A Two Level Dependent Session Type Theory

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1 Syntax

$$
\begin{array}{rlcl}
\text{variables} & x, y, z & & \\
\text{channels} & c, d & & \\
\text{sorts} & s, r, t & ::= & \mathrm{U} \mid \mathrm{L} \\
\text{roles} & \rho & ::= & + \mid - \\
\text{terms} & m, n, A, B, C & ::= & x \mid c \mid s \\
 & & \mid & \Pi_t(x : A).B \mid \Pi_t\{x : A\}.B \mid \Sigma_t(x : A).B \mid \Sigma_t\{x : A\}.B \\
 & & \mid & \lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid \langle m, n\rangle_t \mid \{m, n\}_t \\
 & & \mid & m\ n \mid \mathrm{R}^{\Sigma}_{[z]A}(m, [x, y]n) \mid \mu(x : A).m \\
 & & \mid & 1 \mid () \mid 2 \mid \mathrm{true} \mid \mathrm{false} \mid \mathrm{R}^{2}_{[z]A}(m, n_1, n_2) \\
 & & \mid & \mathrm{T}\ A \mid \mathrm{return}\ m \mid \mathrm{let}\, x \leftarrow m\ \mathrm{in}\ n \\
 & & \mid & \mathrm{proto} \mid \mathrm{end}\rho \mid \rho\{x : A\}.B \mid \rho(x : A).B \mid \rho\mathrm{Ch}\, A \\
 & & \mid & \mathrm{fork}\,(x : A).m \mid \mathrm{recv}\,\{m\} \mid \mathrm{recv}\, m \mid \mathrm{send}\,\{m\} \mid \mathrm{send}\, m \\
 & & \mid & \mathrm{close}\, m \mid \mathrm{wait}\, m \\
\text{values} & u, v & ::= & x \mid c \mid \lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid \langle u, v\rangle_t \mid \{v, m\}_t \\
 & & \mid & () \mid \mathrm{true} \mid \mathrm{false} \mid \mathrm{return}\ v \mid \mathrm{let}\, x \leftarrow v\ \mathrm{in}\ m \\
 & & \mid & \mathrm{fork}\,(x : A).m \mid \mathrm{recv}\,\{v\} \mid \mathrm{recv}\, v \mid \mathrm{send}\,\{v\} \mid \mathrm{send}\, v \\
 & & \mid & \mathrm{send}\,\{v\}\ m \mid \mathrm{send}\, u\ v \mid \mathrm{close}\, v \mid \mathrm{wait}\, v
\end{array}
$$

2 Static Fragment

Sort Order

$$\frac{}{\mathrm{U} \sqsubseteq s} \qquad \frac{}{\mathrm{L} \sqsubseteq \mathrm{L}}$$

Static Context

$$\frac{}{\epsilon \vdash} \qquad \frac{\Gamma \vdash \quad \Gamma \vdash A : s \quad x \in \mathit{fresh}(\Gamma)}{\Gamma, x : A \vdash}$$

Core Typing

$$\frac{\Gamma \vdash}{\Gamma \vdash s : \mathrm{U}} \qquad \frac{\Gamma, x : A \vdash}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t(x : A).B : t} \qquad \frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t\{x : A\}.B : t}$$

$$\frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t(x : A).m : \Pi_t(x : A).B} \qquad \frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B} \qquad \frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash m : A}{\Gamma \vdash \mu(x : A).m : A}$$

$$\frac{\Gamma \vdash m : \Pi_t(x : A).B \quad \Gamma \vdash n : A}{\Gamma \vdash m\ n : B[n/x]} \qquad \frac{\Gamma \vdash m : \Pi_t\{x : A\}.B \quad \Gamma \vdash n : A}{\Gamma \vdash m\ n : B[n/x]}$$

