

# A Two Level Linear Dependent Type Theory

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November 6, 2022

## 1 Syntax

variables	$x, y, z, p$	
sorts	$s, r, t$	$::= \text{U} \mid \text{L}$
terms	$m, n, A, B, C, H, P$	$::= x \mid s$ $\mid \Pi_t(x : A).B \mid \Pi_t\{x : A\}.B \mid \Sigma_t(x : A).B \mid \Sigma_t\{x : A\}.B \mid A \&_t B$ $\mid \lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid \langle m, n \rangle_t \mid \{m, n\}_t \mid (m, n)_t$ $\mid m \ n \mid \mathbf{R}_{[z]A}^\Sigma(m, [x, y]n) \mid \pi_1 m \mid \pi_2 m$ $\mid m \equiv_A n \mid \text{refl } m \mid \mathbf{R}_{[x, y, p]A}^\equiv([z]H, P) \mid \square$
values	$u, v$	$::= x \mid \lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid \langle u, v \rangle_t \mid \{v, m\}_t \mid (m, n)_t$

## 2 Static Fragment

### Sort Order

$$\overline{U \sqsubseteq s}$$

$$\overline{L \sqsubseteq L}$$

### Static Context

$$\overline{\epsilon \vdash} \quad \frac{\Gamma \vdash \quad \Gamma \vdash A : s \quad x \in \text{fresh}(\Gamma)}{\Gamma, x : A \vdash}$$

### Static Typing

$$\begin{array}{c} \frac{\Gamma \vdash}{\Gamma \vdash s : U} \quad \frac{\Gamma, x : A \vdash}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t(x : A).B : t} \quad \frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t\{x : A\}.B : t} \\[10pt] \frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t(x : A).m : \Pi_t(x : A).B} \quad \frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B} \quad \frac{\Gamma \vdash m : \Pi_t(x : A).B \quad \Gamma \vdash n : A}{\Gamma \vdash m \ n : B[n/x]} \\[10pt] \frac{\Gamma \vdash m : \Pi_t\{x : A\}.B \quad \Gamma \vdash n : A}{\Gamma \vdash m \ n : B[n/x]} \quad \frac{s \sqsubseteq t \quad r \sqsubseteq t \quad \Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t(x : A).B : t} \\[10pt] \frac{s \sqsubseteq t \quad \Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t\{x : A\}.B : t} \quad \frac{\Gamma \vdash \Sigma_t(x : A).B \quad \Gamma \vdash m : A \quad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \langle m, n \rangle_t : \Sigma_t(x : A).B} \\[10pt] \frac{\Gamma \vdash \Sigma_t\{x : A\}.B \quad \Gamma \vdash m : A \quad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \{m, n\}_t : \Sigma_t\{x : A\}.B} \\[10pt] \frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \quad \Gamma \vdash m : \Sigma_t(x : A).B \quad \Gamma, x : A, y : B \vdash n : C[\langle x, y \rangle_t / z]}{\Gamma \vdash R_{[z]C}^\Sigma(m, [x, y]n) : C[m/z]} \\[10pt] \frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \quad \Gamma \vdash m : \Sigma_t\{x : A\}.B \quad \Gamma, x : A, y : B \vdash n : C[\{x, y\}_t / z]}{\Gamma \vdash R_{[z]C}^\Sigma(m, [x, y]n) : C[m/z]} \\[10pt] \frac{\Gamma \vdash A : s \quad \Gamma \vdash B : r}{\Gamma \vdash A \ \&_t B : t} \quad \frac{\Gamma \vdash m : A \quad \Gamma \vdash n : B}{\Gamma \vdash (m, n)_t : A \ \&_t B} \quad \frac{\Gamma \vdash m : A \ \&_t B}{\Gamma \vdash \pi_1 m : A} \quad \frac{\Gamma \vdash m : A \ \&_t B}{\Gamma \vdash \pi_2 m : B} \\[10pt] \frac{\Gamma \vdash A : s \quad \Gamma \vdash m : A \quad \Gamma \vdash n : A}{\Gamma \vdash m \equiv_A n : U} \quad \frac{\Gamma \vdash m : A}{\Gamma \vdash \text{refl } m : m \equiv_A m} \\[10pt] \frac{\Gamma, x : A, y : A, p : x \equiv_A y \vdash B : s \quad \Gamma, z : A \vdash H : B[z/x, z/y, \text{refl } z/p] \quad \Gamma \vdash P : m \equiv_A n}{\Gamma \vdash R_{[x, y, p]B}^\equiv([z]H, P) : B[m/x, n/y, P/p]} \\[10pt] \frac{\Gamma \vdash B : s \quad \Gamma \vdash m : A \quad A \simeq B}{\Gamma \vdash m : B} \end{array}$$

