Calibrating a Segmented Taper Equation with Two Diameter Measurements

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ABSTRACT

Recent advances in laser technology help make possible accurate and affordable measurements of upper-stem diameters. These measurements can be used to calibrate results from a taper equation to improve the accuracy of diameter predictions along the tree bole. Felled-tree data from a loblolly pine (*Pinus taeda* L.) plantation were used to evaluate two methods for calibrating outputs from a segmented taper equation with parameters either obtained from the data in this study or originally published by Max and Burkhart (1976, Segmented polynomial regression applied to taper equations, *For. Sci.* 22:283–289). For outside-bark diameters, although a simple calibration for dbh gave desirable results, a better calibration involving both dbh and an upper-stem diameter provided significant improvements in predicting tree taper. Results varied depending on where the diameter was measured, with optimum gains obtained when the upper-stem diameter was measured at the midpoint between breast height and the tree tip. For inside-bark diameters, the calibration for inside-bark dbh actually produced inferior predictions, whereas the calibration based on both dbh and an upper-stem diameter offered only modest improvements over the unadjusted predictions.

Keywords: *Pinus taeda*, loblolly pine, segmented regression model

Taper equations are developed to describe tree profiles and are particularly useful for computing the top diameter limit for any utilization standard. Cao et al. (1980) evaluated six taper equations and concluded that the Max and Burkhart (1976) taper equation produced superior results in predicting tree taper. This segmented taper equation consists of three different quadratic functions joined at two join points and therefore is adequately flexible to describe complicated tree profiles. The same equation form has been used by many authors for taper prediction (Valenti and Cao 1986, Figueiredo-Filho et al. 1996, Williams and Reich 1997, Muhairwe 1999, Sharma and Burkhart 2003, Coble and Hilpp 2006, Trincardo and Burkhart 2006).

With the advent of better and affordable laser dendrometers, upper-stem diameter measurements can be obtained in a manner that was not previously possible and can be used to improve taper predictions. Czaplewski and McClure (1988) conditioned the Max and Burkhart (1976) taper equation for dbh and an upper-stem diameter (at 17.3 ft) to obtain a 10–25% reduction in root mean squared error as compared with the unconditioned model. The objective of this study was to develop and evaluate two methods for calibrating predictions from a segmented taper equation, one for dbh and the other for both dbh and an upper-stem diameter measurement.

Data

Data used in this study were from a loblolly pine plantation at the Hill Farm Research Station, Homer, Louisiana. Site index ranged from 61 to 76 ft (base age, 25 years) for the study area. Twenty 0.5-ac blocks, each with a 0.25-ac measurement plot, were established in 1958. By age 7 years, the plots were thinned to 1,000, 600, 300, 200, and 100 trees/ac. Measurements from 133 trees felled in a thinning at age 21 years were used in this study. The dbh ranged from 4.5 to 19.0 in. and total height ranged from 30 to 76 ft. Outside- and inside-bark diameters were taken at 25-in. intervals starting from the stump to the tree tip, totaling 3,233 observations.

Methods

The Max and Burkhart (1976) taper equation can be written in the following modified form:

$$\hat{y}(z) = b_1 z + b_2 z^2 + b_3 I_1 (z - a_1)^2 + b_4 I_2 (z - a_2)^2, \quad (1)$$

where $\hat{y}(z)$ = predicted value of y for a given value of z; $y = d^2/D^2$; D = dbh; d = diameter outside bark (dob) at any given height h; $z = 1 - h/H$ = relative height from the tree tip; H = total height; a_1 and a_2 = join points to be estimated from the data; $I_i = 1$ if $z > a_i$, and 0 otherwise, $I = 1, 2$; and b_j = regression coefficients, $j = 1, 2, 3, 4$.

Note that this taper equation does not result in dbh when $h = 4.5$ ft. Two calibration procedures are proposed, one for dbh only and the other for both dbh and an upper-stem diameter measurement at height h_0.

Calibration for dbh

The constraint that the predicted diameter at breast height is dbh can be expressed as

$$y{*}(z_{BH}) = y(z_{BH}), \quad (2)$$

where $y^* = d^{*2}/D^2$; d^* = calibrated dob; $z_{BH} = 1 - 4.5/H$; $y(z_{BH}) = \text{dbh}^2/\text{dbh}^2 = 1$.