$$\frac{s \sqsubseteq t \quad r \sqsubseteq t \quad \Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t(x : A).B : t} \qquad \frac{r \sqsubseteq t \quad \Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t\{x : A\}.B : t}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).B \quad \Gamma \vdash m : A \quad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \langle m, n\rangle_t : \Sigma_t(x : A).B} \qquad \frac{\Gamma \vdash \Sigma_t\{x : A\}.B \quad \Gamma \vdash m : A \quad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \{m, n\}_t : \Sigma_t\{x : A\}.B}$$

$$\frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \quad \Gamma \vdash m : \Sigma_t(x : A).B \quad \Gamma, x : A, y : B \vdash n : C[\langle x, y\rangle_t/z]}{\Gamma \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \quad \Gamma \vdash m : \Sigma_t\{x : A\}.B \quad \Gamma, x : A, y : B \vdash n : C[\{x, y\}_t/z]}{\Gamma \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma \vdash B : s \quad \Gamma \vdash m : A \quad A \simeq B}{\Gamma \vdash m : B}$$

Data Typing

$$\frac{\Gamma \vdash}{\Gamma \vdash 1 : \mathrm{U}} \qquad \frac{\Gamma \vdash}{\Gamma \vdash () : 1} \qquad \frac{\Gamma \vdash}{\Gamma \vdash 2 : \mathrm{U}} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathrm{true} : 2} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathrm{false} : 2}$$

$$\frac{\Gamma, z : 2 \vdash A : s \qquad \Gamma \vdash m : 2 \qquad \Gamma \vdash n_1 : A[\mathrm{true}/z] \qquad \Gamma \vdash n_2 : A[\mathrm{false}/z]}{\Gamma \vdash \mathrm{R}^2_{[z]A}(m, n_1, n_2) : A[m/z]}$$

Monadic Typing

$$\frac{\Gamma \vdash A : s}{\Gamma \vdash \mathrm{T}\,A : \mathrm{L}} \qquad \frac{\Gamma \vdash m : A}{\Gamma \vdash \mathrm{return}\ m : \mathrm{T}\,A} \qquad \frac{\Gamma \vdash B : s \qquad \Gamma \vdash m : \mathrm{T}\,A \qquad \Gamma, x : A \vdash n : \mathrm{T}\,B}{\Gamma \vdash \mathrm{let}\, x : A \leftarrow m \ \mathrm{in}\ n : \mathrm{T}\,B}$$

Session Typing

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathrm{proto} : \mathrm{U}} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathrm{end}\rho : \mathrm{proto}} \qquad \frac{\Gamma, x : A \vdash B : \mathrm{proto}}{\Gamma \vdash \rho\{x : A\}.B : \mathrm{proto}} \qquad \frac{\Gamma, x : A \vdash B : \mathrm{proto}}{\Gamma \vdash \rho(x : A).B : \mathrm{proto}} \qquad \frac{\Gamma \vdash A : \mathrm{proto}}{\Gamma \vdash \rho\mathrm{Ch}\, A : \mathrm{L}}$$

$$\frac{\Gamma \vdash \qquad \epsilon \vdash A : \mathrm{proto}}{\Gamma \vdash c : \rho\mathrm{Ch}\, A} \qquad \frac{\Gamma, x : +\mathrm{Ch}\, A \vdash m : \mathrm{T}\, 1}{\Gamma \vdash \mathrm{fork}\,(x : A).m : \mathrm{T}\,(-\mathrm{Ch}\, A)}$$

$$\frac{\Gamma \vdash m : \rho_1\mathrm{Ch}\,(\rho_2\{x : A\}.B) \qquad \rho_1\,\mathrm{xor}\,\rho_2 = ?}{\Gamma \vdash \mathrm{recv}\,\{m\} : \mathrm{T}\,(\Sigma_{\mathrm{L}}\{x : A\}.\rho_1\mathrm{Ch}\, B)} \qquad \frac{\Gamma \vdash m : \rho_1\mathrm{Ch}\,(\rho_2(x : A).B) \qquad \rho_1\,\mathrm{xor}\,\rho_2 = ?}{\Gamma \vdash \mathrm{recv}\, m : \mathrm{T}\,(\Sigma_{\mathrm{L}}(x : A).\rho_1\mathrm{Ch}\, B)}$$

