Macroeconomic Theory
Course Summary

Uppsala Masters in Economics 2021-2023[1]

Winter 2021-2022

[1]Course with Professor Christoph Hedtrich.

Contents

Chapter 1

Growth and Solow Model

1.1 Growth rates

We have by the chain rule that

$$\frac{d\ln x(t)}{dt} = \frac{1}{x(t)}\frac{dx(t)}{dt} = \frac{\dot{x}(t)}{x(t)} \tag{1.1}$$

We call $\frac{\dot{x}(t)}{x(t)}$ the growth rate of x. Suppose growth rate is constant at g, then

$$\begin{aligned}
\frac{d\ln(x(t))}{dt} = \frac{\dot{x}(t)}{x(t)} = g &\implies d\ln(x(t)) = g\ dt && (1.2)\\
&\implies \int d\ln(x(t)) = \int g\ dt && (1.3)\\
&\implies \ln(x(t)) = gt + c && (1.4)\\
&\implies x(t) = e^{gt+c} = e^{gt}e^{c} = x(0)e^{gt} && (1.5)\\
&\implies x(t) = x(0)e^{gt} && (1.6)
\end{aligned}$$

1.2 Solow Model

Firm production is a function of capital $K(t)$, and **effective labor** $A(t)L(t)$. $A(t)$ is technology and $L(t)$ is labor, which also represents **population**:

$$Y(t) = F(K(t), A(t)L(t)) \tag{1.7}$$

Assume production function is homogeneous of degree one:

$$F(aK, aAL) = aF(K, AL) \tag{1.8}$$

Define capital and output **per effective labor** $k(t), y(t)$ as

$$\begin{aligned}
k(t) &= \frac{K}{AL}, && (1.9)\\
y(t) &= \frac{Y}{AL} = \frac{F(K, AL)}{AL} = F\left(\frac{K}{AL}, \frac{AL}{AL}\right) = F\left(\frac{K}{AL}, 1\right) = F(k, 1) = f(k), && (1.10)
\end{aligned}$$

which are also called **the intensive form** of capital and output, where $f'(k) > 0, \quad f''(k) < 0$.

1.3 Dynamics of model

Labor and technology have constant growth rates

$$\frac{\dot{A}(t)}{A(t)} = g, \quad \frac{\dot{L}(t)}{L(t)} = n \tag{1.11}$$

Capital changes with **fixed exogenous rates** for savings s and depreciation δ:

$$\dot{K}(t) = sY(t) - \delta K(t). \tag{1.12}$$

Output that is not saved is consumed:

$$C(t) = (1 - s)Y(t). \tag{1.13}$$

Then we have for effective capital

$$k = \frac{K}{AL} \implies \ln k = \ln K - (\ln A + \ln L) \tag{1.14}$$

$$\implies \underbrace{\frac{\dot{k}}{k}}_{=d\ln(k)/dt} = \frac{\dot{K}}{K} - \left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L}\right) \tag{1.15}$$

$$\implies \dot{k} = \underbrace{\frac{1}{AL}}_{=k/K}(sY - \delta K) - (g + n)\,k \tag{1.16}$$

$$\implies \dot{k} = sf(k) - (g + n + \delta)k \tag{1.17}$$

$$\implies \dot{y} = \frac{d}{dt}f(k) = f'(k)\dot{k}, \quad f'(k) > 0. \tag{1.18}$$

Equations (1.17) and (1.18) are the intensive form of law of motion for capital and income.

1.4 Balanced Growth Path

On the **balanced growth path (BGP)** we have that all growth rates are constant. Let $\frac{\dot{k}(t)}{k(t)} = g_k(t)$. Then on the BGP, constant $g_k(t)$ implies $\dot{g}_k(t) = 0$, which implies $\dot{k}(t) = 0$:

$$\frac{\dot{k}}{k} = g_k(t) = s\frac{f(k)}{k} - (g + n + \delta) \implies \overbrace{\frac{\dot{g}_k(t)}{g_k(t)}}^{\frac{d}{dt}\ln g_k(t)} = \frac{f'(k)\dot{k}}{f(k)} - \frac{\dot{k}}{k} = \left(\frac{f'(k)}{f(k)} - \frac{1}{k}\right)\dot{k} = 0 \tag{1.19}$$

$$\implies \dot{k} = \frac{0}{(f'(k)/f(k) - 1/k)} = 0, \quad \frac{f'(k)}{f(k)} - \frac{1}{k} \neq 0. \tag{1.20}$$

Then with the BGP level of capital k^*, we have

$$\dot{k}(t) = 0 \implies sf(k^*) = sy^* = (g + n + \delta)k^* \tag{1.21}$$

$$\implies y^* = \frac{g + n + \delta}{s}k^*, \quad \dot{y}^* = \frac{g + n + \delta}{s}\dot{k}^* = 0, \tag{1.22}$$

and **income per capita** is equal to technology growth

$$\tilde{y} = \frac{Y}{L} = yA \implies \ln(\tilde{y}) = \ln y + \ln A \tag{1.23}$$

$$\implies \frac{\dot{\tilde{y}}}{\tilde{y}} = \frac{\dot{y}}{y} + \frac{\dot{A}}{A} = 0 + g = g. \tag{1.24}$$

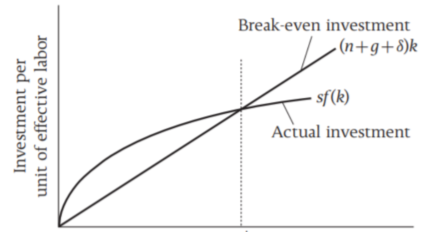


FIGURE 1.2 Actual and break-even investment

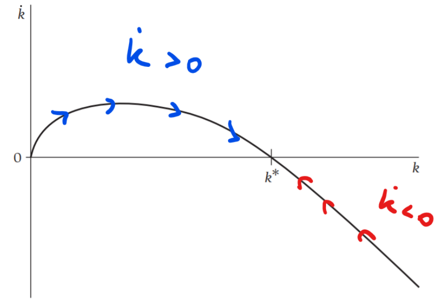


FIGURE 1.3 The phase diagram for k in the Solow model

1.5 Golden-rule level of capital

Golden-rule level of savings and capital maximizes consumption on the BGP path. Assume Cobb-Douglas production,

$$f(k) = k^{\alpha}. \tag{1.25}$$

Then on the BGP,

$$f(k) = sk^{*\alpha} = (g+n+\delta)k^* \implies s = (g+n+\delta)k^{*1-\alpha} \tag{1.26}$$

BGP consumption given by

$$c^* = (1-s)y^* = [1-(g+n+\delta)k^{*1-\alpha}]k^{*\alpha} \tag{1.27}$$
$$= k^{*\alpha} - (g+n+\delta)k^*. \tag{1.28}$$

Maximizing c^* with respect to k^*, we have that

$$\frac{\partial c^*}{\partial k^*} = \alpha k_{GR}^{\alpha-1} - (g+n+\delta) = 0 \tag{1.29}$$
$$\implies k_{GR} = \left(\frac{g+n+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} \tag{1.30}$$
$$\implies s_{GR} = \alpha. \tag{1.31}$$

Chapter 2

Ramsay-Cass-Koopmans Model

Growth rates of technology and labor (also population) remain exogenous and constant as in Solow model,

$$\frac{\dot{A}(t)}{A(t)} = g, \quad \frac{\dot{L}(t)}{L(t)} = n \implies A(t) = A(0)e^{gt}, \quad L(t) = L(0)e^{nt} \tag{2.1}$$

However, law of motion of capital is determined by **optimizing** households (with no depreciation of capital):

$$\dot{K}(t) = Y(t) - \zeta(t), \tag{2.2}$$

where $\zeta(t) = C(t)L(t)$ is the aggregate consumption in the economy.

2.1 Firms

Representative firm has homogeneous of degree one production function,

$$Y = F(K, AL), \quad F(cK, cAL) = cF(K, AL) \tag{2.3}$$

where $K(t), A(t), L(t)$ are capital, technology and labor. Output per effective labor is then

$$y = \frac{F(K, AL)}{AL} = F\left(\frac{K}{AL}, 1\right) = f(k) \quad s.t. \quad f'(k) > 0,\ f''(k) < 0. \tag{2.4}$$

Firms continuously maximize instantaneous profit by choosing capital and labor

$$\max_{K(t), L(t)} \Pi(t) = F(K, AL) - [rK + WL] \tag{2.5}$$

where $r(t)$ is the interest rate and $W(t)$ is wage rate. With FOC with respect to $K(t)$, we have that

$$\frac{\partial \Pi(t)}{\partial K(t)} = 0 \implies r(t) = \frac{\partial F(K, AL)}{\partial K} = AL \frac{\partial F\left(\frac{K}{AL}, 1\right)}{\partial K} \tag{2.6}$$

$$= AL \frac{\partial f(k)}{\partial K} = AL \underbrace{\frac{\partial f(k)}{\partial k}}_{f'(k)} \underbrace{\frac{\partial k}{\partial K}}_{1/(AL)} = f'(k) \tag{2.7}$$

$$\implies r(t) = f'(k), \tag{2.8}$$

which is the equilibrium interest rate. With FOC with respect to $L(t)$, we have

$$\frac{\partial \Pi(t)}{\partial L(t)} = 0 \implies W(t) = \frac{\partial F(K, AL)}{\partial L} = \left[\frac{\partial F(K, AL)}{\partial AL}\right] \frac{\partial AL}{\partial L} \tag{2.9}$$

$$= A\left[\frac{\partial AL \cdot f(k)}{\partial AL}\right] = Af(k) + A\underbrace{\frac{\partial f(k)}{\partial k}}_{f'(k)} \underbrace{\frac{\partial k}{\partial AL}}_{-\frac{k}{AL}} \tag{2.10}$$

$$\implies W(t) = A[f(k) - kf'(k)] \tag{2.11}$$

$$\implies \frac{W(t)}{A(t)} = w(t) = f(k) - kf'(k) \tag{2.12}$$

which is the equilibrium wage per effective labor.

2.2 Households

Each of H identical households chooses consumption to maximize present value of utility:

$$\max_{C(t)} U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H} dt \tag{2.13}$$

$$s.t.$$

$$\int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt \leq \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} W(t) \frac{L(t)}{H} dt \tag{2.14}$$

where $C(t)$ is consumption per worker, ρ is the time discount rate and $R(t) = \int_{\tau=0}^{t} r(\tau) d\tau$ accounts for continuous compounding interest. We can rewrite the budget constraint in (2.14) as

$$\lim_{s\to\infty} \left[\frac{K(0)}{H} + \int_{t=0}^{s} e^{-R(t)} \left[W(t) - C(t)\right] \frac{L(t)}{H} dt\right] \geq 0 \tag{2.15}$$

Household wealth at time s is then

$$\frac{K(s)}{H} = e^{R(s)} \left[\frac{K(0)}{H} + \int_{t=0}^{s} e^{-R(t)} \left[W(t) - C(t)\right] \frac{L(t)}{H}\right] \tag{2.16}$$

$$\implies \lim_{s\to\infty} e^{-R(s)} \frac{K(s)}{H} = \lim_{s\to\infty} \left[\frac{K(0)}{H} + \int_{t=0}^{s} e^{-R(t)} \left[W(t) - C(t)\right] \frac{L(t)}{H}\right] \tag{2.17}$$

$$\implies \lim_{s\to\infty} e^{-R(s)} \frac{K(s)}{H} \geq 0. \tag{2.18}$$

Equation (2.18) is called the **no ponzi-game condition**, imposing that present value of assets cannot be negative in the limit.

Let $c(t) = \frac{C(t)}{A(t)}$ be consumption per effective labor, and assume CRRA instantaneous utility function. Since $A(t) = A(0)e^{gt}$ and $L(t) = L(0)e^{nt}$, we rewrite objective function (2.13) as

$$U = \int_{t=0}^{\infty} e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \frac{L(t)}{H} dt, \quad \theta > 0 \tag{2.19}$$

$$= \int_{t=0}^{\infty} e^{-\rho t} \left[A(0)^{1-\theta} e^{(1-\theta)gt} \frac{c(t)^{1-\theta}}{1-\theta}\right] \frac{L(0)e^{nt}}{H} dt \tag{2.20}$$

$$= B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt \tag{2.21}$$

where $B = A(0)^{1-\theta}\frac{L(0)}{H}$ and $\beta = \rho - n - (1-\theta)g > 0$. Rewrite (2.14) as

$$\frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}[W(t) - C(t)]\frac{L(t)}{H}dt \geq 0 \tag{2.22}$$

$$\implies k(0)\frac{A(0)L(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}[w(t) - c(t)]\frac{A(t)L(t)}{H}dt \geq 0 \tag{2.23}$$

$$\implies k(0)\frac{A(0)L(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}[w(t) - c(t)]e^{(g+n)t}\frac{A(0)L(0)}{H}dt \geq 0 \tag{2.24}$$

$$\implies k(0) + \int_{t=0}^{\infty} e^{-R(t)+(g+n)t}[w(t) - c(t)]dt \geq 0 \tag{2.25}$$

Forming the Lagrangian,

$$\mathcal{L} = B\int_{t=0}^{\infty} e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta}dt + \lambda\left(k(0) + \int_{t=0}^{s} e^{-R(t)+(n+g)t}\left[w(t) - c(t)\right]\right) \tag{2.26}$$

we have with FOC with respect to $c(t)$ that

$$\frac{\partial \mathcal{L}}{\partial c(t)} = 0 \implies Bc(t)^{-\theta} = e^{-R(t)+(n+g+\beta)t} \tag{2.27}$$

$$\implies \frac{d}{dt}\left(\ln B - \theta \ln c(t)\right) = \frac{d}{dt}\left(-R(t) + (n+g+\beta)t\right) \tag{2.28}$$

$$\implies -\theta\frac{\dot{c}(t)}{c(t)} = -r(t) + (n + g + [\rho - n - (1-\theta)g]) \tag{2.29}$$

$$\implies \frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta}. \tag{2.30}$$

Equation (2.30) is the optimal consumption path, which can also be expressed in $C(t)$ as

$$C(t) = A(t)c(t) \implies \frac{\dot{C}(t)}{C(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{\dot{c}(t)}{c(t)} = g + \frac{r(t) - \rho - \theta g}{\theta} = \frac{r(t) - \rho}{\theta}. \tag{2.31}$$

2.3 Dynamics of model

From optimal consumption path (2.30) and equilibrium interest (2.8), we have on the **balanced growth path (BGP)** that

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta} = \frac{f'(k(t)) - \rho - \theta g}{\theta} = 0. \tag{2.32}$$

Growth rate of capital per effective labor on the BGP is then

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L}\right) = \frac{1}{K}\overbrace{(Y - ALc)}^{\dot{K}=Y-\zeta} - (g+n) = 0 \tag{2.33}$$

$$\implies \dot{k} = \underbrace{\frac{k}{K}}_{1/(AL)}(Y - ALc) - (g+n)k = 0 \tag{2.34}$$

$$\implies \dot{k}(t) = y(t) - c(t) - (g+n)k(t) = 0. \tag{2.35}$$

With $f''(k(t)) < 0$, we have that

$$\frac{\partial \dot{c}(t)}{\partial k(t)} = \frac{c(t)}{\theta} f''(k(t)) < 0, \quad \frac{\partial \dot{k}(t)}{\partial c(t)} = -1 < 0, \tag{2.36}$$

which means $\dot{c}(t)$ decreases with $k(t)$ and $\dot{k}(t)$ decreases with $c(t)$. We can then draw a phase diagram with the saddle path that converges to the BGP.

