Recitations 16

[Definitions used today]

- Best correspondence, Nash Equilibrium, Minimax Theorem

Question 1

1/2	L	R
T	3,1	0,0
B	0,0	1,3

- Define: pure actions, mixed actions, best correspondences
- Find all Nash Equilibria

Solution 1

pure strategies: $A^1 = \{T, B\}$, $A^2 = \{L, R\}$,$A = A^1 \times A^2$
mixed strategies:

$$S = S^1 \times S^2 = \Delta(A^1) \times \Delta(A^2) = \{((p, 1-p), (q, 1-q)) \,|\, p, q \in [0,1]\}$$

We can solve for the best responses as follows: Mr 1 best response:

$$BR^1((q, 1-q)) : \left\{ \begin{array}{cc} T & B \\ 3(q) + 0(1-q) & 0(q) + 1(1-q) \end{array} \right\}$$

Equality only holds when $q = \frac{1}{4}$. $T > B \iff q > \frac{1}{4}$. $T < B \iff q < \frac{1}{4}$ Therefore, player 1 sets $p = 1$ if $q > \frac{1}{4}$ and sets $p = 0$. She picks $p \in [0,1]$ where is indifferent between T and B. otherwise.

$$BR^1((q, 1-q)) = \begin{cases} 0 & \text{if } q < \frac{1}{4} \\ [0,1] & \text{if } q = \frac{1}{4} \\ 1 & \text{if } q > \frac{1}{4} \end{cases}$$

Mr 2 best response:

$$BR^2((p, 1-p)) : \left\{ \begin{array}{cc} L & R \\ p + 0(1-p) & 0(p) + 3(1-p) \end{array} \right\}$$

Equality only holds when $p = \frac{3}{4}$. $L > R \iff p > \frac{3}{4}$,$L < R \iff p < \frac{3}{4}$ Similarly, player 2 sets $q = 1$ if $p > \frac{3}{4}$ and sets $q = 0$ otherwise.

$$BR^2((p, 1-p)) = \begin{cases} 0 & \text{if } p < \frac{3}{4} \\ [0,1] & \text{if } p = \frac{3}{4} \\ 1 & \text{if } p > \frac{3}{4} \end{cases}$$

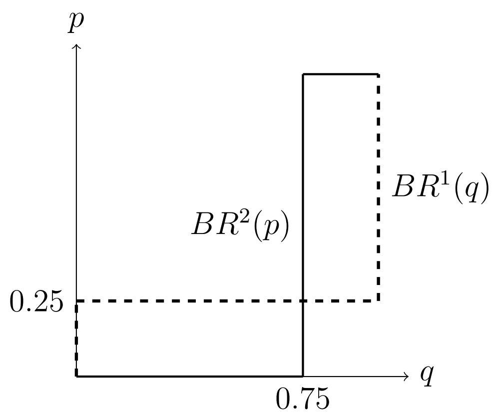

Figure 1: Best Responses

These best responses can be graphed :

The points of interesection

$$\left(\frac{3}{4},\frac{1}{4}\right),(1,1),(0,0)$$

yield the set of Nash equilibria

$$\mathrm{NE}=\left\{\big((1,0),(1,0)\big),\big((0,1),(0,1)\big),\big((\frac{3}{4},\frac{1}{4}),(\frac{1}{4},\frac{3}{4})\big)\right\}.$$

Question 2 [153 III.1 Spring 2013 majors]

A two players finite action normal form game is zero sum if the sum of the utilities of the two players is equal to 0 for any action profile, so $u^1=-u^2$. **The Minimax Theorem** states that in this case

$$\min_{\alpha^2\in\Delta(A^2)}\max_{\alpha^1\in\Delta(A^1)}u\left(\alpha^1,\alpha^2\right)=\max_{\alpha^1\in\Delta(A^1)}\min_{\alpha^2\in\Delta(A^2)}u\left(\alpha^1,\alpha^2\right)\equiv v$$

Prove the minimax theorem. You can use Nash equilibrium existence theorem.

Solution 2

We will do it in two parts: First we will prove that $\geq$ holds. Secondly that $\leq$ holds.