$$\frac{\Gamma \vdash m : \rho_1\mathrm{Ch}\,(\rho_2\{x : A\}.B) \qquad \rho_1\,\mathrm{xor}\,\rho_2 = !}{\Gamma \vdash \mathrm{send}\,\{m\} : \Pi_{\mathrm{L}}\{x : A\}.\mathrm{T}\,(\rho_1\mathrm{Ch}\, B)} \qquad \frac{\Gamma \vdash m : \rho_1\mathrm{Ch}\,(\rho_2(x : A).B) \qquad \rho_1\,\mathrm{xor}\,\rho_2 = !}{\Gamma \vdash \mathrm{send}\, m : \Pi_{\mathrm{L}}(x : A).\mathrm{T}\,(\rho_1\mathrm{Ch}\, B)}$$

$$\frac{\Gamma \vdash m : \rho_1\mathrm{Ch}\,(\mathrm{end}\rho_2) \qquad \rho_1\,\mathrm{xor}\,\rho_2 = ?}{\Gamma \vdash \mathrm{wait}\, m : \mathrm{T}\, 1} \qquad \frac{\Gamma \vdash m : \rho_1\mathrm{Ch}\,(\mathrm{end}\rho_2) \qquad \rho_1\,\mathrm{xor}\,\rho_2 = !}{\Gamma \vdash \mathrm{close}\, m : \mathrm{T}\, 1}$$

3 Dynamic Fragment

Dynamic Context

$$\frac{}{\epsilon;\epsilon\vdash} \qquad \frac{\Gamma;\Delta\vdash \qquad \Gamma\vdash A:s \qquad x\in \mathit{fresh}(\Gamma)}{\Gamma,x:A;\Delta,x:_s A\vdash} \qquad \frac{\Gamma;\Delta\vdash \qquad \Gamma\vdash A:s \qquad x\in \mathit{fresh}(\Gamma)}{\Gamma,x:A;\Delta\vdash}$$

Context Merge

$$\frac{}{\epsilon\uplus\epsilon=\epsilon} \qquad \frac{\Delta_1\uplus\Delta_2=\Delta \qquad x\in\mathit{fresh}(\Delta)}{(\Delta_1,x:_{\mathrm{U}} A)\uplus(\Delta_2,x:_{\mathrm{U}} A)=(\Delta,x:_{\mathrm{U}} A)}$$

$$\frac{\Delta_1\uplus\Delta_2=\Delta \qquad x\in\mathit{fresh}(\Delta)}{(\Delta_1,x:_{\mathrm{L}} A)\uplus\Delta_2=(\Delta,x:_{\mathrm{L}} A)} \qquad \frac{\Delta_1\uplus\Delta_2=\Delta \qquad x\in\mathit{fresh}(\Delta)}{\Delta_1\uplus(\Delta_2,x:_{\mathrm{L}} A)=(\Delta,x:_{\mathrm{L}} A)}$$

Context Constraint

$$\frac{}{\epsilon\triangleright s} \qquad \frac{\Delta\triangleright\mathrm{U}}{\Delta,x:_{\mathrm{U}} A\triangleright\mathrm{U}} \qquad \frac{\Delta\triangleright\mathrm{L}}{\Delta,x:_s A\triangleright\mathrm{L}}$$

