Migration Policy in a Spatial Equilibrium Model

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June 4, 2024

Abstract

This paper studies efficiency of Rosen-Roback model with inelastic housing. Optimal labor tax is characterized. Efficient allocation has higher utility by 0.1 % of consumption equivalent. Correcting for externality housing consumption increases by 2.6% and goods and output decrease by 1.2% each.

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1 Introduction

There is a large interest in place-based policies. These policy proposals seek to.... (Cite some papers, Austin et al. (2018)). It is clear that these policies will have distributional effects. blah. However, the literature has not explored whether these policies have efficiency effects.

This paper explores whether the benchmark Rosen-Roback model is efficient. We begin by defining a simple, benchmark model. Cities are heterogeneous in amenities, land, and efficiency. Cities have two sectors. The first sector turns labor into a tradeable good, with the amount governed by its efficiency. The second sector turns land and goods into housing. Unlike the good, housing is not tradeable across cities. The only other agents are workers, who are ex-ante homogeneous. They supply labor to cities, in exchange for wages which they spend on housing and goods. Their utility functions input amenities, goods, and housing. They are free to move across cities supplying labor to whatever city offers the highest level of utility. Hence, in equilibrium all workers have the same utility level across cities.

The main congestion force is the housing sector. The housing sector inputs land and tradeable goods – it has decreasing returns to scale in goods. This causes the price of housing in a ctiy to be increasing with its population. This force prevents all workers from locating in the single most productive city.

Two factors are key to understanding the competitive equilibrium. First, how much a worker consumes is directly tied to the efficiency of their city. This follows from the fact that a city's efficiency determines their wages. Second, when a worker moves to a city, they increase the price of housing for everyone: this is a negative externality that workers do not internalize. Hence, many workers move to the most productive cities, to capture their higher wages, but this raises the prices in these cities for everyone.

Having discussed the competitive equilibrium, we turn to the social planner's problem. We solve this problem to characterize the efficient allocation, and we find that the competitive equilibrium is inefficient. The reason is the negative externality workers have on housing prices. The social planner finds it optimal to reallocate both labor and resources away from the most productive cities to the less productive ones. This has two effects. First, due to the congestion forces from the housing sector, this increases utility in the most productive cities while decreasing that in the less productive cities. Second, the reallocation of resources decreases utility in the most productive cities while increasing that in the less productive cities. The overall effect of both these forces is positive, increasing the utility of workers across all cities.

Having found the competitive equilibrium is inefficient, we turn to measuring this inefficiency. We calibrate the model to the US, and we find that moving to an efficient allocation increases welfare by 0.1 percent, as measured by a subsidy to goods in the competitive equilbrium.

We make XXX contributions to the literature. XXXX

This paper is organized as follows. Section 2 characterizes the competitive equilibrium of the benchmark Rosen-Roback model. Section 3 characterizes the Pareto efficient allocation, and shows that the competitive equilibrium is inefficient. Finally, section 4 calibrates the model to measure this inefficiency.

2 The Competitive Equilibrium

The model is a standard spatial equilibrium model (Rosen, 1974; Roback, 1982). Our congestion force is inelastic housing supply; this causes the price of housing to increase with population, thus preventing all workers from locating in the most productive city.

Environment. The model is static. There are three types of agents: workers, housing sectors, and tradeable goods sectors. The mass of workers is N. There is a discrete set of cities $\mathcal{J}$. Cities $j \in \mathcal{J}$ are a tuple of amenities, land, and efficiency (A_j, L_j, Z_j). Each city operates a housing sector and a tradeable goods sector. The tradeable goods sectors use labor from workers to produce a tradeable good. This tradeable good is the numeraire and can be traded at no cost across cities. The goods sector operates a constant returns to scale technology:

$$Y_j = Z_j n_j$$

where n_j denotes labor and Z_j denotes efficiency.

The second sector that operates in each city is the housing sector. Its technology turns tradeable goods x_j and land L_j into housing. Housing cannot be traded across cities. The technology is constant-returns-to-scale.

$$H_j = x_j^{\sigma} L_j^{1-\sigma} \tag{1}$$

where σ is the elasticity of housing supply to the tradeable good.

Workers preferences turn their city's amenities A_j, tradeable goods c_j, and housing h_h into utility. We assume a log Cobb-Douglas functional form:

$$u(A_j, c_j, h_j) = \log(A_j c_j^{1-\psi} h_j^{\psi})$$

where ψ is a weight on housing. To understand this functional form, looking ahead to the equilibrium the share of expenditure spent on housing is constant and equal to ψ. This keeps the model tractable and matches the data.

Tradeable Goods Sector Problem. In each city, the tradeable goods sector take prices as given and chooses labor inputs to maximize profits:

$$\max_{n_j} Z_j n_j - w_j n_j$$

Looking ahead to the equilibrium, because the technology is constant returns to scale, the tradeable goods sector has zero profits.

Housing Sector Problem. In each city j, the housing sector owns the land L_j. They take prices as given and choose the tradeable goods inputs x_j to maximize profits:

$$\Pi_j = \max_{x_j} p_j x_j^{\sigma} L_j^{1-\sigma} - x_j$$

Looking ahead to the equilibrium, the housing sector has positive profits because it own land L_j. We assume that profits are equally distributed to residents of the city:

$$\pi_j = \frac{\Pi_j}{n_j}.$$

Note, this model is isomorphic to one where the workers own the land. In such a setting, the housing sector has zero profits, and pays rents to workers for the land.

The Worker Problem. The worker problem can be split into two steps. Given the choice to live in city, the worker takes wages w_j, housing prices h_j, and profits π_j as given and chooses how much tradeable goods and housing to consume to maximize utility:

$$v_j(w_j, p_j, \pi_j) = \max_{c,h} \{u(A_j, c, h) \ \mid \ c + p_j h \leq w_j + \pi_j\} \tag{2}$$

Given the distribution of amenities, housing prices, and wages across cities, workers chooses where to live:

$$\max_{j \in \mathcal{J}} \{v_j(w_j, p_j, \pi_j)\}. \tag{3}$$

Equilibrium. Bold font denotes a vector. An equilibrium is wages $\boldsymbol{w}$, housing prices $\boldsymbol{p}$, housing sector profits $\boldsymbol{\pi}$, and employment $\boldsymbol{n}$ such that 1) workers maximize utility, 2) tradable goods sectors maximize profits, 3) housing sectors maximize profits, and 4) markets for labor, housing, and the tradeable good clear:

$$\sum_j n_j = N \tag{4}$$

$$n_j h_j = H_j \quad \forall j \tag{5}$$

$$\sum_j n_j c_j + x_j = \sum_j y_j \tag{6}$$

2.1 Simplifying the equilibrium

We solve the model step by step in the Appendix Section A. In brief, we can use optimality conditions to rewrite utility in terms of only prices:

$$v_j(w_j, p_j, \pi_j) = \log \left(A_j(w_j + \pi_j) p_j^{-\psi} \psi^{\psi} (1-\psi)^{1-\psi} \right). \tag{7}$$

As expected, utility is decreasing in housing prices p_j and increasing in amenities A_j, wages w_j, and redistributed profits π_j. Next we can use optimality conditions to solve for prices and simplify further:

$$w_j = Z_j$$

$$\pi_j = Z_j \frac{\psi(1-\sigma)}{1-\psi(1-\sigma)}$$

$$p_j = L_j^{\sigma-1} Z_j^{1-\sigma} n_j^{1-\sigma} \sigma^{-\sigma} \left(\frac{\psi}{1-\psi+\psi\sigma} \right)^{1-\sigma}$$

Wages follow from the fact that the tradeable goods sector is constant returns to scale. Housing prices and profits follow from combining the housing sectors optimality conditions with housing market clearing (5).

