1. Normal distribution

A set of grades on a Discrete Math examination in an inferior school (not UC!) are approximately normally distributed with a mean of 64 and a standard deviation of 7.1.

(Note: you may assume that if X is normal with mean 0 and variance 1, then $\Pr[X \leq 1.3] \approx 0.9$ and $\Pr[X \leq 1.65] \approx 0.95$.)

a. Find the lowest passing grade if the bottom 5% of the students fail the class.

Answer: Let X denote the grade distribution of the class. Then, $\frac{X-64}{7.1}$ is a standard normal variable. Since the normal distribution is symmetric about μ,

$$\Pr\left[\frac{X-64}{7.1} \leq 1.65\right] \approx 0.95$$
$$\Pr\left[\frac{X-64}{7.1} \geq -1.65\right] \approx 0.95$$
$$\Pr[X \geq 64 - 1.65 \cdot 7.1] \approx 0.95$$
$$\Pr[X \geq 52.285] \approx 0.95$$

The lowest passing grade is approximately 52.285.

b. Find the grade of the highest B if the top 10% of the students are given A's.

Answer: Similarly to part a,

$$\Pr\left[\frac{X-64}{7.1} \leq 1.3\right] \approx 0.9$$
$$\Pr[X \leq 64 + 1.3 \cdot 7.1] \approx 0.9$$
$$\Pr[X \leq 73.23] \approx 0.9$$

The highest B is approximately 73.23.

2. Guessing Age

You meet someone at a party, and judging by how he looks, you guess that he is either 19, 20, or 21 with probability $1/2$, $1/3$, and $1/6$ respectively. He then tells you that he is a 3rd-year, and you know that 30% of 19-year-olds, 60% of 20-year-olds, and 40% of 21-year-olds are 3rd-years. What is the MAP estimate for his age?

Answer: Let θ be his age, and $\Theta = \{19, 20, 21\}$ because he is either 19, 20, or 21 years old. Let X be a random variable corresponding to what year he is in college. We are given that $X = 3$, and the following probabilities:

$$P_{19}(X = 3) = .3$$
$$P_{20}(X = 3) = .6$$
$$P_{21}(X = 3) = .4$$

To find the MAP estimator, we use

$$\hat{\theta}_{\text{MAP}} = \underset{\theta \in \Theta}{\operatorname{argmax}} P_\theta(X) \cdot \Pr(\theta)$$

Plugging in our three values of θ, we find

$$\begin{aligned}
P_{19}(X) \cdot \Pr(\theta = 19) &= .3 \times \frac{1}{2} = \frac{3}{20} \\
P_{20}(X) \cdot \Pr(\theta = 20) &= .6 \times \frac{1}{3} = \frac{1}{5} \\
P_{21}(X) \cdot \Pr(\theta = 21) &= .4 \times \frac{1}{6} = \frac{1}{15}
\end{aligned}$$

Thus, we find that $\theta = 20$ maximizes the above expression, so $\hat{\theta}_{\text{MAP}} = 20$.

3. MAP Estimation with Coins

a. Suppose you have a coin that you suspect to be biased, i.e. there is some p such that $\Pr(\text{Heads}) = p$. However, you have no information about how biased the coin is, meaning that your belief about the bias is that it is uniform on the interval $[0,1]$. You flip the coin and it comes up heads. What is the MAP estimate for p? Recall that in the continuous case,

$$\hat{p}_{\text{MAP}} = \underset{p \in [0,1]}{\operatorname{argmax}} \Pr(X \mid p) \cdot f(p)$$

where $f(p)$ is the probabilty density function of your belief about p.

Answer: $\Pr(X \mid p) = p$ because the probability of getting heads given p is just p. Since $f(p) = 1$, as our belief about p is just a uniform distribution, we have

$$\begin{aligned}
\hat{p}_{\text{MAP}} &= \underset{p \in [0,1]}{\operatorname{argmax}} \Pr(X \mid p) \cdot f(p) \\
&= \underset{p \in [0,1]}{\operatorname{argmax}}\, p \\
&= 1
\end{aligned}$$

Intuitively, we believe that since the coin came up heads, and we have no other information about its bias, it will always come up heads.

b. Your friend tells you that the likelihood of the bias decreases linearly to 0 as p moves away from $\frac{1}{2}$. In other words,

$$f(p) = 1 - 2\left|p - \frac{1}{2}\right|$$

What is the MAP estimate for p?

