Homework set 1

There are two problems with 4 points each. The points you gain from the homework may substitute some problems in the final exam

1. Let S^n denote the vector space of $n \times n$ real symmetric matrices. Let $K := S^n_+$ (resp. S^n_{++}) denote the set of positive semidefinite matrices, i.e., the set of real symmetric matrices having nonnegative (resp. strictly positive) eigenvalues.

   We equip the vector space S^n with the (trace) inner product $\langle A, B\rangle := \mathrm{Tr}AB$, which induces a norm in S^n given by $\|A\| := \sqrt{\langle A, A\rangle}$, and so a distance $\mathrm{d}(A, B) = \|A - B\|$.

   Show that K is a closed convex cone in S^n with interior S^n_{++} and if $0 \neq A \in K$ implies that $-A \notin K$. Show also that $(K)^+ = K$ where

$$K^+ := \{B \in S^n : \langle A, B\rangle \geq 0, \forall A \in K\}.$$

2. Show that

$$L = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \, : \, t \geq \|x\|_2\}$$

   is a convex cone, and the function $f(x, t) = -\log(t^2 - x^T x)$ is convex on this convex cone.