A Two Level Dependent Session Type Theory

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1 Syntax

variables	x, y, z		
channels	c, d		
sorts	s, r, t	$::=$	$\mathrm{U} \mid \mathrm{L}$
roles	ρ	$::=$	$+ \mid -$
terms	m, n, A, B, C	$::=$	$x \mid c \mid s$
		$\mid$	$\Pi_t(x : A).B \mid \Pi_t\{x : A\}.B \mid \Sigma_t(x : A).B \mid \Sigma_t\{x : A\}.B$
		$\mid$	$\lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid \langle m, n \rangle_t \mid \{m, n\}_t$
		$\mid$	$m\, n \mid m\, \{n\} \mid \mathrm{R}^{\Sigma}_{[z]A}(m, [x, y]n) \mid \mu(x : A).m$
		$\mid$	$1 \mid () \mid 2 \mid \text{true} \mid \text{false} \mid \mathrm{R}^{2}_{[z]A}(m, n_1, n_2)$
		$\mid$	$\mathrm{T}\, A \mid \text{return}\, m \mid \text{let}\, x \leftarrow m \text{ in } n$
		$\mid$	$\text{proto} \mid \text{end}\rho \mid \rho\{x : A\}.B \mid \rho(x : A).B \mid \rho\mathrm{Ch}\, A$
		$\mid$	$\text{fork}\,(x : A).m \mid \text{recv}\, m \mid \underline{\text{recv}}\, m \mid \text{send}\, m \mid \underline{\text{send}}\, m$
		$\mid$	$\text{close}\, m \mid \text{wait}\, m$
values	u, v	$::=$	$x \mid c \mid \lambda_t(x : A).m \mid \lambda_t\{x : A\}.m \mid \langle u, v \rangle_t \mid \{v, m\}_t$
		$\mid$	$() \mid \text{true} \mid \text{false} \mid \text{return}\, v \mid \text{let}\, x \leftarrow v \text{ in } m$
		$\mid$	$\text{fork}\,(x : A).m \mid \text{recv}\, v \mid \underline{\text{recv}}\, v \mid \text{send}\, v \mid \underline{\text{send}}\, v$
		$\mid$	$\text{send}\, u\, v \mid \underline{\text{send}}\, v\, \{m\} \mid \text{close}\, v \mid \text{wait}\, v$
process	o, p, q	$\mid$	$\langle m \rangle \mid p \mid q \mid \nu cd.p$

2　Static Fragment

Sort Order

$$\overline{\mathrm{U} \sqsubseteq s} \qquad\qquad \overline{\mathrm{L} \sqsubseteq \mathrm{L}}$$

Static Context

$$\overline{\epsilon \vdash} \qquad\qquad \frac{\Gamma \vdash \qquad \Gamma \vdash A : s \qquad x \in \mathit{fresh}(\Gamma)}{\Gamma, x : A \vdash}$$

Core Typing

$$\frac{\Gamma \vdash}{\Gamma \vdash s : \mathrm{U}} \qquad \frac{\Gamma, x : A \vdash}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t(x : A).B : t} \qquad \frac{\Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Pi_t\{x : A\}.B : t}$$

$$\frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t(x : A).m : \Pi_t(x : A).B} \qquad \frac{\Gamma, x : A \vdash m : B}{\Gamma \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B} \qquad \frac{\Gamma, x : A \vdash m : A}{\Gamma \vdash \mu(x : A).m : A}$$

$$\frac{\Gamma \vdash m : \Pi_t(x : A).B \qquad \Gamma \vdash n : A}{\Gamma \vdash m\,n : B[n/x]} \qquad \frac{\Gamma \vdash m : \Pi_t\{x : A\}.B \qquad \Gamma \vdash n : A}{\Gamma \vdash m\,\{n\} : B[n/x]}$$

