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Question 1 [261 IV.2 Spring 2018 majors]

Dynamic Moral Hazard

Consider a two-period principal-agent problem. In each period, there are m possible output levels for the principal, $q_i \in \mathbb{R}$, and n effort levels for the agent, $a_j \in \mathbb{R}$.
Let $\mathbb{P}(q|a)$ be the probability of q given a. The principal's per-period payoff is expected output minus payments to the agent. The agent's utility is

$$U(w, a) = v(w) - c(a)$$

where $v' > 0, v'' < 0, c' > 0, c'' > 0, v(w)$ corresponds to the agent's utility from payments by the principal and $c(a)$ corresponds to effort costs.

Time elapses as follows. First, the principal makes a take-it-or-leave-it offer to the agent, who has an outside option worth $\bar{U}$. Once everyone agrees to the contract, there are no more opportunities to quit throughout the two-period relationship.
In period 1, the agent exerts effort a_1, output q_1 realizes, and the agent is paid $w_1(q_1)$. Everyone observes q_1 at the end of period 1.
In period 2,the agent exerts effort a_2, output q_2 realizes, and the agent is paid $w_2(q_1, q_2)$.
Everyone's overall payoff is the sum of per-period payoffs.

1. Write down formally the principal's problem of
   (a) minimizing the cost of implementing a given effort profile $\mathbf{a} = (a_1, a_2)$, where $a_1 \in \mathbb{R}$ is period- 1 effort and $a_2 \in \mathbb{R}^m$ is the effort plan in period 2 contingent on output in period 1 subject to a lifetime participation constraint,
   (b) maximizing expected profit.
2. Show that, in general, w_2 will depend not only on q_2, but also on q_1.
3. Derive and interpret the inverse Euler equation from 1(a):

$$\frac{1}{v'(w_1(q_1))} = \mathbb{E}[\frac{1}{v'(w_2(q_1, q_2))}|q_1] \equiv \sum_{q_2} \frac{1}{v'(w_2(q_1, q_2))}\mathbb{P}(q_2|a_2(q_1))$$

Solution 1

This problem comes from - Rogerson (Econometrica, 1985).Let's summarize set up

- Time $t \in \{1, 2\}$
- Effort $a_t \in A \subseteq \mathbb{R}$ at cost $c(a_t)$
- Output $q_t \sim f(q \mid a_t)$
- Agent's utility $U(w, a) = \sum_t [u(w_t) - c(a_t)]$ preferences are time-separable.
- Reservation utility $\bar{U}$ in each period.
- Contract: $w_1(q_1), w_2(q_1, q_2)$
- Principal's profit $= \sum_t (q_t - w_t) = [q_1 - w_1(q_1)] + [q_2 - w_2(q_1, q_2)]$
- Links between periods:
  1. No technological link or changes in preferences of the agent.
  2. Principal can use w_2 to reward a_1.
- Agent has no access to credit markets (i.e., cannot borrow or save).
- Action at $t = 2$ will depend on q_1; i.e., $a_2(q_1)$.
- b)Principal's maximization problem:

$$\begin{aligned}
\max_{w_1,w_2,a_1,a_2} \quad & \mathbb{E}[q_1 - w_1(q_1) + q_2 - w_2(q_1, q_2) \mid \mathbf{a}] \\
\text{s.t. IC} \quad & \{a_1, a_2\} \in \arg\max_{\tilde{a}_1,\tilde{a}_2} \mathbb{E}[u(w_1(q_1)) - c(\tilde{a}_1) + u(w_2(q_1, q_2)) - c(\tilde{a}_2(q_1)) \mid \tilde{a}_1, \tilde{a}_2] \\
\text{IR} \quad & \mathbb{E}[u(w_1(q_1)) - c(a_1) + u(w_2(q_1, q_2)) - c(a_2(q_1)) \mid a_1, a_2] \geq 2\bar{u}
\end{aligned}$$

- Instead of period by period IC, IR we have one
- To get Inverse Euler Equation we will solve P problem
- a) Cost minimization - We can separate P's profit into part depending only on wages (w_1, w_2) and part depending only on effort (a_1, a_2) and solve them step by step.
- **Step 1**- Solve for optimal cost as function of effort

$$C(a) \equiv \min_{w_1,w_2} \mathbb{E}[w_1(q_1) + w_2(q_1, q_2) \mid \mathbf{a}]$$

$$\text{s.t. IC and IR holds}$$

- **Step 2**

$$\max_{a_1,a_2} \mathbb{E}\left[q_1+q_2-C(\mathbf{a}) \mid \mathbf{a}\right]$$

3. Along the optimal path:

$$\frac{1}{u'\left(w_1\left(q_1\right)\right)}=\mathbb{E}\left[\frac{1}{u'\left(w_2\left(q_1, q_2\right)\right)} \mid q_1\right]$$

