CS 70 Discrete Mathematics and Probability Theory
Spring 2015 Vazirani Discussion 11M

1. **Balls in Bins**

   You have n bins and you throw balls into them one by one randomly. A collision is when a ball is thrown into a bin which already has another ball.

   a. What is the probability that the first ball thrown will cause the first collision?

   Answer: 0

   b. What is the probability that the second ball thrown will cause the first collision?

   Answer: $\frac{1}{n}$

   c. What is the probability that, given the first two balls are not in collision, the third ball thrown will cause the first collision?

   Answer: $\frac{2}{n}$

   d. What is the probability that the third ball thrown will cause the first collision?

   Answer: The probability that the first two balls don't colide and the third ball does is

$$\Pr[\text{Ball 1, 2 do not collide}] \cdot \Pr[\text{Ball 3 collides} \mid \text{Balls 1, 2 do not collide}] = \frac{n-1}{n} \cdot \frac{2}{n}$$

   e. What is the probability that, given the first $m-1$ balls are not in collision, the m^{th} ball thrown will cause the first collision?

   Answer: $\frac{m-1}{n}$

   f. What is the probability that the m^{th} ball thrown will cause the first collision?

   Answer: Similarly to part d,

$$\frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-m+2}{n} \cdot \frac{m-1}{n} = \frac{m-1}{n} \cdot \prod_{i=0}^{m-2} \frac{n-i}{n}$$

2. **Birthdays**

   Suppose you record the birthdays of a large group of people, one at a time until you have found a match, i.e., a birthday that has already been recorded. (Assume there are 365 days in a year.)

   a. What is the probability that it takes more than 20 people for this to occur?

   Answer:

$$\begin{aligned}
\Pr[\text{it takes more than 20 people}] &= \Pr[\text{20 people don't have the same birthday}] \\
&= \frac{\frac{365!}{(365-20)!}}{365^{20}} \\
&= \boxed{\frac{365!}{345!} \frac{1}{365^{20}}} \\
&\approx .589
\end{aligned}$$

Another explanation that does not use counting:
Let b_i be the birthday of the i-th person.

$$\begin{aligned}
\Pr[\text{it takes more than 20 people}] &= \Pr[b_{20} \text{ different from } b_1 \cdots b_{19}\,] \times \cdots \times \Pr[b_2 \neq b_1] \\
&= \frac{365-19}{365} \times \frac{365-18}{365} \times \cdots \times \frac{363}{365} \times \frac{364}{365} \\
&\approx .589
\end{aligned}$$

b. What is the probability that it takes exactly 20 people for this to occur?

Answer:

$\Pr[\text{it takes exactly 20 people}] =$
$\Pr[\text{first 19 have different birthdays } \textbf{and } 20^{\text{th}} \text{ person shares a birthday with one of the first 19}]$

The probability of the first 19 people having different birthdays can be computed similar the previous part: $\frac{365!}{346!}\frac{1}{365^{19}}$. The probability of the 20$^{\text{th}}$ person sharing a birthday with one of then first 19 is $\frac{19}{365}$. Thus the total probability is $\boxed{\frac{365!}{346!}\frac{19}{365^{20}}} \approx .032$

c. Suppose instead that you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur?

Answer:
We want the first 19 people to have a birdhday different from yours, and there are 365^19 different ways of doing that. Thus $\Pr[\text{it takes exactly 20 people}] = \boxed{\frac{364^{19}}{365^{20}}} \approx .0026$

Another explanation that does not use counting:

$$\begin{aligned}
\Pr[\text{it takes exactly 20 people}] =& \Pr[\text{the 1st person does not have the same birthday as yours}]\times \\
& \Pr[\text{the 2nd person does not have the same birthday as yours}]\times \\
& \cdots \times \Pr[\text{the 19th person does not have the same birthday as yours}]\times \\
& \Pr[\text{the 20th person has the same birthday as yours}] \\
=& \frac{364}{365} \times \frac{364}{365} \times \cdots \frac{364}{365} \times \frac{1}{365} \\
=& \frac{364^{19} \times 1}{365^{20}} \\
\approx& 0.0026
\end{aligned}$$

3. Independence in balls and bins

You have k balls and n bins labelled $1, 2, \ldots, n$, where $n \geq 2$. You drop each ball uniformly at random into the bins.

a. What is the probability that bin n is empty?

Answer: $(\frac{n-1}{n})^k$

b. What is the probability that bin 1 is non-empty?

Answer: $1 - (\frac{n-1}{n})^k$

c. What is the probability that both bin 1 and bin n are empty?

Answer: $(\frac{n-2}{n})^k$

d. What is the probability that bin 1 is non-empty and bin n is empty?

Answer:

$$\Pr[\text{bin 1 nonempty} \cap \text{bin } n \text{ empty}] = \Pr[\text{bin } n \text{ empty}] - \Pr[\text{bin 1 empty} \cap \text{bin } n \text{ empty}] = \left(\frac{n-1}{n}\right)^k - \left(\frac{n-2}{n}\right)^k$$

e. What is the probability that bin 1 is non-empty given that bin n is empty?

Answer:

$$\Pr[\text{bin 1 nonempty} \mid \text{bin } n \text{ empty}] = \frac{\Pr[\text{bin 1 nonempty} \cap \text{bin } n \text{ empty}]}{\Pr[\text{bin } n \text{ empty}]} = \frac{\left(\frac{n-1}{n}\right)^k - \left(\frac{n-2}{n}\right)^k}{\left(\frac{n-1}{n}\right)^k} = 1 - \left(\frac{n-2}{n-1}\right)^k$$

f. What does this tell us about the independence of the two events, A: bin 1 is non-empty and B: bin n is non-empty?

Answer: If two events A and B are independent, we have $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$, which is equivalent to $\Pr[A] = \frac{\Pr[A \cap B]}{\Pr[B]} = \Pr[A \mid B]$. Since $\Pr[A \mid B] \neq \Pr[A]$ in this case, it shows us that these two events are not independent.