Quantum Programming Languages

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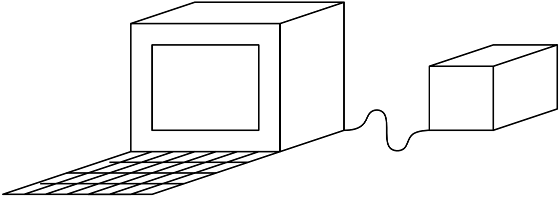
1st QCOMICAL School

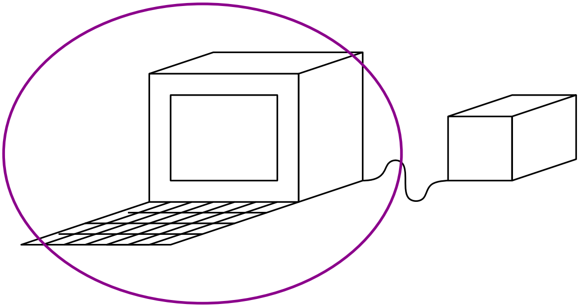
Plan

Plan

Plan

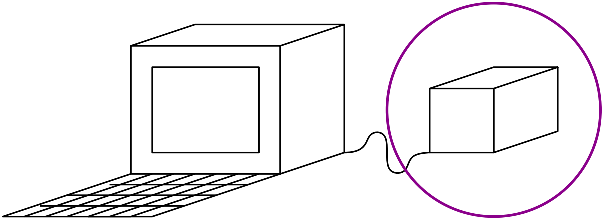
Model of Computation: Co-processor





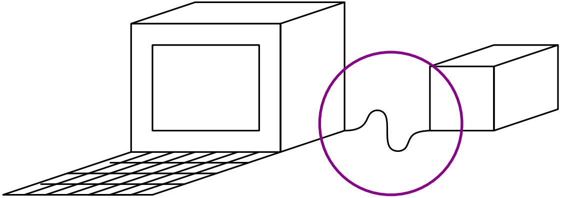
The program lives here

Model of Computation: Co-processor



This only holds the quantum memory

Model of Computation: Co-processor



Series of instructions/feedbacks

The Quantum Memory

A quantum memory

- Contains individually addressable quantum registers (qbits)
- State of n qbits: complex combination of strings of n bits
- E.g. for $n = 3$:

$$\begin{array}{rcl} & -\frac{1}{2} & \cdot\, 0\,0\,0 \\ + & \frac{1}{2} & \cdot\, 0\,0\,1 \\ + & \frac{i}{2} & \cdot\, 1\,1\,0 \\ - & \frac{i}{2} & \cdot\, 1\,1\,1 \end{array}$$

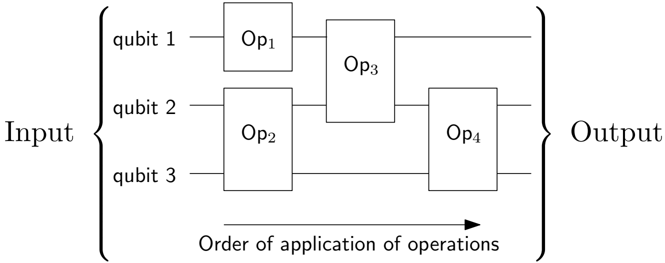
- With a norm condition.

Unlike probabilistic distributions,

all are available at the same time.

Quantum Circuit Model

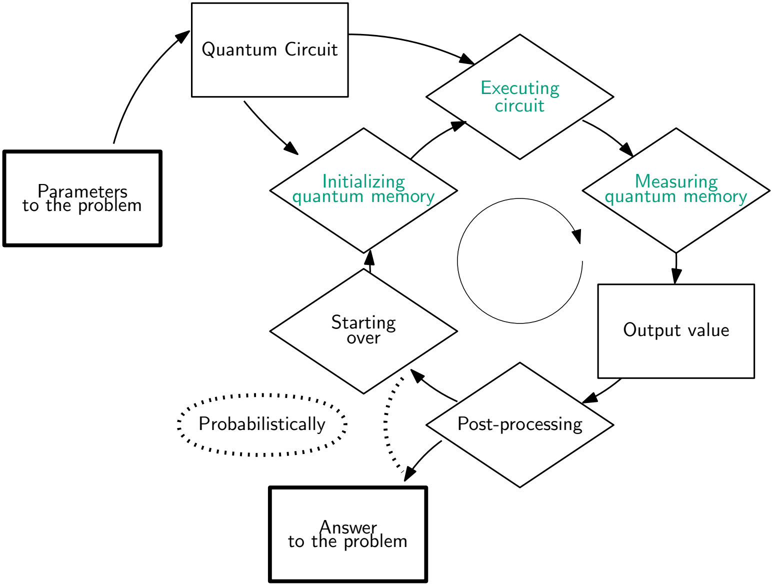
Stream of instructions: a series of elementary gates applied on the quantum memory, that are described by a quantum circuit.



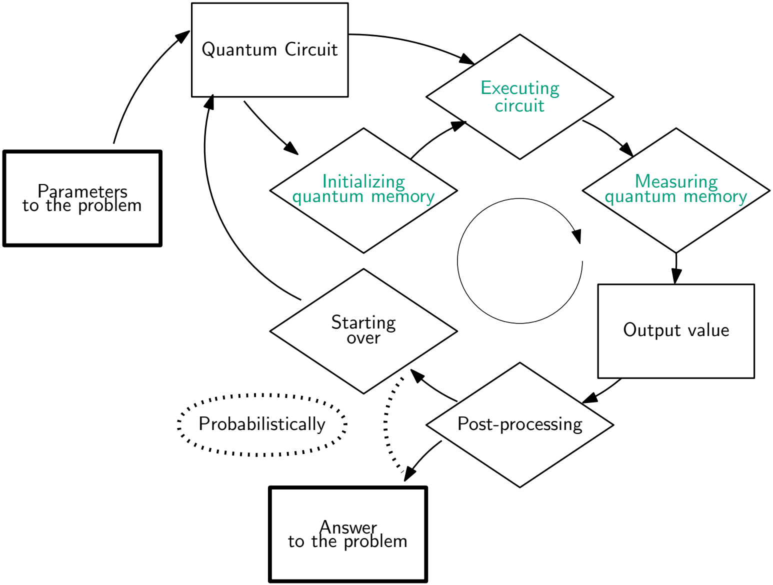
- Each operation is reversible, unitary on the space of states
- Wire $\equiv$ quantum bit $\equiv$ a quantum register
- No "quantum loop", "conditional stop" nor "branching point"

Plan

Structure of (Static) Quantum Algorithms



Structure of (Variational) Quantum Algorithms



Quantum Algorithm, Probabilistic Algorithm

Simple probabilistic algorithm to factor 289884400687823

- Fair draw of a number among $2, 3, 4, 5, \ldots$
- Test: Euclidian division
- Found a factor: success. Otherwise: start over.

Very poor probability of success!

