

Recitations 5

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MINI

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RECITATION 5

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ZOOM

Today:

- Time-separable u
- state-separable g E u
- Topics again

Office Hours: 5PM's, 7 topics,
Milgram, Shannon

Next time: Risk, Uncertainty
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Recitation 5

[Definitions used today]

- Topkis theorem, Supermodularity, Increasing Differences
- Recursive and dynamically consistent family of utility function, time separability, ICC axiom

Question 1

Consider the following utility functions

a $u(c_1, c_2, c_3) = \min\{2c_1 + c_2 + c_3, c_1 + c_2 + 2c_3\}$

b $u(c_1, c_2, c_3) = c_1 + \sqrt{c_2} + \sqrt{c_3}$

1. Show that (a) does not have state-separable representation
2. Show that (b) does not have expected utility representation
3. Find $\pi \in \Delta \subset \mathbb{R}^3$ s.t. $u(c_1, c_2, c_3)$ is strictly risk averse with respect to π
4. Show that there is no $\pi \Delta \subset \mathbb{R}^3$ s.t. $u(c_1, c_2, c_3)$ is strictly risk averse with respect to π

Question 2 [Properties of state separable u]

- a Prove that every recursive family of utility functions $\{U_t\}$ is dynamically consistent if the aggregator function $G(\cdot, \cdot)$ is strictly increasing in continuation
- b $S \geq 3$ and \succeq increasing and continuous. Prove that \succeq has state-separable representation then ICC holds.
- c For $S = 2$ all increasing functions obey ICC. Show that for $u(c_1, c_2) = c_1\sqrt{c_2} + c_1 + c_2$ it does not have state separable utility function

Question 3 [Topkis theorem]

If S is a lattice, f is supermodular in x for fixed t , and f has nondecreasing differences in $(x; t)$, then $\varphi^*(t) = \arg \max_{x \in S} f(x, t)$ is monotone nondecreasing in t .

Question 4 254 [I.1 Spring 2018 majors]

Consider the problem of finding a Pareto optimal allocation of aggregate resources $\omega \in \mathbb{R}_+^n$ in an economy with two agents:

$$\begin{aligned} & \max_x \mu_1 u_1(x) + \mu_2 u_2(\omega - x) \\ & \text{subject to } 0 \leq x \leq \omega \end{aligned}$$

where $u_i : \mathbb{R}_+^n \rightarrow \mathbb{R}$ are agents' utility functions (assumed continuous) and $\mu_i > 0$ are welfare weights for $i = 1, 2$. Let $x^*(\mu_1, \mu_2)$ be the set of solutions.

- a State a definition of utility function u_i being supermodular. Show that if u_i is supermodular, then $u_i(\omega - x)$ is supermodular in x
- b Show that, if u_1 and u_2 are strictly increasing and supermodular in x then $x^*(\mu_1, \mu_2)$ is non-decreasing in μ_1 . You may assume that $x^*(\mu)$ is single-valued. Is $x^*(\mu_1, \mu_2)$ non-increasing in μ_2 ? Justify your answer. If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.
- c Under what conditions on u_1 and u_2 is the solution $x^*(\mu_1, \mu_2)$ unique. Justify your answer.

$$\Rightarrow c = (c_0, c_1 \dots c_t, \dots) \quad c \in l_\infty$$

$$\sup_t |c_t| < \infty$$

Def. \succ has time separable utility

$$: f \exists v_t: \mathbb{R}_+ \rightarrow \mathbb{R}$$

$$\forall c, c' \underset{P}{\succ} c \Leftrightarrow \sum_{t=0}^{\infty} v_t(c'_t) \geq \sum_{t=0}^{\infty} v_t(c_t)$$

