Recitations 24

Question 1 VCG [Final 2014]

There are n individuals considering whether or not to place a sculpture (already sculptured so it costs nothing) in the park. Each individual i has utility function:

$$u_i(k,t) = \theta_i k + t_i$$

for some number $\theta_i \in \mathbb{R}$ that reflects the taste of an individual for the sculpture, where $k \in \{0,1\}$ is an indicator of statue being placed in the park, the quantity $t_i \in \mathbb{R}$ is money transfer and $t \in \mathbb{R}^n$ is the money allocation.

1. Describe a Vickrey-Clarke-Groves (VCG) mechanism for this economy formally and intuitively.
2. Suppose $\theta_i \sim N(0,1)$. Show that for every individual i the probability that i will be pivotal (i.e. will end up paying some money) converge to 0 as $n \to \infty$.

Solution 1

Set-up

- n agents, each agent has utility $u_i(k,\theta_i) = v_i(k,\theta_i) + t_i$
- $k \in K$ where K is the set of non-monetary allocation
- $t_i \in \mathbb{R}$ monetary transfer
- The allocation is (k,t) where $t = (t_1,\dots,t_n)$
- $\theta_i \in \Theta_i$ type of agents, $\theta = (\theta_1,\dots,\theta_n) \in \prod_{i=1}^n \Theta_i$
- Set of feasible allocation $X = \{(k,t) \mid k \in K, t_i \in \mathbb{R}, \sum_i t_i \leq 0\}$;
- Let $k : \Theta \to K$, where $\Theta = \Theta_1, \cdots \times \Theta_n = [0,1]^n$, K
- $t_i : \Theta \to \mathbb{R}$,
- we even specify it to $v_i(k,\theta_i) = \theta_i k$
- The social choice function is $f(\theta) = (k(\theta), t(\theta))$ where $t(\theta) = (t_i(\theta))_{i=1}^n$.

Definition 0.1. *A social choice function f satisfies full ex-post efficiency if:*

1. *it is ex-post efficient:* $\sum_{i=1}^n v_i(k(\theta),\theta_i) \geq \sum_{i=1}^n v_i(k,\theta_i) \, \forall k,\theta$;
2. *it is balanced budget:* $\sum_{i=1}^n t_i(\theta) = 0$

3. *dominant strategy incentive compatible (DSIC) or strategy-proof if* $\forall i, \theta_i, \theta_i', \theta_{-i}$,

$$v_i(k(\theta), \theta_i) + t_i(\theta) \geq v_i(k(\theta_i', \theta_{-i}), \theta_i) + t(\theta_i', \theta_{-i})$$

Theorem 0.2. *Let* $k^* : \prod_{i=1}^n \Theta_i \to K$ *be ex-post efficient. If* $\forall i, \exists h_i : \forall \theta$,

$$t_i(\theta) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j) + h_i(\theta_{-i})$$

then, $f = (k^*, t)$ *is DSIC.*

Example 1. *(Clarke mechanism). Consider to construct* $h_i : \prod_{j \neq i} \Theta_j \to \mathbb{R}$

$$h_i(\theta_{-i}) = -\sum_{j \neq i} v_j\left(k_{-i}^*(\theta_{-i}), \theta_j\right)$$

where $k_{-i}^*(\theta_{-i}) \in \arg\max_{k \in K} \sum_{j \neq i} v_j(k, \theta_j) \, \forall \theta_{-i}$

k_{-i}^* maximizes welfare anibgs everyone but i

Example 2. *(pivotal mechanism)*

$$t_i(\theta) = \sum_{j \neq i} (v_j(k^*(\theta), \theta_j) - v_j\left(k_{-i}^*(\theta_{-i}), \theta_j\right))$$

If $k(\theta) = k_{-i}^*(\theta_{-i})$ then i pays nothing.
If $k(\theta) \neq k_{-i}^*(\theta_{-i})$ then i pays a tax equal to his effect on others.
In an auction setting, this is just like a Vickery auction (2nd price sealed action)

