Recitations 9

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MIN II

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05/11/20

RECITATION 9

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ZOOM

Today: - Negishi, Arrow

Good luck tomorrow!

Recitations 9

[Definitions used today]

- Arrow problem, Negishi problem, Social Planner Problem, Pareto efficient allocation, Competetive Equilibrium

Question 1 [Midterm 2018]

Consider a pure exchange economy with two agents, $i \in \{1,2\}$ and two goods $m \in \{1,2\}$. There exists a total of 1 unit of each good. Agent 1's preference is represented by utility function $u_1(c_{1,1}, c_{1,2}) = \sqrt{c_{1,1}} + \sqrt{c_{1,2}}$. Agent 2 's preference is represented by the utility function $u(c_{2,1}, c_{2,2}) = 0$. Both agents have an unbounded ability to eat any non-negative amount of either good.

a) Sketch the utility possibilities frontier for this economy.

b) Set up the two "Arrow Problems" for this economy,

c) Are all Pareto efficient allocations a solution to each Arrow problem?

d) Are all solutions to each Arrow problem Pareto efficient? If so, prove why so. If not, argue why not.

e) Set up the class of "Negishi Problems" for this economy.

f) Are all Pareto efficient allocations a solution to a Negishi problem? (If so, which ones.)

g) Are all solutions to a Negishi problem Pareto efficient?

Question 2

Consider the following pure exchange economies with two agents (both of the agent consume nonnegative amount of goods):

1. 1 good world with total endowment $e = 4$. Person 1's utility function is $u_1(c_1) = c_1$, Person 2's utility function is $u_2(c_2) = [c_2]$.
2. 2 goods world with total endowment $e_1 = e_2 = 4$. Each person has the utility function $u_i(c_{i1}, c_{i2}) = c_{i1}^2 + c_{i2}^2$
3. 2 goods world with total endowment $e_1 = e_2 = 4$. Each person has the utility function $u_i(c_{i1}, c_{i2}) = \sqrt{c_{i1}} + \sqrt{c_{i2}}$. Agent 1 can eat any non-negative amount of both goods. Agent 1 however, cannot eat more than 1 unit of each good.

Answer following questions:

a) What is the utility possible set / frontier?

b) Show whether every solution to the Pareto Problem is Pareto efficient.

c) Show whether every Pareto efficient allocation is a solution to the Pareto problem.

d) Show whether every solution to the Negishi Problem is Pareto efficient.

e) Show whether every Pareto efficient allocation is a solution to the Negishi problem for some specification of Pareto weights λ_i

Question 3 [Prelim QII Fall 2020]

Consider the following pure exchange economies with two agents $i \in \{1, 2\}$ and two goods $m \in \{1, 2\}$. There exists a total of 25 units of each good in the economy. Agent 1 has preferences represented by utility $u_1(c_{11}, c_{12}) = \sqrt{c_{11}} + \sqrt{c_{12}}$. Agent 2 has preferences represented by utility $u_2(c_{21}, c_{22}) = \sqrt{\min\{c_{21}, 16\}} + \sqrt{\min\{c_{22}, 16\}}$. Both agents have an unbounded ability to eat any non-negative amount of wither good.

a) Carefully characterize the set of Pareto Efficient allocations for this economy and sketch utility possibility frontier for this economy

b) Set up Social Planner's (or Negishi Problem) for this economy

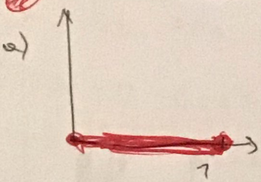
c) Are all Pareto Efficient allocations solutions to the Social Planner's problem? Are akk solutions to the Social Planner's problem Pareto Efficient? Explain

d) Define a Competetive Equilibrium and give the set of Competetive Equilibtria for this economy for all possible endowment specifications subject to the aggregate endowment beign 25 for each goood/ are they all Pareto Efficient? Are any Pareto Efficient. If not why not?

e) Can all Pareto Efficient Allocations in this environment be supported as a Competetive Equilibrium for some set of endowments (again where the aggregate endownment is 25 for each good)? If not, what assumption of the 2nd Welfare Theorem is violated?

Recitations 9

1. a)



b)

Arrow p6

① $\max \sqrt{c_{11}} + \sqrt{c_{12}}$

FEAS $\begin{cases} c_{11} + c_{21} \le 1 \\ c_{12} + c_{22} \le 1 \\ 0 \le c_{ij} \\ 0 \ge \bar{u}_2 \end{cases}$

② max 0

FEAS

$\sqrt{c_{11}} + \sqrt{c_{12}} \ge \bar{u}_1$

c) Yes There is only one PE $(1,1,0,0) = c$

Take $\bar{u}_2 = 0$ $\bar{u}_1 \ge 2$

- c feasible
- $(1,1)$ optimal for $\bar{u}_2 = 0$
  $(0,0)$ optimal $\bar{u}_1 \le 2$

(d) No Take $\bar{u}_2 = 0$

$\bar{u}_1 = 0$ → sol to Arrow is $c_{11} = c_{12}$

f.e. $\begin{cases} c_{11} = c_{12} = 0.3 \\ c_{21} = c_{22} = 0.7 \end{cases}$

It is not PE

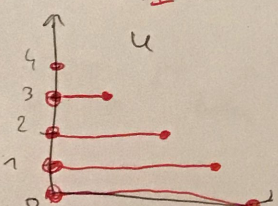
(e) Given λ $\max \sum \lambda_i u_i(c_{ij})$

FEAS

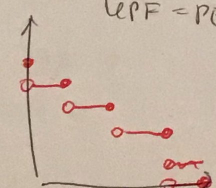
(f) Yes $\lambda_1 = 0$ $-\frac{\lambda_1}{\lambda_2}$ slope $\varepsilon u_1 + (1-\varepsilon) u_2{}^{.5}$ $\varepsilon = 0$ —— $\varepsilon = 1$ | $\varepsilon = \frac{1}{2}$ \

