# Theory of Game Theory

# ECON 8103 notes

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## Contents



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*These notes are intended to summarize the main concepts, definitions and results covered in the first year of micro sequence for the Economics PhD of the University of Minnesota. The material is not my own. Please let me know of any errors that persist in the document. E-mail: pawel042@umn.edu .

# 1 Normal Form Games

## 1.1 Games on Consequences

**Definition 1** (Games on Consequences)**.** *consists of:*

- $I = \{1, \dots, n\}$ *is the finite set of players.*
- $A^i$ *is the (finite) set of actions for player* $i$.
- $A \equiv X_{i \in I} A^i$ *is the (finite) set of action profiles*
- $C$ *is finite the set of consequences,* $C = \{c^1, \dots, c^m\}$.
- $\succeq_i$ *preference relation of Mr* $i$ *over* $C$
- $g : A \to C$ *mapping of actions to consequences*

*This will be compactly denoted as a* $\left\langle I, (A^i)_{i \in I}, (\succeq^i)_{i \in I}, C, g \right\rangle$.

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**Example 1.**

Mr 2

| | | L | R |
|---|---|---|---|
| Mr 1 | T | $c^1$ | $c^2$ |
| | B | $c^3$ | $c^4$ |

Table above induces $g$

$A^1 = \{T, B\}$, $A^2 = \{L, R\}$

$c^1 = (10, 5)$, $c^1 = (1, 2)$, $c^1 = (3, 2)$, $c^1 = (4, 3)$

Mr 1 : $c^1 \succeq_1 c^3$ and $\succeq_1 c^2 \succeq_1 c^4$

Mr 2 : $c^2 \succeq_2 c^1$ and $\succeq_2 c^4 \succeq_2 c^3$

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## 1.2 Preferences on lotteries

**Definition 2** (Simplex).

$$\Delta(C) \equiv \left\{ p = \left(p^1, \ldots, p^m\right) \mid \forall i \quad p^i \geq 0 \quad \sum_{i=1}^{m} p^i = 1 \right\}$$

**Definition 3** (Lottery). *$L \in \Delta(C)$ is a simple lottery, where*

$$L = \begin{pmatrix} p^1 & \cdots & p^i & \cdots & p^m \\ c^1 & \cdots & c^i & \cdots & c^m \end{pmatrix}$$

**Example 2** (Degenerated lottery). $\delta_{c^i} \in \mathcal{L}$

$$\delta_{c^i} = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ c^1 & \cdots & c^i & \cdots & c^m \end{pmatrix}$$

**Definition 4.** *$\mathcal{L} \equiv \Delta(C)$ is the set of (simple) lotteries.*

**Definition 5.** *$G = \left(q^1 L^1, \ldots, q^K L^K\right) \in \Delta(\mathcal{L})$ is a compound lottery, where*

$$G = \begin{pmatrix} q^1 & \cdots & q^k \\ L^1 & \cdots & L^k \end{pmatrix}$$

$L^k \in \mathcal{L} \quad \forall k = 1, \ldots, K, q^k \geq 0$ *and* $\sum_{k=1}^{K} q^k = 1$

**Definition 6.** *$\mathcal{G} \equiv \Delta(\mathcal{L})$ is the set of compound lotteries.*

Note that all simple lotteries can be viewed as compound lotteries with degenerate distributions. For example, the simple lottery $L = \left(p^1, \ldots p^m\right)$ can be viewed as a compound lottery $L = \left(p^1 \delta_{c^1}, \ldots, p^m \delta_{c^m}\right)$, where $\delta_{c^i}$ is a degenerate lottery giving fully probability to consequence $c^i$

**Definition 7** (Reduction of a lottery). *. For every $G \in \mathcal{G}, R(G) \in \mathcal{L}$ is the reduction of G, and gives probability $\sum_{k=1}^{K} q^k p_k^i$ to consequence $c^i$*

**Definition 8** (Convex combination). *For any $F, G$ and $\alpha \in [0, 1]$, denote the convex combination as $F\alpha G \equiv \alpha F + (1 - \alpha)G$*

## 1.3 Assumptions on $\succeq$

We are interested in the binary preference relation $\succeq_i$ on $\mathcal{L}$.

**Definition 9** ( Complete (C)). *$\forall F, G \in \mathcal{G}$ either $F \succeq G$ or $G \succeq F$*

**Definition 10** ( Reflexive (R)). *$\forall F \in \mathcal{G}$ $F \succeq F$*

**Definition 11** ( Transitive (T)). *$\forall F, G, H \in \mathcal{G}$ such that $F \succeq G$, $G \succeq H$ then $F \succeq H$*

**Definition 12** ( Weak Order (WO)-A1). *$\succeq$ is compete , reflexive, and transitive.*

**Definition 13** (Independence (I)-A2). *$\forall F, G, H \in \mathcal{G}$ and $\alpha \in (0,1)$ : such that*

$$F \succ G \Rightarrow F\alpha H \succ G\alpha H$$

**Definition 14** ( Continuity (Cty)-A3). *$\forall F, G, H \in \mathcal{G}$ such that $F \succeq G \succeq H, \forall \alpha \in [0,1]$ such that $\{\alpha | F\alpha H \geq G\}$ and $\{\beta | F\beta H \leq G\}$ are closed sets.*

Alternative definition of Cty

**Definition 15** ( Continuity (Cty2)). *$\forall F, G, H \in \mathcal{G}$ such that $F \succeq G \succeq H, \exists \alpha \in [0,1]$ such that $F\alpha H \sim G$*

**Lemma 1.** *If $[C, T, Cty]$ holds then $Cty2$ holds too.*

*Proof.* Suppose $F \succeq G$. Define A=$\{\alpha | F\alpha H \geq G\}$ and B=$\{\beta | F\beta H \leq G\}$. Observe that:

- $A, B \subset [0,1]$
- $1 \in A$ , $0 \in B$
- $A, B$ are closed (by Cty)
- $A \cup B = [0,1]$
- $[0,1]$ is a connected set

(1)-(5) implies that $A \cap B \neq \emptyset$. So $\exists \alpha \in A \cap B$ s.t. $F\alpha G \succeq H \succeq F\alpha G$ . Thus $F\alpha G \sim H$. □

**Lemma 2.** *Suppose [WOI] hold then:*

$$\forall_{F \in \mathcal{L}} \quad \delta_{c^1} \succeq F \succeq \delta_{c^m}$$

*Proof.* Since $C$ is finite then $\exists$ best and worst outcome $\delta_{c^b}$ and $\delta_{c^w}$. WTS $\forall L \quad \delta_{c^b} \succeq L \succeq \delta_{c^w}$. I will use (easy to prove) corollary

**Corollary 1.** *Let $L_0, \ldots L_K$ be (1+K) lotteries $\alpha_k \geq 0 : \sum_k \alpha_k = 1$ :*

$$\begin{aligned} \text{If} \quad \forall k \quad L_k \succeq L_0 \quad &\Rightarrow \quad \sum_k \alpha_k L_k \succeq L_0 \\ \text{If} \quad \forall k \quad L_0 \succeq L_k \quad &\Rightarrow \quad L_0 \succeq \sum_k \alpha_k L_k \end{aligned}$$

Now let lottery $L^k$ yields outcome $k$ with probability 1. Then $\delta_{c^b} \succeq L \succeq \delta_{c^w}$ and any $L$ can be represented as $L = \sum_k p_k L^k$ so by corollary $\delta_{c^b} \succeq L \succeq \delta_{c^w}$ □

**Definition 16** ( Monotonicity (M))**.** *$\forall F, G \in \mathcal{G}$ such that $F \succ G$, then for $\alpha, \beta \in (0, 1)$ :*

$$\alpha > \beta \Leftrightarrow F\alpha G \succ F\beta G$$

**Lemma 3.** *If I holds and $F \succ G$ $\forall \alpha \in (0, 1) \quad \Rightarrow F \succ F\alpha G \succ G$*

*Proof.*

$$F = \alpha F + (1-\alpha)F \succ^I \alpha F + (1-\alpha)G = F\alpha G = \alpha F + (1-\alpha)G \succ^I \alpha G + (1-\alpha)G = M$$

□

**Lemma 4.** *Prove that WO, Cty, I imply M.*

*Proof.* $\Rightarrow$ Suppose $\alpha > \beta$. Observe that

$$F = \alpha F + (1-\alpha)G = \gamma F + (1-\gamma)[\beta F + (1-\beta)G]$$

after rearrangement $\gamma = \frac{\alpha-\beta}{1-\beta} \in (0, 1)$ By lemma 3 $F \succ G$: $F \succ F\beta G$

$$F\alpha G = F\gamma(F\beta G) \succ^I (F\beta G)\gamma(F\beta G) = F\beta G$$

Now $\Leftarrow$ part. Suppose $F \succ G$ and $F\alpha G \succ F\beta G$. WTS: $\alpha > \beta$.
Suppose not. So either $\alpha = \beta$ or $\alpha < \beta$. If $\alpha = \beta$ then we have $\lightning$with $F\alpha G \succ F\beta G$. If $\alpha < \beta$ by $\Rightarrow$ part $F\beta G \succ F\alpha G$ $\lightning$. □

**Definition 17** ( Reduction (R))**.** $\forall G \in \mathcal{G}, R(G) \sim G$

**Definition 18** ( Substitution (S):). *$\forall G \in \mathcal{G}$, if $G = \begin{pmatrix} q^1 & \dots & q^j & \dots & q^K \\ L^1 & \dots & L^j & \dots & L^K \end{pmatrix}$ is modified by substituting $L^j$ for $M^j$, where $M^j \sim L^j$, then $G \sim H$, where $H = \begin{pmatrix} q^1 & \dots & q^j & \dots & q^K \\ L^1 & \dots & M^j & \dots & L^K \end{pmatrix}$*

## 1.4 Utility representation

**Definition 19** (Utility representation)**.** *The function $u : \mathcal{G} \to \mathbb{R}$ is a representation of $\succeq$ if and only if:*

$$F \succeq G \Leftrightarrow u(F) \geq u(G)$$

Recall:

$$F \succ G \Leftrightarrow F \succeq G \text{ and not } G \succeq F$$
$$F \sim G \Leftrightarrow F \succeq G \text{ and } G \succeq F$$

**Lemma 5.** *If $u$ represents $\succeq$ and $T : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $T(u(\cdot)) : \mathcal{G} \to \mathbb{R}$ is a representation of $\succeq$*

**Lemma 6** (Recap from MINI 1)**.** *If $\succeq$ satisfies WO and C, then $\succeq$ has some (continuous) utility representation.*

**Definition 20** (Linear utility)**.** *If $u$ is linear then $u(F\alpha G) = u(F)\alpha u(G)$, where $\alpha \in [0,1]]$*

Alternative definition of linearity:

**Definition 21** (Linear utility)**.** *$u$ is linear if and only if $u(L) = \sum_{i=1}^{m} p^i u\left(c^i\right)$, where $L = \left(p^1, \ldots, p^m\right)$*

**Corollary 2.** *If $u$ represents $\succeq$ and is linear, then if $A > 0$ and $B \in \mathbb{R}$, $Au(\cdot) + B$ also represents $\succeq$ and is linear.*

**Corollary 3.** *$\succeq$ satisfies* WO, Cty, *and* M *if and only if $\forall F \in \mathcal{G} \quad \exists u(F) \in [0,1]$ such that $F \sim \delta_{c^1} u(F) \delta_{c^m}$ and $u(F)$ is unique. In particular, $\forall c^i \in C \quad \exists u\left(c^i\right) \in [0,1]$ such that $c^i \sim c^1 u\left(c^i\right) c^m$.*

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**Theorem 1** (von Neumann-Morgenstern (I))**.** ***1. (existence)** $\succeq$ **on** $\mathcal{L}$ **satisfies** WO, Cty, I **if and only if there exists a linear** $u : \mathcal{G} \to \mathbb{R}$ **that represents** $\succeq$*

***2. (uniqueness) If** $u, v$ **are linear representations of** $\succeq$**, then** $\exists A > 0, B \in \mathbb{R}$ **such that** $u(\cdot) = Av(\cdot) + B$*

*Proof.* We will proceed in three steps: 1) (existence): $\Rightarrow$; 2)(existence): $\Leftarrow$; 3)(uniqueness)

- (existence): $\Rightarrow$

By lemma 2: $\exists \delta_{c^1}, \delta_{c^m} : \forall F : \delta_{c^1} \succeq F \succeq \delta_{c^m}$ and $\delta_{c^1} \succ \delta_{c^m}$.
Define $u(F)$ :$\delta_{c^1} u(F) \delta_{c^m} \sim F$. By lemma 1 we know that such $u(F)$ is well defined. Our goal is to show for $\alpha = u(F)$ that this is representation, it is unique and linear. We do it with two lemmas. We want to avoid $\alpha \neq \beta$ $\delta_{c^1} \alpha \delta_{c^m} \sim \delta_{c^1} \beta \delta_{c^m}$, we want $\delta_{c^1} \alpha \delta_{c^m} \succ \delta_{c^1} \beta \delta_{c^m} \iff \alpha > \beta$ .

**Lemma 7.** *$u(F) : \delta_{c^1} u(F) \delta_{c^m} \sim F$ is unique*

*Proof.* Let $\bar{u}(F)$ and $u(F)$ be two different values and WLOG $\bar{u}(F) > u(F)$.

$$\delta_{c^1} u(F) \delta_{c^m} \sim F \sim \delta_{c^1} \bar{u}(F) \delta_{c^m}$$

by applying lemma 4 ($\delta_{c^1} \succ \delta_{c^m}$), $\bar{u}(F) > u(F)$):

$$\delta_{c^1} u(F) \delta_{c^m} \succ \delta_{c^1} \bar{u}(F) \delta_{c^m}$$

$\lightning$.

$\square$

By last lemma $F \succeq G \iff \delta_{c^1} u(F) \delta_{c^m} \succeq \delta_{c^1} u(G) \delta_{c^m}$ by lemma 4 $\iff u(F) \geq u(G)$. So $u : \mathcal{L} \to \mathbb{R}$ represents $\succeq$. Let's show now that following mixing is allowed

**Lemma 8.** *For $\forall a, b, \mathcal{L} \quad \alpha \in [0,1] \quad a \sin b$ then $a\alpha c \sim b\alpha c$*

*Proof.* $a \sim b$ so $\delta_c^1 u(a) \delta_c^m \sim \delta_c^1 u(b) \delta_c^m$ and by lemma 7 $u(a) = u(b)$. Next observe that for every $c$ by lemma 7 again applied to $\alpha u(a) + (1-\alpha) u(c) = \alpha u(b) + (1-\alpha) u(c)$:

$$\delta_c^1 (\alpha u(a) + (1-\alpha) u(c)) \delta_c^m \sim \delta_c^1 (\alpha u(b) + (1-\alpha) u(c)) \delta_c^m$$

so $a\alpha c \sim b\alpha c$ $\square$

**Lemma 9.** *$u(\cdot)$ is linear*

*Proof.* By definition of $u$

$$F \sim \delta_{c^1} u(F) \delta_{c^m}$$

$$G \sim \delta_{c^1} u(G) \delta_{c^m}$$

by I (and rearrangement and lemma 8) :

$$F\alpha G \sim (\delta_{c^1} u(F) \delta_{c^m})\alpha G \sim (\delta_{c^1} u(F) \delta_{c^m})\alpha(\delta_{c^1} u(G) \delta_{c^m}) \sim \delta_{c^1}(u(F)\alpha u(G))\delta_{c^m}$$

Thus $u(F\alpha G) = u(F)\alpha u(G)$ $\square$

- (existence):$\Leftarrow$

Let's show that $\succeq$ satisfy weak order (WO). Let's start with completeness.

$$\forall F, G \in \mathcal{L} \quad u(F) \geq u(G) \quad \text{or} \quad u(F) \leq u(G) \quad \iff \quad F \succeq G \quad \text{or} \quad G \succeq F$$

since it is order on real line.

Transitivity. WLOG $F \succeq G$ and $G \succeq H$. Observe that since $u$ represents preferences:

$$u(F) \geq u(G) \iff F \succeq G$$

$$u(G) \geq u(H) \iff G \succeq H$$

$$u(F) \geq u(H) \iff F \succeq H$$

we have $u(F) \geq u(G)$,$u(G) \geq u(H) \Rightarrow u(F) \geq u(H)$ comes from linear order on real line. So $F \succeq H$.

Now we show continuity. Consider any sequence $\{\alpha_i\}_{i=1}^{\infty} \to \alpha$, (where $\forall i, \alpha_i \in [0,1]$) and $\alpha_i F + (1-\alpha_i)\, G \succsim H, \forall i$ Then,

$$U\left(\alpha_i F + (1-\alpha_i)\, G\right) \geq U(H), \forall i$$

and using the linearity of $U$

$$\alpha_i U(F) + (1-\alpha_i)\, U\,(G) \geq U(H), \forall i$$

which implies (taking limit as $i \to \infty$)

$$\alpha U(F) + (1-\alpha) U\,(G) \geq U(H)$$

so that $\alpha F + (1-\alpha) G \succsim H$.

Next, we show independence. Consider $F, G, H \in \mathcal{L}$ and $\alpha \in (0,1)$ Need to show:

$F \succsim G \Longleftrightarrow \alpha F + (1-\alpha)H \succsim \alpha G + (1-\alpha)H$ Suppose $F \succsim G$ Then, $U(F) \geq U(G)$ so that

$$\alpha U(F) + (1-\alpha)U(H) \geq \alpha U(G) + (1-\alpha)U(H)$$

which implies

$$\alpha F + (1-\alpha)H \succsim \alpha G + (1-\alpha)H$$

Suppose that $\alpha F + (1-\alpha)H \succsim \alpha G + (1-\alpha)H$ Then,

$$U(\alpha F + (1-\alpha)H) \geq U(\alpha G + (1-\alpha)H)$$

and using linearity of $U$,

$$\alpha U(F) + (1-\alpha)U(H) \geq \alpha U(G) + (1-\alpha)U(H)$$

which implies that $U(F) \geq U(G)$

- (uniqueness):

Let $u, v$ be linear representations of $\succeq$ and take $F$ such that $F \sim c^1 \alpha c^m$ for some $\alpha \in [0,1]$. Then, by linearity:

$$u(F) = u\left(c^1 \alpha c^m\right) = \alpha u\left(c^1\right) + (1-\alpha)u\left(c^m\right)$$

$$\text{and} \quad v(F) = v\left(c^1 \alpha c^m\right) = \alpha v\left(c^1\right) + (1-\alpha)v\left(c^m\right)$$

$$\alpha = \frac{u(F) - u\left(c^m\right)}{u\left(c^1\right) - u\left(c^m\right)} = \frac{v(F) - v\left(c^m\right)}{v\left(c^1\right) - v\left(c^m\right)} \Longrightarrow u(F) = \frac{u\left(c^1\right) - u\left(c^m\right)}{v\left(c^1\right) - v\left(c^m\right)} v(F) - \frac{u\left(c^1\right) - u\left(c^m\right)}{v\left(c^1\right) - v\left(c^m\right)} v\left(c^m\right) + u\,(c$$

$$u(F) = Av(F) + B$$

where $A \equiv \frac{u(c^1) - u(c^m)}{v(c^1) - v(c^m)}$ and $B \equiv u\left(c^m\right) - \frac{u(c^1) - u(c^m)}{v(c^1) - v(c^m)} v\left(c^m\right)$

$\square$

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Theorem is true under alternative set of axioms. We present proof of it for pedagogical reasons.

**Theorem 2** (von Neumann-Morgenstern (M,S,R)). *1. (existence) $\succeq$ on $\mathcal{L}$ satisfies* $\mathrm{WO}, \mathrm{Cty}, \mathrm{M}, \mathrm{S}, \mathrm{R}$ *if and only if there exists a linear $u : \mathcal{G} \to \mathbb{R}$ that represents $\succeq$*

*2. (uniqueness) If $u, v$ are linear representations of $\succeq$, then $\exists A > 0, B \in \mathbb{R}$ such that $u(\cdot) = Av(\cdot) + B$*

*Proof.* Below we prove theorem when $\succeq$ on $\mathcal{G}$ satisfies WO, Cty, M, R*and*S. We show only (existence )$\Rightarrow$ part. Uniqueness remains the same and $\Leftarrow$ of existence is easy exercise left for a reader.

(existence): $\Rightarrow$

By WO, Cty, and M, we know there exists $u : C \to \mathbb{R}$ and thus $c^i \sim c^1 u\left(c^i\right) c^m$ implies $\bar{u}(L) \equiv \sum_{i=1}^{m} p^i u\left(c^i\right)$, where $L = \left(p^1, \ldots, p^m\right)$ and $L \sim c^1 u(L) c^m$

**Lemma 10.** $\bar{u}(L) = u(L)$

Proof: Recall $c^2 \sim c^1 u\left(c^2\right) c^m$ and construct

$$L' = \begin{pmatrix} p^1 & p^2 & \ldots & p^m \\ c^1 & c^1 u\left(c^2\right) c^m & \ldots & c^m \end{pmatrix}$$

where $L' \sim L$ by substitution. Repeat this substitution process for all but $c^1$ and $c^m$. Now take the reduction

$$R\left(L'\right) = \begin{pmatrix} p^1 + p^2 u\left(c^2\right) + p^3 u\left(c^3\right) \ldots & 0 & \ldots & 1 - \left(p^1 + \ldots\right) \\ c^1 & c^2 & \ldots & c^m \end{pmatrix}$$

and note $R\left(L'\right) \sim L$ by reduction. Then $u(L) = \sum_{i=1}^{m} p^i u\left(c^i\right) = \bar{u}(L)$. $\square$

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**Definition 22 ( Sure Thing Principle).** *For lotteries $L, M, N, R \in \mathcal{L}$ and $\alpha \in (0,1]$*

$$L\alpha M \succ N\alpha M \Leftrightarrow L\alpha R \succ N\alpha R$$

**Lemma 11.** *If $\succeq$ satisfies the vNM axioms, then $\succeq$ satisfies the Sure Thing Principle.*

*Proof.* Since $\succeq$ satisfies the vNM axioms, there exists a linear utility representation $u(\cdot)$. Thus, $\forall \alpha \in (0,1]$ :

$$\begin{aligned} L\alpha M \succ N\alpha M &\Leftrightarrow u(L\alpha M) > u(N\alpha M) \\ &\Leftrightarrow \alpha u(L) + (1-\alpha)u(M) > \alpha u(N) + (1-\alpha)u(M) \\ &\Leftrightarrow u(L) > u(N) \\ &\Leftrightarrow \alpha u(L) + (1-\alpha)u(R) > \alpha u(N) + (1-\alpha)u(R) \\ &\Leftrightarrow u(L\alpha R) > u(N\alpha R) \\ &\Leftrightarrow L\alpha R \succ N\alpha R \end{aligned}$$

$\square$

From a game on consequences, we elicit $\succeq_i$ for each player.

We then use the von Neumann-Morgenstern Theorem to construct utility functions $u^i : C \to \mathbb{R}$

Then we construct utility functions $\hat{u}^i : A \to \mathbb{R}$ defined by $\hat{u}^i = u^i(g(a))$.

