Question 1

$$\begin{aligned} CDF &= P(W < w) \\ &= P(\frac{Y}{X} < w) \\ &= P(Y < wX) \\ &= \int_{x=0}^{x=+\infty} \int_{y=0}^{y=wx} f(x,y)dxdy \\ &= \int_{x=0}^{x=+\infty} \int_{y=0}^{y=wx} \lambda\mu e^{-\lambda x - \mu y} dxdy \\ &= \int_{x=0}^{x=+\infty} \left[(-\lambda)e^{-\lambda x - \mu y} \right]_0^{wx} dx \\ &= \int_{x=0}^{x=+\infty} \left[(-\lambda)e^{-\lambda x - \mu wx} - (-\lambda)e^{-\lambda x} \right] dx \\ &= (-\lambda) \int_{x=0}^{x=+\infty} -e^{-\lambda x}(1 - e^{-\mu wx})dx \end{aligned}$$

$$\begin{aligned} Let\ u &= 1 - e^{-\mu wx}\ ; \\ u' &= \mu w e^{-\mu wx}\ ; \\ v' &= \lambda e^{-\lambda x}\ ; \\ v &= \int \lambda e^{-\lambda x} dx = -e^{-\lambda x} \end{aligned}$$

$$\begin{aligned} CDF &= \int_{x=0}^{x=+\infty} uv' dx \\ &= \left[uv \right]_0^{+\infty} - \int_0^{+\infty} u'v dx \\ &= \left[(1 - e^{-\mu wx})(-e^{-\lambda x}) \right]_0^{+\infty} - \int_0^{+\infty} (\mu w e^{-\mu wx})(-e^{-\lambda x})dx \\ &= \left[(1-0) \cdot 0 \right] - \left[(1-1)(-1) \right] + \int_0^{+\infty} \mu w e^{(-\lambda - \mu w)x} dx \\ &= 0 + \left[\mu w \cdot \tfrac{1}{-\lambda - \mu w} \cdot e^{(-\lambda - \mu w)x} \right]_0^{+\infty} \\ &= 0 - \mu w \cdot \frac{1}{-\lambda - \mu w} \\ &= \frac{\mu w}{\lambda + \mu w} \\ &= \frac{1}{\frac{\lambda}{\mu w} + 1} \end{aligned}$$

$$
\begin{aligned}
PDF &= \frac{d(CDF)}{dw} \\
&= ln(\frac{\lambda}{\mu w} + 1) \cdot (\frac{\lambda}{\mu w} + 1)' \\
&= ln(\frac{\lambda}{\mu w} + 1) \cdot \frac{\lambda}{\mu} \cdot ln(w)
\end{aligned}
$$

b)

$$
\begin{aligned}
P(X < Y) &= P(\frac{Y}{x} > 1) \\
&= P(w > 1) \\
&= CDF(w > 1) \\
&= \frac{\mu w}{\lambda + \mu w}|_1^\infty \\
&= \frac{1}{\frac{\lambda}{\mu w} + 1}|_1^\infty \\
&= 1 - \frac{1}{\frac{\lambda}{\mu} + 1} \\
&= 1 - \frac{\mu}{\lambda + \mu} \\
&= \frac{\lambda}{\lambda + \mu}
\end{aligned}
$$

Question 2

a)

Given 3 independent binary classifiers ($C_i, i=1,2,3$) using Majority voting, if the ensemble ($H(\vec{x})$) receiving more than half of the votes predicts a label otherwise the prediction is rejected.
The ensemble predicts the wrong answer is in such condition: There are k classifiers predicting the right answer while $k \leq 1$.
Based on uncorrelated error rates, the equation for the expected error rate of the ensemble as shown below:

$$P(H(\vec{x}) \neq y) = \sum_{k=0}^{1} C_3^k (1-\epsilon)^k \epsilon^{3-k}$$

Specifically, the equation could be written as below:

$$P(H(\vec{x}) \neq y) = e_1 e_2 e_3 + (1 - e_1)e_2 e_3 + (1 - e_2)e_1 e_3 + (1 - e_3)e_2 e_1 = 0.132$$

b)

If the assumption of independence is relaxed on the errors, we need to apply the total probability theorem to calculate.