(g) No Check $\lambda = 0$ $c = (0,0,1,1)$ is sol to Negishi

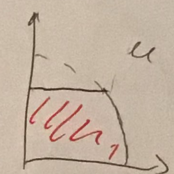
2. - 3.1 a

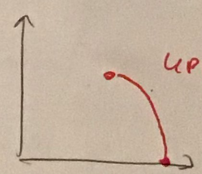


UPF = PE

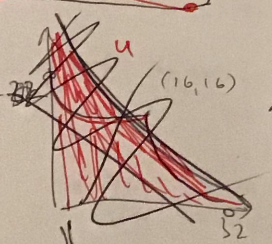


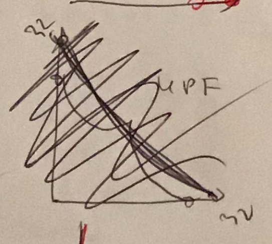
c

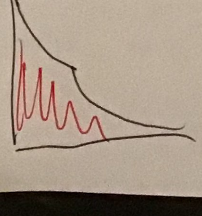


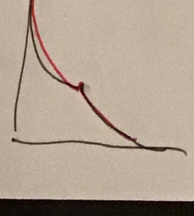


b





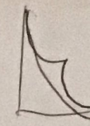




Pictures on mail !

R 9.1

How to find UPF? (b)



No

• Give one everything (monotone u_i):

a) $c_{11}=0$ $c_{21}=4$ $0\le c_{12}\le 4$ $u_1=c_{12}^2$ $u_2=16+(4-c_{12})^2$

b) $c_{11}=4$ $c_{21}=0$ $0\le c_{12}\le 4$ $u_1=16+c_{12}^2$ $u_2=0+(4-c_{12})^2$

~~Then~~ Here Arrow does not work

(c) Arrow pb: $\max \sqrt{c_{11}}+\sqrt{c_{12}}$

Forget about min

(λ_1) $c_{11}+c_{21}\le 4$

(λ_2) $c_{12}+c_{22}\le 4$

(μ) $\sqrt{c_{21}}+\sqrt{c_{22}}\ge u_2$

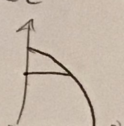
u is str. increasing $\lambda_1,\lambda_2>0$ BC = , FOCS

$\frac{1}{2}c_{11}^{-1/2}=\lambda_1=\frac{1}{2}\mu c_{21}^{-1/2}$ etc $\Rightarrow c_{11}=c_{12}=\bar{c}_1$

$u(c_{11}c_{12})=2\sqrt{\bar{c}_1}\Rightarrow \left(\frac{\bar{u}_1}{2}\right)^2=c_1$ the same $\left(\frac{\bar{u}_2}{2}\right)=c_2$

$\bar{c}_1+\bar{c}_2=4$ $\bar{u}_1^2+\bar{u}_2^2=16$

Now add $c_{21}\le 1$ $c_{22}\le 1$ $\Rightarrow u_2\le 2$



(2)

(2) Arrow pb sol $\Rightarrow$ PE?

(a) No take $\bar{u}_1=2.5$ $\max [c_2]$

FEAS $\Rightarrow^{sol}$ $c_2\in[1,2)$

$c_1\ge 2.5$

$c_1=2.5$ $c_2=1$ solves but $c_1=2.9$, $c_2=1$ Pareto Dominates $\boxed{c_1=2.5\ c_2=1 \text{ solves}}$

(b) Yes. Proof by contradiction. $\exists(\bar{c}_1,\bar{c}_2)$ solving Arrow Pb given $\bar{u}_2,\bar{u}_1$

$\max u_1(c_1)$ FEAS $u_2(c_2)\ge \bar{u}_2$ $\max u_2(c_2)$ FEAS $u_1(c_1)\ge\bar{u}_1$

Then suppose it is not PE $\exists(c_1,c_2)$

Pareto dominating and feasible $u_1(c_1)\ge u(\bar{c}_1)$ $u_2(c_2)\ge u(\bar{c}_2)$

And one strict >. If (1) then c_1 is sol to AP1 if (2) then AP(2)

thus $(\bar{c}_1,\bar{c}_2)$. It is true bc u_1,u_2 are increasing ↯

5

RAQ-2

(c) No. Take ~~u66t~~ $u_1 = 2$ max $\min\{1, \sqrt{c_{21}}\} + \min\{1, \sqrt{c_{22}}\}$

FEAS $\sqrt{c_{21}} + \sqrt{c_{22}} \geq u_1$

$c_1 = (1,1)$ $c_2 = (1,1)$ solves

and $c_1 = (3,3)$ | $c_2 = (1,1)$ solves and Pareto Dominates

(3) PE ⇒ Sol to Amow

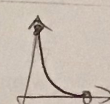
Yes. Proof: Take For (c_1, c_2) take $u_i = u(c_i)$

Suppose ∃ $\hat{c}$ feasible and solving higher $u(\hat{c}_1) > u(c_1)$ $u(\hat{c}_2) \geq u(c_2)$

$\hat{c}$ feasible better than $\hat{c}$ contradicts Pareto efficiency

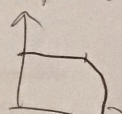
Every PE allocation is sol of Negishi for some λ?

No Only two PE allocations are sol. to Negishi



Every Negishi for some λ_i ⇒ PE?

No.



(4) Every solution to Negishi ⇒ PE

(a) Consider cases of Negishi

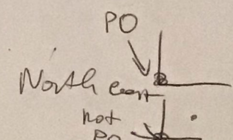
$\alpha = \frac{1}{2}$ (4,0) (3,1) (2,2) (1,3) (0,4)

$\alpha < \frac{1}{2}$ (0,4)

$\alpha > \frac{1}{2}$ (4,0)

PE : (4,0) (3,1) (2,2) (1,3) (0,4)

So Yes



(b) $\wedge$ $\alpha = 0$ — (0,0,4,4) max $u(c_2)$

Yes $\alpha = 1$ | (4,4,0,0) max $u(c_1)$

For $\alpha \in (0,1)$ No sol to Negishi

$\alpha \in (0,1)$ Let assume that (c_1, c_2) sol to Negishi is not PE ∃ feas. $(\hat{c}_1, \hat{c}_2)$ PD (c_1, c_2)

~~We have ... so monotone~~

$u(\hat{c}_i) \geq u(c_i)$ $\forall i$ with one strict

$\alpha u(\hat{c}_1) + (1-\alpha) u(\hat{c}_2) > \alpha u(c_1) + (1-\alpha) u(c_2)$ $\alpha, 1-\alpha > 0$ ↯

