Recitation 5

[Definitions used today]

- Topkis theorem, Supermodularity, Increasing Differences
- Recursive and dynamically consistent family of utility function, time separablity, ICC axiom

Question 1

Consider the following utility functions

a $u(c_1, c_2, c_3) = \min\{2c_1 + c_2 + c_3, c_1 + c_2 + 2c_3\}$

b $u(c_1, c_2, c_3) = c_1 + \sqrt{c_2} + \sqrt{c_3}$

1. Show that (a) does not have state-separable representation
2. Show that (b) does not have expected utility representation
3. Find $\pi \in \Delta \subset \mathbb{R}^3$ s.t. $u(c_1, c_2, c_3)$ is strictly risk averse with respect to π
4. Show that there is no $\pi\Delta \subset \mathbb{R}^3$ s.t. $u(c_1, c_2, c_3)$ is strictly risk averse with respect to π

Question 2 [Properties of state separable u]

a Prove that every recursive family of utility functions $\{U_t\}$ is dynamically consistent if the aggregator function $G(\cdot, \cdot)$ is strictly increasing in continuation

b $S \geq 3$ and $\succeq$ increasing and continuous. Prove that $\succeq$ has state-separable representation then ICC holds.

c For $S = 2$ all increasing functions obey ICC. Show that for $u(c_1, c_2) = c_1\sqrt{c_2} + c_1 + c_2$ it does not have state separable utility function

Question 3 [Topkis theorem]

If S is a lattice, f is supermodular in x for fixed t, and f has nondecreasing differences in $(x; t)$, then $\varphi^*(t) = \arg\max_{x\in S} f(x, t)$ is monotone nondecreasing in t.

Solution 3

Monotone comparative statics is based on mathematical theories of super-modularity and vector lattices developed by D.M. Topkis and others (see the book by Topkis (1998), Topkis (1978) or Milgrom Shannon (1994).

Definition 0.1. *For two vectors $x, y \in \mathbb{R}^n$, the* ***lattice operations*** *are the* ***supremum- join****:*

$$x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$$

and the ***infimum- meet****:*

$$x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}).$$

Note: $x + y = x \vee y + x \wedge y$.

Definition 0.2. *A set $S \subseteq \mathbb{R}^n$ is said to be a **lattice** if $x \vee y \in S$ and $x \wedge y \in S$ for all $x, y \in S$.*

Interval $[a, b] \subset \mathbb{R}^n$ and $\mathbb{R}^n_+$ is a lattice.

Definition 0.3. *Let $X \subset \mathbb{R}^n$ be a lattice. A function $f : X \to \mathbb{R}$ is **supermodular** on X if*

$$f(x \vee y) - f(x) \geq f(y) - f(x \wedge y),\ \forall x, y \in X \tag{0.1}$$

Note: *An equivalent formulation:*

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y),\ \forall x, y \in X$$

Let X be a subset of $\mathbb{R}^n$. We will generally assume that either $X = \mathbb{R}^n$ or $X = \mathbb{R}^n_+$. Let $T \subseteq \mathbb{R}^m$. For a function $f : X \times T \to \mathbb{R}^n$ and a set $S \subseteq X$, consider the problem

$$\max_{x \in S} f(x, t)$$

Let the correspondence $\varphi^*(t)$ denote the set of solutions for a given t, i.e.,

$$\varphi^*(t) = \arg\max_{x \in S} f(x, t)$$

Definition 0.4. *$\leq_{sso}$ is the **strong set order** if for every $x \in \varphi^*(t)$ and $x' \in \varphi^*(t')$, $x \wedge x' \in \varphi^*(t)$ and $x \vee x' \in \varphi^*(t')$. Note that if $\varphi^*(t)$ and $\varphi^*(t')$ are singletons, the strong set order is the same as the usual order on vectors.*

*The correspondence φ^* is **monotone nondecreasing in** t if*

$$\varphi^*(t) \leq_{sso} \varphi^*(t'),\ \forall t \leq t'$$

Definition 0.5. *A function $f : X \times T \to \mathbb{R}$ has **nondecreasing differences in** $(x; t)$ if the difference $f(x', t) - f(x, t)$ is monotone nondecreasing in t for every $x' \geq x$, i.e.,*

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t),\ \forall x' \geq x,\ \forall t' \geq t.$$

Theorem 0.6. Topkis Theorem

If S is a lattice, f is supermodular in x, and f has nondecreasing differences in $(x; t)$, then $\varphi^*(t) = \arg\max_{x \in S} f(x, t)$ is monotone nondecreasing in t.

