# Recitations 17

**[Definitions used today]**

- Weak and Strict Dominance,IESDS, Rationalizability

**Question 1**

Find all the solutions obtained by IESDS.

| | L | R |
|---|---|---|
| T | 3,0 | 0,1 |
| M | 0,0 | 3,1 |
| B | 1,1 | 1,0 |

**Solution 1**

In this game, $A_1^1 = \{T, M, B\}$ and $A_1^2 = \{L, R\}$. No (pure) strategy dominates any other (pure) strategy for both players. However, the mixed strategy $s_1(T) = s_1(M) = \frac{1}{2}$ and $s_1(B) = 0$ strictly dominates $B$ since $\forall q \in [0, 1]$

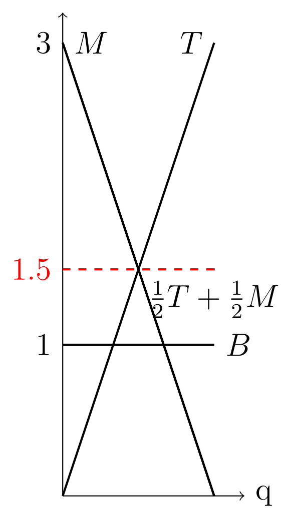


Figure 1: Dominant strategies: mixed strategies

$$u_1(s_1, q) = 3q\frac{1}{2} + 3(1-q)\frac{1}{2} = \frac{3}{2} > 1 = u_1(B, q)$$

So $B$ is eliminated from player 1's set of actions. Given that player 2 knows this, $s^2 = (0, 1)DL$. Thus $L$ is eliminated from player 2's action set. Finally, given that player 2 will only play $R$, $M$ dominates $T$. Thus player 1 will eliminate $T$ as well. This leads to a final action set $C_T = \{M\} \times \{R\}$.

Since each player only has one action now, no more actions can be eliminated. This is referred to as a **complete** IESDA. Note that we have need to allow dominance by mixed strategies for this to work;

neither $T$ nor $M$ alone strictly dominates $B$.

**Question 2**

Find all the solutions using IESDS and IEWDS

a)

| | L | R |
|---|---|---|
| T | 1,2 | 2,2 |
| B | 1,2 | 1,1 |

b)

| | L | R |
|---|---|---|
| T | 1,1 | 0,0 |
| B | 0,0 | 0,0 |

c)

| | L | C | R |
|---|---|---|---|
| T | 4,3 | 5,1 | 6,2 |
| M | 2,1 | 8,4 | 3,6 |
| B | 3,0 | 9,5 | 2,6 |

**Solution 2**

a) Observe that complete IESDS results in a unique outcome which we prove below. It is not true for IEWDS. Let's ilustrate it with this example.

- Procedure 1: T weakly dominates B:eliminate B then Mr2 is indifferent between L and R so we get $((1,0) \times (q, 1-q))$
- Procedure 2: L weakly dominates R:eliminate R then Mr1 is indifferent between T and B so we get $((p, 1-p) \times (1,0))$

b) Furthermore IEWDS can eliminate a NE Observe that $\{TL, BR\}$ are pure NE.
Let's do IEWDS for this game: For Mr2 L weakly dominates R so elimiate R. For Mr1 T weakly dominates B so elimiate B so we elimated our NE, we elimintaed $(B, R)$.

c) One way to organize our work is put it in table. Observe that

| | |
|---|---|
| $C_1^0 = \{T, M, B\}$ | $C_2^0 = \{L, C, R\}$ |
| $C_1^1 = \{T, M, B\}$ | $C_2^1 = \{L, R\}$ |
| $C_1^2 = \{T\}$ | $C_2^2 = \{L, R\}$ |
| $C_1^3 = \{T\}$ | $C_2^3 = \{L\}$ |
| ... | ... |
| $C_1^\infty = \{T\}$ | $C_2^\infty = \{L\}$ |

So $\{(T, L)\}$ is our final result of IESDS.

