Recitations 6

Jakub Pawelczak

MINI

FALL 2020

10/15/20 RECITATION 6

me: JAKUB PAWELCZAK

mail: PAWEL042@UMN.EDU

www: JAKUBPAWELCZAK.COM

office hours: MONDAY 5:30-6:30
ZOOM

Today:
- Risk
- Uncertainty
Knight 1921

Office Hours: Instead additional meeting

Next time: Exam !

Recitation 6

[Definitions used today]

- Risk aversion and strict risk aversion, Jensen's inequalities.
- FOSD, SOSD, more risky than, Pratt's theorem

Question 1 [Risk compensation] 107 [I.2 Fall 2010 majors]

Consider an agent with expected utility function $E[v(\cdot)]$, where the von Neumann-Morgenstern utility function v is strictly increasing. Consider risk compensation $\rho(w, t\tilde{z})$ as a function of scale factor t for arbitrary $t \in \mathbb{R}_+$.

a State a definition of risk compensation $\rho(w, \tilde{z})$ for risky gamble $\tilde{z}$ with $E(\tilde{z}) = 0$ at deterministic wealth w

b Show that $\rho(w, t\tilde{z})$ is a strictly increasing function of t that takes zero value at $t = 0$ for every w and $\tilde{z}$ with $E(\tilde{z}) = 0$, if and only if the agent is strictly risk averse.

Question 2 [Pratt] 91 [II.1 Fall 2009 majors]

Consider an agent whose preferences over real-valued random variables (or state-contingent consumption plans) are represented by an expected utility function with strictly increasing and twicedifferentianble vN-M utility $v : \mathbb{R} \to \mathbb{R}$. Let $\rho(w, \tilde{z})$ denote the risk compensation for random variable $\tilde{z}$ with $\mathbb{E}(z) = 0$ at risk-free initial wealth w. Let $A(w)$ denote the Arrow-Pratt measure of risk aversion at w.

a Prove that A is an increasing function of w if and only if risk compensation ρ is an increasing function of w for every $\tilde{z}$ with $\mathbb{E}(\tilde{z}) = 0$ and $\tilde{z} \neq 0$.

b Derive an explicit expression for risk compensation for quadratic utility $v(x) = -(\alpha - x)^2$ where $\alpha > 0$. Prove that this quadratic utility is, up to an increasing linear transformation, the only utility function with risk compensation of the form you derived.

c Give an example of two vN-M utility function v_1 and v_2 such that neither v_1 is more risk averse than v_2, nor v_2 is risk averse than v_1 in the sense of the Theorem of Pratt.

Question 3

There are three states with equal probabilities $\pi_s = \frac{1}{3}$ for $s \in \{1, 2, 3\}$. Consider two state contingent consumption plans $z = (8, 2, 2)$, and $y = (3, 3, 6)$

a Does y FOSD dominate z?

b Is z more risky than y?

Question 4 124 [I.2 Fall 2011 majors]

Consider two real-valued random variables $\tilde{y}$ and $\tilde{z}$ on some state space (i.e. probability space). Let F_y and F_z be their cumulative distribution functions, and $E(\tilde{z})$ and $E(\tilde{y})$ their expected values.

a State a definition of $\tilde{z}$ first-order stochastically dominating (FSOD) $\tilde{y}$. Show that if $\tilde{z}$ FSOD $\tilde{y}$, then $E(\tilde{z}) \geq E(\tilde{y})$

b Show that, if $\tilde{z}$ FSOD $\tilde{y}$ and $E(\tilde{z}) = E(\tilde{y})$, then $\tilde{y}$ and $\tilde{z}$ have the same distribution, i.e., $F_y(t) = F_z(t)$ for every $t \in \mathbb{R}$. If you find it convenient, you may assume in your proof that random variables $\tilde{y}$ and $\tilde{z}$ have densities, or alternatively that $\tilde{y}$ and $\tilde{z}$ are discrete random variables (i.e., take finitely many values).

c State a definition of $\tilde{z}$ second-order stochastically dominating (SSOD) $\tilde{y}$. Show that if $\tilde{z}$ ssn $\tilde{y}$, then $E(\tilde{z}) \geq E(\tilde{y})$

d Show that if $\tilde{z}$ FSOD $\tilde{y}$, then $\tilde{z}$ SSD $\tilde{y}$.

e State a definition of $\tilde{y}$ being more risky than $\tilde{z}$. Give a brief justification for why it is a sensible definition of more risky.

