

Recitations 1

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MINI

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RECITATION 1

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Recitation 1

[Definitions used today]

- (strictly) convex, concave, quasi convex, quasi concave functions
- production set Y , input requirement set V , transformation function T , production function f
- DRS, IRS, CRS of production function
- NIRS, NDRS, CRS of production set
- Meet and Joint, Lattice, Supermodularity of a function, Increasing Differences function

Question 1 [Production function/set]

- (a) Show that if $f(x)$ is concave \Rightarrow production set Y is convex.
- (b) Prove that for a convex production set $Y \Rightarrow$ input requirement set V is convex. Prove that converse is not true.
- (c) Show that $f(x)$ is quasi concave function \iff input requirement set V is convex.
- (d) Show that if $f(x)$ is strictly concave and $f(0) = 0 \Rightarrow f$ exhibits DRS

Question 2 [Properties of Y, f]

Let $f(x)$ be a production function and Y a production set associated with f . Show the following propositions hold

- (a) if f exhibits DRS then Y exhibits NIRS
- (b) if f exhibits IRS then Y exhibits NDRS
- (c) if f exhibits CRS then Y exhibits CRS

Question 3 [Supermodularity] 89 [I.1 Fall 2009 majors]

Show that following functions are **supermodular**

- (a) the Cobb-Douglas production function $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, where $\forall_i \alpha_i > 0$, and $\sum_i \alpha_i < 1$
- (b) the Leontief function $f(x) = \min_i \{\alpha_i x_i\} \quad \forall_i \alpha_i > 0$

Question 4 [Properties of Y]

Prove following properties

- (a) Assume that for Y closed and convex, $Y \subset \mathbb{R}^L$ s.t. $0 \in Y$. Free disposal property $Y - \mathbb{R}_+^L \subset T \iff \mathbb{R}_-^L \subset Y$
- (b) If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient
- (c) If Y is a convex set, then supply correspondence $s^*(p)$ is a convex set.

Question 5 165 [I.1 Fall 2013 minors]

Consider a production function $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ with n inputs and one output. Assume that $f(0) = 0$.

- (a) State a definition of f having (strictly) IRS.
- (b) Prove that if f exhibits IRS, then, for any strictly positive input prices w_i (where $i = 1, \dots, n$) and strictly positive output price p , either the firm's output at the profit-maximizing production plan is zero or otherwise the profit-maximizing production plan is not well defined (i.e. it does not exist).
- (c) Consider the following example of production function with two inputs:

$$f(x_1, x_2) = [\min\{x_1, x_2\}]^2$$

Does this f exhibit increasing returns to scale?

- (d) Does the cost-minimization problem for production function f of (c) have a solution for arbitrary prices $w_1 > 0, w_2 > 0$ and output level $y > 0$? Justify your answer

Definitions

Def. (Convex set) A set $S \subseteq \mathbb{R}^n$ is **convex** if

$$\forall x, y \in S \quad \forall \alpha \in [0, 1] \quad \alpha x + (1-\alpha)y \in S$$

Def. (Convex function) $f: X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is **(strictly) convex** if:

$$\forall x, y \in X \quad \forall \alpha \in [0, 1] \quad f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

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Def. (Concave function) $f: X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is **(strictly) concave** if:

$$\forall x, y \in X \quad \forall \alpha \in [0, 1] \quad f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)$$

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Def. (Quasi-convex function).

$f: X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is **quasi-convex** if

$$\forall x, y \in X \quad \forall \alpha \in (0, 1) \quad f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}$$

Def. (Quasi-concave function).

$f: X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is **quasi-concave** if

$$\forall x, y \in X \quad \forall \alpha \in (0, 1) \quad f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\}$$

Def. (Upper contour set) of $f: X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ at $\alpha \in \mathbb{R}$

$$M(\alpha) = \{x \in X : f(x) \geq \alpha\}$$

Def. (Production set) - denoted by Y , $Y \subseteq \mathbb{R}^L$ is:

a) NON EMPTY if $\exists y \in \mathbb{R}^L \quad y \in Y$

b) CLOSED $(\lim_{n \rightarrow +\infty} y_n = y \quad \forall n \in \mathbb{N}) \Rightarrow y \in Y$

c) NO FREE PRODUCTION $Y \cap \mathbb{R}_+^L = \{0\}$

d) POSSIBILITY OF INACTION $\{0\} \subseteq Y$

e) FREE DISPOSAL $Y - \mathbb{R}_+^L \subseteq Y$
 $(\forall t \in \mathbb{R}_+^L \quad \forall y \in Y \quad y - t \in Y)$