Note that housing prices p_j are increasing in population n_j. This is key to the model having a well-defined solution; this force prevents all workers from living in the single city with the highest efficiency (or more precisely, the highest amalgam of efficiency, land, and amenities, which we elaborate on below.). This follows from the fact that the housing sector has a decreasing returns to scale technology $\sigma < 1$. To clearly see this, observe that optimality conditions of the housing sector imply that the elasticity of housing price with respect of housing inputs is $\frac{\partial \log p_j}{\partial \log x_j} = 1 - \sigma$. With $\sigma < 1$, housing prices is increasing in goods used to make housing x_j; and housing sector inputs x_j are increasing with population n_j, because people consume housing. So, with $\sigma < 1$ we get that housing prices are increasing in n_j, and this force prevents everyone from living in the city with the highest efficiency. If the housing sector is constant returns to scale, $\sigma = 1$, then the price of housing is constant and independent of x_j and thus n_j.[1] In this case, the model will not have a well defined solution. Using prices, utility simplifies to

$$v_j = \log(\Phi_j) - (\psi - \sigma\psi)\log(n_j) + \log(\chi), \tag{8}$$

$$\Phi_j \equiv A_j L_j^{\psi(1-\sigma)} Z_j^{1+\psi\sigma-\psi} \tag{9}$$

$$\chi \equiv \frac{(1-\psi)^{1-\psi}(\psi\sigma)^{\psi\sigma}}{(1-\psi+\psi\sigma)^{1-\psi+\psi\sigma}} \tag{10}$$

where Φ_j is an amalgam of a land, amenities, and efficiency. χ is a book-keeping constant. Intuitively, a city's utility is increasing in α_j but decreasing in n_j.

Following from Equation (3), all workers supply labor to the city that offers the highest utility. So, in equilibrium, all cities offer the same utility. Let v denote the utility level of workers:

$$v = v_j(w_j, p_j, \pi_j), \quad \forall j \in \mathcal{J}. \tag{11}$$

This equilibrium condition is key to solving Rosen-Roback models. It implies that utility is equalized across cities, so their is no gain for the marginal worker to move between cities. Following from this result and Equation (8), as Φ_j increase across cities, labor supply n_j also increases. Following from Equations (11) and the labor market clearing condition (4), we solve for a closed form solution for the utility expression. Then we solve for equilibrium allocations using optimality conditions:

[1] The optimality condition of the housing sector is $p_j L_j^{1-\sigma}\sigma x_j^{\sigma-1} = 1$. Rearrange to get $p_j = L_j^{\sigma-1}\sigma^{-1}x_j^{1-\sigma}$. Thus the elasticity of housing price with respect of housing inputs is $\frac{\partial \log p_j}{\partial \log x_j} = 1 - \sigma$. If $\sigma = 1$, then housing prices is pinned by the housing sector technology, $p_j = 1$.

$$v = \log\left(\chi N^{\psi(\sigma-1)}\left[\sum_j \Phi_j^{\frac{1}{\psi(1-\sigma)}}\right]^{\psi(1-\sigma)}\right), \tag{12}$$

$$n_j = N\frac{\Phi_j^{\frac{1}{\psi(1-\sigma)}}}{\sum_{k\in\mathcal{J}}\Phi_k^{\frac{1}{\psi(1-\sigma)}}} \tag{13}$$

$$c_j = Z_j\frac{1-\psi}{1-\psi+\psi\sigma} \tag{14}$$

$$h_j = Z_j^{\sigma}L_j^{1-\sigma}n_j^{\sigma-1}\left(\frac{\sigma\psi}{1-\psi+\psi\sigma}\right)^{\sigma} \tag{15}$$

Intuitively, utility is decreasing in total population N — this is analagous to how an increase in labor supply will decrease wages and thus utility in a typical real business cycle model. Utility is increasing in an aggregation of the Φ_j terms. Also intuitively, city j's share of labor is proportional to Φ_j. For a clearer interpretation of tradeable goods and housing allocation, we express them as shares of total economy output, $Y \equiv \sum_j Z_j n_j$.

$$\frac{c_j n_j}{Y} = \frac{n_j Z_j}{Y}\frac{1-\psi}{1-\psi+\psi\sigma} \tag{16}$$

$$\frac{x_j}{Y} = \frac{n_j Z_j}{Y}\frac{\psi\sigma}{1-\psi+\psi\sigma} \tag{17}$$

The share of output consumed in city j in the form of tradeable goods, $c_j n_j$ is a fraction of city j's share of total output $\frac{n_j Z_j}{Y}$. What is left over of city j's output is allocated towards housing, x_j. Importantly, the amount of goods and housing consumed in city j is directly tied to the productivity of city j, Z_j. In the next section, we show that breaking this link allows overall welfare to increase.

2.2 Remarks

We conclude this section with three remarks. First, we emphasize that our model is equivalent to much of the reduced form models in the literature. These models typically define utility with an indirect function which is increasing in wages and decreasing in housing prices. They then assume a reduced form specification for housing prices which is increasing in a city's population. This specification prevents everyone from locating in a single city. In our model, we have shown that utility is increasing in wages and decreasing in housing prices. And, housing prices are increasing with population.

Second, our main point of departure from most of the literature is to specify where profits from the housing sector go. The importance of this decision is self-evident in the context of discussing efficiency. Several other papers assume profits are paid to an invisible investor.

Third, the model implies that utility is equal across cities, which may seem like a strong implication. It follows from the facts that i) all workers have identical utility specification across cities, and ii) workers are free to move across cities. So, if a city offers less utility than other cities, no one would live there. Likewise, if a city offers more utility than other cities, everyone would live there. As is well known in the literature, this implication can be weakened by simply adding idiosyncratic taste shocks to each worker's city choice problem. As before, workers still locate in the city that offers them the highest utility, but due to the taste shock this city is not the same for everyone. This creates variation in mean utility levels across cities, while keeping the model tractable. We discuss this in Appendix Section A.1. (Further, this implication may not actually be that strong. While wages vary across cities, so do local prices, and they are positively correlated. XXX finds that college workers consume similar bundles across cities.)

3 The Social Planner's Problem

Having discussed the Competitive Equilibrium, we turn to discussing whether it is efficient. To do this, we first solve the social planner problem, then we show that the CE is not efficient. The social planner chooses a utility level u and allocations to maximize u, subject to resource constraints and offering each worker a utility level equal to or greater than u. That is, the SPP is

$$\max_{u,\boldsymbol{c},\boldsymbol{h},\boldsymbol{n}} u \tag{18}$$

subject to the tradeable goods resource constraint (6), the housing constraints (5), the labor constraint (4), and the following utility constraint:

$$u \leq \log(A_j c_j^{1-\psi} h_j^{\psi}) \tag{19}$$

We leave details to solving this system in Appendix Section B. We find that the key difference between the CE and SPP is the amount allocated to each city. In the CE, the amount of resources allocated to a city is equal to the amount it produces, $Z_j n_j$. This occurs as workers have no means by which to trade goods across cities. In the SPP, each city j is allocated an amount equal to the amount it produces plus a mean reversion term, $\tilde{Z}_j n_j$, where $\tilde{Z}_j$ is defined as

$$\tilde{Z}_j \equiv Z_j + (\psi - \psi\sigma)(\bar{Z} - Z_j)$$

where $\bar{Z} \equiv \sum_j \frac{n_j}{N} Z_j$. In other words, the social planner reallocates goods away from the most productive cities and towards the least productive cities. Likewise, people are reallocated from more productive cities to less productive cities. Altogether, the efficient allocation is:

$$\tilde{\Phi}_j \equiv A_j L_j^{\psi(1-\sigma)} \tilde{Z}_j^{1-\psi+\psi\sigma} \tag{20}$$

$$v = \log\left(\chi N^{\psi(\sigma-1)} \left[\sum_j \tilde{\Phi}_j^{\frac{1}{\psi(1-\sigma)}}\right]^{\psi(1-\sigma)}\right), \tag{21}$$

$$n_j = N \frac{\tilde{\Phi}_j^{\frac{1}{\psi(1-\sigma)}}}{\sum_{k\in\mathcal{J}} \tilde{\Phi}_k^{\frac{1}{\psi(1-\sigma)}}} \tag{22}$$

$$c_j = \tilde{Z}_j \frac{1-\psi}{1-\psi+\psi\sigma} \tag{23}$$

$$h_j = \tilde{Z}_j^{\sigma} L_j^{1-\sigma} n_j^{\sigma-1} \left(\frac{\sigma\psi}{1-\psi+\psi\sigma}\right)^{\sigma} \tag{24}$$

The reason the CE is inefficient is that workers do not internalize their effect on housing prices when they decide where to live. On the other hand, the SPP internalizes this effects, and decides to allocate less people to the productive cities. In turn, due to congestion effects, this increases the utility of people in productive cities while decreasing that in unproductive cities. The social planner corrects for this effect by reallocating resources from the productive cities to the unproductive cities. So, essentially, the SPP pays the marginal workers to leave the productive cities because of their negative impact on the housing market. This transfer system actually increases utility overall.