(c) No Take SPP with $\alpha = 0$

(0,0,1,1) solves Negishi

But (3,3,1,1) PD (0,0,1,1)

R20.3

(5) Every PE solves SPP

(a) Yes $PE = \{(0,4) \dots \}$ are solutions to SPP

~~(b) No PE ∈ UPF~~ : $\left(\left(\frac{u_1}{2}\right)^{1/2} + \left(\frac{u_2}{2}\right)^{1/2}\right) = 4$

take $u_1 = u_2 = 16$ $\quad 2\left(\frac{u_1}{2}\right)^{1/2} = 4$ $\quad$ ~~$u_1 = 8 \quad u_2 = 8$~~

Lemma. c is PE $\Leftrightarrow$ $C \in UPF$

Proof. If $(u_1(c_1) \dots u_I(c_I)) \notin UPF$ then $\exists u' \in U$ $u_i' \geq u_i(c_i)$ and for some $u_i' > u_i(x_i)$. Then u' PDominates u.

Conversely if (x, y) is not a PE then it is PD by a c' feasible

So $u_i(c_i') \geq u_i(c_i)$ $\forall i$ $\exists g >$. Hence $(u_1(c_1), \dots u_I(c_I)) \notin UPF$

Prop. If u is sol to Negishi $\Rightarrow$ $u \in UPF$.

(b) Take $(0,4), (4,3)$ $\rightarrow$ PE $\quad$ check it on your own

$u(c_1) = 1 \quad u(c_2) = 25$ $\quad$ Take $(0,0,4,4)$ $\quad (0,4,0,4)$
$(4,4,0,0)$ $\quad (2,2,2,2)$

So $w = \alpha \cdot 1 + 25(1-\alpha) < \alpha \cdot 0 + (1-\alpha) \cdot 32$ $\quad (1)$

and $w = \alpha \cdot 1 + 25(1-\alpha) < (1-\alpha) \cdot 0 + \alpha \cdot 32$ $\quad (2)$

(1) $\alpha < 7 - 7\alpha$ $\quad \alpha < \frac{7}{8} = \frac{49}{56}$

(2) $25 - 24\alpha < 32\alpha \Rightarrow \frac{25}{56} < \alpha$

(3) $w < \alpha 16 + (1-\alpha) 16 = 16$ $\quad 9 < 24\alpha \quad \frac{9}{24} < \alpha$

(4) , $w < 8 \Rightarrow \alpha + 25(1-\alpha) < 8$ $\quad 17 < 24\alpha \quad \frac{17}{24} < \alpha$

(5) $(1,1) \quad (3,3)$

$\alpha + 25(1-\alpha) < 2\alpha + (1-\alpha) \cdot 18$ $\quad 7 < (-16 + 24)\alpha \quad \frac{7}{8} < \alpha$

(1) & (5)

Only PE which is SPP ~~are~~ $(0,0,4,4)$ $\quad (4,4,0,0)$ ✓

$32(1-\alpha)$ ~~32α~~ $\geq \alpha(x^2 + y^2) + (1-\alpha)((4-x)^2 + (4-y)^2)$

$= \alpha(x^2 + y^2) + (1-\alpha)[32 + x^2 + y^2 - 8(x+y)]$

$= x^2 + y^2 + (1-\alpha)[32 - 8(x+y)(1-\alpha)]$

If $\alpha = 1$ $\quad 0 \geq x^2 + y^2$ $\quad x = y = 0$ $\quad$ only $(0,0,4,4)$

~~If $\alpha = 0$~~ 32α

if $\alpha = 0$ $\quad 0 \geq x^2 + y^2 + 32 - 8(x+y)$

RE 8.4

Wrap up

IMPORTANT

① Pareto pb (Arrow pb)

$$\max u_i(c_i)$$
$$\forall_m \sum c_{i,m} \le e_m$$
$$\forall j \ne i \quad u_j(c_j) \geq \overline{u_j}$$

② Pareto Efficient allocation $\hat{c} = \{\{c_{i,m}\}\}$

$\neg \exists\, c$ feasible & $\forall i \quad u_i(c_i) \geq u_i(\hat{c}_i)$

$\exists j \quad u_j(c_j) > u_j(\hat{c}_j)$

③ Walras Equilibrium $(p \in \mathbb{R}^m_+)$

- $c \in C_i \subseteq \mathbb{R}^m_+$, $\sum_i c_{i,m} \le \sum e_{i,m} \ \forall m$ – feasibility
- $\sum_m p_m \cdot c_{i,m} \le \sum_m p_m \cdot e_i \quad \forall i$ affordable
- $\forall i \forall \hat{c}_m \quad \hat{c}_i > c_i \Rightarrow \sum p_m \hat{c_{im}} > \sum p_m c_{im}$

or $\forall i \quad \max u_i(c_i)$

$$\sum_m p_m c_{im} \le \sum_m p_m e_{im} \quad \forall i$$

④ Negishi Pb (SPP) for given $\{\lambda_i\}_{i=1}^{I}$

$$\max \sum \lambda_i u_i(c_i)$$
$$\sum c_{i,m} \le \sum e_{i,m} \ \forall m$$

⑤ $U = \{U_i(c_i) \mid c_i \in C \quad \sum c_{i,m} \le \sum e_{i,m} \ \forall m\}$

$UPF = \{u \in U \mid \neg \exists u' \in U \quad u_i' \geq u_i \ \forall i \ \exists j \ u_j' > u_j\}$

~~Pareto U border~~

Relationship Between ① ② ③ ④ ⑤

② ⇔ ⑤ always → Easy

③ ⇒ ② LNS – 1st Welfare Thm

③ ⇐ ② LNS, X_i (Y_i if prod) conv – 2nd Welfare Thm
$\succsim_i$ cont, convex

① ⇒ ② – LNS

② ⇒ ① Always

⑤ ⇒ ④ convexity of UP set (not related to convexity of pref.)