**Question 3**

Show that following three statements are equivalent:

$$\begin{aligned} u^i\left(s^i, a^{-i}\right) &> u^i\left(a^i, a^{-i}\right) \quad \forall a^{-i} \in A^{-i} \\ u^i\left(s^i, s^{-i}\right) &> u^i\left(a^i, s^{-i}\right) \quad \forall s^{-i} \in S^{-i} \\ u^i\left(s^i, \mu^{-i}\right) &> u^i\left(a^i, \mu^{-i}\right) \quad \forall \mu^{-i} \in \Delta\left(A^{-i}\right) \end{aligned}$$

**Solution 3**

$(1) \Longrightarrow (3)$ :

$$u^i\left(s^i, \mu^{-i}\right) - u^i\left(a^i, \mu^{-i}\right) = \sum_{a^{-i} \in A^{-i}} \mu\left(a^{-i}\right)\left[u^i\left(s^i, a^{-i}\right) - u^i\left(a^i, a^{-i}\right)\right],$$

and the first term is greater than or equal to zero and the second is strictly greater than 0 by hypothesis. Thus the difference is strictly greater than 0 .
(2)$\Longrightarrow$ (1) Since $A^{-i} \subseteq S^{-i} \equiv \prod_{j \neq i} \Delta\left(A^j\right)$, the result is immediate
(3)$\Longrightarrow$ (2) Since $S^{-i} \equiv \prod_{j \neq i} \Delta\left(A^j\right) \subseteq \Delta(A^{-i})$ and thus the result follows immediately.

**Question 4**

Let $C_T$ be the outcome of a complete IESDS and let $R$ be the unique maximal rationalizable set. Show that $C_T = R$

**Solution 4**

- $R \subseteq C_T$

We proceed by induction on the elimination stages of IESDS. Note in $t = 0, \forall i R^i \subseteq C_0^i \equiv A^i$. From this, assume $\forall i R^i \subseteq C_t^i$. Then $\forall i, \forall a^i \in R^i$ it must be that:

$$\begin{aligned}
&\exists \mu \in \Delta\left(R^{-i}\right) \text{ such that } \forall b^i \in A^i, u^i\left(a^i, \mu\right) \geq u^i\left(b^i, \mu\right) \quad \text{(by definition)}\\
\Longrightarrow\ &\exists \mu \in \Delta\left(C_t^{-i}\right) \text{ such that } \forall b^i \in A^i, u^i\left(a^i, \mu\right) \geq u^i\left(b^i, \mu\right) \quad \text{(by hypothesis)}\\
\Longrightarrow\ &\exists \mu \in \Delta\left(C_t^{-i}\right) \text{ such that } \forall b^i \in C_t^i, u^i\left(a^i, \mu\right) \geq u^i\left(b^i, \mu\right) \quad \left(\text{ since } C_t^i \subseteq A^i\right)\\
\Longrightarrow\ &\nexists s^i \in \Delta\left(C_t^i\right) \text{ such that } \forall b^{-i} \in C_t^{-i}, u^i\left(s^i, b^{-i}\right) > u^i\left(a^i, b^{-i}\right)
\end{aligned}$$

Thus $\forall i, \forall a^i \in R^i, a^i \in C_{t+1}^i$, so $R^i \subseteq C_{t+1}^i$. Then, by induction, $R \subseteq C_T$.

- $C_T \subseteq R$

Since $C_T$ is the outcome of a complete IESDS, $\forall i, \forall a^i \in C_T^i$ it must be that:

$$\begin{aligned}
&\nexists s^i \in \Delta\left(C_T^i\right) \text{ such that } \forall b^{-i} \in C_T^{-i}, u^i\left(s^i, b^{-i}\right) > u^i\left(a^i, b^{-i}\right)\\
\Longrightarrow &\nexists s^i \in \Delta\left(A^i\right) \text{ such that } \forall b^{-i} \in C_T^{-i}, u^i\left(s^i, b^{-i}\right) > u^i\left(a^i, b^{-i}\right)\\
\Longrightarrow &\exists \mu \in \Delta\left(C_T^{-i}\right) \text{ such that } \forall b^i \in A^i, u^i\left(a^i, \mu\right) \geq u^i\left(b^i, \mu\right)
\end{aligned}$$

with the first implication following from the fact that $\forall a^i \in A^i \backslash C_T^i, a^i$ is strictly dominated.

