Recitations 5

JAKUB PAWELCZAK

MINI I

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RECITATION 5

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Today:
- Time-separable u
- state-separable, EU
- Topkis again

Office Hours: SPM's, Topkis, Milgrom Shannon

Next time: Risk, Uncertainty

Recitation 5

[Definitions used today]

- Topkis theorem, Supermodularity, Increasing Differences
- Recursive and dynamically consistent family of utility function, time separablity, ICC axiom

Question 1

Consider the following utility functions

a $u(c_1, c_2, c_3) = \min\{2c_1 + c_2 + c_3, c_1 + c_2 + 2c_3\}$

b $u(c_1, c_2, c_3) = c_1 + \sqrt{c_2} + \sqrt{c_3}$

1. Show that (a) does not have state-separable representation
2. Show that (b) does not have expected utility representation
3. Find $\pi \in \Delta \subset \mathbb{R}^3$ s.t. $u(c_1, c_2, c_3)$ is strictly risk averse with respect to π
4. Show that there is no $\pi\Delta \subset \mathbb{R}^3$ s.t. $u(c_1, c_2, c_3)$ is strictly risk averse with respect to π

Question 2 [Properties of state separable u]

a Prove that every recursive family of utility functions $\{U_t\}$ is dynamically consistent if the aggregator function $G(\cdot, \cdot)$ is strictly increasing in continuation

b $S \geq 3$ and $\succeq$ increasing and continuous. Prove that $\succeq$ has state-separable representation then ICC holds.

c For $S = 2$ all increasing functions obey ICC. Show that for $u(c_1, c_2) = c_1\sqrt{c_2} + c_1 + c_2$ it does not have state separable utility function

Question 3 [Topkis theorem]

If S is a lattice, f is supermodular in x for fixed t, and f has nondecreasing differences in $(x; t)$, then $\varphi^*(t) = \arg\max_{x\in S} f(x, t)$ is monotone nondecreasing in t.

Question 4 254 [I.1 Spring 2018 majors]

Consider the problem of finding a Pareto optimal allocation of aggregate resources $\omega \in \mathbb{R}^n_+$ in an economy with two agents:

$$\max_x \mu_1 u_1(x) + \mu_2 u_2(\omega - x)$$
$$\text{subject to } 0 \leq x \leq \omega$$

where $u_i : \mathbb{R}^n_+ \to \mathbb{R}$ are agents' utility functions (assumed continuous) and $\mu_i > 0$ are welfare weights for $i = 1, 2$. Let $x^*(\mu_1, \mu_2)$ be the set of solutions.

a State a definition of utility function u_i being supermodular. Show that if u_i is supermodular, then $u_i(\omega - x)$ is supermodular in x

b Show that, if u_1 and u_2 are strictly increasing and supermodular in x then $x^*(\mu_1, \mu_2)$ is non-decreasing in μ_1. You may assume that $x^*(\mu)$ is single-valued. Is $x^*(\mu_1, \mu_2)$ non-increasing in μ_2? Justify your answer. If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.

c Under what conditions on u_1 and u_2 is the solution $x^*(\mu_1, \mu_2)$ unique. Justify your answer.

$\succcurlyeq \quad c = (c_0, c_1, \dots, c_t, \dots) \qquad c \in l_\infty$

$\sup_t |c_t| < +\infty$

Def. $\succcurlyeq$ has time separable utility

if $\exists\, v_t : \mathbb{R}_+ \to \mathbb{R}$

$\forall\, c, c' \qquad c' \succcurlyeq_p c \iff \sum_{t=0}^{\infty} v_t(c'_t) \geq \sum_{t=0}^{\infty} v(c_t)$

Def. (ICC) Irrelevant of Common Consequences

$c_{-t}\, y = (c_0, \dots, c_{t-1}, \underbrace{\boxed{y}}_{th}, c_{t+1}, \dots)$

$\forall\, c, d \in l_\infty$, $y, w \in \mathbb{R}_+$ $\qquad c_{-t}\, y \succcurlyeq_p d_{-t}\, y \iff$

