Part 1.

Q1. Can rewrite Var(2) as Var(1) where

$$\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} = \begin{pmatrix} A & B \\ I_2 & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_t \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix} + \begin{pmatrix} d \\ 0 \end{pmatrix} t + \begin{pmatrix} u \\ 0 \end{pmatrix}$$

or equivalently

$$\xi_t = F \xi_{t-1} + \alpha + \delta t + \varepsilon_t,$$

where F is the companion matrix.

Then trend covar. stationary if eigenvals of F lie inside unit circle:

$$\left(\forall \lambda \in \mathbb{C} : \left| I_2 \lambda^2 - A\lambda - B \right| = 0\right) : |\lambda| < 1.$$

Q2. Cov. stationarity means $\forall t \in \mathbb{Z}, E[y_t] = \mu$. Then

$$E[y_t] = A\, E[y_{t-1}] + B E[y_{t-2}] + c + \underbrace{E[u_t]}_{=0 \text{ (white noise)}}$$

$$\Rightarrow \mu = A\mu + B\mu + c$$

$$\mu = (I_2 - A - B)\, c.$$

Q3. From Q2 we have $c = (I_2 - A - B)\mu$, then

$$y_t = c + A y_{t-1} + B y_{t-2} + u_t$$

$$= \mu - A\mu - B\mu + A y_{t-1} + B y_{t-2} + u_t$$

$$\Rightarrow (y_t - \mu) = A(y_{t-1} - \mu) + B(y_{t-2} - \mu) + u_t.$$

Q4. (i) Calc Sample mean: $\bar{y}_T = \frac{1}{T}\sum_{t=1}^{T} y_t$

and (ii) Sample covar matrix: $\Sigma_T = \frac{1}{T}\sum_{t=1}^{T}(Y_t - \bar{Y}_T)(Y_t - \bar{Y}_T)'$

Then test

$$Q_{wald} = (\bar{y}_T - m)'\hat{\Sigma}_T(\bar{y}_T - m) \sim \chi^2(k).$$

If p-val $1 - F_{\chi^2(k)}(Q_{wald}) < p$ for some threshold $p \in \{0.1, 0.05, 0.01, \text{etc.}\}$, we can reject null $H_0: \mu = m$.

Q5. We can write the VAR(2) as

$$(I - AL - BL^2)(Y_t - \mu) = u_t$$

with VMA representation

$$\begin{aligned} Y_t - \mu &= (I - AL - BL^2)^{-1} u_t \\ &= \Psi(L) u_t = (I + \Psi_1 L + \Psi_2 L^2 + \dots) u_t \end{aligned}$$

$$\Rightarrow \quad (I - AL - BL^2)^{-1} = \Psi(L)$$

Then
$$\begin{aligned} I &= (I - AL - BL^2)^{-1}(I - AL - BL^2) \\ &= \Psi(L)(I - AL - BL^2) \\ &= I - AL - BL^2 \\ &\quad + \Psi_1 L - \Psi_1 L AL - \Psi_1 L BL^2 \\ &\quad + \Psi_2 L^2 - \Psi_2 L^2 AL - \Psi_2 L^2 BL^2 \\ &\quad + \Psi_3 L^3 - \Psi_3 L^3 AL - \Psi_3 B L^4 \\ &\quad + \Psi_4 L^4 - \Psi_4 L^4 AL - \Psi_4 L^4 BL^2 \\ &\quad + \dots \end{aligned}$$

$\Rightarrow \Psi_1 = A,$

$\Psi_2 = B + \Psi_1 A = B + A^2,$

$\Psi_3 = \Psi_1 B + \Psi_2 A = AB + (B + A^2)A = AB + BA + A^3$

$\Psi_4 = \Psi_2 B + \Psi_3 A = B^2 + A^2 B + ABA + BA^2 + A^4.$

Q6. We have that

$$\Delta y_t = y_t - y_{t-1}$$

$$= (A-I)y_{t-1} + By_{t-2} + c + \delta t + u_t$$

$$= (A-I+B)y_{t-1} - B\Delta y_{t-1} + c + \delta t + u_t$$

$$= \Pi y_{t-1} - \Gamma \Delta y_{t-1} + c + \delta t + u_t,$$

$$\boxed{\Pi = A + B - I, \quad \Gamma = -B.}$$

Q7. $\text{rank}(\Pi) = r < n$ is the cointegration rank, and $n-r$ is the number of unit roots so

$$y_t \sim I(n-r).$$

Q8. If y_{1t} and y_{2t} are cointegrated, then $r=1$. If $r=0$, no cointegration vector. If $r=2$, Π is full rank and no unit root.

