Recitations 6

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MINI

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Today: – Risk

– Uncertainty

Knight 1921

Office Hours: Instead

Next time: Exam !

Q 2(a) Before that

$v_1, V_2 \in C^2$, str. increasing v NM

Thm. Pratt - Following are equivalent

① $A_1(x) \geq A_2(x) \quad \forall x$

② $p_1(x, z) \geq p_2(x, z) \quad \forall x,$
$\forall z \in \mathbb{R}^S \; Ez=0$

③ $\exists f$ str. inc, str. concave
$v_1 = f(v_2) \qquad [\forall x \;\; v_1(x) = f(V_2(x))]$

$$A(x) = -\frac{u''(x)}{u'(x)} \qquad \text{ARA}$$

$$R(x) = \frac{-u''(x)}{u'(x)} \cdot x \qquad \text{RRA}$$

$\Rightarrow$ Consider $v(x)$
$u(x) \equiv V(x+\varepsilon) \quad \varepsilon > 0$

$$A_u(x) = A_V(x+\varepsilon) = -\frac{v''(x+\varepsilon)}{v'(x+\varepsilon)} \geq A_v(x)$$

$A_v(x)$ is ~~us~~ increasing

By Pr-ett (by $Au(x) \geq Av(x)$)

we get $pu(x,z) \geq pv(x,z)$

but $pu(x,z) := pv(x+\varepsilon, z)$

$pv(x+\varepsilon, z) \geq pv(x,z)$ $\quad \varepsilon > 0$

pv is increasing in x.

$\Leftarrow$ Let's assume that

$pv(x+\varepsilon, z) = \underline{pu(x,z)} \geq \underline{pv(x,z)}$

by Prett Thm

$$Au(x) = \underbrace{Av(x+\varepsilon) \geq Av(x)}$$

$Av(\cdot)$ is increasing function

(b) $\quad v(x - p(x,z)) = E\, v(x+z)$

$\forall z \in \mathbb{R}^S \quad Ez = 0$

$v(x) = -(\alpha - x)^2$

$E\, v(x+z) = \underline{E\left(-(\alpha - (x+z))^2\right)}$

$= v(x - p(x,z)) = -\underbrace{(\alpha - (x - p(x,z)))}^2$

$$\mathbb{E}\left[f\left(\alpha^2 - 2\alpha(x+z) + (x+z)^2\right)\right]$$

$$= f\left[\alpha^2 - 2\alpha\left(x - p(x,z)\right) + \left(x - p(x,z)\right)^2\right]$$

$$\cancel{\alpha^2} - 2\alpha(\mathbb{E}x + \overset{0}{\widetilde{\mathbb{E}z}}) + \mathbb{E}(x^2 + \overset{0}{\widetilde{2xz}} + z^2)$$

$$= \cancel{\alpha^2} - 2\alpha\left(x - p(x,z)\right) + \left(x - p(x,z)\right)^2$$

$$p(x,z)^2 + 2(\alpha - x)\,p(x,z) = \mathbb{E}z^2$$

$$p(x,z) = -(\alpha - w) \pm \sqrt{(\alpha - w) + \mathbb{E}z^2}$$

since $f \cdot v$ is concave by Ex 1

$$p(x,z) \geq 0 \qquad \underbrace{-(\alpha - w) + \sqrt{(\alpha - w) + \mathbb{E}z^2}}_{= p(w,z)}$$

Let say that we have $\exists u$ vNM

$$A_u(x) = A_v(x) = -\frac{v''(x)}{v'(x)} =$$

$$= \frac{1}{\alpha - w}$$

$$\log u'(x) = \log(x - \alpha) + a_0 \qquad a_0 \in \mathbb{R}$$

$$[\ln f'(x)]' = \frac{f''(x)}{f'(x)}$$

$$u'(x) = e^{a_0}(x-\alpha)$$

$$u(x) = \frac{e^{a_0}}{2}x^2 - e^{a_0}\alpha x + a_1$$

$$u'(x) = \frac{e^{a_0}}{2}\cdot 2x - e^{a_0}\alpha = e^{a_0}(x-\alpha)$$

$$= -\frac{e^{a_0}}{2}\left(\underline{-\underline{(\alpha-x)}^2}\right) + \underline{\left(a_1 - \frac{e^{a_1}}{2}\alpha^2\right)}$$

$$= \underline{A}\cdot v(x) + B$$

(c) $v_1(x) = -e^{-x}$ $\qquad$ $v_2(x) = \log x$

$$A_1(x) = -\frac{-e^{-x}}{e^{-x}} = 1 \qquad \underline{A_2(x)} = -\frac{-\frac{1}{x^2}}{\frac{1}{x}} = \frac{1}{x}$$

