Recitations 3

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MINI

FALL 2020

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RECITATION 3

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ZOOM

CHECK OUT
MATERIALS
ON MY
WEBPAGE

- AFRIAT & DEBREU'S

Next time:

- HW2
- Topkis
- Midterms

OH, RA on Monday

5:40 - 7:40

CMSch L110

- HW1, HW2
- solve previous midterm Q's

Debreu's Theorem proof by Ariel Rubinstein and friends

In what follows, we will need the mathematical concept of a dense set.

Definition 0.1. *A set Y is said to be dense in X if every non-empty open set $B \subset X$ contains an element in Y*

Corollary 0.2. *Any set $X \subset \mathbb{R}^m$ has a countable dense subset.*

Proof. The standard topology in $\mathbb{R}^n$ has a countable base, that is, any open set is the union of subsets of the countable collection of open sets: $B(a, \frac{1}{m})$ $a \in \mathbb{R}^m$ and all its componenets are rationalnumbers; m is a natural number.
For every set $B(q, \frac{1}{m}$ that intersects X, pick a point $y_{q,m} \in X \cap B(q, \frac{1}{m})$. The set that contains all of the points $y_{q,m}$ is a countable dense set in X. □

Theorem 0.3. ***Debreu's**. Let $\succeq$ be a continuous preference relation on X, which is a convex subset of $\mathbb{R}^n$. Then $\succeq$ has a continuous utility representation.*

Before we prove this theorem few lemmas

Lemma 0.4. *If $x \succ y$ then there $\exists z \in X$ s.t. $x \succ z \succ y$*

Proof. Assume not. Let I be the interval between x and y. By the convexity of X, $I \subset X$.
Construct inductively two sequences of points in I, $\{x_t\}$ and $\{y_t\}$, in the following manner:
First, define $x_0 = x$ and $y_0 = y$. Assume that the two points $x_t, y_t \in I$, and satisfy $x_t \succeq x$ and $y \succeq y_t$. Consider m, the middle point between x_t and y_t. Either $m \succeq x$ or $y \succeq m$. In the former case, define $x_{t+1} = m$ and $y_{t+1} = y_t$, and in the latter case define $x_{t+1} = x_t$ and $y_{t+1} = m$.

The sequences $\{x_t\}$ and $\{y_t\}$ are converging, and they must converge to the same point z because the distance between x_t and y_t converges to zero. By the continuity of $\succeq$, we have $z \succeq x$ and $y \succeq z$ and thus, by transitivity, $y \succeq x$, which contradicts the assumption that $x \succ y$.

Another simple proof would fit the more general case, in which the assumption that the set X is convex is replaced by the weaker assumption that X is a connected subset of $\mathbb{R}^n$: If there is no z such that $x \succ z \succ y$, then X is the union of two disjoint sets $\{a | a \succ y\}$ and $\{a | x \succ a\}$, which are open by the continuity of the preference relation.

This contradicts the connectness of X (a connected set cannot be covered by two nonempty disjoint open sets).

□

Lemma 0.5. *Let Y be dense in X. Then for every $x, y \in X$ if $x \succ y$ then there $\exists z \in X$ s.t. $x \succ z \succ y$*

Proof. By Lemma 1, there exists $z \in X$ such that $x \succ z \succ y$. By continuity, there is a ball around z such that any point in the ball is sandwiched between x and y and, by the denseness of Y, the ball contains an element of Y. □

Lemma 0.6. *Let E be the set of $\succeq$ -maxima and $\succeq$ -minima in X. Let Y be a countable dense set in XE. Then, $\succeq$ has a utility representation on Y, u with a range that consists of all dyadic rational numbers in* $(0,1)$ *(namely all numbers that can be expressed as $k/2^l$ where k and l are natural numbers and $k < 2^l$).*

Proof. By **Lemma 1**, XE is an infinite set and therefore Y is as well. Let $Y = \{y_n\}$.
Construct u by induction as follows:
Start with $u(y_1) = 0.5$. Let $P(y_n) = \{y_1, \dots y_{n1}\}$, i.e., the set of elements that precedes y_n in the enumeration of Y. If $y_n \sim y_m$ for some $y_m \in P(y_n)$, let $u(y_n) = u(y_m)$. If $y_n \succ y_k$ where y_k is maximal in $P(y_n)$, set $u(y_n) = (1 + u(y_k))/2$. If $y_k \succ y_n$ where y_k is minimal in $P(y_n)$, set $u(y_n) = u(y_k)/2$.

Otherwise, there are $y_i, y_j \in P(y_n)$ such that y_i is minimal among the elements in $P(y_n)$ that are preferred to y_n and y_j is maximal among the elements in $P(y_n)$ that are inferior to y_n. Let $u(y_n) = (u(y_i) + u(y_j))/2$.

Note that by **Lemma 2**, for every element in the sequence there will always eventually be one element in the sequence that is above it and one that is below it and for every two elements in the sequence there will eventually be an element in the sequence that is sandwiched between the two.

Therefore, the range of u is exactly all dyadic numbers in $(0,1)$. □

Proof. Here we comlpete the proof of Debreu's Theorem (0.3)

Let Y be a countable dense set in XE. Define u on Y according to **Lemma 3** .
The function u can be extended to X by:

1. assigning the value 1 to all maxima points in X and the value 0 to all minima points
2. defining $u(x) = \sup\{u(y)|x \succ yandy \in Y\} \quad \forall x \notin Y \cup E$.

This function represents the preference relation since by definition if $x \sim z$ we have $u(x) = u(z)$ and if $x \succ z$ then by **Lemma 2** there are y_1 and y_2 in Y such that $x \succ y_1 \succ y_2 \succ z$ and thus $u(x) \geq u(y_1) \geq u(y_2) \geq u(z)$.
In order to prove the continuity of u, consider a point $x \notin E$ (a similar proof applies to extreme points). Let $\epsilon > 0$. By Lemma 3, there are y_1 and y_2 in Y such that

$$u(x) - \epsilon < u(y_1) < u(x) < u(y_2) < u(x) - \epsilon$$

By twice applying the definition of the continuity of $\succeq$, we obtain a ball B around x that is between y_1 and y_2 with respect to the preference relation. By definition, elements in this ball receive u values between $u(y_1)$ and $u(y_2)$ and thus are not further than ϵ from $u(x)$. □

Afriat's Theorem proof by Fostel, Scarf, and Todd (2004)

A version of this proof was provided by Ichiro Obara.

