USING NONLINEAR QUANTILE REGRESSION TO ESTIMATE THE SELF-THINNING BOUNDARY CURVE

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Abstract--The relationship between tree size (quadratic mean diameter) and tree density (number of trees per unit area) has been a topic of research and discussion for many decades. Starting with Reineke in 1933, the maximum size-density relationship, on a log-log scale, has been assumed to be linear. Several techniques, including linear quantile regression, have been employed to obtain parameters of the self-thinning line. Some authors recently considered that restriction on the maximum diameter at lower spatial densities resulted in a curvilinear relationship. In this study, a nonlinear quantile regression based on the 99th quantile was used to characterize this upper boundary. The resulting self-thinning curve fit the curvilinear boundary much better than did the Reineke's self-thinning line.

INTRODUCTION

The reciprocal relationship between tree size and stand density has been a topic of research and discussion since Reineke (1933) expressed the logarithm of maximum quadratic mean diameter at breast height (Q_m) as a linear function of the logarithm of number of trees per unit area (N). He found that a slope of -1.605 adequately described the relationship for 12 out of 14 species examined. Other researchers have since statistically fit slopes to log-transformed values of Q_m and N and found that they varied, ranging in values from -1.707 to -1.505 (Bailey 1972, Drew and Flewelling 1977, Harms 1981, MacKinney and Chaiken 1935, Williams 1996).

While the relationship between tree size and tree density on a log-log scale can be considered linear within some range of stand density, it might actually be curvilinear throughout the entire range of tree density because trees are ultimately limited in size by their weight, thus restricting the maximum diameter associated with lower spatial densities (Westoby 1984). The curvilinear boundary was evident for southern pines (Zeide 1987) and slash pines (Cao and others 2000) in particular.

Quantile regression (Koenker and Bassett 1978) has recently been employed by scientists from various backgrounds to address research problems in medicine (Austin and Schull 2003), economics (Machado and Mata 2005), education and policy (Haile and Nguyen 2008), and natural resource management (Cade and others 2005). In forestry, quantile regression has been applied to compute stand density index (Ducey and Knapp 2010) or evaluate the spread rate of forest diseases (Evans and Finkral 2010).

One advantage of quantile regression over ordinary least squares regression is that the quantile regression estimates are more robust against outliers. It is particularly useful in estimating the quantiles (or percentiles) of the response variable, for example, tree diameter percentiles (Mehtätalo and others 2008) or maximum diameters in self-thinning stands (Zhang and others 2005).

The objective of this study was to apply nonlinear quantile regression in modeling the self-thinning boundary curve.

MATERIALS AND METHODS

Data

Data from 147 permanent plots of direct-seeded slash pine (*Pinus elliottii* Engelm.) stands were used for this study. These stands were established on cutover sites located in Natchitoches and Rapides Parishes (central Louisiana) and in Washington Parish (southeast Louisiana). Baldwin (1985) and Lohrey (1987) described these data in detail. Plot size ranged from 0.040 to 0.048 ha. Stand age ranged from 8 to 28 years, stand density from 445 to 12,108 trees/ha, basal area from 2.6 to 52.6 m^2/ha, and site index (base age 25 years) from 9 to 23 m. Some plots were precommercially thinned at age 3 or 4 years. Each plot was measured from 3 to 6 times, at 3 to 10 years apart. There was a total of 615 measurements encompassing 468 growth periods. The trajectories of stand density and quadratic mean diameter for these measurements are shown in figure 1.

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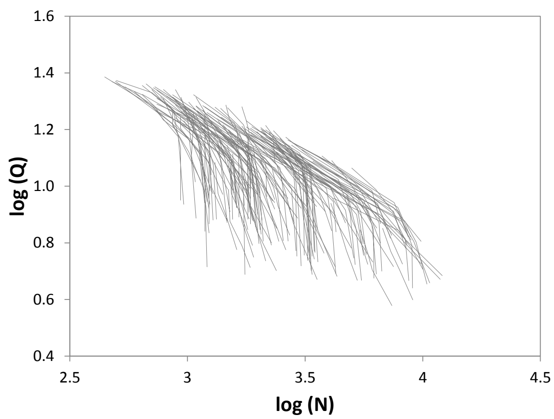

Figure 1--Trajectories of quadratic mean diameter (Q) stand density (N) for measurements of direct-seeded slash pines.

Self-thinning Boundary

The maximum number of trees that can survive on a given land area depends on the quadratic mean diameter and decreases predictably with increasing mean diameter as a stand self-thins. Reineke (1933) described this boundary with the simple linear equation:

$$\log(N) = a + b \log(Q_m) \tag{1}$$

where $b = -1.605$ for many species. This equation can be rewritten as:

$$Q_m = = b_1 N^{-0.623} \tag{2}$$

where $b_1 = 10^a$ and $-0.623 = 1/(-1.605)$.

