Recitations 4

JAKUB PAWELCZAK

MINI

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RECITATION 4

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office hours: MONDAY 5:30-6:30
ZOOM

Today: - Midterm
- Toplcis Thm

Next time: ?

Recitation 4

[Definitions used today]

- Topkis theorem, Supermodularity, Increasing Differences

Question 1 [Midterm]

Suppose that a firm with production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ such that $f(0) = 0$ chooses its production plan $(x; z)$ at prices $w \in \mathbb{R}^n_{++}$ of inputs and $q \in \mathbb{R}_{++}$ of the output in such a way that minimizes the cost of producing z at prices w, and the marginal cost $\frac{\partial C^*}{\partial z}(w; z)$ equals the output price q:

a Under what conditions on f is the firm maximizing its production? Be as general as you can. Prove you answer.

b Suppose that cost function C^* is strictly concave in$z$. Show that the firm makes a loss (strictly negative profit) when following the marginal cost rule whenever the output is non-zero.

Question 2 [Topkis theorem]

If S is a lattice, f is supermodular in x, and f has nondecreasing differences in $(x; t)$, then $\varphi^*(t) = \arg\max_{x \in S} f(x, t)$ is monotone nondecreasing in t.

Question 3 [Midterm 2017] or $\sim$ 82,89 [II.1 Spring 2009 majors]

Consider a profit maximizing firm with single output and n inputs, with production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ assumed strictly increasing, continuous (but possibly nondifferentiable), and $f(0) = 0$. Let $q \in \mathbb{R}_{++}$ be the price of output and $w \in \mathbb{R}^n_{++}$ be the vector of prices of inputs. The firm's profit maximization problem is

$$\max_{x \geq 0} [qf(x) - wx]$$

a Show that if the production function f is supermodular, then the firm's input demand x is monotone non-increasing in input prices, that is if $w \leq w'$ for $w, w \in \mathbb{R}^N_{++}$ then $x(w, q) \geq x(w, q)$. You may assume that input demand x is single valued. Production function is strictly increasing but need not be differentiable.

b Under what conditions on f is the solution $x(w, q)$ unique? Be as general as you can and prove your answer

c Give an example of strictly increasing function that is not supermodular.

Question 4

Consider a $C \subset \mathbb{R}^L$, $T \subset \mathbb{R}$. Define function F in following way:

$$F : \mathbb{R}^L \times T \to \mathbb{R} \quad F(x, t) = \bar{F}(x) + f(x, t)$$

where $f : \mathbb{R} \times T \to \mathbb{R}$ is supermodular and $\bar{F} : \mathbb{R}^L \to \mathbb{R}$. Assume that:

$$\forall \quad t'' > t' \quad x'' \in \operatorname*{argmax}_{x \in C} F(x, t'') \quad x' \in \operatorname*{argmax}_{x \in C} F(x, t')$$

Show that if $x'_i > x''_i$ then

$$\forall \quad t'' > t' \quad x'' \in \operatorname*{argmax}_{x \in C} F(x, t') \quad x' \in \operatorname*{argmax}_{x \in C} F(x, t'')$$

Question 5

Let $\{f(s, t)\}$ $t \in T$ be a family of density functions on $S \subset R$. T is a poset (partially ordered set). Consider

$$v(x, t) = \int_S u(x, s) f(s, t) ds$$

Prove the following statement. Suppose u has increasing differences and that $\{f(\cdot, t)\}$ $t \in T$ are ordered with t by first order stochastic dominance. Then v has increasing differences in (x, t).

Question 6 Suppose that utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ is supermodular, strictly concave, and locally non-satiated. Then the Walrasian demand function $x^*(\cdot)$ is a nondecreasing function of income, i.e.,

$$x^*(p, w') \geq x^*(p, w), \; \forall w' \geq w \geq 0, \; \forall p \gg 0.$$

In other words, the demand for every good is normal.

$\bar{x} = 61$

Q	1	2	3
$\bar{x}$	9.6	25	26
sd	8	4.7	9.96
skewness	1.24	-1.16	-0.23

Midterm Q1.

a) f concave $\rightarrow$ production set $\underline{Y}$ is convex

$\Rightarrow \pi(p) \in \partial \underline{Y}$

C(w,z) is convex so

$$\pi(q,w) = \sup_{z \geq 0} q \cdot z - C(w,z)$$

use envelope

b) C str concave

Taylor expansion

for concave function

~~applied~~ ~~to~~

$$f(y) \leq f(x) + f'(x)(y-x)$$

$C(w,z)$ $z=0$

$$0 \leq C(w,0) \leq C(w,z) - z \cdot \frac{\partial C(w,z)}{\partial z}$$

$$\boxed{q = \frac{\partial C(w,z)}{\partial z}}$$

$\pi(p) \leq 0$

$y \neq 0 \quad y \in Y$

Q3. $p_1, p_2 > 0$ ~~$p = (0,1)$~~

$p_1', p_2' > 0$

Topkis & Monotone Comparative Statics

$$x^*(q) = \arg\max_{q \in \Theta} f(x, q) \quad x \in S(q)$$

- If we can use IFT

$$x'(q) = -\frac{f_{xq}(x,q)}{f_{xx}(x,q)} \geq 0$$

$$\begin{cases} \cdot\ f_x(x,q) = 0 \\ \cdot\ \text{rank condition} \end{cases}$$

