

Tutorial 6

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LDA, QDA, Naive Bayes and KNN

Suppose an anesthetist needs to determine whether an anesthetic is safe for a patient who is having a heart operation. The anesthesiologist may know certain things about this patient such as their age, gender, race, blood pressure and weight. Based on these kinds of data, the anesthesiologist would like to identify if the patient is safe or unsafe for anesthetic. This is a classification problem, we can employ one of many Classification methods for this. Last week, we started on logistic regression, this week we explore the use of Linear discriminant analysis (LDA), quadratic discriminant analysis (QDA) and K nearest approach (KNN).

Both LDA and QDA are based on the Gaussian mixture model framework, which assumes that the data is a mixture of K normal distributions, where K is the number of components. The probability of x_i belong to component k is,

$$\Pr(y = k|x_i) = \frac{\pi_k f_k(x_i)}{\sum_{l=1}^K \pi_l f_l(x_i)}$$

where π_k is the “weight” of the component and $f_k()$ is a Gaussian distribution with mean of μ_k and variance σ_k .

The key difference between LDA and QDA is in the variance term. In LDA, it assumes variance is the same for all classes (or components) while QDA allows covariance matrix to vary between classes (i.e. k). This makes the boundaries (discriminant function) between classes more flexible (in quadratic form).

The discriminant function for LDA is

$$\hat{\delta}_k(x) = x \frac{\hat{\mu}_k}{\hat{\sigma}^2} - \frac{\hat{\mu}_k^2}{2\hat{\sigma}^2} + \log(\hat{\pi}_k)$$

and QDA

$$\delta_k(x) = -\frac{1}{2}x^T \Sigma_k^{-1}x + x^T \Sigma_k^{-1} \mu_k - \frac{1}{2} \mu_k^T \Sigma_k^{-1} \mu_k + \log \pi_k$$

Here is just a snapshot of what we covered in the lecture. Now, it's your turn to work through the questions below in Group Two.

Discuss the following topics:

- Explain how LDA, QDA, Naive Bayes and KNN work

These are classification methods.

LDA and QDA use Gaussian Maximum Likelihood Classification, where they look at the probability of an observation being in one class or another (each class has a Multivariate Gaussian (Normal) Distribution). For LDA it assumes that all classes have the same covariance.

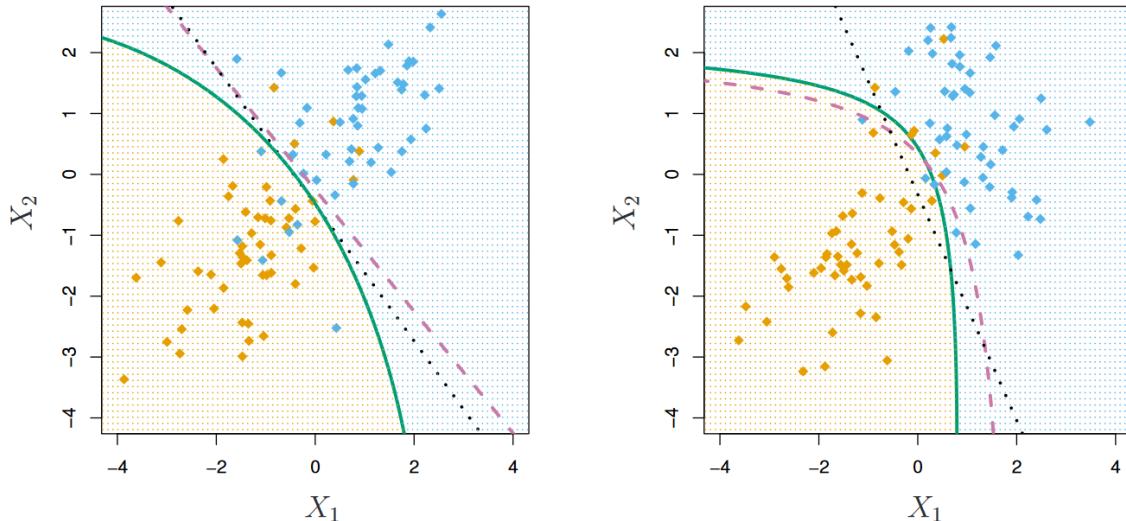
<https://www.youtube.com/watch?v=IMfLXE0ksGc>

- What are the Key assumption behind LDA?

It assumes all classes have the same covariance matrix ($\Sigma_1 = \Sigma_2 = \dots = \Sigma_k$).

- What are the key difference between LDA and QDA?

LDA assumes all classes have the same covariance matrix ($\Sigma_1 = \Sigma_2$) while QDA uses class-specific covariance matrices ($\Sigma_1 \neq \Sigma_2$). The decision boundaries between classes are linear for LDA, but quadratic (curved) for QDA:



left: $\Sigma_1 = \Sigma_2$

right: $\Sigma_1 \neq \Sigma_2$

where:

- Green - QDA
- black dotted - LDA
- purple - Naive Bayes

- Why does QDA use class-specific covariance matrices?

QDA uses class-specific covariance matrices to model the variability of data within each class more accurately. This flexibility is useful when the assumption of equal covariance matrices (as in LDA) is not valid.

- When would LDA be a better choice than QDA?

When the assumption of covariance matrices being equal ($\Sigma_1 = \Sigma_2$) is valid.

- What challenges might arise when using QDA with high-dimensional data?

QDA is computationally expensive when K (no of classes) and P (no of predictors) are large.

- What does k represent in K-nearest neighbor (KNN)?

"K" is not number of classes, but the number of "neighboring points" to be considered.