Thus we transform a game on consequences into a **normal form game**

**Definition 23** (Normal Form Game (NFG))**.** *is a tuple* $\left(I, \left(A^i\right)_{i\in I}, \left(u^i\right)_{i\in I}\right)$

## 1.5 Strategies of Normal Form Games

**Definition 24.** *A mixed strategy for player $i$ is $s^i \in \Delta\left(A^i\right)$; we denote the mixed strategies of all players $j \neq i$ as $s^{-i} \in \Delta\left(A^{-i}\right)$*

**Definition 25.** *The set of mixed strategy profiles for player $i$ is $S^i \equiv \Delta\left(A^i\right)$; we denote the set for all players $j \neq i$ as $S^{-i} \equiv \prod_{j \neq i} \Delta\left(A^j\right)$. Equivalently,*

$$S^i = \left\{ \left\{ s^i\left(a^i\right) \right\}_{a^i \in A^i} \mid \sum_{a^i \in A^i} s^i\left(a^i\right) = 1; \forall a^i \in A^i, s^i\left(a^i\right) \geq 0 \right\}$$

[Note: $S^i = \text{co}\left(A^i\right)$, and so $S^i$ is convex. If $A^i$ is finite, then $S^i = \overline{co}\left(A^i\right)$

**Definition 26.** *A mixed strategy for all players is $s \in S$, where $S \equiv \prod_{i \in I} S^i$ is the set of all mixed strategy profiles.*

**Definition 27.** *Fully mixed strategy A mixed strategy $s^i \in \Delta\left(A^i\right)$ is a fully mixed strategy if $\forall a^i \in A^i, s^i\left(a^i\right) > 0$*

## 1.6 Nash Equilibrium

**Definition 28.** *A normal form game* (NFG) *is a tuple* $\left\langle I, \left(A^i, u^i\right\rangle_{i\in I}\right)$ *, where* $\forall i \quad u^i : A \to \mathbb{R}$

**Definition 29** (Mixed extension of NFG). *For a NFG* $\left\langle I, \left(A^i, u^i\right)_{i\in I}\right\rangle$*, the mixed extension is* $\left\langle I, \left(S^i, u^i\right)_{i\in I}\right\rangle$ *where* $\forall i \quad s^i \in S^i$ *and* $u^i : S \to \mathbb{R}$

In general, for any $s \in S$ we have an element $\mathrm{Pr}_s \in \Delta(A)$ defined by

$$\mathrm{Pr}_s(a) = \mathrm{Pr}_s\left(a^1, \ldots, a^n\right) = s^1\left(a^1\right)s^2\left(a^2\right)\cdots s^n\left(a^n\right) = \prod_{i\in I} s^i\left(a^i\right)$$

We define agent $i$ 's expected utility over mixed strategy profiles as $u^i : S \to \mathbb{R}$, where:

$$\begin{aligned} u^i(s) &= \sum_{a\in A} \mathrm{Pr}_s(a)u^i(a) \\ &= \sum_{a^i\in A^i} s^i\left(a^i\right) \sum_{a^{-i}\in A^{-i}} \mathrm{Pr}_{s^{-i}}\left(a^{-i}\right)u^i\left(a^i, a^{-i}\right) \\ &= \sum_{a^i\in A^i} s^i\left(a^i\right)u^i\left(a^i, s^{-i}\right) \\ &= u^i\left(s^i, s^{-i}\right) \end{aligned}$$

We will use this representation extensively.

**Definition 30** (Pure action best response correspondence). *The action best response correspondence of player* $i, BR^i_{A^i} : S \rightrightarrows A^i$*, is:*

$$\begin{aligned} BR^i_{A^i}(s) &\equiv \left\{a^i \in A^i \mid \forall b^i \in A^i u^i\left(a^i, s^{-i}\right) \geq u^i\left(b^i, s^{-i}\right)\right\} \\ &= \underset{a^i\in A^i}{\arg\max} u^i\left(a^i, s^{-i}\right) \end{aligned}$$

**Definition 31** (Best response correspondence). *The best response correspondence of player* $i, BR^i : S \rightrightarrows S^i$*, is:*

$$\begin{aligned} BR^i\left(s^{-i}\right) = BR^i(s) &\equiv \left\{r^i \in S^i \mid \forall t^i \in S^i u^i\left(r^i, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right)\right\} \\ &= \left\{r^i \in S^i \mid u^i\left(r^i, s^{-i}\right) = \max_{t^i\in S^i} u^i\left(t^i, s^{-i}\right)\right\} \\ &= \underset{s^i\in S^i}{\arg\max} u^i\left(s^i, s^{-i}\right) \end{aligned}$$

The only difference between those two Best responses is on domain of correspondences.

**Definition 32** ( Best reply correspondence). *The best reply correspondence $BR : S \rightrightarrows S$ is defined by:*

$$BR(s) = \prod_{i \in I} BR^i(s)$$

**Definition 33** (Nash equilibrium). *If $\left(I, \left(S^i, u^i\right)_{i \in I}\right)$ is the mixed extension of a NFG, then $\hat{s} \in S$ is a Nash equilibrium if and only if $\forall i \hat{s}^i \in BR^i(\hat{s})$.*

---

**Example 3.** *Consider following game (called Battle of Sexes):*

| | | Mr 2 | |
|---|---|---|---|
| | | L | R |
| Mr 1 | T | 3,1 | 0,0 |
| | B | 0.0 | 1.3 |

- Define: pure actions, mixed actions, best correspondences
- Find all Nash Equilibria

pure strategies: $A^1 = \{T, B\}$, $A^2 = \{L, R\}$,$A = A^1 A^2$

mixed strategies:

$$S = S^1 \times S^2 = \Delta(A^1) \times \Delta(A^2) = \{((p, 1-p), (q, 1-q)) \,|\, p, q \in [0, 1]\}$$

We can solve for the best responses as follows: Mr 1 best response:

$$BR^1((q, 1-q)) : \left\{ \begin{array}{cc} T & B \\ 3(q) + 0(1-q) & 0(q) + 1(1-q) \end{array} \right\}$$

Equality only holds when $q = \frac{1}{4}$. $T > B \iff q > \frac{1}{4}$ . $T < B \iff q < \frac{1}{4}$ Therefore, player 1 sets $p = 1$ if $q > \frac{1}{4}$ and sets $p = 0$. She picks $p \in [0, 1]$ where is indifferent between T and B. otherwise.

$$BR^1((q, 1-q)) = \begin{cases} 0 & \text{if } q < \frac{1}{4} \\ [0, 1] & \text{if } q = \frac{1}{4} \\ 1 & \text{if } q > \frac{1}{4} \end{cases}$$

Mr 2 best response:

$$BR^2\left((p, 1-p)\right) : \left\{ \begin{array}{cc} L & R \\ p + 0\,(1-p) & 0\,(p) + 3\,(1-p) \end{array} \right\}$$

Equality only holds when $p = \frac{3}{4}$. $L > R \iff p > \frac{3}{4}$ ,$L < R \iff p < \frac{3}{4}$ Similarly, player 2 sets $q = 1$ if $p > \frac{3}{4}$ and sets $q = 0$ otherwise.

$$BR^2\left((p, 1-p)\right) = \begin{cases} 0 & \text{if } p < \frac{3}{4} \\ [0, 1] & \text{if } p = \frac{3}{4} \\ 1 & \text{if } p > \frac{3}{4} \end{cases}$$

These best responses can be graphed :

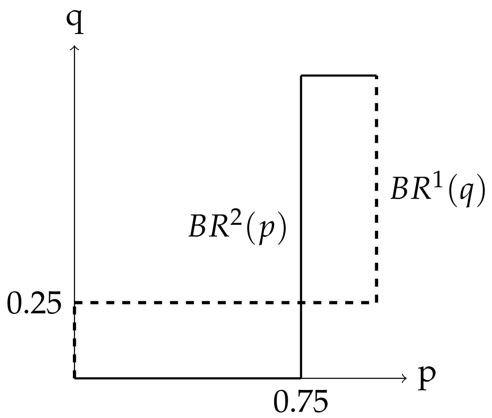


Figure 1: Best Responses

The points of interesection

$$\left(\frac{3}{4}, \frac{1}{4}\right), (1, 1), (0, 0)$$

yield the set of Nash equilibria

$$\text{NE} = \left\{ \left((1,0), (1,0)\right), \left((0,1), (0,1)\right), \left((\frac{3}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{3}{4})\right) \right\}.$$

**Corollary 4.** *A NE exists if and only if the best response correspondence $BR : S \rightrightarrows S$ has a fixed point (i.e. $s \in BR(s)$ )*

**Lemma 12.** *Show that* $\mathrm{BR}_i(s) = \mathrm{co}\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right)$

*Proof.* • $\mathrm{BR}_i(s) \subset \mathrm{co}\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right)$

We present here small but important result: if strategy is not best response in pure best response, corresponding probability in best response in mixed strategies is zero. Let $s^i \in \mathrm{BR}^i(s)$.

**Lemma 13.**

$$\forall b^i \notin BR_{A^i}(s), b^i \in A^i \Rightarrow s^i(b^i) = 0$$

*Proof.* Suppose not. if the strategy $s^i \in \mathrm{BR}^i(s)$ uses some pure action $b^i \in A^i$ which $\notin \mathrm{BR}_{A^i}(s)$, i.e. $s^i(b^i) > 0$ then

$$\forall c^i \in \mathrm{BR}_{A^i}(s): \quad u^i(c^i, s^{-i}) > u^i(b^i, s^{-i})$$

Consider another mixed strategy $r^i$, defined as follows:

$$\begin{cases} r^i(a^i) = s^i(a^i) & \forall a^i \in A^i / \{b^i, c^i\} \\ r^i(b^i) = 0 \\ r^i(c^i) = s^i(b^i) + s^i(c^i) \end{cases}$$

then

$$u^i(r^i, s) = \sum_{a^i \in A^i} r^i(a^i) u(a^i, s^{-i}) + r^i(b^i) u^i(b^i, s^{-i}) + r^i(c^i) u^i(c^i, s^{-i}) =$$

$$= \sum_{a^i \in A^i} s^i(a^i) u^i(a^i, s) + [s^i(b^i) + s^i(c^i)] u^i(c^i, s^{-i}) >$$

$$\sum_{a^i \in A^i} s^i(a^i) u^i(a^i, s^{-i}) + s^i(b^i) u^i(b^i, s^{-i}) + s^i(c^i) u^i(c^i, {}^{-i}s) = u^i(s^i, s^{-i})$$

$\lightning$with $s^i \in \mathrm{BR}^i(s)$. □

$\mathrm{BR}_i(s) \subset \mathrm{co}\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right)$ comes straight from lemma (our mixed best response has zeros when it is not in pure best response).

• $\mathrm{BR}_i(s) \supset \mathrm{co}\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right)$

BR is convex valued. We need to show that $\left(\left\{\delta_{b^i} : b^i \in \mathrm{BR}^i_{A^i}(s)\right\}\right) \subset \mathrm{BR}^i(s)$ Suppose not Let $b^i \in \mathrm{BR}^i_A(s)$ and suppose $\delta_{b^i} \notin \mathrm{BR}^i(s)$ then

$$\exists s^i \in \Delta(A^i) \quad u^i(s^i, s^{-i}) > u^i(b^i, s^{-i})$$

$$\sum_{a^i \in A^i} s^i(a) u^i(a^i, s^{-i}) > u^i(b^i, s^{-i}) = \sum_{a^i \in A^i} s^i(a^i) u^i(b^i, s^{-i})$$

for at least one $a^i$ $u^i(a^i, s^{-i}) > u^i(b^i, s^{-i})$ $\lightning$with $b^i \in \mathrm{BR}^i_{A^i}(s)$ □

---

**Lemma 14.** $\forall i \quad \forall s^{-i} \quad u^i\left(\cdot, s^{-i}\right) : s^i \to u^i\left(s^i, s^{-i}\right)$ *is linear, and thus it is continuous.*

**Lemma 15.** $\forall i \quad u^i : S \to \mathbb{R}$ *is continuous and linear in each argument, fixing other arguments.*

**Lemma 16.** *If $A^i$ is finite then $S^i$ is closed.*

*Proof.* Let $A^i$ be finite. Take any $\left\{s^i_n\right\}_{n \in \mathbb{N}} \in S^{i^{\mathbb{N}}}$ such that $s^i_n \to s^i$. Then $\forall n \sum_{a^i \in A^i} s^i_n\left(a^i\right) = 1$. Taking limits:

$$\begin{aligned} \lim_{n \to \infty} \sum_{a^i \in A^i} s^i_n\left(a^i\right) &= \lim_{n \to \infty} 1 \\ \Longrightarrow \sum_{a^i \in A^i} \lim_{n \to \infty} s^i_n\left(a^i\right) &= 1 \\ \Longrightarrow \sum_{a^i \in A^i} s^i\left(a^i\right) &= 1 \end{aligned}$$

Also $\forall n \forall a^i \in A^i s^i_n\left(a^i\right) \geq 0$. Taking limits again, clearly $s^i\left(a^i\right) \geq 0$. Thus $S^i$ is closed.

□

## 1.7 Correspondences

Let $\Theta \subseteq \mathbb{R}^n, X \subseteq \mathbb{R}^n$.

**Definition 34.** *A correspondence* $\Gamma : \Theta \rightrightarrows X$ *is a map s.t.* $\Gamma(\Theta) \subseteq X$. $\left(\Gamma : \Theta \to 2^X\right)$

**Definition 35.** ***(Graph of correspondence)**.* $Gr(\Gamma) = \{(\theta, x) : \theta \in \Theta, x \in \Gamma(\theta)\}$

**Definition 36.** ***(Properties of correspondences)**.*

1. ***not empty valued*** *if* $\Gamma(\theta) \neq \emptyset \quad \forall\theta$

2. ***single valued*** *if* $|\Gamma(\theta)| = 1 \quad \forall\theta$

3. ***closed valued*** *if* $\Gamma(\theta)$ *is closed set* $\forall\theta$

4. ***compact valued*** *if* $\Gamma(\theta)$ *is compact set* $\forall\theta$

5. ***convex valued*** *if* $\Gamma(\theta)$ *is convex set* $\forall\theta$

6. ***closed (graph)*** *if* $\mathrm{Gr}(\Gamma)$ *is closed subset of* $\mathbb{E} \times X$

7. ***convex (graph)*** *if* $Gr(\Gamma)$ *is convex on* $\Theta \times X$

**Lemma 17.** *$Gr(\Gamma)$ is closed graph* $\iff \forall_{\theta:\theta_n \to \theta} \forall_{x_n \to x} : x_n \in \Gamma\left(\theta_n\right) \Rightarrow \quad x \in \Gamma\left(\theta\right)$

**Lemma 18.** $\mathrm{Gr}(\Gamma)$ *is convex graph* $\iff \forall_{\theta}, \theta', x \in \Gamma(\theta), x' \in \Gamma\left(\theta'\right)$ *it holds that* $\lambda x + (1 - \lambda)x' \in \Gamma\left(\theta\lambda + (1-\lambda)\theta'\right) \forall_{x \in [0,1]}$

**Lemma 19.** $\Gamma : \Theta \rightrightarrows X$ *has closed* graph $\Rightarrow$ *it is closed valued. If* $X$ *is compact, than* $\Gamma$ *is also compact valued.*

**Definition 37.** ***(Upper Hemi-Continuity)*** *Let* $\Gamma : \Theta \rightrightarrows X$ *be a correspondence.*

- $\Gamma$ *is said to be* ***upper hemi-continuous (uhc)*** *at a point* $\theta \in \Theta$ *if and only if for all open sets* $V \subseteq X$ *such that* $\Gamma(\theta) \subseteq V$*, there exists an open set* $U \subseteq \Theta$ *such that* $\theta \in U$ *and for all* $\theta' \in U$ *it holds that* $\Gamma\left(\theta'\right) \subseteq V$

- *A compact valued correspondence* $\Gamma : \Theta \rightrightarrows X$ *is u.h.c. at* $\theta \in \Theta$ *if and only if for every* $\{\theta_n\} \subset \Theta$ *such that* $\theta_n \to \theta$ *and every sequence* $\{x_n\} \subset X$ *such that* $x_n \in \Gamma\left(\theta_n\right)$ *there exits a convergent subsequence* $\{x_{n_k}\}$ *such that* $x_{n_k} \to x \in \Gamma(\theta)$

$$\forall_{\theta_n \to \theta} \forall_{x_n \in \Gamma(\theta_n)} \exists_{\{x_{n_k}\}} x_{n_k} \to x \in \Gamma(\theta)$$

**Definition 38.** ***(Lower Hemi-Continuity)***. *Let $\Gamma : \Theta \rightrightarrows X$ be a correspondence.*

- *$\Gamma$ is said to be **lower hemi-continuous (1hc)** at a point $\theta \in \Theta$ if and only if for all open sets $V \subseteq X$ such that $\Gamma(\theta) \cap V \neq \emptyset$, there exists an open set $U \subseteq \Theta$ such that $\theta \in U$ and for all $\theta' \in U$ it holds that $\Gamma(\theta') \cap V \neq \emptyset$*
- *A correspondence $\Gamma : \Theta \rightrightarrows X$ is l.h.c. at $\theta \in \Theta$ if for all $x \in \Gamma(\theta)$ and all sequences $\{\theta_n\} \subset \theta$ such that $\theta_n \to \theta$ there exits a sequence $\{x_n\} \subset X$ such that $x_n \in \Gamma(\theta_n)$ and $x_n \to x$*

$$\forall_{\theta_n \to \theta} \forall_{x \in \Gamma(\theta)} \exists_{x_n \in \Gamma(\theta_n)} x_n \to x$$

**Definition 39.** ***(Continuity)*** *$\Gamma$ is said to be continuous at a point $\theta \in \Theta$ if it is both UHC an LHC.*

**Lemma 20.** ***(u.h.c and Closed graph)*** *Let $\Gamma : \Theta \rightrightarrows X$. If $\Gamma$ is u.h.c, then $\Gamma$ is closed (has a closed graph).*

**Lemma 21.** ***(Closed graph and u.h.c.)*** *Let $\Gamma : \Theta \rightrightarrows X$. If $X$ is compact and $\Gamma$ is closed (has a closed graph), then $\Gamma$ is u.h.c.*

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**Theorem 3.** ***(Berge (1961) of Maximum) Let $\Theta \subseteq \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$, let $f : \Theta \times X \to \mathbb{R}$ be a continuous function and $\Gamma : \Theta \rightrightarrows X$ a nonempty, compact valued, continuous correspondence. Define:***

$$v(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta) \quad G(\theta) = \{x \in \Gamma(\theta) \mid f(x, \theta) = v(\theta)\}$$

***Then***

- ***$v : \Theta \to X$ is continuous***
- ***$G : \Theta \rightrightarrows X$ is nonempty and compact valued, and UHC***

*Proof.* The proof is divided in three parts. First it is proven that $G$ is nonempty and compact valued, then that it is u.h.c. and finally that $v$ is continuous.

1. $G$ is nonempty valued and compact valued.

- Let $\theta \in \Theta$, by hypothesis $\Gamma(\theta)$ is compact and nonempty. since $f(\cdot, \theta)$ is continuous a maximum is attained on $\Gamma(\theta)$ by the extreme value theorem (Weierstrass). This proves that $G(\theta)$ is nonempty for arbitrary $\theta$.
- Let $\theta \in \Theta$, by hypothesis $\Gamma(\theta)$ is compact and nonempty. since $G(\theta) \subseteq \Gamma(\theta)$ it follows that $G(\theta)$ is bounded, it is left to show closedness to establish compactness. Let $x_n \rightarrow x$ and $x_n \in G(\theta)$ for all $n$. Clearly $x_n \in \Gamma(\theta)$ for all $n$, since $\Gamma$ is closed valued it follows that $x \in \Gamma(\theta)$, so its feasible. By definition of $G$ we have $v(\theta) = f(x_n, \theta)$ for all $n$, since $f$ is continuous we get $v(\theta) = \lim f(x_n, \theta) = f(x, \theta)$, then by definition $x \in G(\theta)$, which proves closedness.

2. $G$ is u.h.c. Consider $\theta \in \Theta$, a sequence in $\Theta$ such that $\theta_n \rightarrow \theta$ and a sequence in $X$ such that $x_n \in G(\theta_n)$ for all $n$. Note that $x_n \in \Gamma(\theta_n)$. since $\Gamma$ is u.h.c. there exists a subsequence $x_{n_k} \rightarrow x \in \Gamma(\theta)$ Now consider $z \in \Gamma(\theta)$. since $\Gamma$ is l.h.c. there exists a sequence in $X$ such that $z_n \in \Gamma(\theta_n)$ and $z_n \rightarrow z$. In particular the subsequence $\{z_{n_k}\}$ also converges to $z$ since $x_n \in G(\theta_n)$ and $z_n \in \Gamma(\theta_n)$ it follows that $f(x_n, \theta_n) \geq f(z_n, \theta_n)$. since $f$ is continuous in both arguments we get by taking limits: $f(x, \theta) \geq f(z, \theta)$. since the inequality holds for arbitrary $z \in \Gamma(\theta)$ we get the result: $x \in G(\theta)$. This proves u.h.c.

3. $v$ is continuous. Let $\theta \in \Theta$ and $\theta_n \rightarrow \theta$ an arbitrary sequence converging to $\theta$. Consider an arbitrary sequence in $X$ such that $x_n \in G(\theta_n)$ for all $n$. Let $\bar{v} = \limsup v(\theta_n)$. By proposition 2.9 there is a subsequence $\{\theta_{n_k}\}$ such that $v(\theta_{n_k}) \rightarrow \bar{v}$. since $G$ is u.h.c. there exists a subsequence of $\{x_{n_k}\}$ ( call it $\{x_{n_{kl}}\}$) converging to a point $x \in G(\theta)$. Then

$$\bar{v} = \lim v\left(\theta_{k_l}\right) = \lim f\left(x_{k_l}, \theta_{k_l}\right) = f(x, \theta) = v(\theta)$$

where the second equality follows from $x_{k_l} \in G\left(\theta_{k_l}\right)$, the third one from $f$ being continuous and the final one from $x \in G(\theta)$. Let $\underline{v} = \liminf v(\theta_n)$ and by a similar argument we get $v(\theta) = \underline{v}$ since $v(\theta) = \liminf v(\theta_n) = \limsup v(\theta_n)$ we get $v(\theta) = \lim v(\theta_n)$ for arbitrary $\{\theta_n\}$ converging to $\theta$. This proves continuity.

