



Recitation 3

[Definitions used today]

- (weakly/strongly) convex, continuous, monotone preferences, locally non-satiated utility function
- utility maximization, Debreu theorem, lexicographic preferences
- WARP, GWARP, GARP, Topkis theorem, Afriat theorem

Question 1 [Weak vs strong continuity] 182 [Question I.1 Fall 2014 majors]

Let \succeq be a transitive and complete preference relation on (connected) set $X \subseteq \mathbb{R}_+^N$:
Prove that the following statements are equivalent

- \succeq on X is **weakly continuous** if $\forall x \in X$ the preferred-to- x set $U(x) = \{y \in X : y \succeq x\}$ and lower contour set $L(x) = \{y \in X : x \succeq y\}$ are closed.
- \succeq on X is **strongly continuous** if for all sequences $\{x_n\} \{y_n\} \in X$ such that $x_n \rightarrow x, y_n \rightarrow y$, if $\forall n, x_n \succeq y_n$, then $x \succeq y$.

Solution 1

TBD

Question 2 [Properties of preferences]

Prove following statements

1. If a preorder \succeq is monotone in \mathbb{R}^l , then it is locally nonsatiated.
2. If a preorder \succeq is transitive, weakly monotone, and locally nonsatiated then it is monotone
3. A preorder \succeq is weakly convex \iff the upper contour sets $U(x) = \{y \in X : y \succeq x\}$ are convex for all $x \in X$
4. If a preorder \succeq is continuous and strictly convex then it is convex

Solution 2

The following definitions are following from Debreu(1987).

Definition 0.1. A **preorder** \succeq on X is a binary relation which is reflexive ($\forall x \in X, x \succeq x$) and transitive ($\forall x, y, z \in X \ni (x \succeq y \wedge y \succeq z) \Rightarrow x \succeq z$). A **complete preorder** is a preorder that is complete ($\forall x, y \in X, x \succeq y \vee y \succeq x$).

0.1 Monotonicity and Nonsatiation

Definition 0.2. A preorder \succeq is **weakly monotone** on a set X if $\forall x, y \in X, x \geq y \Rightarrow x \succeq y$.

A preorder \succeq is **monotone** on a set X if $\forall x, y \in X, x \gg y \Rightarrow x \succ y$.

A preorder \succeq is **strongly monotone** on a set X if $\forall x, y \in X, (x \geq y \wedge x \neq y) \Rightarrow x \succ y$.

Definition 0.3. A preorder \succeq is **locally nonsatiated** on a set X if $\forall x \in X$ and $\forall \epsilon > 0, \exists y \in X \ni \|x - y\| < \epsilon$ and $y \succ x$.

0.2 Convexity

Definition 0.4. A preorder \succeq is **weakly convex** on a set X if $\forall x, y \in X, \forall \lambda \in (0, 1)$,

$$x \succeq y \Rightarrow \lambda x + (1 - \lambda)y \succeq y$$

Definition 0.5. A preorder \succeq is **convex** on a set X if $\forall x, y \in X, \forall \lambda \in (0, 1)$,

$$x \succ y \Rightarrow \lambda x + (1 - \lambda)y \succ y$$

Definition 0.6. A preorder \succeq is **strongly/strictly convex** on a set X if $\forall x, y \in X, \forall \lambda \in (0, 1)$,

$$x \sim y \wedge x \neq y \Rightarrow \lambda x + (1 - \lambda)y \succ y$$

0.3 Continuity

Definition 0.7. (Sequential definition/ weak continuity) A preorder \succeq is continuous on a set X if $\forall \{x_n\}, \{y_n\} \subseteq X$,

$$\forall n \in \mathbb{N}, (x_n \succeq y_n) \wedge (x_n \rightarrow x) \wedge (y_n \rightarrow y) \Rightarrow x \succeq y$$

Definition 0.8. (Set definition/ strong continuity) A preorder \succeq is continuous on a set X if $\forall x \in X$, the upper contour set $U(x) = \{y \in X : y \succeq x\}$ and the lower contour set $L(x) = \{y \in X : x \succeq y\}$ are closed in X .

Proposition 0.9. Equivalence of Weak and Strong Continuity

If X is connected (e.g. $X = \mathbb{R}_+^l$), then definitions of weak and strong continuity are equivalent.

0.4 Relationships Among Preorder Properties

Proposition 0.10. If a preorder \succeq is strictly monotone, then it is monotone.