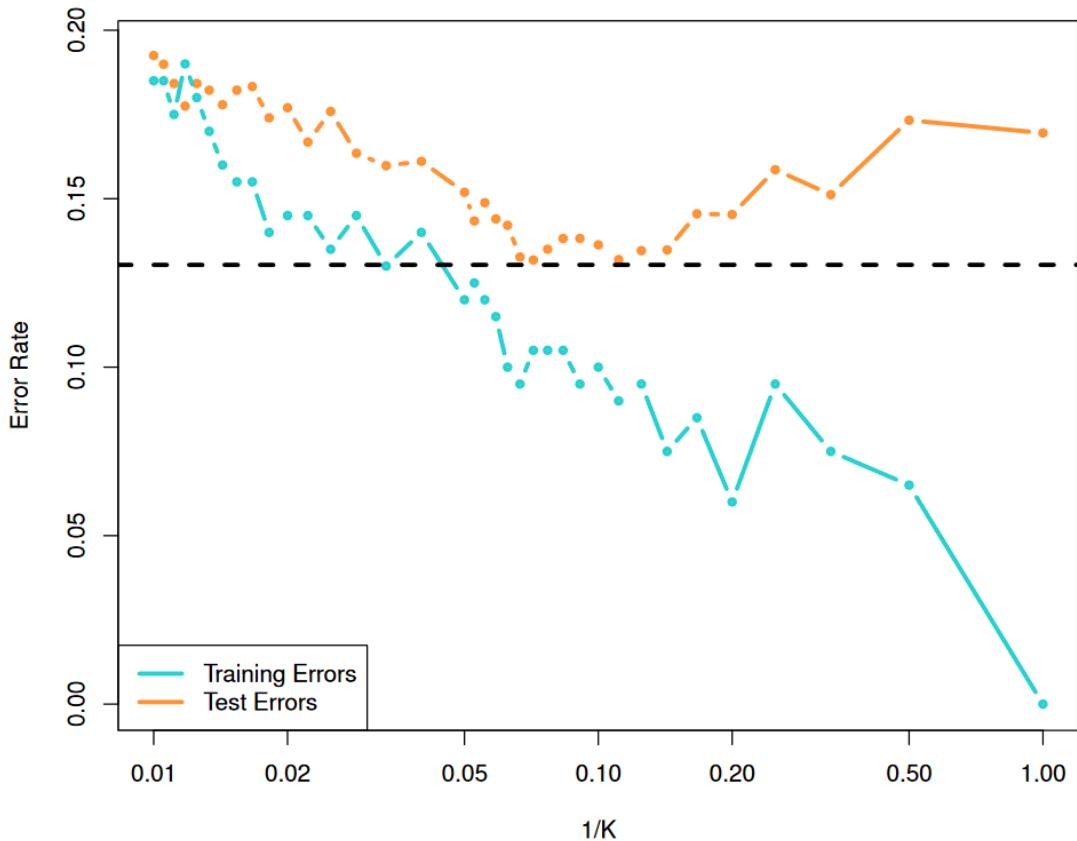
- How does the choice of k affect the performance of the KNN algorithm?

- A small K (e.g. K=1) is too flexible resulting in low bias but high variance.
- As K increases, bias increases but variance decreases.

Trade-off between BIAS and VARIANCE.

- How would you choose an optimal value for k in KNN?

1. Split data into training and test sets
2. Use test set to find the optimal K



- Can KNN be used for regression tasks, and if so, how?

Yes, KNN can be used for regression tasks. It is like a moving average, where it takes the average of the K closest observations. This means that the average of those observations will serve for prediction.

Labs - Classification Methods

4.7.1 The Stock Market Data

We will begin by examining some numerical and graphical summaries of the Smarket data, which is part of the ISLR2 library. This data set consists of percentage returns for the S&P 500 stock index over 1, 250 days, from the beginning of 2001 until the end of 2005. For each date, we have recorded the percentage returns for each of the five previous trading days, Lag1 through Lag5. We have also recorded Volume (the number of shares traded on the previous day, in billions), Today (the percentage return on the date in question) and Direction (whether the market was Up or Down on this date). Our goal is to predict Direction (a qualitative response) using the other features.

```
library(ISLR2)
```

```
## Warning: package 'ISLR2' was built under R version 4.3.3
```

```
names(Smarket)
```

```
## [1] "Year"      "Lag1"       "Lag2"       "Lag3"       "Lag4"       "Lag5"  
## [7] "Volume"    "Today"     "Direction"
```