□

**Theorem 4.** *(ToM under convexity) Let $\Theta \subseteq \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$, let $f : \Theta \times X \to \mathbb{R}$ be a continuous function and $\Gamma : \Theta \Rightarrow X$ a nonempty, compact valued, continuous correspondence. Define:*

$$v(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta) \quad G(\theta) = \{x \in \Gamma(\theta) \mid f(x, \theta) = v(\theta)\}$$

*a If $f(\cdot, \theta)$ is concave in $x$ for all $\theta$ and $\Gamma$ is convex valued then $G$ is convex valued.*

*b If $f(\cdot, \theta)$ is strictly concave in $x$ for all $\theta$ and $\Gamma$ is convex valued then $G$ is single valued, hence a continuous function.*

*c If $f$ is concave on $\Theta \times X$ and $\Gamma$ has a convex graph then $v$ is concave and $G$ is convex valued.*

*d If $f$ is strictly concave on $\Theta \times X$ and $\Gamma$ has a convex graph then $v$ is strictly concave and $G$ is single valued, hence a continuous function.*

**Theorem 5.** *Kakutani's Fixed Point Theorem – u.h.c. correspondence*

*Let $S \subset \mathbb{R}^n$ be nonempty, compact, and convex, and $\Gamma : S \rightrightarrows S$ be a nonempty, convex-valued, and u.h.c. correspondence.*
*Then $\Gamma$ has a fixed point in S, i.e. $\exists x^* \in S : x^* \in \Gamma(x^*)$*

Since $S$ is compact, u.h.c. is equivalent to $\Gamma$ having a closed graph.

**Example 4.** *Under standard assumptions, prove the following properties of $\mathrm{BR}^i_{A^i}(s)$ :*

*i) non-empty valued,*

*ii) compact valued,*

*iii) upper hemi continuous.*

*iv) Is it convex-valued?*

**Example 5.** *Under standard assumptions, prove the following properties of $\mathrm{BR}_i(s)$ :*

*i) non-empty valued,*

*ii) compact valued,*

*iii) upper hemi continuous.*

*iv) Is it convex-valued?*

*Proof.* (i)Take any $s \in S$. Then $BR^i(s) = \arg\max_{r^i \in S^i} u^i\left(r^i, s^{-i}\right)$. Since $u^i\left(\cdot, s^{-i}\right)$ is continuous and $S^i = \Delta\left(A^i\right)$ is compact, by the Weierstrass Theorem $u^i$ achieves a maximum on $S^i$. Hence, $BR^i(s)$ is nonempty. Since $s$ has been arbitrary, $BR^i(\cdot)$ is nonempty-valued.

(ii) Fix $s \in S$ arbitrarily and take any sequence $\left(r^i_m\right) \in BR^i(s)^\infty$ that converges in $S^i$, i.e. $r^i_m \to r^i \in S^i$. By definition we have $u^i\left(r^i_m, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right) \forall t^i \in S^i, m \in \mathbb{N}$. Then since $u^i\left(\cdot, s^{-i}\right)$ is continuous,

$$u^i\left(r^i, s^{-i}\right) = u^i\left(\lim_{m\to\infty} r^i_m, s^{-i}\right) = \lim_{m\to\infty} u^i\left(r^i_m, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right) \quad \forall t^i \in S^i$$

Hence, $r^i \in BR^i(s)$. Since $s$ has been arbitrary, $BR^i(\cdot)$ is closed-valued.

(iii) Since $S^i$ (the range of $BR^i(\cdot)$) is compact, it is sufficient to establish that $BR^i(\cdot)$ has a closed graph. Fix $s \in S$ arbitrarily and take any sequences $(s_m) \in S^\infty$ and $\left(r^i_m\right) \in S^{i\infty}$ with $s_m \to s \in S, r^i_m \to r^i \in S^i$ and $r^i_m \in BR^i\left(s_m\right) \forall m \in \mathbb{N}$. Then $u^i\left(r^i_m, s^{-i}_m\right) \geq u^i\left(t^i, s^{-i}_m\right), \forall t^i \in S^i$. Since $u^i(\cdot, \cdot)$ is continuous it follows that $\forall t^i \in S^i$

$$\begin{aligned} u^i\left(r^i, s^{-i}\right) = u^i\left(\lim_{m\to\infty} r^i_m, \lim_{m\to\infty} s^{-i}_m\right) &= \lim_{m\to\infty} u^i\left(r^i_m, s^{-i}_m\right) \\ &\geq \lim_{m\to\infty} u^i\left(t^i, s^{-i}_m\right) \\ &= u^i\left(t^i, \lim_{m\to\infty} s^{-i}_m\right) \\ &= u^i\left(t^i, s^{-i}\right) \end{aligned}$$

Hence, $r^i \in BR^i(s)$ and $BR^i(\cdot)$ is closed at $s$. Since $s$ has been arbitrary, $BR^i(\cdot)$ has a closed graph.

(iv) Fix $s \in S$ arbitrarily and take any $r^i_a, r^i_b \in BR^i(s)$ and $\lambda \in (0,1)$. Then it must be that $u^i\left(r^i_a, s^{-i}\right) = u^i\left(r^i_b, s^{-i}\right) \geq u^i\left(r^i, s^{-i}\right) \forall r^i \in S^i$. Or, equivalently,

$$\sum_{a^i \in A^i} r^i_a\left(a^i\right) u^i\left(a^i, s^{-i}\right) = \sum_{a^i \in A^i} r^i_b\left(a^i\right) u^i\left(a^i, s^{-i}\right) \geq \sum_{a^i \in A^i} r^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) \quad \forall r^i \in S^i$$

Now consider the mixed strategy $\lambda r_a^i + (1-\lambda) r_b^i$. The utility of this strategy profile is

$$\begin{aligned}
u^i\left[\lambda r_a^i + (1-\lambda) r_b^i, s^{-i}\right] &= \sum_{a^i \in A^i} \left[\lambda r_a^i\left(a^i\right) + (1-\lambda) r_b^i\left(a^i\right)\right] u^i\left(a^i, s^{-i}\right) \\
&= \lambda \sum_{a^i \in A^i} r_a^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) + (1-\lambda) \sum_{a^i \in A^i} r_b^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) \\
&= \sum_{a^i \in A^i} r_a^i\left(a^i\right) u^i\left(a^i, s^{-i}\right) \\
&\geq u^i\left(r^i, s^{-i}\right) \quad \forall r^i \in S^i,
\end{aligned}$$

where the third line follows from (2) and the inequality holds since $r_a^i \in BR^i(s)$. Hence, $\lambda r_a^i + (1-\lambda) r_b^i \in BR^i(s)$ and, since $s$ has been arbitrary, $BR^i(\cdot)$ is convex-valued. □

**Lemma 22** (Properties of Best Response Correspondence). *$BR^i : S \rightrightarrows S^i$ is nonempty-valued, compact-valued, convex-valued, and upper hemicontinuous.*

*Proof.* Assume $A^i$ is nonempty and finite. Then recall $BR^i$ is the argmax of the problem (for a given $s^{-i}$)

$$\max_{s^i \in S^i} u^i\left(s^i, s^{-i}\right)$$

then by Berge theorem we have that $BR^i : S \rightrightarrows S^i$ is nonempty-valued, compact-valued, convex-valued, and upper hemicontinuous. □

---

**Theorem 6** (Existence of Nash Equilibrium 1950). ***The correspondence*** $BR : S \rightrightarrows S$ ***defined by*** $BR(s) = X_{i \in I} BR^i(s)$ ***is***

***(1) nonempty-valued***

***(2) closed-valued***

***(3) convex-valued***

***(4) upper hemicontinous. Thus by Kakutani fixed point theorem it has fixed point*** $s \in BR(s)$.

*Proof.* Fix $s = \left(s^1, s^2, \ldots, s^n\right) \in S$ arbitrarily.

(1) $BR$ maps $s$ into the set $BR^1(s) \times BR^2(s) \times \cdots \times BR^n(s)$. Since each $BR^i(s), i \in I$, is nonempty and $I$ is finite, we can choose an element $r^i \in BR^i(s)$ for each $i \in I$. Then $\left(r^1, r^2, \ldots, r^n\right) \in BR^1(s) \times \cdots \times BR^n(s) = BR(s)$. Then, since $s$ has been arbitrary,

$BR(s)$ is nonempty for all $s \in S$. Hence, $BR$ is nonempty-valued.

(2) Take any $r_a, r_b \in BR(s)$ and $\lambda \in (0,1)$. Then

$$\lambda r_a + (1-\lambda) r_b = \left(\lambda r_a^1 + (1-\lambda) r_b^1, \ldots, \lambda r_a^n + (1-\lambda) r_b^n\right)$$

Since for each $i \in I$ the set $BR^i(s)$ is convex, $\lambda r_a^i + (1-\lambda) r_b^i \in BR^i(s) \forall i \in I$. Then $\lambda r_a + (1-\lambda) r_b \in BR(s)$ and, hence, $BR(s)$ is a convex set for all $s \in S$, i.e., $BR$ is convex- valued.

(3) Take any point $v = (v^1, \ldots, v^n) \notin BR(s)$. Then for some $i \in I, v^i \notin BR^i(s)$. Since $BR^i(S)$ is closed in $S^i, v^i$ is not a limit point of $BR^i(s)$. That is, there exists an open set [1]$U^i \subset S^i$ containing $v^i$ that contains no more than a finite number of points of $BR^i(s)$. Now, $\forall j \neq i$, choose any $U^j \subset S^j$. Then the neighborhood $U = X_{i \in I} U^i$ of $v$ contains no more than a finite number of points of $BR(s)$, i.e. $v$ is not a limit point of $BR(s)$. Since $v$ has been arbitrary, for all $v \notin BR(s) v$ is not a limit point of $BR(s)$, which implies that $BR(s)$ contains all of its limit points and is, hence, closed in $S$.
Since $S^i \subset \mathbb{R}_+^{m_i}, \forall i \in I$, where $m_i$ is the cardinality of $A^i$, I consider each $S^i$ as a metric subspace of $\mathbb{R}^{m_i}$ with the Euclidean metric. Then $S = X_{i \in I} S^i$ is considered as a metric subspace with the usual product metric.

(4) Take any sequences $(s_m), (r_m) \in S^\infty$ such that $s_m \to s$ and $r_m \in BR(s_m) \forall m$.[2] Then for all $i \in I$, $(s_m^i), (r_m^i) \in S^{i\infty}, s_m^i \to s^i$, and $r_m^i \in BR^i(s_m) \forall m$. Since $BR^i$ is u.h.c., this implies that there exists a subsequence $r_{m_k}^i \to r^i \in BR^i(s)$. Then the sequence $r_{m_k} = \left(r_{m_k}^1, \ldots, r_{m_k}^n\right)$ of $r_m$ converges to $r = (r^1, \ldots, r^n) \in BR^1(s) \times \cdots \times BR^n(s)$. Hence, $BR$ is upper hemicontinuous □

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Now let's characterize Nash equilibria in terms of inequalities. We will use lemma 13.

**Theorem 7** (NE inequality). ***A mixed strategy*** $s \in S$ ***is a*** N.E. ***if, and only if***

$$\forall i \in I, a, b \in A^i \quad u^i\left(a, s^{-i}\right) < u^i\left(b, s^{-i}\right) \quad \textbf{\textit{implies}} \quad s^i(a) = 0$$

*Proof.* $\Longrightarrow$ Suppose $s \in S$ is a N.E. but the implication is false. Then $\exists i \in I, a^i, b^i \in A^i$ with $u^i\left(a^i, s^{-i}\right) < u^i\left(b^i, s^{-i}\right)$ but $s^i\left(a^i\right) > 0$. Then we have

$$u^i\left(a^i, s^{-i}\right) = \sum_{a^{-i}\in A^{-i}}\left(\prod_{j\in I\setminus\{i\}} s^j\left(a^j\right)\right) u^i\left(a^i, a^{-i}\right) < \sum_{a^{-i}\in A^{-i}}\left(\prod_{j\in I\setminus\{i\}} s^j\left(a^j\right)\right) u^i\left(b^i, a^{-i}\right) =$$

$$= u^i\left(b^i, s^{-i}\right)$$

Define $A_a, A_b \subset A$ by $A_a := \cup\left\{\left(a^i, a^{-i}\right) : a^{-i} \in A^{-i}\right\}$ and $A_b := \left\{\left(b^i, a^{-i}\right) : a^{-i} \in A^{-i}\right\}$. Also, define the mixed strategy $\hat{s}^i \in S^i$ by $\hat{s}^i\left(a^i\right) = 0, \hat{s}^i\left(b^i\right) = s^i\left(b^i\right) + s^i\left(a^i\right)$ and $\hat{s}^i\left(c^i\right) = s^i\left(c^i\right) \forall c^i \neq a^i, b^i$ Then

$$\begin{aligned}
u^i\left(s^i, s^{-i}\right) &= \sum_{a\in A}\left(\prod_{j\in I} s^j\left(a^j\right)\right) u^i(a) \\
&= \sum_{a\in A_a}\left(\prod_{j\in I} s^j\left(a^j\right)\right) u^i\left(a^i, a^{-i}\right) + \\
&+ \sum_{a\in A_b}\left(\prod_{j\in I} s^j\left(a^j\right)\right) u^i\left(b^i, a^{-i}\right) + \sum_{a\in A\setminus(A_a\cup A_b)}\left(\prod_{j\in I} s^j\left(a^j\right)\right) u^i(a) = \\
&= \sum_{a\in A_b}\left(s^i\left(a^i\right)\prod_{j\in I\setminus\{i\}} s^j\left(a^j\right)\right) u^i\left(a^i, a^{-i}\right) + \sum_{a\in A_b}\left(s^i\left(b^i\right)\prod_{j\in I\setminus\{i\}} s^j\left(a^j\right)\right) u^i\left(b^i, a^{-i}\right) \\
&+ \sum_{a\in A_b}\left(\hat{s}^i\left(a^i\right)\prod_{j\in I\setminus\{i\}} s^j\left(a^j\right)\right) u^i\left(a^i, a^{-i}\right) + \sum_{a\in A_b}\left(\hat{s}^i\left(b^i\right)\prod_{j\in I\setminus\{i\}} s^j\left(a^j\right)\right) u^i\left(b^i, a^{-i}\right) \\
&= \sum_{a\in A_b}\left(\hat{s}^i\left(b^i\right)\prod_{j\in I\setminus\{i\}} s^j\left(a^j\right)\right) u^i\left(b^i, a^{-i}\right) + \sum_{a\in A\setminus(A_a\cup A_b)}\left(\hat{s}^i\left(a^i\right)\prod_{j\in I} s^j\left(a^j\right)\right) u^i(a) \\
&= u^i\left(\hat{s}^i, s^{-i}\right)
\end{aligned}$$

where the inequality holds from above and the fact that $\hat{s}^i\left(a^i\right) < \hat{s}^i\left(b^i\right)$. But then $s^i \notin BR^i(s)$, which implies $s \notin NE$ ↯.

$\Longleftarrow$ Suppose not. $\exists i, t^i : u(s^i.s^{-i}) < u(t^i, s^{-i})$ observe that it is equivalent to

$$\sum_a^i s^i(a)u(a, s^{-i}) < \sum_a^i t^i(b)u(b, s^{-i}) \leq \sum_a^i u(b, s^{-i}) = \sum_a^i s^i(a)u(b, s^{-i})$$

$$\sum_a^i s^i(a)[u(a, s^{-i}) - u(b, s^{-i})] < 0$$

where it is sum of non negative numbers (if $u(a, s^{-i}) - u(b, s^{-i})$ then $s^i(a) =$). ↯. □

## 1.8 Zero sum games

**Definition 40.** *A two players finite action normal form game is zero sum if the sum of the utilities of the two players is equal to 0 for any action profile, so $u^1 = -u^2$.*

**Theorem 8** (Minimax- von Neumann 1928)**.** ***For any 2-player zero-sum game,***

$$\min_{\alpha^2\in\Delta(A^2)}\max_{\alpha^1\in\Delta(A^1)} u\left(\alpha^1,\alpha^2\right) = \max_{\alpha^1\in\Delta(A^1)}\min_{\alpha^2\in\Delta(A^2)} u\left(\alpha^1,\alpha^2\right) \equiv v$$

*Proof.* We will do it in two steps: First we will prove that $\geq$ holds. Secondly that $\leq$ holds.

$\geq$. Note that for any $\bar{s}^1 \in \Delta\left(A^1\right)$ and $\bar{s}^2 \in \Delta\left(A^2\right)$ it holds that:

$$u\left(\bar{s}^1,\bar{s}^2\right) \geq \min_{s^2\in\Delta(A^2)} u\left(\bar{s}^1,s^2\right)$$

Then by taking maximum at both sides with respect to $s^1$ :

$$\max_{s^1\in\Delta(A^1)} u\left(s^1,\bar{s}^2\right) \geq \max_{s^1\in\Delta(A^1)}\min_{s^2\in\Delta(A^2)} u\left(s^1,s^2\right)$$

Note that the RHS is now constant, and a lower bound to the LHS across $s^2$, then:

$$\min_{s^2\in\Delta(A^2)}\max_{s^1\in\Delta(A^1)} u\left(s^1,s^2\right) \geq \max_{s^1\in\Delta(A^1)}\min_{s^2\in\Delta(A^2)} u\left(s^1,s^2\right) \tag{1}$$

$\leq$. Note that for any $\bar{s}^1 \in \Delta\left(A^1\right)$ it holds that:

$$\max_{s^1\in\Delta(A^1)}\min_{s^2\in\Delta(A^2)} u\left(s^1,s^2\right) \geq \min_{s^2\in\Delta(A^2)} u\left(\bar{s}^1,s^2\right)$$

In particular for $\hat{s}^1$ a NE of the game the inequality must hold. We assume that such NE exists in mixed strategies. Note that if $(\hat{s}^1,\hat{s}^2)$ it is defined as an strategy profile such that:

$$u\left(\hat{s}^1,\hat{s}^2\right) = \max_{s^1\in\Delta(A^1)} u\left(s^1,\hat{s}^2\right) \qquad -u\left(\hat{s}^1,\hat{s}^2\right) = \max_{s^2\in\Delta(A^2)} -u\left(\hat{s}^1,s^2\right)$$

The second condition implies:

$$u\left(\hat{s}^1,\hat{s}^2\right) = \min_{s^2\in\Delta(A^2)} u\left(\hat{s}^1,s^2\right) = \max_{s^1\in\Delta(A^1)} u\left(s^1,\hat{s}^2\right)$$

thus

$$\begin{aligned}
\min_{s^2\in\Delta(A^2)} u^1\left(\hat{s}^1,s^2\right) &= u^1\left(\hat{s}^1, \underset{s^2\in\Delta(A^2)}{\operatorname{argmin}} u^1\left(\hat{s}^1,s^2\right)\right)\\
&= u^1\left(\hat{s}^1, \underset{s^2\in\Delta(A^2)}{\operatorname{argmax}} u^2\left(\hat{s}^1,s^2\right)\right)\\
&= u^1\left(\hat{s}^1,\hat{s}^2\right)\\
&= \max_{s^1\in\Delta\left(A^1\right)} u^1\left(s^1,\hat{s}^2\right)\\
&\geq \min_{s^2\in\Delta(A^2)} \max_{s^1\Delta\left(A^1\right)} u^1\left(s^1,s^2\right)
\end{aligned}$$

Then by taking max over $\Delta\left(A^1\right)$:

$$\max_{s^1\in\Delta\left(A^1\right)} \min_{s^2\in\Delta(A^2)} u\left(s^1,s^2\right) \geq \min_{s^1\in\Delta\left(A^1\right)} u\left(s^1,\hat{s}^2\right) \geq \min_{s^2\in\Delta(A^2)} \max_{s^1\in\Delta\left(A^1\right)} u\left(s^1,s^2\right) \tag{2}$$

Inequalities (1) and (2) gives us thesis of minimax theorem. □

**Definition 41.** *For a zero sum game of two players define the value of the game as* $V : \mathbb{R}^{nm} \to \mathbb{R}$ *(where* $n = \#A^1$ *and* $m = \#A^2$*) :*

$$V(u) = \max_{s^1\in\Delta\left(A^1\right)} \min_{s^2\in\Delta(A^2)} U\left(s^1,s^2 \mid u\right)$$

*where for a given strategy profile* $s^1 = (p_1,\ldots,p_n), s^2 = (q_1,\ldots,q_n)$ *and payoffs* $u \in \mathbb{R}^{nm}$ *we define*

$$U\left(s^1,s^2 \mid u\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j u_{ij}$$

---

**Example 6.** *Show that* ***The value of a game*** *is*

*a) continuous*

*b) non-decreasing*

*c) homogenous of degree one in payoffs.*

- Consider the problem:

$$v\left(s^1,u\right) = \min_{s^2\in\Delta(A^2)} U\left(s^1,s^2 \mid u\right)$$

note that $U$ is continuous in $s_1, s_2$ and $u$ and that the minimum is being taken over $s^2$ in a compact set that does not vary with $s^1$ or $u$. By the theorem of the maximum the value of this problem, as a function of $s^1$ and $u$ is a continuous function. Now consider:

$$V(u) = \max_{s^1 \in \Delta\left(A^1\right)} \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u\right) = \max_{s^1 \in \Delta\left(A^1\right)} v\left(s^1, u\right)$$

again since $v$ is continuous and $s^1$ varies in a compact set independent of $u$ by the theorem of the maximum $V$ is a continuous function of $u$.