Proof. • Perturbation argument. No need for KKT

- Suppose $w_1\left(q_1\right)$ and $w_2\left(q_1, q_2\right)$ is optimal.
- Consider modifying the contract such that $u\left(\hat{w}_1\left(q_1\right)\right)=u\left(w_1\left(q_1\right)\right)-\epsilon$ and $u\left(\hat{w}_2\left(q_1, q_2\right)\right)= u\left(w_2\left(q_1, q_2\right)\right)+\epsilon$
- The agent's (IC) and (IR) are unaffected. Increases principal's profit by:

$$\Delta=\mathbb{E}\left[w_1\left(q_1\right)-\hat{w}_1\left(q_1\right)+w_2\left(q_1, q_2\right)-\hat{w}_2\left(q_1, q_2\right) \mid q_1\right]$$

- Apply the Taylor

$$u(\hat{w})=u(w)+u'(\bar{w})(\hat{w}-w) \quad \text{for some} \quad \bar{w} \in(w, \hat{w}) \Longrightarrow \hat{w}-w=\frac{u(\hat{w})-u(w)}{u'(\bar{w})}$$

To get for small $\epsilon>0$:

$$\begin{aligned} & w_1\left(q_1\right)-\hat{w}_1\left(q_1\right)=\tfrac{\epsilon}{u'\left(\bar{w}_1\left(q_1\right)\right)} \\ & w_2\left(q_1, q_2\right)-\hat{w}_2\left(q_1, q_2\right)=-\tfrac{\epsilon}{u'\left(\bar{w}_2\left(q_1, q_2\right)\right)}\end{aligned}$$

- Then:

$$\Delta=\epsilon \mathbb{E}\left[\frac{1}{u'\left(\bar{w}_1\left(q_1\right)\right)}-\frac{1}{u'\left(\bar{w}_2\left(q_1, q_2\right)\right)} \mid q_1\right]$$

- Because we can pick $\epsilon \gtrless 0$, the optimal contract must satisfy $(\bar{w} \rightarrow_{e \rightarrow 0} w)$

$$\frac{1}{u'\left(w_1\left(q_1\right)\right)}=\mathbb{E}\left[\frac{1}{u'\left(w_2\left(q_1, q_2\right)\right)} \mid q_1\right]$$

□

Implications

1. 2) Contract has memory.
2. w_2 Depends not only on q_2 but also on q_1. whole point of paper is to show that agents are better off when they have long term contracts instead of period by period contracts.

- Suppose $w_2(q_1, q_2) = w_2(q_2)$. Then $\mathbb{E}\left[\frac{1}{u'(w_2(q_1,q_2))} \mid q_1\right] =$ constant (independent of q_1).
- Hence $\frac{1}{u'(w_1(q_1))} =$ constant $\Longrightarrow$ no incentives in period 1

3. The principal front-loads consumption.

- Jensen's inequality

$$\Longrightarrow \mathbb{E}\left[\frac{1}{u'(w_2(q_1,q_2))} \mid q_1\right] \geq \frac{1}{\mathbb{E}\left[u'(w_2(q_1,q_2)) \mid q_1\right]} \Longrightarrow u'(w_1(q_1)) \leq$$

$\mathbb{E}\left[u'(w_2(q_1,q_2)) \mid q_1\right]$

- Intuition: The principal forces the agent to consume more in the first period to keep his continuation wealth low, so that the marginal utility for money remains high.
- The agent would like to save (not borrow).

4. What does "front-loading consumption" mean?

- Suppose the agent has $\$W$ that he can consume over two periods. Then he solves

$$\max_{w_1}\left\{u(w_1) + u(W - w_1)\right\}$$

The first order condition implies that $u'(w_1) = u'(W - w_1)$, so that $w_1 = \frac{W}{2}$.

- Because $u'(\cdot)$ is decreasing, we say that consumption is front-loaded if $u'(w_1) \leq u'(w_2)$ (because $w_1 \geq w_2$), and back-loaded if $u'(w_1) \geq u'(w_2)$ (because $w_1 \leq w_2$).

