *C is open and f is differentiable on $C.f$ is concave if and only if for all x and $\bar{x}$ in C,*

$$f(x) \leq f(\bar{x}) + \nabla f(\bar{x}) \cdot (x - \bar{x})$$

Proposition 19

Lemma 19 C *is open and f is twice continuously differentiable*

on $C.f$ is concave if and only if for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ *is negative semidefinite, that is, for all $x \in C$*

$$v \mathrm{H} f(x) v^T \leq 0, \forall v \in \mathbb{R}^n$$

Definition 58 *(Strictly concave function) f is strictly concave if for all $t \in]0, 1[$ and for all x and $\bar{x}$ in C with $x \neq \bar{x}$*

$$f(tx + (1 - t)\bar{x}) > tf(x) + (1 - t)f(\bar{x})$$

Lemma 20 *C is open and f is differentiable on C.f is strictly concave if and only if for all x and $\bar{x}$ in C with $x \neq \bar{x}$,*

$$f(x) < f(\bar{x}) + \nabla f(\bar{x}) \cdot (x - \bar{x})$$

Lemma 21 *C is open and f is twice continuously differentiable on C. If for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative definite, that is, for all $x \in C$*

$$v\mathrm{H}f(x)v^T < 0, \forall v \in \mathbb{R}^n, v \neq 0$$

then f is strictly concave

Definition 59 *(Quasi-concave function) f is quasi-concave if and only if for all $\alpha \in \mathbb{R}$ the set*

$$\{x \in C : f(x) \geq \alpha\}$$

is a convex subset of $\mathbb{R}^n$. The set above is called upper contour set of f at α.

Lemma 22 *f is quasi-concave if and only if for all $t \in [0, 1]$ and for all x and $\bar{x}$ in C,*

$$f(tx + (1 - t)\bar{x}) \geq \min\{f(x), f(\bar{x})\}$$

Lemma 23 *C is open and f is differentiable on C.f is quasiconcave if and only if for all x and $\bar{x}$ in C,*

$$f(x) \geq f(\bar{x}) \Longrightarrow \nabla f(\bar{x}) \cdot (x - \bar{x}) \geq 0$$

Lemma 24 *C is open and f is differentiable on C. If f is quasiconcave and $\nabla f(x) \neq 0$ for all $x \in C$, then for all x and $\bar{x}$ in C with $x \neq \bar{x}$,*

$$f(x) > f(\bar{x}) \Longrightarrow \nabla f(\bar{x}) \cdot (x - \bar{x}) > 0$$

Lemma 25 *C is open and f is twice continuously differentiable on C. If f is quasi-concave, then for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative semidefinite on $Ker\nabla f(x)$, that is, for all $x \in C$*

$$v \in \mathbb{R}^n and \nabla f(x) \cdot v = 0 \Longrightarrow v\mathrm{H}f(x)v^T \leq 0$$

Definition 60 *(Strictly quasi-concave function) f is strictly quasi-concave if and only if for all $t \in]0, 1[$ and for all x and $\bar{x}$ in C with $x \neq \bar{x}$,*

$$f(tx + (1 - t)\bar{x}) > \min\{f(x), f(\bar{x})\}$$

Lemma 26 *C is open and f is differentiable on C. 1. If for all x and $\bar{x}$ in C with $x \neq \bar{x}$,*

$$f(x) \geq f(\bar{x}) \Longrightarrow \nabla f(\bar{x}) \cdot (x - \bar{x}) > 0$$

then f is strictly quasi-concave.

2. If f is strictly quasi-concave and $\nabla f(x) \neq 0$ for all $x \in C$, then for all x and $\bar{x}$ in C with $x \neq \bar{x}$

$$f(x) \geq f(\bar{x}) \Longrightarrow \nabla f(\bar{x}) \cdot (x - \bar{x}) > 0$$

Lemma 27 *C is open and f is twice continuously differentiable on C. If for all $x \in C$ the Hessian matrix $\mathrm{H}f(x)$ is negative definite on $Ker\nabla f(x)$, that is, for all $x \in C$*

$$v \in \mathbb{R}^n, v \neq 0 and \nabla f(x) \cdot v = 0 \Longrightarrow v\mathrm{H}f(x)v^T < 0$$

then f is strictly quasi-concave

Lemma 28 *We remark that f linear or affine $\Rightarrow$ f concave $\Leftarrow$ f strictly concave*

$$\downarrow$$

$$f quasi-concave \Leftarrow f strictly quasi-concave$$

We remind the definitions and some properties of negative definite/semidefinite matrices. Let H be a $n \times n$ symmetric matrix.

Definition 61 *1.* H *is negative semidefinite if* $v\mathrm{H}v^T \leq 0$ *for all* $v \in \mathbb{R}^n$
2. H *is negative definite if* $v\mathrm{H}v^T < 0$ *for all* $v \in \mathbb{R}^n$ *with* $v \neq 0$

Theorem 9 *1.* H *has* n *real eigenvalues. We denote* $\lambda_1, \ldots, \lambda_n$ *the eigenvalues of* H.