Question 5 [Stochastic Dominance and Risk]

Consider two real-valued random variables y and z on some finite state space with $\mathbb{E}[y] = \mathbb{E}[z]$.

a Prove that if y is more risky than z, then $\mathbb{E}[v(z)] \geq \mathbb{E}[v(y)]$ for every nondecreasing continuous and concave function $v : \mathbb{R} \to \mathbb{R}$. You may assume v is twice differentiable.

b Give an example of two random variables y and z such that $y \neq z, \mathbb{E}[y] = \mathbb{E}[z]$ and neither z is more risky than y nor y is more risky than z.

Question 6

Consider an optimal portfolio choice problem with one risky asset with return $\tilde{r}$ and a risk-free asset with return r_f. Suppose that the agent's vNM utility function is $v(x) = -(\alpha - x)^2$ for some $\alpha > 0$. Assume that $\alpha > wr_f$ where $w > 0$ is agent's wealth. Negative investment (i.e. short selling) is permitted for both assets.

a Find the optimal investment in the risky asset as a function of expected return and the variance of the risky return.

b Suppose that the return $\tilde{r}$ on the risky asset is changed to a more risky return $\tilde{r}'$ with the same expectation $\mathbb{E}[\tilde{r}'] = \mathbb{E}[\tilde{r}]$. Assume $\mathbb{E}[\tilde{r}] > r_f$. Prove that the optimal investment in the risky asset with more risky return $\tilde{r}'$ is smaller than the optimal investment with return $\tilde{r}$, all else unchanged.

Q 2(a) Before that

$v_1, v_2 \in C^2$, str. increasing v NM

Thm. Pratt - Following are equivalent

① $A_1(x) \geq A_2(x) \quad \forall x$

② $p_1(x,z) \geq p_2(x,z) \quad \forall x,$

$\forall z \in \mathbb{R}^S \; Ez=0$

③ $\exists f$ str. inc, str. concave

$v_1 = f(v_2) \quad [\forall x \; v_1(x) = f(v_2(x))]$

$A(x) = -\frac{u''(x)}{u'(x)}$ ARA

$R(x) = \frac{-u''(x)}{u'(x)} \cdot x$ RRA

$\Rightarrow$ Consider $v(x)$

$u(x) \equiv v(x+\varepsilon) \quad \varepsilon > 0$

$\underline{A_u(x)} = A_v(x+\varepsilon) = -\frac{v''(x+\varepsilon)}{v'(x+\varepsilon)} \geq \underline{A_v(x)}$

$A_v(x)$ is ~~us~~ increasing

By Pratt (by $Au(x) \geq Av(x)$)

we get $pu(x,z) \geq pv(x,z)$

but $pu(x,z) := pv(x+\varepsilon, z)$

$pv(x+\varepsilon, z) \geq pv(x,z)$ $\quad \varepsilon > 0$

pv is increasing in x.