Assume $f_{xx}(x,q) < 0$

$\boxed{f_{xq}(x,q) \geq 0} \Rightarrow$ Generalize this

$\boxed{\begin{array}{c}\text{SPM} \\ f \in C^2\end{array}} \rightarrow \frac{\partial^2 f}{\partial x \partial q}(x,q) \geq 0$

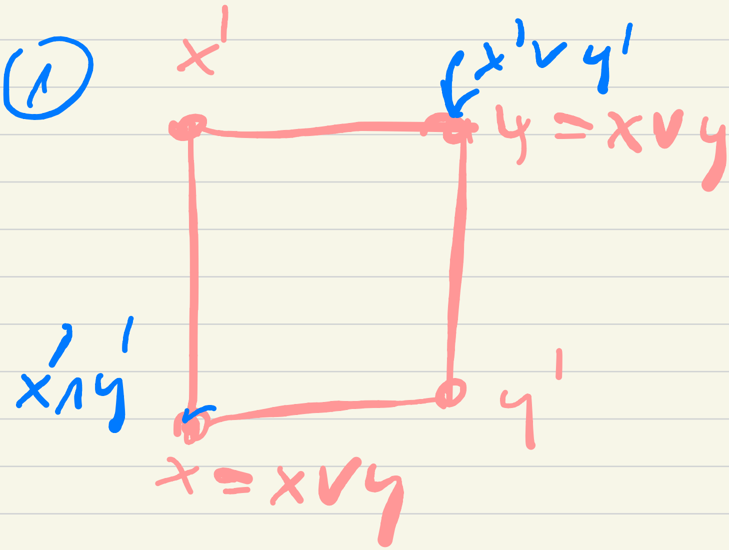
- Topkis (1978), (1998)
- Milgrom, Shannon (1994)
- Quah

Def. $x \vee y$, $x \wedge y$

Def. Lattice S $\forall x, y \in S$
$x \wedge y \in S$ $x \vee y \in S$

Examples $S \subseteq \mathbb{R}^n$

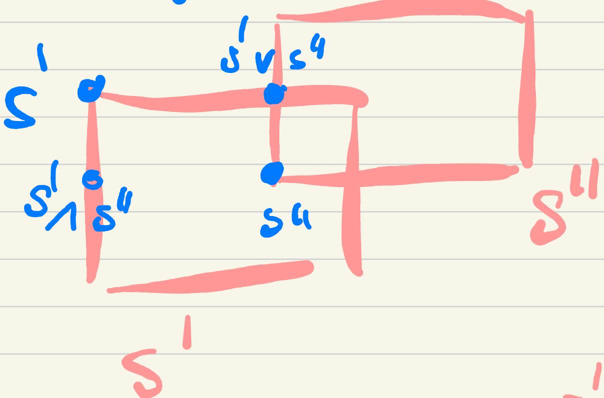
(1)



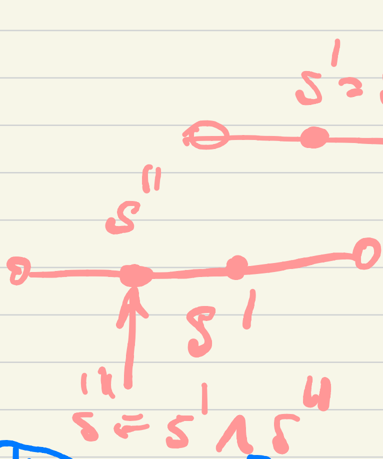
Def. SSO $\quad S'' \geq_{SSO} S'$

$\forall\, s' \in S',\ s'' \in S''$,
$s' \vee s'' \in S''$ and $s' \wedge s'' \in S'$

Examples.

