A Two Level Linear Dependent Type Theory

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November 11, 2022

1 Syntax

variables	x, y, z, p		
sorts	s, r, t	$::=$	$\mathrm{U} \mid \mathrm{L}$
terms	m, n, A, B, C, H, P	$::=$	$x \mid s$
		$\mid$	$\Pi_t(x : A).B \mid \Pi_t\{x : A\}.B \mid \Sigma_t(x : A).B \mid \Sigma_t\{x : A\}.B \mid A \mathbin{\&_t} B$
		$\mid$	$\lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid \langle m, n\rangle_t \mid \{m, n\}_t \mid (m, n)_t$
		$\mid$	$m\ n \mid \mathrm{R}^{\Sigma}_{[z]A}(m, [x, y]n) \mid \pi_1\, m \mid \pi_2\, m$
		$\mid$	$m \equiv_A n \mid \mathrm{refl}\ m \mid \mathrm{R}^{\equiv}_{[x,y,p]A}([z]H, P) \mid \square$
values	u, v	$::=$	$x \mid \lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid \langle u, v\rangle_t \mid \{v, m\}_t \mid (m, n)_t$

2 Static Fragment

Sort Order

$$\frac{}{\mathrm{U} \sqsubseteq s} \qquad \frac{}{\mathrm{L} \sqsubseteq \mathrm{L}}$$

Static Context

$$\frac{}{\epsilon \vdash} \qquad \frac{\Gamma \vdash \qquad \Gamma \vdash A : s \qquad x \in \mathit{fresh}(\Gamma)}{\Gamma, x : A \vdash}$$

Static Typing

$$\frac{\Gamma \vdash}{\Gamma \vdash s : \mathrm{U}} \qquad \frac{\Gamma, x : A \vdash}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t(x : A).B : t} \qquad \frac{\Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t\{x : A\}.B : t}$$

$$\frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t(x : A).m : \Pi_t(x : A).B} \qquad \frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B} \qquad \frac{\Gamma \vdash m : \Pi_t(x : A).B \qquad \Gamma \vdash n : A}{\Gamma \vdash m\ n : B[n/x]}$$

$$\frac{\Gamma \vdash m : \Pi_t\{x : A\}.B \qquad \Gamma \vdash n : A}{\Gamma \vdash m\ n : B[n/x]} \qquad \frac{s \sqsubseteq t \qquad r \sqsubseteq t \qquad \Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t(x : A).B : t}$$

$$\frac{s \sqsubseteq t \qquad \Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t\{x : A\}.B : t} \qquad \frac{\Gamma \vdash \Sigma_t(x : A).B \qquad \Gamma \vdash m : A \qquad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \langle m, n\rangle_t : \Sigma_t(x : A).B}$$

$$\frac{\Gamma \vdash \Sigma_t\{x : A\}.B \qquad \Gamma \vdash m : A \qquad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \{m, n\}_t : \Sigma_t\{x : A\}.B}$$

$$\frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \qquad \Gamma \vdash m : \Sigma_t(x : A).B \qquad \Gamma, x : A, y : B \vdash n : C[\langle x, y\rangle_t/z]}{\Gamma \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x,y]n) : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \qquad \Gamma \vdash m : \Sigma_t\{x : A\}.B \qquad \Gamma, x : A, y : B \vdash n : C[\{x, y\}_t/z]}{\Gamma \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x,y]n) : C[m/z]}$$

$$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash B : r}{\Gamma \vdash A \,\&_t B : t} \qquad \frac{\Gamma \vdash m : A \qquad \Gamma \vdash n : B}{\Gamma \vdash (m, n)_t : A \,\&_t B} \qquad \frac{\Gamma \vdash m : A \,\&_t B}{\Gamma \vdash \pi_1\, m : A} \qquad \frac{\Gamma \vdash m : A \,\&_t B}{\Gamma \vdash \pi_2\, m : B}$$

$$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash m : A \qquad \Gamma \vdash n : A}{\Gamma \vdash m \equiv_A n : \mathrm{U}} \qquad \frac{\Gamma \vdash m : A}{\Gamma \vdash \mathrm{refl}\ m : m \equiv_A m}$$

$$\frac{\Gamma, x : A, y : A, p : x \equiv_A y \vdash B : s \qquad \Gamma, z : A \vdash H : B[z/x, z/y, \mathrm{refl}\ z/p] \qquad \Gamma \vdash P : m \equiv_A n}{\Gamma \vdash \mathrm{R}^{\equiv}_{[x,y,p]B}([z]H, P) : B[m/x, n/y, P/p]}$$

$$\frac{\Gamma \vdash B : s \qquad \Gamma \vdash m : A \qquad A \simeq B}{\Gamma \vdash m : B}$$

