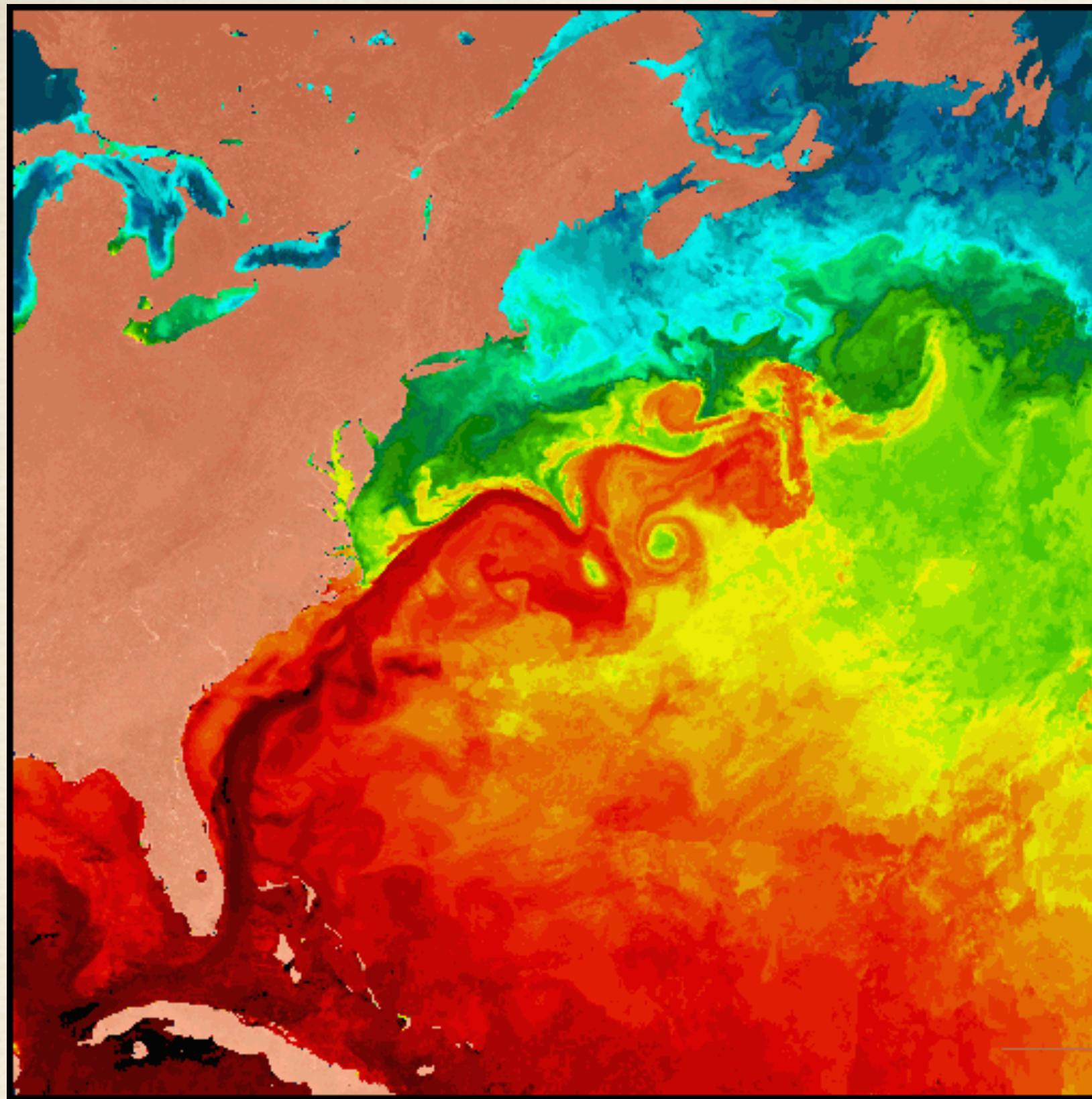


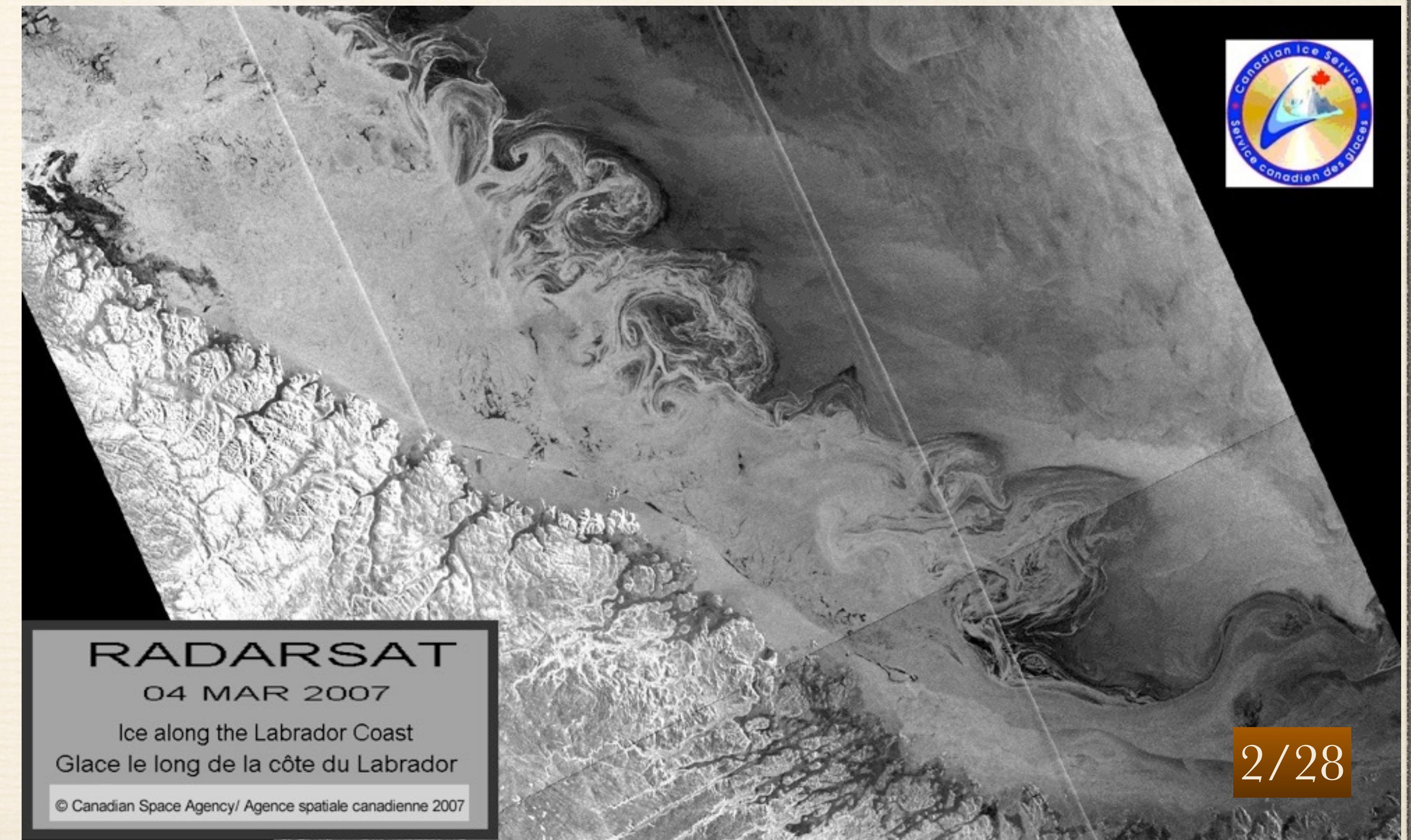
# Shallow Water, the f-plane and dispersive corrections