Shor's factorization algorithm

- Probabilistic sampling performed with measurement
- The quantum circuit build a "good" probability distribution.
  $\rightarrow$ boosts factors!

Quantum programming means building a circuit

(In case you're wondering: $315697 \cdot 918236159$)

Internals of Current Quantum Algorithms

The techniques used to described quantum algorithms are diverse.

1. Quantum primitives.
   – Quantum Fourier Transform
     Assuming $\omega = 0.\mathtt{xy}$, we want

$$\begin{array}{cl} & (e^{2\pi i\omega})^0 \cdot 00 \\ + & (e^{2\pi i\omega})^1 \cdot 01 \\ + & (e^{2\pi i\omega})^2 \cdot 10 \\ + & (e^{2\pi i\omega})^3 \cdot 11 \end{array} \qquad \longmapsto \qquad 1 \cdot \mathtt{xy}$$

Internals of Current Quantum Algorithms

The techniques used to described quantum algorithms are diverse.

1. Quantum primitives.
   - Phase estimation.
   - Amplitude amplification.
     Qubit 3 in state 1 means good.

$$\begin{array}{rl} & \alpha_0 \cdot 000 \\ + & \alpha_1 \cdot 011 \\ + & \alpha_2 \cdot 100 \\ + & \alpha_3 \cdot 110 \end{array} \longmapsto \begin{array}{rl} & {\scriptstyle \alpha_0} \cdot 000 \\ + & \large{\alpha_1} \cdot 011 \\ + & {\scriptstyle \alpha_2} \cdot 100 \\ + & {\scriptstyle \alpha_3} \cdot 110 \end{array}$$

Internals of Current Quantum Algorithms

The techniques used to described quantum algorithms are diverse.

1. Quantum primitives.
   - Quantum Fourier Transform
   - Amplitude amplification.
   - Quantum walk.

$$\begin{tikzcd}
0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \arrow[d] \\
15 \arrow[u] & 14 \arrow[l] & 13 \arrow[l] & 12 \arrow[l] & 11 \arrow[l] & 10 \arrow[l] & 9 \arrow[l] & 8 \arrow[l]
\end{tikzcd}$$

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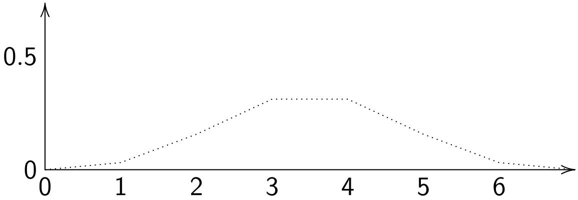
     After 5 steps of a probabilistic walk:



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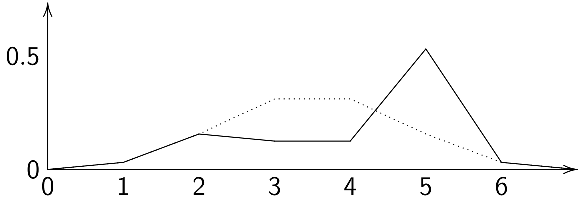


Internals of Current Quantum Algorithms

The techniques used to described quantum algorithms are diverse.

1. Quantum primitives.
   - Quantum Fourier Transform
   - Amplitude amplification
   - Quantum walk
   - Hamiltonian simulation
   - ...

They are given as circuit templates

Internals of Current Quantum Algorithms

The techniques used to described quantum algorithms are diverse.

2. Oracles.
   - Take a classical function $f : \texttt{Bool}^n \to \texttt{Bool}^m$.
   - Construct

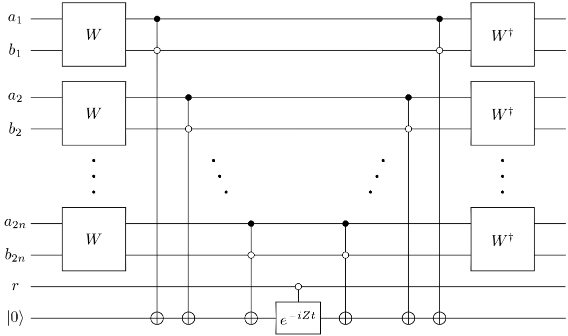
$$\begin{array}{rccc} \overline{f}: & \texttt{Bool}^{n+m} & \longrightarrow & \texttt{Bool}^{n+m} \\ & (x, y) & \longmapsto & (x, y \oplus f(x)) \end{array}$$

   - Build the unitary U_f acting on $n + m$ qubits computing $\overline{f}$.

Building the circuit depends on how f is given

The techniques used to described quantum algorithms are diverse.

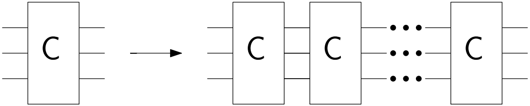
3. Blocks of loosely-defined low-level circuits.



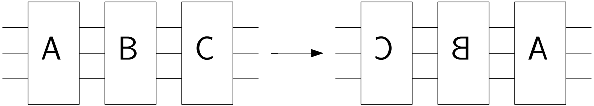
This is not a formal specification!

The techniques used to described quantum algorithms are diverse.

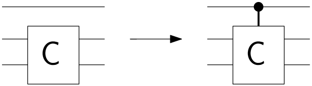
4. High-level operations on circuit:
   - Repetition



   - Inversion



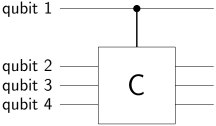
   - Control



Internals of Current Quantum Algorithms

The techniques used to described quantum algorithms are diverse.

4. High-level operations on circuit:
   - Control : conditional action of a circuit



C is applied on qubits 2-4 only when qubit 1 is true:
Suppose that C flips its input bits. Then the above circuit does

$$\begin{array}{rl} \texttt{qbit} & 1\,234 \\ \frac{1}{\sqrt{2}} & 1\,010 \\ +\,\frac{1}{\sqrt{2}} & 0\,110 \end{array} \quad \longmapsto \quad \begin{array}{rl} \texttt{qbit} & 1\,234 \\ \frac{1}{\sqrt{2}} & 1\,101 \\ +\,\frac{1}{\sqrt{2}} & 0\,110 \end{array}$$

This acts as a form of “quantum test”

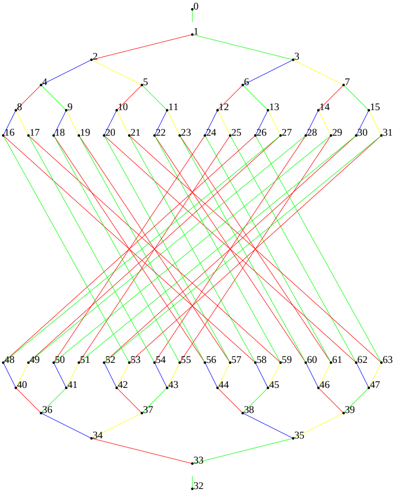
Internals of Current Quantum Algorithms

The techniques used to described quantum algorithms are diverse.

