Recitations 10

I was sloppy today with dimensions of Z and p so to be clear. There are I agents and M goods. So aggregate excess demand is from $\bar{\Delta} \subset \mathbb{R}^M$ to $\mathbb{R}^M$ and we are suming excess demand (what is more above endowments) over I agents. $\bar{\Delta}$ is in those notes closure of simplex (so $p_m \geq 0$, and Δ is interior of it ($p_m > 0$). Sorry for confusion.

> From the time of Adam Smith's Wealth of Nations in 1776, one recurrent theme of economic analysis has been the remarkable degree of coherence among the vast numbers of individual and seemingly separate decisions about the buying and selling of commodities. In everyday, normal experience, there is something of a balance between the amounts of goods and services that some individuals want to supply and the amounts that other, differerent individuals want to sell [sic]. Would-be buyers ordinarily count correctly on being able to carry out their intentions, and would-be sellers do not ordinarily find themselves producing great amounts of goods that they cannot sell. This experience of balance is indeed so widespread that it raises no intellectual disquiet among laymen; they take it so much for granted that they are not disposed to understand the mechanism by which it occurs.
>
> Kenneth Arrow (1973)

[Definitions used today]

- Existence of Equilibrium, Excess Aggregate Demand, Boundary Condition, Welfare Theorems,

Definition 0.1. ***Aggregate excess demand*** *$Z : \bar{\Delta} \to \mathbb{R}^M$ is defined as*

$$Z(p) = \sum_{i \in I} x_i(p, e_i) - \sum_{i \in I} e_i.$$

Common assumptions imposed on aggregate excess demand function (correspondence). $Z : \bar{\Delta} \to \mathbb{R}^M$ is a:

1. continuous function [1] on Δ
2. homogenous of degree 1
3. Walras law $p \cdot Z(p) = 0$
4. the boundary condition: $\forall \{p_n\} : p_n \to p \in \partial\Delta \Rightarrow \|Z(p_n)\| \to \infty$
5. $\det Dz(p) \neq 0$
6. $\sum_{i=1}^{n-1} p_i z_i(p)$ is nondecreasing function of p_n
7. $Z(p)$ is bounded from below

[1] or even differentiable function

0.1 Recap of properties

Proposition 0.2. *$B(\cdot, e_i) : \Delta \to X$ is a continuous (u.h.c. and l.h.c) correspondence for all $e_i \in \mathbb{R}^M_+$.*

Theorem 0.3. ***Well-defined demand correspondence***
If $B(p, e_i)$ is compact and $\succeq_i$ is a complete and transitive preorder with upper contour sets $U_i(x) = \{y \in X_i : \ y \succeq_i x\}$ that are closed for all $x \in B(p, e_i)$ then demand is well-defined (nonempty).

Proposition 0.4. *Let $\succeq_i$ be a complete, continuous and monotone preorder and let $e_i \in \mathbb{R}^M_+$. Then $x_i(\cdot, e_i)$ is a nonempty, compact-valued and u.h.c. correspondence on Δ.*

Proposition 0.5. *If $\succeq_i$ is complete, continuous, monotone and strictly convex, then $x_i(p, e_i) : \Delta \to X$ is a continuous function of p.*

Proposition 0.6. *If $\succeq_i$ is locally nonsatiated then $p \cdot x = p \cdot e_i, \ \forall p \in \Delta, \ \forall x \in x_i(p, e_i)$.*

Proposition 0.7. *If $e_i \geqq 0$, $e_i \neq 0$, and $\succeq_i$ is complete, continuous and strictly monotone, then for any $p_n \to p \in \partial\Delta$, $\|x_i(p_n, e_i)\| \to \infty$.*

Proposition 0.8. *If $\succeq_i$ is locally nonsatiated for all i, then $Z(p)$ satisfies Walras' law: $\forall p \in \Delta$, $p \cdot Z(p) = 0$.*

Proposition 0.9. *$Z(p)$ is bounded from below.*

Proof. This comes directly from the fact that for all i, the consumption set $X_i = \mathbb{R}^M_+$. This implies that consumers cannot have negative demand, so demand for each consumer is bounded from below by the zero vector. Then $Z(p)$ is bounded from below by $-\sum_{i \in I} e_i$. □

Definition 0.10. ***Boundary condition for correspondences***

Function *Let $Z : \Delta \to \mathbb{R}^l$ be an excess demand function. Z satisfies the boundary condition if $\forall \{p_n\} \subset \Delta, p_n \to p \in \partial\Delta$ implies $\|Z_n\| \to \infty$*

