Part 1

Q1

$$\Delta X_t = \alpha\beta' X_{t-1} + \Gamma_1 \overset{x_{t-1}-x_{t-2}}{\Delta X_{t-1}} + \Gamma_2 \overset{x_{t-2}-x_{t-3}}{\Delta X_{t-2}} + \varepsilon_t$$

$$= (\alpha\beta' + \Gamma_1) X_{t-1} + (\Gamma_2 - \Gamma_1) X_{t-2} - \Gamma_2 X_{t-3} + \varepsilon_t$$

$$= \begin{pmatrix} \alpha\beta' + \Gamma_1 \\ \Gamma_2 - \Gamma_1 \\ -\Gamma_2 \end{pmatrix}' \begin{pmatrix} X_{t-1} \\ X_{t-2} \\ X_{t-3} \end{pmatrix} + \varepsilon_t$$

$$= \Pi' \tilde{X}_{t-1} + \varepsilon_t$$

$$\Rightarrow \Delta \tilde{X}_t = \tilde{\Pi} \tilde{X}_{t-1} + B \varepsilon_t$$

$$= \begin{pmatrix} \Delta X_t \\ \Delta X_{t-1} \\ \Delta X_{t-2} \end{pmatrix} = \begin{pmatrix} \Pi_1 & \Pi_2 & \Pi_3 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \\ X_{t-3} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \varepsilon_t$$

Then $\tilde{\Pi} = \begin{pmatrix} \alpha\beta' + \Gamma_1 & \Gamma_2 - \Gamma_1 & -\Gamma_2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Q2.

$$\tilde{\Pi} = \tilde{\alpha} \tilde{\beta}' \qquad \beta' = \begin{pmatrix} \beta & 0 & 0 \\ \times & \times & 0 \\ 0 & \times & \times \end{pmatrix}$$

$$\begin{pmatrix} \alpha\beta' + \Gamma_1 & \Gamma_2 - \Gamma_1 & -\Gamma_2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} \alpha & \tilde{\alpha}_{12} & \tilde{\alpha}_{13} \\ 0 & \tilde{\alpha}_{22} & 0 \\ 0 & 0 & \tilde{\alpha}_{33} \end{pmatrix} \begin{pmatrix} \beta & 0 & 0 \\ \tilde{\beta}'_{21} & \tilde{\beta}'_{22} & 0 \\ 0 & \tilde{\beta}'_{32} & \tilde{\beta}'_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha\beta + \tilde{\alpha}_{12}\tilde{\beta}'_{21} & \tilde{\alpha}_{12}\tilde{\beta}'_{22} + \tilde{\alpha}_{13} & \tilde{\alpha}_{13}\tilde{\beta}'_{33} \\ \tilde{\alpha}_{22}\tilde{\beta}'_{21} & \tilde{\alpha}_{22}\tilde{\beta}'_{22} & 0 \\ 0 & \tilde{\alpha}_{33}\tilde{\beta}'_{32} & \tilde{\alpha}_{33}\tilde{\beta}'_{33} \end{pmatrix}$$

Then

$$\left.\begin{aligned}
\Gamma_1 &= \tilde{\alpha}_{12}\tilde{\beta}'_{21} = -\tilde{\alpha}_{12}\tilde{\beta}'_{22} \\
\Gamma_2 - \Gamma_1 &= \tilde{\alpha}_{12}\tilde{\beta}'_{22} + \tilde{\alpha}_{13}, \\
\Gamma_2 &= \tilde{\alpha}_{13}\tilde{\beta}'_{33}, \Rightarrow -\tilde{\alpha}_{13}\tilde{\beta}'_{32}
\end{aligned}\right\} \quad \tilde{\alpha}_{12}\tilde{\beta}'_{22} - \tilde{\alpha}_{13}\tilde{\beta}'_{32} = \tilde{\alpha}_{12}\tilde{\beta}'_{22} + \tilde{\alpha}_{13}$$

$$\left.\begin{aligned}
1 &= \tilde{\alpha}_{22}\tilde{\beta}'_{21}, \\
-1 &= \tilde{\alpha}_{22}\tilde{\beta}'_{22},
\end{aligned}\right\} \quad \tilde{\beta}'_{21} = -\tilde{\beta}'_{22} \qquad \Rightarrow \tilde{\beta}'_{32} = -1,$$

$$\tilde{\beta}'_{33} = 1$$

$$\left.\begin{aligned}
1 &= \tilde{\alpha}_{33}\tilde{\beta}'_{32} \\
-1 &= \tilde{\alpha}_{33}\tilde{\beta}'_{33}
\end{aligned}\right\} \Rightarrow \tilde{\beta}'_{32} = -\tilde{\beta}'_{33} \qquad \begin{aligned} \tilde{\alpha}_{13} &= \Gamma_2 \\ \tilde{\alpha}_{12}\tilde{\beta}'_{22} &= -\Gamma_1 \\ \tilde{\alpha}_{12}\tilde{\beta}_{21} &= \end{aligned}$$

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Take-Home Re-Exam of the Course
Time Series Econometrics 2ST111 Fall 2020

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Saturday, 28 November 2020

UPPSALA UNIVERSITET

Instructions

- This is a 48 hours individual take-home exam starting from 8:00, 28 November.
- You are not supposed to cooperate with other students or other people.
- Please read carefully and answer all questions. The answers shall be clearly written, concise and relevant, and all steps shall be well explained.
- The final report shall be submitted in PDF format by uploading onto studentportalen (where you download the exam) no later than 12:00, 30 November.
- Please do not write your name or anything related to your identity on the final report. Do it in an anonymous way!
- Note that you will not get any credit from the exam until you pass all the assignments.
- You can use, for example, word or latex to make the pdf. Handwriting is not accepted.
- To save time, you do not have to replicate the formulas and the derivations given in the lecture notes. However, it is a good idea to make a reference to the appropriate chapter and formula when answering the questions.
- The total score of the exam is 100 points.