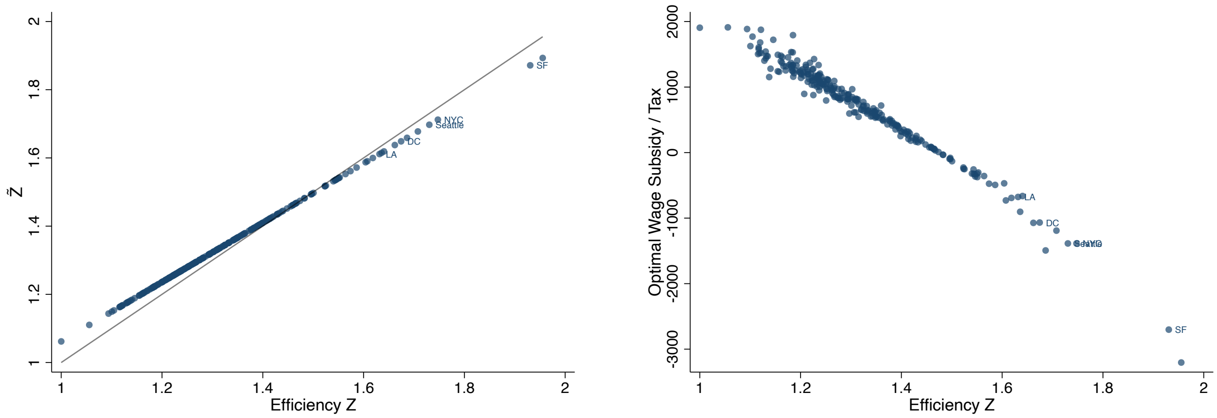
Comparing the CE to the SPP leads us to our main result: the CE is not efficient. This can be seen by directly comparing the SPP and CE allocations.

Proposition 1. *The competitive equilibrium is inefficient.*

Proposition 2. *The optimal tax rate is equal*

$$T(w_j) = T(Z_j) = \tilde{Z}_j.$$

Figure 1: Efficiency Z_j vs $\tilde{Z}_j$ and optimal tax



Notes: Subfigure (a) shows the relationship between Z and $\tilde{Z}$ across MSAs. The grey line is a 45 degree line. Subfigure (b) shows the relationship between efficiency Z and the optimal tax /subsidy across MSAs: a positive number is a subsidy while a negative number is a tax. The optimal tax/subsidy makes the competitive equilibrium efficient. Each point is a MSA from our sample. Efficiencies are estimated by Equation (25). Data is from the 2018-2020 American Community Survey. We filter to privately employed workers between ages 25 and 64, who live in MSAs, as detailed in Section 4.1. To adjust for inflation, dollar values are reported in 2010 terms. Efficiency terms are normalized by the efficiency term from the MSA with the lowest value. Note that the figures show MSAs, not cities: Seattle is seattle-tacoma-bellevue, wa; SF is san francisco-oakland-hayward, ca; DC is washington-arlington-alexandria, dc-va-md-wv; LA is los angeles-long beach-anaheim, CA; and NYC is new york-newark-jersey city, ny-nj-pa.

where $T(w_j)$ is the after-tax wage rate, which can be greater or less than w_j.

The optimal tax rate, which makes the CE efficient, follows immediately from the expression for $\tilde{Z}_j$. Because $Z_j = w_j$ in the CE, and we can express $\tilde{Z}_j$ as just a function of Z_js and parameters, the optimal tax rate $T(Z_j)$ is

$$T(w_j) = T(Z_j) = \tilde{Z}_j.$$

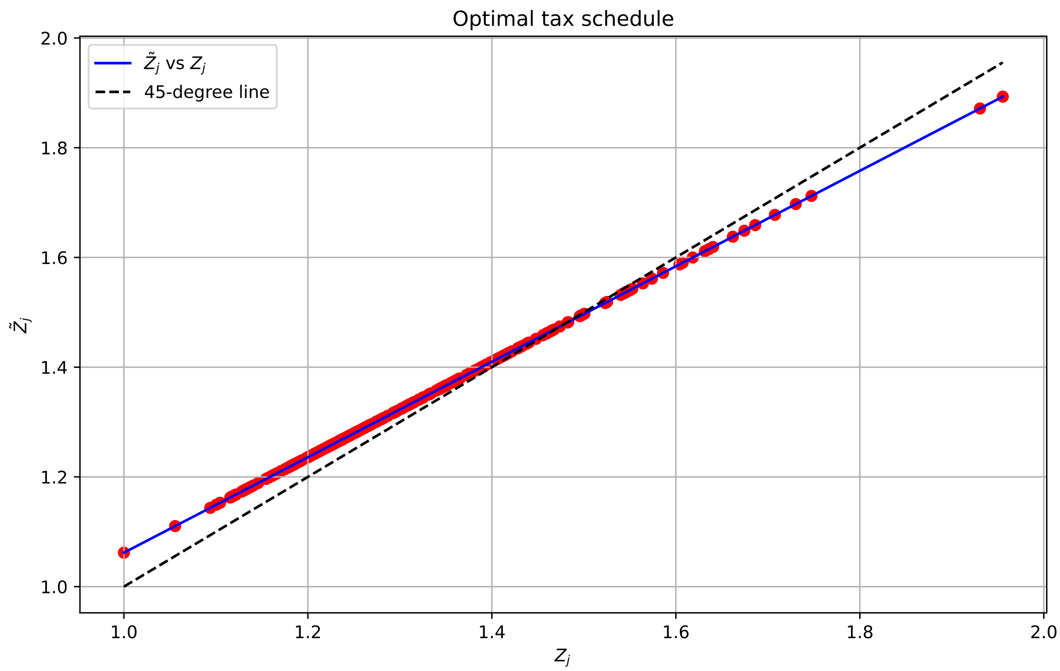
4 Moving to Efficiency: Estimating Welfare Gains

Having shown that the competitive equilibrium is inefficient, we estimate the magnitude of this inefficient. To do this, we need to calibrate the model. Subsection 4.1 calibrates the model, while subsection 4.2 discusses our quantitative results.

4.1 Calibration

Our calibration strategy follows. We calibrate efficiency Z_j to match the MSA component of wages from the data. We calibrate the two elasticities, σ and ψ, externally. Finally we

Figure 2: Efficient tax system



Notes: XXX.

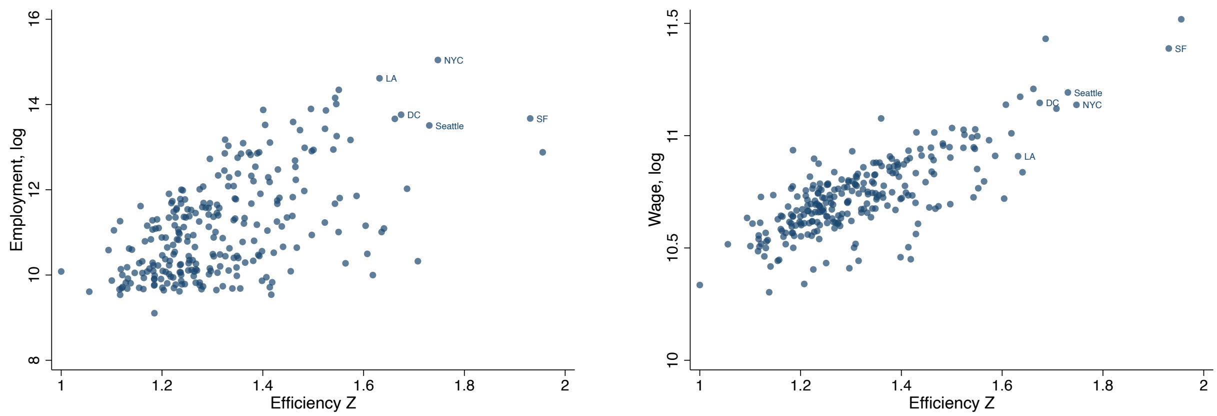
internally calibrate the distribution of land L_j and amenities A_j to match the employment distribution from the data. The key observation is that we do not need to separately identify land L_j and amenities A_j; instead, we estimate the amalgam $A_j L_j^{\psi}$ for each city as to match its employment from the data.

