Time Series Econometrics
Supplementary Lecture 4

Yukai Yang

Department of Statistics
Uppsala University

1 Exercise 8.3

Derive result [8.2.28].

This result is

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbf{x}_t u_t \xrightarrow{L} N(\mathbf{0},\sigma^2\mathbf{Q})$$

where $\mathbf{x}_t' = (1, y_{t-1}, \ldots, y_{t-p})$ and

$$\mathbf{Q} = \begin{pmatrix} 1 & \mu & \cdots & \mu \\ \mu & \gamma_0+\mu^2 & \cdots & \gamma_{p-1}+\mu^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mu & \gamma_{p-1}+\mu^2 & \cdots & \gamma_0+\mu^2 \end{pmatrix}$$

This is in the context of an autoregression, so we have by Assumption 8.4

Assumption (8.4). *The regression model is*

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

with roots of $(1-\phi_1 z - \cdots - \phi_p z^p) = 0$ *outside the unit circle and with* $\{\varepsilon_t\}$ *an i.i.d. sequence with mean zero, variance* σ^2 *and finite fourth moment* μ_4*.*

The convergence is shown by applying Proposition 7.9, so what needs to be done is to show that its conditions are satisfied.

Proposition (7.9). *Let* $\{\mathbf{Y}_t\}_{t=1}^{\infty}$ *be an n-dimensional vector martingale difference sequence with* $\bar{Y}_T = (1/T)\sum_{t=1}^{T}\mathbf{Y}_t$*. Suppose that*

(a) $E(\mathbf{Y}_t\mathbf{Y}_t') = \mathbf{\Omega}_t$*, a positive definite matrix with* $(1/T)\sum_{t=1}^{T}\mathbf{\Omega}_t \to \mathbf{\Omega}$*, a positive definite matrix;*

(b) $E(Y_{i,t}Y_{j,t}Y_{l,t}Y_{m,t}) < \infty$ *for all* t *and all* i, j, l *and* m*, where* $Y_{i,t}$ *is the* i*th element of* $\mathbf{Y}_t$*;*

(c) $(1/T)\sum_{t=1}^{T}\mathbf{Y}_t\mathbf{Y}_t' \xrightarrow{p} \mathbf{\Omega}$.

Then, $\sqrt{T}\bar{\mathbf{Y}}_T \xrightarrow{L} N(\mathbf{0}, \mathbf{\Omega})$.

Let us define $\mathbf{Y}_t = \mathbf{x}_t u_t$. This is a vector MDS, since $E(\mathbf{Y}_t) = \mathbf{0}$ and $E(\mathbf{Y}_t|\mathbf{Y}_{t-1}, \dots) = E(\mathbf{Y}_t|\mathbf{x}_{t-1}, \dots, \mathbf{x}_1, u_{t-1}, \dots, u_1) = E(\mathbf{Y}_t|y_{t-2}, \dots, y_1, u_{t-1}, \dots, u_1) = \mathbf{0}$, since

$$E(\mathbf{Y}_t|y_{t-2}, \dots, y_1, u_{t-1}, \dots, u_1) = \begin{pmatrix} E(u_t|y_{t-2}, \dots, y_1, u_{t-1}, \dots, u_1) \\ E(y_{t-1}u_t|y_{t-2}, \dots, y_1, u_{t-1}, \dots, u_1) \\ y_{t-2}E(u_t|y_{t-2}, \dots, y_1, u_{t-1}, \dots, u_1) \\ \vdots \\ y_{t-p}E(u_t|y_{t-2}, \dots, y_1, u_{t-1}, \dots, u_1) \end{pmatrix}$$

and the expectation of the error conditional on the history is zero, the covariance between u_t and y_{t-1} is zero and, again, the expectation of the error conditional on the history is zero.

To verify condition (a),

$$E(\mathbf{Y}_t\mathbf{Y}_t') = E(u_t^2\mathbf{x}_t\mathbf{x}_t') = E(u_t^2)E(\mathbf{x}_t\mathbf{x}_t')$$

where the last equality is motivated by u_t being independent of previous values of y, which is what $\mathbf{x}_t$ contains. $E(u_t^2) = \sigma^2$, so we have

$$E(\mathbf{Y}_t\mathbf{Y}_t') = \sigma^2 \begin{pmatrix} 1 & E(y_{t-1}) & E(y_{t-2}) & \cdots & E(y_{t-p}) \\ E(y_{t-1}) & E(y_{t-1}^2) & E(y_{t-1}y_{t-2}) & \cdots & E(y_{t-1}y_{t-p}) \\ E(y_{t-2}) & E(y_{t-2}y_{t-1}) & E(y_{t-2}^2) & \cdots & E(y_{t-2}y_{t-p}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E(y_{t-p}) & E(y_{t-p}y_{t-1}) & E(y_{t-p}y_{t-2}) & \cdots & E(y_{t-p}^2) \end{pmatrix}$$

