Growth theory

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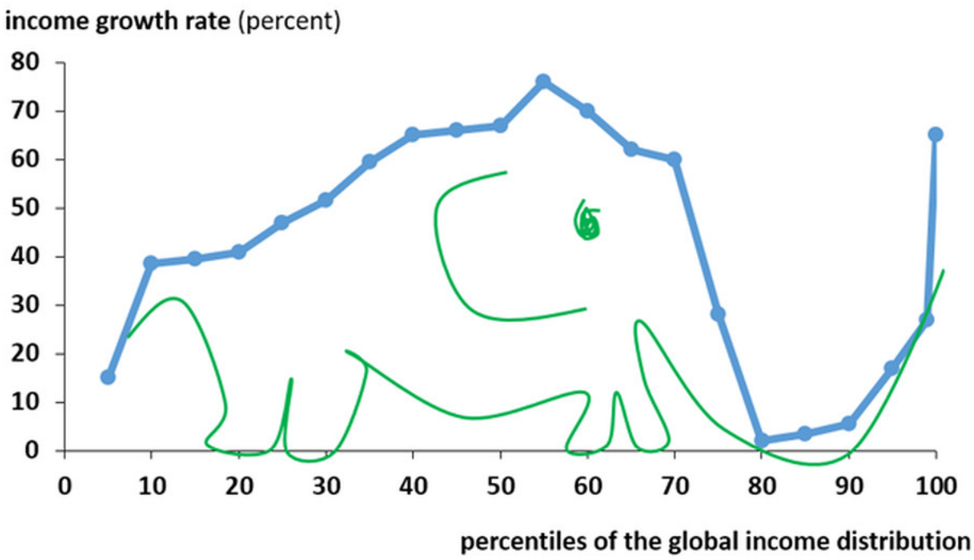
Lecture notes



SGH

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CONTENTS

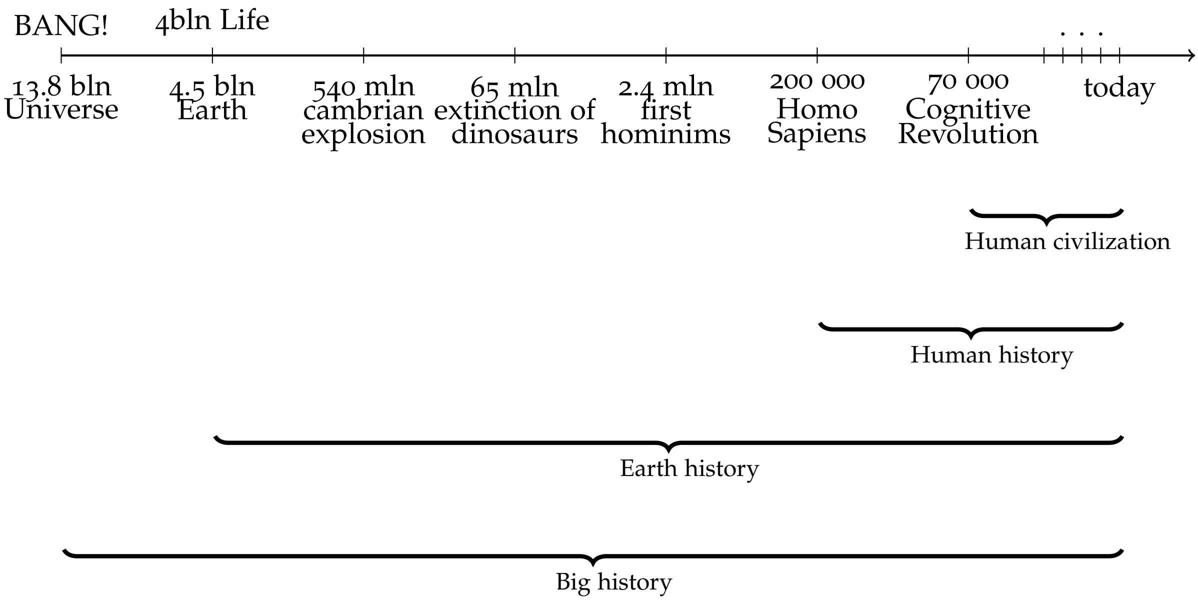
Part I

EMPIRICS OF ECONOMIC GROWTH

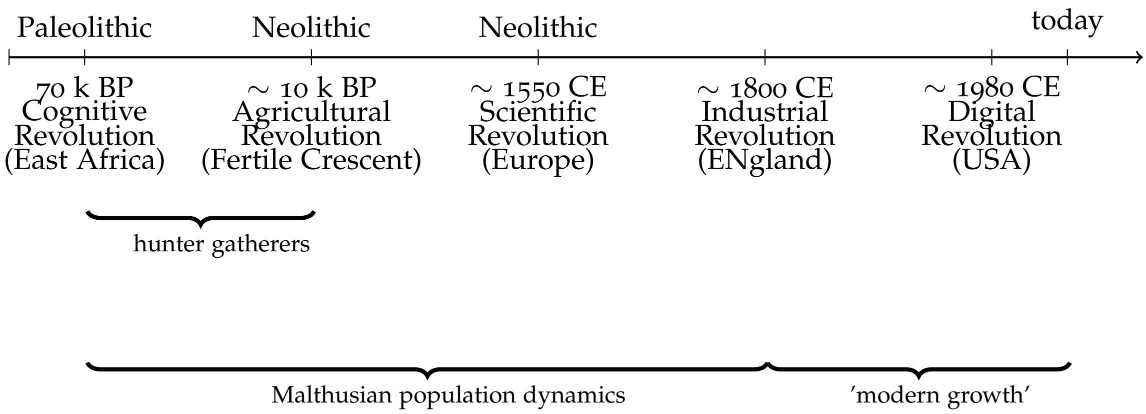
1

OVERVIEW OF GROWTH THEORY

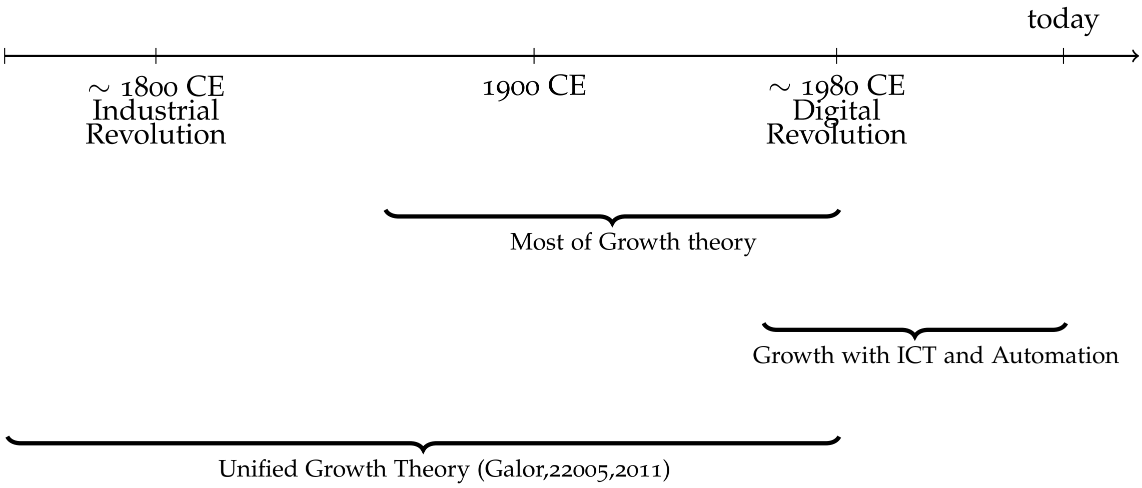
Timeline (not to scale!)



Zoom in



to scale



1.1 HOW MANY INDUSTRIAL REVOLUTIONS??

1. ~ 1800 CE, steam engine, Railroads, Loom **Energy**
2. ~ 1870 CE, Electricity, Internal combusting engine, telephone
3. ~ 1960 CE (Gordon 2016), ICT (computers, cell phones, Internet) **Data processing**
4. ~ 2000 CE(Schwab 2016), Cyber physical systems, Internet of things, 5G,3D printing

Growth Theory vs Development Economics

- Sources of growth in rich countries
- Focus on the world technology frontier
- R&D Technological progress
- Institutions as a source of growth

- Growth and catch-up in poorer countries
- Focus on distance to frontier
- Technology diffusion, adoption foreign direct investment, spillovers
- institutional failures ('Why Nations Fail?')

Cross-country perspective

- Wealth of Nations (Adam Smith)
- 'Why do some countries produce so much more output per worker tan others?' (Hall & Jones 1999)

1. Some region takes off, other stay behind $\Rightarrow$ DIVERGENCE
2. Forces of catch-up, tech diffusion $\Rightarrow$ CONVERGENCE
3. New tech breakthrough, acceleration at WTF $\Rightarrow$ DIVERGENCE
4. . . .

Possibility of leapfrogging!

Leaders in population density

- East Africa ($\sim$ 20 000 BP onwards)
  ⋮
- Fertile Crescent (Mezopotamia, Indus Valley, Nile Valley) ($\sim$ 10 000 BP onwards)
- Mediteranean Basin (Archaemenid Empire $\sim$ 480 BCE 49.4 mln, 44% of world population)
- China (1500 CE : 125 mln, 28.5% of world pop)

Leaders in GDP percapita (Maddison 2008)

1. Italy 1500 CE, 1100$ **Scientific revolution**
2. Netherlands, 1600 CE, 1400 $
3. UK, 1870 CE 3200$ **Industrial revolution**
4. USA 1913 CE ,5300$, 2008, 31200$

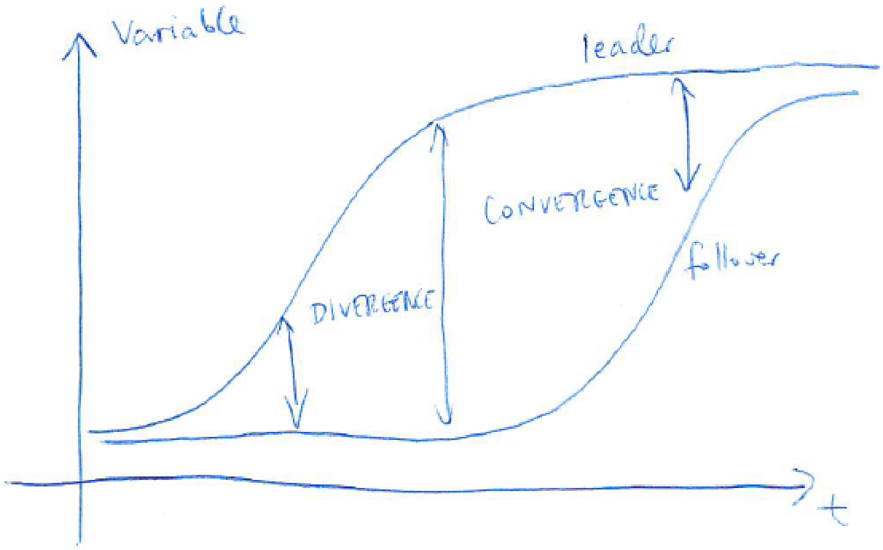


Figure 1

1.2 CONVERGENCE CONCEPTS

1.2.1 *β-convergence*

$$\text{Growth}_t = \beta \cdot \text{GDP}_{t-1} + \cdots + \varepsilon_t$$

$\beta < 0$ and growth $\approx 2\%$ per annum

Types of β-convergence:

- absolute
- conditional

1.2.2 *σ-convergence*

$$\text{VAR}(\text{GDP}_t) \searrow t$$

$\beta < 0$ and growth $\approx 2\%$ per annum

Types:

- absolute
- club convergence (Quah 1995)

You may have absolute β-divergence with conditional β convergence

You may have absolute σ-divergence with conditional σ convergence

---

1.3 Sources of growth? (globally and at frontier)

1. Technological progress
   - Ideas are non-rivaralrous and therefore a source of increasing returns to scale (Romer 1990)
2. Factor accumulation
   - K
   - Human capital
   - Computer software ?
3. Raw materials? Energy? Data?

Gapmider.org
(Mark a Countrym slide the year)
(Income vs Life Expectancy)
(Income vs $C)_2$ Emissions)
(Income vs Total Fertility Rate)
(Income vs Child Mortality)
(Income vs Expected Growth 10 years) UK vs US
Global Income Distribution

Part II

EXOGENOUS GROWTH MODELS

2

SOLOW AND MANKIW-ROMER-WEIL MODELS

2.1 WHAT CAN BE A SOURCE OF GROWTH?

Observation: 1

Decreasing returns + depreciation lead to a steady state. Growth must come to a stop

2.1.1 *Physical capital (Solow 1956)*

$$Y = F(K, L)$$

with constant returns to (K, L) and decreasing returns to K alone

$$y = \frac{Y}{L} = \frac{F(K,L)}{L} = F(\frac{K}{L}, \frac{L}{L}) := f(k), \qquad \text{where} \quad k = \frac{K}{L}$$

- Assume a constant saving rate (Solow model) and a standard capital equation of motion
- Denote by $\dot{X} = \frac{dX(t)}{dt}$ and $\hat{X} = \frac{\dot{X}}{X}$

$$\dot{K} = sY - \delta K$$

$$\frac{\dot{K}}{L} = sy - \delta k$$

$$\dot{k} = \frac{\dot{K}}{L} = \frac{\dot{K}L - \dot{L}K}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L}\underbrace{\frac{\dot{L}}{L}}_{1} \quad \text{or} \quad \hat{k} = \hat{K} - \hat{L}$$

Assuming a steady population growth rate

$$\hat{L} = n \quad \dot{L} = nL$$

we have:

$$\dot{k} = \frac{\dot{K}}{L} - kn = sy - (\delta + n)k$$

where $y = f(k)$ -increasing and concave

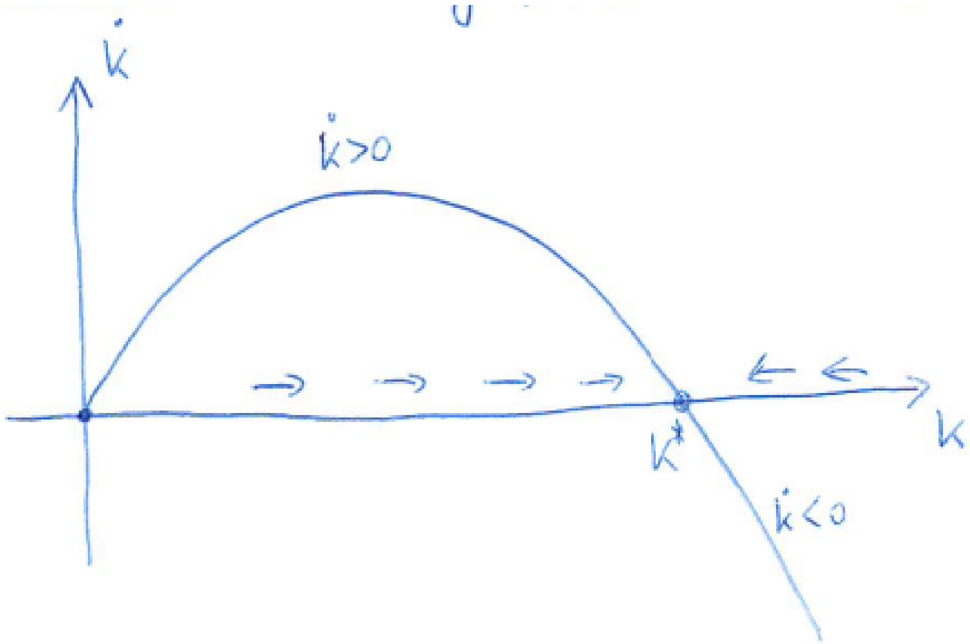


Figure 2

Definition: 1: Cobb-Douglas production

$$F(K,L) = K^{\alpha}L^{1-\alpha} \quad \Rightarrow \quad f(k) = k^{\alpha}$$

Definition: 2: CES production

$$F(K,L) = (\alpha K^{\theta} + (1-\alpha)L^{\theta})^{\frac{1}{\theta}} \quad \Rightarrow \quad f(k) = (\alpha k^{\theta} + 1 - \alpha)^{\frac{1}{\theta}}$$

Decreasing returns to K and a steady state are guaranteed for the GROSS COMPLEMENTARITY Case ($\theta < 0$) but not the GROSS SUBSTITUTABILITY Case ($\theta > 0$).