$$\frac{s \sqsubseteq t \qquad r \sqsubseteq t \qquad \Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t(x : A).B : t} \qquad \frac{r \sqsubseteq t \qquad \Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : r}{\Gamma \vdash \Sigma_t\{x : A\}.B : t}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).B \qquad \Gamma \vdash m : A \qquad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \langle m, n \rangle_t : \Sigma_t(x : A).B} \qquad \frac{\Gamma \vdash \Sigma_t\{x : A\}.B \qquad \Gamma \vdash m : A \qquad \Gamma \vdash n : B[m/x]}{\Gamma \vdash \{m, n\}_t : \Sigma_t\{x : A\}.B}$$

$$\frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \qquad \Gamma \vdash m : \Sigma_t(x : A).B \qquad \Gamma, x : A, y : B \vdash n : C[\langle x, y \rangle_t/z]}{\Gamma \vdash \mathrm{R}^\Sigma_{[z]C}(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \qquad \Gamma \vdash m : \Sigma_t\{x : A\}.B \qquad \Gamma, x : A, y : B \vdash n : C[\{x, y\}_t/z]}{\Gamma \vdash \mathrm{R}^\Sigma_{[z]C}(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma \vdash B : s \qquad \Gamma \vdash m : A \qquad A \simeq B}{\Gamma \vdash m : B}$$

Data Typing

$$\frac{\Gamma \vdash}{\Gamma \vdash 1 : \mathrm{U}} \qquad \frac{\Gamma \vdash}{\Gamma \vdash () : 1} \qquad \frac{\Gamma \vdash}{\Gamma \vdash 2 : \mathrm{U}} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathrm{true} : 2} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathrm{false} : 2}$$

$$\frac{\Gamma, z : 2 \vdash A : s \qquad \Gamma \vdash m : 2 \qquad \Gamma \vdash n_1 : A[\mathrm{true}/z] \qquad \Gamma \vdash n_2 : A[\mathrm{false}/z]}{\Gamma \vdash \mathrm{R}^2_{[z]A}(m, n_1, n_2) : A[m/z]}$$

Monadic Typing

$$\frac{\Gamma \vdash A : s}{\Gamma \vdash \mathrm{T}\, A : \mathrm{L}} \qquad \frac{\Gamma \vdash m : A}{\Gamma \vdash \mathrm{return}\, m : \mathrm{T}\, A} \qquad \frac{\Gamma \vdash B : s \qquad \Gamma \vdash m : \mathrm{T}\, A \qquad \Gamma, x : A \vdash n : \mathrm{T}\, B}{\Gamma \vdash \mathrm{let}\ x : A \leftarrow m\ \mathrm{in}\ n : \mathrm{T}\, B}$$

Session Typing

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathrm{proto} : \mathrm{U}} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathrm{end}\rho : \mathrm{proto}} \qquad \frac{\Gamma, x : A \vdash B : \mathrm{proto}}{\Gamma \vdash \rho\{x : A\}.B : \mathrm{proto}} \qquad \frac{\Gamma, x : A \vdash B : \mathrm{proto}}{\Gamma \vdash \rho(x : A).B : \mathrm{proto}} \qquad \frac{\Gamma \vdash A : \mathrm{proto}}{\Gamma \vdash \rho\mathrm{Ch}\, A : \mathrm{L}}$$

$$\frac{\Gamma \vdash \qquad \epsilon \vdash A : \mathrm{proto}}{\Gamma \vdash c : \rho\mathrm{Ch}\, A} \qquad\qquad \frac{\Gamma, x : +\mathrm{Ch}\, A \vdash m : \mathrm{T}\, 1}{\Gamma \vdash \mathrm{fork}\, (x : A).m : \mathrm{T}\, (-\mathrm{Ch}\, A)}$$