$A_2(x) > A_1(x)$ $\qquad$ $x \in (0,1)$

$A_2(x) < A_1(x)$ $\qquad$ $x > 1$

neither v_1 more r.A than v_2

nor v_2 more r.a. than v_1

$$RRA_2(x) = -\frac{u''(x)}{u'(x)}\cdot x = 1$$

CRRA (1)

Q 1 (a) $\underline{p(w, z)}$

$$\mathbb{E} v(w+z) = v(w - p(w,z)) \quad (*)$$

So consumer is willing to sacrifice $p(w,z)$ amount to get sure outcome that gives him/her as much utility as Expected utility for accepting gamble z ($w+z \to$ new wealth)

This holds for $\forall z \in \mathbb{R}^S$ $\mathbb{E}z = 0$

(b) $\Rightarrow$ Suppose $p(w, tz)$ is str. incr. in $t \geq 0$. WTS: agent is str. r. a.

- $p(w, 0) = 0$
- $p(w, tz) > p(w, 0 \cdot z) = 0 \quad t \geq 0$

so $w - 0 > w - p(w, tz)$

V is str. increasing so

$$V(w) > v(w - p(w,tz)) \cancel{\mathbb{E}}$$

$$V(\underbrace{\mathbb{E}\, w+z}_{\mathbb{E} w+z = w}) > \underbrace{\mathbb{E} v(w+tz)}_{\text{def. risk compensation}}$$

$\mathbb{E} w+z = w$

def. risk compensation

$\underbrace{\mathbb{E}c \succsim_{\phi} c}_{\succ} \iff \underbrace{u(\mathbb{E}c) \geq \mathbb{E}u(c)}_{>}$

take $y = \omega + z$

For all ω, z $(\mathbb{E}z = 0) \rightarrow$

For all y

$\underline{v(\mathbb{E}y) > \mathbb{E}v(y)}$

So indeed agent is str. RA

$\Leftarrow$ Consider $0 \leq t_2 < t_1$

$\alpha = \frac{t_2}{t_1} \quad \& < 1$

$$w + t_2 z = \underbrace{\frac{t_2}{t_1}}_{\alpha}(w + t_1 z) + \underbrace{\left(1 - \frac{t_2}{t_1}\right)}_{1-\alpha} \cdot w$$

Lemma: If agent is str. RA

$\Rightarrow$ v is str. concave.

Proof: p_v, p_u

$$v(w - p_v(w,z)) = \mathbb{E}\, v(w+z) < v(\mathbb{E}\, w+z)$$

$v(w)$ $\parallel$

$\downarrow$ stv. RA of agent

v ist stv. incr.

$$p_v(w,z) > 0 .$$

Now let $u(w) = w$

then $u(w - p_u(w,z)) = \mathbb{E}\, u(w+z) =$

$$= \mathbb{E}\, w + z = \underline{\underline{w}}$$

$$w - p_u(w,z) = w \Rightarrow p_u(w,z) = 0 .$$

$$p_v(w,z) > 0 = p_u(w,z)$$

by Pratt Thm.

$\exists$ $\underline{f}$ stv. increasing, $\underline{\underline{\text{stv. concave}}}$ s.t.

$$\underline{v(w)} = f(u(w)) = \underline{\underline{f(w)}}$$

then v is stv. concave. $\boxtimes$

by our lemma

$$\mathbb{E}v(w+t_2 z) = \mathbb{E}v\left(\frac{t_2}{t_1}(w+t_1 z)+\left(1-\frac{t_2}{t_1}\right)w\right)$$

$$> \underbrace{\frac{t_2}{t_1}\mathbb{E}v(w+t_1 z)+\mathbb{E}\left(1-\frac{t_2}{t_1}\right)v(w)}$$

(**)

since

$$\mathbb{E}v(w)=v(w) \underset{①}{=} v(\mathbb{E}w+t_1 z) \underset{②}{>} \mathbb{E}v(w+t_1 z)$$

① $w=\mathbb{E}w+\mathbb{E}t_1 z$

② str. aversion of v

$$\mathbb{E}v(w+t_2 z) > \mathbb{E}\frac{t_2}{t_1}v(w+t_1 z)+\left(1-\frac{t_2}{t_1}\right)\mathbb{E}v(w)$$

$$> \frac{t_2}{t_1}\mathbb{E}v(w+t_1 z)+\left(1-\frac{t_2}{t_1}\right)\mathbb{E}v(w+t_1 z)$$

$$= \mathbb{E}v(w+t_1 z)$$

$$\mathbb{E}v(w+t_2 z) > \mathbb{E}v(w+t_1 z) \quad (***)$$

Before def. of concavity should be either

$$v(w+t_2 z s) > \frac{t_2}{t_1}v(w+t_1 z s) + \left(1-\frac{t_2}{t_1}\right)v(w)$$

or

$$\mathbb{E}v(w+t_2 z) > \frac{t_2}{t_1}\mathbb{E}v(w+t_1 z) + \left(1-\frac{t_2}{t_1}\right)\mathbb{E}v(w)$$