*Marek Stastna*

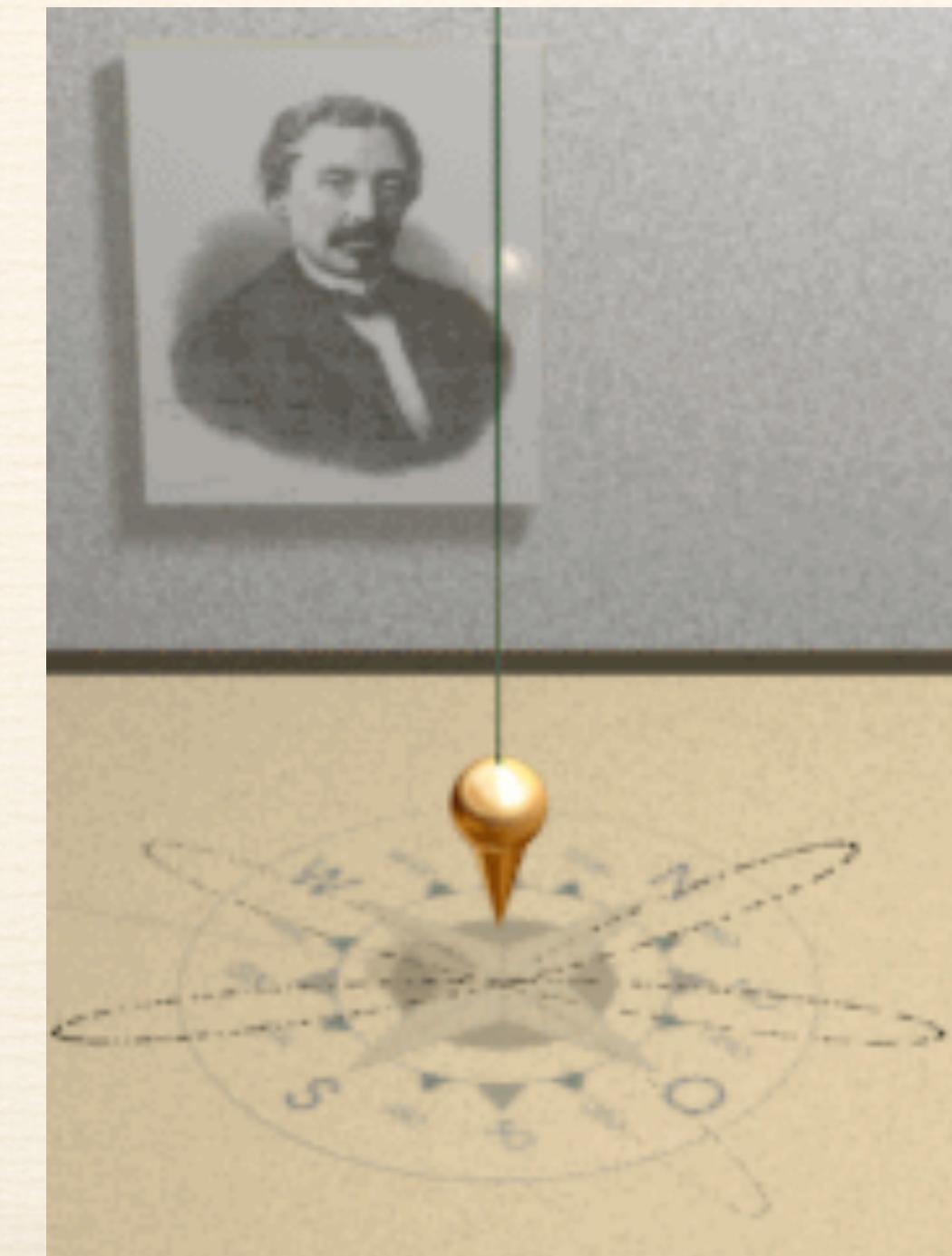


Real world geophysical flows from space:  
top left: false color of Gulf stream sea surface temperature  
bottom left: true color of a hurricane  
bottom right: Radarsat of sea ice edge



# Rotation

- ❖ Any measurement made on Earth is one made in a rotating reference frame due to the rotation of the planet about its axis.
- ❖ Rotation matters little for small scales, but even on moderate length scales, long time scales imply that rotation cannot be discounted (Foucault pendulum).
- ❖ The size and shape of the Earth also imply that in order to use Cartesian coordinates certain approximations must be made (see diagram on next slide).



