Shallow Water, the f-plane and dispersive corrections

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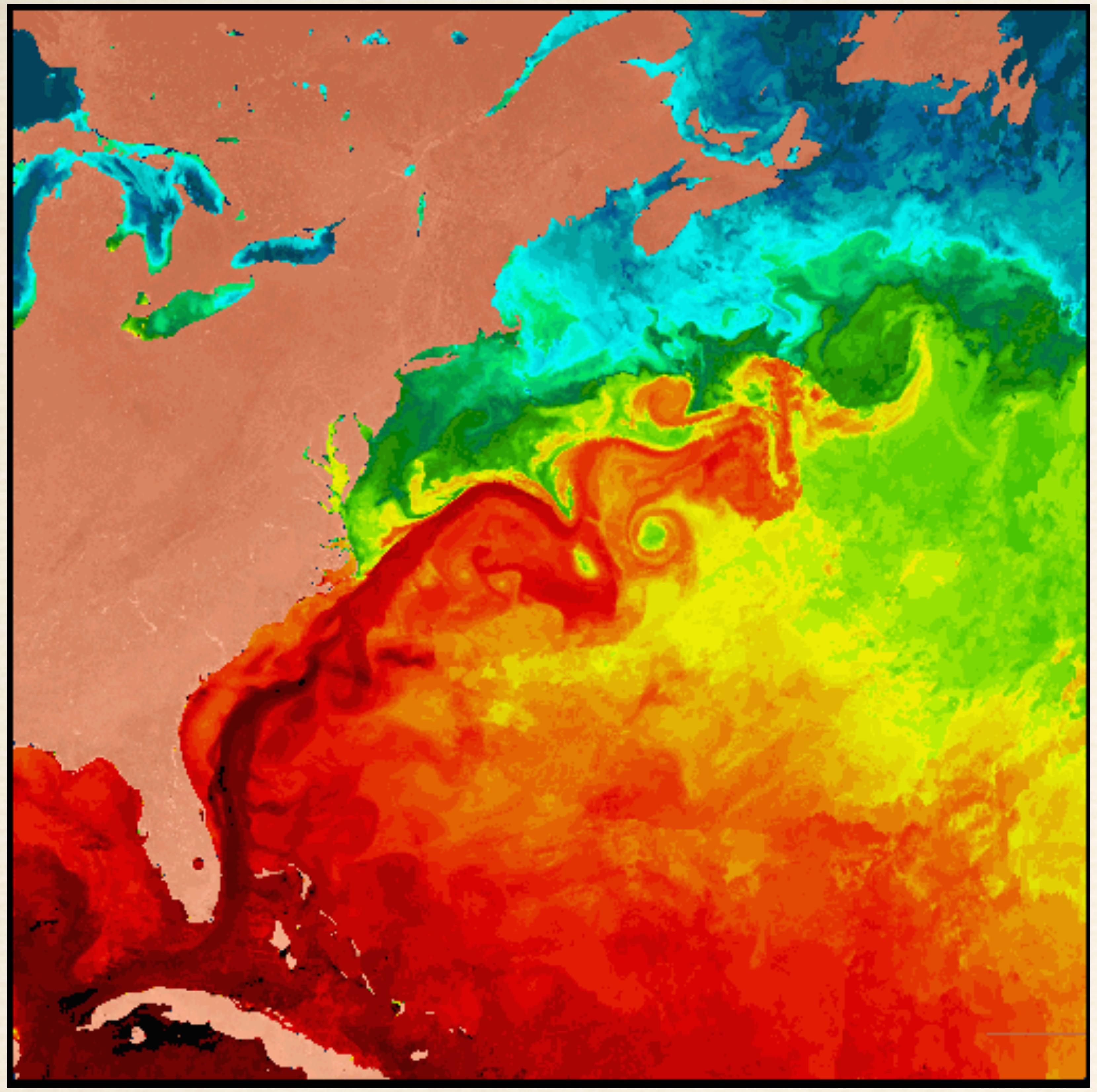
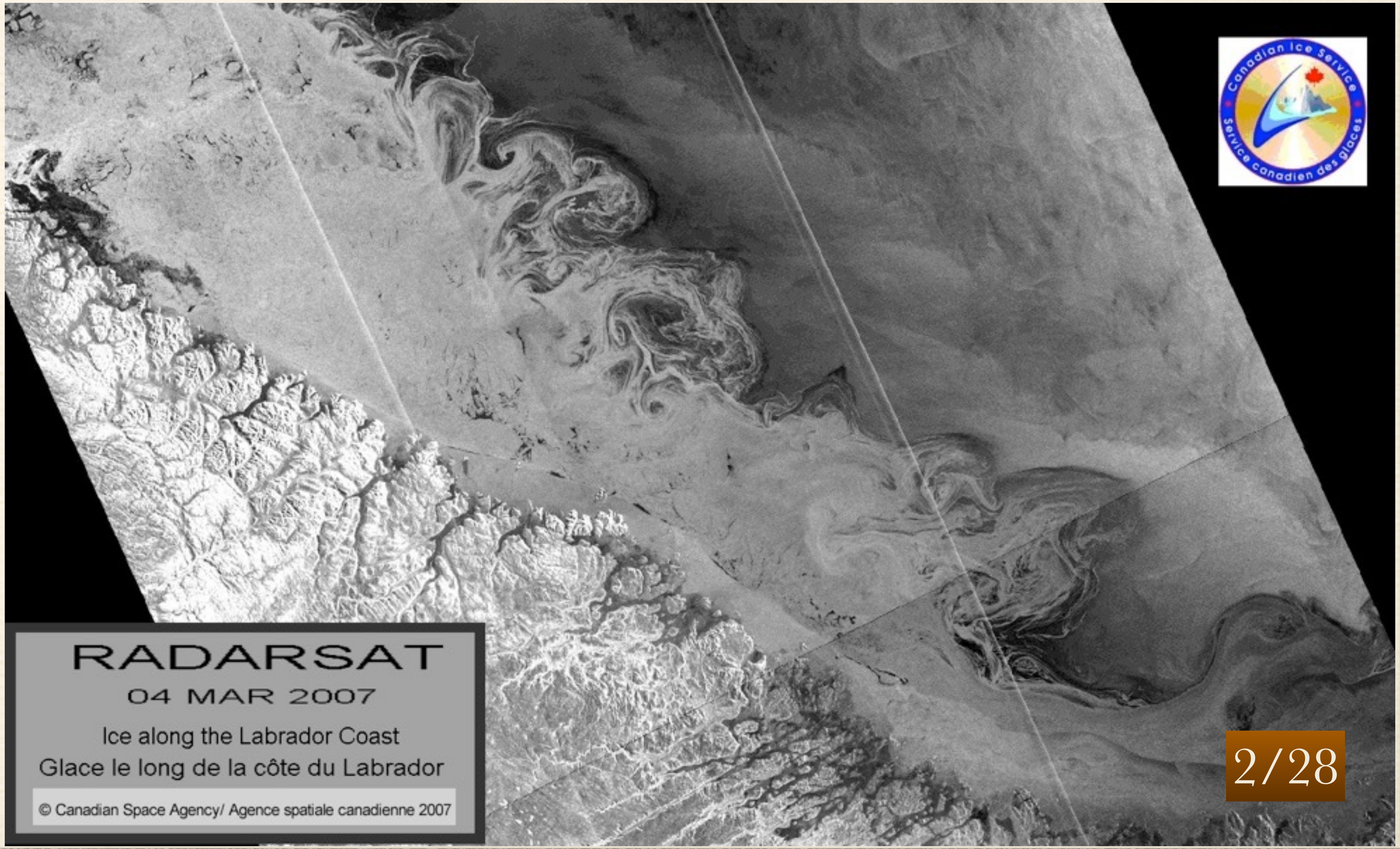