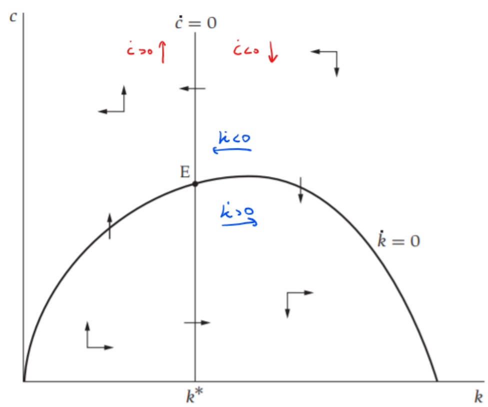


FIGURE 2.3 The dynamics of *c* and *k*

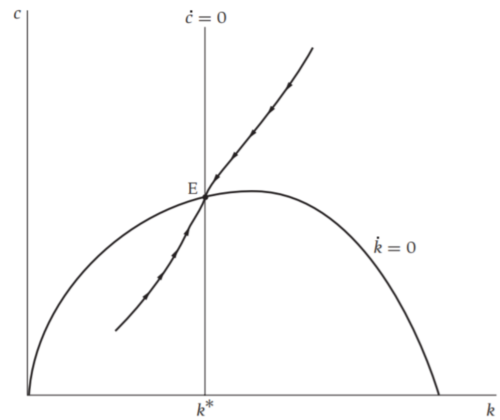


FIGURE 2.5 The saddle path

We can also see effects of changes to other parameters, for example the discount rate

$$\frac{\partial \dot{c}(t)}{\partial \rho} = -\frac{1}{\theta} < 0, \quad \frac{\partial \dot{k}(t)}{\partial \rho} = 0, \tag{2.37}$$

which implies a fall in discount rate ρ will positively shift the locus of k^* where $\dot{c} = 0$ while the $\dot{k} = 0$ path remains unchanged.

2.3.1 Derivations that $\dot{c}(t) = 0$ and $\dot{k}(t) = 0$ on the BGP

To see why $\frac{\dot{c}(t)}{c(t)} = 0$ on the BGP, let $g_c(t) = \frac{\dot{c}(t)}{c(t)}$. On the BGP, for $g_c(t)$ to be constant we must have that $\dot{g}_c(t) = 0$ and the second time derivative of consumption $\frac{d^2 c(t)}{dt} = \ddot{c}(t) = 0$. (?) Then we have that on the BGP:

$$\dot{c}(t) = \frac{f'(k) - \rho - \theta g}{\theta} c(t) = g_c(t) c(t) \tag{2.38}$$

$$\implies \ddot{c}(t) = \dot{g}_c(t) c(t) + g_c(t) \dot{c}(t) = 0 \tag{2.39}$$

$$\implies \frac{\dot{c}(t)}{c(t)} = -\frac{\dot{g}_c(t)}{g_c(t)} = 0. \tag{2.40}$$

Then for capital on the BGP, the second time derivative $\ddot{k}(t) = 0$, and then:

$$\ddot{k}(t) = f'(k)\dot{k}(t) - \dot{c}(t) - (g+n)\dot{k}(t) = 0 \implies \dot{k}(t) = \frac{\dot{c}(t)}{f'(k) - (g+n)} = 0. \tag{2.41}$$

2.4 Golden-rule level of capital

The **golden-rule level** capital stock k_{GR} maximizes consumption $c(t)$. BGP capital stock k^* in the RCK model will never exceed the golden-rule level k_{GR}.

From (2.32) we have that on the BGP

$$f'(k^*) = \rho + \theta g. \tag{2.42}$$

From (2.35) we get the golden-rule capital level k_{GR}, since when $\dot{k} = 0$ we have

$$c = f(k) - (g+n)k, \quad \frac{\partial c}{\partial k} = 0 \implies f'(k_{GR}) = g + n. \tag{2.43}$$

We have from (2.21) that $\beta = \rho - n - (1-\theta)g > 0$, then

$$\begin{aligned} f'(k^*) - f'(k_{GR}) = \rho - n - (1-\theta)g > 0 &\implies f'(k^*) > f'(k_{GR}) && (2.44) \\ &\implies k^* < k_{GR} && (2.45) \end{aligned}$$

since $f'(k)$ is decreasing in k ($f''(k) < 0$). Since $\beta > 0$, we have $k^* < k_{GR}$. However, $k^* \geq k_{GR}$ is possible in Solow growth model with sufficiently high savings rate.

Chapter 3

Diamond Model

Diamond model accounts for turnover of population. Technology and labor (population) follow laws of motion

$$L_t = (1+n)L_{t-1}, \quad A_t = (1+g)A_{t-1} \implies \frac{L_t}{L_{t+1}} = \frac{1}{1+n}, \quad \frac{A_t}{A_{t+1}} = \frac{1}{1+n}, \tag{3.1}$$

and capital follows law of motion

$$K_{t+1} = (w_t A_t - C_{1,t})L_t \tag{3.2}$$

where $W_t = w_t A_t$ is labor income and $C_{1,t}$ is consumption of young individuals at time t.

3.1 Firms

Similar to RCK model, firms production function and intensive are then

$$Y_t = F(K_t, A_t L_t), \quad y_t = \frac{Y_t}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, 1\right) = f(k_t) \tag{3.3}$$

with

$$r_t = f'(k_t), \quad w_t = \frac{W_t}{A_t} = f(k_t) - k_t f'(k_t), \tag{3.4}$$

which have the same derivations that result in (2.8) and (2.12).

3.2 Households

Assume CRRA utility and that individuals live for two discrete periods. Let $C_{1,t}$ and $C_{2,t+1}$ be consumption of young and old individuals, individuals born in period t maximizes lifetime utility

$$\max_{C_{1,t}, C_{2,t+1}} U_t = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta}, \quad \theta > 0, \quad \rho > -1. \tag{3.5}$$

$$s.t.$$

$$w_t A_t = C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}}. \tag{3.6}$$

Form the Lagrangian

$$\mathcal{L} = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta} + \lambda\left(w_t A_t - C_{1,t} - \frac{C_{2,t+1}}{1+r_{t+1}}\right), \tag{3.7}$$

and with the first order conditions we have

$$\frac{\partial \mathcal{L}}{\partial C_{1,t}} = 0 \implies C_{1,t}^{-\theta} = \lambda, \tag{3.8}$$

$$\frac{\partial \mathcal{L}}{\partial C_{2,t+1}} = 0 \implies \frac{1}{1+\rho}C_{2,t+1}^{-\theta} = \frac{\lambda}{1+r_{t+1}} \tag{3.9}$$

Dividing the two FOC results gives us the **Euler equation** for consumption, which determines the optimal intertemporal choice:

$$\frac{C_{2,t+1}}{C_{1,t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}. \tag{3.10}$$

Substituting back into the budget constraint (3.6), we have

$$w_t A_t = C_{1,t} + \frac{1}{1+r_{t+1}}C_{2,t+1} = C_{1,t} + \frac{1}{1+r_{t+1}}\left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta} C_{1,t} \tag{3.11}$$

$$\implies w_t A_t = \frac{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}} C_{1,t} \tag{3.12}$$

$$\implies C_{1,t} = \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}} w_t A_t \tag{3.13}$$

Define savings rate $s(r)$ as

$$s(r_{t+1}) = \frac{w_t A_t - C_{1,t}}{w_t A_t} = \frac{(1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}}, \tag{3.14}$$

which simplifies our consumption function:

$$C_{1,t} = [1 - s(r_{t+1})] w_t A_t. \tag{3.15}$$

3.3 Dynamics of model

We have for capital that

$$K_{t+1} = (w_t A_t - C_{1,t}) L_t = [s(r_{t+1}) w_t A_t] L_t \tag{3.16}$$

$$\implies k_{t+1} = \frac{K_{t+1}}{A_{t+1}L_{t+1}} = s(r_{t+1}) w_t \frac{A_t}{A_{t+1}}\frac{L_t}{L_{t+1}} = \frac{s(f'(k_{t+1}))}{(1+g)(1+n)}[f(k_t) - k_t f'(k_t)] \tag{3.17}$$

where $r_{t+1} = f'(k_{t+1})$ and $w(t) = f(k_t) - k_t f'(k_t)$.

On the balanced growth path, we have that $k_{t+1} = k_t = k^*$, where capital per effective labor becomes **globally stable across periods**. Consider the special case with Cobb-Douglas production $f(k) = k^\alpha$ and $\theta = 1$. We then have

$$s(r_{t+1}) = \frac{1}{2+\rho}, \quad k_{t+1} = \frac{(1-\alpha)k_t^\alpha}{(1+g)(1+n)(2+\rho)}. \tag{3.18}$$

Effective capital will converge to k^*.

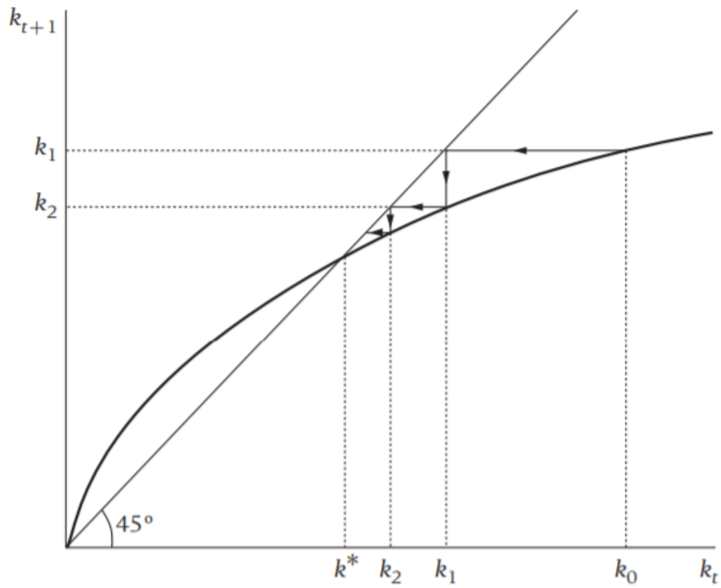


FIGURE 2.10 The dynamics of *k*

3.3.1 Dynamic inefficiency

Dynamic inefficiency, or Pareto inefficiency, is possible in the Diamond model if the equilibrium capital level exceeds the golden rule level $k^* > k_{GR}$. This this case consumption per worker can be increased in every generation if the capital is reduced to $k^* = k_{GR}$.

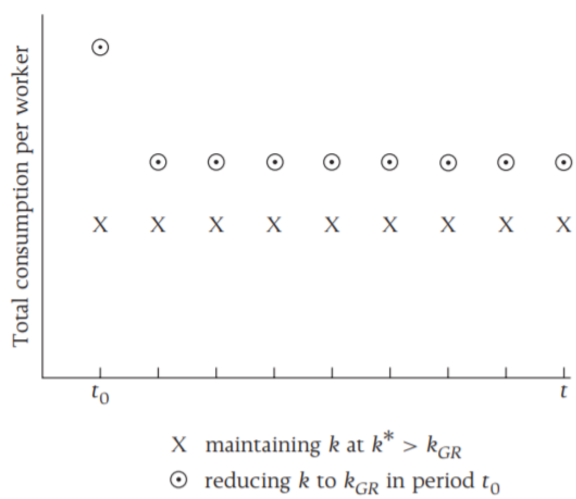


FIGURE 2.13 How reducing *k* to the golden-rule level affects the path of consumption per worker

For RCK, dynamic inefficiency is not possible as the equilibrium capital level is always less than the golden rule level (see section 2.4). There would be a trade off in consumption between generations if BGP capital level was raised to golden rule.

Chapter 4

Endogenous Growth Model

Technology growth $\dot{A}$ is **endogenous** instead of being set exogenously with $\frac{\dot{A}}{A} = g$ in previous models.

Capital $K(t)$ and effective labor $A(t)L(t)$ are production factors. Agents allocate factors between productions of output $Y(t)$ and knowledge growth $\dot{A}(t)$:

$$Y(t) = [(1-a_K)K(t)]^\alpha [A(t)(1-a_L)L(t)]^{1-\alpha}, \quad \alpha \in [0,1) \tag{4.1}$$

$$\dot{A}(t) = B[a_K K(t)]^\beta [a_L L(t)]^\gamma A(t)^\theta, \quad B > 0, \quad \beta, \gamma \geq 0 \tag{4.2}$$

where a_K, a_L are shares of capital and labor used to knowledge technology growth. B is a "shift parameter" for knowledge production growth. θ is a productivity factor of technology growth.

Capital grows with savings without depreciation, and growth rate for labor (or population) remains exogenous:

$$\dot{K}(t) = sY(t), \quad \frac{\dot{L}(t)}{L(t)} = n \geq 0, \tag{4.3}$$

where s is the savings rate.