Dynamic Typing

$$\frac{\Gamma,x:A;\Delta,x:_s A\vdash \qquad \Delta\triangleright\mathrm{U}}{\Gamma,x:A;\Delta,x:_s A\vdash x:A} \qquad \frac{\Gamma,x:A;\Delta,x:_s A\vdash m:B \qquad \Delta\triangleright t}{\Gamma;\Delta\vdash\lambda_t(x:A).m:\Pi_t(x:A).B} \qquad \frac{\Gamma,x:A;\Delta\vdash m:B \qquad \Delta\triangleright t}{\Gamma;\Delta\vdash\lambda_t\{x:A\}.m:\Pi_t\{x:A\}.B}$$

$$\frac{\Gamma;\Delta_1\vdash m:\Pi_t(x:A).B \qquad \Gamma;\Delta_2\vdash n:A}{\Gamma;\Delta_1\uplus\Delta_2\vdash m\ n:B[n/x]} \qquad \frac{\Gamma;\Delta\vdash m:\Pi_t\{x:A\}.B \qquad \Gamma\vdash n:A}{\Gamma;\Delta\vdash m\ n:B[n/x]}$$

$$\frac{\Gamma\vdash\Sigma_t(x:A).B:t \qquad \Gamma;\Delta_1\vdash m:A \qquad \Gamma;\Delta_2\vdash n:B[m/x]}{\Gamma;\Delta_1\uplus\Delta_2\vdash\langle m,n\rangle_t:\Sigma_t(x:A).B}$$

$$\frac{\Gamma\vdash\Sigma_t\{x:A\}.B:t \qquad \Gamma;\Delta\vdash m:A \qquad \Gamma\vdash n:B[m/x]}{\Gamma;\Delta\vdash\{m,n\}_t:\Sigma_t\{x:A\}.B}$$

$$\frac{\Gamma,z:\Sigma_t(x:A).B\vdash C:s \qquad \Gamma;\Delta_1\vdash m:\Sigma_t(x:A).B \qquad \Gamma,x:A,y:B;\Delta_2,x:_{r1} A,y:_{r2} B\vdash n:C[\langle x,y\rangle_t/z]}{\Gamma;\Delta_1\uplus\Delta_2\vdash \mathrm{R}^{\Sigma}_{[z]C}(m,[x,y]n):C[m/z]}$$

$$\frac{\Gamma,z:\Sigma_t\{x:A\}.B\vdash C:s \qquad \Gamma;\Delta_1\vdash m:\Sigma_t\{x:A\}.B \qquad \Gamma,x:A,y:B;\Delta_2,x:_r A\vdash n:C[\{x,y\}_t/z]}{\Gamma;\Delta_1\uplus\Delta_2\vdash \mathrm{R}^{\Sigma}_{[z]C}(m,[x,y]n):C[m/z]}$$

$$\frac{\Gamma;\Delta\vdash m:A \qquad \Gamma;\Delta\vdash n:B \qquad \Delta\triangleright t}{\Gamma;\Delta\vdash(m,n)_t:A\ \&_t B} \qquad \frac{\Gamma;\Delta\vdash m:A\ \&_t B}{\Gamma;\Delta\vdash\pi_1\,m:A} \qquad \frac{\Gamma;\Delta\vdash m:A\ \&_t B}{\Gamma;\Delta\vdash\pi_2\,m:B}$$

$$\frac{\Gamma,x:A,p:m=_A x\vdash B:s \qquad \Gamma;\Delta\vdash H:B[m/x,\mathrm{refl}\ m/p] \qquad \Gamma\vdash P:m=_A n}{\Gamma;\Delta\vdash\mathbf{R}^{=}_{[x,p]B}(H,P):B[n/x,P/p]}$$

$$\frac{\Gamma\vdash B:s \qquad \Gamma;\Delta\vdash m:A \qquad A\equiv B}{\Gamma;\Delta\vdash m:B}$$

