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Question 1 The Social Value of Public Information [221 IV.2 Spring 2016 majors]

There is a continuum of agents, uniformly distributed on $[0,1]$. Each agent $i \in [0,1]$ chooses $a_i \in R$. Let a be the action profile. Agent i has utility function

$$u_i(a,\theta) = -\left[(1-r)\left(a_i-\theta\right)^2 + r\left(L_i - \bar{L}\right)\right]$$

where $r \in (0,1)$ is a constant, θ represents the state of the economy,

$$L_i = \int_0^1 \left(a_j - a_i\right)^2 dj \quad \text{and} \quad \bar{L} = \int_0^1 L_j dj$$

Intuitively, agent i wants to minimize the distance between his action and the true state θ, and also minimize the distance between his action and the actions of others. The parameter r represents the trade-off between these two objectives. Social welfare (normalized) is

$$W(a,\theta) = \frac{1}{1-r}\int_0^1 u_i(a,\theta)di = -\int_0^1 \left(a_i-\theta\right)^2 di$$

Agent i forms expectations $E_i[\cdot] = E\left[\cdot|\mathcal{I}_i\right]$ conditional on his information $\mathcal{I}_i$ and maximizes expected utility.

1. Show that each agent i 's optimal action is given by

$$a_i = (1-r)E_i[\theta] + rE_i[\bar{a}]$$

where $\pi = \int_0^1 a_j dj$ is the average action. Show that if θ is common knowledge then $a_i = \theta$ for every i is an equilibrium.

Solution

We can write the problem of agent i in the following way:

$$\max_{a_i} \mathbb{E}_i\left[-\left[(1-r)\left(a_i-\theta\right)^2 + r\left(\int_0^1 \left(a_j-a_i\right)^2 dj - \bar{L}\right)\right]\right]$$

Or

$$\max_{a_i} \mathbb{E}_i\left[-\left[(1-r)\left(a_i-\theta\right)^2 + r\left(\int_0^1 \left(a_j^2+a_i^2-2a_ja_i\right) dj - \bar{L}\right)\right]\right]$$

FOC:

$$\mathbb{E}_i\left[-2(1-r)(a_i-\theta)-r\left(2a_i-2\int_0^1 a_j dj\right)\right]=0$$

that gives the desired result

$$a_i=(1-r)\mathbb{E}_i[\theta]+r\mathbb{E}_i[\bar{a}]$$

where $\bar{a}=\int_0^1 a_j dj$. Note that the true state of the economy θ and the average action $\bar{a}$ are not observable by the agent i, and thus he/she assigns some positive weights on the expectations over these values.

If θ is common knowledge, then in the equilibrium all the agents simply choose $a_i=\theta$, and thus $\bar{a}=\theta$ as well. In this case $u_i(a,\theta)=0, \forall i$, i.e. it attains a maximum. Moreover, $W(a,\theta)=0$ in this case, i.e. the social welfare is also attains a maximum. Therefore, under the assumption of perfect information there is no trade-off between the socially optimal and individually rational actions.

2. Suppose that θ is drawn heuristically from a uniform prior over the real line. Agents observe a public signal

$$y=\theta+\eta$$

where $\eta\sim N\left(0,\sigma^2\right)$. Therefore, $\theta|y\sim N\left(y,\sigma^2\right)$. Now, agents maximize expected utility $E\left[u_i|y\right]$ given the same public information y. Show that $a_i(y)=y$ for every i is an equilibrium. Derive the following expression for welfare given θ :

$$E[W|\theta]=-\sigma^2$$

Solution

Now the problem of agent i is

$$\max_{a_i}\mathbb{E}\left[-\left[(1-r)(a_i-\theta)^2+r\left(\int_0^1\left(a_j^2+a_i^2-2a_ja_i\right)dj-\bar{L}\right)\right]\mid y\right]$$

FOC:

$$\mathbb{E}\left[-2(1-r)(a_i-\theta)-r\left(2a_i-2\int_0^1 a_j dj\right)\mid y\right]=0$$

which can be simplified to

$$a_i(y)=(1-r)\mathbb{E}[\theta\mid y]+r\int_0^1\mathbb{E}\left[a_j\mid y\right]dj$$

Note that $\mathbb{E}[\theta\mid y]=y$, and $\mathbb{E}\left[a_j\mid y\right]=a_j(y)$ since the strategies of the agents are measurable with respect to y. Therefore, in the unique equlibrium we have

$$a_i(y) = (1-r)y + ra_i(y)$$

which gives

$$a_i(y) = y$$

Regarding expected welfare, we have the following:

$$\mathbb{E}[W \mid \theta] = -\mathbb{E}\left[\int_0^1 (y-\theta)^2 di \mid \theta\right] = -\mathbb{E}\left[(y-\theta)^2 \mid \theta\right] = -\mathbb{E}\left[\eta^2 \mid \theta\right] = -\sigma^2$$

3. Assume now that, in addition to the public signal, each agent i observes a private signal

$$x_i = \theta + \epsilon_i$$

where $\epsilon_i \sim N(0, \tau^2)$ is (heuristically) independent across i and of θ and η. Let $\alpha = 1/\sigma^2$ and $\beta = 1/\tau^2$

a) Show that

$$E_i[\theta] = E[\theta | x_i, y] = \frac{\alpha y + \beta x_i}{\alpha + \beta}$$

Solution

We can treat α as the precision of public information, and β as the precision of private information (they are both reciprocal to corresponding variances). Note that now the information set $\mathcal{I}_i$ is given by the realizations of x_i and y.