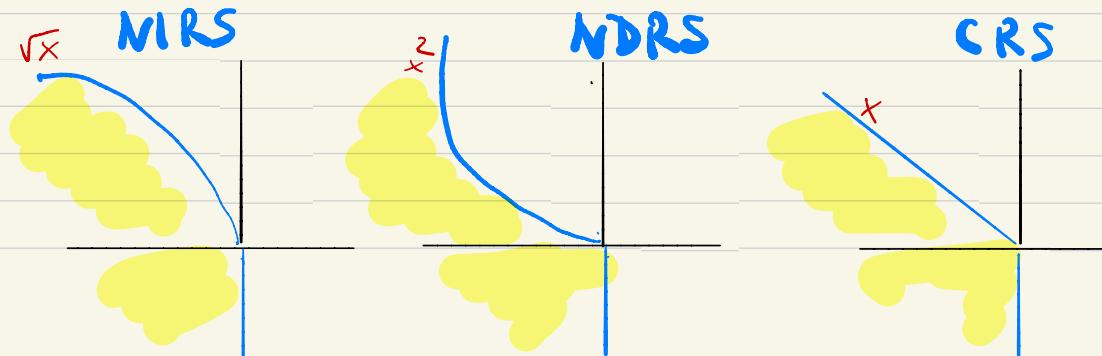
f) IRREVERSIBILITY $\forall y \in Y \quad \forall y \neq 0 \Rightarrow -y \notin Y$

Y exhibits

i) NIRS $\forall y \in Y \quad \forall \alpha \in [0,1] \Rightarrow \alpha y \in Y$

ii) NDRS $\forall y \in Y \quad \forall \alpha \geq 1 \Rightarrow \alpha y \in Y$

iii) CRS $\forall y \in Y \quad \forall \alpha > 0 \Rightarrow \alpha y \in Y$



Def. (Production function) $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ exhibits:

(i) IRS if:

$$\forall \lambda > 1 \quad \forall x \geq 0 \quad f(x) \neq 0 : \quad f(\lambda x) > \lambda f(x)$$

(ii) DRS if

$$\forall \lambda > 1 \quad \forall x \geq 0 \quad f(x) \neq 0 : \quad f(\lambda x) < \lambda f(x)$$

(iii) CRS if

$$\forall \lambda > 1 \quad \forall x \geq 0 \quad f(x) \neq 0 : \quad f(\lambda x) = \lambda f(x)$$

Def. (Meet and Joint). Given $x, y \in \mathbb{R}^n$

$$(\text{meet}) \quad x \wedge y := (\min\{x_1, y_1\}, \dots, \min\{x_n, y_n\})$$

$$(\text{join}) \quad x \vee y := (\max\{x_1, y_1\}, \dots, \max\{x_n, y_n\})$$

Def. $X \subseteq \mathbb{R}^n$ is a lattice if:

$$\forall x, y \in X : x \vee y \in X \text{ and } x \wedge y \in X$$

Def (Supermodularity) $f: Y \rightarrow \mathbb{R}$ is SPM if:

$$\forall y, y' \in Y \quad f(y_1y') + f(y_2y') \geq f(y_1) + f(y_2)$$

Def. (Increasing Differences) $f: X \times \mathbb{D} \rightarrow \mathbb{R}$
exhibits ID in (x, \mathbb{D}) if:

$$\forall x, x' \in X \quad x > x' \quad \forall \theta, \theta' \geq \theta' \quad f(x, \theta) - f(x', \theta) \geq f(x', \theta) - f(x', \theta')$$

Def. Production set induced by production function

$$Y_f = \{(x, z) \in \mathbb{R}^L : x \leq 0, f(x) \geq z \geq 0\}$$

Def. (Transformation function) $T: \mathbb{R}^L \rightarrow \mathbb{R}$ is transformation function if we can represent Y s.t.:

$$Y = \{y \in \mathbb{R}^L : T(y) \leq 0\}$$

Cov. Observe that $T(y) = z - f(x)$ for $y = (x, z)$

Def. (Input requirement set) is defined as follows for every output z :

$$V(z) = \{x \in \mathbb{R}^{L-1} : f(x) \geq z\}$$

Cov. In other "words":

$$V(z) = \{x \in \mathbb{R}_+^{L-1} : (x, z) \in Y\}$$

Exs

Question 1 [Production function/set]