4 Erasure

Erasure Relation

$$\frac{\Gamma, x : A; \Delta, x :_s A \vdash \qquad \Delta \triangleright \mathrm{U}}{\Gamma, x : A; \Delta, x :_s A \vdash x \sim x : A} \qquad \frac{\Gamma, x : A; \Delta, x :_s A \vdash m \sim m' : B \qquad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t(x : A).m \sim \lambda_t(x : \Box).m' : \Pi_t(x : A).B}$$

$$\frac{\Gamma, x : A; \Delta \vdash m \sim m' : B \qquad \Delta \triangleright t}{\Gamma; \Delta \vdash \lambda_t\{x : A\}.m \sim \lambda_t\{x : \Box\}.m' : \Pi_t\{x : A\}.B} \qquad \frac{\Gamma; \Delta_1 \vdash m \sim m' : \Pi_t(x : A).B \qquad \Gamma; \Delta_2 \vdash n \sim n' : A}{\Gamma; \Delta_1 \cup \Delta_2 \vdash m\ n \sim m'\ n' : B[n/x]}$$

$$\frac{\Gamma; \Delta \vdash m \sim m' : \Pi_t\{x : A\}.B \qquad \Gamma \vdash n : A}{\Gamma; \Delta \vdash m\ n \sim m'\ \Box : B[n/x]}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).B : t \qquad \Gamma; \Delta_1 \vdash m \sim m' : A \qquad \Gamma; \Delta_2 \vdash n \sim n' : B[m/x]}{\Gamma; \Delta_1 \cup \Delta_2 \vdash \langle m, n\rangle_t \sim \langle m', n'\rangle_t : \Sigma_t(x : A).B}$$

$$\frac{\Gamma \vdash \Sigma_t\{x : A\}.B : t \qquad \Gamma; \Delta \vdash m \sim m' : A \qquad \Gamma \vdash n : B[m/x]}{\Gamma; \Delta \vdash \{m, n\}_t \sim \{m', \Box\}_t : \Sigma_t\{x : A\}.B}$$

$$\frac{\begin{array}{c}\Gamma, z : \Sigma_t(x : A).B \vdash C : s \\ \Gamma; \Delta_1 \vdash m \sim m' : \Sigma_t(x : A).B \qquad \Gamma, x : A, y : B; \Delta_2, x :_{r1} A, y :_{r2} B \vdash n \sim n' : C[\langle x, y\rangle_t/z]\end{array}}{\Gamma; \Delta_1 \cup \Delta_2 \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) \sim \mathrm{R}^{\Sigma}_{\Box}(m', [x, y]n') : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \qquad \Gamma; \Delta_1 \vdash m \sim m' : \Sigma_t\{x : A\}.B \qquad \Gamma, x : A, y : B; \Delta_2, x :_r A \vdash n \sim n' : C[\{x, y\}_t/z]}{\Gamma; \Delta_1 \cup \Delta_2 \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) \sim \mathrm{R}^{\Sigma}_{\Box}(m', [x, y]n') : C[m/z]}$$

$$\frac{\Gamma; \Delta \vdash m \sim m' : A \qquad \Gamma; \Delta \vdash n \sim n' : B \qquad \Delta \triangleright t}{\Gamma; \Delta \vdash (m, n)_t \sim (m', n')_t : A \mathbin{\&_t} B} \qquad \frac{\Gamma; \Delta \vdash m \sim m' : A \mathbin{\&_t} B}{\Gamma; \Delta \vdash \pi_1\, m \sim \pi_1\, m' : A} \qquad \frac{\Gamma; \Delta \vdash m \sim m' : A \mathbin{\&_t} B}{\Gamma; \Delta \vdash \pi_2\, m \sim \pi_2\, m' : B}$$

$$\frac{\Gamma, x : A, p : m =_A x \vdash B : s \qquad \Gamma; \Delta \vdash H \sim H' : B[m/x, \mathrm{refl}\ m/p] \qquad \Gamma \vdash P : m =_A n}{\Gamma; \Delta \vdash \mathbf{R}^{=}_{[x,p]B}(H, P) \sim \mathbf{R}^{=}_{\Box}(H', \Box) : B[n/x, P/p]}$$

$$\frac{\Gamma \vdash B : s \qquad \Gamma; \Delta \vdash m \sim m' : A \qquad A \equiv B}{\Gamma; \Delta \vdash m \sim m' : B}$$