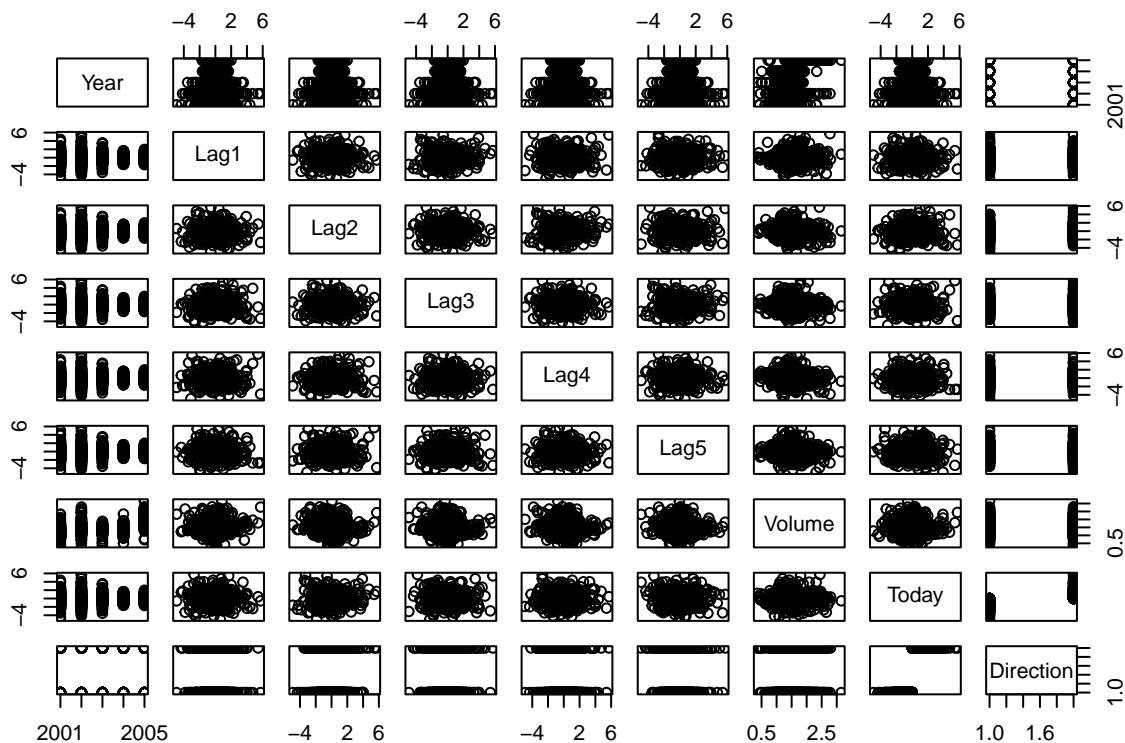
```
dim(Smarket)
```

```
## [1] 1250     9
```

```
summary(Smarket)
```

```
##      Year        Lag1        Lag2        Lag3  
##  Min.   :2001   Min.   :-4.922000   Min.   :-4.922000   Min.   :-4.922000  
##  1st Qu.:2002  1st Qu.:-0.639500  1st Qu.:-0.639500  1st Qu.:-0.640000  
##  Median :2003  Median : 0.039000  Median : 0.039000  Median : 0.038500  
##  Mean   :2003  Mean   : 0.003834  Mean   : 0.003919  Mean   : 0.001716  
##  3rd Qu.:2004 3rd Qu.: 0.596750  3rd Qu.: 0.596750  3rd Qu.: 0.596750  
##  Max.   :2005  Max.   : 5.733000  Max.   : 5.733000  Max.   : 5.733000  
##      Lag4        Lag5        Volume        Today  
##  Min.   :-4.922000  Min.   :-4.922000  Min.   :0.3561  Min.   :-4.922000  
##  1st Qu.:-0.640000  1st Qu.:-0.640000  1st Qu.:1.2574  1st Qu.:-0.639500  
##  Median : 0.038500  Median : 0.038500  Median :1.4229  Median : 0.038500  
##  Mean   : 0.001636  Mean   : 0.00561   Mean   :1.4783  Mean   : 0.003138  
##  3rd Qu.: 0.596750  3rd Qu.: 0.59700   3rd Qu.:1.6417  3rd Qu.: 0.596750  
##  Max.   : 5.733000  Max.   : 5.733000  Max.   :3.1525  Max.   : 5.733000  
##      Direction  
##  Down:602  
##  Up :648  
##  
##
```

```
pairs(Smarket)
```



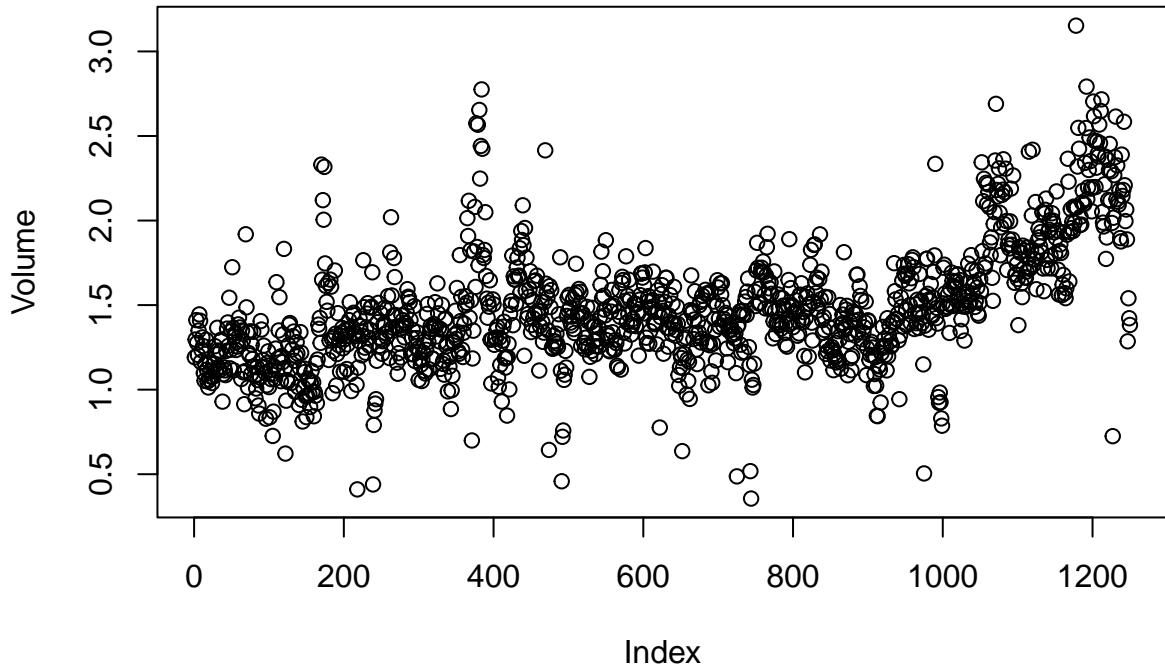
The `cor()` function produces a matrix that contains all of the pairwise correlations among the predictors in a data set. The first command below gives an error message because the `Direction` variable is qualitative.

```
# cor(Smarket) results in "Error in cor(Smarket) : 'x' must be numeric"
cor(Smarket[, -9])
```

```
##          Year      Lag1      Lag2      Lag3      Lag4
## Year  1.00000000  0.029699649  0.030596422  0.033194581  0.035688718
## Lag1  0.02969965  1.000000000 -0.026294328 -0.010803402 -0.002985911
## Lag2  0.03059642 -0.026294328  1.000000000 -0.025896670 -0.010853533
## Lag3  0.03319458 -0.010803402 -0.025896670  1.000000000 -0.024051036
## Lag4  0.03568872 -0.002985911 -0.010853533 -0.024051036  1.000000000
## Lag5  0.02978799 -0.005674606 -0.003557949 -0.018808338 -0.027083641
## Volume 0.53900647  0.040909908 -0.043383215 -0.041823686 -0.048414246
## Today  0.03009523 -0.026155045 -0.010250033 -0.002447647 -0.006899527
##          Lag5      Volume     Today
## Year   0.029787995  0.53900647  0.030095229
## Lag1  -0.005674606  0.04090991 -0.026155045
## Lag2  -0.003557949 -0.04338321 -0.010250033
## Lag3  -0.018808338 -0.04182369 -0.002447647
## Lag4  -0.027083641 -0.048414245 -0.006899527
## Lag5   1.000000000 -0.02200231 -0.034860083
## Volume -0.022002315  1.000000000  0.014591823
## Today  -0.034860083  0.01459182  1.000000000
```