Proof. Let $x, y \in X$ such that $y \gg x$. Then $y \geq x$ and $y \neq x$. Since \succeq is strictly monotone, this implies that $y \succ x$. Thus, \succeq is monotone. \square

Proposition 0.11. If a preorder \succeq is monotone in \mathbb{R}^l , then it is locally nonsatiated.

Proof. Let $x \in X$ and fix $\epsilon > 0$. Let e denote the unit vector in \mathbb{R}^l . Pick $y = x + \frac{\epsilon}{2\sqrt{l}}e$. Note that $\|x - y\| = \epsilon/2 < \epsilon$, and $y \gg x$. Since \succeq is monotone, $y \succ x$. Thus, \succeq is l.n.s. \square

Proposition 0.12. If a preorder \succeq is transitive, weakly monotone, and locally nonsatiated then it is monotone.

Proof. Let $x, y \in X$ such that $y \gg x$. Let $\epsilon = \min\{y_1 - x_1, \dots, y_l - x_l\}$. Then for all $z \in X$ such that $\|x - z\| < \epsilon$, $y \gg z$. By local nonsatiation, $\exists z' \in X$ such that $\|x - z'\| < \epsilon$ and $z' \succ x$. Since $y \gg z'$, weak monotonicity implies that $y \succeq z'$. By transitivity, $y \succ x$. Thus \succeq is monotone. \square

Proposition 0.13. A preorder \succeq is weakly convex \iff the upper contour sets $U(x) = \{y \in X : y \succeq x\}$ are convex for all $x \in X$.

Proof. Suppose \succeq is weakly convex. Let $x \in X$, let $y, y' \in U(x)$ such that $y' \succeq y$, and let $\lambda \in [0, 1]$. By weak convexity, $\lambda y + (1 - \lambda)y' \succeq y$. By definition of $U(x)$, $y \succeq x$. By transitivity, $\lambda y + (1 - \lambda)y' \succeq x$, so $\lambda y + (1 - \lambda)y' \in U(x)$. Thus $U(x)$ is convex.

Now suppose that $U(x)$ are convex for all $x \in X$. Let $x, y \in X$ such that $y \succeq x$ and let $\lambda \in [0, 1]$. By reflexivity, $x \succeq x$, so $x, y \in U(x)$. Since $U(x)$ is convex, $\lambda x + (1 - \lambda)y \in U(x)$. By definition of $U(x)$, $\lambda x + (1 - \lambda)y \succeq x$. Thus \succeq is weakly convex. Therefore \succeq is convex $\iff U(x)$ is convex for all $x \in X$. \square

Proposition 0.14. If a preorder \succeq is continuous and strictly convex then it is convex.

Proof. Let $x, y \in X$ such that $y \succ x$ and $x \neq y$, and let $\lambda \in (0, 1)$. Then $y \succeq x$ and $x \neq y$, so strict convexity implies that $\lambda y + (1 - \lambda)x \succ x$. Thus we have $y \succ x \Rightarrow \lambda y + (1 - \lambda)x \succ x$, so \succeq is convex. \square

Question 4

Give an example of preferences/utility function such that :

1. satisfy non-satiation, but not weak monotonicity
2. satisfy non-satiation, but not local non-satiation
3. satisfy local non-satiation, strict monotonicity, but not quasi-concave
4. does not satisfy continuous but it is representable by a utility function

Solution 4

TBD

Question 3

Consider the following preference relations on \mathbb{R}_+^2

1. $x \succeq y \iff \min\{x_1, x_2\} \geq \min\{y_1, y_2\}$
2. $x \succeq y \iff \max\{x_1, x_2\} \geq \max\{y_1, y_2\}$

are they convex? Are they strictly convex?

Solution 3

a) It is convex but not strictly convex

$$(2, 1) \succeq (1, 1) \quad (3, 1) \succeq (1, 1)$$

but

$$\frac{1}{2}(2, 1) + \frac{1}{2}(3, 1) \succeq (2.5, 1) \not\succeq (1, 1)$$

b) it is not even convex. Take

$$(5, 3) \succeq (4.5, 4.5) \quad (3, 5) \succeq (4.5, 4.5)$$

but

$$\frac{1}{2}(5, 3) + \frac{1}{2}(3, 5) \succeq (4, 4) \not\succeq (4.5, 4.5)$$

Question 5 [Utility representation] 157 [I.1 Fall 2013 majors]

Consider preference relation \succeq on the consumption set \mathbb{R}_+^L . Suppose that \succeq is reflexive and complete.