## Assorted Demos on Youtube

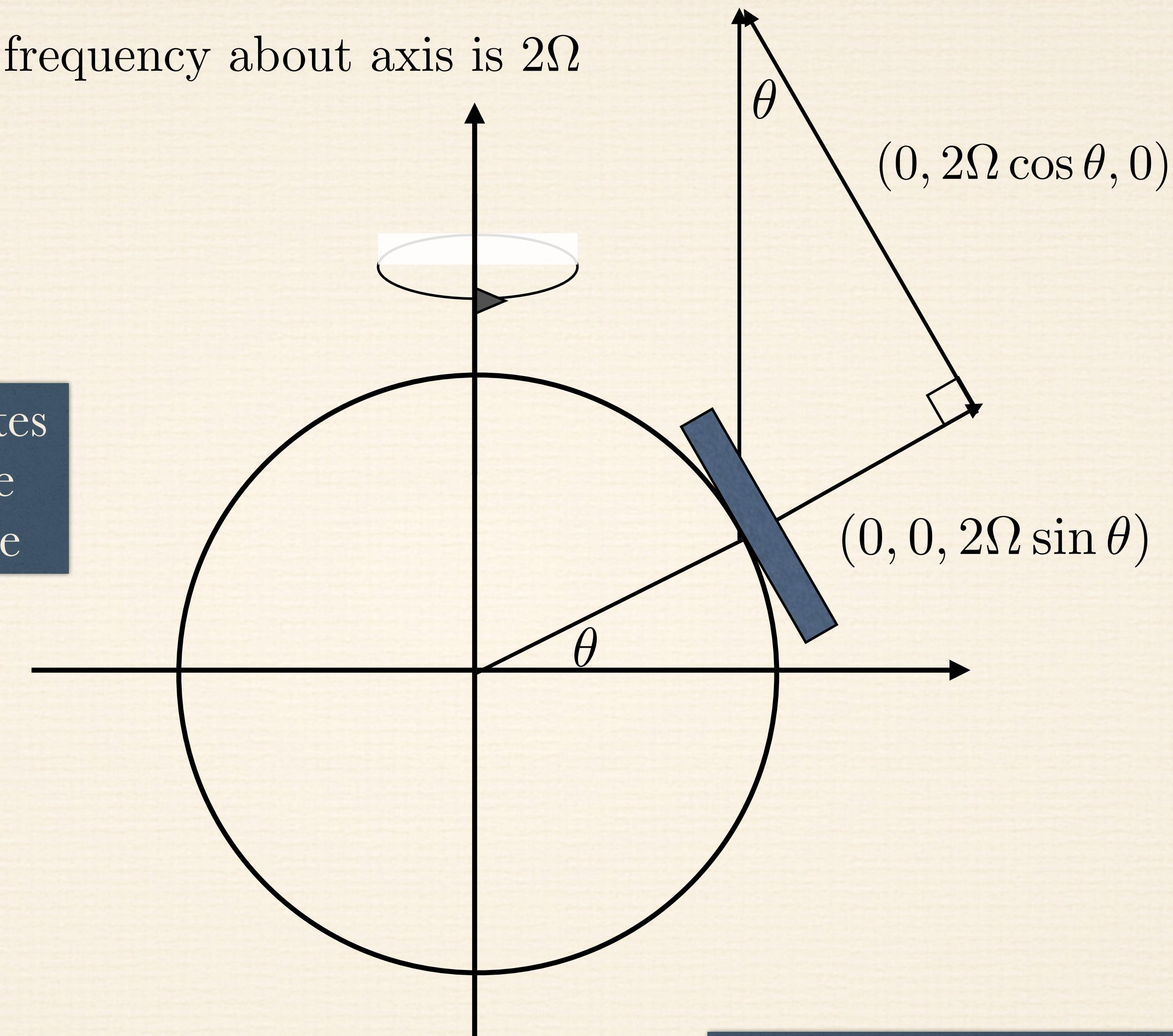
[https://www.youtube.com/watch?  
v=pKGgICawAKc](https://www.youtube.com/watch?v=pKGgICawAKc)

<https://www.youtube.com/watch?v=QAZPJakovabA>

<https://www.youtube.com/watch?v=7TjOy56-x8Q>

Rotation frequency about axis is  $2\Omega$

To get Cartesian coordinates  
use a tangent plane to the  
sphere at the local latitude



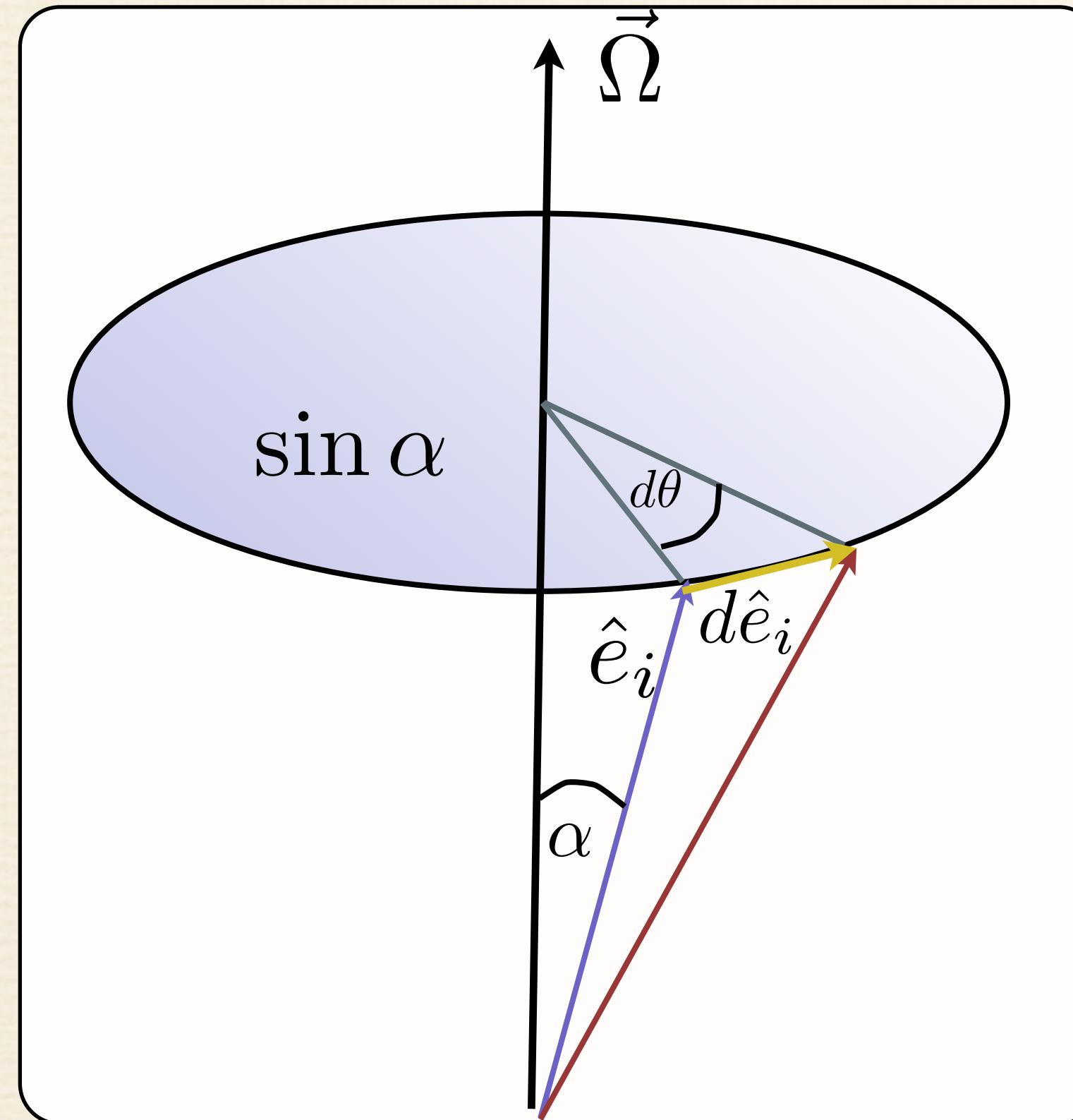
In local Cartesian coordinates the rotation  
vector has two nonzero components:  $(0, f^*, f)$