5 Static Semantics

Static Reduction

$$\frac{A \rightsquigarrow A'}{\Pi_t(x:A).B \rightsquigarrow \Pi_t(x:A').B} \qquad \frac{B \rightsquigarrow B'}{\Pi_t(x:A).B \rightsquigarrow \Pi_t(x:A).B'} \qquad \frac{A \rightsquigarrow A'}{\Pi_t\{x:A\}.B \rightsquigarrow \Pi_t\{x:A'\}.B}$$

$$\frac{B \rightsquigarrow B'}{\Pi_t\{x:A\}.B \rightsquigarrow \Pi_t\{x:A\}.B'} \qquad \frac{A \rightsquigarrow A'}{\lambda_t(x:A).m \rightsquigarrow \lambda_t(x:A').m} \qquad \frac{m \rightsquigarrow m'}{\lambda_t(x:A).m \rightsquigarrow \lambda_t(x:A).m'}$$

$$\frac{A \rightsquigarrow A'}{\lambda_t\{x:A\}.m \rightsquigarrow \lambda_t\{x:A'\}.m} \qquad \frac{m \rightsquigarrow m'}{\lambda_t\{x:A\}.m \rightsquigarrow \lambda_t\{x:A\}.m'} \qquad \frac{m \rightsquigarrow m'}{m\ n \rightsquigarrow m'\ n} \qquad \frac{n \rightsquigarrow n'}{m\ n \rightsquigarrow m\ n'}$$

$$\frac{}{(\lambda_t(x:A).m)\ n \rightsquigarrow m[n/x]} \qquad \frac{}{(\lambda_t\{x:A\}.m)\ n \rightsquigarrow m[n/x]} \qquad \frac{A \rightsquigarrow A'}{\Sigma_t(x:A).B \rightsquigarrow \Sigma_t(x:A').B}$$

$$\frac{B \rightsquigarrow B'}{\Sigma_t(x:A).B \rightsquigarrow \Sigma_t(x:A).B'} \qquad \frac{A \rightsquigarrow A'}{\Sigma_t\{x:A\}.B \rightsquigarrow \Sigma_t\{x:A'\}.B} \qquad \frac{B \rightsquigarrow B'}{\Sigma_t\{x:A\}.B \rightsquigarrow \Sigma_t\{x:A\}.B'}$$

$$\frac{m \rightsquigarrow m'}{\langle m,n\rangle_t \rightsquigarrow \langle m',n\rangle_t} \qquad \frac{n \rightsquigarrow n'}{\langle m,n\rangle_t \rightsquigarrow \langle m,n'\rangle_t} \qquad \frac{m \rightsquigarrow m'}{\{m,n\}_t \rightsquigarrow \{m',n\}_t} \qquad \frac{n \rightsquigarrow n'}{\{m,n\}_t \rightsquigarrow \{m,n'\}_t}$$

$$\frac{A \rightsquigarrow A'}{\mathrm{R}^{\Sigma}_{[z]A}(m,[x,y]n) \rightsquigarrow \mathrm{R}^{\Sigma}_{[z]A'}(m,[x,y]n)} \qquad \frac{m \rightsquigarrow m'}{\mathrm{R}^{\Sigma}_{[z]A}(m,[x,y]n) \rightsquigarrow \mathrm{R}^{\Sigma}_{[z]A}(m',[x,y]n)}$$

$$\frac{n \rightsquigarrow n'}{\mathrm{R}^{\Sigma}_{[z]A}(m,[x,y]n) \rightsquigarrow \mathrm{R}^{\Sigma}_{[z]A}(m,[x,y]n')} \qquad \frac{}{\mathrm{R}^{\Sigma}_{[z]A}(\langle m_1,m_2\rangle_t,[x,y]n) \rightsquigarrow n[m_1/x,m_2/y]}$$

