Learning Notation: Seminar Two
Set Theory

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1 Sets and Membership

1.1 Elements and membership

Definition 1.1 (Naive set)**.** *A **set** is an unordered collection of unique objects. The objects in a set are called its **elements.***

For example, if some unique variables a, b, c are the elements of the set S, we can write

$$S = \{a, b, c\}. \tag{1.1}$$

Then a, b, c are elements of S, which we can express as

$$a, b, c \in S. \tag{1.2}$$

The "$\in$" symbol is read "is an element of" or "are elements of." We can also say that a, b, c are **members** of S, a, b, c **belong** to S, and S **contains** a, b, and c. If $S = \{a, b, c\}$, then for some other unique element d not in the set, we can write

$$d \notin S = \{a, b, c\}. \tag{1.3}$$

The elements of a set must be **unique**, meaning a set cannot contain more than one of the same thing. For example, it would be slightly improper to write

$$S = \{a, a, b, c\}, \tag{1.4}$$

since the element a is not unique in S. Alternatively, we can say that

$$\{a, a, b, c\} = \{a, b, c\}, \tag{1.5}$$

by ignoring the repeated element a.

Exercise 1. *For a set A, does $a_1, a_2, a_3 \in A$ imply $A = \{a_1, a_2, a_3\}$?*

1.2 Set equality and element order

Two sets are **equal** if they have the same elements, regardless of the order of the elements. For example,

$$\{a, b, c\} = \{c, b, a\}. \tag{1.6}$$

1.3 Cardinality

A set is **infinite** if it has infinitely many members. Otherwise it is **finite.** For example, the set

$$\mathbb{N} = \{0, 1, 2, 3, ...\}, \tag{1.7}$$

containing all natural numbers (non-negative integers) is infinite.

The **cardinality** or **size** of a set is the number of elements it has. For a set S, we write its cardinality as $|S|$. For example, if $S = \{a, b, c\}$ then

$$|S| = |\{a, b, c\}| = 3. \tag{1.8}$$

Two sets are **equicardinal** if they have the same cardinality. For example, if

$$A = \{1, 2, 3, 4, 5\}, \quad B = \{3, 4, 5, 6, 7\}, \tag{1.9}$$

then $|A| = |B| = 5$, so A and B are equicardinal. This is a very informal definition of cardinality. We will look at a more formal definition of equicardinality when discussing functions.

1.4 Empty set, sets as elements

The **empty set** is a special set that contains no elements. We denote it by $\varnothing$ or $\{\}$. A set can have other sets as its members. For example, the following are valid sets:

$$A = \{a, b, c\}, \quad B = \{1, 2, 3\}, \tag{1.10}$$

$$C = \{\varnothing, A, B\} = \{\{\}, \{a, b, c\}, \{1, 2, 3\}\}, \tag{1.11}$$

$$D = \{\varnothing, a, b, c, 1, 2, 3, C\} = \{\{\}, a, b, c, 1, 2, 3, \{\{\}, \{a, b, c\}, \{1, 2, 3\}\}\}. \tag{1.12}$$

Exercise 2. *What are $|D|$, $|\varnothing|$, and $|\{\varnothing\}|$?*

1.5 Number systems

We've seen the empty set $\varnothing$, which is a special concept in set theory. Now we will look at some other commonly used sets and their conventional notation.

The set of all **natural numbers**, which has only positive whole number elements (and zero), is denoted as

$$\mathbb{N} = \{0, 1, 2, 3, ...\}. \tag{1.13}$$

The set of all **integers**, which contains all positive and negative whole numbers and zero, is denoted as

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4...\}. \tag{1.14}$$

The set of all **rational numbers**, which contains all integers that can be written as a fraction of integers, is denoted as

$$\mathbb{Q} = \left\{ ..., -\frac{2}{3}, -\frac{1}{4}, -\frac{3}{1}, -\frac{1}{3}, -\frac{2}{1}, -\frac{1}{2}, -\frac{1}{1}, \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, ... \right\} \tag{1.15}$$

The set of all **real numbers** is written as

$$\mathbb{R} = (-\infty, \infty), \tag{1.16}$$

which contains numbers like π, e, and $\sqrt{2}$ that cannot be expressed as fractions of integers.