As one would expect, the correlations between the lag variables and to-day's returns are close to zero. In other words, there appears to be little correlation between today's returns and previous days' returns. The only substantial correlation is between Year and Volume. By plotting the data, which is ordered chronologically, we see that Volume is increasing over time. In other words, the average number of shares traded daily increased from 2001 to 2005.

```
attach(Smarket)
plot(Volume)
```



4.7.2 Logistic Regression

Next, we will fit a logistic regression model in order to predict Direction using Lag1 through Lag5 and Volume. The `glm()` function can be used to fit `glm()` many types of generalized linear models, including logistic regression. The generalized linear models syntax of the `glm()` function is similar to that of `lm()`, except that we must pass in the argument `family = binomial` in order to tell R to run a logistic regression rather than some other type of generalized linear model.

```
glm.fits <- glm(Direction ~ Lag1 + Lag2 + Lag3 + Lag4 + Lag5 + Volume, data = Smarket , family = binomial)
summary(glm.fits)
```

```
##
## Call:
## glm(formula = Direction ~ Lag1 + Lag2 + Lag3 + Lag4 + Lag5 +
##     Volume, family = binomial, data = Smarket)
```

```

## 
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -0.126000  0.240736 -0.523   0.601
## Lag1        -0.073074  0.050167 -1.457   0.145
## Lag2        -0.042301  0.050086 -0.845   0.398
## Lag3         0.011085  0.049939  0.222   0.824
## Lag4         0.009359  0.049974  0.187   0.851
## Lag5         0.010313  0.049511  0.208   0.835
## Volume       0.135441  0.158360  0.855   0.392
##
## (Dispersion parameter for binomial family taken to be 1)
##
## Null deviance: 1731.2 on 1249 degrees of freedom
## Residual deviance: 1727.6 on 1243 degrees of freedom
## AIC: 1741.6
##
## Number of Fisher Scoring iterations: 3

```

The smallest p-value here is associated with Lag1. The negative coefficient for this predictor suggests that if the market had a positive return yesterday, then it is less likely to go up today. However, at a value of 0.15, the p-value is still relatively large, and so there is no clear evidence of a real association between Lag1 and Direction. We use the `coef()` function in order to access just the coefficients for this fitted model. We can also use the `summary()` function to access particular aspects of the fitted model, such as the p-values for the coefficients.

```
coef(glm.fits)
```

```

## (Intercept)      Lag1      Lag2      Lag3      Lag4      Lag5
## -0.126000257 -0.073073746 -0.042301344  0.011085108  0.009358938  0.010313068
## Volume
##  0.135440659

```

```
summary(glm.fits)$coef
```

```

##              Estimate Std. Error     z value Pr(>|z|)
## (Intercept) -0.126000257 0.24073574 -0.5233966 0.6006983
## Lag1        -0.073073746 0.05016739 -1.4565986 0.1452272
## Lag2        -0.042301344 0.05008605 -0.8445733 0.3983491
## Lag3         0.011085108 0.04993854  0.2219750 0.8243333
## Lag4         0.009358938 0.04997413  0.1872757 0.8514445
## Lag5         0.010313068 0.04951146  0.2082966 0.8349974
## Volume       0.135440659 0.15835970  0.8552723 0.3924004

```

```
summary(glm.fits)$coef[, 4]
```

```

## (Intercept)      Lag1      Lag2      Lag3      Lag4      Lag5
## 0.6006983    0.1452272   0.3983491   0.8243333   0.8514445   0.8349974
## Volume
## 0.3924004

```

The predict() function can be used to predict the probability that the market will go up, given values of the predictors. The type = “response” option tells R to output probabilities of the form $P(Y = 1|X)$, as opposed to other information such as the logit. If no data set is supplied to the predict() function, then the probabilities are computed for the training data that was used to fit the logistic regression model. Here we have printed only the first ten probabilities. We know that these values correspond to the probability of the market going up, rather than down, because the contrasts() function indicates that R has created a dummy variable with a 1 for Up.

```
glm.probs <- predict(glm.fits , type = "response")
glm.probs [1:10]
```

```
##      1       2       3       4       5       6       7       8
## 0.5070841 0.4814679 0.4811388 0.5152224 0.5107812 0.5069565 0.4926509 0.5092292
##      9      10
## 0.5176135 0.4888378
```

```
contrasts(Direction)
```

```
##      Up
## Down 0
## Up   1
```

In order to make a prediction as to whether the market will go up or down on a particular day, we must convert these predicted probabilities into class labels, Up or Down. The following two commands create a vector of class predictions based on whether the predicted probability of a market increase is greater than or less than 0.5.

```
glm.pred <- rep("Down" , 1250)
glm.pred[glm.probs > .5] = "Up"
```

The first command creates a vector of 1,250 Down elements. The second line transforms to Up all of the elements for which the predicted probability of a market increase exceeds 0.5. Given these predictions, the table() function table() can be used to produce a confusion matrix in order to determine how many observations were correctly or incorrectly classified.

```
table(glm.pred , Direction )
```

```
##          Direction
## glm.pred Down Up
##      Down 145 141
##      Up   457 507
```

```
(507 + 145) / 1250
```

```
## [1] 0.5216
```

```
mean(glm.pred == Direction)
```

```
## [1] 0.5216
```

The diagonal elements of the confusion matrix indicate correct predictions, while the off-diagonals represent incorrect predictions. Hence our model correctly predicted that the market would go up on 507 days and that it would go down on 145 days, for a total of $507 + 145 = 652$ correct predictions. The `mean()` function can be used to compute the fraction of days for which the prediction was correct. In this case, logistic regression correctly predicted the movement of the market 52.2 % of the time.

