Recitations 2

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RECITATION 2

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CHECK OUT MATERIALS ON MY WEBPAGE

Next time:

- Homework 1
- Consumer theory

Recitation 2

[Definitions used today]

- (conditional) factor demand, cost function, Shephard's lemma, Hotelling's lemma
- Δ-monotone, homogeneous, positive definite matrix, correspondence, upper hemicontinuity (UHC)

Question 1 [Properties of π^* and s^*] 33 [I.1 Fall 2006 majors]

Suppose that production set Y is closed. Let $s^*(p)$ denote supply at price level p and by $\pi^*(p)$ corresponding profit level. Then the following properties hold:

1. π^* is homogeneous of deg. 1 in prices p
2. π^* is a convex function in prices p
3. **correspondence** s^* is homogeneous of deg. 0
4. s^* is Δ-monotone, that is:
$$[s^*(p) - s^*(p')] \cdot [p - p'] \geq 0 \quad \forall p, p'$$
5. **Hotelling's Lemma**: If π^* is differentiable at p (this holds iff s is single-valued at p), then
$$D\pi^*(p) = s^*(p)$$
6. Assuming that π^*,s^* are differentiable at $p \in \mathbb{R}^n$ prove comparative statics **law of supply**:
$$\frac{\partial s_i}{\partial p_i}(p) \geq 0$$
7. If Y is compact, then π^* is a continuous function and s^* is an upper hemicontinuous (UHC) correspondence.

Question 2 [Zero profit CRS]

If Y exhibits CRTS, then $\pi^*(p) = 0$ whenever it is well-defined.

Question 3 [Properties of C and x]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a production function that is strictly increasing (continuous) and satisfies $f(0) = 0$. Let $C^*(w, z)$ be the (minimum) cost function, where $w \in \mathbb{R}^n$ is a vector of input prices and $z > 0$ is an output level. Let $x^*(w, z)$ be the optimizer of cost minimization problem. Prove following properties:

1. $C^*(w, z)$ is homogeneous of degree 1 in factor prices w
2. $C^*(w, z)$is a concave function of w
3. $x^*(w, z)$ is homogeneous of degree zero in w.
4. $x^*(w, z)$ is Δ-monotone for fixed z, in following way:
$$[x^*(w, z) - x^*(w', z)] \cdot [w - w'] \leq 0 \quad \forall w, w' \gg 0$$
5. **Shephard's Lemma** If $C^*(w, z)$ is differentiable at w (this holds $\iff$ $x^*(w, z)$ is single-valued) then
$$D_w C^*(w, z) = x^*(w, z)$$
6. Assuming that C^*, x^* are differentiable at $w \in \mathbb{R}^n$ prove comparative statics property of factor demand:
$$\frac{\partial x_i}{\partial w_i}(w, z) \leq 0$$

7. Show that cost function C is a non-decreasing function of output level z, for every $w \gg 0$.

8. If production function f is concave, then cost function C is a convex function of output level z, for every $w \gg 0$

Question 4 [Aggregation]

Consider two closed production sets $Y_1, Y_2 \subseteq \mathbb{R}^L$ such that $0 \in Y_1$ and $0 \in Y_2$. Let π_1^* and π_2^* denote the profit functions associated with Y_1 and Y_2. Let π^* be the profit functions associated with Y.

1. Let $Y = Y_1 + Y_2$ be the (algebraic) sum of the two production sets. Prove that $\pi_1(p) + \pi_2(p) = \pi(p)$ for every $p \in \mathbb{R}^L$

2. Prove that $Y_1 \subseteq Y_2$ if and only if $\pi_1(p) \leq \pi_2(p)$

3. Let $Y = \text{co}\{Y_1, Y_2\}$ be the convex hull of the two production sets (that is, the set of all convex combinations of elements of Y_1 and Y_2). Prove that $\pi(p) = \max\{\pi_1(p), \pi_2(p)\}$for every for every $p \in \mathbb{R}^L$

Question 5 [Midterm 2006]

Consider the following supply function of a firm

$$s(p_1, p_2) = \left(- \frac{2p_2}{p_1}, \frac{p_2}{p_1} \right)$$

Show that this supply function can not result from profit maximization on any production set.

