

A Two Level Dependent Session Type Theory

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1 Syntax

variables	x, y, z	
channels	c, d	
sorts	s, r, t	$::= U \mid L$
roles	ρ	$::= + \mid -$
terms	m, n, A, B, C	$::= x \mid c \mid s$ $\mid \Pi_t(x : A).B \mid \Pi_t\{x : A\}.B \mid \Sigma_t(x : A).B \mid \Sigma_t\{x : A\}.B$ $\mid \lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid \langle m, n \rangle_t \mid \{m, n\}_t$ $\mid m \ n \mid R_{[z]A}^\Sigma(m, [x, y]n) \mid \mu(x : A).m$ $\mid 1 \mid () \mid 2 \mid \text{true} \mid \text{false} \mid R_{[z]A}^2(m, n_1, n_2)$ $\mid T A \mid \text{return } m \mid \text{let } x \leftarrow m \text{ in } n$ $\mid \text{proto} \mid \text{end}\rho \mid \rho\{x : A\}.B \mid \rho(x : A).B \mid \rho\text{Ch } A$ $\mid \text{fork}(x : A).m \mid \text{recv } m \mid \text{recv } \{m\} \mid \text{send } m \mid \text{send } \{m\}$ $\mid \text{close } m \mid \text{wait } m$
values	u, v	$::= x \mid c \mid \lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid \langle u, v \rangle_t \mid \{v, m\}_t$ $\mid () \mid \text{true} \mid \text{false} \mid \text{return } v \mid \text{let } x \leftarrow v \text{ in } m$ $\mid \text{fork}(x : A).m \mid \text{recv } v \mid \text{recv } \{v\} \mid \text{send } v \mid \text{send } \{v\}$ $\mid \text{send } u \ v \mid \text{send } \{v\} \ m \mid \text{close } v \mid \text{wait } v$
process	o, p, q	$\mid \langle m \rangle \mid p \mid q \mid \nu cd.p$

2 Static Fragment

Sort Order

$$\overline{U \sqsubseteq s}$$

$$\overline{L \sqsubseteq L}$$

Static Context

$$\overline{\epsilon \vdash} \quad \frac{\Gamma \vdash \quad \Gamma \vdash A : s \quad x \in \text{fresh}(\Gamma)}{\Gamma, x : A \vdash}$$

Core Typing

$$\begin{array}{c} \frac{\Gamma \vdash}{\Gamma \vdash s : U} \quad \frac{\Gamma, x : A \vdash}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t(x : A).B : t} \quad \frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t\{x : A\}.B : t} \\[10pt] \frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t(x : A).m : \Pi_t(x : A).B} \quad \frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B} \quad \frac{\Gamma, x : A \vdash m : A}{\Gamma \vdash \mu(x : A).m : A} \\[10pt] \frac{\Gamma \vdash m : \Pi_t(x : A).B \quad \Gamma \vdash n : A}{\Gamma \vdash m \ n : B[n/x]} \quad \frac{\Gamma \vdash m : \Pi_t\{x : A\}.B \quad \Gamma \vdash n : A}{\Gamma \vdash m \ n : B[n/x]} \\[10pt] \frac{s \sqsubseteq t \quad r \sqsubseteq t \quad \Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t(x : A).B : t} \quad \frac{r \sqsubseteq t \quad \Gamma \vdash A : s \quad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t\{x : A\}.B : t} \\[10pt] \frac{\Gamma \vdash \Sigma_t(x : A).B \quad \Gamma \vdash m : A \quad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \langle m, n \rangle_t : \Sigma_t(x : A).B} \quad \frac{\Gamma \vdash \Sigma_t\{x : A\}.B \quad \Gamma \vdash m : A \quad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \{m, n\}_t : \Sigma_t\{x : A\}.B} \\[10pt] \frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \quad \Gamma \vdash m : \Sigma_t(x : A).B \quad \Gamma, x : A, y : B \vdash n : C[\langle x, y \rangle_t/z]}{\Gamma \vdash R_{[z]C}^\Sigma(m, [x, y]n) : C[m/z]} \\[10pt] \frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \quad \Gamma \vdash m : \Sigma_t\{x : A\}.B \quad \Gamma, x : A, y : B \vdash n : C[\{x, y\}_t/z]}{\Gamma \vdash R_{[z]C}^\Sigma(m, [x, y]n) : C[m/z]} \\[10pt] \frac{\Gamma \vdash B : s \quad \Gamma \vdash m : A \quad A \simeq B}{\Gamma \vdash m : B} \end{array}$$

