Recitations 25

1 Static Mirrlees taxation

Standard assumption in the Ramsey literature is that lump sum taxes are not allowed.Why aren't lump sum taxes used in practice? One reason for this is they require truthful elicitation of agents characteristics, which might not be publicly observable. Moreover, agents might not have an incentive to reveal these characteristics truthfully. We will next consider a mechanism design problem in which agents true ability types are private and allow the designer to use arbitrary mechanisms and transfer schedules to achieve efficiency. Next, will consider implementations/decentralizations.

1.1 A Two Type Example

Consider an environment with a continuum of HHs characterized by a productivity level $\theta \in \Theta = \{\theta_{\mathrm{H}}, \theta_{\mathrm{L}}\}$ with $\theta_{\mathrm{H}} > \theta_{\mathrm{L}}$. A household of type θ who works l hours can produce $y = \theta l$ of output. Let $\pi(\theta)$ denote the probability that given household is of type θ. By the LLN, this is also the fraction of HHs with productivity θ. Household preferences are given by $u(c) - v(l)$ but we will use $l = \frac{y}{\theta}$ to define the preferences as $\mathrm{U}(\mathrm{c}, \mathrm{y}, \theta) = \mathrm{u}(\mathrm{c}) - v\left(\frac{\mathrm{y}}{\theta}\right)$ Suppose first that HH productivities are public information. Under full information a utilitarian planner (cares about all types equally) solves

$$\max_{c(\theta), y(\theta)} \pi\left(\theta_H\right)\left[u\left(c\left(\theta_H\right)\right) - v\left(\frac{y\left(\theta_H\right)}{\theta_H}\right)\right] + \pi\left(\theta_L\right)\left[u\left(c\left(\theta_L\right)\right) - v\left(\frac{y\left(\theta_L\right)}{\theta_L}\right)\right]$$

subject to

$$\pi\left(\theta_{\mathrm{H}}\right) \mathrm{c}\left(\theta_{\mathrm{H}}\right) + \pi\left(\theta_{\mathrm{L}}\right) c\left(\theta_{\mathrm{L}}\right) \leqslant \pi\left(\theta_{\mathrm{H}}\right) \mathrm{y}\left(\theta_{\mathrm{H}}\right) + \pi\left(\theta_{\mathrm{L}}\right) \mathrm{y}\left(\theta_{\mathrm{L}}\right)$$

Let μ be the multiplier on the RC. Then the the focs imply

$$\begin{gathered}
u'(c(\theta)) = \tfrac{1}{\theta} v'(l(\theta)) \\
c\left(\theta_H\right) = c\left(\theta_L\right) \\
\tfrac{v'\left(l\left(\theta_H\right)\right)}{v'\left(l\left(\theta_L\right)\right)} = \tfrac{\theta_H}{\theta_L} > 1
\end{gathered}$$

where the last equation implies that $l\left(\theta_H\right) > l\left(\theta_L\right)$ since v is convex. Now suppose that θ is private information. It is easy to see that the above allocation is not incentive compatible. A high type households strictly prefers to pretend to be a low type since the consumption levels are the same but hours worked is lower. From the revelation principle, that we can restrict ourselves to direct revelation mechanisms.

Definition 1.1 (Direct revelation mechanism)**.** *consists of action/message sets $A_i, i \in [0,1]$ such that for each $i, A_i = \Theta_i$ and outcome functions (c, y) where $c, y : \Theta \to \mathbb{R}_+$.*

Since there is no aggregate uncertainty (LLN), we will consider mechanisms that treat households anonymously, i.e. mechanisms that are independent of i.

Definition 1.2 (Revelation mechanism)**.** *is*

1. *Incentive compatible if and only if*

$$u(c(\theta)) - v\left(\frac{y(\theta)}{\theta}\right) \geqslant u(c(\hat{\theta})) - v\left(\frac{y(\hat{\theta})}{\theta}\right) \text{ for all } \theta, \hat{\theta} \in \Theta \tag{1.1}$$

2. *Resource feasible if and only if*

$$\pi(\theta_{\mathrm{H}})\, c(\theta_{\mathrm{H}}) + \pi(\theta_{\mathrm{L}})\, c(\theta_{\mathrm{L}}) \leqslant \pi(\theta_{\mathrm{H}})\, \mathrm{y}(\theta_{\mathrm{H}}) + \pi(\theta_{\mathrm{L}})\, \mathrm{y}(\theta_{\mathrm{L}}) \tag{1.2}$$

Then, the Planner's/Mechanism designer's problem is

$$\max_{\mathfrak{c}(\theta), \dagger(\theta)} \pi(\theta_{\mathrm{H}}) \left[\mathrm{u}(\mathrm{c}(\theta_{\mathrm{H}})) - v\left(\frac{\mathrm{y}(\theta_{\mathrm{H}})}{\theta_{\mathrm{H}}}\right)\right] + \pi(\theta_{\mathrm{L}}) \left[\mathrm{u}(\mathrm{c}(\theta_{\mathrm{L}})) - v\left(\frac{\mathrm{y}(\theta_{\mathrm{L}})}{\theta_{\mathrm{L}}}\right)\right]$$

subject to (1) and (2) Notice that there are two incentive compatibility constraints, one for the high type and one for the low type. We will first consider a "relaxed problem" in which we drop the incentive constraint for the low type. We will show in the resulting mechanism that this constraint is indeed slack.