4.1 Dynamics of Capital

Let $g_K(t) = \frac{\dot{K}(t)}{K(t)}$. We have that

$$\begin{aligned}
\dot{K} = sY &= s(1-a_K)^\alpha (1-a_L)^{1-\alpha} K^\alpha (AL)^{1-\alpha} && (4.4)\\
\implies \frac{\dot{K}}{K} = g_K(t) &= s(1-a_K)^\alpha (1-a_L)^{1-\alpha} \frac{AL}{K}^{1-\alpha} && (4.5)\\
\implies \frac{\dot{g}_K(t)}{g_K(t)} &= (1-\alpha)\left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L} - \frac{\dot{K}}{K}\right) && (4.6)\\
&= (1-\alpha)\left(g_A(t) + n - g_K(t)\right). && (4.7)
\end{aligned}$$

Then on the **balanced growth path (BGP)**, constant $g_K(t)$ implies $\dot{g}_K(t) = 0$, then we have

$$\frac{\dot{g}_K}{g_K} = 0 \implies g_K^* = g_A^* + n. \tag{4.8}$$

4.2 Dynamics of Knowledge

Let $g_A(t) = \frac{\dot{A}(t)}{A(t)}$. Then we have

$$\frac{\dot{A}}{A} = g_A(t) = B[a_K K(t)]^{\beta}[a_L L(t)]^{\gamma} A(t)^{\theta-1} \tag{4.9}$$

$$\implies \frac{\dot{g}_A(t)}{g_A(t)} = \beta\frac{\dot{K}}{K} + \gamma\frac{\dot{L}}{L} + (\theta-1)\frac{\dot{A}}{A} \tag{4.10}$$

$$= \beta g_K + \gamma n + (\theta-1)g_A. \tag{4.11}$$

Then on the BGP, constant g_A implies $\dot{g}_A = 0$, when

$$\frac{\dot{g}_A}{g_A} = 0 \implies (1-\theta)g_A^* = \beta g_K^* + \gamma n \tag{4.12}$$

$$= \beta(g_A^* + n) + \gamma n \tag{4.13}$$

$$\implies g_A^* = \frac{\beta+\gamma}{1-(\theta+\beta)}n, \quad g_K^* = \frac{(1-\theta)+\gamma}{1-(\theta+\beta)}n. \tag{4.14}$$

Capital and knowledge will always return to constant growth set by the exogenous parameters.

4.3 Illustration of BGP

Assume simpler production function with only labor where $\alpha, \beta = 0$, then we have

$$Y = A(1-a_L)L, \quad \dot{A} = Ba_L L^{\gamma} A^{\theta}, \tag{4.15}$$

$$\dot{g}_A = \gamma n g_A + (\theta-1)g_A^2, \quad \dot{g}_A = 0 \implies g_A^* = \frac{\gamma}{1-\theta}n, \tag{4.16}$$

Then we have that

$$\frac{\partial \dot{g}_A}{\partial g_A} = \gamma n + 2(\theta-1)g_A, \quad \frac{\partial^2 \dot{g}_A}{\partial g_A^2} = 2(\theta-1) \tag{4.17}$$

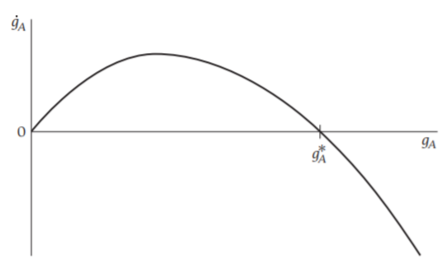


FIGURE 3.1 The dynamics of the growth rate of knowledge when $\theta < 1$

Chapter 5

Consumption

5.1 Permanent income hypothesis (PIH)

Individual lives for T periods, lifetime utility is given by

$$U = \sum_{t=0}^{T} \frac{u(C_t)}{(1+\rho)^t}, \quad u'(\cdot) > 0, \quad u'(\cdot) < 0, \tag{5.1}$$

subject to the constraint

$$\sum_{t=0}^{T} \frac{C_0}{\prod_{s=1}^{t}(1+r_s)} \leq A_0 + \sum_{t=1}^{T} \frac{Y_t}{\prod_{s=0}^{t}(1+r_s)}, \tag{5.2}$$

where A_0 and ρ are initial wealth and time discount rate, and C_t, Y_t, and r_t are consumption, income, and interest rate at time t. Form the Lagrangian

$$\mathcal{L}\left(\{C_t\}_{t=0}^{T}, \lambda\right) = \sum_{t=0}^{T} \frac{u(C_t)}{(1+\rho)^t} + \lambda \left(A_0 + \sum_{t=0}^{T} \frac{Y_t - C_t}{\prod_{s=0}^{t}(1+r_s)} \right), \tag{5.3}$$

Then with FOC with respect to consumption C_t in any period t, we have the Euler equation

$$\frac{\partial \mathcal{L}}{\partial C_t} = \frac{u'(C_t^*)}{(1+\rho)^t} - \frac{\lambda}{\prod_{s=0}^{t}(1+r_s)} \tag{5.4}$$

$$\implies u'(C_t^*) = \lambda \frac{(1+\rho)^t}{\prod_{s=0}^{t}(1+r_s)} \implies \frac{u'(C_t)}{u'(C_{t+1})} = \frac{1+r_{t+1}}{1+\rho}. \tag{5.5}$$

Assume $r = \rho$ for all periods, then we have that

$$u'(C_t) = \frac{1+r_{t+1}}{1+\rho} u'(C_{t+1}) = u'(C_{t+1}) \implies C_t^* = C_{t+1}^* = \bar{C}. \tag{5.6}$$

From the budget constraint (5.2) (and assuming no free disposal), we have that

$$\sum_{t=1}^{T} C_t^* = \sum_{t=1}^{T} \bar{C} = T\bar{C} = A_0 + \sum_{t=1}^{T} Y_t \tag{5.7}$$

$$\implies C_t^* = \bar{C} = \frac{1}{T}\left(A_0 + \sum_{s=1}^{T} Y_s \right) = \frac{1}{T} \sum_{s=1}^{T} Y_s + \frac{A_0}{T}. \tag{5.8}$$

This implies optimal consumption is not determined by income in any period t but only by the total (or average) income across time. This is the **permanent-income hypothesis.** Individual savings is then

$$S_t = Y_t - C_t = Y_t - \frac{1}{T}\left(A_0 + \sum_{s=1}^{T} Y_s\right) \tag{5.9}$$

$$= \left(Y_t - \frac{1}{T}\sum_{s=1}^{T} Y_s\right) - \frac{A_0}{T}. \tag{5.10}$$

5.2 Income and consumption fluctuations

Income Y_{it} for individual i at time t can be represented as

$$Y_{it} = Y_i^P + Y_{it}^T \quad s.t. \quad \sum_{t=1}^{T} Y_{it}^T = 0, \quad \text{Cov}\left(Y_{it}^T, Y_i^P\right) = 0. \tag{5.11}$$

where Y_i^P is the average across time or **permanent income**, and $Y_{it}^T = Y_{it} - Y_i^P$ is the **transitory income**. Assume exogenous, unexpected fluctuations in consumption. Then individual i maximizes lifetime utility

$$U_i = \sum_{t=1}^{T} u\left(C_{it} - e_{it}\right) \quad s.t. \quad \sum_{t=1}^{T}\left(C_{it} - e_{it}\right) \leq A_0 + \sum_{t=1}^{T} Y_t, \tag{5.12}$$

where e_{it} is the **transitory consumption** (sort of like a "consumption shock") with $\sum_{t=1}^{T} e_{it} = 0$. Then forming the Lagrangian, we have that

$$\mathcal{L} = \sum_{t=1}^{T} u\left(C_{it} - e_{it}\right) + \lambda\left[A_0 + \sum_{t=1}^{T} Y_t - \sum_{t=1}^{T}\left(C_{it} - e_{it}\right)\right] \tag{5.13}$$

$$\frac{\partial \mathcal{L}}{\partial C_{it}} = u'\left(C_{it}^* - e_{it}\right) - \lambda = 0 \implies u'\left(C_{it}^* - e_{it}\right) = \lambda, \tag{5.14}$$

which implies $C_{it}^* - e_{it} = \bar{C}_e$ is constant across time also. Then with the budget constraint in (5.12), we have that

$$\sum_{t=1}^{T}\left(C_{it} - e_{it}\right) = A_0 + \sum_{t=1}^{T} Y_{it} \tag{5.15}$$

$$\implies \sum_{t=1}^{T} C_e = A_0 + \sum_{t=1}^{T}(Y_i^P + Y_{it}^T) \tag{5.16}$$

$$= A_0 + \sum_{t=1}^{T} Y_i^P + \sum_{t=1}^{T} Y_{it}^T \tag{5.17}$$

$$\implies \sum_{t=1}^{T} \bar{C}_e = T\bar{C}_e = A_0 + T \cdot Y_i^P. \tag{5.18}$$

Assuming $A_0 = 0$, we have that

$$T\bar{C}_e = T \cdot Y_i^P \tag{5.19}$$

$$\implies \bar{C}_e = Y_i^P \tag{5.20}$$

$$\implies C_{it} - e_{it} = Y_i^P, \quad t = 1, ..., T \tag{5.21}$$

Rearranging, we have the consumption function

$$C_{it} = Y_i^P + e_{it}, \quad t = 1, ..., T. \tag{5.22}$$

5.3 Cross-sectional regression

Running an cross-section OLS regression on C_{it} with Y_{it}, we have

$$C_{it} = a + bY_{it} + u_{it} \tag{5.23}$$

where with permanent income (5.11) and consumption (5.22), we have

$$\mathrm{Cov}\,(x, y) = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y}) \tag{5.24}$$

$$\mathrm{Var}\,(y) = \frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{y})^2 \tag{5.25}$$

$$\hat{b} = \frac{\mathrm{Cov}\,(Y_{it}, C_{it})}{\mathrm{Var}\,(Y_{it})} = \frac{\mathrm{Cov}\left(Y_i^P + Y_{it}^T, Y_i^P + e_{it}\right)}{\mathrm{Var}\left(Y_i^P + Y_{it}^T\right)} = \frac{\mathrm{Var}\left(Y_i^P\right)}{\mathrm{Var}\left(Y_i^P\right) + \mathrm{Var}\left(Y_{it}^T\right)} \tag{5.26}$$

and

$$\hat{a} = \mathbb{E}\left[C_{it}\right] - \hat{b}\mathbb{E}\left[Y_{it}\right] = \left(1 - \hat{b}\right)\mathbb{E}\left[Y_i^P\right]. \tag{5.27}$$

5.4 Precautionary savings

Two reasons why PIH might fail are

- **precautionary savings**, which is a result of risk averse utility, and
- **liquidity constraints**, which imposes an additional constraint on consumer's borrowing.

Precautionary savings occurs in the presence of uncertain income and where marginal utility $u'(C_t) > 0$ is decreasing and convex, meaning that

$$u''(C_t) < 0, \quad u'''(C_t) > 0. \tag{5.28}$$

CRRA utility with $\theta > 0$ has such properties:

$$u(C_t) = \frac{C_t^{1-\theta}}{1-\theta}, \quad u'(C_t) = C_t^{-\theta} > 0, \tag{5.29}$$

$$u''(C_t) = -\theta C_t^{-\theta-1} < 0, \quad u''(C_t) = (\theta^2 + \theta)C_t^{-\theta-2} > 0. \tag{5.30}$$

Assume consumer has certain income in period 1 Y_1 and period 2 $Y_2 = \bar{Y}_2$. Assume zero discount and interest rates, then consumer maximizes

$$U = u(C_1) + u(C_2) \quad s.t. \quad C_1 + C_2 = Y_1 + Y_2. \tag{5.31}$$

Euler equation implies that consumption is the same in both periods:

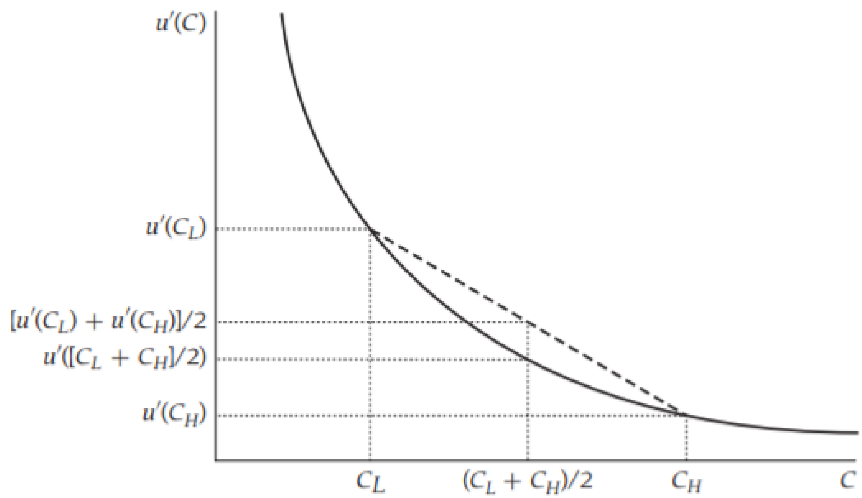
$$\frac{u'(C_1)}{u'(C_2)} = 1 \implies C_1 = C_2 = \frac{Y_1 + \bar{Y}_2}{2}. \tag{5.32}$$

Let period 2 income be uncertain, either high with probability p or low, and $E[Y_2] = pY_H + (1-p)Y_L = \bar{Y}_2$. Then with the Euler equation, we have at the optimal consumption choices that

$$u'(C_1) = \mathbb{E}\left[u'(C_2)\right] \geq u'(\mathbb{E}\left[C_2\right]) \tag{5.33}$$

$$\implies u'(C_1) \geq u'(\mathbb{E}\left[C_2\right]) \implies \underbrace{C_1 \leq \mathbb{E}\left[C_2\right]}_{\text{since } u''(\cdot)<0}. \tag{5.34}$$

$\mathbb{E}\left[u'(C_2)\right] \leq u'(\mathbb{E}\left[C_2\right])$ is due to the convexity of $u'(\cdot)$:



(a)

5.5 Random walk hypothesis

Assume wage is uncertain, and consumer maximizes expected utility

$$\mathbb{E}\left[U\right] = \mathbb{E}\left[\sum_{t=0}^{T}\left(C_t - \frac{a}{2}C_t^2\right)\right], \quad a > 0, \tag{5.35}$$

$$s.t. \quad \sum_{t=0}^{T}\mathbb{E}\left[C_t\right] \leq \mathbb{E}\left[A_0 + \sum_{t=0}^{T} Y_t\right]. \tag{5.36}$$

Form Lagrangian

$$\mathbb{E}\left[\sum_{t=0}^{T}\left(C_t - \frac{a}{2}C_t^2\right)\right] + \lambda\mathbb{E}\left[A_0 + \sum_{t=0}^{T}(Y_t - C_t),\right] \tag{5.37}$$

With the FOC with respect to C_t, we have

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \implies \mathbb{E}\left[1 + aC_t\right] = \lambda \implies \mathbb{E}\left[1 + aC_t\right] = \mathbb{E}\left[1 + aC_{t+1}\right] \tag{5.38}$$

$$\implies C_t = \mathbb{E}_t\left[C_{t+1}\right] = C_{t+1} - e_{t+1}, \quad \mathbb{E}_t\left[e_{t+1}\right] = 0 \tag{5.39}$$

$$\implies C_{t+1} = C_t + e_{t+1}, \tag{5.40}$$

implying consumption follows a random walk.