Def. (ICC) Irrelevant of Common Consequences

$$c_{-t} y = (c_0, \dots, c_{t-1}, \boxed{y}, c_{t+1}, \dots,)$$

$$y, w \in \mathbb{R}_+$$

$$c_{-t} y \underset{P}{\succ} d_{-t} y \Leftrightarrow$$

$$\Leftrightarrow c_{-t} w \underset{P}{\succ} d_{-t} w$$

2(b) NTS: \succ time separable \Rightarrow ICC holds

We have $S \geq 3$

Proof: ~~time separable~~

$$c \succ c' \Leftrightarrow \sum_{t=0}^{\infty} v_t(c_t) \geq \sum_{t=0}^{\infty} v_t(c'_t)$$

$$\text{take } c_t = c'_t = y$$

$$\sum_{S \neq t}^{\infty} V_S(c_S) + V_t(y) \geq \sum_{S \neq t}^{\infty} V_S(c'_S) + V_t(y)$$

let's add to both sides $V_t(w) - V_t(y)$

$$\sum_{S \neq t}^{\infty} V_S(c_S) + V_t(w) \geq \sum_{S \neq t}^{\infty} V_S(c'_S) + V_t(w)$$

$$\Leftrightarrow c_{-t}w \geq c_{-t}^l w$$

Wrap up:

$$c_{-t}y \geq c_{-t}^l y \Rightarrow c_{-t}w \geq c_{-t}^l w$$

$| \subset$

2(b) $S=2$ all stru. incr. functions

satisfy $| \subset$

It will be necessary but not sufficient

\rightarrow time separability

$$u(c_1, c_2) = c_1 S c_2 + c_1 + c_2$$

$$\text{Take } \alpha \in \mathbb{R} \quad u(0, \alpha) = \alpha = u(\alpha, 0)$$

Let's assume that we have time-sep utility rep. $\rightarrow v_1, v_2$

$$V_0(0) + V_1(\alpha) = V_0(\alpha) + V_1(0)$$

$$V_1(c\alpha) \quad V_1(c)$$

$$c_0=0, c_1=\alpha \quad c_0=\alpha, c_1=0$$

$$V_1(\alpha) - V_0(\alpha) = V_1(0) - V_0(0) = \text{constant}$$

Take $C = (1, 4)$ $\alpha = (4, 1)$

$$\alpha(1, 4) = 1 \cdot 2 + 1 + 4 = 7$$

$$\alpha(4, 1) = 4 \cdot 1 + 1 + 4 = 9$$

$$(4, 1) > (1, 4)$$

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$$V_0(4) + V_1(1) > V_0(1) + V_1(4)$$

$$\underbrace{V_1(1) - V_0(1)}_{V_1(\alpha) - V_0(\alpha)} > \underbrace{V_1(4) - V_0(4)}_{V_1(\alpha) - V_0(\alpha)}$$

$$0 > 0$$

$S \rightarrow$ no linear separable α -repr.

$$\text{Ex-1. (2)} \quad u(c_1, c_2, c_3) = \checkmark$$

$$= \min \left\{ 2c_1 + c_2 + c_3, c_1 + c_2 + 2c_3 \right\}$$

By T. 1 \Rightarrow str. inv., continuous, S.P. 3

Time sep \Leftrightarrow ICC

It is enough to show ICC does not hold

$$\text{Let } c_1 = y = 1 \quad c_2 = 2 \quad c_3 = 1$$

$$d_1 = y = 1 \quad d_2 = 1 \quad d_3 = 2$$

(ICC):

$$(c_{-1}, y) = (1, 2, 1) \succ (1, 1, 2) = (d_{-1}, y)$$

$$u(1, 2, 1) = \min \{ 2+2+1, 1+2+2 \}$$

$$u(1, 1, 2) = \min \{ 2 \cancel{+} 1 + 1+2, \cancel{1+1+2} \stackrel{=}{=} 5 \}$$

$$(c_{-1}, y) \succ (d_{-1}, y) \stackrel{=}{=} 5$$

Now take $w = 5$

$$c_1 = 5 \quad c_2 = 2 \quad c_3 = 1$$

$$d_1 = 5 \quad d_2 = 1 \quad d_3 = 2$$

$$(c_{-1}, y) = (5, 2, 1) \quad \overline{\succ} \quad \overline{(5, 1, 2)} = (d_{-1}, w)$$

? ? ?