Answer:

$$\begin{aligned}
\hat{p}_{\text{MAP}} &= \underset{p \in [0,1]}{\operatorname{argmax}} \Pr(X \mid p) \cdot f(p) \\
&= \underset{p \in [0,1]}{\operatorname{argmax}}\, p \left(1 - 2\left|p - \frac{1}{2}\right|\right)
\end{aligned}$$

This is maximized by $p = \frac{1}{2}$.

4. Vegas

On the planet Vegas, everyone carries a coin. Many people are honest and carry a fair coin (heads on one side and tails on the other), but a fraction p of them cheat and carry a trick coin with heads on both sides. You want to estimate p with the following experiment: you pick a random sample of n people and ask each one to flip his or her coin. Assume that each person is independently likely to carry a fair or a trick coin.

a. Suppose you find through your experiment that a fraction q of n coin flip are heads, how should you use this to estimate p?

Answer:

For a coin to come up tails, we need to pick a non-cheating person as well as get heads on the flip. The chance of this happening is $\frac{1-p}{2}$. Thus the chance of getting heads is $1-\frac{1-p}{2} = 1/2+p/2$, which is the same as flipping a coin biased with this probability. This is now similar to the estimation problem described in Note 18, we can then estimate this probability with $q = 1/2+p/2$, thus $p = 2q-1$.

b. How many people do you need to ask to be 95% sure that your answer is off by at most 0.05? Compute this using two different methods, i.e. Central Limit Theorem and Chebyshev's inequality, and then compare the answers.

Note: If Z is a standard normal random variable, $\Pr[Z \leq 1.96] \approx 0.975$.

Answer:

1. Using Central Limit Theorem:
Let X_i's be i.i.d. coin flips with bias $1/2+p/2$. Clearly, $\mathrm{E}[X_i] = 1/2+p/2$ and $\mathrm{Var}[X_i] = \mathrm{E}[X_i^2] - \mathrm{E}[X_i]^2 = 1/4-p^2/4$. By the central limit theorem, as $n \to \infty$, the fraction of heads $q = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges to a normal distribution with mean $\mu = 1/2+p/2$ and variance $\sigma^2 = (1/4-p^2/4)/n$. We are asking for n such that $\Pr[|\hat{p}-p| \leq 0.05] \geq 0.95$, where $\hat{p} = 2q-1$ is our estimate of p. Note that

$$\begin{aligned}\Pr[|\hat{p}-p| \leq 0.05] &= \Pr[|2q-1-p| \leq 0.05] = \Pr[|2q-2\mu| \leq 0.05] = \Pr[2|q-\mu| \leq 0.05] \\ &= \Pr[|q-\mu| \leq 0.025] = \Pr\left[\frac{|q-\mu|}{\sigma} \leq \frac{0.025}{\sigma}\right] = \Pr\left[-\frac{0.025}{\sigma} \leq \frac{q-\mu}{\sigma} \leq \frac{0.025}{\sigma}\right] \\ &= 1-2\Pr\left[\frac{q-\mu}{\sigma} > \frac{0.025}{\sigma}\right] \geq 0.95,\end{aligned}$$

so we are looking for n such that $\Pr[\frac{q-\mu}{\sigma} > \frac{0.025}{\sigma}] \leq 0.025$. Since $\frac{q-\mu}{\sigma}$ is approximately the standard normal random variable Z, this is equivalent to $\Pr[Z > \frac{0.025\sqrt{n}}{\sqrt{1/4-p^2/4}}] \leq 0.025$. While we do not know the value of p, it would suffice to show that $\Pr[Z > \frac{0.025\sqrt{n}}{\sqrt{1/4}}] \leq 0.025$ because $\Pr[Z > \frac{0.025\sqrt{n}}{\sqrt{1/4-p^2/4}}] \leq \Pr[Z > \frac{0.025\sqrt{n}}{\sqrt{1/4}}]$ for any $p \in [0,1]$. Hence, we need $0.05\sqrt{n} \approx 1.96$, so $n \approx 1537$.

2. Using Chebyshev:
Again, we want $\Pr[|\hat{p}-p| \leq 0.05] = \Pr[|q-\mu| \leq 0.025] \geq 0.95$. Using Chebyshev,

$$\Pr[|q-\mu| > 0.025] \leq \frac{\mathrm{Var}[q]}{(0.025)^2} = \frac{1/4-p^2/4}{n(0.025)^2} \leq \frac{1/4}{n(0.025)^2} = \frac{400}{n}.$$

Setting $\frac{400}{n} = 0.05$, we get $n = 8000$. We can see that the normal approximation provides a much tighter bound than Chebyshev's inequality.