Definition 0.1. *We say that the observations satisfy the Generalized Axiom of Revealed Preference **(GARP)** if for every ordered subset* $\{i, j, k, \dots r\} \subset \mathbb{N}$*:*

$$p_i \cdot x_j \leq p_i \cdot x_i$$

$$p_j \cdot x_k \leq p_j \cdot x_j$$

$$\vdots$$

$$p_r \cdot x_i \leq p_r \cdot x_r$$

it must be true that each inequality is, in fact, an equality

Consider a finite data set $D = \left\{(p^t, x^t) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_{+}, t = 1, \dots, T\right\}$. This note proves the following proposition, which is skipped in the class.

Theorem 0.2. *Suppose that a finite data set D satisfies GARP. Then there exists $\lambda^t > 0, U^t, t = 1, \dots, T$ such that*

$$U^j \leq U^i + \lambda^i p^i \cdot \left(x^j - x^i\right) \text{ for all } i, j \in \{1, \dots, T\}$$

Denote by M^T the set of $T \times T$ matrices where all diagonal elements are 0. We say that $A \in M^T$ satisfies GA if the following condition is satisfied.

$$\begin{aligned} &\text{(GA) For any } \left\{a_{t(n)t(n+1)}\right\}_{n=1}^{N} \\ &\text{if } a_{t(n)t(n+1)} \leq 0 \text{ for } n = 1 \dots N-1 \\ &\text{then } a_{t(N)t(1)} \geq 0 \end{aligned}$$

where $a_{N,N+1} = a_{N,1}$. Note that the $T \times T$ matrix where ij entry is given by $a_{ij} = p^i \cdot \left(x^j - x^i\right)$ satisfies GA if D satisfies GARP.
First we show that GA is equivalent to the following condition.

$$\begin{aligned} &(\text{GA}^*) \text{ For any } \left\{a_{t(n)t(n+1)}\right\}_{n=1}^{N} \\ &\text{if } a_{t(n)t(n+1)} \leq 0 \text{ for } n = 1 \dots N \\ &\text{then } a_{t(n)t(n+1)} = 0 \text{ for } n = 1 \dots N \end{aligned}$$

Lemma 0.3. *GA and GA* are equivalent.*

Proof. Suppose that GA is satisfied. If $a_{t(n)t(n+1)} \leq 0$ for all n, then any $a_{t(n)t(n+1)}$ can be regarded as the tail of a cycle (the one starting at $a_{t(n+1)t(n+2)}$) in GA. So $a_{t(n)t(n+1)} \geq 0$, hence $a_{t(n)t(n+1)} = 0$ for all n

Conversely, suppose that GA* is satisfied. If $a_{t(n)t(n+1)} \leq 0$ for $n = 1 \dots N-1$, then $a_{t(N)t(1)} < 0$ cannot be the case because then GA* implies $a_{t(N)t(1)} = 0$, a contradiction. Hence $a_{t(N)t(1)} \geq 0$ □

Now we prove the above proposition.

*Proof. of **Theorem** (0.2).*

The proof is based on induction. We show that we can find such $\lambda^t > 0, U^t, t = 1, \dots, T$ for every $A \in M^T$ that satisfies GA* if we can find them for every $A \in M^{T-1}$ that satisfies GA*. since this is trivially true for $T = 1$ this proves that the same property holds for every T. We start with the following lemma.

Lemma 0.4. *Suppose that $A \in M^T$ satisfies GA^*. Then there exists t^* such that $a_{t^*t} \geq 0$ for $t = 1, \ldots, T$*

Proof. Suppose not. Then we can construct a cycle $\{a_{t(n)t(n+1)}, n = 1, \ldots, N\}$ such that $q_{t(n)t(n+1)} < 0$ for all n. This contradicts GA*. Suppose that $A \in M^T$ satisfies GA and, without loss of generality, assume that $a_{Tt} \geq 0$ for all t. Now define $(T-1) \times (T-1)$ matrix A' as follows.

$$a'_{ij} = \begin{cases} a_{ij} \text{ if } a_{Tj} > 0 \\ \min\{a_{ij}, a_{iT}\} \text{ if } a_{Tj} = 0 \end{cases}$$

(Note that any diagonal element is zero $(a'_{ii} = a_{ii} = 0)$, so $A' \in M^{T-1}$. If not, then $a'_{ii} = a_{iT} < 0$. But $a_{iT} < 0$ and $a_{Ti} = 0$ violates GA*). □

Lemma 0.5. *If $A \in M^T$ satisfies GA^*, then $A' \in M^{T-1}$ satisfies GA^*.*

Proof. Suppose not. Then we can find a cycle $\left\{a'_{t(n)t(n+1)}, n = 1, \ldots, N\right\}$ such that $a'_{t(n)t(n+1)} \leq 0$ for all n such that at least one inequality is strict. If every a'_{ij} is a_{ij} in this cycle, then this contradicts the assumption that A satisfies GA *. So suppose that there exists a'_{ij} within this cycle such that $a'_{ij} = a_{iT} \leq 0$ and $a_{Tj} = 0$. Then we can replace a'_{ij} with a_{iT} and a_{Tj} in the original cycle. In this way, we can eliminate such a'_{ij} and guarantee that each element of this cycle is from A. Thus again we reach a contradiction. Hence A' must satisfy GA* □

Now we can complete the proof. By the inductive assumption, there exists $\lambda^t > 0, U^t, t = 1, \ldots, T-1$ such that

$$U^j \leq U^i + \lambda^i a'_{ij} \text{ for all } i, j \in \{1, \ldots, T-1\}$$

By definition of a'_{ij}, we have

$$U^j \leq U^i + \lambda^i a_{ij} \text{ for all } i, j \in \{1, \ldots, T-1\}$$

Define U^T and $\lambda^T > 0$ as follows.

$$U^T = \min_{i \in \{1, \ldots, T-1\}} \left\{U^i + \lambda^i a_{iT}\right\}$$
$$\lambda^T = \max\left\{1, \max_{j: a_{Tj} \neq 0} \left\{\frac{U^j - U^T}{a_{TJ}}\right\}\right\}$$

