1. Midterm 2 Short Answers

Provide brief justifications of your answers. In parts (d) – (f) you can leave your answers as factorials, n choose k, etc., but do explain your calculations clearly.

(a) Are there integers x, y such that $21x + 55y = 3$?

(b) Is the set of all C programs countable?

(c) Consider a function f that takes as input a program P, and outputs:

$$f(P) = \begin{cases} 1 & \text{if program } P \text{ on input } P \text{ does not halt within the first 1000 steps,} \\ 0 & \text{otherwise.} \end{cases}$$

Is f computable?

(d) How many seven-card hands are there with three pairs? That is, there are two cards each of three different ranks, and one card of a different rank. For example, $(2\clubsuit, 2\heartsuit, 5\diamondsuit, 5\spadesuit, 6\clubsuit, 10\diamondsuit, 10\spadesuit)$ is one such hand with three pairs, but $(2\clubsuit, 2\heartsuit, 5\diamondsuit, 5\spadesuit, 5\clubsuit, 10\diamondsuit, 10\spadesuit)$ is not. Here the ordering does not matter.

(e) How many different ways are there to rearrange the letters of DIAGONALIZATION without the two N's being adjacent?

(f) How many non-decreasing sequences of k numbers from $\{1, \ldots, n\}$ are there? For example, for $n = 12$ and $k = 7$, $(2, 3, 3, 6, 9, 9, 12)$ is a non-decreasing sequence, but $(2, 3, 3, 9, 9, 6, 12)$ is not.

For the remaining two parts: Three people each independently choose a random number between 1 and 50. Let $A_{i,j}$ be the event that persons i and j choose the same number. Let $A_{1,2,3}$ be the event that all three people choose the same number.

(g) Are $A_{1,2}$ and $A_{2,3}$ independent?

(h) Are $A_{1,2}$ and $A_{1,2,3}$ independent?

2. Coin flipping

Suppose that a fair coin is flipped n times. We are curious how likely it is to get a sequence of k consecutive heads.

a. Let i be an integer between 1 and $n-k+1$ (inclusive). What is the probability that we get k consecutive heads starting with the i-th flip?

b. Suppose $k = \lceil \log n \rceil + 1$. Prove that the probability of getting a sequence of k consecutive heads is at most $1/2$.