

Full-Stack Quantum Computing

1 Quantum Tunneling

Quantum tunneling refers to the effect that microscopic particles have a chance of going through a region with higher potentials. In this problem, we are going to explore the mathematical formalism that makes quantum tunneling possible.

- (a) Consider the 1D time independent Schrodinger's Equation

$$-\frac{\hbar}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

Show that the ansatz $\psi(x) = Ae^{-ikx} + Be^{ikx}$ is a solution to this equation. What's k? You can think of the two components as the parts of the wave moving to the right vs moving to the left.

- (b) Consider a potential where

$$V(x) = \begin{cases} V_0, & \text{if } 0 < x < L \\ 0, & \text{if } x < 0 \text{ or } L < x \end{cases} \quad (1)$$

Sketch the potential, and write down the generalized wave functions for the following:

- i. $x < 0$
- ii. $0 < x < L$
- iii. $L < x$

Hint: it should look like $Ae^{ik_1x} + Be^{-ik_1x}$ for region i, and $Ce^{k_2x} + De^{-k_2x}$ for region ii.

- (c) Consider now that we are shooting a particle from left to right (x is increasing). In this case, you can set the coefficient of the left moving component to be 0 in region 3. Consider the fact that wave functions need to be continuous. Write down boundary conditions at 0 and L.
- (d) Use the boundary conditions at L to determine the relationship between C and D. In particular, find $\frac{|C|}{|D|}$, and discuss under what kind of condition can we get $|C| \gg |D|$ and approximate D to 0.
- (e) Take the approximation that D is 0, and find the reflection coefficient defined as $\frac{|B|^2}{|A|^2}$. In classical cases, we would expect the reflection coefficient to be 1, since when we throw a ball at the wall, it's always going to bounce back. Show that in this case, the reflection coefficient is less than 1. Explain what's happening.
- (f) Sketch qualitatively the wave function in all three regions.

2 Hermitian and Unitary

In class we covered that a Hermitian H and a Unitary U are defined as

$$H = H^\dagger \quad UU^\dagger = I$$

Where dagger A^\dagger denotes the complex conjugate transpose of a matrix

$$A^\dagger = \overline{A^T}$$

(a) Show the following matrices are Hermitian and Unitary using the definitions above

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(b) Compute the eigenvalue decomposition of $X = V\Lambda V^{-1}$. Can you represent the decomposition components V and Λ using only matrices we defined above?