$$u(5, 2, 1) = \min \{ 2 \cdot 5 + 2 + 1, 5 + 2 + 2 \cdot 1 \} \\ = \min \{ 13, 8 \} = 8$$

$$u(5, 1, 2) = \min \{ 2 \cdot 5 + 1 + 2, 5 + 1 + 2 \cdot 2 \} \\ = \min \{ 13, 10 \} = 10$$

$$(5, 1, 2) = (d-1, \omega) > (c-1, \omega) = (5, 2, 1)$$

contradiction with ICC

$$c_1, \omega \quad \omega, y$$

$$c - t y \geq d - t y \text{ true}$$

$$c - t \omega < d - t \omega \quad \text{so ICC violated.}$$

So $\ell(c_1, c_2, c_3)$ does not have time sep. rep.

$$(b) u(c_1, c_2, c_3) = c_1 + \sqrt{c_2} + \sqrt{c_3}$$

What does it mean for u to have EU rep.

$$\pi_1, \pi_2, \pi_3 \quad \sum_{i=1}^3 \pi_i = 1 \quad \text{if } i > 0 \text{ then } JV$$

$$u(c_1, c_2, c_3) \geq u(d_1, d_2, d_3)$$

\Leftrightarrow

$$\pi_1 v(c_1) + \pi_2 v(c_2) + \pi_3 v(c_3)$$

\Rightarrow

$$\pi_1 v(d_1) + \pi_2 v(d_2) + \pi_3 v(d_3)$$

In particular take c, d

$$(0, 0, 0) \quad (1, 1, 1)$$

$$u(0, 0, 0) = 0 + 0 + 0 = 0$$

$$u(1, 1, 1) = 1 + 1 - 1 = 1$$

$$(1, 1, 1) > (0, 0, 0)$$

Now plug it to EU

$$\underbrace{(\pi_1 + \pi_2 + \pi_3)}_{v(0) < v(1)} \cdot v(0) < \underbrace{(\pi_1 + \pi_2 + \pi_3)}_{v(1)} v(1)$$

$$\text{Take } (0, 0, 0) \quad (2, 2, 2) \Rightarrow v(0) < v(2)$$

Now take $\mathbf{e} = (1, 0, 0) \quad (0, 0, 1)$

$$u(1, 0, 0) = 1 + 0 + 0 = 1$$

$$u(0, 0, 1) = 0 + 0 + 1 = 1$$

$$(1, 0, 0) \sim_I (0, 0, 1)$$

$$\pi_1 \cdot v(1) + \underline{\pi_2 v(0)} + \pi_3 v(0) =$$

$$= \pi_1 \cdot v(0) + \underline{\pi_2 v(0)} + \pi_3 v(1)$$

$$(\pi_1 - \pi_3)(v(1) - v(0)) = 0$$

$$v(1) > v(0) \Rightarrow \pi_1 = \pi_3$$

Take $(2, 2, 0) \& (0, 2, 2)$

$$u(2, 2, 0) = 2 + \sqrt{2} + 0$$

$$u(0, 2, 2) = 0 + \sqrt{2} + \sqrt{2} = 2\sqrt{2}$$

$$2\sqrt{2} < 2 + \sqrt{2}$$

$$\therefore (2, 2, 0) >_P (0, 2, 2)$$

Write $\mathbb{E} u$, so:

$$\underline{\pi_1 v(2)} + \underline{\pi_2 v(2)} + \underline{\pi_3 v(0)} >$$

$$> \underline{\pi_1 v(0)} + \underline{\pi_2 v(2)} + \underline{\pi_3 v(2)}$$

$$(\pi_1 - \pi_3)(v(2) - v(0)) > 0$$

$$v(2) > v(0) \quad \pi_1 = \pi_3$$

$$0 > 0$$

So $u(-)$ does not have $\mathbb{E} u$ rep.

Def. Let $\{\pi_i\}$ given $E_C = \sum_{i=1}^S \pi_i c_i$

$$E_d = (E_{C_1}, E_{C_2}, \dots, E_{C_S})$$

we say that π_i are Risk averse

wrt $\{\pi_i\}$ if $E_d \geq_p C$

Thm. $S \geq 3$, π_i str. incr., continuous.