2. H *is negative semidefinite if and only* $\lambda_i \leq 0$ *for every* $i = 1, ..., n$

3. H *is negative definite if and only* $\lambda_i < 0$ *for every* $i = 1, ..., n$

Theorem 10 *1. If H is negative semidefinite, then tr(H) 0 and det(H)* $\geq$ *0 if n is even, det(H)* $\leq$ *0 if n is odd*

2. If H is negative definite, then tr(H) ¡ 0 and det(H) ¿ 0 if n is even, det(H) ¡ 0 if n is odd
We remark that if n = 2, then the conditions stated in the proposition above also are sufficient conditions, that is
1. H is negative semidefinite if and only if tr(H) $\leq$ *0 and det(H)* $\geq$ *0.*
2. H is negative definite if and only if tr(H) ¡ 0 and det(H) ¿ 0.

4 Optimization

4.1 Karush-Kuhn-Tucker Conditions

In this section, we focus on necessary and sufficient conditions in terms of first-order conditions for solving a maximization problem with inequality constraints.
In this section, we assume that $C \subseteq \mathbb{R}^n$ is convex and open
- the following functions f and g_j with $j = 1, \ldots, m$ are differentiable on C

$$f : x \in C \subseteq \mathbb{R}^n \longrightarrow f(x) \in \mathbb{R} \text{ and}$$
$$g_j : x \in C \subseteq \mathbb{R}^n \longrightarrow g_j(x) \in \mathbb{R}, \forall j = 1, \ldots, m$$

Maximization problem

$$\max_{x \in C} f(x)$$

$$\text{subject to } g_j(x) \geq 0, \forall j = 1, \ldots, m$$

where f is the objective function, and g_j with $j = 1, \ldots, m$ are the constraint functions.

The Karush-Kuhn-Tucker conditions associated with problem (1) are given below

$$\begin{cases} \nabla f(x) + \sum_{j=1}^{m} \lambda_j \nabla g_j(x) = 0 \\ \lambda_j g_j(x) = 0, \forall j = 1, \ldots, m \\ g_j(x) \geq 0, \forall j = 1, \ldots, m \\ \lambda_j \geq 0, \forall j = 1, \ldots, m \end{cases}$$

where for every $j = 1, \ldots, m, \lambda_j \in \mathbb{R}$ is called Lagrange multiplier associated with the inequality constraint g_j

Definition 62 *Let* $x^* \in C$, *we say that the constraint* j *is binding at* x^* *if f* $g_j(x^*) = 0$. *We denote*

1. $B(x^*)$ *the set of all binding constraints at* x^*, *that is*

$$B(x^*) := \{j = 1, \ldots, m : g_j(x^*) = 0\}$$

2. $m^* \leq m$ *the number of elements of* $B(x^*)$ *and*

3. $g^* := (g_j)_{j \in B(x^*)}$ *the following mapping*

$$g^* : x \in C \subseteq \mathbb{R}^n \longrightarrow g^*(x) = (g_j(x))_{j \in B(x^*)} \in \mathbb{R}^{m^*}$$

Theorem 11 *(Karush-Kuhn-Tucker are necessary conditions) Let x^* be a solution to problem (1). Assume that one of the following conditions is satisfied.*

1. *For all $j=1,\ldots,m, g_j$ is a linear or affine function.*

2. *Slater's Condition :*

for all $j=1,\ldots,m, g_j$ is a concave function or g_j is a quasiconcave function with $\nabla g_j(x) \neq 0$ for all $x \in C$, and
o there exists $\bar{x} \in C$ such that $g_j(\bar{x})>0$ for all $j=1,\ldots,m$

3. *Rank Condition : rank $\mathrm{J} g^*\left(x^*\right)=m^* \leq n$ Then, there exists $\lambda^*=\left(\lambda_1^*,\ldots,\lambda_j^*,\ldots,\lambda_m^*\right) \in \mathbb{R}_{+}^m$ such that $\left(x^*,\lambda^*\right)$ satisfies the Karush-Kuhn-Tucker Conditions (2).*

Theorem 12 *(Karush-Kuhn-Tucker are sufficient conditions) Suppose that there exists $\lambda^*=\left(\lambda_1^*,\ldots,\lambda_j^*,\ldots,\lambda_m^*\right) \in \mathbb{R}_{+}^m$ such that $\left(x^*,\lambda^*\right) \in C \times \mathbb{R}_{+}^m$*

satisfies the Karush-Kuhn-Tucker Conditions (2). Assume that

1. *f is a concave function or f is a quasi-concave function with $\nabla f(x) \neq 0$ for all $x \in C$, and*

2. *g_j is a quasi-concave function for all $j=1,\ldots,m$*

Then, x^* is a solution to problem (1).

