

# A Two Level Dependent Session Type Theory

Qiancheng Fu<sup>1</sup> and Hongwei Xi<sup>1</sup>

<sup>1</sup>Boston University

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## 1 Syntax

variables	$x, y, z$	
channels	$c, d$	
sorts	$s, r, t$	$::= \text{U} \mid \text{L}$
roles	$\rho$	$::= + \mid -$
terms	$m, n, A, B, C$	$::= x \mid c \mid s$ $\mid \Pi_t(x : A).B \mid \Pi_t\{x : A\}.B \mid \Sigma_t(x : A).B \mid \Sigma_t\{x : A\}.B$ $\mid \lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid \langle m, n \rangle_t \mid \{m, n\}_t$ $\mid m \ n \mid m \ \{n\} \mid \mathbf{R}_{[z]A}^\Sigma(m, [x, y]n) \mid \mu(x : A).m$ $\mid 1 \mid () \mid 2 \mid \text{true} \mid \text{false} \mid \mathbf{R}_{[z]A}^2(m, n_1, n_2)$ $\mid \mathbf{T} \ A \mid \text{return } m \mid \text{let } x \Leftarrow m \text{ in } n$ $\mid \text{proto} \mid \text{end}\rho \mid \rho(x : A).B \mid \rho\{x : A\}.B \mid \rho\text{Ch } A$ $\mid \text{fork } (x : A).m \mid \text{recv } m \mid \underline{\text{recv}} \ m \mid \text{send } m \mid \underline{\text{send}} \ m$ $\mid \text{close } m \mid \text{wait } m$
values	$u, v$	$::= x \mid c \mid \lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid \langle u, v \rangle_t \mid \{v, m\}_t$ $\mid () \mid \text{true} \mid \text{false} \mid \text{return } v \mid \text{let } x \Leftarrow v \text{ in } m$ $\mid \text{fork } (x : A).m \mid \text{recv } v \mid \underline{\text{recv}} \ v \mid \text{send } v \mid \underline{\text{send}} \ v$ $\mid \text{send } v \ u \mid \underline{\text{send}} \ v \ \{m\} \mid \text{close } v \mid \text{wait } v$
process	$o, p, q$	$\mid \langle m \rangle \mid p \mid q \mid \nu cd.p$

## 2 Static Fragment

### Sort Order

$$\overline{U \sqsubseteq s}$$

$$\overline{L \sqsubseteq L}$$

### Static Context

$$\overline{\epsilon \vdash} \quad \frac{\Gamma \vdash \quad \Gamma \vdash A : s \quad x \in \text{fresh}(\Gamma)}{\Gamma, x : A \vdash}$$

### Core Typing

$$\begin{array}{c} \frac{\Gamma \vdash}{\Gamma \vdash s : U} \quad \frac{\Gamma, x : A \vdash}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t(x : A).B : t} \quad \frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t\{x : A\}.B : t} \\[10pt] \frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t(x : A).m : \Pi_t(x : A).B} \quad \frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B} \quad \frac{\Gamma, x : A \vdash m : A}{\Gamma \vdash \mu(x : A).m : A} \\[10pt] \frac{\Gamma \vdash m : \Pi_t(x : A).B \quad \Gamma \vdash n : A}{\Gamma \vdash m \ n : B[n/x]} \quad \frac{\Gamma \vdash m : \Pi_t\{x : A\}.B \quad \Gamma \vdash n : A}{\Gamma \vdash m \ \{n\} : B[n/x]} \\[10pt] \frac{s \sqsubseteq t \quad r \sqsubseteq t \quad \Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t(x : A).B : t} \quad \frac{r \sqsubseteq t \quad \Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t\{x : A\}.B : t} \\[10pt] \frac{\Gamma \vdash \Sigma_t(x : A).B \quad \Gamma \vdash m : A \quad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \langle m, n \rangle_t : \Sigma_t(x : A).B} \quad \frac{\Gamma \vdash \Sigma_t\{x : A\}.B \quad \Gamma \vdash m : A \quad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \{m, n\}_t : \Sigma_t\{x : A\}.B} \\[10pt] \frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \quad \Gamma \vdash m : \Sigma_t(x : A).B \quad \Gamma, x : A, y : B \vdash n : C[\langle x, y \rangle_t / z]}{\Gamma \vdash R_{[z]C}^\Sigma(m, [x, y]n) : C[m/z]} \\[10pt] \frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \quad \Gamma \vdash m : \Sigma_t\{x : A\}.B \quad \Gamma, x : A, y : B \vdash n : C[\{x, y\}_t / z]}{\Gamma \vdash R_{[z]C}^\Sigma(m, [x, y]n) : C[m/z]} \\[10pt] \frac{\Gamma \vdash B : s \quad \Gamma \vdash m : A \quad A \simeq B}{\Gamma \vdash m : B} \end{array}$$

