CS 70	Discrete Mathematics and Probability Theory	
Spring 2015	Vazirani	Discussion 14W

1. **Induction**

   Prove that $2^n < n!$ for all integers $n \geq 4$.

   Answer:

   Base case: when $n = 4$, $2^n = 16 < 24 = n!$

   Induction hypothesis: Assume that the statement holds up to n.

   Induction step: We show that the statement holds for $n+1$. Note that $2^{n+1} = 2 \cdot 2^n$ and $(n+1)! = (n+1) \cdot n!$. Since $n! > 2^n$ by induction hypothesis and $n+1 > 2$, we have $2^{n+1} < (n+1)!$.

   It follows by induction that $2^n < n!$ for all integers $n \geq 4$.

2. **GCD**

   Find $gcd(2n+1, 3n+2)$, where n is a positive integer.

   Answer: $gcd(3n+2, 2n+1) = gcd(2n+1, n+1) = gcd(n+1, n) = gcd(n, 1) = 1$

3. **Marbles**

   Box A contains 1 black and 3 white marbles, and box B contains 2 black and 4 white marbles. A box is selected at random, and a marble is drawn at random from the selected box.

   a. What is the probability that the marble is black?

      Answer: $\Pr(Black) = \Pr(Black|A)\Pr(A) + \Pr(Black|B)\Pr(B) = \frac{1}{4} \cdot \frac{1}{2} + \frac{2}{6} \cdot \frac{1}{2} = \frac{7}{24}$.

   b. Given that the marble is white, what is the probability that it came from box A?

      Answer: $\Pr(A|white) = \frac{\Pr(A \cap white)}{\Pr(white)} = \frac{\Pr(white|A)\Pr(A)}{\Pr(white)} = \frac{3/4 \cdot 1/2}{17/24} = \frac{9}{17}$.

4. **Hospital**

   In a hospital unit there are 8 nurses and 5 physicians; 7 nurses and 3 physicians are females. If a staff person is selected, find the probability that the subject is a nurse or a male.

   Answer: $\Pr[\text{nurse or male}] = \Pr[\text{nurse}] + \Pr[\text{male}] - \Pr[\text{male,nurse}] = 8/13 + 3/13 - 1/13 = 10/13$

5. **Palindromes**

   How many 5-digit palindromes are there? (A palindrome is a number that reads the same way forwards and backwards. For example, 27872 and 48484 are palindromes, but 28389 and 12541 are not.)

   Answer: We construct the number from left-to-right. We have 9 choices for the first digit (since it can't be 0), then 10 choices for the second digit, then 10 choices for the third digit. But now we are out of choices because the fourth digit must match the second, and the last digit must match the first. Therefore, there are 9 * 10 * 10 = 900 such numbers.

6. Bit String

How many bit strings of length 10 contain at least five consecutive 0's.

Answer: One counting strategy is based on where the run of 0's begins. (There cannot be more than one such run, because they would have to be separated by at least one 1 and the length of the string is 10.) The run of 0's can begin somewhere between the first digit and the sixth digit, inclusively. If the run begins with the first digit, the first five digits are 0, and there are $2^5 = 32$ choices for the other 5 digits. If the run begins after the first digit, then it must be preceded by a 1. The other four digits can be freely chosen with $2^4 = 16$ possibilities. Thus the total number of 10-bit strings with at least five consecutive 0's is $2^5 + 5 \cdot 2^4 = 112$.

7. Combinatorial Proof

Prove $\binom{2n}{n} = 2\binom{2n-1}{n-1}$.

Hint: If you want to pick n out of $2n$ items, you can either pick the first item or not pick it.

Answer: LHS: Choose n elements from $2n$ elements.

RHS: Suppose we pick the first item, then we pick $n-1$ elements from the remaining $2n-1$. Suppose we don't pick the first item, then we pick n elements from the remaning $2n-1$. Notice that the two events are mutually exclusive, hence the number of ways on the RHS is $\binom{2n-1}{n-1} + \binom{2n-1}{n} = \binom{2n-1}{n-1} + \binom{2n-1}{n-1} = 2\binom{2n-1}{n-1}$.

8. Balls and Bins

We throw n balls into n bins randomly.

1. What is the probability that the first bin is empty?

   Answer: $\left(\frac{n-1}{n}\right)^n$.
2. What is the probability that the first k bins are empty?

   Answer: $\left(\frac{n-k}{n}\right)^n$.
3. What is the probability that the second bin is empty given that the first one is empty?

   Answer: $\frac{\left(\frac{n-2}{n}\right)^n}{\left(\frac{n-1}{n}\right)^n} = \left(\frac{n-2}{n-1}\right)^n$.
4. Are the events that "the first bin is empty" and "the first two bins are empty" independent?

   Answer: They are dependent. Knowing the latter means the former happens with probability 1.

9. Random Variables

You have a die which has one side with a 0, one side with a 2, and four sides with 1s. (So the six sides are 0,1,1,1,1,2.) You roll the die twice. Let X be the product of the two rolls.

a. What is $E[X]$?

Answer: Let Y denote the result of the second roll. Clearly, $\mathrm{E}[Y] = 1$.

$$\mathrm{E}[X] = \frac{1}{6} \cdot 0 + \frac{4}{6} \cdot \mathrm{E}[Y] + \frac{1}{6} \cdot \mathrm{E}[2Y] = 1$$

b. What is $Var[X]$?

Answer: We compute $\mathrm{E}[X^2]$. If Y still denotes the result of the second roll, clearly $\mathrm{E}[Y^2] = \frac{4}{3}$.

$$\mathrm{E}[X^2] = \frac{1}{6} \cdot 0 + \frac{4}{6} \cdot \mathrm{E}[1^2 \cdot Y^2] + \frac{1}{6} \cdot \mathrm{E}[2^2 \cdot Y^2] = \frac{8}{9} + \frac{8}{9} = \frac{16}{9}.$$

Hence $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2 = \frac{16}{9} - 1^2 = \frac{7}{9}$.

10. Accidents

The number of accidents (per month) at a certain factory has a Poisson distribution. If the probability that there is at least one accident is $1/2$, what is the probability that there are exactly two accidents?

Answer: Let X be the number of accidents this month. Since it follows a Poisson distribution, $\Pr[X = i] = \frac{\lambda^i}{i!} e^{-\lambda}$. Hence,

$$\Pr[X \geq 1] = 1 - \Pr[X = 0] = 1 - e^{-\lambda} = \frac{1}{2},$$

which means that $\lambda = \ln 2$. Then $\Pr[X = 2] = \frac{(\ln 2)^2}{2} e^{-\ln 2} \approx 0.12$.