$\mathbb{E}\, v(w + t_2 z) > \mathbb{E}\, v(w + t_1 z)$

$v(w - p(w, t_2 z)) > v(w - p(w, t_1 z))$

by def of compensation

observe that v is str. incr.

$$w - p(w, t_2 z) > w - p(w, t_1, z)$$

$$p(w, t_2 z) < p(w, t_1, z)$$

$\underline{t_1 > t_2}$ so p is str. incr in $t \geq 0$

Q.4. $z \geq_{FOSD} y$ Def.

(a)

$\Rightarrow F_z(t) \le F_y(t) \quad t \in [a,b]$

[Equivalent to $\mathbb{E}v(z) \geq \mathbb{E}v(y)$]

$\forall$ cont, non decreasing $\underline{v}$.

~~W~~TS $\mathbb{E}z \geq \mathbb{E}y$

$\circ\ \mathbb{E}z = \int_a^b t\, dF_z(t)\, dt = $ (Int. by parts)

$= b - \int_a^b F_z(t)\,dt$

$\circ\ \mathbb{E}y = \int_a^b t\,dF_y(t)\,dt = b - \int_a^b F_y(t)\,dt$

Let's integrate def. over (a,b)

$$\int_a^b F_z(t)\,dt \le \int_a^b F_y(t)\,dt$$

$$\mathbb{E}z = b - \int_a^b F_z(t)\,dt \geq \underbrace{b - \int_a^b F_y(t)\,dt}_{\mathbb{E}y}$$

Indeed

$\mathbb{E}z \geq \mathbb{E}y$

$\mathbb{E}z - \mathbb{E}y = b - b + \int_a^b F_y(t) - F_z(t) = 0$

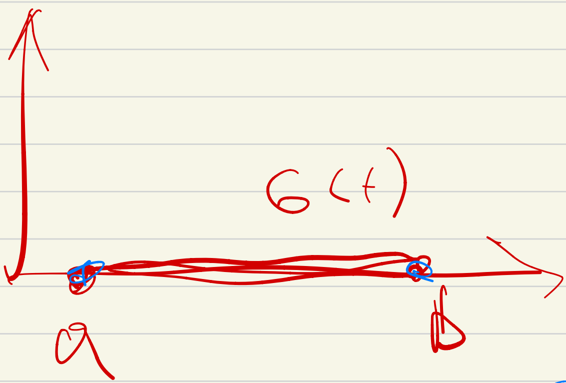
we know that $F_z(t) \le F_y(t)$

Define $\underline{\underline{G(t)}} = \int_a^t F_z(s) - F_y(s)\,ds$

$\underline{G(b)} = \underline{0} = \mathbb{E}z - \mathbb{E}y$

$G(a) = \int_a^a \ldots = 0$

$G'(t) = F_z(t) - F_y(t) \le 0$



$G'(t) = 0$ since $G(a) = G(b)$ and $G'(t) \le 0$

$F_z(t) - F_y(t) = 0$

So $F_z(t) = F_y(t)$. a.s.

(c) • $z \geq_{SOSD} y$ $\quad \int_a^t F_z(s)\,ds \leq \int_a^t F_y(s)\,ds$

$\forall t \in [a,b]$

~~• Eq~~

• y more risky than z

• Equivalent

$$\boxed{z \geq_{SOSD} y \iff \underline{\mathbb{E}v(z) \geq \mathbb{E}v(y)} \quad \forall v \text{ cont, non decr., concave f.'s}}$$

(d) $z \geq_{FOSD} y$

$$F_z(s) \leq F_y(s) \quad \text{integrate over } (a,t)$$

$$\underbrace{\int_a^t F_z(s)\,ds \leq \int_a^t F_y(s)\,ds}_{z \geq_{SOSD} y}$$

In (c) WTS: $\mathbb{E}z \geq \mathbb{E}y$

Take $t = b$

$$\mathbb{E}z = b - \int_a^b F_z(t) \geq b - \int_a^b F_y(t) = \mathbb{E}y$$

y is more risky than z $\Leftrightarrow$

$z \geq_{SOSD} y$ & $\mathbb{E}\, y = \mathbb{E}\, z$