0.1 Syntax

$$
\begin{array}{ll}
Ar & ::= s_k \mid (x :_U M) \to Ar \\
Pos & ::= X\overrightarrow{m} \mid (x :_s M) \to Pos \mid (x :_s M) \multimap Pos \\
Co_U & ::= X\overrightarrow{m} \mid (\_ :_U P) \to Co'_U \mid (x :_U M) \to Co'_U \\
Co'_U & ::= X\overrightarrow{m} \mid (\_ :_U P) \multimap Co'_U \mid (x :_U M) \multimap Co'_U \\
Co_L & ::= X\overrightarrow{m} \mid (\_ :_s P) \to Co'_L \mid (x :_U M) \to Co'_L \mid (\_ :_L M) \to Co'_L \\
Co'_L & ::= X\overrightarrow{m} \mid (\_ :_s P) \multimap Co'_L \mid (x :_U M) \multimap Co'_L \mid (\_ :_L M) \multimap Co'_L
\end{array}
$$

0.2 Introduction Rules

$$
\frac{\begin{array}{c}(\forall i = 1...n) \qquad arity(A, s_k) \qquad constructor_s(C_i, X) \\ |\Gamma| \qquad \Gamma \vdash A : U_{k+1} \qquad \Gamma, X :_U A \vdash C_i : U_k\end{array}}{\Gamma \vdash Ind_s(X : A)\{C_1|...|C_n\} : A}
\qquad
\frac{|\Gamma| \qquad \Gamma \vdash Ind_s(X : A)\{C_1|...|C_n\} : A \qquad 1 \le i \le n}{\Gamma \vdash Constr(i, I_s) : C_i[I_s/X]}
$$

0.3 Non-Dependent Elimination

$$
\begin{array}{ll}
((x :_{s'} M) \to P)\{s\} & = U \\
((x :_{s'} M) \multimap P)\{s\} & = L \\
(X\ \overrightarrow{m})\{s\} & = s
\end{array}
$$

$$
\begin{array}{lll}
((\_ :_U P) \to C)\{X, Q, s\} & = (\_ :_U P) \to (\_ :_{s'} P[Q/X]) \multimap C\{X, Q, s\} & (where\ s' = P\{s\}) \\
((\_ :_U P) \multimap C)\{X, Q, s\} & = (\_ :_U P) \multimap (\_ :_{s'} P[Q/X]) \multimap C\{X, Q, s\} & (where\ s' = P\{s\}) \\
((\_ :_L P) \to C)\{X, Q, s\} & = (\_ :_{s'} P[Q/X]) \to C\{X, Q, s\} & (where\ s' = P\{s\}) \\
((\_ :_L P) \multimap C)\{X, Q, s\} & = (\_ :_{s'} P[Q/X]) \multimap C\{X, Q, s\} & (where\ s' = P\{s\}) \\
((x :_s M) \to C)\{X, Q, s\} & = (x :_s M) \to C\{X, Q, s\} & \\
((x :_s M) \multimap C)\{X, Q, s\} & = (x :_s M) \multimap C\{X, Q, s\} & \\
(X\ \overrightarrow{m})\{X, Q, s\} & = Q\ \overrightarrow{m} &
\end{array}
$$

$$
\begin{array}{ll}
((\_ :_s P) \to C)\{X, Q\} & = (\_ :_s P) \multimap C\{X, Q\} \\
((\_ :_s P) \multimap C)\{X, Q\} & = (\_ :_s P) \multimap C\{X, Q\} \\
((x :_s M) \to C)\{X, Q\} & = (x :_s M) \multimap C\{X, Q\} \\
((x :_s M) \multimap C)\{X, Q\} & = (x :_s M) \multimap C\{X, Q\} \\
(X\ \overrightarrow{m})\{X, Q\} & = Q\ \overrightarrow{m}
\end{array}
$$

$$
\frac{\begin{array}{c}(\forall i =\ 1...n) \qquad |\Gamma_2| \qquad \Gamma_1 \ddagger \Gamma_2 \ddagger \Gamma \\ \Gamma_1 \vdash c : (I_s\ \overrightarrow{a}) \qquad \Gamma_2 \vdash Q : (\overrightarrow{x} :_U \overrightarrow{A}) \to s' \qquad \Gamma_2 \vdash f_i : C_i\{I_s, Q, s'\}\end{array}}{\Gamma \vdash Rec(c, Q)\{f_1|...|f_n\} : (Q\overrightarrow{a})}
$$

$$
\frac{\begin{array}{c}(\forall i =\ 1...n) \qquad \Gamma_1 \ddagger \Gamma_2 \ddagger \Gamma \\ \Gamma_1 \vdash c : (I_s\ \overrightarrow{a}) \qquad \overline{\Gamma_2} \vdash Q : (\overrightarrow{x} :_U \overrightarrow{A}) \to s' \qquad \Gamma_2 \vdash f_i : C_i\{I_s, Q\}\end{array}}{\Gamma \vdash Elim(c, Q)\{f_1|...|f_n\} : (Q\overrightarrow{a})}
$$