- Let $u_1 \leq u_2$. Clearly for all $s^1, s^2$:

$$U\left(s^1, s^2 \mid u_1\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j u_{ij}^1 \leq U\left(s^1, s^2 \mid u_1\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j u_{ij}^2$$

so $U\left(s^1, s^2 \mid u_1\right) \leq U\left(s^1, s^2 \mid u_2\right)$. Then:

$$\begin{gathered} \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u_1\right) \leq \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u_2\right) \\ V\left(u_1\right) = \max_{s^1 \in \Delta\left(A^1\right)} \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u_1\right) \leq \\ \leq \max_{s^1 \in \Delta\left(A^1\right)} \min_{s^2 \in \Delta(A^2)} U\left(s^1, s^2 \mid u_2\right) = V\left(u_2\right) \end{gathered}$$

- Let $\lambda \in \mathbb{R}$, note that $U\left(s^1, s^2 \mid \lambda u\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j \lambda u_{ij} = \lambda U\left(s^1, s^2 \mid u\right)$ and $\max_x \lambda f(x) = \lambda \max_x f(x)$. Thus $V(\lambda u) = \lambda V(u)$

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## 1.9 Dominance

**Definition 42** (Weak dominance). *An action $a^i \in A^i$ is weakly dominated if $\exists s^i \in \Delta\left(A^i\right)$ such that:*

$$\forall b^{-i} \in A^{-i}, \quad u^i\left(s^i, b^{-i}\right) \geq u^i\left(a^i, b^{-i}\right)$$
$$\exists c^{-i} \in A^{-i}, \quad u^i\left(s^i, c^{-i}\right) > u^i\left(a^i, c^{-i}\right)$$

**Definition 43** (Strict dominance). *An action $a^i \in A^i$ is strictly dominated if $\exists s^i \in \Delta\left(A^i\right)$ such that:*

$$\forall b^{-i} \in A^{-i}, u^i\left(s^i, b^{-i}\right) > u^i\left(a^i, b^{-i}\right)$$

**Definition 44** (Weakly undominated). *A strategy profile $s \in S$ is weakly undominated if and only if $\forall i \in I, s^i$ isn't weakly dominated.*

**Definition 45** (Strictly undominated). *A strategy profile $s \in S$ is strictly undominated if and only if $\forall i \in I, s^i$ isn't strictly dominated*

**Example 7.** *Consider following game*

Mr 2

| | | L | R |
|---|---|---|---|
| | T | 3,0 | 0,1 |
| Mr 1 | M | 0,0 | 3,1 |
| | B | 1,1 | 1,0 |

In this game, $A_1^1 = \{T, M, B\}$ and $A_1^2 = \{L, R\}$. No (pure) strategy dominates any other (pure) strategy for both players. However, the mixed strategy $s_1(T) = s_1(M) = \frac{1}{2}$ and $s_1(B) = 0$ strictly dominates $B$ since $\forall q \in [0,1]$

$$u_1\left(s_1, q\right) = 3q\frac{1}{2} + 3(1-q)\frac{1}{2} = \frac{3}{2} > 1 = u_1(B, q)$$

**Definition 46** (Belief). *We call $\mu^{-i}$ player $i$ 's belief if and only if $\mu^{-i} \in \Delta\left(A^{-i}\right)$.*

[ Note: $u^i\left(a^i, \mu\right) = \sum_{a^{-i} \in A^{-i}} \mu\left(a^{-i}\right) u^i\left(a^i, a^{-i}\right)$]

**Definition 47** (Never a best response). *An action $a^i \in A^i$ is never a best response if $\nexists \mu \in \Delta\left(A^{-i}\right)$ such that $a^i \in BR^i_{\Delta i}(\mu)$.*

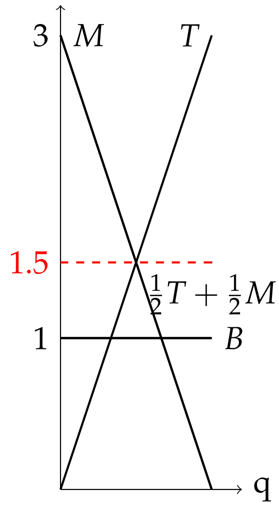


Figure 2: Dominant strategies: mixed strategies

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**Example 8.** *Consider following game*

|  |  | Mr 2 |  |
|---|---|---|---|
|  |  | L | R |
|  | T | 3,0 | 0,1 |
| Mr 1 | M | 0,0 | 3,1 |
|  | B | $x, -x$ | $x, -x-1$ |

If $x > 0$, then no action dominates $B$. Suppose $\mu = \left(\frac{1}{2}, \frac{1}{2}\right) \in \Delta(\{L, R\})$. Then if $x < \frac{3}{2}, B$ is never *a* best response. If, $x \geq \frac{3}{2}$, then $B$ is sometimes a best response, depending on $q$. If $x = \frac{3}{2}$, then $B \in BR^i_{A^i}((1/2, 1/2))$

Let $s^i = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$. Then $s^i$ gives $\frac{3}{2}$ to player $i = 1$ for all $a^{-i} \in A^{-i}$. Suppose now that $x < \frac{3}{2}$. Then

1. $\nexists \mu \in \Delta\left(A^{-i}\right)$ s.t. $B \in BR^i_{A^i}(\mu)$
2. $\exists s^i \in \Delta\left(A^i\right)$ s.t. $\forall a^{-i} \in A^{-i}, u^i\left(s^i, a^{-i}\right) \geq u^i\left(a, a^{-i}\right)$

Conversely, if $x \geq \frac{3}{2}$, then

1. $\exists \mu \in \Delta\left(A^{-i}\right)$ s.t. $B \in BR^i_{A^i}(\mu)$

2. $\nexists s^i \in \Delta\left(A^i\right)$ s.t. $\forall a^{-i} \in A^{-i}, u^i\left(s^i, a^{-i}\right) \geq u^i\left(a, a^{-i}\right)$

---

More generally, we have the following definitions and theorem.

**Theorem 9.** *The following three statements are equivalent:*

$$\begin{aligned} u^i\left(s^i, a^{-i}\right) &> u^i\left(a^i, a^{-i}\right) & \forall a^{-i} \in A^{-i} \\ u^i\left(s^i, s^{-i}\right) &> u^i\left(a^i, s^{-i}\right) & \forall s^{-i} \in S^{-i} \\ u^i\left(s^i, \mu^{-i}\right) &> u^i\left(a^i, \mu^{-i}\right) & \forall \mu^{-i} \in \Delta\left(A^{-i}\right) \end{aligned}$$

*Proof.* $(1) \implies (3)$ :

$$u^i\left(s^i, \mu^{-i}\right) - u^i\left(a^i, \mu^{-i}\right) = \sum_{a^{-i} \in A^{-i}} \mu\left(a^{-i}\right)\left[u^i\left(s^i, a^{-i}\right) - u^i\left(a^i, a^{-i}\right)\right],$$

and the first term is greater than or equal to zero and the second is strictly greater than 0 by hypothesis. Thus the difference is strictly greater than 0 .

(2)$\implies$ (1) Since $A^{-i} \subseteq S^{-i} \equiv \prod_{j \neq i} \Delta\left(A^j\right)$ , the result is immediate

(3)$\implies$ (2) Since $S^{-i} \equiv \prod_{j \neq i} \Delta\left(A^j\right) \subseteq \Delta(A^{-i})$ and thus the result follows immediately. □

**Theorem 10** (Theorem of the Alternative). .

$$\exists x \textbf{ s.t.} \left\{ \begin{array}{l} Ax \gg a \\ Bx \geq b \\ Cx = c \end{array} \right\} \iff \nexists \mu \geq 0, \lambda \geq 0, \nu \textbf{ s.t.} \left\{ \begin{array}{c} \mu A + \lambda B + \nu c = 0 \\ \mu a + \lambda b + \nu c \geq 0 \\ \mu(a+c) + \lambda b + \nu c > 0 \end{array} \right\}$$

**Theorem 11.** *A strategy $b^i \in A^i$ is strictly dominated $\iff$ it is never a best response.*

*Proof.* $\Rightarrow$ Define

$$U := \begin{bmatrix} u^i\left(a_1^i, a_1^{-i}\right) & \cdots & u^i\left(a_{\#A^i}^i, a_1^{-i}\right) \\ \vdots & \ddots & \vdots \\ u^i\left(a_1^i, a_{\#A^{-i}}^{-i}\right) & \cdots & u^i\left(a_{\#A^i}^i, a_{\#A^{-i}}^{-i}\right) \end{bmatrix}$$

Take any $b^i \in A^i$ and define

$$u := \begin{bmatrix} u^i\left(b^i, a_1^{-i}\right) \\ \vdots \\ u^i\left(b^i, a_{\#A^{-i}}^{-i}\right) \end{bmatrix}$$

So $b^i$ is never a best response if $\nexists \mu = [\mu_1, \ldots, \mu_{\#A^{-i}}]^T \in \Delta\left(A^{-i}\right)$ such that $\mu^T U \leq \mu^T u e^T$, i.e.

$$\begin{bmatrix} u^i\left(a^i, \mu\right) \\ \vdots \\ u^i\left(a^i_{\#A^i}, \mu\right) \end{bmatrix} \leq \begin{bmatrix} u^i\left(b^i, \mu\right) \\ \vdots \\ u^i\left(b^i, \mu\right) \end{bmatrix}$$

Moreover, $b^i$ is strictly dominated if $\exists s^i = \left[s^i\left(a^i_1\right), \ldots, s^i\left(a^i_{\#A^i}\right)\right]^T$ such that $Us^i \gg u, Is^i \geq 0$, and $e^T s^i = 1$, where $I$ is the $\#A^i$ dimensional identity matrix. The first condition gives dominance while the second two ensure that $s^i$ is a mixed strategy. Now, suppose $b^i$ is never a best response but is not dominated. Then $\nexists s^i$ such that

$$\left\{ \begin{matrix} Us^i \gg u \\ Is^i \geq 0 \\ e^T s^i = 1 \end{matrix} \right\}$$

Then by the Theorem of the Alternative, $\exists \mu \geq 0, \lambda \geq 0, \nu$ such that

$$\left\{ \begin{matrix} \mu^T U + \lambda I + \nu e^T = 0 \\ \mu^T u + \lambda \cdot 0 + \nu \cdot 1 \geq 0 \\ \mu^T (u+1) + \lambda \cdot 0 + \nu \cdot 1 > 0 \end{matrix} \right\}$$

Notice that if $\mu = 0$ then $\lambda I + \nu e^T = 0$ and $\nu > 0$, which $\lightning \lambda \geq 0$. So $\mu \geq 0, \mu \neq 0$. Now, normalize $\mu, \lambda$ and $\nu$ so that $\mu \in \Delta\left(A^{-i}\right.$. Then (8) reduces to

$$\mu^T U + \nu e^T \leq 0 \quad \text{and} \quad \mu^T u + \nu \geq 0$$

which implies

$$\mu^T U \leq \mu^T u e^T$$

which $\lightning b^i$ as a never best response.

$\Leftarrow$

If $a^i$ is strictly dominated then $\exists s^i \in S^i$ such that $\forall a^{-i} \in A^{-i}, u^i\left(s^i, a^{-i}\right) > u^i\left(a^i, a^{-i}\right)$. Take any $\mu \in \Delta\left(A^{-i}\right)$. Then

$$u^i\left(s^i, a^{-i}\right) \mu\left(a^{-i}\right) \geq u^i\left(a^i, a^{-i}\right) \mu\left(a^{-i}\right) \forall a^{-i}$$

with strict inequality if $\mu\left(a^{-i}\right) > 0$. Since $\mu\left(a^{-i}\right) > 0$ for some $a^{-i}$

$$\sum_{a^{-i} \in A^{-i}} u^i\left(s^i, a^{-i}\right) \mu\left(a^{-i}\right) > \sum_{a^{-i} \in A^{-i}} u^i\left(a^i, a^{-i}\right) \mu\left(a^{-i}\right)$$

There must be $b^i \in \text{support}\left(s^i\right) \subseteq A^i$ such that

$$\sum_{a^{-i}\in A^{-i}} u^i\left(b^i, a^{-i}\right)\mu\left(a^{-i}\right) \geq \sum_{a^{-i}\in A^{-i}} u^i\left(s^i, a^{-i}\right)\mu\left(a^{-i}\right)$$

Then

$$\sum_{a^{-i}\in A^{-i}} u^i\left(b^i, a^{-i}\right)\mu\left(a^{-i}\right) > \sum_{a^{-i}\in A^{-i}} u^i\left(a^i, a^{-i}\right)\mu\left(a^{-i}\right)$$

so $u^i\left(b^i, \mu\right) > u^i\left(a^i, \mu\right)$. Thus $a^i$ is never a best response. $\square$

Let's see one more encounter with Nash equilibria, this time we will show that they are strictly undominated.

**Corollary 5.** *If $s \in S$ is a* NE *and $a^i \in A^i$ strictly dominated, then $s^i\left(a^i\right) = 0$*

*Proof.* Since $a^i$ is strictly dominated, it is never a best response. Then it must be $s^i\left(a^i\right) = 0$ $\square$

## 1.10 Iterated elimination

**Definition 48** (Iterated elimination of strictly dominated strategies (IESDS))**.** *An IESDS is a sequence $C_t = (C_t^1, \ldots, C_t^i, \ldots, C_t^n)$ for $t = 0, \ldots, T$, where:*

1. $\forall i, \quad C_0^i = A^i$
2. $\forall i \forall t, \quad C_{t+1}^i \subseteq C_t^i$
3. $\forall i \forall a^i \forall t, \quad a^i \in C_t^i \backslash C_{t+1}^i$ *if and only if* $\exists s^i \in \Delta\left(C_t^i\right)$ *such that*

$$\forall b^{-i} \in C_t^{-i}, u^i\left(s^i, b^{-i}\right) > u^i\left(a^i, b^{-i}\right)$$

**Definition 49** (Iterated elimination of weakly dominated strategies (IEWDS))**.** *An IEWDS is a sequence $C_t = (C_t^1, \ldots, C_t^i, \ldots, C_t^n)$ for $t = 0, \ldots, T$, where:*

1. $\forall i, \quad C_0^i = A^i$
2. $\forall i \forall t, \quad C_{t+1}^i \subseteq C_t^i$
3. $\forall i \forall a^i \forall t, \quad a^i \in C_t^i \backslash C_{t+1}^i$ *if and only if* $\exists s^i \in \Delta\left(C_t^i\right)$ *such that*

$$\forall b^{-i} \in C_t^{-i}, u^i\left(s^i, b^{-i}\right) \geq u^i\left(a^i, b^{-i}\right)$$

$$\exists c^{-i} \in C_t^{-i}, u^i\left(s^i, c^{-i}\right) > u^i\left(a^i, c^{-i}\right)$$

---

**Example 9.** *Find all the solutions obtained by IESDS*

| | | Mr 2 | |
|---|---|---|---|
| | | L | R |
| | T | 3,0 | 0,1 |
| Mr 1 | M | 0,0 | 3,1 |
| | B | 1,1 | 1,0 |

In this game, $A_1^1 = \{T, M, B\}$ and $A_1^2 = \{L, R\}$. No (pure) strategy dominates any other (pure) strategy for both players. However, the mixed strategy $s_1(T) = s_1(M) = \frac{1}{2}$ and $s_1(B) = 0$ strictly dominates $B$ since $\forall q \in [0, 1]$

$$u_1(s_1, q) = 3q\frac{1}{2} + 3(1-q)\frac{1}{2} = \frac{3}{2} > 1 = u_1(B, q)$$

So $B$ is eliminated from player 1's set of actions. Given that player 2 knows this, $s^2 = (0,1)DL$.

Thus $L$ is eliminated from player 2's action set. Finally, given that player 2 will only play $R$, $M$ dominates $T$. Thus player 1 will eliminate $T$ as well. This leads to a final action set $C_T = \{M\} \times \{R\}$.

Since each player only has one action now, no more actions can be eliminated. This is referred to as a **complete** IESDA. Note that we have need to allow dominance by mixed strategies for this to work; neither $T$ nor $M$ alone strictly dominates $B$.

---

**Example 10.** *Find all the solutions obtained by IESDS*

Mr 2

| | | L | C | R |
|---|---|---|---|---|
| | T | 4,3 | 5,1 | 6,2 |
| Mr 1 | M | 2,1 | 8,4 | 3,6 |
| | B | 3,0 | 9,5 | 2,6 |

One way to organize our work is put it in table. Observe that

| | |
|---|---|
| $C_1^0 = \{T, M, B\}$ | $C_2^0 = \{L, C, R\}$ |
| $C_1^1 = \{T, M, B\}$ | $C_2^1 = \{L, R\}$ |
| $C_1^2 = \{T\}$ | $C_2^2 = \{L, R\}$ |
| $C_1^3 = \{T\}$ | $C_2^3 = \{L\}$ |
| ... | ... |
| $C_1^\infty = \{T\}$ | $C_2^\infty = \{L\}$ |

So $\{(T, L)\}$ is our final result of IESDS.

---

**Example 11.** *Find all the solutions obtained by IESDS and IEWDS*

|  |  |  | Mr 2 |  |
|---|---|---|---|---|
|  |  | L | C | R |
|  | T | 1,2 | 2,3 | 0,3 |
| Mr 1 | M | 2,2 | 2,1 | 3,2 |
|  | B | 2,1 | 0,0 | 1,0 |

IESDS: nothing to rule out in pure strategies

IEWDS: For Mr1 M weakly dominates T and B and For Mr2 R weakly dominates C. Consider 3 procedures

- Procedure 1: Mr1 eliminates T
- Procedure 2: Mr1 eliminates B
- Procedure 3: Mr2 eliminates R

For example in Procedure 3 we can have following solution. Mr1 can eliminate T or B.
If he eliminates T , Mr2 can eliminate R or B. If we eliminated C, T and R then Mr1 eliminates B and we end up in (M,R).
If we eliminated C, T and B then Mr2 can not eliminate and we end up in (M,(l,r)).

In total we have 4 outcomes

---

We say IESDS is complete if no elimination is possible in the $C_T$ game

Observe that complete IESDS results in a unique outcome which we prove below. It is not true for IEWDS. Let's ilustrate it with example.

**Example 12.**

- Procedure 1: T weakly dominates B:eliminate B then Mr2 is indifferent between L and R so we get $((1,0) \times (q, 1-q))$

|  |  | Mr 2 |  |
|---|---|---|---|
|  |  | L | R |
| Mr 1 | T | 1,2 | 2,2 |
|  | B | 1,2 | 1,1 |

- Procedure 2: L weakly dominates R:eliminate R then Mr1 is indifferent between T and B so we get $((p, 1-p) \times (1, 0))$

---

Furthermore IEWDS can eliminate a NE

**Example 13.**

|  |  | Mr 2 |  |
|---|---|---|---|
|  |  | L | R |
| Mr 1 | T | 1,1 | 0,0 |
|  | B | 0,0 | 0,0 |

Observe that $\{(T, L), (B, R))\}$ are pure NE.

Let's do IEWDS for this game: For Mr2 L weakly dominates R so elimiate R. For Mr1 T weakly dominates B so elimiate B so we elimated our NE.

---

**Theorem 12.** ***For any normal-form game*** $\left\langle I, (A^i)_{i \in I}, (u^i)_{i \in I} \right\rangle$***, the outcome of a complete IESDS is unique.***

*Proof.* Let $(C_t^i)_{i \in I, t=0,1,\ldots,T}^i$ be a complete IESDS. We show by induction that, $\forall i \in I$, if a strategy $c^i$ cannot be eliminated at $C_T^i$, then it cannot be eliminated at $C_t^i$ for any $t \in \{0, 1, \ldots, T-1\}$. Fix $i \in I$ and take any $c^i \in C_T^i$ Basis Step: By definition.

Induction Step: Suppose $c^i \in C_{t+1}^i, t \in \{0, 1, \ldots, T-1\}$, and that it cannot be eliminated at this stage. We want to show that $c^i$ cannot be eliminated earlier. WLOG we can consider $C_t^i = C_{t+1}^i \cup \{d^i\}$. Suppose for $\not\exists$that $c^i$ can be eliminated at $C_t^i$. Then

$$\exists \hat{s}^i \in \Delta\left(C_t^i\right) \quad \text{s.t.} \quad \forall c^{-i} \in C_t^{-i}, \quad u^i\left(\hat{s}^i, c^{-i}\right) > u^i\left(c^i, c^{-i}\right)$$

Define $s^i \in \Delta\left(C_{t+1}^i\right)$ as follows. If $a^i \in C_{t+1}^i$, then let $s^i\left(a^i\right) = \hat{s}^i\left(a^i\right)$. If $d^i \in C_t^i \backslash C_{t+1}^i$, then $d^i$ must have been eliminated at $t$. Then $\exists r^i \in \Delta\left(C_t^i\right)$ that strictly dominates $d^i$.

Notice that since $r^i\left(d^i\right)=0$, we have supp $\left(r^i\right)\subseteq C_t^i$. Then define

$$s^i\left(a^i\right)\equiv\hat{s}^i\left(a^i\right)+r^i\left(a^i\right)\hat{s}^i\left(d^i\right)\quad\forall a^i\in C_{t+1}^i$$

Clearly $s^i\left(a^i\right)\geq 0\forall a^i\in C_{t+1}^i$. Moreover,

$$\begin{aligned}\sum_{a^i\in C_{t+1}^i}s^i\left(a^i\right)&=\sum_{a^i\in C_{t+1}^i}\hat{s}^i\left(a^i\right)+r^i\left(a^i\right)\hat{s}^i\left(d^i\right)\\&=\hat{s}^i\left(C_{t+1}^i\right)+\hat{s}^i\left(d^i\right)\sum_{a^i\in C_{t+1}^i}r^i\left(a^i\right)\\&=\hat{s}^i\left(C_{t+1}^i\right)+\hat{s}^i\left(d^i\right)\\&=1-\hat{s}^i\left(d^i\right)+\hat{s}^i\left(d^i\right)\\&=1\end{aligned}$$

where the second line follows since supp $\left(r^i\right)\subseteq C_{t+1}^i$. So we have that $s^i\in\Delta\left(C_{t+1}^i\right)$. Now, we have that

$$\begin{aligned}&\forall c^{-i}\in C_t^{-i},\quad u^i\left(\hat{s}^i,c^{-i}\right)>u^i\left(c^i,c^{-i}\right)\\\Longrightarrow&\forall c^{-i}\in C_{t+1}^{-i},\quad u^i\left(\hat{s}^i,c^{-i}\right)>u^i\left(c^i,c^{-i}\right)\\\Rightarrow&\forall c^{-i}\in C_{t+1}^{-i},\quad\sum_{a^i\in C_{t+1}^i}\hat{s}^i\left(a^i\right)u^i\left(a^i,c^{-i}\right)>u^i\left(c^i,c^{-i}\right)\\\Longrightarrow&\forall c^{-i}\in C_{t+1}^{-i},\quad\sum_{a^i\in C_{t+1}^i}\left(\hat{s}^i\left(a^i\right)+r^i\left(a^i\right)\hat{s}^i\left(d^i\right)\right)u^i\left(a^i,c^{-i}\right)>u^i\left(c^i,c^{-i}\right)\\\Longrightarrow&\forall c^{-i}\in C_{t+1}^{-i},\quad\sum_{a^i\in C_{t+1}^i}s^i\left(a^i\right)u^i\left(a^i,c^{-i}\right)>u^i\left(c^i,c^{-i}\right)\\\Longrightarrow&\forall c^{-i}\in C_{t+1}^{-i},\quad u^i\left(s^i,c^{-i}\right)>u^i\left(c^i,c^{-i}\right)\end{aligned}$$

where the second line holds since $C_{t+1}^i\subseteq C_t^i$ and the third since $d^i$ is strictly dominated. Then $s^2$ strictly dominates $c^i$ in $C_{t+1}^i$, which is a $\lightning$. Hence $c^i$ cannot be eliminated at $C_t^i$. By induction, $c^i$ cannot be eliminated at any $\left(C_t^i\right)_{t=0,1,\ldots,T}$. Since $i\in I$ has been arbitrary, IESDS is unique. $\square$

The non-empty set resulting from IESDS is called the set of *sophisticated equilibria*. Check H. Moulin 'Dominance Solvable Voting Schemes', Econometrica 47, 1979.

**Example 14.** *This game is called Guess the average*

- *Each player $i \in I$ picks simultaneously an integer $x_i$ between 1 and 999. Hence, $A_i = \{1, \ldots, 999\}$.*

- *Given $x = (x_1, \ldots, x_n) \in \{1, \ldots, 999\}^n$, let*

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

- *The winners are those players whose ballots are closest to $\frac{2}{3}\bar{x}$.*

*Observe that every strategy $x_i > 666$ is dominated by 666. Hence, for every $i \in I, C_i^1 = \{1, \ldots, 666\}$. Now, for every $i \in I, C_i^2 = \{1, \ldots, 444\}$. Proceeding this way, for every $i \in I, C_i^\infty = \{1\}$.*

Let's conclude with final step of IESDS algoirthm, namely property that guarantees that NE are inside set which is result of applying of IESDS.