Q9. Regress $y_{1t} = \alpha + \beta y_{2t} + \eta_t$ and test if $\hat{\eta}_t \sim I(1)$.
If $\hat{\eta}_t \sim I(0)$ then cointegrated.

Written Exam of the Course
Time Series Econometrics 2ST111 2021

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UPPSALA UNIVERSITET

This is a four hours open-book exam. Please read carefully and answer all questions. The answers shall be clearly written, concise and relevant, and all steps shall be well explained. The total score is 100 points.

You can bring the textbook, the printed materials offered by the teachers (slides, notes), and any paper or printed articles that are relevant to the course. You are not allowed to use any calculator, computer, smart-phone or any devices with internet or bluetooth connection. You can bring paper dictionary, but the electronic dictionary is not allowed. You can use pen, pencil, eraser and ruler. You are not allowed to share any books, notes, papers, tools or devices with others during the exam.

Part 1: Vector AutoRegressive Model (60 points)

Consider the VAR(2) model:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} y_{1,t-2} \\ y_{2,t-2} \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} t + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}, \tag{1}$$

and its matrix version

$$\boldsymbol{y}_t = \boldsymbol{A}\boldsymbol{y}_{t-1} + \boldsymbol{B}\boldsymbol{y}_{t-2} + \boldsymbol{c} + \boldsymbol{d}t + \boldsymbol{u}_t, \tag{2}$$

where $\boldsymbol{u}_t$ is a white noise process with zero mean and positive definite covariance matrix $\boldsymbol{\Omega}$. The covariance matrix has the structure

$$\boldsymbol{\Omega} = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}. \tag{3}$$

Define the set of some parameters of the VAR(2) model $\theta = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{d}\}$.

1. (4 points) Speficy the sufficient (trend) covariance stationarity condition(s) for the VAR(2) model. The answer shall be in details in the sense that, if you mention for example the "companion matrix", then you have to speficy explicitly what the companion matrix is in this case.

Now assume for simplicity that $\boldsymbol{d} = \mathbf{0}$ and that the VAR(2) model is weakly or covariance stationary.

2. (4 points) Compute the unconditional expectation $\boldsymbol{\mu}$ of $\boldsymbol{y}_t$ in terms of the parameters in θ.

3. (4 points) Give the steady-state form of the VAR(2) model. The steady-state form is sometimes called the process in deviations of the mean, that is, to rewrite (or reparameterize) the model in terms of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{\mu}$.

4. (8 points) Suppose that you have the sample $\{\boldsymbol{y}_t\}_{t=1}^{T}$. Propose a feasible scheme to test the hypothesis $H_0 : \boldsymbol{\mu} = \boldsymbol{m}$ for some known values $\boldsymbol{m}$. Describe the scheme in details.

5. (8 points) The VAR(2) model has an infinite VMA representation with the lag polynomial $\boldsymbol{\Psi}(L)$ in the moving average part. Derive the close forms for $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$ and $\boldsymbol{\Psi}_4$ given $\boldsymbol{A}$ and $\boldsymbol{B}$.

Assume now that the vector system (2) contains unit root(s) and that $\boldsymbol{d} \neq 0$. Furthermore, $\boldsymbol{y}_t \sim$ I(1).