We are done if we can show

$$U^T \leq U^i + \lambda^i a_{iT} \text{ for all } i$$
$$U^j \leq U^T + \lambda^T a_{Tj} \text{ for all } j$$

The first inequalities are satisfied by definition. As for the second inequality, it follows from the definition for any j when $a_{Tj} > 0$. If $a_{Tj} = 0$, then

$$U^j \leq U^i + \lambda^i a'_{ij} = U^i + \lambda^i a_{iT} \text{ for any } i \in \{1, \ldots, T-1\}$$

by definition of a'_{ij}. Hence we get

$$\begin{aligned} U^j &\leq \min_{i \in \{1, \ldots, T-1\}} \left\{U^i + \lambda^i a_{iT}\right\} \\ &= U^T \\ &= U^T + \lambda^T a_{Tj} \end{aligned}$$

This proves that we can find such $\lambda^t > 0, U^t, t = 1, \ldots, T$ for every $A \in M^T$ that satisfies GA*. since GA* is equivalent to GA by the lemma and GA is satisfied when D satisfies GARP, the proposition is proved. □

Recitation 3

[Definitions used today]

- (weakly/strongly) convex, continuous, monotone preferences, locally non-satiated utility function
- utility maximization, Debreu theorem, lexicographic preferences
- WARP, GWARP, GARP, Topkis theorem, Afriat theorem

Question 1 [Weak vs strong continuity] 182 [Question I.1 Fall 2014 majors]

Let $\succeq$ be a transitive and complete preference relation on (connected) set $X \subseteq \mathbb{R}_+^N$:
Prove that the following statements are equivalent

- $\succeq$ on X is **weakly continuous** if $\forall x \in X$ the preferred-to-x set $U(x) = \{y \in X : y \succeq x\}$ and lower countur set $L(x) = \{y \in X : x \succeq y\}$ are closed.
- $\succeq$ on X is **strongly continuous** if for all sequences $\{x_n\}\{y_n\} \in X$ such that $x_n \to x$, $y_n \to y$, if $\forall n$, $x_n \succeq y_n$, then $x \succeq y$.

Question 2 [Properties of preferences]

Prove following statements

1. If a preorder $\succeq$ is monotone in $\mathbb{R}^l$, then it is locally nonsatiated.
2. If a preorder $\succeq$ is transitive, weakly monotone, and locally nonsatiated then it is monotone
3. A preorder $\succeq$ is weakly convex $\iff$ the upper contour sets $U(x) = \{y \in X : y \succeq x\}$ are convex for all $x \in X$
4. If a preorder $\succeq$ is continuous and strictly convex then it is convex

Question 3

Consider the following preference relations on $\mathbb{R}_+^2$

1. $x \succeq y \quad \iff \quad \min\{x_1, x_2\} \geq \min\{y_1, y_2\}$
2. $x \succeq y \quad \iff \quad \max\{x_1, x_2\} \geq \max\{y_1, y_2\}$

are they convex? Are they strictly convex?

Question 4

Give an example of preferences/utility function such that :

1. satisfy non-satiation, but not weak monotonicity
2. satisfy non-satiation, but not local non-satiation
3. satisfy local non-satiation, strict monotonicity, but not quasi-concave
4. does not satisfy continuous but it is representable by a utility function

Question 5 [Utility representation] 157 [I.1 Fall 2013 majors]

Consider preference relation $\succeq$ on the consumption set $\mathbb{R}_+^L$. Suppose that $\succeq$ is reflexive and complete.

1. State a definition of $\succeq$ having a utility representation. Is utility representation, if it exists, unique?
2. State a theorem providing sufficient conditions on $\succeq$ to have a utility representation. Be as general as you can and clearly define any extra properties of $\succeq$ that you use
3. [**Debreu Theorem**] Let $\succeq$ be a complete, transitive and continuous, strictly increasing (i.e. strongly monotone) preference relation on $\mathbb{R}_+^L$, show that it has a continuous utility representation

$\boxed{\Leftarrow}$ Take $y \in X$ $\quad y_n = y$ $\quad x_n \geq y_n$

Take $x_n \to x$ $\quad x_n \in U(y) =$

$= \{x \in X : x \succsim_p y\}$

Obviously $\quad x_n \succsim_p y_n = y \Rightarrow \lim$

$\Rightarrow \underbrace{y \in L(x)}_{?} = \{y \in X : x \geq y\}$

$x_n \to x \quad y_n \to y$

$x \geq y$ so $\underline{L(x) \text{ is closed}}$

$y \in X \quad x_n \in L(y) = \{x \in X \; y \geq x\}$

$x_n \to x \quad y_n = y$ so we get

$y_n \geq x_n \Rightarrow y_n \in U(x)$

$\Rightarrow y_n \to y \quad y \in U(x) \to$ closed

$\boxed{\Rightarrow}$

$\Rightarrow$ Take $x_n \to x$ $y_n \to y$

$x_n \succsim y_n$ WTS: $x \succsim y$

Suppose not so $x \prec y$

By Assumption $U(y)$, $L(x)$ are closed. So $U^c(y) = \{x \in X \ y \succ x\}$

$L^c(x) = \{y \in X \ y \succ x\}$ are open

Since $y \succ x \Rightarrow y \in L^c(x)$, $x \in U^c(y)$

If $x \in U$ is open $\exists \varepsilon$ $B(x, \varepsilon) \subseteq U$

$\exists \varepsilon_1$ $\forall x' \in B(x, \varepsilon_1)$ $x' \in U^c(y)$

$B(x, \varepsilon_1) \subseteq U^c(y)$

$\exists \varepsilon_2$ $\forall y' \in B(y, \varepsilon_2)$ $y' \in L^c(x)$

$B(y, \varepsilon_2) \subset L^c(x)$

Since $x_n \to x$ $\exists N_{\varepsilon_1}$ $\forall n \geq N_{\varepsilon_1}$

$x_n \in B(x, \varepsilon_1) \subset U^c(y)$

Since $y_n \to y$ $\exists N_{\varepsilon_2}$ $\forall n \geq N_{\varepsilon_2}$

$y_n \in B(y, \varepsilon_2) \subset L^c(x)$

Take $N = \max\{N_{\varepsilon_1}, N_{\varepsilon_2}\}$

$x_n \in U^c(y)$ $\quad y \succ x_n \Rightarrow y \in L^c(x_n)$

$y_n \in L^c(x)$ $\quad y_n \succ x \Rightarrow x \in U^c(y_n)$

By similar reasoning we will get:

$\exists M$ $\quad \forall m \geq M$

$y_m \in L^c(x_n) \Rightarrow y_m \succ x_n$

$x_m \in U^c(y_n) \Rightarrow y_n \succ x_m$

$$y_m \succ x_n \succsim y_n \succ x_m$$

From T $\quad y_m \succ x_m$

$x_m \succsim y_m$ $\quad ↯$

Q. 2_1.

WM $\quad x, y \quad \forall i: x_i \geq y_i \Rightarrow x \succeq_p y$

M $\quad x, y \; \forall i \; x_i > y_i \Rightarrow x \succ_p y$

SM $\quad x \neq y \; \forall \; x_i \geq y_i \Rightarrow x \succ_p y$

LNS $\forall x \in X \; \forall \varepsilon > 0 \; \exists y$

$\|x - y\| < \varepsilon \quad y \succ_p x$

Monotone $\Rightarrow$ LNS

Take $x \in X$ fix $\varepsilon > 0$ $\quad y$ as follows

$e = (\underbrace{1, 1, 1, \ldots 1}_{\ell})' \qquad X \subseteq \mathbb{R}^\ell$

$$y = \begin{pmatrix} x_1 + \frac{\varepsilon}{2\sqrt{\ell}} \\ x_2 + \frac{\varepsilon}{2\sqrt{\ell}} \\ \vdots \\ x_\ell \quad \frac{\varepsilon}{2\sqrt{\ell}} \end{pmatrix} = x + \frac{\varepsilon}{2\sqrt{\ell}} \cdot e$$

$$\|y - x\| = \left\| \frac{\varepsilon}{2\sqrt{\ell}} e \right\|$$

$$= \left(\sum_{i=1}^{\ell} \frac{\varepsilon^2}{4\ell} \right)^{1/2} = \frac{\varepsilon}{2} < \varepsilon$$

$y >> x$ $\quad y = x + \frac{\varepsilon}{2\sqrt{i}} \cdot e >> x$

since $\succsim_i$ is MON $y >> x$

$y > x$.

$\|y - x\| < \varepsilon$ & $y > x$ LNS

$\boxtimes$

Q5.

Thm. (Debreu) $X \subseteq \mathbb{R}^L_+$ C, T, (+)

& (str. monotone) $\Longleftrightarrow$

$\exists$ utility representing $\succsim$,

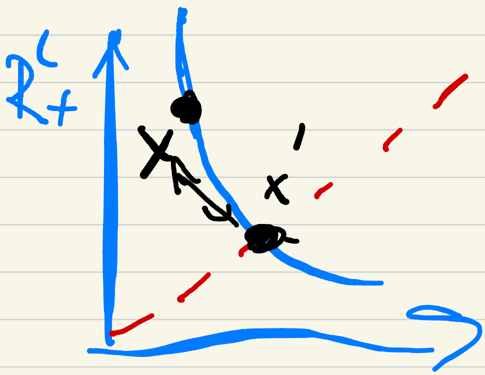
$u : X \to \mathbb{R}$ continuous

Proof: Rader 64,

Debreu 54

Hildebrand 1981

Q: How do we find u ?



$\underline{\underline{e}} = \underbrace{(1, 1, \ldots 1)}_{L}$

$u(x) = u(x') = \alpha(x)$

$\exists\, \alpha(x) \cdot e \sim x$

Proof..

1. Existence $\alpha(x)$

2. Uniqueness

3. $u(x) = \alpha(x)$ indeed represents $\succsim$

4. Continuity of u

1. $\succsim$ c, m, cty , $e = (\underbrace{1, \dots, 1}_{L})$

$\bar{\alpha}$ $\quad \bar{\alpha} \cdot e \geq x \quad \bar{\alpha} \geq x_i \ \forall i$

Since $\succsim$ are m. Take $0 = \alpha$

$0 \cdot e \leq x$

Define A^+

$A^+ = \{ \alpha \in \mathbb{R} \quad \alpha \cdot e \succsim_p x \}$

$A^- = \{ \beta \in \mathbb{R} \quad x \succsim_p \beta \cdot e \}$

From C $\quad A^+ \cup A^- = \mathbb{R}$

From Cty $\quad A^+, A^-$ are closed

$0 \in A^-$, $\bar{\alpha} \in A^+$

There is $\alpha(x) \in A^+ \cap A^-$

We are $\alpha(x) \cdot e \sim x$

Step 2.

Suppose that we have $\alpha' \in A^+ \cap A^-$

$\alpha' \neq \alpha(x)$ WLOG $\underline{\alpha' > \alpha(x)}$

Since $\alpha', \alpha(x) \in \mathbb{R}$

Recall that from Mon

$\alpha' \cdot e \succ \alpha(x) \cdot e \sim x$

$\alpha' \cdot e \succ_p x$ From T ($\succ_p$)

Contradiction with $\alpha' \in A^+ \cap A^-$

So we have only one $\alpha(x)$!

Def. $u : X \to \mathbb{R}$ represents $\succsim$ (preorder) $\iff$ $\forall x, y \in X$

$$u(x) \geq u(y) \iff x \succsim_p y$$

$$\Rightarrow$$

$$u(x) = \alpha(x), \; u(y) = \alpha(y)$$

$\Rightarrow u(x) = \alpha(x) \geq \alpha(y) = u(y)$

Then by MON $\alpha(x)\cdot e \succsim_p \alpha(y)\cdot e$

But we know that

$$\underbrace{x \sim \underbrace{\alpha(x)\cdot e}} \succsim_p \underbrace{\underbrace{\alpha(y)\cdot e} \sim y}$$

From T $x \succsim_p y$

$\Leftarrow$ x, y $y \succsim_p x$. We constructed $\alpha(y), \alpha(x)$

$\alpha(y)\cdot e \sim y \succsim_p x \sim \alpha(x)\cdot e$

From (T) $\alpha(y)\cdot e \succsim_p \alpha(x)\cdot e$

$$\underset{\mathbb{R}}{\underset{\Uparrow}{\alpha(y)}} \geq \underset{\mathbb{R}}{\underset{\Uparrow}{\alpha(x)}}$$ by MON

$u(y) \geq u(x)$ when $y \succsim_p x$

Def.