Data Typing

$$\begin{array}{c}
\frac{\Gamma \vdash}{\Gamma \vdash 1 : \mathsf{U}} \quad \frac{\Gamma \vdash}{\Gamma \vdash () : 1} \quad \frac{\Gamma \vdash}{\Gamma \vdash 2 : \mathsf{U}} \quad \frac{\Gamma \vdash}{\Gamma \vdash \text{true} : 2} \quad \frac{\Gamma \vdash}{\Gamma \vdash \text{false} : 2} \\
\\
\frac{\Gamma, z : 2 \vdash A : s \quad \Gamma \vdash m : 2 \quad \Gamma \vdash n_1 : A[\text{true}/z] \quad \Gamma \vdash n_2 : A[\text{false}/z]}{\Gamma \vdash \mathsf{R}_{[z]A}^2(m, n_1, n_2) : A[m/z]}
\end{array}$$

Monadic Typing

$$\frac{\Gamma \vdash A : s}{\Gamma \vdash \mathsf{T} A : \mathsf{L}} \quad \frac{\Gamma \vdash m : A}{\Gamma \vdash \text{return } m : \mathsf{T} A} \quad \frac{\Gamma \vdash B : s \quad \Gamma \vdash m : \mathsf{T} A \quad \Gamma, x : A \vdash n : \mathsf{T} B}{\Gamma \vdash \text{let } x : A \leftarrow m \text{ in } n : \mathsf{T} B}$$

Session Typing

$$\begin{array}{c}
\frac{\Gamma \vdash}{\Gamma \vdash \text{proto} : \mathsf{U}} \quad \frac{\Gamma \vdash}{\Gamma \vdash \text{end}\rho : \text{proto}} \quad \frac{\Gamma, x : A \vdash B : \text{proto}}{\Gamma \vdash \rho\{x : A\}.B : \text{proto}} \quad \frac{\Gamma, x : A \vdash B : \text{proto}}{\Gamma \vdash \rho(x : A).B : \text{proto}} \quad \frac{\Gamma \vdash A : \text{proto}}{\Gamma \vdash \rho \mathsf{Ch} A : \mathsf{L}} \\
\\
\frac{\Gamma \vdash \quad \epsilon \vdash A : \text{proto}}{\Gamma \vdash c : \rho \mathsf{Ch} A} \quad \frac{\Gamma, x : +\mathsf{Ch} A \vdash m : \mathsf{T} 1}{\Gamma \vdash \text{fork}(x : A).m : \mathsf{T} (-\mathsf{Ch} A)} \\
\\
\frac{\Gamma \vdash m : \rho_1 \mathsf{Ch} (\rho_2(x : A).B) \quad \rho_1 \text{ xor } \rho_2 =?}{\Gamma \vdash \text{recv } m : \mathsf{T} (\Sigma_{\mathsf{L}}(x : A).\rho_1 \mathsf{Ch} B)} \quad \frac{\Gamma \vdash m : \rho_1 \mathsf{Ch} (\rho_2\{x : A\}.B) \quad \rho_1 \text{ xor } \rho_2 =?}{\Gamma \vdash \text{recv } \{m\} : \mathsf{T} (\Sigma_{\mathsf{L}}\{x : A\}.\rho_1 \mathsf{Ch} B)} \\
\\
\frac{\Gamma \vdash m : \rho_1 \mathsf{Ch} (\rho_2(x : A).B) \quad \rho_1 \text{ xor } \rho_2 =!}{\Gamma \vdash \text{send } m : \Pi_{\mathsf{L}}(x : A).\mathsf{T} (\rho_1 \mathsf{Ch} B)} \quad \frac{\Gamma \vdash m : \rho_1 \mathsf{Ch} (\rho_2\{x : A\}.B) \quad \rho_1 \text{ xor } \rho_2 =!}{\Gamma \vdash \text{send } \{m\} : \Pi_{\mathsf{L}}\{x : A\}.\mathsf{T} (\rho_1 \mathsf{Ch} B)} \\
\\
\frac{\Gamma \vdash m : \rho_1 \mathsf{Ch} (\text{end}\rho_2) \quad \rho_1 \text{ xor } \rho_2 =?}{\Gamma \vdash \text{wait } m : \mathsf{T} 1} \quad \frac{\Gamma \vdash m : \rho_1 \mathsf{Ch} (\text{end}\rho_2) \quad \rho_1 \text{ xor } \rho_2 =!}{\Gamma \vdash \text{close } m : \mathsf{T} 1}
\end{array}$$