$\Leftarrow$ Let's assume that

$pv(x+\varepsilon, z) = \underline{pu(x,z)} \geq \underline{pv(x,z)}$

by Pratt Thm

$$Au(x) = \underbrace{Av(x+\varepsilon) \geq Av(x)}$$

$Av(\cdot)$ is increasing function

(b) $v(x - p(x,z)) = E\,v(x+z)$ $\quad \forall z \in \mathbb{R}^s \quad Ez = 0$

$v(x) = -(\alpha - x)^2$

$E\,v(x+z) = \underline{E(-(\alpha - (x+z))^2)}$

$= v(x - p(x,z)) = -\underbrace{(\alpha - (x - p(x,z)))}^2$

$$\mathbb{E}\left[f\left(\alpha^2 - 2\alpha(x+z) + (x+z)^2\right)\right]$$

$$= f\left[\alpha^2 - 2\alpha(x - p(x,z)) + (x - p(x,z))^2\right]$$

$$\cancel{\alpha^2} - 2\alpha(\mathbb{E}x + \overset{0}{\widetilde{\mathbb{E}z}}) + \mathbb{E}(x^2 + \overset{0}{\widetilde{2xz}} + z^2)$$

$$= \cancel{\alpha^2} - 2\alpha(x - p(x,z)) + (x - p(x,z))^2$$

$$p(x,z)^2 + 2(\alpha - \cancel{x})\,p(x,z) = \mathbb{E}z^2$$

$$p(x,z) = -(\alpha - w) \pm \sqrt{(\alpha - w) + \mathbb{E}z^2}$$

since $f \cdot v$ is concave by Ex 1

$$p(x,z) \geq 0 \qquad -(\alpha - w) + \underbrace{\sqrt{(\alpha - w) + \mathbb{E}z^2}}_{\substack{\| \\ p(w,z)}}$$

Let say that we have $\exists u$ rNM

$$A_u(x) = A_v(x) = -\frac{v''(x)}{v'(x)} =$$

$$= \frac{1}{\alpha - w}$$

$$\log u'(x) = \log(x - \alpha) + \overset{0}{a_0} \qquad a_0 \in \mathbb{R}$$

$$[\ln f'(x)]' = \frac{f''(x)}{f'(x)}$$

$$u'(x) = e^{a_0}(x-\alpha)$$

$$u(x) = \frac{e^{a_0}}{2}x^2 - e^{a_0}\alpha x + a_1$$

$$u'(x) = \frac{e^{a_0}}{2}\cdot 2x - e^{a_0}\alpha = e^{a_0}(x-\alpha)$$

$$= -\frac{e^{a_0}}{2}\left(\underline{-(\alpha-x)^2}\right) + \underline{\left(a_1 - \frac{e^{a_1}}{2}\alpha^2\right)}$$

$$= \underline{A}\cdot v(x) + B$$

(c) $v_1(x) = -e^{-x}$ $\qquad$ $v_2(x) = \log x$

$$A_1(x) = -\frac{-e^{-x}}{e^{-x}} = 1 \qquad \underline{A_2(x)} = -\frac{-\frac{1}{x^2}}{\frac{1}{x}} = \frac{1}{x}$$

$A_2(x) > A_1(x)$ $\qquad x \in (0,1)$

$A_2(x) < A_1(x)$ $\qquad \underline{x > 1}$

neither v_1 more r.A than v_2

nor v_2 more r.a. than v_1

$$RRA_2(x) = -\frac{u''(x)}{u'(x)}\cdot x = \underline{1}$$

CRRA (1)

Q1 (a) $\underline{p(w, z)}$

$$\mathbb{E}v(w+z) = v(w - p(w,z)) \quad (*)$$

So consumer is willing to sacrifice $p(w,z)$ amount to get sure outcome that gives him/her as much ~~u~~ utility as Expected utility for accepting gamble z ($w+z \to$ new wealth)

This holds for $\forall z \in \mathbb{R}^S$ $\mathbb{E}z = 0$

(b) $\Rightarrow$ Suppose $p(w, tz)$ is str. incr. in $t \geq 0$. WTS: agent is str. r. a.