④ ⇒ ⑤ always

② ⇐ ⑤ str. convex set ~~str. decreasing~~

not $\pi(u)$

R ~~4~~ 8 5

GENERAL EQUILIBRIUM NOTES

$e = (e_1 \dots e_M)$ societal

Def Price equilibrium with transfers

(x^*, y^*) allocation, $p \in \mathbb{R}^M$ prices, PEwTr if $\exists$ assignment of wealth levels $(w_1, \dots w_I)$ with

~~$\sum_i w_{i,m} = p\left(\sum_i e_{i,m} +\right.$~~

$$\sum_i w_i = p \cdot \left(e + \sum_j y_j^* \right) \text{ s.t.}$$

① $\forall j$: $\quad p y_j^* \geq p y_j \quad \forall y_j \in Y_j \quad y_j^* \in Y_j$

② $\forall i$ $\quad x_i^* \succsim_i x_i \quad x_i \in B(p, w_i) \quad x_i^* \in B(p, w_i^*)$

③ $\sum x_i^* = \sum y_j^* + e$

CE $\subseteq$ PEwTr : $w_i = p \cdot e_i + \sum_j \theta_{ij} \, p \cdot y_j^*$

Thm. 1st Welfare Thm.

If $\succsim_i$ are LNS, (x^*, y^*, p) is a price equilibrium with transfers $\Rightarrow$ (x^*, y^*) is Pareto optimal.

In particular any Walrasian (CE) is PO

Proof: Suppose (x^*, y^*, p) is PEwT and $w = (w_1 \dots w_I)$ s.t.

$\sum w_i = p \cdot e + \sum_j p \cdot y_j^*$. From LNS:

If $x_i \succ_i x_i^* \Rightarrow p x_i > w_i$

If $x_i \succsim x_i^* \Rightarrow p \cdot x_i \geq w_i$

Now let (x, y) be feasible and Pareto dominates (x^*, y^*)

Then $x_i \succsim_i x_i^*$ & $\exists i$ $x_i \succ x_i^*$ so

$$\sum_i p \cdot x_i > \sum_i w_i = p \cdot e + \sum_j p \cdot y_j^*$$

But y_j^* comes from Π-max so $p \cdot e + \sum_j p y_j^* \geq p \cdot e + \sum_j p \cdot y_j$

Thus $\sum_i p x_i > p \cdot e + \sum p \cdot y_j$ so (x, y) it can not be feasible. Contradiction. So (x^*, y^*) is PO

$\boxtimes$

1. Pure exchange economy

Def. An allocation (x,y) $x_i \in X_i \ \forall i \in I$ $y_j \in Y_j$ $j \in J$
is <u>feasible</u> if $(e_m = \sum_i e_{im})$

$$\forall m \ \sum_i x_{i,m} = e_m + \sum_j y_{j,m}$$

Denote by $A = \{(x,y) \in X_1 \times \dots \times X_I \times Y_1 \times \dots \times Y_J : \sum x_i = e + \sum y_j\}$
set of feasible allocations

Def. A feasible allocation (x,y) is (strong) Pareto optimal (efficient)
if $\neg \exists (\bar{x},\bar{y})$ feasible $(\bar{x},\bar{y})$ (weakly) Pareto dominates (x,y)
$[\forall i \ \bar{x}_i \succsim_i x_i \quad \exists k \ \bar{x}_k \succ x_k]$

- In PO there is no waste
- It is impossible to make any consumer strictly better off without making some other consumer worse off
- Note that PO itself does not concern itself with distributional issues (society & which one gets everything due to very strong monotone pref is PO)
- PO ~~def~~ defined on (x,y) but cares about x only

2. Private ownership economy

shares to profit of the firms (j) $\theta_{i,j} \in [0,1], \sum_i \theta_{i,j} = 1 \ \forall j$

Def Walrasian (competitive) equilibrium

Given $\mathcal{E} = \{ (X_i, \succsim_i)_{i=1}^{I}, (Y_j)_{j=1}^{J}, (e_i, (\theta_{ij})_{j=1}^{J})_{i=1}^{I} \}$
an allocation (x^*, y^*) and price vector $p \in \mathbb{R}^m$ is a CE:

① $\forall j$ given p : $p y_j^* \geq p y_j \quad \forall y_j \in Y_j$ $y_j^* \in Y_j$
profit max

② $\forall i$ given p $x_i^* \succsim_i x_i \quad \forall x_i \in B(p, w_i)$ $x_i^* \in B(p, w_i)$
Consumer max
$w_i = e_i + \sum_j \theta_{ij} y_j^*$

③ MCC $\forall m \sum_i x_{i,m}^* = \sum_j y_{j,m}^* +$ ~~em~~ tem

Def. Price Quasi Equilibrium with transfers

(x^*, y^*, p) $w: \sum_i w_i = p \cdot e + \sum_j p \cdot y_j^*$:

① $\forall j$ $\quad p \cdot y_j^* \geq p \cdot y_j \quad \forall y_j \in Y_j$

② $\forall i$ $\quad x_i \succ x_i^* \Rightarrow p x_i \geq w$

③ $\sum_i x_i^* = \sum_j y_j^* + e$

② If x_i^* comes from max over $B(p, w_i)$ then no $x_i \succ x_i^*$ can emerge for $(p x_i < w_i)$. Hence PEwT is PQEwT.
When $\succsim$ are LNS $\Rightarrow p x_i \geq w$ for $(x_i \succ x_i^* \Rightarrow p x_i \geq w)$
So we get $\sum p \cdot x_i^* = p \cdot e + \sum p y_j^* = \sum w_i$) so under LNS $p \cdot x_i = w_i$

Thm. 2nd Welfare Thm.

Y_j is convex, $\succsim_i$ is convex [i.e. $\{x_i' \in X_i : x_i' \succsim x_i\}$ convex for every $x_i \in X_i$] and $\succsim_i$ are LNS. Then for every Pareto optimal allocation (x^*, y^*) there is a price vector $p \in \mathbb{R}^M$ $p \neq 0$ s.t. (x^*, y^*, p) is a Price Quasi Equilibrium with Transfers

Proof:
Define $V_i = \{x_i \in X_i : x_i \succ x_i^*\} \subseteq \mathbb{R}^M$

$V = \sum_i V_i = \{\sum x_i \in \mathbb{R}^M : x_1 \in V_1, \dots, x_I \in V_I\}$

$Y = \sum Y_j = \{\sum y_j \in \mathbb{R}^M : y_1 \in Y_1, \dots, y_J \in Y_J\}$ Lee Look at $Y + \{e\}$

We prove it in 9 steps

① $\forall i$ V_i is convex ② V, $Y + \{e\}$ are convex ③ $V \cap \{Y + \{e\}\} = \emptyset$

④ $\exists p \neq 0 \in \mathbb{R}^M$ $p \cdot z \geq p \cdot v$ $\forall z \in V$ $v \in Y + \{e\}$ ⑤ If $x_i \succsim x_i^* \Rightarrow p(\sum x_i) \geq v$

⑥ $p \sum x_i^* = p(\bar{\omega} + \sum y_j^*) = v$ ⑦ $\forall j$ $p y_j \leq p y_j^*$ ⑧ $\forall i$ $x_i \succ x_i^* \Rightarrow p x_i \geq p x_i^*$

⑨ $w_i = p x_i^*$ support (x^*, y^*, p) as PQEwTr.