One could also write  
 $2\Omega \sin \theta = 2\Omega_{local}$

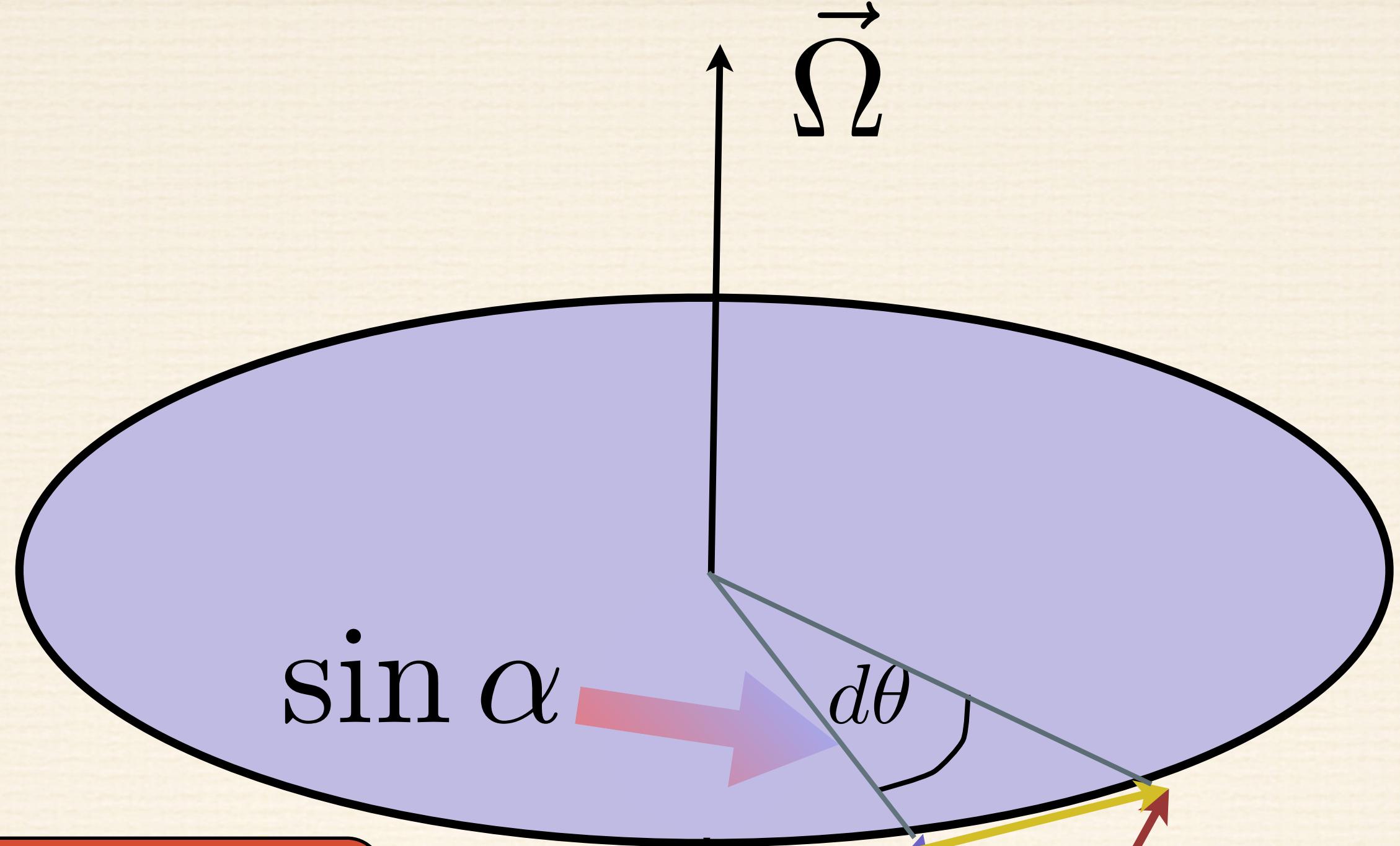
$\vec{P} = p_i \hat{e}_i$  For a given set of coordinates

$$\left( \frac{d\vec{P}}{dt} \right)_{Fixed} = \frac{dp_i}{dt} \hat{e}_i + p_i \frac{d\hat{e}_i}{dt} \text{ For time dependent basis}$$

$$\left( \frac{d\vec{P}}{dt} \right)_{Fixed} = \left( \frac{d\vec{P}}{dt} \right)_{Rotating} + p_i \frac{d\hat{e}_i}{dt} \text{ Need to evaluate extra term}$$



- ❖ Each basis vector traces a cone around the rotation axis.
- ❖ Its radius is the sine of the angle the vector makes with the rotation axis (labelled alpha on the left).
- ❖ The angle of the nose of the basis vector in its plane of rotation is labelled theta on the left.



$$|d\hat{e}_i| = \sin \alpha d\theta$$

$$\left| \frac{d\hat{e}_i}{dt} \right| = |\vec{\Omega}| \sin \alpha$$

$$\vec{\Omega} \cdot d\hat{e}_i = 0$$

$$\frac{d\hat{e}_i}{dt} = \vec{\Omega} \times \hat{e}_i$$

$$\begin{aligned} & \hat{e}_i \\ & \alpha \\ & \hat{e}_i + d\hat{e}_i \end{aligned}$$

$$\left( \frac{d\vec{P}}{dt} \right)_{Fixed} = \left( \frac{d\vec{P}}{dt} \right)_{Rotating} + \vec{\Omega} \times \vec{P}$$

$$\left( \frac{d\vec{r}}{dt} \right)_F = \left( \frac{d\vec{r}}{dt} \right)_R + \vec{\Omega} \times \vec{r} \text{ where } \vec{r} \text{ is the position}$$

$$\vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r} \text{ for the velocity}$$

$$\frac{d\vec{u}_F}{dt} = \frac{d}{dt}(\vec{u}_R + \vec{\Omega} \times \vec{r})_R + \vec{\Omega} \times (\vec{u}_R + \vec{\Omega} \times \vec{r})$$

$$\frac{d\vec{u}_F}{dt} = \left( \frac{d\vec{u}_R}{dt} \right)_R + \vec{\Omega} \times \left( \frac{d\vec{r}}{dt} \right)_R + \vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$\vec{a}_F = \vec{a}_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