At first glance, it appears that the logistic regression model is working a little better than random guessing. However, this result is misleading 4.7 Lab: Classification Methods 175 because we trained and tested the model on the same set of 1, 250 observations. In other words, $100\% - 52.2\% = 47.8\%$, is the training error rate. As we have seen previously, the training error rate is often overly optimistic—it tends to underestimate the test error rate. In order to better assess the accuracy of the logistic regression model in this setting, we can fit the model using part of the data, and then examine how well it predicts the held out data. This will yield a more realistic error rate, in the sense that in practice we will be interested in our model's performance not on the data that we used to fit the model, but rather on days in the future for which the market's movements are unknown. To implement this strategy, we will first create a vector corresponding to the observations from 2001 through 2004. We will then use this vector to create a held out data set of observations from 2005.

```
train <- (Year < 2005)
Smarket.2005 <- Smarket[!train, ]
dim(Smarket.2005)
```

```
## [1] 252    9
```

```
Direction.2005 <- Direction[! train]
```

The object `train` is a vector of 1,250 elements, corresponding to the observations in our data set. The elements of the vector that correspond to observations that occurred before 2005 are set to TRUE, whereas those that correspond to observations in 2005 are set to FALSE. The object `train` is a Boolean vector, since its elements are TRUE and FALSE. Boolean vectors can be used to obtain a subset of the rows or columns of a matrix. For instance, the command `Smarket[train,]` would pick out a submatrix of the stock market data set, corresponding only to the dates before 2005, since those are the ones for which the elements of `train` are TRUE. The `!` symbol can be used to reverse all of the elements of a Boolean vector. That is, `!train` is a vector similar to `train`, except that the elements that are TRUE in `train` get swapped to FALSE in `!train`, and the elements that are FALSE in `train` get swapped to TRUE in `!train`. Therefore, `Smarket[!train,]` yields a submatrix of the stock market data containing only the observations for which `train` is FALSE—that is, the observations with dates in 2005. The output above indicates that there are 252 such observations. We now fit a logistic regression model using only the subset of the observations that correspond to dates before 2005, using the `subset` argument. We then obtain predicted probabilities of the stock market going up for each of the days in our test set—that is, for the days in 2005.

```
glm.fits <- glm(Direction ~ Lag1 + Lag2 + Lag3 + Lag4 + Lag5 + Volume ,
data = Smarket, family = binomial, subset = train)
glm.probs <- predict(glm.fits, Smarket.2005, type = "response")
```

Notice that we have trained and tested our model on two completely separate data sets: training was performed using only the dates before 2005, and testing was performed using only the dates in 2005. Finally, we compute the predictions for 2005 and compare them to the actual movements of the market over that time period.

```
glm.pred <- rep("Down", 252)
glm.pred[glm.probs > .5] <- "Up"
table(glm.pred, Direction.2005)
```

```

##          Direction.2005
## glm.pred Down Up
##     Down    77 97
##     Up      34 44

mean(glm.pred == Direction.2005)

## [1] 0.4801587

mean(glm.pred != Direction.2005)

## [1] 0.5198413

```

The `!=` notation means not equal to, and so the last command computes the test set error rate. The results are rather disappointing: the test error rate is 52 %, which is worse than random guessing! Of course this result is not all that surprising, given that one would not generally expect to be able to use previous days' returns to predict future market performance. (After all, if it were possible to do so, then the authors of this book would be out striking it rich rather than writing a statistics textbook.) We recall that the logistic regression model had very underwhelming p-values associated with all of the predictors, and that the smallest p-value, though not very small, corresponded to Lag1. Perhaps by removing the variables that appear not to be helpful in predicting Direction, we can obtain a more effective model. After all, using predictors that have no relationship with the response tends to cause a deterioration in the test error rate (since such predictors cause an increase in variance without a corresponding decrease in bias), and so removing such predictors may in turn yield an improvement. Below we have refit the logistic regression using just Lag1 and Lag2, which seemed to have the highest predictive power in the original logistic regression model.

```

glm.fits <- glm(Direction ~ Lag1 + Lag2 , data = Smarket, family = binomial , subset = train)
glm.probs <- predict(glm.fits , Smarket.2005 , type = "response")
glm.pred <- rep("Down", 252)
glm.pred[glm.probs > .5] <- "Up"
table(glm.pred , Direction.2005)

```

```

##          Direction.2005
## glm.pred Down Up
##     Down    35 35
##     Up      76 106

mean(glm.pred == Direction.2005)

## [1] 0.5595238

106 / (106 + 76)

## [1] 0.5824176

```

Now the results appear to be a little better: 56% of the daily movements have been correctly predicted. It is worth noting that in this case, a much 4.7 Lab: Classification Methods 177 simpler strategy of predicting that the market will increase every day will also be correct 56% of the time! Hence, in terms of overall error rate, the logistic regression method is no better than the naive approach. However, the confusion matrix shows that on days when logistic regression predicts an increase in the market, it has a 58% accuracy rate.

This suggests a possible trading strategy of buying on days when the model predicts an increasing market, and avoiding trades on days when a decrease is predicted. Of course one would need to investigate more carefully whether this small improvement was real or just due to random chance. Suppose that we want to predict the returns associated with particular values of Lag1 and Lag2. In particular, we want to predict Direction on a day when Lag1 and Lag2 equal 1.2 and 1.1, respectively, and on a day when they equal 1.5 and -0.8. We do this using the predict() function.

```
predict(glm.fits , newdata = data.frame(Lag1 = c(1.2, 1.5), Lag2 = c(1.1, -0.8)), type = "response")
```

```
##           1          2
## 0.4791462 0.4960939
```

4.7.3 Linear Discriminant Analysis

Now we will perform LDA on the Smarket data. In R, we fit an LDA model using the lda() function, which is part of the MASS library. Notice that the lda() syntax for the lda() function is identical to that of lm(), and to that of glm() except for the absence of the family option. We fit the model using only the observations before 2005.

```
library(MASS)
```

```
## Warning: package 'MASS' was built under R version 4.3.3
```

```
##
## Attaching package: 'MASS'
```

