Mechanism Design

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In this section will deconstruct following powerful picture:

$$
\begin{tikzcd}
\Theta \arrow[rr, "f"] \arrow[dr, dash] & & X \\
& (\Sigma, g) \arrow[ur] &
\end{tikzcd}
$$

Figure: Hurwicz mechanism design diagram

In this section we present first of three canonical examples of mechanism design which show following issues :

- Example 1: There are SCF which are not Nash implementable
- Example 2: Allocation rules are monotone. There are transfers which are not truthfully implementable.
- Example 3: in 2nd price auction SCF is truthfully implementable in DS by direct mechanism

Today **Example 1**

King Solomon's dilemma

- Two mothers Ann and Beth came to King Solomon with a baby and both claimed to be the baby's genuine mother.
- King Solomon faced the problem of finding out which of two women was the true mother of the baby.
- He proposes to split the baby in half and give one half to each woman.
- This macabre proposal prompts one of the women to scream while the other remains silent.
- King Solomon decrees that the true mother would never stand by while her baby was murdered, and thus gives the baby to woman who screamed.

Furthermore he proposed following solution :

Γ give baby to mother which cried, cut the kid in half when neither of them cried or if both of them screamed

Translate the problem to Mechanism Design

Let's define components of the problem:

- agents $N = \{Ann, Beth\}$
- outcomes $X = \{A, B, C, D\}$
  - $A =$ give baby to Ann
  - $B =$ give baby to Beth
  - $C =$ cut the baby to halves
  - $D =$ death to everyone
- types $\Theta = \{\alpha, \beta\}$
- preference profiles $\Omega = \{P^{\alpha}, P^{\beta}\}$

- in state α their preferences are

$$\begin{array}{ll} \text{Ann} & A \succ B \succ C \succ D \\ \text{Beth} & B \succ C \succ A \succ D \\ \text{utils} & 4 \succ 3 \succ 2 \succ 1 \end{array}$$

- in state β their preferences are

$$\begin{array}{ll} \text{Ann} & A \succ C \succ B \succ D \\ \text{Beth} & B \succ A \succ C \succ D \\ \text{utils} & 4 \succ 3 \succ 2 \succ 1 \end{array}$$

- Γ is a sort of meta-game that induces two NFG corresponding to the states α and β.
- Let $NE(\Gamma, \theta)$ denote the set of Nash equilibria in the game induced by Γ when the state of the world is θ.
- We say that Γ fully implements f^* in Nash equilibrium (aka Nash implements) if $g(NE(\Gamma, \alpha)) = \{f^*(P^\alpha)\}$ and $g(NE(\Gamma, \beta)) = \{f^*(P^\beta)\}$.
- We say that Γ truthfully implements f^* in Nash equilibrium (aka truthful Nash implements) if $g(NE(\Gamma, \alpha)) \ni f^*(\alpha)\}$ and $g(NE(\Gamma, \beta)) \ni f^*(P^\beta)$.

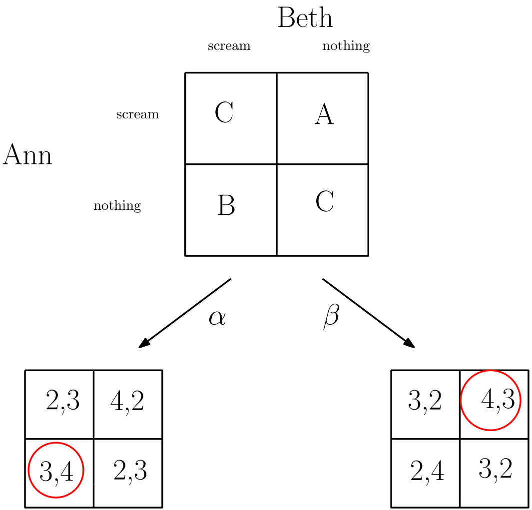

Figure: Γ

So which mechanism was played?

- Let's take a look at Nash equilibria in all 2 games.
- Consider cases corresponding to types α and β:

The game induced by Γ_1 when the state is α.