fixed frame acceleration

$$\vec{a}_F$$

rotating frame acceleration

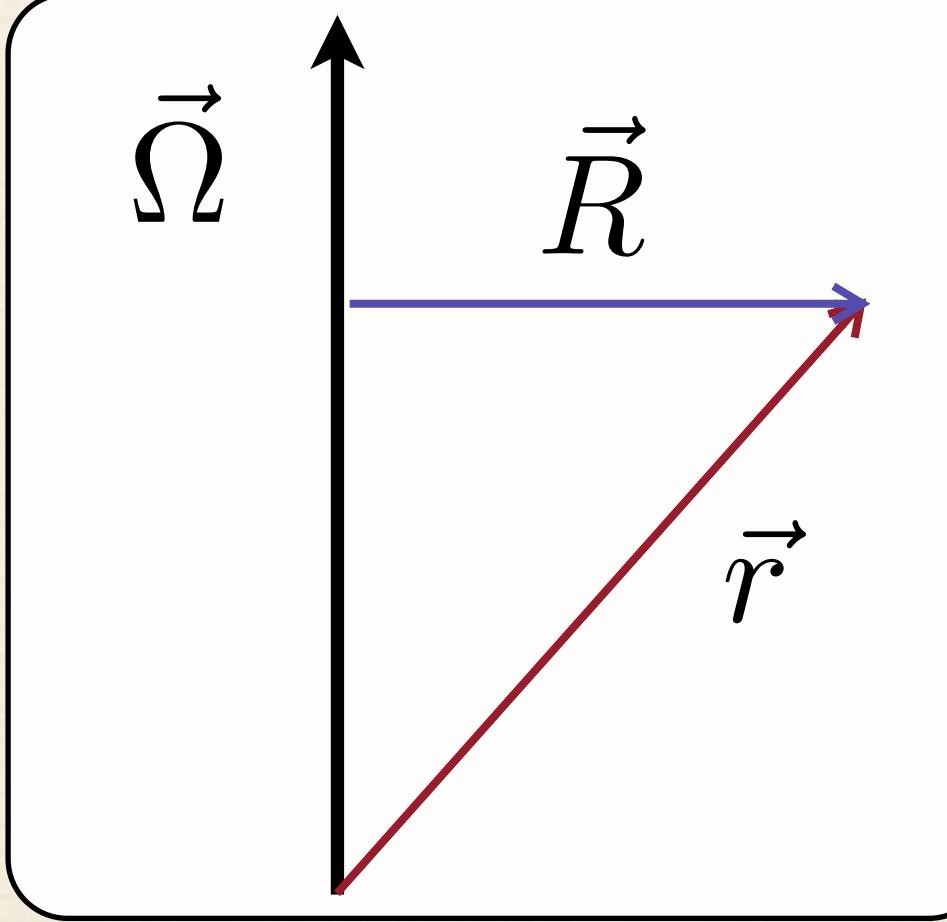
$$\vec{a}_R$$

Coriolis acceleration

$$2\vec{\Omega} \times \vec{u}_R$$

centripetal acceleration

$$-|\vec{\Omega}|^2 \vec{R}$$



$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = \vec{\Omega} \times (\vec{\Omega} \times \vec{R}) = -|\vec{\Omega}|^2 \vec{R}$$

often neglected

# Momentum and Vorticity Equations with Rotation

- ❖ If centripetal effects are neglected only the Coriolis term appears.
- ❖ This term is perpendicular to both the rotation axis and the velocity vector.
- ❖ It thus does not contribute to the mechanical energy balance.
- ❖ It does contribute a new term (and a new mechanism) to the vorticity balance.

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_0} \nabla p - 2\vec{\Omega} \times \vec{u} + \nu \nabla^2 \vec{u}$$

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_0} \nabla p - (-fv, fu, 0)$$

On f-plane  $2\vec{\Omega} = (0,0,f)$   
and we usually assume inviscid

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \nabla \vec{u} + 2\vec{\Omega} \cdot \nabla \vec{u} + \nu \nabla^2 \vec{u}$$

$$\frac{D\vec{\omega}}{Dt} = -\vec{\omega} \cdot \nabla \vec{u} + (fu_z, fv_z, -f[u_x + v_y])$$

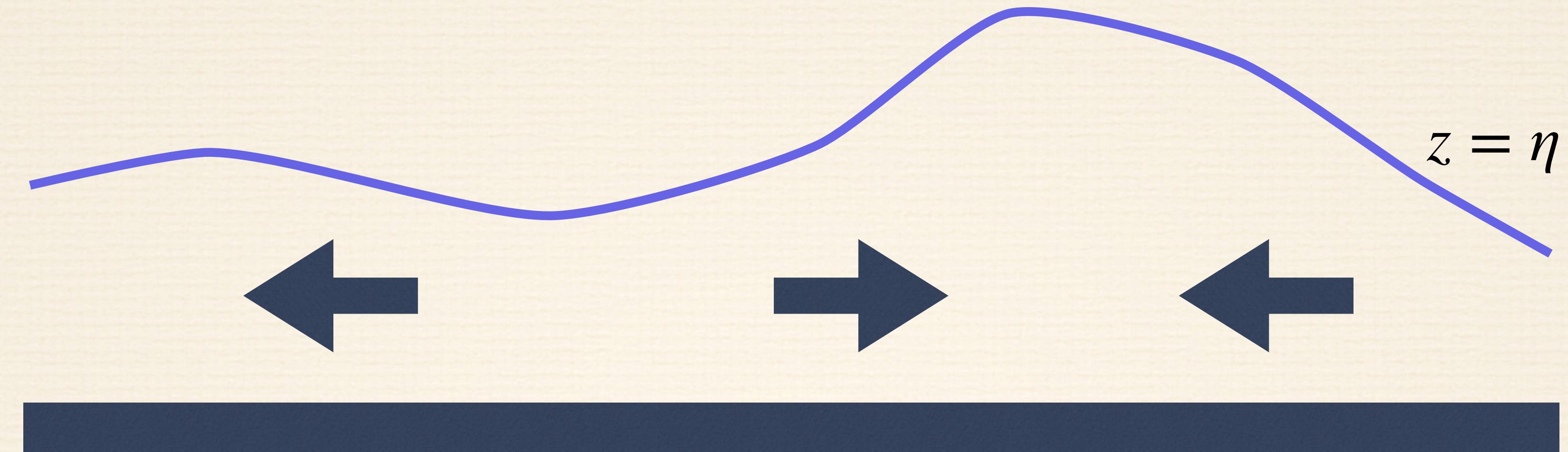
# Vorticity Equation with Rotation

- ❖ Let the z axis line up with the rotation axis.
- ❖ Neglect all other terms aside from the rotation term in the vorticity equation.

$$\frac{D\omega_x}{Dt} = 2\Omega_{local} \frac{\partial u}{\partial z} \text{ THE X COMPONENT}$$
$$\frac{D\omega_y}{Dt} = 2\Omega_{local} \frac{\partial v}{\partial z} \text{ THE Y COMPONENT}$$
$$\frac{D\omega_z}{Dt} = 2\Omega_{local} \frac{\partial w}{\partial z} \text{ THE Z COMPONENT}$$

- ❖ Rotation leads to vorticity production by vertical (or along the axis of rotation) shear.
- ❖ Production of a particular vorticity component is due to the vertical shear of the corresponding velocity component.