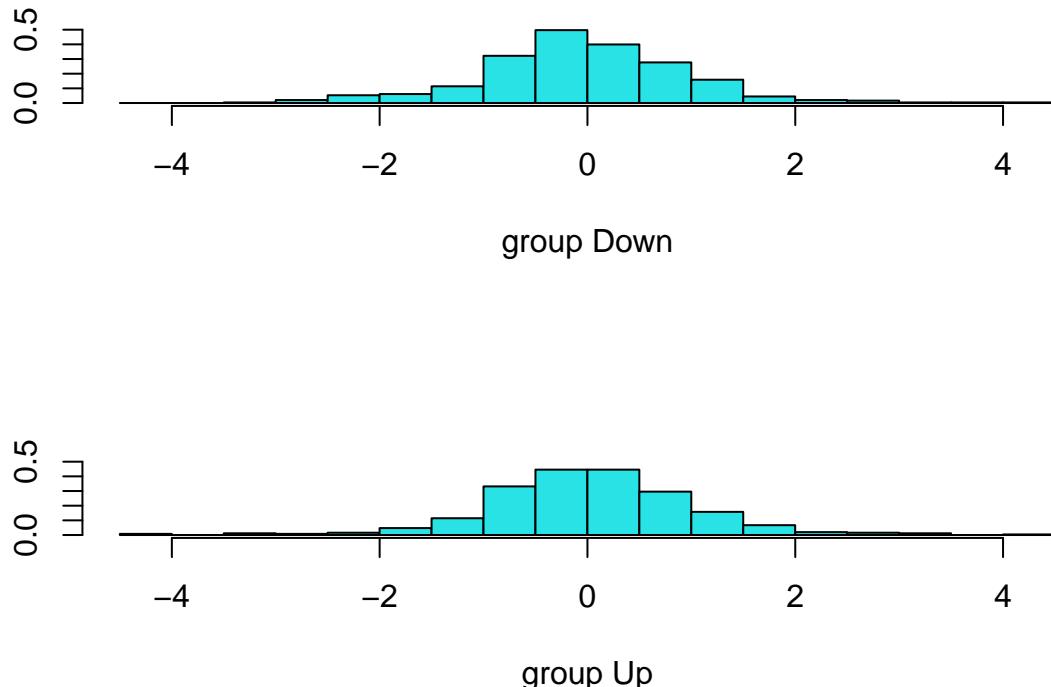
```
## The following object is masked from 'package:ISLR2':
```

```
##
## Boston
```

```
lda.fit <- lda(Direction ~ Lag1 + Lag2, data = Smarket, subset = train)
lda.fit
```

```
## Call:
## lda(Direction ~ Lag1 + Lag2, data = Smarket, subset = train)
##
## Prior probabilities of groups:
##     Down      Up
## 0.491984 0.508016
##
## Group means:
##             Lag1      Lag2
## Down  0.04279022 0.03389409
## Up   -0.03954635 -0.03132544
##
## Coefficients of linear discriminants:
##             LD1
## Lag1 -0.6420190
## Lag2 -0.5135293
```

```
plot(lda.fit)
```



The LDA output indicates that $\hat{\pi}_1 = 0.492$ and $\hat{\pi}_2 = 0.508$; in other words, 49.2 % of the training observations correspond to days during which the market went down. It also provides the group means; these are the average of each predictor within each class, and are used by LDA as estimates of μ_k . These suggest that there is a tendency for the previous 2 days' returns to be negative on days when the market increases, and a tendency for the previous days' returns to be positive on days when the market declines. The coefficients of linear discriminants output provides the linear combination of Lag1 and Lag2 that are used to form the LDA decision rule. In other words, these are the multipliers of the elements of $X = x$ in (4.24). If $-0.642 \times \text{Lag1} - 0.514 \times \text{Lag2}$ is large, then the LDA classifier will predict a market increase, and if it is small, then the LDA classifier will predict a market decline. The `plot()` function produces plots of the linear discriminants, obtained by computing $-0.642 \times \text{Lag1} - 0.514 \times \text{Lag2}$ for each of the training observations. The Up and Down observations are displayed separately. The `predict()` function returns a list with three elements. The first element, `class`, contains LDA's predictions about the movement of the market. The second element, `posterior`, is a matrix whose k th column contains the posterior probability that the corresponding observation belongs to the k th class, computed from (4.15). Finally, `x` contains the linear discriminants, described earlier.

```
lda.pred <- predict(lda.fit , Smarket.2005)
names(lda.pred)
```

```
## [1] "class"      "posterior"   "x"
```

As we observed in Section 4.5, the LDA and logistic regression predictions are almost identical.

```

lda.class <- lda.pred$class
table(lda.class , Direction.2005)

##          Direction.2005
## lda.class Down Up
##      Down   35 35
##      Up    76 106

mean(lda.class == Direction.2005)

## [1] 0.5595238

```

Applying a 50 % threshold to the posterior probabilities allows us to recreate the predictions contained in lda.pred\$class.

```

sum(lda.pred$posterior[, 1] >= .5)

## [1] 70

sum(lda.pred$posterior[, 1] < .5)

## [1] 182

```

Notice that the posterior probability output by the model corresponds to the probability that the market will decrease:

```

lda.pred$posterior [1:20 , 1]

##      999     1000     1001     1002     1003     1004     1005     1006
## 0.4901792 0.4792185 0.4668185 0.4740011 0.4927877 0.4938562 0.4951016 0.4872861
##      1007     1008     1009     1010     1011     1012     1013     1014
## 0.4907013 0.4844026 0.4906963 0.5119988 0.4895152 0.4706761 0.4744593 0.4799583
##      1015     1016     1017     1018
## 0.4935775 0.5030894 0.4978806 0.4886331

lda.class [1:20]

##  [1] Up   Up
## [16] Up   Up   Down Up   Up
## Levels: Down Up

```

If we wanted to use a posterior probability threshold other than 50 % in order to make predictions, then we could easily do so. For instance, suppose that we wish to predict a market decrease only if we are very certain that the market will indeed decrease on that day—say, if the posterior probability is at least 90 %.

```

sum(lda.pred$posterior[, 1] > .9)

## [1] 0

```

No days in 2005 meet that threshold! In fact, the greatest posterior probability of decrease in all of 2005 was 52.02 %.