$\iff c_{-t}\, w \succcurlyeq_p d_{-t}\, w$

2(b) NTS: $\succcurlyeq$ time separable $\Rightarrow$ ICC holds

we have $S \geq 3$

Proof:

$c \succcurlyeq c' \iff \sum_{t=0}^{\infty} v_t(c_t) \geq \sum_{t=0}^{\infty} v_t(c'_t)$

take $c_t = c'_t = y$

$$\sum_{s\neq t}^{\infty} V_s(c_s) + V_t(y) \geq \sum_{s\neq t}^{\infty} V_s(c_s') + V_t(y)$$

let's add to both sides $V_t(w) - V_t(y)$

$$\sum_{s\neq t}^{\infty} V_s(c_s) + V_t(w) \geq \sum_{s\neq t} V_s(c_s') + V_t(w)$$

$$\Longleftrightarrow \quad c_{-t}w \geq c'_{-t}w$$

wrap up:

$$c_{-t}y \geq c'_{-t}y \Rightarrow c_{-t}w \geq c'_{-t}w$$

ICC

2(b) $S=2$ all str. incr. functions satisfy ICC

It will be necessary but not sufficient

time separability

$u(c_1, c_2) = c_1\sqrt{c_2} + c_1 + c_2$

Take $\alpha \in \mathbb{R}$ $\quad u(0,\alpha) = \alpha = u(\alpha, 0)$

Let' ~~s~~ assume that we have time-sep utility rep. $\exists\, V_1, V_2$

$$V_0(0) + V_1(\alpha) = V_0(\alpha) + V_1(0)$$

$V_t(c_t)$ $\qquad$ $V_t(c_t)$

$c_0 = 0,\ c_1 = \alpha$ $\qquad$ $c_0 = \alpha \quad c_1 = 0$

$$V_1(\alpha) - V_0(\alpha) = V_1(0) - V_0(0) = \text{constant}$$

Take $c = (1,4)$ $\qquad$ $d = (4,1)$

$$u(1,4) = 1 \cdot 2 + 1 + 4 = 7$$

$$u(4,1) = 4 \cdot 1 + 1 + 4 = 9$$

$$(4,1) > (1,4)$$

$$\Downarrow$$

$$V_0(4) + V_1(1) > V_0(1) + V_1(4)$$

$$\underbrace{V_1(1) - V_0(1)}_{V_1(\alpha) - V_0(\alpha)} > \underbrace{V_1(4) - V_0(4)}_{V_1(\alpha) - V_0(\alpha)}$$

$0 > 0$ ↯

So no time separable u-repr.

Ex. 1. (a) $u(c_1, c_2, c_3) =$ ✓

$$= \min\{2c_1 + c_2 + c_3,\ c_1 + c_2 + 2c_3\}$$

By T. 1 $\geq$ str. incr, continuous, $S \geq 3$

Time sep $\Longleftrightarrow$ ICC

It is enough to show ICC does not hold

Let $c_1 = y = 1$ $\quad c_2 = 2$ $\quad c_3 = 1$

$d_1 = y = 1$ $\quad d_2 = 1$ $\quad d_3 = 2$

(ICC):

$$(c_{-1}, y) = (1,2,1) \geq (1,1,2) = (d_{-1}, y)$$

$$u(1,2,1) = \min\{2+2+1,\ 1+2+2\} = 5$$

$$u(1,1,2) = \min\{2\cdot 1 + 1 + 1\cdot 2,\ 1+1+2\cdot 2\} = 5$$

$$(c_{-1}, y) \geq (d_{-1}, y)$$

Now take $w = 5$

$c_1 = 5$ $\quad c_2 = 2$ $\quad c_3 = 1$

$d_1 = 5$ $\quad d_2 = 1$ $\quad d_3 = 2$

$$(c_{-1}, w) = (5,2,1) \ \boxed{\geq}\ (5,1,2) = (d_{-1}, w)$$

???