To estimate the efficiencies of each city Z_j, we leverage the first order condition which relates wages to efficiency $w_j = Z_j$. This condition implies that we can estimate the efficiency of a city using its wage rate. However, much of the variation in wages across cities is due to differences in human capital or skill. We control for these differences using worker level survey data from the American Community Survey. We pool together survey data from the years 2018-2020, then filter to privately employed workers between the ages of 25 and 65, not living in group quarters.

For our geographic delineation, we use Metro Statistical Areas (MSAs). A MSA is typical a city or a cluster of cities plus their surrounding suburban area. Hence, this forces us to drop observations in rural areas outside of the identified MSAs, but this step only decreases our sample size by 30 percent. Our sample covers 1,690,834 million people across 260 cities.

With our sample in hand, we estimate each MSA's efficiency by regressing wages on MSA

Figure 3: Efficiencies Z_j vs Wages and Employment



Notes: Subfigure (a) shows the relationship between efficiencies and employment across MSAs. Subfigure (b) shows the relationship between efficiency and mean wages across MSAs. Each point is a MSA from our sample. Efficiencies are estimated by Equation (25). Data is from the 2018-2020 American Community Survey. We filter to privately employed workers between ages 25 and 64, who live in MSAs, as detailed in Section 4.1. To adjust for inflation, dollar values are reported in 2010 terms. Efficiency terms are normalized by the efficiency term from the MSA with the lowest value. Note that the figures show MSAs, not cities: Seattle is seattle-tacoma-bellevue, wa; SF is san francisco-oakland-hayward, ca; DC is washington-arlington-alexandria, dc-va-md-wv; LA is los angeles-long beach-anaheim, CA; and NYC is new york-newark-jersey city, ny-nj-pa.

fixed effects while controlling for demographics:

$$\log(w_{ij}) = \alpha + \Theta_j + \beta X_i + \epsilon_{ij}. \tag{25}$$

where w_{ij} is wage of person i, X_i are the demographic controls, and Θ_j is the MSA fixed effect. The demographics are dummies for race, gender, age, occupation, and industry. For occupation, we use 2-digit SOC codes. For industry we use 2-digit sectors. Once we have the fixed effects, we can back out the efficiency terms, $\Theta_j = \log(Z_j)$. We normalize Z_j so that the MSA with the lowest efficiency has it equal to 1.

Table 1 displays the MSAs with highest and lowest efficiency terms. As expected, the MSAs centered around San Jose, San Francisco, New York, and Seattle have the highest efficiency level, being almost double that of the MSA with the lowest efficiency: Las Cruces, NM. Midland, TX with its booming oil industry rounds out the top five. In terms of dollars, we find that living in the San Jose MSA boosts wages by a large $20,000, while living in Las Cruces, NM is associated with a wage penalty of almost the same magnitude. Figure 3 displays the relationships between efficiency, employment, and wages across MSAs. As expected, as efficiency increases across MSAs, so do wages and employment.

Next we externally calibrating the housing weight parameter ψ. Optimality conditions

Table 1: MSAs with the Highest and Lowest Efficiencies

			Wage	
Rank	Metro Statistical Area	Z	Mean	MSA FE
1	san jose-sunnyvale-santa clara, ca	1.96	100500	20500
2	san francisco-oakland-hayward, ca	1.93	88300	17100
3	new york-newark-jersey city, ny-nj-pa	1.75	68700	7500
4	seattle-tacoma-bellevue, wa	1.73	72600	7300
5	midland, tx	1.71	67500	5900
6	bridgeport-stamford-norwalk, ct	1.69	92200	7000
7	washington-arlington-alexandria, dc-va-md-wv	1.67	69300	4800
8	boston-cambridge-newton, ma-nh	1.66	73700	4600
9	vallejo-fairfield, ca	1.64	50900	2600
10	trenton, nj	1.64	71200	3400
...				
256	springfield, mo	1.1	40500	-16600
257	johnstown, pa	1.1	36600	-15200
258	erie, pa	1.09	41500	-17600
259	muncie, in	1.06	36900	-17500
260	las cruces, nm	1	30800	-17200

Notes: The table shows the MSAs with the highest and lowest efficiencies Z_j. Efficiencies are estimated by regressing wages on MSA-level fixed effects and demographic dummies, as described in Equation (25). Data is from the 2018-2020 American Community Survey, as described in Section 4.1. We filter to privately employed workers between ages 25 and 64 who live in MSAs. The Mean Wage is the MSA's mean wage taken from our sample. The MSA Fixed Effect column displays the city's contribution to the mean wage level: we estimate this effect by using the regression described in Equation (25) to predict each cities mean wage without the city component. Then we subtract these predicted values from the real mean wages from the data. To adjust for inflation, dollar values are reported in 2010 terms. Efficiency terms are normalized by the MSA with the lowest value, so that for Las Cruces, NM equals one by construction. The city component of wages is not strictly increasing with efficiency also by construction, as efficiency enters into wages multiplicatively.

imply that the housing weight ψ is equal to the share of expenditure on housing. Using data from NIPA, we estimate that this share is roughly equal to 20. percent.[2] Thus, we set $\psi = 0.20$.

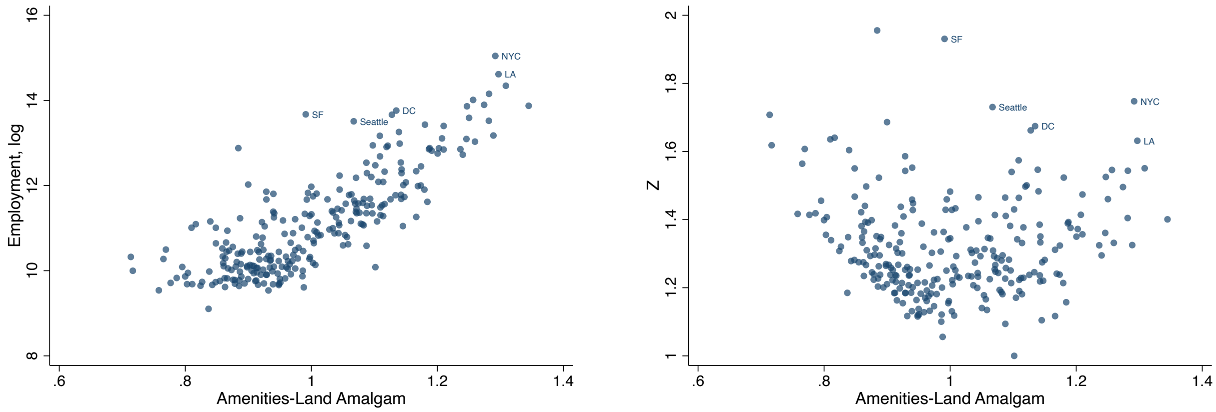
Calibration of σ is taken from Saiz (2010).[3] Saiz (2010) regresss housing prices on population (and construction costs) to get the inverse elasticity of housing supply $\beta = \frac{\partial \log p_j}{\partial \log n_j} = 0.65$. From Equation (29) we get that $\beta = 1 - \sigma$ and thus $\sigma = 0.35$.

Finally, we calibrate the amenities land amalgam $A_j L_j^{\psi(1-\sigma)}$ internally so that the employment distribution in the model equals that from the data. Figure 5 shows scatter plots

[2] See Table 2.3.5 Personal Consumption Expenditures by Major Type of Product.

[3] See page 1267 Equation (3).

Figure 4: Amenities-Land Amalgam $A_j L_j^{\psi(1-\sigma)}$ vs Employment and Efficiency Z



Notes: Subfigure (a) shows the relationship between the amenities-land amalgam $A_j L_j^{\psi(1-\sigma)}$ and employment. Subfigure (b) shows the relationship between this amalgam and efficiency Z. Each point is a MSA from our sample. The amenities-land amalgam $A_j L_j^{\psi(1-\sigma)}$ is calibrated internally to match each city's share of employment from the data. Efficiencies are estimated by Equation (25); Efficiency terms are normalized by the efficiency term from the MSA with the lowest value. Data is from the 2018-2020 American Community Survey. We filter to privately employed workers between ages 25 and 64, who live in MSAs, as detailed in Section 4.1. Note that the figures show MSAs, not cities: Seattle is seattle-tacoma-bellevue, wa; SF is san francisco-oakland-hayward, ca; DC is washington-arlington-alexandria, dc-va-md-wv; LA is los angeles-long beach-anaheim, CA; and NYC is new york-newark-jersey city, ny-nj-pa.

of our estimated amenities-land amalgams against employment and efficiencies.