Correspondence *Let $\phi : \Delta \rightrightarrows \mathbb{R}^l$ be an excess demand correspondence. ϕ satisfies the boundary condition if $\forall \{p_n\} \subset \Delta, p_n \to p \in \partial\Delta$ implies $\forall Z_n \in \phi(p_n), \|Z_n\| \to \infty$*

Question 1 [Very Very Easy Existence Theorem]

Let $Z : \bar{\Delta} \to \mathbb{R}^l$ is a differentiable function that satisfies Walras' law ($\forall p \in \bar{\Delta} : p \cdot Z(p) = 0$), z is homogenous of degree 1 in p, $\det Dz(p) \neq 0$, the boundary condition: $p_n \to p \in \partial\Delta \Rightarrow \|Z(p_n)\| \to \infty$ and $\sum\limits_{i=1}^{n-1} p_i z_i(p)$ is nondecreasing function of p_n. Then $\exists p^* \in \bar{\Delta}$ such that $Z(p^*) \leq 0$. Further, $Z(p^*) = 0$ only if $p^* \in \Delta$.

Solution 1

Let $p \in \Delta$. Define $w = \sum\limits_{i=1}^{M-1} z_i(p)^2$, $B = \{p \in \Delta : \sum\limits_{i=1}^{M-1} z_i(p)^2 \leq 2w\}$ B is non empty $(p \in B)$, bounded $(p \in \Delta)$ closed (1 and 4). Additionally $\forall \bar{p} \in B$ first $n-1$ coordinates of $\bar{p}$ must be strictly > 0 (by 4). So $\sum\limits_{i=1}^{M-1} z_i(p)^2$ is continuous on B. Consider following problem

$$\min_p \sum_{i=1}^{M-1} z_i(p)^2 \qquad \text{s.t.} \quad p \in B$$

A solution exists. Call it p. It is equivalent to solve

$$\min_p \sum_{i=1}^{M-1} z_i(p)^2$$

$$\text{s.t.} \quad p_i \geq 0 \tag{0.1}$$

$$\sum_{i=1}^{n} p_i = 1 \tag{0.2}$$

$$\sum_{i=1}^{M-1} z_i(p)^2 \leq 2w \tag{0.3}$$

If $z_i(p) = 0$ the argument is done. (3) holds with $>$. $\lambda_2 = 0$ by Assumption 2. take FOCs:

$$0 = \frac{\partial z_i}{\partial p_j} \cdot z_i(p) = \det Dz(p) \cdot z(p) \quad \forall i = 1, \dots M-1$$

Thus $z_i(p) = 0$, $\forall i = 1, \dots M-1$. If $p_n \neq 0$ $z_M(p) = 0$ i.e. p is in equilibrium.
If $p_M = 0$ and $z_M(p) < 0$ then equilibrium exists. What about $z_M(p) > 0$ or $z_M(p) = +\infty$? Take $p'_M \neq 0$ but close to zero and observe

$$0 > -p'_M z_n(p_{-M}, p'_M) = \sum_{i=1}^{M-1} z_i(p_{-M}, p'_M) \geq 0$$

where first inequality comes from 1 and second from 6th assumption. Contradiction. So we showed existence.

Question 2 [Very Easy Existence Theorem]

Let $Z : \bar{\Delta} \to \mathbb{R}^l$ is a continuous function that satisfies Walras' law ($\forall p \in \bar{\Delta} : p \cdot Z(p) = 0$), then $\exists p^* \in \bar{\Delta}$ such that $Z(p^*) \leq 0$. Further, $Z(p^*) = 0$ only if $p^* \in \Delta$.

Solution 2

Steps

1. $\bar{\Delta}$ is nonempty, compact, and convex
2. Define $F : \bar{\Delta} \to \bar{\Delta}$ such that F is continuous.
3. Use Brouwer's Theorem to get a fixed point
4. Claim that the fixed point is a CE

Proof. **Step 1**- obvious

Step2

Define $G : \bar{\Delta} \to \mathbb{R}^M$ by

$$G(p) = (p_1 + \max(0, Z_1(p)), ..., p_l + \max(0, Z_l(p)))$$

Divide each component of $G(p)$ by the sum of all components:

$$\sum_{k=1}^{M}(p_k + \max(0, Z_k(p))) = \sum_{k=1}^{l} p_k + \sum_{k=1}^{M} \max(0, Z_k(p)) = 1 + \sum_{k=1}^{M} \max(0, Z_k(p))$$

Then we have $F : \bar{\Delta} \to \bar{\Delta}$

$$F(p) = \left(\frac{p_1 + \max(0, Z_1(p))}{1 + \sum_{k=1}^{M} \max(0, Z_k(p))}, ..., \frac{p_l + \max(0, Z_l(p))}{1 + \sum_{k=1}^{M} \max(0, Z_k(p))} \right)$$

Since $Z(\cdot)$ is continuous, $Z_i(\cdot)$ and $\max(0, Z_i(\cdot))$ are continuous. Also, $1 + \sum_{k=1}^{l} \max(0, Z_k(p)) \geq 1 > 0 \forall p \in \bar{\Delta}$. Thus, $F(p)$ is continuous $\forall p \in \bar{\Delta}$.