$\geq$. Note that for any $\bar{s}^1\in\Delta\left(A^1\right)$ and $\bar{s}^2\in\Delta\left(A^2\right)$ it holds that:

$$u\left(\bar{s}^1,\bar{s}^2\right)\geq\min_{s^2\in\Delta(A^2)}u\left(\bar{s}^1,s^2\right)$$

Then by taking maximum at both sides with respect to s^1 :

$$\max_{s^1\in\Delta(A^1)}u\left(s^1,\bar{s}^2\right)\geq\max_{s^1\in\Delta(A^1)}\min_{s^2\in\Delta(A^2)}u\left(s^1,s^2\right)$$

Note that the RHS is now constant, and a lower bound to the LHS across s^2, then:

$$\min_{s^2\in\Delta(A^2)}\max_{s^1\in\Delta(A^1)}u\left(s^1,s^2\right)\geq\max_{s^1\in\Delta(A^1)}\min_{s^2\in\Delta(A^2)}u\left(s^1,s^2\right)\tag{0.1}$$

$\leq$. Note that for any $\bar{s}^1 \in \Delta\left(A^1\right)$ it holds that:

$$\max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} u\left(s^1, s^2\right) \geq \min_{s^2 \in \Delta(A^2)} u\left(\bar{s}^1, s^2\right)$$

In particular for $\hat{s}^1$ a NE of the game the inequality must hold. We assume that such NE exists in mixed strategies. Note that if $(\hat{s}^1, \hat{s}^2)$ it is defined as an strategy profile such that:

$$u\left(\hat{s}^1, \hat{s}^2\right) = \max_{s^1 \in \Delta(A^1)} u\left(s^1, \hat{s}^2\right) \qquad -u\left(\hat{s}^1, \hat{s}^2\right) = \max_{s^2 \in \Delta(A^2)} -u\left(\hat{s}^1, s^2\right)$$

The second condition implies:

$$u\left(\hat{s}^1, \hat{s}^2\right) = \min_{s^2 \in \Delta(A^2)} u\left(\hat{s}^1, s^2\right) = \max_{s^1 \in \Delta(A^1)} u\left(s^1, \hat{s}^2\right)$$

thus

$$\begin{aligned}
\min_{s^2 \in \Delta(A^2)} u^1\left(\hat{s}^1, s^2\right) &= u^1\left(\hat{s}^1, \operatorname*{argmin}_{s^2 \in \Delta(A^2)} u^1\left(\hat{s}^1, s^2\right)\right) \\
&= u^1\left(\hat{s}^1, \operatorname*{argmax}_{s^2 \in \Delta(A^2)} u^2\left(\hat{s}^1, s^2\right)\right) \\
&= u^1\left(\hat{s}^1, \hat{s}^2\right) \\
&= \max_{s^1 \in \Delta(A^1)} u^1\left(s^1, \hat{s}^2\right) \\
&\geq \min_{s^2 \in \Delta(A^2)} \max_{s^1 \Delta(A^1)} u^1\left(s^1, s^2\right)
\end{aligned}$$

Then by taking max over $\Delta\left(A^1\right)$:

$$\max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} u\left(s^1, s^2\right) \geq \min_{s^1 \in \Delta(A^1)} u\left(s^1, \hat{s}^2\right) \geq \min_{s^2 \in \Delta(A^2)} \max_{s^1 \in \Delta(A^1)} u\left(s^1, s^2\right) \tag{0.2}$$

Inequalities (0.1) and (0.2) gives us thesis of minimax theorem.

Question 3

For a zero sum game of two players define the value of the game as $V : \mathbb{R}^{nm} \to \mathbb{R}$ (where $n = \#A^1$ and $m = \#A^2$) :

$$V(u) = \max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u\right)$$

where for a given strategy profile $s^1 = (p_1, \ldots, p_n), s^2 = (q_1, \ldots, q_n)$ and payoffs $u \in \mathbb{R}^{nm}$ we define

$$U\left(s^1, s^2 \mid u\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j u_{ij}$$

Show that **The value of a game** is

a) continuous

b) non-decreasing

c) homogenous of degree one in payoffs.

Solution 3

- Consider the problem:

$$v\left(s^1, u\right) = \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u\right)$$

note that U is continuous in s_1, s_2 and u and that the minimum is being taken over s^2 in a compact set that does not vary with s^1 or u. By the theorem of the maximum the value of this problem, as a function of s^1 and u is a continuous function. Now consider:

$$V(u) = \max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u\right) = \max_{s^1 \in \Delta(A^1)} v\left(s^1, u\right)$$

again since v is continuous and s^1 varies in a compact set independent of u by the theorem of the maximum V is a continuous function of u.