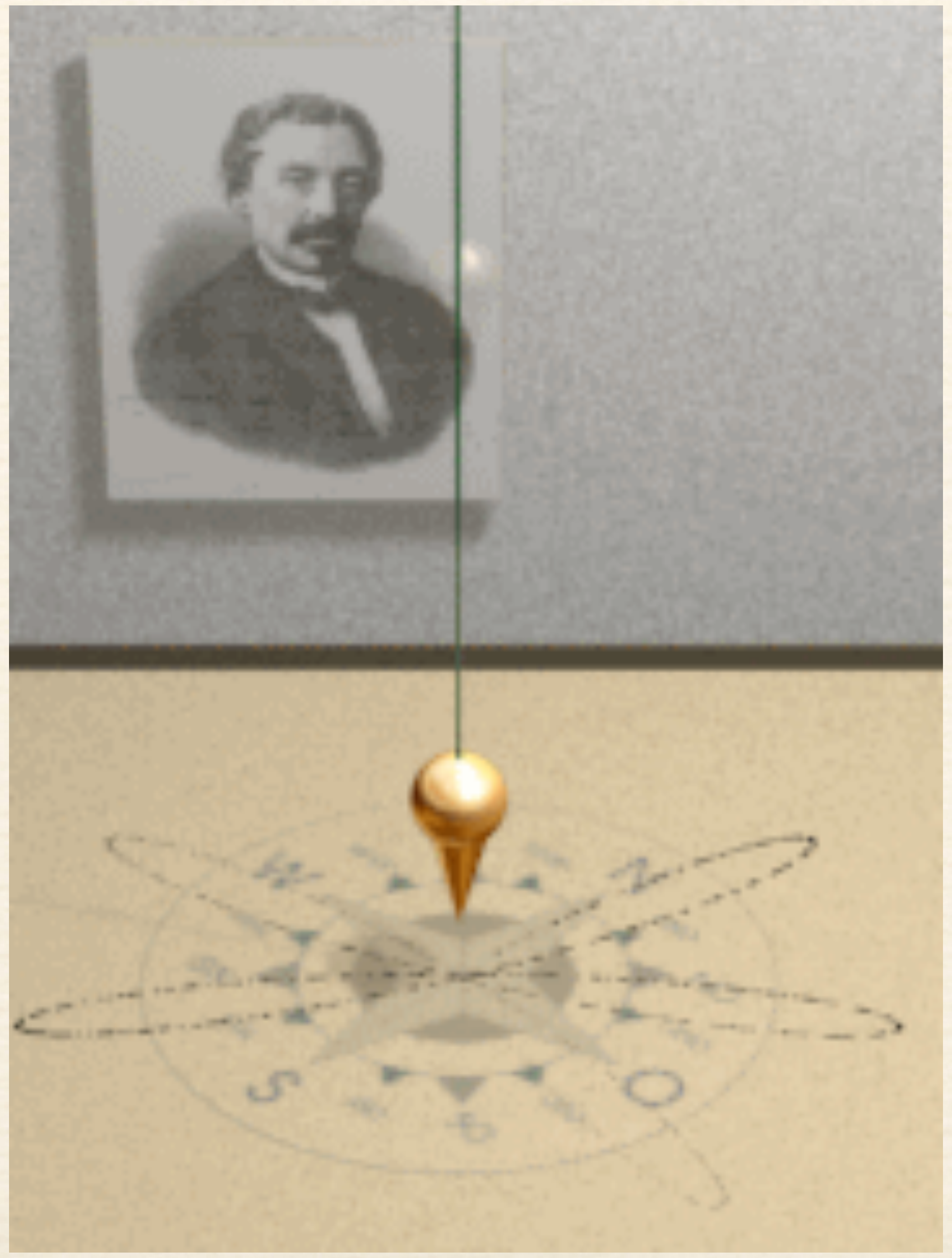
Real world geophysical flows from space:
top left: false color of Gulf stream sea surface temperature
bottom left: true color of a hurricane
bottom right: Radarsat of sea ice edge