1.2 The relaxed problem

$$\max_{\mathfrak{c}(\theta), y(\theta)} \pi(\theta_H) \left[u(c(\theta_H)) - v\left(\frac{y(\theta_H)}{\theta_H}\right)\right] + \pi(\theta_L) \left[u(c(\theta_L)) - v\left(\frac{y(\theta_L)}{\theta_L}\right)\right]$$

subject to

$$u(c(\theta_H)) - v\left(\tfrac{y(\theta_H)}{\theta_H}\right) \geqslant u(c(\theta_L)) - v\left(\tfrac{y(\theta_L)}{\theta_H}\right)$$
$$\pi(\theta_H)\, c(\theta_H) + \pi(\theta_L)\, c(\theta_L) \leqslant \pi(\theta_H)\, y(\theta_H) + \pi(\theta_L)\, c(\theta_L)$$

Let λ be the multiplier on the first constraint and μ on the second. The focs are

$$\pi(\theta_{\mathrm{H}})\, u'(c(\theta_{\mathrm{H}})) + \lambda u'(c(\theta_{\mathrm{H}})) - \pi(\theta_{\mathrm{H}})\, \mu = 0 \tag{1.3}$$

$$\pi(\theta_{\mathrm{L}})\, u'(c(\theta_{\mathrm{L}})) - \lambda u'(c(\theta_{\mathrm{L}})) - \pi(\theta_{\mathrm{L}})\, \mu = 0 \tag{1.4}$$

$$-\frac{\pi(\theta_{\mathrm{H}})}{\theta_{\mathrm{H}}} v'\left(\frac{\mathrm{y}(\theta_{\mathrm{H}})}{\theta_{\mathrm{H}}}\right) - \lambda \frac{1}{\theta_{\mathrm{H}}} v'\left(\frac{\mathrm{y}(\theta_{\mathrm{H}})}{\theta_{\mathrm{H}}}\right) + \pi(\theta_{\mathrm{H}})\, \mu = 0 \tag{1.5}$$

$$-\frac{\pi\left(\theta_{\mathrm{L}}\right)}{\theta_{\mathrm{L}}}v'\left(\frac{\mathrm{y}\left(\theta_{\mathrm{L}}\right)}{\theta_{\mathrm{L}}}\right)+\lambda\frac{1}{\theta_{\mathrm{H}}}v'\left(\frac{\mathrm{y}\left(\theta_{\mathrm{L}}\right)}{\theta_{\mathrm{H}}}\right)+\pi\left(\theta_{\mathrm{L}}\right)\mu=0 \tag{1.6}$$

Combining (3) and 5) we obtain

$$u'\left(c\left(\theta_H\right)\right)=\frac{1}{\theta_H}\nu'\left(\frac{y\left(\theta_H\right)}{\theta_H}\right)$$

This says that the just as in the unconstrained problem, for the high type, the marginal utility of consumption equals the marginal disutility of working. In particular, the allocation for the high type households is ex-post efficient. This is sometimes referred to as "no-distortion at the top". The mechanical reason for this is that, no type wants to pretend to be the high type and thus the planner does not need to distort his allocation. This will not be true for the low type. Next, if we combine (3) and 4) we obtain

$$\frac{u'\left(c\left(\theta_H\right)\right)}{u'\left(c\left(\theta_L\right)\right)}=\frac{\pi\left(\theta_H\right)}{\pi\left(\theta_L\right)}\frac{\left[\pi\left(\theta_L\right)-\lambda\right]}{\left[\pi\left(\theta_H\right)+\lambda\right]}=\frac{\pi\left(\theta_H\right)\pi\left(\theta_L\right)-\lambda\pi\left(\theta_H\right)}{\pi\left(\theta_H\right)\pi\left(\theta_L\right)+\lambda\pi\left(\theta_L\right)}<1$$

$$\Rightarrow u'\left(c\left(\theta_H\right)\right)<u'\left(c\left(\theta_L\right)\right)\qquad\Rightarrow c\left(\theta_H\right)>c\left(\theta_L\right)$$