**Theorem 13.** ***Let** $C \equiv X_{i \in I} C_T^i$ **be the set resulting from complete IESDS and let** $s \in S$ **be a NE. Then,*** $\operatorname{supp}(s) = \{a \in A : \prod_{i \in I} s^i\left(a^i\right) > 0\} \subseteq C$.

*Proof.* The proof is by induction. Basis Step: Since $C_0 = A$, obviously $\operatorname{supp}(s) \subseteq C_0$. Induction Step: Note first that if $a \in \operatorname{supp}(s)$, then $a^i \in \operatorname{supp}\left(s^i\right), \forall i \in I$. Suppose $\operatorname{supp}\,(s) \subset C_k$ but that $\operatorname{supp}(s) \nsubseteq C_{k+1}$. Then $\exists a \in \operatorname{supp}(s)$ such that $a \notin C_{k+1}$, i.e. for some $i \in I, a^i$ is strictly dominated by some $t^i \in \Delta\left(C_k^i\right)$

$$\sum_{a^{-i} \in C_k^{-i}} u^i\left(a^i, a^{-i}\right) s^{-i}\left(a^{-i}\right) < \sum_{a^{-i} \in C_k^{-i}} u^i\left(t^i, a^{-i}\right) s^{-i}\left(a^{-i}\right)$$

where $s^{-i}\left(a^{-i}\right) = \prod_{j \in I \setminus \{i\}} s^j\left(a^j\right)$. Since $s$ is a Nash equilibrium and $\operatorname{supp}(s) \subseteq C_k, s^{-i}\left(a^{-i}\right) = 0$

$\forall a^{-i} \notin C_k^{-i}$ since these strategies were strictly dominated. Then line above is equivalent to

$$\sum_{a^{-i} \in A^{-i}} u^i\left(a^i, a^{-i}\right) s^{-i}\left(a^{-i}\right) < \sum_{a^{-i} \in A^{-i}} u^i\left(t^i, a^{-i}\right) s^{-i}\left(a^{-i}\right)$$

But this contradicts $s^i \in BR^i(s)$ since $a^i \in \operatorname{supp}\left(s^i\right)$, i.e. $s \notin NE$. Hence, $\operatorname{supp}(s) \subseteq C_{k+1}$. Then by Induction, $\operatorname{supp}(s) \subseteq C$. □

## 1.11 Rationalizability

**Definition 50** (Rationalizable sets)**.** *A tuple* $R = (R^1, \ldots, R^n)$ *where* $\forall i R^i \subseteq A^i$*, is rationalizable if and only if* $\forall i, \forall a^i \in R^i, \exists \mu \in \Delta(R^{-i})$ *such that* $\forall b^i \in A^i, u^i(a^i, \mu) \geq u^i(b^i, \mu)$

**Lemma 23.** *If* $R$ *and* $S$ *are two rationalizable sets, then* $R \cup S = (R^1 \cup S^1, \ldots, R^n \cup S^n)$ *is rationalizable as well.*

*Proof.* For any $i \in I$ and any $a^i \in R^i$, since $R^i$ is rationalizable we know $\exists \mu \in \Delta(R^{-i})$ such that $\forall b^i \in A^i, u^i(a^i, \mu) \geq u^i(b^i, \mu)$ .
$\Delta(R^{-i}) \subsetneq \Delta((R \cup S)^{-i})$ Therefore $\exists \mu \in \Delta((R \cup S)^{-i})$ such that $\forall b^i \in A^i, u^i(a^i, \mu) \geq u^i(b^i, \mu)$ , and thus $R \cup S$ is rationalizable. □

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**Example 15.** *Consider following game once again: From previous parts we know that final*

| | $L$ | $R$ |
|---|---|---|
| $T$ | 3,0 | 0,1 |
| $M$ | 0,0 | 3,1 |
| $B$ | 1,1 | 1,0 |

Figure 3: A game that illustrates IESDS

*result of IESDS is following action set* $C_T = \{M\} \times \{R\}$.
*However, from observation we can see that* $(R^1, R^2) = \{(M), (R)\}$ *is not rationalizable.*

---

**Lemma 24.** *There is a unique maximal rationalizable set* $R$*, i.e.* $\nexists S \supset R$ *where* $S$ *is rationalizable.*

*Proof.* Suppose not, i.e. both sets $R$ and $S$ are rationalizable, maximal, and $R \neq S$. Then, by the above lemma, $R \cup S$ is rationalizable as well and $R \cup S \supset R$, which $\lightning R$ being maximal. □

**Theorem 14.** ***Let*** $C_T$ ***be the outcome of a complete IESDS and let*** $R$ ***be the unique maximal rationalizable set. Then*** $R \subseteq C_T$***.***

*Proof.* We proceed by induction on the elimination stages of IESDS. Note in $t = 0, \forall i R^i \subseteq C_0^i \equiv A^i$. From this, assume $\forall i R^i \subseteq C_t^i$. Then $\forall i, \forall a^i \in R^i$ it must be that:

$$\exists \mu \in \Delta\left(R^{-i}\right) \text{ such that } \forall b^i \in A^i, u^i\left(a^i, \mu\right) \geq u^i\left(b^i, \mu\right) \qquad \text{(by definition)}$$

$$\implies \exists \mu \in \Delta\left(C_t^{-i}\right) \text{ such that } \forall b^i \in A^i, u^i\left(a^i,\mu\right) \geq u^i\left(b^i,\mu\right) \qquad \text{(by hypothesis)}$$

$$\begin{aligned} &\implies \exists \mu \in \Delta\left(C_t^{-i}\right) \text{ such that } \forall b^i \in C_t^i, u^i\left(a^i,\mu\right) \geq u^i\left(b^i,\mu\right) \\ &\implies \nexists s^i \in \Delta\left(C_t^i\right) \text{ such that } \forall b^{-i} \in C_t^{-i}, u^i\left(s^i,b^{-i}\right) > u^i\left(a^i,b^{-i}\right) \end{aligned} \qquad \left(\text{since } C_t^i \subseteq A^i\right)$$

Thus $\forall i, \forall a^i \in R^i, a^i \in C_{t+1}^i$, so $R^i \subseteq C_{t+1}^i$. Then, by induction, $R \subseteq C_T$. $\square$

**Theorem 15.** ***Let $C_T$ be the outcome of a complete IESDS and let $R$ be the unique maximal rationalizable set. Then $C_T = R$***

*Proof.* Since $C_T$ is the outcome of a complete IESDS, $\forall i, \forall a^i \in C_T^i$ it must be that:

$$\begin{aligned} &\nexists s^i \in \Delta\left(C_T^i\right) \text{ such that } \forall b^{-i} \in C_T^{-i}, u^i\left(s^i,b^{-i}\right) > u^i\left(a^i,b^{-i}\right) \\ \implies &\nexists s^i \in \Delta\left(A^i\right) \text{ such that } \forall b^{-i} \in C_T^{-i}, u^i\left(s^i,b^{-i}\right) > u^i\left(a^i,b^{-i}\right) \\ \implies &\exists \mu \in \Delta\left(C_T^{-i}\right) \text{ such that } \forall b^i \in A^i, u^i\left(a^i,\mu\right) \geq u^i\left(b^i,\mu\right) \end{aligned}$$

with the first implication following from the fact that $\forall a^i \in A^i \backslash C_T^i, a^i$ is strictly dominated.

Since $i$ and $a^i$ were arbitrarily taken, it follows that $C_T$ is rationalizable, and recall by the previous theorem $R \subseteq C_T$. Further, since $R$ is the unique maximal rationalizable set, by the above lemma, it must be that $C_T = R$. $\square$

**Theorem 16.** ***If a strategy profile $s \in S$ is a perfect equilibrium then it is (weakly) undominated.***

*Proof.* Let $s \in S$ be a perfect equilibrium and suppose $s$ is weakly dominated. Then $\exists i \in I, r^i \in S^i$ such that

$$\begin{aligned} \forall a^{-i} \in A^{-i}, \quad & u^i\left(r^i,a^{-i}\right) \geq u^i\left(s^i,a^{-i}\right) \\ \exists b^{-i} \in A^{-i}, \quad & u^i\left(r^i,b^{-i}\right) > u^i\left(s^i,b^{-i}\right) \end{aligned}$$

Since $s$ is a perfect equilibrium, by Theorem 33 $\exists\left(s_n\right) \in S^\infty$ s.t. $\forall n s_n$ is fully mixed, $s_n \to s$ and $\forall (i,n) s^i \in BR^i\left(s^i, s_n^{-i}\right)$. Since $s^n$ is fully mixed for each $n \in \mathbb{N}, \Pr_{s_n}\left(a^{-i}\right) > 0$ for all $a^{-i} \in A^{-i}$ By multiplying $(6)-(7)$ by $\Pr_{s_n}\left(a^{-i}\right)$ and summing across $A^{-i}$ we have

$$u^i\left(r^i,s_n^{-i}\right) = \sum_{a^{-i}\in A^{-i}} u^i\left(r^i,a^{-i}\right)\Pr_{s_n}\left(a^{-i}\right) > \sum_{a^{-i}\in A^{-i}} u^i\left(s^i,a^{-i}\right)\Pr_{s_n}\left(a^{-i}\right) = u^i\left(s^i,s_n^{-i}\right)$$

for each $n \in \mathbb{N}$. But this $\lightning s^i$ being a best response to $s_n^{-i}$ for all $n \in \mathbb{N}$. Hence, $s$ is undominated. $\square$

# 2 Extensive Form Games

**Definition 51** (Extensive Form Game)**.** *consists of*

- *The set of all nodes is* $\mathcal{X}$.
- *The set of all final nodes is* $Z = \left(z^1, z^2, \ldots\right)$ *, where z are the consequences of the EFG.*
- *The initial node is* $\alpha$. *Nature, sometimes denoted player 0 , chooses* $\alpha$ *with* $p \in \Delta(IS(\alpha))$
- *The set of move nodes for player i is* $X^i$*; also called player i 's partition. Note* $\forall i \neq j$, $X^i \cap X^j = \emptyset$ *and* $\cup_{i \in I} X^i \equiv X = \mathcal{X} \backslash \{\alpha, Z\}$
- *Let* $\succeq$ *be an asymmetric partial order over* $\mathcal{X}$*, where for* $x, y \in \mathcal{X}, x \succeq y$ *means x comes after y. Note that* $\forall x \in \mathcal{X}, x \succeq \alpha$
- $\forall x, y \in \mathcal{X}$*, let* $x \succeq_c y$ *mean x follows action c played at y.*
- $\forall x \in \mathcal{X}$*, the set of predecessor nodes is* $P(x) \equiv \{y \in \mathcal{X} \mid x \neq y, x \succeq y\}$.
- $\forall x \in \mathcal{X}$*, the set of immediate predecessor nodes is*

$$IP(x) \equiv \{z \in P(x) \mid \nexists y \neq z, y \neq x, z \preceq y \preceq x\}$$

- $\forall x \in \mathcal{X}$*, the set of successor nodes is* $S(x) \equiv \{y \in \mathcal{X} \mid x \neq y, x \preceq y\}$.
- $\forall x \in \mathcal{X}$*, the set of immediate successor nodes is*

$$IS(x) \equiv \{z \in S(x) \mid \nexists y \neq z, y \neq x, z \succeq y \succeq x\}$$

  *Observe that* $Z = \{x \in \mathcal{X} \mid S(x) = \emptyset\}$.

- $\forall i, u^i : Z \rightarrow \mathbb{R}$ *is a vNM utility function.*
- *An information set for player i is* $I^i_k$*, where* $k = 1, \ldots, K^i$. *Note* $\forall k \neq j, I^i_k \cap I^i_j = \emptyset$ *and* $\cup_{k=1}^{K^i} I^i_k = X^i$. *Player i 's set of information sets is* $\mathcal{I}^i \equiv X_{k=1}^{K^i} I^i_k$
- *For each* $I^i_k \in \mathcal{I}^i$*, an action for player i is* $c^i_{I^i_k}$*, or equivalently* $c^i_k$. *The set of actions for player i is* $C^i_{I^i_i}$*, or equivalently* $C^i_k$.

**Definition 52** (A pure strategy for player $i$ ). *is $s^i = \cup_{k=1}^{K^i} c_k^i$. A pure strategy can also be viewed as a* map $s^i : \mathcal{I}^i \to \cup_{k=1}^{K^i} c_k^i$ *such that $\forall k s^i\left(I_k^i\right) \in C_k^i$. The set of pure strategies for player $i$ is $S^i = X_{k=1}^{K^i} C_k^i$. A pure strategy profile for all players is $s = \left(s^1, \ldots, s^n\right)$*

**Definition 53** (A mixed strategy for player $i$ ). *is $\sigma^i \in \Sigma^i = \Delta\left(S^i\right)$, where $\Sigma^i$ is the set of mixed strategies for player $i$. A mixed strategy profile for all players is $\sigma = \left(\sigma^1, \ldots, \sigma^n\right)$.*

**Definition 54.** *The probability of reaching final node $z$ under pure strategy profile $s$ is $\Pr_s(z) \in \Delta(Z)$ The probability of reaching final node $z$ under pure strategy profile $s$ is $\Pr_s(z) \in \Delta(Z)$.*

**Definition 55.** *The probability of the pure strategy profile $s$ being played under the mixed strategy profile $\sigma$ is $\Pr_\sigma^s(s) = \prod_{i \in I} \sigma^i\left(s^i\right)$*

**Definition 56.** *The probability of reaching final node $z$ under mixed strategy profile $\sigma$ is $\Pr_\sigma(z) = \sum_{s \in S} \Pr_\sigma^s(s) \Pr_s(z)$*

**Definition 57** (Player $i$ 's expected utility). *from playing pure strategy $s$ is $E_{\Pr_s}\left[u^i\right] = \sum_{z \in Z} \Pr_s(z) u^i(z)$.*

**Definition 58** (Extended form game (EFG)). *An EFG is $G = \left(I, \mathcal{X}, \succeq, p, \left(X^i, u^i, (I_k^i, C_k^i)_{k=1,\ldots,K^i}\right)_{i \in I}\right)$*

**Definition 59** (Associated NFG). *The pure strategy NFG associated with an EFG is $\left(I, \left(S^i, E_{Pr.}\left[u^i\right]\right)_{i \in I}\right)$*

**Definition 60** (Associated mixed extension NFG). *The mixed extension NFG associated with an EFG is $\left(I, \left(\Sigma^i, E_{Prr}\left[u^i\right]\right)_{i \in I}\right)$.*

**Definition 61** (Nash equilibrium of EFG ). *A NE of an EFG is a NE of the associated mixed extension NFG, and vice-versa.*

**Definition 62** (Normal form perfect equilibrium). *A normal form PE of an EFG is a PE of the associated mixed extension NFG.*

**Theorem 17.** ***In any finite EFG, $\{x \in X \mid IS(x) \subseteq Z\} \neq \emptyset$.***

*Proof.* Suppose not, i.e. $\forall x \in X, \exists y \in X$ such that $y \in IS(x) \backslash Z$. Then, since $y \in X$, we know $\exists w \in X$ such that $w \in IS(y) \backslash Z$. By induction, for any move node there will always be a following move node that itself has a following move node, and thus there will be an infinite amount of move nodes, $\lightning$the EFG being finite. $\square$

**Theorem 18.** ***For any finite EFG, the set of NE is nonempty.***

*Proof.* Immediate by Nash's Existence Theorem □

## 2.1 Strategies of Extensive Form Games

**Definition 63** (Behavioral strategy). *A behavioral strategy for player i is* $\beta^i = \left(\beta^i_{I^i_1}, \ldots, \beta^i_{I^i_{Ki}}\right)$, *or equivalently* $\beta^i = \left(\beta^i_1, \ldots, \beta^i_{K^i}\right)$ *where* $\beta^i_{I^i_k} \in \Delta\left(C^i_{I^i_k}\right)$ $\forall k$. *The set of behavioral strategies for player i is* $B^i \equiv X^{K^i}_{k=1}\Delta\left(C^i_{I^i_k}\right)$

[Note: A player using mixed strategy $\sigma^i$ randomizes once over the set of all pure strategies. A player using behavioral strategy $\beta^i$ randomizes at each information set over only the available choices at that information set.]

**Definition 64** (General strategy). *A general strategy for player i is* $\pi^i$.
*The set of general strategies for player i is* $\Gamma^i = \Delta\left(B^i\right)$

## 2.2 Kuhn and Dalkey Theorems

**Definition 65** (Equivalence). *A behavioral strategy $\beta^i \in B^i$ and a mixed strategy $\sigma^i \in \Sigma^i$ are equivalent, denoted as $\beta^i \sim \sigma^i$, if and only if they induce the same probability on final nodes $Z$ for any given $\pi^{-i} \in \Gamma^{-i}$, i.e.:*

$$\forall \pi^{-i} \in \Gamma^{-i}, \forall z \in Z, \Pr_{\left(\beta^i, \pi^{-i}\right)}(z) = \Pr_{\left(\sigma^i, \pi^{-i}\right)}(z)$$

**Definition 66** (Linear game). *An EFG is linear if no information set intersects a path more than once, i.e.*

$$\forall i \in I, \forall I_k^i \in \mathcal{I}^i, \forall z \in Z, \# \left\{ P(z) \cap I_k^i \right\} \leq 1$$

[Note: Intuitively, every player always knows if they've moved or not.

**Definition 67** (Games of perfect recall (PR)). *An EFG is perfect recall if and only if $\nexists I_k^i, I_l^i \in \mathcal{I}^i, x, y \in I_l^i$ such that $x$ follows some $c_k^i \in C_k^i$ but $y$ does not. [Alternatively, $\nexists I_k^i, I_l^i \in \mathcal{I}^i, x, y \in I_l^i, w \in I_k^i, c_k^i \in C_k^i$, such that $x \succeq_c w$ but not $y \succeq_c w$.*

[Note: In general, for linear games not of perfect recall, $\{\Pr_\sigma \mid \sigma \in \Sigma\} = \Delta(z)$ but $\left\{\Pr_\beta \mid \beta \in B\right\} \subset \Delta(z)$ and so $\forall \sigma^i \in \Sigma^i, \nexists \beta^i \in B^i$ such that $\beta^i \sim \sigma^i$.]

**Definition 68** (Relevant information sets). *The set of pure strategies for player $i$ that lead to $I_k^i \in \mathcal{I}^i$ for some given strategy $s^{-i} \in S^{-i}$ of the other players, i.e. the set of pure strategies relevant for $I_k^i$, is:*

$$\operatorname{Rel}\left(I_k^i\right) = \left\{ s^i \in S^i \mid \exists s^{-i} \in S^{-i}, \Pr_{\left(s^i, s^{-i}\right)}\left(\left\{ z \in Z \mid \Pr(z) \cap I_k^i \neq \varnothing \right\}\right) > 0 \right\}$$

Further, the set of pure strategies relevant for $I_k^i$ that play action $c \in C_k^i$ is:

$$\operatorname{Rel}\left(I_k^i, c\right) = \left\{ s^i \in \operatorname{Rel}\left(I_k^i\right) \mid s^i\left(I_k^i\right) = c \right\} \subseteq \operatorname{Rel}\left(I_k^i\right)$$

**Theorem 19.** ***Every game of perfect recall is linear.***

*Proof.* Suppose not, i.e. there is some perfect recall EFG that is not linear. Then we know there exists some $i \in I$ and $I_k^i \in \mathcal{I}^i$ such that for some $z \in Z, \# \left\{P(z) \cap I_k^i\right\} > 1$. Now take $x, y \in P(z) \cap I_k^i$ such that $x \succeq_c y$ for some action $c_k^i \in C_k^i$, i.e. $x$ follows $c_k^i$

but $y$ does not. Now let $I_l^i = I_k^i$ and $y' = y$. Then clearly there exists $c_k^i \in C_k^i$ such that $x \succeq_c y'$ but not $y \succeq_c y'$, since a node cannot come after itself. Therefore the game is not of perfect recall, which $\lightning$ the hypothesis. □

**Theorem 20** (Dalkey). ***In any linear EFG, for any behavioral strategy $\beta^i \in B^i$ there is a mixed strategy $\sigma^i \in \Sigma^i$ such that $\beta^i \sim \sigma^i$.***

*Proof.* Consider any linear EFG and note that $\forall i, \forall \beta^i \in B^i$ it is possible to construct $\sigma_{\beta^i}^i\left(s^i\right) = \prod_{k=1}^{K^i} \beta_k^i\left(s^i\left(I_k^i\right)\right)$. Further note that, clearly, $\sigma_{\beta^i}^i\left(s^i\right) \in [0,1] \forall s^i \in S^i$. Since the game is linear, we know each path intersects each information set only once, and thus $\sum_{s^i \in S^i} \sigma_{\beta^i}^i\left(s^i\right) = \sum_{s^i \in S^i} \prod_{k=1}^{K^i} \beta_k^i\left(s^i\left(I_k^i\right)\right) = 1$, so the constructed $\sigma_{\beta^i}^i$ is a mixed strategy.

Now take any $z \in Z$ and any $\pi^{-i} \in \Gamma^{-i}$. Consider first the cases where $z$ is always reached or $z$ is never reached, regardless of player $i$'s actions. In these cases, $\Pr(z) = 1$ and $\Pr(z) = 0$, respectively, for any $\beta^i \in B^i$ and for any $\sigma^i \in \Sigma^i$, so $\beta^i \sim \sigma_{\beta^i}^i$ trivially. Consider now the case where $\Pr(z) \in (0,1)$ and depends on player $i$'s actions. Define $\tilde{c}_{I_k^i}^i(z)$ as the action of player $i$ at information set $I_k^i$ that leads to final node $z$ and $\tilde{I}^i(z) \equiv \left\{I_k^i \in \mathcal{I}^i \mid P(z) \cap I_k^i \neq \emptyset\right\}$ as the set of player $i$'s information sets in the path of $z$. Then the probability on $z$ induced by $\beta^i$ is $\Pr_{\left(\beta^i, \pi^{-i}\right)}(z) = \prod_{I_k^i \in \tilde{I}^i(z)} \beta_k^i\left(\tilde{c}_{I_k^i}^i(z)\right)$. Now define $\tilde{S}^i(z)$ as the set of player $i$'s pure strategies that result in $z$, i.e. $\forall s^i \in \tilde{S}^i(z), \forall I_k^i \in \tilde{I}^i(z), c_k^i = \tilde{c}_k^i(z)$. Then the probability on $z$ induced by $\sigma_{\beta^i}^i$ is:

$$
\begin{aligned}
\Pr_{\left(\sigma^i, \pi^{-i}\right)}(z) &= \sum_{s^i \in \tilde{S}^i(z)} \sigma_{\beta^i}^i\left(s^i\right) \\
&= \sum_{s^i \in \tilde{S}^i(z) I_k^i \in \mathcal{I}^i} \beta_k^i\left(s^i\left(I_k^i\right)\right) \\
&= \sum_{s^i \in \tilde{S}^i(z)} \prod_{I_k^i \in \tilde{I}^i(z)} \beta_k^i\left(\tilde{c}_{I_k^i}^i(z)\right) \prod_{I_k^i \notin \tilde{I}^i(z)} \beta_k^i\left(s^i\left(I_k^i\right)\right) \\
&= \prod_{I_k^i \in \tilde{I}^i(z)} \beta_k^i\left(\tilde{c}_{I_k^i}^i(z)\right) \quad \left(\text{since } \forall s^i \in \tilde{S}^i(z), \forall I_k^i \in \tilde{I}^i(z), s_k^i = \tilde{c}_k^i(z)\right) \\
&= \Pr_{\left(\beta^i, \pi^{-i}\right)}(z)
\end{aligned}
$$

Thus $\sigma^i$ and $\beta^i$ induce the same probability on $z$, and this is true $\forall z \in Z$. Thus $\beta^i \sim \sigma_{\beta^i}^i$ □

**Theorem 21** (Kuhn). ***For games of perfect recall,*** $\forall i, \forall \sigma^i \in \Sigma^i, \exists \beta^i \in B^i$ ***such that*** $\sigma^i \sim \beta^i$.