5 Correspondences

Definition 63 $(\Theta \subseteq \mathbb{R}^n, .X \subseteq \mathbb{R}^n$ *A correspondence* $\Gamma: \Theta \rightrightarrows X$ *is a map s.t.* $\Gamma(\Theta) \subseteq X$. $\left(\Gamma:(\Theta) \rightarrow 2^X\right)$

Definition 64 *(Graph of correspondence).* $Gr(\Gamma=\{(\theta, x): \theta \in \Theta, x \in \Gamma(\Theta)\}$

Definition 65 *(Properties of correspondences).*

1. *not empty valued if* $\Gamma(\Theta) \neq \emptyset \forall_{\Theta}$
2. *single valued if* $|\Gamma(\Theta)|=1 ; \forall \Theta$
3. *closed valued if* $\Gamma(\Theta)$ *is closed set* $\forall_{\Theta}$
4. *compact valued if* $\Gamma(\Theta)$ *is compact set* $\forall_{\Theta}$
5. *convex valued if* $\Gamma(\Theta)$ *is convex set* $\forall_{\Theta}$
6. *closed (graph) if* $Gr(\Gamma)$ *is closed subset of* $\mathbb{E} \times X$ *(graph) if* $Gr(\Gamma)$ *is convex on* $\Theta \times X$

Lemma 29 *$Gr(\Gamma)$ is closed graph* $\Longleftrightarrow \forall_{\Theta: \Theta_n \rightarrow \Theta} \forall_{x_n \rightarrow x}: x_n \in \Gamma\left(\Theta_n\right)$

Lemma 30 *$Gr(\Gamma)$ is convex graph* $\Longleftrightarrow \forall_{\Theta}, \Theta', x \in \Gamma(\Theta), x' \in \Gamma\left(\Theta'\right)$
it holds that $\lambda x+(1-\lambda) x' \in \Gamma\left(\Theta \lambda+(1-\lambda) \Theta'\right) \forall_{x \in[0,1]}$

Lemma 31 $\Gamma:(\Theta \Rightarrow X$ *has closed graph* $\Rightarrow$ *it is closed valued. If X is compact, than Γ is also compact valued.*

Definition 66 *(UHC,LHC)*

- Γ *is said to be upper-hemi continuous at* $\Theta \in(\mathbb{M}) \Longleftrightarrow \forall_{open V \subseteq X s.t. \Gamma\left(\Theta_n\right) \subseteq V} \exists_{U \subseteq(\mathbb{M})}, \Theta \in U^{\forall} \Theta' \in U^{\Gamma}\left(\Theta'\right) \subseteq V$
- *compact valued* Γ *is UHC at* $\Theta \in \mathbb{M} \Longleftrightarrow \forall_{\Theta_n: \Theta_n \rightarrow \Theta} \forall_{x_n: x_n \in \Gamma\left(\Theta_n\right)} \exists_{x_{n_k}} x_{n_k} \rightarrow x \in \Gamma(\Theta)\ V \neq \emptyset$
- Γ *is* LHC *at* $\Theta \in \mathbb{M} \Longleftrightarrow \forall_{x \in \Gamma(\Theta)} \forall \theta_n \rightarrow \Theta \exists_{x_n}: x_n \in \Gamma(\Theta) x_n \rightarrow x$

Lemma 32 *(UHC & Closed graph).* Γ *is* $UHC \Rightarrow \Gamma$ *has closed graph.*

5.1 Powerful theorems of analysis

Theorem 13 *(Inverse Function Theorem) Let V be open in $\mathbb{R}^n$ and $f : V \rightarrow \mathbb{R}^n$ be $\mathcal{C}^1$ on V. If $\Delta_f(a) \neq 0$ for some $a \in V$, then there exists an open set W containing a such that*

- *f is $1-1$ on W*
- *f^{-1} is $\mathcal{C}^1$ on $f(W)$, and*
- *for each $y \in f(W)$*
$$D\left(f^{-1}\right)(y) = \left[Df\left(f^{-1}(y)\right)\right]^{-1}$$

Notation: $[\cdot]^{-1}$ represents matrix inversion, $\Delta_f(a) = det(Df(a))$ (the Jacobian of f at a)

Theorem 14 *(Mean Value Theorem on $\mathbb{R}^n$) Let V be open in $\mathbb{R}^n$ and suppose that $f : V \rightarrow \mathbb{R}^m$ is differentiable on V. If $x, a \in V$ and $L(x; a) \subseteq V$, then for each $u \in \mathbb{R}^m$, there is a $c \in L(x; a)$ such that*
$$u \cdot (f(x) - f(a)) = u \cdot (Df(c)(x - a))$$

Notation: $L(a, b) := \{(1-t)\mathbf{a} + tb : t \in [0, 1)\}$ is called line segment

Corollary: Let $V \subset \mathbb{R}^n$ be open and convex, and let $f : V \rightarrow \mathbb{R}$ be a function that is differentiable everywhere on V. Then, for any $a, b \in V$, there is $\lambda \in (0, 1)$ such that
$$f(b) - f(a) = Df((1-\lambda)a + \lambda b) \cdot (b - a)$$