- In the game α, screaming strictly dominates doing nothing for the fake mother Beth and the true mother's best response when Beth screams is to do nothing.
- Thus $NE(\Gamma, \alpha) = ($ nothing,scream$)$, so $g(NE(\Gamma, \alpha)) = B \neq f^*(P^\alpha) = A$.
- In other words, when Anna is the true mother, the mechanism designed by King Solomon causes him to end up allocating the baby to the fake mother Beth

Similarly, in the game β,

- Screaming strictly dominates doing nothing for the fake mother Anna and doing nothing is a best response for the true mother Beth.
- Thus $NE(\Gamma, \alpha) =$ (nothing, scream) and $g(NE(\Gamma, \alpha)) = A \neq f^*(P^\beta) = B$, i.e., the fake mother Anna gets the baby.
- This means that King Solomon's mechanism does exactly the opposite of what he was intending.

In the parlance of mechanism design, Γ does not Nash-implement f^*.

It must be something that Nash implements f.

It turns out that there is necessary condition for Nash implementability.

Let's look at this problem from Mechanism Design perspective.

Formally problem consists of :

- agents $N = \{Ann, Beth\}$
- outcomes $X = \{A, B, C, D\}$
- types $\Theta = \{\alpha, \beta\}$
- type dependent preference profiles $\Omega = \{P^{\alpha}, P^{\beta}\}$ in short
- King Solomon has social choice function $f^* : A \to X$
- social choice function is such that $f^* : \Theta \to X$ such that $f^*(P^{\alpha}) = A$ and $f^*(P^{\beta}) = B$.
- To impose that SCF king Solomon introduces the mechanism $\Gamma = (\Sigma, g)$

- the mechanism $\Gamma = (M, g)$ where $\Sigma =_A \times \Sigma_B$ is an action (message) space
- $g : \Sigma :\rightarrow X$ is outcome rule that determines which alternative in X is chosen based on the actions of the players.
- Notice that this is not really game- each induces 2 games!
- This is game form (aka **mechanism**) each becomes a game when coupled with a preference profile.
- mechanism induces 2 games (Γ, P^{α}), (Γ, P^{β})
- $\mathrm{NE}(\Gamma, P)$ = set of pure strategy Nash equilirbia

- we look at Nash equilibria of $NE(\Gamma, \alpha)$ and $NE(\Gamma, \beta)$
- g(NE$(\Gamma, P)) =$ set of Nash equilibrium outcomes
- in example we saw:
    - g(NE$(\Gamma, P^{\alpha})) = \{b\}$
    - g(NE$(\Gamma, P^{\beta})) = \{a\}$
- We say that Γ fully implements f^* in Nash equilirbium (aka Nash implements) if
    - $g(NE(\Gamma, \alpha)) = \{f^*(P^{\alpha})\}$
    - $g(NE(\Gamma, \beta)) = \{f^*(P^{\beta})\}$.
- neither seems to be what was played

Precise definitions are presented later on.

It turns out that this SCF does not satisfy necessary condition for Nash implementability (which is Maskin monotonicity).

As *The Rolling Stones* sang: You can't always get what you want!

Game theory and Social Choice Theory kicks in

Definition (Mechnism design problem consists of)

- *finite set of agents* $I = \{1, \dots N\}$
- *agents take collective choice from set of possible alternatives* X
- *each agent has private type* $\theta_i \in \Theta_i$
  $\theta = (\theta_1, \dots, \theta_N) \in \prod_{i=1}^{N} \Theta_i$
- *types are privately observed before collective choice*
- *agent* i *has utility* $u_i(x, \theta_i)$
- *agents are assumed to be an expected utility maximizers*
- ϕ *pdf over* Θ, Θ *and* $u_i(\cdot, \theta_i)$ *are common knowledge but specific values of each agent* i *are observed only by* i
- *collective action depend on* θ

Definition (Social choice function)

$f : \Theta_1 \times \ldots \times \Theta_I \to X$ chooses an outcome $f(\theta) \in X$, given types $\theta = (\theta_1, \ldots, \theta_I)$.