*Proof.* TBD $\square$

## 2.3 Backward Induction and Subgame Perfect Equilibrium

**Definition 69.** *An extensive form game is a game of perfect information if $u^i$ is a singleton $\forall u^i \in U^i, i \in I$ The following procedure is useful for finding pure strategy NE*

**Definition 70** (The backward induction procedure). *in a game with perfect information is as follows:*

1. *For each node $x \in X$ such that $IS(x) \subseteq Z$, choose an action $s^i(x) \in C^i_x$ of the player $i \in I$ with $x \in P^i$ such that $s^i(x)$ leads to a node $z \in IS(x)$ with $u^i(z) \geq u^i(z') \, \forall z' \in IS(x)$*

2. *Replace $x$ with a final node with utilities $u_x$, where $u_x$ is the vector of utilities resulting from $s^i(x)$*

3. *Repeat $(1)-(2)$ until an action $s^i(x)$ has been assigned to every $x \in P^i$ for all $i \in I$.*

*The resulting pure strategy profile $\left(s^i(x)\right)_{x\in P^i, i\in I}$ is called a solution to the backwards induction procedure.*

**Theorem 22.** ***The backward induction procedure produces N.E. profiles in pure strategies.***

**Definition 71** (Subgame). *Let $G = \left(I, \mathcal{X}, \succeq, p, \left(X^i, u^i, \mathcal{I}^i, (C^i_k)_{k=1,\ldots,K^i}\right)_{i\in I}\right)$ be a finite EFG and $x \in X$ such that:*

1. *For the player $i$ such that $x \in X^i$, the $I^i_k$ containing $x$ is a singleton.*

2. *$\forall i \in I, \forall I^i_k \in \mathcal{I}^i$, either $I^i_k \subset S(x) \cup \{x\}$ or $I^i_k \cap (S(x) \cup \{x\}) = \emptyset$.*

*Then* $G_x = \left(I, \mathcal{X}_x, \succeq_x, \left(X^i_x, u^i_x, \mathcal{I}^i_x, \left(C^i_{I^i_k}\right)_{I^i_k \in \mathcal{I}^i_x}\right)_{i \in I}\right)$ *is the subgame following* $x$ *where:*

$$\begin{aligned}
&\mathcal{X}_x = S(x) \cup \{x\} \\
&\succeq_x = \succeq \mid \mathcal{X}_x \\
&X^i_x = X^i \cap \mathcal{X}_x \\
&Z_x = Z \cap S(x) \\
&u^i_x : Z_x \to \mathbb{R} \\
&\mathcal{I}^i_x = \left\{ I^i_k \in \mathcal{I}^i \mid I^i_k \subseteq S(x) \right\}
\end{aligned}$$

[Note: For any finite EFG, the full game is a subgame of itself. All others (if there are any) are called proper subgames.

**Definition 72** ( Minimal subgame)**.** *A minimal subgame is a subgame with no subgames other than itself.*

**Definition 73** (Games of perfect information)**.** *An EFG is perfect information if* $\forall i, \forall I^i_k \in \mathcal{I}^i, I^i_k$ *is a singleton.*

[Note: In an EFG of perfect information, every $x \in \mathcal{X} \backslash Z$ induces a subgame.

**Definition 74** (Subgame perfect equilibrium (SPE))**.** *For a finite EFG, a behavioral strategy profile* $\beta \in B$ *is a subgame perfect equilibrium if and only if* $\forall x \in \mathcal{X}$ *for which* $G_x$ *is a well defined EFG the restriction of* $\beta$ *to* $G_x$ *is a* NE *of* $G_x$*. Intuitively, a SPE is a NE that is also a NE in every subgame.*

[Note: SPE make sure behavior on and off the equilibrium path is rational, in contrast to NE which only make sure behavior on the equilibrium path is rational.

**Theorem 23.** ***For any finite EFG of perfect information, the BIP produces a set of*** **NE** ***in pure strategies, but this set does not necessarily contain all NE in pure strategies.***

**Theorem 24** (Zermelo)**.** ***Every finite EFG of perfect information has at least one pure strategy SPE.***

*Proof.* Let $G$ be any finite EFG of perfect information. Apply the BIP, first considering each $x \in P(z)$ and breaking ties arbitrarily. By the BIP, each node $x \in P(z)$ will then be reduced to a single action. Repeat this until only a single node $x$ is left. Then the record generated by the BIP will constitute an action for $i$ at each $x \in X^i \forall i \in I$, i.e. $s = (s^1, \ldots, s^n)$. Since $s$ is a NE of all subgames by construction, including the full game, it is by definition a pure strategy SPE. □

**Theorem 25.** ***For any finite EFG, the set of SPE is nonempty.***

*Proof.* Let $G$ be any finite EFG. If $G$ is of perfect information, then we know by Zermelo's theorem that at least one SPE exists. If $G$ is of imperfect information and has no proper subgames, then every NE is also a SPE, and thus by Nash's Existence Theorem, the set of SPE is nonempty. □

Consider now the case where $G$ is of imperfect information and has proper subgames. [TODO]

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**Example 16.**

Find all SPE and NE of following games

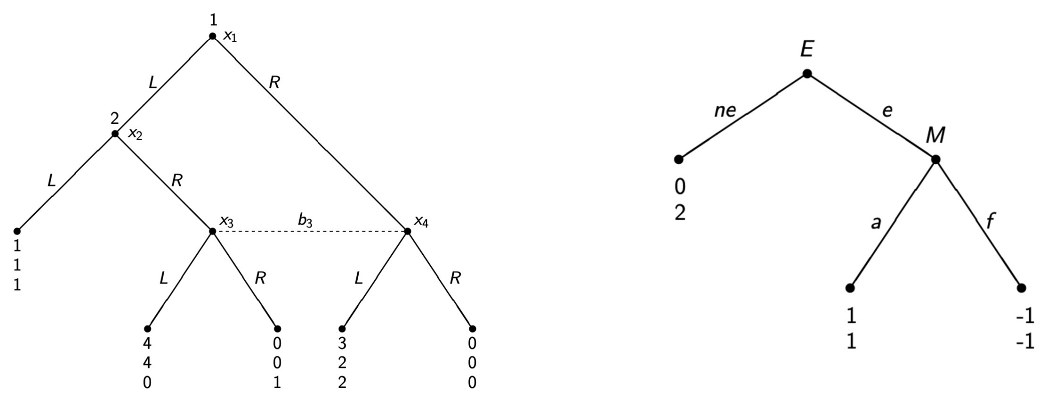


Figure 4:

a)

- Since there is only one subgame (the game itself), all Nash equilibria are SPE.
- Critical point: behavior out-of-equilibrium path.

- Let $p_i$ be the probability that player $i$ chooses $L$.
- There are two types of equilibria:
  1. Type 1: $p_1 = 1, p_2 = 1$ and $p_3 \in \left[0, \frac{1}{4}\right]$
  2. Type 2: $p_1 = 0, p_2 \in \left[\frac{1}{3}, 1\right]$ and $p_3 = 1$.
- Consider first the particular equilibrium of type 2 : $(p_1, p_2, p_3) = (0, 1, 1)$
- The same argument will work for all other equilibria of type 2 , but it will be less transparent. Given $p_1 = 0$ and $p_3 = 1$, is it reasonable to think that player 2 will play $p_2 = 1$? NO. Or is (0,1,1) an stable agreement? Suppose they agree on playing (0,1,1)
- Player 2 arrives home (he does not have to play) but suddenly, the telephone rings and says: "It is your turn, decide between $L$ and $R''$.
- He knows that he is at $x_2$ (player 1 did a mistake), but given $p_3 = 1$, player 2 cannot play $p_2 = 1$ but rather he has to play $R$. Type 2 equilibria are not sensible since they disappear as soon as there is a probability that players make mistakes when implementing their strategies.
- Consider now the type 1 equilibrium $(p_1, p_2, p_3) = (1, 1, 0)$. Now, suppose player 3 is called to play (an out-of-equilibrium play).
- $p_3 = 0$ is still rational since he can be either at $x_3$ or at $x_4$ (the mistake may come from either player 1 or player 2 ). Even with a probability of mistakes, (1,1,0) is still rational.
- $p_3 = 0$ is still rational since he can be either at $x_3$ or at $x_4$ (the mistake may come from either player 1 or player 2 ). Even with a probability of mistakes, (1,1,0) is still rational.
-

b)

Two Nash pure equilibria: (ne, f ), (e, a). And we have mixed too

| E/M | a | f |
|---|---|---|
| ne | 0,2 | 0,2 |
| e | 1,1 | -1,-1 |

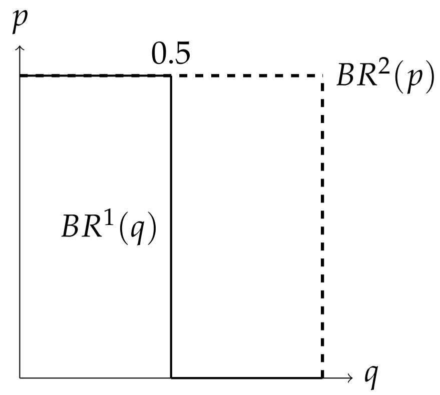


Figure 5: Best Responses

yield the set of Nash equilibria

$$\text{NE} = \left\{ \big((0,1),(1,0)\big), \big((1,0),(q,1-q)\big), \forall 0 \geq q \leq \frac{1}{2} \right\}.$$

Let's focus on two NE.

- Consider the Nash equilibrium (ne, f ).

Entrant plays not to enter (ne) and the telephone rings: Monopolist, it is your turn!!!
M's information set is out of equilibrium path. To play f was part of the optimal behavior because this information set was not reached. Strategy gives plan for moves in all information sets, even though some of them won't be reached. The threat of playing f is what makes optimal for the entrant to play ne.

OK but, f is a non-credible threat, so E should not believe that M will play f (it is not rational for him) if he plays e. **Subgame Perfect Equilibrium** requires rational behavior even in information sets that are not reached in equilibrium (equilibrium should not be based on incredible threats).

- Obtain (e,a) as the unique Subgame Perfect Equilibrium (which coincides with the one obtained by backwards induction).

(e,a) could be obtained also in the normal form as applying the principal ìn ever a dominated strategy since f is dominated by a.
But this is not always true.

(e,a) is the unique SPE of $\Gamma$ and (ne,f) is not a SPE since f is not a Nash equilibrium of the subgame starting at the unique node that belongs to M.

In general, SPE requires two different things:

- SPE gives a solution everywhere (in all subgames), even in subgames where the solution says that they will not be reached (information sets with zero probability).
- SPE imposes rational behavior everywhere, even in the subgames of the game that SPE says that cannot be reached. In out-of-equilibrium subgames, the solution is disapproved, yet players evaluate their actions taking as given the behavior of the other players, that have been demonstrated incorrect since we are in an out-of-equilibrium path.

## 2.4 Sequential Equilibrium

**Definition 75** (System of beliefs)**.** *A system of beliefs is* $\mu = \left(\mu^i_{I^i_k}\right)_{i\in I, I^i_k \in \mathcal{I}^i}$ *where* $\mu^i_{I^i_k} \in \Delta\left(I^i_k\right) \forall i, \forall k$*. Alternatively, a system of beliefs is a map* $\mu : X \to [0,1]$ *such that* $\forall i, \forall I^i_k \in \mathcal{I}^i, \sum_{x\in I^i_k} \mu(x) = 1$

**Definition 76** ( Consistency)**.** *A system of beliefs* $\mu$ *is consistent with behavioral strategy profile* $\beta$ *if* $\exists \{(\beta_n, \mu_n)\}_{n\in\mathbb{N}}$ *such that:*

1. $\forall n \in \mathbb{N}, \beta_n$ *is fully mixed*
2. $\lim_{n\to\infty} \beta_n = \beta$
3. $\forall n \in \mathbb{N}, \mu_n$ *is induced by* $\beta_n$ *according to Bayes' rule*
4. $\lim_{n\to\infty} \mu_n = \mu$

**Definition 77** (Optimality)**.** *A behavioral strategy* $\beta^i \in B^i$ *for player i is optimal with respect to* $\mu$ *at* $I^i_k \in \mathcal{I}^i$*, given* $\beta^{-i}$*, if:*

$$\beta^i \in \arg\max_{b^i\in B^i} V^i\left(I^i_k, \left(b^i, \beta^{-i}\right), \mu\right) = \arg\max_{b^i\in B^i} \sum_{x\in I^i_k} \mu^i_{I^i_k}(x) \sum_{z\in Z} \Pr_{\left(x,\left(b^i,\beta^{-i}\right)\right)}(z) u^i(z)$$

**Definition 78** (Sequential rationality)**.** *A behavioral strategy profile* $\beta \in B$ *is sequentially rational with respect to* $\mu$ *if* $\forall i \in I, \forall I^i_k \in \mathcal{I}^i, \beta^i$ *is optimal with respect to* $\mu$ *given* $\beta^{-i}$*.*

**Definition 79** (Sequential equilibrium (SE))**.** *A sequential equilibrium is a pair* $(\beta, \mu)$*, where* $\beta \in B$ *and* $\mu$ *is a system of beliefs, such that:*

1. $\mu$ *is consistent with* $\beta$
2. $\beta$ *is sequentially rational with respect to* $\mu$

Note that a sequential equilibrium is a pair, not just a strategy profile. Hence, in order to identify a sequential equilibrium, one must identify a strategy profile $\beta$ which describes what a player does at every information set, and a belief assessment $\mu$, which describes what a player believes at every information set. In order to check that that $(\beta, \mu)$ is a sequential equilibrium, one must check that

1. (Sequential Rationality) $s$ is a best response to belief $\mu(\cdot \mid I)$ and the belief that the other players will follow $s$ in the continuation games in every information set$I$, and

2. (Consistency) there exist trembling probabilities that go to zero such that the conditional probabilities derived from Bayes rule under the trembles approach $\mu(\cdot \mid I)$ at every information set $I$. If all the information sets are reached under strategy $\beta$, we just need to use the Bayes rule in order to check consistency. If not do trembling hand.

[Note: SE make sure no strictly dominated strategies are played; weakly dominated strategies may still be played under certain beliefs, e.g. when some nodes in information sets are reached with zero probability

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**Example 17.** *Find Sequential equilibria of following Selten's horse game With coresponding*

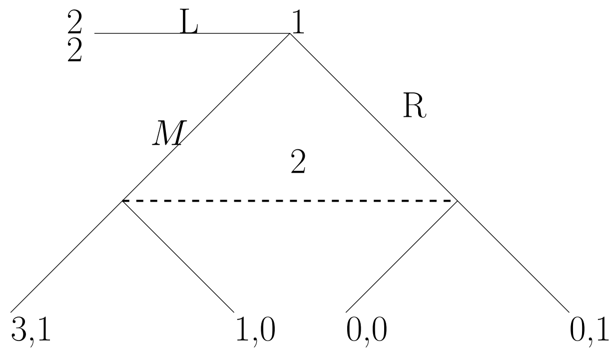


Figure 6:

*Normal Form Game Consider the above example. An assessment is the following object:*

| | | Mr 2 | |
|---|---|---|---|
| | | l | r |
| | L | 2,2 | 2,2 |
| Mr 1 | M | 3,1 | 1,0 |
| | R | 0,0 | 1,0 |

$$(\beta, \mu) = ((\beta_1(\emptyset), \beta_2(\{M, R\})), \mu(\{M, R\}))$$

*where $\beta_1(\emptyset)$, player 1's behavioral strategy, is a probability distribution over player 1's strategies, $L, M$, and $R$ $\beta_2(\{M, R\})$, player 2 's behavioral strategy, is a probability distribution* *over player 2 's strategies, $\ell$ and $r$, and - $\mu(\{M, R\})$ is player 2 's belief conditional on 2 being asked to play over the two histories, M and R, that lead to her information set.*

*A sequential equilibrium is an assessment that is sequentially rational and consistent. Sequential rationality precludes that player 1 assigns positive probability on R, since R is strictly dominated for* 1. *Hence, in any sequential equilibrium, $\beta_1(\emptyset)(R) = 0$ There are two types of sequential equilibria. Sequential equilibrium where $\beta_1(\emptyset)(M) > 0$. In this case, 2 's information set is reached with positive probability, and the consistency of the assessment entails that $\mu$ $(\{M, R\}$ is determined from $\beta_1(\emptyset)$ using Bayes' rule:*

$$\mu(\{M,R\})(M) = \frac{\beta_1(\emptyset)(M)}{\beta_1(\emptyset)(M)+\beta_1(\emptyset)(R)} = 1$$
$$\mu(\{M,R\})(R) = \frac{\beta_1(\emptyset)(R)}{\beta_1(\emptyset)(M)+\beta_1(\emptyset)(R)} = 0$$

*With these beliefs, $\beta_2(\{M, R\})$ is part of a sequentially rational assessment if and only if*

$$\beta_2(\{M,R\})(\ell) = 1$$
$$\beta_2(\{M,R\})(r) = 0$$

*This in turn means that $\beta_1(\emptyset)$ is part of a sequentially rational assessment if and only if*

$$\beta_1(\emptyset)(L) = 0$$
$$\beta_1(\emptyset)(M) = 1$$
$$\beta_1(\emptyset)(R) = 0$$

*Hence, there is unique sequential equilibrium where $\beta_1(\emptyset)(M) > 0$ :*

$$(\beta, \mu) = ((\beta_1(\emptyset), \beta_2(\{M,R\})), \mu(\{M,R\})) = (((0,1,0),(1,0)),(1,0))$$

*Sequential equilibrium where $\beta_1(\emptyset)(M) = 0$. Hence, $\beta_1(\emptyset)$ is given as follows:*

$$\beta_1(\emptyset)(L) = 1$$
$$\beta_1(\emptyset)(M) = 0$$
$$\beta_1(\emptyset)(R) = 0$$

*This behavioral strategy of player 1 is part of a sequentially rational assessment if and only if $\beta_2(\{M, R\})$ satisfies*

$$\beta_2(\{M,R\})(\ell) = 1 - p$$
$$\beta_2(\{M,R\})(r) = p$$

*with* $p \geq \frac{1}{2}$*, because otherwise player 1 would want to have* $\beta_1(\emptyset)(M) = 1$*. If* $\frac{1}{2} \leq p < 1$*, then player 2 's behavioral strategy is part of a sequentially rational assessment if and only if*

$$\mu(\{M,R\})(M) = \tfrac{1}{2}$$
$$\mu(\{M,R\})(R) = \tfrac{1}{2}$$

*To check that this is part of a consistent assessment, consider*

$$\beta^{\epsilon} = ((\beta_1^{\epsilon}(\emptyset), \beta_2^{\epsilon}(\{M,R\})))$$

*where* $\epsilon$ *is a "small" positive number and*

$$\begin{aligned} \beta_1^{\epsilon}(\emptyset)(L) &= 1-2\epsilon & \beta_2^{\epsilon}(\{M,R\})(\ell) &= \beta_2(\{M,R\})(\ell), \\ \beta_1^{\epsilon}(\emptyset)(M) &= \epsilon & \beta_2^{\epsilon}(\{M,R\})(r) &= \beta_2(\{M,R\})(r), \\ \beta_1^{\epsilon}(\emptyset)(R) &= \epsilon \end{aligned}$$

*Then* $\mu$ $(\{M,R\}$ *is determined from* $\beta_1^{\epsilon}(\emptyset)$ *using Bayes' rule:*

$$\mu(\{M,R\})(M) = \frac{\beta_1^{\epsilon}(\emptyset)(M)}{\beta_1^{\epsilon}(\emptyset)(M) + \beta_1^{\epsilon}(\emptyset)(R)} = \frac{\epsilon}{\epsilon+\epsilon} = \frac{1}{2}$$

$$\mu(\{M,R\})(R) = \frac{\beta_1^{\epsilon}(\emptyset)(R)}{\beta_1^{\epsilon}(\emptyset)(M) + \beta_1^{\epsilon}(\emptyset)(R)} = \frac{\epsilon}{\epsilon+\epsilon} = \frac{1}{2}$$

*Furthermore,* $\beta^{\epsilon} \to \beta$ *as* $\epsilon \to 0$*. This shows consistency and establishes that*

$$(\beta,\mu) = ((\beta_1(\emptyset), \beta_2(\{M,R\}))\,, \mu(\{M,R\})) = \left(((1,0,0),(1-p,p)), \left(\frac{1}{2},\frac{1}{2}\right)\right)$$

*is a sequential equilibrium if and only if* $p \geq \frac{1}{2}$*. If* $\beta_2(\{M,R\})(r) = 1$*, then player 2's behavioral strategy is part of a se- quentially rational assessment if and only if*

$$\mu(\{M,R\})(M) = 1-q$$
$$\mu(\{M,R\})(R) = q$$

*with* $q \geq \frac{1}{2}$*. To check that this is part of a consistent assessment, consider*

$$\beta^{\epsilon} = ((\beta_1^{\epsilon}(\emptyset), \beta_2^{\epsilon}(\{M,R\})))$$

*where* $\epsilon$ *is a "small" positive number and*

$$\begin{aligned} \beta_1^{\epsilon}(\emptyset)(L) &= 1-\epsilon & \beta_2^{\epsilon}(\{M,R\})(\ell) &= \epsilon \\ \beta_1^{\epsilon}(\emptyset)(M) &= (1-q)\epsilon & \beta_2^{\epsilon}(\{M,R\})(r) &= 1-\epsilon \end{aligned}$$

*$\beta_1^\epsilon(\emptyset)(R) = q\epsilon$ Then $\mu(\{M,R\}$ is determined from $\beta_1^\epsilon(\emptyset)$ using Bayes' rule:*

$$\mu(\{M,R\})(M) = \frac{\beta_1^\epsilon(\emptyset)(M)}{\beta_1^\epsilon(\emptyset)(M)+\beta_1^\epsilon(\emptyset)(R)} = \frac{1-q}{1-q+q} = 1-q$$
$$\mu(\{M,R\})(R) = \frac{\beta_1^\epsilon(\emptyset)(R)}{\beta_1^\epsilon(\emptyset)(M)+\beta_1^\epsilon(\emptyset)(R)} = \frac{q}{1-q+q} = q$$