Chapter 6

Real Business Cycle Model

6.1 Stylized facts

Business-Cycle Facts

- Fluctuations do not exhibit any simple regular or cyclical pattern
- Output declines and their patterns vary considerably in size and spacing
- Fluctuations are distributed very unevenly over the components of output
- Output growth is distributed roughly symmetrically around its mean, but periods of extremely low growth quickly followed by extremely high growth are much more common than periods exhibiting the reverse pattern
- During recessions:
  - employment falls and unemployment rises
  - length of the average workweek falls
  - productivity output per worker-hour almost always declines
  - real wage tends to fall slightly

6.2 Production with real shocks

Large number of identical competitive (price-taking) firms produce Y_t in period t,

$$\begin{aligned} Y_t &= K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}, \quad \alpha \in (0,1), && (6.1) \\ &= I_t + C_t + G_t && (6.2) \end{aligned}$$

where capital K_t, technology A_t, and labor L_t are production factors. Output Y_t is allocated between investment I_t, household consumption C_t, and government spending G_t. **Capital** K_t, with depreciation rate δ, follows the law of motion

$$\begin{aligned} K_{t+1} &= K_t - \delta K_t + I_t && (6.3) \\ &= (1-\delta)K_t + (Y_t - C_t - G_t) && (6.4) \end{aligned}$$

6.2.1 Real Shocks

Technology follows the law of motion

$$\ln(A_t) = \bar{A} + gt + \tilde{A}_t \tag{6.5}$$

$$\implies A_t = e^{\bar{A}+\tilde{A}_t}e^{gt}, \quad \frac{\dot{A}_t}{A_t} = g + \frac{\dot{\tilde{A}}_t}{\tilde{A}_t}, \tag{6.6}$$

where $\tilde{A}_t$ is a AR(1) process "technology shock"

$$\tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{A,t}, \quad \rho_A \in (-1,1), \quad \mathbb{E}\left[\varepsilon_{A,t}\right] = 0. \tag{6.7}$$

and $\varepsilon_{A,t}$ are independently and identically distributed (i.i.d.) across periods.

Government spending is financed by lump-sum taxes on production that equal to spending each period (so taxes do not impact model[1]), and follows the law of motion

$$\ln(G_t) = \bar{G} + (n+g)t + \tilde{G}_t, \tag{6.8}$$

$$\tilde{G}_t = \rho_G \tilde{G}_{t-1} + \varepsilon_{G,t}, \quad \rho_G \in (-1,1), \tag{6.9}$$

and $\mathbb{E}\left[\varepsilon_{G,t}\right] = 0$ and are i.i.d. across time.

Perfectly competitive firms maximize profits in each period,

$$\Pi_t = Y_t - w_t L_t - (r_t + \delta_t)K_t. \tag{6.10}$$

First order conditions imply factor costs are equal to marginal products,

$$\frac{\partial \Pi_t}{\partial L_t} = 0 \implies w_t = \frac{\partial Y_t}{\partial L_t} = (1-\alpha)\left(\frac{K_t}{A_t L_t}\right)^{\alpha}, \tag{6.11}$$

$$\frac{\partial \Pi_t}{\partial K_t} = 0 \implies r_t = \frac{\partial Y_t}{\partial K_t} = \alpha\left(\frac{A_t L_t}{K_t}\right)^{1-\alpha}. \tag{6.12}$$

6.3 Households

Let N_t be the population in H identical households. Member per household is then N_t/H. Population grows exogenously at a constant continuous rate:

$$\ln N_t = \bar{N} + nt \implies N_t = e^{\bar{N}}e^{nt}. \tag{6.13}$$

Per capita consumption and labor is

$$c_t = \frac{C_t}{N_t}, \quad \ell_t = \frac{L_t}{N_t} \in [0,1], \tag{6.14}$$

where labor ℓ_t is normalized to percentage of time endowment, and leisure per capita is defined as $1-\ell_t$. The representative individual's instantaneous utility is given by

$$u_t(c_t, \ell_t) = \ln c_t + b\ln(1-\ell_t), \quad b > 0, \tag{6.15}$$

[1] Government imposes lump-sum taxes $T_t = G_t$, then after tax output is $Y_t - T_t = C_t + I_t + G_t - T_t = C_t + I_t$.

where each period is discounted by the parameter $\rho > n$. Altogether, the **representative household** (with N_t/H members) maximizes expected utility

$$U = \sum_{t=0}^{\infty} e^{-\rho t}\Big(\ln c_t + b\ln(1-\ell_t)\Big)\frac{N_t}{H} \tag{6.16}$$

$$s.t.$$

$$\sum_{t=0}^{\infty} \frac{c_t}{\prod_{s=1}^{t}(1+r_s)}\frac{N_t}{H} \leq \sum_{t=0}^{\infty} \frac{w_t\ell_t}{\prod_{s=1}^{t}(1+r_s)}\frac{N_t}{H}. \tag{6.17}$$

Form the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} e^{-\rho t}\Big(\ln c_t + b\ln(1-\ell_t)\Big)\frac{N_t}{H} + \lambda\left(\sum_{t=0}^{\infty} \frac{w_t\ell_t - c_t}{\prod_{s=1}^{t}(1+r_s)}\frac{N_t}{H}\right) \tag{6.18}$$

and we have the following FOC for each period t that

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \implies \frac{1}{c_t}\frac{N_t}{H}e^{-\rho t} = \frac{\lambda}{\prod_{s=1}^{t}(1+r_s)}\frac{N_t}{H}, \tag{6.19}$$

$$\frac{\partial \mathcal{L}}{\partial \ell_t} = 0 \implies \frac{b}{1-\ell_t}\frac{N_t}{H}e^{-\rho t} = \frac{\lambda w_t}{\prod_{s=1}^{t}(1+r_s)}\frac{N_t}{H}. \tag{6.20}$$

Combining the above for two periods t and $t+1$, we have the Euler equations:

$$\frac{c_{t+1}}{c_t} = e^{-\rho}(1+r_{t+1}), \tag{6.21}$$

$$\frac{1-\ell_{t+1}}{1-\ell_t} = e^{-\rho}(1+r_{t+1})\frac{w_t}{w_{t+1}} \tag{6.22}$$

See Appendix A6.1 for details of derivations. Suppose that

6.4 Uncertainty

By (6.11) and (6.12), future equilibrium wages w_t and interest rates r_t are uncertain, as they depend on A_t which has a stochastic component. At period t, the expected utility continuation utility is

$$U_t = \mathbb{E}_t\left[\sum_{s=0}^{\infty} e^{-\rho(t+s)}\Big(\ln c_{t+s} + b\ln(1-\ell_{t+s})\Big)\frac{N_t}{H}\right] \tag{6.23}$$

$$s.t.\quad \mathbb{E}_t\left[\sum_{t=0}^{\infty} \frac{c_{t+s}}{\prod_{\tau=1}^{t+1}(1+r_\tau)}\frac{N_t}{H}\right] \leq \mathbb{E}_t\left[\sum_{t=0}^{\infty} \frac{w_{t+s}\ell_{t+s}}{\prod_{\tau=1}^{t+1}(1+r_\tau)}\frac{N_t}{H}\right]. \tag{6.24}$$

Consider consumption budget of two consecutive periods t and $t+1$. By (6.24), the household consumption budget is

$$c_t\frac{N_t}{H} + \mathbb{E}_t\left[\frac{c_{t+1}}{1+r_{t+1}}\frac{N_{t+1}}{H}\right] = (c_t - \Delta c)\frac{N_t}{H} + \mathbb{E}_t\left[\frac{(c_{t+1} + (1+r_{t+1})e^{-n}\Delta c)}{(1+r_{t+1})}\frac{N_{t+1}}{H}\right] \tag{6.25}$$

Thus, a change of $-\Delta c$ in c_t should result in a change of $(1+r_{t+1})e^{-n}\Delta c$ in c_{t+1} within the budget constraint. See Appendix A6.2 for details of the derivation.

On the optimal path, allowing changes in c_t and c_{t+1} within the budget constraint, we have the "stochastic" Euler equation:

$$\frac{1}{c_t} = e^{-\rho}\mathbb{E}_t\left[\frac{1+r_{t+1}}{c_{t+1}}\right] \tag{6.26}$$

See Appendix (A6.3) for details.

6.5 Optimal consumption-labor choices

In any period, the one period budget constraint is $dc = \Delta c = w_t \Delta \ell$. Then at the optimal single-period choice we have

$$du = \frac{\partial u}{\partial c_t}dc_t + \frac{\partial u}{\partial \ell_t}d\ell_t = 0 = \frac{1}{c_t}dc_t - \frac{b}{1-\ell_t}d\ell_t \tag{6.27}$$

$$= \frac{1}{c_t}w_t\Delta\ell - \frac{b}{1-\ell_t}\Delta\ell = 0 \tag{6.28}$$

$$\implies \frac{1}{c_t}w_t\Delta\ell = \frac{b}{1-\ell_t}\Delta\ell \tag{6.29}$$

$$\implies \frac{1}{c_t} = \frac{b}{w_t(1-\ell_t)}. \tag{6.30}$$

Which gives us a relationship for the optimal within-period consumption and labor choice. Then with optimal intertemporal consumption choice in (6.26), we have that

$$\frac{1}{c_t} = e^{-\rho}\mathbb{E}_t\left[(1+r_{t+1})\frac{1}{c_{t+1}}\right] \tag{6.31}$$

$$\implies \frac{b}{w_t(1-\ell_t)} = e^{-\rho}\mathbb{E}_t\left[(1+r_{t+1})\frac{b}{w_{t+1}(1-\ell_{t+1})}\right] \tag{6.32}$$

$$\implies \frac{1}{(1-\ell_t)} = e^{-\rho}\mathbb{E}_t\left[\frac{w_t}{w_{t+1}}\frac{1+r_{t+1}}{1-\ell_{t+1}}\right]. \tag{6.33}$$

6.6 Model Results

Baseline model predicts volatility of output, and relative to volatility of consumption, and investment. Labor predictions are less successful in terms of volatility and correlation to GDP per capita.

TABLE 5.4 A calibrated real-business-cycle model versus actual data

	U.S. data	Baseline real-business-cycle model
σ_Y	1.92	1.30
σ_C/σ_Y	0.45	0.31
σ_I/σ_Y	2.78	3.15
σ_L/σ_Y	0.96	0.49
Corr(L,Y/L)	−0.14	0.93

Source: Hansen and Wright (1992).

A6 Appendix

A6.1 Euler equation derivations eqs (6.21) and (6.22)

By FOCs (6.19) and (6.20),

$$\frac{c_{t+1}}{c_t} = \frac{e^{-\rho(t+1)}\left[\prod_{s=1}^{t+1}(1+r_s)\right]/\lambda}{e^{-\rho t}\left[\prod_{s=1}^{t}(1+r_s)\right]/\lambda} \tag{6.34}$$

$$= e^{-\rho}(1+r_{t+1}) \tag{6.35}$$

$$\frac{1-\ell_{t+1}}{1-\ell_t} = \frac{-be^{-\rho(t+1)}\left[\prod_{s=1}^{t+1}(1+r_s)\right]/(\lambda w_{t+1})}{-be^{-\rho t}\left[\prod_{s=1}^{t}(1+r_s)\right]/(\lambda w_t)} \tag{6.36}$$

$$= e^{-\rho}(1+r_{t+1})\frac{w_t}{w_{t+1}} \tag{6.37}$$

A6.2 Inter-temporal budget change derivation eq (6.25)

Note that $\frac{N_t}{N_{t+1}} = e^{-n}$ by (6.13). At period t, the present value of consumption in periods t and $t+1$ is

$$c_t\frac{N_t}{H} + \mathbb{E}_t\left[\frac{c_{t+1}}{1+r_{t+1}}\frac{N_{t+1}}{H}\right] = \left(c_t\frac{N_t}{H} - \Delta c\frac{N_t}{H}\right) + \mathbb{E}_t\left[\frac{c_{t+1}N_{t+1}}{(1+r_{t+1})H}\right] + \Delta c\frac{N_t}{H} \tag{6.38}$$

$$= (c_t - \Delta c)\frac{N_t}{H} + \mathbb{E}_t\left[\frac{c_{t+1}N_{t+1}}{(1+r_{t+1})H} + \Delta c\frac{N_t}{H}\right] \tag{6.39}$$

$$= (c_t - \Delta c)\frac{N_t}{H} + \mathbb{E}_t\left[\frac{c_{t+1}N_{t+1} + (1+r_{t+1})\Delta c N_t}{(1+r_{t+1})H}\right] \tag{6.40}$$

$$= (c_t - \Delta c)\frac{N_t}{H} + \mathbb{E}_t\left[\frac{\left(c_{t+1} + (1+r_{t+1})\frac{N_t}{N_{t+1}}\Delta c\right)}{(1+r_{t+1})}\frac{N_{t+1}}{H}\right] \tag{6.41}$$

$$= (c_t - \Delta c)\frac{N_t}{H} + \mathbb{E}_t\left[\frac{(c_{t+1} + (1+r_{t+1})e^{-n}\Delta c)}{(1+r_{t+1})}\frac{N_{t+1}}{H}\right] \tag{6.42}$$