- Show that if $f(x)$ is concave $\Rightarrow f$ production set Y is convex
- Prove that for a convex production set $Y \rightarrow$ input requirement sets V is convex. Prove that converse is not true.
- Show that $f(x)$ is quasi concave function \iff input requirement sets V is convex.
- Show that if $f(x)$ is strictly concave and $f(0) = 0 \Rightarrow f$ exhibits DRS

Take $f: \mathbb{R}_+^n \rightarrow \mathbb{R}$

(a) We want to show (\therefore WTS) :

$$\forall y \in (-x, z), y' \in (x', z') \in Y, \forall \alpha \in [0, 1] : \alpha y + (1-\alpha)y' \in Y$$

Pick $y, y' \in Y, \alpha \in [0, 1]$ since f is a production function:

$$0 \leq z \leq f(x) \quad 0 \leq z' \leq f(x')$$

$$\lambda \geq 0 \Rightarrow 0 \leq \alpha z \leq \alpha f(x) \quad 0 \leq (1-\alpha)z' \leq (1-\alpha)f(x')$$

$$\begin{aligned} \text{Sum it up: } 0 &\leq \alpha z + (1-\alpha)z' \leq \alpha f(x) + (1-\alpha)f(x') \\ &\leq f(\alpha x + (1-\alpha)x') \end{aligned}$$

Where last \leq comes from concavity of f



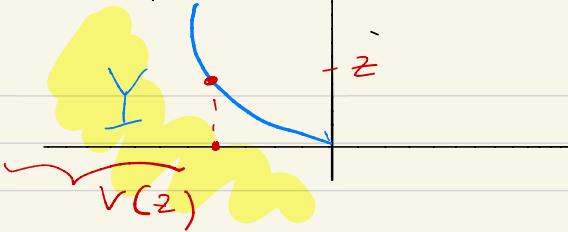
$$(b) \text{ WTS: } \forall -x', -x'' \in V(z) \quad \forall \alpha \in [0, 1] : \alpha x' + (1-\alpha)x'' \in V(z)$$

Pick $(-x', z), (-x'', z) \in Y$ then since Y is convex

$$\forall \alpha \in [0, 1] \quad (-(\alpha x' + (1-\alpha)x''), \alpha z + (1-\alpha)z) \in Y$$

$$\text{Hence } (-(\alpha x' + (1-\alpha)x''), z) \in Y \Leftrightarrow -[\alpha x' + (1-\alpha)x''] \in V(z)$$

\Leftarrow Counter example.



$$f(x) = x^2$$

$V(z)$ convex

V not convex

(c) Lemma. f is Quasi-concave \Leftrightarrow

upper contour set $M_f(z)$ is convex

$V(z) = \{x : f(x) \geq z\}$ is upper contour set

$V(z)$ is convex $\forall z \Leftrightarrow f$ is Q-concave

(d) WTS: $\forall \alpha > 1 \ \forall x \geq 0 \ f(x) \neq 0 \quad f(\alpha x) < \alpha f(x)$

Pick $x \geq 0, x' = 0, \alpha \in (0, 1)$ from str. concavity of f :

$$f(\alpha x + (1-\alpha)x') > \alpha f(x) + (1-\alpha)f(x')$$

$$f(\alpha x) > \alpha f(x) + (1-\alpha)f(0) = \alpha f(x) \text{ (*)}$$

Denote $y := \alpha x$ and $\lambda := \frac{1}{\alpha} \quad (\alpha > 0)$

Then $x = \frac{1}{\alpha}y = \lambda y$. From (*):

$$\frac{1}{\alpha}f(\alpha x) > f(x), \text{ so}$$

$$\lambda f(y) > f(\lambda y)$$

Question 2 [Properties of Y, f]

Let $f(x)$ be a production function and Y a production set associated with f . Show the following propositions holds

- (a) if f exhibits DRS then Y exhibits NIRS
- (b) if f exhibits IRS then Y exhibits NDRS
- (c) if f exhibits CRS then Y exhibits CRS

$$(a) \text{ WTS: } \forall y \in Y \quad \forall \alpha \in [0,1] \Rightarrow \alpha y \in Y$$

