Homework 2: Ordinary differential equations

Due April 24.

1. Write a program (or programs) to integrate a set of an arbitrary number of coupled differential equations using the Euler method, fourth order Runge-Kutta, and Leapfrog. (Note: Leapfrog only applies to special cases.) For Runge-Kutta, you may use a packaged routine such as available in scipy or Numerical Recipes. If you do used a packaged routine, **be sure to use one with a fixed timestep and order so that testing the convergence can be easily performed.**

2. Use your program to solve the differential equation for $x(t)$:
$$\frac{d^2x}{dt^2} + x = 0;$$
with the initial conditions $x(0) = 1, x'(0) = 0$. (Prime indicates a derivative with respect to t.) Note that this has the analytical solution: $x = \cos(t)$

   (a) Integrate the equation for $0 \leq t \leq 30$ using each of the methods, and step sizes of 1, .3, .1, .03, and .01. Comment on the behavior of the solutions.

   (b) Plot $\log(|x_{numerical}(30) - x_{exact}(30)|)$ as a function of $\log(stepsize)$ and check for the expected convergence of the error term.

3. Now try the two dimensional orbit described by the potential:
$$\Phi = -\frac{1}{\sqrt{1 + x^2 + y^2}}.$$
From $\frac{d^2\vec{r}}{dt^2} = -\nabla\Phi$, the orbits are given by the coupled differential equations:
$$\frac{d^2x}{dt^2} = -\frac{x}{(1 + x^2 + y^2)^{3/2}},$$
$$\frac{d^2y}{dt^2} = -\frac{y}{(1 + x^2 + y^2)^{3/2}}.$$

   (a) Integrate this for $0 \leq t \leq 100$ for the initial conditions $x = 1, y = 0, x' = 0, y' = .3$. Try this with either Leapfrog or Runge-Kutta and step sizes from .01 to 1. Plot x vs. y for these integrations.

   (b) Plot the energy $E = (x'^2 + y'^2)/2 + \Phi(x, y)$ as a function of time for your integrations.

   (c) Integrate this same orbit (same force law and same initial conditions) for $0 \leq t \leq 40000$ and a fixed step size of 0.4 with both Leapfrog and Runge-Kutta, and plot the energy, E, as a function of time for these integrations. Comment on the behavior of $E(t)$ for these integrations, and the wall clock time each method took.