~~Step~~ 1

Prop. Suppose $\forall i: X_i$ is convex (free disposal) $0 \in X_i$ and $\succsim_i$ is continuous. Then any PQEWT is a PEWT

Lemma. X_i convex, $\succsim_i$ continuous. If $x_i \succ x_i^* \Rightarrow p x_i \geq w_i$
Then if there is $x_i' \in X_i$ $p x_i' < w$ (cheaper) then
$x_i \succ x_i \Rightarrow p x_i > w_i$.

16.E Pareto Optimality and Social Welfare Optima

Def. (UPS) Utility Possibility Set

$$\mathcal{U} = \{ (u_1, \dots u_I) \in \mathbb{R}^I : c_i \in C_i \quad \sum_i c_{i,m} \leq e_m \quad u_i \leq u_i(c_i) \}$$

Lemma. Corresponding points on UPS of Pareto Efficient allocation $\in$ UPF

Lemma. For LNS Every

Def. (UPF) Pareto Frontier:

$$UPF = \{ u = (u_1, \dots u_I) \in \mathcal{U} : \neg \exists\, u' \in \mathcal{U} \quad u_i' \geq u_i \ \forall i, \ \exists j \ u_j' > u_j \}$$

Lemma: (x, y) is PO $\Leftrightarrow (u_1(x_1), \dots u_I(x_I)) \in UPF$.

Lemma. X_i convex, Y_j convex, u_i convex $\Rightarrow \mathcal{U}$ convex

Define Welfare f. (given λ_i) $\quad W(u_1 \dots u_I) = \sum \lambda_i u_i$

Lemma. If $u \in$ sol to Negishi $\Rightarrow u \in UPF$

Lemma. If $\mathcal{U}$ convex $\Rightarrow \forall \bar{u} \in UPF \ \exists \lambda$ it is Negishi problem solution

Lemma. $A \neq \emptyset$ compact, $u_i \in C^0 \Rightarrow \mathcal{U}$ is ~~Compact~~ closed and bounded from above.

WELFARE THEOREMS

Def. ~~Weakly Pareto~~ $x \in \mathbb{R}_+^{Ln}$ is feasible if
$$\sum_{i=1}^{L} x_i = \sum_{i=1}^{L} e_i$$

Def. A feasible allocation x is Weakly Pareto Dominates y if
$\forall i \ x_i \succsim y_i, \ \exists j \ x_j \succ y_j$ $\quad (x >_{WPD} y)$

Def. A feasible allocation x Strongly Pareto Dominates y if
$\forall i \ x_i \succ y_i$ $\quad (x >_{SPD} y)$

Def. An allocation x is Weakly Pareto Optimal if
- it's feasible $\sum x_i = \sum e_i$
- $\neg \exists$ feasible y $\quad y >_{SPD} x$

Def. An allocation x is Strongly Pareto Optimal if
- it's feasible $\sum x_i = \sum e_i$
- $\neg \exists$ feasible y $\quad y >_{WPD} x$

Lemma $SPO \subseteq WPO$

Proof: ~~$x \in SPO$ If $y \in SPD$ $y >_{SPD}$~~
If $y >_{SPD} x$ then $y >_{WPD} x$
Let $x \in SPO$ then $\neg\exists y : y >_{WPD} x$ $\quad$ y feasible
so $\neg\exists y \ y >_{SPD} x$ so $x \in WPO$

Lemma If $\succsim_i$ continuous & monotone then
$WPO \Leftrightarrow SPO$

FIRST WELFARE THEOREM

Let $\mathcal{E} = \{\{\succsim_i, e_i\}\}$ be a pure exchange economy $e_i \in \mathbb{R}^l$ $\sum_i e_i \gg 0$. $\succsim_i$ are:

- complete preordered, (continuous), LNS on X_i.

If (x^*, p^*) is a COMPETETIVE EQUILIBRIUM then x^* is (Strongly) PARETO OPTIMAL

$CE \Rightarrow PO$

Proof Let (x^*, p^*) be CE suppose it is not PO,
$\exists y \in \mathbb{R}^{ln}$ feasible $y_i \succsim_i x_i \; \forall i$, $\exists j \; y_j \succ_j x_j$

Lemma { Since $x_j^* \in X_j(p^*, e_j)$ it must be that $y_j \notin B(p^*, e_j)$ so $p^* y_j > p^* e_j$

Proof: Suppose by contradiction that $p^* y_j \leq p^* e_j$
Then $y_j \in B(p^*, e_j)$ & $y_j \succ x_j$ contradicts optimality of x_j

KERTE

Lemma Since $x_i^* \in x_j(p^*, e_i) \Rightarrow$ ~~[illegible]~~ so $p^* y_i \geq p^* e_i$

Proof: If not. $p^* y_i < p^* e_i$ so $y_i \in B(p, e_i)$ is feasible
$y_i \succsim x_i$ ~~contradicts~~ optimality of x
So $y_i \sim x_i$. By LNS $\forall \varepsilon \; \exists y_i' \; \|y_i - y_i'\| < \varepsilon$ & $y_i' \succ y_i$ $p^* y_i' < p^* e$

Since $\forall i \; p^* y_i \geq p^* e_i$ $\exists j \; p^* y_j > p^* e_j$

sum them up:

$$p^* \sum y_i = \sum_{i=1}^{n} p^* y_i > \sum p^* e_i = p^* \sum e_i = p^* \sum y_i$$

feasibility

So we obtain a contradiction.
thus x^* is SPO. (Pareto optimal)