Except for the top row and column, all elements follow the pattern of $E(y_{t-j}y_{t-k})$ for j and k integers between 1 and p. Hence,

$$E(y_{t-j}y_{t-k}) = E(y_{t-j}y_{t-k}) - E(y_{t-j})E(y_{t-k}) + E(y_{t-j})E(y_{t-k}) = \gamma_{|j-k|} + \mu^2,$$

and putting this into the matrix we obtain

$$E(\mathbf{Y}_t\mathbf{Y}_t') = \sigma^2 \begin{pmatrix} 1 & \mu & \mu & \cdots & \mu \\ \mu & \gamma_0 + \mu^2 & \gamma_1 + \mu^2 & \cdots & \gamma_{p-1} + \mu^2 \\ \mu & \gamma_1 + \mu^2 & \gamma_0 + \mu^2 & \cdots & \gamma_{p-2} + \mu^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu & \gamma_{p-1} + \mu^2 & \gamma_{p-2} + \mu^2 & \cdots & \gamma_0 + \mu^2 \end{pmatrix}$$

This is the given $\mathbf{Q}$ matrix (multiplied by σ^2), and in Proposition 7.9 it's called $\mathbf{\Omega}_t$. It is the same for all t, so $\mathbf{\Omega}_t = \mathbf{\Omega}$. Since it's the usual covariance matrix, we'll assume it's positive definite.

To verify condition (b), we can use Proposition 7.10:

Proposition (7.10). *Let X_t be a strictly stationary stochastic process with $E(X_t^4) = \mu_4 < \infty$. Let $Y_t = \sum_{j=0}^{\infty} h_j X_{t-j}$, where $\sum_{j=0}^{\infty} |h_j| < \infty$. Then Y_t is a strictly stationary stochastic process with $E|Y_t Y_s Y_u Y_v| < \infty$ for all t, s, u, v.*

By assumption 8.4, the fourth moment of our error term u_t exists, which corresponds to the X_t term in the Proposition. The u_t sequence is an iid sequence, so it is strictly stationary. Thus, we can replace X_t in the proposition by u_t. Since the AR process is stationary it has an $MA(\infty)$ representation with absolutely summable weights. It therefore follows by Proposition 7.10 that y_t has finite fourth moments. The fourth moments of $\mathbf{Y}_t = \mathbf{x}_t u_t$ are of the form $E(u_t^4)E(y_{t-i}y_{t-j}y_{t-k}y_{t-l})$ since $i, j, k, l > 0$ and future values of u are independent of previous values of y. By Assumption 8.4, $E(u_t^4) < \infty$ and by the proposition $E(y_{t-i}y_{t-j}y_{t-k}y_{t-l}) < \infty$, so their product is also finite. Thus, condition (b) holds too.

Lastly, to apply Proposition 7.9 we need to confirm condition (c). This matrix is

$$\frac{1}{T}\sum_{t=1}^{T} \mathbf{Y}_t \mathbf{Y}_t' = \frac{1}{T}\sum_{t=1}^{T} \begin{pmatrix} u_t^2 & y_{t-1}u_t^2 & \cdots & y_{t-p}u_t^2 \\ y_{t-1}u_t^2 & y_{t-1}^2 u_t^2 & \cdots & y_{t-1}y_{t-p}u_t^2 \\ \vdots & \vdots & \ddots & \vdots \\ y_{t-p}u_t^2 & y_{t-1}y_{t-p}u_t^2 & \cdots & y_{t-p}^2 u_t^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{T}\sum_{t=1}^{T} u_t^2 & \frac{1}{T}\sum_{t=1}^{T} y_{t-1}u_t^2 & \cdots & \frac{1}{T}\sum_{t=1}^{T} y_{t-p}u_t^2 \\ \frac{1}{T}\sum_{t=1}^{T} y_{t-1}u_t^2 & \frac{1}{T}\sum_{t=1}^{T} y_{t-1}^2 u_t^2 & \cdots & \frac{1}{T}\sum_{t=1}^{T} y_{t-1}y_{t-p}u_t^2 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{T}\sum_{t=1}^{T} y_{t-p}u_t^2 & \frac{1}{T}\sum_{t=1}^{T} y_{t-1}y_{t-p}u_t^2 & \cdots & \frac{1}{T}\sum_{t=1}^{T} y_{t-p}^2 u_t^2 \end{pmatrix}.$$