## Data Typing

$$\begin{array}{c}
\frac{\Gamma \vdash}{\Gamma \vdash 1 : \mathsf{U}} \quad \frac{\Gamma \vdash}{\Gamma \vdash () : 1} \quad \frac{\Gamma \vdash}{\Gamma \vdash 2 : \mathsf{U}} \quad \frac{\Gamma \vdash}{\Gamma \vdash \text{true} : 2} \quad \frac{\Gamma \vdash}{\Gamma \vdash \text{false} : 2} \\
\\
\frac{\Gamma, z : 2 \vdash A : s \quad \Gamma \vdash m : 2 \quad \Gamma \vdash n_1 : A[\text{true}/z] \quad \Gamma \vdash n_2 : A[\text{false}/z]}{\Gamma \vdash \mathsf{R}_{[z]A}^2(m, n_1, n_2) : A[m/z]}
\end{array}$$

## Monadic Typing

$$\frac{\Gamma \vdash A : s}{\Gamma \vdash \mathsf{T} A : \mathsf{L}} \quad \frac{\Gamma \vdash m : A}{\Gamma \vdash \text{return } m : \mathsf{T} A} \quad \frac{\Gamma \vdash B : s \quad \Gamma \vdash m : \mathsf{T} A \quad \Gamma, x : A \vdash n : \mathsf{T} B}{\Gamma \vdash \text{let } x \Leftarrow m \text{ in } n : \mathsf{T} B}$$

## Session Typing

$$\begin{array}{c}
\frac{\Gamma \vdash}{\Gamma \vdash \text{proto} : \mathsf{U}} \quad \frac{\Gamma \vdash}{\Gamma \vdash \text{end}\rho : \text{proto}} \quad \frac{\Gamma, x : A \vdash B : \text{proto}}{\Gamma \vdash \rho\{x : A\}.B : \text{proto}} \quad \frac{\Gamma, x : A \vdash B : \text{proto}}{\Gamma \vdash \rho(x : A).B : \text{proto}} \quad \frac{\Gamma \vdash A : \text{proto}}{\Gamma \vdash \rho \mathsf{Ch} A : \mathsf{L}} \\
\\
\frac{\Gamma \vdash \quad \epsilon \vdash A : \text{proto}}{\Gamma \vdash c : \rho \mathsf{Ch} A} \quad \frac{\Gamma, x : +\mathsf{Ch} A \vdash m : \mathsf{T} 1}{\Gamma \vdash \text{fork}(x : A).m : \mathsf{T} (-\mathsf{Ch} A)} \\
\\
\frac{\Gamma \vdash m : \rho_1 \mathsf{Ch} (\rho_2(x : A).B) \quad \rho_1 \text{ xor } \rho_2 = ?}{\Gamma \vdash \text{recv } m : \mathsf{T} (\Sigma_{\mathsf{L}}(x : A). \rho_1 \mathsf{Ch} B)} \quad \frac{\Gamma \vdash m : \rho_1 \mathsf{Ch} (\rho_2\{x : A\}.B) \quad \rho_1 \text{ xor } \rho_2 = ?}{\Gamma \vdash \underline{\text{recv}} m : \mathsf{T} (\Sigma_{\mathsf{L}}\{x : A\}. \rho_1 \mathsf{Ch} B)} \\
\\
\frac{\Gamma \vdash m : \rho_1 \mathsf{Ch} (\rho_2(x : A).B) \quad \rho_1 \text{ xor } \rho_2 = !}{\Gamma \vdash \text{send } m : \Pi_{\mathsf{L}}(x : A). \mathsf{T} (\rho_1 \mathsf{Ch} B)} \quad \frac{\Gamma \vdash m : \rho_1 \mathsf{Ch} (\rho_2\{x : A\}.B) \quad \rho_1 \text{ xor } \rho_2 = !}{\Gamma \vdash \underline{\text{send}} m : \Pi_{\mathsf{L}}\{x : A\}. \mathsf{T} (\rho_1 \mathsf{Ch} B)} \\
\\
\frac{\Gamma \vdash m : \rho_1 \mathsf{Ch} (\text{end}\rho_2) \quad \rho_1 \text{ xor } \rho_2 = ?}{\Gamma \vdash \text{wait } m : \mathsf{T} 1} \quad \frac{\Gamma \vdash m : \rho_1 \mathsf{Ch} (\text{end}\rho_2) \quad \rho_1 \text{ xor } \rho_2 = !}{\Gamma \vdash \text{close } m : \mathsf{T} 1}
\end{array}$$

