Quantitative Types

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Introduction

Quantitative Types

Topic of this course

non-idempotent intersection types

a.k.a. **quantitative types**

a.k.a. **multi-types**

a.k.a. **tensor types**

Comparison (in one slide)

"Typical" type systems

▶ guarantee properties of programs   (typable $\implies$ has property P)
▶ capture qualitative properties of programs
    (termination, productivity, deadlock-freeness, . . .)
▶ each fragment of a program is typed exactly once
▶ type inference is decidable   (useful for static analysis)

Quantitative type systems

▶ characterise properties of programs   (typable $\iff$ has property P)
▶ capture quantitative properties of programs
    (reduction length, size of the normal form, $\#$ memory accesses, . . .)
▶ each fragment of a program is typed zero, one, or more times
    (as many times as it is used in runtime)
▶ type inference is undecidable   (but they are useful as models)

Failure of subject expansion

Consider the following interpretation in the **simply typed** λ-calculus:

$$\llbracket t \rrbracket = \{A \mid \, \vdash t : A\}$$

Does $t =_\beta s$ imply $\llbracket t \rrbracket = \llbracket s \rrbracket$? We require:

- ▶ Subject reduction: $t \to_\beta s$ and $\vdash t : A$ implies $\vdash s : A$.
- ▶ Subject expansion: $t \to_\beta s$ and $\vdash s : A$ implies $\vdash t : A$.

Failure of subject expansion

Does $\vdash p\{x := q\} : A$ imply $\vdash (\lambda x.\, p)\, q : A$?

Problem: $p\{x := q\}$ may produce zero, one, or more copies of q.

$$(\lambda x.\, \mathtt{id})\ \Omega\ \to_\beta \mathtt{id}$$
$$???$$

$$(\lambda x.\, x\, x)\ \mathtt{id}\ \to_\beta\ \mathtt{id}\quad \mathtt{id}$$
$$??? \qquad A \to A \quad A$$

Idea: the identity on the left could be typed with $(A \to A) \cap A$.

Intersection type systems Coppo & Dezani-Ciancaglini (1980)

Syntax

$$\text{TERMS} \quad t, s, \ldots ::= x \mid \lambda x.\, t \mid t\, s$$
$$\text{TYPES} \quad A, B, \ldots ::= \alpha \mid \{A_1, \ldots, A_n\} \to B \qquad (n \geq 1)$$

- $\{A_1, \ldots, A_n\}$ is a non-empty **set** of types.
- Intuitively, it represents a finite *intersection* $A_1 \cap \ldots \cap A_n$.

Typing rules of $\lambda_\cap^{\mathrm{CD}}$

$$\overline{\Gamma, x : \{A_1, \ldots, A_i, \ldots, A_n\} \vdash x : A_i}$$

$$\frac{\Gamma, x : \{A_1, \ldots, A_n\} \vdash t : B}{\Gamma \vdash \lambda x.\, t : \{A_1, \ldots, A_n\} \to B}$$

$$\frac{\Gamma \vdash t : \{A_1, A_2, \ldots, A_n\} \to B \quad \Gamma \vdash s : A_1 \quad \Gamma \vdash s : A_2 \quad \ldots \quad \Gamma \vdash s : A_n}{\Gamma \vdash t\, s : B}$$

Intersection type systems

Coppo & Dezani-Ciancaglini (1980)

Example

Let:

- $\mathtt{id} = \lambda x.\, x$
- $A = \{\alpha\} \to \alpha$
- $B = \{A\} \to A = \{\{\alpha\} \to \alpha\} \to \{\alpha\} \to \alpha$

Then:

$$
\cfrac{
  \cfrac{
    \cfrac{}{x : \{A, B\} \vdash x : \underbrace{\{A\} \to A}_{B}}
    \qquad
    \cfrac{}{x : \{A, B\} \vdash x : A}
  }{
    \cfrac{x : \{A, B\} \vdash x\, x : A}{\vdash \lambda x.\, x\, x : \{A, B\} \to A}
  }
  \qquad
  \cfrac{\cfrac{}{x : \{\alpha\} \vdash x : \alpha}}{\vdash \mathtt{id} : A}
  \qquad
  \cfrac{\cfrac{}{x : \{A\} \vdash x : A}}{\vdash \mathtt{id} : B}
}{
  \vdash (\lambda x.\, x\, x)\, \mathtt{id} : A
}
$$

"Finitistic" polymorphism.