### 3 Dynamic Fragment

#### Dynamic Context

$$\frac{}{\epsilon; \epsilon \vdash} \quad \frac{\Gamma; \Delta \vdash \quad \Gamma \vdash A : s \quad x \in \text{fresh}(\Gamma)}{\Gamma, x : A; \Delta, x :_s A \vdash} \quad \frac{\Gamma; \Delta \vdash \quad \Gamma \vdash A : s \quad x \in \text{fresh}(\Gamma)}{\Gamma, x : A; \Delta \vdash}$$

#### Context Merge

$$\frac{}{\epsilon \cup \epsilon = \epsilon} \quad \frac{\Delta_1 \cup \Delta_2 = \Delta \quad x \in \text{fresh}(\Delta)}{(\Delta_1, x :_{\text{U}} A) \cup (\Delta_2, x :_{\text{U}} A) = (\Delta, x :_{\text{U}} A)}$$

$$\frac{\Delta_1 \cup \Delta_2 = \Delta \quad x \in \text{fresh}(\Delta)}{(\Delta_1, x :_{\text{L}} A) \cup \Delta_2 = (\Delta, x :_{\text{L}} A)} \quad \frac{\Delta_1 \cup \Delta_2 = \Delta \quad x \in \text{fresh}(\Delta)}{\Delta_1 \cup (\Delta_2, x :_{\text{L}} A) = (\Delta, x :_{\text{L}} A)}$$

#### Context Constraint

$$\frac{}{\epsilon \triangleright s} \quad \frac{\Delta \triangleright \text{U}}{\Delta, x :_{\text{U}} A \triangleright \text{U}} \quad \frac{\Delta \triangleright \text{L}}{\Delta, x :_s A \triangleright \text{L}}$$

#### Dynamic Typing

$$\frac{\Gamma, x : A; \Delta, x :_s A \vdash \quad \Delta \triangleright \text{U}}{\Gamma, x : A; \Delta, x :_s A \vdash x : A} \quad \frac{\Gamma, x : A; \Delta, x :_s A \vdash m : B \quad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t(x : A).m : \Pi_t(x : A).B} \quad \frac{\Gamma, x : A; \Delta \vdash m : B \quad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B}$$

$$\frac{\Gamma; \Delta_1 \vdash m : \Pi_t(x : A).B \quad \Gamma; \Delta_2 \vdash n : A}{\Gamma; \Delta_1 \cup \Delta_2 \vdash m \ n : B[n/x]} \quad \frac{\Gamma; \Delta \vdash m : \Pi_t\{x : A\}.B \quad \Gamma \vdash n : A}{\Gamma; \Delta \vdash m \ n : B[n/x]}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).B : t \quad \Gamma; \Delta_1 \vdash m : A \quad \Gamma; \Delta_2 \vdash n : B[m/x]}{\Gamma; \Delta_1 \cup \Delta_2 \vdash \langle m, n \rangle_t : \Sigma_t(x : A).B}$$

$$\frac{\Gamma \vdash \Sigma_t\{x : A\}.B : t \quad \Gamma; \Delta \vdash m : A \quad \Gamma \vdash n : B[m/x]}{\Gamma; \Delta \vdash \{m, n\}_t : \Sigma_t\{x : A\}.B}$$

$$\frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \quad \Gamma; \Delta_1 \vdash m : \Sigma_t(x : A).B \quad \Gamma, x : A, y : B; \Delta_2, x :_{r_1} A, y :_{r_2} B \vdash n : C[\langle x, y \rangle_t / z]}{\Gamma; \Delta_1 \cup \Delta_2 \vdash \text{R}_{[z]C}^\Sigma(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \quad \Gamma; \Delta_1 \vdash m : \Sigma_t\{x : A\}.B \quad \Gamma, x : A, y : B; \Delta_2, x :_r A \vdash n : C[\{x, y\}_t / z]}{\Gamma; \Delta_1 \cup \Delta_2 \vdash \text{R}_{[z]C}^\Sigma(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma; \Delta \vdash m : A \quad \Gamma; \Delta \vdash n : B \quad \Delta \triangleright t}{\Gamma; \Delta \vdash (m, n)_t : A \&_t B} \quad \frac{\Gamma; \Delta \vdash m : A \&_t B}{\Gamma; \Delta \vdash \pi_1 m : A} \quad \frac{\Gamma; \Delta \vdash m : A \&_t B}{\Gamma; \Delta \vdash \pi_2 m : B}$$

$$\frac{\Gamma \vdash B : s \quad \Gamma; \Delta \vdash m : A \quad A \simeq B}{\Gamma; \Delta \vdash m : B}$$