By the results on p.192-193, we get the following:

$$\frac{1}{T}\sum_{t=1}^{T} u_t^2 \xrightarrow{p} \sigma^2$$

$$\frac{1}{T}\sum_{t=1}^{T} y_{t-j}^2 u_t^2 \xrightarrow{p} (\gamma_0 + \mu^2)\sigma^2, \quad j = 1, \ldots, p$$

$$\frac{1}{T}\sum_{t=1}^{T} y_{t-j} y_{t-k} u_t^2 \xrightarrow{p} (\gamma_{|j-k|} + \mu^2)\sigma^2, \quad j, k = 1, \ldots, p$$

From these results, all elements of the matrix have been covered except for the first row and column. These elements are of the form $T^{-1}\sum_{t=1}^{T} y_{t-j}u_t^2$ for $j = 1, \ldots, p$. By Example 7.15, these converge to

$$\frac{1}{T}\sum_{t=1}^{T} y_{t-j}u_t^2 \xrightarrow{p} \sigma^2 E(y_{t-j}) = \sigma^2 \mu.$$

All taken together, this means that

$$\frac{1}{T}\sum_{t=1}^{T} \mathbf{Y}_t \mathbf{Y}_t' \xrightarrow{p} \sigma^2 \mathbf{Q}$$

which in Proposition 7.9 is called $\mathbf{\Omega}$, and which also appeared in condition (a). These are the same, as they should, so at this point we are ready to apply the Proposition to establish the convergence. By Proposition 7.9, we therefore have that

$$\sqrt{T}\bar{\mathbf{Y}}_T = \sqrt{T}\sum_{t=1}^{T} \frac{\mathbf{x}_t u_t}{T} = \sum_{t=1}^{T} \frac{\mathbf{x}_t u_t}{\sqrt{T}} \xrightarrow{p} N(\mathbf{0}, \sigma^2 \mathbf{Q}),$$

which is the result of [8.2.28].

2 Exercise 10.2(a)

Let $\mathbf{y}_t = (X_t, Y_t)'$ be given by

$$\begin{aligned} X_t &= \varepsilon_t + \theta\varepsilon_{t-1} \\ Y_t &= h_1 X_{t-1} + u_t \end{aligned}$$

where $(\varepsilon_t, u_t)'$ is vector white noise with contemporaneous variance-covariance matrix given by

$$\begin{pmatrix} E(\varepsilon_t^2) & E(\varepsilon_t u_t) \\ E(\varepsilon_t u_t) & E(u_t^2) \end{pmatrix} = \begin{pmatrix} \sigma_\varepsilon^2 & 0 \\ 0 & \sigma_u^2 \end{pmatrix}.$$

Calculate the autocovariance matrices $\{\mathbf{\Gamma}_k\}_{k=-\infty}^{\infty}$ for this process.

The autocovariance matrices are given by [10.2.1]:

$$\mathbf{\Gamma}_j = E\left[(\mathbf{y}_t - \boldsymbol{\mu})(\mathbf{y}_{t-j} - \boldsymbol{\mu})'\right].$$

First we find $\boldsymbol{\mu}$:

$$\boldsymbol{\mu} = \begin{pmatrix} E(X_t) \\ E(Y_t) \end{pmatrix} = \begin{pmatrix} E(\varepsilon_t + \theta\varepsilon_{t-1}) \\ E(h_1 X_{t-1} + u_t) \end{pmatrix} = \begin{pmatrix} E(\varepsilon_t) + \theta E(\varepsilon_{t-1}) \\ h_1 E(X_{t-1}) + E(u_t) \end{pmatrix} = \mathbf{0}.$$

This means that

$$\mathbf{\Gamma}_j = E\left(\mathbf{y}_t \mathbf{y}'_{t-j}\right).$$

Before we start with this, we can note one thing: X_t is a regular MA(1) process, and Y_t is just a function of it with no lags of itself. As we will see, this simplifies things as we can use what we know about MA(1) autocovariances.