Note: $\lambda x.\, x\, x$ is SN but not typable using simple types.

Theorem (Characterisation of Strong Normalisation)

The following are equivalent:

1. **Typability.**
   There exist Γ, A such that $\Gamma \vdash t : A$ holds in $\lambda_{\cap}^{\mathrm{CD}}$.

2. **Strong $\rightarrow_\beta$-normalisation.**
   There are no infinite reduction sequences $t \rightarrow_\beta t_1 \rightarrow_\beta t_2 \ldots$

Note: connection with denotational semantics

▶ Any Scott $\mathcal{D}_\infty$ model can be described as a filter model $\mathcal{F}^{\mathrm{TT}}$ for some intersection type theory TT.

▶ In a filter model, $[\![t]\!] = \{A \mid \vdash_{\mathrm{TT}} t : A\}$ holds for closed t.

For a survey, see Barendregt *et al.* 's *Lambda Calculus with Types* (2010)

Linear Logic

Sequent calculi usually include **structural rules**:

Weakening	Contraction	Exchange

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\text{LW} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\text{LC} \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\text{LX}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\text{RW} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\text{RC} \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\text{RX}$$

Linear Logic

► **Resource-aware** logic.
► No weakening: $(A \otimes B) \multimap A$ is not a theorem.
► No contraction: $A \multimap (A \otimes A)$ is not a theorem.
► Exchange: contexts can be understood as **multisets** of formulae.
  (Not completely equivalent).
► Intuitively, each hypothesis must be used exactly once.

Linear Logic

MLL (Multiplicative fragment)

$$\text{FORMULAE} \quad A, B, \ldots \quad ::= \alpha \mid \overline{\alpha} \mid A \otimes B \mid A \,\gamma\!\!\!\!\gamma\, B$$

$$\alpha^{\perp} := \overline{\alpha} \quad (A \otimes B)^{\perp} := A^{\perp} \,\gamma\!\!\!\!\gamma\, B^{\perp}$$
$$\overline{\alpha}^{\perp} := \alpha \quad (A \,\gamma\!\!\!\!\gamma\, B)^{\perp} := A^{\perp} \otimes B^{\perp}$$

$A \multimap B$ abbreviates $A^{\perp} \,\gamma\!\!\!\!\gamma\, B$.

Contexts $(\Gamma, \Delta, \ldots)$ are **multisets** of formulae (implicit exchange).

Inference rules

$$\frac{}{\vdash A, A^{\perp}}\text{ax} \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \,\gamma\!\!\!\!\gamma\, B}\gamma\!\!\!\!\gamma$$

- ▶ No implicit weakening in the rule ax.
- ▶ No implicit contraction in the rule $\otimes$.
- ▶ The rule $\otimes$ requires to choose how to split the context.
  For example: $\vdash A^{\perp}, B^{\perp}, C^{\perp}, B \otimes (C \otimes A)$.

Linear Logic

Definition (Approximation)

A formula in MLL approximates an intuitionistic formula according to the inductive definition[1]:

$$\frac{}{\alpha \sqsubset \alpha} \qquad \frac{A_1 \sqsubset X \quad \ldots \quad A_n \sqsubset X \quad B \sqsubset Y}{(A_1 \otimes \ldots \otimes A_n) \multimap B \sqsubset X \to Y}$$

---

[1] More precisely: MLL with units and minimal logic.

Theorem (Girard's translation + approximation theorem)

If X is a valid intuitionistic formula, there is a valid MLL formula $A \sqsubset X$.

$$\alpha \multimap \mathbf{1} \multimap \alpha \qquad \sqsubset \quad \alpha \to \beta \to \alpha$$

$$(\alpha \multimap \alpha \multimap \beta) \multimap (\alpha \otimes \alpha) \multimap \beta \qquad \sqsubset \quad (\alpha \to \alpha \to \beta) \to \alpha \to \beta$$

Quantitative type systems embody approximation theorems.