1. State a definition of \succeq having a utility representation. Is utility representation, if it exists, unique?
2. State a theorem providing sufficient conditions on \succeq to have a utility representation. Be as general as you can and clearly define any extra properties of \succeq that you use
3. **[Debreu Theorem]** Let \succeq be a complete, transitive and continuous, strictly increasing (i.e. strongly monotone) preference relation on \mathbb{R}_+^L , show that it has a continuous utility representation

Solution 5

Check out Debreu's proof by Ariel Rubinstein on my webpage!

Existence theorems: Rader RES 1964, Hildebrand, Mathematical Economics: Debreu 1981,

Definition 0.15. A function $u : X \rightarrow \mathbb{R}$ is a **utility representation** of a preorder \succeq if for all $x, y \in X$,

$$u(y) \geq u(x) \iff y \succeq x.$$

Theorem 0.16. (Existence of Utility Representation)

If a preorder \succeq is complete, continuous and monotone on $X = \mathbb{R}_+^l$, then it has a continuous utility representation.

Proof. Step-1 Constructing $\alpha(x)$

Let \succeq be a complete, continuous and monotone preorder on \mathbb{R}_+^l , and let $x \in \mathbb{R}_+^l$. Let $e = (1, 1, \dots, 1) \in \mathbb{R}_{++}$.

Then $\alpha e \in \mathbb{R}_+^l \forall \alpha \in \mathbb{R}_+$. Choose $\bar{\alpha}$ such that $\bar{\alpha}e \geq x$. Since \succeq is monotone, $\bar{\alpha}e \succeq x$. Also note that $0e \leq x$, so $x \succeq 0e$.

Define A^+ and A^- as

$$A^+ = \{\alpha \in \mathbb{R}_+ : \alpha e \succeq x\} \quad A^- = \{\alpha \in \mathbb{R}_+ : x \succeq \alpha e\}$$

Since \succeq is continuous, A^+ and A^- are closed. $\bar{\alpha} \in A^+$ and $0 \in A^-$, so both sets are also nonempty. By completeness of \succeq , $\mathbb{R}_+ = A^+ \cup A^-$.

Since \mathbb{R}_+ is connected, $A^+ \cap A^- \neq \emptyset$. Let $\alpha(x) \in A^+ \cap A^-$. Then $\alpha(x)e \sim x$.

Step-2 Uniqueness

Suppose that $\exists \alpha' \in A^+ \cap A^-$ such that $\alpha' \neq \alpha(x)$. Then either $\alpha' > \alpha(x)$ or $\alpha(x) > \alpha'$. If the former case is true, then since \succeq is monotone, $\alpha'e \succ \alpha(x)e \sim x$. If the latter, then $\alpha'e \prec \alpha(x)e \sim x$. In either case, it cannot be true that $\alpha' \in A^+ \cap A^-$. Therefore $\alpha(x)$ is unique.

Step-3 Utility Representation

(\Rightarrow) Let $x, y \in \mathbb{R}_+^l$ such that $y \succeq x$. Then by definition of $\alpha(x)$, we have $\alpha(y)e \sim y \succeq x \sim \alpha(x)e$.

By transitivity, $\alpha(y)e \succeq \alpha(x)e$, and so by monotonicity, $\alpha(y) \geq \alpha(x)$.

(\Leftarrow) Suppose that $\alpha(y) \geq \alpha(x)$. Then by monotonicity of \succeq , $\alpha(y)e \succeq \alpha(x)e$.

By definition of $\alpha(x)$, we have $y \sim \alpha(y)e \succeq \alpha(x)e \sim x$. By transitivity, $y \succeq x$.

Thus we have $y \succeq x \iff \alpha(y) \geq \alpha(x)$, so $u(x) = \alpha(x)$ is a utility representation of \succeq .

Step-4 Continuity

Note that $u : \mathbb{R}_+^l \rightarrow \mathbb{R}_+$. For any $x \in \mathbb{R}_+^l$, we can write $u^{-1}([0, \alpha(x)])$ and $u^{-1}([\alpha(x), \infty))$ as

$$\begin{aligned} u^{-1}([0, \alpha(x)]) &= \{y \in \mathbb{R}_+^l : \alpha(x) \geq u(y)\} = \{y \in \mathbb{R}_+^l : \alpha(x)e \succeq y\} \\ u^{-1}([\alpha(x), \infty)) &= \{y \in \mathbb{R}_+^l : u(y) \geq \alpha(x)\} = \{y \in \mathbb{R}_+^l : y \succeq \alpha(x)e\} \end{aligned}$$

By continuity of \succeq , both of these sets are closed. So we have for every $x \in \mathbb{R}_+^l$, $[0, \alpha(x)]$ and $u^{-1}([0, \alpha(x)])$ are closed, and $[\alpha(x), \infty)$ and $u^{-1}([\alpha(x), \infty))$ are closed.