SECOND WELFARE THEOREM

Let $\mathcal{E} = \{ \succsim_i, e_i \}$ be a pure exchange economy e.t.l. $\sum e_i \gg 0$

$\succsim_i$ are

- complete preorder, ~~...~~ continuous, strictly monotone, strictly convex (*) on X_i

[or u is continuous str. quasi concave, str. monotone]

If $\bar{x} \in \mathbb{R}^{Ln}$ is (strongly) Pareto Optimum then $\bar{x}$ is a Competitive Equilibrium for $\hat{\mathcal{E}}$ where $\hat{\mathcal{E}} = \{ \succsim_i, \bar{x}_i \}$ and we can construct prices $p^* \geq 0$ $p^* \neq 0$

(*) we can weaken
- strict convexity to convexity
- str. monotonicity to LNS

~~CE => P~~

PO => CE

Proof: Let $\bar{x} \in SPO$ $\bar{x}$ is feasible $\sum \bar{e}_i = \sum \bar{x}_i = \sum e_i$

Since $\succsim_i$ continuous & convex $\forall i$ so upper contour set $U(\bar{x}_i) = \{ x_i \in \mathbb{R}^L_+ \; x_i \succ_i \bar{x}_i \}$ are open, convex

Define $U = \sum_i U_i(\bar{x}_i)$. It is open & convex as well

It represents commodity bundles that can be (potentially) redistributed

Define $E = \sum \bar{x}_i$ total resources available

Observe that $\bar{x}_i \notin U(\bar{x}_i)$ so $E \notin U$

U is convex $E \notin U$ $\exists$ separating hyperplane $p^*, p^* \neq 0$ s.t. $\underset{z \in U}{p^* z} \geq \underset{x \in E}{p^* \cdot x}$

WTS: p^* is equilibrium price and $p^* \geq 0$

Step 1 $p^* \geq 0$

Fix j, let u_j be $(0,\dots,\overset{j\text{th}}{1},0,\dots,0)$. Let $\bar{y}_i = \bar{x}_i + \frac{1}{n} u_j$

By strict monotonicity $\bar{y}_i \in U(\bar{x}_i)$ Then

$\sum \bar{y}_i = \sum \bar{x}_i + \frac{1}{n} u_j = E + u_j \in U(\bar{x})$

Since U is str. convex by seperating hyperplane

$p^* \cdot (E + u_j - E) = p^* u_j = p_j^* \geq 0$

j was arbitrary so $p^* \geq 0$

Step 2: $\forall i \; x_i \succ \bar{x}_i \Rightarrow p^* \cdot x_i \geq p^* \cdot \bar{x}_i$

Fix j. By continuity of $\succsim_i$, $x \in \mathbb{R}^n_{++}$ $\exists \varepsilon > 0$ $\tilde{x}_j = x_j - \varepsilon e$

$\tilde{x}_j \succ_j x_j$ ($e = (1,\dots,1)$)

By monotonicity for $i \neq i$

$\tilde{x}_i = \bar{x}_i - \frac{\varepsilon}{n-1} \cdot e$, Then $\tilde{x} \in U(\bar{x})$ $\bar{x} \in U(\bar{x})$

So by seperating hyperplane: $(\tilde{x} \in U, E)$

$p^* \sum \tilde{x}_i \geq p^* \cdot \sum \bar{x}_i$ we rewrite it

$p^* \cdot [x_j - \varepsilon \cdot e + \sum_{i \neq j} \bar{x}_i + \sum_{i \neq j} \frac{\varepsilon}{n-1} \cdot e] \geq p^* \cdot \sum \bar{x}_i$

~~$p^* x_i \geq p^* \bar{x}_i$~~

$p^* x_i \geq p^* \cdot \bar{x}_i$

Step 3: $\forall i \; x_i \succ \bar{x}_i \Rightarrow p^* x_i > p^* \bar{x}_i$

S! $p^* x_i \leq p^* \bar{x}_i$ by continuity $\exists \lambda \in (0,1)$ $\lambda x_i \succ \bar{x}_i$

By step 2 ~~$p^* x_i \not< p^* \bar{x}_i$~~ $p^* (\lambda x_i) \geq p^* \bar{x}_i > 0$

By step 2 $p^* x_i \not< p^* \bar{x}_i$ and by assumption $p^* x_i \leq p \bar{x}_i$

so it has to be $p^* x_i = p^* \bar{x}_i$. Since $\lambda \in (0,1)$

$\lambda (p^* x_j) < p^* \bar{x}_j$. $\lightning$ So it has to be

$p^* x_j > p^* \bar{x}_j = p^* e_i$

Let $B_i(p^*, \bar{e}_i)$ $\quad \bar{e}_i = \bar{x}_i$ since $\bar{x} = e$
then $\quad p^* \cdot \bar{x} = p^* e$

Then we have to show that $\forall i \in I \quad \bar{x}_i \in B_i(p^*, e_i)$
& $x_i \succ \bar{x}_i$ implies $x_i \notin B_i(p^*, \bar{e}_i)$

Since $\bar{x}$ is feasible this means $\exists p^* \geq 0 \quad p^* \neq 0$
s.t. $(\bar{x}, p^*)$ form CE of $\mathcal{E}$.

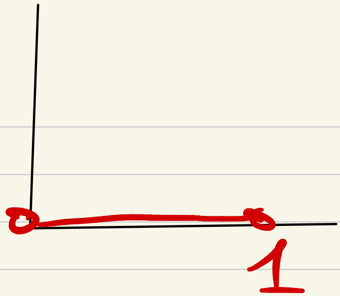
Since $\succ$ is str. convex $\forall i$, demand is single valued
that is $\bar{x}$ is unique CE.