Question 2

Consider the following principal-agent problem. Let $\Pi = \{\pi_1, \ldots, \pi_n\} \subset \mathbb{R}$ be the set of possible output levels, where $\pi_j \in \mathbb{R}$ for every j. An agent is able to exert effort at two levels, e_L and e_H. In addition, suppose that there is another random variable, or signal, y, with possible values $Y = \{y_1, \ldots, y_\ell\}$. Let

$$p(\pi, y \mid e)$$

denote the probability of output level π and signal value y when the agent's effort is e. Assume that the (marginal) distribution of output given e_H first- order stochastically dominates that given e_L, and that the agent is strictly risk averse, with preferences that are separable in effort and output.

A risk-neutral principal only cares about maximizing expected revenue net of wage costs. Assume that if the principal offers to the agent a profit-maximizing wage schedule, the agent will accept the contract and exert effort at level e_H.

a) Describe the principal's problem formally.

b) Write down the optimality conditions of this problem.

c) Show that if $p(\pi, y \mid e) = p_1(\pi \mid e)p_2(y)$ then the profit-maximizing wage schedule does not depend on y.

d) In general, any conditional probability mass function (CPMF) $p(\pi, y \mid e)$ can be reinterpreted as $p(\pi, y \mid e) = p_1(\pi \mid e)p_2(y \mid \pi, e)$ for some CPMFs p_1 and p_2. Show that if $p_2(y \mid \pi, e) = p_2(y \mid \pi)$ then the profit-maximizing wage schedule does not depend on y.

Solution 2

a) Comparing to notes I will expand here conditional expectation, so you will see non convexities in constraints.

P is facing following problem

$$\max_{w,e} \sum_y \sum_\pi p(y, \pi|e) \cdot (\pi - w(\pi, y))$$

$$\text{s.t.} \quad \text{IR} \quad \gamma \quad \sum_y \sum_\pi p(y, \pi|e) \cdot v(w(\pi, y)) - g(e) \geq \bar{U}$$

$$\text{IC} \quad \mu \quad e \in \arg\max_{\tilde{e}} \sum_y \sum_\pi p(y, \pi|\tilde{e}) \cdot v(w(\pi, y)) - g(\tilde{e})$$

b) Define as in notes

$$C(e) = \min_w \sum_y \sum_\pi p(y, \pi|e) \cdot w(\pi, y)$$

$$\text{IR} \quad \gamma \quad \sum_y \sum_\pi p(y, \pi|e) \cdot v(w(\pi, y)) - g(e) \geq \bar{U}$$

$$\text{IC} \quad \mu \quad \sum_y \sum_y \sum_\pi p(y, \pi|e) \cdot v(w(\pi, y)) - g(e) \geq \sum_\pi p(y, \pi|\tilde{e}) \cdot v(w(\pi, y)) - g(\tilde{e})$$

To take FOCs and apply KKT we need to transform problem to one with affine constraints beacuse without it IC is not even quasi concave.

let's switch from wages to utils : $\varphi = v^{-1} : \varphi(v(\pi, y)) = w(\pi, y)$ so we face following transformed problem:

$$C(e) = \min_{\bar{v}} \sum_y \sum_\pi p(y, \pi|e) \cdot \varphi(\bar{v}(\pi, y))$$

$$\text{IR} \quad \gamma \quad \sum_y \sum_\pi p(y, \pi|e) \cdot \bar{v}(\pi, y) - g(e) \geq \bar{U}$$

$$\text{IC} \quad \mu \quad \sum_y \sum_y \sum_\pi p(y, \pi|e) \cdot \bar{v}(\pi, y) - g(e) \geq \sum_\pi p(y, \pi|\tilde{e}) \cdot \bar{v}(\pi, y) - g(\tilde{e})$$

$$\frac{dL}{d\bar{v}}: \quad -\varphi'(\bar{v}(\pi,y))p(y,\pi|e) + p(y,\pi|e)\gamma + \mu\cdot[p(y,\pi|e) - p(y,\pi|\tilde{e})] \geq 0$$

so

$$-\varphi'(\bar{v}(\pi,y)) \geq \gamma + \mu\cdot[1 - \frac{p(y,\pi|e)}{p(y,\pi|\tilde{e})}]$$

go back to wages $\varphi = v^{-1} : \varphi(v\pi,y)) = w(\pi,y)$ and $\varphi' = \frac{1}{v'}$

$$\frac{1}{v'(w(\pi,y))} \geq \gamma + \mu\cdot[1 - \frac{p(y,\pi|e)}{p(y,\pi|\tilde{e})}]$$

and inequality holds for $C(e_H)$.