Let $y = (x, z) \in Y$ where x inputs, z output

f exhibits DRS when ($x \geq 0$ from now on)

$$\forall y \in Y \quad \forall \lambda > 1 \quad f(\lambda x) < \lambda f(x) \quad (1)$$

and since f is production function

$$0 \leq z \leq f(x) \quad (2)$$

Pick any $\alpha \in [0,1]$ & $y = (x, z) \in Y$

$$\text{WTS: } \alpha y \in Y \Leftrightarrow 0 \leq \alpha z \leq f(\alpha x) \quad (3)$$

$\alpha \geq 0$ so multiply (2) by α to get

$$0 \leq \alpha z \leq \alpha f(x)$$

Now consider 3 cases

$$(a) \alpha = 0 \quad (b) \alpha = 1 \quad (c) \alpha > 1$$

$$(a) \alpha = 0 \text{ Then } 0 = \alpha z = \alpha f(x) \text{ so } \alpha y \in Y \quad \checkmark$$

$$(b) \alpha = 1 \text{ Then } 0 \leq 1 \cdot z \leq 1 \cdot f(x) = f(1 \cdot x) \quad 1 \cdot y \in Y \quad \checkmark$$

$$(c) \alpha \in (0,1), \text{ take } \alpha = \frac{1}{\lambda} \in (0,1), x' = \alpha x \text{ for } f(x') \in Y \\ \lambda f(x') > f(\lambda x) \Rightarrow \frac{1}{\lambda} f(\alpha x) > f\left(\frac{1}{\alpha} \cdot \alpha x\right) = f(x) \quad (4)$$

$$\text{by (2) \& (4)} \quad 0 \leq z \leq f(x) < \frac{1}{\alpha} f(\alpha x) \quad / \cdot \alpha \\ 0 \leq \alpha z \leq f(\alpha x)$$

Q.E.D. $\left\{ \begin{array}{l} \text{Quad Evet Demonstrandum} \end{array} \right.$

(b)
(c) } Try solve them on your own

Question 3 [Supermodularity] 89 [I.I Fall 2009 majors]

Show that following functions are supermodular

- the Cobb-Douglas production function $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, where $\forall_i \alpha_i > 0$, and $\sum_i \alpha_i < 1$
- the Leontief function $f(x) = \min\{\alpha_i x_i\} \quad \forall_i \alpha_i > 0$

(a) SPM for $f \in C^2$ functions - twice continuously differentiable

continuously differentiable \Leftrightarrow

$$f: f_{ij} \quad \frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$$

for Cobb-Douglas $f(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

$$\frac{\partial f}{\partial x_j} = \alpha_j x_1^{\alpha_1} \cdots x_j^{\alpha_j-1} \cdot x_{j+1}^{\alpha_{j+1}} \cdots x_n^{\alpha_n}$$

$$\frac{\partial f}{\partial x_i \partial x_j} = \alpha_i \alpha_j \cdot x_1^{\alpha_1} \cdots x_i^{\alpha_i-1} \cdots x_j^{\alpha_j-1} \cdots x_n^{\alpha_n}$$

in other words : $\frac{\partial f}{\partial x_i \partial x_j} = \alpha_i \alpha_j \frac{f(x)}{x_i x_j}$

if $\alpha_i, \alpha_j \geq 0$ then $\frac{\partial f}{\partial x_i \partial x_j} \geq 0$

(b) Lemma 1. f is non decreasing in x

Proof: WTS: $x \geq x' \Rightarrow f(x) \geq f(x')$

$x \geq x' \Rightarrow x_i \geq x'_i \ \forall i$ so for $\alpha_i > 0$:

so taking

$$f(x) = \min \{ \alpha_i x_i \} \geq \min \{ \alpha_i x'_i \} = f(x')$$

Observe that $x \geq x \wedge x'$ so $f(x) - f(x \wedge x') \geq 0$

Consider two cases

$$(1) f(x) - f(x \wedge x') = 0 \quad (2) f(x) - f(x \wedge x') > 0$$

(1) Then $f(x) - f(x \wedge x') = 0$. Additionally $x \vee x' \geq x'$ so

$$f(x \vee x') - f(x) \geq 0 = f(x) - f(x \wedge x')$$

$$\text{So } f(x \vee x') - f(x') \geq f(x) - f(x \wedge x')$$

And f is SPOT

$$(2) f(x) - f(x \wedge x') > 0$$

Let j be $j \in \arg \min_i \{ \alpha_i x_i \}$, so

$$f(x) = \alpha_j x_j \text{ Then}$$

$$f(x) = \alpha_j x_j > f(x \wedge x') = \min_k \{ \alpha_k \min \{ x_k, x'_k \} \}$$

$$\text{So } \exists m: \alpha_j x_j > \alpha_m \min \{ x_m, x'_m \}$$

Lemma 2. Under (2) $\min \{x_m, x_m'\} = x_m'$

Proof: Suppose not. Then $\min \{x_m, x_m'\} = x_m$

And by previous $a_j x_j > a_m x_m$ and j is not minimizer and m is. Contradiction \leftarrow