After calibration, Equation 1 can be written as

$$y^*(z) = b_1^* z + b_2 z^2 + b_3 I_1 (z - a_1)^2 + b_4 I_2 (z - a_2)^2, \quad (3)$$

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Table 1. **Parameter estimates for the segmented taper equation.[a]**

Parameter	Estimate	Standard error
Outside bark		
b_1	0.2415	0.0579
b_2	1.4095	0.2505
b_3	−1.2261	0.2350
b_4	7.3182	0.5196
a_1	0.2566	0.0373
a_2	0.7963	0.0078
Inside bark		
b_1	0.2033	0.0438
b_2	1.2456	0.1574
b_3	−1.4223	0.1442
b_4	21.2897	1.6763
a_1	0.3140	0.0271
a_2	0.8709	0.0048

[a] $\hat{y}(z) = b_1z + b_2z^2 + b_3I_1(z - a_1)^2 + b_4I_2(z - a_2)^2$, where $y = d^2/D^2$; D = dbh; d = diameter outside or inside bark at any given height h; $z = 1 - h/H$; H = total height; and $I_i = 1$ *if* $z > a_i$, and 0 otherwise, i = 1, 2.

where

$$b_1^* = \frac{y(z_{BH}) - \hat{y}(z_{BH}) + b_1z_{BH}}{z_{BH}}. \tag{4}$$

Note that there is more than one way to constrain the taper equation. The parameter b_1 was modified in this case, but other parameters could be selected instead. Furthermore, alternative methods such as a mixed-model approach could also be used to account for variability.

Calibration for dbh and Upper-Stem Diameter

In addition to the dbh constraint (Equation 2), the predicted diameter at a height h_0 also has to equal d_0, which is the dob measured at that height. The second constraint is expressed as

$$y^*(z_0) = y_0 \tag{5}$$

where $z_0 = 1 - h_0/H$; $y_0 = d_0^2/D^2$.

The calibrated taper equation is as follows:

$$y^*(z) = b_1^*z + b_2^*z^2 + b_3I_1(z - a_1)^2 + b_4I_2(z - a_2)^2, \tag{6}$$

where

$$b_2^* = \frac{\left\{\begin{matrix} z_0[y(z_{BH}) - \hat{y}(z_{BH}) + b_2z_{BH}^2] \\ + z_{BH}[\hat{y}(z_0) - y_0 - b_2z_0^2] \end{matrix}\right\}}{z_0z_{BH}(z_{BH} - z_0)}, \tag{7}$$

and

$$b_1^* = \frac{y(z_{BH}) + z_{BH}^2(b_2 - b_2^*) - \hat{y}(z_{BH}) + b_1z_{BH}}{z_{BH}}. \tag{8}$$

Results and Discussion

Regression estimates for parameters of Equation 1 are presented in Table 1. The fit was good, with an R^2 value of 0.967 and root mean squared error of 0.058 in. However, Figure 1 shows that variations in tree taper resulted in a wide band that encompassed the regression line. Two examples of calibration for dbh and an upper-stem diameter are also shown in Figure 1.

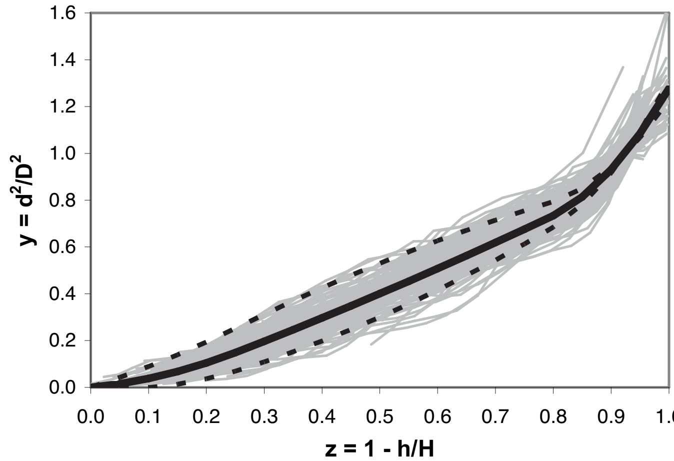

Figure 1. Graphs of observed (gray curves) and predicted (black curve) outside-bark diameters. Each dotted curve denotes tree taper after being calibrated for dbh and an upper-stem diameter.

Table 2. Evaluation statistics[a] for the regression model before and after calibration (for dbh only and for both dbh and a diameter measured at z_0).