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Head reduction

Remark. Every λ-term is of exactly one of the two following forms:

1. $\lambda x_1 \ldots x_n . y \, t_1 \ldots t_m$
2. $\lambda x_1 \ldots x_n . (\lambda y . p) \, q \, t_1 \ldots t_m$

Nomenclature

$$\underbrace{\lambda x_1 \ldots x_n . y \, t_1 \ldots t_m}_{\text{head normal form}}$$

HNF is the set of head normal forms

$$\lambda x_1 \ldots x_n . \underbrace{s}_{\text{subterm in head position}} t_1 \ldots t_m$$

$$\lambda x_1 \ldots x_n . \underbrace{y}_{\text{head variable}} t_1 \ldots t_m \qquad \lambda x_1 \ldots x_n . \underbrace{(\lambda y . p) \, q}_{\text{head redex}} t_1 \ldots t_m$$

$$\lambda x_1 \ldots x_n . (\lambda y . p) \, q \, t_1 \ldots t_m \quad \underbrace{\rightarrow_{\mathrm{h}}}_{\text{head reduction}} \quad \lambda x_1 \ldots x_n . p\{y := q\} \, t_1 \ldots t_m$$

t is head-normalising $\overset{\text{def}}{\Longleftrightarrow} \exists s . t \rightarrow_{\mathrm{h}}^* s \in \mathsf{HNF}$

System $\mathcal{H}$

$$
\begin{array}{lll}
\text{TERMS} & t, s, \ldots ::= x \mid \lambda x.\, t \mid t\, s \\
\text{TYPES} & A, B, \ldots ::= \alpha \mid \mathcal{M} \to A \\
\text{MULTI-TYPES} & \mathcal{M}, \mathcal{N}, \ldots ::= [A_i]_{i \in I}
\end{array}
$$

▶ A multi-type is a (possibly empty) **finite multiset** of types.

▶ $\mathcal{M} + \mathcal{N}$ is the union of multi-types.

▶ A context $(\Gamma, \Delta, \ldots)$ is a function mapping variables to multi-types.

▶ We use sequential notation to write contexts. For instance:

$$
\Gamma \quad = \quad (\, x : [[\alpha] \to \beta, \alpha], y : [\beta, \beta, \gamma] \,)
$$

is the context that maps:

$$
x \mapsto [[\alpha] \to \beta, \alpha] \qquad y \mapsto [\beta, \beta, \gamma] \qquad z \mapsto [\,] \qquad \ldots
$$

▶ We assume that contexts are of **finite support**.

▶ $\Gamma + \Delta$ is the context defined by $(\Gamma + \Delta)(x) = \Gamma(x) + \Delta(x)$.

System $\mathcal{H}$ Gardner (1994), de Carvalho (2007)

We have two forms of judgment:

$$\Gamma \vdash t : A \qquad \Gamma \Vdash t : \mathcal{M}$$

Typing rules of System $\mathcal{H}$

$$\frac{}{x : [A] \vdash x : A}\text{var} \qquad \frac{\Gamma, x : \mathcal{M} \vdash t : A}{\Gamma \vdash \lambda x.\, t : \mathcal{M} \to A}\text{lam}$$

$$\frac{\Gamma \vdash t : \mathcal{M} \to A \quad \Delta \Vdash s : \mathcal{M}}{\Gamma + \Delta \vdash t\, s : A}\text{app} \qquad \frac{\Gamma_1 \vdash t : A_1 \quad \ldots \quad \Gamma_n \vdash t : A_n}{\Gamma_1 + \ldots + \Gamma_n \Vdash t : [A_1, \ldots, A_n]}\text{many}$$

- ▶ "Linear logic in disguise".
- ▶ Rules are multiplicative: no implicit weakening nor contraction.
- ▶ Rules are logically sound w.r.t. the translation to MLL (with units):

$$\underline{\mathcal{M} \to A} = \underline{\mathcal{M}} \multimap \underline{A}$$

$$\underline{[A_1, \ldots, A_n]} = \underline{A_1} \otimes \ldots \otimes \underline{A_n}$$

