3.1. $Y_t = (1+2.4L+0.8L^2)\varepsilon_t$, $\quad E(\varepsilon_t\varepsilon_\tau) = \begin{cases} 1, & t=\tau \\ 0, & t\neq\tau \end{cases}$

Covariance Stationary $\Leftrightarrow E(Y_t)=\mu \quad \forall t \in \mathbb{Z},$

$$E(Y_t-\mu)(Y_{t-j}-\mu) = \gamma_j \quad \forall t\in\mathbb{Z},\ j\in\mathbb{N}.$$

$$E(Y_t) = E(1+2.4L+0.8L^2)\varepsilon_t$$
$$= E(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2})$$
$$E(\varepsilon_t)=0,\ E(\varepsilon_t^2)=1,\ E(\varepsilon_t\varepsilon_{t-j})=0,\ j\neq 0$$

Autocovar: $Cov(Y_t, Y_{t-j}) = Cov(\varepsilon_t + 2.4\varepsilon_{t-1}+0.8\varepsilon_{t-2},$
$\quad \varepsilon_{t-j}+2.4\varepsilon_{t-j-1}+0.8\varepsilon_{t-j-2})$

$$= Cov(\varepsilon_t, \varepsilon_{t-j}+2.4\varepsilon_{t-j-1}+0.8\varepsilon_{t-j-2}) \quad ①$$
$$+ 2.4Cov(\varepsilon_{t-1}, \varepsilon_{t-j}+2.4\varepsilon_{t-j-1}+0.8\varepsilon_{t-j-2}) \quad ②$$
$$+ 0.8Cov(\varepsilon_{t-2}, \varepsilon_{t-j}+2.4\varepsilon_{t-j-1}+0.8\varepsilon_{t-j-2}) \quad ③$$

$$① = \begin{cases} 1, & j=0 \\ 0, & j\geq 1 \end{cases} \quad ② = \begin{cases} (2.4)^2 & j=0 \\ 2.4, & j=1 \\ 0, & j\geq 2 \end{cases} \quad ③ = \begin{cases} (0.8)^2, & j=0 \\ (2.4)0.8, & j=1 \\ 0.8, & j=2 \\ 0, & j\geq 3. \end{cases}$$

3.8. Let $Y_t = (1+\lambda_1 L)(1+\lambda_2 L)\tilde{\varepsilon}_t$ be not invertible

s.t. $|\lambda_1|<1$, $|\lambda_2|>1$ and $E(\tilde{\varepsilon}_t\tilde{\varepsilon}_\tau) = \begin{cases} \tilde{\sigma}^2, & t=\tau, \\ 0, & t\neq\tau. \end{cases}$

Then the invertible representation is

$$Y_t = (1+\lambda_1 L)(1+\lambda_2^{-1}L)\varepsilon_t \quad \text{where} \quad E(\varepsilon_t\varepsilon_\tau) = \begin{cases} \lambda_2^2\tilde{\sigma}^2, & t=\tau, \\ 0, & t\neq\tau. \end{cases}$$

Find λ roots s.t. $0=\phi(z) = 1+2.4z+0.8z^2$.

Using $z=\lambda^{-1}$, $z^2=\lambda^{-2}$, $0 = z^{-2}(1+2.4z+0.8z^2)$

$$= z^{-2}+2.4z^{-1}+0.8$$
$$= \lambda^2+2.4\lambda+0.8$$
$$\Rightarrow \lambda = \frac{-2.4\pm\sqrt{(2.4)^2-4(0.8)}}{2} = \frac{-2.4\pm1.6}{2}. \quad |\lambda_1|=0.4,\ |\lambda_2|=2.$$

$\therefore$ non-invertible.

3.2. $(1-1.1L+0.18L^2)Y_t = \varepsilon_t$, $E(\varepsilon_t \varepsilon_\tau) = \begin{cases} 1, & t=\tau, \\ 0, & t \neq \tau, \end{cases}$ $\varepsilon_t \sim N(0,\sigma^2)$ iid.

$\phi(z)=0 \Rightarrow Y_t = 1.1 Y_{t-1} - 0.18 Y_{t-2} + \varepsilon_t$

For autocovariances, $\gamma_j = Cov(Y_t, Y_{t-j})$, $\gamma_0 = Cov(Y_t, Y_t) = Var(Y_t)$.