A6.3 Stochastic Euler equation derivation eq (6.26)

By (6.25), a change of $-\Delta c$ in c_t results in a change of $(1+r_{t+1})e^{-n}\Delta c$ in c_{t+1} within the budget constraint. On the optimal path, utility remains constant with a slight shift in c_t within the budget constraint. Then we have that

$$dU_t = 0 = \frac{\partial U_t}{\partial c_t} dc_t + \frac{\partial U_t}{\partial c_{t+1}} dc_{t+1} \tag{6.43}$$

$$= \frac{1}{c_t} \frac{N_t}{H} e^{-\rho t} dc_t + \mathbb{E}_t \left[\frac{1}{c_{t+1}} \frac{N_{t+1}}{H} e^{-\rho(t+1)} dc_{t+1} \right] \tag{6.44}$$

$$= \frac{1}{c_t} \frac{N_t}{H} e^{-\rho t} \Delta c + \mathbb{E}_t \left[\frac{1}{c_{t+1}} \frac{N_{t+1}}{H} e^{-\rho(t+1)} (-(1 + r_{t+1}) e^{-n} \Delta c) \right] \tag{6.45}$$

$$\implies \frac{1}{c_t} \frac{N_t}{H} e^{-\rho t} \Delta c = \mathbb{E}_t \left[\frac{1 + r_{t+1}}{c_{t+1}} \right] \frac{N_{t+1}}{H} e^{-\rho(t+1)} e^{-n} \Delta c \tag{6.46}$$

$$\implies \frac{1}{c_t} = \mathbb{E}_t \left[\frac{1 + r_{t+1}}{c_{t+1}} \right] \frac{N_{t+1}}{N_t} e^{-n} e^{-\rho} = \mathbb{E}_t \left[\frac{1 + r_{t+1}}{c_{t+1}} \right] e^{-\rho}. \tag{6.47}$$

Chapter 7

Nominal Rigidity

7.1 IS-LM curve with fixed prices

Assume **firms** produce output using only labor,

$$Y = F(L), \quad F'(\cdot) > 0, \quad F''(\cdot) \leq 0. \tag{7.1}$$

All output is consumed $Y_t = C_t$ since no capital or government spending. Household gains utility from holding money, and maximizes

$$\mathcal{U} = \sum_{t=0}^{\infty} \beta^t \left[U(C_t) + \Gamma\left(\frac{M_t}{P_t}\right) - V(L_t) \right], \tag{7.2}$$

$$s.t. \quad \sum_{t=0}^{\infty} \frac{C_t}{\prod_{s=0}^{t}(1+r_s)} \leq \sum_{t=0}^{\infty} \frac{W_t L_t}{\prod_{s=0}^{t}(1+r_s)}, \tag{7.3}$$

$$U'(\cdot),\ \Gamma'(\cdot) > 0, \quad U''(\cdot),\ \Gamma''(\cdot) < 0, \quad V'(\cdot),\ V''(\cdot) > 0, \tag{7.4}$$

where C_t is consumption, L_t is labor, W_t is wages, M_t is money holdings, P_t is fixed aggregate price, $U(\cdot), \Gamma(\cdot)$ are increasing concave functions, and $V(\cdot)$ is increasing and convex.
We call M_t/P_t the **real money demand**. The **real interest rate** is defined as

$$1 + r_t = \frac{1+i}{1+\pi_t}, \quad 1 + \pi = \frac{P_{t+1}}{P_t} \tag{7.5}$$

where i_t is **nominal interest** and π_t is the **inflation rate**. Household holds wealth in two assets: money M_t which pays no interest, and bonds which pay i_t. Evolution of wealth A is then

$$A_{t+1} = M_t + (1+i_t)(A_t + W_t L_t - P_t C_t - M_t). \tag{7.6}$$

where C_t is consumption, L_t is labor, and W_t is wages. Assume CRRA utility for $U(\cdot)$ and $\Gamma(\cdot)$:

$$U(C_t) = \frac{C_t^{1-\theta}}{1-\theta}, \quad \Gamma\left(\frac{M_t}{P_t}\right) = \frac{(M_t/P_t)^{1-\chi}}{1-\chi}, \quad \theta,\ \chi > 0. \tag{7.7}$$

7.1.1 Optimal intertemporal consumption choice

Maximizing consumption, Euler equation is

$$C_t^{-\theta} = \beta(1+r_t)C_{t+1}^{-\theta} \tag{7.8}$$

See appendix (A7.1) for calculation. Taking logs, we have that

$$\ln C_t = -\frac{\beta + \ln(1 + r_{t+1})}{\theta} + \ln C_{t+1} \tag{7.9}$$

With $Y_t = C_t$ and taking $r \simeq \ln(1 + r)$ as equal for small interest rates. Then

$$\ln Y_t = a + \ln Y_{t+1} - \frac{r_t}{\theta}, \quad a = -\frac{\ln \beta}{\theta} \tag{7.10}$$

Equation (7.10) is the **New Keynesian IS (Investment-Savings) Curve**, which implies an inverse relationship between output Y_t and real interest rate r_t.[1]

7.1.2 Optimal real money demand and consumption choice

From the wealth evolution (7.6) and the household optimization problem (7.2), optimal money holdings is given by the condition

$$\Gamma'\left(\frac{M_t}{P_t}\right) = \frac{i_t}{1 + i_t} U'(C_t) \tag{7.11}$$

Then with CRRA utility for $\Gamma(\cdot)$ and $C_t = Y_t$, we have

$$\frac{M_t}{P_t} = Y_t^{\theta/\chi}\left(\frac{1 + i_t}{i_t}\right)^{1/\chi}, \tag{7.12}$$

which gives us the is the **LM (liquidity-money supply) curve**. See calculations in (A7.2). Note that nominal interest rate $i_t \simeq r_t + \pi_t$ for small values. Then in the LM curve, Y_t is directly proportional to r_t (can show that $\frac{\partial M_t/P_t}{\partial r_t} > 0$). We also have that

7.1.3 Optimal labor choices

Optimizing between within-period consumption and labor, we have that

$$C_t^{-\theta}\frac{W_t}{P_t} = V'(L_t), \quad V''(\cdot) > 0. \tag{7.13}$$

Optimal intertemporal labor choices follows:

$$\frac{V'(L_t)}{V'(L_{t+1})} = \frac{W_t}{W_{t+1}}\beta(1 + r_{t+1}). \tag{7.14}$$

7.2 Nominal rigidities

In absence of nominal rigidity or imperfections, a change in money supply leads to change in prices and wages with real wages remaining the same:

$$\frac{M\uparrow}{P\uparrow} = Y^{\theta/\chi}\left(\frac{1+i}{i}\right)^{1/\chi} \implies C^{-\theta}\frac{W\uparrow}{P\uparrow} = V'(L). \tag{7.15}$$

However, if prices are fixed $P = \bar{P}$ for all periods, Then by Figure 6.2 we have

$$\frac{M\uparrow}{\bar{P}} = Y^{\theta/\chi}\uparrow\left(\frac{1+i}{i}\right)^{1/\chi} \tag{7.16}$$

This movement along the NKIS curve is a demand effect. Why do firms supply additional output? Consider four cases.

[1] Since $\frac{\partial \ln Y_t}{\partial r_t} = \frac{1}{Y_t}\frac{\partial Y_t}{\partial r_t} = -\frac{1}{\theta} \implies \frac{\partial Y_t}{\partial r_t} = -\frac{Y_t}{\theta} < 0$.

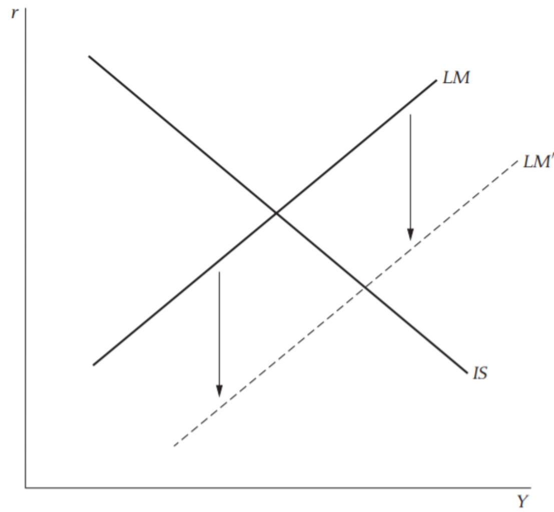


FIGURE 6.2 The effects of a temporary increase in the money supply with completely fixed prices

Case 1: Keynes's General Theory

Goods market competitive, but rigidities in labor market.

Assume flexible prices and wages are fixed above natural level. Output is a function of labor, and firms maximize $F(L) - \frac{W}{P}L$.

$$W = \bar{W}, \quad Y = F(L) \implies F'(L) = \frac{\bar{W}}{P}, \quad F''(\cdot) \leq 0. \tag{7.17}$$

With wages above natural level, we have equilibrium at point E in Figure 6.3.

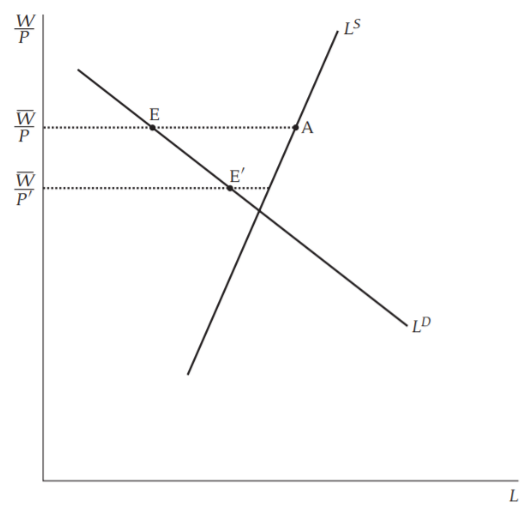


FIGURE 6.3 The labor market with sticky wages, flexible prices, and a competitive goods market

Then with a money supply increase, good prices increase, real wages fall, and employment rises to E' in Figure 6.3:

$$\frac{M\uparrow}{P\uparrow} = Y^{\theta/\chi}\left(\frac{1+i}{i}\right)^{1/\chi} \implies \frac{\bar{W}}{P\uparrow} = F'(L)\downarrow,\ F''(L) \leq 0 \implies F'(L\uparrow) \tag{7.18}$$

$$\implies Y\uparrow = F(L\uparrow). \tag{7.19}$$

Case 2: Sticky prices, flexible wages, competitive labor market

Labor market competitive, but rigidities in goods market.

Assume prices fixed, wages flexible. Workers maximize within-period consumption labor trade off as in (7.13). In market equilibrium, $C = Y = F(L)$. Then we have

$$\frac{W}{\bar{P}} = F(L)^{\theta}V'(L). \tag{7.20}$$

Then if money supply increases with fixed prices, goods demand increases, labor demand increases to meet the goods demand, and wages rise, illustrated in Figure 6.4:

$$\frac{M\uparrow}{\bar{P}} = Y^{\theta/\chi}\uparrow\left(\frac{1+i}{i}\right)^{1/\chi} \implies Y\uparrow = F(L\uparrow),\ F'(\cdot) > 0 \implies F(L)^{\theta}\uparrow V'(L) = \frac{W\uparrow}{\bar{P}}. \tag{7.21}$$

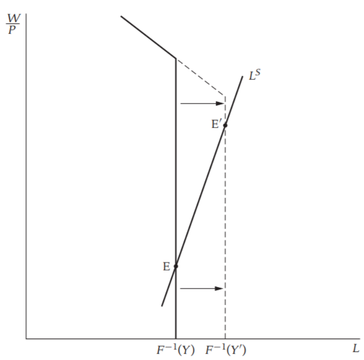


FIGURE 6.4 A competitive labor market when prices are sticky and wages are flexible

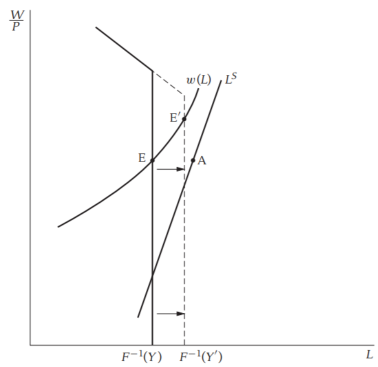


FIGURE 6.5 A non-Walrasian labor market when prices are sticky and nominal wages are flexible

Case 3: Sticky prices, flexible wages, real labor market imperfections

Same setup and mechanisms as Case 2, but with non-Walrasian labor market where real wage remains above level that equates supply and demand (Figure 6.5).

Case 4: Sticky wages, flexible prices, imperfect competition

Assume wages are rigid $W = \bar{W}$. Prices are flexible but imperfectly competitive, given by

$$P = \mu(L)\frac{\bar{W}}{F'(L)} \implies \frac{\bar{W}}{P} = \frac{F'(L)}{\mu(L)} \tag{7.22}$$

where $\mu(L)$ is a markup function. Since $F''(L) \leq 0$, labor demand then depends on $\mu'(\cdot)$, whether markup increases or decreases with labor demand:

$$\frac{M\uparrow}{P\uparrow} = Y^{\theta/\chi}\left(\frac{1+i}{i}\right)^{1/\chi} \implies \frac{\bar{W}}{P\uparrow} = \frac{F'(L)}{\mu(L)}\downarrow \implies F(L) = Y\uparrow\downarrow? \tag{7.23}$$

The markup function $\mu(L)$ is a feature of imperfect competition, which we examine next.