$f: X \to Y$ cont. in topological sense

$\mathcal{F}_Y$ closed

$\forall u \in \mathcal{J}_Y \to$ open set in Y

$f^{-1}(u) \in \mathcal{J}_X$ - open set in X

$\mathcal{F}_X$ closed

$f: X \to \mathbb{R}$

Take open set $\mathbb{R}$ so $\underbrace{[0, \alpha(x)]}_{\text{closed}}$

$u^{-1}([0, \alpha(x)])$:

$= \{ y \in \mathbb{R}^L_+ \ \underbrace{\alpha(x) \cdot e}\ \succsim\ \underbrace{y} \}$

$= \{ y \in \mathbb{R}^L_+ \ \alpha(x) \geq u(y) \}$

$u(x) = \alpha(x)$

$u(y)$

$u(\alpha(x) \cdot e) = \alpha(x) = u(x)$

$u^{-1}([0, \alpha(x)])$ is a closed set in $\mathbb{R}^L_+$ by Q 1.

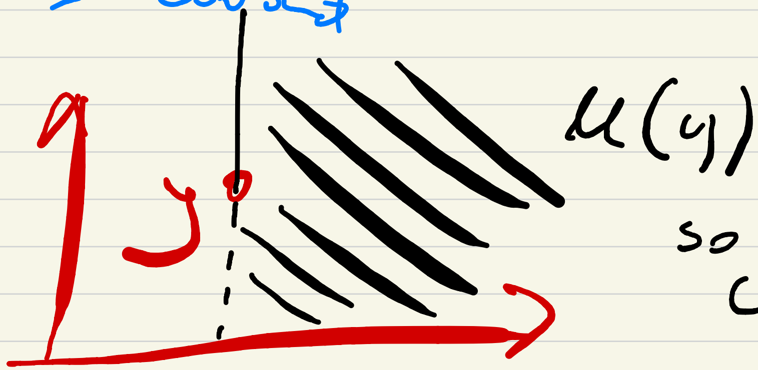
$u^{-1}([\alpha(x), +\infty)] = \{ y \in \mathbb{R}^L :$
$u(y) \geq \alpha(x) \} = \{ y \in \mathbb{R}^L \ y \succsim u(x) e \}$

this is closed from cty by Q1.

$$u^{-1}([\alpha(x),\alpha(y)]) = \underbrace{u^{-1}([\alpha(x),+\infty))}_{\text{closed}} \cap \underbrace{u^{-1}([0,\alpha(y)])}_{\text{closed}}$$

$\cap$

closed



so not cty

Idiotic preferences $x \in [0,1]$

$$x \succsim_P y \iff x \geq y$$

T, C, not C^0. But we can represent them by function

$$\begin{cases} u(0) = 0 \\ u(x) = 1 \end{cases}$$

Q7. Def. utility $u: \mathbb{R}^L_+ \to \mathbb{R}$ rationalizes observations $D = \{(p_1, x_1), \dots, (p_T, x_T)\}$

$p_t \in \mathbb{R}^L_+ \quad x_t \in \mathbb{R}^L_+$

$\forall t \in \{1, \dots T\} \ \forall x$

$$p^t x \leq p^t x^t \Rightarrow u(x) \leq u(x^t)$$

Def. (GARP) - General (Strong) Axiom of Revealed Pref.

If D satisfy GARP:

$\forall \{t_1, \dots t_n\} \subseteq \{1 \dots T\}$

$$\begin{cases} p_{t_1} x_{t_2} \leq p_{t_1} \cdot x_{t_1} \\ p_{t_2} x_{t_3} \leq p_{t_2} \cdot x_{t_2} \\ p_{t_3} x_{t_4} \leq p_{t_3} \cdot x_{t_3} \\ \\ p_{t_{n-1}} \cdot x_{t_n} \leq p_{t_{n-1}} \cdot x_{t-1} \end{cases} \Rightarrow p_{t_n} \cdot x_{t_n} \leq p_{t_n} \cdot x_{t_1}$$

GARP $\{t_1, t_n\} = \{t, s\}$ GWARP

Def. GWARP $\{(p_t, x_t), (p_s, x_s)\}$

$$p_t x_s \leq p_t x_t \Rightarrow p_s x_s \leq p_s x_t$$

2. GARP doesn't hold (by contradiction)

$\exists \{t_1, \dots, t_n\} \subseteq \{1 \dots T\}$

$$\begin{cases} p_{t_1} x_{t_2} \leq p_{t_1} \cdot x_{t_1} \\ p_{t_2} x_{t_3} \leq p_{t_2} \cdot x_{t_2} \\ p_{t_3} x_{t_4} \leq p_{t_3} \cdot x_{t_3} \\ \vdots \\ p_{t_{n-1}} \cdot x_{t_n} \leq p_{t_{n-1}} \cdot x_{t-1} \end{cases} \Rightarrow p_{t_n} \cdot x_{t_n} > p_{t_n} \cdot x_{t_1}$$

Since u rationalizes D

$$\begin{cases} u(x_{t_2}) \leq u(x_{t_1}) \\ u(x_{t_3}) \leq u(x_{t_2}) \\ \vdots \\ u(x_{t_n}) \leq u(x_{t_{n-1}}) \end{cases}$$

$$\Rightarrow u(x_{t_1}) \geq u(x_{t_n})$$

Lemma. $\boxed{u \text{ is LNS}}$ $p_{t_n} \cdot x_{t_n} > p_{t_n} \cdot x_{t_1} \Rightarrow$

$\Rightarrow u(x_{t_n}) > u(x_{t_1})$

Proof: Suppose not $u(x_{t_n}) \leq u(x_{t_1})$

$\langle p, x \rangle = p \cdot x$ is continous

$\exists \varepsilon > 0 \quad \forall \bar{x} \in B(x_{t_1}, \varepsilon)$

$p_{t_n} x_{t_n} \geq p_{t_n} \cdot \bar{x}$

By LNS: $\forall \varepsilon \ \exists \bar{\bar{x}} \in B(x_{t_1}, \varepsilon)$

$p_{t_n} x_{t_n} > p_{t_n} \cdot \bar{\bar{x}}$ & by LNS for $\bar{\bar{x}}, x_{t_1}$