We have the following:

$$y = \theta + \eta$$
$$x_i = \theta + \varepsilon_i$$

Indeed, $\mathbb{E}[\theta \mid x_i, y]$ is a linear MMSE estimator which can be obtained from the following linear combination of y and x_i (also it is useful to recall the properties of Bayes updating with normal random variables):

$$\mathbb{E}[\theta \mid x_i, y] = \omega_1(y - \mathbb{E}(\theta)) + \omega_2(x_i - \mathbb{E}(\theta)) + \mathbb{E}(\theta)$$

with the weights

$$\omega_1 = \frac{1/\sigma^2}{1/\sigma^2 + 1/\tau^2 + 1/\operatorname{var}(\theta)}, \quad \omega_2 = \frac{1/\tau^2}{1/\sigma^2 + 1/\tau^2 + 1/\operatorname{var}(\theta)}$$

Moreover, note that since θ is drawn heuristically from a uniform prior over the real lin e^2, then $\mathbb{E}(\theta) = 0, \operatorname{var}(\theta) = +\infty$. Combining all these arguments, we get the desired expression:

$$\mathbb{E}_i[\theta] = \mathbb{E}\left[\theta \mid x_i, y\right] = \frac{\alpha}{\alpha+\beta} y + \frac{\beta}{\alpha+\beta} x_i = \frac{\alpha y + \beta x_i}{\alpha+\beta}$$

b) Suppose that there is a number κ such that for every agent j

$$a_j\left(x_j, y\right) = \kappa x_j + (1-\kappa) y$$

Compute the value of $E_i[\bar{a}]$ and show that following κ defines an equilibrium.

$$\kappa = \frac{\beta(1-r)}{\alpha+\beta(1-r)}$$

Solution

Suppose that $\exists \kappa$ such that $\forall j$

$$a_j\left(x_j, y\right) = \kappa x_j + (1-\kappa) y$$

Then agent's i conditional estimate of the average expected action across all the agents is

$$\begin{aligned}\mathbb{E}_i[\bar{a}] = \mathbb{E}_i\left[\int_0^1 a_j\left(x_j, y\right) dj\right] = \mathbb{E}_i\left[\int_0^1\left(\kappa x_j + (1-\kappa) y\right) dj\right] = \\ = \kappa \mathbb{E}_i\left[\int_0^1 x_j dj\right] + (1-\kappa) y = \kappa \mathbb{E}_i\left[\int_0^1\left(\theta + \varepsilon_j\right) dj\right] + (1-\kappa) y =\end{aligned}$$

$$= \kappa \mathbb{E}_i[\theta] + (1-\kappa) y = \kappa \frac{\alpha y + \beta x_i}{\alpha+\beta} + (1-\kappa) y = \left(\frac{\kappa \beta}{\alpha+\beta}\right) x_i + \left(1 - \frac{\kappa \beta}{\alpha+\beta}\right) y$$

where we use the result from part 3(a) in the third line. Recall from part 1 that the optimum is characterized by

$$a_i\left(x_i, y\right) = (1-r) \mathbb{E}_i[\theta] + r \mathbb{E}_i[\bar{a}].$$

Plugging in the expressions that we get above, we get

$$\begin{gathered}a_i\left(x_i, y\right) = (1-r) \mathbb{E}_i[\theta] + r \mathbb{E}_i[\bar{a}] = \\ = (1-r) \cdot\left[\tfrac{\alpha}{\alpha+\beta} y + \tfrac{\beta}{\alpha+\beta} x_i\right] + r \cdot\left[\left(\tfrac{\kappa \beta}{\alpha+\beta}\right) x_i + \left(1 - \tfrac{\kappa \beta}{\alpha+\beta}\right) y\right] = \\ = \tfrac{\beta(1+r\kappa-r)}{\alpha+\beta} x_i + \left(1 - \tfrac{\beta(1+r\kappa-r)}{\alpha+\beta}\right) y\end{gathered}$$

Note that since we assume that $\exists \kappa$ such that for any agent j the following holds $a_j\left(x_j, y\right) = \kappa x_j + (1-\kappa) y -$ then, comparing with the equation for $a_i\left(x_i, y\right)$ above, we get

$$\kappa = \frac{\beta(1+r\kappa-r)}{\alpha+\beta}$$

Solving it for κ, one can easily get

$$\kappa = \frac{\beta(1-r)}{\alpha + \beta(1-r)}$$

which is a desired result.

