ECON 8104 notes
preliminary version

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*These notes are intended to summarize the main concepts, definitions and results covered in the first year of micro sequence for the Economics PhD of the University of Minnesota. The material is not my own. Please let me know of any errors that persist in the document. E-mail: pawel042@umn.edu .

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1 Social Choice Theory

1.1 Arrow's Theorem

- N individuals
- A alternatives
- individual preferences $R_i \subset A \times A$ are complete and transitive

Preferences satisfy following axioms

Definition 1 (Social Welfare Function). *Function f aggregates preferences of agents, $R = f(R_1, \ldots R_N) = f(\{R_i\})$.*

Definition 2 (Universal). *Every $\{R_i\}$ is plausible*

Definition 3 (Rational). *R is rational (complete and transitive)*

Definition 4 (Unanimous).

$$\forall a, b \in A \quad aP_ib \quad \forall i \in N \Rightarrow aPb$$

Definition 5 (Independent (IIA)). *Given $\{R_i\}$ $\{R_i^{'}\}$ and $a, b \in A$*

$$R_i|_{\{a,b\}} = R_i^{'}|_{\{a,b\}} \Rightarrow R|_{\{a,b\}} = R^{'}|_{\{a,b\}}$$

Definition 6 (Dictatorial). $\exists i \in N \quad aP_ib \Rightarrow aPb \quad \forall a, b \in A$

Theorem 1 (Arrow). ***If $|N| < \infty$, $|A| > 2$ then Universal, Rational, Unanimous and Independent $\Rightarrow$ Dictatorial***

Proof. TBD $\square$

1.2 Gibbard Satterthwaite Theorem

Assume $|N| < \infty$, $|A| < \infty$

Definition 7 (Social Choice Function (SCF)). *$f : \mathcal{L}^N \rightarrow A$ where $\mathcal{L}$ is set of strict linear orders.*

Definition 8. *Preference profile $P = (P_1, \ldots P_N) \in \mathcal{L}^N$, $f(P) \in A$*

A SCF f is :

Definition 9 (Pareto Efficient (PE))**.** *If $f(P) = a$ whenever $a \in A$ is at the top $\forall i P_i$*

Definition 10 (Dictatorial)**.** *If $\exists i$ s.t. $f(P) = a \quad \iff \quad a$ is at top of P_i*

Definition 11 (Strategy proof (SP))**.**

$$\text{If} \quad i \in N, P \in \mathcal{L}^N P_i^{'} \mathcal{L} \quad f(P_i^{'}, P_{-i}) \neq f(P) \quad \Rightarrow f(P) P_i f(P_i^{'}, P_{-i})$$

Definition 12 (Maskin Monotone (MM))**.** *If whenever $f(P) = a$ and $\forall i, b$ $P^{'}_i$ ranks a above b if P_i does then $f(P') = a$. In other words, define $B(a, P_i) = \{b \in A | a P_i b\}$*

$$f(P_1, \ldots P_N) = a \quad \text{and} \quad i \quad B(a, P_i) \subset B(a, P_i^{'}) \Rightarrow f(P_1^{'}, \ldots P_N^{'}) = a$$

Theorem 2 (Gibbard Satterthwaite)**.** ***If $|A| > 2$, $f : \mathcal{L}^N \to A$ is onto and SP $\Rightarrow f$ is Dictatorial***

Gibbard (1973), Satterthwaite (1975)

Corollary 1. *Dictatorial $\Rightarrow$ SP and onto*

Proof. PE $\Rightarrow$ onto

PR and SP $\Rightarrow$ dictatorial

TBD □

2 Mechanisms of Mechanism Design

In this section will deconstruct following powerful picture:

$$\begin{tikzcd} \Theta \arrow[rr, "f"] \arrow[dr] & & X \\ & (M,g) \arrow[ur] & \end{tikzcd}$$

Figure 1:

In SCT we studied aggregation of preferences but here preferences are not publicly observable. Individuals must be relied upon to reveal private information. We study here how this information can be elicited and the extent to which the information revelation problem constraints the ways in which social decisions can respond to individual preferences. This is **mechanism design problem**.

The interpretation of x, θ and t (up to sign adjustments) in next examples are as follows

1. **Price discrimination** x is the consumer's purchase and t is the price paid to the monopolist; θ indexes the consumer's surplus from consumption

2. **Income tax** x is the agent's income, and t is the amount of tax paid by the agent; θ is a technological parameter indexing the cost function

3. **Public good provision** x is the amount of public good supplied and t_i is i's consumer monetary contribution to its financing;θ_i is consumer i surplus from the public good

4. **Auction** x_i is the probability that consumer i buys the good ($\sum_i x_i = 1$) and t_i is the amount paid by consumer i, θ_i is consumer i willingness to pay for the good that is auctioned off

5. **Bargaining** x is the quantity sold by a seller to buyer, t_1 is the transfer to the seller and t_2 is negative transfer to buyer, s.t. $t_1 + t_2 = 0$, $\theta_1 = c$ indexes the seller's cost of producing the good and $\theta_2 = v$ is buyer's willingness to pay

2.1 Three examples

In this section we present three canonical examples of mechanism design which show following issues

- Example 1: There are SCF which are not Nash implementable
- Example 2: Allocation rules are monotone. There are transfers which are not truthfully implementable.
- Example 3: in 2nd price auction SCF is truthfully implementable in DS by direct mechanism

Example 1. ***King Solomon's dilemma***
Two mothers Ann and Beth came to King Solomon with a baby and both claimed to be the baby's genuine mother. King Solomon faced the problem of finding out which of two women was the true mother of the baby.
He proposes to split the baby in half and give one half to each woman. This macabre proposal prompts one of the women to scream while the other remains silent. King Solomon decrees that the true mother would never stand by while her baby was murdered, and thus gives the baby to woman who screamed.

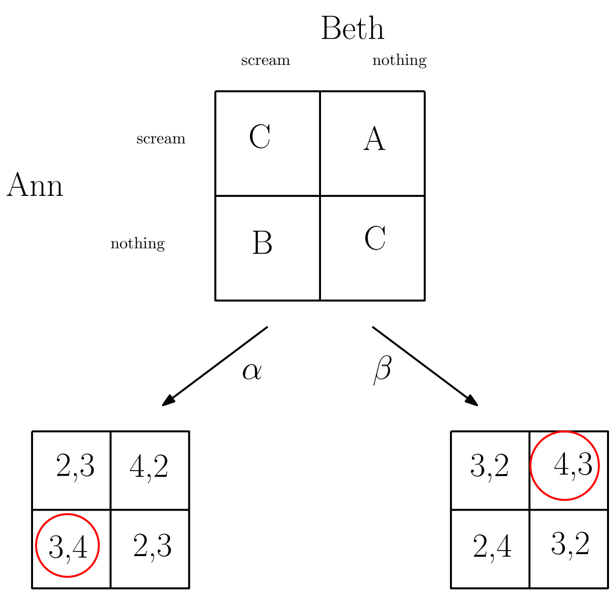

Figure 2: Γ_1

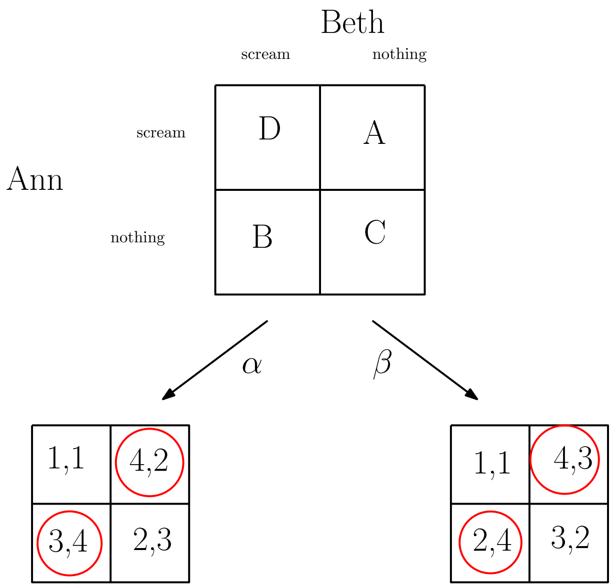

Figure 3: Γ_2

Furthermore he proposed two solutions (remember that it was 10th century BCE kings were cruel back then):

- Γ_1 *give baby to mother which cried, cut the kid in half when neither of them cried or if both of them screamed -Figure 1 2*
- Γ_2 *give baby to mother which cried, cut the kid in half when neither of them cried and if both screamed kill both women -Figure 8.3 3*

Let's define components of the problem:

- *agents* $N = \{Ann, Beth\}$
- *outcomes* $X = \{A, B, C, D\}$
  - $A =$ *give baby to Ann*
  - $B =$ *give baby to Beth*
  - $C =$ *cut the baby to halves*
  - $D =$ *death to everyone*
- *types* $\Theta = \{\alpha, \beta\}$
- *preference profiles* $\Omega = \{P^\alpha, P^\beta\}$

- *in state α their preferences are*

$$\begin{array}{cc} \textit{Ann} & A \succ B \succ C \succ D \\ \textit{Beth} & B \succ C \succ A \succ D \\ \textit{utils} & 4 \succ 3 \succ 2 \succ 1 \end{array}$$

- *in state β their preferences are*

$$\begin{array}{cc} \textit{Ann} & A \succ C \succ B \succ D \\ \textit{Beth} & B \succ A \succ C \succ D \\ \textit{utils} & 4 \succ 3 \succ 2 \succ 1 \end{array}$$

- Γ_i *are a sort of meta-game that induces two NFG corresponding to the states α and β.*
- *Let $NE(\Gamma,\theta)$ denote the set of Nash equilibria in the game induced by Γ when the state of the world is θ.*
- *We say that Γ fully implements f^* in Nash equilibrium (aka Nash implements) if $g(NE(\Gamma,\alpha)) = \{f^*(P^\alpha)\}$ and $g(NE(\Gamma,\beta)) = \{f^*(P^\beta)\}$.*
- *We say that Γ truthfully implements f^* in Nash equilibrium (aka truthful Nash implements) if $g(NE(\Gamma,\alpha)) \ni f^*(\alpha)\}$ and $g(NE(\Gamma,\beta)) \ni f^*(P^\beta)$.*

So which mechanism was played? *Let's take a look at Nash equilibria in all 4 games. Consider cases corresponding to types α and β:*

Case 1 *The game induced by Γ_1 when the state is α.*

In the game α, screaming strictly dominates doing nothing for the fake mother Beth and the true mother's best response when Beth screams is to do nothing. Thus $NE(\Gamma,\alpha) =$ (nothing,scream), so $g(NE(\Gamma,\alpha)) = B \neq f^*(P^\alpha) = A$. In other words, when Anna is the true mother, the mechanism designed by King Solomon causes him to end up allocating the baby to the fake mother Beth.

Similarly, in the game β, screaming strictly dominates doing nothing for the fake mother Anna and doing nothing is a best response for the true mother Beth. Thus $NE(\Gamma,\alpha) =$ (nothing, scream) and $g(NE(\Gamma,\alpha)) = A \neq f^*(P^\beta) = B$, i.e., the fake mother Anna gets the baby.

This means that King Solomon's mechanism does exactly the opposite of what he was intending.
In the parlance of mechanism design, Γ does not Nash-implement f^*.

Case 2
The game induced by Γ_2 when the state is α we have two NE in pure strategies $NE(\Gamma, \alpha)$ =\{(nothing,scream),(scream, nothing)\}, so $g(NE(\Gamma, \alpha)) = \{B, A\} \neq f^*(P^\alpha) = \{A\}$ In other words, when Anna is the true mother, the mechanism designed by King Solomon causes him to end up allocating the baby either to true mother Ann or to the fake mother Beth.

Similarly, in the game β, we have two NE in pure strategies $NE(\Gamma, \alpha)$ =\{(nothing,scream),(scream, nothing)\}, so $g(NE(\Gamma, \alpha)) = \{B, A\} \neq f^*(P^\beta) = \{B\}$. Thus Γ_2 does not Nash implements, though it implements it in truthfully in Nash equilibrium.

It must be something that Nash implements f. It turns out that there is necessary condition for Nash implementability Let's look at this problem from Mechanism Design perspective.

***Formally problem consists of** (let's skip index i taking $i = 1$):*

- *agents $N = \{Ann, Beth\}$*
- *outcomes $X = \{A, B, C, D\}$*
- *types $\Theta = \{\alpha, \beta\}$*
- *type dependent preference profiles $\Omega = \{P^\alpha, P^\beta\}$ in short*
- *King Solomon has social choice function $f^* : A \to X$*
- *social choice function is such that $f^* : \Theta \to X$ such that $f^*(P^\alpha) = A$ and $f^*(P^\beta) = B$.*
- *To impose that SCF king Solomon introduces the mechanism $\Gamma = (M, g)$*
- *where $M = M_A \times M_B$ is an action (message) space*
- *$g : M \to X$ is outcome rule that determines which alternative in X is chosen based on the actions of the players.*
- *Notice that this is not really game- each induces 2 games!*

- *This is game form (aka **mechanism**) each becomes a game when coupled with a preference profile.*
- *mechanism induces 2 games* (Γ, P^{α})*,* (Γ, P^{β})
- $NE(\Gamma, P) =$ *set of pure strategy Nash equilirbia*
- *we look at Nash equilibria of* $NE(\Gamma, \alpha)$ *and* $NE(\Gamma, \beta)$
- *g(*$NE(\Gamma, P)) =$ *set of Nash equilibrium outcomes*
- *in example we saw: g(*$NE(\Gamma, P^{\alpha})) = \{b\}$*, g(*$NE(\Gamma, P^{\beta})) = \{a\}$
- *We say that* Γ *fully implements* f^* *in Nash equilirbium (aka Nash implements) if* $g(NE(\Gamma, \alpha)) = \{f^*(P^{\alpha})\}$ *and* $g(NE(\Gamma, \beta)) = \{f^*(P^{\beta})\}$.
- *neither seems to be what was played*

Precise definitions are presented later on.

It turns out that this SCF does not satisfy necessary condition for Nash implementability (which is Maskin monotonicity). As The Rolling Stones sang: You can't always get what you want!

Example 2. ***Provision of public good***

Society consists of agents decides about conducting public investment. It needs to set up transfers between its members*.*

Let's introduce environment and necessary definitions

Set-up

1. *N agents*
2. *each agent has private type* $\theta_i \in \Theta_i \quad \theta = (\theta_1, \ldots, \theta_N) \in \prod_{i=1}^{N} \Theta_i$
3. *agent i has utility* $u_i(k, \theta_i) = v_i(k, \theta_i) + t_i$
4. *we even specify it to* $v_i(k, \theta_i) = \theta_i k$ *so*
$$u_i(k, \theta_i) = \theta_i k + t_i$$
5. $k \in K$ *where K is the set of non-monetary allocation, let's take* $K = \{0, 1\}$

6. $t_i \in \mathbb{R}$ *monetary transfer*

7. *The allocation is* (k, t) *where* $t = (t_1, \ldots, t_N)$

8. *Set of feasible allocation* $X = \{(k, t) \quad | \; k \in K, t_i \in \mathbb{R}, \quad \sum_i t_i \leq 0\};$

9. *Let* $k : \Theta \rightarrow K$, *where* $\Theta = \Theta_1, \cdots \times \Theta_n = [0, 1]^n$

10. $t_i : \Theta \rightarrow \mathbb{R}$,

11. *The social choice function is* $f(\theta) = (k(\theta), t(\theta))$ *where* $t(\theta) = (t_i(\theta))_{i=1}^{n}$.

