A Two Level Linear Dependent Type Theory

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1 Syntax

$$
\begin{array}{rlcl}
\text{variable} & x, y, z, p & & \\
\text{sorts} & s, r, t & ::= & \mathrm{U} \mid \mathrm{L} \\
\text{expressions} & m, n, A, B, C, H, P & ::= & x \mid s \\
 & & \mid & \Pi_t(x : A).B \mid \Pi_t\{x : A\}.B \mid \lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid m\ n \\
 & & \mid & \Sigma_t(x : A).B \mid \Sigma_t(x : A).\{B\} \mid (m, n)_t \mid (m, \{n\})_t \\
 & & \mid & \text{let } (x, y) \text{ as } A := m \text{ in } n \\
 & & \mid & A \mathbin{\&_t} B \mid [m, n]_t \mid \pi_1\, m \mid \pi_2\, m \\
 & & \mid & m \equiv_A n \mid \mathrm{refl}\ m \mid \mathrm{J}\, A\, H\, P \mid \square
\end{array}
$$

2 Static Fragment

Sort Order

$$\frac{}{\mathrm{U} \sqsubseteq s} \qquad \frac{}{\mathrm{L} \sqsubseteq \mathrm{L}}$$

Static Context

$$\frac{}{\epsilon \vdash} \qquad \frac{\Gamma \vdash \qquad \Gamma \vdash A : s \qquad x \notin \mathit{fresh}(\Gamma)}{\Gamma, x : A \vdash}$$

Static Typing

$$\frac{\Gamma \vdash}{\Gamma \vdash s : \mathrm{U}} \qquad \frac{\Gamma, x : A \vdash}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t(x : A).B : t} \qquad \frac{\Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t\{x : A\}.B : t}$$

$$\frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t(x : A).m : \Pi_t(x : A).B} \qquad \frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B} \qquad \frac{\Gamma \vdash m : \Pi_t(x : A).B \qquad \Gamma \vdash n : A}{\Gamma \vdash m\ n : B[n/x]}$$

$$\frac{\Gamma \vdash m : \Pi_t\{x : A\}.B \qquad \Gamma \vdash n : A}{\Gamma \vdash m\ n : B[n/x]} \qquad \frac{s \sqsubseteq t \qquad r \sqsubseteq t \qquad \Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t(x : A).B : t}$$

$$\frac{s \sqsubseteq t \qquad \Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t(x : A).\{B\} : t} \qquad \frac{\Gamma \vdash \Sigma_t(x : A).B \qquad \Gamma \vdash m : A \qquad \Gamma \vdash n : B[m/x]}{\Gamma \vdash (m, n)_t : \Sigma_t(x : A).B}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).\{B\} \qquad \Gamma \vdash m : A \qquad \Gamma \vdash n : B[m/x]}{\Gamma \vdash (m, \{n\})_t : \Sigma_t(x : A).\{B\}}$$

$$\frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \qquad \Gamma \vdash m : \Sigma_t(x : A).B \qquad \Gamma, x : A, y : B \vdash n : C[(x, y)_t/z]}{\Gamma \vdash \text{let } (x, y) \text{ as } C := m \text{ in } n : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t(x : A).\{B\} \vdash C : s \qquad \Gamma \vdash m : \Sigma_t(x : A).\{B\} \qquad \Gamma, x : A, y : B \vdash n : C[(x, \{y\})_t/z]}{\Gamma \vdash \text{let } (x, y) \text{ as } C := m \text{ in } n : C[m/z]}$$

$$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash B : r}{\Gamma \vdash A \,\&_t B : t} \qquad \frac{\Gamma \vdash m : A \qquad \Gamma \vdash n : B}{\Gamma \vdash [m, n]_t : A \,\&_t B} \qquad \frac{\Gamma \vdash m : A \,\&_t B}{\Gamma \vdash \pi_1\, m : A} \qquad \frac{\Gamma \vdash m : A \,\&_t B}{\Gamma \vdash \pi_2\, m : B}$$

$$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash m : A \qquad \Gamma \vdash n : A}{\Gamma \vdash m \equiv_A n : \mathrm{U}} \qquad \frac{\Gamma \vdash m : A}{\Gamma \vdash \text{refl } m : m \equiv_A m}$$

$$\frac{\Gamma, x : A, y : A, p : x \equiv_A y \vdash B : s \qquad \Gamma, z : A \vdash H : B[z/x, z/y, \text{refl } z/p] \qquad \Gamma \vdash P : m \equiv_A n}{\Gamma \vdash \mathrm{J}\, B\, H\, P : B[m/x, n/y, P/p]}$$

$$\frac{\Gamma \vdash B : s \qquad \Gamma \vdash m : A \qquad A = B}{\Gamma \vdash m : B}$$