3 Dynamic Fragment

Dynamic Context

$$\frac{}{\epsilon; \epsilon \vdash} \quad \frac{\Gamma; \Delta \vdash \quad \Gamma \vdash A : s \quad x \in \text{fresh}(\Gamma)}{\Gamma, x : A; \Delta, x :_s A \vdash} \quad \frac{\Gamma; \Delta \vdash \quad \Gamma \vdash A : s \quad x \in \text{fresh}(\Gamma)}{\Gamma, x : A; \Delta \vdash}$$

Context Merge

$$\frac{}{\epsilon \cup \epsilon = \epsilon} \quad \frac{\Delta_1 \cup \Delta_2 = \Delta \quad x \in \text{fresh}(\Delta)}{(\Delta_1, x :_{\text{U}} A) \cup (\Delta_2, x :_{\text{U}} A) = (\Delta, x :_{\text{U}} A)}$$

$$\frac{\Delta_1 \cup \Delta_2 = \Delta \quad x \in \text{fresh}(\Delta)}{(\Delta_1, x :_{\text{L}} A) \cup \Delta_2 = (\Delta, x :_{\text{L}} A)} \quad \frac{\Delta_1 \cup \Delta_2 = \Delta \quad x \in \text{fresh}(\Delta)}{\Delta_1 \cup (\Delta_2, x :_{\text{L}} A) = (\Delta, x :_{\text{L}} A)}$$

Context Constraint

$$\frac{}{\epsilon \triangleright s} \quad \frac{\Delta \triangleright \text{U}}{\Delta, x :_{\text{U}} A \triangleright \text{U}} \quad \frac{\Delta \triangleright \text{L}}{\Delta, x :_s A \triangleright \text{L}}$$

Core Typing

$$\frac{\epsilon; \Gamma, x : A; \Delta, x :_s A \vdash \quad \Delta \triangleright \text{U}}{\epsilon; \Gamma, x : A; \Delta, x :_s A \vdash x : A} \quad \frac{\Theta; \Gamma, x : A; \Delta, x :_s A \vdash m : B \quad \Theta \triangleright t \quad \Delta \triangleright t}{\Theta; \Gamma; \Delta \vdash \lambda_t(x : A).m : \Pi_t(x : A).B}$$

$$\frac{\Theta; \Gamma, x : A; \Delta \vdash m : B \quad \Theta \triangleright t \quad \Delta \triangleright t}{\Theta; \Gamma; \Delta \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B} \quad \frac{\epsilon; \Gamma, x : A; \Delta, x :_{\text{U}} A \vdash m : A \quad \Delta \triangleright \text{U}}{\epsilon; \Gamma; \Delta \vdash \mu(x : A).m : A}$$

