UNIVERSITY OF GHANA

BSc/BA, FIRST SEMESTER EXAMINATIONS: 2019/2020

DEPARTMENT OF MATHEMATICS

MATH 123: VECTORS AND GEOMETRY(3 credits)

INSTRUCTION:

ANSWER ALL QUESTIONS FROM SECTION A
AND ANY ONE QUESTION FROM SECTION B

TIME ALLOWED:

TWO AND HALF HOURS $\left(2\frac{1}{2} \text{ hours}\right)$

Note: Bold letters indicate vectors

Section A (Multiple Choice): Answer all questions by indicating the letter for the correct answer in your answer booklet (75 marks)

Vectors $\boldsymbol{OA}$, $\boldsymbol{OB}$ in the $xy-$plane make angles measured counterclockwise of $\frac{\pi}{6}$ and $\frac{2\pi}{3}$ radians respectively with the $\boldsymbol{OX}$ axis. Use this information to answer questions 1 and 2.

1. The direction cosines of $\boldsymbol{OA}$ relative to the $Oxy-$axes are

   (a) $\frac{-1}{2}, \frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}, \frac{\sqrt{3}}{2}$ (c) $\frac{\sqrt{3}}{2}, \frac{1}{2}$ (d) $\frac{\sqrt{3}}{2}, \frac{-1}{2}$

2. If $2\boldsymbol{OA} + 3\boldsymbol{OB} = 6i + 4j$, what is the magnitude of $\boldsymbol{OB}$?

   (a) $\frac{2\sqrt{3}-3}{3}$ (b) $\frac{2+3\sqrt{3}}{2}$ (c) $\frac{2\sqrt{3}-3}{2}$ (d) $\frac{2\sqrt{3}+3}{3}$

3. If the vector $(a-4)\,\boldsymbol{i}-b\boldsymbol{j}$ is parallel to the $\boldsymbol{OY}$ axis and the vector $(a-8)\,\boldsymbol{i}+b\boldsymbol{j}$ is parallel to the vector $-\boldsymbol{i}-\boldsymbol{j}$, the values of a, b are

(a) $0, 8$ (b) $4, -4$ (c) $-4, 4$ (d) $0, -8$

4. If $\boldsymbol{a}$, $\boldsymbol{b}$ are given vectors and $\boldsymbol{p}=\boldsymbol{a}+\boldsymbol{b}$, $\boldsymbol{q}=2\boldsymbol{a}+3\boldsymbol{b}$ and $\boldsymbol{r}=5\boldsymbol{a}+9\boldsymbol{b}$, then

(a) $\boldsymbol{q}=\dfrac{\boldsymbol{p}+3\boldsymbol{r}}{4}$ (b) $\boldsymbol{r}=2\boldsymbol{q}+\boldsymbol{p}$ (c) $\boldsymbol{q}=\dfrac{2\boldsymbol{p}+\boldsymbol{r}}{3}$ (d) $\boldsymbol{q}=\dfrac{3\boldsymbol{p}+\boldsymbol{r}}{4}$

5. The vector equation of a plane is given as $\boldsymbol{r}=(1+3s+4t)\,\boldsymbol{i}+(-4+4s+3t)\,\boldsymbol{j}+(1+s+12t)\,\boldsymbol{k}$. What is the normal-vector equation of the plane?

(a) $\boldsymbol{r}\cdot(45\boldsymbol{i}-32\boldsymbol{j}-7\boldsymbol{k})=166$
(b) $\boldsymbol{r}\cdot(3\boldsymbol{i}+4\boldsymbol{j}+\boldsymbol{k})=-12$
(c) $\boldsymbol{r}\cdot(4\boldsymbol{i}+3\boldsymbol{j}+12\boldsymbol{k})=4$
(d) $\boldsymbol{r}\cdot(45\boldsymbol{i}-32\boldsymbol{j}-7\boldsymbol{k})=-90$

Consider the plane defined as $\boldsymbol{r}\cdot(2\boldsymbol{i}-6\boldsymbol{j}+3\boldsymbol{k})=8$ and the line defined as $\boldsymbol{r}=(\boldsymbol{i}-7\boldsymbol{j}+5\boldsymbol{k})+\lambda(3\boldsymbol{i}+4\boldsymbol{j}+12\boldsymbol{k})$. Use this information to answer question 6 to 9.

