Part 1.

Q1. Yes, Strict stationarity $\Rightarrow$ cov stationarity but not the other way around.

Q2. Stationarity $\not\Rightarrow$ ergodic in mean

$$\left(\begin{array}{l}\forall t\in\mathbb{Z},\ E(y_t)=\mu,\\ \quad \forall j\in\mathbb{Z}, Cov(y_t, y_{t+j})=\gamma_j\end{array}\right) \qquad \left(\frac{1}{T}\sum_{t=1}^{T} y_t \xrightarrow{p} E(y_t)\right)$$

Counter example: $Y_t = U_t + Z$, $U_t \overset{iid}{\sim} N(0,\sigma_U^2)$,

$Z \overset{iid}{\sim} N(0,1)$,

$U_t \perp Z$

$$E(Y_t) = E(U_t) + E(Z) = 0$$

$$Cov(Y_t, Y_{t-1}) = 0$$ by iid of U_t, Z and $U_t \perp Z$.

But $\frac{1}{T}\sum_{t=1}^{T} y_t = \frac{1}{T}\sum_{t=1}^{T}(U_t + Z) \xrightarrow{p} Z$, and

$$\forall \varepsilon>0,\ P(|Z-E(y_t)|\geq\varepsilon) = P(|Z-0|\geq\varepsilon) = P(|Z|\geq\varepsilon)\neq 0.$$

Therefore $\frac{1}{T}\sum_{t=1}^{T} y_t \not\xrightarrow{p} E(y_t)$, not ergodic in mean.

Q3. Yes estimator is unbiased.

$E(\hat{\beta}) = \beta.$

$$\min \sum_{t=1}^{T} (y_t - b y_{t-1})^2 \Rightarrow -2\sum(y_t - b y_{t-1}) = 0$$

$$\Rightarrow \sum(\beta y_{t-1} + u_t - b y_{t-1}) = 0$$

$$\Rightarrow (b-\beta)\sum y_{t-1} = \sum u_t$$

$$\Rightarrow b = \beta + \frac{\sum u_t}{\sum y_{t-1}}$$

$E(b) = \beta$, so unbiased.

By LLN $\beta + \frac{\sum u_t}{\sum y_{t-1}} \to E\left(\beta + \frac{\sum u_t}{\sum y_{t-1}}\right) = \beta$,

So consistent.

Q4. Now $b = \min_b \sum(y_t - b y_{t-1})^2 \Rightarrow 2\sum(\beta y_{t-1} + \theta u_{t-1} + \varepsilon_t - b y_{t-1}) = 0$

$$\Rightarrow (b-\beta)\sum y_{t-1} = \theta \sum u_{t-1} + \sum \varepsilon_t$$

$$\Rightarrow b - \beta = \frac{\sum_t^T \sum_{i=0}^{\infty} \theta^i \varepsilon_{t-i}}{\sum y_{t-1}}$$

$$\Rightarrow E(b) = \beta + E\left[\frac{\frac{1}{T}\sum_t^T \left(\sum_{i=0}^{\infty} \theta^i \varepsilon_{t-i}\right)}{\frac{1}{T}\sum_t y_{t-1}}\right] = \beta$$

By LLN, $b \to E(b) = \beta$.

Q5. Yes, stationarity $\Rightarrow I(0)$.

Q6. Standard Brownian motion $W: [0,1] \to \mathbb{R}$

Continuous but nowhere differentiable

Q7. We have

$$\Delta Z_t = \delta + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\Rightarrow \quad Z_t = \delta + \varepsilon_t + \theta\varepsilon_{t-1} + Z_{t-1}$$

$$= \delta + \varepsilon_t + \theta\varepsilon_{t-1} + (\delta + \varepsilon_{t-1} + \theta\varepsilon_{t-1} + Z_{t-2})$$

$$= \dots$$

$$= \underbrace{\delta t}_{\text{lin. trend}} + \underbrace{\varepsilon_t}_{\text{cycle}} + \underbrace{(1+\theta)\sum_{s=1}^{t} \varepsilon_{t-s}}_{\text{stoch. trend}} + \underbrace{Z_0 - \varepsilon_0}_{\text{initial cond.}}$$