$$\frac{\Theta_1; \Gamma; \Delta_1 \vdash m : \Pi_t(x : A).B \quad \Theta_2; \Gamma; \Delta_2 \vdash n : A}{\Theta_1 \cup \Theta_2; \Gamma; \Delta_1 \cup \Delta_2 \vdash m \ n : B[n/x]} \quad \frac{\Theta; \Gamma; \Delta \vdash m : \Pi_t\{x : A\}.B \quad \Gamma \vdash n : A}{\Theta; \Gamma; \Delta \vdash m \ n : B[n/x]}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).B : t \quad \Theta_1; \Gamma; \Delta_1 \vdash m : A \quad \Theta_2; \Gamma; \Delta_2 \vdash n : B[m/x]}{\Theta_1 \cup \Theta_2; \Gamma; \Delta_1 \cup \Delta_2 \vdash \langle m, n \rangle_t : \Sigma_t(x : A).B}$$

$$\frac{\Gamma \vdash \Sigma_t\{x : A\}.B : t \quad \Gamma \vdash m : A \quad \Theta; \Gamma; \Delta \vdash n : B[m/x]}{\Theta; \Gamma; \Delta \vdash \{m, n\}_t : \Sigma_t\{x : A\}.B}$$

$$\frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \quad \Theta_1; \Gamma; \Delta_1 \vdash m : \Sigma_t(x : A).B \quad \Theta_2; \Gamma, x : A, y : B; \Delta_2, x :_{r1} A, y :_{r2} B \vdash n : C[\langle x, y \rangle_t / z]}{\Theta_1 \cup \Theta_2; \Gamma; \Delta_1 \cup \Delta_2 \vdash R_{[z]C}^{\Sigma}(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \quad \Theta_1; \Gamma; \Delta_1 \vdash m : \Sigma_t\{x : A\}.B \quad \Theta_2; \Gamma, x : A, y : B; \Delta_2, y :_r B \vdash n : C[\{x, y\}_t / z]}{\Theta_1 \cup \Theta_2; \Gamma; \Delta_1 \cup \Delta_2 \vdash R_{[z]C}^{\Sigma}(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma \vdash B : s \quad \Theta; \Gamma; \Delta \vdash m : A \quad A \simeq B}{\Theta; \Gamma; \Delta \vdash m : B}$$

Data Typing

$$\begin{array}{c}
\frac{\Gamma; \Delta \vdash \quad \Delta \triangleright U}{\epsilon; \Gamma; \Delta \vdash () : 1} \quad \frac{\Gamma; \Delta \vdash \quad \Delta \triangleright U}{\epsilon; \Gamma; \Delta \vdash \text{true} : 2} \quad \frac{\Gamma; \Delta \vdash \quad \Delta \triangleright U}{\epsilon; \Gamma; \Delta \vdash \text{false} : 2} \\
\\
\frac{\Gamma, z : 2 \vdash A : s \quad \Theta_1; \Gamma; \Delta_1 \vdash m : 2 \quad \Theta_2; \Gamma; \Delta_2 \vdash n_1 : A[\text{true}/z] \quad \Theta_2; \Gamma; \Delta_2 \vdash n_2 : A[\text{false}/z]}{\Theta_1 \cup \Theta_2; \Gamma; \Delta_1 \cup \Delta_2 \vdash R_{[z]A}^2(m, n_1, n_2) : A[m/z]}
\end{array}$$

Monadic Typing

$$\frac{\Theta; \Gamma; \Delta \vdash m : A}{\Theta; \Gamma; \Delta \vdash \text{return } m : T A} \quad \frac{\Gamma \vdash B : s \quad \Theta_1; \Gamma; \Delta_1 \vdash m : T A \quad \Theta_2; \Gamma, x : A; \Delta_2, x :_r A \vdash n : T B}{\Theta_1 \cup \Theta_2; \Gamma; \Delta_1 \cup \Delta_2 \vdash \text{let } x \leftarrow m \text{ in } n : T B}$$