### 3 Dynamic Fragment

#### Dynamic Context

$$\frac{}{\epsilon; \epsilon \vdash} \quad \frac{\Gamma; \Delta \vdash \quad \Gamma \vdash A : s \quad x \in \text{fresh}(\Gamma)}{\Gamma, x : A; \Delta, x :_s A \vdash} \quad \frac{\Gamma; \Delta \vdash \quad \Gamma \vdash A : s \quad x \in \text{fresh}(\Gamma)}{\Gamma, x : A; \Delta \vdash}$$

#### Context Merge

$$\frac{}{\epsilon \cup \epsilon = \epsilon} \quad \frac{\Delta_1 \cup \Delta_2 = \Delta \quad x \in \text{fresh}(\Delta)}{(\Delta_1, x :_{\text{U}} A) \cup (\Delta_2, x :_{\text{U}} A) = (\Delta, x :_{\text{U}} A)}$$

$$\frac{\Delta_1 \cup \Delta_2 = \Delta \quad x \in \text{fresh}(\Delta)}{(\Delta_1, x :_{\text{L}} A) \cup \Delta_2 = (\Delta, x :_{\text{L}} A)} \quad \frac{\Delta_1 \cup \Delta_2 = \Delta \quad x \in \text{fresh}(\Delta)}{\Delta_1 \cup (\Delta_2, x :_{\text{L}} A) = (\Delta, x :_{\text{L}} A)}$$

#### Context Constraint

$$\frac{}{\epsilon \triangleright s} \quad \frac{\Delta \triangleright \text{U}}{\Delta, x :_{\text{U}} A \triangleright \text{U}} \quad \frac{\Delta \triangleright \text{L}}{\Delta, x :_s A \triangleright \text{L}}$$

#### Core Typing

$$\frac{\epsilon; \Gamma, x : A; \Delta, x :_s A \vdash \quad \Delta \triangleright \text{U}}{\epsilon; \Gamma, x : A; \Delta, x :_s A \vdash x : A} \quad \frac{\Theta; \Gamma, x : A; \Delta, x :_s A \vdash m : B \quad \Theta \triangleright t \quad \Delta \triangleright t}{\Theta; \Gamma; \Delta \vdash \lambda_t(x : A).m : \Pi_t(x : A).B}$$

$$\frac{\Theta; \Gamma, x : A; \Delta \vdash m : B \quad \Theta \triangleright t \quad \Delta \triangleright t}{\Theta; \Gamma; \Delta \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B} \quad \frac{\epsilon; \Gamma, x : A; \Delta, x :_{\text{U}} A \vdash m : A \quad \Delta \triangleright \text{U}}{\epsilon; \Gamma; \Delta \vdash \mu(x : A).m : A}$$

$$\frac{\Theta_1; \Gamma; \Delta_1 \vdash m : \Pi_t(x : A).B \quad \Theta_2; \Gamma; \Delta_2 \vdash n : A}{\Theta_1 \cup \Theta_2; \Gamma; \Delta_1 \cup \Delta_2 \vdash m \ n : B[n/x]} \quad \frac{\Theta; \Gamma; \Delta \vdash m : \Pi_t\{x : A\}.B \quad \Gamma \vdash n : A}{\Theta; \Gamma; \Delta \vdash m \ \{n\} : B[n/x]}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).B : t \quad \Theta_1; \Gamma; \Delta_1 \vdash m : A \quad \Theta_2; \Gamma; \Delta_2 \vdash n : B[m/x]}{\Theta_1 \cup \Theta_2; \Gamma; \Delta_1 \cup \Delta_2 \vdash \langle m, n \rangle_t : \Sigma_t(x : A).B}$$

$$\frac{\Gamma \vdash \Sigma_t\{x : A\}.B : t \quad \Gamma \vdash m : A \quad \Theta; \Gamma; \Delta \vdash n : B[m/x]}{\Theta; \Gamma; \Delta \vdash \{m, n\}_t : \Sigma_t\{x : A\}.B}$$