But then given the IC holds with equality, it must be that

$$v\left(\frac{\mathrm{y}\left(\theta_{\mathrm{H}}\right)}{\theta_{\mathrm{H}}}\right)>v\left(\frac{\mathrm{y}\left(\theta_{\mathrm{L}}\right)}{\theta_{\mathrm{H}}}\right)$$

$$\Rightarrow\mathrm{y}\left(\theta_{\mathrm{H}}\right)>\mathrm{y}\left(\theta_{\mathrm{L}}\right)$$

To see the allocation for the low type is distorted, combine (4) and 6) to obtain

$$\begin{aligned}
u'\left(c\left(\theta_L\right)\right)&=\frac{1}{\theta_L}v'\left(\frac{y\left(\theta_L\right)}{\theta_L}\right)+\frac{\lambda}{\pi\left(\theta_L\right)}\left[u'\left(c\left(\theta_L\right)\right)-\frac{1}{\theta_H}v'\left(\frac{y\left(\theta_L\right)}{\theta_H}\right)\right]\\
&>\frac{1}{\theta_L}v'\left(\frac{y\left(\theta_L\right)}{\theta_L}\right)+\frac{\lambda}{\pi\left(\theta_L\right)}\left[u'\left(c\left(\theta_H\right)\right)-\frac{1}{\theta_H}v'\left(\frac{y\left(\theta_H\right)}{\theta_H}\right)\right]\\
&=\frac{1}{\theta_L}v'\left(\frac{y\left(\theta_L\right)}{\theta_L}\right)
\end{aligned}$$

Finally, we will check that that the solution to the relaxed problem satisfies the incentive constraint for the low type. Suppose not and that this constraint was violated. Then

$$
\begin{aligned}
& u\left(c\left(\theta_L\right)\right)-v\left(\frac{y\left(\theta_L\right)}{\theta_L}\right)<u\left(c\left(\theta_H\right)\right)-v\left(\frac{y\left(\theta_H\right)}{\theta_L}\right) \\
\Longrightarrow & v\left(\frac{y\left(\theta_H\right)}{\theta_L}\right)-v\left(\frac{y\left(\theta_L\right)}{\theta_L}\right)<u\left(c\left(\theta_H\right)\right)-u\left(c\left(\theta_L\right)\right) \\
\Longrightarrow & \frac{1}{\theta_L} \int_{y\left(\theta_L\right)}^{y\left(\theta_H\right)} v^{\prime}\left(\frac{y}{\theta_L}\right) d y<u\left(c\left(\theta_H\right)\right)-u\left(c\left(\theta_L\right)\right) \\
\Longrightarrow & \frac{1}{\theta_H} \int_{y\left(\theta_L\right)}^{y\left(\theta_H\right)} v^{\prime}\left(\frac{y}{\theta_H}\right) d y<\frac{1}{\theta_L} \int_{y\left(\theta_L\right)}^{y\left(\theta_H\right)} v^{\prime}\left(\frac{y}{\theta_L}\right) d y<u\left(c\left(\theta_H\right)\right)-u\left(c\left(\theta_L\right)\right) \\
\Longrightarrow & v\left(\frac{y\left(\theta_H\right)}{\theta_H}\right)-v\left(\frac{y\left(\theta_L\right)}{\theta_H}\right)<u\left(c\left(\theta_H\right)\right)-u\left(c\left(\theta_L\right)\right) \\
\Longrightarrow & u\left(c\left(\theta_L\right)\right)-v\left(\frac{y\left(\theta_L\right)}{\theta_H}\right)<u\left(c\left(\theta_H\right)\right)-v\left(\frac{y\left(\theta_H\right)}{\theta_H}\right)
\end{aligned}
$$

which contradicts the IC for the high type holding with equality. And so the solution to the relaxed problem is the solution to the original constrained problem.

1.3 Don't tax at the top

Above tells us nothing about implementation, i.e. whether there exist tax systems, for example such that the equilibrium, given the tax system gives efficient allocation. We turn to this next. In particular, we will show that a non-linear income tax schedule can implement the efficient allocation.

Denote the optimal mechanism by $\left(c^*, y^*\right)$. Define a tax function $T(y)=y-c$ if $\mathrm{y} \in \left\{\mathrm{y}^*\left(\theta_{\mathrm{H}}\right), \mathrm{y}^*\left(\theta_{\mathrm{L}}\right)\right\}$ and $\mathrm{T(y)} = \mathrm{y}$ otherwise. Given this tax function, the household of type θ solves:

$$\max_{c,y} u(c)-v\left(\frac{y}{\theta}\right)$$

subject to

$$c \leqslant \mathrm{y}-\mathrm{T(y)}$$

The first order condition is

$$u^{\prime}(c)\left(1-T^{\prime}(y)\right)=\frac{1}{\theta} v^{\prime}\left(\frac{y}{\theta}\right)$$

Comparing this equation to the relevant one in the planning problem implies that $T^{\prime}\left(y_H^*\right)=0$ and $\mathrm{T}^{\prime}\left(\mathrm{y}_{\mathrm{L}}^*\right)>0$