5. Classical processing.
   - Generating the circuit...
   - Computing the input to the circuit.
   - Processing classical feedback in the middle of the computation.
   - Analyzing the final answer (and possibly starting over).

Plan

Case study: BWT algorithm



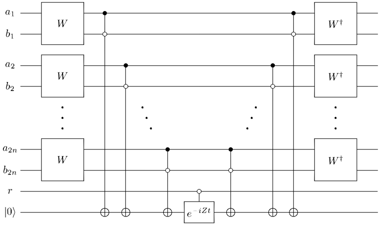
- Start at entrance, look for exit
- Description of the graph:

$$
\begin{array}{lll}
I & : & \texttt{Node} \\
G & : & \texttt{Color} \times \texttt{Node} \to \texttt{Maybe Node} \\
O & : & \texttt{Node} \to \texttt{Bool}
\end{array}
$$

- Random/Quantum walk
- Parameters:
  height of tree ; number of steps.

Case study: BWT algorithm

- Initialization of a register to the input node (using I)
- 10^6 iterations:
  - Diffuse
  - Call oracle for red
  - Diffuse
  - Call oracle for green
  - Diffuse
  - Call oracle for blue
  - Diffuse
  - Call oracle for yellow
- Measure the node we sit on
- Test with O that we reached the output node.



Case study: QLS algorithm

Considering a vector $\vec{b}$ and the system

$$A \cdot \vec{x} = \vec{b},$$

compute the value of $\quad \langle\, \vec{x} \,|\, \vec{r} \,\rangle \quad$ for some vector $\vec{r}$.

Practical situation: the matrix A corresponds to the finite-element approximation of the scattering problem:

Case study: QLS algorithm

For more precision: `arXiv:1505.06552`

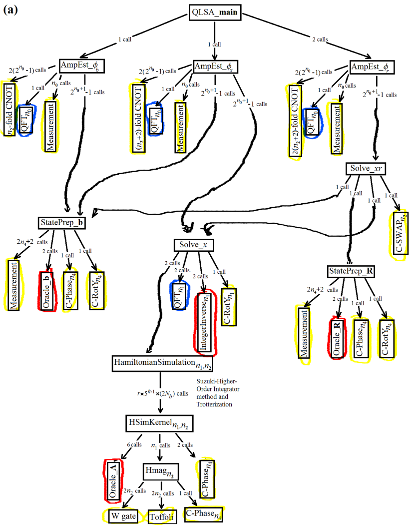
Three oracles:

- for $\vec{r}$ and for $\vec{b}$: input an index, output (the representation of) a complex number
- for A: input two indexes, output also a complex number

It uses many quantum primitives

- Amplitude estimation
- Phase estimation
- Amplitude amplification
- Hamiltonian simulation

Case study: QLS algorithm



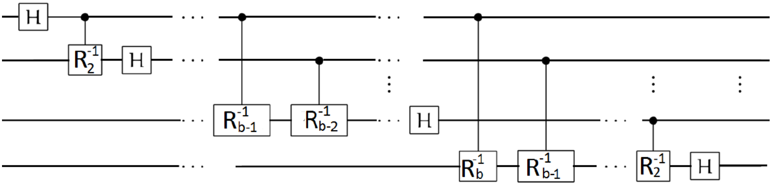
- Yellow: Elementary gates.
- Red: Oracles.
- Blue: QFT's.
- Black: Subroutines.
- *Parameters:*
  Dimensions of the space;
  Precision for each of the vectors;
  Allowed error;
  Various parameters for A...
  In total, 19 parameters.

Case study: QLS algorithm

Oracle R is given by the function

```
calcRweights y nx ny lx ly k theta phi =
    let (xc',yc') = edgetoxy y nx ny in
    let xc = (xc'-1.0)*lx - ((fromIntegral nx)-1.0)*lx/2.0 in
    let yc = (yc'-1.0)*ly - ((fromIntegral ny)-1.0)*ly/2.0 in
    let (xg,yg) = itoxy y nx ny in
    if (xg == nx) then
        let i = (mkPolar ly (k*xc*(cos phi)))*(mkPolar 1.0 (k*yc*(sin phi)))*
                ((sinc (k*ly*(sin phi)/2.0)) :+ 0.0) in
        let r = ( cos(phi) :+ k*lx )*((cos (theta - phi))/lx :+ 0.0) in i * r
    else if (xg==2*nx-1) then
        let i = (mkPolar ly (k*xc*cos(phi)))*(mkPolar 1.0 (k*yc*sin(phi)))*
                ((sinc (k*ly*sin(phi)/2.0)) :+ 0.0) in
        let r = ( cos(phi) :+ (- k*lx))*((cos (theta - phi))/lx :+ 0.0) in i * r
    else if ( (yg==1) && (xg<nx) ) then
        let i = (mkPolar lx (k*yc*sin(phi)))*(mkPolar 1.0 (k*xc*cos(phi)))*
                ((sinc (k*lx*(cos phi)/2.0)) :+ 0.0) in
        let r = ( (- sin phi) :+ k*ly )*((cos(theta - phi))/ly :+ 0.0) in i * r
    else if ( (yg==ny) && (xg<nx) ) then
        let i = (mkPolar lx (k*yc*sin(phi)))*(mkPolar 1.0 (k*xc*cos(phi)))*
                ((sinc (k*lx*(cos phi)/2.0)) :+ 0.0) in
        let r = ( (- sin phi) :+ (- k*ly) )*((cos(theta - phi)/ly) :+ 0.0) in i * r
    else 0.0 :+ 0.0
```

The algorithms create circuits whose sizes and shapes depend on the parameters. E.g. the size of the input register:



(QFT)

Case study: circuit snippets

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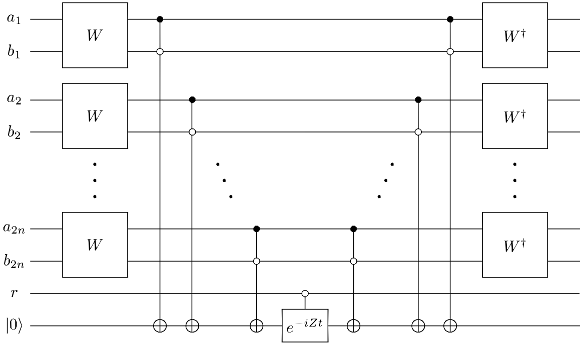

(diffusion step in BWT)

Case study: circuit snippets

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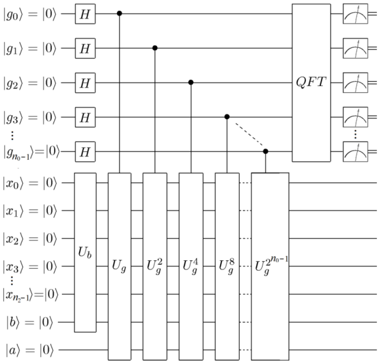

(piece of one subroutine of QLS)