Also, for $[\alpha(x), \alpha(y)]$ which is closed, we have

$$u^{-1}([\alpha(x), \alpha(y)]) = u^{-1}([\alpha(x), \infty)) \cap u^{-1}([0, \alpha(y)])$$

which is closed.

Thus by the topological definition of continuity, $u(x) = \alpha(x)$ is continuous. □

Proposition 0.17. If \succeq is a lexicographic preference then \succeq has no utility representation.

Proof. Let \succeq be a lexicographic preference relation on \mathbb{R}_+^2 and suppose for contradiction that $\exists u : X \rightarrow \mathbb{R}$ such that u is a utility representation of \succeq . For each $x \in \mathbb{R}_+$, we can find a rational number $r(x)$ such that $u(x, 2) > r(x) > u(x, 1)$. Let $x, x' \in \mathbb{R}_+$ such that $x' > x$. Then

$$u(x', 2) > r(x') > u(x', 1) > u(x, 2) > r(x) > u(x, 1).$$

Thus $r(x') > r(x)$, so we have a one-to-one map $r : \mathbb{R}_+ \rightarrow \mathbb{Q}$. This is a contradiction since \mathbb{R}_+ is uncountable and \mathbb{Q} is countable. Therefore \succeq has no utility representation. □

Question 6 [Lexicographic preference]

Consider the following lexicographic preferences on the consumption set \mathbb{R}_+^2 : the value $x_1 + x_2$ has the first priority, the value of x_2 has the second priority.

1. Is this preference relation continuous? Prove or give a counter example.
2. Does this preference relation have the utility representation? Prove or give a counter example.
3. Consider the lexicographic preferences on \mathbb{R}_{++}^N such that the first priority is described by an increasing and continuous utility function $u_1(x)$ and the second priority is described by another increasing and continuous utility function $u_2(x)$. Show that, if u_1 is strictly concave, then the Walrasian demand of the lexicographic preference coincides with the Walrasian demand of u_1 for every $p \in \mathbb{R}_+^N$, $p \neq 0$ and $w > 0$.

Question 7 [Midterm 2018]

Consider a list of observations $\{(p_1, x_1), \dots, (p_T, x_T)\}$ where $p_t \in \mathbb{R}_+^N$ and $x_t \in \mathbb{R}_+^N$ are price vector and a corresponding consumption plan of a consumer respectively, for every $t \in \{1, \dots, T\}$.

1. State the Generalized Weak Axiom of Revealed Preference (GWARP) and Generalized (strong) Axiom of Revealed Preference (GARP) for these observations.
2. Show that if a locally non-satiated utility function rationalized observations then GARP holds.
3. Suppose that the observations are generated by a demand function $d(p, w)$ that is $x_t = d(p_t, w_t)$ for every t . Function d is given as

$$d(p, w) = \begin{cases} \left(\frac{w}{p_1}, 0\right) & \text{if } p_1 \geq p_2 \\ \left(\frac{w}{p_1 + p_2}, \frac{w}{p_1 + p_2}\right) & \text{if } p_2 > p_1 \end{cases}$$

Does GWARP hold for arbitrary observations generated by d ? Can demand d be rationalized by a locally non-satiated utility function?

4. Show that if a locally non-satiated utility function rationalized observations then GWARP holds.
5. Show that the assumption of local non-satiation in the previous point cannot be dispensed with - i.e. give an example of a utility function that rationalizes a set of pairs of prices and consumption bundles that violates GWARP

Question 8 [Properties of Walrasian Demand]

Prove following claims

1. [Walras Law] Show that if a preference relation \succeq is continuous and locally non-satiated then $p \cdot x^*(p, w) = w$, for all $x^*(p, w)$ that belong to the Walrasian Demand correspondence.
2. [GWARP] Show that if a preference relation \succeq is continuous and locally non-satiated then for all $w > 0$

$$w' > 0, p \gg 0 \text{ and } p' \gg 0 : \quad p \cdot x^*(p', w') \leq w \Rightarrow p' \cdot x^*(p, w) \geq w'$$

Question 9 230 [I.1 Fall 2016 minors]