Step 3

Theorem 0.11. ***Brouwer's Fixed Point Theorem – continuous function***
Let $S \subset \mathbb{R}^n$ be nonempty, compact, and convex, and $f : S \to S$ be a continuous function. Then f has (at least) a fixed point in S, i.e. $\exists x^* \in S : x^* = f(x^*)$

By Brouwer's fixed point theorem, there exists p^* such that $p^* = F(p^*)$.

Step 4

$Z(p^*) \leq 0$ Suppose not. Then $\exists l$ such that $Z_l(p^*) > 0$. We have 2 cases:

- **Case 1:** $p_l^* > 0$

So $p_l^* Z_l(p^*) > 0$. By Walras' law, $\exists l' \neq l$ such that $p_{l'} Z_{l'} < 0$, which implies that $Z_{l'} < 0$ as $p_{l'} \geq 0$. Thus,

$$F_{l'}(p^*) = \frac{p_{l'}^*}{1 + \sum_{k=1}^{M} \max(0, Z_k(p^*))} < p_{l'}^*$$

which contradicts p^* being a fixed point.

- **Case 2:** $p_l^* = 0$

Then

$$F_{l'}(p^*) = \frac{Z_{l'}(p^*)}{1 + \sum_{k=1}^{M} \max(0, Z_k(p^*))} > 0 = p_{l'}^*$$

which also contradicts p^* being a fixed point.

Thus, $Z(p^*) \leq 0$.

If $p^* \in \Delta$, then $p_k > 0, \forall k \in \{1, ..., l\}$. Because Walras' Law implies that $p^* \cdot Z(p^*) = 0$, it must be that $Z_k(p^*) = 0, \forall k$. Thus, $Z(p^*) = 0$. □

Question 3 [Easy Existence Theorem]

Let $Z: \Delta \to \mathbb{R}^l$ be a continuous function that is bouded from below, satisfying Walras' Law and the boundary condition: $p_n \to p \in \partial\Delta \Rightarrow \|Z(p_n)\| \to \infty$. Then $\exists p^* \in \Delta$ such that $Z(p^*) = 0$.

Solution 3

Steps

- Z is defined on Δ not $\bar{\Delta}$ Define $\mu : \bar{\Delta} \rightrightarrows \bar{\Delta}$.
- Show that is nonempty, convex-valued, u.h.c.
- Use Kakutani's to find a fixed point in $\bar{\Delta}$
- Argue that the fixed point is in Δ and it is CE.

Proof. **Step 1.** Define $\mu : \bar{\Delta} \rightrightarrows \bar{\Delta}$ by

$$\mu(p) = \begin{cases} \{\bar{q} \in \bar{\Delta} | \bar{q} \in argmax_{q \in \bar{\Delta}} q \cdot Z(p)\}, & \text{if } p \in \Delta \\ \{\bar{q} \in \bar{\Delta} | \bar{q} \cdot p = 0\}, & \text{if } p \in \partial\Delta \end{cases}$$

Note that $\bar{\Delta}$ is nonempty, compact, and convex. We need to show that μ is nonempty, convex-valued, and has a closed graph, to use Kakutani's Thm.

Step 2

Nonempty-valued Let $p \in \Delta$, then since $q \cdot Z(p)$ is continuous for $q \in \bar{\Delta}$ and $\bar{\Delta}$ is compact, by Weierstrass' Theorem, $argmax_{q \in \bar{\Delta}} q \cdot Z(p) \neq \emptyset$. So $\mu(p) \neq \emptyset, \forall p \in \Delta$.

Let $p \in \partial\Delta$, then $\exists q \in \bar{\Delta}$ such that for each k with $p_k > 0, q_k = 0$. So $q \cdot p = 0, q \in \mu(p)$. Thus, $\mu(p) \neq \emptyset, \forall p \in \partial\Delta$.