Assume that A represents binary classifiers $C1$ predicting the wrong answer, B represents binary classifiers $C2$ predicting the wrong answer and C represents binary classifiers $C3$ predicting the wrong answer.

The equation could be written as below:

$$P(H(\vec{x}) \neq y) = P(A \cap B \cap C) + P(A \cap \bar{B} \cap C) + P(A \cap B \cap \bar{C}) + P(\bar{A} \cap B \cap C)$$
$$= P(A) \cdot P(B|A) \cdot P(C|A \cap B) + P(A) \cdot P(\bar{B}|A) \cdot P(C|A \cap \bar{B}) + P(A) \cdot P(B|A) \cdot P(\bar{C}|A \cap B) + P(\bar{A}) \cdot P(B|\bar{A}) \cdot P(C|\bar{A} \cap B)$$

Question 3

$$P(Y = N) = 0.5;\ P(Y = Y) = 0.5;$$
$$P(X_1 = S|Y = N) = \frac{2}{3};\ P(X_1 = S|Y = Y) = 0;$$
$$P(X_3 = N|Y = N) = \frac{2}{3};\ P(X_3 = N|Y = Y) = \frac{1}{3};$$
$$P(X_4 = F|Y = N) = \frac{1}{3};\ P(X_4 = F|Y = Y) = 1;\ P(X_4 = T|Y = N) = \frac{2}{3};\ P(X_4 = T|Y = Y) = 0$$
$$If\ Y = N, \bar{x} = \frac{1}{3}(42.5 + 39.2 + 15.4) = 32.4;\ s^2 = \frac{1}{2}((42.5 - 32.4)^2 + (39.2 - 32.4)^2 + (15.4 - 32.4)^2) = 218.6;$$
$$If\ Y = Y, \bar{x} = \frac{1}{2}(33.6 + 22.8) = 28.2;\ s^2 = (33.6 - 28.2)^2 + (22.8 - 28.2)^2 = 58.3$$

```
import scipy.stats
mu = 32.4
std = 218.6
prob = scipy.stats.norm(mu, std).pdf(25)
print('P(X2=25|Y=N)=', prob)
mu = 28.2
std = 58.3
prob_2 = scipy.stats.norm(mu, std).pdf(25)
print('P(X2=25|Y=Y)=', prob_2)
mu = 32.4
std = 218.6
prob_3 = scipy.stats.norm(mu, std).pdf(36.4)
print('P(X2=36.4|Y=N)=', prob_3)
mu = 28.2
std = 58.3
prob_4 = scipy.stats.norm(mu, std).pdf(36.4)
print('P(X2=36.4|Y=Y)=', prob_4)
```

```
P(X2=25|Y=N)= 0.00182394219241013
P(X2=25|Y=Y)= 0.006832620522681207
P(X2=36.4|Y=N)= 0.0018246820576027104
P(X2=36.4|Y=Y)= 0.006775567911678318
```

$$ID_7:$$

$$P(Y=N)\cdot P(X_2=25|Y=N)\cdot P(X_3=N|Y=N)\cdot P(X_4=T|Y=N)=0.5*0.00182*\frac{2}{3}*\frac{2}{3}=0.0004$$

$$P(Y=Y)\cdot P(X_2=25|Y=Y)\cdot P(X_3=N|Y=Y\cdot P(X_4=T|Y=Y)=0.5*0.00683*\frac{1}{3}*0=0$$

$$ID_8:$$

$$P(Y=N)\cdot P(X_1=S|Y=N)\cdot P(X_2=36.4|Y=N)\cdot P(X_4=F|Y=N)=0.5*\frac{2}{3}*0.00182*\frac{1}{3}=0.0002$$

$$P(Y=Y)\cdot P(X_1=S|Y=Y)\cdot P(X_2=36.4|Y=Y)\cdot P(X_4=F|Y=Y)=0.5*0*0.00678*1=0$$

Therefore, for ID_7, Y=N. For ID_8, Y=N