- Production economy $\mathcal{E} = \{(\succ_i, e_i, \{\theta_{ij}\})\}$
Feasibility set $F^P(e) = \{(x, y) \in \mathbb{R}^{ln}_+ \quad x_i \in X_i, y_j \in Y_j$
$\forall i \; \forall j \quad \sum_i x_i = \sum_i e_i + \sum_j y_j$

$WPO^P(x, y, e) = \{(\hat{x}, \hat{y}) \in F^P(e) \; \forall i \; \hat{x}_i \succeq x_i, \exists j \; \hat{x}_j \succ x_j\}$
nothing about production! only consumers

Def. Competetive Equilibrium in production economy
$(x^*, y^*, p^*) \in \mathbb{R}^{ln}_+ \times \mathbb{R}^{LK} \times \mathbb{R}^l_+ \setminus \{0\}$
- $\forall j \quad y_j \in \arg\max_{y \in Y_j} p^* y_j$
- $x_i^* \in B_i^P(p^*, e) = \{x \in \mathbb{R}^l_+ \; p^* x \leq p^* e_i + \sum_{k=1}^{K} \theta_{ij} \pi_j$
- $\hat{x} \succ x_i^* \Rightarrow \hat{x} \notin B^P(p^*, e)$
- $(x^*, y^*) \in F^P(e)$

1. a)



$$\alpha u_1 + (1-\alpha) u_2$$

$\alpha = 0$

$\alpha = 1$

b) Arrow (1)

$$\max_c \; u_1(c_{11}, c_{12})$$

FEAS $\begin{cases} \sum_i c_{i,m} \le \sum e_{i,m} \quad \forall m \\ \text{nonnegative } c_{i,m} \\ c_{j m} \in \{1, 2\} \end{cases}$

(2) $\max_c \; u_2(c_{21}, c_{22})$ FEAS

- $u_2(c_{21}, c_{22}) \geq \bar{u}_2$

$u_1(c_{11}, c_{12}) \geq \bar{u}_1$

$u_2 = 0$ $\quad u_1 = \sqrt{c_{11}} + \sqrt{c_{12}}$

(c) Only PE allocation

~~$c = (0, 0, 1, 1)$~~

$c = (1, 1, 0, 0)$

$\bar{u}_1 \geq 2 \qquad \bar{u}_2 = 0$

FEAS easy ✓

$u_1(1,1) = 2 \geq 2$ ✓

$u_2(0,0) = 0 = 0$ ✓

$(1,1)$ is optimal in (1) $\overline{u_2} = 0$

$(0,0)$ is optimal in (2) $\overline{u_1} \cancel{\neq} 2$

(d) $\overline{u_2} = \cancel{B}\, 0$

$\overline{u_1} = 0$

Sol. Arrow $\Rightarrow \max \sqrt{c_{11}} + \sqrt{c_{12}}$ FEAS

$\leftarrow \cancel{\sqrt{c_{11}}}\ 0 \geq 0$

$\Rightarrow \max 0$ FEAS

$\swarrow \sqrt{c_{11}} + \sqrt{c_{12}} \geq \cancel{2}\ 0$

$c_{11} = c_{12}$ f.e.

$$\begin{cases} c_{11} = c_{12} = 0.3 \\ c_{21} = c_{22} = 0.7 \end{cases} \rightarrow PE$$

(e) Negishi, (SPP) given λ

$$\max_c \sum \lambda_i u_i(c_{i1}, c_{i2})$$

s.t. FEAS

(f) $$W = \sum_{i=1}^{I} \lambda_i u_i(c_{i1}, c_{i,2})$$

(f) Yes PE $\Rightarrow$ sol. Neg ish for some α

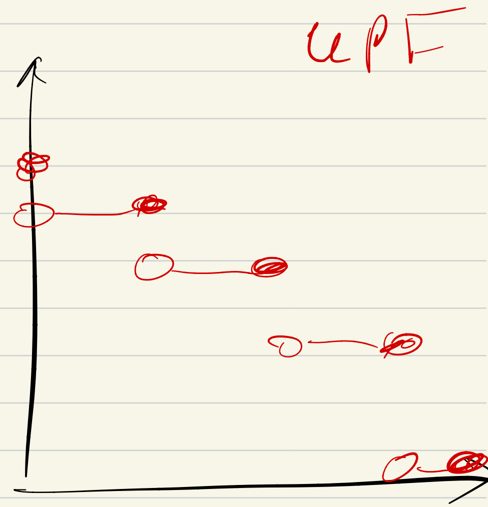
(g) No $\alpha = 1$ $\alpha \in (0,1)$

Neg ish $\Rightarrow$ PE

No. Take $\alpha \to 0$ $\boxed{\alpha}$ ll

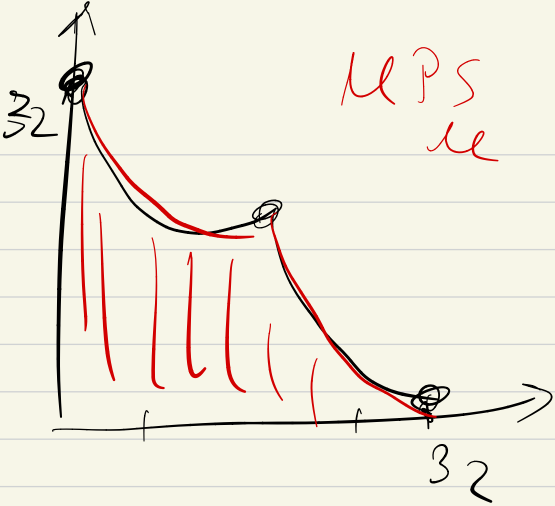
Q2

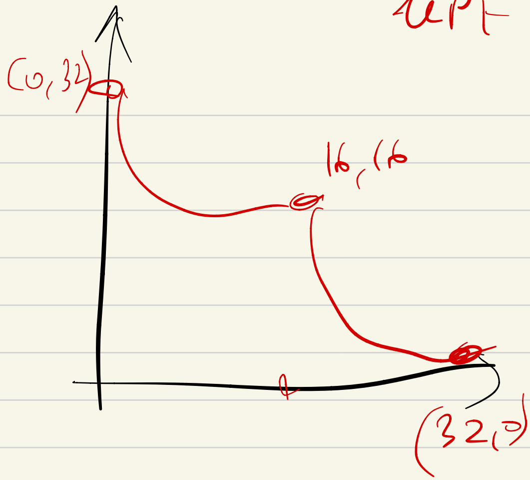
(1)



Lemma: $u \in {}^{u}PF \Leftrightarrow$ PE all

②





① Dres

③

$$u_1(c_{11}, c_{12}) = \sqrt{\max\{1, c_{11}\}} + \sqrt{\min\{1, c_{12}\}}$$

④ $\max \sqrt{c_{11}} + \sqrt{c_{12}} = u_1$

FEAS

$\sqrt{c_1}$

How to draw solution (CUPS)

solve Arr P6

$$\overline{u_1} = \max \sqrt{c_{11}} + \sqrt{c_{12}}$$

$\to \lambda_1$ $\quad c_{11} + c_{21} \leq 4 \quad (1)$

$\to \lambda_2$ $\quad c_{12} + c_{22} \leq 4 \quad (2)$

$(\mu) \quad \sqrt{c_{21}} + \sqrt{c_{22}} \geq \overline{u_2}$ (3)