We also have the set of complex numbers, denoted by $\mathbb{C}$, which can be constructed from all pairs of real numbers $x, y \in \mathbb{R}$. We have x for the real component, and add it to y times $i = \sqrt{-1}$, the imaginary component. We can write the complex numbers using set builder notation:

$$\mathbb{C} = \left\{ x + yi \ : \ x, y \in \mathbb{R}, \ i = \sqrt{-1} \right\} \tag{1.17}$$

1.6 Set builder notation

Up to now we have mostly been explicitly writing sets with **set roster notation**. The can be a little vague if we are not careful. Instead we can use **set builder notation** to precisely specify what elements we want in a defined set. Set notation specifies the elements of the set by using **predicate logic**:

$$(x \in X) \wedge P(x) \iff x \in \{y \in X : P(y)\}. \tag{1.18}$$

The set $\{y \in X : P(y)\}$ contains all objects y in the set X where $P(y)$ is a true statement. Here $P(y)$ is called the **predicate** and X is the **universe of discourse**. We discuss predicate logic in our seminar on logic and proofs.

For example, we can write the set of all positive real numbers $\mathbb{R}$ as

$$(0, \infty) = \{x \in \mathbb{R} : x > 0\}, \tag{1.19}$$

a set we can also denote with $\mathbb{R}^+$. The ":" symbol is read **"where"** or **"such that."** We also sometimes use "|" instead of ":".

Exercise 3. *Construct $\mathbb{Z}$ and $\mathbb{Q}$ from $\mathbb{N}$ with set builder notation.*

1.7 Tuple and Cartesian product

Definition 1.2 (Tuple)**.** *Informally, a **tuple** is a finite, ordered list (or sequence) of elements. A **n-tuple** is a tuple of n elements.*

$$(a_1, a_2, a_3, ..., a_n). \tag{1.20}$$

Definition 1.3 (Cartesian product)**.** *The **Cartesian product** of two sets A and B is the set of all pairs (or 2-tuples) of elements (a, b) from $a \in A$ and $b \in B$.*

$$A \times B = \{(a, b) : a \in A, b \in B\} \tag{1.21}$$

We can generalize to the definition of a n-ary Cartesian product of arbitrary many sets:

$$\prod_{i=1}^{n} X_n = \{(x_1, ..., x_n) : \forall i \in \{i, ..., n\},\ \ x_i \in X_i\} \tag{1.22}$$

1.8 Russell's paradox

So far we have allowed pretty much anything to be a member of a set, including other sets. However, if we are not careful with our formulation of what sets are, we can run into some bizarre problems. The logician Bertrand Russell came up with a famous paradox that arises from a naive definition of a set.

Paradox 1 (Russell)**.** *Let R be the set of all sets which are not members of themselves.*

$$R = \{S : S \notin S\}. \tag{1.23}$$

Then we have that R is a member of itself if and only if it is not a member of itself.

$$R \in R \iff R \notin R. \tag{1.24}$$

The definition of R states for any set S in the universe of all sets, we have $S \notin S \implies S \in R$. Since R is also a set, if we let $S = R$, then $R \notin R \implies R \in R$, which is the converse in (1.24).

The (logically equivalent) contrapositive of the definition is that for any sets S, $S \in R \implies S \notin R$. Then letting $S = R$ means $R \in R \implies R \notin R$, the conditional statement in (1.24).

Then both the implication $R \notin R \implies R \in R$ and converse $R \in R \implies R \notin R$ are true, which means the biconditional statement $R \notin R \iff R \in R$ is true. But this biconditional statement is a contradiction, which is always false.

Together this shows that the set R which contains all sets other except itself can neither contain itself or not contain itself—a paradox.