Theorem 15 *(Taylor Theorem on $\mathbb{R}^n$) Let $p \in \mathbb{N}$, let V be open in $\mathbb{R}^n$, let $x, a \in V$, and suppose that $f : V \rightarrow \mathbb{R}$. If the pth total differential of f exists on V and $L(x; a) \subseteq V$, then there is a point $c \in L(x, a)$ such that*
$$f(x) = f(a) + \sum_{k=1}^{p-1} \frac{1}{k!} D^{(k)} f(a; h) + \frac{1}{p!} D^{(p)} f(c, h)$$

for $h := x - a$ Remark: These hypotheses are met if V is convex and f is $\mathcal{C}^P$ on V. Proof: See Wade(2010) 11.37. Remark: For multi-variable version Newton method, use Jacobian Matrix to replace $f'(x_n)$

Theorem 16 *(Implicit Function Theorem) Let $F : S \subset \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a $\mathcal{C}^1$ function, where S is open. Let (x^*, y^*) be a point in S such that $DF_y(x^*, y^*)$ is invertible, and let $F(x^*, y^*) = c$. Then, there is a neighborhood $U \subset \mathbb{R}^m$ of x^* and a $\mathcal{C}^1$ function $g : U \rightarrow \mathbb{R}^n$ such that*

- *$(x, g(x)) \in S, \forall x \in U$*
- *$g(x^*) = y^*$*
- *$F(x, g(x)) \equiv c, \forall x \in U$*
- *$Dg(x) = (DF_y(x, y))^{-1} \cdot DF_x(x, y)$*

Theorem 17 *(The Maximum Theorem) Let $f : S \times \Theta \rightarrow \mathbb{R}$ be a continuous function, and $D : \Theta \rightarrow P(S)$ be a compact-valued, continuous correspondence. Let $f^* : \Theta \rightarrow \mathbb{R}$ and $D^* : \Theta \rightarrow P(S)$ be defined by*
$$f^*(\theta) = \max\{f(x, \theta) \mid x \in D(\theta)\} D^*(\theta) = \arg\max\{f(x, \theta) \mid x \in D(\theta)\}$$

Then f^* is a continuous function on Θ, and D^* is a compact-valued, uhc correspondence on Θ.

Proof: since D is compact-valued and f is continuous, a maximum is always attained. So f^* is well-defined and $D^*(\theta)$ is non-empty $\forall \theta \in \Theta$. We'll divide the proof to 3 parts:
(1) To show D^* is compact-valued:
$\forall \theta \in \Theta, D^*(\theta) \subset D(\theta)$, which is compact, $D^*(\theta)$ is also bounded. For closeness, let $s_n \rightarrow s$ and $s_n \in D^*(\theta)$. Clearly, $s_n \in D(\theta)$ which is compact, therefore $s \in D(\theta)$, so this limit is feasible. By definition of $D^*, f(s_n, \theta) \geq f(x, \theta)$ forall $x \in D(\theta)$, since f is continuous,
$$f(s, \theta) = \lim f(s_n, \theta) \geq f(x, \theta), hences \in D^*(\theta)$$

(2) To show D^* uhc: since D^* is compact-valued, we can use the sequential definition. Take any $\theta \in \Theta$, any sequence $\theta_n \rightarrow \theta$ and any $\{s_n\}$ s.t. $s_n \in D^*(\theta_n)$. since D is uhc, then exists a subsequence $s_{n_k} \rightarrow s \in D(\theta)$. The proof will be completed if we can show $s \in D^*$.

Let take any arbitrary $\bar{s} \in D(\theta)$, since D is lhc, there exists a sequence $\bar{s}_{n_k} \rightarrow \bar{s}$ with $\bar{s}_{n_k} \in D(\theta_{n_k})$. since $f(s_{n_k}, \theta_{n_k}) \geq f(\bar{s}_{n_k}, \theta_{n_k})$ and f is continuous, it follows $f(s, \theta) \geq f(\bar{s}, \theta)$ for all $\bar{s} \in D(\theta)$. Therefore $s \in D^*(\theta)$ and D^* uhc.

(3) To show f^* is continuous: First notice for each $\theta, f^*(\theta)$ must be a singleton, hence f^* is a function. We'll use limsup and liminf to proof the continuity of a function.

Fix $\theta \in \Theta$, take an arbitrary sequence $\theta_n \rightarrow \theta$. Consider an arbitrary sequence $s_n \in D^*(\theta_n)$ Let $\bar{f} = \limsup f^*(\theta_n)$. By previous theorem, there exist a sub-sequence θ_{n_k} s.t. $f^*(\theta_{n_k}) = \bar{f}$ since D^* is uhc, there exist a subset of $\{s_{n_k}\}$ (let's call it $\{s_{n_{kl}}\}$) converging to a point $s \in D^*(\theta)$. Then

$$\bar{f} = \lim f^*(\theta_{n_{kl}}) = \lim f(s_{n_{kl}}, \theta_{n_{kl}}) = f(s, \theta) = f^*(\theta)$$