12. *we are looking for* $\Gamma = (M, g)$ *s.t.* Γ *implements* f *in Dominant Strategies*

To wrap it up f *is dominant strategy incentive compatible (DSIC) or strategy-proof if* $\forall i, \theta_i, \theta_i', \theta_{-i}$,

$$k(\theta)\theta_i + t_i(\theta) \geq k\left(\theta_i', \theta_{-i}\right)\theta_i + t\left(\theta_i', \theta_{-i}\right)$$

The first thing we want to know is what kind of allocation functions are implementable? We borrow from auction setting and we can think of K as

$$K = \left\{(y_1, \ldots, y_n) : \forall i, y_i \in \{0, 1\}, \sum_{i \in I} y_i = 1\right\}$$

Example. Consider $N = 2$

Suppose that $N = 2$ *and* $k(\cdot)$ *is such that for some* θ_2 *and* $\theta_1 < \theta_1', y_1(\theta_1) = 1$ *and* $y_1(\theta_1') = 0$ *Suppose that* $k(\cdot)$ *is implementable. Then there exists* $t(\cdot)$ *such that*

$$\theta_1 + t_1(\theta_1, \theta_2) \geq 0 + t_1\left(\theta_1', \theta_2\right)$$

and

$$0 + t_1\left(\theta_1', \theta_2\right) \geq \theta_1' + t_1(\theta_1, \theta_2)$$

Adding these inequalities together, we get $\theta_1 \geq \theta_1'$ *which is a contradiction. This leads us to the observation that* $k(\cdot)$ *is implementable only if* $\forall i \in I$ *and* $\forall \theta_{-i} \in \Theta_{-i}, \exists \bar{\theta}_i$ *such that*

$$y_i(\theta_i, \theta_{-i}) = \begin{cases} 1 & \theta_i > \bar{\theta}_i \\ 0 & \theta_i < \bar{\theta}_i \end{cases}$$

We call this type of allocation function a monotone allocation function. This naturally leads one to ask if all monotone allocation functions are implementable. Answer to this is positive what will be presented later.

Finally we want to have some notion of efficiency, control over level of transfers:

Definition 13. *A social choice function f satisfies full ex-post efficiency if:*

1. *it is ex-post efficient:* $\sum_{i=1}^{n} v_i\left(k(\theta), \theta_i\right) \geq \sum_{i=1}^{n} v_i\left(k, \theta_i\right) \forall k, \theta;$
2. *it is balanced budget:* $\sum_{i=1}^{n} t_i(\theta) = 0$

Let's solve it in more general set up with N agents and with $0 < c < N$.
In this case condition for ex-post efficiency is equivalent to:

$$\sum_{i=1}^{n} k(\theta)\theta_i \geq \sum_{i=1}^{n} k \cdot \theta_i \quad \forall k \in \{0,1\}, \theta$$

which is equivalent to

$$(k(\theta) - k) \cdot \sum_{i=1}^{n} \theta_i \geq 0 \quad \forall k \in \{0,1\}, \theta$$

Consider following SCF $f(\theta) = (k(\theta), t_1(\theta), \ldots, t_I(\theta))\ \forall\theta,$

$$k(\theta) = \begin{cases} 1 & \text{if } \sum_i \theta_i \geq c \\ 0 & \text{otherwise} \end{cases}$$

with

$$\sum_i t_i(\theta) = \begin{cases} -c & \text{if } k(\theta) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$k(\theta)$ is either one or zero.
Fix $\sum_i \theta_i \geq c$, in this case 1, then EPIC is satisfied. Analogous $\sum_i \theta_i < c$ with $k \in \{0,1\}$ is obviously satisfied.

Example. Equal transfers
Suppose that the agents want to implement this ex post efficient SCF with average transfer,i.e.

$$t_i(\theta) = -ck(\theta)/n$$

Suppose that $\Theta_i = \{\bar{\theta}_i\}$ for $i \neq 1$ and $\Theta_1 = [0, +\infty)$ and assume $c > \sum_{i \neq 1} \bar{\theta}_i > c(n-1)/n$.

This implies that, firstly, with this social function agent 1's type is critical for whether the bridge is built (note that if $\theta_1 \geq c - \sum_{i \neq 1} \bar{\theta}_i$, it is. while if $\theta_1 < c - \sum_{i \neq 1} \bar{\theta}_i$, it is not), and, secondly, that the sum of the utilities of the agents $2, \ldots, n$ is strictly greater if the bridge is built than if it is not built (since $\sum_{i \neq 1} \bar{\theta}_i - c(n-1)/I > 0$)

What are the incentives of agent 1 to truthfully reveal her type when $\theta_1 = c - \sum_{i \neq 1} \bar{\theta}_i + \varepsilon$ *for some* $\varepsilon > 0$*?*
If agent 1 reveals her true preferences, then the bridge will be built because

$$\left(c - \sum_{i \neq 1} \bar{\theta}_i + \varepsilon\right) + \sum_{i \neq 1} \bar{\theta}_i > c$$

In this case agent 1 's utility is

$$\theta_1 - \frac{c}{n} = \left(c - \sum_{i \neq 1} \bar{\theta}_i + \varepsilon\right) + 0 - \frac{c}{n} = \left(\frac{c(n-1)}{n} - \sum_{i \neq 1} \bar{\theta}_i + \varepsilon\right) + 0$$

However, for $\varepsilon > 0$ *small enough, the utility of agent 1 is less than 0 (in fact, this is her utility if she instead claims that* $\theta_1 = 0$*, a claim that results in the bridge not being built). Therefore, agent 1 will not truthfully reveal her type.*

Under this allocation rule, when agent 1 causes the bridge to be built he has a positive externality on the other agents. Because he fails to internalize this effect, he has an incentive to understate his benefit from this project.

We have considered just a couple of examples which illustrate the two issues:

- *monotonicity of allocation in agent types*
- *not truthful revelation of private information*

The central question that we impose is the following: What social choice functions can be implemented when agents' types are private information?
In general, we need to consider not only the possibility of directly implementing social choice function by asking agents to reveal their types but also their indirect implementation through the design of institutions in which the agents interact.

Example 3. ***Allocation of single unit of indivisible private good***
There is a single unit of an indivisible private good to be allocated to one of N ***agents. Monetary transfers can also be made.***

Here $\theta_i \in \mathbb{R}$ *can be viewed as agent i 's valuation of the good, and we take the set of possible valuations for agent i to be* $\Theta_i = \left[\theta_i, \bar{\theta}_i\right] \subset \mathbb{R}$.

Two special cases ubiquitous in the literature deserve mention. The first is the case of ***bilateral trade****. In this case we have* $N = 2$ *agent 1 is the seller and agent 2 is the buyer. Consider cases*

- *When* $\underline{\theta}_2 > \bar{\theta}_1$ *there are certain to be gains from trade regardless of the realizations of* θ_1 *and* θ_2
- *when* $\underline{\theta}_1 > \bar{\theta}_2$ *there are certain to be no gains from trade*
- *finally, if* $\underline{\theta}_2 < \bar{\theta}_1$ *and* $\underline{\theta}_1 < \bar{\theta}_2$ *then there may or may not be gains from trade, depending on the realization of* θ

The second special case is the ***auction*** *setting. Here, one agent, whom we shall designate as agent* 0 *is interpreted as the seller of the good (the auctioneer) and is assumed to derive no value from it (more generally, the seller might have a known value* $\theta_0 = \bar{\theta}_0$ *different from zero). The other agents,* $1, \ldots, I$*, are potential buyers (the bidders). Let's present auction in form of 2nd price sealed auctioned aka Vickery auction.*

Definition 14 (2nd price auction)**.** *is such incomplete information game in which*

- *Bidders are asked to submit sealed bids* $b_1(\theta_1), \ldots, b_n(\theta_n)$*. The bidder who submits the highest bid is awarded the object, and pays the amount of the second highest bid.*
- $K = \{0, 1, \ldots N\}$
- *k- who gets the object if anyone*
- *utility*

$$u_i\left(k, \theta_i\right) = \theta_i k + t_i$$

$$v_i(k, \theta_i) = \begin{cases} \theta_i & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$$

- *utility of auctioneer* $u_0 = t_1 + t_2$
- *define transfers in following way*

$$t_1(\theta) = -\theta_2 k(\theta)$$

$$t_2(\theta) = -\theta_1 k(\theta)$$

$$t_0(\theta) = -\left(t_1(\theta) + t_2(\theta)\right)$$

- *SCF is $f(\theta) = (k(\theta), t(\theta))$*

Lemma 1. *In a second price auction, it is a weakly dominant strategy to bid one's value, $b_i(\theta_i) = \theta_i$*

Proof. Suppose i 's value is θ_i, and she considers bidding $b_i > \theta_i$. Let $\hat{b}$ denote the highest bid of the other bidders $j \neq i$ (from i 's perspective this is a random variable). Consider three possible outcomes from i's perspective:

1. $\hat{b} > b_i, \theta_i$
2. $b_i > \hat{b} > \theta_i$
3. $b_i, \theta_i > \hat{b}$

- In the event of the 1st or 3rd outcome i would have done equally well to bid θ_i rather than $b_i > \theta_i$
- In 1st he won't win regardless, and in 2nd she will win, and will pay $\hat{b}$ regardless.
- However, 2nd case i will win and pay more than her value if she bids $\hat{b}$, something that won't happen if she bids θ_i. Thus, i does better to bid $s\theta_i$ than $b_i > \theta_i$.
- A similar argument shows that i also does better to bid θ_i than to bid $b_i < \theta_i$

□

Since everyone is bidding their true value, seller will receive second highest value. The truthfull equilibrium described above is unique symmetric Bayesian Nash equilibrium .

Thus, this social choice function is implementable even though the buyers' valuations arc private information: it suffices to simply ask each buyer to report his type, and then to choose $f(\theta)$ We will show later that :

- *$k^*(\theta)$ ex post efficient allocation give it to whoever values it most*
- *i is pivotal when she is the highest bidder. His tax is the 2nd highest bid.*

Now in general one can ask: What SCF can be implemented directly when agents types are private information? And given SCF which institutions (mechanisms) may guarantee indirect implementation?

2.2 Game theory and Social Choice Theory kicks in

In this section we provide a notation and results on intersection of mechanism design, social choice theory and game theory.

Definition 15 (Mechnism design problem). *consists of:*

- *finite set of agents* $I = \{1, \ldots N\}$
- *agents take collective choice from set of possible alternatives* X
- *each agent has private type* $\theta_i \in \Theta_i \quad \theta = (\theta_1, \ldots, \theta_N) \in \prod_{i=1}^{N} \Theta_i$
- *types are privately observed before collective choice*
- *agent* i *has utility* $u_i\left(x, \theta_i\right)$
- *agents are assumed to be an expected utility maximizers*
- ϕ *pdf over* Θ, Θ *and* $u_i(\cdot, \theta_i)$ *are common knowledge but specific values of each agent* i *are observed only by* i
- *collective action depend on* θ

Definition 16 (Social choice function). $f : \Theta_1 \times \ldots \times \Theta_I \rightarrow X$ *chooses an outcome* $f(\theta) \in X$*, given types* $\theta = (\theta_1, \ldots, \theta_I)$.

Definition 17 (Ex post efficiency). *The social choice function* $f : \Theta_1 \times \cdots \times \Theta_I \rightarrow X$ *is* ***ex post efficient*** *(EPE or Paretian) if for no profile* $\theta = (\theta_1, \ldots, \theta_I)$ *is there an* $x \in X$ *such that*

$$\forall i \quad u_i(x, \theta_i) \geq u_i\left(f(\theta), \theta_i\right)$$

$$\exists j \quad u_j\left(x, \theta_j\right) > u_j\left(f(\theta), \theta_j\right)$$

SCF is ex post efficient if it selects, for every profile θ an alternative $f(\theta) \in X$ that is Pareto optimal given the agents' utility functions $u_1\left(\cdot, \theta_1\right), \ldots, u_I\left(\cdot, \theta_I\right)$ The problem is that the θ_i 's are not publicly observable, and so for the social choice f to be chosen when the agents' types are $\left(\theta_1, \ldots, \theta_1\right)$, each agent i must be relied upon to disclose his type θ_i. However, for a given social choice function $f(\cdot)$, an agent may not find it to be in his best interest to reveal this information truthfully. We illustrated this information revelation problem in Example 2. In Example 3 we showed that in auction setting agent reveal information truthfully.

The mechanism design problem is to implement rules of a game or meta game by defining possible strategies and the method used to select an outcome based on agent strategies, to implement the solution to the social choice function despite agent's selfinterest.

Definition 18 (Mechanism). *$\mathcal{M} = (\Sigma, g(\cdot))$ consists of*

- *set of strategies $\Sigma = \Sigma_1 \times \ldots \times \Sigma_I$*
- *an outcome rule $g : \Sigma_1 \times \ldots \times \Sigma_I \to X$*
- *such that $g(s)$ is the outcome implemented by the mechanism for strategy profile $s = (s_1, \ldots, s_I)$.*

In words, a mechanism defines the strategies available (e.g., bid at least the ask price, etc.) and the method used to select the final outcome based on agent strategies (e.g., the price increases until only one agent bids, then the item is sold to that agent for its bid price).

Let's define remaining parts of incomplete information game: strategies and equilibrium concepts:

Definition 19 (Strategy). *A strategy of player i is a mapping $\sigma_i : \Theta_i \to \Sigma_i$.*

Definition 20 (Preferences). *of agent i are function of outcome and private type:*

$$u_i(s_i, \theta_i) : \Sigma_i \times \Theta \to \mathbb{R}$$

Formally the mechanism $\mathcal{M}$ induces Bayesian game of incomplete information

Definition 21 (Bayesian game of incomplete information). *is $\{I, \{S_i, \bar{u}_i(\cdot)\}, \Theta, \phi\}$ where we have:*

- *finite set of agents $I = \{1, \ldots N\}$*
- *strategy sets S_i*
- $\bar{u}_i(s_1, \ldots, s_N, \theta_i) = u_i(g(s_1, \ldots, s_N), \theta_i)]$
- *expected payoff (expectation with ϕ as pdf):*

$$\hat{u}_i(s_1(\theta_1), \ldots, s_N(\theta_N)) = \mathbb{E}_\theta[\bar{u}_i(s_1(\theta_1), \ldots, s_N(\theta_N), \theta_i)]$$

Game theory is used to analyze the outcome of a mechanism. Given mechanism $\mathcal{M}$ with outcome function $g(\cdot)$, we say that a mechanism implements social choice function $f(\theta)$ if the outcome computed with equilibrium agent strategies is a solution to the social choice function for all possible agent preferences.

Definition 22 (Mechanism implementation). *$\mathcal{M} = (\Sigma_1, \ldots, \Sigma_I, g(\cdot))$ implements social choice function $f(\theta)$ in Z -equilibrium if $\exists$ a Z -equilibrium s^* such that*

$$\forall \theta \in \Theta \quad \mathbb{P}(\theta) > 0, \quad g\left(s_1^*\left(\theta_1\right), \ldots, s_I^*\left(\theta_I\right)\right) = f(\theta)$$

$\Theta_1 \times \ldots \times \Theta_I$, where strategy profile $(s_1^*, \ldots, s_I^*)$ is an equilibrium solution of game induced by $\mathcal{M}$.

As an Z- equilibrium concept we will consider one of following

- Nash
- Dominant Strategy
- Bayes-Nash

Note that we allow for multiple equilirbium strategy profiles from which at least one satisfy condition from definition above.

Definition 23. *Γ strongly implements f in Z -equilibrium there exists a Z -equilibrium, and for all Z -equilibrium s^* and for all $\theta \in \Theta$ with $\mathbb{P}(\theta) > 0, g\left(s^*\left(\theta_i\right)_{i \in I}\right) = f(\theta)$.*

Picking right mechanism is a daunting task. Looking at space of whole possible mechanism may be exhausting task. The mechanism asks agents to report their types, and then simply implements the solution to the social choice function that corresponds with their reports. Very naive mechanisms gives no good reason for self-interested to truthfully report their types.

Definition 24 (Direct Mechanism). *Given SCF f, the direct mechanism is Γ_{direct} where $M_i = \Theta_i$ and $g = f$.*

Note that all other mechanisms are indirect.

2.2.1 Incentive Compatibility

Definition 25 (Incentive Compatibile)**.** *The SCF f is truthfully implementable (or **Incentive Compatible**) if the direct revelation mechanism $\Gamma = (\Theta_1, \ldots, \Theta_I, f(\cdot))$ has an equilibrium $\left(s_1^*(\cdot), \ldots, s_I^*(\cdot)\right)$ in which*

$$s_i^*\left(\theta_i\right) = \theta_i \quad \forall i, \theta_i \in \Theta_i$$

That is, if truth telling by each agent i constitutes an equilibrium of $\Gamma = (\Theta_1, \ldots, \Theta_I, f)$.