Example 3. *(Vickery auction)*

- $K = \{0, 1, \ldots n\}$
- k- *who gets the object if anyone*
- *utility*

$$v_i(k, \theta_i) = \begin{cases} \theta_i & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$$

- $k^*(\theta)$ *give it to whoever values it most*
- i *is pivotal when she is the highest bidder. His tax is the 2nd highest bid.*

Theorem 0.3. *Suppose, for each* $i, \{v_i(\cdot, \theta_i) \mid \theta_i \in \Theta_i\} = \mathbb{R}^K$. *If* $f = (k^*, t)$ *satisfies ex-post efficiency and DSIC, then,* $\forall i, \exists h_i : \forall \theta$,

$$t_i(\theta) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j) + h_i(\theta_{-i})$$

b) A special version of VCG is the pivotal mechanism with

$$h_i(\theta_{-i}) = -\sum_{j\neq i} v_j\left(k^*_{-i}(\theta_{-i}), \theta_j\right)$$

where $k^*_{-i}(\theta_{-i})$ is ex post efficient rule in the absence of agent i. Transfers are zero if agent is pivotal and equal to the change in other agent's payoffs due to his action The expost efficient rule is

$$k(\theta) = \begin{cases} 1 & \text{if } \sum_i \theta_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The more general proof is in next question.
Agent is pivotal if his type changes the decision:

$$\{\theta : \theta_i + \sum_{j\neq i}\theta_j \geq 0 \quad \text{and} \quad \sum_{j\neq}\theta_j < 0\}$$

or

$$\{\theta : \theta_i + \sum_{j\neq i}\theta_j < 0 \quad \text{and} \quad \sum_{j\neq}\theta_j \geq 0\}$$

since $v_i \sim N(0,1)$ then $\sum_{j\neq}\theta_j \sim N(0, n-1)$ The two event above are disjoint so probability is sum :

$$\mathbb{P}(\theta_i + \sum_{j\neq i}\theta_j \geq 0 \quad \text{and} \quad \sum_j \theta_j < 0) = \mathbb{P}(\theta_i + \sum_{j\neq i}\theta_j \geq 0 | \sum_{j\neq i}\theta_j < 0)\mathbb{P}(\sum_{j\neq i}\theta_j < 0) =$$

$$= \mathbb{P}(\sum_{j\neq}\theta_j \geq -\theta_i | \sum_{j\neq i}\theta_j < 0)\cdot\frac{1}{2}$$

use truncated normal

$$f(x|a_1 < x < a_2) = \frac{\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})}{\frac{a_2-\mu}{\sigma} - \frac{a_1-\mu}{\sigma}}$$

where $a_2 = 0, \mu = 0, \sigma = \sqrt{n-1}, a_1 = -\infty$

$$f(x|x<0) = \frac{\frac{1}{\sqrt{n-1}}\phi(\frac{x}{\sqrt{n-1}})}{\phi(0) - \phi(-\infty)} = \frac{2}{\sqrt{n-1}}\phi(\frac{x}{\sqrt{n-1}})$$

$$\mathbb{P}(\sum_{j\neq i}\theta_j \geq -\theta_i | \sum_{j\neq i}\theta_j < 0)\cdot\frac{1}{2} = \int_0^{+\infty}\int_{-\infty}^{\theta_i} f(x|x<0)dxd\phi(\theta_i) =$$

$$= \int_0^{+\infty}\int_{-\infty}^{\theta_i}\frac{2}{\sqrt{n-1}}\phi(\frac{x}{\sqrt{n-1}})dxd\phi(\theta_i) = \frac{2}{\sqrt{n-1}}\int_0^{+\infty}\int_{-\infty}^{\theta_i}\phi(\frac{x}{\sqrt{n-1}})dxd\phi(\theta_i) \to 0$$

