Recitations 7

JAKUB PAWELCZAK

MINI

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RECITATION 7

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Today: PRACTICE,
PRACTICE
PRACTICE!

- PS3
- PS4
- PS5

Good luck tomorrow!

Recitations 7 - Additional meeting

Question 1 247 [I.2 Fall 2017 majors]

There are two assets: a risk-free asset with return r_f and a risky asset with return $\tilde{r}$. Returns are per-dollar returns, that is, total returns for one dollar invested. An agent has expected utility function with von Neumann (or Bernoulli) utility index $v(x) = -(\alpha - x)^2$, for $\alpha > 0$, and initial wealth $w > 0$ Assume that $\alpha > wr_f$. Negative investment (i.e., short selling) is permitted for both assets.

a Find the optimal investment in the risky asset as a function of the expected return and the variance of the return on the risky asset, the risk-free return, and the agent's wealth. Consider the following comparative statics exercise. The return $\tilde{r}$ is changed to a more risky return $\tilde{r}'$. The expected return on $\tilde{r}'$ is the same as on $\tilde{r}$, that is, $E(\tilde{r}') = E(\tilde{r})$

b State a definition of more risky return (or more risky random variable).

c Suppose that $E(\tilde{r}) > r_f$. Show that the optimal investment in the risky asset with more risky return $\tilde{r}'$ is smaller then the optimal investment with less risky return $\tilde{r}$, everything else being unchanged? Prove your answer.

Question 2

There are three states with equal probabilities $\pi_s = \frac{1}{3}$ for $s \in \{1, 2, 3\}$. Consider two state contingent consumption plans $z = (8, 2, 2)$, and $y = (3, 3, 6)$

a Does y FOSD dominate z?

b Is z more risky than y?

Question 3 [Stochastic Dominance and Risk]

Consider two real-valued random variables y and z on some finite state space with $\mathbb{E}[y] = \mathbb{E}[z]$.

a Prove that if y is more risky than z, then $\mathbb{E}[v(z)] \geq \mathbb{E}[v(y)]$ for every nondecreasing continuous and concave function $v : \mathbb{R} \to \mathbb{R}$. You may assume v is twice differentiable.

b Give an example of two random variables y and z such that $y \neq z, \mathbb{E}[y] = \mathbb{E}[z]$ and neither z is more risky than y nor y is more risky than z.

Question 4 [Pratt]

Consider an agent whose preferences over real-valued random variables (or state-contingent consumption plans) are represented by an expected utility function with strictly increasing and twicedifferentianble vN-M utility $v : \mathbb{R} \to \mathbb{R}$. Let $\rho(w, \tilde{z})$ denote the risk compensation for random variable $\tilde{z}$ with $\mathbb{E}(z) = 0$ at risk-free initial wealth w. Let $A(w)$ denote the Arrow-Pratt measure of risk aversion at w.

a Prove that A is an increasing function of w if and only if risk compensation ρ is an increasing function of w for every $\tilde{z}$ with $\mathbb{E}(\tilde{z}) = 0$ and $\tilde{z} \neq 0$.

b Derive an explicit expression for risk compensation for quadratic utility $v(x) = -(\alpha - x)^2$ where $\alpha > 0$. Prove that this quadratic utility is, up to an increasing linear transformation, the only utility function with risk compensation of the form you derived.

c Give an example of two vN-M utility function v_1 and v_2 such that neither v_1 is more risk averse than v_2, nor v_2 is risk averse than v_1 in the sense of the Theorem of Pratt.

Question 5

Suppose that $\tilde{z}$ takes two values z_1 or z_2, where $z_1 \leq z_2$, with probabilities π and $1 - \pi$, respectively. Let $\tilde{y}$ takes two values y_1 or y_2, where again $y_1 \leq y_2$, with probabilities π and $1 - \pi$. Assume that $0 < \pi < 1$

a Under what conditions (necessary and sufficient) on z_i and y_i does $\tilde{z}$ dominate $\tilde{y}$ in the sense of the First-Order Stochastic Dominance? Justify your answer.

b Under what conditions does $\tilde{z}$ dominate $\tilde{y}$ in the sense of the Second-Order Stochastic Dominance? Under what conditions is $\tilde{y}$ more risky than $\tilde{z}$?