Rotation

- Any measurement made on Earth is one made in a rotating reference frame due to the rotation of the planet about its axis.
- Rotation matters little for small scales, but even on moderate length scales, long time scales imply that rotation cannot be discounted (Foucault pendulum).
- The size and shape of the Earth also imply that in order to use Cartesian coordinates certain approximations must be made (see diagram on next slide).



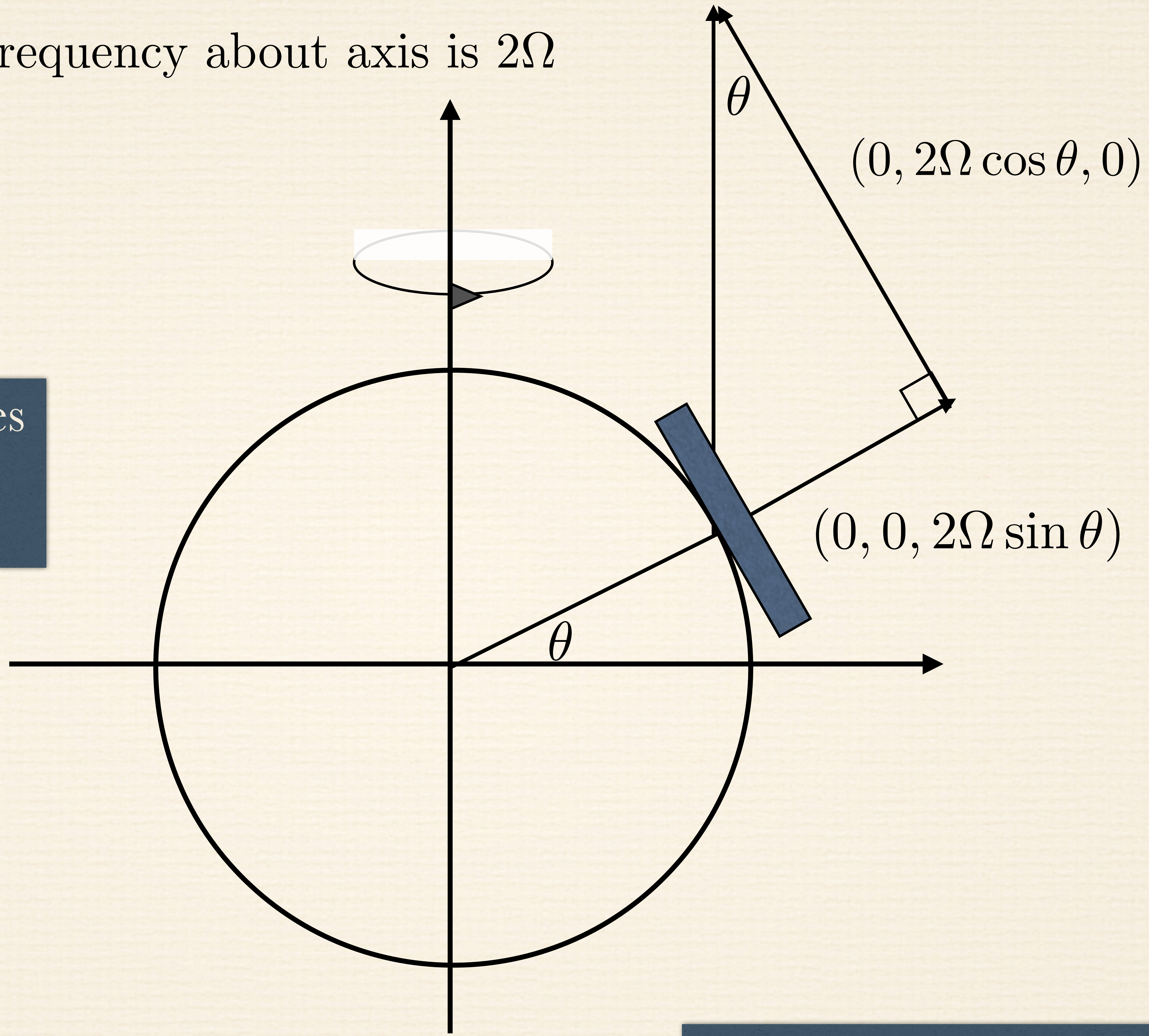
Assorted Demos on Youtube

https://www.youtube.com/watch?v=pKGgICawAKc

https://www.youtube.com/watch?v=QAZPJakvabA

https://www.youtube.com/watch?v=7TjOy56-x8Q

Rotation frequency about axis is 2Ω



To get Cartesian coordinates use a tangent plane to the sphere at the local latitude

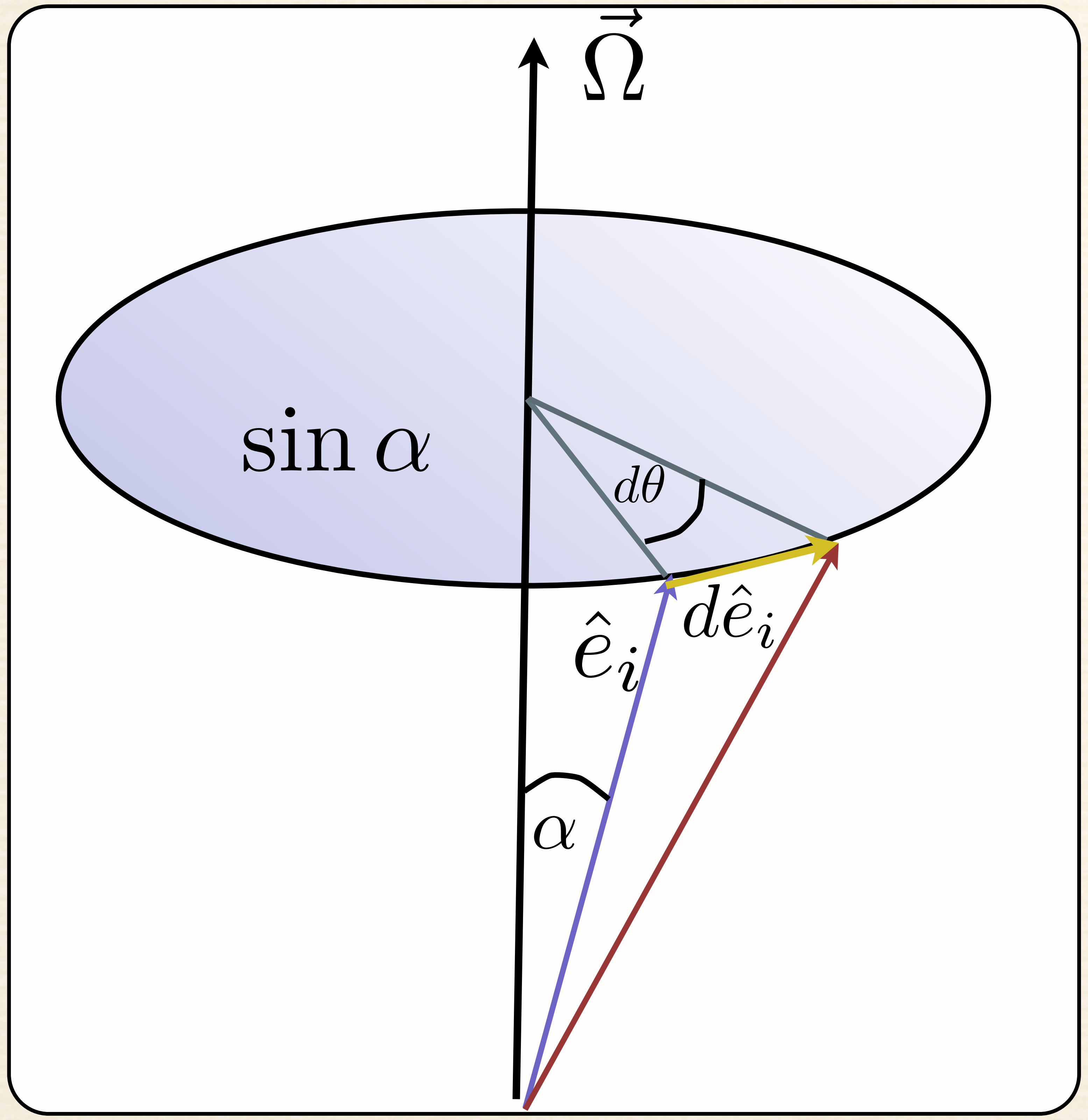
In local Cartesian coordinates the rotation vector has two nonzero components: $(0, f^*, f)$