3 Dynamic Fragment

Dynamic Context

$$\frac{}{\epsilon;\epsilon\vdash}\qquad \frac{\Gamma;\Delta\vdash \qquad \Gamma\vdash A:s \qquad x\in \mathit{fresh}(\Gamma)}{\Gamma,x:A;\Delta,x:_s A\vdash}\qquad \frac{\Gamma;\Delta\vdash \qquad \Gamma\vdash A:s \qquad x\in \mathit{fresh}(\Gamma)}{\Gamma,x:A;\Delta\vdash}$$

Context Merge

$$\frac{}{\epsilon\uplus\epsilon=\epsilon}\qquad \frac{\Delta_1\uplus\Delta_2=\Delta \qquad x\in \mathit{fresh}(\Delta)}{(\Delta_1,x:_{\mathrm{U}}A)\uplus(\Delta_2,x:_{\mathrm{U}}A)=(\Delta,x:_{\mathrm{U}}A)}$$

$$\frac{\Delta_1\uplus\Delta_2=\Delta \qquad x\in \mathit{fresh}(\Delta)}{(\Delta_1,x:_{\mathrm{L}}A)\uplus\Delta_2=(\Delta,x:_{\mathrm{L}}A)}\qquad \frac{\Delta_1\uplus\Delta_2=\Delta \qquad x\in \mathit{fresh}(\Delta)}{\Delta_1\uplus(\Delta_2,x:_{\mathrm{L}}A)=(\Delta,x:_{\mathrm{L}}A)}$$

Context Constraint

$$\frac{}{\epsilon\rhd s}\qquad \frac{\Delta\rhd\mathrm{U}}{\Delta,x:_{\mathrm{U}}A\rhd\mathrm{U}}\qquad \frac{\Delta\rhd\mathrm{L}}{\Delta,x:_s A\rhd\mathrm{L}}$$

Dynamic Typing

$$\frac{\Gamma,x:A;\Delta,x:_s A\vdash \qquad \Delta\rhd\mathrm{U}}{\Gamma,x:A;\Delta,x:_s A\vdash x:A}\qquad \frac{\Gamma,x:A;\Delta,x:_s A\vdash m:B \qquad \Delta\rhd t}{\Gamma;\Delta\vdash \lambda_t(x:A).m:\Pi_t(x:A).B}\qquad \frac{\Gamma,x:A;\Delta\vdash m:B \qquad \Delta\rhd t}{\Gamma;\Delta\vdash \lambda_t\{x:A\}.m:\Pi_t\{x:A\}.B}$$

$$\frac{\Gamma;\Delta_1\vdash m:\Pi_t(x:A).B \qquad \Gamma;\Delta_2\vdash n:A}{\Gamma;\Delta_1\uplus\Delta_2\vdash m\ n:B[n/x]}\qquad \frac{\Gamma;\Delta\vdash m:\Pi_t\{x:A\}.B \qquad \Gamma\vdash n:A}{\Gamma;\Delta\vdash m\ n:B[n/x]}$$

$$\frac{\Gamma\vdash\Sigma_t(x:A).B:t \qquad \Gamma;\Delta_1\vdash m:A \qquad \Gamma;\Delta_2\vdash n:B[m/x]}{\Gamma;\Delta_1\uplus\Delta_2\vdash\langle m,n\rangle_t:\Sigma_t(x:A).B}$$

$$\frac{\Gamma\vdash\Sigma_t\{x:A\}.B:t \qquad \Gamma;\Delta\vdash m:A \qquad \Gamma\vdash n:B[m/x]}{\Gamma;\Delta\vdash\{m,n\}_t:\Sigma_t\{x:A\}.B}$$