Folowing condition holds

Theorem: 1: Steady state of the Solow model

$$\dot{k} = 0 \qquad \Longleftrightarrow \qquad sy = (\delta + n)k$$

-E.g. Cobb-Douglas

$$sk^{\alpha} = (\delta + n)k \quad \Rightarrow \quad k^{\alpha - 1} = \frac{\delta + n}{\delta}$$

$$k^* = (\frac{\delta}{\delta + n})^{\frac{1}{1-\alpha}}$$

-E.g. CES production ($\theta < 0$) :

$$s(\alpha k^{\theta} + 1 - \alpha)^{\frac{1}{\theta}} = (\delta + n)k$$

$$s^{\theta}\alpha k^{\theta} + s^{\theta}(1-\alpha) = (\delta + n)^{\theta}k^{\theta}$$

$$(s^\theta \alpha - (\delta + n)^\theta) k^\theta = -s^\theta (1-\alpha)$$

$$k^\theta = \frac{s^\theta (1-\alpha)}{(\delta+n)^\theta - s^\theta \alpha}$$

$$k^* = \frac{s(1-\alpha)^{1/\theta}}{((\delta+n)^\theta - s^\theta \alpha)^{1/\theta}}$$

Under the assumption that $(n+\delta)^\theta > \alpha s^\theta \iff \delta + n < \alpha^{1/\theta} s \iff s > \frac{\delta+n}{\alpha^{1/\theta}}$

2.2 HUMAN CAPITAL

simplified model

$$Y = C = hl$$

where h- human capital per worker
l- hours worked per worker $l \in [0,1]$
$L \equiv 1$- number of workres

- Assume a la Solow that $l \equiv$const.
- Let $\phi(l,h)$ be the 'education function' with decreasing returns to h

$$\dot{h} = \phi(l,h) - \delta h$$

E.g. Cobb Douglas $\phi(l,h) = (1-l)h^\gamma$ where $\gamma \in (0,1)$ decreasing retrns

$$\dot{h} = (1-l)h^\gamma - \delta h$$

Steady state

$$\dot{h} = 0 \iff (1-l)h^{\gamma-1} = \delta \iff h^* = (\frac{1-l}{\delta})^{\frac{1}{1-\gamma}}$$

2.3 MANKIW-ROMER-WEIL MODEL (1992)

with both physical and human capital

$$Y = F(K,H,L) = K^\alpha H^\beta L^{1-\alpha-\beta} \qquad \alpha + \beta < 1$$

assumed immediately by MRW

$$y = k^\alpha h^\beta$$

- Assume identical production function for physical and human capital as well as the consumption good. And equal depreciation rates
- Assume constant savings rate a la Solow (s_k, s_h)

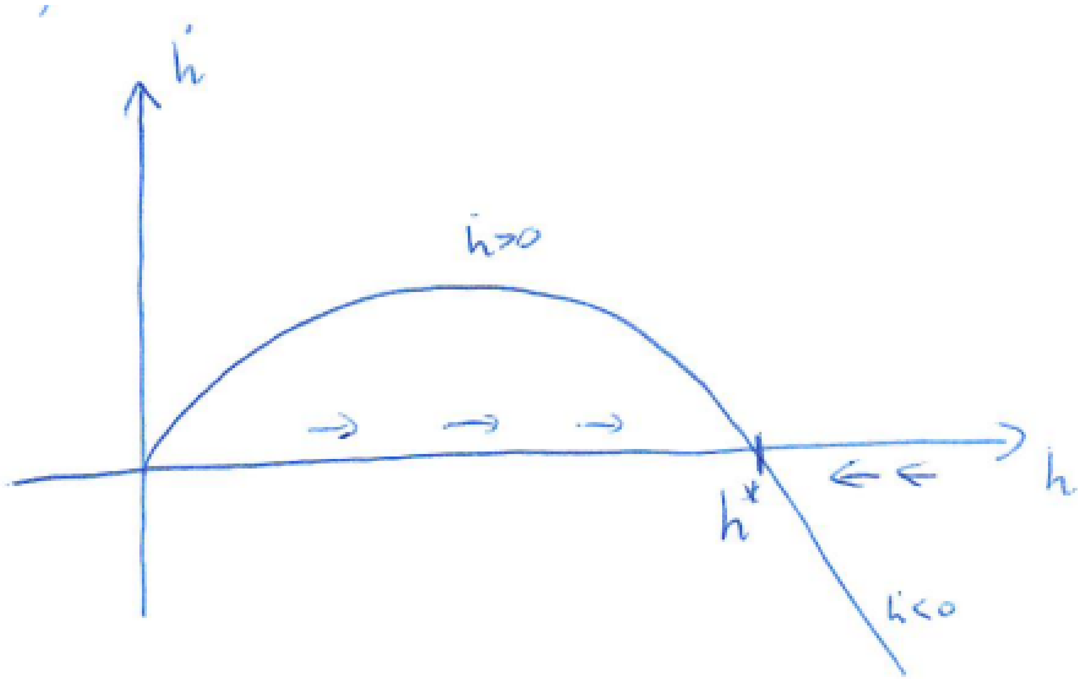


Figure 3

$$\begin{cases} \dot{k} = s_k y - (\delta + n)k \\ \dot{h} = s_h y - (\delta + n)h \end{cases}$$

Steady state

$$\dot{k} = \dot{h} = 0 \iff \begin{cases} k = \frac{s_k y}{\delta+n} \\ h = \frac{s_h y}{\delta+n} \end{cases} \iff \frac{k}{h} = \frac{s_k}{s_h}$$

$$k = \frac{s_k k^\alpha h^\beta}{\delta + n} = \frac{s_k k^\alpha k^\beta \frac{s_h^\beta}{s_k^\beta}}{\delta + n}$$

$$k^{1-\alpha-\beta} = \frac{s_h^\beta s_k^{1-\beta}}{\delta + n} h = \frac{k s_h}{s_k}$$

$$\begin{cases} k^* = \left(\frac{s_h^\beta s_k^{1-\beta}}{\delta+n}\right)^{\frac{1}{1-\alpha-\beta}} \\ h^* = \left(\frac{s_k^\alpha s_h^{1-\alpha}}{\delta+n}\right)^{\frac{1}{1-\alpha-\beta}} \end{cases}$$

2.3.1 *Dynamics around steady state*

Isoclines

$$\dot{k} = 0 \iff s_k k^\alpha h^\beta = (\delta + n)k \iff h = \left(\frac{(\delta + n)k^{1-\alpha}}{s_k}\right)^{\frac{1}{\beta}}$$

$$\dot{h} = 0 \iff s_h k^\alpha h^\beta = (\delta + n)h \iff h = \left(\frac{s_h k^\alpha}{\delta + n}\right)^{\frac{1}{1-\beta}}$$

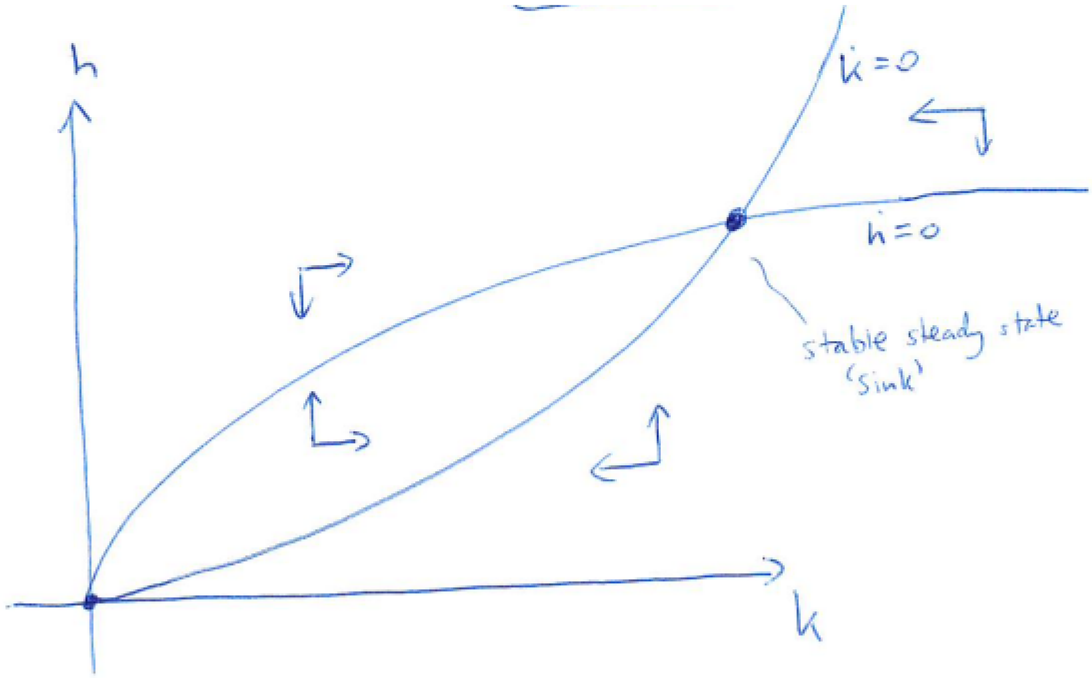


Figure 4: Phase diagram

$$\dot{k} = 0 \text{ concave if } \frac{1-\alpha}{\beta} < 1 \iff 1-\alpha < \beta \iff 1-\alpha-\beta < 0 \text{ impossible so } \quad \dot{k} = 0 \quad \text{convex}$$

$$\dot{h} = 0 \text{ concave if } \frac{\alpha}{1-\beta} < 1 \iff \alpha < 1-\beta \iff 1-\alpha-\beta > 0 \text{ sure so } \quad \dot{h} = 0 \quad \text{concave}$$

Observation: 2

Long run growth can be imposed exogenously (hence, 'exogenous growth models')

2.4 SOLOW MODEL WITH EXOGENOUS GROWTH

$$Y = AF(K, L) \quad \text{or} \quad Y = F(K, AL)$$

$A \approx$ technology level
$\hat{A} \approx g$ technological progress
E.g. Cobb-Douglas : $Y = K^{\alpha}(AL)^{1-\alpha}$ Harrod neutral technological progress

Balanced growth path ($\hat{Y} \equiv$ const, $\hat{K} \equiv$ const)

$$\dot{K} = sY - \delta K$$

$$\hat{K} = s\frac{Y}{K} - \delta \equiv \text{const} \iff \frac{Y}{K} \equiv \text{const} \iff \hat{Y} = \hat{K}$$

Then

$$\hat{Y} = \hat{K} = \alpha\hat{K} + (1-\alpha)(g+n) \iff (1-\alpha)\hat{K} = (1-\alpha)(g+n) \iff \hat{Y} = \hat{K} = g+n$$

so $\hat{k} = \hat{K} - \hat{l} = g + n - n = g$ One may redefine the 'intensive units' as in $k = \frac{K}{AL}$ then $\hat{k} = \hat{K} - g - n = 0$ in steady state (BGP)
Outside of the steady state

$$\frac{\dot{k}}{K} = \frac{\dot{K}}{K} - g - n = s\frac{Y}{K} - \delta - g - n \Rightarrow \dot{k} = sk^{\alpha} - (\delta + g + n)k$$

2.5 HUMAN CAPITAL

$$Y = C = hl$$

$$\dot{h} = A\phi(l,h) - \delta h$$

E.g. Cobb Douglas $\phi(l,h) = (1-l)h^{\gamma}$ where $\gamma \in (0,1)$ decreasing retrns

$$\dot{h} = A(1-l)h^{\gamma} - \delta h$$

with $\hat{A} = g$ -technological progress is the schooling technology

Balanced growth path ($\hat{h} \equiv$ const)

$$\dot{h} = A(1-l)h^{\gamma-1} - \delta$$

$$\hat{h} \equiv \text{const} \iff Ah^{\gamma-1} = \text{const} \iff \hat{=} g = (1\gamma)\hat{h} \iff \hat{h} = \frac{g}{1-\gamma}$$

Then by assumption $\hat{Y} = \hat{C} = \hat{h} = \frac{g}{1-\gamma}$
We can rewrite the model in terms of the stationary variable $Ah^{\gamma-1}$ or better $\frac{h^{1-\gamma}}{A}$ or even $\frac{h}{A^{\frac{1}{1-\gamma}}} = \chi$. Then

$$\hat{\chi} = \hat{h} - \frac{g}{1-\gamma} \quad Ah^{\gamma-1} = \chi^{\frac{1}{\gamma-1}}$$

$$\dot{\chi} = (\hat{h} - \frac{g}{1-\gamma})\chi = ((1-l)\chi^{\frac{1}{\gamma-1}} - \delta - \frac{g}{1-\gamma})\chi$$

$$\dot{\chi} = (1-l)\chi^{\frac{\gamma}{\gamma-1}} - (\delta + \frac{g}{1-\gamma})\chi$$

2.6 MANKIW-ROMER-WEIL MODEL WITH EXOGENOUS GROWTH

$$Y = F(K,H,L) = K^{\alpha}H^{\beta}(AL)^{1-\alpha-\beta} \qquad \alpha + \beta < 1$$

Harrod netural technological progress assumed by MRW and $\hat{A} = g$

$$\begin{cases} \dot{K} = s_k Y - (\delta + n)K \\ \dot{H} = s_h Y - (\delta + n)H \end{cases}$$

Balanced growth path ($\hat{Y} \equiv$ const, $\hat{H} \equiv$ const,$\hat{K} \equiv$ const)

$$\begin{cases} \hat{K} = s_k \frac{Y}{K} - (\delta + n) = \text{const} \\ \hat{H} = s_h \frac{Y}{H} - (\delta + n) = \text{const} \end{cases}$$

so $\frac{Y}{H}$ and $\frac{Y}{K}$ const so $\hat{Y} = \hat{K} = \hat{H}$

$$\hat{Y} = \alpha\hat{Y} + \beta\hat{Y} + (1 - \alpha - \beta)(g + n)$$

$$\hat{Y} = \hat{K} = \hat{H} = g + n$$

$\frac{Y}{L}$, $\frac{K}{L}$, $\frac{H}{L}$ grow at rate g.

- One may redefine 'intensive units' as in $y = \frac{Y}{AL}$, $k = \frac{K}{AL}$, $h = \frac{H}{AL}$ then:

$$\dot{k} = s_k y - (\delta + n + g)k$$
$$\dot{h} = s_k y - (\delta + n + g)h$$

And analysis is analogous

3

PONTRYAGIN MAXIMUM PRINCIPLE

3.1 ECONOMIC GROWTH TOOLBOX

1. Dynamic optimization(with continouos time and infinite time horizon)
2. Monopolistic competition (a la Dixit Stiglitz) (R&D based models feature increasing returns to scale which are inconsistent with perfect competition)
3. General Equilibrium
4. Comparision: Decentralized Equilibrium vs Social Planner

The most basic dynamic optimization problem in Growth Theory:

- 'dynastic model' $t \in [0, \infty)$
- infinite horizon, discounting
- the consumption-savings decision of the houshelod

c -control variable /CONSUMPTION/
a-state variable /ASSETS, CAPITAL/ $\dot{a} = \frac{da}{dt}$

$$\max_{\{c(t)_0^{+\infty}\}} \int_0^{\infty} e^{-\rho t} u(c(t)) dt \qquad \text{s.t.} \quad \dot{a} = ra + w - c$$

we'd like to find
-the Euler equation $\dot{c} = \dots$
- ideally the optimal growth path $c(t) = \dots, a(t) = \dots$

Solution:

3.1.1 *Define Hamiltonian*

$$H(c, a; \lambda) = \overbrace{e^{-\rho t} u(c)}^{\text{term within integral}} + \underbrace{\lambda(ra + W - c)}_{\text{RHS of eq. of motion } \dot{a} = \dots}$$

λ is called co-state variable shadow price of a

3.1.2 *Pontryagin maximum principle (FOCs and TVC)*

First FOCs:

$$\frac{\partial H}{\partial c} = 0 \qquad (max_c H)$$

$$\frac{\partial H}{\partial a} = -\dot{\lambda}$$

$$\frac{\partial H}{\partial \lambda} = \dot{a}$$

Here

$$e^{-\rho t} u'(c) - \lambda = 0$$

$$\lambda r = -\dot{\lambda}$$

$$\dot{a} = ra + w - c$$

3.1.3 *Solve for Euler equation*

Trick: use log derivatives

$$\hat{x} = \frac{\dot{x}}{x} = \frac{\partial \ln x}{\partial t} \quad \text{if} \quad x > 0$$

Rule

$$\text{when} \quad x = a^{\alpha} b^{\beta} \quad \hat{x} = \alpha \hat{a} + \beta \hat{b}$$

$$\begin{cases} \lambda = e^{-\rho t} u'(c) & \hat{\lambda} = -\rho + \hat{u'(c)} = -\rho + \frac{u''(c)\dot{c}}{u'(c)} \\ \hat{\lambda} = -r \\ \dot{a} = ra + w - c \end{cases}$$

$$\Rightarrow -r = -\rho + \frac{u''(c)}{u'(c)} \frac{\dot{c}}{c} = -\rho - \theta(c)\hat{c}$$

$$\hat{c} = \frac{r - \rho}{\theta(c)} \qquad \text{Euler equation!}$$

3.1.4 *Transversality Condtitions*

Use TVC: transversality conditions (also part of Pontryagin maximum principle)

$$\lim_{t \to \infty} \lambda(t) = 0$$

$$\lim_{t \to \infty} H(t) = 0$$

We also frequently use (instead) a stronger single TVC:

$$\lim_{t\to\infty} \lambda(t)a(t) = 0$$

When

$$\underbrace{\lim_{t\to\infty} \hat{\lambda_t a_t} < 0}_{\text{Often suffices in Growth Theory}} \qquad \text{then} \lim_{t\to\infty} \lambda(t)a(t) = 0$$

Here: It depends on the assumptions on $r(t)w(t)$

3.1.5 *Equivalent Approaches*

Note:

Present value Hamiltonian

$$H(c, a; \lambda) = e^{-\rho t}u(c) + \lambda(ra + W - c)$$

$$\frac{\partial H}{\partial c} = 0 \qquad (max_c H)$$

$$\frac{\partial H}{\partial a} = -\dot{\lambda}$$

$$\frac{\partial H}{\partial \lambda} = \dot{a}$$

Current value Hamiltonian

$$H(c, a; \lambda) = u(c) + \mu(ra + W - c)$$

$$\frac{\partial H}{\partial c} = 0 \qquad (max_c H)$$

$$\frac{\partial H}{\partial a} = \rho\mu - \dot{\mu}$$

$$\frac{\partial H}{\partial \mu} = \dot{a}$$

4

RAMSEY MODEL

-Production $F(K, L)$ with constant returns to scale

$$Y = F(K, L) \quad \frac{Y}{L} = y = \frac{F(K, L)}{L} = F(\frac{K}{L}, 1) = f(k) \quad \text{where} k = \frac{K}{L} \quad c = \frac{C}{L}$$

-Equation of motion-for captial

$$\dot{K} = Y - C - \delta K \quad \text{where} \quad \delta \geq 0$$

-Assuming constant population growth, $\frac{\dot{L}}{L} = n \Rightarrow L(t) = L_0 E^{nt}$

-Equation of motion for $k = \frac{K}{L}$

$$\dot{k} = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L} \frac{\dot{L}}{L = \frac{\dot{K}}{L} - nk}$$

$$\dot{k} = y - c - (\delta + n)k$$

4.1 OPTIMAL ECONOMIC GROWTH MODEL (RAMSEY-CASS-KOOPMANS)

$$\max_{\{c(t)_0^{+\infty}\}} \int_0^{\infty} e^{-\rho t} L(t) u(c(t)) dt = L_0 \int_0^{\infty} e^{-(\rho - n)t} u(c(t)) dt$$

$$\text{s.t.} \quad \dot{k} = y - c - (\delta + n)k \quad k_0 \quad \text{given}$$

The current value Hamiltonian

$$H^c = u(c) + \lambda(y - c - (\delta + n)k)$$

FOCs:

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0$$

$$\frac{\partial H}{\partial k} = \lambda(f'(k) - (\delta + n) = (\rho - n)\lambda - \dot{\lambda}$$

$$\frac{\partial H}{\partial \lambda} = \dot{k} = y - c - (\delta + n)k$$

$$\dot{\lambda} = -\lambda(f'(k) - \delta - \rho) \iff \frac{\dot{\lambda}}{\lambda} = -(f'(k) - \delta - \rho)$$

$$\dot{\lambda} = u''(c)\dot{c} \iff \frac{\dot{\lambda}}{\lambda} = \frac{u''(c)\dot{c}}{u'(c)}$$

***Theorem:* 2: The Euler equation**

$-\frac{u''(c)\dot{c}}{u'(c)} = f'(k) - \delta - \rho$

***Definition:* 3: RRA**

Relative risk aversion (RRA) we define as follows:

$$\theta = -\frac{u''(c)\dot{c}}{u'(c)}$$

***Definition:* 4: CRRA**

Function of CRRA class- constant RRA

$$u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta} & \text{if} \quad \theta > 1, \theta \neq 1 \\ \log(c) & \text{if} \quad \theta = 1 \end{cases}$$

Here if we assume CRRA utility, the Euler equation becomes

$$\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho}{\theta}$$

Dynamic system of the Ramsey model

$$\begin{cases} \dot{c} = c \cdot \frac{f'(k)-\delta-\rho}{\theta} \\ \dot{k} = y - c - (\delta + n)k \end{cases}$$

Steady state ($\dot{k} = \dot{c} = 0$):

$$\begin{cases} f'(k) = \delta + \rho \\ c = f(k) - (\delta + n)k \end{cases}$$

a unique (c^*, k^*)

Phase diagram
Isoclines

$$1) \qquad \dot{c} = 0 \iff f'(k) - \delta - \rho = 0$$

-if $c > 0$,
-meaning that $k = k^*$

$$2) \qquad \dot{k} = 0 \iff c = f(k)(\delta + n)k$$

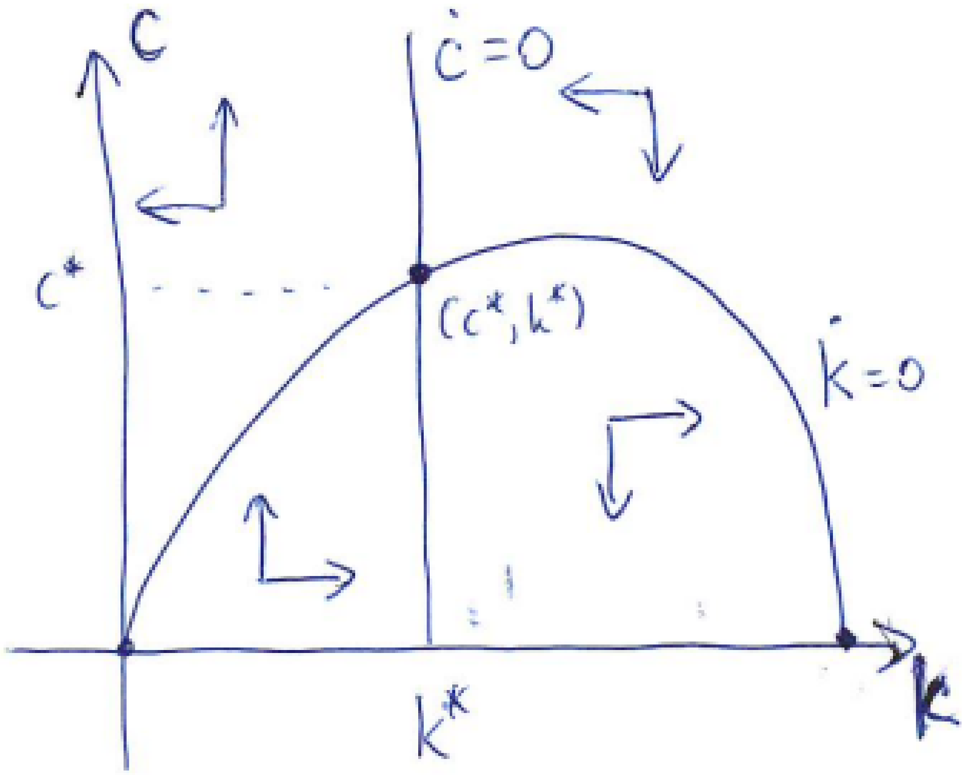


Figure 5: Phase diagram

4.1.1 *Transversality Conditions*

TVC for infinite horizion problems (with continuous time)

$$\lim_{t\to\infty} e^{-\beta t}\lambda(t) = 0$$

$$\lim_{t\to\infty} |e^{-\beta t}\lambda(t)s(t)| < \infty \iff \in_0^\infty e^{-\beta t}u(c(t))dt < \infty \text{ integrability}$$

if λ-associated with the current value Hamiltonian

(back to the Ramsey model)

with $\beta = \rho - n >$

$$\lim_{t\to\infty} e^{-\beta t}\lambda(t) = \lim_{t\to\infty} e^{-\beta t}u'(c) = 0$$

$$\lim_{t\to\infty} |e^{-\beta t}\lambda(t)k(t)| = \lim_{t\to\infty} |e^{-\beta t}u'(c)k(t)| < \infty \text{ should be finite}$$

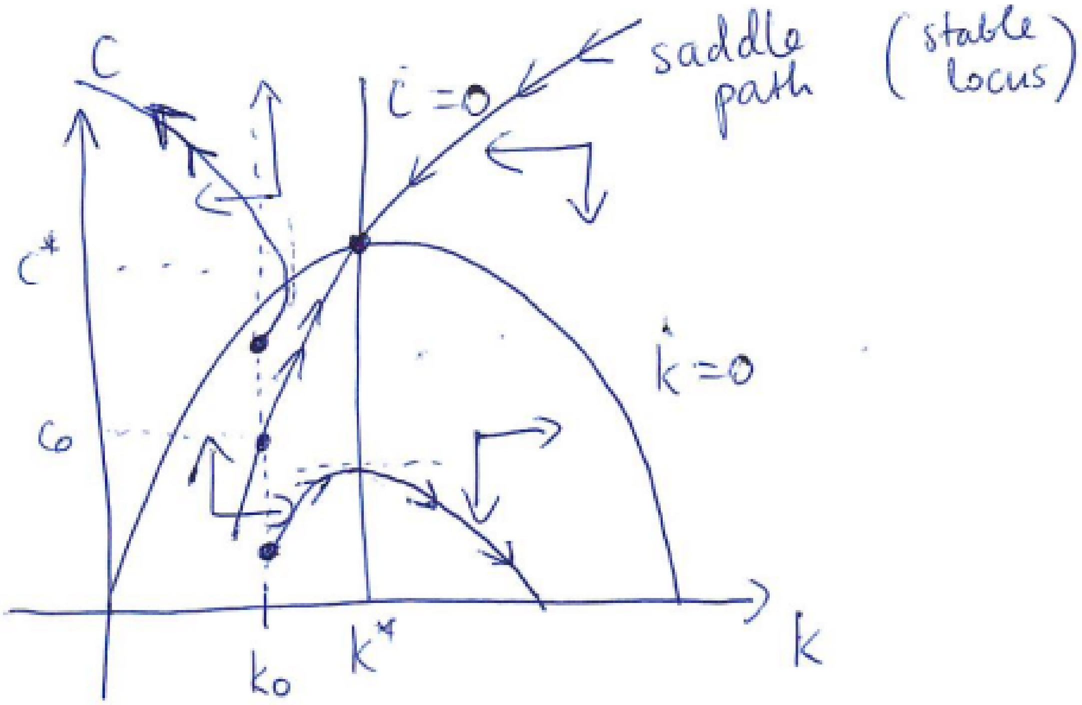


Figure 6: Phase diagram

Only the saddle path is consistent with the TVC. In the optimum, one has to choose c_0 to follow the saddle path.

4.1.2 *Human capital accumulation model*

$$\max_{\{c(t)\}} \int_0^\infty e^{-\rho t}u(c)dt$$

$$\text{s.t. } y = hl \quad c = y \quad \dot{h} = A(1-l)h^\gamma - \delta h$$

with $\rho > 0$ $\gamma \in (0,1)$, $\delta > 0$ $A > 0$ and $h \geq 0$, $l \in [0,1]$.
Let us assume CRRA utility $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, h_0-given.

$$H = e^{-\rho t}u(c) + \lambda(A(1-l)h^\gamma - \delta h) =$$

not the current value H^c

$$e^{-\rho t}\frac{(hl)^{1-\theta}-1}{1-\theta}+\lambda(A(1-l)h^{\gamma}-\delta h)$$

FOCs:

$$\frac{\partial H}{\partial l}=e^{-\rho t}h^{1-\theta}l^{-\theta}-\lambda A h^{\gamma}=0$$

$$\frac{\partial H}{\partial h}=e^{-\rho t}h^{-\theta}l^{1-\theta}+\lambda(A(1-l)\gamma h^{\gamma-1}-\delta)=-\dot{\lambda}$$

From first

$$\lambda=\frac{e^{-\rho t}h^{1-\theta-\gamma}l^{-\theta}}{A}$$

Hence

$$\hat{\lambda}=-\rho+(1-\theta-\gamma)\hat{h}-\theta\hat{l}$$

$$-\hat{\lambda}=A(1-l)\gamma h^{\gamma-1}-\delta+\frac{e^{-\rho t}h^{-\theta}l^{1-\theta}}{\lambda}=lh^{\gamma-1}A$$

$$A(1-l)\gamma h^{\gamma-1}-\delta+lh^{\gamma-1}A=\rho-(1-\theta-\gamma)\hat{h}+\theta\hat{l}$$

$$\theta\hat{l}=A(1-l)\gamma h^{\gamma-1}-\delta+lh^{\gamma-1}A-\rho+(1-\theta-\gamma)(A(1-l)h^{\gamma-1}-\delta)$$

$$\theta\hat{l}=h^{\gamma-1}[(1-\theta)A(1-l)+Al]-\delta(2-\theta-\gamma)-\rho$$

$$\underbrace{\hat{l}=\frac{1}{\theta}[h^{\gamma-1}[(1-\theta)A(1-l)+Al]-\delta(2-\theta-\gamma)-\rho]}_{\text{Euler equation}}$$

<u>Phase diagram</u>

$$\begin{cases}\dot{l}=\frac{l}{\theta}[h^{\gamma-1}[(1-\theta)A(1-l)+Al]-\delta(2-\theta-\gamma)-\rho]\\ \dot{h}=A(1-l)h^{\gamma}-\delta h\end{cases}$$

Steady state ($\dot{h}=\dot{l}=0$)
Isoclines (on figure 3)

$$\dot{l}=0 \quad\iff\quad h^{\gamma-1}(1-\theta+\theta l)A=\delta(2-\theta-\gamma)+\rho$$

$$1-\theta+\theta l=\frac{h^{1-\gamma}}{A}(\delta(2-\theta-\gamma)+\rho)$$

$$l=\frac{1}{\theta}[\frac{\delta(2-\theta-\gamma)+\rho}{A}h^{1-\gamma}-(1-\theta)]$$

$$\dot{h}=0 \quad\iff\quad A(1-l)h^{\gamma-1}=\delta$$

$$1-l=\frac{\delta}{A}h^{1-\gamma}$$

$$l=1-\frac{\delta}{A}h^{1-\gamma}$$

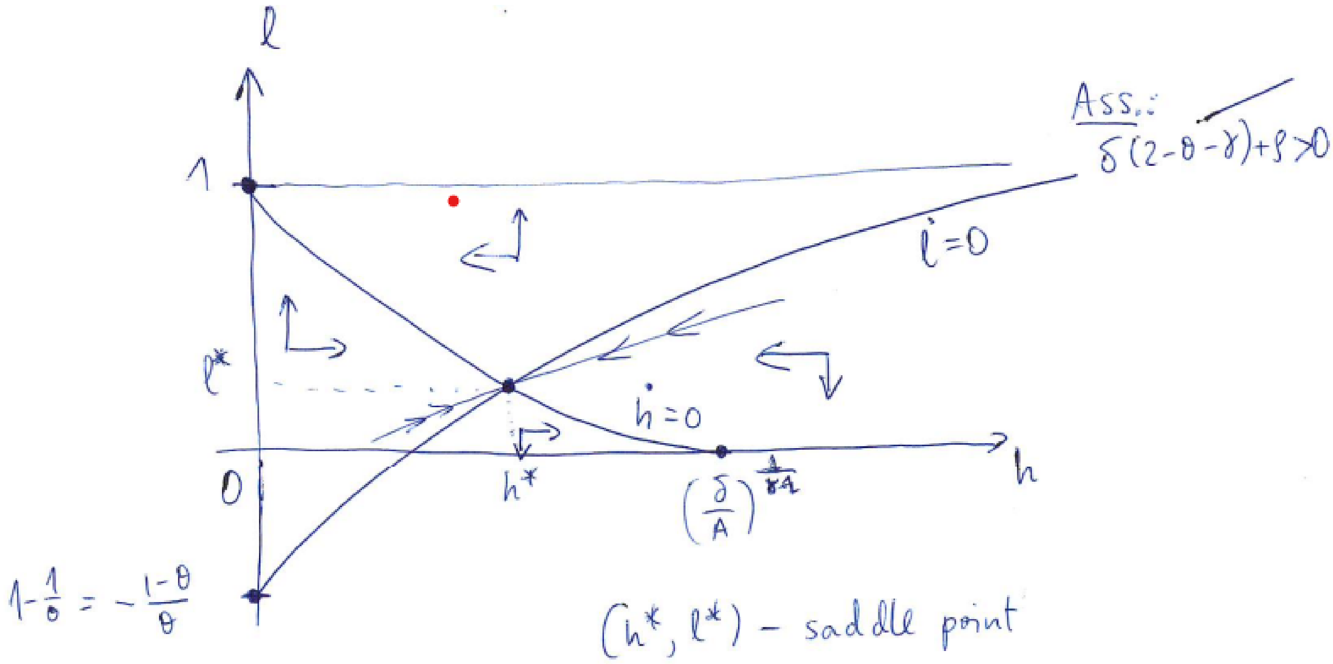


Figure 7

Steady state

$$l^* = \frac{1-\gamma+\rho/\delta}{2-\gamma+\rho/\delta}$$

$$h^* = (\frac{A}{\delta(2-\gamma)+\rho})^{\frac{1}{1-\gamma}}$$

$$\lim_{t\to\infty} \lambda(t) = \lim_{t\to\infty} \frac{e^{-\rho t} l^{-\theta} h^{1-\theta-\gamma}}{A} = 0$$

$$\lim_{t\to\infty} |\lambda(t)h(t)| = \lim_{t\to\infty} \frac{e^{-\rho t} l^{-\theta} h^{2-\theta-\gamma}}{A} < \infty$$

satisfied if $l \to l^*$, $h \to h^*$ (saddle path)

Useful hint

$$\frac{\dot{A^\alpha B^\beta}}{A^\alpha B^\beta} = \hat{A^\alpha B^\beta} = \alpha\hat{A} + \beta\hat{B}$$

$$\hat{e^{-\rho t}} = -\rho$$

Part III

ENDOGENOUS GROWTH MODELS

5

ENDOGENOUS GROWTH. AK MODEL

-Properties if the aggregate production $Y = F(K, L, A)$

Neoclassical growth model

$\Rightarrow$ Constant returns to scale

- replication argument
- firms optimize and thus should be expected to operate at optimal scale
- empirical studies typically don't reject CRS

$\Rightarrow$ Decreasing returns to reproducible inputs (e.g. capital)

Definition: 5:

We say that model features endogenous growth if it possesses a solution along which all key economic variables grow perpetually and the long-run growth rate is pinned down by variables determined within the mode

5.1 TYPES/CLASSES OF ENDOGENOUS GROWTH MODELS

1. models where long-run growth is driven by accumulation of reproducible inputs only
   - AK model, Jones -Manuelli model $[K]$
   - Uzawa-Lucas model $[H]$
   - models where growth is driven by (appropriately specified) externalities
   - models with multiple reproducible inputs
2. models with endogenous technological change
   - to which we return later

Let's assume that the aggregate production function has two inputs only (K, L), constant returns to scale and constant technology

$$Y = F(K, L) = LF(\frac{K}{L}, 1) = Lf(k)$$

$$y = \frac{Y}{L} = f(k)$$

Assume that :
s-'endogenous ' variable
Neoclassical assumptions:

$$f(0) = 0 \qquad f'(k) > 0 \quad f''(k) < 0$$

but relax $\lim_{k\to\infty} f'(k) = 0$ [Inada condition].

