A Two Level Linear Dependent Type Theory

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1 Syntax

$$
\begin{array}{rlcl}
\text{variable} & x, y, z, p & & \\
\text{sorts} & s, r, t & ::= & \mathrm{U} \mid \mathrm{L} \\
\text{terms} & m, n, A, B, C, H, P & ::= & x \mid s \\
& & \mid & \Pi_t(x : A).B \mid \Pi_t\{x : A\}.B \mid \Sigma_t(x : A).B \mid \Sigma_t\{x : A\}.B \mid A \mathbin{\&_t} B \\
& & \mid & \lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid \langle m, n\rangle_t \mid \{m, n\}_t \mid (m, n)_t \\
& & \mid & m\ n \mid \mathrm{R}^{\Sigma}_{[z]A}(m, [x, y]n) \mid \pi_1\, m \mid \pi_2\, m \\
& & \mid & m \equiv_A n \mid \mathrm{refl}\ m \mid \mathrm{R}^{\equiv}_{[x,y,p]A}([z]H, P) \mid \Box
\end{array}
$$

2 Static Fragment

Sort Order

$$\frac{}{\mathrm{U} \sqsubseteq s} \qquad \frac{}{\mathrm{L} \sqsubseteq \mathrm{L}}$$

Static Context

$$\frac{}{\epsilon \vdash} \qquad \frac{\Gamma \vdash \quad \Gamma \vdash A : s \quad x \notin \mathit{fresh}(\Gamma)}{\Gamma, x : A \vdash}$$

Static Typing

$$\frac{\Gamma \vdash}{\Gamma \vdash s : \mathrm{U}} \qquad \frac{\Gamma, x : A \vdash}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t(x : A).B : t} \qquad \frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t\{x : A\}.B : t}$$

$$\frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t(x : A).m : \Pi_t(x : A).B} \qquad \frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B} \qquad \frac{\Gamma \vdash m : \Pi_t(x : A).B \quad \Gamma \vdash n : A}{\Gamma \vdash m\ n : B[n/x]}$$

$$\frac{\Gamma \vdash m : \Pi_t\{x : A\}.B \quad \Gamma \vdash n : A}{\Gamma \vdash m\ n : B[n/x]} \qquad \frac{s \sqsubseteq t \quad r \sqsubseteq t \quad \Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t(x : A).B : t}$$

$$\frac{s \sqsubseteq t \quad \Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t\{x : A\}.B : t} \qquad \frac{\Gamma \vdash \Sigma_t(x : A).B \quad \Gamma \vdash m : A \quad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \langle m, n\rangle_t : \Sigma_t(x : A).B}$$

$$\frac{\Gamma \vdash \Sigma_t\{x : A\}.B \quad \Gamma \vdash m : A \quad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \{m, n\}_t : \Sigma_t\{x : A\}.B}$$

$$\frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \quad \Gamma \vdash m : \Sigma_t(x : A).B \quad \Gamma, x : A, y : B \vdash n : C[\langle x, y\rangle_t/z]}{\Gamma \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \quad \Gamma \vdash m : \Sigma_t\{x : A\}.B \quad \Gamma, x : A, y : B \vdash n : C[\{x, y\}_t/z]}{\Gamma \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) : C[m/z]} \qquad \frac{\Gamma \vdash A : s \quad \Gamma \vdash B : r}{\Gamma \vdash A \,\&_t B : t}$$

$$\frac{\Gamma \vdash m : A \quad \Gamma \vdash n : B}{\Gamma \vdash (m, n)_t : A \,\&_t B} \qquad \frac{\Gamma \vdash m : A \,\&_t B}{\Gamma \vdash \pi_1\, m : A} \qquad \frac{\Gamma \vdash m : A \,\&_t B}{\Gamma \vdash \pi_2\, m : B} \qquad \frac{\Gamma \vdash A : s \quad \Gamma \vdash m : A \quad \Gamma \vdash n : A}{\Gamma \vdash m \equiv_A n : \mathrm{U}}$$