4.2 Results

Having calibrated the model, we measure the welfare gains associated with moving from the competitive equilibrium to the efficient allocation. We measure this change in utility in real consumption units of the tradeable goods. Specifically, we estimate the percent increase in tradeable goods that makes workers indifferent between living in i) the competitive equilibrium with this percent increase and ii) the pareto equilibrium. That is, we solve for the subsidy s that satisfies:

$$\log(A_j(c_j(1+s))^{1-\psi} h_j^{\psi}) = u^{SPP}$$

The subsidy s has a closed form solution:

$$s = e^{\frac{u^{SPP}-u^{CE}}{1-\psi}} - 1. \tag{26}$$

We find that the indifference subsidy is equal to 0.1 percent. (To be clear, not 1 percent, it is 0.1 percent.) Hence, moving to the efficient equilibrium is associated with a small welfare

Table 2: Moving to Efficiency: Changes in Utility, Output and Consumptions

Metric	%Δ, CE to PO
Utility (measured by change in goods consumption s)	0.1
Total Output, $\sum_j Z_j n_j$	-1.24
Goods Consumption, $\sum_j c_j n_j$	-1.24
Housing Consumption, $\sum_j h_j n_j$	2.63

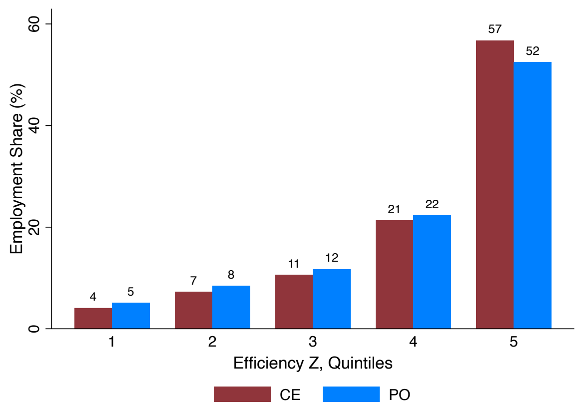
Notes: The Table shows the percent change in utility, total output, goods consumption, and housing consumption associated with moving from the competitive equilibrium to the Pareto optimum. A positive number means the measure is higher in the Pareto optimum than the competitive equilibrium. We measure the change in utility using real consumption units, s, as defined by Equation (26).

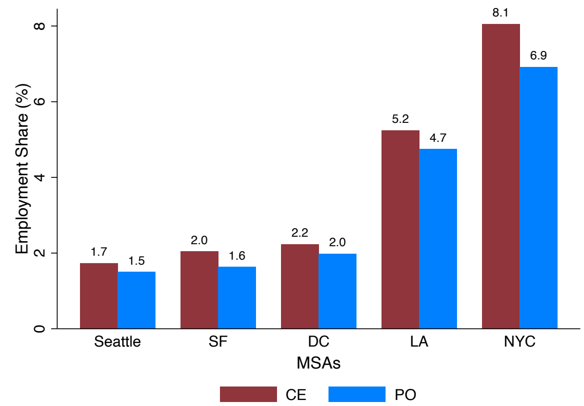
gain.

The increase in welfare from moving to efficiency is due to an increase in housing consumption, as seen in Table 2. The Table shows the changes in total output, total goods consumption, and total housing consumption associated with moving from the competitive equilibrium to an efficient allocation. The efficient allocation actually has lower total output and goods consumption than the competitive equilibrium; these metrics decrease by 1.24 percent. (The change in total output and goods consumption are identical following from optimality conditions, $\frac{\sum_j c_j n_j}{Y} = \frac{1-\psi}{1-\psi+\psi\sigma}$.) However, housing consumption in the efficient allocation is over 2 percent higher than that in the competitive equilibrium. The increase in housing consumption is not due to more resources being allocated to the housing sector — the share or resources allocated to the housing sector is constant across the two equilibriums: $\frac{\sum_j x_j}{Y} = \frac{\psi\sigma}{1-\psi+\psi\sigma}$. Rather, the increase in housing consumption is due to less employment being concentrated in the most productive cities; workers move to places with lower marginal cost of housing, increasing overall housing consumption.

Figure 5 shows the change in employment associated with moving to the efficient allocation across cities. Subfigure (a) displays the change by efficiency quintile. The share of employment in the MSAs in the top quintile of efficiency drops by 5 percentage points. This 5 percentage points of employment is fairly evenly distributed across the four lower quintiles. Subfigure (b) shows the change in employment for the top MSAs that see the largest change in employment. The New York City MSA sees the largest drop in employment at just over one percentage point. The MSAs of Los Angeles, Washington DC, San Francisco, and Seattle round out the top five.

Figure 5: Employment Shares: CE vs PO





Notes: The Figures compares employment shares between the competitive equilibrium and the Pareto optimum. Subfigure (a) groups cities into quintiles by efficiency Z_j. Subfigure (b) looks at the five MSAs with the largest changes in employment between the CE and PO. Note the employment shares in the CE match that from the data described in Section 4.1, while the employment shares in the PO are estimated using the calibrated model. Note that Subfigure (b) plots MSAs, not cities: Seattle is seattle-tacoma-bellevue, wa; SF is san francisco-oakland-hayward, ca; DC is washington-arlington-alexandria, dc-va-md-wv; LA is los angeles-long beach-anaheim, CA; and NYC is new york-newark-jersey city, ny-nj-pa.

5 Conclusion

We find that the benchmark Rosen-Roback model is inefficient. We characterize the optimal taxation rate that can remove the inefficiencies from the competitive equilibrium. We then calibrate the model and find small welfare gains associated with moving to efficiency — a 0.1 percent permanent increase in consumption.

A Solving the Analytical Model Step by Step

Household problem in city j

$$\max_{\{c_j, h_j\}} \log(A_j c_j^{1-\psi} h_j^{\psi})$$

$$\text{s.t.} \quad [\lambda_j] \quad c_j + p_j h_j \leq w_j + \pi_j$$

Firm (general good) in city j

$$\max_{n_j} \underbrace{Z_j n_j}_{Y_j} - w_j n_j$$

Firm (housing good) in city j

$$\Pi_j = \max_{x_j} p_j \underbrace{x_j^{\sigma} L_j^{1-\sigma}}_{H_j} - x_j$$

$$\Pi_j = \frac{\pi_j}{n_j}$$

Market Clearing Conditions:

$$\sum_j n_j = N$$

$$n_j h_j = H_j$$

$$\sum_j (n_j c_j + x_j) = \sum_j Y_j$$

HH FOCs:

$$\frac{1-\psi}{c_j} = \lambda \quad \Rightarrow \quad c_j = (1-\psi)(w_j + \pi_j)$$

$$\frac{\psi}{h_j} = \lambda p_j \quad \Rightarrow \quad h_j = \frac{1}{p_j}\psi(w_j + \pi_j)$$

Solving the good's sectors problem. Following from the good's sectors maximization problem, wages are pinned by productivities:

$$w_j = Z_j. \tag{27}$$

As a city's productivity increases, so does its wage rate. Also, because the traditional firm has a constant returns to scale technology, it has zero profits.

Solving the housing sector. The optimality condition from the housing sector is:

$$\sigma p_j x_j^{\sigma-1} L_j^{1-\sigma} = 1 \tag{28}$$

Intuitively, as the land endowment of a city increases, housing prices decrease. Combining equation (28) with the housing sector's technology (1), housing market clearing (5), and the household's optimality condition (**??**) yields:

$$\begin{aligned}
p_j &= \sigma^{-1} L_j^{\sigma-1} x_j^{1-\sigma} &(29)\\
&= \sigma^{-1} L_j^{\sigma-1} \left(\frac{H_j}{L_j^{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}} &(30)\\
&= \sigma^{-1} L_j^{\frac{\sigma-1}{\sigma}} (n_j h_j)^{\frac{1-\sigma}{\sigma}} &(31)\\
&= \sigma^{-1} L_j^{\frac{\sigma-1}{\sigma}} n_j^{\frac{1-\sigma}{\sigma}} \left(\frac{1}{p_j}\psi(w_j+\pi_j)\right)^{\frac{1-\sigma}{\sigma}} &(32)\\
&= p_j^{-\frac{1-\sigma}{\sigma}} \sigma^{-1} \psi^{\frac{1-\sigma}{\sigma}} L_j^{\frac{\sigma-1}{\sigma}} n_j^{\frac{1-\sigma}{\sigma}} (w_j+\pi_j)^{\frac{1-\sigma}{\sigma}} &(33)\\
p_j &= \sigma^{-\sigma} \psi^{1-\sigma} L_j^{\sigma-1} n_j^{1-\sigma} (w_j+\pi_j)^{1-\sigma} &(34)
\end{aligned}$$