6. (6 points) The VAR(2) model (2) has the VECM form

$$\Delta \boldsymbol{y}_t = \boldsymbol{\Pi} \boldsymbol{y}_{t-1} + \boldsymbol{\Gamma} \Delta \boldsymbol{y}_{t-1} + \boldsymbol{c} + \boldsymbol{d} t + \boldsymbol{u}_t. \tag{4}$$

   Find $\boldsymbol{\Pi}$ and $\boldsymbol{\Gamma}$ in terms of the elements in θ.

7. (6 points) The rank of $\boldsymbol{\Pi}$, say r, has special meaning under the cointegration framework. Briefly describe the meaning of r, and the meaning of $n - r$, where n is the dimension of the vector system (number of the dependent variables).

8. (6 points) If the two variables y_{1t} and y_{2t} can be cointegrated, give the possible value for r, and briefly explain why.

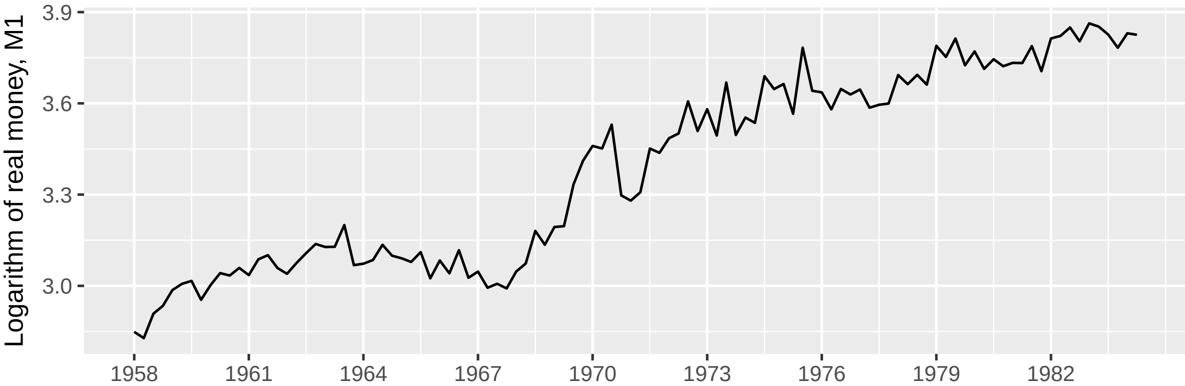
9. (8 points) In order to test and estimate the cointegration relation between the two variables y_{1t} and y_{2t}, the Engle-Granger's Procedure can be applied. Briefly describe the procedure.

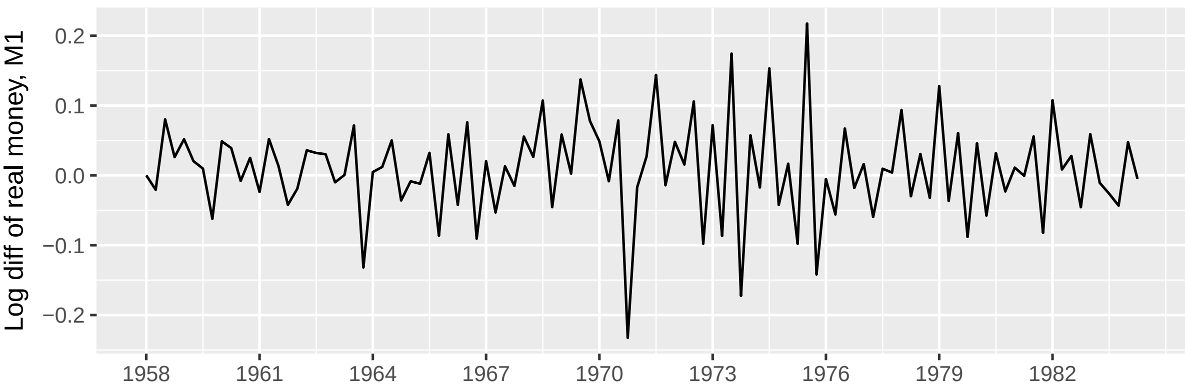
10. (6 points) The Johansen's procedure can also be applied to test and estimate the cointegration relation. Briefly show the difference between the two procedures in this case (two dependent variables).