Session Typing

$$\begin{array}{c}
\frac{\Gamma; \Delta \vdash \quad \epsilon \vdash A : \text{proto} \quad \Delta \triangleright U}{c :_L \rho \text{Ch } A; \Gamma; \Delta \vdash c : \rho \text{Ch } A} \quad \frac{\Theta; \Gamma, x : +\text{Ch } A; \Delta, x :_L +\text{Ch } A \vdash m : T 1}{\Theta; \Gamma; \Delta \vdash \text{fork } (x : A).m : T (-\text{Ch } A)} \\
\\
\frac{\Theta; \Gamma; \Delta \vdash m : \rho_1 \text{Ch } (\rho_2(x : A).B) \quad \rho_1 \text{ xor } \rho_2 =?}{\Theta; \Gamma; \Delta \vdash \text{recv } m : T (\Sigma_L(x : A). \rho_1 \text{Ch } B)} \quad \frac{\Theta; \Gamma; \Delta \vdash m : \rho_1 \text{Ch } (\rho_2\{x : A\}.B) \quad \rho_1 \text{ xor } \rho_2 =?}{\Theta; \Gamma; \Delta \vdash \text{recv } \{m\} : T (\Sigma_L\{x : A\}. \rho_1 \text{Ch } B)} \\
\\
\frac{\Theta; \Gamma; \Delta \vdash m : \rho_1 \text{Ch } (\rho_2(x : A).B) \quad \rho_1 \text{ xor } \rho_2 =!}{\Theta; \Gamma; \Delta \vdash \text{send } m : \Pi_L(x : A).T (\rho_1 \text{Ch } B)} \quad \frac{\Theta; \Gamma; \Delta \vdash m : \rho_1 \text{Ch } (\rho_2\{x : A\}.B) \quad \rho_1 \text{ xor } \rho_2 =!}{\Theta; \Gamma; \Delta \vdash \text{send } \{m\} : \Pi_L\{x : A\}.T (\rho_1 \text{Ch } B)} \\
\\
\frac{\Theta; \Gamma; \Delta \vdash m : \rho_1 \text{Ch } (\text{end} \rho_2) \quad \rho_1 \text{ xor } \rho_2 =?}{\Theta; \Gamma; \Delta \vdash \text{wait } m : T 1} \quad \frac{\Theta; \Gamma; \Delta \vdash m : \rho_1 \text{Ch } (\text{end} \rho_2) \quad \rho_1 \text{ xor } \rho_2 =!}{\Theta; \Gamma; \Delta \vdash \text{close } m : T 1}
\end{array}$$

4 Process Fragment

Process Typing

$$\frac{\Theta; \epsilon; \epsilon \vdash m : \mathsf{T}1}{\Theta \vdash \langle m \rangle} \qquad \frac{\Theta_1 \vdash p \quad \Theta_2 \vdash q}{\Theta_1 \sqcup \Theta_2 \vdash p \mid q} \qquad \frac{\Theta, c :_{\mathsf{L}} \rho \mathsf{Ch} A, d :_{\mathsf{L}} \neg \rho \mathsf{Ch} A \vdash p}{\Theta \vdash \nu c d.p}$$

Structural Congruence

Structural congruence is the least congruence relation with the following properties.

$$\overline{p \mid q \equiv q \mid p} \qquad \overline{o \mid (p \mid q) \equiv (o \mid p) \mid q} \qquad \overline{\nu c d.p \mid q \equiv \nu c d.(p \mid q)}$$

$$\overline{\nu c d.p \equiv \nu d c.p} \qquad \overline{\nu c d.\nu c' d'.p \equiv \nu c' d'.\nu c d.p}$$