Ex. 2

1) Show that $0 \in Y$

2) Show that $\pi^*(p) \geq 0$

3) Show that if $|\pi(p)| < +\infty$
$\Rightarrow \pi^*(p) \leq 0$

1) Let $y \in Y$ by Y CRS $\forall \lambda \geq 0$
$\lambda y \in Y$. So pick $\lambda = 0$, $0 \cdot y = 0 \in Y$

2) Since $\pi^*(p) = \sup_{y \in Y} p \cdot y \geq p \cdot y \ \forall y \in Y$
From step 1 $0 \in Y$ $\pi^*(p) \geq p \cdot 0 = 0$

3) (By contradiction) Suppose not. $\exists p$ $|\pi(p)| < +\infty$ and $\pi(p) > 0$.
Then $\exists y \in Y$ s.t. $0 < \underbrace{\frac{\pi(p)}{2} < py \leq \pi(p)}_{(*)}$

I used in (*) definition of Supremum, i.e.:
$\alpha = \sup A$ then $\forall \varepsilon > 0$ $\exists \beta$ $\alpha - \varepsilon < \beta \leq \alpha$

Take (*), multiply by 3
$$\pi(p) < \frac{3}{2}\pi(p) < 3py \leq 3\pi(p)$$

Now observe that $y \in Y$ $\lambda = 3$ $3 \cdot y \in Y$
$\pi(p) = \sup_{y \in Y} p \cdot y < p \cdot (3y)$. This is a
CONTRADICTION ↯

By step 2 & 3 we conclude that $\pi(p) \leq 0 \leq \pi(p) \Rightarrow \pi^*(p) = 0$

Ex. 3

Def. The problem of cost minimization for a producer with production function $f: \mathbb{R}^n_+ \to \mathbb{R}_+$:

$$c^*(w, z) = \inf \; w \cdot x \quad \text{s.t.} \quad f(x) \geq z, \; x \geq 0$$

- $w = (w_1, \dots, w_n) \gg 0$ Input prices
- $x = (x_1, \dots, x_n) \geq 0$ inputs
- $z \geq 0$ output
- $\forall f$ Assume: closed, (convex), Free disposal

Def. (Conditional) factor demand

$$x^*(w, z) = \operatorname{argmin} \left\{ w \cdot x \;\; \text{s.t.} \; f(x) \geq z, \; x \geq 0 \right\}$$

Def. $g: \mathbb{R}^n \to \mathbb{R}$ it is Homo(k)

$\forall \lambda \geq 0 \; \forall x \in \mathbb{R}^n \quad g(\lambda x) = \lambda^k g(x)$

Q3.3. $x(\lambda w, z) = \{ x^* \in \mathbb{R}^n :$

$(\lambda w)\cdot x^* \leq (\lambda w)\cdot x \;\&\; f(x) \geq z \} =$

$= \{ w \cdot x^* \leq w \cdot x \;\&\; f(x) \geq z \} =$

$= x(w, z)$

Q3.1. $\lambda \geq 0$

$c^*(\lambda w, z) = (\lambda w) \circ x^*(\lambda w, z) \overset{\text{by Q3.3}}{=}$

$= \lambda (w \circ x^*(w, z)) = \lambda \cdot c^*(w, z)$

Q3.8. Take $z_1 \neq z_2$

$\forall x_1 \geq 0 \quad f(x_1) \geq z_1 \qquad (\lambda)$

$\forall x_2 \geq 0 \quad f(x_2) \geq z_2 \qquad (1-\lambda)$

Take $\lambda \in [0, 1]$

$f(\lambda x_1 + (1-\lambda) x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$

$\geq \lambda z_1 + (1-\lambda) z_2$

$$f(\lambda x_1 + (1-\lambda) x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$$
$$\geq \lambda z_1 + (1-\lambda) z_2$$

$$c^*(w, \lambda z_1 + (1-\lambda) z_2) \leq w \cdot (\lambda x_1 + (1-\lambda) x_2)$$
$$= \lambda (w x_1) + (1-\lambda) \cdot (w x_2) \quad (*)$$

x_1, x_2 we arbitrary $\qquad f(x_i) \geq z_i$

~~x_1~~ Take $\quad x_1 = x^*(w, z_1)$

$\qquad\qquad x_2 = x^*(w, z_2)$

Plug it into $(*)$ and get

$$c^*(w, \lambda z_1 + (1-\lambda) z_2) \leq$$
$$\lambda\, c^*(w, z_1) + (1-\lambda) c^*(w, z_2)$$

Q.3.4. $\forall\, w, w' \gg 0$ fixed z

$$[x(w,z) - x(w',z)] \cdot [w - w'] =$$
$$= [w \cdot x(w,z) - w x(w', z)] +$$
$$+ [w' x(w', z) - w' x(w, z)] =$$
$$\leq 0 + 0 \leq 0$$

we know that $\quad w \cdot x(w,z) \leq w \cdot x$

Take $x = x(w', z)$ to obtain 1st

Q 3.7. Take w_1, w_2 $w_1 \geq w_2$, fix z.