$$\frac{\Gamma,z:\Sigma_t(x:A).B\vdash C:s \qquad \Gamma;\Delta_1\vdash m:\Sigma_t(x:A).B \qquad \Gamma,x:A,y:B;\Delta_2,x:_{r1}A,y:_{r2}B\vdash n:C[\langle x,y\rangle_t/z]}{\Gamma;\Delta_1\uplus\Delta_2\vdash \mathrm{R}^{\Sigma}_{[z]C}(m,[x,y]n):C[m/z]}$$

$$\frac{\Gamma,z:\Sigma_t\{x:A\}.B\vdash C:s \qquad \Gamma;\Delta_1\vdash m:\Sigma_t\{x:A\}.B \qquad \Gamma,x:A,y:B;\Delta_2,x:_r A\vdash n:C[\{x,y\}_t/z]}{\Gamma;\Delta_1\uplus\Delta_2\vdash \mathrm{R}^{\Sigma}_{[z]C}(m,[x,y]n):C[m/z]}$$

$$\frac{\Gamma;\Delta\vdash m:A \qquad \Gamma;\Delta\vdash n:B \qquad \Delta\rhd t}{\Gamma;\Delta\vdash(m,n)_t:A\,\&_t B}\qquad \frac{\Gamma;\Delta\vdash m:A\,\&_t B}{\Gamma;\Delta\vdash\pi_1\,m:A}\qquad \frac{\Gamma;\Delta\vdash m:A\,\&_t B}{\Gamma;\Delta\vdash\pi_2\,m:B}$$

$$\frac{\Gamma\vdash B:s \qquad \Gamma;\Delta\vdash m:A \qquad A\simeq B}{\Gamma;\Delta\vdash m:B}$$

4 Erasure

Erasure Relation

$$\frac{\Gamma, x : A; \Delta, x :_s A \vdash \qquad \Delta \triangleright \mathrm{U}}{\Gamma, x : A; \Delta, x :_s A \vdash x \sim x : A} \qquad \frac{\Gamma, x : A; \Delta, x :_s A \vdash m \sim m' : B \qquad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t(x : A).m \sim \lambda_t(x : \square).m' : \Pi_t(x : A).B}$$

$$\frac{\Gamma, x : A; \Delta \vdash m \sim m' : B \qquad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t\{x : A\}.m \sim \lambda_t\{x : \square\}.m' : \Pi_t\{x : A\}.B} \qquad \frac{\Gamma; \Delta_1 \vdash m \sim m' : \Pi_t(x : A).B \qquad \Gamma; \Delta_2 \vdash n \sim n' : A}{\Gamma; \Delta_1 \uplus \Delta_2 \vdash m\ n \sim m'\ n' : B[n/x]}$$

$$\frac{\Gamma; \Delta \vdash m \sim m' : \Pi_t\{x : A\}.B \qquad \Gamma \vdash n : A}{\Gamma; \Delta \vdash m\ n \sim m'\ \square : B[n/x]}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).B : t \qquad \Gamma; \Delta_1 \vdash m \sim m' : A \qquad \Gamma; \Delta_2 \vdash n \sim n' : B[m/x]}{\Gamma; \Delta_1 \uplus \Delta_2 \vdash \langle m, n\rangle_t \sim \langle m', n'\rangle_t : \Sigma_t(x : A).B}$$

$$\frac{\Gamma \vdash \Sigma_t\{x : A\}.B : t \qquad \Gamma; \Delta \vdash m \sim m' : A \qquad \Gamma \vdash n : B[m/x]}{\Gamma; \Delta \vdash \{m, n\}_t \sim \{m', \square\}_t : \Sigma_t\{x : A\}.B}$$

$$\frac{\begin{array}{c}\Gamma, z : \Sigma_t(x : A).B \vdash C : s \\ \Gamma; \Delta_1 \vdash m \sim m' : \Sigma_t(x : A).B \qquad \Gamma, x : A, y : B; \Delta_2, x :_{r1} A, y :_{r2} B \vdash n \sim n' : C[\langle x, y\rangle_t/z]\end{array}}{\Gamma; \Delta_1 \uplus \Delta_2 \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) \sim \mathrm{R}^{\Sigma}_{[z]\square}(m', [x, y]n') : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \qquad \Gamma; \Delta_1 \vdash m \sim m' : \Sigma_t\{x : A\}.B \qquad \Gamma, x : A, y : B; \Delta_2, x :_r A \vdash n \sim n' : C[\{x, y\}_t/z]}{\Gamma; \Delta_1 \uplus \Delta_2 \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) \sim \mathrm{R}^{\Sigma}_{[z]\square}(m', [x, y]n') : C[m/z]}$$