Since $i$ and $a^i$ were arbitrarily taken, it follows that $C_T$ is rationalizable, and recall by the previous theorem $R \subseteq C_T$. Further, since $R$ is the unique maximal rationalizable set, by the above lemma, it must be that $C_T = R$.

**Question 5 [$\sim$ Midterm 2020]**

**Guess the average** game goes as follows:

- Each player $i \in I$ picks simultaneously an integer $x_i$ between 1 and 999. Hence, $A_i = \{1, \ldots, 999\}$.
- Given $x = (x_1, \ldots, x_n) \in \{1, \ldots, 999\}^n$, let

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

- $\theta \in (0,1)$ and ket $\theta = \frac{2}{3}$
- The winners are those players whose ballots are closest to $\theta\bar{x}$. If there is a tie they share equally.

1. for $n = 3$ use **one shot deviation** to find NE.
   One shot deviation principle is based on following observation: A strategy profile is a Nash equilibrium if no player has incentive to deviate from his strategy given that the other players do not deviate. [1]
2. solve it for any $n$ using IESDS
3. Does the value of $\theta$ matter, and if so how?
4. How solution of IESDS changes if we change rule that if a tie happens no one gets anything?

**Solution 5** 1) Suppose all three players announce the same number $k \geq 2$. Then 2/3 of the average is $2/3k$, and each gets \$1/3. Suppose now one of the players deviates to $k-1$. Now 2/3 of the average is $2/3k - 2/9$. We now wish to show that the player with $k-1$ is closer to the new 2/3 of the average than the two whose integers where $k$ :

$$2/3k - 2/9 - (k-1) < k - (2/3k - 2/9) \quad k > 5/6$$

Since $k \geq 2$, the inequality is always true. Therefore, the player with $k-1$ is closer, and thus he can get the entire \$1. We conclude that for any $k \geq 2$, the profile $(k, k, k)$ cannot be a Nash equilibrium.

The strategy profile $(1, 1, 1)$, on the other hand, is NE. (Note that the above inequality works just fine for $k = 1$. However, since we cannot choose 0 as the integer, it is not possible to undercut the other two players with a smaller number.)

Now other cases.First one player names a highest integer. $(k^*, k_1, k_2)$ , where $k^*$ is the highest integer and $k_1 \geq k_2$. 2/3 is $a = 2/9\,(k^* + k_1 + k_2)$ . If $k_1 > a$, then $k^*$ is further from $a$ than $k_1$, and therefore $k^*$ does not win anything. If $k_1 < a$, then the difference between $k^*$ and $a$ is $k^* - a = 7/9k^* - 2/9k_1 - 2/9k_2$. The difference between $k_1$ and $a$ is $a - k_1 = 2/9k^* - 7/9k_1 + 2/9k_2$ The difference between the two is then $5/9k^* + 5/9k_1 - 4/9k_2 > 0$, and so $k_1$ is closer to $a$. Thus $k^*$ does not win and the player who offers it is better off by deviating to $k_1$ and sharing the prize. Thus, no profile in which one player names a highest integer can be Nash equilibrium.
Consider a profile in which two players name highest integers. Denote this profile by $(k^*, k^*, k)$ with $k^* > k$. Then $a = 4/9k^* + 2/9k$. The midpoint of the difference between $k^*$ and $k$ is $1/2\,(k^* + k) > a$. Therefore, $k$ is closer to $a$ and wins the entire \$1. Either of the two other players can deviate by switching to $k$ and thus share the prize. Thus, no such profile can be Nash equilibrium.

This exhausts all possible strategy profiles. We conclude that this game has a unique Nash equilibrium, in which all three players announce the integer 1.

2) Observe that every strategy $x_i > 666$ is dominated by 666. Hence, for every $i \in I, C_i^1 = \{1, \ldots, 666\}$. Now, for every $i \in I, C_i^2 = \{1, \ldots, 444\}$. Proceeding this way, for every $i \in I, C_i^{\infty} = \{1\}$.

[1] The one-shot deviation principle is fundamental to the theory of extensive games. It was originally formulated by David Blackwell (1965) in the context of dynamic programming. As the strategy of other players induces a normal maximization problem for any one player, we can formulate the principle in the context of a single-person decision tree