To resolve such paradoxes, mathematicians limit what are allowed to be valid sets, which is standardized with the Zermelo-Fraenkel axioms (ZF axioms). However, problems with naive set theory apparently rarely come up unless you specifically look for them (like in the case of Russell's paradox), so we will not worry about it.

2 Set Operations and Subsets

2.1 Union

Definition 2.1 (Union). *The **union** of sets A and B are defined as*

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}. \tag{2.1}$$

For example, if $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$ then $A \cup B = \{a, b, c, d, e\}$.

2.2 Intersection

Definition 2.2 (Intersection). *The **intersection** of sets A and B are defined as*

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}. \tag{2.2}$$

For example, if $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$ then $A \cap B = \{b, c\}$.

2.3 Difference and complement

Definition 2.3 (Set difference). *The **difference** between sets A and B are defined as*

$$A \setminus B = \{x : (x \in A) \wedge (x \notin B)\}. \tag{2.3}$$

For example, if $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$, then $A \setminus B = \{a\}$ and $B \setminus A = \{d, e\}$.

2.4 Subset and superset

Definition 2.4 (Subset and superset). *A is a **subset** of S if and only if*

$$A \cap S = A. \tag{2.4}$$

*Then we write that $A \subseteq S$. We also call S the **superset** of A and write $S \supseteq A$.*

Definition 2.5 (Proper subset and superset). *A is a **proper subset** of S if and only if*

$$A \subseteq S \quad \text{and} \quad (S \setminus A) \neq \varnothing. \tag{2.5}$$

*Then we write $A \subset S$. We also call S the **proper superset** of A and write $S \supset A$.*

Definition 2.6 (Complement)**.** *Given a **universal set** U which is a superset of A, we define the **complement** of set A as*

$$A^c = \bar{A} = U \setminus A \tag{2.6}$$

Theorem 1 (De Morgan's laws)**.**

$$A \cup B = (A^c \cap B^c)^c, \tag{2.7}$$
$$A \cap B = (A^c \cup B^c)^c. \tag{2.8}$$

Proof. For any element x, if $p := x \in A$ and $q := x \in B$, then

$$x \in A \cup B \iff (x \in A) \vee (x \in B) \iff p \vee q, \tag{2.9}$$
$$x \in A \cap B \iff (x \in A) \wedge (x \in B) \iff p \wedge q, \tag{2.10}$$

For complements we can show that $r = y \in S \implies \neg r = y \in S^c$:

$$r = y \in S \implies y \in S^c \iff y \notin S \text{ or } \neg(y \in S) \iff \neg r. \tag{2.11}$$

Then write (2.7) and (2.8) as

$$A \cup B = (A^c \cap B^c)^c \iff p \vee q \equiv \neg(\neg p \wedge \neg q), \tag{2.12}$$
$$A \cap B = (A^c \cup B^c)^c \iff p \wedge q \equiv \neg(\neg p \vee \neg q), \tag{2.13}$$

which we've previously proved with truth tables. $\square$

2.5 Power set

Definition 2.7 (Power set)**.** *The **power set** of S, denoted $\mathcal{P}(S)$ or 2^S, is the set of all subsets of S:*

$$\mathcal{P}(S) = \{A : A \subseteq S\}. \tag{2.14}$$

For example, $\mathcal{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$.

Exercise 4. *What is $\mathcal{P}(\varnothing)$? What is $\mathcal{P}(\mathcal{P}(\varnothing))$ and $\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing)))$?*

Exercise 5. *Prove by induction that $|\mathcal{P}(S)| = 2^{|S|}$ for any finite set S.*

3 Functions

/ We can define functions in relation to sets.

Definition 3.1 (Function)**.** *A **function** from a set X to Y maps each element in X to exactly one element in Y. We denote the function as $f : X \to Y$.*

*X is called the **domain** of the function and Y is called its **co-domain**.*

We can also write explicity. For example, $f(x) = \ln x$ and $x \mapsto \ln x$, which are two ways to write the same function.