One could also write $2\Omega \sin\theta = 2\Omega_{local}$

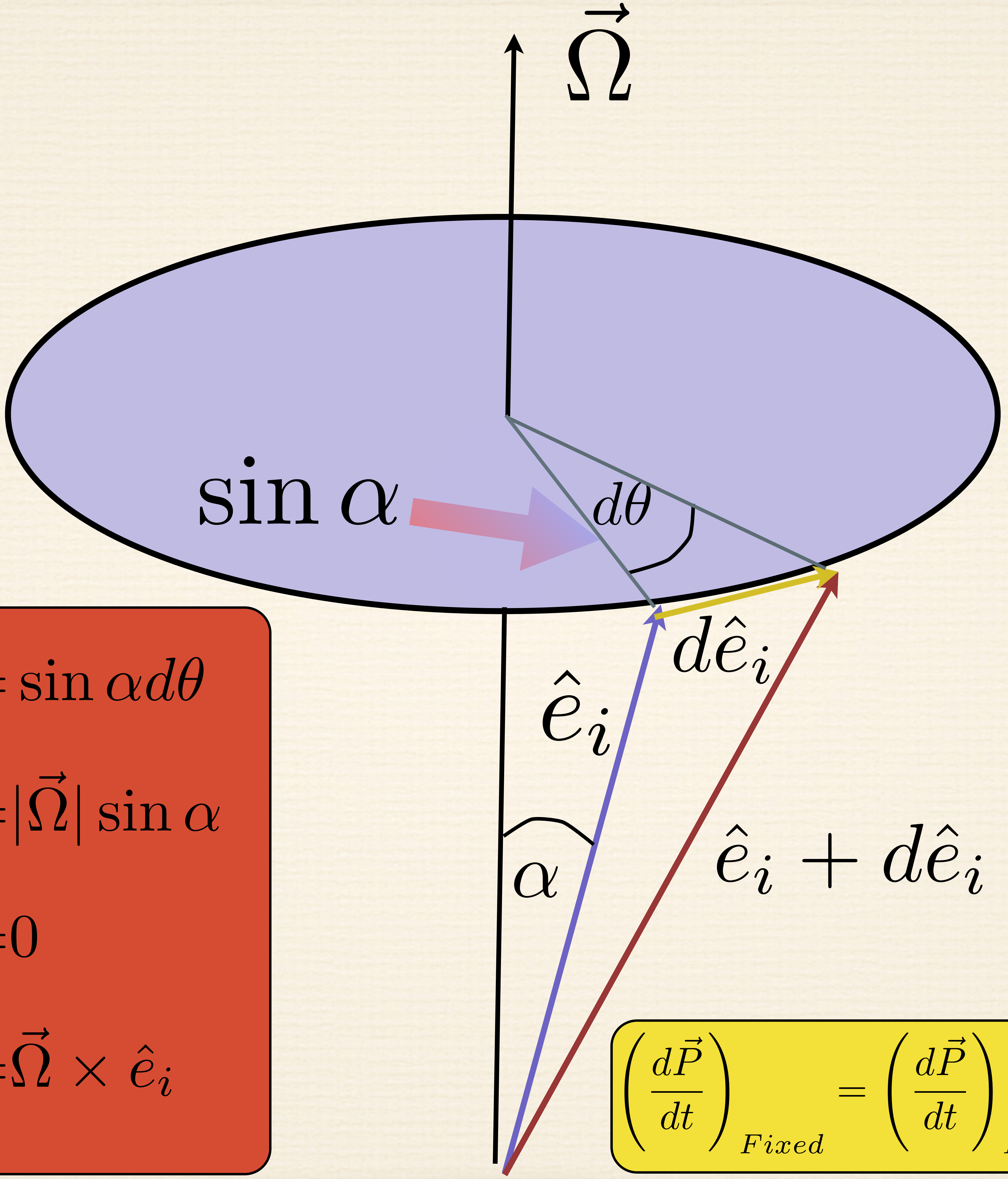
$$\vec{P} = p_i \widehat{e}_i \quad \text{For a given set of coordinates}$$

$$\left(\frac{d\vec{P}}{dt}\right)_{Fixed} = \frac{dp_i}{dt}\widehat{e}_i + p_i\frac{d\widehat{e}_i}{dt} \quad \text{For time dependent basis}$$

$$\left(\frac{d\vec{P}}{dt}\right)_{Fixed} = \left(\frac{d\vec{P}}{dt}\right)_{Rotating} + p_i\frac{d\widehat{e}_i}{dt} \quad \text{Need to evaluate extra term}$$



- Each basis vector traces a cone around the rotation axis.
- Its radius is the sine of the angle the vector makes with the rotation axis (labelled alpha on the left).
- The angle of the nose of the basis vector in its plane of rotation is labelled theta on the left.



$$|d\hat{e}_i| = \sin\alpha\, d\theta$$

$$\left|\frac{d\hat{e}_i}{dt}\right| = |\vec{\Omega}| \sin\alpha$$

$$\vec{\Omega} \cdot d\hat{e}_i = 0$$

$$\frac{d\hat{e}_i}{dt} = \vec{\Omega} \times \hat{e}_i$$

$$\left(\frac{d\vec{P}}{dt}\right)_{Fixed} = \left(\frac{d\vec{P}}{dt}\right)_{Rotating} + \vec{\Omega} \times \vec{P}$$

$$\left(\frac{d\vec{r}}{dt}\right)_F = \left(\frac{d\vec{r}}{dt}\right)_R + \vec{\Omega} \times \vec{r} \text{ where } \vec{r} \text{ is the position}$$

$$\vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r} \text{ for the velocity}$$

$$\frac{d\vec{u}_F}{dt} = \frac{d}{dt}(\vec{u}_R + \vec{\Omega} \times \vec{r})_R + \vec{\Omega} \times (\vec{u}_R + \vec{\Omega} \times \vec{r})$$

$$\frac{d\vec{u}_F}{dt} = \left(\frac{d\vec{u}_R}{dt}\right)_R + \vec{\Omega} \times \left(\frac{d\vec{r}}{dt}\right)_R + \vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$\boxed{\vec{a}_F} = \boxed{\vec{a}_R} + \boxed{2\vec{\Omega} \times \vec{u}_R} + \boxed{\vec{\Omega} \times (\vec{\Omega} \times \vec{r})}$$