$$\frac{\Gamma \vdash m : \rho_1\mathrm{Ch}\, (\rho_2(x : A).B) \qquad \rho_1 \, \mathrm{xor} \, \rho_2 \, = ?}{\Gamma \vdash \mathrm{recv}\, m : \mathrm{T}\, (\Sigma_\mathrm{L}(x : A).\rho_1\mathrm{Ch}\, B)} \qquad \frac{\Gamma \vdash m : \rho_1\mathrm{Ch}\, (\rho_2\{x : A\}.B) \qquad \rho_1 \, \mathrm{xor} \, \rho_2 \, = ?}{\Gamma \vdash \underline{\mathrm{recv}}\, m : \mathrm{T}\, (\Sigma_\mathrm{L}\{x : A\}.\rho_1\mathrm{Ch}\, B)}$$

$$\frac{\Gamma \vdash m : \rho_1\mathrm{Ch}\, (\rho_2(x : A).B) \qquad \rho_1 \, \mathrm{xor} \, \rho_2 \, = !}{\Gamma \vdash \mathrm{send}\, m : \Pi_\mathrm{L}(x : A).\mathrm{T}\, (\rho_1\mathrm{Ch}\, B)} \qquad \frac{\Gamma \vdash m : \rho_1\mathrm{Ch}\, (\rho_2\{x : A\}.B) \qquad \rho_1 \, \mathrm{xor} \, \rho_2 \, = !}{\Gamma \vdash \underline{\mathrm{send}}\, m : \Pi_\mathrm{L}\{x : A\}.\mathrm{T}\, (\rho_1\mathrm{Ch}\, B)}$$

$$\frac{\Gamma \vdash m : \rho_1\mathrm{Ch}\, (\mathrm{end}\rho_2) \qquad \rho_1 \, \mathrm{xor} \, \rho_2 \, = ?}{\Gamma \vdash \mathrm{wait}\, m : \mathrm{T}\, 1} \qquad\qquad \frac{\Gamma \vdash m : \rho_1\mathrm{Ch}\, (\mathrm{end}\rho_2) \qquad \rho_1 \, \mathrm{xor} \, \rho_2 \, = !}{\Gamma \vdash \mathrm{close}\, m : \mathrm{T}\, 1}$$

3 Dynamic Fragment

Dynamic Context

$$\frac{}{\epsilon; \epsilon \vdash} \qquad \frac{\Gamma; \Delta \vdash \quad \Gamma \vdash A : s \quad x \in \mathit{fresh}(\Gamma)}{\Gamma, x : A; \Delta, x :_s A \vdash} \qquad \frac{\Gamma; \Delta \vdash \quad \Gamma \vdash A : s \quad x \in \mathit{fresh}(\Gamma)}{\Gamma, x : A; \Delta \vdash}$$

Context Merge

$$\frac{}{\epsilon \uplus \epsilon = \epsilon} \qquad \frac{\Delta_1 \uplus \Delta_2 = \Delta \quad x \in \mathit{fresh}(\Delta)}{(\Delta_1, x :_{\mathrm{U}} A) \uplus (\Delta_2, x :_{\mathrm{U}} A) = (\Delta, x :_{\mathrm{U}} A)}$$

$$\frac{\Delta_1 \uplus \Delta_2 = \Delta \quad x \in \mathit{fresh}(\Delta)}{(\Delta_1, x :_{\mathrm{L}} A) \uplus \Delta_2 = (\Delta, x :_{\mathrm{L}} A)} \qquad \frac{\Delta_1 \uplus \Delta_2 = \Delta \quad x \in \mathit{fresh}(\Delta)}{\Delta_1 \uplus (\Delta_2, x :_{\mathrm{L}} A) = (\Delta, x :_{\mathrm{L}} A)}$$

Context Constraint

$$\frac{}{\epsilon \rhd s} \qquad \frac{\Delta \rhd \mathrm{U}}{\Delta, x :_{\mathrm{U}} A \rhd \mathrm{U}} \qquad \frac{\Delta \rhd \mathrm{L}}{\Delta, x :_s A \rhd \mathrm{L}}$$