$u(\bar{\bar{x}}) > u(x_{t_1})$

But

$p_{t_n} x_{t_n} > p_{t_n} \cdot x_{t_1}$

$p_{t_n} \cdot x_{t_n} > p_{t_n} \bar{\bar{x}}$

$u(\underbrace{\bar{\bar{x}}}) > u(x_{t_1}) \geq u(t_n)$

By $\bar{\bar{x}}$ feasible

t_n is optimal

$u(x_{t_n}) \geq u(\bar{\bar{x}})$ → from Ral.

iii) w, p, x

$p=(1,1) \quad w=9 \quad x=(9,0)$

$p'=(1,4) \quad w'=11 \quad x'=(2.2,\ 2.2)$

$p\cdot x = 1\cdot 9 + 1\cdot 0 = 9$

$p'\cdot x' = 1\cdot 2.2 + 4\cdot 2.2 = 11$

$p\cdot x' = 1\cdot 2.2 + 1\cdot 2.2 = 4.4$

$p'\cdot x = 1\cdot 9 + 4\cdot 0 = 9$

so $p\cdot x = 9 > 4.4 = p\cdot x'$

$p'\cdot x' = 11 > 9 = p'\cdot x$

Therefore GWARP doesn't hold

iv) Idiotic $u(x)=1$

p, x $\quad p=(5,1) \quad x=(5,1)$

$p'=(1,5) \quad x'=(1,5)$

$p\cdot x = 5\cdot 5 + 1\cdot 1 = 26 \quad p\cdot x' = 5\cdot 1 + 1\cdot 5 = 10$

$p'\cdot x' = 1\cdot 1 + 5\cdot 5 = 26 \quad p'\cdot x = 10 \quad u(x)=1 \geq 1 = u(x')$

$p\cdot x = 26 > 10 = p\cdot x'$ & $u(x)=1 \geq 1 = u(x')$

$p'\cdot x' = 26 > 10 = p'\cdot x$ & $u(x')=1 \geq 1 = u(x)$

Afriat Thm

D - observations satisfy GARP

$\iff$ there exists a LNS utility function rationalizing D.

Proof: N part

So we got u, LNS which R

$\Leftarrow$ GARP holds.

Suppose GARP is violated

$\exists \{t_1, \dots, t_n\} \subseteq \{1 \dots T\}$

$$\begin{cases} p_{t_1} x_{t_2} \leq p_{t_1} \cdot x_{t_1} \\ p_{t_2} x_{t_3} \leq p_{t_2} \cdot x_{t_2} \\ p_{t_3} x_{t_4} \leq p_{t_3} \cdot x_{t_3} \\ \\ p_{t_{n-1}} \cdot x_{t_n} \leq p_{t_{n-1}} \cdot x_{t-1} \end{cases} \Rightarrow p_{t_n} \cdot x_{t_n} > p_{t_n} \cdot x_{t_1}$$

$u(x_{t_1}) \geq u(x_{t_2}) \geq \dots \geq u(x_{t_{n-1}}) \geq u(x_{t_n})$

By LNS $u(x_{t_n}) > u(x_{t_1})$ | $u(x_{t_1}) \geq u(x_{t_1})$

$\mu($

Question 6 [Lexicographic preference]

Consider the following lexicographic preferences on the consumption set $\mathbb{R}^2_+$: the value $x_1 + x_2$ has the first priority, the value of x_2 has the second priority.

1. Is this preference relation continuous? Prove of give a counter example.
2. Does this preference relation have the utility representation? Prove of give a counter example.
3. Consider the lexicographic preferences on $\mathbb{R}^N_{++}$ such that the first priority is described by an increasing and continuous utility function $u_1(x)$ and the second priority is described by another increasing and continuous utility function $u_2(x)$. Show that, if u_1 is strictly concave, then the Walrasian demand of the lexicographic preference coincides with the Walrasian demand of u_1 for every $p \in \mathbb{R}^N_+$, $p \neq 0$ and $w > 0$.

Question 7 [Midterm 2018]

Consider a list of observations $\{(p_1, x_1), \ldots, (p_T, x_T)\}$ where $p_t \in \mathbb{R}^N_+$ and $x_t \in \mathbb{R}^N_+$ are price vector and a corresponding consumption plan of a consumer respectively, for every $t \in \{1, \ldots, T\}$.

1. State the Generalized Weak Axiom of Revealed Preference (GWARP) and Generalized (strong) Axiom of Revealed Preference (GARP) for these observations.
2. Show that if a locally non-satiated utility function rationalized observations then GARP holds.
3. Suppose that the observations are generated by a demand function $d(p, w)$ that is $x_t = d(p_t, w_t)$ for every t. Function d is given as

$$d(p,w) = \begin{cases} (\frac{w}{p_1}, 0) & \text{if } p_1 \geq p_2 \\ (\frac{w}{p_1+p_2}, \frac{w}{p_1+p_2}) & \text{if } p_2 > p_1 \end{cases}$$

   Does GWARP hold for arbitrary observations generated by d? Can demand d be rationalized by a locally non-satiated utility function?
4. Show that if a locally non-satiated utility function rationalized observations then GWARP holds.
5. Show that the assumption of local non-satiation in the previous point cannot be dispensed with - i.e. give an example of a utility function that rationalizes a set of pairs of prices and consumption bundles that violates GWARP

Question 8 [Properties of Walrasian Demand]

Prove following claims

1. **[Walras Law]** Show that if a preference relation $\succeq$ is continuous and locally non-satiated then $p \cdot x^*(p, w) = w$, for all $x^*(p, w)$ that belong to the Walrasian Demand correspondence.
2. **[GWARP]** Show that if a preference relation $\succeq$ is continuous and locally non-satiated then for all $w > 0$

$$w' > 0, p >> 0 \text{ and } p' >> 0: \quad p \cdot x^*(p', w') \leq w \Rightarrow p' \cdot x^*(p, w) \geq w'$$

Question 9 230 [I.1 Fall 2016 minors]

Let d : be a demand function of prices and income satisfying budget equation $pd(p, w) = w$ for every p and w