fixed frame acceleration

$\vec{a}_F$

rotating frame acceleration

$\vec{a}_R$

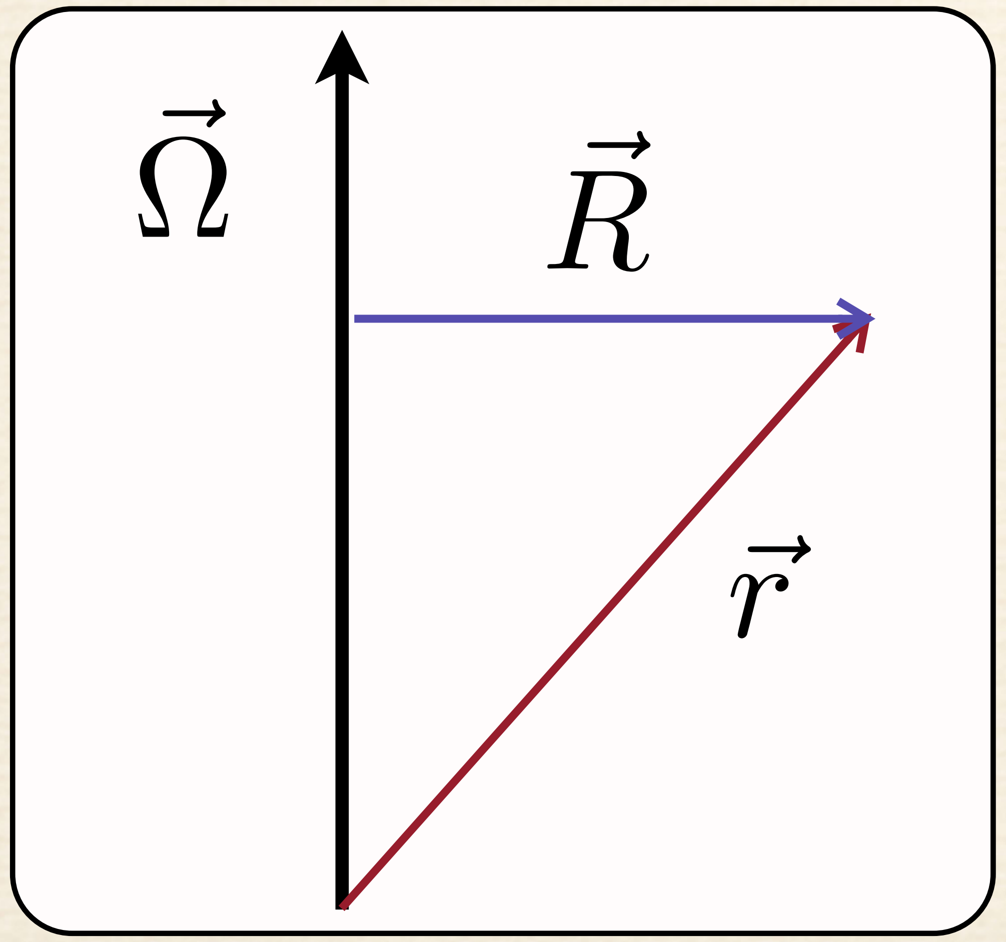
Coriolis acceleration

$2\vec{\Omega} \times \vec{u}_R$

centripetal acceleration

$-|\vec{\Omega}|^2 \vec{R}$

often neglected



$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = \vec{\Omega} \times (\vec{\Omega} \times \vec{R}) = -|\vec{\Omega}|^2 \vec{R}$$

Momentum and Vorticity Equations with Rotation

- If centripetal effects are neglected only the Coriolis term appears.
- This term is perpendicular to both the rotation axis and the velocity vector.
- It thus does not contribute to the mechanical energy balance.
- It does contribute a new term (and a new mechanism) to the vorticity balance.

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_0}\nabla p - \boxed{2\overrightarrow{\Omega}\times\vec{u} +}\,\nu\nabla^2\vec{u}$$

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_0}\nabla p - \boxed{(-fv, fu, 0)}$$

On f-plane $2\overrightarrow{\Omega} = (0,0,f)$
and we usually assume inviscid

$$\frac{D\overrightarrow{\omega}}{Dt} = -\overrightarrow{\omega}\cdot\nabla\vec{u} + \boxed{2\overrightarrow{\Omega}\cdot\nabla\vec{u} +}\,\nu\nabla^2\vec{u}$$

$$\frac{D\overrightarrow{\omega}}{Dt} = -\overrightarrow{\omega}\cdot\nabla\vec{u} + \boxed{(-fu_z, fv_z, f[u_x - v_y])}$$

Vorticity Equation with Rotation

- Let the z axis line up with the rotation axis.
- Neglect all other terms aside from the rotation term in the vorticity equation.

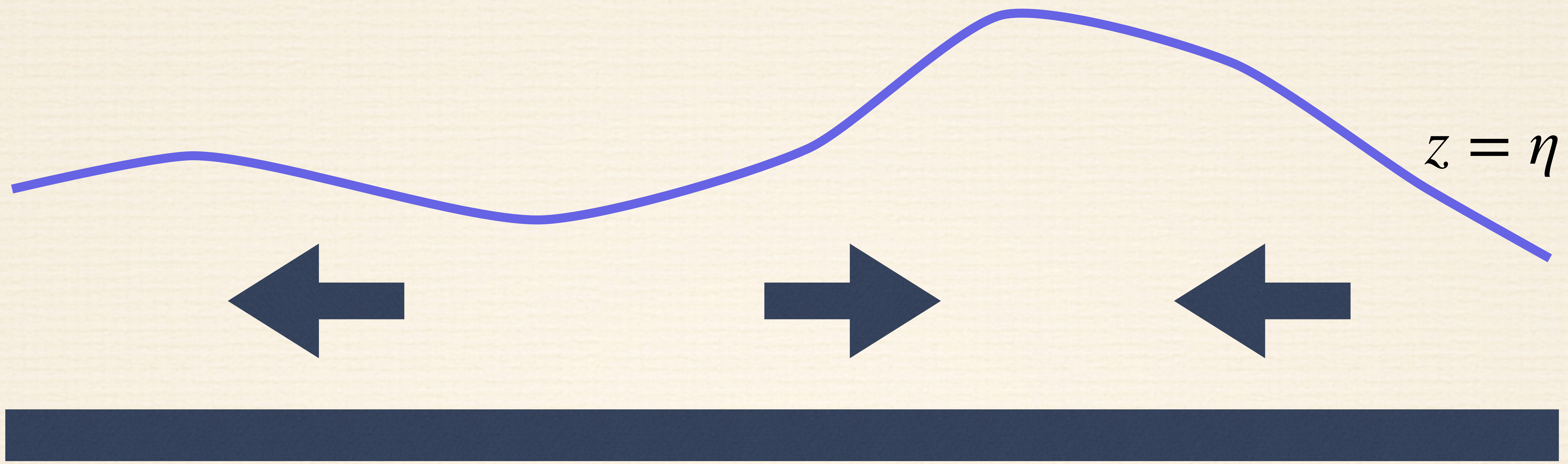
$$\frac{D\omega_x}{Dt} = 2\Omega_{local}\frac{\partial u}{\partial z}$$ THE X COMPONENT

$$\frac{D\omega_y}{Dt} = 2\Omega_{local}\frac{\partial v}{\partial z}$$ THE Y COMPONENT

$$\frac{D\omega_z}{Dt} = 2\Omega_{local}\frac{\partial w}{\partial z}$$ THE Z COMPONENT

- Rotation leads to vorticity production by vertical (or along the axis of rotation) shear.
- Production of a particular vorticity component is due to the vertical shear of the corresponding velocity component.