$$\frac{}{\mathrm{R}^{\Sigma}_{[z]A}(\{m_1,m_2\}_t,[x,y]n) \rightsquigarrow n[m_1/x,m_2/y]} \qquad \frac{A \rightsquigarrow A'}{A \mathbin{\&_t} B \rightsquigarrow A' \mathbin{\&_t} B} \qquad \frac{B \rightsquigarrow B'}{A \mathbin{\&_t} B \rightsquigarrow A \mathbin{\&_t} B'} \qquad \frac{m \rightsquigarrow m'}{(m,n)_t \rightsquigarrow (m',n)_t}$$

$$\frac{n \rightsquigarrow n'}{(m,n)_t \rightsquigarrow (m,n')_t} \qquad \frac{m \rightsquigarrow m'}{\pi_1\, m \rightsquigarrow \pi_1\, m'} \qquad \frac{m \rightsquigarrow m'}{\pi_2\, m \rightsquigarrow \pi_2\, m'} \qquad \frac{}{\pi_1\,(m,n)_t \rightsquigarrow m} \qquad \frac{}{\pi_2\,(m,n)_t \rightsquigarrow n}$$

$$\frac{A \rightsquigarrow A'}{m =_A n \rightsquigarrow m =_{A'} n} \qquad \frac{m \rightsquigarrow m'}{m =_A n \rightsquigarrow m' =_A n} \qquad \frac{n \rightsquigarrow n'}{m =_A n \rightsquigarrow m =_A n'} \qquad \frac{A \rightsquigarrow A'}{\mathbf{R}^{=}_{[x,p]A}(H,P) \rightsquigarrow \mathbf{R}^{=}_{[x,p]A'}(H,P)}$$

$$\frac{H \rightsquigarrow H'}{\mathbf{R}^{=}_{[x,p]A}(H,P) \rightsquigarrow \mathbf{R}^{=}_{[x,p]A}(H',P)} \qquad \frac{P \rightsquigarrow P'}{\mathbf{R}^{=}_{[x,p]A}(H,P) \rightsquigarrow \mathbf{R}^{=}_{[x,p]A}(H,P')} \qquad \frac{}{\mathbf{R}^{=}_{[x,p]A}(H,\mathrm{refl}\ m) \rightsquigarrow H}$$

Conversion

$$\frac{}{A \equiv A} \qquad \frac{A \equiv B \qquad B \rightsquigarrow C}{A \equiv C} \qquad \frac{A \equiv B \qquad C \rightsquigarrow B}{A \equiv C}$$

6 Dynamic Semantics

Value

$$\frac{}{x \text{ value}} \qquad \frac{}{\lambda_t(x:A).m \text{ value}} \qquad \frac{}{\lambda_t\{x:A\}.m \text{ value}} \qquad \frac{u \text{ value} \qquad v \text{ value}}{\langle u, v\rangle_t \text{ value}} \qquad \frac{v \text{ value}}{\{v, m\}_t \text{ value}} \qquad \frac{}{(m, n)_t \text{ value}}$$

Dynamic Reduction

$$\frac{m \rightsquigarrow m'}{m\ n \rightsquigarrow m'\ n} \qquad \frac{n \rightsquigarrow n'}{m\ n \rightsquigarrow m\ n'} \qquad \frac{v \text{ value}}{(\lambda_t(x:A).m)\ v \rightsquigarrow m[v/x]} \qquad \frac{}{(\lambda_t\{x:A\}.m)\ n \rightsquigarrow m[n/x]}$$

$$\frac{m \rightsquigarrow m'}{\langle m, n\rangle_t \rightsquigarrow \langle m', n\rangle_t} \qquad \frac{n \rightsquigarrow n'}{\langle m, n\rangle_t \rightsquigarrow \langle m, n'\rangle_t} \qquad \frac{m \rightsquigarrow m'}{\{m, n\}_t \rightsquigarrow \{m', n\}_t} \qquad \frac{m \rightsquigarrow m'}{\mathrm{R}^{\Sigma}_{[z]A}(m, [x,y]n) \rightsquigarrow \mathrm{R}^{\Sigma}_{[z]A}(m', [x,y]n)}$$