1. Show that if d is a Walrasian demand function of a consumer with strictly increasing utility function, then the Generalized Weak Axiom of Revealed Preference (GWARP) holds for every T -tuple of price-quantity pairs $\{p^t, x^t\}_{t=1}^T$, where $x^t = d(p^t, w^t)$ $p^t \in \mathbb{R}^L_{++}$ and $w^t \in \mathcal{R}_+$ for every $t = 1, \ldots, T$. State GWARP
2. onsider the following demand function for $L = 2$ and show that GWARP does not hold for $\hat{d}$.:

$$\hat{d}(p,w) = \begin{cases} \left(\frac{w}{p_1}, 0\right) & \text{if } p_1 \geq p_2 \\ \left(0, \frac{w}{p_2}\right) & \text{if } p_2 > p_1 \end{cases}$$

3. State the Afriat's Theorem. The proof is not required
4. Prove the necessity of an axiom for rationalizability

Recitation 2

[Definitions used today]

- (conditional) factor demand, cost function, Shephard's lemma, Hotelling's lemma
- Δ-monotone, homogeneous, positive definite matrix, correspondence, upper hemicontinuity (UHC)

Question 1 [Properties of C and x]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a production function that is strictly increasing and satisfies $f(0) = 0$. Let $C^*(w, z)$ be the (minimum) cost function, where $w \in \mathbb{R}^n$ is a vector of input prices and $z > 0$ is an output level. Let $x^*(w, z)$ be the optimizer of cost minimization problem. Prove following properties:

1. C^* is homogeneous of degree 1 in factor prices p
2. $C^*$is a concave function of p
3. $x^*(w, z)$ is homogeneous of degree zero in w.
4. x is Δ-monotone for fixed z, in following way:
$$[x^*(w, z) - x^*(w', z)][w - w'] \leq 0 \quad \forall w, w' \gg 0$$
5. **Shephard's Lemma** If C^* is differentiable at p (this holds $\iff$ x^* is single-valued) then
$$D_w C^*(w, z) = x^*(w, z)$$
6. Assuming that C^*, x^* are differentiable at $w \in \mathbb{R}^n$ prove comparative statics property of factor demand:
$$\frac{\partial x_i}{\partial w_i}(w, z) \leq 0$$
7. Show that cost function C is a non-decreasing function of output level z, for every $w \gg 0$.
8. If production function f is concave, then cost function C is a convex function of output level z, for every $w \gg 0$

Question 2 [Zero profit CRS]

If Y exhibits CRTS, then $\pi^*(p) = 0$ whenever it is well-defined.

Question 3 [Properties of π^* and s^*] 33 [I.1 Fall 2006 majors]

Suppose that production set Y is closed. Let $s^*(p)$ denote supply at price level p and by $\pi^*(p)$ corresponding profit level. Then the following properties hold:

1. π^* is homogeneous of deg. 1 in prices p
2. π^* is a convex function in prices p
3. **correspondence** s^* is homogeneous of deg. 0
4. s^* is Δ-monotone, that is:
$$[s^*(p) - s^*(p')][p - p'] \geq 0 \quad \forall p, p'$$
5. **Hotelling's Lemma**: If π^* is differentiable at p (this holds iff s is single-valued at p), then
$$D\pi^*(p) = s^*(p)$$
6. Assuming that π^*,s^* are differentiable at $p \in \mathbb{R}^n$ prove comparative statics **law of supply**:
$$\frac{\partial s_i}{\partial p_i}(p) \geq 0$$

Ex. 2

1) Show that $0 \in Y$

2) Show that $\pi^*(p) \geq 0$

3) Show that if $|\pi(p)| < +\infty$
$$\Rightarrow \pi^*(p) \leq 0$$

1) Let $y \in Y$ by Y CRS $\forall \lambda \geq 0$ $\lambda y \in Y$. So pick $\lambda = 0$, $0 \cdot y = 0 \in Y$

2) Since $\pi^*(p) = \sup_{y \in Y} p \cdot y \geq p \cdot y \ \forall y \in Y$

From step 1 $0 \in Y$ $\pi^*(p) \geq p \cdot 0 = 0$

3) (By contradiction) Suppose not. $\exists p$ $|\pi(p)| < +\infty$ and $\pi(p) > 0$.

Then $\exists y \in Y$ s.t. $0 < \underbrace{\frac{\pi(p)}{2} < py \leq \pi(p)}_{(*)}$

I used in (*) definition of Supremum, i.e.:
$\alpha = \sup A$ then $\forall \varepsilon > 0$ $\exists \beta$ $\alpha - \varepsilon < \beta \leq \alpha$

Take (*), multiply by 3

$$\pi(p) < \frac{3}{2}\pi(p) < 3py \leq 3\pi(p)$$

Now observe that $y \in Y$ $\lambda = 3$ $3 \cdot y \in Y$
$\pi(p) = \sup_{y \in Y} p \cdot y < p \cdot (3y)$. This is a **CONTRADICTION** ↯

By step 2 & 3 we conclude that $\pi(p) \leq 0 \leq \pi(p) \Rightarrow \pi^*(p) = 0$

Ex. 3

Def. The problem of cost minimization for a producer with production function $f: \mathbb{R}^n_+ \to \mathbb{R}_+$:

$$c^*(w,z) = \inf \; w \cdot x \quad \text{s.t.} \quad f(x) \geq z, \; x \geq 0$$

- $w = (w_1, \ldots, w_n) \gg 0$ Input prices
- $x = (x_1, \ldots, x_n) \geq 0$ inputs
- $z \geq 0$ output
- V_f Assume : closed, (convex), Free disposal

Def. (Conditional) factor demand

$$x^*(w,z) = \arg\min \{ w \cdot x \;\text{ s.t. }\; f(x) \geq z, \; x \geq 0 \}$$

Def. $g: \mathbb{R}^n \to \mathbb{R}$ it is Homo(k)

$$\forall \lambda \geq 0 \; \forall x \in \mathbb{R}^n \quad g(\lambda x) = \lambda^k g(x)$$

Q3.3. $x(\lambda w, z) = \{ x^* \in \mathbb{R}^n :$
$(\lambda w) \cdot x^* \leq (\lambda w) \cdot x \ \& \ f(x) \geq z \} =$
$= \{ w \cdot x^* \leq w \cdot x \ \& \ f(x) \geq z \} =$
$= x(w, z)$

Q3.1. $\lambda \geq 0$

$$c^*(\lambda w, z) = (\lambda w) \cdot x^*(\lambda w, z) \overset{\text{by Q3.3}}{=}$$
$$= \lambda \underbrace{(w \cdot x^*(w, z))}_{} = \lambda \cdot \underbrace{c^*(w, z)}_{}$$