- Let $u_1 \leq u_2$. Clearly for all s^1, s^2:

$$U\left(s^1, s^2 \mid u_1\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j u_{ij}^1 \leq \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j u_{ij}^2 = U\left(s^1, s^2 \mid u_1\right)$$

so $U\left(s^1, s^2 \mid u_1\right) \leq U\left(s^1, s^2 \mid u_2\right)$. Then:

$$\min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u_1\right) \leq \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u_2\right)$$
$$V\left(u_1\right) = \max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u_1\right) \leq \max_{s^1 \in \Delta(A^1)} \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u_2\right) = V\left(u_2\right)$$

- Let $\lambda \in \mathbb{R}$, note that $U\left(s^1, s^2 \mid \lambda u\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j \lambda u_{ij} = \lambda U\left(s^1, s^2 \mid u\right)$ and $\max_x \lambda f(x) = \lambda \max_x f(x)$. Thus $V(\lambda u) = \lambda V(u)$

Question 4 Under standard assumptions, prove the following properties of best response in mixed $\mathrm{BR}_i(s)$:

a) non-empty valued,

b) compact valued,

c) upper hemi continuous.

d) convex-valued

Solution 4

a) Take any $s \in S$. Then $BR^i(s) = \arg\max_{r^i \in S^i} u^i\left(r^i, s^{-i}\right)$. Since $u^i\left(\cdot, s^{-i}\right)$ is continuous and $S^i = \Delta\left(A^i\right)$ is compact, by the Weierstrass Theorem u^i achieves a maximum on S^i. Hence, $BR^i(s)$ is nonempty. Since s has been arbitrary, $BR^i(\cdot)$ is nonempty-valued.

b) that converges in S^i, i.e. $r^i_m \to r^i \in S^i$. By definition we have $u^i\left(r^i_m, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right) \forall t^i \in S^i, m \in \mathbb{N}$. Then since $u^i\left(\cdot, s^{-i}\right)$ is continuous,

$$u^i\left(r^i, s^{-i}\right) = u^i\left(\lim_{m\to\infty} r^i_m, s^{-i}\right) = \lim_{m\to\infty} u^i\left(r^i_m, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right) \quad \forall t^i \in S^i$$

Hence, $r^i \in BR^i(s)$. Since s has been arbitrary, $BR^i(\cdot)$ is closed-valued.

c) Since S^i (the range of $BR^i(\cdot)))$ is compact and u is continuous so $BR^i(S))$ is compact.It is sufficient to establish that $BR^i(\cdot)$ has a closed graph. Fix $s \in S$ arbitrarily and take any sequences $(s_m) \in S^\infty$ and $(r^i_m) \in S^{i\infty}$ with $s_m \to s \in S, r^i_m \to r^i \in S^i$ and $r^i_m \in BR^i\left(s_m\right) \forall m \in \mathbb{N}$. Then $u^i\left(r^i_m, s^{-i}_m\right) \geq u^i\left(t^i, s^{-i}_m\right), \forall t^i \in S^i$. Since $u^i(\cdot,\cdot)$ is continuous it follows that $\forall t^i \in S^i$

$$\begin{aligned}
u^i\left(r^i, s^{-i}\right) = u^i\left(\lim_{m\to\infty} r^i_m, \lim_{m\to\infty} s^{-i}_m\right) &= \lim_{m\to\infty} u^i\left(r^i_m, s^{-i}_m\right) \\
&\geq \lim_{m\to\infty} u^i\left(t^i, s^{-i}_m\right) \\
&= u^i\left(t^i, \lim_{m\to\infty} s^{-i}_m\right) \\
&= u^i\left(t^i, s^{-i}\right)
\end{aligned}$$

Hence, $r^i \in BR^i(s)$ and $BR^i(\cdot)$ is closed at s. Since s has been arbitrary, $BR^i(\cdot)$ has a closed graph.

d) Fix $s \in S$ arbitrarily and take any $r^i_a, r^i_b \in BR^i(s)$ and $\lambda \in [0,1]$. Then it must be that $u^i\left(r^i_a, s^{-i}\right) = u^i\left(r^i_b, s^{-i}\right) \geq u^i\left(r^i, s^{-i}\right) \forall r^i \in S^i$. Or, equivalently,

$$\sum_{a^i \in A^i} r^i_a\left(a^i\right) u^i\left(a^i, s^{-i}\right) = \sum_{a^i \in A^i} r^i_b\left(a^i\right) u^i\left(a^i, s^{-i}\right) \geq \sum_{a^i \in A^i} r^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) \quad \forall r^i \in S^i$$