Cao and others (2000) proposed the following nonlinear relationship for the self-thinning boundary:

$$Q_m = b_1 N^{-0.623}[1 - \exp(b_3 N^{b_4})]. \tag{3}$$

Quantile Regression

Parameters b_1, b_3, and b_4 can be estimated via quantile regression techniques by minimizing:

$$S = \sum_{Q_i \geq \hat{Q}_i} \tau |Q_i - \hat{Q}_i| + \sum_{Q_i < \hat{Q}_i} (1 - \tau)|Q_i - \hat{Q}_i| \tag{4}$$

where Q_i = observed tree diameter at breast height and $\hat{Q}_i$ = predicted τ^{th} quantile of tree diameters. SAS procedure NLP (SAS Institute Inc. 2010) was used for this purpose.

RESULTS AND DISCUSSION

Predicted self-thinning curves for the 90$^{\text{th}}$, 95$^{\text{th}}$, 99$^{\text{th}}$, 100$^{\text{th}}$ quantiles from equation (3) (i.e., τ = 0.90, 0.95, 0.99, and 1.00) have different shapes (fig. 2). Compared to the apparent shape of the boundary seen with the data, the 90$^{\text{th}}$ and 95$^{\text{th}}$ quantiles do not have adequate curvature, and the one based on the 100$^{\text{th}}$ quantile is too high for the data.

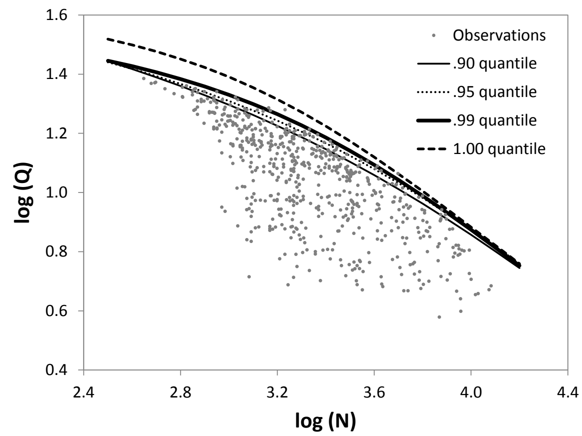

Figure 2--Self-thinning boundary curves predicted from equation (3) by use of quantile regression with four values of τ: 0.90, 0.95, 0.99, and 1.00.

The self-thinning boundary curve based on the 99$^{\text{th}}$ quantile appears appropriate to model the relationship between maximum tree diameter and stand density (fig. 3). The final equation is:

$$\hat{Q}_m = 2326\, N^{-0.623}[1 - \exp(-0.013\, N^{0.645})]. \tag{5}$$

For this data set, the self-thinning curve provides a more realistic reciprocal relationship between maximum tree diameter and stand density than does the self-thinning line proposed in 1933 by Reineke (fig. 3). Cao and others (2000) explained that the self-thinning curve is approximately linear for either a narrow range of low stand densities or a wider range of relatively high stand densities. This also explains different slopes of the self-thinning line reported from data of different stand density ranges; the slope varies with the range of *N* included in the data set.

The wide range of stand density from this direct-seeded slash pine data set (from 445 to 12,108 trees/ha) allowed us to construct a meaningful self-thinning curve. While the quantile regression techniques proved to be appropriate for this task, only the 99$^{\text{th}}$ quantile appeared to be best for representing the boundary curve.

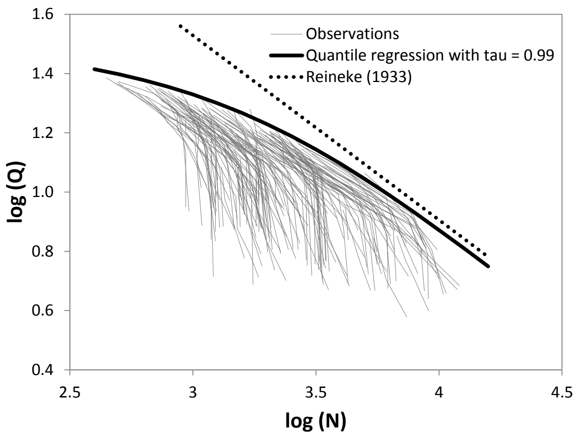

Figure 3--Reineke's (1933) self-thinning line and self-thinning curve from quantile regression with $\tau = 0.99$.

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