Case study: circuit snippets

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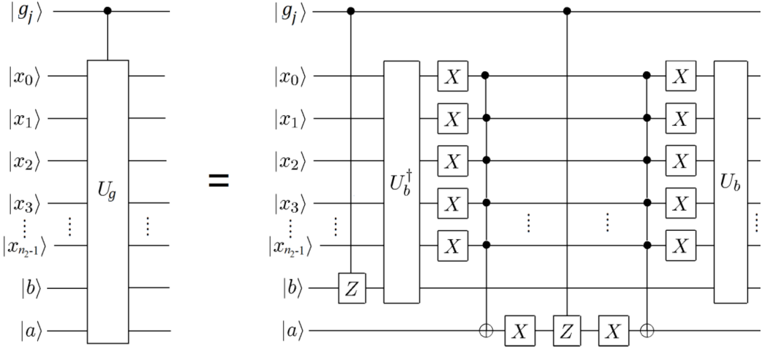

(the subroutine U_g)

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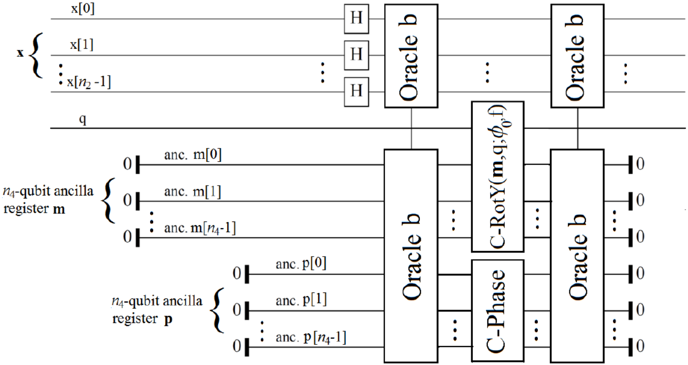


(the subroutine U_b)

Plan

Lessons learned

- Circuit construction
  - Procedural: Instruction-based, one line at a time
  - Declarative: Circuit combinators
    - Inversion
    - Repetition
    - Control
    - Computation/uncomputation
- Circuits as inputs to other circuits
- Regularity with respect to the size of the input
- Distinction parameter / input
- Need for automation for oracle generation

Plan

Programming framework

Two approaches

- » Circuit as a record
  - One type `circuit`
  - Qubits $\equiv$ wire numbers
  - Native: vertical/horizontal concatenation, gate addition
- » Circuit as a function
  - Qubits $\equiv$ first-order objects
  - Input wires $\equiv$ function input
  - Output wires $\equiv$ function output

Circuits as Records

Simplest model: an object holding all of the circuit structure

- Classical wires
- Quantum wires
- List of gates (or directed acyclic graph)
- This is for instance QisKit/QASM model

In this system

- Static circuit
- No high-level hybrid interaction: sequence
  1. circuit generation
  2. circuit evaluation
  3. measure
  4. classical post-processing
  5. back to (1)

Circuits as Records

Procedural construction (QisKit)

```
q = QuantumRegister(5)
c = ClassicalRegister(1)
circ = QuantumCircuit(q,c)

circ.h(q[0])
for i in range(1,5):
  circ.cx(q[0], q[i])
circ.meas(q[4],c[0])
```

- Static ID For registers
- Wires are numbers
- Gate $\equiv$ instruction
- Classical control: Circuit building
- Explicit "run" of circuit

Combinators: return a record `circuit`

- `circ.control(4)`
- `circ.inverse()`
- `circ.append(other-circuit)`

Circuits as Functions

A function (Quipper)

```
a -> Circ b
```

- Inputs something of type `a`
- Outputs something of type `b`
- As a side-effect, generates a circuit snippet.

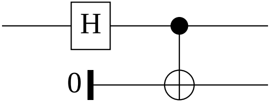
Or

- Inputs a value of type `a`
- Outputs a computation of type `b`

Plan

Circuits as Functions

The circuit



can be typed with

```
Qubit -> Circ (Qubit,Qubit)
```

- Inputs one qubit
- Outputs a pair of qubits
- Spits out some gates when evaluated

The gates are however encapsulated in the function

Circuits as Functions

Representing circuits (Quipper)

Name of circuit

Input: one wire

Indeed a circuit

Two output wires

```
myCircuit :: Qubit -> Circ (Qubit, Qubit)

myCircuit q = do
      ...
      ...
      return (x,y)
```

Start a procedural sequence

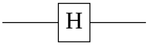
The name of the input wire

The two output wires

Circuits as Functions

Procedural presentation of circuits:

```
prog :: Qubit -> Circ (Qubit,Qubit)
prog q = do
  hadamard_at q
  r <- qinit False
  qnot_at r `controlled` q
  return (q,r)
```



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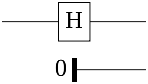

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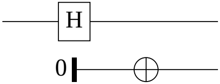

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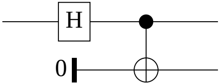

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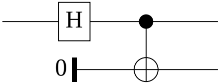
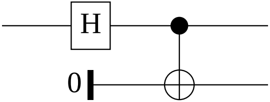


Circuits as Functions




```
import Quipper

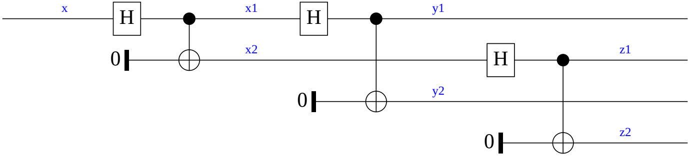
circ ::
   Qubit -> Circ (Qubit,Qubit)
circ x = do
   y <- qinit False
   hadamard_at x
   qnot_at y ‘controlled‘ x
   return (x,y)
```

- Qubits $\equiv$ first-class variable
- Circuit $\equiv$ function
- Wires $\equiv$ inputs and outputs
- Mix classical/quantum

Circuits as Functions

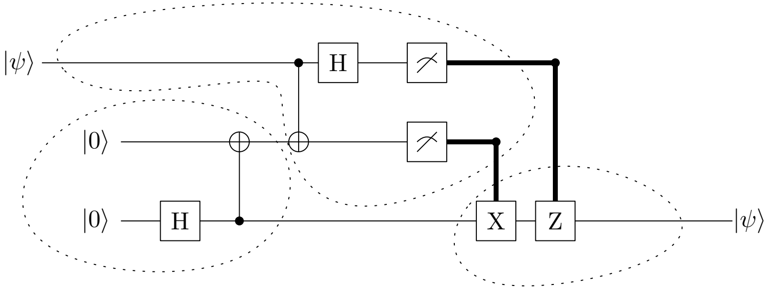
Wires do not have "fixed" location

```
circ2 :: Qubit -> Circ ()
circ2 x = do
   (x1,x2) <- circ x
   (y1,y2) <- circ x1
   (z1,z2) <- circ x2
   return ()
```



- Qubit $\not\equiv$ Wire number
- Circuits as functions: can be applied
- More expressive types

Exercise: Decompose according to the dashed sections

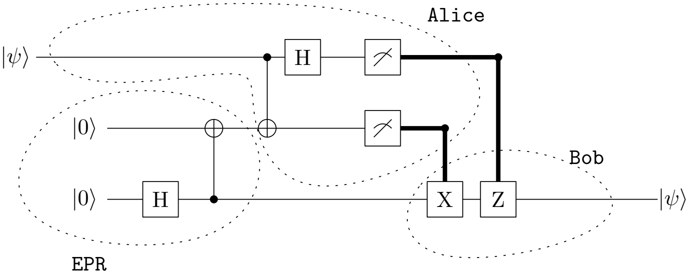


Circuit Combinators: exercise !

