Recitations 25

Question 1 Optimal Auctions [220 IV.1 Spring 2016 majors]

A seller owns an object, and values it at 0. There is a buyer with valuation $v \sim U[0,1]$. The seller does not know the buyer's valuation, and designs an optimal mechanism to fulfill some objective, whereby the seller asks for the buyer's valuation and then awards the object to the buyer with probability $q(v)$ and charges the buyer an amount of money $p(v)$ if the buyer reported a valuation v

1. Assume that the seller wants to maximize own profit, $p(v)$

   a) Show that the seller's virtual surplus can be written as

   $$2v - 1$$

   b) Describe the seller's optimal auction.

2. Assume instead that the seller wants to maximize a weighted average of own profit, $p(v)$ (with weight $\alpha \in [0,1]$), and consumer surplus, $v - p(v)$ (with weight $1 - \alpha$).

   a) Show that the seller's virtual surplus can be written as

   $$(3\alpha - 1)v + 1 - 2\alpha$$

   b) Describe the seller's optimal auction as a function of $\alpha \in [0,1]$

Question 2 Nonlinear pricing [181 IV.3 Spring 2014 majors]

Screening. A monopolist faces a single consumer with utility function $u = \theta q - \frac{1}{2}q^2 - T$, where$\theta$ is private information of the consumer, q is the level of consumption and T is the amount of money that the consumer pays the monopolist. The monopolist's cost of producing q equals $\frac{1}{2}cq^2$ for some constant $c > 0$. The consumer's reservation utility equals 0. The (Pareto) CDF ofθ is $F(\theta) = 1 - \theta^{\alpha}$ for all $\theta \in [1;\infty)$, where $\alpha > 1$.

(a) Derive the monopolist's optimal bundle (q,T) assuming that it knows θ.

(b) Write down the monopolist's nonlinear pricing problem.

(c) Derive the optimal nonlinear pricing schedule and compare it with (a).

(d) What can you say about distortions at the top? (Here top means $\theta \to \infty$.)

Solution

1. The problem of the monopolist for perfect observability is:

$$\max_{q,T} T - \frac{c}{2}q^2 \quad \text{s.t. } \theta q - \frac{1}{2}q^2 - T \geq 0$$

It must be that the constraint is met with equality, or else T can be increased, increasing profits.:

$$T = \theta q - \frac{1}{2}q^2$$

The problem is then:

$$\max_{q} \theta q - \frac{1+c}{2}q^2$$

The FOC:

$$q = \frac{\theta}{1+c}$$

2. Under uncertainty the problem of the monopolist is:

$$\begin{array}{c} \max_{q(\theta),T(\theta)} \int \left(T(\theta) - \frac{c}{2}q(\theta)^2\right) dF(\theta) \\ \text{IR} \quad \theta q(\theta) - \frac{1}{2}q(\theta)^2 - T(\theta) \geq 0 \\ \text{IC} \quad \theta q(\theta) - \frac{1}{2}q(\theta)^2 - T(\theta) \geq \theta q(\theta') - \frac{1}{2}q(\theta')^2 - T(\theta') \end{array}$$

3. Following Myerson's Theorem (1981) the profit maximization problem is equivalent to:

$$\max_{q(\theta),T(\theta)} \int \left(T(\theta) - \frac{c}{2}q(\theta)^2\right) dF(\theta)$$

$$\begin{array}{l} U(\underline{\theta}) \geq 0 \\ U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(x)dx \end{array}$$

q is monotone increasing
where $U(\theta) = \theta q(\theta) - \frac{1}{2}q(\theta)^2 - T(\theta)$. The first condition can be solved by setting $T(\underline{\theta}) = \theta q(\underline{\theta}) - \frac{1}{2}q(\underline{\theta})^2$ so that $U(\underline{\theta}) = 0$, the value of $q(\underline{\theta})$ is found later. With this

$$U(\theta) = \int_{\underline{\theta}}^{\theta} q(x)dx \longrightarrow T(\theta) = \theta q(\theta) - \frac{1}{2}q(\theta)^2 - \int_{\underline{\theta}}^{\theta} q(x)dx$$

which allows to eliminate $T(\theta)$ from the problem. The problem can be then solved ignoring the last restriction (over q) and verifying its monotonicity ex-post.