Convex-valued Let $p \in \Delta$, and let $q', q'' \in \mu(p), \lambda \in [0,1]$. Then $\forall q \in \bar{\Delta}$:

$$q' \cdot Z(p) \geq q \cdot Z(p) \text{ and } q'' \cdot Z(p) \geq q \cdot Z(p)$$

Then

$$\begin{aligned}(\lambda q' + (1-\lambda)q'') \cdot Z(p) &= \lambda q' \cdot Z(p) + (1-\lambda)q'' \cdot Z(p) \\ &= \lambda(q' \cdot Z(p)) + (1-\lambda)(q'' \cdot Z(p)) \\ &\geq \lambda(q \cdot Z(p)) + (1-\lambda)(q \cdot Z(p)) \\ &= q \cdot Z(p)\end{aligned}$$

Thus, $\mu(p)$ is convex-valued $\forall p \in \Delta$.

Let $p \in \partial\Delta$, and let $q', q'' \in \mu(p), \lambda \in [0,1]$. Then $q' \cdot p = q'' \cdot p = 0$. Then

Thus, $\mu(p)$ is convex-valued $\forall p \in \partial\Delta$.

UHC We also want to show that $f(p)$ is upper hemicontinuous. since it is compact valued, it suffices to establish that $\mu(p)$ has a closed graph. We want to show that $\forall \{p^n\}_{n=1}^{\infty} : p^n \to p$, and $\forall \{q^n\}_{n=1}^{\infty} : q^n \to q$, such that $\forall n : q^n \in \mu(p^n)$, it is true that $q \in f(p)$

1. Suppose $p \in \Delta$. Then $p^n \in \Delta$ for large n. since $q^n \cdot z(p^n) \geq q' \cdot z(p^n)\ \forall q' \in \bar{\Delta}$ and since $z(p)$ is continuous, we have that

$$q \cdot z(p) \geq q' \cdot z(p) \Longrightarrow q \in f(p)$$

2. Let $p \in \partial\Delta$. Fix some $i : p_i > 0$. We want to show that $\exists \hat{N} : \forall n > \hat{N} : q_i^n = 0$, and thus $q_i = 0$, so $q \in f(p)$. since $p_i > 0, \exists \varepsilon > 0 : p_i^n > \varepsilon$ for large n.
   - If $p^n \in \partial\Delta$, then $q_i^n = 0$ by construction of $f(p)$ in step 2, so $q_i^n = 0$ for large n, what we needed
   - If $p^n \in \Delta$, then $\exists M : \forall n > M$ we have $z_i(p^n) < \max_{i=1,\ldots,L}\{z_i(p^n)\}$ so $q_i^n = 0$ by construction in step 1. This is true since $\max_{i=1,\ldots,L}\{z_i(p^n)\} \to \infty$, because $p^n \to p \in \partial\Delta$, but $z_i(p^n)$ is bounded:

   $$p_i^n > \varepsilon \Longleftrightarrow \frac{1}{\varepsilon}p_i^n > 1 \Longrightarrow z_i(p^n) \leq z_i(p^n)\frac{1}{\varepsilon}p_i^n$$

   By Walras' Law:

   $$\begin{gathered} z_i(p^n)\tfrac{1}{\varepsilon}p_i^n = -\tfrac{1}{\varepsilon}\textstyle\sum_{j\neq i}^{L} p_j^n z_j(p^n) \\ \text{since } z_i(p) \geq -B, -z_i(p) \leq B, \text{ so:} \\ -\tfrac{1}{\varepsilon}\textstyle\sum_{j\neq i}^{L} p_j^n z_j(p^n) \leq \tfrac{1}{\varepsilon}\sum_{j\neq i}^{L} p_j^n B \leq \tfrac{1}{\varepsilon}\sum_{j=1}^{L} p_j^n B = \tfrac{B}{\varepsilon} \end{gathered}$$

Thus $z_i(p^n) \leq \frac{B}{\varepsilon}$, so it is bounded. Therefore if $p^n \in \Delta$ we have that $z_i(p^n) < \max_{i=1,\ldots,L}\{z_i(p^n)\}$ for large n thus $\mu(p^n) = \{q^n \mid q_i^n = 0\}$. So if $p \in \partial\Delta$, if $p_i > 0$ for some $i = 1, \ldots, M, q_i^n = 0$ for large n, so $q_i = 0$, and hence $q \in f(p)$. This establishes that the correspondence has closed graph (and hence, is uhc).

Step 3

Theorem 0.12. ***Kakutani's Fixed Point Theorem – u.h.c. correspondence*** *Let $S \subset \mathbb{R}^n$ be nonempty, compact, and convex, and $\Gamma : S \rightrightarrows S$ be a nonempty, convex-valued, and u.h.c. correspondence. Then Γ has a fixed point in S, i.e. $\exists x^* \in S : x^* \in \Gamma(x^*)$*

By Kakutani's fixed point theorem, $\exists p^* \in \bar{\Delta}$ such that $p^* \in \mu(p^*)$.