Q8. Require that for $i, j \in \{1,2\}$, $i \neq j$, the estimators $\beta_{i1} = \beta_{i2} = \dots = \beta_{ip} = 0$ from

$$y_{it} = c_i + \sum_{s=1}^{p} \alpha_{is} y_{it} + \sum_{s=1}^{p} \beta_{is} y_{jt} + u_{it}.$$

Written Re-Exam of the Course
Time Series Econometrics 2ST111 2021

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This is a four hours open-book exam. Please read carefully and answer all questions. The answers shall be clearly written, concise and relevant, and all steps shall be well explained. The total score is 100 points.

You can bring the textbook, the printed materials offered by the teachers (slides, notes), and any paper or printed articles that are relevant to the course. You are not allowed to use any calculator, computer, smart-phone or any devices with internet or bluetooth connection. You can bring paper dictionary, but the electronic dictionary is not allowed. You can use pen, pencil, eraser and ruler. You are not allowed to share any books, notes, papers, tools or devices with others during the exam.

Part 1: Theoretical Puzzles in Time Series Econometrics (66 points)

Read and answer the following questions:

1. (6 points) The strict stationarity is a stronger assumption than the weak stationarity. Can we say that "strict stationarity imples weak stationarity"? If no, motivate your answer.

2. (9 points) Is a stationary process ergodic in mean, or for the mean? If no, give an example to show why it is not, and then state sufficient condition(s) to make it ergodic in mean.

3. (12 points) Consider the statistical model

$$y_t = \beta y_{t-1} + u_t, \tag{1}$$

where the error u_t is independent of y_{t-1}. Is the corresponding OLS estimators of β unbiased? If no, show why it is not. Is it consistent? If no, show why it is not.

4. (9 points) Consider the statistical model (1) again but with u_t following an AR(1) process

$$u_t = \theta u_{t-1} + \varepsilon_t, \tag{2}$$

where ε_t is independent of y_{t-1}. Is the corresponding OLS estimators of β from the regression model (1) consistent? If no, show why it is not, and propose a way to consistently estimate it.

5. (6 points) Is a stationary process I(0)? If no, give an example showing why not.

6. (6 points) Is the trajectory of the standard Brownian motian continuous? Is it differentiable?

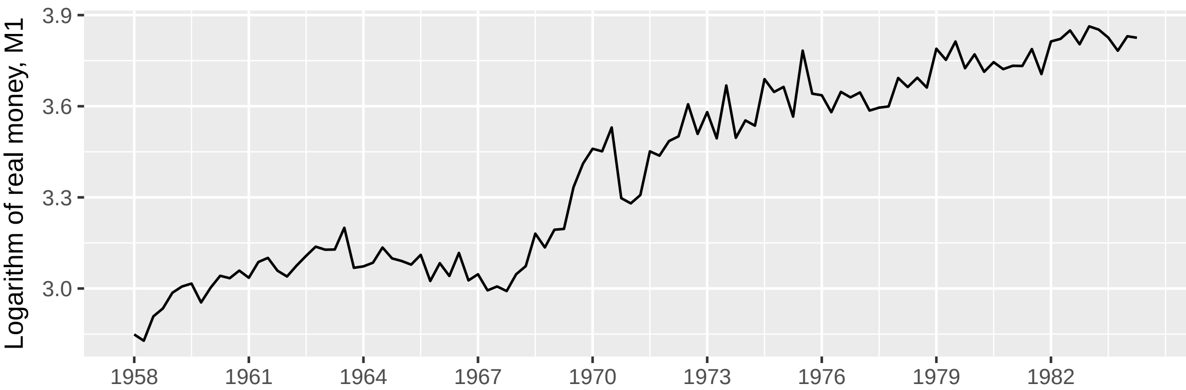
7. (12 points) Let $\Delta z_t = \delta + u_t$ with $u_t = \varepsilon_t + \theta\varepsilon_{t-1}$. Show how you obtain the Beveridge-Nelson decomposition of z_t, and interpret it (show explicitly the trend, stochastic trend, cycle, and initial conditions).