$$\frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \quad \Theta_1; \Gamma; \Delta_1 \vdash m : \Sigma_t(x : A).B \quad \Theta_2; \Gamma, x : A, y : B; \Delta_2, x :_{r1} A, y :_{r2} B \vdash n : C[\langle x, y \rangle_t / z]}{\Theta_1 \cup \Theta_2; \Gamma; \Delta_1 \cup \Delta_2 \vdash \text{R}_{[z]C}^{\Sigma}(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \quad \Theta_1; \Gamma; \Delta_1 \vdash m : \Sigma_t\{x : A\}.B \quad \Theta_2; \Gamma, x : A, y : B; \Delta_2, y :_r B \vdash n : C[\{x, y\}_t / z]}{\Theta_1 \cup \Theta_2; \Gamma; \Delta_1 \cup \Delta_2 \vdash \text{R}_{[z]C}^{\Sigma}(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma \vdash B : s \quad \Theta; \Gamma; \Delta \vdash m : A \quad A \simeq B}{\Theta; \Gamma; \Delta \vdash m : B}$$

## Data Typing

$$\begin{array}{c}
\frac{\Gamma; \Delta \vdash \quad \Delta \triangleright U}{\epsilon; \Gamma; \Delta \vdash () : 1} \quad \frac{\Gamma; \Delta \vdash \quad \Delta \triangleright U}{\epsilon; \Gamma; \Delta \vdash \text{true} : 2} \quad \frac{\Gamma; \Delta \vdash \quad \Delta \triangleright U}{\epsilon; \Gamma; \Delta \vdash \text{false} : 2} \\
\\
\frac{\Gamma, z : 2 \vdash A : s \quad \Theta_1; \Gamma; \Delta_1 \vdash m : 2 \quad \Theta_2; \Gamma; \Delta_2 \vdash n_1 : A[\text{true}/z] \quad \Theta_2; \Gamma; \Delta_2 \vdash n_2 : A[\text{false}/z]}{\Theta_1 \cup \Theta_2; \Gamma; \Delta_1 \cup \Delta_2 \vdash R_{[z]A}^2(m, n_1, n_2) : A[m/z]}
\end{array}$$

## Monadic Typing

$$\frac{\Theta; \Gamma; \Delta \vdash m : A}{\Theta; \Gamma; \Delta \vdash \text{return } m : T A} \quad \frac{\Gamma \vdash B : s \quad \Theta_1; \Gamma; \Delta_1 \vdash m : T A \quad \Theta_2; \Gamma, x : A; \Delta_2, x :_r A \vdash n : T B}{\Theta_1 \cup \Theta_2; \Gamma; \Delta_1 \cup \Delta_2 \vdash \text{let } x \Leftarrow m \text{ in } n : T B}$$

## Session Typing

$$\begin{array}{c}
\frac{\Gamma; \Delta \vdash \quad \epsilon \vdash A : \text{proto} \quad \Delta \triangleright U}{c :_L \rho \text{Ch } A; \Gamma; \Delta \vdash c : \rho \text{Ch } A} \quad \frac{\Theta; \Gamma, x : +\text{Ch } A; \Delta, x :_L +\text{Ch } A \vdash m : T 1}{\Theta; \Gamma; \Delta \vdash \text{fork } (x : A).m : T (-\text{Ch } A)} \\
\\
\frac{\Theta; \Gamma; \Delta \vdash m : \rho_1 \text{Ch } (\rho_2(x : A).B) \quad \rho_1 \text{ xor } \rho_2 =?}{\Theta; \Gamma; \Delta \vdash \text{recv } m : T (\Sigma_L(x : A). \rho_1 \text{Ch } B)} \quad \frac{\Theta; \Gamma; \Delta \vdash m : \rho_1 \text{Ch } (\rho_2\{x : A\}.B) \quad \rho_1 \text{ xor } \rho_2 =?}{\Theta; \Gamma; \Delta \vdash \underline{\text{recv}} m : T (\Sigma_L\{x : A\}. \rho_1 \text{Ch } B)} \\
\\
\frac{\Theta; \Gamma; \Delta \vdash m : \rho_1 \text{Ch } (\rho_2(x : A).B) \quad \rho_1 \text{ xor } \rho_2 =!}{\Theta; \Gamma; \Delta \vdash \text{send } m : \Pi_L(x : A).T (\rho_1 \text{Ch } B)} \quad \frac{\Theta; \Gamma; \Delta \vdash m : \rho_1 \text{Ch } (\rho_2\{x : A\}.B) \quad \rho_1 \text{ xor } \rho_2 =!}{\Theta; \Gamma; \Delta \vdash \underline{\text{send}} m : \Pi_L\{x : A\}.T (\rho_1 \text{Ch } B)} \\
\\
\frac{\Theta; \Gamma; \Delta \vdash m : \rho_1 \text{Ch } (\text{end} \rho_2) \quad \rho_1 \text{ xor } \rho_2 =?}{\Theta; \Gamma; \Delta \vdash \text{wait } m : T 1} \quad \frac{\Theta; \Gamma; \Delta \vdash m : \rho_1 \text{Ch } (\text{end} \rho_2) \quad \rho_1 \text{ xor } \rho_2 =!}{\Theta; \Gamma; \Delta \vdash \text{close } m : T 1}
\end{array}$$