\Rightarrow $c_i \in \mathbb{R}$ & risk averse wrt $\{\pi_i\}_{i=1}^S$

$$\Leftrightarrow \text{our } EV(C) = \sum_{i=1}^S \pi_i v(c_i)$$

has v being concave f.

$$v(c) \quad \pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{2} \right)$$

Consider cases : (1) $c_1 \leq c_3$

$$(2) \quad c_1 > c_3$$

$$(1) \quad \text{Want } c_1 \leq c_3 \quad 1 + c_1 + c_2 + c_3$$

$$2c_1 + c_2 + c_3 \leq c_1 + c_2 + 2c_3$$

$$\min \{1, c_1, c_2, c_3\} = 2c_1 + c_2 + c_3$$

$$E_C = \pi_1 \cdot c_1 + \pi_2 \cdot c_2 + \pi_3 \cdot c_3 =$$
$$\left(\frac{1}{3} (c_1 + c_2 + 2c_3) \right)$$

$$u(\mathbb{E} \mathbf{c}) = \min \{2 \cdot \mathbb{E}_C + \mathbb{E}_C + \mathbb{E}_C\}$$

$$\mathbb{E}_C + \mathbb{E}_C + 2\mathbb{E}_C$$

$$= 4 \cdot \mathbb{E}_C = 4 \cdot \frac{1}{3} (c_1 + c_2 + 2c_3)$$

$$= c_1 + c_2 + 2c_3$$

$$\mathbb{E} \mathbf{c} \succ_p c$$

$$\boxed{u(\mathbb{E} \mathbf{c})} \geq u(c)$$

$$c_1 + c_2 + 2c_3 \geq 2c_1 + c_2 + c_3 \quad \checkmark$$

$$② c_3 < c_1 \quad / + c_1 + c_2 + c_3$$

$$c_1 + c_2 + 2c_3 < 2c_1 + c_2 + c_3$$

$$\min \{c_1 + c_2 + 2c_3, 2c_1 + c_2 + c_3\}$$

$$u(c_1, c_2, c_3) = c_1 + c_2 + 2c_3 \quad (**)$$

$$\mathbb{E} \mathbf{c} \succ_p c$$

$$\boxed{u(\mathbb{E} \mathbf{c})} \geq u(c) \quad (**)$$

$$c_1 + c_2 + 2c_3 \geq c_1 + c_2 + 2c_3$$

1 game $\Pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ u is risk averse wrt Π

$$2(d) \quad \pi \in \Delta \quad \Delta \subseteq \mathbb{R}^3$$

$$(\pi_1, \pi_2, \pi_3) : \pi_1 + \pi_2 + \pi_3 = 1$$

In 2(b) we said that there $\pi_1 \geq 0$
 is no $\pi \in \Delta^3$ for which
 $h(c_1, c_2, c_3) = c_1 + \sqrt{c_2} + \sqrt{c_3}$
 has a \mathbb{E}_μ repr.

In particular it does not have
 concave \mathbb{E}_μ repr.

However this function has simple
 times separable repr.

$$v_1(c_1) = c_1$$

$$v_2(c_2) = \sqrt{c_2}$$

$$v_3(c_3) = \sqrt{c_3}$$

$$v_i(c_i) = 0 \quad i \neq 1, 2, 3$$

$$h(c_1, c_2, c_3) = \sum_{i=0}^{\infty} v_i(c_i)$$

By previous thm we know
 that LCC holds

Then about Ell rep with risk
averse utility wrt \mathcal{U}

Theorem. i.e. ~~Ell~~ ^{Ell} strategy cont. S23

Ell has risk averse utility

\Leftrightarrow has V being concave

Thus we know that
there is no ~~\mathcal{U}~~ for which
Ell is ~~Ell~~ risk averse cont. \mathcal{U} .

$$L(a) \quad \max \mu_1 u_1(x) + \mu_2 u_2(\omega - x)$$

s.t. $0 \leq x \leq \omega$

$u_i(x)$ SPM: $\forall x, x' \in \mathbb{R}^n$

$$u_i(x \wedge x') + u_i(x \vee x') \geq u_i(x) + u_i(x')$$

$\forall \omega \in \mathbb{R}_+$

• $(\omega - x) \wedge (\omega - x') = \omega - x \vee x'$,
 $(\omega - x) \vee (\omega - x') = \omega - x \wedge x'$

