Recitations 23

[Definitions used today]

- dictatorial game, Nakamura number, winning subset, pivotal player

Question 1 [285 IV.1 Fall 2019]

- Let N be a finite set of players. A simple game is a family $\mathbb{D}$ of so-called winning subsets of N.
- For example, simple majority with n voters is a simple game with winning subsets $\{D : |D| > n/2\}$.
- A game is gridlocked if there is a subset of N such that neither it nor its complement is winning. Thus, simple majority is gridlocked if and only if n is even. Individual i is pivotal if it belongs to every winning subset, that is, $i \in D$ for all $D \in \mathbb{D}$.
- A game is free if it has no pivotal individuals, and dictatorial if it has only one. For example, the game $\mathbb{D}_P = \{S \subset N : P \subset S\}$ is free if and only if P is empty, and dictatorial if and only if $|P| = 1$

a) Prove that if a simple game is neither free nor gridlocked then it is dictatorial.

Question 2 [286 IV.2 Fall 2019]

Given a simple game $\mathbb{D}$, define its Nakamura number as

$$\nu(\mathbb{D}) = \min_{\mathcal{F} \subset \mathbb{D}} \{|\mathcal{F}| : \bigcap \{F : F \in \mathcal{F}\} = \emptyset\}$$

with $\nu(\mathbb{D}) = \infty$ if the game is not free. The number ν computes the smallest number of winning subsets with empty intersection. For example, the Nakamura number of simple majority with n voters is 3 if $n = 3$ or $n > 4$.
A simple game $\mathbb{D}$ induces a social welfare function by the definition aPb if $\{i \in N : aP_ib\} \in \mathbb{D}$. Let A be a finite set of alternatives.

a) Prove that P is acyclic whenever P_i is acyclic for every individual $i \iff \nu(\mathbb{D}) > |A|$

b) The UN Security Council consists of fifteen members, five of which are permanent with veto power. Otherwise, majority voting prevails. Model this as a simple game and find its Nakamura number