4.7.4 Quadratic Discriminant Analysis

We will now fit a QDA model to the Smarket data. QDA is implemented in R using the qda() function, which is also part of the MASS library. The qda() syntax is identical to that of lda().

```
qda.fit <- qda(Direction ~ Lag1 + Lag2 , data = Smarket ,
subset = train)
qda.fit

## Call:
## qda(Direction ~ Lag1 + Lag2, data = Smarket, subset = train)
##
## Prior probabilities of groups:
##      Down      Up
## 0.491984 0.508016
##
## Group means:
##           Lag1      Lag2
## Down  0.04279022 0.03389409
## Up   -0.03954635 -0.03132544
```

The output contains the group means. But it does not contain the coefficients of the linear discriminants, because the QDA classifier involves a quadratic, rather than a linear, function of the predictors. The predict() function works in exactly the same fashion as for LDA

```
qda.class <- predict(qda.fit , Smarket.2005)$class
table(qda.class , Direction.2005)
```

```
##          Direction.2005
## qda.class Down  Up
##      Down    30  20
##      Up     81 121

mean(qda.class == Direction.2005)
```

```
## [1] 0.5992063
```

Interestingly, the QDA predictions are accurate almost 60 % of the time, even though the 2005 data was not used to fit the model. This level of accuracy is quite impressive for stock market data, which is known to be quite hard to model accurately. This suggests that the quadratic form assumed by QDA may capture the true relationship more accurately than the linear forms assumed by LDA and logistic regression. However, we recommend evaluating this method's performance on a larger test set before betting that this approach will consistently beat the market!

4.7.5 Naive Bayes

Next, we fit a naive Bayes model to the Smarket data. Naive Bayes is implemented in R using the naiveBayes() function, which is part of the e1071 library. The syntax is identical to that of lda() and qda(). By default, this implementation of the naive Bayes classifier models each quantitative feature using a Gaussian distribution. However, a kernel density method can also be used to estimate the distributions

```

#install.packages("e1071")
library(e1071)

## Warning: package 'e1071' was built under R version 4.3.3

nb.fit <- naiveBayes(Direction ~ Lag1 + Lag2 , data = Smarket ,
subset = train)
nb.fit

## 
## Naive Bayes Classifier for Discrete Predictors
##
## Call:
## naiveBayes.default(x = X, y = Y, laplace = laplace)
##
## A-priori probabilities:
## Y
##   Down      Up
## 0.491984 0.508016
##
## Conditional probabilities:
## Lag1
## Y          [,1]      [,2]
## Down  0.04279022 1.227446
## Up    -0.03954635 1.231668
##
## Lag2
## Y          [,1]      [,2]
## Down  0.03389409 1.239191
## Up    -0.03132544 1.220765

```

The output contains the estimated mean and standard deviation for each variable in each class. For example, the mean for Lag1 is 0.0428 for Direction=Down, and the standard deviation is 1.23. We can easily verify this:

```
mean(Lag1[train ][ Direction[train ] == "Down"])
```

```
## [1] 0.04279022
```

```
sd(Lag1[train ][ Direction[train ] == "Down"])
```

```
## [1] 1.227446
```

The predict() function is straightforward.

```
nb.class <- predict(nb.fit , Smarket.2005)
table(nb.class , Direction.2005)
```

```

##           Direction.2005
## nb.class  Down  Up
##       Down   28  20
##       Up     83 121

```

```
mean(nb.class == Direction.2005)
```

```
## [1] 0.5912698
```

Naive Bayes performs very well on this data, with accurate predictions over 59% of the time. This is slightly worse than QDA, but much better than LDA.

The predict() function can also generate estimates of the probability that each observation belongs to a particular class.

```
nb.preds <- predict(nb.fit , Smarket.2005 , type = "raw")  
nb.preds [1:5, ]
```

```
##           Down        Up  
## [1,] 0.4873164 0.5126836  
## [2,] 0.4762492 0.5237508  
## [3,] 0.4653377 0.5346623  
## [4,] 0.4748652 0.5251348  
## [5,] 0.4901890 0.5098110
```

4.7.6 K-Nearest Neighbors

We will now perform KNN using the knn() function, which is part of the knn() class library. This function works rather differently from the other model-fitting functions that we have encountered thus far. Rather than a two-step approach in which we first fit the model and then we use the model to make predictions, knn() forms predictions using a single command. The function requires four inputs.

1. A matrix containing the predictors associated with the training data, labeled train.X below.
2. A matrix containing the predictors associated with the data for which we wish to make predictions, labeled test.X below.
3. A vector containing the class labels for the training observations, labeled train.Direction below.
4. A value for K, the number of nearest neighbors to be used by the classifier.

We use the cbind() function, short for column bind, to bind the Lag1 and cbind() Lag2 variables together into two matrices, one for the training set and the other for the test set.

```
library(class)  
train.X <- cbind(Lag1 , Lag2)[train , ]  
test.X <- cbind(Lag1 , Lag2)[!train , ]  
train.Direction <- Direction[train]
```

Now the knn() function can be used to predict the market's movement for the dates in 2005. We set a random seed before we apply knn() because if several observations are tied as nearest neighbors, then R will randomly break the tie. Therefore, a seed must be set in order to ensure reproducibility of results.

```
set.seed (1)  
knn.pred <- knn(train.X, test.X, train.Direction , k = 1)  
table(knn.pred , Direction.2005)
```

```

##          Direction.2005
## knn.pred Down Up
##      Down   43 58
##      Up    68 83

```

The results using $K = 1$ are not very good, since only 50 % of the observations are correctly predicted. Of course, it may be that $K = 1$ results in an overly flexible fit to the data. Below, we repeat the analysis using $K = 3$.

```

knn.pred <- knn(train.X, test.X, train.Direction , k = 3)
table(knn.pred , Direction.2005)

```

```

##          Direction.2005
## knn.pred Down Up
##      Down   48 54
##      Up    63 87