$u(5,2,1) = \min\{2\cdot 5+2+1, 5+2+2\cdot 1\}$
$= \min\{13, 9\} = 9$

$u(5,1,2) = \min\{2\cdot 5+1+2, 5+1+2\cdot 2\}$
$= \min\{13, \cancel{\ } 10\} = 10$

$(5,1,2) = (d-1, w) > (c-1, w) = (5,2,1)$
contradiction with ICC

$c, d \qquad w, y$

$c - t y \geq d - t y$ but
$c - t w < d - t w$ so ICC violated.

So ~~th~~ $u(c_1, c_2, c_3)$ does not have time sep. rep.

(b) $u(c_1, c_2, c_3) = c_1 + \sqrt{c_2} + \sqrt{c_3}$

what does it mean for u to have EU rep.

π_1, π_2, π_3 $\quad \sum_{i=1}^{3} \pi_i = 1 \quad \pi_i \geq 0 \ \forall i \quad \exists V$

$$u(c_1, c_2, c_3) \geq u(d_1, d_2, d_3)$$
$$\iff$$
$$\pi_1 V(c_1) + \pi_2 V(c_2) + \pi_3 V(c_3)$$
$$\geq$$
$$\pi_1 V(d_1) + \pi_2 V(d_2) + \pi_3 V(d_3)$$

In particular take c, d

$(0,0,0) \quad (1,1,1)$

$u(0,0,0) = 0+0+0 = 0$

$u(1,1,1) = 1+1+1 = 3$

$(1,1,1) > (0,0,0)$

Now plug it to EU

$$\underbrace{(\pi_1 + \pi_2 + \pi_3)}_{} \cdot V(0) < \underbrace{(\pi_1 + \pi_2 + \pi_3)}_{} V(1)$$

$V(0) < V(1)$

Take $(0,0,0) \quad (2,2,2) \Rightarrow V(0) < V(2)$

Now take $(1,0,0)$ $(0,0,1)$

$u(1,0,0) = 1+0+0 = 1$

$u(0,0,1) = 0+0+1 = 1$

$(1,0,0) \sim_I (0,0,1)$

$\pi_1 \cdot v(1) + \underline{\pi_2 v(0)} + \pi_3 v(0) =$

$= \pi_1 \cdot v(0) + \underline{\pi_2 v(0)} + \pi_3 v(1)$

$(\pi_1 - \pi_3)(v(1) - v(0)) = 0$

$v(1) > v(0) \Rightarrow \pi_1 = \pi_3$

Take $(2,2,0)$ & $(0,2,2)$

$u(2,2,0) = 2 + \sqrt{2} + 0$

$u(0,2,2) = 0 + \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

$2\sqrt{2} < 2 + \sqrt{2}$

$(2,2,0) >_P (0,2,2)$

Write EU, so:

$\underbrace{\pi_1}\, v(2) + \underline{\pi_2 v(2)} + \underbrace{\pi_3}\, v(0) >$

$> \underbrace{\pi_1}\, v(0) + \underline{\pi_2 v(2)} + \underbrace{\pi_3}\, v(2)$

$(\pi_1 - \pi_3)(v(2) - v(0)) > 0$

$v(2) > v(0)$ $\pi_1 = \pi_3$

$0 > 0$

So $u(\cdot)$ doesn't have EU rep.

Def. $\{\pi_s\}$ given $\mathbb{E}c = \sum_{i=1}^{s} \pi_i c_i$

$\mathbb{E}\underline{c} = (\mathbb{E}c_1, \mathbb{E}c_2, \dots, \mathbb{E}c_S)$

we say that $\succsim$ are Risk averse wrt $\{\pi_s\}$ if $\mathbb{E}\underline{c} \succsim_p c$

Thm. $S \geq 3$, $\succsim$ str. incr., continuous.

$\Rightarrow$ ICC & risk averse wrt $\{\pi_i\}_{i=1}^{S}$

$\Leftrightarrow$ our $\mathbb{E}v(c) = \sum_{i=1}^{S} \pi_i v(c_i)$

has v being concave f.