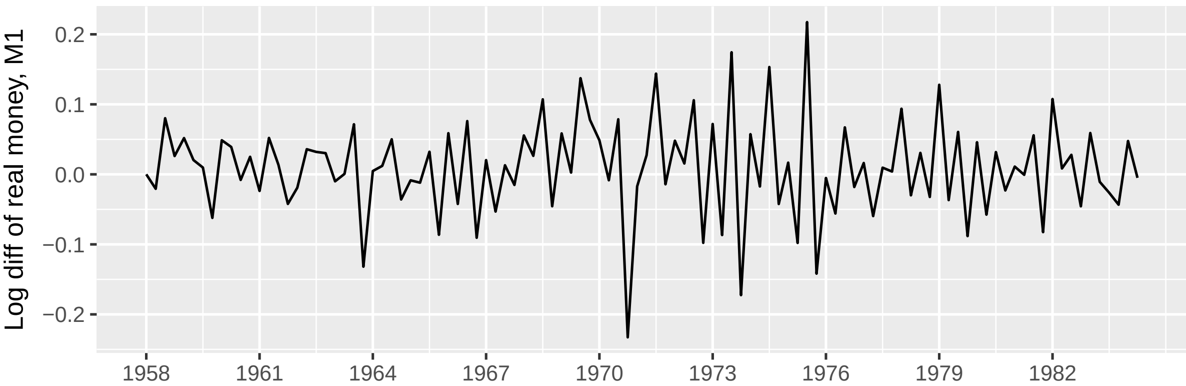
8. (6 points) In a VAR(p) model for y_t partitioned between y_{1t} and y_{2t}, what restrictions do you need to impose if you want that y_{1t} does not Granger-cause y_{2t}, and meanwhile y_{2t} does not Granger-cause y_{1t} ?

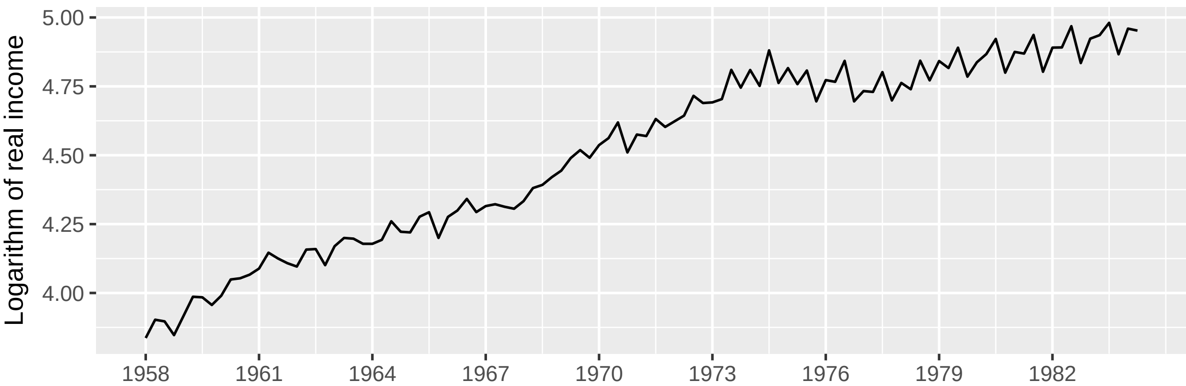
Part 2: Revisit the Finnish Economy on the Demand for Money (34 points)