Question 6 ∼ 223 [I.2 Fall 2016 majors]

Consider two real-valued random variables y and z such that $\mathbb{E}[y] = \mathbb{E}[z]$. Suppose that cumulative distribution functions F_y and F_z have the following (weak) single-crossing property:

$$\exists_{t^* \in \mathbb{R}} \quad \begin{cases} F_y(t) \geqslant F_z(t) & \text{for } t \leqslant t^* \\ F_y(t) \leqslant F_z(t) & \text{for } t \geqslant t^* \end{cases}$$

Show that y is more risky than z. You may assume that y and z are random variables on a state space with S states.

Question 7

Consider two random variables y and z with the same expectations $\mathbb{E}[y] = \mathbb{E}[z]$.

a Show that if y is more risky than z then $\text{Var}[y] \geqslant \text{Var}[z]$.

b Suppose both y and z are normal. Show that y is more risky than z if and only if $\text{Var}[y] \geqslant \text{Var}[z]$.

c Show that if z is more risky than y and y is more risky than z then y and z have the same distribution,i.e. $F_y(t) = F_z(t)$ for all t You may assume y and z take only finitely many values.

Question 8

Consider two random variables y and z on a probability space. You may think about y and z as statecontingent consumption plans on a finite state space. Let $\{y_1, y_2, \ldots, y_k\}$ be the values y can take. Show that if $\mathbb{E}[z \mid y] = 0\,(= \mathbb{E}\left[z \mid y = y_i\right]$ for all $i \in \{1, 2, \ldots, k\})$ then $z + y$ is more risky than y

Question 9 271 [I.2 Spring 2019 majors]

Consider two real-valued random variables $\tilde{y}$ and $\tilde{z}$ on some state space (i.e. probability space). Let F_y and F_z be their cumulative distribution functions, and $E(\tilde{z})$ and $E(\tilde{y})$ their expected values. You may assume that $\tilde{y}$ and $\tilde{z}$ take values in a finite interval $[a, b]$

a State a definition of $\tilde{z}$ first-order stochastically dominating (FSD) $\tilde{y}$. State a definition of second-order stochastic dominance (SSD). Your definitions should be stated in terms of cumulative distribution functions F_y and F_z

b State a definition of $\tilde{y}$ being more risky than $\tilde{z}$. Give a brief justification for why it is a sensible definition of more risky.

c Suppose that $\tilde{z}$ has uniform distribution on an interval $[\underline{z}, \bar{z}]$ while $\tilde{y}$ has uniform distribution on $[\underline{y}, \bar{y}]$. Under what conditions on the bounds $\underline{z}, \bar{z}, \underline{y}, \bar{y}$ does $\tilde{z}$ FSD $\tilde{y}$? Prove your statement.

d Suppose again that $\tilde{z}$ and $\tilde{y}$ have uniform distributions as in (c). Show that if $\underline{y} \leq \underline{z}$ and $z \leq \bar{y}$ and $E(\tilde{z}) = E(\tilde{y})$, then $\tilde{y}$ is more risky than $\tilde{z}$ You may assume some specific (distinet) numerical values of $\underline{z}, \bar{z}, \underline{y}, \bar{y}$ in your proof, if you find it convenient.