We want exert high effort so $e = e_H, \tilde{e} = e_L$,

$$\frac{1}{v'(w(\pi,y))} = \gamma + \mu\cdot[1 - \frac{p(y,\pi|e_L)}{p(y,\pi|e_H)}]$$

We can show that both $\gamma > 0$ and $\mu > 0 \Rightarrow$ Lemma 14 in notes

c) Suppose that $p(,y|e) = p_1(\pi|e)p_2(y)$ then equation above becomes

$$\frac{1}{v'(w(\pi,y))} = \gamma + \mu\cdot[1 - \frac{p_1(\pi|e_L)p_2(y)}{p_1(\pi|e_H)p_2(y)}] = \gamma + \mu\cdot[1 - \frac{p_1(\pi|e_L)}{p_1(\pi|e_H)}]$$

and it is case which does not have signal (so wages wont depend on it). And wage schemes implementaiton follow from what we did in lecture/notes.

d)Now suppose $p(,y|e) = p_1(\pi|e)p_2(y|\pi,e)$.

If $p_2(y|\pi,e) = p_2(y|\pi)$ then from b) above we have

$$\frac{p(,y|e_L)}{p(,y|e_H)} = \frac{p_1(\pi|e_L)p_2(y|pi)}{p_1(\pi|e_H)p_2(y|\pi)} = \frac{p_1(\pi|e_L)}{p_1(\pi|e_H)}$$

and we get exactly what we get in c). Again if this decomposition holds then wage does not depend on signal.

Question 3 [220 IV.1 Spring 2016 majors]

Optimal Auction

A seller owns an object, and values it at 0. There is a buyer with valuation $v \sim U[0,1]$. The seller does not know the buyer's valuation, and designs an optimal mechanism to fulfill some objective, whereby the seller asks for the buyer's valuation and then awards the object to the buyer with probability $q(v)$

and charges the buyer an amount of money $p(v)$ if the buyer reported a valuation v.

1. Assume that the seller wants to maximize own profit, $p(v)$.

   (a) Show that the seller's virtual surplus can be written as

$$2v - 1$$

   (b) Describe the seller's optimal auction.

2. Assume instead that the seller wants to maximize a weighted average of own profit, $p(v)$ (with weight $\alpha \in [0,1]$), and consumer surplus, $v - p(v)$ (with weight $1-\alpha$).

   (a) Show that the seller's virtual surplus can be written as

$$(3\alpha - 1)v + 1 - 2\alpha$$

   (b) Describe the seller's optimal auction as a function of $\alpha \in [0,1]$.

Solution 3

1. (a) There is $v_0 = 0$, and we have a standard setup of the model as in Myerson (1981), where the seller wants to maximize own profit. Virtual value of bidder i with value v_i drawn from F_i is defined as

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

   and the virtual surplus of a single-item auction is just the virtual value of the winner (see, e.g., Hartline, 2012).

   Since $v \sim U[0,1]$, then the PDF is $f(v) = 1$ and the CDF is $F(v) = v$. Therefore the seller's virtual surplus is

$$\phi(v) = v - \frac{1-v}{1} = 2v - 1$$

   (b) Note that the seller's virtual surplus is strictly increasing in v, i.e. the regularity as- sumption holds. From $2v-1=0$, we can derive the seller's "reservation price" which is $\frac{1}{2}$. The optimal auction is

$$q(v) = \begin{cases} 1, & \text{if } \frac{1}{2} \leq v \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   and

$$p(v) = \begin{cases} \frac{1}{2}, & \text{if } \frac{1}{2} \leq v \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Expected seller's revenue in this case is $\frac{1}{4}$. Note that the optimal auction may not be ex post efficient. Since $v_0 = 0$, then ex post efficiency requires that the buyer must always get the object, as long as his/her value estimate is positive. Therefore, the buyer would never admit to more than $\varepsilon \longrightarrow 0^+$ value estimate, because any positive bid would win the object. In this case the seller would have to expect zero revenue.

2. The problem is

$$\max_{p(\cdot)} \int_0^1 [\alpha p(v) + (1-\alpha)(v - p(v))] f(v) dv$$

s.t.

$$q(v) \in [0, 1]$$

$q(v)$ is nondecreasing

$$u(v) = u(0) + \int_0^v q(s) ds$$

Let's normalize $u(0) = 0$.