Core Typing

$$\frac{\epsilon; \Gamma, x : A; \Delta, x :_s A \vdash \quad \Delta \rhd \mathrm{U}}{\epsilon; \Gamma, x : A; \Delta, x :_s A \vdash x : A} \qquad \frac{\Theta; \Gamma, x : A; \Delta, x :_s A \vdash m : B \quad \Theta \rhd t \quad \Delta \rhd t}{\Theta; \Gamma; \Delta \vdash \lambda_t(x : A).m : \Pi_t(x : A).B}$$

$$\frac{\Theta; \Gamma, x : A; \Delta \vdash m : B \quad \Theta \rhd t \quad \Delta \rhd t}{\Theta; \Gamma; \Delta \vdash \lambda_t\{x : A\}.m : \Pi_t\{x : A\}.B} \qquad \frac{\epsilon; \Gamma, x : A; \Delta, x :_{\mathrm{U}} A \vdash m : A \quad \Delta \rhd \mathrm{U}}{\epsilon; \Gamma; \Delta \vdash \mu(x : A).m : A}$$

$$\frac{\Theta_1; \Gamma; \Delta_1 \vdash m : \Pi_t(x : A).B \quad \Theta_2; \Gamma; \Delta_2 \vdash n : A}{\Theta_1 \uplus \Theta_2; \Gamma; \Delta_1 \uplus \Delta_2 \vdash m\ n : B[n/x]} \qquad \frac{\Theta; \Gamma; \Delta \vdash m : \Pi_t\{x : A\}.B \quad \Gamma \vdash n : A}{\Theta; \Gamma; \Delta \vdash m\ \{n\} : B[n/x]}$$

$$\frac{\Gamma \vdash \Sigma_t(x : A).B : t \quad \Theta_1; \Gamma; \Delta_1 \vdash m : A \quad \Theta_2; \Gamma; \Delta_2 \vdash n : B[m/x]}{\Theta_1 \uplus \Theta_2; \Gamma; \Delta_1 \uplus \Delta_2 \vdash \langle m, n \rangle_t : \Sigma_t(x : A).B}$$

$$\frac{\Gamma \vdash \Sigma_t\{x : A\}.B : t \quad \Gamma \vdash m : A \quad \Theta; \Gamma; \Delta \vdash n : B[m/x]}{\Theta; \Gamma; \Delta \vdash \{m, n\}_t : \Sigma_t\{x : A\}.B}$$

$$\frac{\Gamma, z : \Sigma_t(x : A).B \vdash C : s \quad \Theta_1; \Gamma; \Delta_1 \vdash m : \Sigma_t(x : A).B \quad \Theta_2; \Gamma, x : A, y : B; \Delta_2, x :_{r1} A, y :_{r2} B \vdash n : C[\langle x, y \rangle_t / z]}{\Theta_1 \uplus \Theta_2; \Gamma; \Delta_1 \uplus \Delta_2 \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma, z : \Sigma_t\{x : A\}.B \vdash C : s \quad \Theta_1; \Gamma; \Delta_1 \vdash m : \Sigma_t\{x : A\}.B \quad \Theta_2; \Gamma, x : A, y : B; \Delta_2, y :_r B \vdash n : C[\{x, y\}_t / z]}{\Theta_1 \uplus \Theta_2; \Gamma; \Delta_1 \uplus \Delta_2 \vdash \mathrm{R}^{\Sigma}_{[z]C}(m, [x, y]n) : C[m/z]}$$

$$\frac{\Gamma \vdash B : s \quad \Theta; \Gamma; \Delta \vdash m : A \quad A \simeq B}{\Theta; \Gamma; \Delta \vdash m : B}$$