Definition (Ex post efficiency)

The social choice function $f : \Theta_1 \times \cdots \times \Theta_I \to X$ is **ex post efficient** *(EPE or Paretian) if for no profile $\theta = (\theta_1, \ldots, \theta_I)$ is there an $x \in X$ such that*

$$\forall i \quad u_i(x, \theta_i) \geq u_i\left(f(\theta), \theta_i\right)$$

$$\exists j \quad u_j\left(x, \theta_j\right) > u_j\left(f(\theta), \theta_j\right)$$

The mechanism design problem is to implement rules of a game or meta game by defining possible strategies and the method used to select an outcome based on agent strategies, to implement the solution to the social choice function despite agent's selfinterest.

Definition (Mechanism)

$\mathcal{M} = (\Sigma, g)$ consists of

- *set of strategies $\Sigma = \Sigma_1 \times \ldots \times \Sigma_N$*
- *an outcome rule $g : \Sigma_1 \times \ldots \times \Sigma_N \to X$*
- *such that $g(s)$ is the outcome implemented by the mechanism for strategy profile $s = (s_1, \ldots, s_I)$.*

In words, a mechanism defines the strategies available (e.g., bid at least the ask price, etc.) and the method used to select the final outcome based on agent strategies (e.g., the price increases until only one agent bids, then the item is sold to that agent for its bid price).

Definition ((Private) Strategy)

A strategy of player i is a mapping $\sigma_i : \Theta_i \to \Sigma_i$.

Definition (Preferences)

of agent i are function of outcome and private type:

$$u_i(s_i, \theta_i) : \Sigma_i \times \Theta \to \mathbb{R}$$

Definition (Bayesian game of incomplete information)

is $\{I, \{S_i, \bar{u}_i(\cdot)\}, \Theta, \phi\}$ *where we have:*

- *finite set of agents* $I = \{1, \ldots N\}$
- *strategy sets* S_i
- $\bar{u}_i(s_1, \ldots, s_N, \theta_i) = u_i(g(s_1, \ldots, s_N), \theta_i)]$
- *expected payoff (expectation with* ϕ *as pdf):*

$$\hat{u}_i(s_1(\theta_1), \ldots, s_N(\theta_N)) = \mathbb{E}_\theta[\bar{u}_i(s_1(\theta_1), \ldots, s_N(\theta_N), \theta_i)]$$

Given mechanism $\mathcal{M}$ with outcome function $g(\cdot)$, we say that a mechanism implements social choice function $f(\theta)$ if the outcome computed with equilibrium agent strategies is a solution to the social choice function for all preferences.

Definition (Mechanism implementation)

$\mathcal{M} = (\Sigma_1, \ldots, \Sigma_I, g(\cdot))$ *implements social choice function* $f(\theta)$ *in* Z *-equilibrium if* $\exists$ *a* Z *-equilibrium* s^* *such that*

$$\forall \theta \in \Theta \quad \mathbb{P}(\theta) > 0, \quad g\left(s_1^*\left(\theta_1\right), \ldots, s_I^*\left(\theta_I\right)\right) = f\left(\theta\right)$$

$\Theta_1 \times \ldots \times \Theta_I$, *where strategy profile* $\left(s_1^*, \ldots, s_I^*\right)$ *is an equilibrium solution of game induced by* $\mathcal{M}$.

As an Z- equilibrium concept we will consider one of following

- Nash
- Dominant Strategy
- Bayes-Nash

- Picking right mechanism is a daunting task.
- Looking at space of whole possible mechanism may be exhausting task.
- The mechanism asks agents to report their types, and then simply implements the solution to the social choice function that corresponds with their reports.
- Very naive mechanisms gives no good reason for self-interested to truthfully report their types.

Definition (Direct Mechanism)

Given SCF f, the direct mechanism is Γ_{direct} where $M_i = \Theta_i$ and $g = f$.

Note that all other mechanisms are indirect.