Take x: $f(x) \geq z$. Obviously

$$w_1 x \geq w_2 x \quad (*)$$

Now

$$w_1 \cdot x \overset{(1)}{\geq} w_1 \cdot x(w_1, z) \overset{(2)}{\geq} w_2\, x(w_1, z) \overset{(3)}{\geq} w_2 \cdot x(w_2, z)$$

Observe that $f(x) \geq z$ does not depend on w.

(1) comes from optimality of $x(w_1, z)$ at w_1

(2) It is $(*)$ for $x = x(w_1, z)$

(3) Comes from optimality of $x(w_2, z)$ at w_2

Q 3.5 Shephard's lemma

$$f(w, y) = w \cdot y$$

$$c^*(w, z) = w \cdot x^*(w, z)$$

$$D_w c^*(w, z) = D_w f(w, x^*(w, z)) =$$

$$D_w f(w, y)\Big|_{y = x^*(w, z)}$$

$$+ D_z f(w, y(w, z)) \cdot D_w x(w, z)$$

$$D_w f(w, y) = y$$

$$D_w c^*(w, z) = y^*(w, z) = x^*(w, z)$$

$$D_w c^*(w, z) = x^*(w, z) \quad \boxed{D^2_w c = D_w x(w, z)}$$

$$0 \geq \frac{\partial^2 \pi^*(w,z)}{\partial w_i^2} = \frac{\partial x^*(w,z)}{\partial w_i}$$

Q 3.6.

Q.5 Look at S-monotone property

- $\frac{\partial S}{\partial p_i} \geq 0 \quad \forall i$

Q.5

- S - monotone

Pick $p = (1,1)$ $\quad p' = (x,y)$

$p' = \left(\frac{1}{3}, \frac{1}{6}\right)$

$$S(p) = \left(-\frac{2p_2}{p_1}, \frac{p_2}{p_1}\right)$$

$S(p) = (-2,1)$

$S(p') = \left(-1, \frac{1}{2}\right)$

$$[S(p)-S(p')] \cdot [p-p'] = \left(-1, \frac{1}{2}\right) \cdot \left(\frac{2}{3}, \frac{5}{6}\right)$$

$$= -\frac{2}{3} + \frac{5}{12} = -\frac{8}{12} + \frac{5}{12} = -\frac{3}{12} < 0$$

So ~~sa~~ this function can not come from π - maximization

Q.4. b

$Y_1 \subseteq Y_2 \iff \Pi_1(p) \leq \Pi_2(p)$

$\Rightarrow \quad Y_2 = (Y_2 \setminus Y_1) \cup Y_1 \;, \; Y_1 \cap (Y_2 \setminus Y_1) = \emptyset$

$$\max_{x \in Y_2} p \cdot x = \max_{x \in (Y_2 \setminus Y_1) \cup Y_1} p \cdot x$$

$$\overset{(*)}{=} \max \left\{ \max_{x \in (Y_2 \setminus Y_1)} p \cdot x \,,\, \max_{x \in Y_1} p \cdot x \right\}$$

$$\overset{(**)}{\geq} \max_{x \in Y_1} p \cdot x$$

(*) $\max\limits_{\substack{x \in A \cup B \\ A \cap B = \emptyset}} f(x) = \max\left\{ \max\limits_{x \in A} f(x), \max\limits_{x \in B} f(x) \right\}$

(**) $\max\{a, b\} \geq b$

$\Leftarrow$ (By contradiction) Suppose not

$\cancel{Y_1 \subseteq Y_2}$ There will $x \in Y_1$, $x \notin Y_2$

$\{x\}$, Y_2 they are both closed, convex, nonempty

$\{x\}$ is bounded $\Rightarrow$ $\{x\}$ is compact

Apply Seperating Hyperplane Thm

(assume X, Y nonempty, closed, convex, X ~~both~~ bounded) then strict seperation holds:

($\exists$ p - vector, b - scalar)

$$p \cdot x > b > p \cdot y$$

$\forall x \in X$, $y \in Y$

In our case that

$\exists$ p - vector b - scalar

$\pi_1(p) \geq p \cdot x > b > p \cdot y \quad y \in Y_2$

Take sup

$\pi_1(p) > \pi_2(p)$ This is $\lightning$

7. If Y is compact, then π^* is a continuous function and s^* is an upper hemicontinuous (UHC) correspondence.

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