## 4 Erasure

### Erasure Relation

$$\begin{array}{c}
\frac{\Gamma, x : A; \Delta, x :_s A \vdash \quad \Delta \triangleright U}{\Gamma, x : A; \Delta, x :_s A \vdash x \sim x : A} \quad \frac{\Gamma, x : A; \Delta, x :_s A \vdash m \sim m' : B \quad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t(x : A).m \sim \lambda_t(x : \square).m' : \Pi_t(x : A).B} \\
\\
\frac{\Gamma, x : A; \Delta \vdash m \sim m' : B \quad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t\{x : A\}.m \sim \lambda_t\{x : \square\}.m' : \Pi_t\{x : A\}.B} \quad \frac{\Gamma; \Delta_1 \vdash m \sim m' : \Pi_t(x : A).B \quad \Gamma; \Delta_2 \vdash n \sim n' : A}{\Gamma; \Delta_1 \cup \Delta_2 \vdash m \ n \sim m' \ n' : B[n/x]} \\
\\
\frac{\Gamma; \Delta \vdash m \sim m' : \Pi_t\{x : A\}.B \quad \Gamma \vdash n : A}{\Gamma; \Delta \vdash m \ n \sim m' \ \square : B[n/x]} \\
\\
\frac{\Gamma \vdash \Sigma_t(x : A).B : t \quad \Gamma; \Delta_1 \vdash m \sim m' : A \quad \Gamma; \Delta_2 \vdash n \sim n' : B[m/x]}{\Gamma; \Delta_1 \cup \Delta_2 \vdash \langle m, n \rangle_t \sim \langle m', n' \rangle_t : \Sigma_t(x : A).B} \\
\\
\frac{\Gamma \vdash \Sigma_t\{x : A\}.B : t \quad \Gamma; \Delta \vdash m \sim m' : A \quad \Gamma \vdash n : B[m/x]}{\Gamma; \Delta \vdash \{m, n\}_t \sim \{m', \square\}_t : \Sigma_t\{x : A\}.B} \\
\\
\frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \quad \Gamma; \Delta_1 \vdash m \sim m' : \Sigma_t(x : A).B \quad \Gamma, x : A, y : B; \Delta_2, x :_{r1} A, y :_{r2} B \vdash n \sim n' : C[\langle x, y \rangle_t / z]}{\Gamma; \Delta_1 \cup \Delta_2 \vdash R_{[z]C}^\Sigma(m, [x, y]n) \sim R_{[z]\square}^\Sigma(m', [x, y]n') : C[m/z]} \\
\\
\frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \quad \Gamma; \Delta_1 \vdash m \sim m' : \Sigma_t\{x : A\}.B \quad \Gamma, x : A, y : B; \Delta_2, x :_r A \vdash n \sim n' : C[\{x, y\}_t / z]}{\Gamma; \Delta_1 \cup \Delta_2 \vdash R_{[z]C}^\Sigma(m, [x, y]n) \sim R_{[z]\square}^\Sigma(m', [x, y]n') : C[m/z]} \\
\\
\frac{\Gamma; \Delta \vdash m \sim m' : A \quad \Gamma; \Delta \vdash n \sim n' : B \quad \Delta \triangleright t}{\Gamma; \Delta \vdash (m, n)_t \sim (m', n')_t : A \&_t B} \quad \frac{\Gamma; \Delta \vdash m \sim m' : A \&_t B}{\Gamma; \Delta \vdash \pi_1 m \sim \pi_1 m' : A} \quad \frac{\Gamma; \Delta \vdash m \sim m' : A \&_t B}{\Gamma; \Delta \vdash \pi_2 m \sim \pi_2 m' : B} \\
\\
\frac{\Gamma \vdash B : s \quad \Gamma; \Delta \vdash m \sim m' : A \quad A \simeq B}{\Gamma; \Delta \vdash m \sim m' : B}
\end{array}$$