Definition (**Incentive Compatibile**)

The SCF f is IC or truthfully implementable if the direct revelation mechanism $\Gamma = (\Theta_1, \ldots, \Theta_I, f(\cdot))$ has an equilibrium $(s_1^*(\cdot), \ldots, s_I^*(\cdot))$ in which

$$s_i^*(\theta_i) = \theta_i \quad \forall i, \theta_i \in \Theta_i$$

That is, if truth telling by each agent i constitutes an equilibrium of $\Gamma = (\Theta_1, \ldots, \Theta_I, f)$.
Notion of Incentive Compatibility was introduced to economics by Minnesota faculty member Leonid Hurwicz (2007 Nobel), check Hurwicz (1972, 1976).

Dominant Strategy Implementation

Recall that a strategy is weakly dominant strategy if gives a player at least as large a payoff as any of his other possible strategies for every possible strategy that his rival might play.

Definition (Dominant Strategy Equilibrium)

s^* is a DS equilibrium:
$\forall i \in I, \forall \theta_i \in \Theta_i, \forall \sigma_i \in \Sigma_i, \forall \theta_{-i} \in \Theta_{-i}, \forall s_{-i}$

$$u_i(g\left(s^*\left(\theta_i\right), s_{-i}\right), \theta_i) \geq u_i(g\left(\sigma_i, s_{-i}\right), \theta_i)$$

Condition above is equivalent to

$$\mathbb{E}_{\theta_{-i}}[u_i(g\left(s^*\left(\theta_i\right), s^*_{-i}\left(\theta_{-i}\right)\right), \theta_i)|\theta_i] \geq \mathbb{E}_{\theta_{-i}}[u_i(g\left(\sigma_i, s^*_{-i}\left(\theta_{-i}\right)\right), \theta_i)|\theta_i]$$

Definition (Implementation in DS)

Γ *implements* f *in DS if* $\exists s^*$ *such that*

- s^* *is a DS equilibrium*
- *Implementation:* $\forall \theta \in \Theta, g\left(s^*(\theta)\right) = f(\theta)$.

- The concept of dominant strategy implementation is of special interest because if we can find a mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ that implements $f(\cdot)$ in dominant strategics, then this mechanism implements $f(\cdot)$ in a very strong and robust way.
- This implementation will be robust even if agents have incorrect, and perhaps even contradictory, beliefs about this distribution.
- In particular, agent i 's beliefs regarding the distribution of θ_{-i} do not affect the dominance of his strategy $s_i^*(\cdot)$.
- Thus, the same mechanism can be used to implement $f(\cdot)$ for any $\phi(\cdot)$. One advantage of this is that if the mechanism designer is an outsider (say, the "government"), he need not know $\phi(\cdot)$ to successfully implement $f(\cdot)$

Definition (Dominant Strategy Incentive Compatible (DSIC))

The SCF f is truthfully implementable in dominant strategies (or dominant strategy incentive compatible, or strategy-proof, or **DSIC***) if $s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and $i = 1, \ldots, N$ is a dominant strategy equilibrium of the direct revelation mechanism $\Gamma = (\Theta_1, \ldots, \Theta_N, f)$.*
That is, if for all i and all $\theta_i \in \Theta_i$

$$u_i\left(f\left(\theta_i, \theta_{-i}\right), \theta_i\right) \geq u_i\left(f\left(\hat{\theta}_i, \theta_{-i}\right), \theta_i\right) \quad \forall \theta', \theta_{-i}$$

Theorem (Revelation principle for DS equilibrium)

If there exists a mechanism that implements f in DS.-equilibrium, then f can be implemented in DS equilibrium with the direct mechanism, with truth-telling as the domininant strategy (i.e., f is strategy proof).

Proof:
Let $\Gamma = (\Sigma, g)$ be the mechanism that implements f is d.s. equilibrium. Let σ^* be the d.s. equilibrium such that $g\left(s^*(\theta)\right) = f(\theta), \forall \theta \in \Theta$. Then by definition of d.s. equilibrium, $\forall i \in I, \forall \theta_i \in \Theta_i, \forall \sigma_i \in \Sigma_i, \forall \theta_{-i} \in \Theta_{-i}, \forall s_{-i}$

$$u_i(g\left(s_i^*\left(\theta_i\right), s_{-i}\left(\theta_{-i}\right)\right)) \geq u_i(g\left(\sigma_i, s_{-i}\left(\theta_{-i}\right)\right))$$

Since this holds for every possible strategy of the other players, we can substitute s_{-i}^* for s_{-i}. Similarly, since $s_i^*\left(\theta_i'\right) \in \Sigma_i, \forall \theta_i' \in \Theta_i$, we can substitute $s_i^*\left(\theta_i'\right)$ for θ_i.