7.3 Imperfect competition

Assume a continuum of firms $i \in [0,1]$ producing unique goods with production functions

$$Y_i = L_i \tag{7.24}$$

Representative household maximizes utility

$$U = C - \frac{1}{\gamma}L^\gamma, \quad \gamma > 1, \tag{7.25}$$

$$C = \left[\int_{i=0}^{1} C_i^{(\eta-1)/\eta} di\right]^{\eta/(\eta-1)}, \quad \eta > 1. \tag{7.26}$$

where C is the CES consumption index. With no investment or government spending, we use $Y = C$ as measure of output in the economy. Assumption for simplicity output is equal to money demand:

$$Y = \frac{M}{P}. \tag{7.27}$$

7.3.1 Households

Let S be the household's total spending budget. Household's Lagrangian is then

$$\mathcal{L} = \left(\int_{i=0}^{1} C_i^{(\eta-1)/\eta} di\right)^{\eta/(\eta-1)} - \frac{1}{\gamma}L^\gamma + \lambda\left(S - \int_{i=0}^{1} P_i C_i di\right). \tag{7.28}$$

Optimizing consumption, we have the FOC

$$\frac{\partial \mathcal{L}}{\partial C_i} = 0 \implies \frac{\eta}{\eta-1}\left(\int_{j=0}^{1} C_j^{(\eta-1)/\eta} dj\right)^{1/(\eta-1)} C_i^{\frac{-1}{\eta}}\frac{\eta-1}{\eta} = \lambda P_i \tag{7.29}$$

$$\implies \left(\int_{j=0}^{1} C_j^{(\eta-1)/\eta} dj\right)^{1/(\eta-1)} C_i^{\frac{-1}{\eta}} = \lambda P_i \tag{7.30}$$

$$\implies C_i = AP_i^{-\eta} \tag{7.31}$$

for some A which is constant across firms i. Plugging (7.31) into the budget constraint, we have

$$S = \int_{j=0}^{1} P_j C_j dj = \int_{j=0}^{1} P_j(AP_j^{-\eta})dj = A\int_{j=0}^{1} P_j^{1-\eta} dj \tag{7.32}$$

$$\implies A = \frac{S}{\int_{j=0}^{1} P_j^{(1-\eta)} dj}. \tag{7.33}$$

Then we have that

$$C = \left[\int_{i=0}^{1} C_i^{(\eta-1)/\eta} di\right]^{\eta/(\eta-1)} = \frac{S}{P}, \quad P = \left[\int_{i=0}^{1} P_i^{(1-\eta)} di\right]^{\frac{1}{1-\eta}}, \tag{7.34}$$

where P is the price index corresponding to consumption utility. See appendix (A7.3) for full derivations. Then we have by (7.31), (7.33), and (7.34) that

$$C_i = AP_i^{-\eta} = \frac{S}{P^{1-\eta}} P_i^{-\eta} = \frac{S}{P}\left(\frac{P_i}{P}\right)^{-\eta} = C\left(\frac{P_i}{P}\right)^{-\eta}. \tag{7.35}$$

7.3.2 Firms

Our production function is $Y_i = L_i$. Firm i maximizes real profits by choosing prices:

$$\max_{P_i} R_i = \frac{P_i Y_i}{P} - \frac{W L_i}{P} = \frac{P_i Y_i}{P} - \frac{W Y_i}{P} \tag{7.36}$$

$$= \frac{P_i Y \left(\frac{P_i}{P}\right)^{-\eta}}{P} - \frac{W Y \left(\frac{P_i}{P}\right)^{-\eta}}{P} \tag{7.37}$$

$$= Y\left(\frac{P_i}{P}\right)^{1-\eta} - \frac{WY}{P}\left(\frac{P_i}{P}\right)^{-\eta} \tag{7.38}$$

FOC with respect to $\frac{P_i}{P}$:

$$\frac{\partial R_i}{\partial (P_i/P)} = 0 \implies (\eta-1)Y\left(\frac{P_i}{P}\right)^{-\eta} = \eta \frac{WY}{P}\left(\frac{P_i}{P}\right)^{-\eta-1} \tag{7.39}$$

$$\implies \frac{P_i}{P} = \frac{W}{P}\frac{\eta}{\eta-1}. \tag{7.40}$$

In perfect competition the firm would be a price taker maximizing profit by choosing Y_i, which implies $\frac{P_i}{P} = \frac{W}{P}$, the marginal cost. With imperfect competition, $\frac{\eta}{\eta-1}$ becomes the markup factor over the perfect competition price. Notice that if $\eta \to -\infty$ then $\frac{P_i}{P} \to \frac{W}{P}$.

A7 Appendix

A7.1 CRRA euler utility derivation eq (7.8)

$$\mathcal{L}(C_t, M_t, \lambda) = \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\theta}}{1-\theta} + \frac{(M_t/P_t)^{1-\chi}}{1-\chi} - V(L_t) \right] \tag{7.41}$$

$$+ \lambda \left(\sum_{t=0}^{\infty} \frac{W_t L_t}{\prod_{s=0}^{t}(1+r_s)} - \frac{C_t}{\prod_{s=0}^{t}(1+r_s)} \right). \tag{7.42}$$

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \implies \beta^t C_t^{-\theta} = \frac{\lambda}{\prod_{s=0}^{t}(1+r_s)} \tag{7.43}$$

$$\implies \frac{\beta^t C_t^{-\theta}}{\beta^{t+1} C_{t+1}^{-\theta}} = \frac{\lambda \prod_{s=0}^{t+1}(1+r_s)}{\lambda \prod_{s=0}^{t}(1+r_s)} \tag{7.44}$$

$$\implies C_t^{-\theta} = \beta(1+r_{t+1}) C_{t+1}^{-\theta} \tag{7.45}$$

Change time indexing convention for r_{t+1} to one period prior. r_t is the interest to be paid out at $t+1$. Then the Euler equation is

$$C_t^{-\theta} = \beta(1+r_t) C_{t+1}^{-\theta}. \tag{7.46}$$

A7.2 LM Curve derivation eq (7.12)

Holding wealth A_{t+1} constant, from (7.6) we have the following from a small change ΔM in period t money supply:

$$A_{t+1} = M_t + (1+i_t)(A_t + W_t L_t - P_t C_t - M_t) \tag{7.47}$$

$$= [M_t + \Delta m] + (1+i_t) \left(A_t + W_t L_t - P_t \left[C_t - \frac{i_t}{(1+i_t)} \frac{1}{P_t} \Delta m \right] - [M_t + \Delta m] \right) \tag{7.48}$$

Thus a change of $dM = \Delta m$ in M_t results in a change of $dC = -\frac{i_t}{(1+i_t)} \frac{1}{P_t} \Delta m$ in C_t within the same budget. Then on the optimal path, with $C_t = Y_t$

$$d\mathcal{U} = 0 = \Gamma'(M_t/P_t) \frac{1}{P_t} dM + U'(C_t) dC \tag{7.49}$$

$$= (M_t/P_t)^{-\chi} \frac{1}{P_t} dM + C_t^{-\theta} dC \tag{7.50}$$

$$= (M_t/P_t)^{-\chi} \frac{1}{P_t} \Delta m - C_t^{-\theta} \frac{i_t}{(1+i_t)} \frac{1}{P_t} \Delta m \tag{7.51}$$

$$\implies (M_t/P_t)^{-\chi} = Y_t^{-\theta} \frac{i_t}{1+i_t} \tag{7.52}$$

$$\implies \frac{M_t}{P_t} = Y_t^{\theta/\chi} \left(\frac{1+i_t}{i_t} \right)^{1/\chi}. \tag{7.53}$$

A7.3 Optimal price index derivation eq (7.34)

From (7.31) we have $C_i = AP_i^{-\eta}$, and from (7.33) we have $A = \frac{S}{\int_{j=0}^1 P_j^{(1-\eta)} dj}$. Then with the definition of aggregate consumption C from (7.26), we have

$$C = \left[\int_{i=0}^1 C_i^{(\eta-1)/\eta} di\right]^{\eta/(\eta-1)} = \left[\int_{i=0}^1 (AP_i^{-\eta})^{(\eta-1)/\eta} di\right]^{\eta/(\eta-1)} \tag{7.54}$$

$$= \left[\int_{i=0}^1 A^{(\eta-1)/\eta} P_i^{(1-\eta)} di\right]^{\eta/(\eta-1)} \tag{7.55}$$

$$= \left[\int_{i=0}^1 \left(\frac{S}{\int_{j=0}^1 P_j^{(1-\eta)} dj}\right)^{(\eta-1)/\eta} P_i^{(1-\eta)} di\right]^{\eta/(\eta-1)} \tag{7.56}$$

$$= \frac{S}{\int_{j=0}^1 P_j^{(1-\eta)} dj} \left[\int_{i=0}^1 P_i^{(1-\eta)} di\right]^{\eta/(\eta-1)} \tag{7.57}$$

$$= \frac{S}{\left[\int_{i=0}^1 P_i^{(1-\eta)} di\right]^{\frac{1}{1-\eta}}} = \frac{S}{P}, \quad P \equiv \left[\int_{i=0}^1 P_i^{(1-\eta)} di\right]^{\frac{1}{1-\eta}}. \tag{7.58}$$

Chapter 8

Dynamic Stochastic General Equilibrium Models (DSGE)

8.1 Households

Representative household maximizes utility

$$\mathcal{U} = \sum_{t=0}^{\infty} \beta^t [U(C_t) - V(L_t)] \quad \beta \in (0,1), \tag{8.1}$$

$$s.t. \quad \sum_{t=0}^{\infty} \frac{P_t C_t}{\prod_{s=0}^{t}(1+r_s)} \leq \sum_{t=0}^{\infty} \frac{W_t L_t}{\prod_{s=0}^{t}(1+r_s)} \tag{8.2}$$

where

$$U(C_t) = \frac{C_t^{1-\theta}}{1-\theta}, \quad V(L_t) = \frac{B}{\gamma} L_t^{\gamma}, \quad \theta, B, \gamma > 0 \tag{8.3}$$

$$\implies U'(C_t) = C_t^{-\theta}, \quad V'(L_t) = B L_t^{\gamma-1}. \tag{8.4}$$

Forming Lagrangian, we have

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t [U(C_t) - V(L_t)] + \lambda \left(\sum_{t=0}^{\infty} \frac{W_t L_t}{\prod_{s=0}^{t}(1+r_s)} - \sum_{t=0}^{\infty} \frac{C_t}{\prod_{s=0}^{t}(1+r_s)} \right). \tag{8.5}$$

We have the following from the FOCs:

$$\frac{\partial \mathcal{L}}{\partial L_t} = 0 \implies \beta^t V'(L_t) = \frac{\lambda W_t}{\prod_{s=0}^{t}(1+r_s)}, \tag{8.6}$$

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \implies \beta^t U'(C_t) = \frac{\lambda P_t}{\prod_{s=0}^{t}(1+r_s)}, \tag{8.7}$$

$$\implies \frac{W_t}{P_t} = \frac{V'(L_t)}{U'(C_t)} = \frac{B L_t^{\gamma-1}}{C_t^{-\theta}} = B Y_t^{\theta+\gamma-1} \tag{8.8}$$

since we assume that $Y_t = F(L_t) = L_t = C_t$.

8.2 Firms

Firm producing good i has production function

$$Y_{it} = L_{it} \tag{8.9}$$

and by (7.35) is met with demand

$$Y_{it} = Y_t \left(\frac{P_{it}}{Pt}\right)^{-\eta}. \tag{8.10}$$

Firm i sets price p_i in period 0. The firm's real profit in period t is

$$R_{it}(P_i) = \frac{P_i}{P} Y_{it} - \frac{W}{P} L_{it} = \frac{P_i}{P} Y_{it} - \frac{W}{P} Y_{it} \tag{8.11}$$

$$= Y_t \left(\frac{P_i}{P_t}\right)^{1-\eta} - \frac{W_t Y_t}{P_t} \left(\frac{P_i}{P_t}\right)^{-\eta} \tag{8.12}$$

8.3 Sticky Prices

Let q_t be the probability that the price the firm sets in period 0 stays the same in period t. Define the stochastic discount factor (SDF):

$$\lambda_t = \beta^t \frac{U'(C_t)}{U'(C_0)}. \tag{8.13}$$

Firm's problem is to choose p_i at time $t = 0$ to maximize A:

$$\max_{P_i} A = \mathbb{E}\left[\sum_{t=0}^{\infty} q_t \lambda_t R_{it}\right] = \mathbb{E}\left[\sum_{t=0}^{\infty} q_t \lambda_t \left(Y_t \left(\frac{P_i}{P_t}\right)^{1-\eta} - Y_t \frac{W_t}{P_t}\left(\frac{P_i}{P_t}\right)^{-\eta}\right)\right] \tag{8.14}$$

$$= \mathbb{E}\left[\sum_{t=0}^{\infty} q_t \lambda_t Y_t P_t^{\eta-1} \left(P_i^{1-\eta} - \frac{W_t}{P_t} P_i^{-\eta}\right)\right] \tag{8.15}$$

$$= \mathbb{E}\left[\sum_{t=0}^{\infty} q_t \lambda_t Y_t P_t^{\eta-1} F(p_i, p_t^*)\right] \tag{8.16}$$

where P_t^* is the price that optimizes profits in period t. For example, by (7.40) $P_{it}^* = P_t^* = W_t \frac{\eta}{\eta-1}$ for all i. $F(p_t, p_t^*)$ is a function of log prices $p_i = \ln P_i$ and $p_t^* = \ln P_t^*$, where $p_i = p_t^*$ maximizes period-t profits. This implies

$$\frac{\partial F(p_t^*, p_t^*)}{\partial p_i} = 0, \quad \frac{\partial^2 F(p_t^*, p_t^*)}{\partial p_i^2} < 0, \tag{8.17}$$

by existence of maximum, assuming $F(p_i, p_t^*)$ is differentiable everywhere. Assume $F(p_i, p_i^*)$ can be approximated around $p_i = p_i^*$ by the second order Taylor approximation:

$$F(p_i, p_i^*) \simeq F(p_t^*, p_t^*) + \frac{\partial F(p_t^*, p_t^*)}{\partial p_i}(p_i - p_t^*) + \frac{\partial^2 F(p_t^*, p_t^*)}{\partial p_i^2}(p_i - p_t^*)^2 \tag{8.18}$$

$$= F(p_t^*, p_t^*) - K(p_i - p_t^*)^2, \quad K > 0 \tag{8.19}$$

$$\implies F(p_t^*, p_t^*) - F(p_i, p_i^*) = K(p_i - p_t^*)^2. \tag{8.20}$$

Assuming fluctuations in $Y_t P_t^{\eta-1}$ is negligible across periods compared to q_t and $F(p_i, p_t^*)$, we can find the optimal $p_i = \ln P_i$ by minimzing the "distance" between $F(p_i, p_t^*)$ and $F(p_t^*, p_t^*)$:

$$p_i^* = \arg\min_{p_i} \sum_{t=0}^{\infty} q_t \beta^t \mathbb{E}\left[F(p_i, p_t^*) - F(p^*, p^*)\right] \tag{8.21}$$

$$\implies p_i^* = \sum_{t=0}^{\infty} \tilde{\omega}_t \mathbb{E}\left[p_t^*\right], \quad \tilde{\omega}_t = \frac{\beta^t q_t}{\sum_{s=0}^{\infty} \beta^s q_s} \tag{8.22}$$