(a) Consider the objective function:

$$\int_0^1 [\alpha p(v) + (1-\alpha)(v - p(v))] f(v) dv =$$

$$= \int_0^1 (1-\alpha) v f(v) dv + \int_0^1 [\alpha p(v) + (2\alpha - 1) p(v)] f(v) dv =$$

$$= \int_0^1 (1-\alpha) v f(v) dv + \int_0^1 (2\alpha - 1)(p(v) - vq(v)) f(v) dv + \int_0^1 (2\alpha - 1) v q(v) f(v) dv =$$

$$= \int_0^1 [(1-\alpha) v + (2\alpha - 1) v q(v)] f(v) dv + (1 - 2\alpha) \int_0^1 u(v) f(v) dv =$$

$$= \int_0^1 [(1-\alpha)(v - vq(v)) + \alpha v q(v)] f(v) dv + (1 - 2\alpha) \int_0^1 \int_0^v q(s) ds f(v) dv =$$

$$= \int_0^1 (1-\alpha)(v - vq(v)) f(v) dv + \int_0^1 \alpha v q(v) f(v) dv +$$

$$+ (1 - 2\alpha) \left[\int_0^v q(s) ds F(v) \Big|_0^1 - \int_0^1 q(v) F(v) dv \right] =$$

$$\int_0^1 (1-\alpha)(v - vq(v)) f(v) dv + \int_0^1 \alpha v q(v) f(v) dv +$$

$$+ (1 - 2\alpha) \left[\int_0^v q(v) dv F(v) - \int_0^1 q(v) F(v) dv \right] =$$

$$\int_0^1 (1-\alpha)(v - vq(v))f(v)dv + \int_0^1 \alpha v q(v) f(v)dv + (1-2\alpha)\int_0^1 q(v)\frac{1-F(v)}{f(v)}f(v)dv =$$

$$= \int_0^1 q(v) \underbrace{\left[\alpha v + (1-2\alpha)\cdot\frac{1-F(v)}{f(v)}\right]}_{\text{virtual surplus}} f(v)dv + \int_0^1 (1-\alpha)(v-vq(v))f(v)dv$$

Thus, the seller's virtual surplus is

$$\phi(v) = \alpha v + (1-2\alpha)\cdot\frac{1-F(v)}{f(v)}.$$

Since $f(v) = 1$ and $F(v) = v$, then

$$\phi(v) = \alpha v + (1-2\alpha)(1-v) = (3\alpha-1)v + 1 - 2\alpha.$$

(b) In this case the seller's "reservation price" is defined from

$$(3\alpha-1)v + 1 - 2\alpha = 0 \quad \Rightarrow \quad \text{"reservation price"} = \frac{2\alpha-1}{3\alpha-1}.$$

Note that if $\alpha = \frac{1}{3}$, then $\phi(v) = \frac{1}{3}$ and it does not depend on the buyer's valuation. Consider two cases, excluding $\alpha = \frac{1}{3}$. Let $\frac{1}{3} < \alpha \leq 1$. Thus, the seller's virtual surplus is increasing in α, i.e. the regularity assumption holds, and we can proceed as usual. The optimal auction is

$$q(v) = \begin{cases} 1, & \text{if } \frac{2\alpha-1}{3\alpha-1} \leq v \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$p(v) = \begin{cases} \frac{2\alpha-1}{3\alpha-1}, & \text{if } \frac{2\alpha-1}{3\alpha-1} \leq v \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Let $0 \leq \alpha < \frac{1}{3}$. Thus, the seller's virtual surplus is decreasing in α, i.e. the regularity assumption does not hold. To handle this case we need to use the so-called "ironing" procedure (see pages 68 – 70, Myerson, 1981). Check it by yourself.

Question 4 The Social Value of Public Information [221 IV.2 Spring 2016 majors]

There is a continuum of agents, uniformly distributed on $[0, 1]$. Each agent $i \in [0, 1]$ chooses $a_i \in R$. Let a be the action profile. Agent i has utility function

$$u_i(a, \theta) = -\left[(1-r)(a_i - \theta)^2 + r\left(L_i - \bar{L}\right)\right]$$

where $r \in (0,1)$ is a constant, θ represents the state of the economy,

$$L_i = \int_0^1 (a_j - a_i)^2 \, dj \quad \text{and} \quad \bar{L} = \int_0^1 L_j dj$$

Intuitively, agent i wants to minimize the distance between his action and the true state θ, and also minimize the distance between his action and the actions of others. The parameter r represents the trade-off between these two objectives. Social welfare (normalized) is

$$W(a,\theta) = \frac{1}{1-r}\int_0^1 u_i(a,\theta)di = -\int_0^1 (a_i - \theta)^2 \, di$$

Agent i forms expectations $E_i[\cdot] = E[\cdot|\mathcal{I}_i]$ conditional on his information $\mathcal{I}_i$ and maximizes expected utility.

1. Show that each agent i 's optimal action is given by

$$a_i = (1-r)E_i[\theta] + rE_i[\bar{a}]$$

where $\pi = \int_0^1 a_j dj$ is the average action. Show that if θ is common knowledge then $a_i = \theta$ for every i is an equilibrium.