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Sometimes instead of:

$$\dfrac{\Gamma \vdash t : [A_1, \ldots, A_n] \to B \qquad \dfrac{\Delta_1 \vdash s : A_1 \quad \ldots \quad \Delta_n \vdash s : A_n}{\Delta_1 + \ldots + \Delta_n \Vdash s : [A_1, \ldots, A_n]}\text{many}}{\Gamma + \Delta_1 + \ldots + \Delta_n \vdash t\,s : B}\text{app}$$

we write:

$$\dfrac{\Gamma \vdash t : [A_1, \ldots, A_n] \to B \quad \Delta_1 \vdash s : A_1 \quad \ldots \quad \Delta_n \vdash s : A_n}{\Gamma + \Delta_1 + \ldots + \Delta_n \vdash t\,s : B}\text{app}$$

This is just a minor abuse of notation.

Gardner (1994), de Carvalho (2007)

Example (1)

$$\frac{\dfrac{}{x : [[A] \to A] \vdash x : [A] \to A}}{\vdash \mathtt{id} : [[A] \to A] \to [A] \to A} \qquad \frac{\dfrac{}{x : [A] \vdash x : A}}{\vdash \mathtt{id} : [A] \to A}$$
$$\vdash \mathtt{id}\,\mathtt{id} : [A] \to A$$

$$\lessgtr$$

$$\frac{\dfrac{}{x : [A] \vdash x : A}}{\vdash \mathtt{id} : [A] \to A}$$

System $\mathcal{H}$

Gardner (1994), de Carvalho (2007)

Example (2)

Let:

- $A = [\alpha] \to \alpha$
- $B = [A] \to A = [[\alpha] \to \alpha] \to [\alpha] \to \alpha$

$$
\dfrac{\dfrac{\overline{x : [B] \vdash x : B} \quad \dfrac{\overline{x : [B] \vdash x : B} \quad \overline{x : [A] \vdash x : A}}{x : [A, B] \vdash x\,x : A}}{\dfrac{x : [A, B, B] \vdash x\,(x\,x) : A}{\vdash \lambda x.\, x\,(x\,x) : [A, B, B] \to A}} \quad \vdash \mathtt{id\,id} : A \ \vdash \mathtt{id\,id} : B \ \vdash \mathtt{id\,id} : B}{\vdash (\lambda x.\, x\,(x\,x))(\mathtt{id\,id}) : A}
$$

$$\rightsquigarrow$$

$$
\dfrac{\vdash \mathtt{id\,id} : B \quad \dfrac{\vdash \mathtt{id\,id} : B \quad \vdash \mathtt{id\,id} : A}{\vdash \mathtt{id\,id}(\mathtt{id\,id}) : A}}{\vdash \mathtt{id\,id}(\mathtt{id\,id}(\mathtt{id\,id})) : A}
$$

Example (3)

$$\frac{\overline{x : [[\,] \to A] \vdash x : [\,] \to A}}{x : [[\,] \to A] \vdash x\,x : A}$$

$$\frac{\overline{x : [[B] \to A] \vdash x : [B] \to A} \quad \overline{x : [B] \vdash x : B}}{x : [[B] \to A, B] \vdash x\,x : A}$$

$$\frac{\overline{x : [[B, C] \to A] \vdash x : [B, C] \to A} \quad \overline{x : [B] \vdash x : B} \quad \overline{x : [C] \vdash x : C}}{x : [[B, C] \to A, B, C] \vdash x\,x : A}$$

More in general:

$$\vdash \lambda x.\, x\,x : [[B_1, \ldots, B_n] \to A, B_1, \ldots, B_n] \to A$$

However, $\Omega = (\lambda x.\, x\,x)\, \lambda x.\, x\,x$ is **not** typable.
Intuitively, the argument should be typed an infinite number of times.