3.1 Image

Definition 3.2 (Image). *For a function $f : X \to Y$, the* ***image*** *of a subset of the domain $A \subseteq X$ are the set of $f(a)$ for every $a \in A$:*

$$f(A) = \{f(a) : a \in A\}. \tag{3.1}$$

Definition 3.3 (Range). *For a function $f : X \to Y$, the* ***range*** *of the function is the image of the domain:*

$$f(X) = R = \{f(x) : x \in X\}. \tag{3.2}$$

Definition 3.4 (Preimage). *For an element y in the co-domain Y, the set of elements in the domain X hat maps to y is the* ***preimage*** *of y:*

$$f^{-1}(y) = \{x : f(x) = y\}. \tag{3.3}$$

We similarly define the preimage of a subset of the co-domain $B \subseteq Y$ as:

$$f^{-1}(B) = \{x : f(x) \in B\}. \tag{3.4}$$

For example, for the real function $f(x) = x^2$, the preimage $f^{-1}(4) = \{2, -2\}$, and $f^{-1}(\{1, 4\}) = \{-2, -1, 1, 2\}$. What is $f^{-1}(\{n^2 : n \in \mathbb{N}\})$?

3.2 Injection, surjection, and bijection

Definition 3.5 (Injection). *An* ***injective function*** *(or* ***injection****, or* ***one-to-one function****) is a function where each element in the co-domain is assigned at most once.*

$$\forall x, x' \in X,\ f(x) = f(x') \implies x = x'. \tag{3.5}$$

Definition 3.6 (Surjection). *An* ***surjective function*** *(or* ***surjection****) is a function where each element in the co-domain is assigned at least once.*

$$\forall y \in Y,\ \exists x \in X,\ y = f(x). \tag{3.6}$$

Definition 3.7 (Bijection). *A* ***bijective function*** *(or* ***bijection****, or* ***one-to-one correspondence****, or* ***invertible function****) is a function that is both an injection and a surjection.*

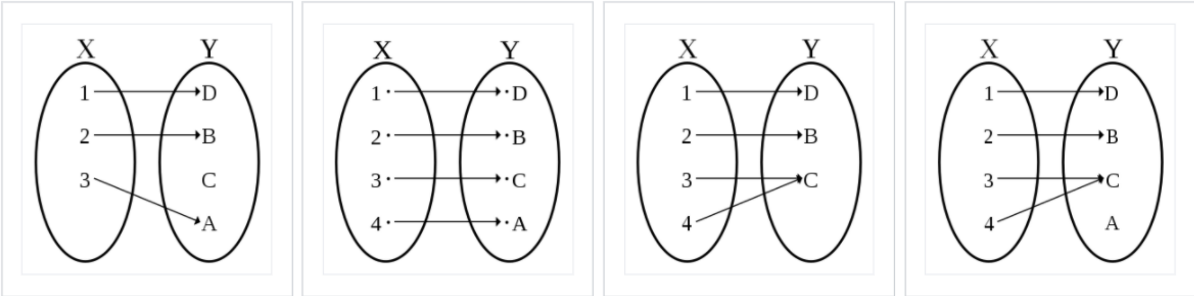
$$\forall y \in Y,\ \exists! x \in X,\ y = f(x), \tag{3.7}$$

where $\exists! x$ means there exists exactly one x.

See Figure 1 for illustrations.

3.3 Equicardinality

Definition 3.8 (Equicardinality). *Two sets are* ***equicardinal*** *if there exists a bijective function between them.*



An **injective** non-surjective function (injection, not a bijection)

An **injective** surjective function (bijection)

A non-injective surjective function (surjection, not a bijection)

A non-injective non-surjective function (also not a bijection)

Figure 1: Illustrations of functions from Wikipedia

Exercise 6. *Are set of all natural numbers $\mathbb{N}$ and the set of all even numbers equicardinal? What about the naturals and odd numbers? How can you prove it?*

4 Countability and infinite sets

4.1 Countable sets

Definition 4.1. *A set is* ***countable*** *if it is either finite or equicardinal to $\mathbb{N}$. Otherwise, it is* ***uncountable.***

We call the cardinality of countably infinite sets $\aleph_0$ ("aleph null"). Both $\mathbb{Z}$ and $\mathbb{Q}$ are countable as they are equicardinal to $\mathbb{N}$.