See appendix (A8.1) for calculations. Then combining equilibrium log-wage w_t from (8.8) and each firm's profit-maximizing log-price in each period p_t^* from (7.40), we have

$$w_t = p_t + \ln B + (\theta + \gamma - 1) y_t, \tag{8.23}$$

$$p_t^* = \ln \frac{\eta}{\eta - 1} + w_t \tag{8.24}$$

$$= \phi m_t + (1 - \phi) p_t + c \tag{8.25}$$

where $m_t = y_t + p_t$ is log-nominal GDP and

$$\phi = \theta + \gamma - 1 > 0, \quad c = \ln \frac{\eta}{\eta - 1} + \ln B = 0 \tag{8.26}$$

See appendix (A8.2) for full derivation. Then firm i's optimal price at time t is

$$p_{it}^* = \sum_{s=0}^{\infty} \tilde{\omega}_{t+s} \mathbb{E}\left[p_{t+s}^*\right] = \sum_{s=0}^{\infty} \tilde{\omega}_{t+s} \mathbb{E}\left[\phi m_{t+s} + (1 - \phi) p_{t+s}\right]. \tag{8.27}$$

8.4 Calvo Model and New Keynesian Phillips Curve

Assume in every period, a random fraction $\alpha \in (0, 1]$ of firms can change prices.[1] Average prices p_t and inflation π_t in period t is then

$$p_t = \alpha x_t + (1 - \alpha) p_{t-1} \tag{8.28}$$

$$\implies p_t - p_{t-1} = \pi_t = \alpha (x_t - p_{t-1}) \implies \frac{\pi_t}{\alpha} = x_t - p_{t-1} \tag{8.29}$$

where p_{t-1} is average old price and z_t is new profit-mazimizing prices[2]

$$x_t = \sum_{j=0}^{\infty} \tilde{\omega}_j \mathbb{E}\left[p_{t+j}^*\right], \quad \tilde{\omega}_j = \frac{\beta^j (1 - \alpha)^j}{\sum_{k=0}^{\infty} \beta^k (1 - \alpha)^k} = [1 - \beta(1 - \alpha)] \beta^j (1 - \alpha)^j q_j \tag{8.30}$$

$$x_t = [1 - \beta(1 - \alpha)] \sum_{j=0}^{\infty} \beta^j (1 - \alpha)^j \mathbb{E}\left[p_{t+j}^*\right] \tag{8.31}$$

$$= [1 - \beta(1 - \alpha)] \mathbb{E}\left[p_t^*\right] + \beta(1 - \alpha)[1 - \beta(1 - \alpha)] \sum_{j=0}^{\infty} \beta^j (1 - \alpha)^j \mathbb{E}\left[p_{(t+1)+j}^*\right] \tag{8.32}$$

$$= [1 - \beta(1 - \alpha)] p_t^* + \beta(1 - \alpha) \mathbb{E}\left[x_{t+1}\right] \tag{8.33}$$

[1] Probability that price stays the same after j periods are then $q_j = (1 - \alpha)^j$.

[2] $y = 1 + z + z^2 + z^3 + \ldots = 1 + z(1 + z + z^2 + z^3 + \ldots) = 1 + zy \implies y = \frac{1}{1-z}$. Then letting $z = \beta(1 - \alpha)$, we have $\left(\sum_{k=0}^{\infty} \beta^k (1 - \alpha)^k\right)^{-1} = 1 - \beta(1 - \alpha)$.

With (8.25) we have that $p_t^* - p_t = \theta y_t$, then

$$x_t - p_t = [1 - \beta(1-\alpha)](p_t^* - p_t) + \beta(1-\alpha)\mathbb{E}\left[x_{t+1} - p_t\right] \tag{8.34}$$

$$= [1 - \beta(1-\alpha)](\phi y_t) + \beta(1-\alpha)\mathbb{E}\left[\frac{\pi_{t+1}}{\alpha}\right] \tag{8.35}$$

We also have that

$$x_t - p_t = (x_t - p_{t-1}) - (p_t - p_{t-1}) = \frac{\pi_t}{\alpha} - \pi_t = \pi_t \frac{1-\alpha}{\alpha} \tag{8.36}$$

Then combining the two

$$\implies \pi_t = \frac{\alpha}{1-\alpha}[1 - \beta(1-\alpha)]\phi y_t + \beta\mathbb{E}\left[\pi_{t+1}\right] \tag{8.37}$$

$$= \kappa y_t + \beta\mathbb{E}_t\left[\pi_{t+1}\right], \quad \kappa = \frac{\alpha}{1-\alpha}[1 - \beta(1-\alpha)]\phi, \tag{8.38}$$

which is the **New Keynsian Phillips Curve**, which relates inflation to output and expected inflation.

A8 Appendix

A8.1 Price difference minimization eq (8.22)

Expectation of the square of a random x is as follows

$$\begin{aligned}
\mathbb{E}\left[x^2\right] &= \mathbb{E}\left[([x-\mu_x]+\mu_x)^2\right] && (8.39)\\
&= \mathbb{E}\left[\mu_x^2 + (x-\mu_x)^2 + 2\mu_x(x-\mu_x)\right] && (8.40)\\
&= \mu_x^2 + \mathbb{E}\left[(x-\mu_x)^2\right] + 2\mu_x\mathbb{E}\left[x-\mu_x\right] && (8.41)\\
&= \mathbb{E}\left[x\right]^2 + \mathrm{Var}\left(x\right). && (8.42)
\end{aligned}$$

Then,

$$\begin{aligned}
p_i^* = \arg\min_{p_i} \sum_{t=0}^{\infty} q_t\beta^t \mathbb{E}\left[F(p_i,p_t^*) - F(p^*,p^*)\right] &= \sum_{t=0}^{\infty} q_t\beta^t\mathbb{E}\left[(p_i-p_t^*)^2\right] && (8.43)\\
&= \sum_{t=0}^{\infty} q_t\beta^t\left[(p_i-\mathbb{E}\left[p_t^*\right])^2 + \mathrm{Var}\left(p_t^*\right)\right]. && (8.44)
\end{aligned}$$

Then finding optimal p_i with the FOC:

$$\begin{aligned}
\frac{\partial D}{\partial p_i} = \sum_{t=0}^{\infty} 2\beta^t q_t(p_i - \mathbb{E}\left[p_t^*\right]) = 0 &\implies 2p_i\sum_{s=0}^{\infty}\beta^s q_s = 2\sum_{t=0}^{\infty} q_t\beta^t\mathbb{E}\left[p_t^*\right] && (8.45)\\
&\implies p_i = \frac{\sum_{t=0}^{\infty} q_t\beta^t\mathbb{E}\left[p_t^*\right]}{\sum_{s=0}^{\infty}\beta^s q_s} = \sum_{t=0}^{\infty}\frac{\beta^t q_t}{\sum_{s=0}^{\infty}\beta^s q_s}\mathbb{E}\left[p_t^*\right] && (8.46)\\
&\implies p_i^* = \sum_{t=0}^{\infty}\tilde{\omega}_t\mathbb{E}\left[p_t^*\right], \quad \tilde{\omega}_t = \frac{\beta^t q_t}{\sum_{s=0}^{\infty}\beta^s q_s} && (8.47)
\end{aligned}$$

A8.2 Revenue maximizing price derivation eq (8.25)

From (8.8) we have

$$w_t = p_t + \ln B + (\theta+\gamma-1)y_t, \tag{8.48}$$

and from (7.40) we have each firm's profit-maximizing log-price in each period,

$$\begin{aligned}
\max_{p_{it}} \ln[R_{it}(P_{it})] = p_t^* &= \ln\frac{\eta}{\eta-1} + w_t && (8.49)\\
&= \ln\frac{\eta}{\eta-1} + p_t + \ln B + (\theta+\gamma-1)y_t && (8.50)\\
&= p_t + c + \phi y_t && (8.51)\\
&= p_t + c + \phi y_t + \phi p_t - \phi p_t && (8.52)\\
&= \phi(y_t+p_t) + (1-\phi)p_t + c && (8.53)\\
&= \phi m_t + (1-\phi)p_t, \quad m_t = y_t + p_t, \quad c = 0 && (8.54)
\end{aligned}$$

where $\phi = \theta+\gamma-1$ and $c = \ln\frac{\eta}{\eta-1} + \ln B$. m_t is log nominal GDP $y_t + p_t$.

Chapter 9

Monetary Policy

9.1 Inflation, money growth, interest rates

Model for LM curve in (7.12) implies real money demand is decreasing in nominal interest i and increasing in real income Y. Write demand for real balances

$$\frac{M}{P} = L(i, Y), \quad \frac{\partial L}{\partial i} < 0, \quad \frac{\partial L}{\partial Y} > 0 \tag{9.1}$$

$$\implies P = \frac{M}{L(i, Y)}, \tag{9.2}$$

where M is money stock and P is price level. Real interest is defined as nominal interest minus expected inflation, which gives us the **Fischer identity**:

$$r \equiv i - \pi^e \implies i \equiv r + \pi^e \tag{9.3}$$

$$\implies P = \frac{M}{L(r + \pi^e, Y)}, \tag{9.4}$$

where r is the real interest rate and π^e is expected inflation. Assume $\pi^e = \frac{\dot{P}}{P} = \frac{d \ln P}{dt}$ is actual inflation, and r and Y are constant at $\bar{r}$ and $\bar{Y}$.

$$\ln P = \ln M - \ln L(\bar{r} + \pi^e, \bar{Y}) \implies \frac{d \ln P}{dt} = \frac{\dot{P}}{P} = \pi^e = \frac{\dot{M}}{M} - \frac{\dot{L}}{L} \tag{9.5}$$

Then if there a permanent increase in growth rate of nominal money supply $\frac{d \ln M}{dt} = \frac{\dot{M}}{M}$ at t_0, we have

$$\frac{\dot{M}}{M} \uparrow \implies \frac{\dot{P}}{P} = \pi^e \uparrow \implies i = \bar{r} + \pi^e \uparrow \implies \frac{\dot{L}}{L} \downarrow, \quad \because \frac{\partial L}{\partial i} < 0 \tag{9.6}$$

The first implication is the **Fischer effect**, which shows that inflation affects nominal rate one-for-one $M \uparrow \implies \pi^e \uparrow \implies i \uparrow$ by the Fischer identity and the assumption that inflation does not affect real interest rate.

9.2 Expectation Theory of Term structure

Let i_t^n be the continuously compounded n-period (zero-coupon) interest rates at period t. If future interests rates are certain, in equilibrium (no arbitrage) we must have that the one-period compounded returns are the same:

$$e^{n\cdot i_t^n} = e^{i_t^1} \times e^{i_{t+1}^1} \times ... \times e^{i_{t+n-1}^1} = e^{i_t^1 + i_{t+1}^1 + ... + i_{t+n-1}^1} \tag{9.7}$$
$$\implies n i_t^n = i_t^1 + i_{t+1}^1 + ... + i_{t+n-1}^1. \tag{9.8}$$

With uncertainty about future one term interest rates, in equilibrium we have

$$i_t^n = \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{E}_t\left[i_{t+j}^1\right] = \frac{i_t^1 + \mathbb{E}_t\left[i_{t+1}^1\right] + ... + \mathbb{E}_t\left[i_{t+n-1}^1\right]}{n} + \theta_{nt} \tag{9.9}$$

where i_t^n is the continuously compounded interest rate for n periods starting from period t, and θ_{nt} is the **term premium** for uncertainty of future one-period interest rates i_{t+j}^1.

9.3 Stabilization policy

Assumed welfare function

$$W_t = -c(y_t^* - y_t)^2 - f(\pi_t - \pi^*), \quad c > 0, \quad f(\cdot) > 0 \tag{9.10}$$

where y_t, π_t are **actual output** and inflation and y_t^*, π^* are **target** output and inflation (perhaps Walrasian[1]). Policy maker's goal is to minimize deviations from target.

Consider the New Keynesian Philips curve (NKPC) from (8.38) and the New Keynesian Investment-Savings curve (NKIS) from (7.10)

$$\pi_t = \beta\mathbb{E}_t\left[\pi_{t+1}\right] + \kappa(y_t - y_t^n), \quad \kappa > 0, \quad \beta \in (0,1) \tag{9.11}$$
$$y_t = \mathbb{E}_t\left[y_{t+1}\right] - \frac{r_t}{\theta} + u_t^{IS} \tag{9.12}$$
$$= \mathbb{E}_t\left[y_{t+1}\right] - \frac{1}{\theta}(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - \rho) + u_t^{IS}, \quad e^{-\rho} = \beta, \theta > 0 \tag{9.13}$$

where y_t^n is natural level of output without sticky prices, and

$$y_t^n = \rho_Y y_{t-1}^n + \varepsilon_t^Y, \quad \rho_Y \in (0,1), \tag{9.14}$$
$$u_t^{IS} = \rho_{IS} u_{t-1} + \varepsilon_t^{IS}, \quad \rho_{IS} \in (0,1). \tag{9.15}$$

9.3.1 Divine coincidence

Assume central bank wants $y_t = y_t^n$ and $\pi_t = 0$ for all periods t. This requires

$$\pi_t = \mathbb{E}_t\left[\pi_{t+1}\right] = 0, \tag{9.16}$$
$$y_t = y_t^n, \quad \mathbb{E}\left[y_{t+1}\right] = \mathbb{E}\left[y_{t+1}^n\right], \tag{9.17}$$

which implies that the NKPC (9.11) holds. Then plugging in the conditions above, the optimal nominal interest rate policy is

$$i_t = \rho + \theta\left(\mathbb{E}_t\left[y_{t+1}^n\right] - y_t^n + u_t^{IS}\right) \tag{9.18}$$
$$= r_t^n + \pi_t = r_t^n. \tag{9.19}$$

[1] A Walrasian model is a model of competitive markets without externalities, asymmetric information, missing markets, or other imperfections.