Similarly, let $f = \liminf f^*(\theta_n)$, by the same argument we can get $f = f^*(\theta)$. Since

$$f^*(\theta) = \limsup f^*(\theta_n) = \liminf f^*(\theta_n)$$

f^* is continuous at θ

Definition 67 *(Meet and Joint) Given $x, y \in \mathbb{R}^n$, the meet of x and y, denoted $x \wedge y$, is*

$$x \wedge y = (\min\{x_1, y_1\}, \cdots, \min\{x_n, y_n\})$$

The joint of x and y, denoted $x \vee y$, is

$$x \vee y = (\max\{x_1, y_2\}, \cdots, \max\{x_n, y_n\})$$

Definition 68 *(Lattice) $X \subset \mathbb{R}^n$ is a lattice of $\mathbb{R}^n$ if $\forall x, y \in X, x \wedge y \in X$ and $x \vee y \in X$*

Remark: A budget set is generally not a lattice of $\mathbb{R}^n$. More for lattice: we can define compact, sup/inf on it:

Definition 69 *(compact lattice) $X \subset \mathbb{R}^n$ is a compact lattice if X is a lattice and X is compact under the Euclidean metric.*

Definition 70 *$x^* \in X$ is a greatest element of lattice X if $x^* \geq x, \forall x \in X, \hat{x} \in X$ is a least element of lattice X if $\hat{x} \leq x, \forall x \in X$*

Definition 71 *(Uniqueness of greatest and least element) Suppose $X \subset \mathbb{R}^n$ is a non-empty, compact lattice. Then, X has a greatest element and a least element.*

Definition 72 *(Supermodular) $f : S \times \Theta \rightarrow \mathbb{R}$ is supermodular in (x, θ) if $\forall z = (x, \theta)$ and*

$$z' = (x', \theta') \, inS \times \Theta f(z) + f(z') \leq f(z \vee z') + f(z \wedge z')$$

Theorem 18 *(Supermodularity) $f : S \times \Theta \rightarrow \mathbb{R}$ is supermodular in (x, θ), then for any fixed θ, f is supermodular inx, i.e.*

$$f(x, \theta) + f(x', \theta) \leq f(x \vee x', \theta) + f(x \wedge x', \theta))$$

Proposition 15.1

Definition 73 *(Increasing Differences) $f : S \times \Theta \rightarrow \mathbb{R}$ satisfies increasing differences in (x, θ) if $\forall (x, \theta), (x', \theta') \in S \times \Theta$ such that $x \geq x'$ and $\theta \geq \theta'$*

$$f(x, \theta) - f(x', \theta) \geq f(x, \theta') - f(x', \theta')$$

If the inequality is strict whenever $x > x'$ and $\theta < \theta'$, then f satisfies strictly increasing differences $in(x, \theta)$

Theorem 19 *(Supermodularity vs. Increasing Differences) $f : Z \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is supermodular in z iff f has increasing return in z*

Theorem 20 *(Topkis' Characterization Theorem) Let Z be an open lattice of $\mathbb{R}^n$. A $\mathcal{C}^2$ function $h : Z \rightarrow \mathbb{R}$ is supermodular on Z iff $\forall z \in Z$*

$$\frac{\partial^2 h}{\partial z_i \partial z_j}(z) \geq 0, \forall i \neq j$$

5.2 Parametric Monotonicity

Now let's consider the optimization problem:

$$\max_{x \in S} f(x;\theta)$$

with

$$f^*(\theta) = \max\{f(x;\theta) \mid x \in S\}, \quad D^*(\theta) = argmax\{f(x;\theta) \mid x \in S\}$$

A correspondence $D^*(\theta)$ is nondecreasing in θ if for every $\theta \leq \theta'$

$$D^*(\theta) \leq D^*(\theta')$$

Above inequality between sets means the strong set order: for every $x \in D^*(\theta)$ and $x' \in D^*(\theta')$, it holds $x \vee x' \in D^*(\theta'), x \wedge x' \in D^*(\theta)$

Theorem 21 *(Topkis' Monotonicity Theorem) Let S be compact lattice of $\mathbb{R}^n$, Θ be a lattice of $\mathbb{R}^l$, and $f : S \times \Theta \to \mathbb{R}$ be a continuous function on S, for each fixed θ. Suppose f satisfies increasing differences in (x, θ), and is supermodular in x for each fixed θ. Then D^* is nondecreasing in θ.*[3]

6 Stochastic analysis

Definition 74 *(σ algebra) Let S be a set and let $\mathcal{F}$ be a family of subsets of S.$\mathcal{F}$ is called a σ -algebra if (1) $\phi, S \in \mathcal{F}$ (2) $(A \in \mathcal{S}) \Rightarrow (A^c = S \backslash A \in \mathcal{F})$ (close under complements) (3) $(A_n \in \mathcal{S}, n = 1, 2, \cdots) \Rightarrow (\cup_{n=1}^{\infty} A_n \in \mathcal{F})$ (close under countable unions / intersections)*

Definition 75 *A pair $(S, \mathcal{F})$, where S is a set and $\mathcal{F}$ is a σ -algebra of its subsets is called a measurable space. Any set $A \in \mathcal{F}$ is called an $\mathcal{F}$ -measurable set.*