Assume instead $\lim_{k\to\infty} f'(k) = A > 0$

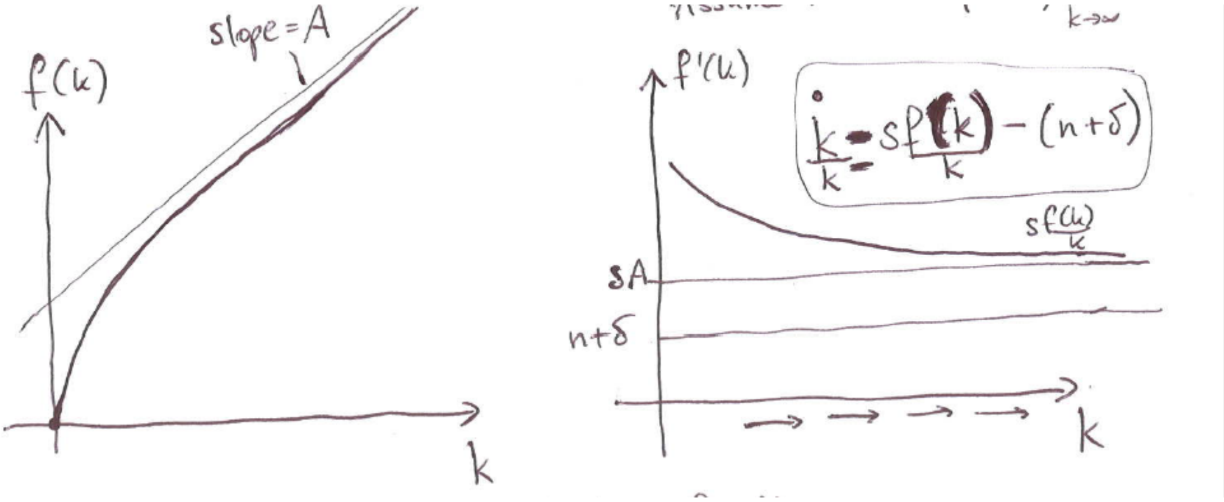


Figure 8

Examples of relevant production functions

1. AK function $f(k) = Ak$, $F(K, L) = AK$
2. Jones -Manuelli function $f(k) = Ak + Bk^{\alpha}$ $F(K, L) = Ak + BK^{\alpha}L^{1-\alpha}$
3. CES production function $\sigma > 1$ $f(k) = A[\pi k^{\frac{\sigma-1}{\sigma}} + 1 - \pi]^{\frac{\sigma}{\sigma-1}}$

Ad 1

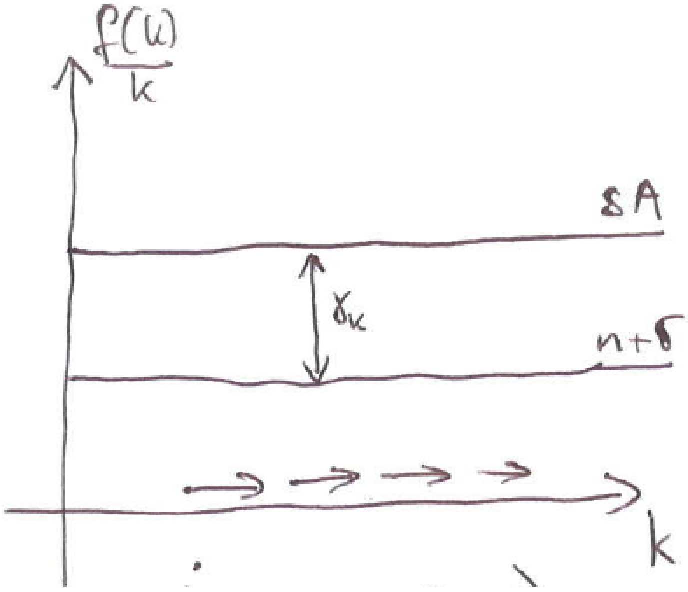
$$f(k) = Ak \quad f'(k) = A \quad f''(k) = 0$$

$$\lim_{k\to\infty} f'(k) = A \qquad s\frac{f(k)}{k} = sA$$

-The growht rate is

$$\gamma_k = \frac{\dot{k}}{k} = sA - (n + \delta)$$

-There are no transitional dynamics -We assume that A is suufficiently large



Ad2

$$f(k) = Ak + Bk^{\alpha} \quad f'(k) = A + \alpha Bk^{\alpha-1} \quad f''(k) = \alpha(\alpha-1)Bk^{\alpha-2} < 0$$

$$\lim_{k\to\infty} f'(k) = A \qquad s\frac{f(k)}{k} = sA + \frac{sBk^{\alpha}}{k}$$

-The growth rate is

$$\gamma_k = \frac{\dot{k}}{k} = sA + sBk^{\alpha-1} - (n+\delta)$$

-Transition dynamics γ decreases with k (in time)
-A has to be sufficiently large for endogenous growth. Otherwise the model converges to steady state

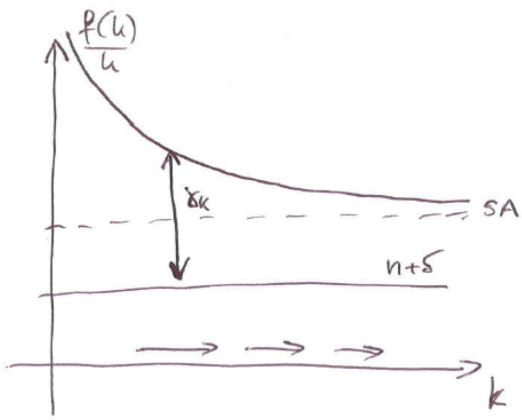


Figure 9

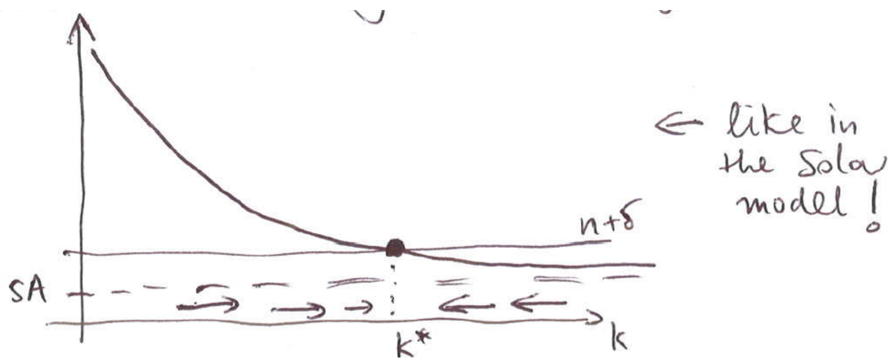


Figure 10

Ad 3

$$f(k) = A[\pi k^{\frac{\sigma-1}{\sigma}} + (1-\pi)]^{\frac{\sigma}{\sigma-1}}$$

constant elasticity of substitution $= \sigma$

$\sigma > 1$	$\iff$	K and L are gross substitutes
$\sigma > 1$	$\iff$	K and L are gross complements
$\sigma = 1$	$\iff$	Cobb Douglas case

$$f'(k) = \frac{\sigma}{\sigma-1}A[\cdot]^{\frac{\sigma}{\sigma-1}-1} \cdot \pi\frac{\sigma-1}{\sigma}k^{\frac{\sigma-1}{\sigma}-1} > 0$$

$$f''(k) = A[\frac{\sigma}{\sigma-1}-1][\cdot]^{\frac{\sigma}{\sigma-1}-1}\pi\frac{\sigma-1}{\sigma}k^{\frac{\sigma-1}{\sigma}-1}\pi k^{\frac{\sigma-1}{\sigma}-1} + [\cdot]^{\frac{\sigma}{\sigma-1}-1}\pi[\frac{\sigma-1}{\sigma}-1]k^{\frac{\sigma-1}{\sigma}-2} =$$

$$A\pi[\cdot]^{\frac{\sigma}{\sigma-1}-1}k^{\frac{\sigma-1}{\sigma}-1}k^{\frac{\sigma-1}{\sigma}-2}[\frac{1}{\sigma}\frac{\pi k^{\frac{\sigma-1}{\sigma}}}{\pi k^{\frac{\sigma-1}{\sigma}}+(1-\pi)} - \frac{1}{\sigma}] < 0$$

$$\lim_{k\to\infty} f'(k) = \lim_{k\to\infty} A[\cdot]^{\frac{1}{\sigma-1}}k^{-\frac{1}{\sigma}} = \lim_{k\to\infty} A[\pi k^{\frac{\sigma-1}{\sigma}} + 1 - \pi]^{\frac{1}{\sigma-1}}k^{-\frac{1}{\sigma}\frac{\sigma-1}{\sigma-1}} =$$

$$\lim_{k\to\infty} A[\pi + (1-\pi)k^{-\frac{\sigma-1}{\sigma}}]^{\frac{1}{\sigma-1}} = \begin{cases} A\pi^{k^{\frac{\sigma}{\sigma-1}}} & \text{if} \quad \sigma > 1 \\ 0 \quad \text{if} \quad \sigma >< \end{cases}$$

Consider cases

$\sigma > 1$
-The growth rate is

$$\gamma_k = \frac{\dot{k}}{k} = sA[\pi + (1-\pi)k^{\frac{-\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}} - (n+\delta)$$

-There are transitional dynamics, γ_k converges to

$$\lim_{k\to\infty} \gamma_k = sA\pi^{\frac{\sigma}{\sigma-1}} - (n+\delta)$$

-We assume that A is 'sufficiently large'.
Otherwise the model converges to a steady state

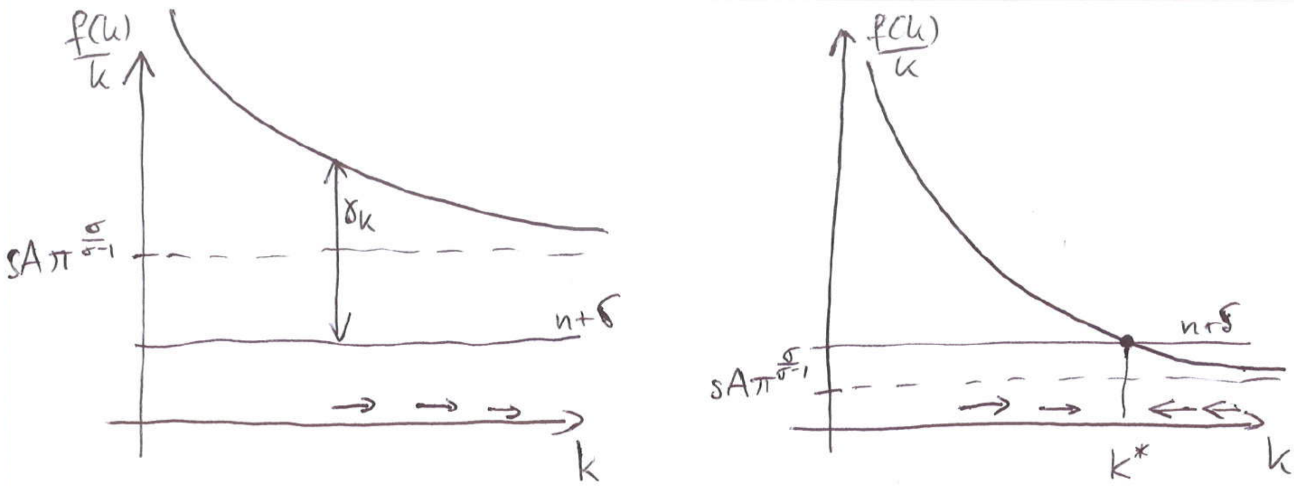


Figure 11

$\sigma < 1$

Note that $\lim_{k\to\infty} f'(k) = 0$ and

$$\lim_{k\to\infty} s\frac{f(k)}{k} = \lim_{k\to\infty} sA[\pi + (1-\pi)k^{\frac{-\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}} = sA\pi^{\frac{\sigma}{\sigma-1}}$$

[Inada condition at 0 doesn't hold]

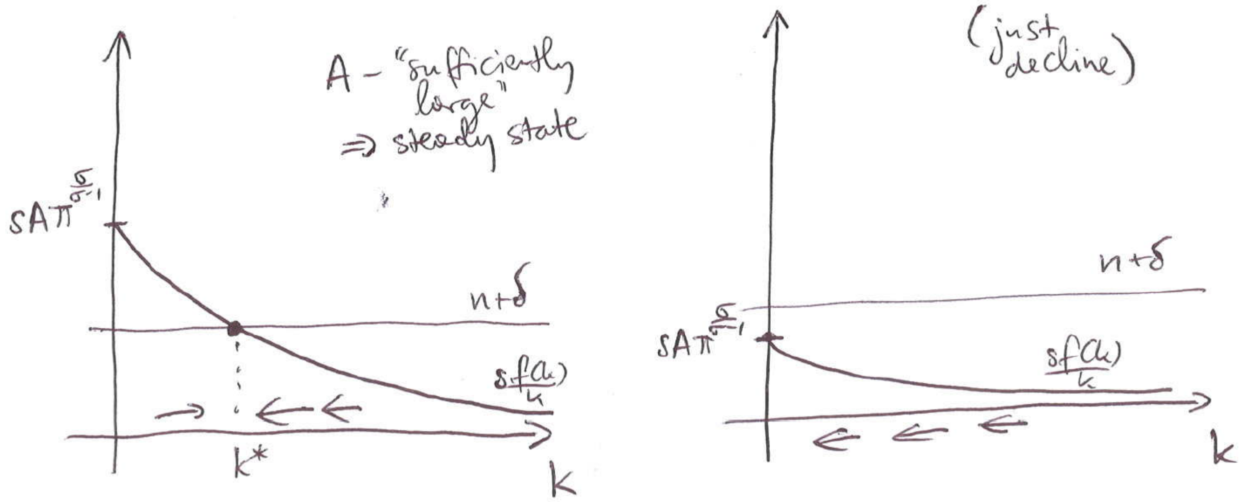


Figure 12

- In any case $\sigma < 1$ precludes endogenous growth.
- As long there is no other sources of growth, e.g. TECHNOLOGICAL PROCESS.

5.2 THE AK ENDOGENOUS GROWTH MODEL

-Key missing element (so for): endogenous saving rate s

-Households

$$\max \int_0^\infty e^{-(\rho-n)t} u(c)dt \quad \text{where} u(c) = \frac{c^{1-\theta}-1}{1-\theta} \quad \text{CRRA}$$

$$\text{s.t.} \quad \dot{a} = (r-n)a + w - c$$

We set up the Hamiltonian:

$$H = \frac{c^{1-\theta}-1}{1-\theta} \cdot e^{-(\rho-n)t} + \lambda((r-n)a + w - c)$$

$$\frac{\partial H}{\partial c} = c^{-\theta} e^{-(\rho-n)t} - \lambda \qquad \lambda = c^{-\theta} e^{-(\rho-n)t}$$

$$\frac{\partial H}{\partial a} = \lambda(r-n) = -\dot{\lambda} \qquad \hat{\lambda} = -(r-n)$$

$$\hat{\lambda} = -\theta\hat{c} - (\delta - n) \Rightarrow \quad \hat{c} = \frac{r-\rho}{\theta}$$

We also require the transversality condition

$$\lim_{a\to\infty} \lambda a = \lim_{a\to\infty} a(t) e^{-\int_0^t (r(v)-n)dv} = 0$$

-Firms (perfect competition)

$$\max_{K,L} \{F(K,L) - \tilde{r}K - wL\}$$

implies

$$\tilde{r} = \frac{\partial F}{\partial K} \qquad w = \frac{\partial F}{\partial L}$$

Here $F(K,L) = AK$ so $\tilde{r} = A$ - gross rental price of K, $w = 0$ (no labor in production)
Note that profits are zero.

Equilibrium

-Capital market clears $a = k$

$$\begin{cases} \dot{k} = y - c + (\delta + n)k = (A - \delta - n)k - c \\ \dot{a} = (r-n)a + w - c = (r-n)k - c \end{cases}$$

$$\text{So} \qquad r = A - \delta \qquad \underbrace{\tilde{r}}_{\text{GROSS}} = \underbrace{r}_{\text{NET}} + \delta$$

Hence $\hat{c} = \frac{\dot{c}}{c} = \frac{A-\delta-\rho}{\theta}$ - Growth rate of the economy!

- Observe that the growth rate does not depend on k and is fixed throughout (no transitional dynamics)
-Solving, we obtain a closed -form solution for $c(t)$

$$c(t) = c(0)e^{(\frac{A-\delta-\rho}{\theta})t}$$

-It is also easy to compute

$$\underbrace{\frac{c}{k}}_{\text{cons}} = (a-\delta-n) - \underbrace{\frac{\dot{k}}{k}}_{\text{const}} = (A-\delta-n) - \frac{\dot{c}}{c} = \underbrace{\frac{A-\delta}{\theta}(\theta-1) + \frac{\rho}{\theta} - n}_{\varphi > 0}$$

-And so $c(0)) = \phi k(0)$ where $k(0)$ is given

The saving rate is

$$s = \frac{y-c}{y} = \frac{Ak-c}{AK} = \frac{Ak-\varphi k}{Ak} = \frac{A-\varphi}{A} = \frac{A-\rho+\theta n+(\theta-1)\delta}{\theta A}$$

The transversality condition ($g = \frac{A-\delta-rho}{\theta}$)

$$\lim_{t\to\infty} k(0)e^{gt}e^{-\int_0^t (A-\delta-n)dv} = k(0)e^{(g-A+\delta+n)t} = k(0)e^{(\frac{A-\delta}{\theta}(1-\theta)-\frac{\rho}{\theta}+n)t} = \lim_{t\to\infty} k(0)e^{\varphi t} = 0$$

because we have assumed that $\phi > 0$

6

JONES-MANUELLI MODEL. UZAWA-LUCAS MODEL. GROWTH WITH EXTERNALITIES

6.1 Jones & Manelli (1990) model

- The household's problem is exactly the same

Firms (perfect competition)

$$\tilde{r} = \frac{\partial F}{\partial K} \qquad w = \frac{\partial F}{\partial L}$$

Here

$$F(K, L) = AK + BK^{\alpha}L^{1-\alpha}$$

$$\tilde{r} = A + BK^{\alpha-1}L^{1-\alpha}$$

$$w = B(1-\alpha)K^{\alpha}L^{-\alpha}$$

$$\begin{cases} y = f(k) = Ak + Bk^{\alpha} \\ \tilde{r} = A + Bk^{\alpha-1} \\ w = B(1-\alpha)k^{\alpha} \end{cases}$$

So

$$\begin{cases} y = \tilde{r}k + w \\ Y = \tilde{r}K + wL \end{cases}$$

Zero profit

Equilibrium

-Capital market clears $a = k$

$$r = \tilde{r} - \delta = A + B\alpha k^{\alpha-1} - \delta$$

-Hence

$$\hat{c} = \frac{\dot{c}}{c} = \frac{r - \delta}{\theta} = \frac{A + B\alpha k^{\alpha-1} - \delta - \rho}{\theta}$$

it is growth rate of the economy

Note that $\hat{c}$ depends on k and

$$\lim_{k\to\infty} \hat{c} = \frac{A - \delta - \rho}{\theta}$$

as in the AK model
We have

$$\frac{\dot{k}}{k} = \frac{y}{k} - \frac{c}{k} - (n+\delta) = A + Bk^{\alpha-1} - \frac{c}{k} - (n+\delta)$$

$$\text{if} \quad k \to \infty \quad \text{then} BK^{\alpha-1} \to 0 \text{so} \quad \frac{\dot{k}}{k} \to A - \frac{c}{k} - (n+\delta)$$

and so endogenous growth implies asymptotical constancy of $\frac{c}{k}$ and thus as $k \to \infty$ $\hat{c} = \hat{k} = \hat{y} = \frac{A-\delta-\rho}{\theta}$
It follows that

$$\lim_{k\to\infty} \frac{c}{k} = \underbrace{A - n - \delta - \frac{A-\delta-\rho}{\theta} = \frac{A-\delta}{\theta}(\theta-1) + \frac{\rho}{\theta} - n}_{\varphi>0}$$

Also $\frac{y}{k} \to A$.