Thus we have $\forall i \in I, \forall \theta_i, \theta_i' \in \Theta_i, \forall \theta_{-i} \in \Theta_{-i}$

$$u_i(g\left(s_i^*\left(\theta_i\right), s_{-i}^*\left(\theta_{-i}\right)\right)) \geq u_i(g\left(s_i^*\left(\theta_i'\right), s_{-i}^*\left(\theta_{-i}\right)\right))$$

By definition of s^*, this implies that
$\forall i \in I, \forall \theta_i, \theta_i' \in \Theta_i, \forall \theta_{-i} \in \Theta_{-i}$

$$u_i(f(\theta)) \geq u_i(f\left(\theta_i', \theta_{-i}\right))$$

Thus f is strategy-proof, so f can be implemented by the direct mechanism with truth-telling as the d.s. equilibrium. ∎

The intuitive idea behind the revelation principle for dominant strategies can be put as follows: Suppose that the indirect mechanism Γ implements f in DS, and that in this indirect mechanism each agent i finds playing $s_i^* (\theta_i)$ when his type is θ_i better than playing any other $s_i \in S_i$ for any choices $s_{-i} \in S_{-i}$ by agents $j \neq i$.

Theorem (Gibbard-Satterthwaite once again)

If $|X| \geq 3, f$ *is onto,* $\{\succeq_i(\theta) : \theta \in \Theta\} = P$*, and* f *is truthfully implementable in DS , then* f *is dictatorial, i.e.,* $\exists i \in I$ *such that* $\forall \theta \in \Theta$,

$$f(\theta) \in \{x \in X : x \succeq_i (\theta) y, \forall y \in X\}$$

Given this negative conclusion, if we are to have any hope of implementing desirable social choice functions, we must either weaken the demands of our implementation concept by accepting implementation by means of less robust equilibrium notions (such as Bayesian Nash equilibria) or we must focus on more restricted environments (s.a. Quasi Linear preferences).

Bayes Nash Implementation

Definition

Definition (Bayesian Nash equilibrium)

of G is a strategy profile $\sigma^* = (\sigma_i^*)_{i \in I}$ (where $\sigma_i^* : \Theta_i \to M_i$) such that $\forall i \in I, \forall \theta_i \in \Theta_i, \forall s_i \in \Sigma_i$

$$\mathbb{E}_{\theta_{-i}}\left[u_i\left(g\left(\sigma_i^*\left(\theta_i\right), \sigma_{-i}^*\left(\theta_{-i}\right)\right), \theta_i\right) \mid \theta_i\right] \geq \mathbb{E}_{\theta_{-i}}\left[u_i\left(g\left(m_i, \sigma_{-i}^*\left(\theta_{-i}\right)\right), \theta_i\right)\right.$$

We will use $BNE(G)$ to denote the set of Bayesian Nash equilibria of G.

Definition (Implementation)

Γ *implements social choice function* $f : \Theta \to X$ *in Bayesian Nash equilibrium if* $\text{BNE}(G) \neq \emptyset$ *and* $\exists \sigma^* \in BNE(G)$ *such that*

$$g\left(\sigma^*(\theta)\right) = f(\theta) \quad \forall \theta \in \text{support}(\phi)$$

Definition (Bayes-Nash Incentive Compatible (BNIC))

The SCF f truthfully implementable in Bayesian Nash equilibrium (or BNIC) if $s_i^*(\theta_i) = \theta_i$ and s^* is BN equilibrium of the direct revelation mechanism $\Gamma = (\Theta, f)$. That is:

$$\mathbb{E}_{\theta_{-i}}\left[u_i\left(f\left(\theta_i, \theta_{-i}\right), \theta_i\right) \mid \theta_i\right] \geq \mathbb{E}_{\theta_{-i}}\left[u_i\left(f\left(\theta_i', \theta_{-i}\right), \theta_i\right) \mid \theta_i\right]$$

Theorem (Revelation principle for BN implementation)

If f is BN -implementable, then f is BN implementable by the direct mechanism $\Gamma_{direct} = (\Theta, f)$ with truth-telling as a BNE.