$$\frac{\Gamma; \Delta \vdash m \sim m' : A \qquad \Gamma; \Delta \vdash n \sim n' : B \qquad \Delta \triangleright t}{\Gamma; \Delta \vdash (m, n)_t \sim (m', n')_t : A \&_t B} \qquad \frac{\Gamma; \Delta \vdash m \sim m' : A \&_t B}{\Gamma; \Delta \vdash \pi_1\, m \sim \pi_1\, m' : A} \qquad \frac{\Gamma; \Delta \vdash m \sim m' : A \&_t B}{\Gamma; \Delta \vdash \pi_2\, m \sim \pi_2\, m' : B}$$

$$\frac{\Gamma \vdash B : s \qquad \Gamma; \Delta \vdash m \sim m' : A \qquad A \simeq B}{\Gamma; \Delta \vdash m \sim m' : B}$$

5 Static Semantics

Static Reduction

$$\frac{A \leadsto A'}{\Pi_t(x : A).B \leadsto \Pi_t(x : A').B} \qquad \frac{B \leadsto B'}{\Pi_t(x : A).B \leadsto \Pi_t(x : A).B'} \qquad \frac{A \leadsto A'}{\Pi_t\{x : A\}.B \leadsto \Pi_t\{x : A'\}.B}$$

$$\frac{B \leadsto B'}{\Pi_t\{x : A\}.B \leadsto \Pi_t\{x : A\}.B'} \qquad \frac{A \leadsto A'}{\lambda_t(x : A).m \leadsto \lambda_t(x : A').m} \qquad \frac{m \leadsto m'}{\lambda_t(x : A).m \leadsto \lambda_t(x : A).m'}$$

$$\frac{A \leadsto A'}{\lambda_t\{x : A\}.m \leadsto \lambda_t\{x : A'\}.m} \qquad \frac{m \leadsto m'}{\lambda_t\{x : A\}.m \leadsto \lambda_t\{x : A\}.m'} \qquad \frac{m \leadsto m'}{m\ n \leadsto m'\ n} \qquad \frac{n \leadsto n'}{m\ n \leadsto m\ n'}$$

$$\frac{}{(\lambda_t(x : A).m)\ n \leadsto m[n/x]} \qquad \frac{}{(\lambda_t\{x : A\}.m)\ n \leadsto m[n/x]} \qquad \frac{A \leadsto A'}{\Sigma_t(x : A).B \leadsto \Sigma_t(x : A').B}$$

$$\frac{B \leadsto B'}{\Sigma_t(x : A).B \leadsto \Sigma_t(x : A).B'} \qquad \frac{A \leadsto A'}{\Sigma_t\{x : A\}.B \leadsto \Sigma_t\{x : A'\}.B} \qquad \frac{B \leadsto B'}{\Sigma_t\{x : A\}.B \leadsto \Sigma_t\{x : A\}.B'}$$

$$\frac{m \leadsto m'}{\langle m, n\rangle_t \leadsto \langle m', n\rangle_t} \qquad \frac{n \leadsto n'}{\langle m, n\rangle_t \leadsto \langle m, n'\rangle_t} \qquad \frac{m \leadsto m'}{\{m, n\}_t \leadsto \{m', n\}_t} \qquad \frac{n \leadsto n'}{\{m, n\}_t \leadsto \{m, n'\}_t}$$

$$\frac{A \leadsto A'}{\mathrm{R}^{\Sigma}_{[z]A}(m, [x, y]n) \leadsto \mathrm{R}^{\Sigma}_{[z]A'}(m, [x, y]n)} \qquad \frac{m \leadsto m'}{\mathrm{R}^{\Sigma}_{[z]A}(m, [x, y]n) \leadsto \mathrm{R}^{\Sigma}_{[z]A}(m', [x, y]n)}$$

$$\frac{n \leadsto n'}{\mathrm{R}^{\Sigma}_{[z]A}(m, [x, y]n) \leadsto \mathrm{R}^{\Sigma}_{[z]A}(m, [x, y]n')} \qquad \frac{}{\mathrm{R}^{\Sigma}_{[z]A}(\langle m_1, m_2\rangle_t, [x, y]n) \leadsto n[m_1/x, m_2/y]}$$

