1. Sanity Check!

Define a random variable X to be the result of rolling two standard dice and summing the results.

(a) What is the distribution of X?

Answer: The distribution of X is

$\Pr[X=2]=\frac{1}{36}$; $\Pr[X=3]=\frac{2}{36}$; $\Pr[X=4]=\frac{3}{36}$; $\Pr[X=5]=\frac{4}{36}$; $\Pr[X=6]=\frac{5}{36}$; $\Pr[X=7]=\frac{6}{36}$; $\Pr[X=8]=\frac{5}{36}$; $\Pr[X=9]=\frac{4}{36}$; $\Pr[X=10]=\frac{3}{36}$; $\Pr[X=11]=\frac{2}{36}$; $\Pr[X=12]=\frac{1}{36}$.

(b) Directly compute the expectation of X.

Answer: The expectation of X is

$$\begin{aligned}
\mathrm{E}[X] &= 2\times\Pr[X=2]+3\times\Pr[X=3]+\cdots+12\times\Pr[X=12]\\
&= \frac{2\times1+3\times2+4\times3+5\times4+6\times5+7\times6+8\times5+9\times4+10\times3+11\times2+12\times1}{36}\\
&= 7.
\end{aligned}$$

(c) Compute the expectation of X using linearity of expectation.

Answer: Let Y be the random variable corresponding to the result of the first die and Z the result of the second die. Clearly,

$$\mathrm{E}[Y]=\mathrm{E}[Z]=\frac{1}{6}(1+2+3+4+5+6)=\frac{21}{6}=3.5$$

Hence, $\mathrm{E}[X]=\mathrm{E}[Y+Z]=\mathrm{E}[Y]+\mathrm{E}[Z]=7$.

2. Parking Meter

Eve drives to school everyday. Tired of always paying for parking, she decides one day not to pay her parking fees. Assume that there is a probability of 0.05 that she gets caught. The parking fee is \$0.25 and if she is caught, her parking ticket is \$10.

(a) How does the expected cost of parking 10 times without paying the meter compare with the cost of paying the meter each time?

Answer: If she pays each time, it costs \$2.50. We can see that if Alice doesn't pay, she has a probability $p=0.05$ of getting caught on any particular day, so over 10 days, the expected number of times she'll get caught is $10\times0.05=0.5$. We multiply that by the cost to see how much she can expect to pay in 10 days if she never pays the meter, which is $0.5\times\$10=\$5$. Therefore, It looks like she's better off just paying each time.

(b) If she parks at the meter 10 times, what is the probability that she will have to pay more than the total amount she could end up saving by not paying the meter?

Answer: This is the probability of Alice getting caught at least one of the 10 days (since the parking ticket is \$10). This is just $1-(0.95)^{10} \approx 0.40$.

3. Quadruply-repeated ones

We say that a string of bits has k *quadruply-repeated ones* if there are k positions where four consecutive 1's appear in a row. For example, the string 0100111110 has two quadruply-repeated ones.

What is the expected number of quadruply-repeated ones in a random n-bit string, when $n \geq 3$ and all n-bit strings are equally likely?

Answer: The probability of four 1's in a string of length 4 is $\frac{1}{2^4} = \frac{1}{16}$. Since there are $n-3$ substrings of length 4, by linearity of expectation, the answer is $\frac{n-3}{16}$.

4. Inversions

Consider a random permutation $a_1, a_2, \ldots, a_n$ of the numbers $1, 2, \ldots, n$. A pair (i, j) is an inversion if $i < j$ and $a_i > a_j$. What is the expected number of inversions in the permutation?

Answer: Let I_{ij} be the indicator random variable for the inversion at (i, j). The probability for an inversion is $\frac{1}{2}$. There are $(n-1)n/2$ terms, so the expectation is $\frac{n(n-1)}{4}$.

5. Coin flip

What is the expected number of times that a person must toss a fair coin to get a head for the first time?

Answer: Let X be the number of throws to first head. With probability $1/2$, we achieve the head in the first throw. If we get a tail in the first throw, we have wasted one throw and we are back to the same situation as before. Hence,

$$\mathrm{E}[X] = \frac{1}{2} \cdot 1 + \frac{1}{2}(1 + \mathrm{E}[X])$$

and solving this, we get $\mathrm{E}[X] = 2$.