*Furthermore, $\beta^\epsilon \to \beta$ as $\epsilon \to 0$. This shows consistency and establishes that*

$$(\beta,\mu) = ((\beta_1(\emptyset),\beta_2(\{M,R\})),\mu(\{M,R\})) = (((1,0,0),(0,1)),(1-q,q))$$

*is a sequential equilibrium if and only if $q \geq \frac{1}{2}$. A final question: Are the sequential equilibria of the second type reason- able?*

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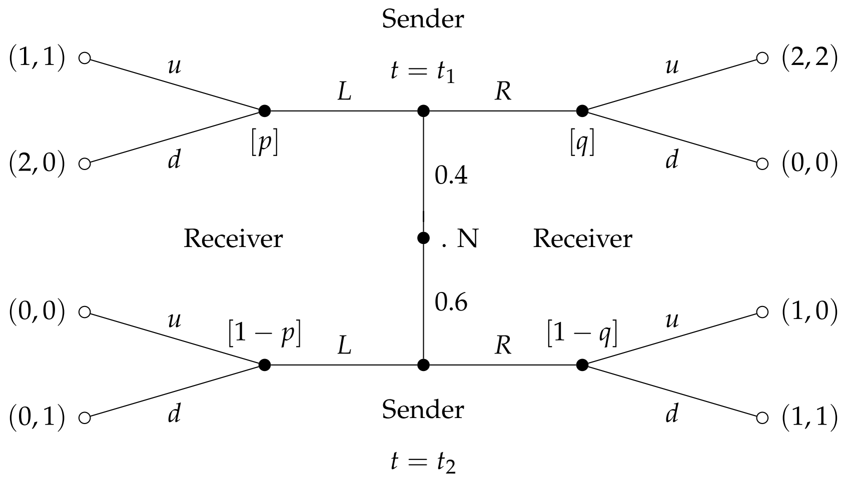
**Example 18.** *Consider beer and quiche game:*

Figure 7: Signaling Game

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**Theorem 26.** ***Every SE is a SPE.***

*Proof.* Suppose not, i.e. there exists some $\beta \in B$ that's a part of an SE of some finite game $G$ but that's not a SPE for $G$. Then we know there exists some subgame $G_x$ such that $\beta_x$, the restriction of $\beta$ to $G_x$, is not a NE. Thus there exists some $b_x^i \in B_x^i$ for some player $i$ such that:

$$\sum_{z\in Z_x} u^i(z) \prod_{x\in \text{Path}(z)\cap \mathcal{I}_x^i} b_x^i(x,c_z) \prod_{x\in \text{Path}(z)\setminus \mathcal{I}_x^i} \beta_x^{-i}(x,c_z) > \sum_{z\in Z_x} u^i(z) \prod_{x\in \text{Path}(z)} \beta_x(x,c_z)$$

Note that the l.h.s. of the above inequality can be expressed as some system of beliefs $\mu$ induced by $\beta_x$. This implies $b_x^i$ is optimal with respect to $\mu$ given $\beta_x^{-i}$, but since $\beta$ is a SE, $\beta^i_x$ is optimal with respect to $\mu$ given $\beta^{-i}_x$. But this is $\not\downarrow$of the above inequality. So it must be that for $\beta \in B$ that's a part of an SE it is also a SPE. $\square$

**Theorem 27.** ***Suppose that $\Gamma^e$ is an extensive-form game with perfect recall and the behavioralstrategy profile $b \in B$ is a perfect equilibrium of $\Gamma^e$. Then there exists a system of beliefs $\mu$ such that $(b, \mu)$ is a sequential equilibrium of $\Gamma^e$***

*Proof.* Since $b$ is a perfect equilibrium, by Theorem 2.11$\exists\left(b_n\right) \in B^{\infty}$ such that (i) $b_n$ is fully mixed $\forall n \in \mathbb{N},($ ii $) b_n \rightarrow b$, and ( iii $) b^i \in \operatorname{argmax}_{d^i \in B^i} u^i\left(d^i, b_n^{-i}\right) \forall n \in \mathbb{N}, i \in I . \forall n \in \mathbb{N}, v \in U^i, y \in u$ define

$$\mu_n(y)=\frac{\operatorname{Pr}\left(y \mid b_n, \theta\right)}{\sum_{y^{\prime} \in v} \operatorname{Pr}\left(y^{\prime} \mid b_n, \theta\right)}$$

Notice that since $b_n$ is fully mixed, $\operatorname{Pr}\left(y \mid b_n, \theta\right)>0$ for every $y \in v$, so $\sum_{y^{\prime} \in v} \operatorname{Pr}\left(y^{\prime} \mid b_n, \theta\right)>0$. Let

$$\mu(y)=\lim _{n \rightarrow \infty} \mu_n(y)$$

Then by (i) and (ii), $\mu$ is a system of beliefs fully consistent with $b$. Let $V^{(i, v)}(\cdot)$ denote the utility function of agent $(i, v)$. When this agent uses a randomized strategy $d^{(i, v)} \in \Delta\left(C_v^i\right)$, his expected payoff is

$$\begin{gathered}
V^{(i, v)}\left(b_n^{-(i, v)}, d^{(i, v)}\right)=\sum_{x \in v} \operatorname{Pr}\left(x \mid b_n^{-(i, v)}, d^{(i, v)}, \theta\right) U^{(i, v)}\left(b_n^{-(i, v)}, d^{(i, v)} \mid x\right)+ \\
+\sum_{z \notin S(v)} \operatorname{Pr}\left(z \mid b_n^{-(i, v)}, d^{(i, v)}, \theta\right) u^{(i, v)}(z)
\end{gathered}$$

where $b_n^{-(i, v)}=b_n \backslash\left\{b_n^{(i, v)}\right\}, U^{(i, v)}(b \mid x)=\sum_{z \in Z} \operatorname{Pr}(z \mid b, x) u^{(i, v)}(z)$ and $S(v)= \cup_{x \in v} S(x)$. Note that for any $x \in v, \operatorname{Pr}\left(x \mid b_n^{-(i, v)}, d^{(i, v)}, \theta\right)=\operatorname{Pr}\left(x \mid b_n, \theta\right)$ since the probability of the node $x$ occurring depends only on the strategies of the agents who move before $v$ occurs. Then

$$\begin{aligned}
V^{(i, v)}\left(b_n^{-(i, v)}, d^{(i, v)}\right) & =\sum_{x \in v} \operatorname{Pr}\left(x \mid b_n, \theta\right) U^{(i, v)}\left(b_n^{-(i, v)}, d^{(i, v)} \mid x\right)+\sum_{z \notin S(v)} \operatorname{Pr}\left(z \mid b_n, \theta\right) u^{(i, v)}(z) \\
& =\left(\sum_{x \in v} \mu_n(x) U^{(i, v)}\left(b_n^{-(i, v)}, d^{(i, v)} \mid x\right)\right)\left(\sum_{x \in v} \operatorname{Pr}\left(x \mid b_n, \theta\right)\right)+ \\
& +\sum_{z \notin S(v)} \operatorname{Pr}\left(z \mid b_n, \theta\right) u^{(i, v)}(z)
\end{aligned}$$

Since $(b_n)$ supports $b$ as a perfect equilibrium of the multi-agent representation of $\Gamma^e$ we have that

$$b^{(i,v)} \in \underset{d^{(i,v)} \in \Delta\left(C_v^i\right)}{\operatorname{argmax}} \; V^{(i,v)} \left(b_n^{-(i,v)}, d^{(i,v)}\right)$$

which implies that

$$b^{(i,v)} \in \underset{d^{(i,v)} \in \Delta\left(C_v^i\right)}{\operatorname{argmax}} \sum_{x \in v} \mu_n(x) U^{(i,v)} \left(b_n^{-(i,v)}, d^{(i,v)} \mid x\right)$$

because these two objectives differ only by a strictly increasing affine transformation whose coeeficients are independent of $d^{(i,v)}$. Then by the upper-hemicontinuity of the best-response correspon- dence we have

$$b^{(i,v)} \in \underset{d^{(i,v)} \in \Delta\left(C_v^i\right)}{\operatorname{argmax}} \sum_{x \in v} \mu(x) U^{(i,v)} \left(b^{-(i,v)}, d^{(i,v)} \mid x\right)$$

This implies that $\forall v \in U^i$ (the information partition),

$$b^i \in \underset{d^i \in B^i}{\operatorname{argmax}} V^i \left[v, \mu, \left(d^i, b^{-i}\right)\right]$$

Since $i \in I$ has been arbitrary $b$ is sequentially rational given the system of beliefs $\mu$. Hence, $(b, \mu)$ is a sequential equilibrium.

$\square$

# 3 Equilibrium Refinements

## 3.1 Perfect Equilibria

**Example 19.** *Let $I = \{1,2\}$ and consider the game $G$ defined by*

| | | Mr 2 | |
|---|---|---|---|
| | | L | R |
| Mr 1 | T | 1,1 | 0,0 |
| | B | 0.0 | x.y |

where $1 > x > 0$ and $y > 0$. Suppose that $s^1 = (p, 1-p)$ and $s^2 = (q, 1-q)$ so that $s = ((p, 1-p), (q, 1-q))$. Then we can view each player's best response as a function of the other player's mixed strategy. In particular, if player 2 plays $L$, his expected utility is $p$. If he plays $R$ it is $(1-p)y$. So his best response depends on the value of $p$. Similarly for player 1. Then $G$ has three NE.

$$NE = \left\{ ((1,0),(1,0)), ((0,1),(0,1)), \left( \left( \frac{y}{1+y}, 1 - \frac{y}{1+y} \right), \left( \frac{x}{1+x}, 1 - \frac{x}{1+x} \right) \right) \right\}$$

Typically $|NE|$ is odd. However, not in general. For instance, in $G$ let $x = y = 0$. Compute equilibrium. Show it is strange in that it gives positive probability to a weakly dominated strategy. Motivate perfect equilibria/perterbations by show that if player 2 plays $L$ with some small but positive probability, this strange equilibrium goes away.

**Definition 80** (Utility robust NE). *Given a* NE$s_u$ *of* $\left(I, S^i, u^i\right)$, $s_u$ *for* $u$ *is utility robust if* $\forall \delta \exists \bar{\epsilon} > 0$ *such that* $\forall v$ *such that* $\|v - u\| < \epsilon$ *where* $\epsilon < \bar{\epsilon}$, $\exists s_v$ *such that* $\|s_v - s_u\| < \delta$

**Definition 81** (Perturbation). *A perturbation is* $\epsilon = \left(\epsilon^i\right)_{i \in I}$, *where* $\forall i \in I \epsilon^i = \left(\epsilon^i\left(a^i\right)\right)_{a^i \in A^i}$, *such that:*

$$\forall i \in I \quad \forall a^i \in A^i, \quad \epsilon^i\left(a^i\right) > 0 \quad \wedge \quad \sum_{a^i \in A^i} \epsilon^i\left(a^i\right) < 1$$

Perturbation is not a mixed strategy.

**Definition 82** (Perturbed strategy set ). *The perturbed strategy set for player $i$ is*

$$S^i_{\epsilon^i} \equiv \left\{ s^i \in S^i \mid \forall a^i \in A^i, s^i\left(a^i\right) \geq \epsilon^i\left(a^i\right) \right\}$$

The perturbed strategy set for all players is $S_\epsilon \equiv \prod_{i\in I} S^i_\epsilon$

**Definition 83.** *NE of $\epsilon$ -perturbed game $s \in S_\epsilon$ is a* NE *of the $\epsilon$ -perturbed game if*

$$\forall i \in I, \forall t^i \in S^i_{\epsilon i}, \quad u^i\left(s^i, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right)$$

[A NE of the $\epsilon$ -perturbed game is $\hat{s} \in S_\epsilon$ such that $\forall i \in I \hat{s}^i \in BR^i_{S^i_\epsilon}(\hat{s})$ or $NE(\Gamma_\varepsilon) := \{s \in S_\varepsilon : s \in BR_\varepsilon(s)\}$

**Definition 84.** *Perfect equilibrium Let $\left(I, \left(S^i, u^i\right)_{i\in I}\right)$ be a NFG. Then $s \in S$ is a* PE *if $\exists \{\epsilon_m\}_{m\in\mathbb{N}}, \{s_m\}_{m\in\mathbb{N}}$ such that $\epsilon_m \to 0, s_m \to s$, and $s_m$ is a* NE *of the $\epsilon_m$ -perturbed game ( for each $m$)*

[$s \in S$ is PE if it is the limit of a sequence of NE of some $\epsilon$ -perturbed game, where $\epsilon \to 0$.

**Theorem 28.** ***The set of* PE *is nonempty***

*Proof.* As proved in Theorem 2.2, for any finite game the set of NE is nonempty. It follows immediately that for any $\epsilon$ -perturbation of a finite game, the set of NE is nonempty. Then, for any sequence of perturbations $\epsilon_n \to 0$, there exists $s_n \in S_{\epsilon_n}$ such that $s_n$ is a NE of the $\epsilon_n$ -perturbed game. Then $s_n$ is a sequence in $S$, and since $S$ is compact, there exists a convergent subsequence $s_{n_k} \to s \in S$. Then $s$ is a perfect equilibrium by definition, and thus the set of PE is nonempty. $\square$

**Theorem 29.** ***If $s \in S$ is a PE, then it is also a NE.***

*Proof.* Let $s \in S$ be a PE. Then $\exists \{\epsilon_m\}_{m\in\mathbb{N}}, \{s_m\}_{m\in\mathbb{N}}$ such that $\epsilon_m \to 0, s_m \to s$, and $\forall m \in \mathbb{N}, s_m$ is a NE of the $\epsilon_m$ -perturbed game. Take any $i \in I$ and any $t^i \in S^i$. Since $\epsilon_m \to 0$, it follows that $\epsilon^i_m \to 0$, and thus there exists a sequence $t^i_m \in S^i_{\epsilon^i_i}$ such that $t^i_m \to t^i$. Take such a sequence. Then, since $s_m$ is a NE of the $\epsilon_m$ -perturbed game, it follows that

$$u^i\left(s^i_m, s^{-i}_m\right) \geq u^i\left(t^i_m, s^{-i}_m\right) \quad \forall m \in \mathbb{N}$$

Since $u^i(\cdot)$ is continuous $\forall i \in I$, then

$$\lim u^i\left(s^i_m, s^{-i}_m\right) \geq \lim u^i\left(t^i_m, s^{-i}_m\right)$$
$$\implies u^i\left(s^i, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right)$$

Since $t^i \in S^i$ was taken arbitrarily, $s^i \in BR^i\left(s^{-i}\right)$. Since $i \in I$ was taken arbitrarily, $s \in BR(s)$, so $s$ is a NE $\square$

**Theorem 30.** ***If*** $s \in S$ ***is a fully mixed*** **NE*****, then it is also a*** **PE.**

*Proof.* Let $s \in S$ be a fully mixed NE for some finite NFG, i.e. $\forall i \in I, \forall a^i \in A^i$, $s^i\left(a^i\right) > 0$. From this, note there exists

$$\bar{s}^i \equiv \min_{a^i \in A^i} s^i\left(a^i\right) \forall i \in I \text{ and } \bar{s} \equiv \min_{i \in I} \bar{s}^i$$

and that $\bar{s} > 0$. It follows that, for any sequence of perturbations $\{\epsilon_n\}_{n \in \mathbb{N}}$ such that $\epsilon_n \to 0, \exists N \in \mathbb{N}$ such that, $\forall m \geq N$

$$\forall i \in I \forall a^i \in A^i, \quad e^i_m\left(a^i\right) < \bar{s}$$

so $\forall m \geq N, s \in S_{\epsilon_m}$. Now recall that since $s$ is a NE of the original game,

$$\forall i \in I, \forall t^i \in S^i, \quad u^i\left(s^i, s^{-i}\right) \geq u^i\left(t^i, s^{-i}\right)$$

Note that $S^i_{\epsilon^i_m} \subseteq S^i$, so since $\forall m \geq N, s \in S_{\epsilon_m}$, we know that $\forall m \geq N, s$ is a NE of the $\epsilon_m$ -perturbed game. Now take a sequence $\{s_m\}$ such that $s_m = s \forall m \in \mathbb{N}$ and construct a new sequence of perturbations $\{\hat{\epsilon}_m\} = \{\epsilon_m\}_{m \geq N}$. Then $s$ is a PE by definition. $\square$

**Theorem 31.** $s_\varepsilon \in NE\left(\Gamma_\varepsilon\right)$ ***if, and only if,***

$$\forall i \in I, a, b \in A^i, \quad u^i\left(a, s_\varepsilon^{-i}\right) < u^i\left(b, s_\varepsilon^{-i}\right) \implies s^i_\varepsilon(a) = \varepsilon^i(a)$$

*Proof.* The argument is identical to that provided in the proof of theorem 7 $\square$

**Definition 85.** *Let $\eta > 0$. A mixed-strategy profile $s \in S$ is $\eta-$ perfect if, and only if,*

1. *it is fully mixed, i.e. $\forall i \in I, a \in A^i \quad s^i(a) > 0$,*
2. $\forall i \in I, a^i, b^i \in A^i \quad u^i\left(a^i, s^{-i}\right) < u^i\left(b^i, s^{-i}\right) \implies s^i\left(a^i\right) \leq \eta$

**Definition 86.** *Let $\eta > 0$. A mixed-strategy profile $s \in S$ is $\eta-$ proper if, and only if,*

1. *it is fully mixed, and*

2. $\forall i \in I, a^i, b^i \in A^i \quad u^i\left(a^i, s^{-i}\right) < u^i\left(b^i, s^{-i}\right) \Longrightarrow s^i\left(a^i\right) \leq \eta s^i\left(b^i\right)$

**Theorem 32.** ***If a strategy profile $s \in S$ is $\eta$ - proper, then $s$ is $\eta$ - perfect.***

*Proof.* Let $s \in S$ be $\eta$ -proper, $\eta > 0$. Then $s$ is fully mixed and $\forall i \in I, a^i, b^i \in A^i$

$$u^i\left(a^i, s^{-i}\right) < u^i\left(b^i, s^{-i}\right) \Longrightarrow s^i\left(a^i\right) \leq \eta s^i\left(b^i\right) \Longrightarrow s^i\left(a^i\right) \leq \eta$$

since $s^i\left(b^i\right) \in (0,1)$. Hence, $s$ is $\eta$ -perfect. $\square$

---

**Example 20.** *Consider the following game:*

| | | Mr 2 L | Mr 2 R |
|---|---|---|---|
| Mr 1 | T | 1,10 | 0,0 |
| Mr 1 | M | 1,2 | 1,2 |
| Mr 1 | B | 3,-10 | 0,0 |

- *Find all the Nash equilibria*
- *Find all the perfect equilibria*
- *Take a perfect equilibrium $s$ of any normal form finite game, and remove the action of a player $i$ which is not a best response to the strategy of the others, $s^i$. Is the restriction of the strategy profile $s$ to the new game a perfect equilibrium of the new game?*

---

**Definition 87.** *A strategy profile $s \in S$ is a perfect-2 equilibrium if, and only if, $\exists\,(\eta_n) \in \mathbb{R}^\infty_{++}$ with $\eta_n \to 0$ and a corresponding sequence of mixed strategy profiles $\left(s_{\eta_n}\right) \in S^\infty$ such that such that $s_{\eta_n}$ is $\eta_n$ -perfect for all $n \in \mathbb{N}$ and $s_{\eta_n} \to s$*

**Definition 88.** *A strategy profile $s \in S$ is a proper equilibrium if, and only if, $\exists\,(\eta_n) \in \mathbb{R}^\infty_{++}$ with $\eta_n \to 0$ and a corresponding sequence of mixed strategy profiles $\left(s_{\eta_n}\right) \in S^\infty$ such that $s_{\eta_n}$ is $\eta_n$ -proper for all $n \in \mathbb{N}$ and $s_{\eta_n} \to s$*

**Theorem 33.** ***Let $s \in S$ be a strategy profile. Then the following are equivalent:***

1. ***$s$ is a perfect equilibrium;***

2. ***$s$ is a perfect-2 equilibrium;***

**3.** $\exists (s_n) \in S^\infty$ **s.t. (i)** $s_n$ ***is fully mixed,* (ii)** $s_n \to s$ ***and*** ( ***iii*** )$\forall i, n \quad s^i \in BR^i_{s^i}\left(s_n^{-i}\right)$

**Theorem 34.** ***If a mixed-strategy profile*** $s \in S$ ***is a perfect equilibrium of*** $\Gamma$***, then it is a Nash equilibrium of*** $\Gamma$***, i.e.*** $s \in NE(\Gamma)$**.**

*Proof.* Since $s$ is a perfect equilibrium, by previous theorem there exists a sequence $(s_n) \in S^\infty$ such that $s_n$ is fully mixed for each $n \in \mathbb{N}, s_n \to s$, and $s^i \in BR^i\left(s^i, s_n^{-i}\right) \forall i \in I$.

Fix $i \in I$ arbitrarily and define a sequence $(s_m) \in S^\infty$ by $s_m^i = s^i$ for all $m \in \mathbb{N}$ and $s_m^{-i} = s_n^i$ for all $m = n$. Then $s_m \to s$ and $s_m^i \in BR^i\left(s_m\right) \forall m \in \mathbb{N}$.