Solution

We can write the problem of agent i in the following way:

$$\max_{a_i} \mathbb{E}_i \left[- \left[(1-r)(a_i - \theta)^2 + r \left(\int_0^1 (a_j - a_i)^2 \, dj - \bar{L} \right) \right] \right]$$

Or

$$\max_{a_i} \mathbb{E}_i \left[- \left[(1-r)(a_i - \theta)^2 + r \left(\int_0^1 \left(a_j^2 + a_i^2 - 2a_j a_i\right) dj - \bar{L} \right) \right] \right]$$

FOC:

$$\mathbb{E}_i \left[-2(1-r)(a_i - \theta) - r\left(2a_i - 2\int_0^1 a_j dj\right) \right] = 0$$

that gives the desired result

$$a_i = (1-r)\mathbb{E}_i[\theta] + r\mathbb{E}_i[\bar{a}]$$

where $\bar{a} = \int_0^1 a_j dj$. Note that the true state of the economy θ and the average action $\bar{a}$ are not observable by the agent i, and thus he/she assigns some positive weights on the expectations over these values.

If θ is common knowledge, then in the equilibrium all the agents simply choose $a_i = \theta$, and thus

$\bar{a} = \theta$ as well. In this case $u_i(a, \theta) = 0, \forall i$, i.e. it attains a maximum. Moreover, $W(a, \theta) = 0$ in this case, i.e. the social welfare is also attains a maximum. Therefore, under the assumption of perfect information there is no trade-off between the socially optimal and individually rational actions.

2. Suppose that θ is drawn heuristically from a uniform prior over the real line. Agents observe a public signal

$$y = \theta + \eta$$

where $\eta \sim N(0, \sigma^2)$. Therefore, $\theta | y \sim N(y, \sigma^2)$. Now, agents maximize expected utility $E[u_i | y]$ given the same public information y. Show that $a_i(y) = y$ for every i is an equilibrium. Derive the following expression for welfare given θ :

$$E[W | \theta] = -\sigma^2$$

Solution

Now the problem of agent i is

$$\max_{a_i} \mathbb{E}\left[-\left[(1-r)(a_i - \theta)^2 + r\left(\int_0^1 \left(a_j^2 + a_i^2 - 2a_j a_i \right) dj - \bar{L} \right) \right] \mid y \right]$$

FOC:

$$\mathbb{E}\left[-2(1-r)(a_i - \theta) - r\left(2a_i - 2\int_0^1 a_j dj \right) \mid y \right] = 0$$

which can be simplified to

$$a_i(y) = (1-r)\mathbb{E}[\theta \mid y] + r\int_0^1 \mathbb{E}[a_j \mid y] \, dj$$

Note that $\mathbb{E}[\theta \mid y] = y$, and $\mathbb{E}[a_j \mid y] = a_j(y)$ since the strategies of the agents are measurable with respect to y. Therefore, in the unique equlibrium we have

$$a_i(y) = (1-r)y + r a_i(y)$$

which gives

$$a_i(y) = y$$

Regarding expected welfare, we have the following:

$$\mathbb{E}[W \mid \theta] = -\mathbb{E}\left[\int_0^1 (y-\theta)^2 di \mid \theta \right] = -\mathbb{E}\left[(y-\theta)^2 \mid \theta \right] = -\mathbb{E}\left[\eta^2 \mid \theta \right] = -\sigma^2$$

3. Assume now that, in addition to the public signal, each agent i observes a private signal

$$x_i = \theta + \epsilon_i$$

where $\epsilon_i \sim N(0, \tau^2)$ is (heuristically) independent across i and of θ and η. Let $\alpha = 1/\sigma^2$ and $\beta = 1/\tau^2$

a) Show that

$$E_i[\theta] = E[\theta | x_i, y] = \frac{\alpha y + \beta x_i}{\alpha + \beta}$$

Solution

We can treat α as the precision of public information, and β as the precision of private information (they are both reciprocal to corresponding variances). Note that now the information set $\mathcal{I}_i$ is given by the realizations of x_i and y.