6. What is the angle between the line and the plane?

(a) $\sin^{-1}\left(\dfrac{18}{91}\right)$ (b) $\cos^{-1}\left(\dfrac{18}{91}\right)$ (c) $\cos^{-1}\left(\dfrac{66}{91}\right)$ (d) $\sin^{-1}\left(\dfrac{66}{91}\right)$

7. What is the position vector of the point of intersection between the line and the plane?

(a) $\dfrac{-1}{6}(27\boldsymbol{i}+86\boldsymbol{j}+102\boldsymbol{k})$
(b) $\dfrac{-1}{11}(7\boldsymbol{i}+101\boldsymbol{j}+17\boldsymbol{k})$
(c) $\dfrac{-1}{6}(45\boldsymbol{i}+110\boldsymbol{j}+174\boldsymbol{k})$
(d) $\dfrac{-1}{17}(\boldsymbol{i}+143\boldsymbol{j}-13\boldsymbol{k})$

8. Which of these pairs of points lie on opposite sides of the plane?

(a) $(1,1,1),(0,0,1)$
(b) $(3,0,2),(0,-2,0)$
(c) $(2,1,2),(1,1,1)$
(d) $(1,-1,1),(1,1,1)$

9. The point $\boldsymbol{i}-7\boldsymbol{j}+5\boldsymbol{k}$ lies on the line. What is the position vector of the foot of the perpendicular from this point in the plane?

(a) $\dfrac{1}{49}(151\boldsymbol{i}+649\boldsymbol{j}+398\boldsymbol{k})$
(b) $\dfrac{1}{49}(151\boldsymbol{i}-649\boldsymbol{j}+398\boldsymbol{k})$
(c) $\dfrac{1}{49}(-53\boldsymbol{i}-649\boldsymbol{j}+92\boldsymbol{k})$
(d) $\dfrac{1}{49}(-53\boldsymbol{i}-37\boldsymbol{j}+92\boldsymbol{k})$

10. Which of the following pairs of lines are coplanar?

(a) $\boldsymbol{r}=(-7-8t)\,\boldsymbol{i}+(5+3t)\,\boldsymbol{j}+(4+t)\,\boldsymbol{k}$; $\boldsymbol{r}=(-4+4t)\,\boldsymbol{i}+(3t)\,\boldsymbol{j}+(19-2t)\,\boldsymbol{k}$
(b) $\boldsymbol{r}=(6+2t)\,\boldsymbol{i}+(-4+5t)\,\boldsymbol{j}+(2+t)\,\boldsymbol{k}$; $\boldsymbol{r}=(-1-4t)\,\boldsymbol{i}+(9+5t)\,\boldsymbol{j}+(5+7t)\,\boldsymbol{k}$
(c) $\boldsymbol{r}=(2-2t)\,\boldsymbol{i}+(-5-2t)\,\boldsymbol{j}-(4t)\,\boldsymbol{k}$; $\boldsymbol{r}=(-1+t)\,\boldsymbol{i}+(6+t)\,\boldsymbol{j}+(-3+2t)\,\boldsymbol{k}$
(d) $\boldsymbol{r}=(t)\,\boldsymbol{i}+(-1+2t)\,\boldsymbol{j}+(-2+4t)\,\boldsymbol{k}$; $\boldsymbol{r}=(2-t)\,\boldsymbol{i}+(6+4t)\,\boldsymbol{j}+(3+2t)\,\boldsymbol{k}$

11. Which of these lines is not perpendicular to the line $r = (1+2t)\,i + (1-t)\,j + (1-t)\,k$?

(a) $r = (3+t)\,i - (1+2t)\,j + (2+4t)\,k$
(b) $r = (-7-8t)\,i + (5+3t)\,j + (4+t)\,k$
(c) $r = (-7-t)\,i + (5+2t)\,j + (4-4t)\,k$
(d) $r = (3-2t)\,i + (5-3t)\,j + (4-t)\,k$