Proof. **Step 1:** Show that $x \vee x' \in \varphi^*(t')$

Let $t \leq t'$, let $x \in \varphi^*(t)$, and let $x' \in \varphi^*(t')$. First, we will show that $x \vee x' \in \varphi^*(t')$. Supermodularity in x implies that

$$f(x \vee x', t') \geq f(x', t') + f(x, t') - f(x \wedge x', t'). \tag{0.2}$$

Nondecreasing differences in $(x; t)$ implies

$$f(x, t') - f(x \wedge x', t') \geq f(x, t) - f(x \wedge x', t). \tag{0.3}$$

Since S is a lattice, $x \wedge x' \in S$. Using this and the fact that $x \in \varphi^*(t)$, we have

$$f(x, t) - f(x \wedge x', t) \geq 0 \tag{0.4}$$

Combining (0.2), (0.3), and (0.4), we get

$$f(x \vee x', t') - f(x', t') \geq 0$$

and so

$$f(x \vee x', t') \geq f(x', t') \tag{0.5}$$

Again, since S is a lattice, $x \vee x' \in S$. Since $x' \in \varphi^*(t')$, (0.5) implies that $x \vee x' \in \varphi^*(t')$.

Step 2: Show that $x \wedge x' \in \varphi^*(t)$.

Supermodularity in x implies that

$$f(x \wedge x', t) \geq f(x, t) + f(x', t) - f(x \vee x', t). \tag{0.6}$$

Nondecreasing differences in $(x; t)$ implies

$$f(x \vee x', t') - f(x', t') \geq f(x \vee x', t) - f(x', t). \tag{0.7}$$

We can rearrange this as

$$f(x', t) - f(x \vee x', t) \geq f(x', t') - f(x \vee x', t'). \tag{0.8}$$

Since S is a lattice, $x \vee x' \in S$. Using this and the fact that $x' \in \varphi^*(t')$, we have

$$f(x', t') - f(x \vee x', t') \geq 0$$

$$f(x', t') - f(x \vee x', t') \geq 0 \tag{0.9}$$

Combining (0.6) and (0.9), we get

$$f(x \wedge x', t) \geq f(x, t). \tag{0.10}$$

Again, since S is a lattice, $x \wedge x' \in S$. Since $x \in \varphi^*(t)$, (0.10) implies that $x \wedge x' \in \varphi^*(t)$. □

Question 4 254 [I.1 Spring 2018 majors]

Consider the problem of finding a Pareto optimal allocation of aggregate resources $\omega \in \mathbb{R}^n_+$ in an economy with two agents:

$$\max_x \mu_1 u_1(x) + \mu_2 u_2(\omega - x)$$
$$\text{subject to } 0 \leq x \leq \omega$$

where $u_i : \mathbb{R}^n_+ \to \mathbb{R}$ are agents' utility functions (assumed continuous) and $\mu_i > 0$ are welfare weights for $i = 1, 2$. Let $x^*(\mu_1, \mu_2)$ be the set of solutions.

a State a definition of utility function u_i being supermodular. Show that if u_i is supermodular, then $u_i(\omega - x)$ is supermodular in x

b Show that, if u_1 and u_2 are strictly increasing and supermodular in x then $x^*(\mu_1, \mu_2)$ is non-decreasing in μ_1. You may assume that $x^*(\mu)$ is single-valued. Is $x^*(\mu_1, \mu_2)$ non-increasing in μ_2? Justify your answer. If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.

c Under what conditions on u_1 and u_2 is the solution $x^*(\mu_1, \mu_2)$ unique. Justify your answer.