Example: (a) The power set of S; (b) The family $\{\varnothing, S\}$ (trivial σ -algebra) (c) For the set $S = \{1, 2, 3, 4\}, \mathcal{F} = \{\{1, 3\}, \{2, 4\}, \varnothing, S\}$ is also a σ -algebra

Definition 76 *Given a set S and a collection $\mathcal{A}$ of subsets of S, the intersection of all σ -algebras (which is also a σ -algebras) containing $\mathcal{A}$ is called the σ -algebra generated by $\mathcal{A}$.*

Let $\mathcal{B}$ be the open ball in $\mathbb{R}^l, \sigma$ -algebra generated by $\mathcal{B}$ is called Borel-algebra generated by $\mathcal{B}$. Similarly, it can also be defined by open rectangles (or closed intervals, half-open intervals if in $\mathbb{R}$

Definition 77 *(Measure) Let $(S, \mathcal{F})$ be a measurable space. A measure is an extended real-valued function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ such that (1) $\mu(\emptyset) = 0$ (2) $\mu(A) \geq 0$ for all $A \in \mathcal{F}$ (3) μ is countably additive: if $\{A_n\}_{n=1}^{\infty}$ is a countable, disjoint sequence in $\mathcal{A}$, then:*

$$\mu(\cup A_n) = \sum \mu(A_n)$$

Definition 78 *If furthermore $\mu(S) < \infty$, then μ is said to be a finite measure and if $\mu(S) = 1$ then μ is said to be a probability measure.*

Definition 79 *A triple $(S, \mathcal{F}, \mu)$ where S is a set, $\mathcal{F}$ is a σ -algebra of its subsets and μ is a measure on $\mathcal{F}$ is called a measure space. The triple is called a probability space if μ is a probability measure*

Definition 80 *(Measurable Functions) Given a measurable space $(S, \mathcal{F})$, a real-valued function $f : S \to \mathbb{R}$ is measurable with respect to $\mathcal{F}$ (or $\mathcal{F}$ -measurable) if*

$$\{s \in S \mid f(s) \leq a\} \in \mathcal{F}, \forall a \in \mathbb{R}$$

If the space is a probability space, then f is called a random variable.

Definition 81 *(Simple Function) Let $(S, \mathcal{F})$ be a measurable space, a function $\phi : S \to \mathbb{R}$ is called a simple function if it is of the form*

$$\phi(s) = \sum_{i=1}^{n} a_i \chi_{A_i}(s)$$

where $a_1, \cdots, a_n$ are distinct real numbers, $\{A_i\}$ is a partition of S, and χ_{A_i} are indicator functions. A simple function is measurable if and only if $A_i \in \mathcal{F}$.

Theorem 22 *(Pointwise convergence preserves measurability) Let* $(S, \mathcal{F})$ *be a measurable space, and let* $\{f_n\}$ *be a sequence of* $\mathcal{F}$ *-measurable functions converging pointwise to* f. *Then* f *is also measurable.*

Theorem 23 *(Approximation of measurable functions by simple functions) Let* $(S, \mathcal{F})$ *be measurable space. If* $f : S \to \mathbb{R}$ *is* $\mathcal{F}$ *-measurable, then there is a sequence of measurable simple functions* $\{\phi_n\}$, *such that* $\phi_n \to f$ *pointwise. If* $0 \leq f$, *then the sequence can be chosen so that*

$$0 \leq \phi_n \leq \phi_{n+1} \leq f, \forall n$$

If f *is bounded, then the sequence can be chosen so that* $\phi_n \to f$ *uniformly.*

Definition 82 *Let* $(S, \mathcal{S})$ *and* $(T, \mathcal{T})$ *be measurable spaces. Then the function* $f : S \to T$ *is measurable if the inverse image of every measurable set is measurable, i.e. if* $\{s \in \mathcal{S} : f(s) \in A\} \in \mathcal{T}$ *for all* $A \in \mathcal{T}$

Definition 83 *(Measurable Selection from a Correspondence) Let* $(S, \mathcal{S})$ *and* $(T, \mathcal{T})$ *be measurable spaces, and let* Γ *be a correspondence of* S *into* T. *Then the function* $h : S \to T$ *is a measurable selection from* Γ *if* h *is measurable and* $h(s) \in \Gamma(s), \forall s \in S$.

