



Recitations 16

[Definitions used today]

- Best correspondence, Nash Equilibrium, Minimax Theorem

Question 1

1/2	L	R
T	3,1	0,0
B	0,0	1,3

- Define: pure actions, mixed actions, best correspondences
- Find all Nash Equilibria

Solution 1

pure strategies: $A^1 = \{T, B\}$, $A^2 = \{L, R\}$

mixed strategies:

$$S = \{(p, 1-p), (q, 1-q) \mid p, q \in [0, 1]\}$$

We can solve for the best responses as follows:

$$BR^2((p, 1-p)) : \left\{ \begin{array}{cc} L & R \\ p + 0(1-p) & 0(p) + 3(1-p) \end{array} \right\}$$

Equality only holds when $p = \frac{3}{4}$. $L > R \iff p > \frac{3}{4}$, $L < R \iff p < \frac{3}{4}$

$$BR^1((q, 1-q)) : \left\{ \begin{array}{cc} T & B \\ 3(q) + 0(1-q) & 0(q) + 1(1-q) \end{array} \right\}$$

Equality only holds when $q = \frac{1}{4}$. $T > B \iff q > \frac{1}{4}$, $T < B \iff q < \frac{1}{4}$

Therefore, player 1 sets $p = 1$ if $q \geq \frac{1}{4}$ and sets $p = 0$ otherwise. Similarly, player 2 sets $q = 1$ if $p > \frac{3}{4}$ and sets $q = 0$ otherwise.

These best responses can be graphed :

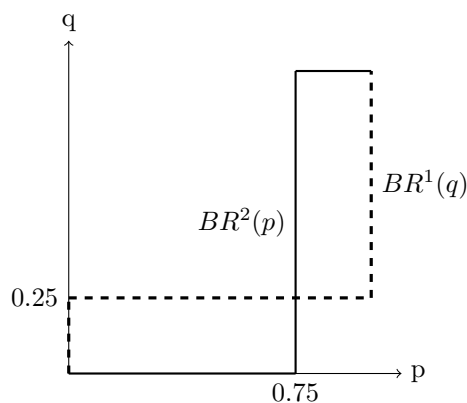


Figure 1: Best Responses

The points of intersection

$$\left(\frac{3}{4}, \frac{1}{4}\right), (1, 1), (0, 0)$$

yield the set of Nash equilibria

$$NE = \left\{ ((1, 0), (1, 0)), ((0, 1), (0, 1)), \left(\frac{3}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{3}{4}\right) \right\}.$$

Question 2 [153 III.1 Spring 2013 majors]

A two players finite action normal form game is zero sum if the sum of the utilities of the two players is equal to 0 for any action profile, so $u^1 = -u^2$. **The Minimax Theorem** states that in this case

$$\min_{\alpha^2 \in \Delta(A^2)} \max_{\alpha^1 \in \Delta(A^1)} u(\alpha^1, \alpha^2) = \max_{\alpha^1 \in \Delta(A^1)} \min_{\alpha^2 \in \Delta(A^2)} u(\alpha^1, \alpha^2) \equiv v$$

Prove the minimax theorem. You can use Nash equilibrium existence theorem.

Solution 2

We will do it in three two: First we will prove that \geq holds. Secondly that \leq holds.

\geq . Note that for any $\bar{s}^1 \in \Delta(A^1)$ and $\bar{s}^2 \in \Delta(A^2)$ it holds that:

$$u(\bar{s}^1, \bar{s}^2) \geq \min_{s^2 \in \Delta(A^2)} u(\bar{s}^1, s^2)$$

Then by taking maximum at both sides with respect to s^1 :

$$\max_{s^1 \in \Delta(A^1)} u(s^1, \bar{s}^2) \geq \max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} u(s^1, s^2)$$

Note that the RHS is now constant, and a lower bound to the LHS across s^2 , then:

$$\min_{s^2 \in \Delta(A^2)} \max_{s^1 \in \Delta(A^1)} u(s^1, s^2) \geq \max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} u(s^1, s^2) \quad (0.1)$$