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\infty} \left(\theta q(\theta) - \frac{1+c}{2}q(\theta)^2 - \int_{\underline{\theta}}^{\theta} q(x)dx\right) dF(\theta)$$

Note the following by defining $Q(\theta)=\int_{\underline{\theta}}^{\theta} q(x)dx$:

$$\int_{\underline{\theta}}^{\infty}\int_{\underline{\theta}}^{\theta} q(x)dxdF(\theta)=\int_{\underline{\theta}}^{\infty} Q(\theta)dF(\theta)=[Q(\theta)F(\theta)]_{\underline{\theta}}^{\infty}-\int_{\underline{\theta}}^{\infty}\frac{\partial Q(\theta)}{\partial\theta}F(\theta)d\theta$$

By the fundamental theorem of calculus: $\frac{\partial Q(\theta)}{\partial\theta}=q(\theta)$, then:

$$\int_{\underline{\theta}}^{\infty}\int_{\underline{\theta}}^{\theta} q(x)dxdF(\theta)=\int_{\underline{\theta}}^{\infty} q(x)dx-\int_{\underline{\theta}}^{\infty} q(\theta)F(\theta)d\theta=\int_{\underline{\theta}}^{\infty}(1-F(\theta))q(\theta)d\theta$$
$$\int_{\underline{\theta}}^{\infty}\int_{\underline{\theta}}^{\theta} q(x)dxdF(\theta)=\int_{\underline{\theta}}^{\infty}\frac{1-F(\theta)}{f(\theta)}q(\theta)dF(\theta)$$

recalling $dF(\theta)=f(\theta)d\theta$. Replacing:

$$\max_{q(\theta)}\int_{\underline{\theta}}^{\infty}\left(\left(\theta-\frac{1-F(\theta)}{f(\theta)}\right)q(\theta)-\frac{1+c}{2}q(\theta)^2\right)dF(\theta)$$

The optimal q for each θ is obtained from the FOC against $q(\theta)$

$$q(\theta)=\frac{1}{1+c}\left(\theta-\frac{1-F(\theta)}{f(\theta)}\right)$$

Since θ is Pareto:

$$F(\theta)=1-\theta^{-\alpha}\quad f(\theta)=\alpha\theta^{-\alpha-1}$$

Replacing:

$$q(\theta)=\frac{1}{1+c}\left(\theta-\frac{\theta^{-\alpha}}{\alpha\theta^{-\alpha-1}}\right)=\frac{\theta}{1+c}\left(\frac{\alpha-1}{\alpha}\right)$$

This function is monotone increasing as required.

4. Under the Pareto distributions there are always distortions, the bundle offered to any type is different than the one offered under perfect information. Under the Pareto distribution there is no actual "top" type.

Question 3 from Myerson 1981 [Final 2017]

Correlated Types. A seller owns an object, and values it at 0. There are two buyers, 1 and 2, each with valuation $v_i\in\{10,100\}$ for $i\in\{1,2\}$.

The seller does not know the buyers' valuations, and designs an optimal mechanism to maximize revenue, whereby the seller asks for each buyer's valuation and then awards the object to buyer i with probability $q_i(v)$ and charges the buyer an amount of money $p_i(v)$, where $v=(v_1,v_2)$ is the profile of valuations reported to the seller by the buyers. Buyers' outside option is normalized to zero.

1. Assume that buyers' valuations are stochasticaly independent and identically distributed, where $\Pr(v_i=10)=\Pr(v_i=100)=\frac{1}{2}$ for every buyer i. Describe the seller's optimal auction.

2. Henceforth, assume instead that buyers' valuations are correlated:

$$\Pr(10,10) = \Pr(100,100) = \frac{1}{3}, \quad \Pr(10,100) = \Pr(100,10) = \frac{1}{6}$$

   a) Calculate the expected revenue to the seller from the optimal auction you derived in part (a).

   b) Consider the following auction: if both bidders have high valuation (100), then sell the object to one of them for the price of 100, randomizing equally between the buyers. If one bidder has a high valuation (100) and the other has a low valuation (10), then sell the object to the high bidder for 100, and charge the low bidder 30 (but do not give him the object). If both bidders have a low valuation (10), then give 15 units of money to one of them, and give 5 units of money and the object to the other, again choosing the recipient of the object at random. Write down both the probability that a buyer gets the good and the expected payment by each buyer conditional on each profile of valuations.

   c) Show that honesty is a Nash equilibrium in this auction

   d) Is this auction efficient?

   e) Calculate each buyer's expected payoff from this auction and show that it is individually rational.

   f) Calculate the seller's expected revenue from this auction and compare it with the revenue from the auction in part (1). Determine whether or not this auction is optimal for the seller.