1. Continuous Probability

Consider the following game: you spin a spinner and wait until it comes to rest at some angle $\theta \in [0, 360)$. The amount of money you win is $\theta/36 - 6$ dollars (if this number is negative, you lose money). Let X be the random variable for your winnings.

a. What is the probability density function for X?

Answer:

Distribution for X:

$$f(x) = \begin{cases} 0 & : x < -6 \\ 1/10 & : -6 \leq x < 4 \\ 0 & : x \geq 4 \end{cases}$$

b. What is the expectation?

Answer:

$$\int_{-\infty}^{\infty} x f(x) dx = \int_{-6}^{4} \frac{x}{10} dx = \left. \frac{x^2}{20} \right|_{-6}^{4} = \frac{1}{20}(16 - 36) = -1.$$

c. What is the variance?

Answer:

$$\int_{-\infty}^{\infty} (x - \mathbf{E}[X])^2 f(x) dx = \int_{-6}^{4} \frac{(x+1)^2}{10} dx = \left. \frac{(x+1)^3}{30} \right|_{-6}^{4} = \frac{1}{30}(125 - (-125)) = \frac{25}{3}.$$

2. Exponential Distribution

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

a. Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

Answer: Let X be the life span of the lightbulb. Since the mean of the exponential distribution with parameter λ is $1/\lambda$, X follows the exponential distribution with parameter $\lambda = 1/50$.

$$\Pr[X < 30] = \int_{0}^{30} \frac{1}{50} \cdot e^{\frac{-x/50}{d}} x = \left. (-e^{-x/50}) \right|_{0}^{30} = 1 - e^{-30/50} \approx 0.4511$$

b. Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probablity that the new bulb will last at least 30 days?

Answer: Let Y be the life span of the new lightbulb, which still follows the exponential distribution with parameter $\lambda = 1/50$. Since X and Y are identically distributed,

$$\Pr[Y \geq 30] = 1 - \Pr[Y < 30] = 1 - \Pr[X < 30] = e^{-30/50} \approx 0.5488$$

c. Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?

Answer: The probability we are after is $\Pr[X \geq 60 \mid X \geq 30]$. Note that

$$\Pr[X \geq 60 \mid X \geq 30] = \frac{\Pr[X \geq 60 \cap X \geq 30]}{\Pr[X \geq 30]} = \frac{\Pr[X \geq 60]}{\Pr[X \geq 30]}.$$

Moreover, from the previous parts we can see that $\Pr[X \geq a] = e^{-a/50}$. Hence,

$$\Pr[X \geq 60 \mid X \geq 30] = \frac{\Pr[X \geq 60]}{\Pr[X \geq 30]} = \frac{e^{-60/50}}{e^{-30/50}} = e^{-30/50} \approx 0.5488.$$

3. The Central Limit Theorem

We have a game in which I have a bag full of 1 dollar and 5 dollar bills, and each round you draw a single bill. I advertise that the average profit of a single round is 3 dollars and the variance is 4.

Note: if $\phi(x)$ is a standard normal density function,

$$\int_{-\infty}^{-5} \phi(x)dx \approx 0.00000029 \qquad \int_{-\infty}^{-4.5} \phi(x)dx \approx 0.00000340$$
$$\int_{-\infty}^{-4} \phi(x)dx \approx 0.00003167 \qquad \int_{-\infty}^{-3.5} \phi(x)dx \approx 0.00023263$$
$$\int_{-\infty}^{-3} \phi(x)dx \approx 0.00134990 \qquad \int_{-\infty}^{-2.5} \phi(x)dx \approx 0.00620967$$

a. Suppose you play 100 rounds of this game. Using the central limit theorem, what is the (approximate) probability that your average profit is less than 2.5?

Answer: Let X_i denote the profit from the i-th round. By the central limit theorem, we know that $\frac{\sum_{i=1}^{n} X_i - n\cdot 3}{\sqrt{4}\cdot\sqrt{n}}$ converges to the standard normal distribution as $n \to \infty$. Assuming this approximation is good enough at $n = 100$,

$$\Pr\left[\frac{1}{100}\sum_{i=1}^{100} X_i < 2.5\right] = \Pr\left[\frac{\sum_{i=1}^{100} X_i - 300}{2\sqrt{100}} < \frac{250-300}{2\sqrt{100}}\right] = \Pr\left[\frac{\sum_{i=1}^{100} X_i - 300}{2\sqrt{100}} < -2.5\right]$$

would be close to the probability that a standard normal random variable is less than -2.5, which according to the given table is 0.0062.

b. Now, suppose you play 400 rounds. What is the (approximate) probability that your average profit is less than 2.5?

Answer: Similarly to the previous part,

$$\Pr\left[\frac{1}{400}\sum_{i=1}^{400} X_i < 2.5\right] = \Pr\left[\frac{\sum_{i=1}^{400} X_i - 1200}{2\sqrt{400}} < \frac{1000-1200}{2\sqrt{400}}\right] = \Pr\left[\frac{\sum_{i=1}^{400} X_i - 1200}{2\sqrt{400}} < -5\right]$$

would be well approximated by the probability that a standard normal random variable is less than -5, which according to the given table is 0.00000029.