\leq . Note that for any $\bar{s}^1 \in \Delta(A^1)$ it holds that:

$$\max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} u(s^1, s^2) \geq \min_{s^2 \in \Delta(A^2)} u(\bar{s}^1, s^2)$$

In particular for \hat{s}^1 a NE of the game the inequality must hold. We assume that such NE exists in mixed strategies. Note that if (\hat{s}^1, \hat{s}^2) it is defined as an strategy profile such that:

$$u(\hat{s}^1, \hat{s}^2) = \max_{s^1 \in \Delta(A^1)} u(s^1, \hat{s}^2) \quad - \quad u(\hat{s}^1, \hat{s}^2) = \max_{s^2 \in \Delta(A^2)} -u(\hat{s}^1, s^2)$$

The second condition implies:

$$u(\hat{s}^1, \hat{s}^2) = \min_{s^2 \in \Delta(A^2)} u(\hat{s}^1, s^2) = \max_{s^1 \in \Delta(A^1)} u(s^1, \hat{s}^2)$$

thus

$$\begin{aligned} \min_{s^2 \in \Delta(A^2)} u^1(\hat{s}^1, s^2) &= u^1\left(\hat{s}^1, \operatorname{argmin}_{s^2 \in \Delta(A^2)} u^1(\hat{s}^1, s^2)\right) \\ &= u^1\left(\hat{s}^1, \operatorname{argmax}_{s^2 \in \Delta(A^2)} u^2(\hat{s}^1, s^2)\right) \\ &= u^1(\hat{s}^1, \hat{s}^2) \\ &= \max_{s^1 \in \Delta(A^1)} u^1(s^1, \hat{s}^2) \\ &\geq \min_{s^2 \in \Delta(A^2)} \max_{s^1 \in \Delta(A^1)} u^1(s^1, s^2) \end{aligned}$$

Then by taking max over $\Delta(A^1)$:

$$\max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} u(s^1, s^2) \geq \min_{s^1 \in \Delta(A^1)} u(s^1, \hat{s}^2) \geq \min_{s^2 \in \Delta(A^2)} \max_{s^1 \in \Delta(A^1)} u(s^1, s^2) \quad (0.2)$$

Inequalities (0.1) and (0.2) gives us thesis of minimax theorem.

Question 3

For a zero sum game of two players define the value of the game as $V : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ (where $n = \#A^1$ and $m = \#A^2$) :

$$V(u) = \max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} U(s^1, s^2 | u)$$

where for a given strategy profile $s^1 = (p_1, \dots, p_n)$, $s^2 = (q_1, \dots, q_m)$ and payoffs $u \in \mathbb{R}^{nm}$ we define

$$U(s^1, s^2 | u) = \sum_{i=1}^n \sum_{j=1}^m p_i q_j u_{ij}$$

Show that **The value of a game** is

- a) continuous
- b) non-decreasing
- c) homogenous of degree one in payoffs.

Solution 3

- Consider the problem:

$$v(s^1, u) = \min_{s^2 \in \Delta(A^2)} U(s^1, s^2 | u)$$

note that U is continuous in s_1, s_2 and u and that the minimum is being taken over s^2 in a compact set that does not vary with s^1 or u . By the theorem of the maximum the value of this problem, as a function of s^1 and u is a continuous function. Now consider:

$$V(u) = \max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} U(s^1, s^2 | u) = \max_{s^1 \in \Delta(A^1)} v(s^1, u)$$

again since v is continuous and s^1 varies in a compact set independent of u by the theorem of the maximum V is a continuous function of u .

- Let $u_1 \leq u_2$. Clearly for all s^1, s^2 :

$$U(s^1, s^2 | u_1) = \sum_{i=1}^n \sum_{j=1}^m p_i q_j u_{ij}^1 \leq U(s^1, s^2 | u_2) = \sum_{i=1}^n \sum_{j=1}^m p_i q_j u_{ij}^2$$

so $U(s^1, s^2 | u_1) \leq U(s^1, s^2 | u_2)$. Then:

$$\begin{aligned} \min_{s^2 \in \Delta(A^2)} U(s^1, s^2 | u_1) &\leq \min_{s^2 \in \Delta(A^2)} U(s^1, s^2 | u_2) \\ V(u_1) = \max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} U(s^1, s^2 | u_1) &\leq \max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} U(s^1, s^2 | u_2) = V(u_2) \end{aligned}$$