$$\frac{u \text{ value} \qquad v \text{ value}}{\mathrm{R}^{\Sigma}_{[z]A}(\langle u, v\rangle_t, [x,y]n) \rightsquigarrow n[u/x, v/y]} \qquad \frac{v \text{ value}}{\mathrm{R}^{\Sigma}_{[z]A}(\{v, m\}_t, [x,y]n) \rightsquigarrow n[v/x, m/y]} \qquad \frac{m \rightsquigarrow m'}{\pi_1\, m \rightsquigarrow \pi_1\, m'}$$

$$\frac{m \rightsquigarrow m'}{\pi_2\, m \rightsquigarrow \pi_2\, m'} \qquad \frac{}{\pi_1\ (m, n)_t \rightsquigarrow m} \qquad \frac{}{\pi_2\ (m, n)_t \rightsquigarrow n} \qquad \frac{}{\mathbf{R}^{=}_{[x,p]A}(H, P) \rightsquigarrow H}$$

7 Meta Theory

Static Meta Theory

Theorem 1 (Confluence). *If $m \rightsquigarrow^* m_1$ and $m \rightsquigarrow^* m_2$, then there exists n such that $m_1 \rightsquigarrow^* n$ and $m_2 \rightsquigarrow^* n$.*

Theorem 2 (Equality). *Definitional equality $\equiv$ is an equivalence relation.*

Theorem 3 (Static Validity). *For any static typing $\Gamma \vdash m : A$, there exists sort s such that $\Gamma \vdash A : s$ is derivable.*

Theorem 4 (Sort Uniqueness). *If there are static typings $\Gamma \vdash A : s$ and $\Gamma \vdash A : t$, then $s = t$.*

Theorem 5 (Static Subject Reduction). *If there is static typing $\Gamma \vdash m : A$ and static reduction $m \rightsquigarrow n$, then $\Gamma \vdash n : A$ is derivable.*

Theorem 6 (Static Normalization). *For any m with static typing $\Gamma \vdash m : A$, it is strongly normalizing.*

Dynamic Meta Theory

Theorem 7 (Dynamic Reflection). *For any dynamic typing $\Gamma; \Delta \vdash m : A$, static typing $\Gamma \vdash m : A$ is derivable.*

Theorem 8 (Value Stability). *If there is value v with dynamic typing $\Gamma; \Delta \vdash v : A$ and $\Gamma \vdash A : s$, then $\Delta \triangleright s$.*

Theorem 9 (Dynamic Subject Reduction). *If there is dynamic typing $\epsilon; \epsilon \vdash m : A$ and dynamic reduction $m \rightsquigarrow\!\!\!\!\!\!\twoheadrightarrow n$, then $\epsilon; \epsilon \vdash n : A$ is derivable.*

Theorem 10 (Dynamic Progress). *If there is dynamic typing $\epsilon; \epsilon \vdash m : A$, then m is a value or there exists n such that $m \rightsquigarrow\!\!\!\!\!\!\twoheadrightarrow n$.*

Erasure-Dynamic Meta Theory

Theorem 11 (Erasure Existence). *For any dynamic typing $\Gamma; \Delta \vdash m : A$, there exists m' such that erasure relation $\Gamma; \Delta \vdash m \sim m' : A$ is derivable.*

Theorem 12 (Erasure Subject Reduction). *For any erasure relation $\epsilon; \epsilon \vdash m \sim m' : A$ and dynamic reduction $m' \rightsquigarrow\!\!\!\!\!\!\twoheadrightarrow n'$, there exists n such that the following diagram commutes.*

$$\begin{array}{ccccc} \epsilon;\epsilon \vdash & m & \sim & m' & : A \\ & \Downarrow & & \Downarrow & \\ \epsilon;\epsilon \vdash & n & \sim & n' & : A \end{array}$$

Theorem 13 (Erasure Progress). *For any erasure relation $\epsilon; \epsilon \vdash m \sim m' : A$, then m' is a value or there exists n' such that $m' \rightsquigarrow\!\!\!\!\!\!\twoheadrightarrow n'$.*