Observe that pontiwise limit

$$\lim_{n\to\infty}\frac{2}{\sqrt{n-1}}\phi(\frac{x}{\sqrt{n-1}}) = 0\cdot\phi(0) = 0$$

function in limit integral is integrable. We know that $\frac{2}{\sqrt{n-1}}\int_0^{+\infty}\int_{-\infty}^{\theta_i}\phi(\frac{x}{\sqrt{n-1}})dxd\phi(\theta_i) < \infty$ because it is integral of normal pdf which is well defined. From theorem 33 from Math appendix (Lebesgue dominated convergence theorem) we know that indeed we can get inside with limit and $\int\int f_n \to \int\int 0 = 0$. The other case is symmetrical. So probabilities goes to 0.

$$\lim_{n\to\infty} \mathbb{P}(\theta_i + \sum_{j\neq i} \theta_j \geq 0 \text{and} \quad \sum_{j\neq i} \theta_j < 0) = 0$$

Question 2 [88 IV.3 Spring 2009 majors]

Consider a quasilinear environment where two agents are to contribute to a public project. Let $K = \{0, 1\}$ be the possible levels of the project, with 1 meaning that the project is "done", and 0 "not done". Agent i's private valuation of the project is denoted by θ_i, which is independently drawn from a uniform distribution on $[0, 1]$. The project costs c to finish, where $0 < c < 2$. Let $k : \Theta \to K$, where $\Theta = \Theta_1 \times \Theta_2 = [0, 1] \times [0, 1]$, denote the following allocation function:

$$k(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_1 + \theta_2 \geq c \\ 0 & \text{otherwise} \end{cases}$$

A transfer rule $t : \Theta \to \mathbb{R}^2$ specifies, for each agent i, the amount of monetary transfer received by agent i at each $\theta = (\theta_1, \theta_2) \in \Theta$. Writing $t(\theta)$ as $(t_1(\theta), t_2(\theta))$, we say that the transfer rule t balances the budget if, for any θ

$$t_1(\theta) + t_2(\theta) = \begin{cases} -c & \text{if } k(\theta) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Construct a budget-balancing transfer rule t that Bayesian-implements the allocation rule k; i.e., construct a budget-balancing transfer rule t such that the social choice function (k, t) is Bayesian incentive compatible.

Solution 2

Let's solve it in more general set up with n agents and with $0 < c < n$.

In this case condition for ex-post efficiency is equivalent to:

$$\sum_{i=1}^{n} k(\theta)\theta_i \geq \sum_{i=1}^{n} k \cdot \theta_i \quad \forall k \in \{0, 1\}, \theta$$

which is equivalent to

$$(k(\theta) - k) \cdot \sum_{i=1}^{n} \theta_i \geq 0 \quad \forall k \in \{0, 1\}, \theta$$

Consider following SCF $f(\theta) = (k(\theta), t_1(\theta), \ldots, t_I(\theta)) \; \forall\theta$,

$$k(\theta) = \begin{cases} 1 & \text{if } \sum_i \theta_i \geq c \\ 0 & \text{otherwise} \end{cases}$$

with

$$\sum_i t_i(\theta) = \begin{cases} -c & \text{if } k(\theta) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$k(\theta)$ is either one or zero.
Fix $\sum_i \theta_i \geq c$, in this case 1, then EPIC is satisfied. Analogous $\sum_i \theta_i < c$ with $k \in \{0, 1\}$ is obviously satisfied.
We are asked to find SCF (transfers in particular) s.t. it is Bayesian incentive compatibile.

Definition 0.4. *A social choice function f is Bayesian incentive compatible if, for all i, θ_i, θ_i', and θ_{-i},*

$$E\left[v_i\left(k(\theta), \theta_i\right) + t_i(\theta) \mid \theta_i\right] \geq E\left[v_i\left(k\left(\theta_i', \theta_{-i}\right), \theta_i\right) + t_i\left(\theta_i', \theta_{-i}\right) \mid \theta_i\right]$$

Instead of going straight to solution let's observe that not all transfer works. Consider following example of egalitarian transfer.