Shallow Water Equations (with Rotation)



- The simplest set of equations that apply to large scale motions.
- Assume the fluid moves up and down in columns and thus consider depth averaged velocity, only.
- Neglect viscosity, assume the bottom is flat (at z=-H) and let eta denote the free surface. Particles at the surface stay there and w(z=-H)=0.
- Assume that the pressure is hydrostatic so that the weight of the overlying fluid, determined by eta, drives motion through its horizontal gradients.
- The resulting conservation of mass equation states that the free surface rises and falls as response to local convergence or divergence of the mass flux.

$z = \eta(x, y, t)$ defines the free surface

$\frac{D}{Dt}[z - \eta(x, y, t)] = 0$ particles on free surface stay there

$w = \frac{D\eta}{Dt}$ at $z = \eta$

$\int_{-H}^{\eta} \nabla \cdot \vec{u} dz = 0$ integrate the continuity equation

$w|_{z=-H}^{z=\eta} = -\nabla_H \cdot (\bar{u}, \bar{v})$ integrate and rearrange

$\frac{\partial\eta}{\partial t} + \nabla \cdot ([H + \eta]u, [H + \eta]v) = 0$ drop bars and subscripts

Conservation of Mass

$$\frac{Du}{Dt} - fv = -g\frac{\partial\eta}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -g\frac{\partial\eta}{\partial y}$$

Layer averaged momentum equations with pressure assumed to be hydrostatic:
$p = p_{reference} - \rho_0 g\eta$

Vorticity in Shallow Water Equations with Rotation

- We can get an idea of the vorticity dynamics by considering the linearized equations (this simplifies the algebra a little bit).

$$\vec{u}_t = -\nabla\eta + 2\Omega(v, -u) \text{ Linearized Momentum Eq'n}$$

$$h = H + \eta \text{ Layer thickness}$$

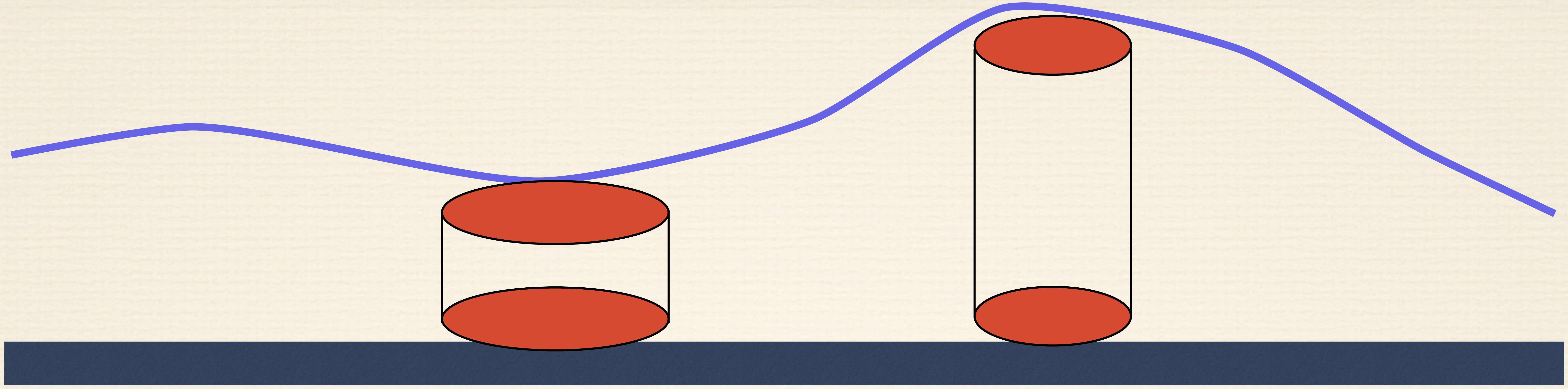
$$h_t = -H\nabla\cdot\vec{u} \text{ Cons. of Mass}$$

$$\omega = v_x - u_y \text{ z component of vorticity}$$

$$\omega_t = -2\Omega\nabla\cdot\vec{u} \text{ Vort. Equation}$$

$$\boxed{\omega_t = \frac{2\Omega}{H}h_t} \text{ Vort.-Thickness Equation}$$

- We thus see that an increase in layer thickness leads to an increase in vorticity.
- This is like the general vortex stretching term but here we are stretching with respect to the rotation axis (i.e. stretching by the background vorticity).



- As the free surface rises, the vortices are stretched and vorticity is increased.
- Notice how much simpler this is than the general vortex stretching argument.
- This is because the rotation axis breaks the symmetry of the general vorticity equation (the vertical is a “preferred” direction).
- In some sense in GFD the z-direction is always more important as far as the vorticity equation is concerned.

The Shallow Water Equations Wave Speed

$$\left.\begin{aligned}\frac{Du}{Dt} &= -g\frac{\partial \eta}{\partial x}\\ \frac{Dv}{Dt} &= -g\frac{\partial \eta}{\partial y}\\ \frac{\partial \eta}{\partial t} + \nabla\cdot(\vec{u}[H+\eta]) &= 0\end{aligned}\right\}\ \text{Linearize}$$

$$\left.\begin{aligned}u_t &= -g\eta_x\\ v_t &= -g\eta_y\end{aligned}\right\}\ \text{Take div}\quad \nabla\cdot\vec{u}_t = -g\nabla^2\eta$$

$$\frac{\partial \eta}{\partial t} + H\nabla\cdot\vec{u} = 0 \implies \eta_{tt} = gH\nabla^2\eta$$

Classical wave equation with wave speed $c = \sqrt{gH}$

- If you substitute in the wave ansatz $\eta = \eta_0 \exp(i(kx + ly - \sigma t))$ you get the dispersion relation: $\sigma^2 = gH(k^2 + l^2)$ or $\sigma = \pm\sqrt{gH}\,|\vec{k}|$.
- In the main slides I use the notation $c_0 = \sqrt{gH}$
- Frequency is a linear function of the magnitude of the wave number meaning shallow water waves are non-dispersive, and $\vec{c}_p = \vec{c}_g$.