Definition

If f is strategy proof, we say that f is d.s. incentive compatible. If f is BN-implementable, we say that f is Baysesian incentive compatible. Note that d.s. IC implies BN IC, but not the other way around.

In terms of set notation, BN IC $\supset$ d.s. IC.

Definition

Let ϕ be a product measure and suppose that $t(\cdot)$ implements $k(\cdot) = (y_1(\cdot), \ldots, y_n(\cdot))$ For each $i \in I$, define $\bar{y}_i(\cdot)$ and $\bar{y}_i(\cdot)$ as

$$\bar{t}_i(\theta_i) := E_{\theta_{-i}}\left[t_i(\theta_i, \theta_{-i})\right]$$

and

$$\bar{y}_i(\theta_i) := E_{\theta_{-i}}\left[y_i(\theta_i, \theta_{-i})\right]$$

BNIC

$$\bar{y}_i(\theta_i)\,\theta_i + \bar{t}_i(\theta_i) \geq \bar{y}_i(\theta_i')\,\theta_i + \bar{t}_i(\theta_i'), \forall \theta_i' \in \Theta_i$$

no separate IC condition for each θ_{-i} as in the DS case. Instead, one constraint given the averages $\bar{y}$ and $\bar{t}$ implied by ϕ.

Note that we don't have a separate IC condition for each θ_{-i} as we did in the DS implementation case. Instead, we just have one constraint given the averages $\bar{y}$ and $\bar{t}$ implied by ϕ.

Theorem

$k(\cdot)$ is BN -implementable only if it is "monotone" in the sense that $\forall i \in I, \bar{y}_i(\cdot)$ is weakly increasing.

To summarize, $\bar{y}_i(\cdot)$ is BN implementable $\iff$ it is "monotone" (in the BN sense). Moreover, if Θ_i is a connected subset of $\mathbb{R}$, then any $\bar{t}_i(\cdot)$ that BN-implements $\bar{y}_i(\cdot)$ must take the form

$$\bar{t}_i\left(\theta_i\right) = \bar{t}_i\left(\underline{\theta}_i\right) - \bar{y}_i\left(\theta_i\right)\theta_i + \int_0^{\theta_i} \bar{y}_i(x)\mathrm{d}x$$

Optimal Mechanisms

Set-up

- One seller has an object and n bidders, set of agents is $A = \{0, 1, \dots, n\}$.
- The utility function for i is

$$v_i\left(a, \theta_i\right) = \begin{cases} \theta_i & \text{if } a = i \\ 0 & \text{otherwise} \end{cases}$$

- For each $i = 1, \dots, n, \theta_i$ is independently drawn from a continuous PDF $\phi_i\left(\theta_i\right)$ with support $\Theta_i\ \left[\underline{\theta}_i, \bar{\theta}_i\right]$
- The mechanism is (q, t) where: $-q_i(\theta) = \Pr(a = i \mid \theta)$ is the probability of agent i get the object; $-t_i(\theta) : \Theta \to \mathbb{R}$ is the transfer from buyers to the seller where $\Theta = \prod_{i=1}^{n} \Theta_i$

- Denote $\bar{q}_i(\theta_i) = E\left[q_i(\theta_i, \theta_{-i}) \mid \theta_{-i}\right]$ and $\bar{t}_i(\theta_i) = E\left[t_i(\theta_i, \theta_{-i}) \mid \theta_i\right]$
- By revelation principal, restrict our focus on direct mechanism.
- $q_i(\theta)$ needs to be probabilities (PR) : $\sum_{i=1}^{n} q_i(\theta) \leq 1$ and $q_i(\theta) \geq 0 \forall i$.
- For $i = 1, \ldots, n$, given the mechanism (q, t), the expected utility is

$$
\begin{aligned}
U_i(\theta_i) &= \int_{\Theta_{-i}} (\theta_i q(\theta) - t_i(\theta))\, \phi_{-i}(\theta_{-i})\, d\theta_{-i} \\
&= \theta_i \bar{q}_i(\theta_i) - \bar{t}_i(\theta_i)
\end{aligned}
$$