$$\frac{\Gamma \vdash m : A}{\Gamma \vdash \mathrm{refl}\ m : m \equiv_A m}$$

$$\frac{\Gamma, x : A, y : A, p : x \equiv_A y \vdash B : s \quad \Gamma, z : A \vdash H : B[z/x, z/y, \mathrm{refl}\ z/p] \quad \Gamma \vdash P : m \equiv_A n}{\Gamma \vdash \mathrm{R}^{\equiv}_{[x,y,p]B}([z]H, P) : B[m/x, n/y, P/p]}$$

$$\frac{\Gamma \vdash B : s \quad \Gamma \vdash m : A \quad A \simeq B}{\Gamma \vdash m : B}$$

3 Dynamic Fragment

Dynamic Context

$$\frac{}{\epsilon; \epsilon \vdash} \qquad \frac{\Gamma; \Delta \vdash \quad \Gamma \vdash A : s \quad x \in \mathit{fresh}(\Gamma)}{\Gamma, x : A; \Delta, x :_s A \vdash} \qquad \frac{\Gamma; \Delta \vdash \quad \Gamma \vdash A : s \quad x \in \mathit{fresh}(\Gamma)}{\Gamma, x : A; \Delta \vdash}$$

Context Merge

$$\frac{}{\epsilon \uplus \epsilon = \epsilon} \qquad \frac{\Delta_1 \uplus \Delta_2 = \Delta \quad x \in \mathit{fresh}(\Delta)}{(\Delta_1, x :_{\mathrm{U}} A) \uplus (\Delta_2, x :_{\mathrm{U}} A) = (\Delta, x :_{\mathrm{U}} A)}$$

$$\frac{\Delta_1 \uplus \Delta_2 = \Delta \quad x \in \mathit{fresh}(\Delta)}{(\Delta_1, x :_{\mathrm{L}} A) \uplus \Delta_2 = (\Delta, x :_{\mathrm{L}} A)} \qquad \frac{\Delta_1 \uplus \Delta_2 = \Delta \quad x \in \mathit{fresh}(\Delta)}{\Delta_1 \uplus (\Delta_2, x :_{\mathrm{L}} A) = (\Delta, x :_{\mathrm{L}} A)}$$

Context Constraint

$$\frac{}{\epsilon \triangleright s} \qquad \frac{\Delta \triangleright \mathrm{U}}{\Delta, x :_{\mathrm{U}} A \triangleright \mathrm{U}} \qquad \frac{\Delta \triangleright \mathrm{L}}{\Delta, x :_s A \triangleright \mathrm{L}}$$

Dynamic Typing

$$\frac{\Gamma, x : A; \Delta, x :_s A \vdash \quad \Delta \triangleright \mathrm{U}}{\Gamma, x : A; \Delta, x :_s A \vdash x : A} \qquad \frac{\Gamma, x : A; \Delta, x :_s A \vdash m : B \quad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t(x : A).m : \Pi_t(x : A).B} \qquad \frac{\Gamma, x : A; \Delta \vdash m : B \quad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B}$$

$$\frac{\Gamma; \Delta_1 \vdash m : \Pi_t(x : A).B \quad \Gamma; \Delta_2 \vdash n : A}{\Gamma; \Delta_1 \uplus \Delta_2 \vdash m\ n : B[n/x]} \qquad \frac{\Gamma; \Delta \vdash m : \Pi_t\{x : A\}.B \quad \Gamma \vdash n : A}{\Gamma; \Delta \vdash m\ n : B[n/x]}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).B : t \quad \Gamma; \Delta_1 \vdash m : A \quad \Gamma; \Delta_2 \vdash n : B[m/x]}{\Gamma; \Delta_1 \uplus \Delta_2 \vdash \langle m, n \rangle_t : \Sigma_t(x : A).B}$$