Method	z_0	MD	MAD	FI
Regression		−0.039	0.413	0.967
Calibrated for dbh		***−0.029***[b]	***0.394***	***0.969***
Calibrated for dbh and a diameter measured at z_0	0.1	**0.033**	**0.384**	**0.967**
	0.2	0.045	***0.351***	***0.970***
	0.3	***0.014***	***0.336***	***0.972***
	0.4	***0.008***[c]	***0.296***	***0.977***
	$z_{1/2}$[d]	***−0.021***	***0.288***	***0.981***
	0.5	***−0.035***	***0.295***	***0.981***
	0.6	***−0.035***	***0.320***	***0.976***
	0.7	***−0.003***	***0.373***	***0.967***
	0.8	−0.067	0.576	0.920
	0.9	0.848	1.510	0.478

[a] MD = mean difference between observed and predicted bole diameters, MAD = mean absolute difference, and FI = fit index (analogous to R^2 in linear regression).
[b] Statistics in bold, italic are better than those from the unadjusted regression model.
[c] An underlined number denotes the best for that evaluation statistic (column).
[d] $z_{1/2}$ = relative distance from the tip to the midpoint between the tip and breast height.

The performance of the taper model before and after calibration was evaluated by use of three evaluation statistics: mean difference (MD), between observed and predicted bole diameters); mean absolute difference (MAD); and fit index (FI), which is analogous to R^2 in linear regression. MD measures the average bias of the prediction, and MAD measures the magnitude of the bias.

Compared with the unadjusted regression equation, the calibration for dbh improved all evaluation statistics (Table 2). Absolute value of MD decreased from 0.039 to 0.029 (or 26%) and MAD decreased from 0.413 to 0.394 (5%), whereas FI slightly increased from 0.967 to 0.969 (0.2%).

The success from calibration for both dbh and an upper-stem diameter depended on z_0, the relative distance from the tip to where the upper-stem diameter was measured. Table 2 shows that the calibration improved taper prediction when $z_0 < 0.8$ and resulted in inferior predictions when based on a diameter taken near the ground level (z_0 = 0.8 or 0.9). Because the calibrated diameter was constrained at the tree tip (zero) and breast height (dbh), it makes sense that the optimal point to measure a diameter for calibration purposes should be at $z_{1/2}$, or the midpoint between the tree tip and breast height. This was indeed the case because it produced the best

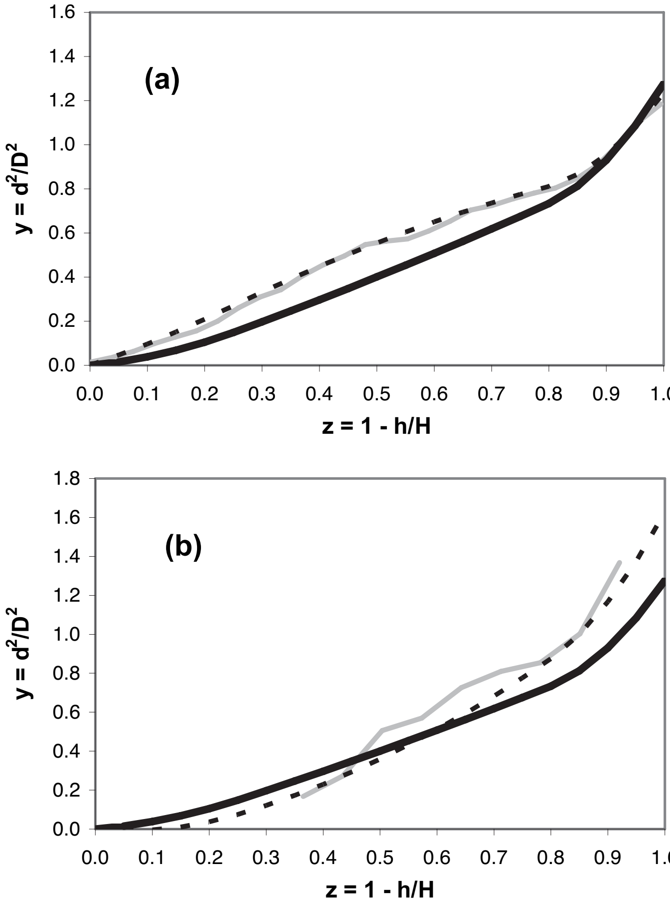

Figure 2. Graphs of observed (gray) and predicted (black) outside-bark diameters for (a) tree 431 and (b) tree 79. Each dotted curve denotes tree taper after being calibrated for dbh and the upper-stem diameter at $z_0 = 0.43$.