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System $\mathcal{H}$

Gardner (1994), de Carvalho (2007)

Example (4)

$$\cfrac{\cfrac{\cfrac{\cfrac{}{x : [[\,] \to A] \vdash x : [\,] \to A}}{x : [[\,] \to A] \vdash x\,\Omega : A}}{\vdash \lambda x.\,x\,\Omega : [[\,] \to A] \to A}}{\cfrac{\vdash \lambda y.\,\lambda x.\,x\,y : [\,] \to [[\,] \to A] \to A}{\vdash (\lambda y.\,\lambda x.\,x\,y)\,\Omega : [[\,] \to A] \to A}}$$

$$\lessgtr$$

$$\cfrac{\cfrac{\cfrac{}{x : [[\,] \to A] \vdash x : [\,] \to A}}{x : [[\,] \to A] \vdash x\,\Omega : A}}{\vdash \lambda x.\,x\,\Omega : [[\,] \to A] \to A}$$

System $\mathcal{H}$

We shall show that System $\mathcal{H}$ characterises **head normalising** terms.
Three key lemmas:

Lemma 1 (Weighted Subject Reduction)

If $t \to_{\mathrm{h}} s$ is a head step and $\Gamma \vdash t : A$ then $\Gamma \vdash s : A$.
Moreover, the **size** of the typing derivation decreases.
(The size is the number of inference rules, not counting the `many` rule).

Lemma 2 (Subject Expansion for head steps)

If $t \to_{\mathrm{h}} s$ is a head step and $\Gamma \vdash s : A$ then $\Gamma \vdash t : A$.

Lemma 3 (Typability of head normal forms)

If t is a head normal form, then t is typable.

System $\mathcal{H}$ Gardner (1994), de Carvalho (2007)

Assuming the lemmas on the previous slide, we have:

Theorem (System $\mathcal{H}$ characterises head normalisation)

The following are equivalent:

1. t is typable in System $\mathcal{H}$.
2. t is head normalising.

Proof of Soundness $(1 \implies 2)$.

▶ Let $D \rhd \Gamma \vdash t : A$ for some Γ, A.

▶ Proceed by induction on the size of D.

▶ If t is a head normal form, we are done.

▶ Otherwise, consider the head step $t \rightarrow_{\mathrm{h}} s$.

▶ By Weighted Subject Reduction, there is a typing derivation D' that concludes $\Gamma \vdash s : A$ and such that $\mathrm{sz}(D) > \mathrm{sz}(D')$.

▶ By IH, s is head normalising.

▶ Hence t is also head normalising.

System $\mathcal{H}$

Theorem (System $\mathcal{H}$ characterises head normalisation)

The following are equivalent:

1. t is typable in System $\mathcal{H}$.
2. t is head normalising.

Proof of Completeness $(2 \implies 1)$.

- ▶ Let $t \to_{\mathrm{h}} t_1 \to_{\mathrm{h}} t_2 \ldots \to_{\mathrm{h}} t_n$ with t_n a head normal form.
- ▶ Proceed by induction on n.
- ▶ If $n = 0$, t is a head normal form, so it is typable (by Lemma 3).
- ▶ If $n > 0$, by IH there exist Γ, A such that $\Gamma \vdash t_1 : A$.
- ▶ But $t \to_{\mathrm{h}} t_1$, so by Subject Expansion $\Gamma \vdash t : A$.

System $\mathcal{H}$ Gardner (1994), de Carvalho (2007)

Lemma 1 (Weighted Subject Reduction)

If $t \to_{\mathrm{h}} s$ is a head step and $\Gamma \vdash t : A$ then $\Gamma \vdash s : A$.
Moreover, the **size** of the typing derivation decreases.

Proof.

▶ The head step is of the form:

$$t = (\lambda x_1 \ldots x_n. (\lambda y. p) \, q \, t_1 \ldots t_m) \to_{\mathrm{h}} (\lambda x_1 \ldots x_n. \, p\{x := q\} \, t_1 \ldots t_m) = s$$

▶ It is easy to reduce the general case to the root case ($n = m = 0$):

$$t = (\lambda y. p) \, q \to_{\mathrm{h}} p\{x := q\} = s$$

$$
\cfrac{
  \cfrac{
    \cfrac{D_1}{\Gamma, x : \mathcal{M} \vdash p : A}
  }{\Gamma \vdash \lambda y. p : \mathcal{M} \to A}\mathtt{lam}
  \quad
  \cfrac{D_2}{\Delta \Vdash q : \mathcal{M}}
}{\Gamma + \Delta \vdash (\lambda y. p) \, q : A}\mathtt{app}
\quad\leadsto\quad
\cfrac{D'}{\Gamma + \Delta \vdash p\{x := q\} : A}
$$

▶ The property is reduced to a Substitution Lemma.