## 4 Process Fragment

### Process Typing

$$\frac{\Theta; \epsilon; \epsilon \vdash m : \mathsf{T} \ 1}{\Theta \vdash \langle m \rangle} \quad \frac{\Theta_1 \vdash p \quad \Theta_2 \vdash q}{\Theta_1 \sqcup \Theta_2 \vdash p \mid q} \quad \frac{\Theta, c :_{\mathsf{L}} \rho \mathsf{Ch} \ A, d :_{\mathsf{L}} \neg \rho \mathsf{Ch} \ A \vdash p}{\Theta \vdash \nu cd.p}$$

### Structural Congruence

Structural congruence is the least congruence relation with the following properties.

$$\begin{array}{lll} p \mid q \equiv q \mid p & o \mid (p \mid q) \equiv (o \mid p) \mid q & p \mid \langle \mathsf{return} \ () \rangle \equiv p \\ \nu cd.p \mid q \equiv \nu cd.(p \mid q) & \nu cd.p \equiv \nu dc.p & \nu cd.\nu c'd'.p \equiv \nu c'd'.\nu cd.p \end{array}$$

### Reductions

$$\frac{p \Rightarrow q}{o \mid p \Rightarrow o \mid q} \quad \frac{p \Rightarrow q}{\nu cd.p \Rightarrow \nu cd.q} \quad \frac{p \equiv p' \quad p' \Rightarrow q' \quad q' \equiv q}{p \Rightarrow q} \quad \frac{m \rightsquigarrow m'}{\langle m \rangle \Rightarrow \langle m' \rangle}$$

$$\overline{\langle \mathsf{let} \ x \leftarrow \mathsf{return} \ v \ \mathsf{in} \ m \rangle \Rightarrow \langle m[v/x] \rangle} \quad \overline{\langle \mathsf{let} \ x \leftarrow \mathsf{fork} \ (y : A).m \ \mathsf{in} \ n \rangle \Rightarrow \nu cd.(\langle n[c/x] \rangle \mid \langle m[d/y] \rangle)}$$

$$\overline{\nu cd.(\langle \mathsf{let} \ x \leftarrow \mathsf{send} \ c \ v \ \mathsf{in} \ n_1 \rangle \mid \langle \mathsf{let} \ y \leftarrow \mathsf{recv} \ d \ \mathsf{in} \ n_2 \rangle) \Rightarrow \nu cd.(\langle \mathsf{let} \ x \leftarrow \mathsf{return} \ c \ \mathsf{in} \ n_1 \rangle \mid \langle \mathsf{let} \ y \leftarrow \mathsf{return} \ \langle v, d \rangle_L \ \mathsf{in} \ n_2 \rangle)}$$

$$\overline{\nu cd.(\langle \mathsf{let} \ x \leftarrow \mathsf{send} \ c \ \{m\} \ \mathsf{in} \ n_1 \rangle \mid \langle \mathsf{let} \ y \leftarrow \mathsf{recv} \ d \ \mathsf{in} \ n_2 \rangle) \Rightarrow \nu cd.(\langle \mathsf{let} \ x \leftarrow \mathsf{return} \ c \ \mathsf{in} \ n_1 \rangle \mid \langle \mathsf{let} \ y \leftarrow \mathsf{return} \ \{m, d\}_L \ \mathsf{in} \ n_2 \rangle)}$$

$$\overline{\nu cd.(\langle \mathsf{let} \ x \leftarrow \mathsf{close} \ c \ \mathsf{in} \ m \rangle \mid \langle \mathsf{let} \ y \leftarrow \mathsf{wait} \ d \ \mathsf{in} \ n \rangle) \Rightarrow \langle \mathsf{let} \ x \leftarrow \mathsf{return} \ () \ \mathsf{in} \ m \rangle \mid \langle \mathsf{let} \ x \leftarrow \mathsf{return} \ () \ \mathsf{in} \ n \rangle}$$