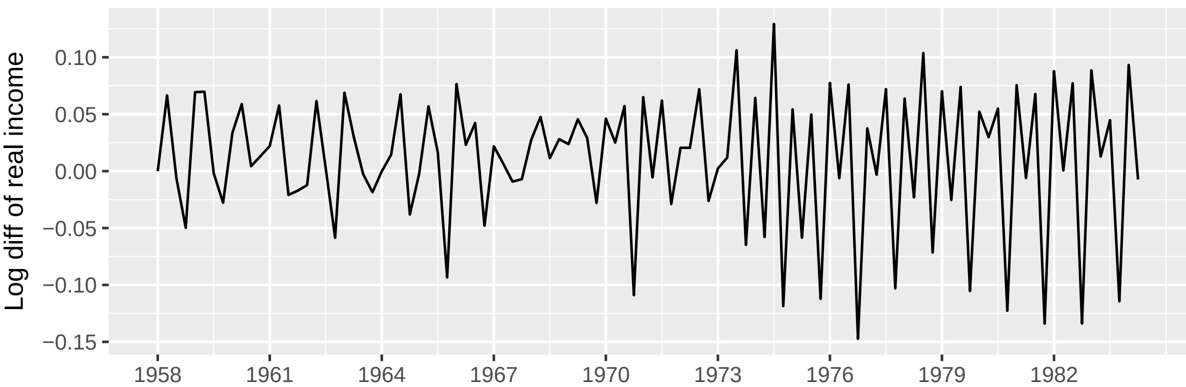
We consider again the example and the corresponding data set given by Johansen and Juselius (1990) about the Finnish economy on the demand for money. The economic theory suggests that the money demand m can be represented by a function $m = f(y, p, c)$, where y is the real income, p the price level and c the cost of holding money.

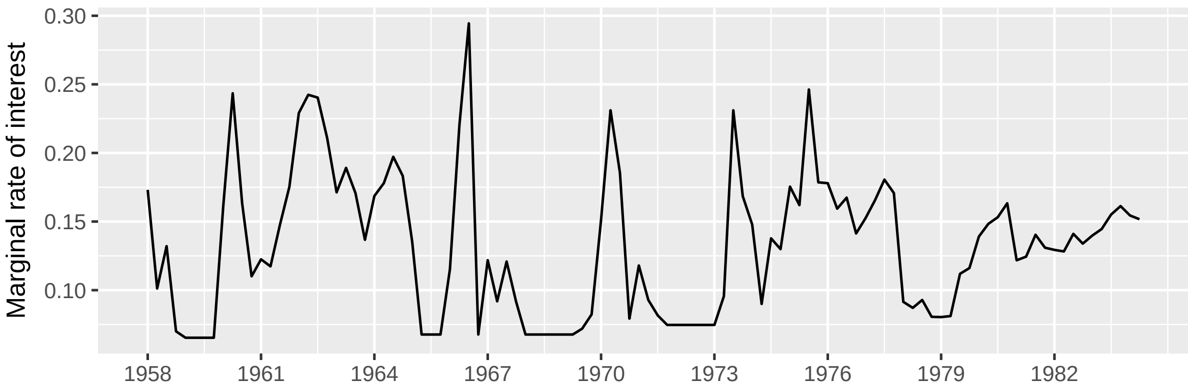
The Finnish data spans from 1958Q1 to 1984Q3. It uses the M1 (mainly the total cash in circulation excluding the bank reserves) as the measure of the money demand, the inflation rate Δp, and the marginal rate of interest i^m of the Bank of Finland as a proxy for the actual costs of holding money. By presumming multiplicative effects, money, income and prices are measured in logatithms. All the time series are plotted in the following:

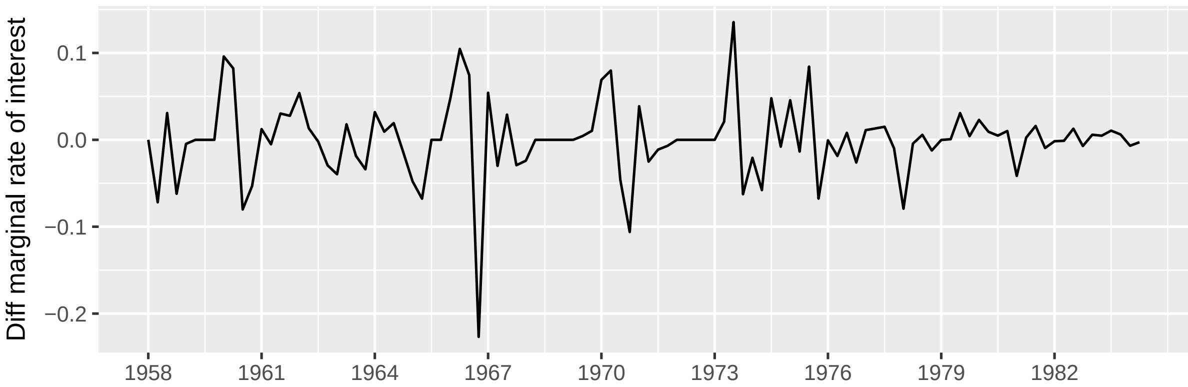

Logarithm of real money, M1
3.9
3.6
3.3
3.0
1958
1961
1964
1967
1970
1973
1976
1979
1982

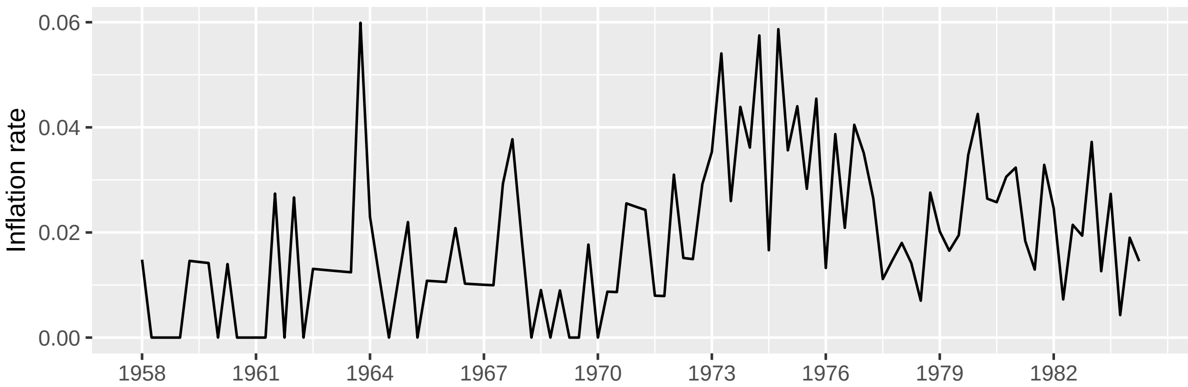

Log diff of real money, M1
0.2
0.1
0.0
−0.1
−0.2
1958
1961
1964
1967
1970
1973
1976
1979
1982