What could be the corresponding operations ?

1. `(a -> Circ b) -> (b -> Circ c) -> (a -> Circ c)`
2. `(a -> Circ b) -> (b -> Circ a)`
3. ```
   (a -> Circ b) -> (c -> Circ d)
                 -> ((a,c) -> Circ (b,d))
   ```
4. `(a -> Circ b) -> ((a,Qubit) -> Circ (b,Qubit))`
5. `(a -> Circ b) -> (Qubit -> a -> Circ (b,Qubit))`
6. `(a -> Circ b) -> (Qubit -> a -> Circ b)`
   ```

Circuits Combinators: Coming back to Teleportation



can be typed as

» `EPR :: Circ (Qubit, Qubit)`

» `Alice :: Qubit -> Qubit -> Circ (Bit,Bit)`

» `Bob :: Qbit -> (Bit,Bit) -> Qubit)`

Composing, we get

```
Circ (Qubit -> Circ (Bit,Bit), (Bit,Bit) -> Circ Qubit)
```

Plan

Families of Circuits

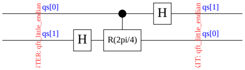
A program

- Inputs classical parameters
- Construct a circuit from these parameters
- Run the circuit

Circuits are parametrized families!

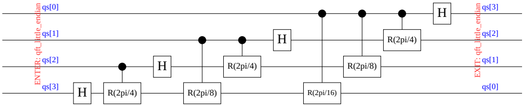
Families of Circuits

Example: QFT



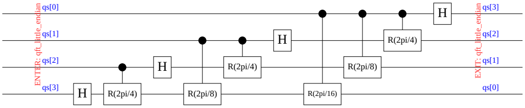
Families of Circuits

Example: QFT



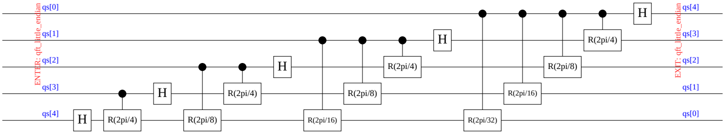
Families of Circuits

Example: QFT



Families of Circuits

Example: QFT



Families of Circuits

With the help of lists:

Name of circuit

Input a **list of wires**

Indeed a circuit

Output a **list of wires**

```
myCircuit :: [Qubit] -> Circ [Qubit]

myCircuit qs = do
     ...
     ...
     return ...
```

Start a procedural sequence

The name of the input list

The output list

List combinators, e.g.

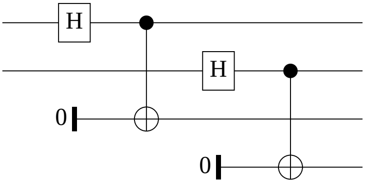
```
mapM ::  (a -> Circ b) -> [a] -> Circ [b]
```

Mixed presentation of circuits:

List of size 2:

```
prog :: Qubit -> Circ (Qubit,Qubit)
prog q = do
  hadamard_at q
  r <- qinit False
  qnot_at r ‘controlled‘ q
  return (q,r)

prog2 :: [Qubit] -> Circ [(Qubit,Qubit)]
prog2 l = mapM prog l
```



Families of Circuits

List combinators, e.g.

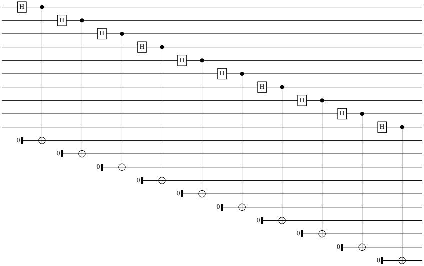
```
mapM ::  (a -> Circ b) -> [a] -> Circ [b]
```

Mixed presentation of circuits:

```
prog :: Qubit -> Circ (Qubit,Qubit)
prog q = do
  hadamard_at q
  r <- qinit False
  qnot_at r `controlled` q
  return (q,r)

prog2 :: [Qubit] -> Circ [(Qubit,Qubit)]
prog2 l = mapM prog l
```

List of size 10:



Plan

Example: Quipper Code

```
import Quipper

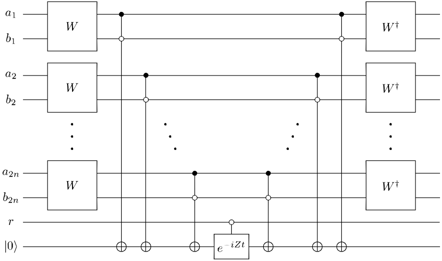
w :: (Qubit,Qubit) -> Circ (Qubit,Qubit)
w = named_gate "W"

toffoli :: Qubit -> (Qubit,Qubit) -> Circ Qubit
toffoli d (x,y) =
  qnot d `controlled` x .==. 1 .&&. y .==. 0

eiz_at :: Qubit -> Qubit -> Circ ()
eiz_at d r =
  named_gate_at "eiZ" d `controlled` r .==. 0

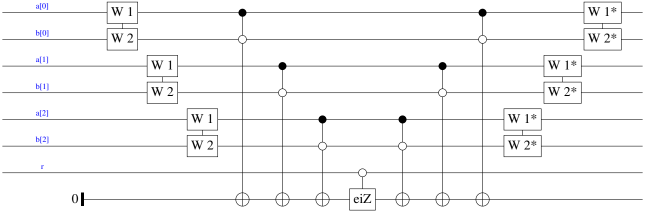
circ :: [(Qubit,Qubit)] -> Qubit -> Circ ()
circ ws r = do
  label (unzip ws,r) (("a","b"),"r")
  d <- qinit 0
  mapM_ w ws
  mapM_ (toffoli d) ws
  eiz_at d r
  mapM_ (toffoli d) (reverse ws)
  mapM_ (reverse_generic w) (reverse ws)
  return ()

main = print_generic EPS circ (replicate 3 (qubit,qubit)) qubit
```



Example: BWT

Result (3 wires):



Example: BWT

Result (30 wires):

Plan

Design Choices: Summary

Requirements for coding Circuits

- Classical structures!
- Hierarchical Representation
- Parametricity
- Non-trivial Combinators

A natural view

- Low Key, First-Order Functions
- Circuit construction seen as a monad

Circuit Construction as a Monad?