Housing prices are decreasing in a city's land endowment L_j. Housing prices are increasing in i) the share of expenditure workers spend on housing ψ and ii) a city's income $n_j(w_j+\pi_j)$. Intuitivelly, as a city's income increases, demand for housing increases and so does its price. Note, the key congestion force in the model is the fact that the housing sector has a decreasing returns to scale technology. If $\sigma = 1$, then population would not enter into the housing price expression. With $\sigma < 1$, a one percent increase in inputs to the housing sector increases housing supply be less than one percent. A marginal increases in housing supply requires more resources as housing supply increases. Next, we solve for the housing sector's profits.

$$
\begin{aligned}
\pi_j &= p_j x_j^{\sigma} L_j^{1-\sigma} - x_j \\
&= (p_j x_j^{\sigma-1} L_j^{1-\sigma} - 1) x_j \\
&= (\frac{1}{\sigma} - 1)(\sigma^{-1} p_j^{-1} L_j^{\sigma-1})^{\frac{1}{\sigma-1}} \\
&= (\frac{1}{\sigma} - 1)\sigma^{\frac{1}{1-\sigma}} p_j^{\frac{1}{1-\sigma}} L_j \\
&= (\frac{1}{\sigma} - 1)\sigma^{\frac{1}{1-\sigma}} \sigma^{-\frac{\sigma}{1-\sigma}} \psi L_j^{-1}(w_j + \pi_j) n_j L_j \\
&= (1-\sigma)\psi(w_j n_j + \pi_j^1)
\end{aligned}
$$

then

$$
(1-(1-\sigma)\psi)(\pi_j) = (1-\sigma)\psi(w_j n_j)
$$

and rearange and plug in wage to get:

$$
\begin{aligned}
\pi_j &= \frac{(1-\sigma)\psi Z_j n_j}{1-(1-\sigma)\psi} \\
p_j &= \sigma^{-\sigma}\psi^{1-\sigma} L_j^{\sigma-1}(w_j n_j + \pi_j)^{1-\sigma} \\
&= \sigma^{-\sigma}\psi^{1-\sigma} L_j^{\sigma-1}(Z_j n_j + \frac{(1-\sigma)\psi Z_j n_j}{1-(1-\sigma)\psi})^{1-\sigma} \\
p_j &= \sigma^{-\sigma}\psi^{1-\sigma} L_j^{\sigma-1}[\frac{Z_j n_j}{1-(1-\sigma)\psi})]^{1-\sigma}
\end{aligned}
$$

$$
w_j = Z_j
$$

$$
\pi_j = \frac{(1-\sigma)\psi Z_j}{1-(1-\sigma)\psi}
$$

so we derived prices and profits plus rents as functions of parameters and n_j. Now with expressions for profits, wages, and housing prices, we can turn back to the worker's utility and the economy wide allocations.

Now

$$
w_j + \pi_j = \frac{\psi Z_j}{1-(1-\sigma)\psi}
$$

let's plug it into consumption and housing demand:

$$c_j = \frac{(1-\psi)Z_j}{1-(1-\sigma)\psi}$$

$$h_j = \frac{\psi Z_j}{1-(1-\sigma)\psi} \cdot \frac{1}{p_j} =$$

$$= \frac{\psi Z_j}{1-(1-\sigma)\psi} \sigma^{\sigma} \psi^{\sigma-1} L_j^{1-\sigma} [\frac{Z_j n_j}{1-(1-\sigma)\psi})]^{\sigma-1}$$

$$= Z_j^{\sigma} n_j^{\sigma-1} L_j^{1-\sigma} (\frac{\sigma\psi}{1-(1-\sigma)\psi})^{\sigma}$$

Utility. Utility of each agent v_i equalize in equilibrium ($v_j = v$). Substitution Equations (29) and (27) into (7) yields:

$$e^v = e^{v_j} = A_j c_j^{1-\psi} h_j^{\psi} =$$

$$A_j \frac{(1-\psi)^{1-\psi} Z_j^{1-\psi}}{(1-(1-\sigma)\psi)^{1-\psi}} \cdot Z_j^{\sigma\psi} n_j^{(\sigma-1)\psi} L_j^{\psi(1-\sigma)} (\frac{\sigma\psi}{1-(1-\sigma)\psi})^{\sigma\psi} =$$

$$\underbrace{A_j Z_j^{1-\psi(1-\sigma)} L_j^{(1-\sigma)\psi}}_{\Phi_j} n_j^{(\sigma-1)\psi} \underbrace{\frac{(1-\psi)^{1-\psi}(\sigma\psi)^{\sigma\psi}}{(1-\psi(1-\sigma))^{1-\psi(1-\sigma)}}}_{\chi}$$

where χ is a constant made up of parameters. Thus, the utilities of all workers can be expressed only with parameters and employment. Intuitively, utility is increasing in amenities, housing supply and productivity. However, it is decreasing in population. This is due to the inelastic housing supply — as the population increases, demand for housing increases. This increases the price of housing, and decreases utility.

$$e^v = \Phi_j n_j^{(\sigma-1)\psi} \chi$$

Solving Equation (8) for employment n_j, and plugging into the labor market clearing condition (4) yields an expression for utility v in terms of only model primitives:

$$n_j = (e^v \Phi_j^{-1} \chi^{-1})^{\frac{1}{(\sigma-1)\psi}}$$

use MCC

$$N = \sum_j n_j = (e^V \chi^{-1})^{\frac{1}{(\sigma-1)\psi}} \sum_j \Phi_j^{\frac{1}{(1-\sigma)\psi}} \quad \Rightarrow \quad (e^V \chi^{-1})^{\frac{1}{(\sigma-1)\psi}} = \frac{N}{\sum_k \Phi_k^{\frac{1}{(1-\sigma)\psi}}}$$

Utility of workers is decreasing in the mass of workers, but increasing in an aggregate of each cities' productivity, amenities, and housing. With the utility level in hand, we can back out employment from Equation (8):

$$n_j = (e^V \chi^{-1})^{\frac{1}{(\sigma-1)\psi}} B_j^{\frac{1}{(1-\sigma)\psi}} = \frac{N B_j^{\frac{1}{(1-\sigma)\psi}}}{\sum_k B_k^{\frac{1}{(1-\sigma)\psi}}} =$$

$$= N \cdot \frac{A_j^{\frac{1}{(1-\sigma)\psi}} Z_j^{\frac{1-(1-\sigma)\psi}{(1-\sigma)\psi}} L_j}{\sum_k A_k^{\frac{1}{(1-\sigma)\psi}} Z_k^{\frac{1-(1-\sigma)\psi}{(1-\sigma)\psi}} L_k}$$

As expected, employment in a city is increasing in its amenities, productivity, and housing supply. Each city's share of employment is its share of an aggregate measure of economy-wide amenities, productivity, and housing supply. To finish solving the equilibrium, we solve for utility level and consumptions:

Plug back

$$e^v = A_j Z_j^{1-\psi(1-\sigma)} L^{(1-\sigma)\psi} \chi N^{(\sigma-1)\psi} \cdot \frac{A_j^{-1} Z_j^{-1+(1-\sigma)\psi} L_j^{(\sigma-1)\psi}}{\left(\sum_k A_k^{\frac{1}{(1-\sigma)\psi}} Z_k^{\frac{1-(1-\sigma)\psi}{(1-\sigma)\psi}} L_k\right)^{(\sigma-1)\psi}}$$

$$= \frac{\chi N^{(\sigma-1)\psi}}{\left(\sum_k A_k^{\frac{1}{(1-\sigma)\psi}} Z_k^{\frac{1-(1-\sigma)\psi}{(1-\sigma)\psi}} L_k\right)^{(\sigma-1)\psi}}$$

$$\frac{(1-\psi)^{1-\psi}(\sigma\psi)^{\sigma\psi}}{(1-\psi(1-\sigma))^{1-\psi(1-\sigma)}} \cdot \frac{N^{(\sigma-1)\psi}}{\left(\sum_k A_k^{\frac{1}{(1-\sigma)\psi}} Z_k^{\frac{1-(1-\sigma)\psi}{(1-\sigma)\psi}} L_k\right)^{(\sigma-1)\psi}}$$