Question 10 255 [I.2 Spring 2018 majors]

Consider two real-valued random variables $\tilde{y}$ and $\tilde{z}$ on some state space (i.e. probability space). Let F_y and F_z be their cumulative distribution functions, and $E(\tilde{z})$ and $E(\tilde{y})$ their expected values. You may assume that $\tilde{y}$ and $\tilde{z}$ take values in a finite interval $[a, b]$

a State a definition of $\tilde{z}$ second-order stochastically dominating (SSD) $\tilde{y}$. Your definition should be stated in terms of cumulative distribution functions F_y and F_z

b State a definition of $\tilde{y}$ being more risky than $\tilde{z}$. Provide a justification for why it is a sensible definition of more risky.

c Extend the definition of more risky to random variables that may have different expected values as follows: $\tilde{y}$ is more risky than $\tilde{z}$ if and only if $\tilde{y} - E(\tilde{y})$ is more risky than $\tilde{z} - E(\tilde{z})$. Show that if $\tilde{y}$ is more risky than $\tilde{z}$ and $E(\tilde{z}) \geq E(\tilde{y})$, then $\tilde{z}$ second-order stochastically dominates $\tilde{y}$

d Show that $\lambda\tilde{z}$ is more risky than $\tilde{z}$ for every $\lambda \geq 1$ and every $\tilde{z}$. Note that $E(\tilde{z})$ may be different from 0

PS3 Q1 a) two constrainst

$$x_1^A + x_1^B \leq x_1$$

$$\circ z \leq x_2$$

$$x_1^A = \left(\frac{2\mu}{1+\mu}\right) x_1 \quad x_1^B = \left(\frac{1-\mu}{1+\mu}\right) x_1$$

(b)

$$\max u^h(x_1, x_2)$$

$$p_1 x_1 + p_2 x_2 \leq w$$

$$x_1 = \frac{(1+\mu)}{3} \frac{w}{p_1} \quad x_2 = \frac{(2-\mu)}{3} \frac{w}{p_2}$$

(c) $f(\underline{p}, \underline{w}, \underline{\mu}) = \left(\downarrow \quad , \quad \downarrow \right)$

$$\sigma_{ij} = \frac{\partial f_i}{\partial p_j} + \frac{\partial f_i}{\partial w} \cdot f_j$$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \quad \& \ \sigma_{12} = \sigma_{21}$$

$$\boxed{\sigma_{12}^2 = \sigma_{11} \cdot \sigma_{22}} \quad \det \Sigma = \sigma_{11}\sigma_{22} - \sigma_{12} \cdot \sigma_{21} = 0$$

$$\boxed{\sigma_{11} < 0} \ \& \ \det \Sigma = 0$$

(d) $\mu = \frac{p_2}{w}$ ...

PS4.1(c) $U_t(c) = G(u(c_t), U_{t+1}(c))$

$c = (1, 1, 2, 3, 3, 3, \ldots)$

$U_0(c) = 1 + \beta + 2\beta^2 + 3\beta^3 + 3\beta^4 + \ldots + 1$

$U_1(c) = 1 + 2\beta + 3\beta^2 + 3\beta^3 + \ldots + 1$

$d = (1, 2+2\beta, 0, 3, \ldots)$

$U_0(d) = 1 + (2\beta + 2)\beta + 3\beta^2 + 3\beta^3 + \ldots + 0$

$U_1(d) = 2 + 2\beta + 3\beta + 3\beta^2 + \ldots + 0$

$U_0(c) > U_0(d)$

~~$U_0(c) = U_0(d)$~~

$U_1(c) = U_1(d)$ / $\to \leq$

$U_0(c) - U_0(d) = 2 + \beta - 1 - 2\beta = 1 - \beta > 0$

$U_1(c) = U_1(d)$ $\qquad U_0(c) > U_0(d)$ $\qquad U_0(d)$

small u $\to U_0(c_0) = u(d_0)$

$U_0(c) = G(u(c_0), U_1(c)) = G(u_0(d), U_1(d))$

PS5 2(a) $\quad p(w,z) = \alpha\sigma(z)$

$z \in R^S \quad E z = 0 \quad \circ\ \sigma(\underbrace{z}_{\smile} + \underbrace{w}_{\text{const}}) = \sigma(z)$