Data Typing

$$\frac{\Gamma;\Delta \vdash \qquad \Delta \triangleright U}{\epsilon;\Gamma;\Delta \vdash () : 1} \qquad\qquad \frac{\Gamma;\Delta \vdash \qquad \Delta \triangleright U}{\epsilon;\Gamma;\Delta \vdash \text{true} : 2} \qquad\qquad \frac{\Gamma;\Delta \vdash \qquad \Delta \triangleright U}{\epsilon;\Gamma;\Delta \vdash \text{false} : 2}$$

$$\frac{\Gamma, z:2 \vdash A:s \qquad \Theta_1;\Gamma;\Delta_1 \vdash m:2 \qquad \Theta_2;\Gamma;\Delta_2 \vdash n_1 : A[\text{true}/z] \qquad \Theta_2;\Gamma;\Delta_2 \vdash n_2 : A[\text{false}/z]}{\Theta_1 \uplus \Theta_2;\Gamma;\Delta_1 \uplus \Delta_2 \vdash \text{R}^2_{[z]A}(m,n_1,n_2) : A[m/z]}$$

Monadic Typing

$$\frac{\Theta;\Gamma;\Delta \vdash m:A}{\Theta;\Gamma;\Delta \vdash \text{return}\, m : \text{T}\, A} \qquad\qquad \frac{\Gamma \vdash B:s \qquad \Theta_1;\Gamma;\Delta_1 \vdash m:\text{T}\, A \qquad \Theta_2;\Gamma,x:A;\Delta_2, x:_r A \vdash n:\text{T}\, B}{\Theta_1 \uplus \Theta_2;\Gamma;\Delta_1 \uplus \Delta_2 \vdash \text{let}\, x \leftarrow m\, \text{in}\, n : \text{T}\, B}$$

Session Typing

$$\frac{\Gamma;\Delta \vdash \qquad \epsilon \vdash A : \text{proto} \qquad \Delta \triangleright U}{c :_{\text{L}} \rho\text{Ch}\, A;\Gamma;\Delta \vdash c : \rho\text{Ch}\, A} \qquad\qquad \frac{\Theta;\Gamma, x:+\text{Ch}\, A;\Delta, x:_{\text{L}} +\text{Ch}\, A \vdash m:\text{T}\, 1}{\Theta;\Gamma;\Delta \vdash \text{fork}\,(x:A).m : \text{T}\,(-\text{Ch}\, A)}$$

$$\frac{\Theta;\Gamma;\Delta \vdash m : \rho_1\text{Ch}\,(\rho_2(x:A).B) \qquad \rho_1 \,\text{xor}\, \rho_2 = ?}{\Theta;\Gamma;\Delta \vdash \text{recv}\, m : \text{T}\,(\Sigma_{\text{L}}(x:A).\rho_1\text{Ch}\, B)} \qquad\qquad \frac{\Theta;\Gamma;\Delta \vdash m : \rho_1\text{Ch}\,(\rho_2\{x:A\}.B) \qquad \rho_1 \,\text{xor}\, \rho_2 = ?}{\Theta;\Gamma;\Delta \vdash \underline{\text{recv}}\, m : \text{T}\,(\Sigma_{\text{L}}\{x:A\}.\rho_1\text{Ch}\, B)}$$

$$\frac{\Theta;\Gamma;\Delta \vdash m : \rho_1\text{Ch}\,(\rho_2(x:A).B) \qquad \rho_1 \,\text{xor}\, \rho_2 = !}{\Theta;\Gamma;\Delta \vdash \text{send}\, m : \Pi_{\text{L}}(x:A).\text{T}\,(\rho_1\text{Ch}\, B)} \qquad\qquad \frac{\Theta;\Gamma;\Delta \vdash m : \rho_1\text{Ch}\,(\rho_2\{x:A\}.B) \qquad \rho_1 \,\text{xor}\, \rho_2 = !}{\Theta;\Gamma;\Delta \vdash \underline{\text{send}}\, m : \Pi_{\text{L}}\{x:A\}.\text{T}\,(\rho_1\text{Ch}\, B)}$$

$$\frac{\Theta;\Gamma;\Delta \vdash m : \rho_1\text{Ch}\,(\text{end}\rho_2) \qquad \rho_1 \,\text{xor}\, \rho_2 = ?}{\Theta;\Gamma;\Delta \vdash \text{wait}\, m : \text{T}\, 1} \qquad\qquad \frac{\Theta;\Gamma;\Delta \vdash m : \rho_1\text{Ch}\,(\text{end}\rho_2) \qquad \rho_1 \,\text{xor}\, \rho_2 = !}{\Theta;\Gamma;\Delta \vdash \text{close}\, m : \text{T}\, 1}$$