# Shallow Water Equations (with Rotation)



- ❖ The simplest set of equations that apply to large scale motions.
- ❖ Assume the fluid moves up and down in columns and thus consider depth averaged velocity, only.
- ❖ Neglect viscosity, assume the bottom is flat (at  $z=-H$ ) and let  $\eta$  denote the free surface. Particles at the surface stay there and  $w(z=-H)=0$ .
- ❖ Assume that the pressure is hydrostatic so that the weight of the overlying fluid, determined by  $\eta$ , drives motion through its horizontal gradients.
- ❖ The resulting conservation of mass equation states that the free surface rises and falls as response to local convergence or divergence of the mass flux.

$z = \eta(x, y, t)$  defines the free surface

$\frac{D}{Dt} [z - \eta(x, y, t)] = 0$  particles on free surface stay there

$$w = \frac{D\eta}{Dt} \text{ at } z = \eta$$

$\int_{-H}^{\eta} \nabla \cdot \vec{u} dz = 0$  integrate the continuity equation

$w|_{z=-H}^{z=\eta} = -\nabla_H \cdot (\bar{u}, \bar{v})$  integrate and rearrange

$$\frac{\partial \eta}{\partial t} + \nabla \cdot ([H + \eta]u, [H + \eta]v) = 0$$
 drop bars and subscripts

Conservation of Mass

$$\frac{Du}{Dt} - fv = -g \frac{\partial \eta}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -g \frac{\partial \eta}{\partial y}$$

Layer averaged momentum equations with pressure assumed to be hydrostatic:

$$p = p_{reference} - \rho_0 g \eta$$

# Vorticity in Shallow Water Equations with Rotation

- ❖ We can get an idea of the vorticity dynamics by considering the linearized equations (this simplifies the algebra a little bit).

$$\vec{u}_t = -\nabla\eta + 2\Omega(v, -u) \text{ Linearized Momentum Eq'n}$$

$$h = H + \eta \text{ Layer thickness}$$

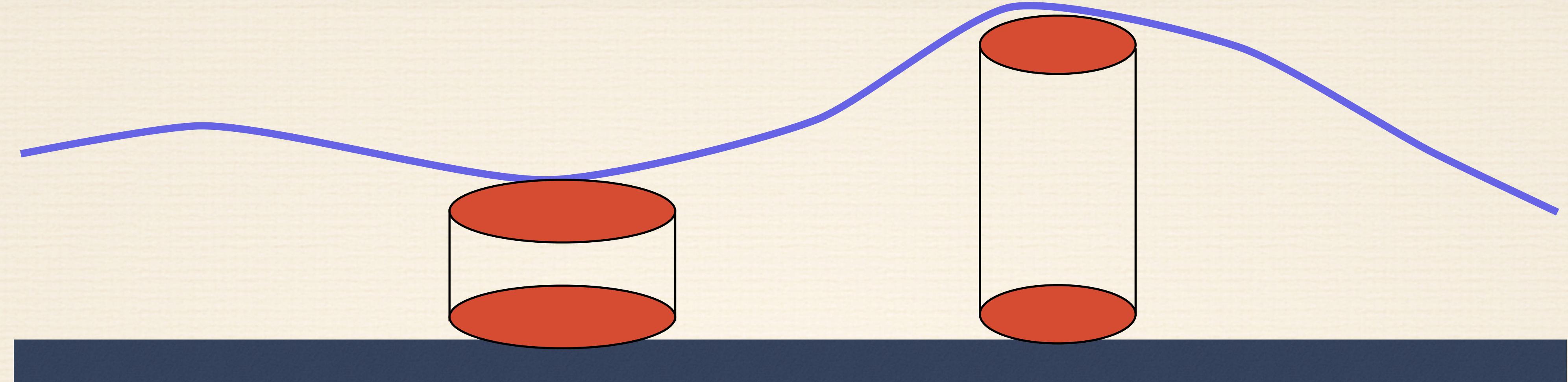
$$h_t = -H\nabla \cdot \vec{u} \text{ Cons. of Mass}$$

$$\omega = v_x - u_y \text{ z component of vorticity}$$

$$\omega_t = -2\Omega\nabla \cdot \vec{u} \text{ Vort. Equation}$$

$$\boxed{\omega_t = \frac{2\Omega}{H} h_t} \text{ Vort.-Thickness Equation}$$

- ❖ We thus see that an increase in layer thickness leads to an increase in vorticity.
- ❖ This is like the general vortex stretching term but here we are stretching with respect to the rotation axis (i.e. stretching by the background vorticity).



- ❖ As the free surface rises, the vortices are stretched and vorticity is increased.
- ❖ Notice how much simpler this is than the general vortex stretching argument.
- ❖ This is because the rotation axis breaks the symmetry of the general vorticity equation (the vertical is a “preferred” direction).
- ❖ In some sense in GFD the z-direction is always more important as far as the vorticity equation is concerned.

# The Shallow Water Equations Wave Speed

$$\begin{aligned}\frac{Du}{Dt} &= -g \frac{\partial \eta}{\partial x} \\ \frac{Dv}{Dt} &= -g \frac{\partial \eta}{\partial y}\end{aligned}$$

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (\vec{u}[H + \eta]) = 0$$

$$\left. \begin{aligned} u_t &= -g\eta_x \\ v_t &= -g\eta_y \end{aligned} \right\} \text{Linearize}$$

$$\frac{\partial \eta}{\partial t} + H \nabla \cdot \vec{u} = 0 \implies \eta_{tt} = gH \nabla^2 \eta$$

Take div

$$\nabla \cdot \vec{u}_t = -g \nabla^2 \eta$$

Classical wave equation with wave speed  
 $c = \sqrt{gH}$

- ❖ If you substitute in the wave ansatz  $\eta = \eta_0 \exp(i(kx + ly - \sigma t))$  you get the dispersion relation:  
 $\sigma^2 = gH(k^2 + l^2)$  or  $\sigma = \pm \sqrt{gH} |\vec{k}|$ .
- ❖ In the main slides I use the notation  $c_0 = \sqrt{gH}$
- ❖ Frequency is a linear function of the magnitude of the wave number meaning shallow water waves are non-dispersive, and  $\vec{c}_p = \vec{c}_g$ .