However there are transitional dynamics
Consider the system

$$\begin{cases} \dot{k} = Ak + Bk^{\alpha} - c - (n+\delta) \\ \dot{c} = \frac{1}{\theta}(A + B\alpha k^{\alpha-1} - \delta - \rho) \end{cases}$$

We have shown that this system doesn't possess a steady state $\dot{k} = \dot{c} = 0$
Let's rewrite it in 'stationary' variables -ones that do possess a steady state.

For example

$$\begin{cases} u = \frac{c}{k} & \text{control like variable} \\ z = \frac{y}{k} & \text{state like variable} \end{cases}$$

there exists a given$z(0)$ but not $u(0)$

$$\begin{cases} \hat{u} = \hat{c} - \hat{k} \\ \hat{z} = \hat{y} - \hat{k} \end{cases}$$

Thus

$$\hat{z} = \hat{y} = \hat{k} = (A + \hat{B}k^{\alpha-1}) = \frac{B(\alpha-1)k^{\alpha-2}\dot{k}}{A + Bk^{\alpha-1'}} = \frac{\frac{1}{k}(\alpha-1)(z-A)}{z} = \hat{k}(\alpha-1)\frac{z-A}{z}$$

Let us also note that

$$A + B\alpha k^{\alpha-1} = \alpha AA + Bk^{\alpha-1}\alpha + A - \alpha A = \alpha z + (1-\alpha)A$$

$$\begin{cases} \hat{u} = \frac{1}{\theta}(\alpha z + (1-\alpha)A - \delta - \rho) - z + u + n + \delta \\ \hat{z} = (z - u - (n+\delta))(\alpha-1)\frac{z-A}{z} \end{cases}$$

The steady state in the (u, z) space:

$$\hat{u} = \hat{z} = 0 \iff \begin{cases} u + z(\frac{\alpha}{\theta} - 1) = -(\frac{1-\alpha}{\theta}A - \frac{\delta+\rho}{\theta} + n + \delta) \\ (z-A)(z-u-n-\delta) = 0 \end{cases}$$

so

$$\underbrace{z = A}_{(*)} \qquad \underbrace{z = u + n + \delta}_{(**)}$$

Case (*)

$$\begin{cases} z = A \\ u + A(1 - \frac{\alpha}{\theta}) = -(\frac{1-\alpha}{\theta}A - \frac{\delta+\rho}{\theta} + n + \delta) \end{cases}$$

$$\begin{cases} z = A \\ u + A(1 - \frac{1}{\theta}) + \frac{\delta+\rho}{\theta} - n - \delta \end{cases}$$

as discussed earlier!

Case (**)

$$\begin{cases} z = u + n + \delta \\ \frac{\alpha}{\theta}(u + n + \delta) = -\frac{1-\alpha}{\theta}A - \frac{\delta+\rho}{\theta} \Rightarrow u^* < 0 \end{cases}$$

contradiction

Isoclines ($\dot{u} = 0$ and $\dot{z} = 0$)

$$\dot{u} = 0 \iff \quad \hat{u} = 0 \iff \quad u = z(1 - \frac{\alpha}{\theta}) - \frac{1-\alpha}{\theta}A + \frac{\delta+\rho}{\theta} - n - \delta$$

$$\dot{z} = 0 \iff \quad \hat{z} = 0 \iff \quad z = A \quad or \quad u = z - n - \delta$$

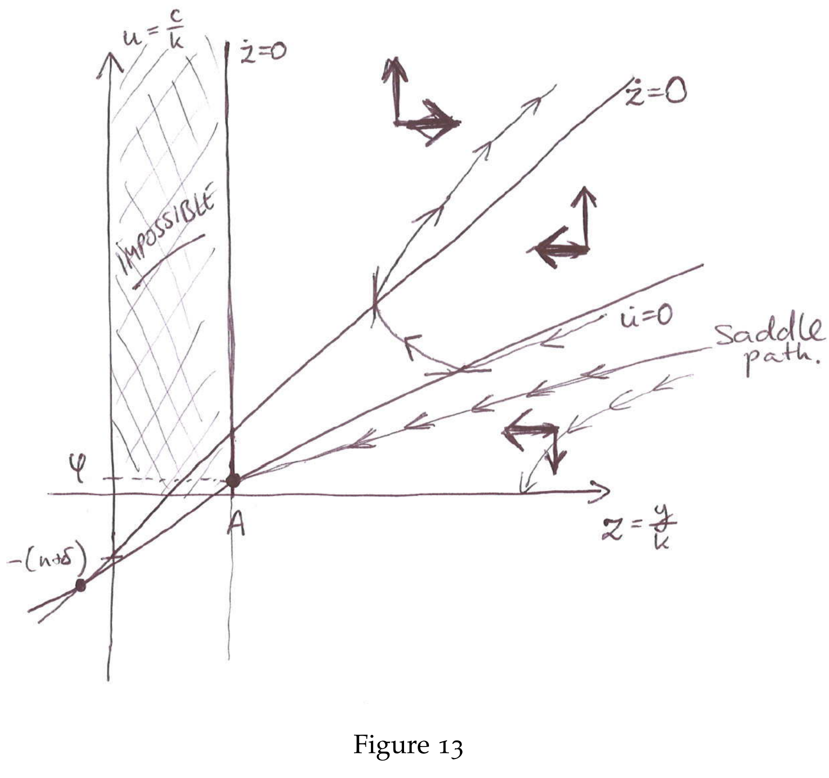


Figure 13

Transversality condition

$$\lim_{t\to\infty} \lambda(t)k(t) = \lim_{t\to\infty} k(t)e^{-\int_0^t (A+Bk(v)^{\alpha-1}-\delta-n)dv}$$

-If system converges to steady state, then

$$\hat{\lambda k} = \hat{\lambda} + \hat{k} = -(r-n) + \frac{r-\rho}{\theta} = r(\frac{1}{\theta}-1) - \frac{\rho}{\theta} + n =$$

$$= (A-\delta)(\frac{1}{\theta}-1) + \underbrace{Bk^{\alpha-1}(\frac{1}{\theta}-1)}_{\to 0} - \frac{\rho}{\theta} + n = -\varphi < 0$$

Negative growth rate implies that the formula tens to 0

-Otherwise, the TVC is violated

-Note that the capital's share of output

$$\pi_k = \frac{\bar{r}k}{Y} = \frac{Ak + B\alpha k^\alpha}{Ak + Bk^\alpha}$$

and so

$$\lim_{k\to 0^+} pi_k = \alpha$$

and

$$\lim_{k\to\infty} pi_k = 0$$

π_k gradually increases from α (the neoclassical case) to 1 (the AK case).

6.2 A SIMPLIFIED VERSION OF UZAWA-LUCAS MODEL

Just to show the mechanism of long run growth driven by human capital accumulation

$$\begin{cases} Y = AK^\alpha H_Y^{1-\alpha} \\ \dot{K} = Y - C - \delta K = sY - \delta K \qquad s\text{-should be endogenous} \\ \dot{H} = \gamma H_H - \delta_H H \qquad H_H\text{-should be endogenous} \\ H = H_H + H_Y \qquad H_H\text{teachers}, \quad H_Y\text{other workers} \end{cases}$$

Let us focus on the Balanced Growth Path (BGP)
-Assumption: at the BGP, all variables grow at fixed rate

$$\hat{H} = \gamma(\frac{H_H}{H}) - \delta_H := \gamma u - \delta_H$$

u-share of researchers in total employment

We hope that in equilibrium

$$\hat{H} = \gamma_u^* - \delta_H > 0$$

Then H accumulation will be the ultimate source of growth.

$$\hat{Y} = \hat{A} + \alpha\hat{K} + (1-\alpha)\hat{H}_Y = 0 + \alpha\hat{K} + (1\alpha)\hat{H} + (1-\alpha)(1 \hat{-} u)$$

Assume s-const, u-const, A-const

$$\hat{K} = s\frac{Y}{K} - \delta \Rightarrow \qquad \frac{Y}{K} = \text{const} \Rightarrow \qquad \hat{Y} = \hat{K} = \hat{C} = g$$

Hence

$$\hat{K} = \alpha\hat{K} + (1-\alpha)\hat{H} \Rightarrow \qquad g = \hat{K} = \hat{H} = \gamma_u - \delta_H$$

-the greater is $u = \frac{H_H}{H}$ the faster is growth
-in the optimal allocation, there will be $u \in (0,1)$ because one needs immediate output (and thus consumption) as well!

6.3 GROWTH AND EXTERNALITIES

- a model based on Romer (1986)
-capital accumulation increases total factor productivity (TFP)
-this effect is external to the firms - 'learning by doing' externality

Key assumptions:
-production function as seen by firm $i \in [0.1]$:

$$Y_i = F(K_i, AL_i)$$

-uponn aggregation

$$\int_0^1 K_i di = K \qquad \int_0^1 L_i di = L = \text{const}$$

-the externality takes the form

$$A = BK \qquad B = \text{const}$$

Firms are perfectly competetive so that

$$\tilde{r} = \frac{\partial Y_i}{\partial K_i} \qquad w = \frac{\partial Y}{\partial L_i}$$

By symmetry

$$\tilde{r} = \frac{\partial F}{\partial K} \qquad w = \frac{\partial F}{\partial L}$$

Using CRS and the definition of externality

$$\frac{Y}{K} = \frac{F(k, BKL)}{K} = F(1, BL) := \tilde{f}(L)$$

$$\frac{Y}{L} = \frac{F(k, BKL)}{L} = F(k, BK) := k\tilde{f}(L)$$

Inserting we obtain

$$\tilde{r} = \tilde{f}(L) - L\tilde{f}'(L) \qquad w = K\tilde{f}'(L)$$

Note:
-the wage rate is straighforward given $Y = K\hat{f}(K)$

$$\tilde{r} \neq \frac{\partial(K\tilde{f}(L))}{\partial K} = \tilde{f}(L)$$

because firms don't take the externality into account!
-instead use the Euler theorem to obtain for CRS

$$Y = \frac{\partial Y}{\partial K}K + \frac{\partial Y}{\partial L}L \Rightarrow \qquad \frac{\partial Y}{\partial K} = \frac{1}{K}(Y - \frac{\partial Y}{\partial L}L) = \tilde{f}(L) - L\frac{K\tilde{f}'(L)}{K} = \tilde{f}(L) - \tilde{f}'(L)$$

Households maximize total discounted utility subject to the CRRA assumption and usual asset dynamics:

$$\max \int_0^\infty e^{-(\rho-n)t}u(c)dt \quad \text{where} u(c) = \frac{c^{1-\theta}-1}{1-\theta} \quad \text{CRRA}$$

$$\text{s.t.} \quad \dot{a} = \underbrace{(r-n)}_{n=0} a + w - c$$

yielding Euler equation:

$$\hat{c} = \frac{r-\rho}{\theta}$$

In equilibrium $a = k = \frac{K}{L}$ where L-fixed

-Comparing with $\dot{K} = Y - C - \delta K$ we obtain

$$\underbrace{r}_{\text{Net return}} = \underbrace{\tilde{r}}_{\text{Gross return}} - \delta$$

and so

$$\hat{y} = \hat{k} = \hat{c} = \frac{f(L) - L\tilde{f}'(L) - \delta - \rho}{\theta}$$

constant growth rate of the economy!

-The transversality condition requires that

$$\lim_{t\to\infty} \lambda(t)a(t)$$

It is sufficient that $\lim \hat{\lambda}\hat{a} < 0$
For this to hold we need

$$-r + \frac{r-\rho}{\theta} < 0 \iff \qquad (1-\theta)r < \rho \qquad \iff$$

$$\iff \qquad (1-\theta)(\tilde{f}(L) - L\tilde{f}'(L) - \delta) < \rho$$

-Note that -just like in the case of the AK model- there is no transitional dynamics $\hat{c}$-const

-Long-run growth driven by physical capital accumulation despite the fact that firms face decreasing returns to capital.

-The capital's share of output is

$$\pi_k = \frac{\partial Y}{\partial K}\frac{K}{Y} = \frac{\tilde{f}(L) - L\tilde{f}'(L)}{\tilde{f}(L)} = 1 - L\frac{\tilde{f}'(L)}{\tilde{f}(L)} \qquad \pi_k \in (0,1)$$

-Scale effect:

$$\hat{y} = \hat{k} = \hat{c}$$

depends on L.

7

ENDOGENOUS TECHNICAL CHANGE. ROMER MODEL

7.1 ENDOGENOUS TECHNOLOGICAL CHANGE

-Romer's (1986,1990) key observation:

- non-rivarly of ideas
- hnce, ideas are a natural candidate to be a source of increasing returns

-Consider $Y = F(A, K, L)$ with CRS with respect to K and L

$$F(A, \lambda K, \lambda L) = \lambda F(A, K, L)$$

Given that if $A_1 < A_2$ then $F(A_1, K, L) < F(A_2, K, L)$ we obtain

$$F(\lambda A, \lambda K, \lambda L) > \lambda F(A, K, L)$$

-how to model endogenous technological change?

- the basic idea is simple
- the ultimate source of long-run growth is shifted to the R&D sector
- in a closed economy, innovations created by R&D translate into the only source of increases in 'A'
- 'A' is Total Factor Productivity/Solow Residual/'Technology level'
- However, finding the market equilibrium can be tedious because:
  1. social planner allocation (without R&D externalities)
  2. increasing variety models (e.g. Romer 1990)
  3. quality ladder 'Schumpeterian' growth models (e.g. Aghion and Howitt 1992)

7.2 'BARE-BONES' R&D-BASED GROWTH MODEL

$$\begin{cases} Y = A^{\alpha} L_Y^{1-\alpha} \\ \dot{K} = Y - C - \delta K = sY - \delta K \qquad \text{capital equation of motion} \\ \dot{A} = \gamma L_A^{\lambda} A \qquad \text{R\&D equation} L = L_A + L_Y \qquad L_A \text{teachers}, \quad L_Y \text{other workers} \end{cases}$$

The dynamics may be complicated but the first key observation follows directly from analyzing the balanced growth path BGP

-Assumption: at the BGP, all variables grow at a fixed rate

$$\hat{A} = \frac{\dot{A}}{A} = \gamma L_A^{\lambda}$$
$$\hat{Y} = \sigma\hat{A} + \alpha\hat{K} + (1-\alpha)\hat{L_Y}$$

Assume s-constant $\frac{L_A}{L}$, $\frac{L_Y}{L}$- constant as well

$$\hat{K} = \frac{Y}{K} - \frac{C}{K} - \delta = s\frac{Y}{K} - \delta$$

-Hence $\frac{Y}{K}$-const, implying $\hat{Y} = \hat{K} = \hat{C} = g$
-from eqution above we have

$$\hat{K} = \sigma\hat{A} + \alpha\hat{K} \Rightarrow \hat{K} = g = \frac{\sigma}{1-\alpha}\hat{A}$$

Finally, the growth rate at BGP is:

$$g = \frac{\sigma}{1-\alpha}\gamma L_A^{\lambda}$$

- the greater is L_A the faster is growth !
- note that in the optimal allocation, there will still be $L_Y > 0$ because it is required to generate immediate output (and consumption)

What is the 'bore-bones' missing?

⇒ dynamics outside of BGP

⇒ endogenization of s, L_A choice variables- soon!