We also express allocations as a share of total economy output Y. They are intuitive.

$$c_j n_j = \frac{(1-\psi)}{1-\psi(1-\sigma)} Z_j n_j = \frac{(1-\psi)N}{1-\psi(1-\sigma)} \cdot \frac{A_j^{\frac{1}{(1-\sigma)\psi}} Z_j^{\frac{1}{(1-\sigma)\psi}} L_j}{\sum_k A_k^{\frac{1}{(1-\sigma)\psi}} Z_k^{\frac{1-(1-\sigma)\psi}{(1-\sigma)\psi}} L_k}$$

$$x_j = \frac{\sigma\psi}{1-\psi(1-\sigma)} Z_j n_j = \frac{\sigma\psi N}{1-\psi(1-\sigma)} \cdot \frac{A_j^{\frac{1}{(1-\sigma)\psi}} Z_j^{\frac{1}{(1-\sigma)\psi}} L_j}{\sum_k A_k^{\frac{1}{(1-\sigma)\psi}} Z_k^{\frac{1-(1-\sigma)\psi}{(1-\sigma)\psi}} L_k}$$

$$Y = \sum_j Y_j = \sum_j Z_j n_j = N \cdot \frac{\sum_j A_j^{\frac{1}{(1-\sigma)\psi}} Z_j^{\frac{1}{(1-\sigma)\psi}} L_j}{\sum_k A_k^{\frac{1}{(1-\sigma)\psi}} Z_k^{\frac{1-(1-\sigma)\psi}{(1-\sigma)\psi}} L_k}$$

Consumption city j share

$$\frac{c_j n_j}{Y} = \frac{(1-\psi)}{1-\psi(1-\sigma)}\frac{Z_j n_j}{\sum_k Z_k n_k} = \frac{(1-\psi)}{1-\psi(1-\sigma)} \cdot \frac{A_j^{\frac{1}{(1-\sigma)\psi}} Z_j^{\frac{1}{(1-\sigma)\psi}} L_j}{\sum_k A_k^{\frac{1}{(1-\sigma)\psi}} Z_k^{\frac{1}{(1-\sigma)\psi}} L_k}$$

$$\frac{x_j}{Y} = \frac{\sigma\psi}{1-\psi(1-\sigma)}\frac{Z_j n_j}{\sum_k Z_k n_k} = \frac{\sigma\psi}{1-\psi(1-\sigma)} \cdot \frac{A_j^{\frac{1}{(1-\sigma)\psi}} Z_j^{\frac{1}{(1-\sigma)\psi}} L_j}{\sum_k A_k^{\frac{1}{(1-\sigma)\psi}} Z_k^{\frac{1}{(1-\sigma)\psi}} L_k}$$

City j housing demand :

$$n_j h_j = H_j = x_j^{\sigma} L_j^{1-\sigma} =$$

$$= \frac{(\frac{\sigma\psi N}{1-\psi(1-\sigma)})^{\sigma} A_j^{\frac{\sigma}{(1-\sigma)\psi}} Z_j^{\frac{\sigma}{(1-\sigma)\psi}} L_j^{\sigma}}{(\sum_k A_k^{\frac{1}{(1-\sigma)\psi}} Z_k^{\frac{1-\psi(1-\sigma)}{(1-\sigma)\psi}} L_k)^{\sigma}} = h_j \cdot N \cdot \frac{A_j^{\frac{1}{(1-\sigma)\psi}} Z_j^{\frac{1-(1-\sigma)\psi}{(1-\sigma)\psi}} L_j}{\sum_k A_k^{\frac{1}{(1-\sigma)\psi}} Z_k^{\frac{1-(1-\sigma)\psi}{(1-\sigma)\psi}} L_k}$$

We conclude this section with derivation of h_j

$$h_j = \frac{(\frac{\sigma\psi}{1-\psi(1-\sigma)})^{\sigma} N^{\sigma-1} A_j^{-\frac{1}{\psi}} Z_j^{1-\frac{1}{\psi}}}{(\sum_k A_k^{\frac{1}{(1-\sigma)\psi}} Z_k^{\frac{1-(1-\sigma)\psi}{(1-\sigma)\psi}} L_k)^{\sigma-1}}$$

A.1 Adding a taste shock to the analytical model

An implication in the baseline spatial equilibrium model is that living in all cities yield the same utility level. This is a strong implication. It follows from the fact that (i) workers are free to move across cities and (ii) workers all yield the same utility in each city. We weaken (ii) by introducing idiosyncratic taste shocks, which creates an equilibrium where there is variance in the utility levels cities offer. Let ϵ_{ij} be the shock to place j for worker i. We assume this shock enters additively into utility, so given the vector of taste shocks $\boldsymbol{\epsilon_i}$ workers solve:

$$\max_{j\in\mathcal{J}} \left\{ v_j(w_j, p_j, \pi_j) + \epsilon_{ij} \right\}.$$

Following the discrete choice literature, we assume that the taste shocks are drawn from a Type 2 extreme Value distribution governed by λ. Immediately following this assumption, the probability a worker locates in city j has a clean closed form solution:

$$Prob(j|s) = \frac{v_j^{\lambda}}{\sum_{k\in\mathcal{J}} v_{ks}^{\lambda}}. \tag{35}$$

where utility $v_j \equiv v_j(w_j, p_j, \pi_j)$ follows from optimality conditions:

$$e^{v_j} = A_j L_j^{\psi} Z_j^{1+(\sigma-1)\psi} n_j^{(\sigma-1)\psi} \chi, \qquad \chi \equiv \frac{(1-\psi)^{1-\psi}(\psi\sigma)^{\psi\sigma}}{(1-\psi+\psi\sigma)^{1-\psi+\psi\sigma}} \tag{36}$$

The probability a worker locates in city j is simply increasing in the utility of living in j, and decreasing in the utility of living in other cities. By the law of large numbers, the supply of labor to j equals the probability a worker chooses to locate in city j. Hence, the labor supply to a city is:

$$\frac{n_j}{N} = \frac{v_j^{\lambda}}{\sum_{l \in \mathcal{J}} v_l^{\lambda}} \tag{37}$$

Thus, the supply of labor to a city is increasing in the utility it offers, and there is variation in utility across cities within skill levels. The equilibrium employment distribution $\boldsymbol{n}_j$ and utilities $\boldsymbol{v}_j$ are pinned by the labor supply equation (37) and the utility equation (36). This is a system of $|\mathcal{J}| \times 2$ equations and unknowns, where $|\cdot|$ denotes the number of objects in the set.

B Solving the Social Planner's Problem Step by Step

This planner does not equalize utility across cities

$$\begin{aligned}
u^* &= \max_{u,c,h,n} \sum_{j \in \mathcal{J}} u_j \\
[\gamma_i] \quad & n_j u_j \leq n_j \log(A_j c_j^{1-\psi} h_j^{\psi}) \\
[\mu] \quad & \sum_j n_j = N \\
[\lambda] \quad & \sum_j (c_j n_j + x_j) = \sum Z_j n_j \\
n_i h_i &= H_j = x_j^{\sigma} L_j^{1-\sigma}
\end{aligned}$$

FOCs:

$$1 = \gamma_j n_j$$

$$\frac{\gamma_j n_j (1-\psi)}{c_j} = \lambda n_j$$

$$\frac{\gamma_j n_j \psi}{h_j} = \lambda \left(\frac{n_j h_j}{L_j^{1-\sigma}}\right)^{\frac{1}{\sigma}} \frac{1}{\sigma} \frac{1}{h_j}$$

$$\gamma_j(\log(A_j c_j^{1-\psi} h_j^{\psi}) - u_j) + \lambda Z_j = \mu + c_j\lambda + \lambda(\frac{n_j h_j}{L_j^{1-\sigma}})^{\frac{1}{\sigma}} \frac{1}{\sigma}\frac{1}{n_j}$$