Step 4

Claim: $p^* \in \Delta$.

Suppose not, i.e. $p^* \in \partial\Delta$, then by $p^* \in \mu(p^*), p^* \cdot p^* = 0$, contradiction!

Therefore, $p^* \cdot Z(p^*) = \max_{q \in \bar{\Delta}} q \cdot Z(p^*)$. If $\exists l\ z_l(p^*) \neq 0$ then there is one for whihc < 0. By maximization $q_l = 0$ so $q \in \partial\Delta$ so $\mu(p^*) = \{q | q_l = 0\} \in \partial\Delta$ but $p \in \mu(p^*) \in \partial\Delta$ contradicts $p \in \Delta$. So $z(p^*) = 0$ for some $p^* \in \Delta$ □

Question 4 [Final 2017] Suppose in an I agent,2 good world, prices are normalized such that $p_1 + p_2 = 1$ (with p_1 and p_2 non-negative) and excess demand for good 1,$x_i(pi) : (0,1) \to \mathbb{R}$ is continuous.Let$P_1(p_1) \subset [0,1]$ specify Debreu's correspondence for proving existence.

a) What is the set $P_1(0)$? What is the set $P_1(l)$?

b) What is the set $P_1(p_i)$ if $x_1(p_1) > 0$? What is the set $P_1(p_1)$ if $x_1(p_1) < 0$?

c) What is the set $P_1(p_1)$ if $x_1(p_1) = 0$?

d) Graph the correspondence assuming $x_1(p_1) > 0$ for some $p_1 \in (0,1)$, $x_1(p_1) < 0$ for some $p_1 \in (0,1)$ and is decreasing.Is this enough for the existence of a fixed point

Solution 4

1. What is the set $P_1(0)$? What is the set $P_1(1)$?

For convenience, let $\Delta \equiv \left\{p \in \mathbb{R}_+^2 \mid p_1 + p_2 = 1\right\}$ since $p_1 + p_2 = 1$, in either case it must be that $p \in \partial\Delta$. Thus the set P_1 is:

$$P_1(0) = \{q \in \Delta \mid q_2 = 0\} = \{1\}$$
$$P_1(1) = \{q \in \Delta \mid q_1 = 0\} = \{0\}$$

2. What is the set $P_1(p_1)$ if $x_1(p_1) > 0$? What is the set $P_1(p_1)$ if $x_1(p_1) < 0$?

First note in either case it must be that $p \in \Delta^\circ$, since $x_1(\cdot)$ is only defined on Δ°. Also note, by Walras' law, if $x_1(p_1) > 0$, then $x_2(p_2) < 0$, and vice-versa. Thus, if $x_1(p_1) > 0$, the set $P_1(p_1)$ is:

$$P_1(p_1) = \{q \in \Delta \mid q_2 = 0\} = \{1\}$$

If $x_1(p_1) < 0$, the set $P_1(p_1)$ is:

$$P_1(p_1) = \{q \in \Delta \mid q_1 = 0\} = \{0\}$$

3. What is the set $P_1(p_1)$ if $x_1(p_1) = 0$?

As in part (c), it must be that $p \in \Delta^\circ$. Further, by Walras' law, since $x_1(p_1) = 0$, it must be that $x_2(p_2) = 0$. Thus the set P_1 is:

$$P_1(p_1) = \{q_1 \in [0,1]\}$$

4. Graph the correspondence assuming $x_1(p_1) > 0$ for some $p_1 \in (0,1)$, $x_1(p_1) < 0$ for some $p_1 \in (0,1)$ and is decreasing. Is this enough for the existence of a fixed point?

If a fixed point p^* exists, then by construction it must be that $p^* \in \Delta^\circ$ and also $x_1(p_1^*) = 0$. Now note from parts (a) – (c) that given any p_1(which is taken from a nonempty, compact, convex subset of $\mathbb{R}$), the set $P_1(p_1)$ is either a singleton or the unit simplex. In either case, these sets are nonempty, compact, and convex, and thus P_1 is nonempty-valued, compact-valued, and convex-valued.

Now take any sequences $\{p^n\}, \{q^n\} \in \Delta^{\mathbb{N}}$ such that $p^n \to p, q^n \to q$, and $q_1^n \in P_1(p_1^n)\ \forall n$. Consider first the case where $p \in \Delta^\circ$. It follows that $\exists, N \in \mathbb{N}$ such that $\forall n \geq N\ p^n \in \Delta^\circ$. Then... P_1 is u.h.c. Thus by the Kakutani Fixed Point Theorem, there exists a fixed point.