Notion of Incentive Compatibility was introduced to economics by Minnesota faculty member Leonid Hurwicz (2007 Nobel prize winner), check Hurwicz(1972, 1976).

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edit this crap To offer a hint as to why we may be able to restrict attention to direct revelation mechanisms that induce truth telling, we briefly verify that the social choice functions that are implemented indirectly through the first-price and second-price sealed-bid auctions of Examples 23.B.5 and 23.B.6 can also be truthfully implemented using a direct revelation mechanism. In fact, for the second-price sealed-bid auction of Example 23.B.6 we have already seen this fact, because the social choice function implemented by the second-price auction is exactly the social choice function that we studied at the end of Example 23.B.4 in which truth telling is a weakly dominant strategy for both buyers. Example 23.B.7 considers the first-price sealed-bid auction. By keeping allocation in Example 3 the same and changing only transfers to

$$t_1(\theta) = -\theta_1 y_1(\theta)$$

$$t_2(\theta) = -\theta_2 y_2(\theta)$$

In this social choice function, the seller gives the good to the buyer with the highest valuation (to buyer 1 if there is a tie) and this buyer gives the seller a payment equal to his valuation (the other, low-valuation buyer makes no transfer payment to the seller). Note that $f(\cdot)$ is not only ex post efficient but also is very attractive for the seller: if $f(\cdot)$ can be implemented, the seller will capture all of the consumption benefits that are generated by the good.

Suppose we try to implement this social choice function. Assume that the buyers are expected utility maximizers. We now ask: If buyer 2 always announces his true value, will buyer 1 find it optimal to do the same? For each value of θ_1, buyer l's problem is to choose the valuation to announce, say $\hat{\theta}_1$, so as to solve

$$\text{Max}_{\hat{\theta}_1} \left(\theta_1 - \hat{\theta}_1\right) \text{Prob} \left(\theta_2 \leq \hat{\theta}_1\right)$$

or

$$\text{Max}_{\hat{\theta}_1} \left(\theta_1 - \hat{\theta}_1\right) \hat{\theta}_1$$

The solution to this problem has buyer 1 set $\hat{\theta}_1 = \theta_1/2$. We see then that if buyer 2 always tells the truth, truth telling is not optimal for buyer 1. A similar point applies to buyer 2. Intuitively, for this social choice function, a buyer has an incentive to understate his valuation so as to lower the transfer he must make in the event that he has the highest announced valuation and gets the good. The cost to him of doing this is that he gets the good less often, but this is a cost worth incurring to at least some degree. [6] Thus, we again see that there may be a problem in implementing certain social choice functions in settings in which information is privately held. (For a similar point in the bilateral trade context, see Exercise 23.B.2.)

Because of the revelation principle, when we explore in Sections 23.C and 23.D the constraints that incomplete information about types puts on the set of implementable social choice functions, we will be able to restrict our analysis to identifying those social choice functions that can be truthfully implemented.

Finally, we note that, in some applications, participation in the mechanism may be voluntary, and so a social choice function must not only induce truthful revelation of information but must also satisfy certain participation (or individual rationality) constraints if it is to be successfully implemented. In Sections 23.C and 23.D, however, we shall abstract from issues of participation to focus exclusively on the information revelation problem. We introduce participation constraints in Section 23.E. **end of edit**

2.2.2 Dominant Strategy Implementation

Recall that a strategy is weakly dominant strategy if gives a player at least as large a payoff as any of his other possible strategies for every possible strategy that his rival migh play.

Definition 26 (Dominant Strategy Equilibrium). *s^* is a DS. equilibrium: $\forall i \in I, \forall \theta_i \in \Theta_i, \forall m_i \in M_i, \forall \theta_{-i} \in \Theta_{-i}, \forall s_{-i}$*

$$u_i(g(s^*(\theta_i), s_{-i}(\theta_{-i})), \theta_i) \geq u_i(g(m_i, s_{-i}(\theta_{-i})), \theta_i)$$

Definition 27 (Implementation in DS). *Γ implements f in DS if $\exists s^*$ such that*

- *s^* is a DS equilibrium*
- *Implementation: $\forall \theta \in \Theta, g(s^*(\theta)) = f(\theta)$.*

The concept of dominant strategy implementation is of special interest because if we can find a mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ that implements $f(\cdot)$ in dominant strategics, then this mechanism implements $f(\cdot)$ in a very strong and robust way. This implementation will be robust even if agents have incorrect, and perhaps even contradictory, beliefs about this distribution. In particular, agent i 's beliefs regarding the distribution of θ_{-i} do not affect the dominance of his strategy $s_i^*(\cdot)$.[18] Third, it follows that if Γ implements $f(\cdot)$ in dominant strategies then it does so regardless of the probability density $\phi(\cdot)$. Thus, the same mechanism can be used to implement $f(\cdot)$ for any $\phi(\cdot)$. One advantage of this is that if the mechanism designer is an outsider (say, the "government"), he need not know $\phi(\cdot)$ to successfully implement $f(\cdot)$

As we noted in Section 23.B, to identify whether a particular social choice function $f(\cdot)$ is implementable, we need, in principle, to consider all possible mechanisms. Fortunately, it turns out that for dominant strategy implementation it suffices to ask whether a particular $f(\cdot)$ is truthfully implementable in the sense introduced in Definition 23.C.3.

Definition 28. *Dominant Strategy Incentive Compatibile The SCF $f(\cdot)$ is truthfully implementable in dominant strategies (or dominant strategy incentive compatible, or strategy-proof, or **DSIC**) if $s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and $i = 1, \ldots, I$ is a dominant strategy equilibrium of the direct revelation mechanism $\Gamma = (\Theta_1, \ldots, \Theta_I, f(\cdot))$. That is, if for all i and all $\theta_i \in \Theta_i$*

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i)$$

for all $\hat{\theta}_i \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$

The ability to restrict our inquiry, without loss of generality, to the question of whether $f(\cdot)$ is truthfully implementable is a consequence of what is known as the revelation principle for dominant strategies.

Theorem 3 (Revelation principle for DS equilibrium)**. .** ***If there exists a mechanism that implements f in DS.-equilibrium, then f can be implemented in DS equilibrium with the direct mechanism, with truth-telling as the domininant strategy (i.e., f is strategy proof).***

Proof. Let $\Gamma = (M, g)$ be the mechanism that implements f is d.s. equilibrium. Let σ^* be the d.s. equilibrium such that $g(s^*(\theta)) = f(\theta), \forall \theta \in \Theta$. Then by definition of d.s. equilibrium, $\forall i \in I, \forall \theta_i \in \Theta_i, \forall m_i \in M_i, \forall \theta_{-i} \in \Theta_{-i}, \forall s_{-i}$

$$g\left(s_i^*\left(\theta_i\right), s_{-i}\left(\theta_{-i}\right)\right) \succeq_i \left(\theta_i\right) g\left(m_i, s_{-i}\left(\theta_{-i}\right)\right)$$

Since this holds for every possible strategy of the other players, we can substitute s_{-i}^* for s_{-i}. Similarly, since $s_i^*\left(\theta_i'\right) \in M_i, \forall \theta_i' \in \Theta_i$, we can substitute $s_i^*\left(\theta_i'\right)$ for m_i. Thus we have $\forall i \in I, \forall \theta_i, \theta_i' \in \Theta_i, \forall \theta_{-i} \in \Theta_{-i}$

$$g\left(s_i^*\left(\theta_i\right), s_{-i}^*\left(\theta_{-i}\right)\right) \succeq_i \left(\theta_i\right) g\left(s_i^*\left(\theta_i'\right), s_{-i}^*\left(\theta_{-i}\right)\right)$$

By definition of s^*, this implies that $\forall i \in I, \forall \theta_i, \theta_i' \in \Theta_i, \forall \theta_{-i} \in \Theta_{-i}$

$$f(\theta) \succeq_i \left(\theta_i\right) f\left(\theta_i', \theta_{-i}\right)$$

Thus f is strategy-proof, so f can be implemented by the direct mechanism with truth-telling as the d.s. equilibrium. □

The intuitive idea behind the revelation principle for dominant strategies can be put as follows: Suppose that the indirect mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ implements $f(\cdot)$ in dominant strategies, and that in this indirect mechanism each agent i finds playing $s_i^*\left(\theta_i\right)$ when his type is θ_i better than playing any other $s_i \in S_i$ for any choices $s_{-i} \in S_{-i}$ by agents $j \neq i$. Now consider altering this mechanism simply by introducing a mediator who says to each agent i : "You tell me your type, and when you say your type is θ_i, I will play $s_i^*\left(\theta_i\right)$ for you." Clearly, if $s_i^*\left(\theta_i\right)$ is agent i 's optimal choice for each $\theta_i \in \Theta_i$ in the initial mechanism Γ for any strategies chosen by the other agents, then agent i will find telling the truth to be a dominant strategy in this new scheme. But this means that we have found a way to truthfully implement $f(\cdot)$. The implication of the revelation principle is that to identify the set of social choice functions that are implementable in dominant strategies, we need only identify those that are truthfully implementable. In principle, for any $f(\cdot)$, this is just a matter of checking the inequalities (23.C.3).

Theorem 4 (Gibbard-Satterthwaite once again). ***If*** $\#X \geq 3, f$ ***is onto,*** $\{\succeq_i (\theta) : \theta \in \Theta\} = P$***, and*** f ***is truthfully implementable in DS , then*** f ***is dictatorial, i.e.,*** $\exists i \in I$ ***such that*** $\forall \theta \in \Theta$***,***

$$f(\theta) \in \{x \in X : x \succeq_i (\theta) y, \forall y \in X\}$$

Given this negative conclusion, if we are to have any hope of implementing desirable social choice functions, we must either weaken the demands of our implementation concept by accepting implementation by means of less robust cquilibrium notions (such as Bayesian Nash equilibria) or we must focus on more restricted environments. In the remainder of this section, we follow the latter course, studying the possibilities for implementing desirable social choice functions in dominant strategics when preferences take a quasilinear form. Section 23.D explores the former possibility: It studies implementation in Bayesian Nash equilibria.

2.3 Properties of Mechanisms

Many properties of a mechanism are stated in terms of the properties of the social choice function that the mechanism implements. A good place to start is to outline a number of desirable properties for social choice functions.

Theorem 3.3 (Taxation principle). Suppose $t\left(\grave{}\right)$ implements $k(\cdot)$. If $k\left(\theta_i, \theta_{-i}\right) = k\left(\theta_i', \theta_{-i}\right)$, then $t_i\left(\theta_i, \theta_{-i}\right) = t_i\left(\theta_i', \theta_{-i}\right)$. Hence t_i can be written as

$$t_i\left(\theta_i, \theta_{-i}\right) = \tau\left(k\left(\theta_i, \theta_{-i}\right), \theta_{-i}\right)$$

Proof. Suppose not. Then player i will lie at either θ_i or either θ_i', i.e., he will just say whichever type will give him a higher transfer. Combined with the monotonicity condition, this implies that transfers must take the following form:

$$t_i\left(\theta_i, \theta_{-i}\right) = \begin{cases} \alpha & \theta_i < \bar{\theta}_i \\ \beta & \theta_i > \bar{\theta}_i \end{cases}$$

Further, it must be that $\alpha > \beta$. Otherwise player i will lie when his type is less than $\bar{\theta}_i$. Even more specifically, $\bar{\theta}_i = \alpha - \beta$ since player i must be indifferent between getting the object and not getting it at $\bar{\theta}_i$ (otherwise he will lie).

To summarize, suppose $k(\cdot)$ is montone and suppose $t(\cdot)$ implements $k(\cdot)$. Then

$t(\cdot)$ takes the following form:

$$\forall i \in I, \forall \theta_{-i} \in \Theta_{-i}, t_i\left(\theta_i, \theta_{-i}\right) = \begin{cases} \alpha & \theta_i < \bar{\theta}_i \\ \beta = \alpha - \bar{\theta}_i & \theta_i > \bar{\theta}_i \end{cases}$$

where $\bar{\theta}$ is a function of θ_{-i}, i.e.

$$\bar{\theta}_i = \bar{\theta}_i\left(\theta_{-i}\right) = \inf\left\{\theta_i \in \Theta_i : y_i\left(\theta_i, \theta_{-i}\right) = 1\right\}$$

Theorem 3.4. $k(\cdot)$ is implementable if and only if it is monotone. If it is monotone, then it can be implemented as above. Example 3.3. Let $k^*(\theta) = \left(y_1^*(\cdot), \ldots, y_n^*(\cdot)\right)$, where

$$y_i^*(\theta) = \begin{cases} 0 & \theta_i < \max_{j \neq i} \theta_j \\ 1 & \theta_i > \max_{j \neq i} \theta_j \end{cases}$$

Note that this describes more than one k^* since you can break ties in many different ways. We can implement this allocation function using the following transfers:

$$t_i\left(\theta, \theta_{-i}\right) = \begin{cases} 0 & \theta_i < \max_{j \neq i} \theta_j \\ 0 - \max_{j \neq i} \theta_j & \theta_i > \max_{j \neq i} \theta_j \end{cases}$$

This is second price auction. Note that the second price auction also satisfies the extra requirement of individual rationality:

$$\forall i \in I, \forall \theta \in \Theta, \theta_i y_i(\theta) + t_i(\theta) \geq 0$$

This ensures than all players are willing to participate.

3 Vickery-Clarke-Groves Mechanisms

Arrow/GS Impossibility theorem. Restrict the domain of preferences to transferable utility or quasi Linear.

We follow the story of public good provision from Example 2. **Set-up**

- N agents, each agent has utility $u_i(k, \theta_i) = v_i(k, \theta_i) + t_i$
- $k \in K$ where K is the set of non-monetary allocation
- $t_i \in \mathbb{R}$ monetary transfer
- The allocation is (k, t) where $t = (t_1, \ldots, t_n)$
- $\theta_i \in \Theta_i$ type of agents, $\theta = (\theta_1, \ldots, \theta_n) \in \prod_{i=1}^n \Theta_i$
- Set of feasible allocation $X = \{(k, t) \mid k \in K, t_i \in \mathbb{R}, \sum_i t_i \leq 0\}$;
- Let $k : \Theta \to K$, where $\Theta = \Theta_1, \cdots \times \Theta_n = [0,1]^n$, K
- $t_i : \Theta \to \mathbb{R}$,
- we even specify it to $v_i(k, \theta_i) = \theta_i k$
- The social choice function is $f(\theta) = (k(\theta), t(\theta))$ where $t(\theta) = (t_i(\theta))_{i=1}^n$.

