Recitations 23

[Definitions used today]

- dictatorial game, Nakamura number, winning subset, pivotal player

Question 1 [285 IV.1 Fall 2019]

- Let N be a finite set of players. A simple game is a family $\mathbb{D}$ of so-called winning subsets of N.
- For example, simple majority with n voters is a simple game with winning subsets $\{D : |D| > n/2\}$.
- A game is gridlocked if there is a subset of N such that neither it nor its complement is winning. Thus, simple majority is gridlocked if and only if n is even. Individual i is pivotal if it belongs to every winning subset, that is, $i \in D$ for all $D \in \mathbb{D}$.
- A game is free if it has no pivotal individuals, and dictatorial if it has only one. For example, the game $\mathbb{D}_P = \{S \subset N : P \subset S\}$ is free if and only if P is empty, and dictatorial if and only if $|P| = 1$

a) Prove that if a simple game is neither free nor gridlocked then it is dictatorial.

Solution 1

WTS: If game $(N, \mathbb{D})$ is neither free nor gridlocked then it is dictatorial.
Suppose it is not dictatorial, i.e. $\neg\exists$ pivotal player or there are at least two pivotal players. Consider cases:

- $\neg\exists$ pivotal plater, it means that it is free
- there are at least two pivotale players $i \neq j$. take any $D \in \mathbb{D}$. Then $i, j \in D$ (from definition of winning subset)

Let's use logic law $\neg(p \Rightarrow q) \equiv p \wedge \neg q$ for definition of gridlock set.
So game is not gridlocked if $\exists S \subseteq N$ that S or $N \setminus S$ are winign. In particular for D it holds. So indeed in this case we have non gridlock game.
To sum this up, this is a contradiction so neither free nor dridlock game is dicatatorial

Question 2 [286 IV.2 Fall 2019]

Given a simple game $\mathbb{D}$, define its Nakamura number as

$$\nu(\mathbb{D}) = \min_{\mathcal{F} \subset \mathbb{D}} \{|\mathcal{F}| : \bigcap \{F : F \in \mathcal{F}\} = \emptyset\}$$

with $\nu(\mathbb{D}) = \infty$ if the game is not free. The number ν computes the smallest number of winning subsets with empty intersection. For example, the Nakamura number of simple majority with n voters is 3 if $n = 3$ or $n > 4$.
A simple game $\mathbb{D}$ induces a social welfare function by the definition aPb if $\{i \in N : aP_i b\} \in \mathbb{D}$. Let A be a finite set of alternatives.

a) Prove that P is acyclic whenever P_i is acyclic for every individual $i \iff \nu(\mathbb{D}) > |A|$

b) The UN Security Council consists of fifteen members, five of which are permanent with veto power. Otherwise, majority voting prevails. Model this as a simple game and find its Nakamura number

Solution 2

a)$\Leftarrow$ Suppose $|A| = r < v(\mathbb{D})$ but for some profile

$$\{x_1, \ldots x_t\} : \quad x_1 P x_2 \ldots P x_t P x_1$$

we have a cycle.
$r < v(\mathbb{D})$ implies that $\exists i \in N$:

$$i \in \cap(x_1 P x_2 \ldots P x_t P x_1)$$

for which

$$x_1 P x_2 \in \mathcal{F}, \ldots x_{t-1} P x_t \in \mathcal{F} \quad x_t P x_1 \in \mathcal{F}$$

then i's pref over $\{x_1, \dots x_t\}$ are:

$$x_1 P_i x_2 \dots P_i x_t P_i x_1$$

which is contradiction with P_i being acycle for all i

$\Rightarrow$ Suppose not. So let $r \geq v(\mathbb{D})$.

We will construc a profile $\mathbb{D}$ of alternatives for which $\mathbb{D}$ generates a cycle. Let start with $F = \{F_1, \dots F_r\}$ family of deceisive coalitions with empty intersection

$$F \subseteq \mathcal{F} \quad \bigcap \{F \subseteq \mathcal{F}\} = \emptyset$$

—F—=r comes from assunption that $r \geq v(\mathbb{D})$. Let $F_k^c = N \setminus F_k$. since $\bigcap F = \emptyset$ then ${}_{k=1}^{r} F_k^c = N$. Therefore, $\exists$ a family of (possibly empty) pairwise disjoint coalitions

$$\{D_1, \dots D_r\} \quad \forall k \quad D_k \subseteq F_k^c \quad {}_{k=1}^{r} D_k^c = N$$

Moreover, by monotonicity of decisive sets $D_k^c = N \setminus D_k \supset F_k$ implies $D_k^c \in \mathcal{F}$.

By construction $k \neq l \quad D_k \cap D_l = \emptyset$. Hence $\exists$ profile of alternatives $\{x_1 \dots x_r\}$ given by following

$$\forall i \in D_1 \quad x_r P_i x_{r-1} \dots P_i x_1$$

$$\forall i \in D_2 \quad x_1 P_i x_r P_i x_{r-1} \dots P_i x_2$$

$$\dots$$

$$\forall i \in D_j \quad x_{j-1} P_i x_{j-2} P_i \dots P_i x_j$$

$$\dots$$

$$\forall i \in D_r \quad x_{r-1} P_i x_{r-2} P_i \dots P_i x_r$$

Then

$$x_1 P x_r = D_1^c \in \mathcal{F}$$

$$\forall j = 1 \dots r-1 x_{j+1} P x_j = D_{j+1}^c \in \mathcal{F}$$

Hence $x_1 P x_r$ and $x_{j+1} P x_j$ for $j = 1 \dots r-1$ but this contradicts $P(x_1, \dots x_r)$ being acyclic (where $P(x_1, \dots x_r)$ is applying $(x_1, \dots x_r)$ with preference P to definition of acyclic preferences).

b) Now, consider the UN Security Counsil with its members $I = \{1, \dots, 15\}$. WLG, assume that if $i \in \{1, \dots, 5\}$ it has veto power. Of course, one option is prefered to other if the majority agrees and there aren't permanent members against it. We can model this game as a simple game if we define the family of winning subsets:

$$\mathbb{D} = \{S \subseteq N : \{1, \dots, 5\} \in S, |S| \geq 8\}$$

Observe that all veto powers have always pivotal voters assigned to permanent member and belongs to every winning subset. The game is not free and its Nakamura number is ∞.