- $p(w, 0) = 0$
- $p(w, tz) > p(w, 0 \cdot z) = 0$ $\quad t \geq 0$

so $w - 0 > w - p(w, tz)$

V is str. increasing so

$$v(w) > v(w - p(w, tz))$$

$$v(\mathbb{E}\, w + z) > \underbrace{\mathbb{E}v(w+tz)}_{\text{def. risk compensation}}$$

$\mathbb{E}\, w + z = w$

$$\underbrace{\mathbb{E}c \succsim_{\phi} c}_{\succ} \iff \underbrace{u(\mathbb{E}c) \geq \mathbb{E}u(c)}_{>}$$

take $y = w + z$

For all w, z $(\mathbb{E}z = 0)$ $\to$

For all y

$$v(\mathbb{E}y) > \mathbb{E}v(y)$$

So indeed agent is str. RA

$\Leftarrow$ Consider $0 \leq t_2 < t_1$

$$\alpha = \frac{t_2}{t_1} < 1$$

$$w + t_2 z = \underbrace{\frac{t_2}{t_1}}_{\alpha}(w + t_1 z) + \underbrace{\left(1 - \frac{t_2}{t_1}\right)}_{1-\alpha} \cdot w$$

Lemma: If agent is str. RA

$\implies$ v is str. concave.

Proof: p_v, p_u

$$v(w - p_v(w,z)) = \mathbb{E}\, v(w+z) < \underbrace{v(\mathbb{E}\, w+z)}_{\substack{\text{stv.} \\ \text{RA of agent}}}$$

$$v(w) = \ldots$$

v ist stv. incr.

$$p_v(w,z) > 0.$$

Now let $u(w) = w$

then $u(w - p_u(w,z)) = \mathbb{E}\, u(w+z) =$

$$= \mathbb{E}\, w + z = \underline{\underline{w}}$$

$$w - p_u(w,z) = w \Rightarrow p_u(w,z) = 0.$$

$$p_v(w,z) > 0 = p_u(w,z)$$

by Pratt Thm.

$\exists$ f stv. increasing, stv. concave s.t.

$$v(w) = f(u(w)) = f(w)$$

then v is stv. concave. $\square$

by our lemma

$$\mathbb{E}v(w+t_2z) = \mathbb{E}v\left(\frac{t_2}{t_1}(w+t_1z)+\left(1-\frac{t_2}{t_1}\right)w\right)$$

$$> \underbrace{\frac{t_2}{t_1}\mathbb{E}v(w+t_1z)+\mathbb{E}\left(1-\frac{t_2}{t_1}\right)v(w)}_{(**)}$$

since

$$\mathbb{E}v(w)=v(w) \underset{①}{=} v(\mathbb{E}w+t_1z) \underset{②}{>} \mathbb{E}v(w+t_1z)$$

① $w=\mathbb{E}w+\mathbb{E}t_1z$

② str. aversion of v

$$\mathbb{E}v(w+t_2z) > \mathbb{E}\frac{t_2}{t_1}v(w+t_1z)+\left(1-\frac{t_2}{t_1}\right)\mathbb{E}v(w)$$

$$> \frac{t_2}{t_1}\mathbb{E}v(w+t_1z)+\left(1-\frac{t_2}{t_1}\right)\mathbb{E}v(w+t_1z)$$

$$= \mathbb{E}v(w+t_1z)$$

$$\mathbb{E}v(w+t_2z) > \mathbb{E}v(w+t_1z) \quad (***)$$

Before def. of concavity should be either

$$v(w+t_2zs) > \frac{t_2}{t_1}v(w+t_1zs) + \left(1-\frac{t_2}{t_1}\right)v(w)$$

or

$$\mathbb{E}v(w+t_2z) > \frac{t_2}{t_1}\mathbb{E}v(w+t_1z) + \left(1-\frac{t_2}{t_1}\right)\mathbb{E}v(w)$$

$\mathbb{E}\, v(w + t_2 z) > \mathbb{E}\, v(w + t_1 z)$

$v(w - p(w, t_2 z)) > v(w - p(w, t_1 z))$

by def of compensation

observe that v is str. incr.

$$w - p(w, t_2 z) > w - p(w, t_1, z)$$

$$p(w, t_2 z) < p(w, t_1, z)$$

$t_1 > t_2$ so p is str. incr in $t \geq 0$

Q.4. $z \geq_{FOSD} y$

(a)

$\circ.\ F_z(t) \leq F_y(t) \qquad t \in [a,b]$ Def.