Theorem 24 *Measurable Selection Theorem)) Let* $S \subset \mathbb{R}^l$ *and* $T \subset \mathbb{R}^m$ *be Borel sets, with their Borel subsets* $\mathcal{S}$ *and* $\mathcal{T}$. *Let* $\Gamma : S \to T$ *be a (nonempty) compact-valued and uhc correspondence. Then there exists a measurable selection from* Γ

Some notion: $M(S, \mathcal{S})$: space of measurable, extended real-valued functions on S $M^+(S, \mathcal{S})$: space of measurable, extended real-valued, non-negative functions on S

Definition 84 *Let* $\phi \in M^+(S, \mathcal{S})$ *be a measurable simple function, with the standard representation* $\phi(s) = \sum_{i=1}^{n} a_i \chi_{A_i}(s)$. *Then the integral of* ϕ *with respect to* μ *is*

$$\int_S \phi(s)\mu(ds) = \sum_{i=1}^{n} a_i \mu(A_i)$$

Definition 85 *For* $f \in M^+(S, \mathcal{S})$, *the integral of* f *with respect to* μ *is*

$$\int_S f(s)\mu(ds) = \sup \int_S \phi(s)\mu(ds)$$

where the supremum is taken over all simple functions ϕ *in* $M^+(S, \mathcal{S})$ *with* $0 \leq \phi \leq f$. *If* $A \in \mathcal{S}$, *then the integral of* f *over* A *with respect to* μ *is*

$$\int_A f(s)\mu(ds) = \int_S f(s)\chi_A(s)\mu(ds)$$

Every $f \in M^+(S, \mathcal{S})$ can be written as the limit of an increasing sequence $\{\phi_n\}$ of simple functions. The next theorem tells us that the integral $\int f d\mu$ is also the unique limit t of $\int \phi_n d\mu$, i.e. it does not depend on the particular sequence $\{\phi_n\}$ chosen.

Theorem 19.4 (Monotone Convergence Theorem) If $\{f_n\}$ is a monotone increasing sequence of functions in $M^+(S, \mathcal{S})$ converging pointwise to f then

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

Now we can think about generate the definition to functions that take negative value. Define the positive parts and negative parts as below: let $f : S \to \mathbb{R}$ be an arbitrary function. We denote

$$f^+(s) = \begin{cases} f(s) & iff(s) \geq 0 \\ 0 & iff(s) < 0 \end{cases}$$

and

$$f^-(s) = \begin{cases} -f(s) & iff(s) \leq 0 \\ 0 & iff(s) > 0 \end{cases}$$

Definition 86 *Let $(S, \mathcal{S}, \mu)$ be a measure space, and let f be a measurable, real-valued function on S. If f^+ and f^- both have finite integrals with respect to μ, then f is integrable and the integral of f with respect to μ is*

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

Definition 87 *If $(S, \mathcal{S}, \mu)$ is a probability space and f is integrable, then call $\int f d\mu$ the expected value of f*

6.1 Transition Functions

Definition 88 *(Transition Function) Let $(Z, \mathcal{Z})$ be a measurable space. A transition function is a function $Q : Z \times \mathcal{Z} \to [0, 1]$ such that*

- *$\forall z \in Z, Q(z, \cdot)$ is a probability measure on $(Z, \mathcal{Z})$, and*
- *$\forall A \in \mathcal{Z}, Q(\cdot, A)$ is a $\mathcal{Z}$ -measurable function.*

Interpretation: $\forall a \in Z, A \in \mathcal{Z}$

$$Q(a, A) = Pr\left\{z_{t+1} \in A \mid z_t = a\right\}$$

A Markov process can be completely described by this transition function, and the most important property of Markov process is the same transition function can be used in all periods, making each period's problem symmetric. Define T to be the operator from $M^+(Z, \mathcal{Z})$

$$(Tf)(z) = \int f\left(z'\right) Q\left(z, dz'\right), \forall z \in Z$$

Interpretation: expected value of f next period if the current state is z, called the Markov operator associated with Q. Define $T^*\lambda$ to be the operator from the set of probability measure on $(Z, \mathcal{Z})$:

$$\left(T^*\lambda\right)(A) = \int Q(z, A)\lambda(dz), \forall A \in \mathcal{Z}$$

Interpretation: probability that the state will be in A next period, given that current values of the state are drawn according to the probability measure λ. Theorem 20.1 (1) T maps the space of bounded $\mathcal{Z}$ -measurable functions, $B(Z; \mathcal{Z})$ into itself. (2) T^* maps the space of probability measures on $(Z; \mathcal{Z})$, that is $\Lambda(Z, \mathcal{Z})$ into itself. (3)

$$\int (Tf)(z)\lambda(dz) = \int f\left(z'\right)\left(T^*\lambda\right)\left(dz'\right)$$

There are other two properties a transition function may have:

Definition 89 *(Feller property) A transition function Q on $(Z, \mathcal{Z})$ has the Feller property if the associated operator T maps the space of bounded continuous functions on Z into itself; that is if $T : C(Z) \to C(Z)$*

Definition 90 *(Monotone) A transition function Q on $(Z, \mathcal{Z})$ is monotone if the associated operator T has the property that for every nondecreasing function $f : Z \to \mathbb{R}$, the function Tf is also nondecreasing.*

6.2 Probability Measures on Space of Sequences

Given a transition function Q on $(Z, \mathcal{Z})$, we want to look at partial (finite) histories of shocks and complete (infinite) histories generated by this transition function:

$$z^t = \left(z_1, \cdots, z_t\right), t = 1, 2, \cdots \quad z^\infty = \left(z_1, z_2, \cdots\right)$$