MAD and FI values among all methods (Table 2). Compared with the unadjusted taper model, MD and MAD decreased by 46 and 30%, respectively, and R^2 increased by 1.4%.

Figure 2a shows that calibration can be effective when taper of the entire stem follows the same smooth trend. The calibrated curve in Figure 2b, in contrast, did not do as good a job because the observed tree profile was uneven.

The foregoing results were based on the assumption that the upper-stem diameter (d_0) was measured without error. A sensitivity analysis was conducted in which d_0 of all trees in the data set was replaced with $(1 + \varepsilon)d_0$, where ε is a relative diameter measurement error. As expected, the performance of the calibrated model deteriorated as the magnitude of ε increased. The MAD and FI values of the unadjusted and adjusted models were approximately equivalent when ε reached $\pm 7\%$. Calibration in cases where measurement error exceeds that level is therefore not warranted.

Validation

Outside-Bark Diameters

Outputs from the Max and Burkhart (1976) outside-bark taper equation with its original parameter estimates were adjusted by use of the two calibration methods. Calibration based on dbh slightly improved the FI from 0.953 to 0.959 (Table 3). Calibration based on both dbh and a diameter measured at $z_0 \leq 0.8$ further improved taper prediction. The best measuring point for upper-stem diameter

Table 3. Evaluation statistics for the Max and Burkhart (1976) model before and after calibration (for dbh only and for both dbh and a dob measured at $z_{1/2}$).

Method	MD	MAD	FI
Outside bark			
Max and Burkhart (1976)	−0.292	0.476	0.953
Calibrated for dbh	0.176	0.458	0.959
Calibrated for dbh and a diameter outside bark measured at $z_{1/2}$	−0.091	0.304	0.975
Inside bark			
Max and Burkhart (1976)	−0.068	0.381	0.967
Calibrated for dbh	−0.016	0.406	0.963
Calibrated for dbh and a diameter outside bark measured at $z_{1/2}$	0.048	0.325	0.974

MD = mean difference between observed and predicted bole diameters, MAD = mean absolute difference, and FI = fit index.

used in calibration was also $z_{1/2}$, the midpoint between breast height and the tree tip. Compared with the original Max and Burkhart (1976) equation, this optimum calibration technique decreased MD from −0.292 to −0.091 (or 68.8%) and MAD from 0.476 to 0.304 (36.1%) and increased FI from 0.953 to 0.975 (2.3%).

Inside-Bark Diameters

It is assumed that only outside-bark diameter measurements are available; inside-bark diameters therefore have to be predicted from their outside-bark counterparts. Inside-bark dbh (dbhib) was predicted from dbh based on the following regression equation:

$$\text{dbhib} = -0.37195 + 0.90384\ \text{dbh} \tag{9}$$

$$n = 132; \quad s_{yx} = 0.32\ \text{in.}; \quad R^2 = 0.983.$$

Inside-bark diameter (dib) at any point on the tree bole was predicted from dob as follows:

$$\text{dib} = 0.02612 + 0.89287\ \text{dob} \tag{10}$$

$$n = 3233; \quad s_{yx} = 0.29\ \text{in.}; \quad R^2 = 0.989$$

The calibration procedures for inside-bark diameters were similar to those for outside-bark diameters, with the following modifications:

- The dbh was replaced with dbhib, predicted from Equation 9.
- The upper-stem diameter d_0 from Equation 5 was replaced with dib, predicted from Equation 10.
- The $y(z_{\mathrm{BH}})$ from Equations 4, 7, and 8 was defined in this case as $(\text{dbhib}/\text{dbh})^2$.

Outputs from the Max and Burkhart (1976) inside-bark taper equation with its original parameter estimates were adjusted by use of the two calibration methods. Calibration based on dbh decreased the FI from 0.967 to 0.963 and increased MAD from 0.381 to 0.406 (Table 3). It is apparent from these results that the adjusted diameters actually became worse instead of better.

The improvement in taper prediction caused by calibration based on both dbh and an outside-bark diameter measured at $z_{1/2}$ was modest. Compared with the original Max and Burkhart (1976) equation, this optimum calibration technique decreased MD from −0.068 to 0.048 (or 29.4%), and MAD from 0.381 to 0.325 (1.5%) and increased FI from 0.967 to 0.974 (0.7%). The adjustment was not as successful as its outside-bark counterpart because both dbhib and dib at $z_{1/2}$ had to be predicted from dbh and dob, respectively.