- Let $\lambda \in \mathbb{R}$, note that $U(s^1, s^2 | \lambda u) = \sum_{i=1}^n \sum_{j=1}^m p_i q_j \lambda u_{ij} = \lambda U(s^1, s^2 | u)$ and $\max_x \lambda f(x) = \lambda \max_x f(x)$. Thus $V(\lambda u) = \lambda V(u)$

Question 4

Under standard assumptions, prove the following properties of best response in mixed $BR_i(s)$:

- a) non-empty valued,
- b) compact valued,
- c) upper hemi continuous.
- d) convex-valued

Solution 4

- a) Take any $s \in S$. Then $BR^i(s) = \arg \max_{r^i \in S^i} u^i(r^i, s^{-i})$. Since $u^i(\cdot, s^{-i})$ is continuous and $S^i = \Delta(A^i)$ is compact, by the Weierstrass Theorem u^i achieves a maximum on S^i . Hence, $BR^i(s)$ is nonempty. Since s has been arbitrary, $BR^i(\cdot)$ is nonempty-valued.
- b) that converges in S^i , i.e. $r_m^i \rightarrow r^i \in S^i$. By definition we have $u^i(r_m^i, s^{-i}) \geq u^i(t^i, s^{-i}) \forall t^i \in S^i, m \in \mathbb{N}$. Then since $u^i(\cdot, s^{-i})$ is continuous,

$$u^i(r^i, s^{-i}) = u^i\left(\lim_{m \rightarrow \infty} r_m^i, s^{-i}\right) = \lim_{m \rightarrow \infty} u^i(r_m^i, s^{-i}) \geq u^i(t^i, s^{-i}) \quad \forall t^i \in S^i$$

Hence, $r^i \in BR^i(s)$. Since s has been arbitrary, $BR^i(\cdot)$ is closed-valued.

- c) Since S^i (the range of $BR^i(\cdot)$) is compact, it is sufficient to establish that $BR^i(\cdot)$ has a closed graph. Fix $s \in S$ arbitrarily and take any sequences $(s_m) \in S^\infty$ and $(r_m^i) \in S^{i\infty}$ with $s_m \rightarrow s \in S, r_m^i \rightarrow r^i \in S^i$ and $r_m^i \in BR^i(s_m) \forall m \in \mathbb{N}$. Then $u^i(r_m^i, s_m^{-i}) \geq u^i(t^i, s_m^{-i}), \forall t^i \in S^i$. Since $u^i(\cdot, \cdot)$ is continuous it follows that $\forall t^i \in S^i$

$$\begin{aligned} u^i(r^i, s^{-i}) &= u^i\left(\lim_{m \rightarrow \infty} r_m^i, \lim_{m \rightarrow \infty} s_m^{-i}\right) = \lim_{m \rightarrow \infty} u^i(r_m^i, s_m^{-i}) \\ &\geq \lim_{m \rightarrow \infty} u^i(t^i, s_m^{-i}) \\ &= u^i\left(t^i, \lim_{m \rightarrow \infty} s_m^{-i}\right) \\ &= u^i(t^i, s^{-i}) \end{aligned}$$

Hence, $r^i \in BR^i(s)$ and $BR^i(\cdot)$ is closed at s . Since s has been arbitrary, $BR^i(\cdot)$ has a closed graph.

- d) Fix $s \in S$ arbitrarily and take any $r_a^i, r_b^i \in BR^i(s)$ and $\lambda \in (0, 1)$. Then it must be that $u^i(r_a^i, s^{-i}) = u^i(r_b^i, s^{-i}) \geq u^i(r^i, s^{-i}) \forall r^i \in S^i$. Or, equivalently,

$$\sum_{a^i \in A^i} r_a^i(a^i) u^i(a^i, s^{-i}) = \sum_{a^i \in A^i} r_b^i(a^i) u^i(a^i, s^{-i}) \geq \sum_{a^i \in A^i} r^i(a^i) u^i(a^i, s^{-i}) \quad \forall r^i \in S^i$$

