Recitations 1

Jakub Pawelczak

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RECITATION 1

me: JAKUB PAWELCZAK

mail: PAWEL042@UMN.EDU

www: JAKUBPAWELCZAK.COM

office hours: MONDAY 5:30-6:30

ZOOM

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Recitation 1

[Definitions used today]

- (strictly) convex, concave, quasi convex, quasi concave functions
- production set Y , input requirement set V, transformation function T, production function f
- DRS, IRS, CRS of production function
- NIRS, NDRS, CRS of production set
- Meet and Joint, Lattice, Supermodularity of a function, Increasing Differences function

Question 1 [Production function/set]

(a) Show that if $f(x)$ is concave $\Rightarrow$ production set Y is convex.

(b) Prove that for a convex production set $Y \Rightarrow$ input requirement set V is convex. Prove that converse is not true.

(c) Show that $f(x)$ is quasi concave function $\iff$ input requirement set V is convex.

(d) Show that if $f(x)$ is strictly concave and $f(0) = 0 \Rightarrow f$ exhibits DRS

Question 2 [Properties of Y, f]

Let $f(x)$ be a production function and Y a production set associated with f. Show the following propositions hold

(a) if f exhibits DRS then Y exhibits NIRS

(b) if f exhibits IRS then Y exhibits NDRS

(c) if f exhibits CRS then Y exhibits CRS

Question 3 [Supermodularity] 89 [I.1 Fall 2009 majors]

Show that following functions are **supermodular**

(a) the Cobb-Douglas production function $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, where $\forall_i\ \alpha_i > 0$, and $\sum_i \alpha_i < 1$

(b) the Leontief function $f(x) = \min_i\{\alpha_i x_i\} \quad \forall_i\ \alpha_i > 0$

Question 4 [Properties of Y]

Prove following properties

(a) Assume that for Y closed and convex , $Y \subset \mathbb{R}^L$ s.t. $0 \in Y$. Free disposal property $Y - \mathbb{R}^L_+ \subset T \iff \mathbb{R}^L_- \subset Y$

(b) If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient

(c) If Y is a convex set, then supply correspondence $s^*(p)$ is a convex set.

Question 5 165 [I.1 Fall 2013 minors]

Consider a production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ with n inputs and one output. Assume that $f(0) = 0$.

(a) State a definition of f having (strictly) IRS.

(b) Prove that if f exhibits IRS, then, for any strictly positive input prices w_i (where $i = 1, \dots, n$) and strictly positive output price p, either the firm's output at the profit-maximizing production plan is zero or otherwise the profit-maximizing production plan is not well defined (i.e. it does not exist).

(c) Consider the following example of production function with two inputs:

$$f(x_1, x_2) = [\min\{x_1, x_2\}]^2$$

Does this f exhibit increasing returns to scale?

(d) Does the cost-minimization problem for production function f of (c) have a solution for arbitrary prices $w_1 > 0, w_2 > 0$ and output level $y > 0$? Justify your answer

Definitions

Def. (Convex set) A set $S \subseteq \mathbb{R}^n$ is **convex** if

$$\forall x, y \in S \;\; \forall \alpha \in [0,1] \qquad \alpha x + (1-\alpha) y \in S$$

Def. (Convex function) $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ is (strictly) **convex** if:

$$\forall x, y \in X \;\; \forall \alpha \in [0,1] \qquad f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y)$$
$$\forall \alpha \in (0,1) \qquad\qquad\qquad\qquad <$$

Def. (Concave function) $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ is (strictly) **concave** if:

$$\forall x, y \in X \;\; \forall \alpha \in [0,1] \qquad f(\alpha x + (1-\alpha) y) \geq \alpha f(x) + (1-\alpha) f(y)$$
$$\forall \alpha \in (0,1) \qquad\qquad\qquad\qquad >$$

Def. (Quasi-convex function).
$f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **quasi-convex** if:

$$\forall x, y \in X \;\; \forall \alpha \in (0,1) \qquad f(\alpha x + (1-\alpha) y) \leq \max\{f(x), f(y)\}$$

Def. (Quasi-concave function).
$f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **quasi-concave** if:

$$\forall x, y \in X \;\; \forall \alpha \in (0,1) \qquad f(\alpha x + (1-\alpha) y) \geq \min\{f(x), f(y)\}$$