3 Dynamic Fragment

Dynamic Context

$$\frac{}{\epsilon;\epsilon \vdash} \qquad \frac{\Gamma;\Delta \vdash \quad \Gamma \vdash A : s \quad x \in \mathit{fresh}(\Gamma)}{\Gamma, x : A; \Delta, x :_s A \vdash} \qquad \frac{\Gamma;\Delta \vdash \quad \Gamma \vdash A : s \quad x \in \mathit{fresh}(\Gamma)}{\Gamma, x : A; \Delta \vdash}$$

Context Merge

$$\frac{}{\epsilon \uplus \epsilon = \epsilon} \qquad \frac{\Delta_1 \uplus \Delta_2 = \Delta \quad x \in \mathit{fresh}(\Delta)}{(\Delta_1, x :_{\mathrm{U}} A) \uplus (\Delta_2, x :_{\mathrm{U}} A) = (\Delta, x :_{\mathrm{U}} A)}$$

$$\frac{\Delta_1 \uplus \Delta_2 = \Delta \quad x \in \mathit{fresh}(\Delta)}{(\Delta_1, x :_{\mathrm{L}} A) \uplus \Delta_2 = (\Delta, x :_{\mathrm{L}} A)} \qquad \frac{\Delta_1 \uplus \Delta_2 = \Delta \quad x \in \mathit{fresh}(\Delta)}{\Delta_1 \uplus (\Delta_2, x :_{\mathrm{L}} A) = (\Delta, x :_{\mathrm{L}} A)}$$

Context Constraint

$$\frac{}{\epsilon \triangleright s} \qquad \frac{\Delta \triangleright \mathrm{U}}{\Delta, x :_{\mathrm{U}} A \triangleright \mathrm{U}} \qquad \frac{\Delta \triangleright \mathrm{L}}{\Delta, x :_s A \triangleright \mathrm{L}}$$

Dynamic Typing

$$\frac{\Gamma, x : A; \Delta, x :_s A \vdash \quad \Delta \triangleright \mathrm{U}}{\Gamma, x : A; \Delta, x :_s A \vdash x : A} \qquad \frac{\Gamma, x : A; \Delta, x :_s A \vdash m : B \quad \Delta \triangleright t}{\Gamma;\Delta \vdash \lambda_t(x : A).m : \Pi_t(x : A).B} \qquad \frac{\Gamma, x : A; \Delta \vdash m : B \quad \Delta \triangleright t}{\Gamma;\Delta \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B}$$

$$\frac{\Gamma;\Delta_1 \vdash m : \Pi_t(x : A).B \quad \Gamma;\Delta_2 \vdash n : A}{\Gamma;\Delta_1 \uplus \Delta_2 \vdash m\ n : B[n/x]} \qquad \frac{\Gamma;\Delta \vdash m : \Pi_t\{x : A\}.B \quad \Gamma \vdash n : A}{\Gamma;\Delta \vdash m\ n : B[n/x]}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).B : t \quad \Gamma;\Delta_1 \vdash m : A \quad \Gamma;\Delta_2 \vdash n : B[m/x]}{\Gamma;\Delta_1 \uplus \Delta_2 \vdash (m, n)_t : \Sigma_t(x : A).B}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).\{B\} : t \quad \Gamma;\Delta \vdash m : A \quad \Gamma \vdash n : B[m/x]}{\Gamma;\Delta \vdash (m, \{n\})_t : \Sigma_t(x : A).\{B\}}$$

$$\frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \quad \Gamma;\Delta_1 \vdash m : \Sigma_t(x : A).B \quad \Gamma, x : A, y : B; \Delta_2, x :_{r1} A, y :_{r2} B \vdash n : C[(x, y)_t/z]}{\Gamma;\Delta_1 \uplus \Delta_2 \vdash \mathrm{let}\ (x, y)\ \mathrm{as}\ C := m\ \mathrm{in}\ n : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t(x : A).\{B\} \vdash C : s \quad \Gamma;\Delta_1 \vdash m : \Sigma_t(x : A).\{B\} \quad \Gamma, x : A, y : B; \Delta_2, x :_r A \vdash n : C[(x, \{y\})_t/z]}{\Gamma;\Delta_1 \uplus \Delta_2 \vdash \mathrm{let}\ (x, y)\ \mathrm{as}\ C := m\ \mathrm{in}\ n : C[m/z]}$$

$$\frac{\Gamma;\Delta \vdash m : A \quad \Gamma;\Delta \vdash n : B \quad \Delta \triangleright t}{\Gamma;\Delta \vdash [m, n]_t : A \,\&_t B} \qquad \frac{\Gamma;\Delta \vdash m : A \,\&_t B}{\Gamma;\Delta \vdash \pi_1\, m : A} \qquad \frac{\Gamma;\Delta \vdash m : A \,\&_t B}{\Gamma;\Delta \vdash \pi_2\, m : B}$$

$$\frac{\Gamma \vdash B : s \quad \Gamma;\Delta \vdash m : A \quad A = B}{\Gamma;\Delta \vdash m : B}$$