▶ The rules on the left (lam, app) are erased — the size decreases.

Lemma 1' (Substitution Lemma)

Let $D_1 \triangleright \Gamma, x : \mathcal{M} \vdash t : A$ and $D_2 \triangleright \Delta \Vdash s : \mathcal{M}$.
Then there exists a derivation D' such that $D' \triangleright \Gamma + \Delta \vdash t\{x := s\} : A$
and $\mathtt{sz}(D') = \mathtt{sz}(D_1) - |\mathcal{M}| + \mathtt{sz}(D_2)$.

Proof.

- ▶ Proceed by induction on D_1.
- ▶ We only show some interesting cases:

System $\mathcal{H}$

The most interesting part is the substitution lemma on the `many` rule:

$$D_1 = \dfrac{\dfrac{D_{1,1}}{\Gamma_1, x : \mathcal{M}_1 \vdash t : A_1} \quad \cdots \quad \dfrac{D_{1,n}}{(\Gamma_n, x : \mathcal{M}_n \vdash t : A_n)}}{(\Gamma_1 + \ldots + \Gamma_n), x : (\mathcal{M}_1 + \ldots + \mathcal{M}_n) \vdash t : [A_1, \ldots, A_n]}\text{many}$$

$$D_2 = \dfrac{\vdots}{\Delta \Vdash s : (+_{i \in I} \mathcal{M}_i)}\text{many}$$

Then there exist contexts $\Delta_1, \ldots, \Delta_n$ and derivations $D_{2,1}, \ldots, D_{2,n}$ s.t.:

$$\dfrac{D_{2,1}}{\Delta_1 \Vdash s : \mathcal{M}_1}\text{many} \quad \cdots \quad \dfrac{D_{2,n}}{\Delta_n \Vdash s : \mathcal{M}_n}\text{many}$$

where $\Delta = +_{i=1}^{n} \Delta_i$ and $\text{sz}(D_2) = +_{i=1}^{n} \text{sz}(D_{2,i})$.

Applying the IH on each pair $D_{1,i}$ / $D_{2,i}$ to obtain $D_{3,i}$, we conclude:

$$D_3 = \dfrac{\dfrac{D_{3,1}}{\Gamma_1 + \Delta_1 \vdash t\{x := s\} : A_1} \quad \cdots \quad \dfrac{D_{3,n}}{\Gamma_n + \Delta_n \vdash t\{x := s\} : A_n}}{(\Gamma_1 + \ldots + \Gamma_n) + (\Delta_1 + \ldots + \Delta_n) \vdash t\{x := s\} : [A_1, \ldots, A_n]}\text{many}$$

System $\mathcal{H}$

Lemma 2 (Subject Expansion)

If $t \to_h s$ is a head step and $\Gamma \vdash s : A$ then $\Gamma \vdash t : A$.

Proof. Similar to the proof of Subject Reduction.
Relies on an Anti-Substitution Lemma.

Lemma 2' (Anti-Substitution Lemma)

If $\Gamma \vdash t\{x := s\} : A$, there exist $\Gamma_1, \Gamma_2, \mathcal{M}$ such that:

- $\Gamma_1, x : \mathcal{M} \vdash t : A$
- $\Gamma_2 \Vdash s : \mathcal{M}$
- $\Gamma = \Gamma_1 + \Gamma_2$

System $\mathcal{H}$

Lemma 3 (Typability of head normal forms)

If t is a head normal form, then t is typable.

Proof. Since t is a head normal form, it is of the form:

$$t = \lambda x_1 \ldots x_n. y\, t_1 \ldots t_m$$

Let $A = \underbrace{[] \to \ldots \to []}_{m \text{ times}} \to \alpha$.