Definition 29. *A social choice function f satisfies :*

1. *it is ex-post efficient: $\sum_{i=1}^n v_i(k(\theta), \theta_i) \geq \sum_{i=1}^n v_i(k, \theta_i)\ \forall k, \theta$;*
2. *it is balanced budget: $\sum_{i=1}^n t_i(\theta) = 0$*
3. *full ex-post efficiency: if it is ex-post efficient with balanced budget*
4. *dominant strategy incentive compatible (DSIC) or strategy-proof if $\forall i, \theta_i, \theta_i', \theta_{-i}$,*

$$v_i(k(\theta), \theta_i) + t_i(\theta) \geq v_i\left(k\left(\theta_i', \theta_{-i}\right), \theta_i\right) + t\left(\theta_i', \theta_{-i}\right)$$

Theorem 5. ***Let** $k^* : \prod_{i=1}^n \Theta_i \to K$ **be ex-post efficient. If** $\forall i, \exists h_i : \forall \theta$,*

$$t_i(\theta) = \sum_{j \neq i} v_j\left(k^*(\theta), \theta_j\right) + h_i(\theta_{-i})$$

***then,** $f = (k^*, t)$ **is** DSIC.*

Proof. Proof. Suppose, for contradiction, $f = (k^*, t)$ is not DSIC, then, $\exists i, \theta_i, \theta_i', \theta_{-i}$:

$$\begin{gathered} v_i\left(k(\theta), \theta_i\right) + t_i(\theta) < v_i\left(k\left(\theta_i', \theta_{-i}\right), \theta_i\right) + t\left(\theta_i', \theta_{-i}\right) \\ \Rightarrow v_i\left(k(\theta), \theta_i\right) + \sum_{j \neq i} v_j\left(k^*(\theta), \theta_j\right) + h_i\left(\theta_{-i}\right) < \\ < v_i\left(k\left(\theta_i', \theta_{-i}\right), \theta_i\right) + \sum_{j \neq i} v_j\left(k^*\left(\theta_i', \theta_{-i}\right), \theta_j\right) + h_i\left(\theta_{-i}\right) \\ \Rightarrow \sum_{i=1}^n v_i\left(k(\theta), \theta_i\right) < \sum_{i=1}^n v_i\left(k\left(\theta_i', \theta_{-i}\right), \theta_i\right) \end{gathered}$$

It contradicts with ex-post efficiency. □

Definition 30 (Clarke mechanism)**.** *Consider to construct* $h_i : \prod_{j \neq i} \Theta_j \to \mathbb{R}$

$$h_i\left(\theta_{-i}\right) = -\sum_{j \neq i} v_j\left(k_{-i}^*\left(\theta_{-i}\right), \theta_j\right)$$

where $k_{-i}^*\left(\theta_{-i}\right) \in \arg\max_{k \in K} \sum_{j \neq i} v_j\left(k, \theta_j\right) \forall \theta_{-i}$

k_{-i}^* maximizes welfare among everyone but i

Intuitively h_i can be thought of as follows:

We have seen that when agents type is $\theta 1_{\ldots \theta_n}$ each agent payment is equal $\xi_i(\theta_i)$. Now if each agent contributes an equal $\frac{1}{n-1}$ share of all of the other agent's payments, payments from a given agent i to each other $n-1$ will total$\frac{1}{n-1} \sum_{j \neq i} \xi_j(\theta_j)$ and agent i will receive from these agents in return payment thta total to $\xi(\theta_i)$. Agent i's net transfer will therefore be $\xi_i(\theta_i) - \frac{1}{n-1} \sum_{j \neq i} \xi_j(\theta_j)$.

Example 4 (Pivotal mechanism)**.**

$$t_i(\theta) = \sum_{j \neq i} \left(v_j\left(k^*(\theta), \theta_j\right) - v_j\left(k_{-i}^*(\theta_{-i}), \theta_j\right)\right)$$

If $k(\theta) = k_{-i}^*(\theta_{-i})$ then i pays nothing.

If $k(\theta) \neq k_{-i}^*(\theta_{-i})$ then i pays a tax equal to his effect on others.

This direct revelation mechanism is known as the expected externality mechanism due to d'Aspremont adn Gerard Varret and Arrow.

In an auction setting, this is just like a Vickery auction (2nd price sealed action)

Example 5. *(Vickery auction)*

- $K = \{0, 1, \ldots n\}$

- *k- who gets the object if anyone*
- *utility*

$$v_i(k,\theta_i) = \begin{cases} \theta_i & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$$

- *$k^*(\theta)$ give it to whoever values it most*
- *i is pivotal when she is the highest bidder. His tax is the 2nd highest bid.*

Theorem 6. ***Suppose, for each** $i, \{v_i(\cdot,\theta_i) \mid \theta_i \in \Theta_i\} = \mathbb{R}^K$. **If** $f = (k^*, t)$ **satisfies ex-post efficiency and** $DSIC$, **then,** $\forall i, \exists h_i : \forall \theta,$*

$$t_i(\theta) = \sum_{j \neq i} v_j\left(k^*(\theta), \theta_j\right) + h_i\left(\theta_{-i}\right)$$

Proof. Let $f = (k^*, t)$ satisfies ex-post efficiency and DSIC, and define $h_i(\theta) = t_i(\theta) - \sum_{j\neq i} v_j\left(k^*(\theta), \theta_j\right)$. WTS $h_i(\theta) = h_i\left(\theta_{-i}\right) \forall \theta_i$

1. Suppose $k^*\left(\theta_i, \theta_{-i}\right) = k^*\left(\theta_i', \theta_{-i}\right)$. Then, by DSIC, $\forall i, \theta_i, \theta_i', \theta_{-i}$

$$v_i\left(k^*(\theta), \theta_i\right) + t_i(\theta) \geq v_i\left(k^*\left(\theta_i', \theta_{-i}\right), \theta_i\right) + t\left(\theta_i', \theta_{-i}\right) \Rightarrow t_i(\theta) \geq t_i\left(\theta_i', \theta_{-i}\right)$$

$$v_i\left(k^*\left(\theta_i', \theta_{-i}\right), \theta_i'\right) + t\left(\theta_i', \theta_{-i}\right) \geq v_i\left(k^*(\theta), \theta_i'\right) + t_i(\theta) \Rightarrow t_i\left(\theta_i', \theta_{-i}\right) \geq t_i(\theta)$$

$$\Rightarrow t_i\left(\theta_i, \theta_{-i}\right) = t_i\left(\theta_i', \theta_{-i}\right) = t_i\left(\theta_{-i}\right) \forall \theta_i, \theta_i'$$

$$\Rightarrow h_i(\theta) = h_i\left(\theta_{-i}\right)$$

where the last implication is due to $k^*\left(\theta_i, \theta_{-i}\right) = k^*\left(\theta_i', \theta_{-i}\right)$.

2. Suppose, $\exists i, \exists \theta_i, \theta_i' : k^*\left(\theta_i, \theta_{-i}\right) \neq k^*\left(\theta_i', \theta_{-i}\right)$, and suppose, per contra, wlog, $h_i\left(\theta_i, \theta_{-i}\right) > h_i\left(\theta_i', \theta_{-i}\right)$ Fix an $\epsilon > 0$, and let $\theta_i^\epsilon \in \Theta_i$ be such that:

$$v_i\left(k, \theta_i^\epsilon\right) = \begin{cases} -\sum_{j\neq i} v_j\left(k^*(\theta), \theta_j\right) & k = k^*(\theta) \\ -\sum_{j\neq i} v_j\left(k^*\left(\theta_i', \theta_{-i}\right), \theta_j\right) + \epsilon & k = k^*\left(\theta_i', \theta_{-i}\right) \\ -\infty & \text{otherwise} \end{cases}$$

First, notice that, $k^*\left(\theta_i^\epsilon, \theta_{-i}\right) = k^*\left(\theta_i', \theta_{-i}\right)$ otherwise ex-post efficiency is violated. By DSIC,

$$v_i\left(k^*\left(\theta_i^\epsilon, \theta_{-i}\right), \theta_i^\epsilon\right) + t_i\left(\theta_i^\epsilon, \theta_{-i}\right) \geq v_i\left(k^*\left(\theta_i, \theta_{-i}\right), \theta_i^\epsilon\right) + t_i\left(\theta_i, \theta_{-i}\right)$$

$$\Rightarrow v_i\left(k^*\left(\theta_i^\epsilon, \theta_{-i}\right), \theta_i^\epsilon\right) + \sum_{j\neq i} v_j\left(k^*\left(\theta_i^\epsilon, \theta_{-i}\right), \theta_j\right) + h_i\left(\theta_i^\epsilon, \theta_{-i}\right) \geq v_i\left(k^*(\theta), \theta_i^\epsilon\right) + \sum_{j\neq i} v_j\left(k^*(\theta), \theta_j\right) +$$

$\Rightarrow h_i\left(\theta_i^\epsilon, \theta_{-i}\right) + \epsilon \geq h_i\left(\theta_i, \theta_{-i}\right)$ Then, since $k^*\left(\theta_i^\epsilon, \theta_{-i}\right) = k^*\left(\theta_i', \theta_{-i}\right)$, by part [1], we have $h_i\left(\theta_i^\epsilon, \theta_{-i}\right) = h_i\left(\theta_i', \theta_{-i}\right)$. Hence,

$$h_i\left(\theta_i', \theta_{-i}\right) + \epsilon \geq h_i\left(\theta_i, \theta_{-i}\right)$$

However, when ϵ small enough, it will contradict with the hypothesis $h_i\left(\theta_i, \theta_{-i}\right) > h_i\left(\theta_i', \theta_{-i}\right)$

□

Theorem 7. ***Suppose, for each*** i, $\{v_i(\cdot, \theta_i) \mid \theta_i \in \Theta_i\} = \mathbb{R}^K$. ***Then, there does not exist a SCF*** f ***that is DSIC and fully ex-post efficient.***

Proof. (Easy and incomplete version: assume twice differentiability and $n = 2$.) Assume $K = \mathbb{R}, \theta_i \in \left[\underline{\theta}_i, \bar{\theta}_i\right] \subseteq \mathbb{R}, \partial^2 v_i / \partial k^2 < 0$, and $\partial^2 v_i / \partial k \partial \theta_i \neq 0$ FOC for reporting θ_i : $\frac{\partial v_i}{\partial k} \frac{\partial k^*}{\partial \theta_i} + \frac{\partial t_i}{\partial \theta_i} = 0 \Rightarrow -\frac{\partial^2 t_i}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 v_i}{\partial k^2} \frac{\partial k^*}{\partial \theta_1} \frac{\partial k^*}{\partial \theta_2} + \frac{\partial v_i}{\partial k} \frac{\partial^2 k^*}{\partial \theta_1 \partial \theta_2}$ Balanced budget $\Rightarrow t_1(\theta) + t_2(\theta) = 0, \forall \theta$, then,

$$\left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right) \frac{\partial k^*}{\partial \theta_1} \frac{\partial k^*}{\theta_2} + \left(\frac{\partial v_1}{\partial k} + \frac{\partial v_2}{\partial k}\right) \frac{\partial^2 k^*}{\partial \theta_1 \partial \theta_2} = 0$$

But, by ex-post efficiency, $\frac{\partial v_1}{\partial k} + \frac{\partial v_2}{\partial k} = 0$. And we can show that $\partial k^* / \partial \theta_i > 0$ by implicit function theorem so that $\left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right) \frac{\partial k^*}{\partial \theta_1} \frac{\partial k^*}{\theta_2} < 0$, contradiction. □

3.1 Expected Externality Mechanism

Each agent has utility $u_i(k, \theta_i) = v_i(k, \theta_i) + t_i$

Definition 31. *SCF* f *is Bayesian incentive compatible if, for all* i, θ_i, θ_i', *and* θ_{-i}

$$E\left[v_i(k(\theta), \theta_i) + t_i(\theta) \mid \theta_i\right] \geq E\left[v_i\left(k\left(\theta_i', \theta_{-i}\right), \theta_i\right) + t_i\left(\theta_i', \theta_{-i}\right) \mid \theta_i\right]$$

Theorem 8. ***Suppose agents' types are statistically independent. Let*** k^* ***satisfy ex-post efficiency and let*** $t_i(\theta) = E\left[\sum_{j \neq i} v_j\left(k^*(\theta), \theta_j\right) \mid \theta_i\right] + h_i\left(\theta_{-i}\right)$. ***Then,*** $f = (k^*, t)$ ***is Bayesian incentive compatible and there exist some*** h_i ***for each*** i ***such that it satisfies balanced budget.***

Proof.

- BIC. For any θ_i, θ_i',

$$\begin{aligned}
&E\left[v_i\left(k^*(\theta), \theta_i\right) + t_i(\theta) \mid \theta_i\right] \\
=&E\left[v_i\left(k^*(\theta), \theta_i\right) \mid \theta_i\right] + E\left[\sum_{j\neq i} v_j\left(k^*(\theta), \theta_j\right) \mid \theta_i\right] + E\left[h_i\left(\theta_{-i}\right)\right] \\
=&E\left[\sum_{i=1}^{n} v_i\left(k^*(\theta), \theta_i\right) \mid \theta_i\right] + E\left[h_i\left(\theta_{-i}\right)\right] \\
\geq&E\left[\sum_{i=1}^{n} v_i\left(k^*\left(\theta_i', \theta_{-i}\right), \theta_i\right) \mid \theta_i\right] + E\left[h_i\left(\theta_{-i}\right)\right] \\
=&E\left[v_i\left(k^*\left(\theta_i', \theta_{-i}\right), \theta_i\right) + t_i(\theta) \mid \theta_i\right]
\end{aligned}$$

- Balanced budget. Let $h_i\left(\theta_{-i}\right) = -\frac{1}{n-1}\sum_{j\neq i}\xi_j\left(\theta_j\right)$ where $\xi_i\left(\theta_i\right) = E\left[\sum_{j\neq i} v_j\left(k^*(\theta), \theta_j\right) \mid \theta_i\right]$. Then

$$\begin{aligned}
\sum_{i=1}^{n} t_i\left(\theta_i\right) &= \sum_{i=1}^{n}\xi_i\left(\theta_i\right) + \sum_{i=1}^{n} h_i\left(\theta_{-i}\right) \\
&= \sum_{i=1}^{n}\xi_i\left(\theta_i\right) - \frac{1}{n-1}\sum_{i=1}^{n}\sum_{j\neq i}\xi_j\left(\theta_j\right) = 0
\end{aligned}$$

□

3.2 Rochet and Vorha Theorems

Suppose we have one agent k is implementable if $\exists t : \Theta \to R$

$$v(\theta, k(\theta)) - t(\theta) \geq \theta, k(\theta')) - t(\theta')\theta, \theta'$$

k is C-MON if $(\theta_1, \ldots \theta_n = \theta_1)$

$$\sum_{i=1}^{m} v(\theta_{i+1}, k(\theta_i) - v(\theta_i, k(\theta_i)) \leq 0$$

k is MON if it holds 2-cycles $(m = 3)$

Theorem 9. ***Rochet 1989*** k ***is implementable*** $\iff$ k ***is C-MON***

Proof. TBD

□

Revenue Equivalence Suppose k is implementable and $\exists t$. k exhibits revenue equivalence if t, t' that implement k $\exists c \in R$:

$$t(\theta) = t'(\theta) + c$$

Theorem 10. ***Suppose k is implementable. k exhibits Revenue Equivalence $\Longleftrightarrow$***

$$V(\theta,\theta^{'}) = -V(\theta^{'},\theta)$$

Proof. TBD □

Theorem 11. ***Let $V = \{v(\theta) \in \mathbb{R}^X : \theta \in \Theta\}$. If k is implementable and V is a convex set then k exhibits revenue equivalence***

Proof. TBD □

Corollary 2. *Grove's scheme uniquely implements an ex post efficient allocation if V is convex*

3.3 Individual Rationality

- In general, without IR, there exists a Groves scheme that would be DSIC
- With IR, there would not exist such a Groves scheme

Example 6. *$K = \{0,1\}, n = 2, \Theta_i = \{\underline{\theta},\bar{\theta}\}, \bar{\theta} > 2\underline{\theta} > 0$. If $k = 1$, there is a cost $c \in (2\underline{\theta},\bar{\theta})$. $v_i(0,\theta_i) = 0$ and $v_i(1,\theta_i) = \theta_i$*

- *The efficient allocation is*

$$k^* = \begin{cases} 1 & \text{if } \theta_1 = \bar{\theta} \text{ or } \theta_2 = \bar{\theta} \\ 0 & \text{if } \theta_1 = \theta_2 = \underline{\theta} \end{cases}$$

- *Ex-post IR: $\underline{\theta} + t_1(\underline{\theta},\bar{\theta}) \geq 0 \Rightarrow t_1(\underline{\theta},\bar{\theta}) \geq -\underline{\theta}$.*
- *DSIC: $\bar{\theta} + t_1(\bar{\theta},\bar{\theta}) \geq \bar{\theta} + t_1(\underline{\theta},\bar{\theta}) \Rightarrow t_1(\bar{\theta},\bar{\theta}) \geq t_1(\underline{\theta},\bar{\theta}) \geq -\underline{\theta}$*
- *By symmetry, $t_2(\bar{\theta},\bar{\theta}) \geq -\underline{\theta}$.*
- *Therefore, $t_1(\bar{\theta},\bar{\theta}) + t_2(\bar{\theta},\bar{\theta}) \geq -2\underline{\theta} > -c$;*
- *But balanced budget requires $t_1(\bar{\theta},\bar{\theta}) + t_2(\bar{\theta},\bar{\theta}) + c \leq 0$, contradiction.*

Example 7. *Same as previous one*

- *But consider a prior $\Pr(\theta_i) = \frac{1}{2}$.*

- *Interim IR:* $\frac{1}{2}\left[\underline{\theta} + t_1(\underline{\theta}, \bar{\theta}) + t_1(\underline{\theta}, \underline{\theta})\right] \geq 0 \Rightarrow t_1(\underline{\theta}, \bar{\theta}) \geq -\underline{\theta}$

- *Interim IC:* $\frac{1}{2}\left[\bar{\theta} + t_1(\bar{\theta}, \underline{\theta}) + \bar{\theta} + t_1(\bar{\theta}, \bar{\theta})\right] \geq \frac{1}{2}\left[\bar{\theta} + t_1(\underline{\theta}, \bar{\theta}) + t_1(\underline{\theta}, \underline{\theta})\right] \Rightarrow t_1(\bar{\theta}, \bar{\theta}) + t_1(\bar{\theta}, \underline{\theta}) \geq -\bar{\theta}$*;*

- *By symmetircity:* $t_2(\bar{\theta}, \underline{\theta}) \geq -\underline{\theta}$ *and* $t_2(\bar{\theta}, \bar{\theta}) + t_2(\underline{\theta}, \bar{\theta}) \geq -\bar{\theta}$*; Therefore,* $\sum_i t_i(\theta) \geq -2\bar{\theta} - 2\underline{\theta}$

- *By the feasibility region:*

$$
\begin{aligned}
t_1(\bar{\theta}, \bar{\theta}) + t_2(\bar{\theta}, \bar{\theta}) &\leq -c \\
t_1(\bar{\theta}, \underline{\theta}) + t_2(\bar{\theta}, \underline{\theta}) &\leq -c \\
t_1(\underline{\theta}, \bar{\theta}) + t_2(\underline{\theta}, \bar{\theta}) &\leq -c \\
t_1(\underline{\theta}, \underline{\theta}) + t_2(\underline{\theta}, \underline{\theta}) &\leq 0
\end{aligned}
$$

- *Therefore,* $-3c \geq \sum_i t_i(\theta) \geq -2\bar{\theta} - 2\underline{\theta} \Rightarrow c \leq \frac{2}{3}(\bar{\theta} + \underline{\theta})$*. So, if* $\bar{\theta} < 3\underline{\theta}$*, then there is a contradiction.*

3.4 Bayesian Nash equilibrium implementation

EDIT WHOLE SECTION

Again, a mechanism is $\Gamma = (M, g)$. Let $G = (\Gamma, \Theta, \phi)$ denote the incomplete information game associated with Γ.