~~$x : \omega - x \quad x' := \omega - x'$~~

$$u_i(\omega - x \vee x') + u_i(\omega - x \wedge x')$$

$$\Rightarrow u_i(\omega - x) + u_i(\omega - x')$$

this means

$u_i(\omega - x)$ is SPM in x

Topcis then ① $X = \mathbb{R}^n$ is lattice ✓

② f has D (x, t)

③ f is SPM in x , fixed t

$$x(t) = \arg \max_{x \in X} f(x, t)$$

If ①-②-③ satisfied then $x(t)$ non decreasing

• Fix μ_2

We need to check (2), (3)

t must be $t \in \mathbb{R}$

(2) $F(x, \mu_1) = \mu_1 u_1(x) + \mu_2 u_2(\omega - x)$

(2) F has DD in (x, μ_1)

WTS: $x' \geq x$ $\mu'_1 \geq \mu_1$ (*)

$$F(x', \mu'_1) - F(x, \mu_1) \geq \bar{F}(x', \mu_1) - \bar{F}(x, \mu_1)$$

$$\mu'_1 u_1(x') + \mu_2 u_2(\omega - x') - ?$$

$$- \mu_1 u_1(x) - \mu_2 u_2(\omega - x) \geq ?$$

$$\mu_1 u_1(x') + \mu_2 u_2(\omega - x') - ?$$

$$- \mu_1 u_1(x) - \mu_2 u_2(\omega - x) - ?$$

$$(\mu'_1 - \mu_1)(u_1(x') - u_1(x)) \geq ?$$

Yes $\mu'_1 \geq \mu_1$ u_1 is str. increasing
 $(x' \geq x)$ ok

• Fix μ_1

$$\bar{F}(x, \mu_2) = \mu_1 u_1(x) + \mu_2 u_2(\omega - x)$$

take $\mu'_2 \geq \mu_2$ $x' \geq x$

WTS: \bar{F} has DD. In (*) \leq instead!

• Go back to case with fixed μ_2

$$\underline{F(x, \mu_1) = \dots}$$

③ with fixed μ_1 WTS it is SPM

Obvious : μ_1, μ_2 fixed

$$\mu_1 \ell_1(x \wedge x') + \ell_{1j}(x \vee x') \geq \ell_j(x) + \ell_j(x')$$

$$\mu_2 \ell_2(\omega - x \wedge x') + \ell_{2j}(\omega - x \vee x') \geq \ell_j(\omega - x) + \ell_j(\omega - x')$$

$\mu_1, \mu_2 > 0$ and get

$$\begin{aligned} & \mu_1 \ell_1(x \wedge x') + \mu_2 \ell_2(\omega - x \wedge x') \\ & + \mu_1 \ell_{1j}(x \vee x') + \mu_2 \ell_{2j}(\omega - x \vee x') \\ & \geq \mu_1 \ell_j(x) + \mu_2 \ell_j(\omega - x) \\ & + \mu_1 \ell_1(x') + \mu_2 \ell_2(\omega - x') \end{aligned}$$

Well

$$\begin{aligned} & F(x \wedge x', \mu_1) + F(x \vee x', \mu_1) \\ & \geq \underbrace{F(x, \mu_1)}_{\text{- } \widehat{F}} + F(x, \mu_2) \quad \checkmark \\ & - \widehat{F} \text{ SPM} \end{aligned}$$

(1), (2) & (3) satisfied so from Toplis
we get $x(\mu_n)$ is non decreasing

\bar{F} has DD (1), (2), (3) satisfied

$\Rightarrow x(\mu_2)$ is non increasing

~~(1) (2)~~

h(c) $\circ u$ is str. quasi concave

\Rightarrow at most one ~~solution~~ solution

to problem $\max_{x \in X} u(x)$

* $\max_{0 \leq x \leq w} u$ u - cont.

$x \in [0, w]$ compact

* Weierstrass Thm $\Rightarrow \exists \bar{x}$

$u(\bar{x}) = \bar{x}$ being maximizer

* Lemma 1

$u_i(x) \rightarrow u_i(w-x)$

Lemmas 2

SQC

$\{u_i\}_{i=1}^n$ SQF $\Rightarrow u = \sum \alpha_i u_i \quad \alpha_i \geq 0$
 u is SQF