Regardless of the shapes of $t_1, \ldots, t_m$:

$$y : [A] \vdash y\, t_1 \ldots t_m : \alpha$$

We consider two cases:

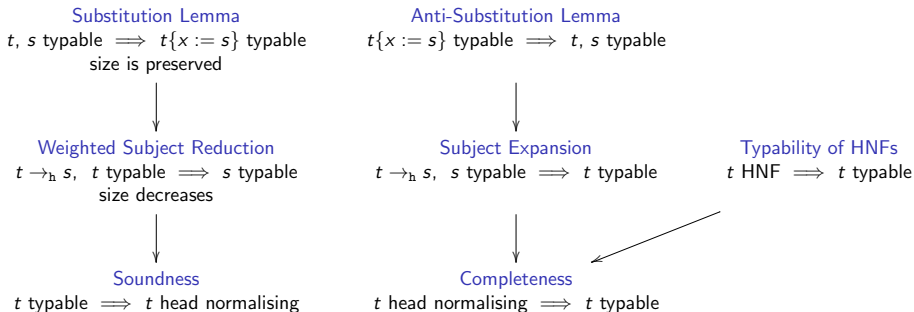1. If $y \notin \{x_1, \ldots, x_n\}$, then:
$$y : [A] \vdash \lambda x_1 \ldots x_n. y\, t_1 \ldots t_m : \underbrace{[] \to \ldots \to []}_{n \text{ times}} \to \alpha$$

2. If $y = x_i$ for some $1 \leq i \leq n$, then:
$$\vdash \lambda x_1 \ldots x_n. x_i\, t_1 \ldots t_m : \underbrace{[] \to \ldots \to []}_{i-1 \text{ times}} \to [A] \to \underbrace{[] \to \ldots \to []}_{n-i \text{ times}} \to \alpha$$

Summary of proof technique

Substitution Lemma
t, s typable $\implies$ $t\{x := s\}$ typable
size is preserved

Anti-Substitution Lemma
$t\{x := s\}$ typable $\implies$ t, s typable

Weighted Subject Reduction
$t \to_h s$, t typable $\implies$ s typable
size decreases

Subject Expansion
$t \to_h s$, s typable $\implies$ t typable

Typability of HNFs
t HNF $\implies$ t typable

Soundness
t typable $\implies$ t head normalising

Completeness
t head normalising $\implies$ t typable

The same techniques are extended to other systems:

Cf. Mazza, Pellissier & Vial (POPL'18)

typable in System $\mathcal{H}$ $\rightsquigarrow$ typable in System X
head normalising $\rightsquigarrow$ $\to_X$ normalising
head normal form $\rightsquigarrow$ $\to_X$ normal form
$\vdots$

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Head normalisation

Remark
Subject reduction and expansion hold for **arbitrary** reduction steps.

$$\text{Let } t \to_\beta s. \text{ Then } \Gamma \vdash t : A \text{ if and only if } \Gamma \vdash s : A.$$

(Only slightly revising the proofs).

Corollary (Head normalisation)
If $t \to_\beta^* s \in \mathsf{HNF}$ then there exists $s' \in \mathsf{HNF}$ such that $t \to_h^* s'$.

Remark
Weighted subject reduction does not hold for arbitrary reduction steps.

Subject reduction may yield a derivation of the same size when the reduction occurs in an **untyped** subterm:

$$\cfrac{\cfrac{}{x : [] \to A \vdash x : [] \to A}\text{var}}{x : [] \to A \vdash x\,t : A}\text{app} \quad\leadsto\quad \cfrac{\cfrac{}{x : [] \to A \vdash x : [] \to A}\text{var}}{x : [] \to A \vdash x\,s : A}\text{app}$$

Quantitative upper bounds

The Weighted Subject Reduction lemma ensures that the size of the typing derivation decreases after each head reduction step.

Theorem (Upper bounds for reduction lengths)

Let $D \rhd \Gamma \vdash t : A$ in System $\mathcal{H}$ and let $t \rightarrow_{\mathrm{h}}^{*} s \in \mathsf{HNF}$.
Then the number of steps in the reduction is at most $\mathrm{sz}(D)$.