A Numerical Example

Let us consider tree 431 depicted in Figure 2a. This tree measured 7.0 in. in dbh and 56.6 ft. in total height. The first constraint involved computation of $z_{BH} = 1 - 4.5/56.6 = 0.9205$, and $\hat{y}(z_{BH}) = 0.9890$ from Equation 1 with parameters from Table 1.

The second constraint, which used an upper-stem diameter $d =$ 4.8 in. measured at $h = 31.5$ ft., yielded $z_0 = z_{1/2} = 1 - 31.5/56.6 = 0.4432$, $y_0 = 4.8^2/7.0^2 = 0.4955$, and $\hat{y}(z_0) = 0.3412$ from Equation 1 with parameters from Table 1.

Parameters b_1 and b_2 were then calibrated from Equations 7 and 8 as follows:

$$b_2^* = \frac{\left\{\begin{array}{l} 0.4432[1 - 0.9890 + 1.4095(0.9205)^2] \\ \quad + 0.9205[0.3412 - 0.4955 - 1.4095(0.4432)^2] \end{array}\right\}}{(0.4432)(0.9205)(0.9205 - 0.4432)}$$

$$= 0.7050,$$

and

$$b_1^* = \frac{\left\{\begin{array}{l} 1 + (0.9295)^2(1.4095 - 0.7050) \\ \quad - 0.9890 + 0.2415(0.9205) \end{array}\right\}}{0.9205} = 0.9019.$$

The calibrated (dotted) taper curve for outside-bark diameters in Figure 2a was finally constructed from Equation 6 by use of the aforementioned values of b_1^* and b_2^*.

Summary and Conclusions

Recent advances in laser technology help make possible accurate and affordable measurements of upper-stem diameters. These measurements can be used to calibrate results from a taper equation to improve the accuracy of diameter predictions along the tree bole. Felled-tree data from a loblolly pine (*P. taeda* L.) plantation was used to evaluate two methods for calibrating outputs from a segmented taper equation with parameters either obtained from the data in this study or originally published by Max and Burkhart (1976).

For outside-bark diameters, although a simple calibration for dbh gave desirable results, a better calibration involving both dbh and an upper-stem diameter provided significant improvements in predicting tree taper. Results varied depending on where the diameter was measured, with optimum gains obtained when the upper-stem diameter was measured at the midpoint between breast height and the tree tip. For inside-bark diameters, the calibration for dbhib actually produced inferior predictions, whereas the calibration based on both dbh and an upper-stem diameter offered only modest improvements over the unadjusted predictions. In this case, the taper equation was constrained to pass through dbhib and dib at $z_{1/2}$; both were not measured but had to be predicted from dbh and dob, respectively.

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Appendix

Calibration for dbh

First, z_{BH} is substituted for z in Equations 3 and 1. Taking the difference of the resulting two equations yields

$$y^*(z_{BH}) - \hat{y}(z_{BH}) = b_1 z_{BH} - b_1^* z_{BH}. \tag{A1}$$

Because $y^*(z_{BH}) = y(z_{BH})$ (from Equation 2), b_1^* is given by

$$b_1^* = \frac{y(z_{BH}) - \hat{y}(z_{BH}) + b_1 z_{BH}}{z_{BH}}, \tag{A2}$$

where $y(z_{BH}) = 1$ for outside-bark diameters and $(\text{dbhib/dbh})^2$ for inside-bark diameters.

Calibration for dbh and an Upper-Stem Diameter

Substituting z_{BH} for z in Equations 6 and 1, and then taking the difference of the resulting two equations yields

$$y(z_{BH}) - \hat{y}(z_{BH}) = z_{BH}(b_1^* - b_1) + z_{BH}^2(b_2^* - b_2). \tag{A3}$$

In a similar operation, substituting z_0 for z results in

$$y_0 - \hat{y}(z_0) + z_0(b_1^* - b_1) + z_0^2(b_2^* - b_2). \tag{A4}$$

The solution of this system of Equations A3 and A4 is

$$b_2^* = \frac{\left\{\begin{array}{l} z_0[y(z_{BH}) - \hat{y}(z_{BH}) + b_2 z_{BH}^2] \\ \quad + z_{BH}[\hat{y}(z_0) - y_0 - b_2 z_0^2] \end{array}\right\}}{z_0 z_{BH}(z_{BH} - z_0)}, \tag{A5}$$

and

$$b_1^* = \frac{y(z_{BH}) + z_{BH}^2(b_2 - b_2^*) - \hat{y}(z_{BH}) + b_1 z_{BH}}{z_{BH}}. \tag{A6}$$