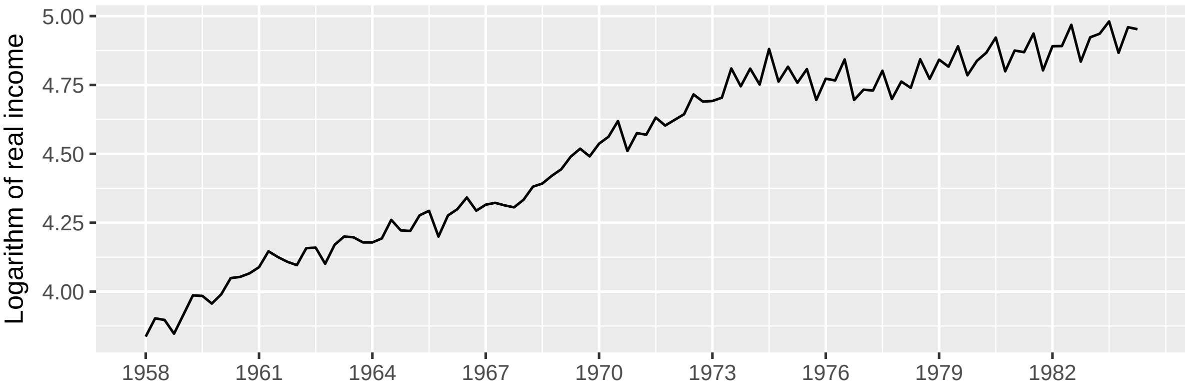
Part 2: Finnish Economy on the Demand for Money (40 points)

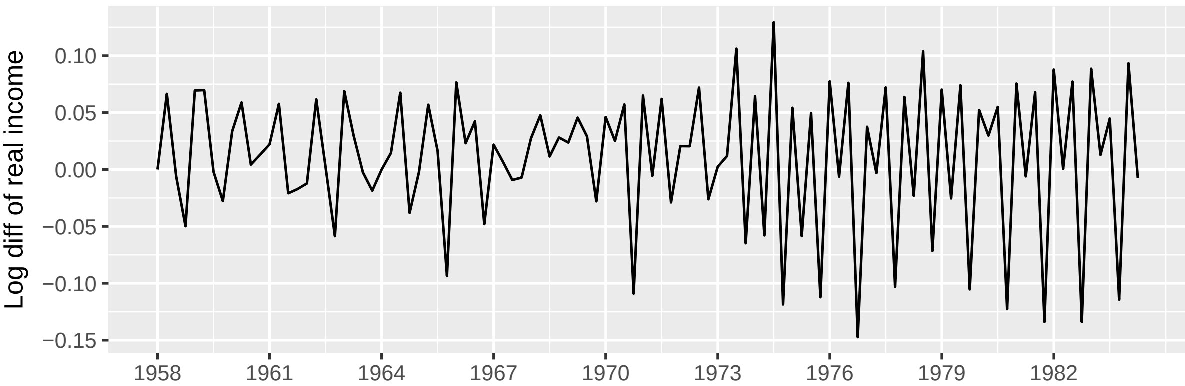
We chose the example and the corresponding data set given by Johansen and Juselius (1990) about the Finnish economy on the demand for money. The economic theory suggests that the money demand m can be represented by a function $m = f(y, p, c)$, where y is the real income, p the price level and c the cost of holding money.