$$\frac{\Gamma \vdash \Sigma_t\{x : A\}.B : t \quad \Gamma; \Delta \vdash m : A \quad \Gamma \vdash n : B[m/x]}{\Gamma; \Delta \vdash \{m, n\}_t : \Sigma_t\{x : A\}.B}$$

$$\frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \quad \Gamma; \Delta_1 \vdash m : \Sigma_t(x : A).B \quad \Gamma, x : A, y : B; \Delta_2, x :_{r1} A, y :_{r2} B \vdash n : C[\langle x, y \rangle_t / z]}{\Gamma; \Delta_1 \uplus \Delta_2 \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \quad \Gamma; \Delta_1 \vdash m : \Sigma_t\{x : A\}.B \quad \Gamma, x : A, y : B; \Delta_2, x :_r A \vdash n : C[\{x, y\}_t / z]}{\Gamma; \Delta_1 \uplus \Delta_2 \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma; \Delta \vdash m : A \quad \Gamma; \Delta \vdash n : B \quad \Delta \triangleright t}{\Gamma; \Delta \vdash (m, n)_t : A \mathbin{\&_t} B} \qquad \frac{\Gamma; \Delta \vdash m : A \mathbin{\&_t} B}{\Gamma; \Delta \vdash \pi_1\, m : A} \qquad \frac{\Gamma; \Delta \vdash m : A \mathbin{\&_t} B}{\Gamma; \Delta \vdash \pi_2\, m : B}$$

$$\frac{\Gamma \vdash B : s \quad \Gamma; \Delta \vdash m : A \quad A \simeq B}{\Gamma; \Delta \vdash m : B}$$

4 Erasure

Erasure Relation

$$\frac{\Gamma, x : A; \Delta, x :_s A \vdash \qquad \Delta \triangleright \mathrm{U}}{\Gamma, x : A; \Delta, x :_s A \vdash x \sim x : A} \qquad \frac{\Gamma, x : A; \Delta, x :_s A \vdash m \sim m' : B \qquad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t(x : A).m \sim \lambda_t(x : \Box).m' : \Pi_t(x : A).B}$$

$$\frac{\Gamma, x : A; \Delta \vdash m \sim m' : B \qquad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t\{x : A\}.m \sim \lambda_t\{x : \Box\}.m' : \Pi_t\{x : A\}.B} \qquad \frac{\Gamma; \Delta_1 \vdash m \sim m' : \Pi_t(x : A).B \qquad \Gamma; \Delta_2 \vdash n \sim n' : A}{\Gamma; \Delta_1 \cup \Delta_2 \vdash m\ n \sim m'\ n' : B[n/x]}$$

$$\frac{\Gamma; \Delta \vdash m \sim m' : \Pi_t\{x : A\}.B \qquad \Gamma \vdash n : A}{\Gamma; \Delta \vdash m\ n \sim m'\ \Box : B[n/x]}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).B : t \qquad \Gamma; \Delta_1 \vdash m \sim m' : A \qquad \Gamma; \Delta_2 \vdash n \sim n' : B[m/x]}{\Gamma; \Delta_1 \cup \Delta_2 \vdash \langle m, n\rangle_t \sim \langle m', n'\rangle_t : \Sigma_t(x : A).B}$$

$$\frac{\Gamma \vdash \Sigma_t\{x : A\}.B : t \qquad \Gamma; \Delta \vdash m \sim m' : A \qquad \Gamma \vdash n : B[m/x]}{\Gamma; \Delta \vdash \{m, n\}_t \sim \{m', \Box\}_t : \Sigma_t\{x : A\}.B}$$