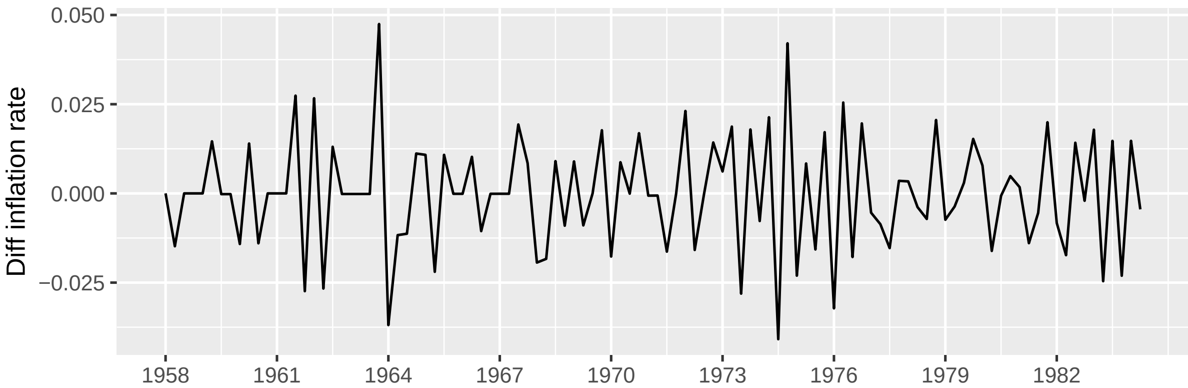

Logarithm of real income
5.00
4.75
4.50
4.25
4.00
1958
1961
1964
1967
1970
1973
1976
1979
1982


Log diff of real income
0.10
0.05
0.00
−0.05
−0.10
−0.15
1958
1961
1964
1967
1970
1973
1976
1979
1982





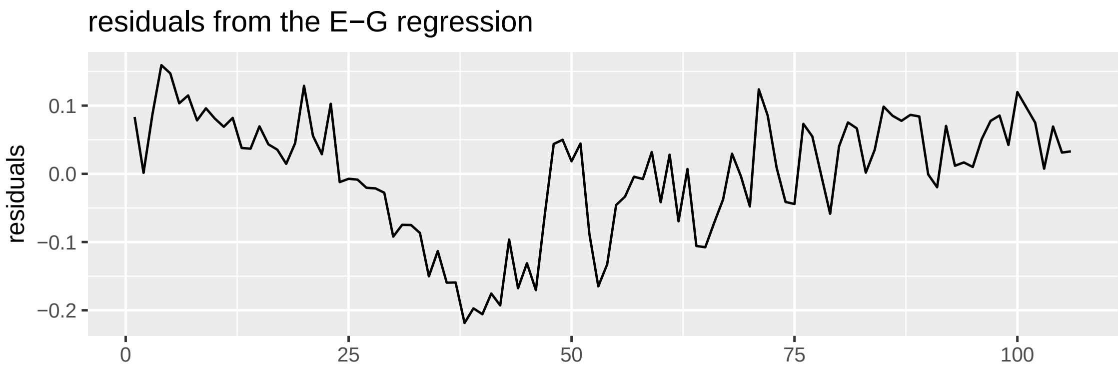


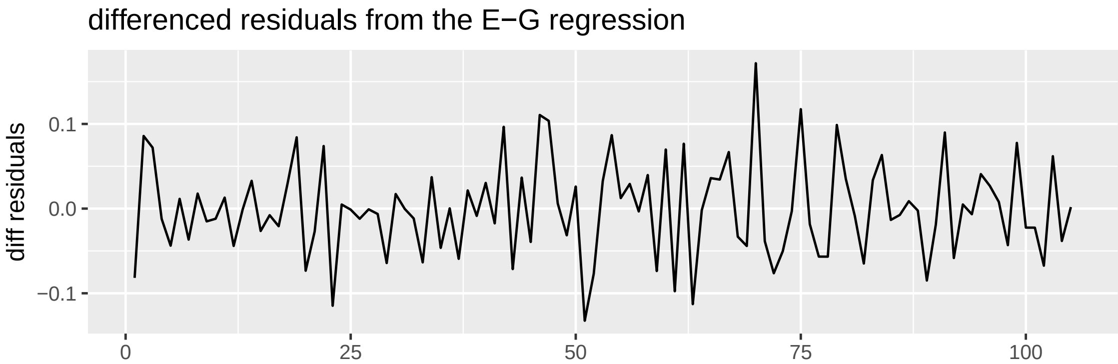


Presumming that the marginal rate of interest and the inflation rate have rejected the null of the corresponding ADF test, and that log M1 and the log real income have accepted. Let us conduct the Engle-Granger's (E-G) procedure.