$$\frac{}{\mathrm{R}^{\Sigma}_{[z]A}(\{m_1, m_2\}_t, [x, y]n) \leadsto n[m_1/x, m_2/y]} \qquad \frac{A \leadsto A'}{A \,\&_t B \leadsto A' \,\&_t B} \qquad \frac{B \leadsto B'}{A \,\&_t B \leadsto A \,\&_t B'} \qquad \frac{m \leadsto m'}{(m, n)_t \leadsto (m', n)_t}$$

$$\frac{n \leadsto n'}{(m, n)_t \leadsto (m, n')_t} \qquad \frac{m \leadsto m'}{\pi_1\, m \leadsto \pi_1\, m'} \qquad \frac{m \leadsto m'}{\pi_2\, m \leadsto \pi_2\, m'} \qquad \frac{}{\pi_1\, (m, n)_t \leadsto m} \qquad \frac{}{\pi_2\, (m, n)_t \leadsto n}$$

$$\frac{A \leadsto A'}{m \equiv_A n \leadsto m \equiv_{A'} n} \qquad \frac{m \leadsto m'}{m \equiv_A n \leadsto m' \equiv_A n} \qquad \frac{n \leadsto n'}{m \equiv_A n \leadsto m \equiv_A n'}$$

$$\frac{A \leadsto A'}{\mathrm{R}^{\equiv}_{[x,y,p]A}([z]H, P) \leadsto \mathrm{R}^{\equiv}_{[x,y,p]A'}([z]H, P)} \qquad \frac{H \leadsto H'}{\mathrm{R}^{\equiv}_{[x,y,p]A}([z]H, P) \leadsto \mathrm{R}^{\equiv}_{[x,y,p]A}([z]H', P)}$$

$$\frac{P \leadsto P'}{\mathrm{R}^{\equiv}_{[x,y,p]A}([z]H, P) \leadsto \mathrm{R}^{\equiv}_{[x,y,p]A}([z]H, P')} \qquad \frac{}{\mathrm{R}^{\equiv}_{[x,y,p]A}([z]H, \mathrm{refl}\ m) \leadsto H[m/z]}$$

Conversion

$$\frac{}{A \simeq A} \qquad \frac{A \simeq B \qquad B \leadsto C}{A \simeq C} \qquad \frac{A \simeq B \qquad C \leadsto B}{A \simeq C}$$

6 Dynamic Semantics

Value

$$\frac{}{x \text{ value}} \qquad \frac{}{\lambda_t(x : A).m \text{ value}} \qquad \frac{}{\lambda_t\{x : A\}.m \text{ value}} \qquad \frac{u \text{ value} \qquad v \text{ value}}{\langle u, v\rangle_t \text{ value}} \qquad \frac{v \text{ value}}{\{v, m\}_t \text{ value}} \qquad \frac{}{(m, n)_t \text{ value}}$$

Dynamic Reduction

$$\frac{m \rightsquigarrow\!\!\!\!\rightarrow m'}{m\ n \rightsquigarrow\!\!\!\!\rightarrow m'\ n} \qquad \frac{n \rightsquigarrow\!\!\!\!\rightarrow n'}{m\ n \rightsquigarrow\!\!\!\!\rightarrow m\ n'} \qquad \frac{v \text{ value}}{(\lambda_t(x : A).m)\ v \rightsquigarrow\!\!\!\!\rightarrow m[v/x]} \qquad \frac{}{(\lambda_t\{x : A\}.m)\ n \rightsquigarrow\!\!\!\!\rightarrow m[n/x]}$$

$$\frac{m \rightsquigarrow\!\!\!\!\rightarrow m'}{\langle m, n\rangle_t \rightsquigarrow\!\!\!\!\rightarrow \langle m', n\rangle_t} \qquad \frac{n \rightsquigarrow\!\!\!\!\rightarrow n'}{\langle m, n\rangle_t \rightsquigarrow\!\!\!\!\rightarrow \langle m, n'\rangle_t} \qquad \frac{m \rightsquigarrow\!\!\!\!\rightarrow m'}{\{m, n\}_t \rightsquigarrow\!\!\!\!\rightarrow \{m', n\}_t} \qquad \frac{m \rightsquigarrow\!\!\!\!\rightarrow m'}{\mathrm{R}^{\Sigma}_{[z]A}(m, [x, y]n) \rightsquigarrow\!\!\!\!\rightarrow \mathrm{R}^{\Sigma}_{[z]A}(m', [x, y]n)}$$

