CS 70	Discrete Mathematics and Probability Theory	
Spring 2015	Vazirani	Discussion 10W

1. Midterm 2 Short Answers

Provide brief justifications of your answers. In parts (d) – (f) you can leave your answers as factorials, n choose k, etc., but do explain your calculations clearly.

(a) Are there integers x, y such that $21x + 55y = 3$?

Answer: Yes. Observe that $\gcd(21, 55) = 1$, so there exist integers a, b such that $21a + 55b = 1$ (we can find a and b via the extended-gcd algorithm). Setting $x = 3a$ and $y = 3b$ gives us $21x + 55y = 3$, as required.

(b) Is the set of all C programs countable?

Answer: Yes. Recall that a C program (and any program in general) can be represented as a finite-length binary string, and the set of all finite-length binary strings $\{0, 1\}^*$ is countable.

(c) Consider a function f that takes as input a program P, and outputs:

$$f(P) = \begin{cases} 1 & \text{if program } P \text{ on input } P \text{ does not halt within the first 1000 steps,} \\ 0 & \text{otherwise.} \end{cases}$$

Is f computable?

Answer: Yes. We can compute f by writing a program that takes as input a program P, runs the program P on input P, and waits for 1000 steps. If the execution $P(P)$ continues for the 1001-st step, then the program f outputs 1, otherwise it outputs 0.

(d) How many seven-card hands are there with three pairs? That is, there are two cards each of three different ranks, and one card of a different rank. For example, $(2\clubsuit, 2\heartsuit, 5\diamondsuit, 5\spadesuit, 6\clubsuit, 10\diamondsuit, 10\spadesuit)$ is one such hand with three pairs, but $(2\clubsuit, 2\heartsuit, 5\diamondsuit, 5\spadesuit, 5\clubsuit, 10\diamondsuit, 10\spadesuit)$ is not. Here the ordering does not matter.

Answer:

$$\binom{13}{3}\binom{10}{1}\binom{4}{2}\binom{4}{2}\binom{4}{2}\binom{4}{1}$$

There are $\binom{13}{3}$ ways to choose 3 distinct ranks for the three pairs, and $\binom{10}{1}$ ways to choose 1 rank for the lone card, which must be different from the three pairs. Once we have fixed the ranks, there are $\binom{4}{2}$ ways to choose which suits for each pair, and $\binom{4}{1}$ ways to choose the suit for the lone card.

There are multiple ways to solve this problem (for example, by choosing 1 rank for the lone card first and then 3 ranks for the three pairs), but the final answer should be the same, and can be simplified into:

$$\frac{13!}{3!10!} \cdot \frac{10!}{1!9!} \cdot \frac{4!}{2!2!} \cdot \frac{4!}{2!2!} \cdot \frac{4!}{2!2!} \cdot \frac{4!}{1!3!} = 13 \cdot 12 \cdot 11 \cdot 10 \cdot 6 \cdot 6 \cdot 4 = 2471040.$$

(e) How many different ways are there to rearrange the letters of DIAGONALIZATION without the two N's being adjacent?

Answer:

$$\frac{15!}{3!3!2!2!} - \frac{14!}{3!3!2!}$$

The word DIAGONALIZATION has 15 letters with 3 A's, 3 I's, 2 N's, and 2 O's, so there are $\frac{15!}{3!3!2!2!}$ ways to rearrange the letters in total. The number of rearrangements where the two N's are adjacent is $\frac{14!}{3!3!2!}$, where we have considered "NN" as a single character. The difference $\frac{15!}{3!3!2!2!} - \frac{14!}{3!3!2!}$ is then equal to the number of rearrangements without the two N's being adjacent.

(f) How many non-decreasing sequences of k numbers from $\{1,\ldots,n\}$ are there? For example, for $n = 12$ and $k = 7$, $(2,3,3,6,9,9,12)$ is a non-decreasing sequence, but $(2,3,3,9,9,6,12)$ is not.

Answer:

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{(n-1)!k!}$$

Each non-decreasing sequence is specified by how many times each element $i \in \{1,\ldots,n\}$ appears in the sequence. Therefore, the number of non-decreasing sequences of length k is equal to the number of solutions to the equation

$$x_1 + x_2 + \cdots + x_n = k$$

where $x_i \geq 0$ is the number of times i appears in the sequence. Each such solution can be represented as a binary string of length $n+k-1$ with exactly k 1's, where the 0's represent the plus signs and the consecutive 1's represent the x_i's. Therefore, the number of such non-decreasing sequences is $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$.

For the remaining two parts: Three people each independently choose a random number between 1 and 50. Let $A_{i,j}$ be the event that persons i and j choose the same number. Let $A_{1,2,3}$ be the event that all three people choose the same number.

(g) Are $A_{1,2}$ and $A_{2,3}$ independent?

Answer: **Yes.** We can compute explicitly $\Pr[A_{1,2}] = \Pr[A_{2,3}] = 50/50^2 = 1/50$, and

$$\Pr[A_{1,2} \cap A_{2,3}] = \Pr[A_{1,2,3}] = \frac{50}{50^3} = \frac{1}{50^2} = \Pr[A_{1,2}] \cdot \Pr[A_{2,3}]$$

Alternatively, we can argue that conditioned on the event $A_{2,3}$ (i.e., persons 2 and 3 choose the same number), person 1 still makes his choice independently, so he has $1/50$ chance of picking the same number as person 2. Therefore,

$$\Pr[A_{1,2} \mid A_{2,3}] = \frac{1}{50} = \Pr[A_{1,2}]$$

(h) Are $A_{1,2}$ and $A_{1,2,3}$ independent?

Answer: **No.** Observe that the event $A_{1,2,3}$ is a subset of the event $A_{1,2}$, so

$$\Pr[A_{1,2,3} \cap A_{1,2}] = \Pr[A_{1,2,3}] \neq \Pr[A_{1,2}] \cdot \Pr[A_{1,2,3}]$$

since $\Pr[A_{1,2}] = 1/50 \neq 1$.

2. Coin flipping

Suppose that a fair coin is flipped n times. We are curious how likely it is to get a sequence of k consecutive heads.

a. Let i be an integer between 1 and $n-k+1$ (inclusive). What is the probability that we get k consecutive heads starting with the i-th flip?

Answer: $1/2^k$.

b. Suppose $k = \lceil \log n \rceil + 1$. Prove that the probability of getting a sequence of k consecutive heads is at most $1/2$.

Answer: Let A_i be the event that we get k consecutive heads starting with the i-th flip. Then, we are interested in the probability $\Pr[A_1 \cup A_2 \cup \cdots \cup A_{n-k+1}]$. Using the union bound,

$$\Pr[A_1 \cup A_2 \cup \cdots \cup A_{n-k+1}] \leq \sum_{i=1}^{n-k+1} \Pr[A_i] = \sum_{i=1}^{n-k+1} \frac{1}{2^k} = \frac{n-k+1}{2^k}.$$

Substituting $k = \lceil \log n \rceil + 1$,

$$\frac{n-k+1}{2^k} = \frac{n - \lceil \log n \rceil - 1 + 1}{2^{\lceil \log n \rceil + 1}} = \frac{n - \lceil \log n \rceil}{2^{\lceil \log n \rceil + 1}} \leq \frac{n}{2^{\log n + 1}} = \frac{n}{2n} = \frac{1}{2}.$$