We have the following:

$$y = \theta + \eta$$
$$x_i = \theta + \varepsilon_i$$

Indeed, $\mathbb{E}[\theta \mid x_i, y]$ is a linear MMSE estimator which can be obtained from the following linear combination of y and x_i (also it is useful to recall the properties of Bayes updating with normal random variables):

$$\mathbb{E}[\theta \mid x_i, y] = \omega_1(y - \mathbb{E}(\theta)) + \omega_2(x_i - \mathbb{E}(\theta)) + \mathbb{E}(\theta)$$

with the weights

$$\omega_1 = \frac{1/\sigma^2}{1/\sigma^2 + 1/\tau^2 + 1/\operatorname{var}(\theta)}, \quad \omega_2 = \frac{1/\tau^2}{1/\sigma^2 + 1/\tau^2 + 1/\operatorname{var}(\theta)}$$

Moreover, note that since θ is drawn heuristically from a uniform prior over the real lin e^2, then $\mathbb{E}(\theta) = 0, \operatorname{var}(\theta) = +\infty$. Combining all these arguments, we get the desired expression:

$$\mathbb{E}_i[\theta] = \mathbb{E}[\theta \mid x_i, y] = \frac{\alpha}{\alpha + \beta} y + \frac{\beta}{\alpha + \beta} x_i = \frac{\alpha y + \beta x_i}{\alpha + \beta}$$

b) Suppose that there is a number κ such that for every agent j

$$a_j(x_j, y) = \kappa x_j + (1 - \kappa) y$$

Compute the value of $E_i[\bar{a}]$ and show that following κ defines an equilibrium.

$$\kappa = \frac{\beta(1-r)}{\alpha+\beta(1-r)}$$

Solution

Suppose that $\exists\kappa$ such that $\forall j$

$$a_j(x_j, y) = \kappa x_j + (1-\kappa)y$$

Then agent's i conditional estimate of the average expected action across all the agents is

$$\begin{gathered}\mathbb{E}_i[\bar{a}] = \mathbb{E}_i\left[\int_0^1 a_j(x_j,y)\,dj\right] = \mathbb{E}_i\left[\int_0^1 (\kappa x_j + (1-\kappa)y)\,dj\right] = \\ = \kappa\mathbb{E}_i\left[\int_0^1 x_j dj\right] + (1-\kappa)y = \kappa\mathbb{E}_i\left[\int_0^1 (\theta+\varepsilon_j)\,dj\right] + (1-\kappa)y = \\ = \kappa\mathbb{E}_i[\theta] + (1-\kappa)y = \kappa\frac{\alpha y + \beta x_i}{\alpha+\beta} + (1-\kappa)y = \left(\frac{\kappa\beta}{\alpha+\beta}\right)x_i + \left(1 - \frac{\kappa\beta}{\alpha+\beta}\right)y\end{gathered}$$

where we use the result from part 3(a) in the third line. Recall from part 1 that the optimum is characterized by

$$a_i(x_i,y) = (1-r)\mathbb{E}_i[\theta] + r\mathbb{E}_i[\bar{a}].$$

Plugging in the expressions that we get above, we get

$$\begin{gathered}a_i(x_i,y) = (1-r)\mathbb{E}_i[\theta] + r\mathbb{E}_i[\bar{a}] = \\ = (1-r)\cdot\left[\tfrac{\alpha}{\alpha+\beta}y + \tfrac{\beta}{\alpha+\beta}x_i\right] + r\cdot\left[\left(\tfrac{\kappa\beta}{\alpha+\beta}\right)x_i + \left(1-\tfrac{\kappa\beta}{\alpha+\beta}\right)y\right] = \\ = \tfrac{\beta(1+r\kappa-r)}{\alpha+\beta}x_i + \left(1 - \tfrac{\beta(1+r\kappa-r)}{\alpha+\beta}\right)y\end{gathered}$$

Note that since we assume that $\exists\kappa$ such that for any agent j the following holds $a_j(x_j,y) = \kappa x_j + (1-\kappa)y-$ then, comparing with the equation for $a_i(x_i,y)$ above, we get

$$\kappa = \frac{\beta(1+r\kappa-r)}{\alpha+\beta}$$

Solving it for κ, one can easily get

$$\kappa = \frac{\beta(1-r)}{\alpha+\beta(1-r)}$$

which is a desired result.

4. Show that expected welfare is given by

$$E[W(a,\theta)|\theta] = -\frac{\alpha+\beta(1-r)^2}{[\alpha+\beta(1-r)]^2}$$

Show that

$$\frac{\partial E[W|\theta]}{\partial \beta} > 0$$

and $\frac{\partial E[W|\theta]}{\partial \alpha} \geq 0$ if and only if $\frac{\beta}{\alpha} \leq \frac{1}{(2r-1)(1-r)}$ Interpret and compare with part (b).