4. Show that expected welfare is given by

$$E[W(a,\theta)|\theta] = -\frac{\alpha + \beta(1-r)^2}{[\alpha + \beta(1-r)]^2}$$

Show that

$$\frac{\partial E[W|\theta]}{\partial \beta} > 0$$

and $\frac{\partial E[W|\theta]}{\partial \alpha} \geq 0$ if and only if $\frac{\beta}{\alpha} \leq \frac{1}{(2r-1)(1-r)}$ Interpret and compare with part (b).

Solution

Plugging the expression for κ into the equilibrium condition for $a_i\left(x_i, y\right)$, we get

$$a_i\left(x_i, y\right) = \frac{\beta\left(1+r\frac{\beta(1-r)}{\alpha+\beta(1-r)}-r\right)}{\alpha+\beta} x_i + \left(1 - \frac{\beta\left(1+r\frac{\beta(1-r)}{\alpha+\beta(1-r)}-r\right)}{\alpha+\beta}\right) y =$$
$$= \frac{(1-r)\left(\beta^2+\alpha\beta\right)}{(\alpha+\beta(1-r))(\alpha+\beta)} x_i + \frac{\alpha(\alpha+\beta)}{(\alpha+\beta(1-r))(\alpha+\beta)} y = \frac{\alpha y+\beta(1-r)x_i}{\alpha+\beta(1-r)}$$

Alternatively, we can rewrite this expression in terms of the initially specified random variables:

$$a_i\left(x_i, y\right) = \frac{\alpha(\theta+\eta) + \beta(1-r)\left(\theta+\varepsilon_i\right)}{\alpha+\beta(1-r)} = \frac{\alpha\eta + \beta(1-r)\varepsilon_i}{\alpha+\beta(1-r)} + \theta$$

Now we can calculate expected welfare

$$\mathbb{E}[W(\mathbf{a},\theta) \mid \theta] = -\mathbb{E}\left[\int_0^1 \left(\frac{\alpha\eta+\beta(1-r)\varepsilon_i}{\alpha+\beta(1-r)} + \theta - \theta\right)^2 di \mid \theta\right] =$$

$$= -\mathbb{E}\left[\left(\frac{\alpha\eta+\beta(1-r)\varepsilon_i}{\alpha+\beta(1-r)}\right)^2 \mid \theta\right] = -\mathbb{E}\left[\frac{\alpha^2\eta^2+\beta^2(1-r)^2\varepsilon_i^2+2\alpha\beta(1-r)\eta\varepsilon_i}{[\alpha+\beta(1-r)]^2} \mid \theta\right] =$$
$$= -\frac{\alpha^2\mathbb{E}\left(\eta^2\right)+\beta^2(1-r)^2\mathbb{E}\left(\varepsilon_i^2\right)}{[\alpha+\beta(1-r)]^2} = -\frac{\alpha^2(1/\alpha)+\beta^2(1-r)^2(1/\beta)}{[\alpha+\beta(1-r)]^2} = -\frac{\alpha+\beta(1-r)^2}{[\alpha+\beta(1-r)]^2}$$

Let's analyze, how expected welfare depends on the precision of private (β) and public (α) signals.

$$\frac{\partial\mathbb{E}[W|\theta]}{\partial\beta} = -\frac{(1-r)^2(\alpha+\beta(1-r))^2-2(1-r)(\alpha+\beta(1-r))\left(\alpha+\beta(1-r)^2\right)}{[\alpha+\beta(1-r)]^4} =$$
$$= \frac{(1-r)\left[\alpha(1+r)+\beta(1-r)^2\right]}{[\alpha+\beta(1-r)]^3} > 0$$

which follows from $\alpha > 0, \beta > 0, r \in (0,1)$ Thus, we can conclude that expected welfare increases as the precision of private infor- mation goes up.

$$\frac{\partial \mathbb{E}[W \mid \theta]}{\partial \alpha} = -\frac{(\alpha+\beta(1-r))^2 - 2(\alpha+\beta(1-r))\left(\alpha+\beta(1-r)^2\right)}{[\alpha+\beta(1-r)]^4} = \frac{a-\beta(1-r)(2r-1)}{[\alpha+\beta(1-r)]^3}$$

Since $\alpha > 0, \beta > 0, r \in (0,1)$, then the denominator is strictly positive. The sign of $\frac{\partial \mathbb{E}[W|\theta]}{\partial \alpha}$ depends on the sign of the numerator.
Indeed, $\frac{\partial \mathbb{E}[W|\theta]}{\partial \alpha} \geq 0 \quad$ if and only if $\quad a-\beta(1-r)(2r-1) \geq 0 \quad$ or $\quad \frac{\beta}{\alpha} \leq \frac{1}{(2r-1)(1-r)}$

Therefore, there exist parameter values under which an increase in the precision of publicsignals may decrease expected welfare. Note that high precision of public information high α) is attractive for expected welfare only when the private information is not very orecise (low β).

This result just partly confirms the idea that we get in part 2, where expected welfare s a strictly increasing function of public information precision (alternatively, a strictly decreasing function of the variance of public information). If we introduce private signals o the model, then the impact of public information on expected welfare decreases, and n the case of

$$\frac{\beta}{\alpha} > \frac{1}{(2r-1)(1-r)}$$

it is even harmful. Solution of this problem comes from Egor Malkov.