So it is that $\min \{x_m, x_m'\} = x_m'$ \square

By lemma 2

$$f(x \vee x') = a_m x_m' \geq \min_i \{a_i x_i'\} = f(x') \quad (3)$$

Since $f(x \vee x') \geq f(x)$ by Claim 1

Sum up (2) & (3) to get

$$f(x \vee x') + f(x \wedge x') \geq f(x) + f(x')$$

So indeed f is SPM

Q.E.D.

Question 4 [Properties of Y]

Prove following properties

- Assume that for Y closed and convex, $Y \subset \mathbb{R}^L$ s.t. $0 \in Y$. Free disposal property $Y - \mathbb{R}_+^L \subset T \iff \mathbb{R}_-^L \subset Y$
- If $y \in Y$ is profit maximizing for some $p > 0$, then y is efficient.
- If Y is a convex set, then supply correspondence $s^*(p)$ is a convex set.

(a) \Rightarrow WTS: $Y - \mathbb{R}_+^L \subset T \Rightarrow \mathbb{R}_-^L \subset Y$

By free disposal $\forall y \in Y \quad \forall t \in \mathbb{R}_+^L \quad y - t \in Y$

Take $0 = y$. Then $\forall t \quad -t \in Y$ so $\mathbb{R}_-^L \subset Y$

\Leftarrow WTS: If $\mathbb{R}_-^L \subset Y \Rightarrow Y - \mathbb{R}_+^L \subset Y$

Observe that $\forall t \in \mathbb{R}_+^L \quad -t \in Y$ take $t = ns$ where $n \in \mathbb{N}$ $s \in \mathbb{R}_+^L$ then $-ns \in Y$

$\forall y \in Y \quad \forall \alpha \in [0, 1] \quad \alpha y + (1-\alpha)(-ns) \in Y$
by concavity of Y

Take in particular $\alpha = 1 - \frac{1}{n} \in (0, 1)$

$$y \Rightarrow \left(1 - \frac{1}{n}\right)y + \left(1 - \left(1 - \frac{1}{n}\right)\right)(-ns) = \underbrace{\left(1 - \frac{1}{n}\right)y - s}_{y_n} \in Y$$

Since Y is closed limit of y_n belongs to Y .

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)y - s = y - s \in Y$$

y, s are arbitrary and $s \in \mathbb{R}_+^L$ so $Y - \mathbb{R}_+^L \subset Y$

(b) $y \in \mathcal{Y}$ is efficient if $\nexists y' \in \mathcal{Y}: y' \geq y$ ($y \neq y'$)

Suppose not. So there is $y' \in \mathcal{Y}$ s.t. $y' \geq y$
and $y' \neq y$. Since $p >> 0$ ($p_i > 0 \forall i$)

Then $p \cdot y' > p \cdot y$. Why?

Then y' is feasible and gives higher Π .

Contradiction with y being Π maximizer.

⊗

So y is profit maximizer.

(c) $s(p) = \arg \max \{ p \cdot y \mid y \in \mathcal{Y} \}$

Consider cases

(1) $s(p) = \emptyset$ ✓ (2) $|s(p)| = 1$ ✓

(3) $\exists y', y'' \in s(p) \quad y' \neq y''$.

Take $\alpha \in [0, 1]$ wts: $\alpha y' + (1-\alpha) y'' \in s(p)$
 $y', y'' \in s(p) \Rightarrow p y' \geq p y \quad \forall y \in \mathcal{Y} \Rightarrow \begin{cases} \alpha p y' \geq \alpha p y \\ (1-\alpha) p y'' \geq (1-\alpha) p y \end{cases}$

$\Rightarrow p(\alpha y' + (1-\alpha) y'') \geq p y \quad \forall y \in \mathcal{Y}$. \mathcal{Y} convex $\Rightarrow \alpha y' + (1-\alpha) y'' \in \mathcal{Y}$
 $\alpha y' + (1-\alpha) y''$ is feasible & max profit so

$\alpha y' + (1-\alpha) y'' \in s(p)$. ⊗