⇒ international technology diffusion, imitation

⇒ 'technology' can in fact be multi-dimensional!
  - investment-specific TC
  - capital- vs labor-augmenting TC (needs to go beyond Cobb-Douglas technology)
  - vintage capital/human capital theory (embodied TC)
  - appropriate technology/ world technology frontier (technologies suited to ant given input mix)
  - there can be spillovers between various R&D sectors
  - technology complexity/ skill-biased TC

7.3 Empirical perspective on TFP

$$A = \frac{Y}{K^{\alpha}L^{1-\alpha}} \text{ or } A_h = \frac{Y}{K^{\alpha}H_Y^{1-\alpha}}$$

-the Solow residual is the 'measure of our ignorance' (includes everything but factor inputs)

⇒ mismeasurement

⇒ production function misspecification

⇒ any non-neutral component of TC is stil reported in 'A'

Growth accounting (source Hoeg (2005))

$$\frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \alpha\frac{\Delta K}{K} + (1-\alpha)\frac{\Delta L}{L}$$

-contributes ~15-40% of total variance in GDP per capita growth rates across countries

Levels accounting (development accounting)

$$\ln(\frac{Y_i}{Y_0}) = \ln(\frac{A_i}{A_0}) + \alpha\ln(\frac{K_i}{K_0}) + (1-\alpha)\ln(\frac{L_i}{L_0})$$

-contributes ~50-65% of total variance in GDP per capita levels across countries

7.3.1 *How to measure 'technological change'*

- TFP
- 'purified' TC measures
  - NET of technological efficiency changes (WTF approach)
  - NET of capacity utilization
  - Accounting for human capital accumulation
- direct measures of R&D (output/inputs)
  - patents filed/granted (e.g. Madsen 2008)
  - patent citations (e.g. Hall, Jaffe, Troytenberg 2001)
  - R&D spendings
  - R&D employment/ R&D share in emplyment

Unit factor productivities

- Let's relax the assumption of a Cobb-Douglas production function

- e.g. CES technology

$$Y = (\pi(A_K K)^{\frac{\sigma-1}{\sigma}} + (1-\pi)(A_L L)^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}}$$

- There is no unique 'TFP'!
- The direction of TC(Acemoglu 2003)
  1. purely LATC: $\hat{A}_K = 0$, $\hat{A}_L > 0$
  2. purely KATC: $\hat{A}_L = 0$, $\hat{A}_K > 0$
  3. a mixture of both
- the distribution of A_K and A_L across the world - Caselli& Coleman's 2006 take at 'the world technology frontier'

Increasing variety models
-intermediate goods (Romer, 1990) - 'division of labor', process innovation!
- final goods (Grossman and Helpman, 1991)- 'love of variety ' preferences , product innovation

Quality ladders models
-Schumpeterian ('Creative destruction' models (Aghion& Howitt 1992)
-innovations increasing the quality of product, introducing new vintages

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7.4 DIXIT & STIGLITZ(1977) MONOPOLISTIC COMPETITION MODEL

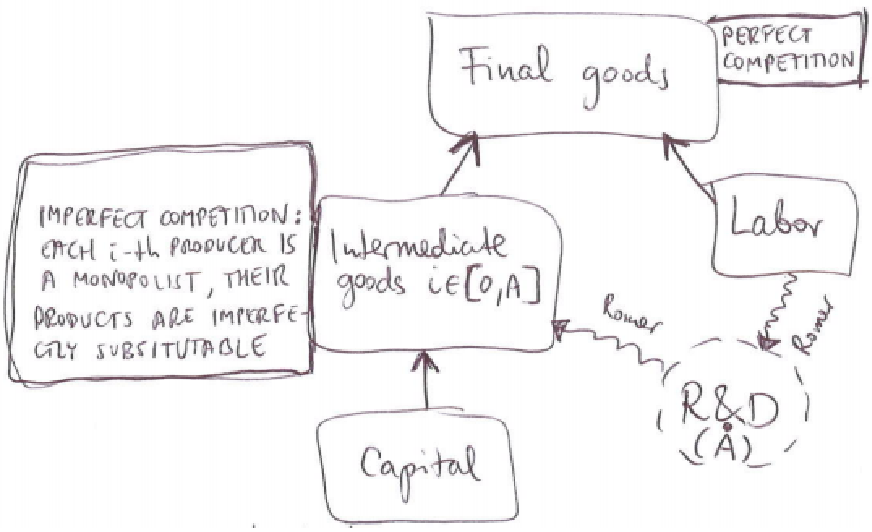

Figure 14

$$\underbrace{Y = X^{\alpha} L_Y^{1-\alpha}}_{\text{Final good prod.f.}} \qquad \underbrace{\forall_{i\in[0,1]} \quad x_i = k_i}_{\text{1-1 prod. f.}} \qquad \underbrace{X = (\int_0^1 x_i^{\theta} di)^{\frac{1}{\theta}}}_{\text{CES Aggregator of intermediates}}$$

7.4.1 *Final goods producers' problem*

$$\max_{\{x_i\}L_Y} \{Y - \int_0^A q_i x_i di - wL\} \qquad \text{where} \quad Y = (\int_0^A x_i^\theta)^{\frac{\alpha}{\theta}} L_Y^{1-\alpha}$$

$$\frac{\partial \Pi}{\partial L_Y} = (1-\alpha)\frac{Y}{L_Y} - w = 0 \quad \Rightarrow \quad w = (1-\alpha)\frac{Y}{L_Y}$$

$$\frac{\partial \Pi}{\partial x_i} = \frac{\alpha}{\theta}(\int_0^A x_i^\theta)^{\frac{\alpha}{\theta}-1} L_Y^{1-\alpha} \theta x_i^{\theta-1} - q_i = 0$$

$$\Rightarrow \qquad q_i = \alpha \frac{Y}{X^\theta} x_i^{\theta-1} \quad \Rightarrow \quad x_i(q_i) = (\frac{\alpha Y}{q_i X^\theta})^{\frac{1}{1-\theta}}$$

7.4.2 *Intermediate goods producers' problem ($i \in [0, A]$)*

$$\max_{q_i} \{q_i x_i - \tilde{r} k_i\}$$

where $x_i = k_i$ using the demand curve above we obtain:

$$\max_{q_i} \{(q_i - \tilde{r}) x_i(q_i)\}$$

$$\frac{\partial \Pi}{\partial q_i} = x_i(q_i) + (q_i - \tilde{r}) x_i'(q_i) = x_i(q_i) + (q_i - \tilde{r}) \frac{-1}{1-\theta} \frac{x_i(q_i)}{q_i} =$$

$$= \underbrace{x_i(q_i)}_{>0}(1 - \frac{1}{1-\theta}\frac{q_i - \tilde{r}}{q_i}) = 0$$

Hence we have

$$\frac{1}{1-\theta}\frac{q_i - \tilde{r}}{q_i}) = 1 \qquad \Rightarrow \qquad 1 - \frac{\tilde{r}}{q_i}) = 1\theta \quad \Rightarrow \quad q_i = \frac{\tilde{r}}{\theta}$$

$$\underbrace{q_i}_{\text{Monopoly prics}} = \underbrace{\frac{1}{\theta}}_{\text{constant markup \%}} \cdot \underbrace{\tilde{r}}_{\text{marginal cost of production}}$$

7.4.3 *Symmetry of intermediate goods producers*

$$q_i = \frac{\tilde{r}}{\theta} \qquad \forall_{i \in [0,A]}$$

hence

$$q_i = q_j = \bar{q} \qquad \text{implying} \quad x_i = \bar{x} \quad \forall_i$$

Monopoly profits are equal to:

$$\bar{\Pi} = (\bar{q} - \tilde{r})\bar{x} = (\frac{\tilde{r}}{\theta} - \tilde{r})\bar{x} = \frac{1-\theta}{\theta}\tilde{r}\bar{x}$$

7.4.4 *General equilibrium (so far)*

a)

$$K = \int_0^A k_i di = \int_0^A x_i di = \int_0^A \bar{x} di = A\bar{x} \tag{1}$$

Hence

$$\bar{x} = \frac{K}{A}$$

b)

$$X = (\int_0^A \bar{x}^\theta di)^{\frac{1}{\theta}} = (A\bar{x}^\theta)^{\frac{1}{\theta}} = A^{\frac{1}{\theta}}\bar{x} = A^{\frac{1}{\theta}}\frac{K}{A} = A^{\frac{1-\theta}{\theta}}K \tag{2}$$

c)

$$Y = X^\alpha L_Y^{1-\alpha} = \underbrace{A^{\frac{\alpha(1-\theta)}{\theta}} K^\alpha L_Y^{1-\alpha}}_{\text{IRS}} \tag{3}$$

d)

$$\bar{q} = \frac{\tilde{r}}{\theta} = \alpha\frac{Y}{X^\theta}\bar{x}^{\theta-1} = \alpha\frac{Y}{\bar{X}\bar{x}^{\theta-1} = \alpha\frac{Y}{K}(4)}$$

hence

$$\tilde{r} = \alpha\theta\frac{Y}{K}$$

e)

$$\int_0^a \Pi_i di = \int_0^A \bar{\Pi} di = A\bar{\Pi} = \frac{1-\theta}{\theta}\tilde{r}A\bar{x} = \alpha(1-\theta)Y \tag{5}$$

f) final output is divided according to:

$$\tilde{r}K + wL_Y + \int_0^A \Pi_i di = \underbrace{w\theta Y}_{\tilde{r}K} + \underbrace{(1-\alpha)Y}_{wL_Y} + \underbrace{\alpha(1-\theta)Y}_{\text{profits}} = Y \tag{6}$$

7.4.5 *Households- Dynamic optimization problem*

$$\max \int_0^\infty e^{-(\rho-n)t}u(c)dt \quad \text{where} u(c) = \frac{c^{1-\theta}-1}{1-\theta} \quad \text{CRRA}$$

$$\text{s.t.} \quad \dot{a} = \underbrace{(r-n)}_{n=0} a + w - c$$

yielding Euler equation:

$$\hat{c} = \frac{r-\rho}{\theta}$$

-Assets are kept in the form of

1. capital
2. shares of intermediate goods firms

-Assets equation (capital market clearing) $a = k + p_a\frac{A}{L}$

-Assets have equal returns:

$$\frac{\dot{p_A}}{p_A} + \frac{\Pi_i}{p_A} = r \qquad \Longleftrightarrow \qquad rp_A = \underbrace{\Pi_i}_{\text{Dividend}} + \underbrace{\dot{p_A}}_{\text{Resale Value}}$$

-Capital follows

$$\dot{k} = y - c - \delta k$$

7.4.6 *R&D firms*

- create new varieties of intermediate goods
- sell the right to produce (patents) to households
- patents have infinite duration and patent protection is perfect
- there is free entry to R&D
- growth effects of innovations are not internalized by R&D firms

$$\max_{L_A}\{p_A \underbrace{\bar{v}L_A}_{\dot{A}} - qL_A\}$$

taking $p_a, \bar{v}, w$ as given. $\bar{v}$-R&D externality

By free entry $w = p_A\bar{v}$.

-Upon aggregation

$$\dot{A} = \gamma L_A^{\lambda} A = \underbrace{\gamma L_A^{\lambda-1} A}_{\bar{v}} L_A$$

7.4.7 *Labor market clears*

$$L = L_A + L_Y \quad \text{equal wage}$$

$$w = (1-\alpha)\frac{Y}{L_Y} = p_A \gamma L_A^{\lambda-1} A$$

Hence

$$p_A \frac{(1-\alpha)Y}{L_Y \gamma L_A^{\lambda-1} A}$$

7.4.8 *Dynamics*

4 key variables $\underbrace{k, A}_{\text{state var.}}, \underbrace{c, L_a}_{\text{control var.}}$

$$\begin{cases} \hat{c} = \frac{1}{\theta}(r-\rho) = \frac{1}{\theta}(\alpha\theta\frac{y}{k} - \delta - \rho) \\ \hat{k} = \frac{y}{k} - \frac{c}{k} - \delta \\ \hat{A} = \gamma L_A^\lambda \\ y = A^{\alpha\frac{1-\theta}{\theta}} k^\alpha \frac{L_Y}{L}^{1-\alpha} \\ \hat{p_A} = r - \frac{\Pi_i}{p_A} \qquad \text{(capital market)} \\ p_A = \frac{(1-\alpha)Y}{L_Y \gamma L_A^{\lambda-1} A} \qquad \text{(labor market)} \\ L_Y = L - L_A \end{cases}$$

One can eliminate p_A, get dynamics of L_A, L_Y

See Arnold (2006)

7.4.9 *BGP equilibrium*

- $\frac{y}{k}$-const, $\frac{c}{k}$-const, $\hat{c} = \hat{k} = \hat{y} := g$-growth rate
- $\frac{L_A}{L}$-const , $\frac{L_Y}{L}$-const, constant savings rate, L_A^*-const, L_Y^*-const
- $\hat{A} = \gamma L_A^{*\lambda}$
- $\hat{y} = \frac{\alpha(1-\theta)}{\theta}\hat{A} + \alpha\hat{y} \Rightarrow \hat{y} = \underbrace{\frac{\alpha(1-\theta)}{\theta}}_{\sigma} \frac{1}{1-\alpha}\gamma L_A^{*\lambda}$ growth rate to the economy:

$$g = \frac{\alpha}{1-\alpha}\frac{1-\theta}{\theta}\gamma L_A^{*\lambda}$$

$$\begin{cases} \bar{p_A} \\ \bar{p_A} = \underbrace{\alpha\theta\frac{y}{k} - \delta}_{r} + \underbrace{\alpha(1-\theta)\frac{Y}{A}}_{\Pi} \cdot \underbrace{\frac{L_Y \gamma L_A^{\lambda-1} A}{(1-\alpha)Y}}_{1/p_A} + \frac{\alpha(1-\theta)}{1-\alpha}\frac{L_Y}{L_A}\underbrace{\gamma L_A^\lambda}_{\hat{A}} \end{cases}$$

with

$$\hat{c} = \frac{r-\rho}{\theta} \Rightarrow \quad r = \theta g + \rho \qquad \theta\hat{c} = \alpha\theta\frac{y}{k} - \delta - rho$$

$$\frac{y}{k} = \frac{\theta g + \delta + g}{\alpha\theta}$$

Hence along the BGP:

$$g - \hat{A} = \theta g + \rho - \frac{\alpha}{1-\alpha}(1-\theta)\frac{L_Y}{L_A}\hat{A}$$

$$-(1-\theta)g + \hat{A} + \rho = \frac{}{1-\alpha}(1-\theta)\frac{L_Y}{L_A}\bar{A}$$

$$\frac{1-L_A^*}{L_A^*}=\frac{L_Y^*}{L_A^*}=\frac{\bar{A}+\rho-(1-\theta)g}{(1-\theta)\bar{A}}\frac{1-\alpha}{\alpha}=\frac{1-\alpha}{\alpha}(\frac{1}{1-\theta}+\rho\frac{\alpha}{1-\alpha}\frac{1-\theta}{\theta}\frac{1}{g}-\frac{\alpha}{1-\alpha}\frac{1-\theta}{\theta})$$

-the equilibrium allocation of labor between L_A and L_Y (R&D and output)

- depends on α-technology, θ-markup parameter, ρ-impatience, γ,λ- R&D technology

8

SCHUMPETERIAN (QUALITY LADDER) GROWTH MODEL

8.1 SCHUMPETERIAN (QUALITY LADDER) GROWTH MODEL

Observation: 3: Key insight

Long-run growth driven by quality improvements within a predefined set of product varieties

8.2 SCHUMPETERIAN 'CREATIVE DESTRUCTION'

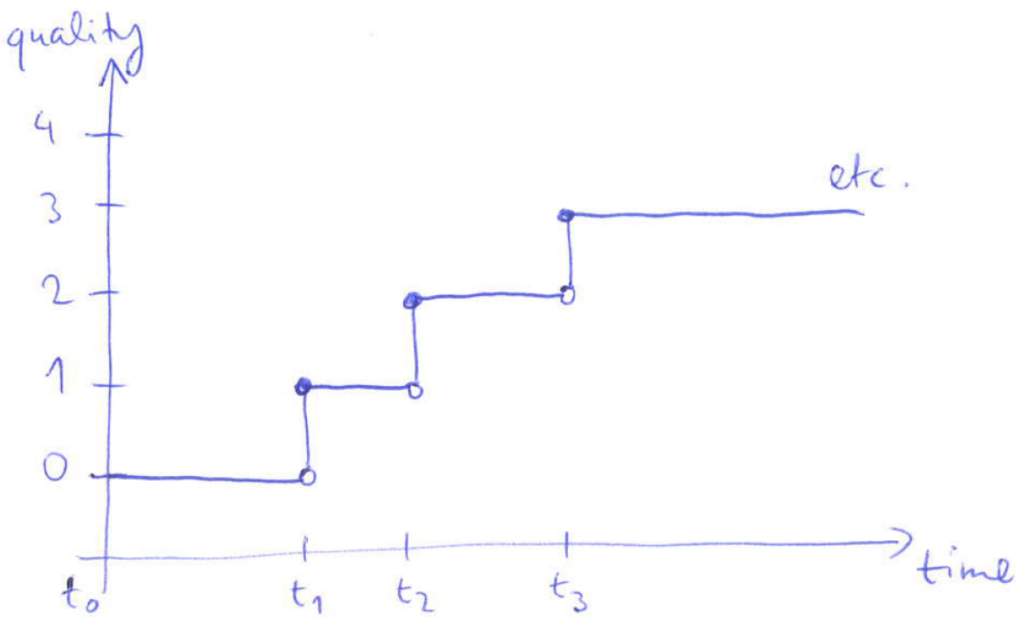

Figure 15

⇒ Timing of quality innovations is uncertain

⇒ Productivity in a given sector is given by g^{κ} where $q > 1$ and κ -rung in the quality ladder

⇒ The researcher responsible for a quality improvement remains the monopoly right to produce the good

⇒ Only the highest available grade of goods will actually be produced in equilibrium

⇒ There are infinitely many sectors indexed by $i \in [0, 1]$

⇒ This helps overcome aggregate uncertainty (law of large numbers)

The quality adjusted amount of i-th intermediate input

$$\tilde{X}_i = q^{k_i} X_i$$

The aggregate production function is

$$Y = AX^{\alpha} L_Y^{1-\alpha} \qquad \text{where} \qquad X = (\int_0^1 (q^{k_i} x_i)^{\theta} di)^{\frac{1}{\theta}}$$

We maintain the Dixit -Stiglitz monopolistic competition setup

8.2.1 *Final goods producers' problem*

$$\max_{\{x_i\} L_Y} \{Y - \int_0^1 q_i x_i di - wL_Y\}$$

using prod.fcts defined above

$$\frac{\partial \Pi}{\partial L_Y} = (1-\alpha)\frac{Y}{L_Y} - w = 0 \quad \Rightarrow \quad w = (1-\alpha)\frac{Y}{L_Y}$$

$$\frac{\partial \Pi}{\partial x_i} = \frac{\alpha}{\theta}(\int_0^1 (q^{k_i} x_i)^{\theta} di)^{\frac{\alpha}{\theta}-1} A L_Y^{1-\alpha} \theta q^{\theta k_i} x_i^{\theta-1} - q_i = 0$$

$$\Rightarrow \qquad q_i = \alpha \frac{Y}{X^{\theta}} q^{\theta k_i} x_i^{\theta-1} \quad \Rightarrow \quad x_i(q_i) = (\frac{\alpha Y q^{\theta k_i}}{q_i X^{\theta}})^{\frac{1}{1-\theta}}$$

which gives demand curve for intermediate goods

8.2.2 *Intermediate goods producers' problem* $i \in [0, 1]$

- it is assumed the marginal cost of production is 1

$$\Pi(k_i) = (q_i - 1)x_i = (q_1 - 1)x(q_i)$$

- monopoly pricing

$$\frac{\partial \Pi(k_i)}{\partial q_i} = x(q_i) + (q_i - 1)x'(q_i) = \underbrace{x(q_i)}_{>0}(1 - \frac{1}{1-\theta}\frac{q_i - 1}{q_i}) = 0$$

hence we obtain

$$\frac{1}{1-\theta}\frac{q_i - 1}{q_i} = 1 \qquad \Rightarrow \qquad q_i = \frac{1}{\theta} \quad \forall_i$$

8.2.3 *By symmetry*

$$q_i = \frac{\tilde{r}}{\theta} \qquad \forall_{i\in[0,1]}$$

- However, output may vary because vectors can be at different rungs of the quality ladder

$$x_i = (\frac{\alpha Y}{X^\theta}\theta q^{\theta k_i})^{\frac{1}{1-\theta}} = (\alpha\theta Y)^{\frac{1}{1-\theta}} \frac{q^{k_i}}{X}^{\frac{\theta}{1-\theta}}$$