[Equivalent to $\mathbb{E}v(z) \geq \mathbb{E}v(y)$]

$\forall$ cont, non decreasing $\underline{\underline{v}}$

WTS $\mathbb{E}z \geq \mathbb{E}y$

$\circ\ \mathbb{E}z = \int_a^b t\, \underline{dF_z(t)}\, dt = \left(\begin{array}{l}\text{int.}\\ \text{by parts}\end{array}\right)$

$= b - \int_a^b F_z(t)dt$

$\circ\ \mathbb{E}y = \int_a^b t\,dF_y(t)\,dt = b - \int_a^b F_y(t)dt$

Let's integrate def. over (a,b)

$$\int_a^b F_z(t)\,dt \leq \int_a^b F_y(t)\,dt$$

$$\mathbb{E}z = b - \int_a^b F_z(t)dt \geq \underbrace{b - \int_a^b F_y(t)dt}_{\mathbb{E}y}$$

Indeed

$\mathbb{E}z \geq \mathbb{E}y$

$$\mathbb{E}z - \mathbb{E}y = b - b + \int_a^b F_y(t) - F_z(t) = 0$$

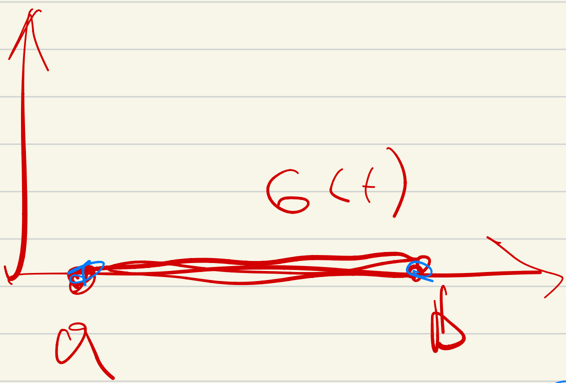
we know that $F_z(t) \le F_y(t)$

Define $\underline{\underline{G(t)}} = \int_a^t F_z(s) - F_y(s)\,ds$

$\underline{G(b)} = \underline{0} = \mathbb{E}z - \mathbb{E}y$

$G(a) = \int_a^a \dots = 0$

$G'(t) = F_z(t) - F_y(t) \le 0$



• $G'(t) = 0$ since $G(a) = G(b)$ and $G'(t) \le 0$

$F_z(t) - F_y(t) = 0$

So $F_z(t) = F_y(t)$. a.s.

(c). • $z \succeq_{SOSD} y$ $\quad \int_a^t F_z(s)\,ds \le \int_a^t F_y(s)\,ds$

$\forall t \in [a,b]$

~~• Exam~~

• y more risky than z

• Equivalent

$z \succeq_{SOSD} y \iff \underline{\mathbb{E}v(z) \ge \mathbb{E}v(y)}$

$\forall v$ cont., non decr., concave f.'s

(d) $z \succeq_{FOSD} y$

$F_z(s) \le F_y(s)$ integrate over (a,t)

$\underbrace{\int_a^t F_z(s)\,ds \le \int_a^t F_y(s)\,ds}$

$z \succeq_{SOSD} y$

In (c) WTS: $\mathbb{E}z \ge \mathbb{E}y$

Take $t = b$

$\mathbb{E}z = b - \int_a^b F_z(t) \ge b - \int_a^b F_y(t) = \mathbb{E}y$

y is more risky than z $\iff$

$z \succeq_{SOSD} y$ & $\mathbb{E}\, y = \mathbb{E}\, z$