The jth autocovariance is

$$\begin{aligned}
\mathbf{\Gamma}_j = E\left(\mathbf{y}_t \mathbf{y}'_{t-j}\right) &= E\begin{pmatrix} X_t X_{t-j} & X_t Y_{t-j} \\ X_{t-j} Y_t & Y_t Y_{t-j} \end{pmatrix} = \begin{pmatrix} E(X_t X_{t-j}) & E(X_t Y_{t-j}) \\ E(X_{t-j} Y_t) & E(Y_t Y_{t-j}) \end{pmatrix} \\
&= \begin{pmatrix} Cov(X_t, X_{t-j}) & Cov(X_t, Y_{t-j}) \\ Cov(X_{t-j}, Y_t) & Cov(Y_t, Y_{t-j}) \end{pmatrix}
\end{aligned}$$

The next step is to rewrite all four elements of this matrix as functions of autocovariances of an MA(1) process. This helps us since we know that the autocovariance will be 0 for orders 2 or higher. To simplify notation and make this more clear, let γ_j denote the jth autocovariance of X_t, i.e. the MA(1) process.

The top left element is $Cov(X_t, X_{t-j}) = \gamma_j$. The top right element is

$$\begin{aligned}
Cov(X_t, Y_{t-j}) &= Cov(X_t, h_1 X_{t-j-1} + u_{t-j}) = h_1 Cov(X_t, X_{t-j-1}) + Cov(X_t, u_{t-j}) \\
&= h_1 \gamma_{j+1}.
\end{aligned}$$

The bottom left element is

$$\begin{aligned}
Cov(X_{t-j}, Y_t) &= Cov(X_{t-j}, h_1 X_{t-1} + u_t) = h_1 Cov(X_{t-j}, X_{t-1}) + Cov(X_{t-j}, u_t) \\
&= h_1 \gamma_{j-1}.
\end{aligned}$$

Lastly, the bottom right element is

$$\begin{aligned}
Cov(Y_t, Y_{t-j}) &= Cov(h_1 X_{t-1} + u_t, h_1 X_{t-j-1} + u_{t-j}) \\
&= h_1^2 Cov(X_{t-1}, X_{t-j-1}) + Cov(u_t, u_{t-j}) \\
&= \begin{cases} h_1^2 \gamma_j, & \text{for } j \neq 0 \\ h_1^2 \gamma_j + \sigma_u^2, & \text{for } j = 0. \end{cases}
\end{aligned}$$

All in all, the matrix is

$$\mathbf{\Gamma}_j = \begin{pmatrix} Cov(X_t, X_{t-j}) & Cov(X_t, Y_{t-j}) \\ Cov(X_{t-j}, Y_t) & Cov(Y_t, Y_{t-j}) \end{pmatrix} = \begin{pmatrix} \gamma_j & h_1 \gamma_{j+1} \\ h_1 \gamma_{j-1} & h_1^2 \gamma_j \end{pmatrix}, \quad j \neq 0$$

Recall for an MA(1):

$$\gamma_j = \begin{cases} \sigma_\varepsilon^2(1+\theta^2), & \text{for } j = 0 \\ \sigma_\varepsilon^2\theta, & \text{for } j = \pm 1 \\ 0, & \text{otherwise.} \end{cases}$$

This makes it straight-forward to calculate $\mathbf{\Gamma}_j$ for various j:

$$\begin{aligned}
\mathbf{\Gamma}_0 &= \begin{pmatrix} \gamma_0 & h_1\gamma_1 \\ h_1\gamma_{-1} & h_1^2\gamma_0 + \sigma_u^2 \end{pmatrix} = \begin{pmatrix} \sigma_\varepsilon^2(1+\theta^2) & h_1\theta\sigma_\varepsilon^2 \\ h_1\theta\sigma_\varepsilon^2 & h_1^2\sigma_\varepsilon^2(1+\theta^2) + \sigma_u^2 \end{pmatrix} \\
\mathbf{\Gamma}_1 &= \begin{pmatrix} \gamma_1 & h_1\gamma_2 \\ h_1\gamma_0 & h_1^2\gamma_1 \end{pmatrix} = \begin{pmatrix} \theta\sigma_\varepsilon^2 & 0 \\ h_1\sigma_\varepsilon^2(1+\theta^2) & h_1^2\theta\sigma_\varepsilon^2 \end{pmatrix} \\
\mathbf{\Gamma}_2 &= \begin{pmatrix} \gamma_2 & h_1\gamma_3 \\ h_1\gamma_1 & h_1^2\gamma_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ h_1\theta\sigma_\varepsilon^2 & 0 \end{pmatrix}
\end{aligned}$$

And for $j > 2$, $\mathbf{\Gamma}_j = \mathbf{0}$. For negative autocovariances, we have

$$\mathbf{\Gamma}_{-j} = \mathbf{\Gamma}_j'.$$