- Profits are given by

$$\Pi(k_i) = \frac{1-\theta}{\theta}\bar{\Pi}(q^{k_i})^{\frac{\theta}{1-\theta}}$$

where

$$\bar{\Pi} = (\frac{\alpha\theta Y}{X^\theta})^{\frac{1}{1-\theta}}$$

8.2.4 *Value of a quality innovation*

- monopoly rights are perpetual
- value of these rights falls to zero when a new quality rung is attained within sector
- The present value of profits for the inventor of rung k_i

$$V(k_i) = \int_{t_{k_i}}^{t_{k_i+1}} \Pi(k_i)e^{-\bar{r}(v,t_{k_i})(v-t_{k_i})}dv$$

where $\bar{r}(v, t_{k_i}) := \frac{1}{r-t_{k_i}}\int_{t_{k_i}}^{v} r(w)dw$ is the average interest rate between t_{k_i} and v

- If the interest rate is fixed this simplifies to

$$V(k_i) = \Pi(k_i)\frac{1-e^{r(t_{k_i+1}-t_{k_i})}}{r}$$

8.2.5 *Aggregation (so far)*

***Definition:* 6: Quality index**

Quality index of the economy is denoted by:

$$Q = (\int_0^1 q^{k_i\frac{\theta}{\theta-1}}di)^{\frac{1-\theta}{\theta}}$$

Using the definition of X we have

$$X^\theta = \int_0^1 (q^{k_i}x_i)^\theta di = \int_0^1 (q^{k_i(1+\frac{\theta}{1-\theta}}\frac{\alpha\theta Y}{X^\theta}^{\frac{1}{1-\theta}})^\theta di =$$

$$= \frac{\alpha\theta Y}{X^\theta} \int_0^! q^{k_i(\frac{\theta}{1-\theta})} di = \frac{\alpha\theta Y}{X^\theta} \cdot Q^{\frac{\theta}{1-\theta}}$$

Hence

$$X = (\alpha\theta Y)^{\frac{1}{1-\theta}} X^{-\frac{\theta}{1-\theta}} Q^{\frac{1}{1-\theta}} \qquad \Rightarrow \qquad X = \alpha\theta Y Q$$

Inserting into the production function

$$Y = (\alpha\theta Y Q)^\alpha L_Y^{1-\alpha} \quad \Rightarrow \quad Y = A^{\frac{1}{1-\alpha}} (\alpha\theta Y Q)^{\frac{\alpha}{1-\alpha}} L_Y$$

8.2.6 *Innovation*

-Recall that $V(k_i)$ is a random variable because the timespan of many k_i is uncertain

$$E[v(k_i)] = \frac{\Pi(k_i)}{r + p(k_i)}$$

where $p(k_i)$ probability density per unit of time or

$$r = \frac{\Pi(k_i) - p(k_i)EV(k_i)}{EV(k_i)}$$

-Assume R&D technology: -$p(k_i)$ depends only on total R&D expenditure $Z(k)i$:

$$p(k_i) = \underbrace{Z(k_i)}_{\text{Linear}} \cdot \underbrace{\phi(k_i)}_{\text{effect of the current tech position}}$$

-Free entry into R&D implies

$$\underbrace{p(k_i)EV(k_i+1)}_{\text{Net expected return per time unit}} = \underbrace{Z(k_i)}_{\text{Cost}}$$

$$Z(k_i)\phi(k_i)E(V(k_i+1)) = Z(k_i) \qquad \Rightarrow \qquad \phi(k_i)E(V(k_i+1)) = 1$$

$$r + p(k_i+1) = \phi(k_i) \underbrace{\bar{\Pi} q^{(k_i+1)\frac{\theta}{1-\theta}} \frac{1-\theta}{\theta}}_{\Pi(k_i+1)}$$

Let us now make a simplifying assumption that

$$\phi(k_i) = \frac{1}{\xi} q^{(k_i+1)\frac{\theta}{1-\theta}} \frac{1-\theta}{\theta}$$

$\xi > 0$-'cost of doing research'
In such case the same for all k_i

$$r + p(k_i+1) = \frac{\bar{\Pi}}{\xi} \qquad \forall_{k_i}$$

And so $p = \frac{\bar{\Pi}}{\xi} - r$ implying

$$Z(k_i) = q^{(k_i+1)\frac{\theta}{1-\theta}} \frac{1-\theta}{\theta}(\bar{\Pi} - r\xi)$$

It is distribution of R&D expenditures across sectors

Aggregate R&D spending is

$$Z = \int_0^1 Z(k_i)do = \int_0^1 q^{\frac{\theta}{1-\theta}} \frac{1-\theta}{\theta}(\bar{\Pi} - r\xi)$$

8.2.7 *Households*

-Usual dynamic optimization problem implies

$$\frac{\dot{c}}{c} = \frac{r-\rho}{\gamma}$$

note that $\hat{c} = \hat{C}$ because L-const.

8.2.8 *Dynamics*

for a very specific parametrization- **Knife-edge** one discussed by Barro&Sala-i-Martin (2003)

Note Aggregate Identities:

$$Y = A^{\frac{1}{1-\alpha}} (\alpha\theta Y Q)^{\frac{\alpha}{1-\alpha}} L_Y$$

$$X = \alpha\theta Y Q = (\alpha\theta)^{\frac{1}{1-\alpha}} A^{\frac{1}{1-\alpha}} L_Y Q^{\frac{1}{1-\alpha}}$$

$$Z = q^{\frac{\theta}{1-\theta}} \frac{1-\theta}{\theta}(\bar{\Pi} - R\xi)^{\frac{\theta}{1-\theta}}$$

$$\bar{\Pi} = \frac{\alpha\theta Y}{X^\theta}^{\frac{1}{1-\theta}} = (\alpha\theta)^{\frac{1}{1-\alpha}} A^{\frac{1}{1-\alpha}} L_Y Q^{\frac{\alpha-\theta}{(1-\alpha)(1-\theta)}}$$

Now assume $\alpha = \theta$

This implies

$$Y \sim X^\alpha \sim Z \sim Q^{\frac{\alpha}{1-\alpha}} \qquad \bar{\Pi}\text{const}$$

If $L_Y \equiv$ const (fixed shares of R&D employment) then:

$$\hat{Y} = \alpha\hat{X} = \hat{Z} = \frac{\alpha}{1-\alpha}\hat{Q} := g$$

also $\hat{C} = g$. Growth rate of the economy is driven by quality innovation

8.2.9 *Dynamics of Q*

$$E(\Delta Q) = ?$$

Let us work with

$$Q^{\frac{\theta}{1-\theta}} = \int_0^1 q^{k_i \frac{\theta}{1-\theta}} di := \bar{Q}$$

$$E(\Delta Q) = \int_0^1 p(k_i)(q^{(k_i+1)\frac{\theta}{1-\theta}} - q^{k_i \frac{\theta}{1-\theta}})di = \int_0^1 p(q^{\frac{\theta}{1-\theta}} - 1)q^{k_i \frac{\theta}{1-\theta}})di = p(q^{\frac{\theta}{1-\theta}} - 1)\bar{Q}$$

and therefore

$$E(\frac{\Delta \bar{Q}}{\bar{Q}}) = p(q^{\frac{\theta}{1-\theta}} - 1)$$

-Law of large numbers allows us to treat $\Delta \bar{Q}$ as deterministic:

$$\frac{\theta}{1-\theta}\hat{Q} = \hat{\bar{Q}} = p(q^{\frac{\theta}{1-\theta}} - 1) = (\frac{\bar{\Pi}}{\xi} - r)(q^{\frac{\theta}{1-\theta}} - 1)$$

8.2.10 *Equilibrium rate of return r and growth rate g*

$$\begin{cases} \hat{c} = g = \frac{r-\rho}{\gamma} \\ g = (\frac{\bar{\Pi}}{\xi} - r)(q^{\frac{\theta}{1-\theta}} - 1) \end{cases}$$

and assume $\alpha = \theta$.
Note that r-fixed $\Rightarrow$ g-fixed $\Rightarrow$ No transitional dynamics, just Balanced growth.

Solving the system implies, growth rate of economy:

$$g = \frac{(\frac{\bar{\Pi}}{\xi} - r)(q^{\frac{\alpha}{1-\alpha}} - 1)}{1 + \gamma(q^{\frac{\alpha}{1-\alpha}} - 1)}$$

and interest rate:

$$r = \frac{\rho + \gamma\frac{\bar{\Pi}}{\xi}(q^{\frac{\alpha}{1-\alpha}} - 1)}{1 + \gamma(q^{\frac{\alpha}{1-\alpha}} - 1)}$$

where $\bar{\Pi} = \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L_Y$ This implies also a constant probability of innovation:

$$p = \frac{\frac{\bar{\Pi}}{\xi} - \rho}{1 + \gamma(q^{\frac{\alpha}{1-\alpha}} - 1)}$$

Comments:

$\Rightarrow$ determinants of long-run growth:

- model parameters $, \gamma, \rho$
- technology level$A$ ($A \nearrow \quad \Rightarrow g \nearrow$)
- population size L_Y ($L_Y \nearrow \quad \Rightarrow g \nearrow$)-scale effect

  - size of the quality innovation rung q ($q \nearrow \Rightarrow g \nearrow$)

$\Rightarrow$ 'Schumpeterian' flavor - creative destruction

$\Rightarrow$ relies on a very specific parametrization $\alpha = \theta$ and of the function$\phi(k_i)$

$\Rightarrow$ playing with $\phi(k_i)$ may destroy the asymptotically balanced growth property, but may also alleviate the scale effect

Alternative definition of $\phi(k_i)$

$$\phi(k_i) = \frac{1}{\xi}\frac{1}{Y(k_i+1)} = \frac{1}{\xi A^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}L_Y q^{(k_i+1)\frac{\alpha}{1-\alpha}}}$$

-Following analogous steps as before we arrive at:

$$g = \frac{(\frac{\alpha(1-\alpha)}{\xi} - \rho)(q^{\frac{\alpha}{1-\alpha}} - 1)}{1 + \gamma(q^{\frac{\alpha}{1-\alpha}} - 1)}$$

-This looks very similar to the previous version, but now there is no scale effect! A scale-free model.

Final notes:

- We have assumed throughout that $L_Y \equiv L$, and thus there was no competition for labor between the production and the R&D sector
- we have skipped physical capital accumulation - the only asset available for households savings are the shares of firms producing intermediate inputs $\{x_i\}$ $i \in [0,1]$
- Adding either of these two possibilities could be a source of transitional dynamics

9

UZAWA BALANCED GROWTH THEOREM

9.1 Uzawa's (1961) steady state growth theorem

...

In a neoclassical gorwth model, the existence of a balanced growth requires that either

- the production function is Cobb-Douglas
- technological progress is purely labor-augmenting

Note:'neoclassical growth model'

$$Y = F(K, L, A)\text{has CRS with respect to K,L} \quad L(t) = L_0 e^{nt} \quad \dot{K} = Y - C - \delta K$$

Note 2: 'balanced growth path' means that

$$Y(t) = Y_0 e^{yt} \quad C(t) = C_0 e^{ct} \quad K(t) = K_0 e^{kt}$$

proof: (Schlicht, 2006)

-Write $Y(t) = F(K(t), L(t), t)$
-From the equation of motion, we have

$$\dot{K}(t) = kK(t) = Y(t) - C(t) - \delta K(t)$$

$$\rightarrow (k + \delta)K_0 = Y_0 e^{(y-k)t} - C_0 e^{(c-k)t} \qquad \forall t \geq 0$$

-Taking time derivatives

$$(y - k)Y_0 e^{(y-k)t} - (c - k)C_0 e^{(c-k)t} = 0$$
$$(y - k)Y_0 e^{(y-k)t} = (c - k)C_0 e^{(c-k)t}$$

-Therefore either $y = k$ and $c = k$ or $y = c$ and $Y_0 = C_0$ and hence $K_0 = 0$. So growth rates of Y, C, K coincide and second part gives contradiction.

-Define $G(K, L) := F(K, L, 0)$
We have $Y_0 = G(K_0, L_0)$, $Y(t) = Y_0 e^{yt}$ $L(t) = L_0 e^{-nt}$, $K_0 = K(t)e^{-kt}$ and G is linear homogenous

-Hence

$$Y(t) = G(K_0, L_0)e^{yt} = G(K(t)e^{-kt}, L(t)e^{-nt})e^{yt} = G(K(t)e^{(y-k)t}, L(t)e^{(y-n)t}) = (K(t), L(t)e^{(y-n)t})$$

because of labor augmenting TC

Comment. Where is the Cobb-Douglas??

-Observe if

$$Y(t) = \underbrace{A(t)}_{\text{any time trend}} \cdot K(t)^{\alpha} L(t)^{1-\alpha}$$

then we can always rewrite it as

$$Y(t) = \cdot K(t)^{\alpha} \underbrace{(\bar{A}(t)L(t))^{1-\alpha}}_{\text{LATC}}$$

here $\bar{A}(t) = A(t)^{\frac{1}{1-\alpha}}$ - observationally equivalent!

-However, we could also, e.g., write it as

$$Y(t) = K(t)^{\alpha} \underbrace{(\tilde{A}(t)L(t))^{1-\alpha}}_{\text{KATC}} \tilde{A}(t) = A(t)^{\frac{1}{\alpha}}$$

-For other, non-mulitplicative production facts, alternative patterns of factor augmentation are not observationally equivalent.

Note nr 1: All growth models with endogenous technical change discussed so far feature Cobb-Douglas production functions. This was not only for simplification

Note nr 2: Empirically, LATC is rather reasonable, given that:

- rates of return to a unit of capital have been broadly stable over time
- wages have been rising roughly exponentially
- labor's share of GDP has been broadly stable over the long run

Those are Kaldor facts (1961) and now are contested!

10

SCALE EFFECTS.JONES CRITIQUE

10.1 SCALE EFFECTS

-Notice that in Romer(1990) and elsewhere

$$g \sim \hat{A} = \gamma L_A^{\gamma} \quad \text{where} \quad L_A - \text{R\&D employment}$$

-These are STRONG **scale effects**:

- provided that the share of R&D employment is the same, bigger countries grow faster:

$$g \sim \gamma L_A^{\gamma} (\underbrace{l_A^{\gamma}}_{\text{fixed}}) \quad \ln g \sim \lambda \ln L$$

- clearly incosistent with empirical evidence!
- Moreover, if there is constant population growth $\hat{L} = n$, then

$$g \sim \gamma (L_0 e^{nt})^{\gamma} (\underbrace{l_A^{\gamma}}_{\text{fixed}})$$

- The growth rate is growing (at a rate λn)

10.2 'JONES CRITIQUE'

-Jones(1995) has shown that also along the US time series, the evidence is inconsistent with strong scale effects: the R&D employment or expenditure increased greatly whereas the long-run growth rate has remained virtually unchanged

Responses to the Jones critique

-Jones (1995) himself+ followers (Kartum, Segerstrom) pose:

$$\dot{A} = \gamma L_A^{\gamma} A^{\phi} \quad \phi < 1$$

$\phi = 1$ imposes strong scale effects,
$\phi \in (0, 1)$ is 'standinn shoulders
$\phi < 0$ is 'fishing out'

-In this case, the long run growth rate is

$$g \sim \hat{A} = \gamma L_A^{\gamma} A^{\phi - 1}$$

-Assuming BGP ($\hat{A}$-const $\Rightarrow \tilde{A} = 0$)we have:

$$0 = \lambda \hat{L_A} + (\phi - 1)\hat{A}$$

$$g \sim \hat{A} = \frac{\lambda}{1-\phi}(\underbrace{\hat{l_A}}_{=0} + \hat{L}) = \frac{\lambda n}{1-\phi}$$

-The long run growth rate is proportional to the population growth rate

-The class of models sharing this property is called SEMI-endogenous growth models

-The long run growth rate does not depend on any endogenous variable

- despide R&D in the model!
- $n = 0 \Rightarrow g = 0$!!!
- Jones foresaw a mojor slowdown in the US economy, already around 2000 (e.g. Jones 2002,AER), perhaps we're observing it just now??

-'Second generation' R&D-based endogenous growth models (e.g. Young 1998, Aghion& Howitt 2000, Peretto 2000):

$$\dot{A} = \gamma(\frac{L_A}{A})^{\lambda} A$$

-hence

$$\underbrace{g \sim \hat{A} = \gamma(\frac{L_A}{A})^{\lambda} = \gamma l_A^{\gamma}}_{\text{Long run growth rate independent of population size and growth}}$$

-Jones (1995) model has 'weak scale effects' (level effects)

-2nd generation endogenous growth models recover the endogeneity of the growth rate (l_A is choice variable)

-These models are also called 'non-scale' models

Knife-edge conditions in growth models -take $\dot{A} = \gamma \frac{L_A^{\gamma}}{L^{\beta}} A^{\phi}$ where L^{β} is 'product proliferation' effect (explained in the increasing variety framework)

-Endogenous growth requires

$\phi = 1$ with scale effects,
or $\phi = 1$ and $\beta = 1$ without scale effects,

-Jones (1999) criticises models based on knife-edge assumptions as implausible

-The 'linearity critique' (Jones 2003, 2005)
Any endogenous growth model has to contain an equation of form

$$\dot{X} = \underbrace{\alpha}_{\text{can be endog.}} X^{\phi} \quad \text{where} X{-}?? \quad \phi = 1$$

$\phi \neq 1$ i.e., any deviation from pure linearity is then leading to qualitatively different model dynamics

-Not entirely true! Take two state variables:

$$\hat{x} = x^{\alpha} y^{\beta}$$

$$\hat{y} = x^{\gamma} y^{\delta}$$

Assuming the BGP $\hat{x}$-const and $\hat{y}$-const ($\tilde{x} = \tilde{y} = 0$)

$$0 = \alpha\hat{x} + \beta\hat{y}$$

$$0 = \gamma\hat{x} + \delta\hat{y}$$

$$\Updownarrow$$

$$\underbrace{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}}_{\text{singular or}} \underbrace{\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}}_{=0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

-Better call this 'Singularity critique'
-Singularity of a given matrix is a knife-edge condition
-General argument (Growiec, 2007)

-For any growth model of form $\dot{X} = F(X)$, $X(t)$-vector, existence of a BGP requires making a knife-edge assumption.