Def. (Upper contour set) of $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ at $a \in X$

$$\mathcal{U}(a) = \{x \in X : f(x) \geq a\}$$

Def. (Production set) - denoted by Y, $Y \subseteq \mathbb{R}^L$ is:

a) NON EMPTY if $\exists y \in \mathbb{R}^L \quad y \in Y$

b) CLOSED $\left(\lim_{n \to +\infty} y_n = y \quad \forall_n \, y_n \in Y\right) \Rightarrow y \in Y$

c) NO FREE PRODUCTION $Y \cap \mathbb{R}^L_+ = \{0\}$

d) POSSIBILITY OF INACTION $\{0\} \in Y$

e) FREE DISPOSAL $Y - \mathbb{R}^L_+ \subset Y$
$(\forall t \in \mathbb{R}^L_+ \quad \forall y \in Y \quad y - t \in Y)$

f) IRREVERSIBILITY $\forall y \in Y \wedge y \neq 0 \Rightarrow -y \notin Y$

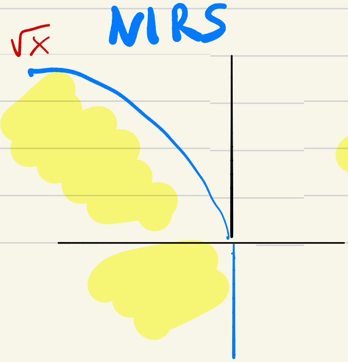
Y exibits

i) NIRS $\forall y \in Y \quad \forall \alpha \in [0,1] \Rightarrow \alpha y \in Y$

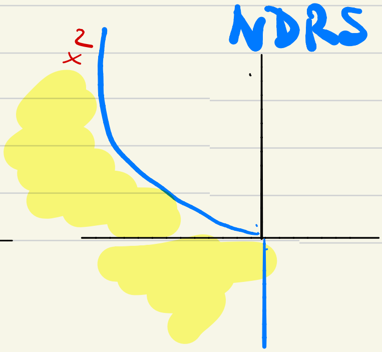
ii) NDRS $\forall y \in Y \quad \forall \alpha \geq 1 \Rightarrow \alpha y \in Y$

iii) CRS $\forall y \in Y \quad \forall \alpha \geq 0 \Rightarrow \alpha y \in Y$

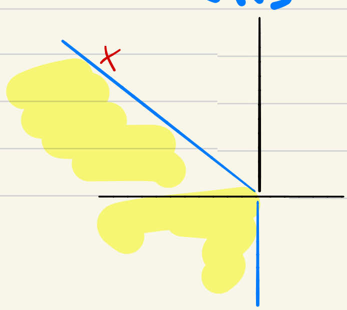
NIRS



NDRS



CRS



Def. (Production function) $f: \mathbb{R}^n_+ \to \mathbb{R}_+$ exhibits:

(i) IRS if:

$$\forall \lambda > 1 \ \forall x \geq 0 \ f(x) \neq 0: \quad f(\lambda x) > \lambda f(x)$$

(ii) DRS if

$$\forall \lambda > 1 \ \forall x \geq 0 \ f(x) \neq 0: \quad f(\lambda x) < \lambda f(x)$$

(iii) CRS if

$$\forall \lambda > 1 \ \forall x \geq 0 \ f(x) \neq 0: \quad f(\lambda x) = \lambda f(x)$$

Def. (Meet and Joint). Given $x, y \in \mathbb{R}^n$

(meet) $\quad x \wedge y := (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})$

(join) $\quad x \vee y := (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$

Def. $X \subseteq \mathbb{R}^n$ is a lattice if:

$$\forall x, y \in X : \quad x \vee y \in X \text{ and } x \wedge y \in X$$

Def (Supermodularity) $f: Y \to \mathbb{R}$ is SPM if:

$$\forall y, y' \in Y \quad f(y \wedge y') + f(y \vee y') \geq f(y) + f(y')$$

Def. (Increasing Differences) $f: X \times \Theta \to \mathbb{R}$ exhibits ID in (x, θ) if:

$$\begin{matrix} \forall x, x' & x \geq x' \\ \forall \theta, \theta' & \theta \geq \theta' \end{matrix} \quad f(x, \theta) - f(x, \theta') \geq f(x', \theta) - f(x', \theta')$$

Def. Production set induced by production function

$$Y_f = \{(x, z) \in \mathbb{R}^l : x \leq 0, \ f(-x) \geq z \geq 0\}$$

Def. (Transformation function) $T: \mathbb{R}^l \to \mathbb{R}$ is transformation function if we can represent Y s.t.:

$$Y = \{y \in \mathbb{R}^l : T(y) \leq 0\}$$

Cor. Observe that $T(y) = z - f(x)$ for $y = (x, z)$

Def. (Input requirement set) is defined as follows for every output z:

$$V(z) = \{-x \in \mathbb{R}^{l-1} \quad f(x) \geq z\}$$

Cor. In other "words":

$$V(z) = \{-x \in \mathbb{R}^{l-1}_{+} \quad (-x, z) \in Y\}$$

Exs

Question 1 [Production function/set]

(a) Show that if $f(x)$ is concave $\Rightarrow$ f production set Y is convex

(b) Prove that for a convex production set $Y \Rightarrow$ input requirement sets V is convex. Prove that converse is not true.