$$\left. \begin{aligned} \frac{Du}{Dt} - fv &= -g \frac{\partial \eta}{\partial x} \\ \frac{Dv}{Dt} + fu &= -g \frac{\partial \eta}{\partial y} \end{aligned} \right\} \quad \frac{D\vec{u}}{Dt} + (-fv, fu) = -g \nabla \eta$$

Layer averaged momentum equations

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (\vec{u}[H + \eta]) = 0$$

Conservation of Mass

$$\vec{u} = U\tilde{u}, \vec{x} = L\tilde{x}, \eta = H\tilde{\eta}, t = \frac{L}{U}\tilde{t}$$

$$\frac{U^2}{L} \frac{D\tilde{u}}{D\tilde{t}} + fU(-\tilde{v}, \tilde{u}) = -\frac{gH}{L} \tilde{\nabla} \tilde{\eta} \implies \frac{D\tilde{u}}{D\tilde{t}} + \frac{fL}{U} (-\tilde{v}, \tilde{u}) = -\frac{gH}{U^2} \tilde{\nabla} \tilde{\eta}$$

The Rossby number measures the importance of rotation compared to inertia

$$Ro = \frac{U}{fL}, Fr = \frac{U}{\sqrt{gH}}$$

The Froude number measures the importance of buoyancy compared to inertia

$$\frac{D\vec{u}}{Dt} + \frac{1}{Ro}(-v, u) = \frac{1}{Fr^2} \nabla \eta$$

Dropping tildes

$$Ro = \frac{U}{fL}$$

Slow speeds or large length scales means  
 $Ro \ll 1$ , or rotation terms matter

$$Fr = \frac{U}{\sqrt{gH}} = \frac{U}{c_0}$$

The Froude number is a ratio of velocities  
 to the natural wave speed  $c_0 = \sqrt{gH}$   
 See extra slide for derivation

$$Ro \frac{D\vec{u}}{Dt} + (-v, u) = - \frac{Ro}{Fr^2} \nabla \eta$$

If both Ro and Fr are small can get a  
 balance between rotation terms and  
 buoyancy or hydrostatic pressure terms

The geostrophic balance:  
 rotation balances pressure

$$(-v_g, u_g) = - \nabla \eta_g$$

$Ro \ll 1, Ro \sim Fr^2$

Dimensionless or “proper” version

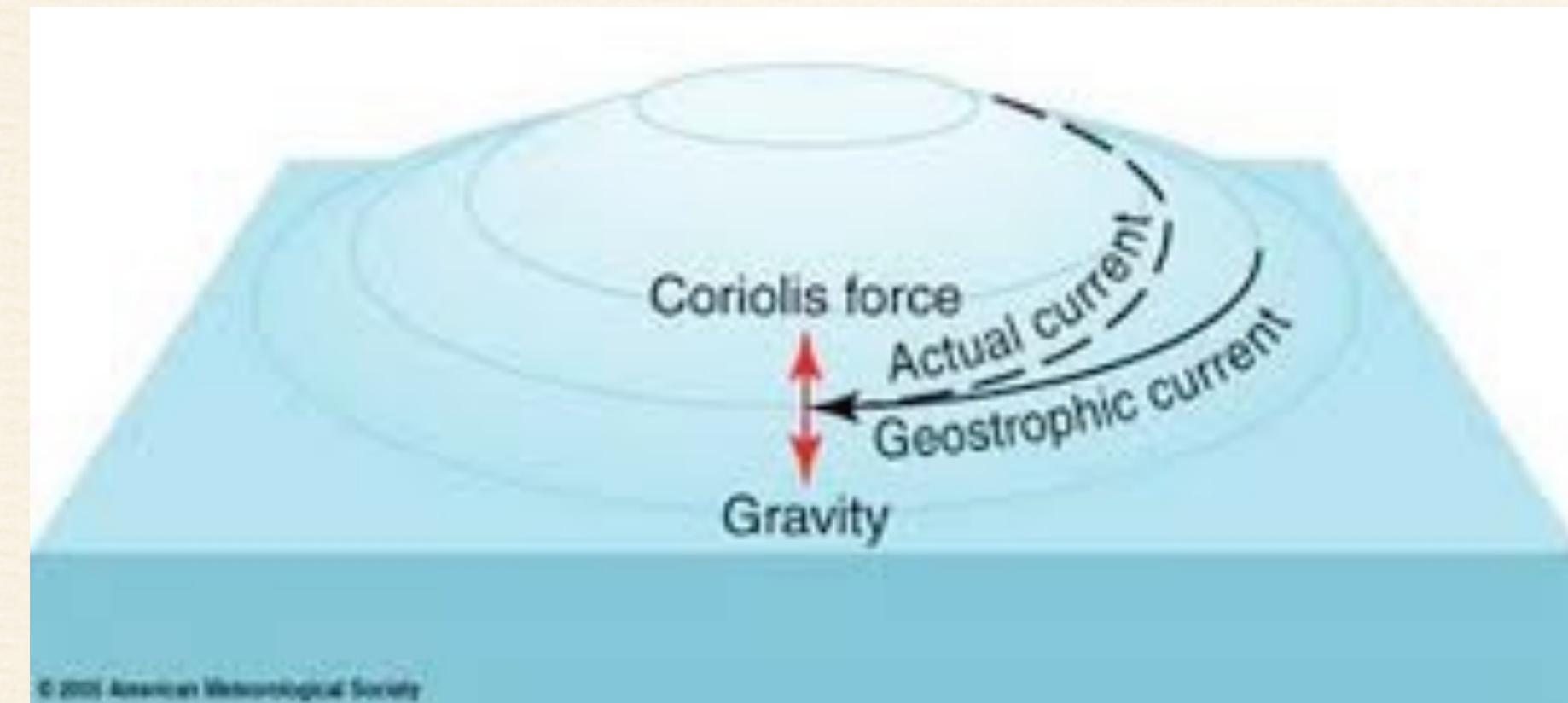
$$(-v_g, u_g) = -\nabla \eta_g$$

Dimensional or “ad hoc” version

$$-fv = -g\eta_x \text{ Linearized East West Momentum}$$

$$fu = -g\eta_y \text{ Linearized North South Momentum}$$

$$(u, v) \cdot f(-v, u) = 0 = -g\nabla\eta \cdot (u, v) \text{ Geostrophy}$$



Geostrophy means flow is NOT from hight to Low Pressure but along lines constant pressure

- ❖ For large scale steady flow the pressure gradient is balanced purely by rotation.
- ❖ The flow is thus along lines of constant pressure (as opposed to from high to low pressure).
- ❖ This is called geostrophic balance or geostrophy and is the first approximation to large scale flow.