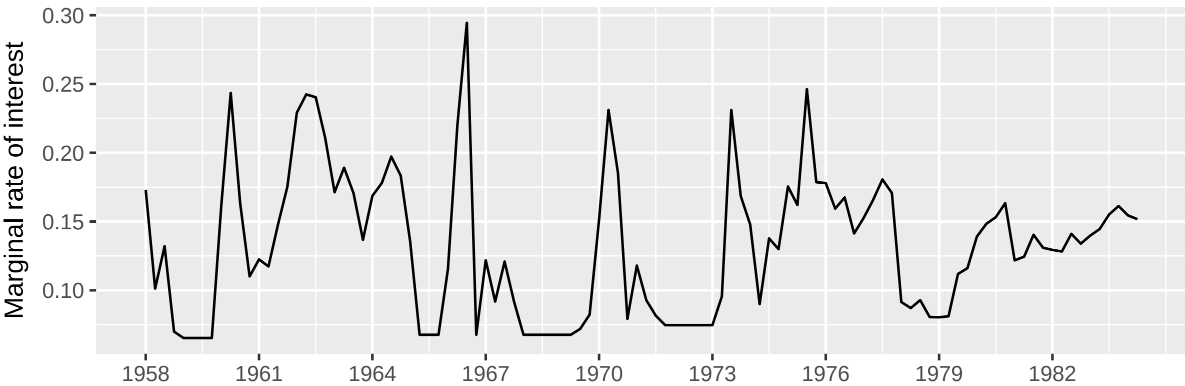
The Finnish data spans from 1958Q1 to 1984Q3. It uses the M1 (mainly the total cash in circulation excluding the bank reserves) as the measure of the money demand, the inflation rate Δp, and the marginal rate of interest i^m of the Bank of Finland as a proxy for the actual costs of holding money. By presumming multiplicative effects, money, income and prices are measured in logatithms. All the time series are plotted in the following:

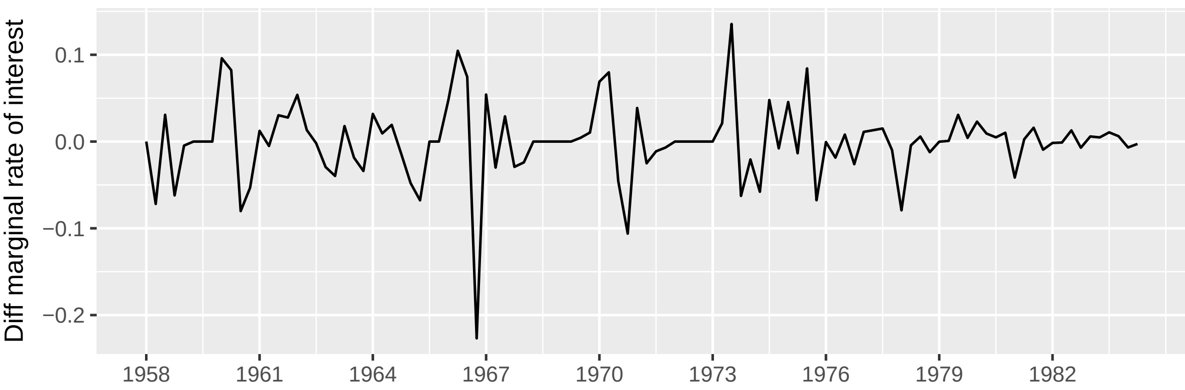


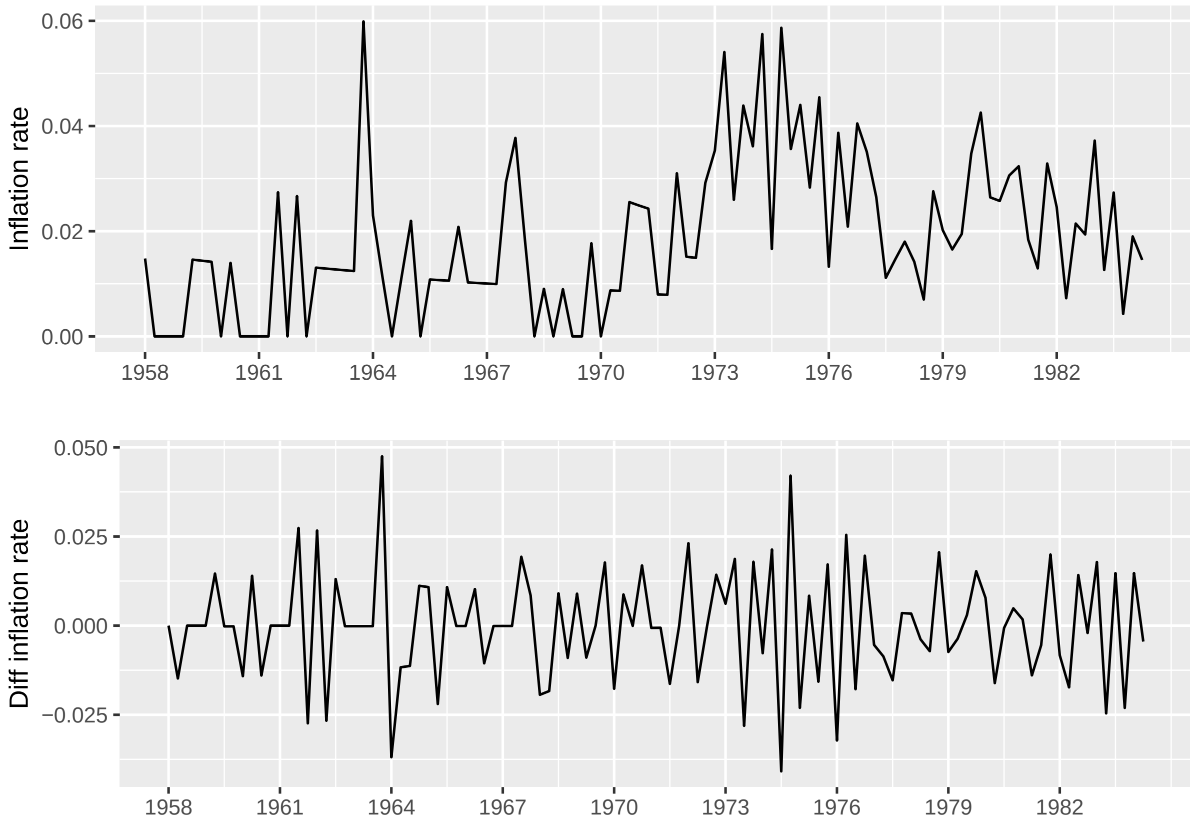



Logarithm of real income
5.00
4.75
4.50
4.25
4.00
1958
1961
1964
1967
1970
1973
1976
1979
1982


Log diff of real income
0.10
0.05
0.00
−0.05
−0.10
−0.15
1958
1961
1964
1967
1970
1973
1976
1979
1982


Marginal rate of interest
0.30
0.25
0.20
0.15
0.10
1958
1961
1964
1967
1970
1973
1976
1979
1982


Diff marginal rate of interest
0.1
0.0
−0.1
−0.2
1958
1961
1964
1967
1970
1973
1976
1979
1982



1. (16 points) The first step of the analysis is to determine the integration orders for the variables. Hamilton (1994) gives four types of augmented Dickey-Fuller tests. Let us assume that the differenced variables are stationary. Propose ADF test(s) for each of the four variables. Briefly explain why. Note that for certain variable, perhaps more than one ADF test will be applied.

2. (8 points) In the following, you find the results from the trace tests based on different hypotheses. The null hypotheses are given in the first column to the left, and the alternatives are the same $H_1 : r = 4$. The column "trace" contains the trace test statstic. The columns "10pct", "5pct" and "1pct" are critical values at significance levels 10%, 5% and 1%, respectively.

   Are the variables cointegrated? Propose a suitable cointegration rank.

```
##          trace 10pct  5pct  1pct
## r <= 3 |  2.25  6.50  8.18 11.65
## r <= 2 | 10.04 15.66 17.95 23.52
## r <= 1 | 39.27 28.71 31.52 37.22
## r = 0  | 79.21 45.23 48.28 55.43
```

3. (16 points) Consider that in this case, if we use the Engle-Granger's procedure, can we get the same result about the cointegration? Briefly explain why or why not.

References

Hamilton, J. D.: 1994, *Time Series Econometrics*, Princeton University Press.

Johansen, S. and Juselius, K.: 1990, Maximum likelihood estimation and inference on cointegration – with applications to the demand for money, *Oxford Bulletin of Economics and Statistics* **52**, 169–210.