Definition 32 (Bayesian Nash equilibrium). *of G is a strategy profile $\sigma^* = (\sigma_i^*)_{i \in I}$ (where $\sigma_i^* : \Theta_i \to M_i$) such that $\forall i \in I, \forall \theta_i \in \Theta_i, \forall m_i \in M_i$*

$$E_{\theta_{-i}}\left[u_i\left(g\left(\sigma_i^*\left(\theta_i\right), \sigma_{-i}^*\left(\theta_{-i}\right)\right), \theta_i\right) \mid \theta_i\right] \geq E_{\theta_{-i}}\left[u_i\left(g\left(m_i, \sigma_{-i}^*\left(\theta_{-i}\right)\right), \theta_i\right) \mid \theta_i\right]$$

We will use $BNE(G)$ to denote the set of Bayesian Nash equilibria of G.

Definition 33 (Implementation). Γ *implements social choice function $f : \Theta \to X$ in Bayesian Nash equilibrium if* $\text{BNE}(G) \neq \emptyset$ *and $\exists \sigma^* \in BNE(G)$ (where $G = (\Gamma, \Theta, \phi)$) such that* $\forall \theta \in \text{support}(\phi)$

$$g\left(\sigma^*(\theta)\right) = f(\theta)$$

As in the d.s. equilibrium section, Γ strictly/fully implements f if the above the above condition holds for all $\sigma^* \in BNE(G)$ rather than just one of them.

Theorem 12 (Revelation principle for BN implementation). . ***If f is BN -implementable, then f is BN implementable by the direct mechanism $\Gamma_{direct} = (\Theta, f)$ with truth-telling as a BNE.***

Proof. Suppose Γ BN-implements f. Let σ^* be the BNE such that $g\left(\sigma^*(\theta)\right) = f(\theta), \forall \theta \in$ support(ϕ). By definition of a BNE, $\forall i \in I, \forall \theta \in \Theta, \forall m_i \in M_i$

$$E_{\theta_{-i}}\left[u_i\left(g\left(\sigma_i^*\left(\theta_i\right), \sigma_{-i}^*\left(\theta_{-i}\right)\right), \theta_i\right) \mid \theta_i\right] \geq E_{\theta_{-i}}\left[u_i\left(g\left(m_i, \sigma_{-i}^*\left(\theta_{-i}\right)\right), \theta_i\right) \mid \theta_i\right]$$

Since $g\left(\sigma^*(\theta)\right) = f(\theta), \forall \theta \in \text{support}(\phi)$, we have $\forall i \in I, \forall \theta \in \Theta, \forall \theta_i' \in \Theta_i$

$$E_{\theta_{-i}}\left[u_i\left(f\left(\theta_i, \theta_{-i}\right), \theta_i\right) \mid \theta_i\right] \geq E_{\theta_{-i}}\left[u_i\left(f\left(\theta_i', \theta_{-i}\right), \theta_i\right) \mid \theta_i\right]$$

Thus truth-telling is a BNE in the direct game, i.e., f is strategy proof. $\square$

Definition 34. *If f is strategy proof, we say that f is d.s. incentive compatible. If f is BN-implementable, we say that f is Baysesian incentive compatible. Note that d.s. IC implies BN IC, but not the other way around. In terms of set notation, BN IC $\supset$ d.s. IC.*

Because BN implementation is a weaker condition, it seems reasonable that a wider variety of social choice functions could be BN-implemented. Exactly which SCFs can be BN-implemented depends on the assumptions about the probability distribution ϕ. If ϕ is a product measure (i.e., θ_i are independently distributed), then the kinds of SCFs that can be BN-implemented are pretty similar to the ones that can be d.s. implemented (in this case there is a similar monotonicty condition that we will see shortly). However, if ϕ is not a product measure, then under weak regularity condition virtually all SCFs can be implemented. In this case, the transfer function that implements the SCF will often take a complex structure and be very sensitive to parameters.

In a mathematical sense product measures are rare (non-generic), and in an economic sense it is often unrealistic to assume that ϕ is a procuct measure. But for better or worse, in most of the mechanism design literature it is assumed that ϕ is a product measure. Thus we will also make this assumption throughout the rest of this section.

Definition 3.11. Let ϕ be a product measure and suppose that $t(\cdot)$ implements $k(\cdot) = (y_1(\cdot), \ldots, y_n(\cdot))$ For each $i \in I$, define $\bar{y}_i(\cdot)$ and $\bar{y}_i(\cdot)$ as

$$\bar{t}_i(\theta_i) := E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})]$$

and

$$\bar{y}_i(\theta_i) := E_{\theta_{-i}}[y_i(\theta_i, \theta_{-i})]$$

These definitions allow us to write the BN incentive compatibility condition as

$$\bar{y}_i(\theta_i)\theta_i + \bar{t}_i(\theta_i) \geq \bar{y}_i(\theta_i')\theta_i + \bar{t}_i(\theta_i'), \forall \theta_i' \in \Theta_i$$

Note that we don't have a separate IC condition for each θ_{-i} as we did in the d.s. implementation case. Instead, we just have one constraint given the averages $\bar{y}$ and $\bar{t}$ implied by ϕ.

Theorem 3.7. $k(\cdot)$ is BN -implementable only if it is "monotone" in the sense that $\forall i \in I, \bar{y}_i(\cdot)$ is weakly increasing. Proof. Suppose not. Then $\exists i \in I$ and $\theta_i, \theta_i' \in \Theta_i$ such that $\theta_i < \theta_i'$ but $\bar{y}_i(\theta) > \bar{y}_i(\theta_i')$. Let $t(\cdot)$ be a transfer function that BN-implements $k(\cdot)$. If player i 's type is θ, then incentive compatibility implies that

$$\bar{y}_i(\theta_i)\theta_i + \bar{t}_i(\theta_i) \geq \bar{y}_i(\theta_i')\theta_i + \bar{t}_i(\theta_i')$$

If his type is θ_i', then IC implies that

$$\bar{y}_i\left(\theta_i'\right)\theta_i' + \bar{t}_i\left(\theta_i'\right) \geq \bar{y}_i\left(\theta_i\right)\theta_i' + \bar{t}_i\left(\theta_i\right)$$

Adding both inequalities together and cancelling out the transfers implies yields

$$\bar{y}_i\left(\theta_i'\right)\left(\theta_i' - \theta_i\right) \geq \bar{y}_i\left(\theta_i\right)\left(\theta_i' - \theta_i\right)$$

Then $\theta_i' - \theta_i$ cancels out as well, implying that

$$\bar{y}_i\left(\theta_i'\right) \geq \bar{y}_i\left(\theta_i\right)$$

This is a contradiction, so it must be that $k(\cdot)$ is "monotone." This leads us to the following question: Are all monotonic $k(\cdot)$ BN-implementable? In the last section, the fact that $k(\cdot)$ was finite implied that y_i was also finite. Here, we are dealing with $\bar{y}_i$, which is a probability. Thus the benefit of dealing with a finite K is gone, i.e., we will now consider the most general case:

$$K = \left\{k = (y_1, \ldots, y_n) \in [0,1]^n : \sum_{i \in I} y_i \leq 1\right\}$$

Given type θ_i and any $m \in M_i$, player i gets expected utility of

$$p(m)\theta_i + t_i(m)$$

which is a linear function of θ_i. Thus player i 's optimal strategy is

$$m^* = \operatorname{argmax}\left\{p(m)\theta_i + t(m) : m \in M_i\right\}$$

Since $M_i = \Theta_i$ in the direct mechanism, this is equivalent to

$$\theta_i^* = \operatorname{argmax}\left\{p\left(\theta_i'\right)\theta_i + t\left(\theta_i'\right) : \theta_i' \in \Theta_i\right\}$$

In other words, player i will claim to have whichever type puts him on the upper envelope of the set of lines given above. Call this upper envelope $U_i\left(\theta_i\right)$.

Since $U_i\left(\theta_i\right)$ is the upper envelope of a set of linear functions, $U_i\left(\theta_i\right)$ is convex and continuous. This implies that is differentiable almost everywhere, and $U_i\left(\theta_i\right)$ is equal to the integral of its own derivatives, i.e.

$$U_i\left(\theta_i\right) = U_i(x) + \int_x^{\theta_i} U_i'(y)\mathrm{d}y$$

Note that $U_i'(y) = p(m)$ for some $m \in M_i$ If $k(\cdot)$ is incentive compatible, then $U_i'(\theta_i) = p(\theta_i) = \bar{y}_i(\theta_i)$, the probability of getting the object by playing type θ_i. Thus

$$U_i(\theta_i) = U_i(x) + \int_x^{\theta_i} \bar{y}_i(z) \mathrm{d}z$$

Assuming $\Theta_i = \left[\underline{\theta}_i, \bar{\theta}_i\right]$, we can also write this as

$$U_i(\theta_i) = \bar{y}_i(\theta_i)\theta_i + \bar{t}_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(z) \mathrm{d}z$$

Rearranging, we have

$$\bar{t}_i(\theta_i) = \bar{t}_i(\underline{\theta}_i) - [\bar{y}_i(\theta_i)\theta_i - \bar{y}_i(\underline{\theta}_i)\underline{\theta}_i] + \int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(x) \mathrm{d}x.$$

Note that if $\underline{\theta}_i = 0$, this simplifies to

$$\bar{t}_i(\theta_i) = \bar{t}_i(\underline{\theta}_i) - \bar{y}_i(\theta_i)\theta_i + \int_0^{\theta_i} \bar{y}_i(x) \mathrm{d}x.$$

Thus if $t(\cdot)$ implements $k(\cdot), \bar{t}_i(\cdot)$ must take this form. Let's check to see that this $\bar{t}_i(\cdot)$ works. If $\theta_i < \theta_i'$, then we need to make sure that player i will not lie regardless of whether his true type is θ_i or θ_i'. Note that

$$U_i(\theta_i') - U\left(i(\theta_i) = \int_{\theta_i}^{\theta_i'} \bar{y}_i(x) \mathrm{d}x.\right.$$

This implies that

$$\bar{y}_i(\theta_i)\left(\theta_i' - \theta_i\right) \leq U_i(\theta_i') - U\left(i(\theta_i) \leq \bar{y}_i(\theta_i')\left(\theta_i' - \theta_i\right)\right.$$

Using the first inequality, we get

$$U_i(\theta_i') \geq U_i(\theta_i) + \bar{y}_i(\theta_i)\left(\theta_i' - \theta_i\right)$$

Using the defintion of $\bar{t}_i(\cdot)$ from above, we get

$$U_i(\theta_i') \geq \bar{t}_i(\theta_i) + \bar{y}_i(\theta_i)\theta_i'$$

Thus player i is better off by telling the truth when his true type is θ_i'. Using the second inequality, we get

$$U_i(\theta_i) \geq U_i(\theta_i') + \bar{y}_i(\theta_i')\left(\theta_i' - \theta_i\right)$$

Using the defintion of $\bar{t}_i(\cdot)$ from above, we get

$$U_i(\theta_i) \geq \bar{t}_i(\theta_i') + \bar{y}_i(\theta_i')\theta_i$$

Thus player i is also better off by telling the truth when his true type is θ_i. Therefore $\bar{t}_i(\cdot)$ defined above does indeed BN-implement $k(\cdot)$

To summarize, $\bar{y}_i(\cdot)$ is BN implementable if and only if it is "monotone" (in the BN sense). Moreover, if Θ_i is a connected subset of $\mathbb{R}$, then any $\bar{t}_i(\cdot)$ that BN-implements $\bar{y}_i(\cdot)$ must take the form

$$\bar{t}_i\left(\theta_i\right) = \bar{t}_i\left(\underline{\theta}_i\right) - \bar{y}_i\left(\theta_i\right)\theta_i + \int_0^{\theta_i} \bar{y}_i(x)\mathrm{d}x$$

3.5.1 Zero-sum transfers in BN-implementation (expected externality mechanism) In the d.s. implementation section, we found that it was impossible to allocate an object between two bidders using zero-sum transfers in a strategy-proof way. In the BN-implementation context, this is no longer the case. Start with a normal second-price auction and modify it so the winner pays a fixed amount to the other player equal to the average he would have paid in the normal second-price auction. In other words, define

$$\tau_i\left(\theta_i\right) = E_{\theta_{-i}}\left[-t_i\left(\theta_i, \theta_{-i}\right)\right], \forall i = 1, 2$$

Then define

$$\hat{t}_i\left(\theta_i, \theta_{-i}\right) = -\tau_i\left(\theta_i\right) + \tau_{-i}\left(\theta_{-i}\right), \forall i = 1, 2$$

This shuts down the incentive to lie and implements the efficient $k^*(\cdot)$ in BN-equilibrium with a balanced budget (tranfers stay "within the system"). Note that in a more general context with more than two players, we can define

$$\hat{t}_i\left(\theta_i, \theta_{-i}\right) = \tau_i\left(\theta_i\right) + \sum_{j \neq i} \frac{1}{n-1}\tau_j\left(\theta_j\right)$$

This will implement the efficient allocation with zero-sum transfers.

4 Optimal Mechanisms

Myerson (1981)

Problem Set-up

- One seller has an object and n bidders, set of agents is $A = \{0, 1, \ldots, n\}$.
- The utility function for i is

$$v_i\left(a, \theta_i\right) = \begin{cases} \theta_i & \text{if } a = i \\ 0 & \text{otherwise} \end{cases}$$

- For each $i = 1, \ldots, n, \theta_i$ is independently drawn from a continuous PDF $\phi_i(\theta_i)$ with support Θ_i $[\underline{\theta}_i, \bar{\theta}_i]$

- The mechanism is (q, t) where: $-q_i(\theta) = \Pr(a = i \mid \theta)$ is the probability of agent i get the object; $-t_i(\theta) : \Theta \to \mathbb{R}$ is the transfer from buyers to the seller where $\Theta = \prod_{i=1}^n \Theta_i$

- Denote $\bar{q}_i(\theta_i) = E[q_i(\theta_i, \theta_{-i}) \mid \theta_{-i}]$ and $\bar{t}_i(\theta_i) = E[t_i(\theta_i, \theta_{-i}) \mid \theta_i]$

- By revelation principal, restrict our focus on direct mechanism.