1(c) $\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$

Consider cases: ① $c_1 \leq c_3$

② $c_1 > c_3$

① ~~2UC...~~ $c_1 \leq c_3$ $/ + c_1 + c_2 + c_3$

$2c_1 + c_2 + c_3 \leq c_1 + c_2 + 2c_3$

$\min\{\cdot, \cdot\} = 2c_1 + c_2 + c_3$

$$\boxed{\mathbb{E}c = \pi_1 \cdot c_1 + \pi_2 \cdot c_2 + \pi_3 \cdot c_3 = \frac{1}{4}(c_1 + c_2 + 2c_3)}$$

$u(\mathbb{E}\mathbf{c}) = \min\{2\cdot\mathbb{E}c + \mathbb{E}c + \mathbb{E}c,$
$\mathbb{E}c + \mathbb{E}c + 2\mathbb{E}c\}$
$= 4\cdot\mathbb{E}c = 4\cdot\frac{1}{4}(c_1 + c_2 + 2c_3)$
$= c_1 + c_2 + 2c_3$ (*)

$\mathbb{E}\mathbf{c} \succcurlyeq_p c$

$\Downarrow$

$\underbrace{u(\mathbb{E}\mathbf{c})} \geq u(c)$

$c_1 + c_2 + 2c_3 \geq 2c_1 + c_2 + c_3$ ✓

(2) $c_3 < c_1$ $\quad / + c_1 + c_2 + c_3$

$c_1 + c_2 + 2c_3 < 2c_1 + c_2 + c_3$

$\min\{c_1 + c_2 + 2c_3, 2c_1 + c_2 + c_3\}$

$u(c_1, c_2, c_3) = c_1 + c_2 + 2c_3$ (**)

$\mathbb{E}\mathbf{c} \succcurlyeq_p c$

$\Updownarrow$

$\underbrace{u(\mathbb{E}\mathbf{c})}_{(*)} \geq u(c)$ (**)

$c_1 + c_2 + 2c_3 \geq c_1 + c_2 + 2c_3$ ✓

I gave $\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ $\quad u$ is risk averse wrt π

2(d) $\quad \pi \in \Delta \qquad \underline{\Delta \subseteq \mathbb{R}^3}$

$(\pi_1, \pi_2, \pi_3) \; : \; \pi_1 + \pi_2 + \pi_3 = 1$

$\pi_i \geq 0$

In 2(b) we said that there is no $\pi \in \Delta^3$ for which

$u(c_1, c_2, c_3) = c_1 + \sqrt{c_2} + \sqrt{c_3}$

has a EU repr.

In particular it does not have concave EU repr.

However this function has simple time separable repr.

$v_1(c_1) = c_1$

$V_2(c_2) = \sqrt{c_2}$

$V_3(c_3) = \sqrt{c_3}$

$V_i(c_i) = 0 \qquad i \neq 1, 2, 3$

$u(c_1, c_2, \cancel{c_3}) = \sum_{i=0}^{\infty} v_i(c_i)$

By previous thm we know that ICC holds

Thm about EU rep with risk averse utility wrt $\{\pi s\}$

Thm. u ~~itu~~ of str incr, cont. $\{2,3\}$

u $\notin CC$, risk averse wrt $\{\pi s\}$

$\iff$ ~~has V being concave~~

Thus we know that there is no ~~ridge~~ π for which u is ~~[illegible]~~ risk averse wrt $\{\pi\}$.