1. (16 points) Briefly decribe the E-G procedure.

2. (18 points) We do the E-G procedure and check if the M1 and the real income can be cointegraded. The resiuals from the E-G regression and their first order difference are plotted as follows.





Let us assume that the differenced variables are stationary.

Though Hamilton (1994) gives four types of augmented Dickey-Fuller tests, in practice people only do three. In the following, the three tests are conducted on the residuals by using the `urca` package, and the results are reported. Note that the lag length is automatically chosen.

The first regression model is Case 1 in Hamilton. `z` is the residual, `z.lag.1` is its first lag, `z.diff.lag` is the first lagged difference of the residual. The test statistic `tau1` which is ρ in Hamilton is given in the end with the critical values at difference significance levels.

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
```

```
##
## Test regression none
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 - 1 + z.diff.lag)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.121308 -0.031402 -0.001702  0.026198  0.157468
##
## Coefficients:
##            Estimate Std. Error t value Pr(>|t|)
## z.lag.1    -0.16861    0.06147  -2.743   0.0072 **
## z.diff.lag -0.13974    0.09696  -1.441   0.1526
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.05247 on 102 degrees of freedom
## Multiple R-squared:  0.1158,Adjusted R-squared:  0.09843
## F-statistic: 6.677 on 2 and 102 DF,  p-value: 0.001883
##
##
## Value of test-statistic is: -2.7429
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau1 -2.58 -1.95 -1.62
```

The second regression model is Case 3 in Hamilton. The test statistics `tau2` which is ρ in Hamilton and `phi1` which is α in Hamilton are given in the end with the critical values at difference significance levels. Note that they are not joint tests.

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression drift
##
```

```
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + z.diff.lag)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.121352 -0.031445 -0.001745  0.026154  0.157425
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  4.349e-05  5.171e-03   0.008  0.99331
## z.lag.1     -1.686e-01  6.178e-02  -2.729  0.00749 **
## z.diff.lag  -1.397e-01  9.744e-02  -1.434  0.15463
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.05273 on 101 degrees of freedom
## Multiple R-squared:  0.1157,Adjusted R-squared:  0.09823
## F-statistic:  6.61 on 2 and 101 DF,  p-value: 0.002006
##
##
## Value of test-statistic is: -2.7292 3.7249
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau2 -3.46 -2.88 -2.57
## phi1  6.52  4.63  3.81
```

The third regression model is Case 4 in Hamilton. The test statistics `tau3` which is ρ in Hamilton, `phi2` which is α in Hamilton, and `phi3` which is δ in Hamilton are given in the end with the critical values at difference significance levels. Note that they are not joint tests.

```
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression trend
##
```

```
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.121068 -0.032574 -0.001728  0.028554  0.156119
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.0038324  0.0106456  -0.360  0.71960
## z.lag.1     -0.1706717  0.0622315  -2.743  0.00723 **
## tt           0.0000724  0.0001736   0.417  0.67753
## z.diff.lag  -0.1400698  0.0978398  -1.432  0.15537
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.05295 on 100 degrees of freedom
## Multiple R-squared:  0.1173,Adjusted R-squared:  0.09079
## F-statistic: 4.429 on 3 and 100 DF,  p-value: 0.005757
##
##
## Value of test-statistic is: -2.7425 2.5209 3.7807
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau3 -3.99 -3.43 -3.13
## phi2  6.22  4.75  4.07
## phi3  8.43  6.49  5.47
```

Explain the results from the three tests. What do you find? From the results of the ADF tests, do you think that they are cointegrated?

References

Johansen, S. and Juselius, K.: 1990, Maximum likelihood estimation and inference on cointegration – with applications to the demand for money, *Oxford Bulletin of Economics and Statistics* **52**, 169–210.