- $q_i(\theta)$ needs to be probabilities (PR) : $\sum_{i=1}^n q_i(\theta) \leq 1$ and $q_i(\theta) \geq 0 \forall i$.

- For $i = 1, \ldots, n$, given the mechanism (q, t), the expected utility is

$$\begin{aligned} U_i(\theta_i) &= \int_{\Theta_{-i}} (\theta_i q(\theta) - t_i(\theta)) \phi_{-i}(\theta_{-i}) d\theta_{-i} \\ &= \theta_i \bar{q}_i(\theta_i) - \bar{t}_i(\theta_i) \end{aligned}$$

- The seller's expected utility is

$$U_0(\theta_0) = \int_\Theta \left[\theta_0 \left(1 - \sum_{i=1}^n q_i(\theta)\right) + \sum_{i=1}^n t_i(\theta)\right] \phi(\theta) d\theta$$

- Individual rationality (IR) requires: $U_i(\theta_i) \geq 0 \forall i = 1, \ldots n, \forall \theta_i \in \Theta_i$.

- Incentive-compatibility (IC) requires: for all $\theta_i' \in \Theta_i$

$$U_i(\theta_i) \geq \int_{\Theta_{-i}} (\theta_i q(\theta_i', \theta_{-i}) - t_i(\theta_i', \theta_{-i})) \phi_{-i}(\theta_{-i}) d\theta_{-i}$$

Definition 35. *A mechanism (p, t) is feasible if it satisfies PR, IR, and IC.*

Lemma 2. *A mechanism (p, t) is feasible $\iff$*

1. *$\bar{q}_i$ is nondecreasing*

2. $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{q}_i(\tau_i) d\tau_i$

3. $U_i(\underline{\theta}_i) \geq 0 \forall i = 1, \ldots, n$

4. $\sum_{i=1}^n q_i(\theta) \leq 1$ *and* $q_i(\theta) \geq 0 \forall i$.

Definition 36. *A mechanism (q,t) is an optimal auction if it maximizes $U_0(\theta_0)$ subject to feasibility.*

Lemma 3. *If $q:\Theta\to\mathbb{R}^n$ maximizes*

$$\int_{\Theta}\left[\sum_{i=1}^{n}\left(\theta_i-\frac{1-\Phi(\theta_i)}{\phi(\theta_i)}-\theta_0\right)q_i(\theta)\right]\phi(\theta)d\theta$$

subject to (1) and (4) from previous lemma, and

$$t_i(\theta)=\theta_i q_i(\theta)-\int_{\underline{\theta}_i}^{\theta_i}q_i(\theta_i,\theta_{-i})\,d\theta_i$$

Then, (q,t) is an optimal auction.

Proof.

$$\begin{aligned}U_0(\theta_0)&=\int_{\Theta}\left[\theta_0\left(1-\sum_{i=1}^{n}q_i(\theta)\right)+\sum_{i=1}^{n}t_i(\theta)\right]\phi(\theta)d\theta\\&=\theta_0+\sum_{i=1}^{n}\int_{\Theta}q_i(\theta)(\theta_i-\theta_0)\phi(\theta)d\theta+\sum_{i=1}^{n}\int_{\Theta}[t_i(\theta)-\theta_i q_i(\theta)]\phi(\theta)d\theta\end{aligned}$$

By feasibility and Lemma ,

$$\begin{aligned}&\int_{\Theta}[t_i(\theta)-\theta_i q_i(\theta)]\phi(\theta)d\theta\\&=-\int_{\underline{\theta}_i}^{\bar{\theta}_i}U_i(\theta_i)\phi_i(\theta_i)\,d\theta_i\\&=-\int_{\underline{\theta}_i}^{\bar{\theta}_i}\left[U_i(\underline{\theta}_i)+\int_{\underline{\theta}_i}^{\theta_i}\bar{q}_i(\tau_i)\,d\tau_i\right]\phi_i(\theta_i)\,d\theta_i\\&=-U_i(\underline{\theta}_i)-\int_{\underline{\theta}_i}^{\bar{\theta}_i}\left[\int_{\underline{\theta}_i}^{\theta_i}\bar{q}_i(\tau_i)\,d\tau_i\right]\phi_i(\theta_i)\,d\theta_i\\&=-U_i(\underline{\theta}_i)-\int_{\underline{\theta}_i}^{\bar{\theta}_i}\int_{\tau_i}^{\bar{\theta}_i}\bar{q}_i(\tau_i)\phi_i(\theta_i)\,d\theta_i d\tau_i\\&=-U_i(\underline{\theta}_i)-\int_{\underline{\theta}_i}^{\bar{\theta}_i}\bar{q}_i(\tau_i)[1-\Phi_i(\tau_i)]\,d\tau_i\\&=-U_i(\underline{\theta}_i)-\int_{\Theta}q_i(\theta)[1-\Phi_i(\theta_i)]\phi_{-i}(\theta_{-i})\,d\theta.\end{aligned}$$

Then,

$$U_0(\theta_0)=\theta_0+\sum_{i=1}^{n}\int_{\Theta}q_i(\theta)(\theta_i-\theta_0)\phi(\theta)d\theta-\sum_{i=1}^{n}\int_{\Theta}q_i(\theta)[1-\Phi_i(\theta_i)]\phi_{-i}(\theta_{-i})\,d\theta-\sum_{i=1}^{n}U_i(\underline{\theta}_i)$$

And,

$$t_i(\theta)=\theta_i q_i(\theta)-\int_{\underline{\theta}_i}^{\theta_i}q_i(\theta_i,\theta_{-i})\,d\theta_i\Rightarrow\bar{t}_i(\underline{\theta}_i)=\underline{\theta}_i\bar{q}_i(\underline{\theta}_i)$$

Therefore,

$$U_i\left(\underline{\theta}_i\right)=\underline{\theta}_i\bar{q}_i\left(\underline{\theta}_i\right)-\bar{t}_i\left(\underline{\theta}_i\right)=0 \forall i=1,\ldots n$$

That is, t_i is chosen to maximize $-\sum_{i=1}^{n} U_i\left(\underline{\theta}_i\right) \leq 0$. Rearrange the objective can be simplified to

$$U_0\left(\theta_0\right)=\int_{\Theta}\left[\sum_{i=1}^{n}\left(\theta_i-\frac{1-\Phi_i\left(\theta_i\right)}{\phi_i\left(\theta_i\right)}-\theta_0\right) q_i(\theta)\right] \phi(\theta) d\theta+\theta_0.$$

where we can drop t_i from the problem. Since θ_0 is constant, if q_i is chosen to maximize

$$\int_{\Theta}\left[\sum_{i=1}^{n}\left(\theta_i-\frac{1-\Phi_i\left(\theta_i\right)}{\phi_i\left(\theta_i\right)}-\theta_0\right) q_i(\theta)\right] \phi(\theta) d\theta$$

subject to PR and nondecreasing, which are the only constraints regarding q_i. Then the solution is feasible and maximizes the objective. □

Corollary 3 (The Revenue-Equivalence Theorem). . *The seller's expected utility from a feasible auction mechanism is completely determined by the probability functions q_i and the numbers $U_i\left(\underline{\theta}_i\right)$ of each $i=1,\ldots,n$*

Regular Case

Definition 37. *The problem is regular if for each i*

$$w_i\left(\theta_i\right)=\theta_i-\frac{1-\Phi_i\left(\theta_i\right)}{\phi_i\left(\theta_i\right)}$$

is strictly monotone.

- Consider the following auction mechanism:
- Seller keeps the object if $\theta_0>\max_i\left(w_i\left(\theta_i\right)\right)$
- Otherwise, give it to $i^*=\arg\min\left\{i \mid w_i\left(\theta_i\right)=\max_j\left(w_j\left(\theta_j\right)\right)\right\}$ Set $t_i(\theta)=\theta_i q_i(\theta)-\int_{\underline{\theta}_i}^{\theta_i} q_i\left(\theta_i, \theta_{-i}\right) d\theta_i$

Theorem 13. ***The auction mechanism (q,t) is optimal.***

Proof. By the construction,

$$q_i\left(\theta_i\right)>0 \Rightarrow w_i\left(\theta_i\right)=\max_j\left(w_j\left(\theta_j\right)\right)$$

Therefore, q maximizes $\sum_{i=1}^{n} (w_i(\theta_i) - \theta_0) q_i(\theta)$, subject to PR (4), and hence the objective. Moreover, to see that $\bar{q}_i$ is non-decreasing, we shall see that $q_i(\theta_i, \theta_{-i})$ is non-decreasing in θ_i for all θ_{-i}. $\theta_i \le \theta_i' \Rightarrow w_i(\theta_i) \le w_i(\theta_i')$ since w_i is nondecreasing, and suppose, for contradiction, $q_i(\theta_i, \theta_{-i}) > q_i(\theta_i', \theta_{-i})$. Then, it must the case that $q_i(\theta_i', \theta_{-i}) = 0$ and $q_i(\theta_i, \theta_{-i}) = 1$ $q_i(\theta_i, \theta_{-i}) = 1 \Rightarrow w_i(\theta_i) = \max_j (w_j(\theta_j))$ and $i = i^*$. However, $q_i(\theta_i', \theta_{-i}) = 0 \Rightarrow w_i(\theta_i') < \max_j (w_j(\theta_j))$ or $i > i^*$, a contradiction. $\square$

- To see $t_i(\theta)$ intuitively,
- Define $z_i(\theta_{-i}) = \inf \{\theta_i \mid w_i(\theta_i) \ge \theta_0 \text{ and } w_i(\theta_i) \ge w_j(\theta_j)\, \forall j\}$, which is the minimum possible winning bid given θ_0 and θ_{-i} for i.
- Then, we can define

$$q_i(\theta) = \begin{cases} 1 & \text{if } \theta_i > z_i(\theta_{-i}) \\ 0 & \text{if } \theta_i < z_i(\theta_{-i}) \end{cases}$$

- Then,

$$\int_{\underline{\theta}_i}^{\theta_i} q_i(\theta_i, \theta_{-i})\, d\theta_i = \begin{cases} \theta_i - z_i(\theta_{-i}) & \text{if } \theta_i \ge z_i(\theta_{-i}) \\ 0 & \text{if } \theta_i < z_i(\theta_{-i}) \end{cases}$$

- Finally,

$$t_i(\theta) = \begin{cases} z_i(\theta_{-i}) & \text{if } q_i(\theta) = 1 \\ 0 & \text{if } q_i(\theta) = 0 \end{cases}$$

- If the bidders are symmetric, i.e. $\Theta_i = [\underline{\theta}, \bar{\theta}]$ and $\phi_i(\theta_i) = \phi(\theta_i)$ for all $i = 1, \ldots, n$, then,

$$z_i(\theta_{-i}) = \max \left\{ w_i^{-1}(\theta_0), \max_{j \neq i} \theta_j \right\}$$

- that is, a modified Vickrey auction where the seller submits a bid or reserved price $w_i^{-1}(\theta_0)$.

Example 8. *$\underline{\theta}_i = 0, \bar{\theta}_i = 100$, and $\phi_i(\theta_i) = \frac{1}{100}$ for all i. Then,*

$$w_i(\theta_i) = \theta_i - \frac{1 - \frac{\theta_i}{100}}{100} = 2\theta_i - 100$$

Suppose, $\theta_0 = 0$, then, the reserved price of the seller is 50. And the risk of keeping the object is $\left(\frac{1}{2}\right)^n$.

Example 9. *For asymmetric bidders, the object may not go to the highest bidder. For example,* $\phi_i(\theta_i) = \frac{1}{\bar{\theta}_i - \underline{\theta}_i}$ *and* $\theta_0 = 0$. *Then,* $w_i(\theta_i) = 2\theta_i - \bar{\theta}_i$. *Therefore, even though* $\theta_i > \theta_j$, *if* $\bar{\theta}_i \ll \bar{\theta}_j$, *then, it is possible that* $2\theta_i - \bar{\theta}_i < 2\theta_j - \bar{\theta}_j$ *so that i could not win the bid.*

In general, the optimal auction may not be ex post efficient.

Example 10. *Interpretation in one buyer case If* $n = 1$, *then set*

$$q_1^*(\theta_1) = \begin{cases} 1 & \text{if } w_1^*(\theta_1) \geq \theta_0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$t_1^*(\theta_1) = q_1^*(\theta_1) \cdot \min\{s_1 : w_1^*(s_1) \geq \theta_0\}$$

- *That is, the seller offer to sell the object at the price* $w_1^{*-1}(\theta_0) = \min\{s_1 : w_1^*(s_1) \geq \theta_0\}$.
- *If i is the only bidder who submit value* θ_i, *then the seller is willing to sell it if and only* $w_i^*(\theta_i)$ *is greater than* θ_0

4.1 Nonlinear pricing

Problem Set-up

- One buyer with private information of preference $\theta \sim F$ on $[\underline{\theta}, \bar{\theta}]$ with pdf $f > 0$.
- The buyer's preference is $u(x, t, \theta) = v(x, \theta) - t$ where x is number of goods and t is the payment.
- Assume v has the following properties:
  1. $v(0, \theta) = 0$ for all θ
  2. strictly increasing and strictly concave in x, and twice differentiable: $\frac{\partial v}{\partial x} > 0$ and $\frac{\partial^2 v}{\partial x^2} < 0$
  3. single-cross property (SCP) of v, i.e. $\frac{\partial^2 v}{\partial x \partial \theta} > 0$
- The marginal cost of producing one unit of good is constant c

Lemma 4. *Single-cross property implies that* $\frac{\partial v}{\partial x} > 0$ *and* $\frac{\partial v}{\partial \theta} > 0$.

Proof.

$$v(x, \theta) = v(0, \underline{\theta}) + \int_{\underline{\theta}}^{\theta} \int_0^x \frac{\partial^2 v}{\partial x \partial \theta}(s, t) ds dt = \int_{\underline{\theta}}^{\theta} \int_0^x \frac{\partial^2 v}{\partial x \partial \theta}(s, t) ds dt \text{ . Therefore,}$$

$$\frac{\partial v}{\partial \theta}(x, \theta) = \int_0^x \frac{\partial^2 v}{\partial x \partial \theta}(s, \theta) ds > 0$$

□

Notations

- $q(\theta)$: amount of non-money allocation for θ;
- $t(\theta)$: amount of money;
- $U(\theta) = v(q(\theta), \theta) - t(\theta)$

The monopoly problem

$$\begin{array}{ll} \max_{q,t} & E[\pi(\theta)] = \int_{\underline{\theta}}^{\bar{\theta}} [t(\theta) - cq(\theta)] f(\theta) d\theta \\ \text{s.t.} & U(\theta) \geq 0 \\ & U(\theta') \geq v(q(\theta), \theta') - t(\theta) \\ & U(\theta) \geq v(q(\theta'), \theta) - t(\theta') \\ \forall \theta & \\ & \forall \theta, \theta' \\ & \forall \theta, \theta' \end{array}$$

Lemma 5. *The mechanism (q, t) is feasible (IR & IC)* $\iff$

1. *q is monotone increasing,*
2. $U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial v}{\partial \theta}(q(\tau), \tau) d\tau,$
3. $U(\underline{\theta}) \geq 0$

Proof. $\Rightarrow$ Note that

$$\begin{aligned} U(\theta) &\geq v\left(q\left(\theta'\right), \theta\right) - t\left(\theta'\right) \\ &= v\left(q\left(\theta'\right), \theta\right) - v\left(q\left(\theta'\right), \theta'\right) + v\left(q\left(\theta'\right), \theta'\right) - t\left(\theta'\right) \\ &= v\left(q\left(\theta'\right), \theta\right) - v\left(q\left(\theta'\right), \theta'\right) + U\left(\theta'\right) \\ \Leftrightarrow U(\theta) - U\left(\theta'\right) &\geq v\left(q\left(\theta'\right), \theta\right) - v\left(q\left(\theta'\right), \theta'\right) \end{aligned}$$

Similarly,

$$\begin{aligned} U(\theta') &\geq v(q(\theta), \theta') - t(\theta) \\ \Leftrightarrow U(\theta') - U(\theta) &\geq v(q(\theta), \theta') - v(q(\theta), \theta) \end{aligned}$$

Therefore,

$$\begin{aligned} v\left(q\left(\theta'\right), \theta'\right) - v\left(q\left(\theta'\right), \theta\right) \geq U\left(\theta'\right) - U(\theta) &\geq v\left(q(\theta), \theta'\right) - v(q(\theta), \theta) \\ \Leftrightarrow \int_{\theta}^{\theta'} \frac{\partial v}{\partial \theta}\left(q\left(\theta'\right), \tau\right) d\tau \geq U\left(\theta'\right) - U(\theta) &\geq \int_{\theta}^{\theta'} \frac{\partial v}{\partial \theta}(q(\theta), \tau) d\tau \\ \Rightarrow \int_{\theta}^{\theta'} \frac{\partial v}{\partial \theta}\left(q\left(\theta'\right), \tau\right) d\tau - \int_{\theta}^{\theta'} \frac{\partial v}{\partial \theta}(q(\theta), \tau) d\tau &\geq 0 \\ \Leftrightarrow \int_{\theta}^{\theta'} \int_{q(\theta)}^{q(\theta')} \frac{\partial^2 v}{\partial x \partial \theta}(q, \tau) dq d\tau &\geq 0 \end{aligned}$$

By SCP of v, the above inequality implies that $\theta \geq \theta' \Rightarrow q(\theta) \geq q(\theta')$, i.e. q is monotone. Moreover, by Envelop Theorem,

$$\begin{aligned} U(\theta) &\geq v\left(q\left(\theta'\right),\theta\right) - t\left(\theta'\right) \forall \theta' \\ \Rightarrow U'(\theta) &= \frac{\partial v}{\partial \theta}(q(\theta),\theta) \text{ a.e.} \end{aligned}$$

Therefore, $U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial v}{\partial \theta}(q(\tau),\tau)d\tau$. Finally, to show IC, $\forall \theta, U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial v}{\partial \theta}(q(\tau),\tau)d\tau \geq 0$ by previous lemma and $U(\underline{\theta}) \geq 0$.