# Waves in shallow water theory with rotation (linearized version)

$u_t - fv = -g\eta_x$  Linearized East West Momentum

$v_t + fu = -g\eta_y$  Linearized North South Momentum

$\frac{\partial \eta}{\partial t} + H\nabla \cdot \vec{u} = 0$  Linearized Mass

❖ Look for plane wave solutions and use linear algebra.

$(u, v, \eta) = (u_0, v_0, a_0) \exp[i\theta]$  wave ansatz

$\theta = kx + ly - \sigma t$ ,  $\vec{w} = (u_0, v_0, a_0)$  define phase and amp.

$(\mathbf{A} - \sigma \mathbf{I}) \vec{w} = 0$  wave eigenvalue problem

$$\mathbf{A} = \begin{pmatrix} 0 & if & kg \\ -if & 0 & lg \\ kH & lH & 0 \end{pmatrix}.$$

$$\sigma = \pm \sqrt{f^2 + gH(k^2 + l^2)} \text{ Poincaré waves}$$

- ❖ It is clear from this equation that Poincare waves are gravity waves that have been modified by rotation.
- ❖ The eigenvectors corresponding to a particular eigenvalue would tell you the amplitudes of  $u$ ,  $v$  and  $\eta$ .
- ❖ There is a lower bound on frequency ( $f$ ) and it is reached for long waves.
- ❖ Motions with a frequency of  $f$  are called inertial motions and are observed in both lakes and coastal oceans.
- ❖ The manner in which rotation modifies the motion of nonlinear waves remains an active area of study.

- ❖ Poincare waves are not that different from gravity waves without rotation.
- ❖ Could ask if there are waves that are **only** possible in rotating systems.
- ❖ Would also like to consider the effects of boundary conditions (like those at ocean coasts).
- ❖ The so-called Kelvin wave fits the bill for all three.
- ❖ Consider a North-South propagating wave on a domain bounded on the right (or east side) by a North-South coastline at  $x=0$ .
- ❖ For simplicity assume no East-West flow ( $u=0$ ).

$-fv = -g\eta_x$  East-West mom.

$v_t = -g\eta_y$  North-South mom.

$\eta_t + Hv_y = 0$  Mass

- ❖ We now want to assume North-South propagating waves with an East-West structure that we solve for.

$\eta = \exp[i(lly - \sigma t)]\phi(x)$  assumed solution form

$0 = -g\eta_{xt} - gf\eta_y$  from the mom. eqns

$0 = i\sigma\phi'(x) - fil\phi(x)$  substituting

$\phi'(x) = \frac{fl}{\sigma}\phi(x)$  Re-arranging

$\phi(x) = \exp(flx/\sigma)$  Solving

- ❖ The waves thus have their largest amplitudes at the coast and their amplitude decays exponentially as we go west.

- ❖ We need to still find the dispersion relation (between sigma and l).
- ❖ Use the North-South Momentum and Mass equations to form a single wave equation.

$\eta_{tt} = gH\eta_{yy}$  North-South wave equation

$c_0 = \sqrt{gH}$  Reference speed

$\sigma(l) = c_0 l$  Non-dispersive!

- ❖ Our new kind of wave, called a Kelvin wave after Lord Kelvin who discovered it, propagates North-South as a non-dispersive gravity wave (i.e. the propagation is not influenced by rotation).
- ❖ Rotation sets the exponentially decaying East-West structure.
- ❖ It is said that the wave “leans” on the coast to its right.

- ❖ Rotation and stratification comprise the two essential components of fluid flow in the atmosphere and the oceans (large lakes, too)
- ❖ The description of these flows is rooted in linear theory, and the geostrophic flow, Poincare waves, and Kelvin waves discussed in these slides are three widely discussed examples.
- ❖ Non-linearity needs either more sophisticated mathematics or numerical simulation (often both).
- ❖ Many problems remain an active area of exploration.

# Dispersive corrections: Basics

- ❖ Consider the SW equations in 1D:
  - ❖  $u_t + uu_x = -g\eta_x, \eta_t + ([H + \eta]u)_x = 0$
- ❖ These extend the simple case with a constant phase and group speed  $c_0 = \sqrt{gH}$
- ❖ That means nonlinearity is the only new piece. This leads to wave steepening and inevitably breaking.
- ❖ However, long before breaking it could be the assumptions of shallow water may be violated.
- ❖ For finite depth waves we have  $\sigma^2 = gk \tanh(kH)$ , and recall for shallow water we have  $\sigma_{SW}^2 = gHk^2$ .

# Dispersive corrections: Approximate dispersion

- ❖ For finite depth waves we have  $\sigma^2 = gk \tanh(kH)$ , and recall for shallow water we have  $\sigma_{SW}^2 = gHk^2$ . How can we do better?
- ❖ Use the Taylor expansion  $\tanh(z) = z - \frac{1}{3}z^3 + O(z^5)$  and find:  
$$\sigma_{cub}^2 = gHk^2 \left(1 - \frac{1}{3}k^2H^2\right)$$
. A bit of algebra gives the phase speed as  
$$c_p^{(cub)} = \pm c_0 \sqrt{1 - k^2H^2/3}$$
 so that as  $k$  increases (shorter waves) waves move faster.
- ❖ How to include this in the SW theory?

# Dispersive corrections: Modifying SW

- ❖ Consider the SW equations in 1D written with the pressure split as  $p = p_H + p_{NH}$
- ❖ It is the second piece that will yield dispersive corrections.
- ❖ Now a long set of calculations tries to relate  $p_{NH}$  to the free surface  $u$  and its derivatives
  - ❖  $u_t + uu_x = -g\eta_x + \beta u_{xxt}, \eta_t + ([H + \eta]u)_x = 0$
  - ❖ Linearize, set  $H$ :  $u_t = -g\eta_x + \beta u_{xxt}, \eta_t + (H_0u)_x = 0$  cross-differentiate and find:  
$$u_{tt} = c_0^2 u_{xx} + \beta u_{xxtt}$$
  - ❖ Try a dispersive wave solution and find  $\sigma^2(k) = \frac{c_0^2 k^2}{1 + \beta k^2}$  and if you Taylor expand you get  $\sigma^2(k) \approx c_0^2 k^2 \left[ 1 - \frac{\beta k^2}{2} \right]$  like on the last page.

# Dispersive corrections: Modified SW f-plane

- ❖ The momentum equations read:  $\frac{D\vec{u}}{Dt} + (fv, -fu) = -g \nabla \eta + \beta \nabla \nabla \cdot \vec{u}_t$
- ❖ And the conservation of mass is unchanged:  $\frac{\partial \eta}{\partial t} + \nabla \cdot ([H + \eta] \vec{u}) = 0$
- ❖ These do not completely preclude breaking but they are much, much friendlier to solve compared to the classical shallow water equations.
- ❖ And the best part they often agree with the beautiful patterns nature makes like those on the next slide.