Now consider the mixed strategy $\lambda r_a^i + (1-\lambda) r_b^i$. The utility of this strategy profile is

$$\begin{aligned}
u^i\left[\lambda r_a^i + (1-\lambda) r_b^i, s^{-i}\right] &= \sum_{a^i \in A^i} \left[\lambda r_a^i\left(a^i\right) + (1-\lambda) r_b^i\left(a^i\right)\right] u^i\left(a^i, s^{-i}\right) \\
&= \lambda \sum_{a^i \in A^i} r_a^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) + (1-\lambda) \sum_{a^i \in A^i} r_b^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) \\
&= \sum_{a^i \in A^i} r_a^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) \\
&\geq u^i\left(r^i, s^{-i}\right) \quad \forall r^i \in S^i,
\end{aligned}$$

where the third line follows from (2) and the inequality holds since $r_a^i \in BR^i(s)$. Hence, $\lambda r_a^i + (1-\lambda) r_b^i \in BR^i(s)$ and, since s has been arbitrary, $BR^i(\cdot)$ is convex-valued.

Question 5 Show that $\mathrm{BR}_i(s) = \mathrm{co}\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right)$

Solution 5

- $\mathrm{BR}_i(s) \subset \mathrm{co}\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right)$

We present here small but important result: if strategy is not best response in pure best response, corresponding probability in best response in mixed strategies is zero.

Lemma 0.1.

$$\textit{Let } s^i \in BR^i(s) \quad \textit{and} \quad \forall b^i \notin BR_{A^i}(s), b^i \in A^i \Rightarrow s^i(b^i) = 0$$

Proof. Suppose not. if the strategy $s^i \in \mathrm{BR}^i(s)$ uses some pure action $b^i \in A^i$ which $\notin \mathrm{BR}_{A^i}(s)$, i.e. $s^i(b^i) > 0$ then

$$\forall c^i \in \mathrm{BR}_{A^i}(s): \quad u^i(c^i, s^{-i}) > u^i(b^i, s^{-i})$$

Consider another mixed strategy r^i, defined as follows:

$$\begin{cases} r^i(a^i) = s^i(a^i) & \forall a^i \in A^i/\{b^i, c^i\} \\ r^i(b^i) = 0 \\ r^i(c^i) = s^i(b^i) + s^i(c^i) \end{cases}$$

then

$$u^i(r^i, s) = \sum_{a^i \in A^i} r^i(a^i)u(a^i, s^{-i}) + r^i(b^i)u^i(b^i, s^{-i}) + r^i(c^i)u^i(c^i, s^{-i}) =$$

$$= \sum_{a^i \in A^i} s^i(a^i)u^i(a^i, s) + [s^i(b^i) + s^i(c^i)]u^i(c^i, s^{-i}) >$$

$$\sum_{a^i \in A^i} s^i(a^i)u^i(a^i, s^{-i}) + s^i(b^i)u^i(b^i, s^{-i}) + s^i(c^i)u^i(c^i, s^{-i}) = u^i(s^i, s^{-i})$$

contradiction with $s^i \in \mathrm{BR}^i(s)$. □

$\mathrm{BR}_i(s) \subset \mathrm{co}\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right)$ comes straight from lemma (our mixed best response has zeros when it is not in pure best response).

- $\mathrm{BR}_i(s) \supset \mathrm{co}\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right)$

BR is convex valued. We need to show that $\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right) \subset \mathrm{BR}^i_A(s)$
Suppose not Let $b^i \in \mathrm{BR}^i(s)$ and suppose $\delta_{b^i} \notin \mathrm{BR}^i(s)$ then

$$\exists s^i \in \Delta(A^i) \quad u^i(s^i, s^{-i}) > u^i(b^i, s^{-i})$$

$$\sum_{a^i \in A^i} s^i(a)u^i(a^i, s^{-i}) = u^i(s^i, s^{-i}) > u^i(b^i, s^{-i}) = 1 \cdot = u^i(b^i, s^{-i}) = \sum_{a^i \in A^i} s^i(a^i)u^i(b^i, s^{-i})$$

for at least one a^i $u^i(a^i, s^{-i}) > u^i(b^i, s^{-i})$ contradicts $b^i \in \mathrm{BR}^i_{A^i}(s)$