Learning Notation: Seminar One
Logic and Proof Techniques

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1 Mathematical Truth

1.1 Statements

Definition 1 (Statement)**.** *A* ***statement*** *or* ***claim*** *is an expression that is either* ***true*** *(T) or* ***false*** *(F), but not both. We call T and F* ***truth values****.*

We can use variables to represent a statement. For example:

$$P := 1 + 1 = 2. \tag{1.1}$$
$$Q := \text{There are infinitely many prime numbers.} \tag{1.2}$$
$$R := \sqrt{2} \text{ is rational.} \tag{1.3}$$
$$S := \text{All horses are the same color.} \tag{1.4}$$

For the above statements, P, Q are true while R, S are false. The truth value of a statement can be conditional on another variable. For example,

$$P(x) := \text{If } x \text{ is an integer, then } 2x \text{ is even.} \tag{1.5}$$
$$Q(x) := \text{If the integer } x \text{ is not } 0 \text{ or } -1\text{, then } x(x+1) \text{ is odd.} \tag{1.6}$$

We call $P(x)$ and $Q(x)$ **predicates**, and the collection of x the **universe of discourse.** For (1.6), the universe of discourse is the set of all integers.

Not all expressions are statements according to Definition 1. For example, the truth values of the following sentences cannot be determined, so they are not mathematical statements.

$$P := \text{Hello world!} \tag{1.7}$$
$$Q := \text{Is } 2 + 2 = 4\text{?} \tag{1.8}$$
$$R := \text{This statement is false.} \tag{1.9}$$
$$S(x) := \text{The integer } x \text{ is even.} \tag{1.10}$$

Theorems, propositions, lemmas, and **corollaries** are all statements which are proven to be true. Terence Tao explains the distinction between them nicely:

> "A lemma is an easily proved claim which is helpful for proving other propositions and theorems, but is usually not particularly interesting in its own right. A proposition is a statement which is interesting in its own right, while a theorem is a more important statement than a proposition which says something definitive on the subject, and often takes more effort to prove than a proposition or lemma. A corollary is a quick consequence of a proposition or theorem that was proven recently."[6]

1.2 Axioms

An **axiom** is a statement that is assumed to be true. An **axiomatic system** uses axioms, definitions, and deductions to derive the truth values of other statements with **proofs**.

A statement is **consistent** in a axiomatic system it does not **contradict** the other axioms or proven statements. To avoid inconsistencies in a axiomatic system, its axioms should be as few and as simple as possible. Though apparently, constructing a consistent system is more difficult than it sounds.[2] See more on Gödel's incompleteness theorem.

1.3 Conjectures

A **conjecture** is a statement that is believed to be true but is not proven. Goldbach's conjecture and the twin prime conjecture are two famous examples that are very simple to state, though demonstratively not simple to prove.

Conjecture 1 (Goldbach)**.** *Every even number is the sum of two primes.*

Conjecture 2 (Twin prime)**.** *There are infinitely many primes p where $p + 2$ is also prime.*

Other famous conjectures are the Riemann hypothesis, the P versus NP problem and the continuum hypothesis. The first two of these are unsolved Millenium Prize Problems.

2 Logic Operation

A **logical operator** (or **connective**) is applied to one or more statements to create a new statement. We start with the unary (i.e. operating on one statement) connective, negation.

2.1 Negation and truth tables

Definition 2 (Negation)**.** *A **negation** $\neg$ (or $\sim$) is a logical operator on a statement that creates a statement of the opposite truth value.*

For example, for a statement P, its negation is $\neg P$, which we also call "not P". We can also negate the negation, $\neg(\neg P)$. Their truth values can be outlined Table 2.1, which is a **truth table**. A truth table shows all possible truth value combinations of statements or propositional variables.

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

Table 1: Negation

2.2 Logical equivalence

Definition 3 (Logical equivalence)**.** *Two statements are **logically equivalent** if they have the same values on the truth table. We express an equivalence between two statements P and Q as $P \equiv Q$.*

For example, from Table 2.1 we can see that $P \equiv \neg(\neg P)$ since they have the same truth values.

2.3 Conjunction and disjunction

Definition 4 (Conjunction)**.** *For two statements P and Q, we define their* ***conjunction*** *$P \wedge Q$ as a statement that is true if both P and Q are true, and false otherwise.*

Definition 5 (Disjunction)**.** *We define their* ***disjunction*** *$P \vee Q$ as a statement that is true if either P is true or Q is true, or both are true.*

We also call $\wedge$ the "and" operator and $\vee$ the "or" operator. The truth table is as follows:

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Table 2: Conjunction and disjunction

2.4 Quantifiers

Quantifiers connect a sequence of statements from a predicate with conjunction or disjunctions.

Definition 6 (Universal quantifier)**.** *The* ***universal quantifier*** *$\forall$ evaluates a conjunction of statements with a predicate $P(x)$ on all elements of x in a set.*

$$\forall x \in \{x_1, x_2, ...\},\ P(x) \ \equiv\ P(x_1) \wedge P(x_2) \wedge ... \tag{2.1}$$

For the above we read, "**For just have to all** x in the set $\{x_1, x_2, ...\}$, $P(x)$ is true." Sometimes we say **"for every"** or **"for any"** instead of "for all"

Definition 7 (Existential quantifier)**.** *The* ***existential quantifier*** *$\exists$ creates a disjunction of $P(x)$ for all elements of x in a set.*

$$\exists x \in \{x_1, x_2, ...\},\ P(x) \ \equiv\ P(x_1) \vee P(x_2) \vee ... \tag{2.2}$$

For the above we read, "**There exists** a x in the set $\{x_1, x_2, ...\}$ where $P(x)$ is true." Sometimes we say **"there is"** instead of "there exists."