## 5 Static Semantics

### Static Reduction

$$\begin{array}{c}
\frac{A \rightsquigarrow A'}{\Pi_t(x : A).B \rightsquigarrow \Pi_t(x : A').B} \quad \frac{B \rightsquigarrow B'}{\Pi_t(x : A).B \rightsquigarrow \Pi_t(x : A).B'} \quad \frac{A \rightsquigarrow A'}{\Pi_t\{x : A\}.B \rightsquigarrow \Pi_t\{x : A'\}.B} \\
\\
\frac{B \rightsquigarrow B'}{\Pi_t\{x : A\}.B \rightsquigarrow \Pi_t\{x : A'\}.B'} \quad \frac{A \rightsquigarrow A'}{\lambda_t(x : A).m \rightsquigarrow \lambda_t(x : A').m} \quad \frac{m \rightsquigarrow m'}{\lambda_t(x : A).m \rightsquigarrow \lambda_t(x : A).m'} \\
\\
\frac{A \rightsquigarrow A'}{\lambda_t\{x : A\}.m \rightsquigarrow \lambda_t\{x : A'\}.m} \quad \frac{m \rightsquigarrow m'}{\lambda_t\{x : A\}.m \rightsquigarrow \lambda_t\{x : A'\}.m'} \quad \frac{m \rightsquigarrow m'}{m \ n \rightsquigarrow m' \ n} \quad \frac{n \rightsquigarrow n'}{m \ n \rightsquigarrow m \ n'} \\
\\
\frac{}{(\lambda_t(x : A).m) \ n \rightsquigarrow m[n/x]} \quad \frac{}{(\lambda_t\{x : A\}.m) \ n \rightsquigarrow m[n/x]} \quad \frac{A \rightsquigarrow A'}{\Sigma_t(x : A).B \rightsquigarrow \Sigma_t(x : A').B} \\
\\
\frac{B \rightsquigarrow B'}{\Sigma_t(x : A).B \rightsquigarrow \Sigma_t(x : A).B'} \quad \frac{A \rightsquigarrow A'}{\Sigma_t\{x : A\}.B \rightsquigarrow \Sigma_t\{x : A'\}.B} \quad \frac{B \rightsquigarrow B'}{\Sigma_t\{x : A\}.B \rightsquigarrow \Sigma_t\{x : A'\}.B'} \\
\\
\frac{m \rightsquigarrow m'}{\langle m, n \rangle_t \rightsquigarrow \langle m', n \rangle_t} \quad \frac{n \rightsquigarrow n'}{\langle m, n \rangle_t \rightsquigarrow \langle m, n' \rangle_t} \quad \frac{m \rightsquigarrow m'}{\{m, n\}_t \rightsquigarrow \{m', n\}_t} \quad \frac{n \rightsquigarrow n'}{\{m, n\}_t \rightsquigarrow \{m, n'\}_t} \\
\\
\frac{A \rightsquigarrow A'}{R_{[z]A}^\Sigma(m, [x, y]n) \rightsquigarrow R_{[z]A'}^\Sigma(m, [x, y]n)} \quad \frac{m \rightsquigarrow m'}{R_{[z]A}^\Sigma(m, [x, y]n) \rightsquigarrow R_{[z]A}^\Sigma(m', [x, y]n)} \\
\\
\frac{n \rightsquigarrow n'}{R_{[z]A}^\Sigma(m, [x, y]n) \rightsquigarrow R_{[z]A}^\Sigma(m, [x, y]n')} \quad \frac{}{R_{[z]A}^\Sigma(\langle m_1, m_2 \rangle_t, [x, y]n) \rightsquigarrow n[m_1/x, m_2/y]} \\
\\
\frac{}{R_{[z]A}^\Sigma(\{m_1, m_2\}_t, [x, y]n) \rightsquigarrow n[m_1/x, m_2/y]} \quad \frac{A \rightsquigarrow A'}{A \ \&_t B \rightsquigarrow A' \ \&_t B} \quad \frac{B \rightsquigarrow B'}{A \ \&_t B \rightsquigarrow A \ \&_t B'} \quad \frac{m \rightsquigarrow m'}{(m, n)_t \rightsquigarrow (m', n)_t} \\
\\
\frac{n \rightsquigarrow n'}{(m, n)_t \rightsquigarrow (m, n')_t} \quad \frac{m \rightsquigarrow m'}{\pi_1 m \rightsquigarrow \pi_1 m'} \quad \frac{m \rightsquigarrow m'}{\pi_2 m \rightsquigarrow \pi_2 m'} \quad \frac{}{\pi_1 (m, n)_t \rightsquigarrow m} \quad \frac{}{\pi_2 (m, n)_t \rightsquigarrow n} \\
\\
\frac{A \rightsquigarrow A'}{m \equiv_A n \rightsquigarrow m \equiv_{A'} n} \quad \frac{m \rightsquigarrow m'}{m \equiv_A n \rightsquigarrow m' \equiv_A n} \quad \frac{n \rightsquigarrow n'}{m \equiv_A n \rightsquigarrow m \equiv_A n'} \\
\\
\frac{A \rightsquigarrow A'}{R_{[x, y, p]A}^\equiv([z]H, P) \rightsquigarrow R_{[x, y, p]A'}^\equiv([z]H, P)} \quad \frac{H \rightsquigarrow H'}{R_{[x, y, p]A}^\equiv([z]H, P) \rightsquigarrow R_{[x, y, p]A}^\equiv([z]H', P)} \\
\\
\frac{P \rightsquigarrow P'}{R_{[x, y, p]A}^\equiv([z]H, P) \rightsquigarrow R_{[x, y, p]A}^\equiv([z]H, P')} \quad \frac{}{R_{[x, y, p]A}^\equiv([z]H, \text{refl } m) \rightsquigarrow H[m/z]}
\end{array}$$