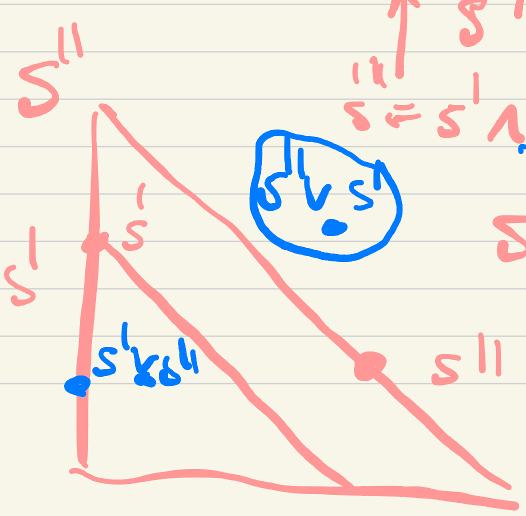


$S'' \geq_{SSO} S'$ ✓



$\mathbb{R}$

✓ $S'' \geq_{SSO} S'$



$\cancel{S'' \geq_{SSO} S}$

Thm. Topkis (1978)

If $f: \underline{\mathbb{R}^n} \times \underline{\mathbb{R}^m} \to \mathbb{R}$ is SPM in $(x(q)) \Rightarrow x^*(q)$ is non decreasing in $\underline{SSO}$.

Proof: Take $x \in X(q)$, $x' \in X(q')$

$\boxed{q' > q}$ WTS: $x \vee x' \in X(q')$ (1)

$x \wedge x' \in X(q)$ (2)

(1) $x \in X(q)$ $x \wedge x' \in \mathbb{R}^n$ so

$$f(\underline{x}, q) - f(\underline{x \wedge x'}, q) \geq 0$$

x being optimal for q.

$$f(x \vee x', q) - f(x', q) \geq 0$$

by SPM $f(x \vee x', q') - f(\underline{x'}, q') \geq 0$

Last means that

$\underline{x \vee x' \in X^*(q')}$

② $x \wedge x' \in \underline{X(q)}$

$x \vee x' \in X(q')$ so

$f(x', q') - f(x \vee x', q') \geq 0$

and $f(x \vee x', q') - f(x', q') \geq 0$

1 means that

$x' \in X(q')$

2nd $x' \vee x \in X^*(q')$

by SPM

$f(x, q) - f(x \wedge x', q) \leq 0$

So $x \wedge x' \in X(q)$

So $X(q') \geq_{SSO} X(q)$

non decreasing in SSO

Q. 4.:

$$F(x,t) = \overline{F}(x) + f(x,t)$$

f is SPM in (x,t)

WTS $\quad F(x'',t') = F(x',t') =$

$$= F(x'',t'') = F(x',t'')$$

$z' = (x', t')$ $\quad z'' = (x'', t'')$

from SPM of f

(*) $f(z' \vee z'') + f(z' \wedge z'') \geq f(z') + f(z'')$

Assume

$t'' > t'$ $\quad x_i'' \leq x_i'$ $\quad \forall i$

$z' \wedge z'' = (x'', t')$

$z' \vee z'' = (x', t'')$

(*) $f(x', t'') + f(x'', t') \geq f(x', t') + f(x'', t'')$

$+ \overline{F}(x'') + \overline{F}(x')$ and I have

*) $f(x, t'') + f(x'', t') \geqslant f(x', t') + f(x'', t'')$

$\bar{F}(x'') + \bar{F}(x')$ and I have

$$F(x'', t') + F(x', t'') \geqslant F(x', t') + F(x'', t'')$$

$$F(x'', t') - F(x', t') \geqslant$$

$$\geqslant F(x'', t'') - F(x', t'') \quad (\bullet\bullet)$$

Now observe that

$x' \in \arg\max_x F(x, t')$

so $F(x', t') \geqslant F(x'', t')$ (:·)

& $x'' \in \arg\max F(x, t'')$

$$F(x'', t'') \geqslant F(x', t'') \quad (::)$$

(••), (:·), (::):

$$0 \geqslant F(x'', t') - F(x', t') \geqslant$$

$$\geqslant F(x'', t'') - F(x', t'') \geqslant 0 \Rightarrow$$

$$F(x'', t') = F(x', t') = F(x'', t'') = F(x', t'')$$

Q. 3. $f(0) = 0$ $\quad f$ str. inc., f SPM

$w' \geq w \Rightarrow x(w', q) \leq x(w, q)$

Ok. f is str. increasing

If f is non decreasing

then $q \cdot f(x) - wx$ ~~is~~

(1) has non decreasing differences

Def. Non Decreasing Differences

$f : \underline{\mathbb{R}^n} \times \underline{\mathbb{R}^n} \to \mathbb{R}$

has NDD when $\forall x' \geq x$

$t' \geq t$

$$\underbrace{f(x', t') - f(x, t')}_{t' \text{ fixed } \underline{x' - x}} \geq \underbrace{f(x', t) - f(x, t)}$$

$\underline{\text{SCP}}$, ND, SPM (QSPM)

- Milgrom - Shannon thm.
- General Topkis thm.

$F(x, t) = t f(x) - w \cdot x$

Next time

- Toplcis
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