4 Erasure

Erasure Relation

$$\frac{\Gamma, x : A; \Delta, x :_s A \vdash \qquad \Delta \triangleright \mathrm{U}}{\Gamma, x : A; \Delta, x :_s A \vdash x \sim x : A} \qquad \frac{\Gamma, x : A; \Delta, x :_s A \vdash m \sim m' : B \qquad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t(x : A).m \sim \lambda_t(x : \square).m' : \Pi_t(x : A).B}$$

$$\frac{\Gamma, x : A; \Delta \vdash m \sim m' : B \qquad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t\{x : A\}.m \sim \lambda_t\{x : \square\}.m' : \Pi_t\{x : A\}.B} \qquad \frac{\Gamma; \Delta_1 \vdash m \sim m' : \Pi_t(x : A).B \qquad \Gamma; \Delta_2 \vdash n \sim n' : A}{\Gamma; \Delta_1 \uplus \Delta_2 \vdash m\ n \sim m'\ n' : B[n/x]}$$

$$\frac{\Gamma; \Delta \vdash m \sim m' : \Pi_t\{x : A\}.B \qquad \Gamma \vdash n : A}{\Gamma; \Delta \vdash m\ n \sim m'\ \square : B[n/x]}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).B : t \qquad \Gamma; \Delta_1 \vdash m \sim m' : A \qquad \Gamma; \Delta_2 \vdash n \sim n' : B[m/x]}{\Gamma; \Delta_1 \uplus \Delta_2 \vdash (m, n)_t \sim (m', n')_t : \Sigma_t(x : A).B}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).\{B\} : t \qquad \Gamma; \Delta \vdash m \sim m' : A \qquad \Gamma \vdash n : B[m/x]}{\Gamma; \Delta \vdash (m, \{n\})_t \sim (m', \{\square\})_t : \Sigma_t(x : A).\{B\}}$$

$$\frac{\begin{array}{c}\Gamma, z : \Sigma_t(x : A).B \vdash C : s \\ \Gamma; \Delta_1 \vdash m \sim m' : \Sigma_t(x : A).B \qquad \Gamma, x : A, y : B; \Delta_2, x :_{r1} A, y :_{r2} B \vdash n \sim n' : C[(x, y)_t/z]\end{array}}{\Gamma; \Delta_1 \uplus \Delta_2 \vdash \text{let } (x, y) \text{ as } C := m \text{ in } n \sim \text{let } (x, y) \text{ as } \square := m' \text{ in } n' : C[m/z]}$$

$$\frac{\begin{array}{c}\Gamma, z : \Sigma_t(x : A).\{B\} \vdash C : s \\ \Gamma; \Delta_1 \vdash m \sim m' : \Sigma_t(x : A).\{B\} \qquad \Gamma, x : A, y : B; \Delta_2, x :_r A \vdash n \sim n' : C[(x, \{y\})_t/z]\end{array}}{\Gamma; \Delta_1 \uplus \Delta_2 \vdash \text{let } (x, y) \text{ as } C := m \text{ in } n \sim \text{let } (x, y) \text{ as } \square := m' \text{ in } n' : C[m/z]}$$

$$\frac{\Gamma; \Delta \vdash m \sim m' : A \qquad \Gamma; \Delta \vdash n \sim n' : B \qquad \Delta \triangleright t}{\Gamma; \Delta \vdash [m, n]_t \sim [m', n']_t : A \,\&_t B} \qquad \frac{\Gamma; \Delta \vdash m \sim m' : A \,\&_t B}{\Gamma; \Delta \vdash \pi_1\, m \sim \pi_1\, m' : A} \qquad \frac{\Gamma; \Delta \vdash m \sim m' : A \,\&_t B}{\Gamma; \Delta \vdash \pi_2\, m \sim \pi_2\, m' : B}$$

$$\frac{\Gamma \vdash B : s \qquad \Gamma; \Delta \vdash m \sim m' : A \qquad A = B}{\Gamma; \Delta \vdash m \sim m' : B}$$