Let $(Z, \mathcal{Z})$ be a measurable space, and for any finite $t = 1, 2, \cdots$, let

$$\left(Z^t, \mathcal{Z}^t\right) = (Z \times \cdots \times Z, \mathcal{Z} \times \cdots \times \mathcal{Z})$$

denote the product space. We can define a measure on $(Z^t, \mathcal{Z}^t)$

$$\mu^t(z_0, \cdot) = \mathcal{Z}^t \to [0,1], \quad t = 1, 2, \ldots$$

as follow: $\forall B = A_1 \times \cdots \times A_t \in \mathcal{Z}^t$

$$\mu^t(z_0, B) = \int_{A_1} \cdots \int_{A_{t-1}} \int_{A_t} 1Q(z_{t-1}, dz_t) Q(z_{t-2}, dz_{t-1}) \cdots Q(z_0, dz_1)$$

This approach can also be used to define probability over infinite sequences $z^\infty = (z_1, z_2, \ldots)$ (So we will work with infinite product space $Z^\infty = Z \times Z \times \cdots$.) Define a finite measurable rectangle $B \subset Z^\infty$:

$$B = A_1 \times A_2 \times \cdots \times A_T \times Z \times Z \times \cdots$$

where $A_t \in \mathcal{Z}, t = 0, 1, 2, \cdots, T < \infty$. Let $\mathcal{C}$ be the family of all finite measurable rectangles, and $\mathcal{A}^\infty$ the family of all finite unions of sets in $\mathcal{C}$. Then we can show that $\mathcal{A}^\infty$ is an algebra. Let $\mathcal{Z}^\infty$ be the σ -algebra generated by $\mathcal{A}^\infty$. Define the measure similar as before;

$$\mu^\infty(z_0, B) = \int_{A_1} \cdots \int_{A_{t-1}} \int_{A_t} Q(z_{t-1}, dz_t) Q(z_{t-2}, dz_{t-1}) \cdots Q(z_0, dz_1)$$

We can check that this measure will satisfy the three conditions imposed on measures on an algebra. By the Caratheodory and Hahn Extension Theorem, exists a unique extension of μ^∞ to $\mathcal{Z}^\infty$. Therefore, $(Z^\infty, \mathcal{Z}^\infty, \mu^\infty)$ is a probability space.

Definition 91 *(Stochastic Process) A stochastic process on $(\Omega, \mathcal{F}, P)$ is an increasing sequence of σ -algebra $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots \subset \mathcal{F}$; a measurable space $(Z, \mathcal{Z})$; and a sequence of functions $\sigma : \Omega \to Z, t = 1, 2, \cdots$ such that each σ_t is $\mathcal{F}_t$ measurable.*

Definition 92 *A stochastic process is called stationary if $P_{t+1,\cdots,t+n}$ is independent of t., i.e*

$$F_{t_1+k, t_2+k, \cdots, t_s+k}(b_1, b_2, \cdots, b_s) = F_{t_1, t_2, \cdots, t_s}(b_1, b_2, \cdots, b_s)$$

for any finite set of indices $\{t_1, t_2, \cdots, t_s\} \subset \mathbb{Z}$ with $s \in \mathbb{Z}^+$, and any $k \in \mathbb{Z}$

Similarly, we can define conditional probability

$$P_{t+1,\cdots,t+n}(C \mid a_{t-s}, \cdots, a_{t-1}, a_t)$$

Definition 93 *A stochastic process is called a (first-order) Markov process if*

$$P_{t+1,\cdots,t+n}(C \mid a_{t-s}, \cdots, a_{t-1}, a_t) = P_{t+1,\cdots,t+n}(C \mid a_t)$$

7 Fixed Point Theorems

Theorem 25 ***Brouwer's Fixed Point Theorem – continuous function***
Let $S \subset \mathbb{R}$ be nonempty, compact, and convex, and $f : S \to S$ be a continuous function. Then f has (at least) a fixed point in S, i.e. $\exists x^* \in S : x^* = f(x^*)$

Theorem 26 ***[Optional] Tarsky's Fixed Point Theorem – weakly increasing functio**ns*
Let $f : [0,1]^n \to [0,1]^n$, where $[0,1]^n = [0,1] \times ... \times [0,1]$, an n-dimensional cube. If f is nondecreasing, then f has a fixed point in $[0,1]^n$.

Theorem 27 ***Kakutani's Fixed Point Theorem – u.h.c. correspondence***
Let $S \subset \mathbb{R}$ be nonempty, compact, and convex, and $\Gamma : S \rightrightarrows S$ be a nonempty, convex-valued, and u.h.c. correspondence. Then Γ has a fixed point in S, i.e. $\exists x^* \in S : x^* \in \Gamma(x^*)$

Since S is compact, u.h.c. is equivalent to Γ having a closed graph.

Theorem 28 ***[Optional] Fixed Point Theorem – l.h.c. correspondence***
Let $S \subset \mathbb{R}$ be nonempty, compact, and convex, and $\Gamma : S \rightrightarrows S$ be a nonempty, convex-valued, closed-valued, and l.h.c. correspondence. Then Γ has a fixed point in S.

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