To show that the integers is countable, we can arrange the its elements in alternating order of negative and positive naturals:

$$\mathbb{Z} = \{0, 1, -1, 2, -2, ...\} = \{z_0, z_1, z_2, z_3, z_4, ...\}. \tag{4.1}$$

Since this arrangement ensures all integers are captured by the pattern, we can assign a natural number to each integer with indexing.

For the rationals, we can also arrange them in an exhaustive, indexable list. We can list all the non-negative rationals by grouping fractions where the numerator and denominator sum to be the same.

$$\mathbb{Q}_{\geq 0} = \left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \left(\frac{2}{2}, \right) \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, ...\right\} = \left\{0, 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, ...\right\} \tag{4.2}$$

Then we can include the negative rationals in the list by inserting them after their positive counterpart, just as in the integers, and index the list.

$$\mathbb{Q} = \left\{0,\ 1, -1,\ \frac{1}{2}, -\frac{1}{2},\ 2, -2,\ \frac{1}{3}, -\frac{1}{3},\ 3, -3, ...\right\} = \left\{q_0,\ q_1,\ q_2,\ q_3,\ q_4,\ q_5,\ q_6, ...\right\} \tag{4.3}$$

More generally, an infinite set is "listable" if (and only if) it is countable.

Other countable sets include those of all

- constructible numbers
- computable numbers: set of reals computable to arbitrary precision
- algebraic numbers: root solutions to polynomials
- finite strings of zeros and ones

4.2 Uncountable sets

Georg Cantor introduced the concept of countability and produced a famous proof that the set of all real numbers is **uncountable**.

The cardinality of the reals is called $\aleph_1$ ("aleph one").

Some other sets with this cardinality are:

- $\mathbb{R} \setminus \mathbb{Q}$, the difference between the reals and rationals
- The interval $[0, 1]$ (or any non-empty real interval)
- $2^{\mathbb{N}}$, the power set of the natural numbers (or any $\aleph_0$ set)
- $\{0, 1\}^{\infty}$, the set of all infinite-length strings of zeros and ones.

Theorem 2. *The set of real numbers in the interval* $[0, 1]$ *is uncountable.*

Proof. (Contradiction) Assume that the real numbers in $[0, 1]$ is countable. Then we should be able to put all of them in a list. Let us represent such list as decimals of infinite-length binary strings, for example like in Table 1.

n	1 2 3 4 5 6 7 8 9 ...
0.	$\underline{\mathbf{0}}$ 0 0 0 0 0 0 0 0 ...
0.	1 $\underline{\mathbf{1}}$ 1 1 1 1 1 1 1 ...
0.	1 0 $\underline{\mathbf{0}}$ 0 0 0 0 0 0 ...
0.	1 1 0 $\underline{\mathbf{0}}$ 0 0 0 0 0 ...
0.	1 0 1 0 $\underline{\mathbf{1}}$ 0 0 0 1 ...
0.	0 1 0 1 1 $\underline{\mathbf{1}}$ 0 1 1 ...
0.	0 1 0 0 1 0 $\underline{\mathbf{0}}$ 0 1 ...
0.	0 1 0 1 0 0 1 $\underline{\mathbf{0}}$ 1 ...
⋮	⋮

Table 1: Cantor's Diagonization.