$\circ\ E\, w + z = w + Ez = w$

(b) Pick a —

$\max \ u\left((w-a) r_f + a \cdot r\right)$

$\cancel{\alpha} = \max\limits_a \ (w-a) + aEr - \alpha\sigma(z) \cdot |a|$

$\sigma(az) = |a|\,\sigma(z)$

$\boxed{\max\limits_x \ ax + b|x|}$

$Er \geq r_f$

$\max \ w r_f + a(Er - r_f) - \alpha|a|\sigma(z)$

~~$Er - r_f > 0$~~

① $Er - r_f > \alpha\sigma(z)$

$a \to \infty$

② $Er - r_f < \alpha\sigma(z)$

$a = 0$

③ $Er - r_f = \alpha\sigma(z)$

$VAR(aZ) = E(aZ)^2 - (EaZ)^2$

$= a^2\, VAR(Z) \to \sqrt{}$

$Sd = \sqrt{VAR}$

$\sqrt{a^2} = |a|$

$Sd(aZ) = \sqrt{VAR(aZ)} = \sqrt{a^2 VAR Z} =$

$= |a|\, Sd(aZ)$

Pb 5.3. $\underline{w}$ – wealth

$\underline{1_R > 1_B} \wedge 1_{B\cup Y} > 1_{R\cup Y}$

$1_R = (w+R, w-1, w-1) >$

$1_B = (w-1, w+R, w-1)$

$\underline{ICC}$ $(w+R, w-1, w+R) >$

$> (w-1, w+R, w+R)$

i.e. $1_{R\cup Y} > 1_{B\cup Y}$

$P = \{ (\pi_R, \pi_B, \pi_Y) : \pi_R = \frac{1}{3}, \pi_B + \pi_Y = \frac{2}{3}$

$\underline{\pi_B \geq \varepsilon} \quad \underline{\pi_Y \geq \varepsilon} \quad \underline{\varepsilon \in [0, \frac{1}{3}]} \}$

$P = \{ (\frac{1}{3}, \pi, \frac{2}{3} - \pi) \quad \pi \in [\varepsilon, \frac{2}{3} - \varepsilon] \quad \varepsilon \in [0, \frac{1}{3}] \}$

$1_R > 1_B$

$$\min_{\pi \in [\varepsilon, \frac{2}{3} - \varepsilon]} \frac{1}{3} u(x+R) + \pi \cdot u(x-1) + \left(\frac{2}{3} - \pi\right) u(x-1)$$

$$> \min_{\pi \in [\varepsilon, \frac{2}{3} - \varepsilon]} \frac{1}{3} u(x-1) + \pi u(x+R) + \left(\frac{2}{3} - \pi\right) u(x-1)$$

$$\left(\frac{1}{3} - \varepsilon\right)\left(u(x+R) - u(x-1)\right) > 0$$

$$\frac{1}{3} - \varepsilon > 0 \qquad \varepsilon < \frac{1}{3}$$

$$\rightarrow \frac{1}{3} u(x+R) + \frac{2}{3} u(x-1) \quad 1R$$

Compare payoff distr.

How to translate it to world

FOSD 1 z yield (unambiguously) higher return than y

SOSD 2 z yield less risk than y

z, y $\rightsquigarrow$ EX, $VAR X$

y, z F_y, F_z cdf's

1 $\underbrace{F_y(x)}_{1-F_y(x)} \leq \underbrace{F_z(x)}_{1-F_z(x)}$ $\forall x$

$1-F_y(x) \geq 1-F_z(x)$

Def 1 is for Jan

2 exp utility max, likes more

$(u \in C^0$, u non decreasing$)$

$\mathbb{E}v(y) \geq \mathbb{E}v(z)$

Def 2 for me

$$\mathbb{E}v(y) = \int v(s)\, dF(s)$$

Lemma 1. 1 & 2 are equivalent.