Example 4. *Suppose that the agents want to implement this ex post efficient SCF with average transfer,i.e.*

$$t_i(\theta) = -ck(\theta)/n$$

Suppose that $\Theta_i = \left\{\bar{\theta}_i\right\}$ for $i \neq 1$ and $\Theta_1 = [0, +\infty)$. Moreover, suppose that $c > \sum_{i \neq 1} \bar{\theta}_i > c(n-1)/n$.

This implies that, firstly, with this social function agent 1's type is critical for whether the bridge is built (note that if $\theta_1 \geq c - \sum_{i \neq 1} \bar{\theta}_i$, it is. while if $\theta_1 < c - \sum_{i \neq 1} \bar{\theta}_i$, it is not), and, secondly, that the sum of the utilities of the agents $2, \ldots, n$ is strictly greater if the bridge is built than if it is not built (since $\sum_{i \neq 1} \bar{\theta}_i - c(n-1)/I > 0\Big)$

What are the incentives of agent 1 to truthfully reveal her type when $\theta_1 = c - \sum_{i \neq 1} \bar{\theta}_i + \varepsilon$ for some $\varepsilon > 0$?
If agent 1 reveals her true preferences, then the bridge will be built because

$$\left(c - \sum_{i \neq 1} \bar{\theta}_i + \varepsilon\right) + \sum_{i \neq 1} \bar{\theta}_i > c$$

In this case agent 1 's utility is

$$\theta_1 - \frac{c}{n} = \left(c - \sum_{i \neq 1} \bar{\theta}_i + \varepsilon\right) + 0 - \frac{c}{n} = \left(\frac{c(n-1)}{n} - \sum_{i \neq 1} \bar{\theta}_i + \varepsilon\right) + 0$$

However, for $\varepsilon > 0$ small enough, the utility of agent 1 is less than 0 (in fact, this is her utility if she instead claims that $\theta_1 = 0$, a claim that results in the bridge not being built). Therefore, agent 1 will not truthfully reveal her type.

We have considered just a couple of examples which illustrate the problems that can arise in the described setting. The central question that we impose is the following: What social choice functions can be implemented when agents' types are private information? In general, we need to consider not only *the possibility of directly implementing social choice function by asking agents to reveal their types but also their indirect implementation through the design of institutions in which the agents interact.*

Instead consider following transfers:

$$t_i(\theta) = \mathbb{E}_{\bar{\theta}_{-i}}[\sum_{j\neq i} v_j(k(\theta_i, \bar{\theta}_{-i}), \bar{\theta}_j) + h_i(\theta_{-i})]$$

note that expectational term represents the expected benefits of agent $j \neq i$ when agent i announces his type to be θ_i and agent j tells truth. Thus the change in agent i transfer when he changes his announced type is exactly equal to the expected externality of this change on agent $j \neq i$. let's focuse on following $h_i(\theta_{-i})$:

$$h_i\left(\theta_{-i}\right) = -\frac{1}{n-1} \sum_{j \neq i} \xi_j\left(\theta_j\right) \quad \text{where} \quad \xi_i\left(\theta_i\right) = E\left[\sum_{j \neq i} v_j\left(k^*(\theta), \theta_j\right) \mid \theta_i\right]$$

Intuitively h_i can be thought of as follows:
We have seen that when agents type is $\theta 1_{\ldots\theta_n}$ each agent payment is equal $\xi_i(\theta_i)$. Now if each agent contributes an equal $\frac{1}{n-1}$ share of all of the other agent's payments, payments from a given agent i to each other $n-1$ will total$\frac{1}{n-1}\sum_{j\neq i}\xi_j(\theta_j)$ and agent i will receive from these agents in return payment thta total to $\xi(\theta_i)$. Agent i's net transfer will therefore be $\xi_i(\theta_i) - \frac{1}{n-1}\sum_{j\neq i}\xi_j(\theta_j)$.
This direct revelation mechanism is known as the expected externality mechanism due to d'Aspremont adn Gerard Varret and Arrow.
Let's prove general result showing that if there is independence of types there is an ex post efficient SCF that is implementable in Bayesian Nash equilibrium