Monad: a type constructor `M` equipped with

- `return ::  a -> M a`
- `app ::  M a -> (a -> M b) -> M b`

Example: the list monad

- Type `[a]` : for lists of elements of type `a`
- `return x = [x]`
- `app [x1, x2, x3] f = (f x1) ++ (f x2) ++ (f x3)`

Circuit Construction as a Monad?

Monad: a type constructor `M` equipped with

- `return :: a -> M a`
- `app :: M a -> (a -> M b) -> M b`

Example: state monad `M a = Int -> (a, Int)`

- `return x =` λ `n . (x, n)`
- `app g f =` λ `n . let (y,m) = g n in f y m`

Special combinators

```
get :: M Int
get = λ n . (n,n)

inc :: Int -> M ()
inc n = λ m . ((), m+n)
```

do-notation

```
double = do
    n <- get
    inc n
```

Circuit Construction as a Monad?

Monad: a type constructor `M` equipped with

- `return ::  a -> M a`
- `app ::  M a -> (a -> M b) -> M b`

Circuit monad: `M a = GateList -> (a, GateList)`

- `return x =` λ `n . (x, n)`
- `app g f =` λ `n . let (y,m) = g n in f y m`

Special combinators

```
addGate ::  Gate -> Wire -> M ()
addGate g w = λ gs . ((), [(g w) added to gs])
```

```
qinit ::  M Wire
qinit = λ gs . ([fresh wire not in gs], gs)
```

Interaction only performed through these combinators

Quantum PL in the wild

Just to name a few

- Quipper (Academic project)
- Q# (Microsoft)
- Silq (ETH Zurich)

And a wealth of Python's libraries

- Cirq (Google)
- myQLM (Eviden)
- Perceval (Quandela)
- Qiskit (IBM)
- and one for about every single company out there

Plan

Context

An oracle:

- classical description f of the problem
- turned into a reversible circuit:

$$U_f : |x\rangle|y\rangle \mapsto |x\rangle|y + f(x)\rangle$$

- How to build U_f ?
  - Small size: circuit synthesis
  - Arithmetic or other studied functions:
    Specific (highly optimized) circuits
  - Other cases?

Context

What about an arbitrary program, for example

```
calcRweights y nx ny lx ly k theta phi =
 let (xc',yc') = edgetoxy y nx ny in
 let xc = (xc'-1.0)*lx - ((fromIntegral nx)-1.0)*lx/2.0 in
 let yc = (yc'-1.0)*ly - ((fromIntegral ny)-1.0)*ly/2.0 in
 let (xg,yg) = itoxy y nx ny in
 if (xg == nx) then
     let i = (mkPolar ly (k*xc*(cos phi)))*(mkPolar 1.0 (k*yc*(sin phi)))*
             ((sinc (k*ly*(sin phi)/2.0))+0.0) in
     let r = ( cos(phi)+k*lx )*((cos (theta - phi))/lx+0.0) in i*r
 else if (xg==2*nx-1) then
     let i = (mkPolar ly (k*xc*cos(phi)))*(mkPolar 1.0 (k*yc*sin(phi)))*
             ((sinc (k*ly*sin(phi)/2.0))+0.0) in
     let r = ( cos(phi)+(- k*lx))*((cos (theta - phi))/lx+0.0) in i*r
 else if ( (yg==1) and (xg<nx) ) then
     let i = (mkPolar lx (k*yc*sin(phi)))*(mkPolar 1.0 (k*xc*cos(phi)))*
             ((sinc (k*lx*(cos phi)/2.0))+0.0) in
     let r = ( (- sin phi)+k*ly )*((cos(theta - phi))/ly+0.0) in i*r
 else if ( (yg==ny) and (xg<nx) ) then
     let i = (mkPolar lx (k*yc*sin(phi)))*(mkPolar 1.0 (k*xc*cos(phi)))*
             ((sinc (k*lx*(cos phi)/2.0))+0.0) in
     let r = ( (- sin phi)+(- k*ly) )*((cos(theta - phi)/ly)+0.0) in i*r
 else 0.0+0.0
```

(For QLS there was 10 matlab files of such functions)

Problem Statement

This is the topic of this section. How to:

- in short time
- and automatically
- get efficient,
- scalable,
- yet guaranteed
- reversible implementation
- of a higher-order, classical function,
- parametrically on the input size.

Plan

Basic idea

Landauer's embedding:

- Record all intermediate results.
- With $(x \wedge y) \wedge z$

$$t \mapsto x \wedge y; \quad u \mapsto t \wedge z; \quad \text{returns } u$$

  while retaining t as "garbage".
- Trace as a partial execution

Basic idea

Example: $x : \texttt{bool} \longmapsto \texttt{let}\ f = \texttt{not}\ \texttt{in}\ (fx)\,\texttt{and}\,(fx) : \texttt{bool}$

Regular execution

» Needs a concrete input, e.g. $x = \texttt{true}$

» Then: rewriting of the term

$$
\begin{array}{rl}
 & \texttt{let}\ f = \texttt{not}\ \texttt{in}\ (f\,\texttt{true})\,\texttt{and}\,(f\,\texttt{true}) \\
\to & (\texttt{not}\,\texttt{true})\,\texttt{and}\,(\texttt{not}\,\texttt{true}) \\
\to & \texttt{false}\,\texttt{and}\,(\texttt{not}\,\texttt{true}) \\
\to & \texttt{false}\,\texttt{and}\,\texttt{false} \\
\to & \texttt{false}
\end{array}
$$

Basic idea

Example: $x : \texttt{bool} \longmapsto \texttt{let}\ f = \texttt{not}\ \texttt{in}\ (fx)\,\texttt{and}\,(fx) : \texttt{bool}$

Trace of a partial execution

- Start with an unknown variable x
- Then: keep the trace of low-level actions to be performed on x

$$
\begin{array}{lll}
 & (\emptyset, & \texttt{let}\ f = \texttt{not}\ \texttt{in}\ (fx)\,\texttt{and}\,(fx)) \\
\to & (\emptyset, & (\texttt{not}\,x)\,\texttt{and}\,(\texttt{not}\,x)) \\
\to & ([y := \texttt{not}\,x] & y\,\texttt{and}\,(\texttt{not}\,x)) \\
\to & ([y \mapsto \texttt{not}\,x; z \mapsto \texttt{not}\,x], & y\,\texttt{and}\,z) \\
\to & ([y \mapsto \texttt{not}\,x; z \mapsto \texttt{not}\,x; t \mapsto y\,\texttt{and}\,z], & t)
\end{array}
$$

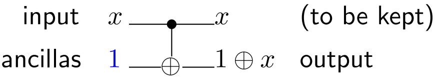
- ...and make this trace reversible: Landauer's embedding

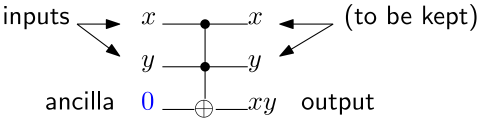
Basic idea

Example: $x : \texttt{bool} \longmapsto \texttt{let}\ f = \texttt{not in}\ (fx)\ \texttt{and}\ (fx) : \texttt{bool}$