$$\frac{u \text{ value} \qquad v \text{ value}}{\mathrm{R}^{\Sigma}_{[z]A}(\langle u, v\rangle_t, [x, y]n) \rightsquigarrow\!\!\!\!\rightarrow n[u/x, v/y]} \qquad \frac{v \text{ value}}{\mathrm{R}^{\Sigma}_{[z]A}(\{v, m\}_t, [x, y]n) \rightsquigarrow\!\!\!\!\rightarrow n[v/x, m/y]} \qquad \frac{m \rightsquigarrow\!\!\!\!\rightarrow m'}{\pi_1\, m \rightsquigarrow\!\!\!\!\rightarrow \pi_1\, m'}$$

$$\frac{m \rightsquigarrow\!\!\!\!\rightarrow m'}{\pi_2\, m \rightsquigarrow\!\!\!\!\rightarrow \pi_2\, m'} \qquad \frac{}{\pi_1\, (m, n)_t \rightsquigarrow\!\!\!\!\rightarrow m} \qquad \frac{}{\pi_2\, (m, n)_t \rightsquigarrow\!\!\!\!\rightarrow n}$$

7 Meta Theory

Static Meta Theory

Theorem 1 (Confluence). *If $m \rightsquigarrow^* m_1$ and $m \rightsquigarrow^* m_2$, then there exists n such that $m_1 \rightsquigarrow^* n$ and $m_2 \rightsquigarrow^* n$.*

Theorem 2 (Conversion). *Conversion $\simeq$ is an equivalence relation.*

Theorem 3 (Static Validity). *For any static typing $\Gamma \vdash m : A$, there exists sort s such that $\Gamma \vdash A : s$ is derivable.*

Theorem 4 (Sort Uniqueness). *If there are static typings $\Gamma \vdash A : s$ and $\Gamma \vdash A : t$, then $s = t$.*

Theorem 5 (Static Subject Reduction). *If there is static typing $\Gamma \vdash m : A$ and static reduction $m \rightsquigarrow n$, then $\Gamma \vdash n : A$ is derivable.*

Theorem 6 (Static Normalization). *For any m with static typing $\Gamma \vdash m : A$, it is strongly normalizing.*

Dynamic Meta Theory

Theorem 7 (Dynamic Reflection). *For any dynamic typing $\Gamma; \Delta \vdash m : A$, static typing $\Gamma \vdash m : A$ is derivable.*

Theorem 8 (Value Stability). *If there is value v with dynamic typing $\Gamma; \Delta \vdash v : A$ and $\Gamma \vdash A : s$, then $\Delta \triangleright s$.*

Theorem 9 (Dynamic Subject Reduction). *If there is dynamic typing $\Gamma; \Delta \vdash m : A$ and dynamic reduction $m \twoheadrightarrow n$, then $\Gamma; \Delta \vdash n : A$ is derivable.*

Theorem 10 (Dynamic Progress). *If there is dynamic typing $\epsilon; \epsilon \vdash m : A$, then m is a value or there exists n such that $m \twoheadrightarrow n$.*

Erasure-Dynamic Meta Theory

Theorem 11 (Erasure Existence). *For any dynamic typing $\Gamma; \Delta \vdash m : A$, there exists m' such that erasure relation $\Gamma; \Delta \vdash m \sim m' : A$ is derivable.*

Theorem 12 (Erasure Subject Reduction). *For any erasure relation $\Gamma; \Delta \vdash m \sim m' : A$ and dynamic reduction $m' \twoheadrightarrow n'$, there exists n such that the following diagram commutes.*

$$\begin{array}{ccccccc} \Gamma;\Delta & \vdash & m & \sim & m' & : & A \\ & & \downarrow^{*} & & \downarrow & & \\ \Gamma;\Delta & \vdash & n & \sim & n' & : & A \end{array}$$

Theorem 13 (Erasure Progress). *For any erasure relation $\epsilon; \epsilon \vdash m \sim m' : A$, then m' is a value or there exists n' such that $m' \twoheadrightarrow n'$.*