$\Leftarrow$ Clearly, IR is satisfied. Suppose, for contradiction, not IC. Then, wlog $\exists \theta > \theta'$:

$$U(\theta) < v\left(q\left(\theta'\right),\theta\right) + t\left(\theta'\right)$$

Then, LHS is:

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial v}{\partial \theta}(q(\tau),\tau)d\tau$$

and the RHS is:

$$\begin{aligned} & v\left(q(\theta),\theta'\right) - t(\theta) \\ =& v\left(q\left(\theta'\right),\theta\right) - v\left(q\left(\theta'\right),\theta'\right) + U\left(\theta'\right) \\ =& v\left(q\left(\theta'\right),\theta\right) - v\left(q\left(\theta'\right),\theta'\right) + U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta'} \frac{\partial v}{\partial \theta}(q(\tau),\tau)d\tau. \end{aligned}$$

Therefore, we have:

$$\begin{aligned} & U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial v}{\partial \theta}(q(\tau),\tau)d\tau \\ & \quad < v\left(q\left(\theta'\right),\theta\right) - v\left(q\left(\theta'\right),\theta'\right) + U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta'} \frac{\partial v}{\partial \theta}(q(\tau),\tau)d\tau \\ \Rightarrow & \int_{\theta'}^{\theta} \frac{\partial v}{\partial \theta}(q(\tau),\tau)d\tau < \int_{\theta'}^{\theta} \frac{\partial v}{\partial \theta}\left(q\left(\theta'\right),\tau\right)d\tau \\ \Rightarrow & \int_{\theta'}^{\theta} \int_{q(\theta')}^{q(\tau)} \frac{\partial^2 v}{\partial x \partial \theta}(q,\tau)dqd\tau < 0 \end{aligned}$$

By SCP of v, the above inequality implies that $\exists \tau \in (\theta',\theta] : q(\tau) < q\left(\theta'\right)$, contradiction to monotonicity. □

By the above Lemma, we could rewrite the monopoly's problem, substituting $t(\theta)$ and $U(\theta)$, as follow:

$$\begin{aligned} & \max_{q,t} E[\pi(\theta)] = \int_{\underline{\theta}}^{\bar{\theta}} \left[v(q(\theta),\theta) - U(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \tfrac{\partial v}{\partial \theta}(q(\tau),\tau)d\tau - cq(\theta) \right] f(\theta)d\theta \\ & \text{s.t. } U(\underline{\theta}) \geq 0 \end{aligned}$$

q is monotone.

Using the similar trick,

$$\begin{aligned}
&\int_{\underline{\theta}}^{\bar{\theta}} \left(\int_{\underline{\theta}}^{\theta} \frac{\partial v}{\partial \theta}(q(\tau), \tau) d\tau \right) f(\theta) d\theta \\
=& \int_{\underline{\theta}}^{\bar{\theta}} \int_{\tau}^{\bar{\theta}} \frac{\partial v}{\partial \theta}(q(\tau), \tau) f(\theta) d\theta d\tau \\
=& \int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial v}{\partial \theta}(q(\tau), \tau)[1 - F(\tau)] d\tau
\end{aligned}$$

Therefore,

$$E[\pi(\theta)] = \int_{\underline{\theta}}^{\bar{\theta}} \left[v(q(\theta), \theta) - U(\underline{\theta}) - \frac{\partial v}{\partial \theta}(q(\theta), \theta) \frac{1 - F(\theta)}{f(\theta)} - cq(\theta) \right] f(\theta) d\theta$$

Pointwise FOC:

$$\frac{\partial v}{\partial q}(q(\theta), \theta) - c - \frac{1 - F(\theta)}{f(\theta)} \frac{\partial^2 v}{\partial x \partial \theta}(q(\theta), \theta) = 0$$

If there is no distribution at top, $\frac{1-F(\bar{\theta})}{f(\bar{\theta})} = 0$, then no distortion for the top, i.e. $\frac{\partial v}{\partial q}(q(\bar{\theta}), \bar{\theta}) = c$.

5 Static Mirrlees taxation

Standard assumption in the Ramsey literature is that lump sum taxes are not allowed. Why aren't lump sum taxes used in practice? One reason for this is they require truthful elicitation of agents characteristics, which might not be publicly observable. Moreover, agents might not have an incentive to reveal these characteristics truthfully. Margaret Thatcher tried in 1989 and it didn't work out well (especially for her political career). We will next consider a mechanism design problem in which agents true ability types are private and allow the designer to use arbitrary mechanisms and transfer schedules to achieve efficiency. Next, will consider implementations/decentralizations.

5.1 A Two Type Example

Consider an environment with a continuum of HHs characterized by a productivity level $\theta \in \Theta = \{\theta_{\mathrm{H}}, \theta_{\mathrm{L}}\}$ with $\theta_{\mathrm{H}} > \theta_{\mathrm{L}}$. A household of type θ who works l hours can produce $y = \theta l$ of output. Let $\pi(\theta)$ denote the probability that given household is of type θ. By the LLN, this is also the fraction of HHs with productivity θ. Household preferences are given by $u(c) - v(l)$ but we will use $l = \frac{y}{\theta}$ to define the preferences as $\mathrm{U}(\mathrm{c}, \mathrm{y}, \theta) = \mathrm{u}(\mathrm{c}) - v\left(\frac{\mathrm{y}}{\theta}\right)$ Suppose first that HH productivities are public information. Under full information a utilitarian planner (cares about all types equally) solves

$$\max_{c(\theta), y(\theta)} \pi\left(\theta_H\right)\left[u\left(c\left(\theta_H\right)\right) - v\left(\frac{y\left(\theta_H\right)}{\theta_H}\right)\right] + \pi\left(\theta_L\right)\left[u\left(c\left(\theta_L\right)\right) - v\left(\frac{y\left(\theta_L\right)}{\theta_L}\right)\right]$$

subject to

$$\pi\left(\theta_{\mathrm{H}}\right) \mathrm{c}\left(\theta_{\mathrm{H}}\right) + \pi\left(\theta_{\mathrm{L}}\right) c\left(\theta_{\mathrm{L}}\right) \leqslant \pi\left(\theta_{\mathrm{H}}\right) \mathrm{y}\left(\theta_{\mathrm{H}}\right) + \pi\left(\theta_{\mathrm{L}}\right) \mathrm{y}\left(\theta_{\mathrm{L}}\right)$$

Let μ be the multiplier on the RC. Then the the focs imply

$$\begin{gathered} u'(c(\theta)) = \tfrac{1}{\theta} v'(l(\theta)) \\ c\left(\theta_H\right) = c\left(\theta_L\right) \\ \tfrac{v'\left(l\left(\theta_H\right)\right)}{v'\left(l\left(\theta_L\right)\right)} = \tfrac{\theta_H}{\theta_L} > 1 \end{gathered}$$

where the last equation implies that $l\left(\theta_H\right) > l\left(\theta_L\right)$ since v is convex. Now suppose that θ is private information. It is easy to see that the above allocation is not incentive compatible. A high type households strictly prefers to pretend to be a low type since the consumption levels are the same but hours worked is lower. From the revelation principle, that we can restrict ourselves to direct revelation mechanisms.

Definition 38 (Direct revelation mechanism)**.** *consists of action/message sets $A_i, i \in [0,1]$ such that for each i, $A_i = \Theta_i$ and outcome functions (c, y) where $c, y : \Theta \to \mathbb{R}_+$.*

Since there is no aggregate uncertainty (LLN), we will consider mechanisms that treat households anonymously, i.e. mechanisms that are independent of i.

Definition 39 (Revelation mechanism)**.** *is*

1. *Incentive compatible if and only if*

$$u(c(\theta)) - v\left(\frac{y(\theta)}{\theta}\right) \geqslant u(c(\hat{\theta})) - v\left(\frac{y(\hat{\theta})}{\theta}\right) \text{ for all } \theta, \hat{\theta} \in \Theta \tag{1}$$

2. *Resource feasible if and only if*

$$\pi(\theta_{\mathrm{H}})\, c(\theta_{\mathrm{H}}) + \pi(\theta_{\mathrm{L}})\, c(\theta_{\mathrm{L}}) \leqslant \pi(\theta_{\mathrm{H}})\, \mathrm{y}(\theta_{\mathrm{H}}) + \pi(\theta_{\mathrm{L}})\, \mathrm{y}(\theta_{\mathrm{L}}) \tag{2}$$

Then, the Planner's/Mechanism designer's problem is

$$\max_{\mathfrak{c}(\theta), \mathfrak{f}(\theta)} \pi(\theta_{\mathrm{H}}) \left[\mathrm{u}(\mathrm{c}(\theta_{\mathrm{H}})) - v\left(\frac{\mathrm{y}(\theta_{\mathrm{H}})}{\theta_{\mathrm{H}}}\right)\right] + \pi(\theta_{\mathrm{L}}) \left[\mathrm{u}(\mathrm{c}(\theta_{\mathrm{L}})) - v\left(\frac{\mathrm{y}(\theta_{\mathrm{L}})}{\theta_{\mathrm{L}}}\right)\right]$$

subject to (1) and (2) Notice that there are two incentive compatibility constraints, one for the high type and one for the low type. We will first consider a "relaxed problem" in which we drop the incentive constraint for the low type. We will show in the resulting mechanism that this constraint is indeed slack.

5.2 The relaxed problem

$$\max_{\mathfrak{c}(\theta), y(\theta)} \pi(\theta_H) \left[u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right)\right] + \pi(\theta_L) \left[u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right)\right]$$

subject to

$$u(c(\theta_H)) - v\left(\tfrac{y(\theta_H)}{\theta_H}\right) \geqslant u(c(\theta_L)) - v\left(\tfrac{y(\theta_L)}{\theta_H}\right)$$
$$\pi(\theta_H)\, c(\theta_H) + \pi(\theta_L)\, c(\theta_L) \leqslant \pi(\theta_H)\, y(\theta_H) + \pi(\theta_L)\, c(\theta_L)$$

Let λ be the multiplier on the first constraint and μ on the second. The focs are

$$\pi(\theta_{\mathrm{H}})\, u'(c(\theta_{\mathrm{H}})) + \lambda u'(c(\theta_{\mathrm{H}})) - \pi(\theta_{\mathrm{H}})\, \mu = 0 \tag{3}$$

$$\pi\left(\theta_{\mathrm{L}}\right)u'\left(c\left(\theta_{\mathrm{L}}\right)\right)-\lambda u'\left(c\left(\theta_{\mathrm{L}}\right)\right)-\pi\left(\theta_{\mathrm{L}}\right)\mu=0 \tag{4}$$

$$-\frac{\pi\left(\theta_{\mathrm{H}}\right)}{\theta_{\mathrm{H}}}v'\left(\frac{\mathrm{y}\left(\theta_{\mathrm{H}}\right)}{\theta_{\mathrm{H}}}\right)-\lambda\frac{1}{\theta_{\mathrm{H}}}v'\left(\frac{\mathrm{y}\left(\theta_{\mathrm{H}}\right)}{\theta_{\mathrm{H}}}\right)+\pi\left(\theta_{\mathrm{H}}\right)\mu=0 \tag{5}$$

$$-\frac{\pi\left(\theta_{\mathrm{L}}\right)}{\theta_{\mathrm{L}}}v'\left(\frac{\mathrm{y}\left(\theta_{\mathrm{L}}\right)}{\theta_{\mathrm{L}}}\right)+\lambda\frac{1}{\theta_{\mathrm{H}}}v'\left(\frac{\mathrm{y}\left(\theta_{\mathrm{L}}\right)}{\theta_{\mathrm{H}}}\right)+\pi\left(\theta_{\mathrm{L}}\right)\mu=0 \tag{6}$$

Combining (3) and 5) we obtain

$$u'\left(c\left(\theta_H\right)\right)=\frac{1}{\theta_H}v'\left(\frac{y\left(\theta_H\right)}{\theta_H}\right)$$

This says that the just as in the unconstrained problem, for the high type, the marginal utility of consumption equals the marginal disutility of working. In particular, the allocation for the high type households is ex-post efficient. This is sometimes referred to as "no-distortion at the top". The mechanical reason for this is that, no type wants to pretend to be the high type and thus the planner does not need to distort his allocation. This will not be true for the low type. Next, if we combine (3) and 4) we obtain

$$\frac{u'\left(c\left(\theta_H\right)\right)}{u'\left(c\left(\theta_L\right)\right)}=\frac{\pi\left(\theta_H\right)}{\pi\left(\theta_L\right)}\frac{\left[\pi\left(\theta_L\right)-\lambda\right]}{\left[\pi\left(\theta_H\right)+\lambda\right]}=\frac{\pi\left(\theta_H\right)\pi\left(\theta_L\right)-\lambda\pi\left(\theta_H\right)}{\pi\left(\theta_H\right)\pi\left(\theta_L\right)+\lambda\pi\left(\theta_L\right)}<1$$
$$\Rightarrow u'\left(c\left(\theta_H\right)\right)<u'\left(c\left(\theta_L\right)\right)\quad\Rightarrow c\left(\theta_H\right)>c\left(\theta_L\right)$$

But then given the IC holds with equality, it must be that

$$v\left(\frac{\mathrm{y}\left(\theta_{\mathrm{H}}\right)}{\theta_{\mathrm{H}}}\right)>v\left(\frac{\mathrm{y}\left(\theta_{\mathrm{L}}\right)}{\theta_{\mathrm{H}}}\right)$$
$$\Rightarrow \mathrm{y}\left(\theta_{\mathrm{H}}\right)>\mathrm{y}\left(\theta_{\mathrm{L}}\right)$$

To see the allocation for the low type is distorted, combine (4) and 6) to obtain

$$\begin{aligned}
u'\left(c\left(\theta_L\right)\right)&=\frac{1}{\theta_L}v'\left(\frac{y\left(\theta_L\right)}{\theta_L}\right)+\frac{\lambda}{\pi\left(\theta_L\right)}\left[u'\left(c\left(\theta_L\right)\right)-\frac{1}{\theta_H}v'\left(\frac{y\left(\theta_L\right)}{\theta_H}\right)\right]\\
&>\frac{1}{\theta_L}v'\left(\frac{y\left(\theta_L\right)}{\theta_L}\right)+\frac{\lambda}{\pi\left(\theta_L\right)}\left[u'\left(c\left(\theta_H\right)\right)-\frac{1}{\theta_H}v'\left(\frac{y\left(\theta_H\right)}{\theta_H}\right)\right]\\
&=\frac{1}{\theta_L}v'\left(\frac{y\left(\theta_L\right)}{\theta_L}\right)
\end{aligned}$$

Finally, we will check that that the solution to the relaxed problem satisfies the incentive constraint for the low type. Suppose not and that this constraint was violated.