- We can use KKT
- $\lambda_1, \lambda_2 > 0$ $\quad$ FOCs: $\frac{\partial \mathcal{L}}{\partial c_{11}} = \frac{1}{2} c_{11}^{-1/2} = \lambda_1 > 0$

$$\frac{\partial \mathcal{L}}{\partial c_{12}} \quad \frac{1}{2} c_{12}^{-1/2} = \lambda_2 > 0$$

So (1), (2)

$$\frac{\partial \mathcal{L}}{\partial c_{21}} : \lambda_1 = \frac{1}{2} \mu c_{21}^{-1/2}$$

$$\frac{\partial \mathcal{L}}{\partial c_{22}} : \lambda_2 = \frac{1}{2} \mu c_{22}^{-1/2}$$

$\Rightarrow$ Observe that $\begin{cases} c_{11} = c_{12} \\ c_{21} = c_{22} \end{cases}$

$$\left.\begin{aligned} \overline{u_1} &= 2\sqrt{c_{11}} \\ \overline{u_2} &= 2\sqrt{c_{21}} \end{aligned}\right\}$$

$$\frac{1}{2} c_{11}^{-1/2} = \lambda_1 = \frac{1}{2} \mu c_{21}^{-1/2}$$

$$\frac{1}{2} c_{12}^{-1/2} = \lambda_2 = \frac{1}{2} \mu c_{22}^{-1/2}$$

$$\underline{c_{11} = \mu^2 c_{21}}$$

$$\underline{c_{11} = \mu^2 c_{21}}$$
$$c_{12} = \mu^2 c_{22}$$

FEAS

$$c_{11} + c_{21} = 4$$
$$\mu^2 c_{21} + c_{21} = 4$$
$$c_{\cancel{1}2} + c_{22} = 4$$
$$\mu^2 c_{22} + c_{22} = 4$$

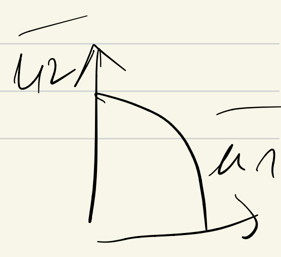
$$\boxed{c_{21} = \frac{4}{1+\mu^2} = c_{22}}$$

$$\underline{\underline{c_{11} = c_{12}}}$$

$$\bar{u}_1 = 2\sqrt{c_{11}}$$
$$\bar{u}_2 = 2\sqrt{c_{21}}$$
$$\Downarrow$$
$$\left(\frac{\bar{u}_1}{2}\right)^2 = c_{11} \qquad \left(\frac{\bar{u}_2}{2}\right)^2 = c_{21}$$

FEAS
$$c_{11} + c_{21} = 4$$

$$\left(\frac{\bar{u}_1}{2}\right)^2 + \left(\frac{\bar{u}_2}{2}\right)^2 = 4$$
$$\bar{u}_1^2 + \bar{u}_2^2 = 16$$



D ① Whet PE ell.

$PE = \{(0,4), (1,3), (2,2), (3,1), (4,0)\}$

Take $\bar{u}_1 = 2.5$

Look at ① Arrow met $[c_2]$

FEAS

$u_1(c_1) = c_1 \geq 2.5$

$\Rightarrow c_2 \in [1, 2)$

$[c_2] = 1$

Let's take $\begin{cases} c_2 = 1 \\ c_1 = 2.5 \end{cases}$ Sol to Arrow Pb

c not PE:

$\exists \hat{c}$ feas $\forall i\ u_i(\hat{c}_i) \geq u_i(c_i)$

$\exists j\ u_j(\hat{c}_j) > u_j(c_j)$

$\hat{c} = \quad \hat{c}_2 = 1 \quad \hat{c}_1 = 2.7$

FEAS $\longrightarrow 1 + 2.7 \leq 4$

$u_2(c_2) = 1 = u_2(\hat{c}_2)$

$u_1(\hat{c}_1) = 2.7 > u_1(c_1) = 2.5$

c_1, c_2 sol Arr 1 2

$$\max u_1(c_1) \quad \text{FEAS} \quad u_2(c_2) \geq \bar{u}_2$$

$$\max u_2(c_2) \quad \text{FEAS} \quad u_1(c_1) \geq \bar{u}_1$$

but c_1, c_2 not PO $\Rightarrow \exists$ FEAS $\hat{c}_1 .. \hat{c}_2$

(hw) PD c.

$$u_1(\hat{c}_1) \geq u_1(c_1) \quad (*)$$

$$u_2(\hat{c}_2) \geq u_2(c_2)$$

~~Ju~~ & (4) one >

If $* >$ $\Rightarrow$ $\hat{c}_1$ FEAS suboptima

For Arr $\underline{1}$

If $** >$ $\Rightarrow$ $\hat{c}_2$ FEAS

$\not\leq$ with Arrow 2.

Rem. Importante was monotonicity on u_1, u_2

③ No $\overline{u_1} = 2$

$c_1 = (1,1)$ $c_2 = (1,1)$ solves Arrow

but $\cancel{c_1 = (3,3)}$

$\hat{c}_1 = (1,1)$ $\hat{c}_2 = (3,3)$ PD (c_1, c_2)

(c) Easy

(d) Negishi $\Rightarrow$ PE

(e) PE $\Rightarrow$ Negishi

(d) Consider cases

① $\alpha = 0$

② $\alpha = 1$

③ $\alpha \in (0,1)$

① $\alpha = \frac{1}{2} \Rightarrow$ sol. of N $= \{(0,4), (1,3), (2,2), (4,0)\}$

$\alpha < \frac{1}{2}$ $(0,4)$

$\alpha > \frac{1}{2}$ $(4,0)$

c) Take $(0,1,4,3) = c$

and $u = (1, 2, 5)$

if $u \in UPF$ so PE (check at home)

Now find α for which

c is not a solution

to Negish.