Now consider the mixed strategy $\lambda r_a^i + (1 - \lambda)r_b^i$. The utility of this strategy profile is

$$\begin{aligned} u^i[\lambda r_a^i + (1 - \lambda)r_b^i, s^{-i}] &= \sum_{a^i \in A^i} [\lambda r_a^i(a^i) + (1 - \lambda)r_b^i(a^i)] u^i(a^i, s^{-i}) \\ &= \lambda \sum_{a^i \in A^i} r_a^i(a^i) u^i(a^i, s^{-i}) + (1 - \lambda) \sum_{a^i \in A^i} r_b^i(a^i) u^i(a^i, s^{-i}) \\ &= \sum_{a^i \in A^i} r^i(a^i) u^i(a^i, s^{-i}) \\ &\geq u^i(r^i, s^{-i}) \quad \forall r^i \in S^i, \end{aligned}$$

where the third line follows from (2) and the inequality holds since $r_a^i \in BR^i(s)$. Hence, $\lambda r_a^i + (1 - \lambda)r_b^i \in BR^i(s)$ and, since s has been arbitrary, $BR^i(\cdot)$ is convex-valued.

Question 5

Show that $BR_i(s) = \text{co}(\{\delta_{b^i} : b^i \in BR_{A^i}^i(s)\})$

Solution 5

- $BR_i(s) \subset \text{co}(\{\delta_{b^i} : b^i \in BR_{A^i}^i(s)\})$

Lemma 0.1.

$$\forall b^i \notin BR_{A^i}(s), b^i \in A^i \Rightarrow s^i(b^i) = 0$$

Suppose not. if the strategy $s^i \in BR_{A^i}(s)$ uses some pure action $b^i \in A^i$ which $\notin BR_{A^i}(s)$, i.e. $s^i(b^i) > 0$ then

$$\forall c^i \in BR_{A^i}(s) : u^i(c^i, s^{-i}) > u^i(b^i, s^{-i})$$

Consider another mixed strategy r^i , defined as follows:

$$r^i(a^i) = s^i(a^i) \quad \forall a^i \in A^i / \{b^i, c^i\}$$

$$r^i(b^i) = 0$$

$$r^i(c^i) = s^i(b^i) + s^i(c^i)$$

then

$$\begin{aligned} u^i(r^i, s) &= \sum_{a^i \in A^i} r^i(a^i) u^i(a^i, s^{-i}) + r^i(b^i) u^i(b^i, s^{-i}) + r^i(c^i) u^i(c^i, s^{-i}) = \\ &= \sum_{a^i \in A^i} s^i(a^i) u^i(a^i, s^{-i}) + [s^i(b^i) + s^i(c^i)] u^i(c^i, s^{-i}) > \\ &\sum_{a^i \in A^i} s^i(a^i) u^i(a^i, s^{-i}) + s^i(b^i) u^i(b^i, s^{-i}) + s^i(c^i) u^i(c^i, s^{-i}) = u^i(s^i, s^{-i}) \end{aligned}$$

contradiction with $s^i \in BR^i(s)$.

$BR_i(s) \subset \text{co}(\{\delta_{b^i} : b^i \in BR_{A^i}^i(s)\})$ comes straight from lemma.

- $BR_i(s) \supset \text{co}(\{\delta_{b^i} : b^i \in BR_{A^i}^i(s)\})$

BR is convex valued. We need to show that $(\{\delta_{b^i} : b^i \in BR_{A^i}^i(s)\}) \subset BR^i(s)$

Suppose not Let $b^i \in BR^i(s)$ and suppose $\delta_{b^i} \notin BR^i(s)$ then

$$\exists s^i \in \Delta(A^i) \quad u^i(s^i, s^{-i}) > u^i(b^i, s^{-i})$$

$$\sum_{a^i \in A^i} s^i(a^i) u^i(a^i, s^{-i}) > u^i(b^i, s^{-i}) = \sum_{a^i \in A^i} s^i(a^i) u^i(b^i, s^{-i})$$

for at least one a^i $u^i(a^i, s^{-i}) > u^i(b^i, s^{-i})$ contradicts $b^i \in BR_{A^i}^i(s)$