$$\left.\begin{aligned}\frac{Du}{Dt} - fv &= -g\frac{\partial \eta}{\partial x}\\ \frac{Dv}{Dt} + fu &= -g\frac{\partial \eta}{\partial y}\end{aligned}\right\} \frac{D\vec{u}}{Dt} + (-fv, fu) = -g\,\nabla\eta$$

Layer averaged momentum equations

$$\frac{\partial \eta}{\partial t} + \nabla\cdot(\vec{u}[H+\eta]) = 0$$

Conservation of Mass

$$\vec{u} = U\tilde{u}, \vec{x} = L\tilde{x}, \eta = H\tilde{\eta}, t = \frac{L}{U}\tilde{t}$$

$$\frac{U^2}{L}\frac{D\tilde{u}}{D\tilde{t}} + fU(-\tilde{v}, u) = -\frac{gH}{L}\tilde{\nabla}\tilde{\eta} \implies \frac{D\tilde{u}}{D\tilde{t}} + \frac{fL}{U}(-\tilde{v}, \tilde{u}) = -\frac{gH}{U^2}\tilde{\nabla}\tilde{\eta}$$

The Rossby number measures the importance of rotation compared to inertia

$$Ro = \frac{U}{fL}, Fr = \frac{U}{\sqrt{gH}}$$

The Froude number measures the importance of buoyancy compared to inertia

$$\frac{D\vec{u}}{Dt} + \frac{1}{Ro}(-v, u) = \frac{1}{Fr^2}\nabla\eta$$

Dropping tildes

$$Ro = \frac{U}{fL}$$

Slow speeds or large length scales means $Ro \ll 1$, or rotation terms matter

$$Fr = \frac{U}{\sqrt{gH}} = \frac{U}{c_0}$$

The Froude number is a ratio of velocities to the natural wave speed $c_0 = \sqrt{gH}$

See extra slide for derivation

$$Ro\frac{D\vec{u}}{Dt} + (-v, u) = -\frac{Ro}{Fr^2}\nabla\eta$$

If both Ro and Fr are small can get a balance between rotation terms and buoyancy or hydrostatic pressure terms

The geostrophic balance: rotation balances pressure

$$(-v_g, u_g) = -\nabla\eta_g$$

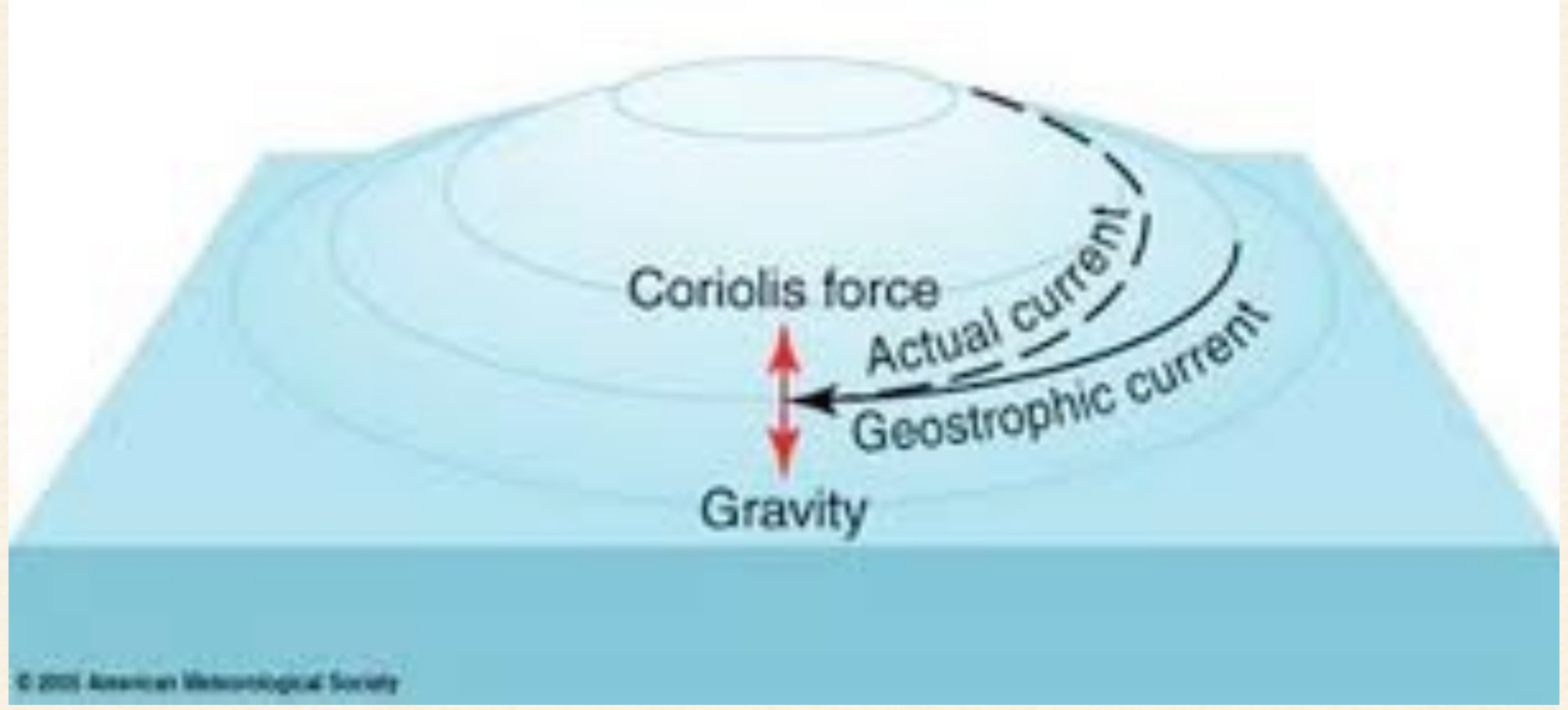
$Ro \ll 1, Ro \sim Fr^2$

Dimensionless or "proper" version

Dimensional or "ad hoc" version

$$(-v_g, u_g) = -\nabla \eta_g$$

$$-fv = -g\eta_x \text{ Linearized East West Momentum}$$
$$fu = -g\eta_y \text{ Linearized North South Momentum}$$
$$(u,v)\cdot f(-v,u) = 0 = -g\nabla\eta\cdot(u,v) \;\; \boxed{\text{Geostrophy}}$$



Geostrophy means flow is NOT from hight to Low Pressure but along lines constant pressure

- For large scale steady flow the pressure gradient is balanced purely by rotation.
- The flow is thus along lines of constant pressure (as opposed to from high to low pressure).
- This is called **geostrophic balance or geostrophy** and is the first approximation to large scale flow.