Proof: $\underbrace{H(x) = F_y(x) - F_z(x)}$

$\int u(x)\, dH(x) = \dots = -\int u'(x) H(x)\, dx \geq 0$

$\Rightarrow$

$\Leftarrow$ $H(x) = F_y(x) - F_z(x)$

$\exists \bar{x}$ $\quad H(\bar{x}) > 0$ (suppose not)

$Eu(y) \geq Eu(z)$ $\quad \forall u \in C^0$, non decr.

$u(x) = \begin{cases} 1 & x > \bar{x} \\ 0 & x \leq \bar{x} \end{cases}$

From ~~FOSD~~ $(*)$

$0 \leq \int u(x)\, dH(x) = -H(\bar{x}) < 0$

so it contradicts $F_y(x) \leq F_z(x)$

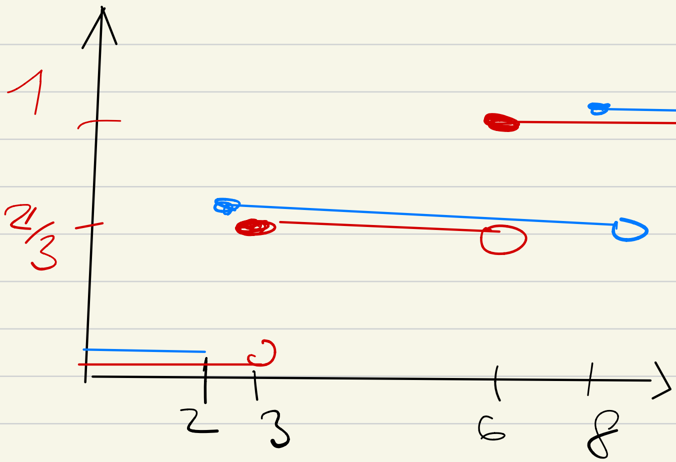
$\forall x \; H(x) \leq 0$.

- $z \succeq_{FOSD} y \Rightarrow Ez \geq Ey$

but $\not\Leftarrow$

Q2.

$$F_y(t) = \begin{cases} 0 & t<3 \\ 2/3 & t\in[3,6) \\ 1 & t\geq 6\end{cases} \qquad F_z(t) = \begin{cases} 0 & t<2 \\ \frac{2}{3} & t\in[2,8) \\ 1 & t\geq 8\end{cases}$$



No. $y \not\geq_{FOSD} z$

$$F_y(7) = 1 > F_z(7) = \frac{2}{3} \qquad \leq$$

but
$$Ey = \frac{1}{3}\cdot 3 + \frac{1}{3}\cdot 3 + \frac{1}{3}\cdot 6 = 4$$
$$Ez = \frac{1}{3}\cdot 2 + \frac{1}{3}\cdot 2 + \frac{1}{3}\cdot 8 = 4$$

Def. $z \geq_{SOSD} y \quad \forall\, v \in C^0$, non decr. concave

$$EV(z) \geq EV(y)$$

Classic stuff by Arrow 1965/1971
Pratt 1964.

SOSD is related to riskiness measure, index

- Auman, Serrano 2008 JPE
riskiness index ~~repi~~ reciprocal of ARA

- Diamond, Stiglitz 1971
Whenever person is taking risk or not depends on following:
(1) atributes of gamble
i.e. how risky the gamble is
(2) atributes of agent
i.e. preferences → risk aversion

Arrow Pratt deals with (2)
But no ~~discu~~ discussion of (1)

(1) → Dealing with it in recent literature

- Machina, Rotschild 2008
- Fundenberg, Strzalecki 2015 ECTA
Risk is what a ~~risky~~ risk averse guy hates