Then let a_n be the nth decimal of the nth real number in the countable list, and let

$$a = 0.\ a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ \ldots = 0.\ \underline{\mathbf{0}}\ \underline{\mathbf{1}}\ \underline{\mathbf{0}}\ \underline{\mathbf{0}}\ \underline{\mathbf{1}}\ \underline{\mathbf{1}}\ \underline{\mathbf{0}}\ \underline{\mathbf{0}}\ \ldots \tag{4.4}$$

These are the underlined decimals in Table 1. Let $b_n = 1$ if $a_n = 0$, and $b_n = 0$ if $a_n = 1$. Then the string

$$b = 0.\ b_1\ b_2\ b_3\ b_4\ b_5\ b_6\ b_7\ b_8\ \ldots = 1.\ \underline{\mathbf{1}}\ \underline{\mathbf{0}}\ \underline{\mathbf{1}}\ \underline{\mathbf{1}}\ \underline{\mathbf{0}}\ \underline{\mathbf{0}}\ \underline{\mathbf{1}}\ \underline{\mathbf{1}}\ \ldots \tag{4.5}$$

is a binary real number in $[0, 1]$ that is not included in the list, since each decimal place is different from at least one other real number in the list. Thus there can never be a countable list of all real numbers between $[0, 1]$, as one can always construct a new real number not included in the list by this **diagonlization** method. □

Exercise 7. *Show for any* $a, b \in \mathbb{R}$*, the cardinality of* $[0, 1]$ *is equal to that of interval* $[a, b]$*.*

Exercise 8. *Show that* $\mathbb{R}$ *and* $\mathbb{R}^+$ *are equicardinal.*

Exercise 9. *See Appendix* (A.1) *for proof that the intervals* $[0, 1]$*,* $[0, 1)$*,* $(0, 1]$*, and* $(0, 1)$ *are all equicardinal. Show that* $[0, 1]$ *is equicardinal to* $\mathbb{R}$*.*

4.3 Cantor's theorem and infinite cardinals

We can show that for any set A, its power set has a greater cardinality. This is true even for uncountable sets, and their power sets, and power sets of their power sets, and so on.

It is easy to see that there always exists an injection between a set A and its power-set $\mathcal{P}(A)$. For example, $x \mapsto \{x\}$ is one such injection. This means that $|A| \leq \mathcal{P}(A)$.

To prove that $|A| < |\mathcal{P}(A)|$, we can show that there cannot exist a bijection—that is, a function that is both injective and surjective. In particular, we can show that there exists no surjection $f : A \to \mathcal{P}(A)$. This theorem is also due to Georg Cantor.

Theorem 3 (Cantor)**.** *There exists no surjective function* $f : A \to \mathcal{P}(A)$.

Proof. (Contradiction) Suppose for the sake of contradiction that there exists a surjection $f : A \to \mathcal{P}(A)$.

Then let $B = \{x \in A : x \notin f(x)\}$, the set of every element in A that is not a member of its own image in the power set. Note that $B \subseteq A$, which means $B \in \mathcal{P}(A)$.

Since $f : A \to \mathcal{P}(A)$ is surjective, meaning every member in $\mathcal{P}(A)$ is mapped to an element in A, there must exists some $y \in A$ such that $f(y) = B$.

Then by definition of B, we have the following implications:

$$y \in f(y) = B \implies y \notin f(y) = B, \tag{4.6}$$
$$y \notin f(y) = B \implies y \in f(y) = B. \tag{4.7}$$

Together this implies that $y \in B \iff y \notin B$. This is a contradiction, so there must not exist any surjection $f : A \to \mathcal{P}(A)$.

Since there always exists a injection $h : A \to \mathcal{P}(A)$, this means that $|\mathcal{P}(A)| > |A|$. □

This implies that there are infinitely many cardinals larger than $\aleph_1$, each successively larger than the next, since by Cantor's theorem,

$$\begin{aligned} \aleph_1 = |\mathbb{R}| &< |\mathcal{P}(\mathbb{R})| = \aleph_2, \\ \aleph_2 = |\mathcal{P}(\mathbb{R})| &< |\mathcal{P}(\mathcal{P}(\mathbb{R}))| = \aleph_3, \\ \aleph_3 = |\mathcal{P}(\mathcal{P}(\mathbb{R}))| &< |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{R})))| = \aleph_4, \\ &\vdots \end{aligned}$$

One famous and important conjecture about infinite cardinalities is the continuum hypothesis (CH).