4 Process Fragment

Process Typing

$$\frac{\Theta; \epsilon; \epsilon \vdash m : \mathrm{T}\,1}{\Theta \vdash \langle m \rangle} \qquad \frac{\Theta_1 \vdash p \qquad \Theta_2 \vdash q}{\Theta_1 \uplus \Theta_2 \vdash p \mid q} \qquad \frac{\Theta, c :_{\mathrm{L}} \rho\mathrm{Ch}\,A, d :_{\mathrm{L}} \neg\rho\mathrm{Ch}\,A \vdash p}{\Theta \vdash \nu cd.p}$$

Structural Congruence

Structural congruence is the least congruence relation with the following properties.

$$p \mid q \equiv q \mid p \qquad\qquad o \mid (p \mid q) \equiv (o \mid p) \mid q \qquad\qquad p \mid \langle \text{return } () \rangle \equiv p$$

$$\nu cd.p \mid q \equiv \nu cd.(p \mid q) \qquad\qquad \nu cd.p \equiv \nu dc.p \qquad\qquad \nu cd.\nu c'd'.p \equiv \nu c'd'.\nu cd.p$$

Reductions

$$\frac{p \Rightarrow q}{o \mid p \Rightarrow o \mid q} \qquad \frac{p \Rightarrow q}{\nu cd.p \Rightarrow \nu cd.q} \qquad \frac{p \equiv p' \qquad p' \Rightarrow q' \qquad q' \equiv q}{p \Rightarrow q} \qquad \frac{m \rightsquigarrow m'}{\langle m \rangle \Rightarrow \langle m' \rangle}$$

$$\frac{}{\langle \text{let } x \leftarrow \text{return } v \text{ in } m \rangle \Rightarrow \langle m[v/x] \rangle} \qquad \frac{}{\langle \text{let } x \leftarrow \text{fork } (y : A).m \text{ in } n \rangle \Rightarrow \nu cd.(\langle n[c/x] \rangle \mid \langle m[d/y] \rangle)}$$

$$\frac{}{\begin{array}{c} \nu cd.(\langle \text{let } x \leftarrow \text{send } c\ v \text{ in } n_1 \rangle \mid \langle \text{let } y \leftarrow \text{recv } d \text{ in } n_2 \rangle) \Rightarrow \\ \nu cd.(\langle \text{let } x \leftarrow \text{return } c \text{ in } n_1 \rangle \mid \langle \text{let } y \leftarrow \text{return } \langle v, d \rangle_L \text{ in } n_2 \rangle) \end{array}}$$

$$\frac{}{\begin{array}{c} \nu cd.(\langle \text{let } x \leftarrow \underline{\text{send}}\ c\ \{m\} \text{ in } n_1 \rangle \mid \langle \text{let } y \leftarrow \underline{\text{recv}}\ d \text{ in } n_2 \rangle) \Rightarrow \\ \nu cd.(\langle \text{let } x \leftarrow \text{return } c \text{ in } n_1 \rangle \mid \langle \text{let } y \leftarrow \text{return } \{m, d\}_L \text{ in } n_2 \rangle) \end{array}}$$

$$\frac{}{\begin{array}{c} \nu cd.(\langle \text{let } x \leftarrow \text{close } c \text{ in } m \rangle \mid \langle \text{let } y \leftarrow \text{wait } d \text{ in } n \rangle) \Rightarrow \\ \langle \text{let } x \leftarrow \text{return } () \text{ in } m \rangle \mid \langle \text{let } x \leftarrow \text{return } () \text{ in } n \rangle \end{array}}$$