8. ① WTS $E[z+y] = E[y]$ ✓

$E z = E\big[\underbrace{E[z|y]}_{0}\big] = E 0 = 0$

so $E[z+y] = Ez + Ey = Ey$

(b) $y \geq_{SOSD} y+z$ Jensen

$Ev(y+z) = E\big[\underbrace{E v(y+z)|y}\big] \leq$

$\leq E\, v(E[y+z|y]) = \{y \in \sigma(y)\}$

$= Ev(y + \underbrace{E[z|y]}_{0}) = Ev(y)$

$y \geq_{SOSD} y+z$ ✓

a & b $\Rightarrow$ $y+z$ is more risky than y

RN

Q5 – Jan's ex's

$y, z \sim N(\cdot,\cdot)$

$VAR\, y = \sigma_y^2$ $VAR\, z = \sigma_z^2$

$\sigma_y^2 \geq \sigma_z^2$

WTS: y is more risky than z ① $\Rightarrow$

$\Leftrightarrow$ $\sigma_y^2 \geq \sigma_z^2$ ② $\Leftarrow$

WTS: $\sigma_y^2 \geq \sigma_z^2 \Rightarrow y$ is more risky than z

By contradiction.

So $\sigma_y^2 \geq \sigma_z^2$ but $Ey = Ez$

$y \not\succ_{SOSD} z$

(*)

$y \sim N(\mu, \sigma_y^2)$

- $F_y(t) = \phi\left(\frac{t-\mu}{\sigma_y^2}\right)$ ϕ is cdf of $N(0,1)$
- $F_z(t) = \phi\left(\frac{t-\mu}{\sigma_z^2}\right)$

$z \sim N(\mu, \sigma_z^2)$

(*) $\exists\, w \in \mathbb{R}$

$$\int_{-\infty}^{w} F_y(t)\,dt < \int_{-\infty}^{w} F_z(t)\,dt \quad (1)$$

if $t-\mu \geq 0$ $\quad \underline{F_y}(t) = \phi\left(\frac{t-\mu}{\sigma_y^2}\right) \overset{(2)}{\leq} \phi\left(\frac{t-\mu}{\sigma_z^2}\right) = \underline{F_z(t)}$

if $t-\mu < 0$ $\quad F_y(t) \overset{(3)}{>} F_z(t)$

Consider cases

1. $w < \mu$ Then 1 $\lightning$ 2
2. Let $w \geq \mu$

$0 = EY - EZ$

Let $\omega \geq \mu$

$$0 = EY - EZ = \int_{-\infty}^{\omega} F_Y(x) - F_Z(x)\,dx$$

$$+ \int_{\omega}^{+\infty} F_Y(x) - F_Z(x)\,dx$$

Observe that by ① < 0

~~but~~ by ② $F_Y(t) - F_Z(t) \leq 0$

So ≤ 0

$0 = \int + \int < 0$ ↯

Q7(a) If y is more risky than z

$\Rightarrow VAR\, y \geq VAR\, z$

Proof: more risky · $Ey = Ez$

· $\forall u \in C^0$, ~~be~~ non dec concave

· $E\, u(z) \geq E\, u(y)$ ($z \geq_{SOSD} y$)

Pick $u(x) = -(x-\alpha)^2$

$E - (z-\alpha)^2 \geq E - (y-\alpha)^2$

$E z^2 + \alpha^2 - 2\alpha Ez \leq E y^2 + \alpha^2 - 2\alpha Ey$

$- (\mathbb{E}y)^2 = -(\mathbb{E}z)^2$

$\mathbb{E}z^2 - (\mathbb{E}z)^2 = VAR(z) \geq VAR(y)$

$= \mathbb{E}y^2 - (\mathbb{E}y)^2$

Q10 $\quad y, z \quad \mathbb{E}y = \mathbb{E}z$

$\tilde{y} - \mathbb{E}\tilde{y}, \quad \tilde{z} - \mathbb{E}\tilde{z}$

(a) for $\mathbb{E}z = 0$

~~WTS~~:

PS5 Last Ex

- $t \geq 1 \quad \mathbb{E}z = 0$
  WTS tz is more risky than z
- $z \in \mathbb{R}^S \quad w \in \mathbb{R}$
  $tz + w$ is more risky than tz