Since $BR^i(\cdot)$ is uhc by Theorem 6 there exists a strictly increasing sequence $(m_k) \in \mathbb{N}^\infty$ such that $s^i_{m_k} \to r^i \in BR^i(s)$. But since every subsequence of $\left(s^i_m\right)$ is the stationary sequence $\left(s^i\right)$, we necessarily have $r^i = s^i$. Hence, $s^i \in BR^i(s)$. Since $i$ has been arbitrary, $s \in NE(\Gamma)$. □

**Example 21.** *Does every normal-form perfect equilibrium of G correspond to a perfect equilibrium of* $\Gamma$*? NO!*

*Look at follwoing game*

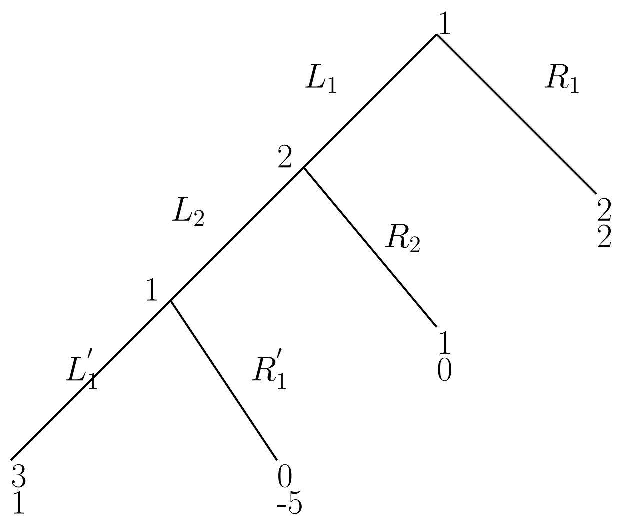


Figure 8: PE of NFG $\neq$ PE of EFG

*With corresponding Normal Form Game with assigned perturbations*

- *Take any sequence* $\{\varepsilon_k\} \to 0$ *and define* $\Sigma_1\left(\frac{\varepsilon_k}{2}\right)$ *and* $\Sigma_2\left(\varepsilon_k\right)$*, where* $\varepsilon_1^k\left(s_1\right) = \frac{\varepsilon_k^2}{2}$ *for all* $s_1 \in S_1$ *and* $\varepsilon_2^k\left(s_2\right) = \varepsilon_k$ *for all* $s_2 \in S_2$

| | | $\epsilon_k$ | $1-\epsilon_k$ |
|---|---|---|---|
| | Mr1\Mrs2 | L_2 | R_2 |
| $\epsilon_k^2/2$ | $L_1L_1^{'}$ | 3,1 | 1,0 |
| $\epsilon_k^2/2$ | $L_1R_1^{'}$ | 0,-5 | 1,0 |
| $\epsilon_k$ | $R_1L_1^{'}$ | 2,2 | 2,2 |
| $1-\epsilon_k-\epsilon_k^2$ | $R_1R_1^{'}$ | 2,2 | 2,2 |

- *Given $\varepsilon_k > 0$ sufficiently small, consider the following strategy $\left(\sigma_1^k, \sigma_2^k\right) \in \Sigma_1\left(\frac{\varepsilon_k^2}{2}\right) \times \Sigma_2\left(\varepsilon_k\right)$ :*

$$\sigma_1^k\left(L_1L_1'\right) = \frac{\varepsilon_k^2}{2}, \sigma_1^k\left(L_1R_1'\right) = \frac{\varepsilon_k^2}{2}, \sigma_1^k\left(R_1L_1\right) = \varepsilon_k, \textit{ and}$$

$\sigma_1^k\left(R_1R_1'\right) = 1-\varepsilon_k-\varepsilon_k^2$

$$\text{-}\ \sigma_2^k\left(L_2\right) = \varepsilon_k \textit{ and } \sigma_2^k\left(R_2\right) = 1-\varepsilon_k$$

- *Consider player 2 :*

$$\text{-}\ H_2\left(\sigma_1^k, L_2\right) = \tfrac{\varepsilon_k^2}{2} - 5\tfrac{\varepsilon_k^2}{2} + 2\varepsilon_k + 2 - 2\varepsilon_k - 2\varepsilon_k^2 = 2 - 4\varepsilon_k^2$$
$$\text{-}\ H_2\left(\sigma_1^k, R_2\right) = 2\varepsilon_k + 2 - 2\varepsilon_k - 2\varepsilon_k^2 = 2 - 2\varepsilon_k^2$$

- *Hence, for all $\varepsilon_k > 0$ sufficiently small, $H_2\left(\sigma_1^k, R_2\right) > H_2\left(\sigma_1^k, L_2\right)$ .*

- *Thus, $\sigma_2^k$ (to play $R_2$ with probability $1-\varepsilon_k$) is the best-reply in $\Sigma_2\left(\varepsilon_k\right)$ against $\sigma_1^k$*

- *Consider now player 1 :*

$$\text{-}\ H_1\left(L_1L_1', \sigma_2^k\right) = 3\varepsilon_k + \left(1-\varepsilon_k\right)$$
$$\text{-}\ H_1\left(L_1R_1', \sigma_2^k\right) = 1-\varepsilon_k$$
$$\text{-}\ H_1\left(R_1L_1', \sigma_2^k\right) = H_2\left(R_1R_1', \sigma_2^k\right) = 2$$

- *Hence, for sufficiently small $\varepsilon_k > 0, \sigma_1^k$ is a best-reply in $\Sigma_1\left(\frac{\varepsilon_k^2}{2}\right)$ against $\sigma_2^k$.*

- *Thus, $\left(R_1R_1', R_2\right)$ is a perfect equilibrium in the normal form .*

However we can fix it by introducing Agent Normal Form Game where every information set corresponds to an agent, and every player controls its agents.

**Definition 89.** *Agent Normal Form Game For EFG $\Gamma$, $\forall i \in I$, let $B_i = \left\{b_i^1, \ldots, b_i^{K_i}\right\}$ and define the set of agents of G as $l^a = \bigcup_{i \in I} \bigcup_{t=1}^{K_i} (i.t)$, and for every $(i.t) \in I^a$, define $S_{(i.t)}^a = C_{b_i^t}$ and $h_{(i.t)}^a = h_i$.*
*Let $G^a = \left(I^a, \left(S_{(i.t)}^a\right)_{(i.t) \in /^a}, \left(h_{(i.t)}^a\right)_{(i.t) \in I^a}\right)$ be the agent-normal form of $\Gamma$*

**Theorem 35.** ***Let $\Gamma$ be a EFG and let $G^a$ be its corresponding agent-normal form of $\Gamma$. Then, $\sigma$ is a perfect equilibrium of $\Gamma$ if and only if $\sigma$ is a perfect equilibrium of $G^a$.***

## 3.2 Correlated Equlibrium

Recall that for a mixed strategy profile $s \in S$ we defined $\Pr_s(a) = \prod_{i \in I} s^i\left(a^i\right), \forall a \in A$ so that $\Pr_s \in \Delta(A)$. We begin by considering what expected payoffs can be generated from such amixed strategy profile. We have the following definition.

**Definition 90.** *The set of feasible payoffs, denoted by $F$, is defined as $F := \{x \in \mathbb{R}^n : \exists s \in S$ s.t. $x = \sum_{a \in A} \Pr_s(a) u(a)\}$*

**Corollary 6.** *Note that $F$ is compact and closed in $\mathbb{R}^n$, but not necessarily convex.*

---

**Example 22.** *Consider game:*

| | | Mr 2 | |
|---|---|---|---|
| | | L | R |
| Mr 1 | T | 3,1 | 0,0 |
| | B | 0,0 | 1,3 |

where $s^1 = (p, 1-p)$ and $s^2 = (q, 1-q)$. Then

$$F = \left\{[3pq + (1-p)(1-q), pq + 3(1-p)(1-q)] : (p,q) \in [0,1]^2\right\}.$$

Clearly this set is not convex.

Now suppose that we do not restrict ourselves to mixed strategy profiles in $S$. In particular, suppose we allow any probability distribution over the elements in $A$. Then we have the following.

---

**Definition 91.** *The set of feasible payoffs in $\Delta(A)$, denoted by $F^*$, is defined as $F^* := \{x \in \mathbb{R}^n : \exists \mu \in \Delta(A)$ s.t. $x = \sum_{a \in A} \mu(a) u(a)\}$.*

Note that $\{Pr_s : s \in S\} \subsetneq \Delta(A)$.

**Definition 92 (Correlated Strategy).** *A correlated strategy (CS) is a probability distribution $\mu \in \Delta(A)$.*

Before introducing the concept of a correlated equilibrium we need some new notation. For all $i \in I$ and $b^i \in A^i$ let

$$\mu\left(b^i\right) := \sum_{a^{-i} \in A^{-i}} \mu\left(b^i, a^{-i}\right), \text{ and}$$

$$\mu\left(a^{-i} \mid b^i\right) := \frac{\mu\left(b^i, a^{-i}\right)}{\mu\left(b^i\right)}$$

**Definition 93** (Correlated Equilibrium)**.** *A correlated equilibrium is a CS* $\mu \in \Delta(A)$ *such that* $\forall i \in I, b^i \in A^i$, $\mu\left(b^i\right) > 0$ *implies* $b^i = \arg\max_{c^i \in A^i} \sum_{a^{-i} \in A^{-i}} u^i\left(c^i, a^{-i}\right) \mu\left(a^{-i} \mid b^i\right)$. *That is,* $b^i$ *maximizes player i 's expected utility when* $b^i$ *has been "recommended."*

---

**Example 23.** *Let* $I = \{1, 2\}$ *and define the game* $\Gamma$

Mr 2

| | | L | R |
|---|---|---|---|
| Mr 1 | T | 5,1 | 0,0 |
| | B | 4,4 | 1.5 |

Consider the following correlated strategy.

$$\mu = \left(\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right) \in \Delta(\{(T,L),(T,R),(B,L),(B,R)\}) = \Delta(A)$$

Then $\mu(T) = \frac{1}{3}, \mu(B) = \frac{2}{3}, \mu(\cdot \mid T) = (1,0) \in \Delta(\{L,R\})$ and $\mu(\cdot \mid B) = \left(\frac{1}{2}, \frac{1}{2}\right) \in (\{L,R\})$. Then, $\mu$ is not a correlated equilibrium. To see this, note that $\mu(B) > 0$ but

$$\begin{aligned} u^1(B) &= u^1(B,L)\mu(L \mid B) + u^1(B,R)\mu(R \mid B) = 4 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{5}{2} \\ &< 5 = u^1(T,L)\mu(L \mid T) + u^1(T,R)\mu(R \mid T) = u^1(T) \end{aligned}$$

Before getting to our next formal result, notice that

$$b^i = \arg\max_{c^i \in A^i} \sum_{a^{-i} \in A^{-i}} u^i\left(c^i, a^{-i}\right) \mu\left(a^{-i} \mid b^i\right) \quad \Longleftrightarrow$$

$$\begin{aligned} &\textstyle\sum_{a^{-i} \in A^{-i}} u^i\left(b^i, a^{-i}\right) \mu\left(a^{-i} \mid b^i\right) - \sum_{a^{-i} \in A^{-i}} u^i\left(c^i, a^{-i}\right) \mu\left(a^{-i} \mid b^i\right) \geq 0, \quad \forall c^i \in A^i \\ &\Longleftrightarrow \textstyle\sum_{a^{-i} \in A^{-i}} \left[u^i\left(b^i, a^{-i}\right) - u^i\left(c^i, a^{-i}\right)\right] \frac{\mu\left(b^i, a^{-i}\right)}{\mu\left(b^i\right)} \geq 0, \quad \forall c^i \in A^i \\ &\Longleftrightarrow \textstyle\sum_{a^{-i} \in A^{-i}} \left[u^i\left(b^i, a^{-i}\right) - u^i\left(c^i, a^{-i}\right)\right] \mu\left(b^i, a^{-i}\right) \geq 0, \quad \forall c^i \in A^i \end{aligned}$$

---

**Theorem 36** (Characterization of CEq)**.** *A* ***CS*** $\mu \in \Delta(A)$ ***is a correlated equilibrium if, and only if,*** $\forall i \in I, b^i \in A^i$, ***if*** $\mu\left(b^i\right) > 0$ ***then*** $\sum_{a^{-i} \in A^{-i}} \left[u^i\left(b^i, a^{-i}\right) - u^i\left(c^i, a^{-i}\right)\right] \mu\left(b^i, a^{-i}\right) \geq 0, \quad \forall c^i \in A^i$

**Example 24.** *Characterize the set of correlated equilibria of the following two player game*

**Lemma 25.** *The set of correlated equilibria (CE) is*

| | | Mr 2 | |
|---|---|---|---|
| | | L | R |
| Mr 1 | T | 5,5 | 1,7 |
| | B | 7,1 | -2,2 |

1. *nonempty,*

2. *closed,*

3. *convex*

*Proof.* 1. To see that $CE$ is nonempty, first recall that the set of Nash equilibria is nonempty. Then for any $s \in NE$, define $\mu = \Pr_s \in CE$. Hence, $\mu$ is a correlated equilibrium.

2. Recall that a correlated strategy $\mu \in \Delta(A)$ is a correlated equilibrium if and only if

$$\sum_{a^{-i} \in A^{-i}} \left[u^i\left(b^i, a^{-i}\right) - u^i\left(a^i, a^{-i}\right)\right] \mu\left(b^i, a^{-i}\right) \geq 0 \quad \forall i \in I, b^i, a^i \in A^i$$

Define a vector-valued function $F : \Delta(A) \to \mathbb{R}^D$, where $D = \left(\#A^1\right)^2 + \left(\#A^2\right)^2 + \cdots + (\#A^n)^2$ by

$$F(\mu) = \begin{pmatrix} \sum_{a^{-i} \in A^{-i}} \left[u^i\left(b^i, a^{-i}\right) - u^i\left(a^i_1, a^{-i}\right)\right] \mu\left(b^i, a^{-i}\right) \\ \vdots \\ \sum_{a^{-i} \in A^{-i}} \left[u^i\left(b^i, a^{-i}\right) - u^i\left(a^i_{\#A^i}, a^{-i}\right)\right] \mu\left(b^i, a^{-i}\right) \end{pmatrix}_{i \in I, b^i \in A^i}$$

Then we can equivalently define the set of correlated equilibria as $CE = \{\mu \in \Delta(A) \mid F(\mu) \geq 0\}$. That is, the upper contour set of $F(\cdot)$ at 0. But since $F(\cdot)$ is clearly continuous in $\mu$, this set is closed.

3. Take any $\mu, \mu' \in CE$ and $\alpha \in (0,1)$. Then

$$\sum_{a^{-i} \in A^{-i}} u^i\left(b^i, a^{-i}\right) \mu\left(b^i, a^{-i}\right) \geq \sum_{a^{-i} \in A^{-i}} u^i\left(a^i, a^{-i}\right) \mu\left(b^i, a^{-i}\right) \quad \forall i \in I, b^i, a^i \in A^i$$

and

$$\sum_{a^{-i} \in A^{-i}} u^i\left(b^i, a^{-i}\right) \mu'\left(b^i, a^{-i}\right) \geq \sum_{a^{-i} \in A^{-i}} u^i\left(a^i, a^{-i}\right) \mu'\left(b^i, a^{-i}\right) \quad \forall i \in I, b^i, a^i \in A^i$$

Multiplying these inequalities by $\alpha$ and $(1-\alpha)$, respectively, and summing gives

$$\begin{aligned}\alpha \textstyle\sum_{a^{-i}\in A^{-i}} \quad & u^i\left(b^i,a^{-i}\right)\mu\left(b^i,a^{-i}\right)+(1-\alpha)\textstyle\sum_{a^{-i}\in A^{-i}} u^i\left(b^i,a^{-i}\right)\mu'\left(b^i,a^{-i}\right)\\ & \geq \alpha \textstyle\sum_{a^{-i}\in A^{-i}} u^i\left(a^i,a^{-i}\right)\mu\left(b^i,a^{-i}\right)+(1-\alpha)\textstyle\sum_{a^{-i}\in A^{-i}} u^i\left(a^i,a^{-i}\right)\mu'\left(b^i,a^{-i}\right)\end{aligned}$$

for all $i\in I, b^i, a^i \in A^i$. Rearranging gives

$$\begin{aligned}&\textstyle\sum_{a^{-i}\in A^{-i}}\left[\alpha\mu\left(b^i,a^{-i}\right)+(1-\alpha)\mu'\left(b^i,a^{-i}\right)\right]u^i\left(b^i,a^{-i}\right)\\ &\geq \textstyle\sum_{a^{-i}\in A^{-i}}\left[\alpha\mu\left(b^i,a^{-i}\right)+(1-\alpha)\mu'\left(b^i,a^{-i}\right)\right]u^i\left(a^i,a^{-i}\right) \quad \forall i\in I, b^i, a^i\in A^i\end{aligned}$$

which implies $\alpha\mu+(1-\alpha)\mu'\in CE$. Since $\mu,\mu'$ and $\alpha$ have been arbitrary, $CE$ is convex.

□

**Corollary 7.** *The set of correlated equilibria outcomes, $CEO=\{x : x=\sum_{a\in A}\mu(a)u(a), \mu\in CE\}$, is (i) nonempty, (ii) closed, and (iii) convex.*

*Proof.* 1. This is immediate from previous lemma

2. Take any sequence of correlated equilibrium outcomes $(x_m)\in CEO^{\infty}$ with $x_m\to x\in\mathbb{R}^n$. Then, for each $m\in\mathbb{N}, \exists\mu_m\in CE$ such that $x_m=\sum_{a\in A}\mu_m(a)u(a)$. Then we have a corresponding sequence $(\mu_m)\in CE^{\infty}$ of probability measures on $A$. Moreover, $x_m\to x$ implies that $\mu_m\to\mu$. Since $CE$ is closed, by Proposition 2.19, $\mu\in CE$. Since $x=\sum_{a\in A}\mu(a)u(a)$, $x\in CEO$. Hence, since $(x_m)$ has been arbitrary, $CEO$ is closed.

3. Take any $x,x'\in CEO$ and $\lambda\in(0,1)$. Then $\exists\mu,\mu'\in CE$ such that

$$x=\sum_{a\in A}\mu(a)u(a) \quad \text{and} \quad x'=\sum_{a\in A}\mu'(a)u(a)$$

Then,

$$\lambda x=\sum_{a\in A}\lambda\mu(a)u(a) \quad \text{and} \quad (1-\lambda)x'=\sum_{a\in A}(1-\lambda)\mu'(a)u(a).$$

Summing these equalities gives

$$\lambda x+(1-\lambda)x'=\sum_{a\in A}\left[\lambda\mu(a)+(1-\lambda)\mu'(a)\right]u(a).$$

Since $CE$ is convex by last $\lambda\mu+(1-\lambda)\mu'\in CE$. Then $\lambda x+(1-\lambda)x'\in CEO$ by definition. Hence, since $x,x'$ and $\lambda$ have been arbitrary, $CEO$ is convex.

□

## 3.3 Communication games

**Definition 94** (Communication mechanism ). *A communication mechanism is a triple $\left\langle X, (X^i)_{i\in I}, \nu\right\rangle$, where $X^i$ is the signal given to player $i$, $X = \times_{i\in I} X^i$, and $\nu \in \Delta(X)$. The communication mechanism functions as follows.*

1. *$x \in X$ is drawn according the the probability distribution $\nu$.*
2. *For each $i \in I$, $x^i$ is communicated to player $i$ only.*
3. *Players choose $a^i \in A^i$ according to a strategy $\sigma^i : X^i \to \Delta\left(A^i\right)$.*
4. *The resulting vector $\left(\sigma^1, \ldots, \sigma^n\right)$ is an equilibrium of this extended game if for all $i \in I$,*

$$\sigma^i = \underset{\rho: X^i \to \Delta\left(A^i\right)}{\operatorname{argmax}} \int_{x\in X} u^i\left[\rho\left(x^i\right), \sigma^{-i}\left(x^{-i}\right)\right] \nu(dx)$$

   Or, equivalently, if $\sigma^i$ solves

$$\int_{x^i\in X^i}\left[\max_{s^i\in\Delta\left(A^i\right)} \int_{x^{-i}\in X^{-i}} u^i\left[s^i, \sigma^{-i}\left(x^{-i}\right)\right] \nu\left(dx^{-i} \mid x^i\right)\right] \nu\left(dx^i\right)$$

**Definition 95.** *If $X^i = X$ for all $i \in I$, then the mechanism is a public communication mechanism.*

**Theorem 37.** ***Let $\left(X, (X^i)_{i\in I}, \nu\right)$ be an extended game and suppose $\left(\sigma^1, \ldots, \sigma^n\right)$, where $\sigma^i : X^i \to \Delta\left(A^i\right) \forall i \in I$, constitutes a general correlated equilibrium (GCE) of this game. Then the correlated strategy $\mu \in \Delta(A)$ defined by***

$$\mu(a) := \sum_{x\in X}\left[\prod_{i\in I} \sigma^i\left(x^i\right)\left(a^i\right)\right] \nu(x), \forall a \in A$$

***is a correlated equilibrium that induces the same outcome as $\left(\sigma^i\right)_{i\in I}$.***

*Proof.* First I establish that $\mu$ produces the same outcome. Let $\left(y^i\right)_{i\in I}$ be a GCE outcome of $\left(\sigma^i\right)_{i\in I}$ and fix $i \in I$ arbitrarily. Then by definition

$$y^i = \sum_{x\in X} u^i[\sigma(x)]\nu(x).$$

Substituting in the definition of $u^i[\sigma(x)]$ gives

$$\begin{aligned} y^i &= \sum_{x\in X}\left(\sum_{a\in A}\left[\prod_{j\in I}\sigma^j\left(x^j\right)\left(a^j\right)\right]u^i(a)\right)\nu(x) \\ &= \sum_{a\in A}\left(\sum_{x\in X}\left[\prod_{j\in I}\sigma^j\left(x^j\right)\left(a^j\right)\right]\nu(x)\right)u^i(a) \\ &= \sum_{a\in A}\mu(a)u^i(a), \end{aligned}$$

which is the definition of a correlated strategy outcome for player $i$ with the measure $\mu$. Hence, since $i$ has been arbitrary, $\mu$ induces the same outcome as $\left(\sigma^i\right)_{i\in I}$. It remains to show that $\mu \in CE$. Fix $i \in I$ and take any $b^i \in A^i$ such that $\mu\left(b^i\right) > 0$. Then by definition of $\mu$

$$\begin{aligned} \mu\left(b^i\right) &= \sum_{a^{-i}\in A^{-i}}\mu\left(b^i, a^{-i}\right) \\ &= \sum_{a^{-i}\in A^{-i}}\left(\sum_{x\in X}\left[\sigma^i\left(x^i\right)\left(b^i\right)\prod_{j\in I\setminus\{i\}}\sigma^j\left(x^j\right)\left(a^j\right)\right]\nu(x)\right) > 0 \end{aligned}$$

that is, the strategy $\left(\sigma^i\right)_{i\in I}$ also puts strictly positive probability on the action $b^i$ for some signal $x^i \in X^i$. Then since $\left(\sigma^i\right)_{i\in I}$ is a GCE we have that, $\forall c^i \in A^i$

$$\begin{aligned} &\textstyle\sum_{x\in X}\left(\sum_{a^{-i}\in A^{-i}}\left[\sigma^i\left(x^i\right)\left(b^i\right)\prod_{j\in I\setminus\{i\}}\sigma^j\left(x^j\right)\left(a^j\right)\right]u^i\left(b^i,a^{-i}\right)\right)\nu(x) \\ &\geq \textstyle\sum_{x\in X}\left(\sum_{a^{-i}\in A^{-i}}\left[\sigma^i\left(x^i\right)\left(b^i\right)\prod_{j\in I\setminus\{i\}}\sigma^j\left(x^j\right)\left(a^j\right)\right]u^i\left(c^i,a^{-i}\right)\right)\nu(x) \\ \iff &\textstyle\sum_{a^{-i}\in A^{-i}}\left(\sum_{x\in X}\left[\sigma^i\left(x^i\right)\left(b^i\right)\prod_{j\in I\setminus\{i\}}\sigma^j\left(x^j\right)\left(a^j\right)\right]\nu(x)\right)u^i\left(b^i,a^{-i}\right) \\ \iff &\textstyle\sum_{a^{-i}\in A^{-i}}u^i\left(b^i,a^{-i}\right)\mu\left(b^i,a^{-i}\right) \geq \sum_{a^{-i}\in A^{-i}}u^i\left(c^i,a^{-i}\right)\mu\left(b^i,a^{-i}\right) \end{aligned}$$

Since $i$ has been arbitrary, Theorem characterizing CS provides that $\mu$ is a correlated equilibrium.

$\square$

$$\text{PE} \subsetneq \text{SE} \subsetneq \text{(weak)PBNE} \subsetneq \text{SPE} \subsetneq \text{NE} \neq \emptyset$$