12. The equation of the line passing through the point $(1,1,2)$ and having direction cosines $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$ is

(a) $r = i + j + 2k + \lambda(2i - 2j + k)$

(b) $r = i + j + 2k + \lambda\left(\cos^{-1}\left(\frac{2}{3}\right)i + \cos^{-1}\left(-\frac{2}{3}\right)j + \cos^{-1}\left(\frac{1}{3}\right)k\right)$

(c) $r = i + j + 2k + \lambda\left(\cos\left(\frac{2}{3}\right)i + \cos\left(-\frac{2}{3}\right)j + \cos\left(\frac{1}{3}\right)k\right)$

(d) $r = \frac{1}{3}(2i - 2j + k) + \lambda(i + j + 2k)$

13. The perpendicular distance from the origin to the line $r = i + j + k + \lambda(2i + 2j - k)$ is

(a) 2 (b) $\frac{\sqrt{2}}{3}$ (c) $\sqrt{2}$ (d) $\frac{2\sqrt{13}}{3}$

Use the vector equations of the two lines $r = 3i - j + 4k + \mu(2i - 3j + 6k)$ and $r = i - 2k + \lambda(i + 4j + 3k)$ to answer questions 14 and 15.

14. What is the position vector of the point of intersection of the two lines?

(a) $15i + 16j - 10k$
(b) $\frac{1}{11}(15i + 16j - 10k)$
(c) $\frac{1}{11}(2i - 36j - 49k)$
(d) $\frac{1}{11}(41i - 23j + 68k)$

15. What is the acute angle between the two lines?

(a) $\cos^{-1}\left(-\frac{8}{7\sqrt{26}}\right)$ (b) $\cos^{-1}\left(\frac{8}{7\sqrt{26}}\right)$ (c) $\cos^{-1}\left(-\frac{5}{\sqrt{130}}\right)$ (d) $\cos^{-1}\left(\frac{5}{\sqrt{130}}\right)$

16. Which one of the following lines represents the bisector of the acute angle between the two lines $r \cdot (3i + 4j) + 5 = 0$ and $r \cdot (5i + 12j) + 13 = 0$?

(a) $r \cdot (14i - 8j) = 0$
(b) $r \cdot (64i + 112j) + 2 = 0$
(c) $r \cdot (64i + 112j) + 130 = 0$
(d) $r \cdot (14i - 8j) + 2 = 0$

17. In the parallelogram $ABCD$,

(a) $AB = CD$ (b) $|AB| = |BC|$ (c) $AB \cdot BC = 0$ (d) $AB = DC$

18. If the vectors $(\boldsymbol{a}+\boldsymbol{b})$ and $(\boldsymbol{a}-\boldsymbol{b})$ are perpendicular, then

(a) $\boldsymbol{a}\cdot\boldsymbol{b}=0$
(b) $\boldsymbol{a}=\boldsymbol{b}$
(c) $|\boldsymbol{a}|^2+|\boldsymbol{b}|^2=2\boldsymbol{a}\cdot\boldsymbol{b}$
(d) $|\boldsymbol{a}|=|\boldsymbol{b}|$

19. If $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ represent the sides $\boldsymbol{BC},\boldsymbol{CA},\boldsymbol{AB}$ of triangle ABC with area Δ, then $|\boldsymbol{b}\times\boldsymbol{c}|=$

(a) Δ (b) 2Δ (c) $\frac{1}{2}\Delta$ (d) $\sin A$

20. The coordinates of the point dividing the line joining the points $(1,2,-1),(2,0,1)$ in the ratio $2:3$ is

(a) $\left(\frac{7}{5},\frac{6}{5},-\frac{2}{5}\right)$ (b) $\left(\frac{8}{5},\frac{4}{5},\frac{1}{5}\right)$ (c) $(-1,6,-5)$ (d) $(4,-4,5)$