- Go back z: $\mathbb{E}z = 0$

$z = \frac{1}{t} \cdot tz + \left(1 - \frac{1}{t}\right) \cdot 0$

$t \geq 1 \quad \frac{1}{t} \in [0, 1]$

- $v(z_s) \geq \frac{1}{t} v(tz_s) + \left(1 - \frac{1}{t}\right) v(0)$
- $\mathbb{E}v(z) \geq \frac{1}{t} \mathbb{E}v(tz) + \left(1 - \frac{1}{t}\right) v(0)$

WTS: $\mathbb{E}v(z) \geq \mathbb{E}v(tz)$

$v(0) = v(Etz) \overset{\text{Jensen}}{\geq} \underline{Ev(tz)}$ $\Big/ \cdot \left(1 - \frac{1}{t}\right)$

$Ev(z) \geq \frac{1}{t} Ev(tz) + \left(1 - \frac{1}{t}\right) Ev(0)$

$\geq \underline{Ev(tz)}.$

(*) We know that for $z : Ez = 0$, $t \geq 1$

$E v(z) \geq Ev(tz)$

$\tilde{z} = z - Ez$ $\quad E\tilde{z} = 0$

$\tilde{y} = y - Ey$ $\quad E\tilde{y} = 0$

~~E~~ WTS:

3. (10 points) Let $\tilde{y}$ and $\tilde{z}$ be arbitrary random variables on some finite state space such that $E(\tilde{y}) = E(\tilde{z})$ and $\tilde{y}$ is more risky than $\tilde{z}$.

(i) Prove that $w + \tilde{y}$ is more risky than $w + \tilde{z}$, for every deterministic w.

(ii) Prove that $\lambda\tilde{y}$ is more risky than $\lambda\tilde{z}$, for every $\lambda > 0$.

Define $v_1(x) = u(x+w)$

$v_2(x) = u(\lambda x)$ $\quad \lambda > 0$

Let's assume that u is C^0, non decreasing and concave.
Clearly v_1, v_2 are continuous. Take $\varepsilon \geq 0$, then:

$v_1(x+\varepsilon) = u(x+w+\varepsilon) \geq u(x+w) = v_1(x)$

$v_2(x+\varepsilon) = u(\lambda x+\lambda\varepsilon) \geq u(\lambda x) = v_2(x)$ so both are non decreasing functions

take $x, x' \in \mathbb{R}^S$ $\lambda \in [0,1]$, $\tilde{\lambda} \in [0,1]$

$v_1(\lambda x+(1-\lambda)x') = u(\lambda(w+x)+(1-\lambda)(w+x')) \geq \lambda u(x+w) + (1-\lambda)u(x'+w) =$
$= \lambda v_1(x) + (1-\lambda)v_1(x')$

$v_2(\tilde{\lambda}x+(1-\tilde{\lambda})x') = u(\lambda(\tilde{\lambda}x+(1-\tilde{\lambda})x')) \geq \tilde{\lambda}u(x\lambda) + (1-\tilde{\lambda})u(\lambda x') =$
$= \tilde{\lambda} v_2(x) + (1-\tilde{\lambda}) v_2(x')$

used concavity of u. Thus v_1, v_2 are concave
we can apply v_1, v_2 to definitions from lecture.

Def. If $\mathbb{E}y = \mathbb{E}z$ we say that y is more risky than z if $\forall u \in C^0$ non decreasing concave function u $\mathbb{E}u(y) \leq \mathbb{E}u(z)$.

we plug v_1 and v_2 respectivly and we obtain that

$\mathbb{E}u(y+w) = \mathbb{E}v(y) \leq \mathbb{E}v(z) = \mathbb{E}u(z+w)$

$\mathbb{E}u(\lambda y) = \mathbb{E}v(y) \leq \mathbb{E}v(z) = \mathbb{E}u(\lambda z)$

thus $w+y$ is more risky then $w+z$ and λy is more risky than λz.