$=$

Q3.8. Take $z_1 \neq z_2$

$\forall x_1 \geq 0 \quad f(x_1) \geq z_1 \quad (\lambda)$

$\forall x_2 \geq 0 \quad f(x_2) \geq z_2 \quad (1-\lambda)$

Take $\lambda \in [0, 1]$

$$f(\lambda x_1 + (1-\lambda) x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$$
$$\geq \lambda z_1 + (1-\lambda) z_2$$

$$f(\lambda x_1+(1-\lambda)x_2) \geq \lambda f(x_1)+(1-\lambda)f(x_2)$$
$$\geq \lambda z_1+(1-\lambda)z_2$$

$$c^*(w, \lambda z_1+(1-\lambda)z_2) \leq w\cdot(\lambda x_1+(1-\lambda)x_2)$$
$$= \lambda\,(w x_1) + (1-\lambda)\cdot(w x_2) \quad (*)$$

x_1, x_2 we arbitrary $\qquad f(x_i) \geq z_i$

~~x_1~~ Take $\quad x_1 = x^*(w, z_1)$
$\qquad x_2 = x^*(w, z_2)$

Plug it into $(*)$ and get

$$c^*(w, \lambda z_1+(1-\lambda)z_2) \leq$$
$$\lambda\, c^*(w, z_1) + (1-\lambda)\, c^*(w, z_2)$$

Q. 3. 4. $\forall\, w, w' \gg 0 \quad$ fixed z

$$[x(w,z) - x(w',z)]\cdot[w-w'] =$$
$$= [w\cdot x(w,z) - w x(w', z)] +$$
$$+ [w' x(w',z) - w' x(w,z)] =$$
$$\leq 0 + 0 \leq 0$$

we know that $\quad w\cdot x(w,z) \leq w\cdot x$

Take $x = x(w',z)$ to obtain 1st

Q 3.7. Take w_1, w_2 $w_1 \geq w_2$, fix z.
Take x : $f(x) \geq z$. Obviously

$$w_1 x \geq w_2 x \quad (*)$$

Now

$$w_1 \cdot x \overset{(1)}{\geq} w_1 \cdot x(w_1, z) \overset{(2)}{\geq} w_2\, x(w_1, z) \overset{(3)}{\geq} w_2 \cdot x(w_2, z)$$

Observe that $f(x) \geq z$ does not depend on w.

(1) comes from optimality of $x(w_1, z)$ at w_1

(2) It is $(*)$ for $x = x(w_1, z)$

(3) Comes from optimality of $x(w_2, z)$ at w_2

Q 3.5 Shephard's lemma

$$f(w, y) = w \cdot y$$

$$c^*(w, z) = w \cdot x^*(w, z)$$

$$D_w c^*(w, z) = D_w f(w, x^*(w, z)) =$$

$$\underline{\underline{D_w}} f(w, y) \Big|_{y = x^*(w, z)}$$

$$+ \underline{\underline{D_z}} f(w, y(w, z)) \cdot D_w x(w, z)$$

$$D_w f(w, y) = y$$

$$D_w c^*(w, z) = y^*(w, z) = x^*(w, z)$$

$$D_w c^*(w, z) = x^*(w, z)$$

$$D_w^2 c = D_w x(w, z)$$

$$0 \geq \frac{\partial^2 \pi^*(w,z)}{\partial w_i^2} = \frac{\partial x^*(w,z)}{\partial w_i}$$

Q 3.6.

Q.5 Look at S-monotone property

$\frac{\partial s}{\partial p_i} \geq 0 \quad \forall i$

Q.5

$\Rightarrow$ S-monotone

Pick $p = (1,1)$ $p' = (x,y)$

$p' = \left(\frac{1}{3}, \frac{1}{6}\right)$

$s(p) = \left(-\frac{2p_2}{p_1}, \frac{p_2}{p_1}\right)$

$s(p) = (-2,1)$

$s(p') = \left(-1, \frac{1}{2}\right)$

$[s(p) - s(p')] \cdot [p - p'] = \left(-1, \frac{1}{2}\right) \cdot \left(\frac{2}{3}, \frac{5}{6}\right)$

$= -\frac{2}{3} + \frac{5}{12} = -\frac{8}{12} + \frac{5}{12} = -\frac{3}{12} < 0$

So ~~so~~ this function can not come from π-maximization

Q.4. b

$Y_1 \subseteq Y_2 \iff \pi_1(p) \leq \pi_2(p)$

$\Rightarrow \quad Y_2 = (Y_2 \setminus Y_1) \cup Y_1 \,, \quad Y_1 \cap (Y_2 \setminus Y_1) = \emptyset$

$$\max_{x \in Y_2} p \cdot x = \max_{x \in (Y_2 \setminus Y_1) \cup Y_1} p \cdot x$$

$$\overset{(*)}{=} \max \left\{ \max_{x \in (Y_2 \setminus Y_1)} p \cdot x \,, \max_{x \in Y_1} p \cdot x \right\}$$

$$\overset{(**)}{\geq} \max_{x \in Y_1} p \cdot x$$

$(*)$ $$\max_{\substack{x \in A \cup B \\ A \cap B = \emptyset}} f(x) = \max \left\{ \max_{x \in A} f(x) \,, \max_{x \in B} f(x) \right\}$$

$(**)$ $\max\{a, b\} \geq b$

$\Leftarrow$ (By contradiction) Suppose not $\cancel{Y_1 \subseteq Y_2}$ There will $x \in Y_1$, $x \notin Y_2$

$\{x\}$, Y_2 they are both closed, convex, nonempty

$\{x\}$ is bounded $\Rightarrow$ $\{x\}$ is compact

Apply Seperating Hyperplane Thm

(J assume X, Y nonempty, closed, convex, X ~~both~~ bounded) then strict seperation holds:

($\exists$ p-vector, b-scalar)

$$p \cdot x > b > p \cdot y \quad \forall x \in X, \; y \in Y$$

| In our case that

$\exists$ p-vector b-scalar

$$\underline{\underline{\pi_1(p) \geq p \cdot x}} > b > p \cdot y \quad y \in Y_2$$

Take sup

$$\pi_1(p) > \pi_2(p)$$ This is $\lightning$