Let d be a demand function of prices and income satisfying budget equation $p \cdot d(p, w) = w$ for every p and w

1. Show that if d is a Walrasian demand function of a consumer with strictly increasing utility function, then the Generalized Weak Axiom of Revealed Preference (GWARP) holds for every T -tuple of price-quantity pairs $\{p^t, x^t\}_{t=1}^T$, where $x^t = d(p^t, w^t)$, $p^t \in \mathbb{R}_{++}^L$ and $w^t \in \mathbb{R}_+$ for every $t = 1, \dots, T$. State GWARP
2. Consider the following demand function for $L = 2$ and show that GWARP does not hold for \hat{d} :

$$\hat{d}(p, w) = \begin{cases} \left(\frac{w}{p_1}, 0\right) & \text{if } p_1 \geq p_2 \\ \left(0, \frac{w}{p_2}\right) & \text{if } p_2 > p_1 \end{cases}$$

3. State the Afriat's Theorem. The proof is not required
4. Prove the necessity of an axiom for rationalizability

Solution 9

Problem 4.7 [F17] Consider a list of observations $(p_1, x_1), \dots, (p_T, x_T)$ where $p_t \in \mathbb{R}_+^N$ and $x_t \in \mathbb{R}_+^N$ are price vector and a corresponding consumption plan of a consumer respectively, for every $t \in \{1, \dots, T\}$.

- State the Generalized Weak Axiom of Revealed Preference (GARP) and Generalized (strong) Axiom of Revealed Preference (GARP) for these observations.
- State the Afriat theorem. Prove the necessity of an axiom for rationalizability.

def. utility function u on \mathbb{R}_+^N rationalizes observations $\{(p_1, x_1), \dots, (p_T, x_T)\}$ if:

$$\forall t \in \{1, \dots, T\} \quad \forall x \in \mathbb{R}_+^N \quad p^t x \leq p^t x^t \Rightarrow u(x^t) \geq u(x)$$

def (GARP) A set of observations satisfies GARP if:

$$\forall s, t \quad \begin{array}{l} p^t x^s \leq p^t x^t \Rightarrow p^s x^t \geq p^s x^s \\ \text{or} \quad x^t R x^s \Rightarrow \sim (x^s P x^t) \quad \text{for } x^t \neq x^s \end{array}$$

def (GARP) A set of observations satisfies GARP if:

For every subset of observations $\{(p^{t_1}, x^{t_1}), \dots, (p^{t_n}, x^{t_n})\}$

$$x^{t_1} R x^{t_2}, x^{t_2} R x^{t_3}, \dots, x^{t_{n-1}} R x^{t_n} \Rightarrow \sim (x^{t_n} P x^{t_1}) \quad \text{for } x^{t_n} \neq x^{t_1}$$

or

$$\begin{cases} p^{t_1} x^{t_1} \geq p^{t_1} x^{t_2} \\ \vdots \\ p^{t_{n-1}} x^{t_{n-1}} \geq p^{t_{n-1}} x^{t_n} \end{cases} \Rightarrow p^{t_n} x^{t_n} \leq p^{t_n} x^{t_1}$$

Theorem (Afriat)

Observations $\{(p^1, x^1), \dots, (p^T, x^T)\}$ satisfy GARP

iff

there exists a lus utility function u that rationalizes them.

ii) WTS: if there exists a lus u that rationalizes the observations then GARP holds.

Suppose for a contradiction that there exists a lus utility function u that rationalizes the observations but GARP doesn't hold. That means:

$$\exists (p^{t_1}, x^{t_1}), \dots, (p^{t_n}, x^{t_n}) \quad \text{s.t.} \quad \begin{cases} p^{t_1} x^{t_1} \geq p^{t_1} x^{t_2} \\ \vdots \\ p^{t_{n-1}} x^{t_{n-1}} \geq p^{t_{n-1}} x^{t_n} \end{cases} \quad \text{and} \quad p^{t_n} x^{t_n} > p^{t_n} x^{t_1}$$

Since utility function is lus:

$$\begin{array}{ll} p^t x^t \geq p^t x^s \Rightarrow u(x^t) \geq u(x^s) & \forall t, s \\ p^t x^t > p^t x^s \Rightarrow u(x^t) > u(x^s). & \forall t, s \end{array}$$

Then:

$$u(x_1) \geq u(x_2) \geq u(x_3) \geq \dots \geq u(x^{n-1}) \geq u(x^n) > u(x_1) \quad \downarrow$$