Consider the following statement: $\exists a \in \mathbb{R},\ \forall x \in \mathbb{R},\ ax = x$. What is a? Also, $\exists b \in \mathbb{R},\ \forall x \in \mathbb{R},\ bx = b$. What is b?

2.5 De Morgan's laws

Two famous identities in propositional logic are De Morgan's Laws.

Theorem 1 (De Morgan's laws)**.** *For two statements P and Q,*

$$P \wedge Q \equiv \neg(\neg P \vee \neg Q), \tag{2.3}$$
$$P \vee Q \equiv \neg(\neg P \wedge \neg Q). \tag{2.4}$$

We will prove the first result (2.3) and leave (2.4) as an exercise.

Proof. To proof that two statements are equivalent, we just have to show that they have the same values on the truth table.

(i) P	(ii) Q	(iii) $P \wedge Q$	(iv) $\neg P$	(v) $\neg Q$	(vi) $(\neg P \vee \neg Q)$	(vii) $\neg(\neg P \vee \neg Q)$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	F	T	F	T	F
F	F	F	T	T	T	F

Table 3: First De Morgan's Law

It can be seen from Table 3 that $p \wedge Q$ in column (iii) and $\neg(\neg P \vee \neg Q)$ in column (vii) have the same truth values for all value combinations of P, Q. Therefore they are logically equivalent by definition. □

Question 1. Prove the second de Morgan's law (2.4), that $P \vee Q \equiv \neg(\neg P \wedge \neg Q)$.

2.6 Conditional and biconditional statements

Definition 8 (Conditional statement)**.** *For two statements P and Q, we can form a new statement*

$$R := \text{If } P \text{ (is true), then } Q \text{ (is true)}, \tag{2.5}$$

where R is a true statement if Q is true under the condition that P is true. We can also say that "P implies Q", or "Q if P", or write $P \implies Q$ (or $P \to Q$).

If $P \implies Q$, we call P the **sufficient condition** for Q, and Q the **necessary condition** for P.

If the reverse implication $Q \implies P$ is true, we can also write $P \impliedby Q$, which we also call the **converse** of $P \implies Q$. Then we can also say that P **only if** Q**.**

Definition 9 (Biconditional statement)**.** *For two statements P and Q, if both $P \implies Q$ and $P \impliedby Q$, then we say that **if and only if** P**, then** Q**.** We also call this a biconditional statement, and write it as $P \iff Q$ or $P \longleftrightarrow Q$. A biconditional statement is equivalent to logical equivalence.*

The truth table for conditional and bi-conditional statements is as follows:[1]

P	Q	$P \implies Q$	$P \impliedby Q$	$P \iff Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Question 2. Show that $P \implies Q$ is logically equivalent to $\neg P \vee (P \wedge Q)$.

Question 3. Express $P \iff Q$ as negations, conjunctions, and disjunctions of P and Q.

Question 4. Which of the following statements are true?

(a) $x < 3 \implies x \leq 4$

(b) $x < 3 \impliedby x \leq 4$

(c) $x > y \implies x \geq y$

(d) $x > y \impliedby x \geq y$

(e) $(|x| < y) \wedge (y > 0) \iff (x > -y) \wedge (x < y)$

(f) $(|x| < y) \wedge (y > 0) \iff (x > -y) \vee (x < y)$

(g) $(|x| > y) \wedge (y > 0) \iff (x > y) \wedge (x < -y)$

(h) $(|x| > y) \wedge (y > 0) \iff (x > y) \vee (x < -y)$

[1]The third case where $F \implies T$ is true is called "vacuous truth".

2.7 Tautology

Definition 10. *A **tautology** is a statement that is always true.*

Definition 11. *A **contradiction** is a statement that is always false.*

For example, $P \vee \neg P$ is a tautology, while $P \wedge \neg P$ is a contradiction.

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg$ P
T	F	T	F
F	T	T	F

Table 4: A tautology and contradiction

Question 5. Is each of the following a tautology, contradiction, or neither?

(a) $(x < 0) \vee (x > 0)$

(b) $(x < 0) \wedge (x > 0)$

(c) $(x < y) \wedge (x \geq y)$

(d) $[(P \implies Q) \wedge (\neg P)] \implies (Q \equiv T)$

(e) $[(P \implies Q) \wedge (\neg P)] \implies Q$

(f) $[(P \implies Q) \wedge P] \implies Q$

(g) $[(P \implies Q) \wedge (\neg Q)] \implies \neg P$

(h) $[(P \vee Q) \wedge (\neg Q)] \implies \neg P$

(i) $(P \wedge Q) \iff Q$

(j) $Q \iff (P \vee Q)$

Question 6. Verify that the following statements are tautologies.

(a) $P \iff \neg(\neg P)$

(b) $P \vee Q \iff \neg(\neg P \wedge \neg Q)$

(c) $P \wedge Q \iff \neg(\neg P \vee \neg Q)$

(d) $(P \implies Q) \iff (\neg Q \implies \neg P)$

(e) $(\neg P \implies Q) \wedge (\neg P \implies \neg Q) \iff P$

(f) $((A \implies B) \wedge (B \implies C)) \implies (A \implies C)$

(g) $[(A \vee B) \wedge (A \implies C) \vee (B \implies C)] \implies C$

3 Proof Techniques

3.1 Direct proof

3.2 Proof by contrapositive

3.3 Proof by contradiction

3.4 Proof by induction

References

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