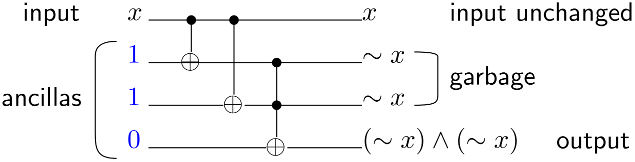
Trace of a partial execution

» `not` becomes a CNOT ; `and` becomes a Toffoli





» And the full trace is



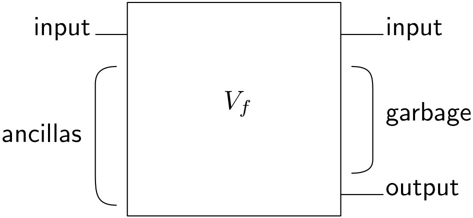
Plan

Composition Procedure

A function f : $\texttt{bool} \longrightarrow \texttt{bool}$
is turned into a map

$$V_f : \texttt{bool} \longrightarrow \mathrm{circuit}(\texttt{bool})$$

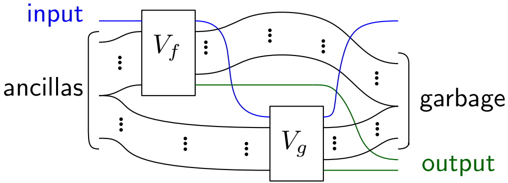
(Note: omit garbage in type)



Composition Procedure

A function $\langle f, g \rangle : \texttt{bool} \longrightarrow (\texttt{bool} \times \texttt{bool})$
is turned into a map

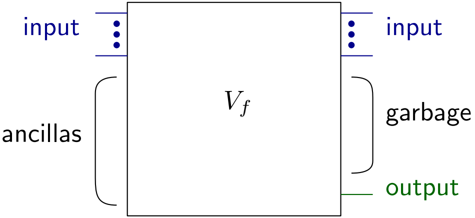
$$V_{\langle f,g \rangle} : \texttt{bool} \longrightarrow \mathrm{circuit}(\texttt{bool}{\times}\texttt{bool})$$



Composition Procedure

A function f : (`bool list`) $\longrightarrow$ `bool`
is turned into a map

$$V_f : (\texttt{bool list}) \longrightarrow \mathrm{circuit}(\texttt{bool})$$

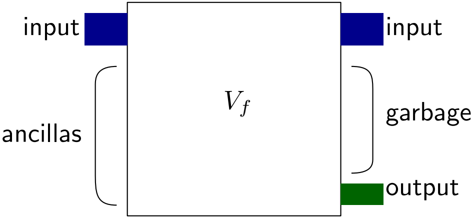


(Parametric circuit !)

Composition Procedure

A function $f : A \longrightarrow B$
is turned into a map

$$f : A \longrightarrow \mathrm{circuit}(\text{“}B\text{”})$$

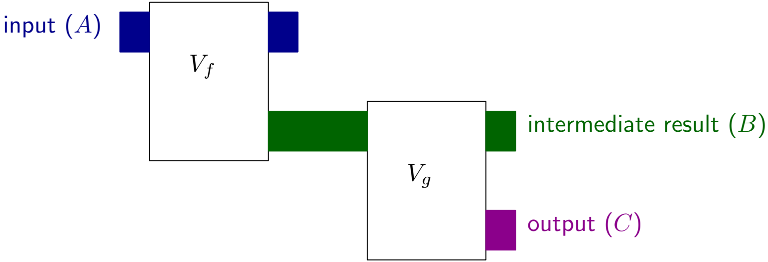


Composition Procedure

Two function $f : A \longrightarrow B$ and $g : B \rightarrow C$ are turned into maps

$$V_f : A \longrightarrow \mathrm{circuit}(\text{“}B\text{”})$$
$$V_g : B \longrightarrow \mathrm{circuit}(\text{“}C\text{”})$$

Composition $g \circ f : A \longrightarrow C$ is turned into $A \longrightarrow \mathrm{circuit}(\text{“}C\text{”})$



Example Try out

$$(x, y) \longmapsto \neg(\neg x) \wedge (\neg y)$$

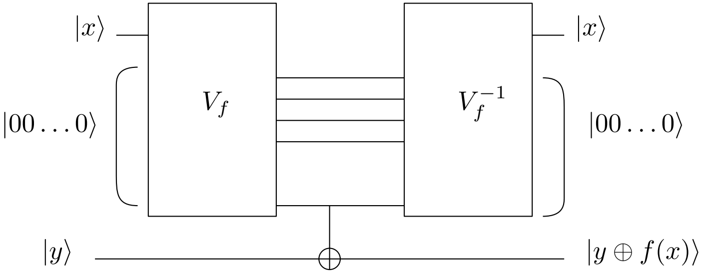
Example Try out

$$x \longmapsto x^8$$

Assume x is a natural number modulo 2^N written as a bitstring of size N, and assume that we already have a very optimized V for the multiplication.

Oracle from V

We construct U as



This scheme is known as compute-uncompute.
It has been implemented in Quipper.

Plan

Example: Adder

```
foldl :: (A → B → A) → A → [B] → A
foldl f a l = let rec g z l' = match (split l') with
                          nil   ↦ z
                        | ⟨h,t⟩ ↦ g (f z h) t
              in g a l

bit_adder : bit → bit → bit → (bit × bit)
bit_adder carry x y =
      let majority a b c = if (xor a b) then c else a in
      let z = xor (xor carry x) y in
      let carry' = majority carry x y in ⟨carry', z⟩

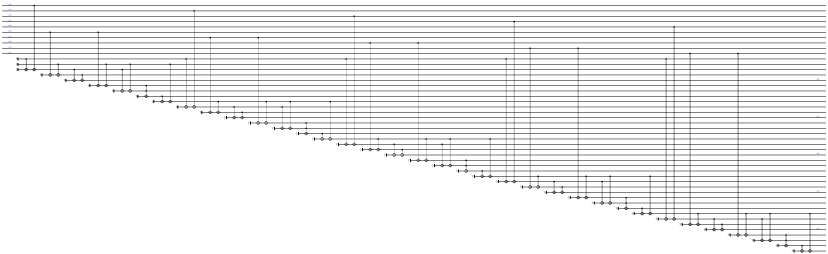
adder_aux : (bit × [bit]) → (bit × bit) → (bit × [bit])
adder_aux ⟨w, cs⟩ ⟨a, b⟩ = let ⟨w', c'⟩ = bit_adder w a b in ⟨w', c'::cs⟩

adder : [bit] × [bit] → [bit]
adder x y = snd (foldl adder_aux ⟨False, nil⟩ (zip y x))
```

adder is lifted to [bit] × [bit] → circuit([bit]).

Example: Adder

For $n = 5$, no optimization:



Size of circuit is proportional to number of low-level bit-operations in all execution paths of `adder`.