- The seller's expected utility is

$$U_0\left(\theta_0\right) = \int_{\Theta}\left[\theta_0\left(1 - \sum_{i=1}^{n} q_i(\theta)\right) + \sum_{i=1}^{n} t_i(\theta)\right]\phi(\theta)\,d\theta$$

- Individual rationality (IR) requires:
$U_i\left(\theta_i\right) \geq 0 \forall i = 1, \ldots n, \forall \theta_i \in \Theta_i$.
- Incentive-compatibility (IC) requires: for all $\theta_i' \in \Theta_i$

$$U_i\left(\theta_i\right) \geq \int_{\Theta_{-i}}\left(\theta_i q\left(\theta_i', \theta_{-i}\right) - t_i\left(\theta_i', \theta_{-i}\right)\right)\phi_{-i}\left(\theta_{-i}\right) d\theta_{-i}$$

Definition

A mechanism (p,t) is feasible if it satisfies PR, IR, and IC.

Lemma

A mechanism (p,t) is feasible $\iff$

1. $\bar{q}_i$ *is nondecreasing*
2. $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{q}_i(\tau_i)\, d\tau_i$
3. $U_i(\underline{\theta}_i) \geq 0 \forall i = 1, \ldots, n$
4. $\sum_{i=1}^{n} q_i(\theta) \leq 1$ *and* $q_i(\theta) \geq 0 \forall i$.

Lemma

If $q : \Theta \to \mathbb{R}^n$ *maximizes*

$$\int_\Theta \left[\sum_{i=1}^n \left(\theta_i - \frac{1 - \Phi(\theta_i)}{\phi(\theta_i)} - \theta_0 \right) q_i(\theta) \right] \phi(\theta) d\theta$$

subject to (1) and (4) from previous lemma, and

$$t_i(\theta) = \theta_i q_i(\theta) - \int_{\underline{\theta}_i}^{\theta_i} q_i(\theta_i, \theta_{-i}) \, d\theta_i$$

Then, (q, t) *is an optimal auction.*

Definition

The problem is regular if for each i

$$w_i(\theta_i) = \theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)}$$

is strictly monotone.

Theorem

The auction mechanism (q, t) is optimal.

Nonlinear pricing [181 IV.3 Spring 2014]

Screening. A monopolist faces a single consumer with utility function $u = \theta q - \frac{1}{2}q^2 - T$, where$\theta$ is private information of the consumer, q is the level of consumption and T is the amount of money that the consumer pays the monopolist.
The monopolist's cost of producing q equals $\frac{1}{2}cq^2$ for some constant $c > 0$. The consumer's reservation utility equals 0.
The (Pareto) CDF ofθ is $F(\theta) = 1 - \theta^{\alpha}$ for all $\theta \in [1; \infty)$, where $\alpha > 1$.

(a) Derive the monopolist's optimal bundle (q, T) assuming that it knows θ.

(b) Write down the monopolist's nonlinear pricing problem.

(c) Derive the optimal nonlinear pricing schedule and compare it with (a).

(d) What can you say about distortions at the top? (Here top means $\theta \to \infty$.)

- The problem of the monopolist for perfect observability is:

$$\max_{q,T} T - \frac{c}{2}q^2 \quad \text{s.t. } \theta q - \frac{1}{2}q^2 - T \geq 0$$

It must be that the constraint is met with equality, or else T can be increased, increasing profits.:

$$T = \theta q - \frac{1}{2}q^2$$

The problem is then:

$$\max_{q} \theta q - \frac{1+c}{2}q^2$$

The FOC:

$$q = \frac{\theta}{1+c}$$

- Under uncertainty the problem of the monopolist is:

$$\begin{gathered}
\max_{q(\theta),T(\theta)} \int \left(T(\theta) - \tfrac{c}{2} q(\theta)^2 \right) dF(\theta) \\
\text{IR} \quad \theta q(\theta) - \tfrac{1}{2} q(\theta)^2 - T(\theta) \geq 0 \\
\text{IC} \quad \theta q(\theta) - \tfrac{1}{2} q(\theta)^2 - T(\theta) \geq \theta q\left(\theta'\right) - \tfrac{1}{2} q\left(\theta'\right)^2 - T\left(\theta'\right)
\end{gathered}$$