The equilibrium is known as **divine coincidence**, since there is no tradeoff between the inflation and output objectives. However, **sunspot equilibria** are possible with the policy $i_t = r_t^n$. Suppose inflation and output exogenously jump at time t, and are expected return to normal $t+1$ (but still above target levels):

$$\pi_t\uparrow > 0, \quad y_t\uparrow > y_t^n, \quad \mathbb{E}_t\left[\pi_{t+1}\right]\uparrow > 0, \quad \mathbb{E}_t\left[y_{t+1}\right]\uparrow > \mathbb{E}_t\left[y_{t+1}\right]^n, \tag{9.20}$$

where the magnitude of blue > red. If the central bank sets $i_t = r_t^n$, then r_t must fall one for one with the rise in $\mathbb{E}_t\left[\pi_{t+1}\right]$ by Fischer's identity (9.3):

$$\mathbb{E}_t\left[\pi_{t+1}\right]\uparrow \implies i_t = r_t^n = r_t\downarrow + \mathbb{E}_t\left[\pi_{t+1}\right]\uparrow \implies r_t\downarrow. \tag{9.21}$$

Then we may have that

$$y_t\uparrow = \mathbb{E}_t\left[y_{t+1}\right]\uparrow - \frac{1}{\theta}(r_t\downarrow - \rho) + u_t^{IS}, \tag{9.22}$$

$$\pi_t\uparrow = \beta\mathbb{E}_t\left[\pi_{t+1}\right]\uparrow + \kappa(y_t\uparrow - y_t^n), \tag{9.23}$$

where the effects of the arrows balance each other out, resulting in a new equilibrium where $\pi_t > 0$ and $y_t > y_t^n$. To prevent sunspot equilibria, the central bank can adopt an interest rate rule of the form

$$i_t = r_t^n + \phi_\pi\mathbb{E}_t\left[\pi_{t+1}\right] + \phi_y\mathbb{E}_t\left[y_{t+1} - y_{t+1}^n\right], \quad \phi_\pi > 1, \quad \phi_y = 0. \tag{9.24}$$

In this case if expected inflation rises, i_t will rise more than expected inflation, then r_t must rise:

$$i = r_t^n + \phi_\pi\mathbb{E}_t\left[\pi_{t+1}\right]\uparrow = r_t\uparrow + \mathbb{E}_t\left[\pi_{t+1}\right]\uparrow \implies r_t\uparrow \tag{9.25}$$

Then the rise in real interest will counteract against the rise in output y_t, which counteracts the rise in inflation π_t.

9.4 Breaking divine coincidence

The Walrasian output y^* may differ from the natural level y^n in some periods due to fluctuations. Walrasian output is (almost always) higher $y^* > y^n$ since it is the level of a perfectly efficient economy.

If the central bank targets y^*, this implies a possible output gap $y_t - y_t^n = y_t^* - y_t^n > 0$, which with the NKPC would imply inflation. In order to keep inflation π_t at zero, they would have to somehow induce $\mathbb{E}_t\left[\pi_{t+1}\right]$ to be negative, though in the next period the central bank would keep the realized inflation $\pi_{t+1} = 0$:

$$\pi_t = \beta\mathbb{E}_t\left[\pi_{t+1}\right]\downarrow + \kappa(y_t - y_t^n)\uparrow. \tag{9.26}$$

The central bank cannot do this consistently, since that requires agents to be fooled every time. This scenario where the central bank *wants* agents to expect a different inflation rate than its own target rate is called **dynamic inconsistency**.

In general, there is always a tradeoff between a desired output-gap and steady inflation, since if the central bank maintains consistent inflation $\pi = \pi_t = \mathbb{E}_t\left[\pi_{t+1}\right]$, then we have

$$\pi = \beta\pi + \kappa(y_t - y_t^n) \implies \pi = \frac{\kappa}{1-\beta}(y_t - y_t^n), \quad \frac{\kappa}{1-\beta} > 0 \tag{9.27}$$

which suggests the central bank cannot raise output without raising inflation.

To prevent dynamic inconsistency, the central bank commits to a rule so that trying to depart systematically from what they want agents to believe is not feasible.

9.5 Taylor Rule

Interest rate policies cannot be passive (constant rate)–leads to instability with backward looking behaviour (?) and sunspots with forward looking behavior. Interest rate adjustments cannot be ad hoc either, since no way of analyzing nominal rate behaviour or expectations.

Solution: *rules* for short term interest rate, adjusting i_t in a predictable way against economic developments. The **Taylor rule** takes the form

$$i_t = a + \phi_\pi \pi_t + \phi_y (y_t - y_t^n), \quad \phi_\pi,\ \phi_y > 0, \tag{9.28}$$

$$\Longrightarrow\ i_t = r_t^n + \phi_\pi (\pi_t - \pi^*) + \phi_y (y_t - y_t^n), \quad \pi^* = \frac{(r_t^n - a)}{\phi_\pi}. \tag{9.29}$$

The rule says that the central bank should raise the real interest rate above its long-run equilibrium level in response to inflation exceeding its target and to output exceeding its natural rate.

Chapter 10

Fiscal Policy

10.1 Government Budget Constraint

Government's budget constraint is

$$\int_{t=0}^{\infty} e^{-R(t)} G(t)dt \leq -D(0) + \int_{t=0}^{\infty} e^{-R(t)} T(t)dt \tag{10.1}$$

where $G(t)$ is government spending, $T(t)$ is tax income, and government debt $D(t)$ follows law of motion:

$$\dot{D}(t) = G(t) - T(t) + r(t)D(t) \tag{10.2}$$

10.2 Ricardian Equivalence and Household constraints

The representative household is faced with the RCK-model budget constraint with lump-sum taxes. Government debt is also passed to households. Then the household budget constraint is

$$\int_{t=0}^{\infty} e^{-R(t)} C(t)dt \leq K(0) + D(0) + \int_{t=0}^{\infty} e^{-R(t)} W(t)dt - \int_{t=0}^{\infty} e^{-R(t)} T(t)dt \tag{10.3}$$

$$= K(0) + \int_{t=0}^{\infty} e^{-R(t)} W(t)dt - \int_{t=0}^{\infty} e^{-R(t)} G(t)dt \tag{10.4}$$

$$= K(0) + \int_{t=0}^{\infty} e^{-R(t)} [W(t) - G(t)]dt, \tag{10.5}$$

where $C(t)$ is consumption, $W(t)$ is wage income, and $K(0)$ is initial capital. This budget constraint implies taxes $T(t)$ do not affect the household's optimization problem. Only government spending $G(t)$ affects the household behaviour, but not the financing of the spending (i.e. taxes). This result is called **Ricardian equivalence (RE)**. Reasons RE may not hold:

- **Infinite lifetimes:** Turnover in population (i.e. finite lived households budget constraint) may cause the math to break.
- **Permanent Income Hypothesis (PIH)** implies that household consumes the same amount every period adjusted for time and interest discounting, saving or borrowing as needed to reach optimal consumption level. But **liquidity constraints** (i.e. borrowing limit) and **precautionary savings** (i.e. household maintaining minimum savings level) will create period specific budget constraints which make timing of taxes matter.

- **Distortionary taxes** (e.g. sales tax on consumption, progressive income tax) instead of lump-sum taxes can change household consumption/labor choices.

10.3 Tax Smoothing

Assume tax rates do have distortionary effects on consumer income, which in each period is

$$\Gamma_t = Y_t f\left(\frac{T_t}{Y_t}\right), \quad f'(\cdot) > 0, f''(\cdot) > 0 \tag{10.6}$$

Assume government's objective is to minimize distortion subject to some required spending:

$$\min_{T_t} \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} Y_t f\left(\frac{T_t}{Y_t}\right) \quad s.t. \quad D(0) + \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} G(t) \leq \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} T(t) \tag{10.7}$$

Formulate Lagrangian for minimization:

$$\mathcal{L} = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} Y_t f\left(\frac{T_t}{Y_t}\right) + \lambda\left(D(0) + \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} G(t) - \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} T(t)\right) \tag{10.8}$$

Then taking FOC with respect to $T(t)$ at some period t, we have:

$$\frac{\partial \mathcal{L}}{\partial T_t} = 0 \implies \frac{1}{(1+r)^t} Y_t f'\left(\frac{T_t}{Y_t}\right) \frac{1}{Y_t} = -\lambda \frac{1}{(1+r)^t} \tag{10.9}$$

$$\implies f'\left(\frac{T_t}{Y_t}\right) = \lambda \tag{10.10}$$

$$\implies f'\left(\frac{T_t}{Y_t}\right) = f'\left(\frac{T_{t+1}}{Y_{t+1}}\right) \implies \frac{T_t}{Y_t} = \frac{T_{t+1}}{Y_{t+1}}, \tag{10.11}$$

implying optimal to impose same tax rate across all periods **(tax smoothing)**.

10.4 Government Debt Crisis

Government pays interest factor $R = (1+r)$ on debt D, financed by taxes T. Assume T is uniformly distributed in an interval:

$$T \sim \text{Uniform}\left(\mu - X, \mu + X\right), \quad X,\ \mu - X \geq 0 \tag{10.12}$$

Let π be the probability the government defaults on its debt, which is occurs if $T < RD$.
Then probability of default is

$$\pi = P(T < RD) = \begin{cases} 0 & R < \frac{\mu - X}{D} \\ \frac{RD - (\mu - X)}{2X} & R \in \left[\frac{\mu - X}{D}, \frac{\mu + X}{D}\right] \\ 1 & R > \frac{\mu + X}{D} \end{cases} \tag{10.13}$$

where T is the tax income, $F(\cdot)$ is the cumulative distribution function.

If in default, government pays nothing. Assume investors are risk-neutral, and indifferent between expected return and risk-free factor $\bar{R}$, then

$$0 \cdot \pi + R \cdot (1 - \pi) = \bar{R} \implies \pi = \frac{R - \bar{R}}{R}. \tag{10.14}$$

In equilibrium, interest and default probabilities (R, π) are where

$$\frac{R - \bar{R}}{R} = P(T < RD). \tag{10.15}$$

Illustrate examples of equilibria in Figure (10.1) with

$$\mu = 2, \quad X = 0.95, \quad \bar{R} = 1.1, \quad D = 1. \tag{10.16}$$

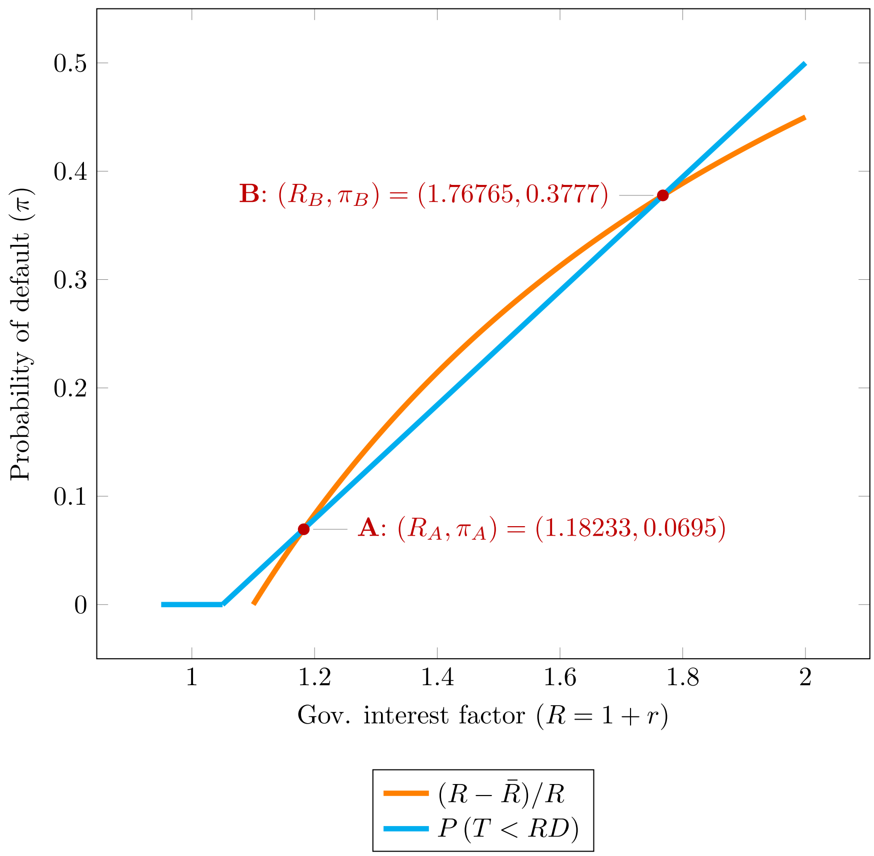


Figure 10.1: (R, π) Equilibria

A third equilibrium exists where $R \to \infty$ and $\pi = 1$, where the investor will not lend at any interest and the government defaults.

Implications

- Intersect **B** is an unstable equilibrium. If investor believes default probability is less than π_B, the interest factor needed for them to invest results in a lower default probability than they expect. Then they may reduce estimate of π, and repeat the process until converging to **A**. Similar process can occur if investor's expectation of π is above B, which through updating will converge to $(R, \pi) = (\infty, 1)$.

- Large differences in fundamentals are not needed for large differences in outcomes. Firstly, due to the convergence mechanism from previous point, expectations about default probability alone can lead to different equilibria. Secondly, small changes in $\bar{R}$ or distribution parameters of tax T can affect equilibria points dramatically.

- Defaults are unexpected, since equilibria are always much less than 1 because of shape of the investor indifference curve.