- note that this case includes also higer-order differential equations
- $X(t)$ contains state variables only (reduced form model)

10.2.1 *Empirical evidence*

-Ha and Howitt(2007) [US data 1950-2000], Madsen (2008) [OECD Panel data] find that the non scale endogenous growth model is better aligned with data than the semi-endogenous growth model

-Caveats

- international technology diffusion
- technology adoption lags
- multi dimensional TC ?

11

CONVERGENCE. TECHNOLOGY DIFFUSION

11.1 CONVERGENCE

Recall the Solow model:

$$\hat{k} = \frac{\dot{k}}{k} = s\frac{f(k)}{k} - \delta$$

hence $\hat{k}$ <u>declines</u> with k (f-concave!) -'Neoclassical convergence'
$\Rightarrow$ Other things equal, richer countries should grow slower
-Absolute convergence (2 countries)
Let $s_1 = s_2$, $\delta_1 = \delta_2$ $f_1 = f_2$ but $k_1 < k_2$ at time $t = 0$. Then

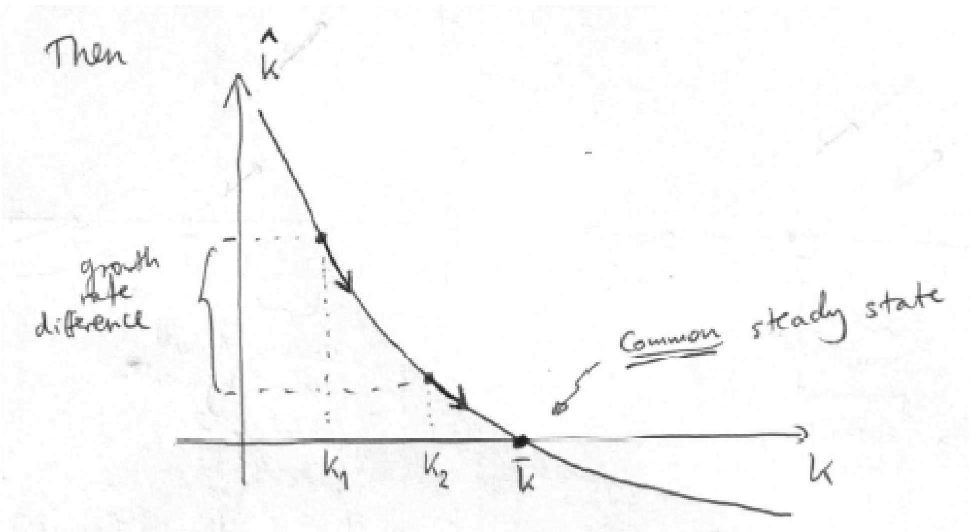

Figure 16

The richer country grows slower, whereas the poorer country catches up (gradually, never fully!).

11.1.1 *Conditional convergence (2 countries)*

$\Rightarrow$ If 'ceteris paribus' doesn't hold then countries converge to different steady states.

$\Rightarrow$ Part of the difference in their output is permanent ('structural', 'fundamental')

$\Rightarrow$ Let $k_1 < k_2$ but either $s_1 \neq s_2$ $\delta_1 \neq \delta_2$ or $f_1 \neq f_2$. For example $s_1 \neq s_2$.

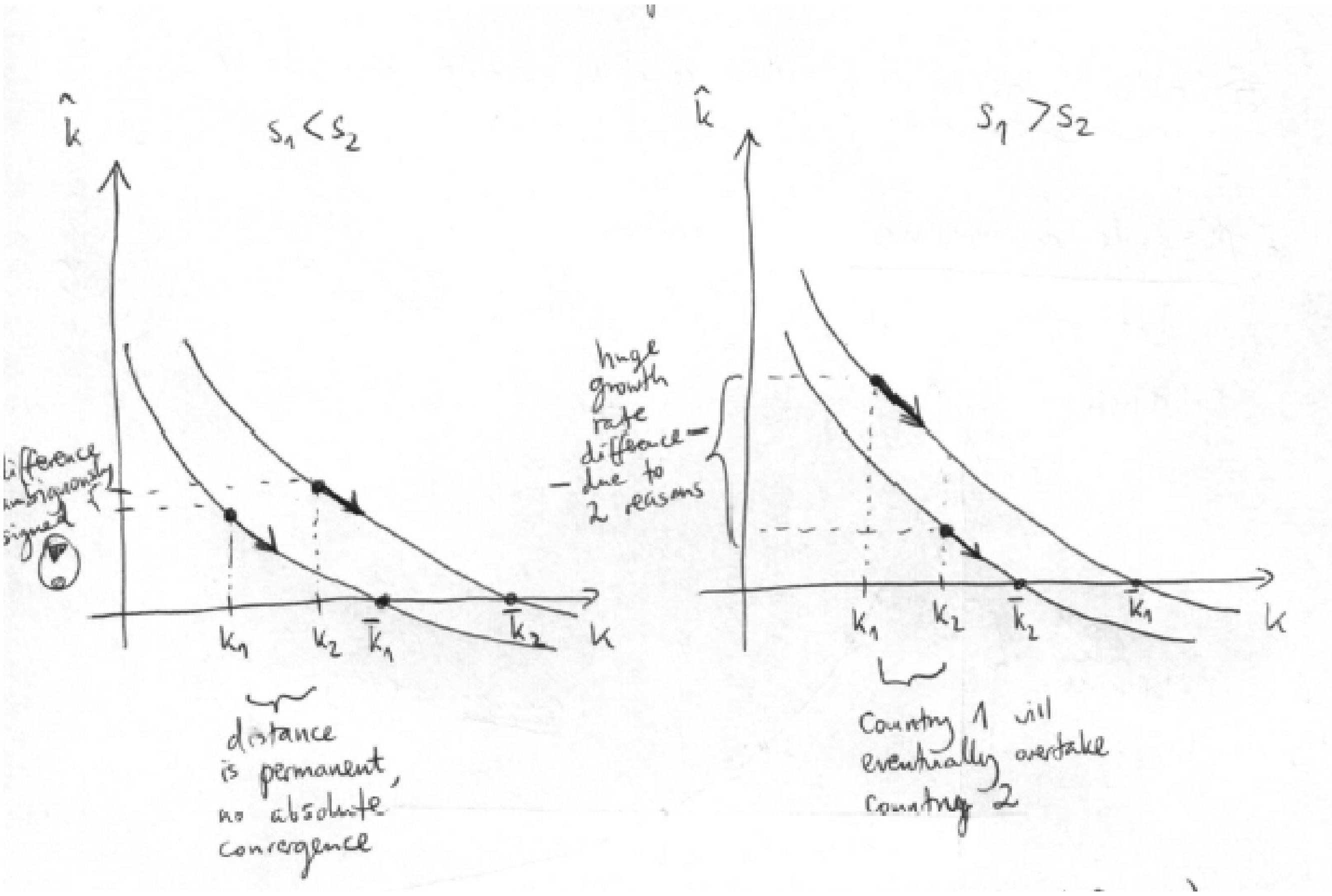

Figure 17

-Conditional on the difference (controlling for the differences) in structural characteristics, richer countries should grow slower.

11.2 SPEED OF CONVERGENCE

Consider the Cobb-Douglas case for simplicity $f(k) = Ak^{\alpha}$

$$\hat{k} = sAk^{\alpha-1} - \delta$$

-The steady state is

$$\hat{k} = 0 \iff \bar{k}^{\alpha-1} = \frac{\delta}{sA} \iff \bar{k} = (\frac{sA}{\delta})^{\frac{1}{1-\alpha}}$$

***Definition:* 7: The speed of convergence**

The speed of convergence is

$$\beta = -\frac{\partial \hat{k}}{\partial \ln k} = \frac{\partial \hat{k}}{\partial k} \cdot \frac{\partial k}{\partial \ln k} = \frac{\partial \hat{k}}{k} \cdot k$$

-In the Solow model

$$\beta = -sA(\alpha - 1)k^{\alpha-1} = (1-\alpha)sAk^{\alpha-1}$$

-In the vicinity of the steady state ($\bar{k}^{-1} = \frac{\delta}{sA}$)

$$\beta = (1-\alpha)\delta$$

Notes

- in Solow model with population growth and technology progress

$$\beta = (1-\alpha)(\delta + n + g)$$

- the saving rate and technology level A affect the level of the steady state, but not the pace of convergence β
- any model with a neoclassical production function predicts β-convergence
- e.g., the AK model doesn't feature transitional dynamics $\Rightarrow$ no β-convergence

11.3 β-convergence vs σ-convergence

***Definition:* 8: Detecting σ-convergence**

σ-convergence is observed if the standard deviation σ of output decreases over time in a group of countries

***Theorem:* 3: Convergence**

σ-convergence implies β-convergence

-Take the Solow model again

Assuume $s_1 = \cdots = s_n$, $\delta_1 = \cdots = \delta_n$, $f_1 = \cdots = f_n$ but $k_1 \leq k_2 \cdots \leq k_n$

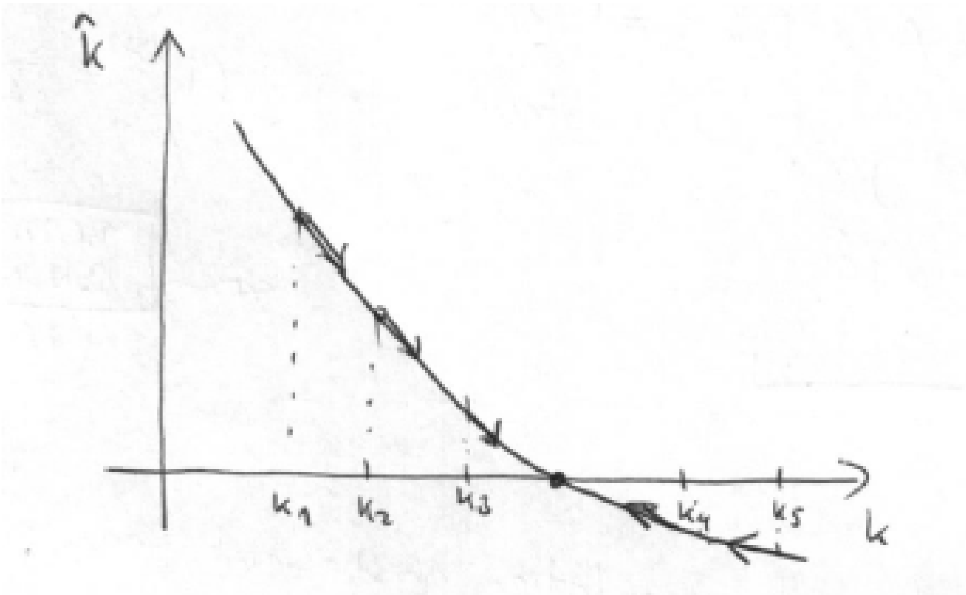

Figure 18

There is -convergence. Eventually $(k_1, \ldots, k_n) \to^{t\to\infty} 0$ as $k_1, \ldots k_n \to \bar{k}$.

-When there is only conditional convergence, σ-convergence need not (and typically will not) hold.

$$(k_1, \ldots, k_n) \to (\bar{k_1}, \ldots, \bar{k_n}) > 0$$

Observation: 4: Add stochastic disturbance

$$\ln(y_{it}) = a + (1-b)\ln(y_{i,t-1}) + u_{it}$$

$$b \in (0,1) \Rightarrow \text{absolute convergence} \quad u_{it} \sim^{\text{iid}} N(0, \sigma_u^2)$$

Compute dispersion

$$D_t = \frac{1}{N}\sum_{i=1}^{N}(\ln y_{it} - \mu_t)^2$$

-For large N we obtain

$$D_t \approx (1-b)^2 D_{t-1} + \sigma_u^2$$

-The steady state $\{D_t\}$ is D^* such that $D^* = (1-b)^2 D^* + \sigma_u^2$

$$D^* = \frac{\sigma_u^2}{1-(1-b)^2} \qquad (b = 0 \Rightarrow \text{Random Walk})$$

$$D_t \to D^* \quad \text{monotonically (could be growing!)}$$

-Galton's fallacy

-Overall β-convergence does not imply σ-convergence.

11.4 TECHNOLOGY DIFFUSION AND TECHNOLOGICAL CATCH-UP

- Nelson-Phelps (1966) model of technology diffusion (m-leader country, i-given country)

$$\frac{\dot{A}_i}{A_i} = g_i + c_i(\frac{A_m}{A_i} - 1)$$

- Solution for $A_i(t)$:

$$A_i(t) = (A_i(0) - \omega A_m(0))e^{(g_i - c_i)t} + \omega A_m(0)e^{g_m t}$$

with $\omega = \frac{c_i}{c_i - g_i + g_m} > 0$

- For example when $g_m = g_i$ then $\omega = 1$. Otherwise $g_m > g_1$ and $\omega < 1$ - diffusion lag.

$$\lim_{t \to \infty} \frac{A_i(t)}{A_m(t)} = \omega$$

- Important implication: eventually all countries grow at the same rate
- g_i, c_i may be functions of human capital (Benhabib & Spiegal, 2005)
- $g_i \approx$ domestic technological progress
- $c_i \approx$ pace of technological adoption

Modified (logistic) Nelson-Phelps diffusion process

$$\frac{\dot{A}_i}{A_i} = g_i + c_i(1 - \frac{A_i}{A_m}) = g_i + c_i\frac{A_i}{A_m}(\frac{A_m}{A_1} - 1)$$

- Solution for $A_i(t)$

$$A_i(t) = \frac{A_i(0)e^{(g_i + c_i)t}}{1 + \frac{A_i(0)}{A_m(0)\tilde{\omega}}(e^{(c_i + g_i - g_m)t} - 1)}$$

- with $\tilde{\omega} = \frac{c_i + g_i - g_m}{c_i} >< 0$ For example when $g_m = g_i$ then $\tilde{\omega} = 1$. Otherwise $g_m > g_i$ adn $\tilde{\omega} < 1 \Rightarrow$ Diffusion lag
- In the limit

$$\lim_{t \to \infty} \frac{A_i(t)}{A_m(t)} = \begin{cases} \tilde{\omega}, & \text{if } \tilde{\omega} > 0 \\ \frac{A_i(0)}{A_m(0)} & \text{if } \tilde{\omega} = 0 \\ 0 & \text{if } \tilde{\omega} < 0 \end{cases}$$

- If the catching-up rate is too low ($c_i + g_i < g_m$) then country in will run away and there will be no catch-up.
- Again g_i, c_i may be functions of human capital

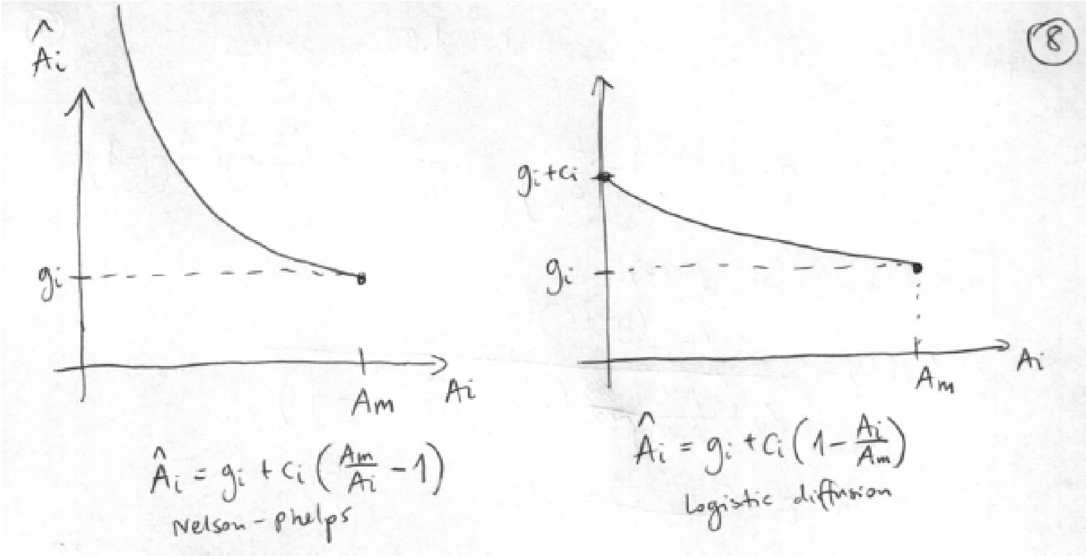

Figure 19

-Historically, after each technological revolution

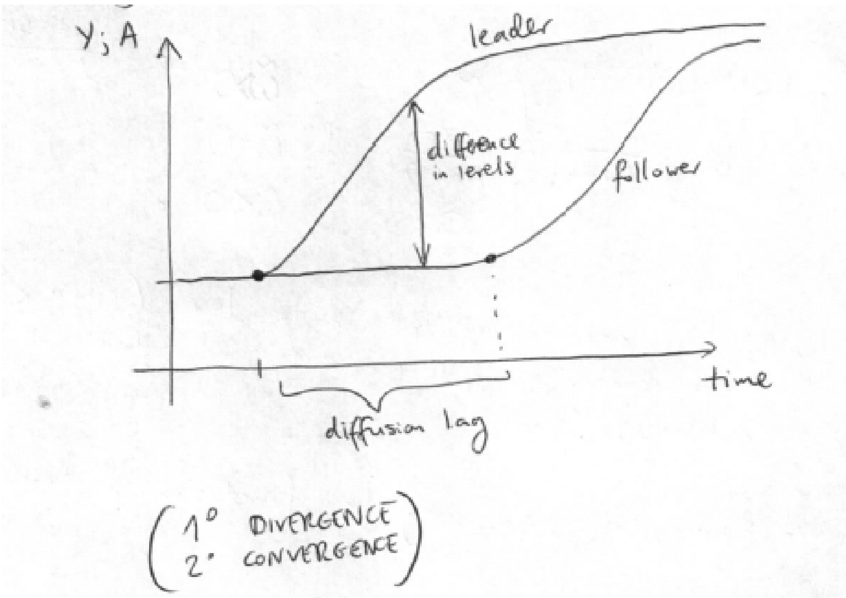

Figure 20

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