Solution

Plugging the expression for κ into the equilibrium condition for $a_i(x_i, y)$, we get

$$a_i(x_i, y) = \frac{\beta\left(1+r\frac{\beta(1-r)}{\alpha+\beta(1-r)}-r\right)}{\alpha+\beta} x_i + \left(1 - \frac{\beta\left(1+r\frac{\beta(1-r)}{\alpha+\beta(1-r)}-r\right)}{\alpha+\beta}\right) y =$$
$$= \frac{(1-r)\left(\beta^2+\alpha\beta\right)}{(\alpha+\beta(1-r))(\alpha+\beta)} x_i + \frac{\alpha(\alpha+\beta)}{(\alpha+\beta(1-r))(\alpha+\beta)} y = \frac{\alpha y+\beta(1-r)x_i}{\alpha+\beta(1-r)}$$

Alternatively, we can rewrite this expression in terms of the initially specified random variables:

$$a_i(x_i, y) = \frac{\alpha(\theta+\eta) + \beta(1-r)(\theta+\varepsilon_i)}{\alpha+\beta(1-r)} = \frac{\alpha\eta + \beta(1-r)\varepsilon_i}{\alpha+\beta(1-r)} + \theta$$

Now we can calculate expected welfare

$$\mathbb{E}[W(\mathbf{a}, \theta) \mid \theta] = -\mathbb{E}\left[\int_0^1 \left(\frac{\alpha\eta + \beta(1-r)\varepsilon_i}{\alpha+\beta(1-r)} + \theta - \theta\right)^2 di \mid \theta\right] =$$

$$= -\mathbb{E}\left[\left(\frac{\alpha\eta+\beta(1-r)\varepsilon_i}{\alpha+\beta(1-r)}\right)^2 \mid \theta\right] = -\mathbb{E}\left[\frac{\alpha^2\eta^2+\beta^2(1-r)^2\varepsilon_i^2+2\alpha\beta(1-r)\eta\varepsilon_i}{[\alpha+\beta(1-r)]^2} \mid \theta\right] =$$
$$= -\frac{\alpha^2\mathbb{E}\left(\eta^2\right)+\beta^2(1-r)^2\mathbb{E}\left(\varepsilon_i^2\right)}{[\alpha+\beta(1-r)]^2} = -\frac{\alpha^2(1/\alpha)+\beta^2(1-r)^2(1/\beta)}{[\alpha+\beta(1-r)]^2} = -\frac{\alpha+\beta(1-r)^2}{[\alpha+\beta(1-r)]^2}$$

Let's analyze, how expected welfare depends on the precision of private (β) and public (α) signals.

$$\frac{\partial\mathbb{E}[W|\theta]}{\partial\beta} = -\frac{(1-r)^2(\alpha+\beta(1-r))^2-2(1-r)(\alpha+\beta(1-r))\left(\alpha+\beta(1-r)^2\right)}{[\alpha+\beta(1-r)]^4} =$$
$$= \frac{(1-r)\left[\alpha(1+r)+\beta(1-r)^2\right]}{[\alpha+\beta(1-r)]^3} > 0$$

which follows from $\alpha > 0, \beta > 0, r \in (0,1)$ Thus, we can conclude that expected welfare increases as the precision of private infor- mation goes up.

$$\frac{\partial\mathbb{E}[W \mid \theta]}{\partial\alpha} = -\frac{(\alpha+\beta(1-r))^2 - 2(\alpha+\beta(1-r))\left(\alpha+\beta(1-r)^2\right)}{[\alpha+\beta(1-r)]^4} = \frac{a-\beta(1-r)(2r-1)}{[\alpha+\beta(1-r)]^3}$$

Since $\alpha > 0, \beta > 0, r \in (0,1)$, then the denominator is strictly positive. The sign of $\frac{\partial\mathbb{E}[W|\theta]}{\partial\alpha}$ depends on the sign of the numerator.
Indeed, $\frac{\partial\mathbb{E}[W|\theta]}{\partial\alpha} \geq 0$ if and only if $a - \beta(1-r)(2r-1) \geq 0$ or $\frac{\beta}{\alpha} \leq \frac{1}{(2r-1)(1-r)}$

Therefore, there exist parameter values under which an increase in the precision of publicsignals may decrease expected welfare. Note that high precision of public information high α) is attractive for expected welfare only when the private information is not very orecise (low β).

This result just partly confirms the idea that we get in part 2, where expected welfare s a strictly increasing function of public information precision (alternatively, a strictly decreasing function of the variance of public information). If we introduce private signals o the model, then the impact of public information on expected welfare decreases, and n the case of

$$\frac{\beta}{\alpha} > \frac{1}{(2r-1)(1-r)}$$

it is even harmful. Solution of last two problem comes from Egor Malkov.