Notice that

$$c_j n_j = \frac{1-\psi}{\lambda}$$

$$x_j = \frac{\psi\sigma}{\lambda}$$

Then $\lambda > 0$ feasibility becomes

$$J\frac{1-\psi(1-\sigma)}{\lambda} = \sum_j Z_j n_j \quad \lambda = \frac{J(1-\psi(1-\sigma))}{\sum_j Z_j n_j}$$

$$Z_j\lambda = \mu + \gamma_j$$

multiply by n_j and sum over j to get

$$\lambda\sum_j Z_j n_j = J(1-\psi(1-\sigma)) = \mu\sum_j n_j + \sum_j \gamma_j n_j = \mu N + J$$

$$\mu = -\frac{J\psi(1-\sigma)}{N}$$

$$\frac{n_k}{n_j} = \frac{c_j}{c_k} = \frac{\gamma_j}{\gamma_k} = \frac{\lambda Z_j - \mu}{\lambda Z_k - \mu} = \frac{\frac{J(1-\psi(1-\sigma))}{\sum_i Z_i n_i} Z_j + \frac{J\psi(1-\sigma)}{N}}{\frac{J(1-\psi(1-\sigma))}{\sum_i Z_i n_i} Z_k + \frac{J\psi(1-\sigma)}{N}} =$$

$$= \frac{Z_j + \psi(1-\sigma)(\bar{Z} - Z_j)}{Z_k + \psi(1-\sigma)(\bar{Z} - Z_k)} = \frac{\tilde{Z}_j}{\tilde{Z}_k}$$

Moreover

$$n_j = \frac{1}{Z_j\frac{J(1-\psi(1-\sigma))}{\sum_i Z_i n_i} + \frac{J\psi(1-\sigma))}{N}} = \frac{\sum_j Z_j n_j}{J\tilde{Z}_j}$$

$$c_j = \frac{1-\psi}{\lambda}\frac{1}{n_j} = \frac{(1-\psi)J\tilde{Z}_j}{\sum_i Z_i n_i}\frac{\sum_k Z_k n_k}{J(1-\psi(1-\sigma)} = \tilde{Z}_j\frac{1-\psi}{1-\psi(1-\sigma)}$$

Now notice that

$$\tilde{Z}_j n_j = \frac{1}{J}\sum_k Z_k n_k$$

which means that under $\tilde{Z}_j$ productivity cities output equalize. When we sum it over$j$

$$\sum_j \tilde{Z}_j n_j = \sum_k Z_k n_k$$

so aggregate production with Z_j and with $\tilde{Z}_j$ equalize.
Let's find e^u, n_j and h_j. Notice that

$$e^u = A_j c_j^{1-\psi} h_j^{\psi}$$

$$(e^u A_j^{-1} c_j^{\psi-1})^{\frac{1}{\psi}} = h_j = x_j^{\sigma} L_j^{1-\sigma} n_j^{-1}$$

$$n_j = (e^u A_j^{-1}(\frac{1-\psi}{1-\psi(1-\sigma)}\tilde{Z}_j)^{\psi-1})^{-\frac{1}{\psi}} \cdot (\frac{\psi\sigma}{J(1-\psi(1-\sigma)}\sum_k Z_k n_k)^{\sigma} L_j^{1-\sigma}$$

$$N = \sum_j n_j = (e^u)^{-\frac{1}{\psi}} \sum_j (A_j^{-1}(\frac{1-\psi}{1-\psi(1-\sigma)}\tilde{Z}_j)^{\psi-1})^{-\frac{1}{\psi}} \cdot (\frac{\psi\sigma}{J(1-\psi(1-\sigma)}\sum_k Z_k n_k)^{\sigma} L_j^{1-\sigma}$$

$$(e^u)^{-\frac{1}{\psi}} = \frac{N}{\sum_j (A_j^{-1}(\frac{1-\psi}{1-\psi(1-\sigma)}\tilde{Z}_j)^{\psi-1})^{-\frac{1}{\psi}} \cdot (\frac{\psi\sigma}{J(1-\psi(1-\sigma)}\sum_k Z_k n_k)^{\sigma} L_j^{1-\sigma}}$$

plug it back to expression for n_j

$$n_j = N \cdot \frac{(A_j^{-1}(\frac{1-\psi}{1-\psi(1-\sigma)}\tilde{Z}_j)^{\psi-1})^{-\frac{1}{\psi}} \cdot (\frac{\psi\sigma}{J(1-\psi(1-\sigma)}\sum Z_i n_i)^{\sigma} L_j^{1-\sigma}}{\sum_k (A_k^{-1}(\frac{1-\psi}{1-\psi(1-\sigma)}\tilde{Z}_k)^{\psi-1})^{-\frac{1}{\psi}} \cdot (\frac{\psi\sigma}{J(1-\psi(1-\sigma)}\sum_l Z_l n_l)^{\sigma} L^{1-\sigma}} =$$

$$= N \cdot \frac{A_j^{\frac{1}{\psi}} \tilde{Z}_j^{\frac{1}{\psi}-1} L_j^{1-\sigma}}{\sum_k A_k^{\frac{1}{\psi}} \tilde{Z}_k^{\frac{1}{\psi}-1} L_k^{1-\sigma}}$$

Let's calculate $\sum_j \tilde{Z}_j n_j$ keep in mind that its equal to $\sum_j Z_j n_j$

$$\sum_j \tilde{Z}_j n_j = N \cdot \frac{\sum_j A_j^{\frac{1}{\psi}} \tilde{Z}_j^{\frac{1}{\psi}} L_j^{1-\sigma}}{\sum_k A_k^{\frac{1}{\psi}} \tilde{Z}_k^{\frac{1}{\psi}-1} L_k^{1-\sigma}}$$

Now solve for e^u:

$$e^u = A_j c_j^{1-\psi} h_j^{\psi}$$

$$h_j = (e^u A_j^{-1} c_j^{-\psi})^{\frac{1}{1-\psi}} = (e^u)^{\frac{1}{1-\psi}} (A_j^{-1} c_j^{-\psi})^{\frac{1}{1-\psi}}$$

$$= \frac{N^{\frac{\psi}{\psi-1}}}{[\sum_j (A_j^{-1}(\frac{1-\psi}{1-\psi(1-\sigma)}\tilde{Z}_j)^{\psi-1})^{-\frac{1}{\psi}} \cdot (\frac{\psi\sigma}{(J(1-\psi(1-\sigma)}\sum_k Z_k n_k)^{\sigma} L_j^{1-\sigma}]^{\frac{\psi}{1-\psi}}} \cdot A_j^{\frac{1}{\psi-1}} \cdot \tilde{Z}_j^{\frac{\psi}{\psi-1}} (\frac{1-\psi}{1-\psi(1-\sigma)})^{\frac{\psi}{\psi-1}} =$$

$$= \frac{N^{\frac{\psi}{\psi-1}} A_j^{\frac{1}{\psi-1}} \tilde{Z}_j^{\frac{\psi}{\psi-1}} (\frac{1-\psi}{1-\psi(1-\sigma)})^{\frac{\psi}{\psi-1}} \cdot (\frac{1-\psi}{1-\psi(1-\sigma)})^{\frac{\psi-1}{\psi}} \cdot (\frac{\psi\sigma}{(J(1-\psi(1-\sigma)})^{\frac{\sigma}{\psi-1}}}{\sum_j (A_j^{\frac{1}{\psi}} \tilde{Z}_j^{\frac{1-\psi}{\psi}} \cdot (\sum_k Z_k n_k)^{\frac{\sigma}{1-\psi}} L_j^{\frac{1-\sigma}{1-\psi}}} =$$

$$= \frac{N^{\frac{\psi}{\psi-1}} A_j^{\frac{1}{\psi-1}} \tilde{Z}_j^{\frac{\psi}{\psi-1}} \left(\frac{1-\psi}{1-\psi(1-\sigma)}\right)^{\frac{\psi}{\psi-1}} \cdot \left(\frac{1-\psi}{1-\psi(1-\sigma)}\right)^{\frac{\psi-1}{\psi}} \cdot \left(\frac{\psi\sigma}{(J(1-\psi(1-\sigma))}\right)^{\frac{\sigma}{\psi-1}}}{\sum_j A_j^{\frac{1}{\psi}} \tilde{Z}_j^{\frac{1-\psi}{\psi}} L_j^{\frac{1-\sigma}{1-\psi}}} \cdot N^{\frac{\sigma}{\psi-1}} \cdot \frac{\left(\sum_k A_k^{\frac{1}{\psi}} \tilde{Z}_k^{\frac{1}{\psi}-1} L_k^{1-\sigma}\right)^{\frac{\sigma}{1-\psi}}}{\left(\sum_j A_j^{\frac{1}{\psi}} \tilde{Z}_j^{\frac{1}{\psi}} L_j^{1-\sigma}\right)^{\frac{\sigma}{1-\psi}}}$$

fixed last two equalities

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