Then

$$
\begin{aligned}
& u\left(c\left(\theta_L\right)\right)-v\left(\frac{y\left(\theta_L\right)}{\theta_L}\right)<u\left(c\left(\theta_H\right)\right)-v\left(\frac{y\left(\theta_H\right)}{\theta_L}\right) \\
\Longrightarrow & v\left(\frac{y\left(\theta_H\right)}{\theta_L}\right)-v\left(\frac{y\left(\theta_L\right)}{\theta_L}\right)<u\left(c\left(\theta_H\right)\right)-u\left(c\left(\theta_L\right)\right) \\
\Longrightarrow & \frac{1}{\theta_L} \int_{y\left(\theta_L\right)}^{y\left(\theta_H\right)} v^{\prime}\left(\frac{y}{\theta_L}\right) d y<u\left(c\left(\theta_H\right)\right)-u\left(c\left(\theta_L\right)\right) \\
\Longrightarrow & \frac{1}{\theta_H} \int_{y\left(\theta_L\right)}^{y\left(\theta_H\right)} v^{\prime}\left(\frac{y}{\theta_H}\right) d y<\frac{1}{\theta_L} \int_{y\left(\theta_L\right)}^{y\left(\theta_H\right)} v^{\prime}\left(\frac{y}{\theta_L}\right) d y<u\left(c\left(\theta_H\right)\right)-u\left(c\left(\theta_L\right)\right) \\
\Longrightarrow & v\left(\frac{y\left(\theta_H\right)}{\theta_H}\right)-v\left(\frac{y\left(\theta_L\right)}{\theta_H}\right)<u\left(c\left(\theta_H\right)\right)-u\left(c\left(\theta_L\right)\right) \\
\Longrightarrow & u\left(c\left(\theta_L\right)\right)-v\left(\frac{y\left(\theta_L\right)}{\theta_H}\right)<u\left(c\left(\theta_H\right)\right)-v\left(\frac{y\left(\theta_H\right)}{\theta_H}\right)
\end{aligned}
$$

which contradicts the IC for the high type holding with equality. And so the solution to the relaxed problem is the solution to the original constrained problem.

5.3 Don't tax at the top

Above tells us nothing about implementation, i.e. whether there exist tax systems, for example such that the equilibrium, given the tax system gives efficient allocation. We turn to this next. In particular, we will show that a non-linear income tax schedule can implement the efficient allocation.

Denote the optimal mechanism by (c^*, y^*). Define a tax function $T(y) = y - c$ if $\mathrm{y} \in \{\mathrm{y}^*(\theta_\mathrm{H}), \mathrm{y}^*(\theta_\mathrm{L})\}$ and $\mathrm{T(y)} = \mathrm{y}$ otherwise. Given this tax function, the household of type θ solves:

$$
\max_{c,y} u(c) - v\left(\frac{y}{\theta}\right)
$$

subject to

$$
c \leqslant \mathrm{y} - \mathrm{T(y)}
$$

The first order condition is

$$
u'(c)\left(1 - T'(y)\right) = \frac{1}{\theta} v'\left(\frac{y}{\theta}\right)
$$

Comparing this equation to the relevant one in the planning problem implies that $T'\left(y_H^*\right) = 0$ and $\mathrm{T}'\left(\mathrm{y}_\mathrm{L}^*\right) > 0$

6 Dynamic Contracting

7 Moral Hazard

7.1 Observed effort

7.2 Unobservable effort

7.3 More actions

7.4 Doubly relaxed problem

7.5 Repeated Partnerships

7.6 Frequent actions

8 Informational Frictions in markets

8.1 Akerlof's market for lemons

- Buyer's valuation:
$$v = \begin{cases} 1 & \text{if peach} \\ 0 & \text{if lemon} \end{cases}$$
- Seller: π fraction are peach, and $(1-\pi)$ are lemon where $0 < \pi < 1$.
- The opportunity cost for seller:
$$c = \begin{cases} \frac{1}{2} & \text{if peach} \\ 0 & \text{if lemon} \end{cases}$$
- The market price is p
- So the buyer's expected payoff is $E(v \mid \text{sale}) = \begin{cases} \pi & \text{if } p \geq \frac{1}{2} \\ 0 & \text{if } p < \frac{1}{2} \end{cases}$.
- If $\pi \geq \frac{1}{2}$, then there will be some trade only because there exist so many peaches;
- If $\pi < \frac{1}{2}$, then there is no trade, and market breaks down.

8.2 Spence's signaling

A worker chooses education level $e \geq 0$ with private cost e/θ where θ is private type and the same as productivity.

- Competitive firm set wage at $w(e) = E(\theta \mid e)$.
- Two types of workers: θ', θ'' s.t. $0 < \theta' < \theta''$ with probability p' and $p'' = 1 - p'$.
- Let σ' and σ'' be some strategies for θ' and θ''

Lemma 6. *If* $\Pr(e' \mid \sigma') > 0$ *and* $\Pr(e'' \mid \sigma'') > 0$, *then* $e' \leq e''$

Proof.

$$\left.\begin{array}{r} w(e') - e'/\theta' \geq w(e'') - e''/\theta' \\ w(e'') - e''/\theta'' \geq w(e') - e'/\theta'' \end{array}\right\} \Rightarrow e''(1/\theta' - 1/\theta'') \geq e'(1/\theta' - 1/\theta'').$$

□

The separating equilibria: so wage is set at θ:

- Type θ' reveal his type and receive wage θ' and choose $e' = 0$;
- Type θ'' chooses e'' and receive wage θ'', for $(e' = 0, e'')$ being a separating equilibrium: $-\theta'' - e''/\theta'' \geq \theta' \Rightarrow e'' \leq \theta''(\theta'' - \theta')$ and;

$$\theta' \geq \theta'' - e''/\theta' \Rightarrow e'' \geq \theta'\left(\theta'' - \theta'\right)$$

- Conversely, suppose $e'' \in [\theta'(\theta'' - \theta'), \theta''(\theta'' - \theta')]$, consider the belief $\Pr(\theta' \mid e) = \begin{cases} 1 & \text{if } e \neq e'' \\ 0 & \text{if } e = e'' \end{cases}$.
- So we have a continuum of separating equilibria.
- The pooling equilibrium: let $\tilde{e} = e' = e''$ and wage is set at $w(\tilde{e}) = p'\theta' + p''\theta''$.
- The belief to support it is $\Pr(\theta' \mid e) = 1$ if $e \neq \tilde{e}$
- So, for $\tilde{e}$ to be a pooling equilibrium:

$$-\theta' \leq p'\theta' + p''\theta'' - \tilde{e}/\theta' \Rightarrow \tilde{e} \leq p''\theta'\left(\theta'' - \theta'\right)$$

- Note that $\theta' < \theta'' \Rightarrow \theta'' < p'\theta' + p''\theta'' < p'\theta' + p''\theta'' - \tilde{e}/\theta''$

8.3 Beer-quiche game

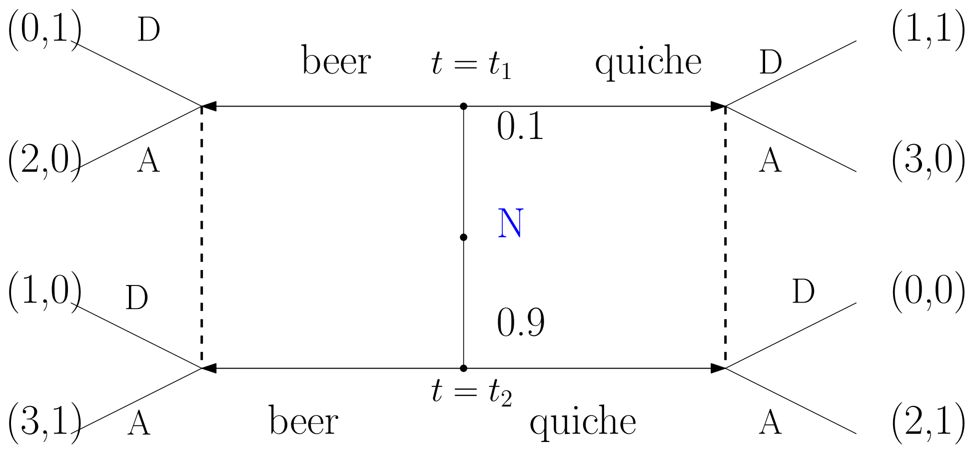

Figure 4: Beer Quiche Game- Cho, Kreps 1984

- There are two pooling equilibria: .9 type s and .1 type w
- Player 1 chooses beer, and Player 2 chooses "Avoid" if beer and chooses "Duel" if quiche, and player 2 believes $\Pr\left(t_w \mid \text{ quiche }\right) = 1$
- Same but beer and quiche are reversed.
- The separating equilibrium: type w chooses beer, and type s chooses quiche;
- Hybrid equilibrium for .1 type s and .9 type $-w$: $1/9$ type w chooses beer and all the type s chooses beer, $8/9$ type w choose quiche,
- player 2 has the correct belief and choose $\left(\frac{1}{2}, \frac{1}{2}\right)$ if beer and dual if quiche.

8.4 Rothschild's and Stiglitz' insurance markets adverse selection

8.5 Grossman's Stiglitz' informational efficiency

8.6 Kyle's information aggregation

8.7 Leland's and Pyle's CAPM

9 Bargaining

9.1 Nash solution

Definition 40. *Two person bargaining* (F, v)

$$F \subset \mathbb{R} : F \cap \{(x_1, x_2) : x_1 \geq v_1, x_2 \geq v_2\} \neq \emptyset, bounded, convex, closed$$

- *F- set of feasible payoff allocations*
- *v- disagreement point*
- (F, v) *is essential* $\iff \exists y \in F : y_1 > v_1, y_2 > v_2$

$\varphi(F, v)$ solution of Nash Bargaining problem satisfy following axioms

Definition 41. *Strong Pareto Efficiency*

$$x \in F if \neg \exists y \in F \quad y \geq x \wedge y_i > x_i for\ some\ i$$

Definition 42. *Weakly Pareto Efficient*

$$z \in F \quad if \quad \neg \exists y \in F \quad y \geq z$$

Definition 43. *Strong Efficiency*

$$\varphi(F, v) \in F x \in F \quad x \geq \varphi(F, v) \quad \Rightarrow \quad x = \varphi(F, v)$$

Definition 44. *Individual Rationality*

$$\varphi(F, v) \geq v$$

Definition 45. *Scale Covariance*

$$\forall \lambda_1, \lambda_2 > 0 \mu_1, \mu_2 \in R \quad w = (\lambda_1 x_1 + v_1, \lambda_2 x_2 + v_2) \quad G = \{(\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2) x \in F\}$$

then

$$\varphi(G, w) = (\lambda_1 \varphi_1(F, v) + \mu_1, \lambda_2 \varphi_2(F, v) + \mu_2)$$

Definition 46. *Independent of Irrelevant Alternatives (IIA) For any closed convex* $G \subset F$

$$if\ G \subset F, \varphi(F, v) \in G \quad \Rightarrow \quad \varphi(F, v) = \varphi(G, v)$$

Definition 47. *Symmetry*

$$if\ v_1 = v_2 \quad F\ symmetric \Rightarrow \varphi_1(F,v) = \varphi_2(F,v)$$

Theorem 14. ***Nash Let** (F,v) **be 2 person bargaining problem** $\varphi(F,v) \in F$ **it's unique solution satisfying SE, IR,SC, IIA and S** $\iff$*

$$\varphi(F,v) \in \arg\max_{y \in F, y \geq v} (y_1 - v_1) \cdot (y_2 - v_2)$$

Proof. TBD □

Example 11. $\Gamma = \{(1,2), c_1, c_2, u_1, u_2\}$

$$F = \{(u_1(\mu), u_2(\mu))\mu\Delta(C)\} \quad u_i(\mu) = \sum_{c \in C} \mu(c) u_i(c)$$

if there is a moral hazard-so no regulation by contracts is possible then

$$F = \{(u_1(\mu), u_2(\mu)) \quad \mu \quad is\ coreelated\ equilibrium\ of\ \Gamma$$

How to pick v?

a) min max $v_1 = \min_{\sigma_2 \in \Delta(C_2)} \max_{\sigma_1 \in \Delta(C_1)} u_1(\sigma_1, \sigma_2)$

b) $(\sigma_1, \sigma_2$ focal equilibrium then $v_i = u_i(\sigma_1, \sigma_2$

c) rational threats

Solution of two bargaining problems

9.2 Interpersonal Comparision of Utilities

Consider two principles

- equal gain - egalitarian solution E^*
- greatest good - utilitarian solution U^*

E^* and U^* need not generally agree.

Let E^* select from (F,v) the unique point that is weakly efficient in F and

$$x_1 - v_1 = x_2 - v_2$$

U^* select from (F,v) x s.t.

$$x_1 + x_2 = \max_{y \in F} y_1 + y_2$$

z is λ-utilitarian solution if

$$\lambda_1(x_1 - v_1) = \lambda_2(x_2 - v_2)$$

Theorem 15. ***Suppose (F,v) is essential and let $x \in F$***

$$x = \varphi(F,v) \iff \exists \lambda > 0 \quad \lambda_1(x_1 - v_1) = \lambda_2(x_2 - v_2) \quad \textbf{\textit{and}} \quad \lambda_1 x_1 + \lambda_2 x_2 = \max_{y \in F} \lambda_1 y_1 + \lambda_2 y_2$$

Example 12. $v = (0,0)$ *Split £30 and P1 is Risk Neutral and P2 is Risk Averse.*

$$F = \{(y_1, y_2) : 0 \leq y_1 \leq 30, 0 \leq y_2 \leq (30 - y_1)^{\frac{1}{2}}\}$$

Nash solution

$$\frac{d}{dy_1}[y_1(30 - y_1)^{\frac{1}{2}}] = (30 - y_1)^{\frac{1}{2}} - \frac{y_1}{2(30 - y_1)^{\frac{1}{2}}} = 0$$

$$30 - y_1 = \frac{y_1}{2} \quad \Rightarrow y_1 = 20$$

$$(20, 10^{\frac{1}{2}}) = (20, 3.162)$$

$$-\frac{x_2}{x_1}\frac{\sqrt{10}}{20} = \frac{\lambda_1}{\lambda_2}$$

$$20\lambda_1 = \sqrt{10}\lambda_2 \quad \lambda_1 = 1\lambda_2 = \sqrt{40}$$

Consider the scaling factors λ_1, λ_2 as above. P2's utility from a monetary gain of £K is $\sqrt{40}K^{\frac{1}{2}}$ instead of $K^{\frac{1}{2}}$. P1 remains unchanged

$$G = \{(y_1, y_2) : 0 \leq y_1 \leq 30, 0 \leq y_2 \leq 6.325(30 - y_1)^{\frac{1}{2}}\}$$

For $(G,(0,0))$ Nash is $(20,20)$ which corresponds to $\$20, \$10$ both utilitarian and egalitarian

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9.3 Rubinstein (1982)

1. 2 players alternating makes decision, start from P1
2. P1 makes an offer (x_1, x_2) , P2 can choose to accept or reject:
   - if accept, game end;
   - if reject: with prob p_1 game end with disagreement and P2 gets v_2, P1 gets $w_1 \leq \max_{y \in F_v} y_1$; with prob $1 - p_1$, game continue.
3. If game continue, P2 makes an offer (y_1, y_2) , P1 A or R :
   - if A, game end;
   - if R, with p_2 game end with disagreement and P1 gets v_1, P2 gets $w_2 \leq \max_{y \in F_v} y_2$; with $1 - p_2$, game continue
4. Repeat 2 and 3