```

```
mean(knn.pred == Direction.2005)
```

```
## [1] 0.5357143
```

The results have improved slightly. But increasing K further turns out to provide no further improvements. It appears that for this data, QDA provides the best results of the methods that we have examined so far. KNN does not perform well on the Smarket data but it does often provide impressive results. As an example we will apply the KNN approach to the Caravan data set, which is part of the ISLR2 library. This data set includes 85 predictors that measure demographic characteristics for 5,822 individuals. The response variable is Purchase, which indicates whether or not a given individual purchases a caravan insurance policy. In this data set, only 6 % of people purchased caravan insurance.

```
dim(Caravan)
```

```
## [1] 5822 86
```

```
attach(Caravan)
summary(Purchase)
```

```
## No Yes
## 5474 348
```

Because the KNN classifier predicts the class of a given test observation by identifying the observations that are nearest to it, the scale of the variables matters. Variables that are on a large scale will have a much larger effect on the distance between the observations, and hence on the KNN classifier, than variables that are on a small scale. For instance, imagine a data set that contains two variables, salary and age (measured in dollars and years, respectively). As far as KNN is concerned, a difference of \$1,000 in salary is enormous compared to a difference of 50 years in age. Consequently, salary will drive the KNN classification results, and age will have almost no effect. This is contrary to our intuition that a salary difference of \$1,000 is quite small compared to an age difference of 50 years. Furthermore, the importance of scale to the KNN classifier leads to another issue: if we measured salary in Japanese yen, or if we measured age in minutes, then we'd get quite different classification results from what we get if these two variables are measured in dollars and years. A good way to handle this problem is to standardize the data so that all variables are given a mean of zero and a standard deviation of one. Then all variables will be on a comparable scale. The scale() function does just this. In standardizing the data, we exclude column 86, because that is the qualitative Purchase variable.

```

standardized.X <- scale(Caravan[, -86])
var(Caravan[, 1])

## [1] 165.0378

var(Caravan[, 2])

## [1] 0.1647078

var(standardized.X[, 1])

## [1] 1

var(standardized.X[, 2])

## [1] 1

```

Now every column of `standardized.X` has a standard deviation of one and a mean of zero. We now split the observations into a test set, containing the first 1,000 observations, and a training set, containing the remaining observations. We fit a KNN model on the training data using $K = 1$, and evaluate its performance on the test data.

```

test <- 1:1000
train.X <- standardized.X[-test, ]
test.X <- standardized.X[test, ]
train.Y <- Purchase[-test]
test.Y <- Purchase[test]
set.seed (1)
knn.pred <- knn(train.X, test.X, train.Y, k = 1)
mean(test.Y != knn.pred)

## [1] 0.118

mean(test.Y != "No")

## [1] 0.059

```

The vector `test` is numeric, with values from 1 through 1, 000. Typing `standardized.X[test,]` yields the submatrix of the data containing the observations whose indices range from 1 to 1, 000, whereas typing `standardized.X[-test,]` yields the submatrix containing the observations whose indices do not range from 1 to 1, 000. The KNN error rate on the 1,000 test observations is just under 12 %. At first glance, this may appear to be fairly good. However, since only 6 % of customers purchased insurance, we could get the error rate down to 6 % by always predicting No regardless of the values of the predictors! Suppose that there is some non-trivial cost to trying to sell insurance to a given individual. For instance, perhaps a salesperson must visit each potential customer. If the company tries to sell insurance to a random selection of customers, then the success rate will be only 6 %, which may be far too low given the costs involved. Instead, the company would like to try to sell insurance only to customers who are likely to buy it. So the overall error rate is not of interest. Instead, the fraction of individuals that are correctly predicted to buy insurance is of interest. It turns out that KNN with $K = 1$ does far better than random guessing among the customers that are predicted to buy insurance. Among 77 such customers, 9, or 11.7 %, actually do purchase insurance. This is double the rate that one would obtain from random guessing.

```
table(knn.pred , test.Y)
```

```
##          test.Y
## knn.pred  No Yes
##       No  873  50
##      Yes   68   9
```

Using $K = 3$, the success rate increases to 19 %, and with $K = 5$ the rate is 26.7 %. This is over four times the rate that results from random guessing. It appears that KNN is finding some real patterns in a difficult data set!

```
knn.pred <- knn(train.X, test.X, train.Y, k = 3)
table(knn.pred , test.Y)
```

```
##          test.Y
## knn.pred  No Yes
##       No  920  54
##      Yes   21   5
```

```
5 / 26
```

```
## [1] 0.1923077
```

```
knn.pred <- knn(train.X, test.X, train.Y, k = 5)
table(knn.pred , test.Y)
```

```
##          test.Y
## knn.pred  No Yes
##       No  930  55
##      Yes   11   4
```

```
4 / 15
```

```
## [1] 0.2666667
```

However, while this strategy is cost-effective, it is worth noting that only 15 customers are predicted to purchase insurance using KNN with $K = 5$. In practice, the insurance company may wish to expend resources on convincing more than just 15 potential customers to buy insurance. As a comparison, we can also fit a logistic regression model to the data. If we use 0.5 as the predicted probability cut-off for the classifier, then we have a problem: only seven of the test observations are predicted to purchase insurance. Even worse, we are wrong about all of these! However, we are not required to use a cut-off of 0.5. If we instead predict a purchase any time the predicted probability of purchase exceeds 0.25, we get much better results: we predict that 33 people will purchase insurance, and we are correct for about 33 % of these people. This is over five times better than random guessing!

```
glm.fits <- glm(Purchase ~ ., data = Caravan, family = binomial , subset = -test)
```

```
## Warning: glm.fit: fitted probabilities numerically 0 or 1 occurred
```

```
glm.probs <- predict(glm.fits, Caravan[test , ], type = "response")
glm.pred <- rep("No", 1000)
glm.pred[glm.probs > .5] <- "Yes"
table(glm.pred , test.Y)
```

```
##          test.Y
## glm.pred  No Yes
##      No  934  59
##      Yes   7   0
```

```
glm.pred <- rep("No", 1000)
glm.pred[glm.probs > .25] <- "Yes"
table(glm.pred , test.Y)
```

```
##          test.Y
## glm.pred  No Yes
##      No  919  48
##      Yes   22  11
```

```
11 / (22 + 11)
```

```
## [1] 0.3333333
```

Exercises