4 (a) $\max \ \mu_1 u_1(x) + \mu_2 u_2(\omega - x)$

s.t. $0 \leq x \leq \omega$

$u_i(x)$ SPM : $\forall x, x' \in \mathbb{R}^n$

$u_i(x \wedge x') + u_i(x \vee x') \geq u_i(x) + u_i(x')$

$\forall \omega \in \mathbb{R}^n_+$

$(\omega - x) \wedge (\omega - x') = \omega - x \vee x'$

$(\omega - x) \vee (\omega - x') = \omega - x \wedge x'$

$X := \omega - x \qquad x' := \omega - x'$

$u_i(\omega - x \vee x') + u_i(\omega - x \wedge x')$

$\geq u_i(\omega - x) + u_i(\omega - x')$

this means

$u_i(\omega - \cancel{x})$ is SPM in x

Topkis thm. ① $X = \mathbb{R}^n$ is lattice ✓

② f has ID (x, t)

③ f is SPM in x, fixed t

$$x(t) = \arg\max_{x \in X} f(x, t)$$

If (1)-(2)-(3) satisfied then $x(t)$ non decreasing

• Fix μ_2 we need to check ②, ③

t must be $t \in \mathbb{R}$

① $F(x, \mu_1) = \mu_1 u_1(x) + \mu_2 u_2(\omega - x)$

② F has ID in (x, μ_1)

wts: $x' \geq x \quad \mu_1' \geq \mu_1$

$$F(x', \mu_1') - F(x, \mu_1') \geq F(x', \mu_1) - F(x, \mu_1) \quad (*)$$

$$\mu_1' u_1(x') + \mu_2 u_2(\omega - x') -$$
$$- \mu_1' u(x) - \mu_2 u_2(\omega - x) \overset{?}{\geq}$$
$$\mu_1 u_1(x') + \mu_2 u_2(\omega - x') -$$
$$- \mu_1 u_1(x) - \mu_2 u_2(\omega - x)$$

$$(\mu_1' - \mu_1)(u_1(x') - u_1(x)) \overset{?}{\geq} 0$$

Yes $\mu_1' \geq \mu_1$ $\quad u_1$ is str. increasing ($x' \geq x$) ok

• Fix μ_1

$\bar{F}(x, \mu_2) = \mu_1 u_1(x) + \mu_2 u_2(\omega - x)$

take $\mu_2' \geq \mu_2 \quad x' \geq x$

WTS: $\bar{F}$ has DD. In $(*)$ $\leq$ instead

- Go back to case with fixed μ_2

$F(x, \mu_1) = \dots$

(3) with fixed μ_1 WTS it is SPM

Obvious : μ_1, μ_2 fixed

μ_1 $u_1(x \wedge x') + u_1(x \vee x') \geq u_1(x) + u_1(x')$

μ_2 $u_2(\omega - x \wedge x') + u_2(\omega - x \vee x') \geq u_2(\omega - x) + u_2(\omega - x')$

$\mu_1, \mu_2 > 0$ and get

$$\mu_1 u_1(x \wedge x') + \mu_2 u_2(\omega - x \wedge x')$$
$$+ \mu_1 u_1(x \vee x') + \mu_2 u_2(\omega - x \vee x')$$
$$\geq \mu_1 u_1(x) + \mu_2 u_2(\omega - x)$$
$$+ \mu_1 u_1(x') + \mu_2 u_2(\omega - x')$$

Well

$$F(x \wedge x', \mu_1) + F(x \vee x', \mu_1)$$
$$\geq \underline{F(x, \mu_1)} + F(x, \mu_2) \quad \checkmark$$

- F SPM

(1), (2) & (3) satisfied so from Topkis we get $x(\mu_1)$ is non decreasing

$\overline{F}$ has DD (1), (2) & (3) satisfied

$\Rightarrow$ $x(\mu_2)$ — is non increasing

4(c): u is str. quasi concave

$\Rightarrow$ at most one ~~max~~ solution to problem $\max_{x \in X} u(x)$

- $\max_{0 \le x \le \omega} u$ u ~ cont.

$x \in [0, \omega]$ compact

- Weierstress Thm $\Rightarrow \exists\, \overline{x}$

~~$u(\overline{x})$~~ $\overline{x}$ being maximizer

- Lemma 1

$u_i(x) \longrightarrow u_i(\omega - x)$

SQC   SQC

Lemma 2

$\{u\}_{i=1}^{n}$ SQF $\Rightarrow u = \sum \alpha_i u_i$ $\alpha_i \geq 0$

u is SQF