(c) Show that $f(x)$ is quasi concave function $\iff$ input requirement sets V is convex.

(d) Show that if $f(x)$ is strictly concave and $f(0) = 0 \Rightarrow f$ exhibits DRS

Take $f : \mathbb{R}^n_+ \rightarrow \mathbb{R}$

(a) We want to show ($\therefore$ WTS) :

$\forall y \in (-x, z), y' \in (-x', z') \in Y, \forall \alpha \in [0,1] \; : \; \alpha y + (1-\alpha) y' \in Y$

Pick $y, y' \in Y, \alpha \in [0,1]$ since f is a production function :

$$0 \leq z \leq f(x) \qquad 0 \leq z' \leq f(x')$$

$\lambda \geq 0 \Rightarrow \quad 0 \leq \alpha z \leq \alpha f(x) \qquad 0 \leq (1-\alpha) z' \leq (1-\alpha) f(x')$

Sum it up : $\quad 0 \leq \alpha z + (1-\alpha) z' \leq \alpha f(x) + (1-\alpha) f(x')$

$$\leq f(\alpha x + (1-\alpha) x')$$

where last $\leq$ comes from concativity of f

$\boxtimes$

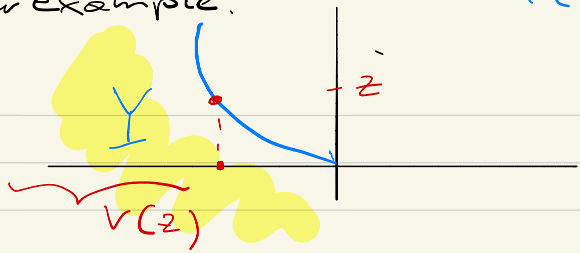
(b) WTS : $\forall -x', -x'' \in V(z) \; \forall \alpha \in [0,1] \; : \; \alpha x' + (1-\alpha) x'' \in V(x)$

Pick $(-x', z), (-x'', z) \in Y$ then since Y is convex

$\forall \alpha \in [0,1] \quad \left(-(\alpha x' + (1-\alpha) x''), \alpha z + (1-\alpha) z \right) \in Y$

Hence $\quad \left(-(\alpha x' + (1-\alpha) x''), z \right) \in Y \iff \left[\alpha x' + (1-\alpha) x'' \right] \in V(z)$

$\Leftarrow$ Counterexample. $f(x) = x^2$



V(z) convex

Y not convex

(c) Lemma. f is Quasi-concave $\Leftrightarrow$ upper contour set $\mathcal{U}_f(a)$ $\forall a$ is convex

$V(z) = \{x : f(x) \geq z\}$ is upper contour set

$V(z)$ is convex $\forall z$ $\Leftrightarrow$ f is Q-Concave

(d) WTS: $\forall \alpha > 1$ $\forall x \geq 0$ $f(x) \neq 0$ $\quad f(\alpha x) < \alpha f(x)$

Pick $x \geq 0$, $x' = 0$ $\alpha \in (0,1)$ from str. concavity of f:

$$f(\alpha x + (1-\alpha) x') > \alpha f(x) + (1-\alpha) f(x')$$

$$f(\alpha x) > \alpha f(x) + (1-\alpha) \cdot f(0) = \alpha f(x) \quad (*)$$

Denote $y := \alpha x$ and $\lambda := \frac{1}{\alpha}$ $\quad$ ($\alpha > 0$!)