Example: Adder

n is the integer-size in bits.

paper	**ancillae**	**size**
VBE (1995)	n	$\sim 8n$
Cuccaro, Drapper & al. (2005)	0	$\sim 7n$
Drapper, Kutin & al. (2008)	$\sim 2n$	$\sim 10n$ (in place)
Drapper, Kutin & al. (2008)	$\sim n$	$\sim 5n$ (not in place)

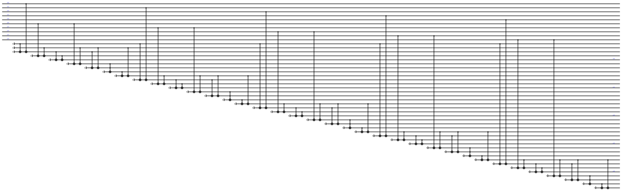
How do we scale against these ?

Example: Adders

If $n = 5$:

paper	**ancillae**	**size**
VBE (1995)	5	~ 40
Cuccaro, Drapper & al. (2005)	0	~ 35
Drapper, Kutin & al. (2008)	~ 10	~ 50 (in place)
Drapper, Kutin & al. (2008)	~ 5	~ 50 (not in place)

Automatically generated (no optimization):



Example: Adders

If $n = 5$:

paper	ancillae	size
VBE (1995)	5	~ 40
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Drapper, Kutin & al. (2008)	~ 10	~ 50 (in place)
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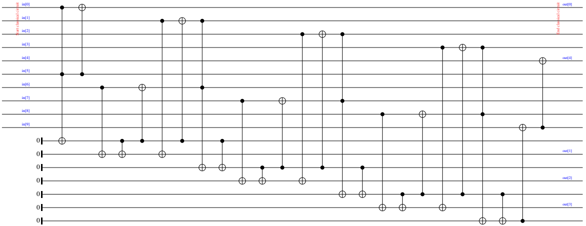
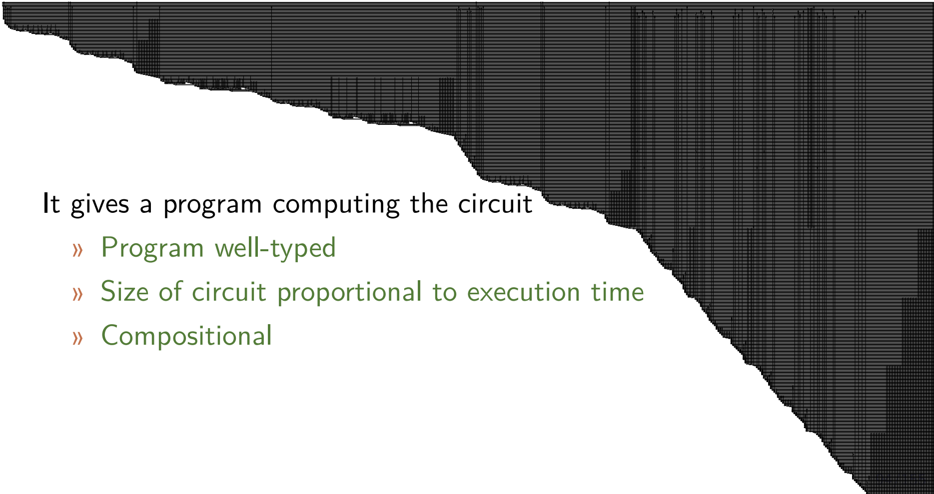
With a bit of (automated) optimization:



Example: The vector b

Hand-made circuits for: adders, multipliers, comparison, square root.
How about the b vector of the QLS algorithm ($Ax = b$) ?

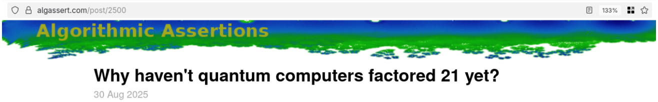


It gives a program computing the circuit

- Program well-typed
- Size of circuit proportional to execution time
- Compositional

Oracle Synthesis Nowadays

A lot of progress! But not quite enough to get there
See for example Gidney's blog:



For Shor's factoring algoritm, aiming at 21:

More broadly

- Realm of FTQC
- Large actors enters the field: Google, Microsoft, IBM, AWS...
- ...and small actors such as Alice&Bob or PsiQuantum

Plan

Plan

Goal

Formalizing

- Functional programming
- Higher-order combinators
- Capable of manipulating quantum information
- And quantum circuits

The first two items

- Realm of lambda-calculus

Lambda-Calculus

- Formal system from Alonzo Church, $\sim$1930
- Concept of Function and Application
  - "Every term is a function!"
  - Core of functional programming
  - For example: Haskell, OCaml, F#, Lisp, Erlang, *etc.*
- Very simple grammar:
  - Variables $x_1, x_2, x_3, \ldots$
  - Application (binary, infix)
  - Abstraction: $\lambda x.t$ (where t is a term)
- $\lambda x.t$: the function "$x \mapsto t$"

Lambda-Calculus

- Extension of a first-order system
  - Can be extended to other first-order symbols
  - Also to other second-order constructions (like μ-calculus...)
  - But universal
- Notation
  - $\lambda x.t_1\, t_2\, t_3 = \lambda x.((t_1\, t_2)\, t_3)$
  - $\lambda xy.t = \lambda x.\lambda y.t$

- Notion of bound and free variables
- In $\lambda y.x(\lambda z.z)$:
  - x is free
  - y, z are bound
- Each bound variable is attached to a λ
  - $\lambda z \;.\; x \; \lambda x \;.\; x \; (\; \lambda x \;.\; x \; z \;)$
- The name of bound variables does not matter
  - $\lambda x.x = \lambda y.y$
  - $\lambda xy.x\,(y\,z) = \lambda ab.a\,(b\,z)$
  - Careful! $\lambda x.y \neq \lambda y.y$

Rewriting Rules

- β-reduction: $(\lambda x.t)u \longrightarrow_\beta t[x := u]$
  (**!** only the x bound by the corresponding λ are replaced)

- A rule that can be added:
  η-reduction: $(\lambda x.tx) \longrightarrow_\eta t$
  (**!** when x is not free in t)

- Congruence:
  Reduction can occur within a term
  If $M \longrightarrow M'$, then $MN \longrightarrow M'N$
  (Context-free rules)

Example

What are the behaviors of

- $\Omega = (\lambda x.xx)(\lambda x.xx)$?
- $ZZ\,V$ when $Z = \lambda zx.x(zzx)$? (Turing's fixed point combinator)

Church numerals are defined as

- $\overline{0} \triangleq \lambda xy.y$
- $\overline{1} \triangleq \lambda xy.xy$
- $\overline{2} \triangleq \lambda xy.x(xy)$
- $\overline{3} \triangleq \lambda xy.x(x(xy))$

When fed with $\overline{m}$ and $\overline{n}$, what are the behaviors of

- $M = \lambda mn.\lambda xy.mx(nxy)$?
- $N = \lambda mn.m(Mn)(\lambda xy.y)$?

Pure lambda calculus is Turing complete