Waves in shallow water theory with rotation (linearized version)

$$u_t - fv = -g\eta_x \text{ Linearized East West Momentum}$$

$$v_t + fu = -g\eta_y \text{ Linearized North South Momentum}$$

$$\frac{\partial \eta}{\partial t} + H\nabla \cdot \vec{u} = 0 \text{ Linearized Mass}$$

- Look for plane wave solutions and use linear algebra.

$$(u, v, \eta) = (u_0, v_0, a_0)\exp[i\theta] \text{ wave ansatz}$$

$$\theta = kx + ly - \sigma t \;, \vec{w} = (u_0, v_0, a_0) \text{ define phase and amp.}$$

$$(\mathbf{A} - \sigma\mathbf{I})\,\vec{w} = 0 \text{ wave eigenvalue problem}$$

$$\mathbf{A} = \begin{pmatrix} 0 & if & kg \\ -if & 0 & lg \\ kH & lH & 0 \end{pmatrix}.$$

$$\sigma = \pm\sqrt{f^2 + gH(k^2 + l^2)} \text{ Poincaré waves}$$

- It is clear from this equation that Poincare waves are gravity waves that have been modified by rotation.
- The eigenvectors corresponding to a particular eigenvalue would tell you the amplitudes of u, v and eta.
- There is a lower bound on frequency (f) and it is reached for long waves.
- Motions with a frequency of f are called inertial motions and are a observed in both lakes and coastal oceans.
- The manner in which rotation modifies the motion of nonlinear waves remains an active area of study.

- Poincare waves are not that different from gravity waves without rotation.
- Could ask if there are waves that are **only** possible in rotating systems.
- Would also like to consider the effects of boundary conditions (like those at ocean coasts).
- The so-called Kelvin wave fits the bill for all three.
- Consider a North-South propagating wave on a domain bounded on the right (or east side) by a North-South coastline at x=0.
- For simplicity assume no East-West flow (u=0).

$$-fv = -g\eta_x \text{ East-West mom.}$$

$$v_t = -g\eta_y \text{ North-South mom.}$$

$$\eta_t + Hv_y = 0 \text{ Mass}$$

- We now want to to assume North-South propagating waves with an East-West structure that we solve for.

$$\eta = \exp[i(ly - \sigma t)]\phi(x) \text{ assumed solution form}$$

$$0 = -g\eta_{xt} - gf\eta_y \text{ from the mom. eqns}$$

$$0 = i\sigma\phi'(x) - fil\phi(x) \text{ substituting}$$

$$\phi'(x) = \frac{fl}{\sigma}\phi(x) \text{ Re-arranging}$$

$$\phi(x) = \exp(flx/\sigma) \text{ Solving}$$

- The waves thus have their largest amplitudes at the coast and their amplitude decays exponentially as we go west.

- We need to still find the dispersion relation (between sigma and l).
- Use the North-South Momentum and Mass equations to form a single wave equation.

$$\eta_{tt} = gH\eta_{yy} \text{ North-South wave equation}$$

$$c_0 = \sqrt{gH} \text{ Reference speed}$$

$$\sigma(l) = c_0 l \text{ Non-dispersive!}$$

- Our new kind of wave, called a Kelvin wave after Lord Kelvin who discovered it, propagates North-South as a non-dispersive gravity wave (i.e. the propagation is not influenced by rotation).
- Rotation sets the exponentially decaying East-West structure.
- It is said that the wave "leans" on the coast to its right.

- Rotation and stratification comprise the two essential components of fluid flow in the atmosphere and the oceans (large lakes, too)

- The description of these flows is rooted in linear theory, and the geostrophic flow, Poincare waves, and Kelvin waves discussed in these slides are three widely discussed examples.

- Non-linearity needs either more sophisticated mathematics or numerical simulation (often both).

- Many problems remain an active area of exploration.

Dispersive corrections: Basics

- Consider the SW equations in 1D:
- $u_t + uu_x = -g\eta_x$, $\eta_t + \left([H+\eta]u\right)_x = 0$
- These extend the simple case with a constant phase and group speed $c_0 = \sqrt{gH}$
- That means nonlinearity is the only new piece. This leads to wave steepening and inevitably breaking.
- However, long before breaking it could be the assumptions of shallow water may be violated.
- For finite depth waves we have $\sigma^2 = gk\tanh(kH)$, and recall for shallow water we have $\sigma_{SW}^2 = gHk^2$.

Dispersive corrections: Approximate dispersion

- For finite depth waves we have $\sigma^2 = gk\tanh(kH)$, and recall for shallow water we have $\sigma_{SW}^2 = gHk^2$. How can we do better?

- Use the Taylor expansion $\tanh(z) = z - \frac{1}{3}z^3 + O(z^5)$ and find: $\sigma_{cub}^2 = gHk^2\left(1 - \frac{1}{3}k^2H^2\right)$. A bit of algebra gives the phase speed as $c_p^{(cub)} = \pm c_0\sqrt{1 - k^2H^2/3}$ so that as k increases (shorter waves) waves move faster.

- How to include this in the SW theory?

Dispersive corrections: Modifying SW

- Consider the SW equations in 1D written with the pressure split as $p = p_H + p_{NH}$
- It is the second piece that will yield dispersive corrections.
- Now a long set of calculations tries to relate p_{NH} to the free surface u and its derivatives
- $u_t + uu_x = -g\eta_x + \beta u_{xxt}$, $\eta_t + \left([H+\eta]u\right)_x = 0$
- Linearize, set H: $u_t = -g\eta_x + \beta u_{xxt}$, $\eta_t + \left(H_0 u\right)_x = 0$ cross-differentiate and find: $u_{tt} = c_0^2 u_{xx} + \beta u_{xxtt}$
- Try a dispersive wave solution and find $\sigma^2(k) = \dfrac{c_0^2 k^2}{1 + \beta k^2}$ and if you Taylor expand you get $\sigma^2(k) \approx c_0^2 k^2 \left[1 - \dfrac{\beta k^2}{2}\right]$ like on the last page.

Dispersive corrections: Modified SW f-plane

- The momentum equations read: $\frac{D\vec{u}}{Dt} + (fv, -fu) = -g\nabla\eta + \beta\nabla\nabla\cdot\vec{u}_t$

- And the conservation of mass is unchanged: $\frac{\partial\eta}{\partial t} + \nabla\cdot([H+\eta]\vec{u}) = 0$

- These do not completely preclude breaking but they are much, much friendlier to solve compared to the classical shallow water equations.

- And the best part they often agree with the beautiful patterns nature makes like those on the next slide.