$$\frac{\begin{array}{c}\Gamma, z : \Sigma_t(x : A).B \vdash C : s \\ \Gamma; \Delta_1 \vdash m \sim m' : \Sigma_t(x : A).B \qquad \Gamma, x : A, y : B; \Delta_2, x :_{r1} A, y :_{r2} B \vdash n \sim n' : C[\langle x, y\rangle_t/z]\end{array}}{\Gamma; \Delta_1 \cup \Delta_2 \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) \sim \mathrm{R}^{\Sigma}_{\Box}(m', [x, y]n') : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \qquad \Gamma; \Delta_1 \vdash m \sim m' : \Sigma_t\{x : A\}.B \qquad \Gamma, x : A, y : B; \Delta_2, x :_r A \vdash n \sim n' : C[\{x, y\}_t/z]}{\Gamma; \Delta_1 \cup \Delta_2 \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) \sim \mathrm{R}^{\Sigma}_{\Box}(m', [x, y]n') : C[m/z]}$$

$$\frac{\Gamma; \Delta \vdash m \sim m' : A \qquad \Gamma; \Delta \vdash n \sim n' : B \qquad \Delta \triangleright t}{\Gamma; \Delta \vdash (m, n)_t \sim (m', n')_t : A \,\&_t B} \qquad \frac{\Gamma; \Delta \vdash m \sim m' : A \,\&_t B}{\Gamma; \Delta \vdash \pi_1\, m \sim \pi_1\, m' : A} \qquad \frac{\Gamma; \Delta \vdash m \sim m' : A \,\&_t B}{\Gamma; \Delta \vdash \pi_2\, m \sim \pi_2\, m' : B}$$

$$\frac{\Gamma \vdash B : s \qquad \Gamma; \Delta \vdash m \sim m' : A \qquad A \simeq B}{\Gamma; \Delta \vdash m \sim m' : B}$$

5 Semantics

Static

$$\frac{A \rightsquigarrow A'}{\Pi_t(x:A).B \rightsquigarrow \Pi_t(x:A').B} \qquad \frac{B \rightsquigarrow B'}{\Pi_t(x:A).B \rightsquigarrow \Pi_t(x:A).B'} \qquad \frac{A \rightsquigarrow A'}{\Pi_t\{x:A\}.B \rightsquigarrow \Pi_t\{x:A'\}.B}$$

$$\frac{B \rightsquigarrow B'}{\Pi_t\{x:A\}.B \rightsquigarrow \Pi_t\{x:A\}.B'} \qquad \frac{A \rightsquigarrow A'}{\lambda_t(x:A).m \rightsquigarrow \lambda_t(x:A').m} \qquad \frac{m \rightsquigarrow m'}{\lambda_t(x:A).m \rightsquigarrow \lambda_t(x:A).m'}$$

$$\frac{A \rightsquigarrow A'}{\lambda_t\{x:A\}.m \rightsquigarrow \lambda_t\{x:A'\}.m} \qquad \frac{m \rightsquigarrow m'}{\lambda_t\{x:A\}.m \rightsquigarrow \lambda_t\{x:A\}.m'} \qquad \frac{m \rightsquigarrow m'}{m\ n \rightsquigarrow m'\ n} \qquad \frac{n \rightsquigarrow n'}{m\ n \rightsquigarrow m\ n'}$$

$$\frac{}{(\lambda_t(x:A).m)\ n \rightsquigarrow m[n/x]} \qquad \frac{}{(\lambda_t\{x:A\}.m)\ n \rightsquigarrow m[n/x]} \qquad \frac{A \rightsquigarrow A'}{\Sigma_t(x:A).B \rightsquigarrow \Sigma_t(x:A').B}$$

$$\frac{B \rightsquigarrow B'}{\Sigma_t(x:A).B \rightsquigarrow \Sigma_t(x:A).B'} \qquad \frac{A \rightsquigarrow A'}{\Sigma_t\{x:A\}.B \rightsquigarrow \Sigma_t\{x:A'\}.B} \qquad \frac{B \rightsquigarrow B'}{\Sigma_t\{x:A\}.B \rightsquigarrow \Sigma_t\{x:A\}.B'}$$