0.4 Dependent Elimination

$$
\begin{array}{ll}
((x :_s M) \to P)\{Q, p\} & = (x :_s M) \to P\{Q, (p\ x)\} \\
((x :_s M) \multimap P)\{Q, p\} & = (x :_s M) \multimap P\{Q, (p\ x)\} \\
(X\ \overrightarrow{m})\{Q, p\} & = Q\ \overrightarrow{m}\ p
\end{array}
$$

$$
\begin{array}{lll}
((\_ :_U P) \to C)\{X, Q, s, c\} & = (p :_U P) \to (\_ :_{s'} P\{Q, p\}) \multimap C\{X, Q, s, (c\ p)\} & (where\ s' = P\{s\}) \\
((\_ :_U P) \multimap C)\{X, Q, s, c\} & = (p :_U P) \multimap (\_ :_{s'} P\{Q, p\}) \multimap C\{X, Q, s, (c\ p)\} & (where\ s' = P\{s\}) \\
((x :_U M) \to C)\{X, Q, s, c\} & = (x :_U M) \to C\{X, Q, s, (c\ x)\} & \\
((x :_U M) \multimap C)\{X, Q, s, c\} & = (x :_U M) \multimap C\{X, Q, s, (c\ x)\} & \\
(X\ \overrightarrow{m})\{X, Q, s, c\} & = (Q\ \overrightarrow{m}\ c) &
\end{array}
$$

$$
\begin{array}{ll}
((\_ :_U P) \to C)\{X, Q, c\} & = (p :_U P) \multimap C\{X, Q, (c\ p)\} \\
((\_ :_U P) \multimap C)\{X, Q, c\} & = (p :_U P) \multimap C\{X, Q, (c\ p)\} \\
((x :_U P) \to C)\{X, Q, c\} & = (x :_U M) \multimap C\{X, Q, (c\ x)\} \\
((x :_U P) \multimap C)\{X, Q, c\} & = (x :_U M) \multimap C\{X, Q, (c\ x)\} \\
(X\ \overrightarrow{m})\{X, Q, c\} & = (Q\ \overrightarrow{m}\ c)
\end{array}
$$

$$
\frac{\begin{array}{c}(\forall i = 1...n) \qquad |\Gamma_2| \qquad \Gamma_1 \ddagger \Gamma_2 \ddagger \Gamma \\ \Gamma_1 \vdash c : (I_U\ \overrightarrow{a}) \qquad \Gamma_2 \vdash Q : (\overrightarrow{x} :_U \overrightarrow{A}) \to (\_ :_U I_U\ \overrightarrow{x}) \to s' \qquad \Gamma_2 \vdash f_i : C_i\{I_U, Q, s', Constr(i, I_U)\}\end{array}}{\Gamma \vdash Rec(c, Q)\{f_1|...|f_n\} : (Q\ \overrightarrow{a}\ c)}
$$

$$
\frac{\begin{array}{c}(\forall i = 1...n) \qquad \Gamma_1 \ddagger \Gamma_2 \ddagger \Gamma \\ \Gamma_1 \vdash c : (I_U\ \overrightarrow{a}) \qquad \overline{\Gamma_2} \vdash Q : (\overrightarrow{x} :_U \overrightarrow{A}) \to (\_ :_U I_U\ \overrightarrow{x}) \to s' \qquad \Gamma_2 \vdash f_i : C_i\{I_U, Q, Constr(i, I_U)\}\end{array}}{\Gamma \vdash Elim(c, Q)\{f_1|...|f_n\} : (Q\ \overrightarrow{a}\ c)}
$$

0.5 Conversion

$$
\begin{array}{ll}
((\_ :_U P) \to C)[X, F, f] & = \lambda p.C[X, F, (f\ p\ \lambda \overrightarrow{x}.(F\ \overrightarrow{m}\ (p\ \overrightarrow{x})))] \\
((\_ :_U P) \multimap C)[X, F, f] & = \lambda p.C[X, F, (f\ p\ \lambda \overrightarrow{x}.(F\ \overrightarrow{m}\ (p\ \overrightarrow{x})))] \\
((\_ :_L P) \to C)[X, F, f] & = \lambda p.C[X, F, (f\ \lambda \overrightarrow{x}.(F\ \overrightarrow{m}\ (p\ \overrightarrow{x})))] \\
((\_ :_L P) \multimap C)[X, F, f] & = \lambda p.C[X, F, (f\ \lambda \overrightarrow{x}.(F\ \overrightarrow{m}\ (p\ \overrightarrow{x})))] \\
((x :_s M) \to C)[X, F, f] & = \lambda x.C[X, F, (f\ x)] \\
((x :_s M) \multimap C)[X, F, f] & = \lambda x.C[X, F, (f\ x)] \\
(X\ \overrightarrow{m})[X, F, f] & = f
\end{array}
$$

$$
\begin{array}{ll}
((\_ :_s P) \to C)[X, f] & = \lambda p.C[X, (f\ p)] \\
((\_ :_s P) \multimap C)[X, f] & = \lambda p.C[X, (f\ p)] \\
((x :_s M) \to C)[X, f] & = \lambda x.C[X, (f\ x)] \\
((x :_s M) \multimap C)[X, f] & = \lambda x.C[X, (f\ x)] \\
(X\ \overrightarrow{m})[X, f] & = f
\end{array}
$$

$$
Fun\_Rec(I_s, Q, \overrightarrow{f}) = \lambda \overrightarrow{x}.\lambda c.Rec(c, Q)\{\overrightarrow{f}\}
$$

$$
\begin{array}{lll}
Rec((Constr(i, I_s)\ \overrightarrow{m}), Q)\{\overrightarrow{f}\} & \to_\iota & (C_i[I_s, Fun\_Rec(I_s, Q, \overrightarrow{f}), f_i]\ \overrightarrow{m}) \\
Elim((Constr(i, I_s)\ \overrightarrow{m}), Q)\{\overrightarrow{f}\} & \to_\iota & (C_i[I_s, f_i]\ \overrightarrow{m})
\end{array}
$$