21. The equation of the plane containing the three points $A(1,2,3),B(-1,2,0),C(2,-1,-1)$ is

(a) $\boldsymbol{r}=(\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})+\lambda(-\boldsymbol{i}+2\boldsymbol{j})+\mu(2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k})$
(b) $\boldsymbol{r}=(\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})+\lambda(-\boldsymbol{i}+2\boldsymbol{j})+\mu(\boldsymbol{i}-3\boldsymbol{j}-4\boldsymbol{k})$
(c) $\boldsymbol{r}=(\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})+\lambda(-2\boldsymbol{i}-3\boldsymbol{k})+\mu(\boldsymbol{i}-3\boldsymbol{j}-4\boldsymbol{k})$
(d) $\boldsymbol{r}=(3\boldsymbol{i}-3\boldsymbol{j}-\boldsymbol{k})+\lambda(-2\boldsymbol{i}-3\boldsymbol{k})+\mu(\boldsymbol{i}-3\boldsymbol{j}-4\boldsymbol{k})$

22. Three points $A(2,1),B(-1,5),C(3,-4)$ lie in a plane. If D is a point in the plane such that $ABCD$ is a parallelogram, then the coordinates of D are

(a) $(6,-8)$ (b) $(6,8)$ (c) $(-6,4)$ (d) $(-6,8)$

23. If the points $A(1,1,2),B(3,2,4),C(a,3,b)$ are collinear, find the values of a and b

(a) $a=5,b=5$ (b) $a=6,b=5$ (c) $a=5,b=6$ (d) $a=6,b=6$

24. If $\boldsymbol{r}+\boldsymbol{j}-2\boldsymbol{k}$ is parallel to the $\boldsymbol{OX}$ axis and $\boldsymbol{r}-2\boldsymbol{i}$ is parallel to the vector $-2\boldsymbol{j}+4\boldsymbol{k}$ then

(a) $\boldsymbol{r}=-2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$ (b) $\boldsymbol{r}=2\boldsymbol{i}-2\boldsymbol{j}+4\boldsymbol{k}$ (c) $\boldsymbol{r}=2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$ (d) $\boldsymbol{r}=2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$

25. Simplify $(\boldsymbol{a}-\boldsymbol{b})\times(\boldsymbol{a}+\boldsymbol{b})$

(a) $2(\boldsymbol{a}\times\boldsymbol{b})$
(b) $\boldsymbol{a}\times\boldsymbol{a}-\boldsymbol{b}\times\boldsymbol{b}$
(c) $-2(\boldsymbol{a}\times\boldsymbol{b})$
(d) $\boldsymbol{a}\times\boldsymbol{a}+2\boldsymbol{a}\times\boldsymbol{b}-\boldsymbol{b}\times\boldsymbol{b}$

Section B (Free Response): Answer any one of the following two questions (25 marks)

1. (a) List two non-parallel vectors in the $xz-$plane.

   (b) Hence show that the normal vector to the $xz-$plane is $\boldsymbol{n} = \lambda \boldsymbol{j}$.

   (c) Hence write down the normal form of the equation of the $xz-$plane.

   (d) The line

   $$\boldsymbol{r} = 4\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$

   intersects the plane

   $$\boldsymbol{r} \cdot \boldsymbol{j} = 0.$$

   i. Find the position vector of the point of intersection between the line and the plane.

   ii. The point $4\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$ is on the line, find the mirror image of the point in the plane.

   iii. Hence find the vector equations of the mirror image of the line in the plane.

   (e) Find the acute angle between the two planes.

   $$\boldsymbol{r} \cdot (5\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) = 5$$

   and

   $$\boldsymbol{r} \cdot \boldsymbol{j} = 0$$

2. (a) The points H and K are the midpoints of the sides BC and CD respectively of a parallelogram $ABCD$. Prove that

   $$3(\boldsymbol{AB} + \boldsymbol{AC} + \boldsymbol{AD}) = 4(\boldsymbol{AH} + \boldsymbol{AK})$$

   (b) Let $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ be the position vectors of three points P, Q, R from the origin O. If

   $$\boldsymbol{p} - 3\boldsymbol{q} + 2\boldsymbol{r} = \boldsymbol{0},$$

   show that P, Q, R are collinear.

   (c) Find the sine of the angle between the vectors $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$.

   (d) Show that the vectors

   $$\boldsymbol{a} = 3\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$$
   $$\boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$$
   $$\boldsymbol{c} = 4\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$$

   are linearly independent.