$$\frac{m \rightsquigarrow m'}{\langle m,n\rangle_t \rightsquigarrow \langle m',n\rangle_t} \qquad \frac{n \rightsquigarrow n'}{\langle m,n\rangle_t \rightsquigarrow \langle m,n'\rangle_t} \qquad \frac{m \rightsquigarrow m'}{\{m,n\}_t \rightsquigarrow \{m',n\}_t} \qquad \frac{n \rightsquigarrow n'}{\{m,n\}_t \rightsquigarrow \{m,n'\}_t}$$

$$\frac{A \rightsquigarrow A'}{\mathrm{R}^{\Sigma}_{[z]A}(m,[x,y]n) \rightsquigarrow \mathrm{R}^{\Sigma}_{[z]A'}(m,[x,y]n)} \qquad \frac{m \rightsquigarrow m'}{\mathrm{R}^{\Sigma}_{[z]A}(m,[x,y]n) \rightsquigarrow \mathrm{R}^{\Sigma}_{[z]A}(m',[x,y]n)}$$

$$\frac{n \rightsquigarrow n'}{\mathrm{R}^{\Sigma}_{[z]A}(m,[x,y]n) \rightsquigarrow \mathrm{R}^{\Sigma}_{[z]A}(m,[x,y]n')} \qquad \frac{}{\mathrm{R}^{\Sigma}_{[z]A}(\langle m_1,m_2\rangle_t,[x,y]n) \rightsquigarrow n[m_1/x,m_2/y]}$$

$$\frac{}{\mathrm{R}^{\Sigma}_{[z]A}(\{m_1,m_2\}_t,[x,y]n) \rightsquigarrow n[m_1/x,m_2/y]} \qquad \frac{A \rightsquigarrow A'}{A \,\&_t B \rightsquigarrow A' \,\&_t B} \qquad \frac{B \rightsquigarrow B'}{A \,\&_t B \rightsquigarrow A \,\&_t B'} \qquad \frac{m \rightsquigarrow m'}{(m,n)_t \rightsquigarrow (m',n)_t}$$

$$\frac{n \rightsquigarrow n'}{(m,n)_t \rightsquigarrow (m,n')_t} \qquad \frac{m \rightsquigarrow m'}{\pi_1\, m \rightsquigarrow \pi_1\, m'} \qquad \frac{m \rightsquigarrow m'}{\pi_2\, m \rightsquigarrow \pi_2\, m'} \qquad \frac{}{\pi_1\,(m,n)_t \rightsquigarrow m} \qquad \frac{}{\pi_2\,(m,n)_t \rightsquigarrow n}$$

$$\frac{A \rightsquigarrow A'}{m \equiv_A n \rightsquigarrow m \equiv_{A'} n} \qquad \frac{m \rightsquigarrow m'}{m \equiv_A n \rightsquigarrow m' \equiv_A n} \qquad \frac{n \rightsquigarrow n'}{m \equiv_A n \rightsquigarrow m \equiv_A n'}$$

$$\frac{A \rightsquigarrow A'}{\mathrm{R}^{\equiv}_{[x,y,p]A}([z]H,P) \rightsquigarrow \mathrm{R}^{\equiv}_{[x,y,p]A'}([z]H,P)} \qquad \frac{H \rightsquigarrow H'}{\mathrm{R}^{\equiv}_{[x,y,p]A}([z]H,P) \rightsquigarrow \mathrm{R}^{\equiv}_{[x,y,p]A}([z]H',P)}$$

$$\frac{P \rightsquigarrow P'}{\mathrm{R}^{\equiv}_{[x,y,p]A}([z]H,P) \rightsquigarrow \mathrm{R}^{\equiv}_{[x,y,p]A}([z]H,P')} \qquad \frac{}{\mathrm{R}^{\equiv}_{[x,y,p]A}([z]H,\mathrm{refl}\ m) \rightsquigarrow H[m/z]}$$