Then $x = \frac{1}{\alpha} y = \lambda y$. From $(*)$:

$\frac{1}{\alpha} f(\alpha x) > f(x)$, so

$\lambda f(y) > f(\lambda y)$

Question 2 [Properties of Y, f]

Let $f(x)$ be a production function and Y a production set associated with f. Show the following propositions holds

(a) if f exhibits DRS then Y exhibits NIRS

(b) if f exhibits IRS then Y exhibits NDRS

(c) if f exhibits CRS then Y exhibits CRS

(a) WTS: $\forall y \in Y \ \forall \alpha \in [0,1] \Rightarrow \alpha y \in Y$

Let $y = (-x, z) \in Y$ where x inputs, z output

f exhibits DRS when ($x \geq 0$ from now on)

$\forall y \in Y \ \forall \lambda > 1 \quad f(\lambda x) < \lambda f(x) \qquad (1)$

and since f is production function

$$0 \leq z \leq f(x) \qquad (2)$$

Pick any $\alpha \in [0,1]$ & $y = (-x, z) \in Y$

WTS: $\alpha y \in Y \iff 0 \leq \alpha z \leq f(\alpha x) \qquad (3)$

$\alpha \geq 0$ so multiply (2) by α to get

$$0 \leq \alpha z \leq \alpha f(x)$$

Now consider 3 cases

(a) $\alpha = 0$ (b) $\alpha = 1$ (c) $\alpha > 1$

(a) $\alpha = 0$ Then $0 = \alpha z = \alpha f(x)$ so $\alpha y \in Y$

(b) $\alpha = 1$ Then $0 \leq 1 \cdot z \leq 1 \cdot f(x) = f(1 \cdot x)$ $1 \cdot y \in Y$

(c) $\alpha \in (0,1)$. Take $\alpha = \frac{1}{\lambda} \in (0,1)$. $x' = \alpha x$ for $(f(x'), \lambda)$:

$$\lambda f(x') > f(\lambda x') \Rightarrow \frac{1}{\alpha} f(\alpha x) > f\left(\frac{1}{\alpha} \cdot \alpha x\right) = f(x) \quad (4)$$

by (2) & (4) $0 \leq z \leq f(x) < \frac{1}{\alpha} f(\alpha x) \quad / \cdot \alpha$

$$0 \leq \alpha z \leq f(\alpha x)$$

Q.E.D. $\{$ Quod Erat Demonstrandum

(b)
(c) $\}$ Try solve them on your own

Question 3 [Supermodularity] 89 [I.1 Fall 2009 majors]

Show that following functions are **supermodular**

(a) the Cobb-Douglas production function $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, where $\forall_i\ \alpha_i > 0$, and $\sum_i \alpha_i < 1$

(b) the Leontief function $f(x) = \min\{\alpha_i x_i\} \quad \forall_i\ \alpha_i > 0$

(a) SPM for $f \in C^2$ functions - twicely continously differentiable $\Leftrightarrow$

$$\forall i \neq j \qquad \frac{\partial^2 f}{\partial x_i \partial x_j} \geqslant 0$$

for Cobb-Douglas $f(x) = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$

$$\frac{\partial f}{\partial x_j} = \alpha_j \; x_1^{\alpha_1} \cdot \ldots \cdot x_j^{\alpha_j - 1} \cdot x_{j+1}^{\alpha_{j+1}} \cdot \ldots \cdot x_n^{\alpha_n}$$

$$\frac{\partial f}{\partial x_i \partial x_j} = \alpha_i \alpha_j \cdot x_1^{\alpha_1} \cdot \ldots \cdot x_i^{\alpha_i - 1} \cdot \ldots \cdot x_j^{\alpha_j - 1} \cdot \ldots \cdot x_n^{\alpha_n}$$

in other 'words' : $\frac{\partial f}{\partial x_i \partial x_j} = \alpha_i \alpha_j \dfrac{f(x)}{x_i x_j}$

if $\alpha_i, \alpha_j \geqslant 0$ then $\frac{\partial f}{\partial x_i \partial x_j} \geqslant 0$

(b) Lemma 1. f is non decreasing in x

Proof: WTS: $x \geq x' \Rightarrow f(x) \geq f(x')$

$x \geq x' \Rightarrow x_i \geq x_i' \;\; \forall i$ so for $\alpha_i > 0$:

$\alpha_i x_i \geq \alpha_i x_i'$

So take min

$$f(x) = \min_i \{\alpha_i x_i\} \geq \min_i \{\alpha_i x_i'\} = f(x')$$

Observe that $x \geq x \wedge x'$ so $f(x) - f(x \wedge x') \geq 0$

Consider two cases

(1) $f(x) - f(x \wedge x') = 0$ (2) $f(x) - f(x \wedge x') > 0$

(1) Then $f(x) - f(x \wedge x') = 0$. Additionaly $x \vee x' \geq x'$ so

$$f(x \vee x') - f(x') \geq 0 = f(x) - f(x \wedge x')$$

So $f(x \vee x') - f(x') \geq f(x) - f(x \wedge x')$

And f is SPM

(2) $f(x) - f(x \wedge x') > 0$

Let j be $j \in \arg\min_i \{\alpha_i x_i\}$, so

$f(x) = \alpha_j x_j$ Then

$$f(x) = \alpha_j x_j > f(x \wedge x') = \min_k \{\alpha_k \min\{x_k, x_k'\}\}$$

So $\exists m$: $\alpha_j x_j > \alpha_m \min\{x_m, x_m'\}$

Lemma 2. Under (2) $\min\{x_m, x_m'\} = x_m'$

Proof: Suppose not. Then $\min\{x_m, x_m'\} = x_m$

And by previous $\alpha_j x_j > \alpha_m x_m$ and j is

not minimizer and m is. Contradiction $\lightning$

So it is that $\min\{x_m, x_m'\} = x_m'$ $\boxtimes$

By lemma 2

$$f(x \wedge x') = \alpha_m x_m' \geq \min_i \{\alpha_i x_i'\} = f(x') \quad (3)$$

Since $f(x \vee x') \geq f(x)$ by Claim 1

Sum up (2) & (3) to get

$$f(x \vee x') + f(x \wedge x') \geq f(x) + f(x')$$

So indeed f is SPM

Q.E.D.

Question 4 [Properties of Y]

Prove following properties

(a) Assume that for Y closed and convex , $Y \subset \mathbb{R}^L$ s.t. $0 \in Y$. Free disposal property $Y - \mathbb{R}^L_+ \subset T \iff \mathbb{R}^L_- \subset Y$

(b) If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient

(c) If Y is a convex set, then supply correspondence $s^*(p)$ is a convex set.

(a) $\Rightarrow$ WTS: $Y - \mathbb{R}^L_+ \subset T \Rightarrow \mathbb{R}^L_- \subseteq Y$

By free disposal $\forall y \in Y \ \forall t \in \mathbb{R}^L_+ \quad y - t \in Y$

Take $0 = y$. Then $\forall t \ -t \in Y$ so $\mathbb{R}^L_- \subseteq Y$

$\Leftarrow$ WTS: If $\mathbb{R}^L_- \subseteq Y \Rightarrow Y - \mathbb{R}^L_+ \subseteq Y$

Observe that $\forall t \in \mathbb{R}^L_+ \quad -t \in Y$ take $t = ns$ where $n \in \mathbb{N}$ $s \in \mathbb{R}^L_+$ then $-ns \in Y$

$\forall y \in Y \quad \forall \alpha \in [0,1] \quad \alpha y + (1-\alpha)(-ns) \in Y$ by concavity of Y

Take in particular $\alpha = 1 - \frac{1}{n} \in (0,1)$

$Y \ni (1-\frac{1}{n})y + (1-(1-\frac{1}{n}))(-ns) = \underbrace{(1-\frac{1}{n})y - s}_{y_n} \in Y$

since Y is closed limit of y_n belongs to Y.

$\lim_{n\to\infty} (1-\frac{1}{n})y - s = y - s \in Y$

y, s are arbitrary and $s \in \mathbb{R}^L_+$ so $Y - \mathbb{R}^L_+ \subseteq Y$

(b) $y \in Y$ is efficient if $\neg \exists y' \in Y: y' \geq y$ $(y' \neq y)$

Suppose not. So there is $y' \in Y$ s.t. $y' \geq y$ and $y' \neq y$. Since $p >> 0$ $(p_i > 0 \ \forall i)$

Then $p \cdot y' > p \cdot y$. Why?

Then y' is feasible and gives higher Π.

Contradiction with y being Π maximizer. $\boxtimes$

So y is profit maximizer.

(c) $S(p) = \arg\max \{ p y \mid y \in Y \}$

Consider cases

(1) $S(p) = \emptyset$ (2) $|S(p)| = 1$

(3) $\exists y', y'' \in S(p)$ $y' \neq y''$.

Take $\alpha \in [0,1]$ WTS: $\alpha y' + (1-\alpha) y'' \in S(p)$

$y', y'' \in S(p) \Rightarrow \begin{matrix} p y' \geq p y \\ p y'' \geq p y \end{matrix} \ \forall y \in Y \Rightarrow \begin{cases} \alpha p y' \geq \alpha p y \\ (1-\alpha) p y'' \geq (1-\alpha) p y \end{cases}$

$\Rightarrow p \cdot (\alpha y' + (1-\alpha) y'') \geq p y \ \forall y \in Y$. Y convex $\Rightarrow \alpha y' + (1-\alpha) y'' \in Y$

$\alpha y' + (1-\alpha) y''$ is feasible & max profit So

$\alpha y' + (1-\alpha) y'' \in S(p)$. $\boxtimes$