Conjecture 1 (Continuum hypothesis)**.** *There does not exists a set whose cardinality is strictly between* $\aleph_0$ *and* $\aleph_1$*.*

In 1940, Kurt Gödel showed that CH **cannot be disproven** from Zermelo-Fraenkel set theory with axiom of choice (ZFC). Then in the 1960s, Paul Cohen showed that CH also **cannot be proven** with ZFC, a result for which he was awarded a Field's Medal.

Gödel also famously showed in his **incompleteness theorems** that there exists truths in mathematics that cannot be proven.

A Appendix

A.1 Equicardinality of intervals

Proposition. *The intervals* $(0, 1)$ *,* $[0, 1)$*,* $(0, 1]$*, and* $[0, 1]$ *are equicardinal.*

Proof. We can show there exist bijective functions from $(0,1)$ to $[0,1), (0,1]$, and $[0,1]$ by Hilbert's Hotel arguments. Let $H = \left\{\frac{1}{n} : n \in \mathbb{N}, n > 1\right\}$, a countably infinite set which is a proper subset of $\left(0, \frac{1}{2}\right]$.

Define $f_1 : (0,1) \to [0,1)$ as

$$f_1(x) = \begin{cases} 0, & x = \frac{1}{2} \\ \frac{1}{n-1}, & x \in \left\{\frac{1}{n} : n \in \mathbb{N}, n > 2\right\} \\ x, & x \notin H. \end{cases} \tag{A.1}$$

Define $f_2 : (0,1) \to (0,1]$ as

$$f_2(x) = \begin{cases} 1, & x = \frac{1}{2} \\ \frac{1}{n-1}, & x \in \left\{\frac{1}{n} : n \in \mathbb{N}, n > 2\right\} \\ x, & x \notin H. \end{cases} \tag{A.2}$$

Define $f_3 : (0,1) \to [0,1]$ as

$$f_3(x) = \begin{cases} 0, & x = \frac{1}{2} \\ 1, & x = \frac{1}{3} \\ \frac{1}{n-2}, & x \in \left\{\frac{1}{n} : n \in \mathbb{N}, n > 3\right\} \\ x, & x \notin H. \end{cases} \tag{A.3}$$

It can be seen that each of these functions is a bijection. Then the open interval is equicardinal to each of the other type of interval, which means they are all equicardinal. □

References

- Ben (https://math.stackexchange.com/users/32139/ben). (2012), *How to define a bijection between* $(0,1)$ *and* $(0,1]$*?* math.stackexchange.com. https://math.stackexchange.com/q/160750

- Hammack, Richard, (2018). *Book of Proof, third edition.* Richard Hammack. https://www.people.vcu.edu/~rhammack/BookOfProof/

- Jagannathan, Krishna. (2015). *Mod-01 Lec-02 CARDINALITY AND COUNTABILITY-1*, lecture. Probability Foundation for Electrical Engineers. Indian Institute Of Technology–Madras. https://youtu.be/gLT58t2z48A

- Jagannathan, Krishna. (2015). *Mod-01 Lec-02 CARDINALITY AND COUNTABILITY-2*, lecture. Probability Foundation for Electrical Engineers. Indian Institute Of Technology–Madras. https://youtu.be/KCEtDSGrVko

- Rayo, Agustin, (2020). *Infinite Cardinalities*, lecture. Paradox and Infinity. MIT Open Learning Library. https://openlearninglibrary.mit.edu/courses/course-v1:MITx+24.118x+2T2020/course

- Tao, Terence. (2003). *Week 1*, lecture notes. Honors Analysis Math131AH. University of California, Los Angeles. https://www.math.ucla.edu/~tao/resource/general/131ah.1.03w/

- Tao, Terence. (2016). *Analysis I, third edition.* Springer Singapore.

- Wikipedia contributors. (2022). *Bijection.* In Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Bijection&oldid=1074730184