- Following Myerson's Theorem (1981) the profit maximization problem is equivalent to:

$$\max_{q(\theta),T(\theta)} \int \left(T(\theta) - \frac{c}{2} q(\theta)^2 \right) dF(\theta)$$

$$U(\underline{\theta}) \geq 0$$
$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(x) dx$$

q is monotone increasing
where $U(\theta) = \theta q(\theta) - \frac{1}{2} q(\theta)^2 - T(\theta)$.

The first condition can be solved by setting $T(\underline{\theta}) = \theta q(\underline{\theta}) - \frac{1}{2}q(\underline{\theta})^2$ so that $U(\underline{\theta}) = 0$, the value of $q(\underline{\theta})$ is found later. With this

$$U(\theta) = \int_{\underline{\theta}}^{\theta} q(x)dx \longrightarrow T(\theta) = \theta q(\theta) - \frac{1}{2}q(\theta)^2 - \int_{\underline{\theta}}^{\theta} q(x)dx$$

which allows to eliminate $T(\theta)$ from the problem. The problem can be then solved ignoring the last restriction (over q) and verifying its monotonicity ex-post.

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\infty} \left(\theta q(\theta) - \frac{1+c}{2}q(\theta)^2 - \int_{\underline{\theta}}^{\theta} q(x)dx \right) dF(\theta)$$

Note the following by defining $Q(\theta) = \int_{\underline{\theta}}^{\theta} q(x)dx$:

$$\int_{\underline{\theta}}^{\infty} \int_{\underline{\theta}}^{\theta} q(x)dxdF(\theta) = \int_{\underline{\theta}}^{\infty} Q(\theta)dF(\theta) =$$

$$= [Q(\theta)F(\theta)]_{\underline{\theta}}^{\infty} - \int_{\underline{\theta}}^{\infty} \frac{\partial Q(\theta)}{\partial \theta} F(\theta)d\theta$$

By the fundamental theorem of calculus: $\frac{\partial Q(\theta)}{\partial \theta} = q(\theta)$, then:

$$\int_{\underline{\theta}}^{\infty} \int_{\underline{\theta}}^{\theta} q(x)dxdF(\theta) = \int_{\underline{\theta}}^{\infty} q(x)dx - \int_{\underline{\theta}}^{\infty} q(\theta)F(\theta)d\theta =$$
$$= \int_{\underline{\theta}}^{\infty} (1 - F(\theta))q(\theta)d\theta$$
$$\int_{\underline{\theta}}^{\infty} \int_{\underline{\theta}}^{\theta} q(x)dxdF(\theta) = \int_{\underline{\theta}}^{\infty} \frac{1-F(\theta)}{f(\theta)} q(\theta)dF(\theta)$$

recalling $dF(\theta) = f(\theta)d\theta$.

Replacing:

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\infty} \left(\left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) q(\theta) - \frac{1+c}{2} q(\theta)^2 \right) dF(\theta)$$

The optimal q for each θ is obtained from the FOC against $q(\theta)$

$$q(\theta) = \frac{1}{1+c} \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right)$$

Since θ is Pareto:

$$F(\theta) = 1 - \theta^{-\alpha} \quad f(\theta) = \alpha \theta^{-\alpha-1}$$

Replacing:

$$q(\theta) = \frac{1}{1+c}\left(\theta - \frac{\theta^{-\alpha}}{\alpha\theta^{-\alpha-1}}\right) = \frac{\theta}{1+c}\left(\frac{\alpha-1}{\alpha}\right)$$

This function is monotone increasing as required and is smaller than q from (a).

- Under the Pareto distributions there are always distortions, the bundle offered to any type is different than the one offered under perfect information. Under the Pareto distribution there is no actual "top" type.

Thank you!