### Conversion

$$\frac{}{A \simeq A} \quad \frac{A \simeq B \quad B \rightsquigarrow C}{A \simeq C} \quad \frac{A \simeq B \quad C \rightsquigarrow B}{A \simeq C}$$

## 6 Dynamic Semantics

### Value

$$\overline{x \text{ value}} \quad \overline{\lambda_t(x : A).m \text{ value}} \quad \overline{\lambda_t\{x : A\}.m \text{ value}} \quad \frac{u \text{ value} \quad v \text{ value}}{\langle u, v \rangle_t \text{ value}} \quad \frac{v \text{ value}}{\{v, m\}_t \text{ value}} \quad \overline{(m, n)_t \text{ value}}$$

### Dynamic Reduction

$$\begin{array}{c} \frac{m \rightsquigarrow m'}{m \ n \rightsquigarrow m' \ n} \quad \frac{n \rightsquigarrow n'}{m \ n \rightsquigarrow m \ n'} \quad \frac{v \text{ value}}{(\lambda_t(x : A).m) \ v \rightsquigarrow m[v/x]} \quad \frac{}{(\lambda_t\{x : A\}.m) \ n \rightsquigarrow m[n/x]} \\[10pt] \frac{m \rightsquigarrow m'}{\langle m, n \rangle_t \rightsquigarrow \langle m', n \rangle_t} \quad \frac{n \rightsquigarrow n'}{\langle m, n \rangle_t \rightsquigarrow \langle m, n' \rangle_t} \quad \frac{m \rightsquigarrow m'}{\{m, n\}_t \rightsquigarrow \{m', n\}_t} \quad \frac{m \rightsquigarrow m'}{R_{[z]A}^\Sigma(m, [x, y]n) \rightsquigarrow R_{[z]A}^\Sigma(m', [x, y]n)} \\[10pt] \frac{u \text{ value} \quad v \text{ value}}{R_{[z]A}^\Sigma(\langle u, v \rangle_t, [x, y]n) \rightsquigarrow n[u/x, v/y]} \quad \frac{v \text{ value}}{R_{[z]A}^\Sigma(\{v, m\}_t, [x, y]n) \rightsquigarrow n[v/x, m/y]} \quad \frac{m \rightsquigarrow m'}{\pi_1 m \rightsquigarrow \pi_1 m'} \\[10pt] \frac{m \rightsquigarrow m'}{\pi_2 m \rightsquigarrow \pi_2 m'} \quad \frac{}{\pi_1 (m, n)_t \rightsquigarrow m} \quad \frac{}{\pi_2 (m, n)_t \rightsquigarrow n} \end{array}$$

### Erasure-Dynamic Reduction

**Theorem 1.** For any dynamic typing  $\Gamma; \Delta \vdash m : A$ , there exists  $m'$  such that  $\Gamma; \Delta \vdash m \sim m' : A$ .

**Theorem 2.** For any erasure relation  $\Gamma; \Delta \vdash m \sim m' : A$  and dynamic reduction  $m' \rightsquigarrow n'$ , there exists  $n$  such that the following diagram commutes.

$$\begin{array}{ccc} \Gamma; \Delta \vdash m & \sim & m' : A \\ \Downarrow & & \Downarrow \\ \Gamma; \Delta \vdash n & \sim & n' : A \end{array}$$

**Theorem 3.** For any erasure relation  $\epsilon; \epsilon \vdash m \sim m' : A$ , then  $m'$  is a value or there exists  $n'$  such that  $m' \rightsquigarrow n'$ .