y, z | y is more risky than z

- $\forall w \in \mathbb{R}$ $\quad y+w$ is more risky than ~~z~~ $z+w$

- $\lambda > 0$ $\quad y\lambda$ is more risky than λz

2. (15 points) Consider two real-valued random variables $\tilde{y}$ and $\tilde{z}$ such that $E(\tilde{y}) = E(\tilde{z})$. Suppose that cumulative distribution functions F_y and F_z have the following (weak) single-crossing property: there exists $t^* \in R$ such that $F_y(t) - F_z(t) \geq 0$ for $t \leq t^*$ and $F_y(t) - F_z(t) \leq 0$ for $t \geq t^*$. Show that $\tilde{y}$ is more risky than $\tilde{z}$. You may assume that $\tilde{y}$ and $\tilde{z}$ are random variables on a state space with S states.

Claim 1. y is mean-preserving spread of z.

Pf: I denote by $a = \min \operatorname{Supp} y$ $\quad b = \max \operatorname{supp} z$ then

$$\mathbb{E}y = \int_a^b t \, dF_y(t) = \int_a^b t \, dF_z(t) = \mathbb{E}z$$ integrating by parts:

$$0 = \int_a^b t\,(dF_y(t) - dF_z(t)) = \left\{ \begin{matrix} f = t & g' = dF_y - dF_z \\ f' = 1 & g = F_y - F_z \end{matrix} \right\} = \Big[t\,(F_y(t) - F_z(t)) \Big]_a^b -$$

$$- \int_a^b F_y(t) - F_z(t)\,dt = b\cdot(1-1) - a\cdot(0-0) - \int_a^b F_y(t) - F_z(t)\,dt$$

thus $\displaystyle\int_a^b F_y(t)\,dt = \int_a^b F_z(t)\,dt \quad (*)$

Now define $z \sim F_z \quad x \sim H$ where $dH = dF_y - dF_z$

then $z + x \sim F_y$ and $\int_a^b x \, dH(x) = 0$, where I provided a new RV x with distribution H (with density $dF_y - dF_z$).

By showing that $\int_a^b x\, dH(x) = 0$ I will get y being mean preserving spread of z.

$$\int_a^b x\,dH(x) = \int_a^b x\,(dF_y - dF_z)(x) = \left\{ \begin{matrix} f = x & g' = dF_y - dF_z \\ f' = 1 & g = F_y - F_z \end{matrix} \right\} = \Big[x\,(F_y(x) - F_z(x)) \Big]_a^b -$$

$$- \int_a^b (F_y(x) - F_z(x))\,dx = b\,(1-1) - a\cdot(0-0) - 0 = 0$$

Claim 2. $\displaystyle\int_a^w F_y(t)\,dt \geq \int_a^w F_z(t)\,dt$

Pf: take $t^* \geq w$ then $F_y(t) - F_z(t)$ and by integrating over (a, w):

$$\int_a^w F_y(t)\,dt \geq \int_a^w F_z(t)\,dt.$$

Consider $t^* < w$ we can decompose integral into parts

$$\int_a^w F_y(t) - F_z(t)\,dt = \int_a^b F_y(t) - F_z(t)\,dt - \int_w^b F_y(t) - F_z(t)\,dt = \int_w^b F_z(t) - F_y(t)\,dt \geq 0$$

where third equality comes from $(*)$ and condition: $F_z(t) - F_y(t) \geq 0$ for $t^* < w$. Now I'll show that for mean spread preserving functions $\mathbb{E}\,u(y) \leq \mathbb{E}\,u(z)$ $\forall u \in C^0$ non decreasing concave. But as we did during lecture showing Claim 2 is equivalent to show $z \geq_{SOSD} y$ & $\mathbb{E}y = \mathbb{E}z$,

$$\max_{a} \mathbb{E}u\left((w-a)\cdot r_f + a\cdot r\right)$$

$a \geq 0$ when shortselling is not allowed

& u is concave

$\Downarrow$

FOCs:

$$\mathbb{E}u'(\cdot)\cdot(r-r_f) = 0$$

$\downarrow$

$a \in$ ...