Theorem 0.5. *Suppose agents' types are statistically independent. Let k^* satisfy ex-post efficiency and let $t_i(\theta) = E\left[\sum_{j \neq i} v_j\left(k^*(\theta), \theta_j\right) \mid \theta_i\right] + h_i\left(\theta_{-i}\right)$. Then, $f = (k^*, t)$ is Bayesian incentive compatible and there exist some h_i for each i such that it satisfies balanced budget.*

Proof. 1. BIC. For any θ_i, θ_i',

$$\begin{aligned}
&E\left[v_i\left(k^*(\theta), \theta_i\right) + t_i(\theta) \mid \theta_i\right] \\
=&E\left[v_i\left(k^*(\theta), \theta_i\right) \mid \theta_i\right] + E\left[\sum_{j \neq i} v_j\left(k^*(\theta), \theta_j\right) \mid \theta_i\right] + E\left[h_i\left(\theta_{-i}\right)\right] \\
=&E\left[\sum_{i=1}^n v_i\left(k^*(\theta), \theta_i\right) \mid \theta_i\right] + E\left[h_i\left(\theta_{-i}\right)\right] \\
\geq&E\left[\sum_{i=1}^n v_i\left(k^*\left(\theta_i', \theta_{-i}\right), \theta_i\right) \mid \theta_i\right] + E\left[h_i\left(\theta_{-i}\right)\right] \\
=&E\left[v_i\left(k^*\left(\theta_i', \theta_{-i}\right), \theta_i\right) + t_i(\theta) \mid \theta_i\right]
\end{aligned}$$

2. Balanced budget. Let $h_i\left(\theta_{-i}\right) = -\frac{1}{n-1}\sum_{j \neq i} \xi_j\left(\theta_j\right)$ where $\xi_i\left(\theta_i\right) = E\left[\sum_{j \neq i} v_j\left(k^*(\theta), \theta_j\right) \mid \theta_i\right]$.

Then,

$$\sum_{i=1}^{n} t_i(\theta_i) = \sum_{i=1}^{n} \xi_i(\theta_i) + \sum_{i=1}^{n} h_i(\theta_{-i})$$
$$= \sum_{i=1}^{n} \xi_i(\theta_i) - \frac{1}{n-1} \sum_{i=1}^{n} \sum_{j \neq i} \xi_j(\theta_j) = 0$$

□

Question 3 [Final 2015]

A seller owns two objects, A and B, and decides to run a marginal product mechanism to sell them. In other words, she solicits from each bidder i his utility function $u_i = (u_i(A), u_i(B), u_i(AB)) \in \mathbb{R}^3$ (where $u_i(A)$ is i 's utility for good A, $u_i(B)$ is i 's utility for good B, $u_i(AB)$ is i 's utility for the bundle consisting of both goods A and B. Normalize $u_i(\emptyset) = 0$), computes the efficient allocation of objects given the bidder's reported utility functions and charges them an amount of money such that each bidder's net payoff always equals his marginal product if preferences are reported truthfully. Everyone has quasi-linear preferences with respect to money.

1. Suppose that there are two bidders with valuation: Completely derive the outcome of the auction in this case.

	A	B	AB
Bidder 1	0	0	12
Bidder 2	10	10	10

2. Suppose now instead there are three bidders with valuations: Completely derive the outcome of the

	A	B	AB
Bidder 1	0	0	12
Bidder 2	10	10	10
Bidder 3	10	10	10

auction in this case.

3. Suppose that the seller does not know whether there are two bidders with preferences like in i) or three bidders with preferences like in ii). Suppose that there are in fact only two bidders, as in i), but bidder 2 is able to pretend to be two separate bidders. What would be bidder's 2 decision and payoff in this case, assuming he chooses optimally whether or not to pretend to be two separate bidders?

4. Repeat three point above assuming that bidder's 1 preferences are (6,6,12) instead if $(0, 0, 12)$. Conjecture a reason what happens in iii) under the two assumptions on bidder's 1 utility function.

