



Recitation 6

[Definitions used today]

- Risk aversion and strict risk aversion, Jensen's inequalities.
- FOSD, SOSD, more risky than, Pratt's theorem

Question 1 [Risk compensation] 107 [I.2 Fall 2010 majors]

Consider an agent with expected utility function $E[v(\cdot)]$, where the von Neumann-Morgenstern utility function v is strictly increasing. Consider risk compensation $\rho(w, t\tilde{z})$ as a function of scale factor t for arbitrary $t \in \mathbb{R}_+$.

- State a definition of risk compensation $\rho(w, \tilde{z})$ for risky gamble \tilde{z} with $E(\tilde{z}) = 0$ at deterministic wealth w
- Show that $\rho(w, t\tilde{z})$ is a strictly increasing function of t that takes zero value at $t = 0$ for every w and \tilde{z} with $E(\tilde{z}) = 0$, if and only if the agent is strictly risk averse.

Question 2 [Pratt] 91 [II.1 Fall 2009 majors]

Consider an agent whose preferences over real-valued random variables (or state-contingent consumption plans) are represented by an expected utility function with strictly increasing and twice differentiable vN-M utility $v: \mathbb{R} \rightarrow \mathbb{R}$. Let $\rho(w, \tilde{z})$ denote the risk compensation for random variable \tilde{z} with $E(z) = 0$ at risk-free initial wealth w . Let $A(w)$ denote the Arrow-Pratt measure of risk aversion at w .

- Prove that A is an increasing function of w if and only if risk compensation ρ is an increasing function of w for every \tilde{z} with $E(\tilde{z}) = 0$ and $\tilde{z} \neq 0$.
- Derive an explicit expression for risk compensation for quadratic utility $v(x) = -(\alpha - x)^2$ where $\alpha > 0$. Prove that this quadratic utility is, up to an increasing linear transformation, the only utility function with risk compensation of the form you derived.
- Give an example of two vN-M utility function v_1 and v_2 such that neither v_1 is more risk averse than v_2 , nor v_2 is risk averse than v_1 in the sense of the Theorem of Pratt.

Question 3

There are three states with equal probabilities $\pi_s = \frac{1}{3}$ for $s \in \{1, 2, 3\}$. Consider two state contingent consumption plans $z = (8, 2, 2)$, and $y = (3, 3, 6)$

- Does y FOSD dominate z ?
- Is z more risky than y ?

Question 4 124 [I.2 Fall 2011 majors]

Consider two real-valued random variables \tilde{y} and \tilde{z} on some state space (i.e. probability space). Let F_y and F_z be their cumulative distribution functions, and $E(\tilde{z})$ and $E(\tilde{y})$ their expected values.

- State a definition of \tilde{z} first-order stochastically dominating (FSOD) \tilde{y} . Show that if \tilde{z} FSOD \tilde{y} , then $E(\tilde{z}) \geq E(\tilde{y})$
- Show that, if \tilde{z} FSOD \tilde{y} and $E(\tilde{z}) = E(\tilde{y})$, then \tilde{y} and \tilde{z} have the same distribution, i.e., $F_y(t) = F_z(t)$ for every $t \in \mathbb{R}$. If you find it convenient, you may assume in your proof that random variables \tilde{y} and \tilde{z} have densities, or alternatively that \tilde{y} and \tilde{z} are discrete random variables (i.e., take finitely many values).
- State a definition of \tilde{z} second-order stochastically dominating (SSOD) \tilde{y} . Show that if \tilde{z} ssn \tilde{y} , then $E(\tilde{z}) \geq E(\tilde{y})$
- Show that if \tilde{z} FSOD \tilde{y} , then \tilde{z} SSD \tilde{y} .
- State a definition of \tilde{y} being more risky than \tilde{z} . Give a brief justification for why it is a sensible definition of more risky.

Question 5 [Stochastic Dominance and Risk]

Consider two real-valued random variables y and z on some finite state space with $\mathbb{E}[y] = \mathbb{E}[z]$.

- a Prove that if y is more risky than z , then $\mathbb{E}[v(z)] \geq \mathbb{E}[v(y)]$ for every nondecreasing continuous and concave function $v : \mathbb{R} \rightarrow \mathbb{R}$. You may assume v is twice differentiable.
- b Give an example of two random variables y and z such that $y \neq z$, $\mathbb{E}[y] = \mathbb{E}[z]$ and neither z is more risky than y nor y is more risky than z .

Question 6

Consider an optimal portfolio choice problem with one risky asset with return \tilde{r} and a risk-free asset with return r_f . Suppose that the agent's vNM utility function is $v(x) = -(\alpha - x)^2$ for some $\alpha > 0$. Assume that $\alpha > wr_f$ where $w > 0$ is agent's wealth. Negative investment (i.e. short selling) is permitted for both assets.

- a Find the optimal investment in the risky asset as a function of expected return and the variance of the risky return.
- b Suppose that the return \tilde{r} on the risky asset is changed to a more risky return \tilde{r}' with the same expectation $\mathbb{E}[\tilde{r}'] = \mathbb{E}[\tilde{r}]$. Assume $\mathbb{E}[\tilde{r}] > r_f$. Prove that the optimal investment in the risky asset with more risky return \tilde{r}' is smaller than the optimal investment with return \tilde{r} , all else unchanged.