Part 1: VECM and the Granger Representation (50 points)

Let X_t be given by the vector autoregressive model

$$\Delta X_t = \alpha\beta' X_{t-1} + \Gamma_1 \Delta X_{t-1} + \Gamma_2 \Delta X_{t-2} + \varepsilon_t \tag{1}$$

where ε_t are i.i.d. $N_p(0, \Omega)$ and α and β are $p \times r$ or rank r.

1. (10 points) Define $\tilde{X}_t = (X_t, X_{t-1}, X_{t-2})'$. Write the model into its companion matrix form

$$\Delta\tilde{X}_t = \tilde{\Pi}\tilde{X}_{t-1} + B\varepsilon_t.$$

by finding explicitly all the elements in matrix $\tilde{\Pi}$ and B.

2. (10 points) The matrix $\tilde{\Pi}$ can be decomposed as $\tilde{\Pi} = \tilde{\alpha}\tilde{\beta}'$. Find the $*$ elements in the following $\tilde{\alpha}$ and $\tilde{\beta}$ in terms of α, β, Γ_1, γ_2 and identity matrix I_p.

$$\tilde{\alpha} = \begin{pmatrix} \alpha & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad \tilde{\beta} = \begin{pmatrix} \beta & * & 0 \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

3. (10 points) Define $p \times (p-r)$ full rank matrices $\alpha_\perp$ and $\beta_\perp$ such that $\alpha'\alpha_\perp = \beta'\beta_\perp = 0$. Given $\tilde{\beta}_\perp = (\beta_\perp', \beta_\perp', \beta_\perp')'$, find $\tilde{\alpha}_\perp$ in terms of $\alpha_\perp$, Γ_1 and Γ_2.

4. (10 points) Define $\Gamma = I_p - \Gamma_1 - \Gamma_2$, $C = \beta_\perp(\alpha_\perp'\Gamma\beta_\perp)^{-1}\alpha_\perp'$ and $\tilde{C} = \tilde{\beta}_\perp(\tilde{\alpha}_\perp'\tilde{\beta}_\perp)^{-1}\tilde{\alpha}_\perp'$. Show that

$$\tilde{C} = \begin{pmatrix} I_p \\ I_p \\ I_p \end{pmatrix} C \ (I_p, -\Gamma_1, -\Gamma_2)$$

5. (10 points) Finally derive the Granger Representation Theorem for (1) from the Granger Representation Theorem for the model with one lag.

Part 2: Spurious Regressions (20 points)

Suppose that there is an I(1) vector sequence $y_t = (y_{1t}, y_{2t})'$ whose first order difference is simply $\Delta Y_t = \varepsilon_t$ where ε_t i.i.d. with covariance I_2. Consider the regression

$$y_{1t} = \alpha + \gamma y_{2t} + \epsilon_t. \tag{2}$$

1. (10 points) What are the true values for α and γ?

2. (10 points) However, it can be shown that, from the linear regression, the estimator follows asymptotically

$$\begin{pmatrix} T^{-1/2}\hat{\alpha}_T \\ \hat{\gamma}_T \end{pmatrix} \xrightarrow{d} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \tag{3}$$

for some random variables h_1 and h_2 with certain non-degenerating distributions. Explain why the two estimators are not consistent.

Part 3: The Multivariate Wiener Processes (30 points)

A standard Brownian motion $W(u)$ is a continuous-time process indexed by $u \in [0, 1]$ with some properties. Let the process $z_t = z_{t-1} + \eta_t$, $t = 1, ..., T$, where η_t i.i.d. $N(0, 1)$, and $z_0 = 0$.

1. (10 points) State the properties of the standard Brownian motion.

2. (10 points) We can approximate the continuous-time stochastic process $W(u)$ in discrete time. Give a scheme to approximate the standard Brownian motion by means of the sequence z_t and η_t.

3. (10 points) The Dickey-Fuller test for the case $y_t = \rho y_{t-1} + \varepsilon_t$ is given by

$$T(\hat{\rho}_T - 1) \xrightarrow{d} \frac{\frac{1}{2}\left(W(1)^2 - 1\right)}{\int_0^1 W(u)^2 \mathrm{d}u} \tag{4}$$

   One can employ the Monte Carlo simulation method to produce the critical values of this distribution. Give a scheme to achieve the algorithm.

   Hint: the most difficult part of the algorithm is to approximate the integral in the denominator, which requires an appropriate approximation of the integral based on the approximation in previous question. Write down the relevant formulas by means of z_t and η_t. And then demonstrate how to obtain the critical values of interest.