Solution 3

The general idea here is to implement Clarke mechanism, where: $h_i : \quad \times_{j\neq i}\Theta_j \to \mathbb{R}$ is defined as:

$$h_i(\theta_{-i}) = -\sum_{j\neq i} v_j\left(k^*_{-i}\left(\theta_{-i}\right), \theta_j\right)$$

$$\text{where} \quad k^*_{-i}\left(\theta_{-i}\right) \in \arg_{k\in K}\sum_{j\neq i} v_j\left(k_1\theta_j\right) \quad \forall\theta_{-i}$$

Note that the payment of agent i is :

$$t_i(\theta) = \sum_{j\neq i} v_j\left(k^*(\theta), \theta_j\right) - \Sigma_{j\neq i} v_j\left(k^*_{-i}\left(\theta_{-i}\right), \theta_j\right)$$

Note that

$$k = \{[(A,1),(B,2)],[(A,2),(B,1)],[(AB,1),(\emptyset,2)],[\emptyset,1),(AB,2)]\}$$

Choose $k \in K$ to maximize $\sum_{i=1,2} v_i(k(\theta),\theta)$ so that k is ex post efficient
$k = (AB,1),(\emptyset,2)$ we allocate A and B to the bidder 1. We will use following theorem:

Theorem 0.6. *Let $k^* : \prod_{i=1}^n \Theta_i \to K$ be ex-post efficient. If $\forall i, \exists h_i : \forall\theta$,*

$$t_i(\theta) = \sum_{j\neq i} v_j\left(k^*(\theta), \theta_j\right) + h_i\left(\theta_{-i}\right)$$

then, $f = (k^*, t)$ is DSIC.

Since k^* is EPE then by thm bidders will reveal their true value.
Bidder 1 gets AB so $u_1(AB) = 12$
Bidder 1 gets nothin so $u_1(\emptyset) = 0$
Observe that $v(k^*_{-i}) = 10$
the monetary transfer is

$$t_1(\theta) = v_2(k^*) - v_2(k_{-1}(\theta_{-1})) = 0 - 10 = -10$$

$$t_2(\theta) = v_1(k^*) - v_1(k_{-2}(\theta_{-2})) = 12 - 12 = 0$$

Seller's revenue is 10
b) There are many optimal k^*. Consider $(A,2),(B,3),(\emptyset,1)$. then

- $u_1(\emptyset) = 0$
- $u_2(A) = 10$
- $u_3(B) = 10$
- $t_1(\theta) = 0$
- $t_2(\theta) = 10 - 12 = -2$
- $t_3(\theta) = 10 - 12 = -2$

- Seller's revenue is 4

c) By a) if bidder does not pretend he gets $u_2(\emptyset) = 0$
By b) if he pretends he gets $u_s = 10 - 4 > 0$. So it is optimal to pretend

d) For k^* consider $(A, 1), (B, 2)$. then

- $u_1 = 6$
- $u_2 = 10$
- $t_1(\theta) = 10 - 10 = 0$
- $t_2(\theta) = 6 - 12 = -6$
- Seller's revenue is 6

For k^* consider $(A, 2), (B, 3), (\emptyset, 1)$. then

- $u_1(\emptyset) = 0$
- $u_2(A) = 10$
- $u_3(B) = 10$
- $t_1(\theta) = 10 + 10 - 10 - 10 = 0$
- $t_2(\theta) = 0 + 10 - 10 - 6 = -6$
- $t_3(\theta) = 0 + 10 - 10 - 6 = -6$
- Seller's revenue is 6

IF the bidder 2 does not pretend $u_2 = 10 - 6 = 4$
If the bidder 2 pretends $u_2 = 10 - 6 - 6 = -2$. So he won't pretend.

This problem implies one of the shortcomings of the VCG mechanism, which is vulnerable to shift of bidding in which a buyer uses multiple identities (by pretending to be 2 separate bidders) in order to maximize his payoff and perhaps lower the revenue of the seller. But shift does not always happen . Whether or not bidder chooses to cheat depend on other bidder's valuation.