$$\gamma_0 = V(Y_t) = V(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t)$$
$$= \phi_1^2 V(Y_{t-1}) + \phi_2^2 V(Y_{t-2}) + V(\varepsilon_t)$$
$$+ 2\phi_1\phi_2 Cov(Y_{t-1}, Y_{t-2})$$
$$+ 2\phi_1 Cov(Y_{t-1}, \varepsilon_t)$$
$$+ 2\phi_2 Cov(Y_{t-2}, \varepsilon_t)$$
$$= \phi_1^2\gamma_0 + \phi_2^2\gamma_0 + \sigma^2 + 2\phi_1\phi_2\gamma_1 \Rightarrow \gamma_0 = \frac{\sigma^2 + 2\phi_1\phi_2\gamma_1}{1-\phi_1^2-\phi_2^2}.$$

$$\gamma_1 = Cov(Y_t, Y_{t-1})$$
$$= Cov(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t, Y_{t-1})$$
$$= \phi_1\gamma_0 + \phi_2\gamma_1 \Rightarrow \gamma_1 = \frac{\phi_1\gamma_0}{1-\phi_2}.$$

$$\gamma_k = Cov(Y_t, Y_{t-k})$$
$$= Cov(\phi_1 Y_{t-k-1} + \phi_2 Y_{t-k-2} + \varepsilon_{t-k}, Y_t)$$
$$= \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2}.$$

3.3. A covariance stationary process

$(1-\phi_1 L - \phi_2 L^2 \dots - \phi_p L^p)(Y_t - \mu) = \varepsilon_t$ has an $MA(\infty)$

$(Y_t - \mu) = \psi(L)\varepsilon_t$ with $\psi(L) = (1-\phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p)^{-1}$,

i.e. $1 = (1-\phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p)^{-1}(\psi_0 + \psi_1 L + \psi_2 L^2 + \dots)$

$$= \left(1 - \sum_{i=1}^{p}\phi_i L^i\right)\left(\psi_1 + \sum_{k=1}^{\infty}\psi_k L^k\right)$$
$$= \psi_1 - \psi_1\sum_{i=1}^{p}\phi_i L^i + \sum_{k=1}^{\infty}\psi_k L^k - \left(\sum_{i=1}^{p}\phi_i L^i\right)\left(\sum_{k=1}^{\infty}\phi_k L^k\right), \quad \psi_1 = 1$$

$$0 = \sum_{k=1}^{\infty} \psi_k L^k - \sum_{i=1}^{p} \phi_i L^i - \left(\sum_{i=1}^{p} \phi_i L^i\right)\left(\sum_{k=1}^{\infty} \psi_k L^k\right).$$

Coeffs: $L^1 : \psi_1 - \phi_1 = 0$ $\quad L^2 : \psi_2 - \phi_2 - \phi_1\psi_1 = 0$

$$\Rightarrow \psi_2 = \phi_2 + \phi_1\psi_1.$$

For L^s, $s > p$, we have

$$(\psi_s - \phi_1\psi_{s-1} - \phi_2\psi_{s-2} - \dots - \phi_p\psi_{s-p}) L^s = 0$$

Let $\psi_{-1} = \psi_{-2} = \dots = \psi_{-p+1} = 0$. Can write as

$$\begin{pmatrix} \psi_s \\ \psi_{s-1} \\ \vdots \\ \psi_{s-p} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_p \\ 1 & & & 0 \\ & 1 & & \\ 0 & & \ddots & 1 \end{pmatrix} \begin{pmatrix} \psi_{s-1} \\ \psi_{s-2} \\ \vdots \\ \psi_{s-p-1} \end{pmatrix}$$

$$\xi_s = F \xi_{s-1}$$

Time Series Econometrics
Supplementary Lecture 2

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1 Exercise 3.1

Is the following $MA(2)$ process covariance-stationary?

$$Y_t = (1 + 2.4L + 0.8L^2)\varepsilon_t$$
$$E(\varepsilon_t \varepsilon_\tau) = \begin{cases} 1, & \text{for } t = \tau \\ 0, & \text{otherwise.} \end{cases}$$

If so, calculate its autocovariances.

A process $\{y_t\}$ is (covariance-)stationary if (p. 45)

$$\begin{aligned} E(Y_t) &= \mu, && \text{for all } t \\ E(Y_t - \mu)(Y_{t-j} - \mu) &= \gamma_j, && \text{for all } t \text{ and any } j. \end{aligned}$$

In our case,

$$\begin{aligned} E(Y_t) = E[(1 + 2.4L + 0.8L^2)\varepsilon_t)] &= E(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2}) \\ &= E(\varepsilon_t) + 2.4E(\varepsilon_{t-1}) + 0.8E(\varepsilon_{t-2}) \\ &= 0. \end{aligned}$$

So $E(Y_t) = \mu = 0$ for all t, i.e. the mean does not depend on t.

For the autocovariances, note that $E(Y_t - \mu)(Y_{t-j} - \mu) = E(Y_t Y_{t-j}) = Cov(Y_t, Y_{t-j})$.

$$\begin{aligned} Cov(Y_t, Y_{t-j}) &= Cov(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2}, \varepsilon_{t-j} + 2.4\varepsilon_{t-j-1} + 0.8\varepsilon_{t-j-2}) \\ &= Cov(\varepsilon_t, \varepsilon_{t-j} + 2.4\varepsilon_{t-j-1} + 0.8\varepsilon_{t-j-2}) && (*) \\ &+ 2.4Cov(\varepsilon_{t-1}, \varepsilon_{t-j} + 2.4\varepsilon_{t-j-1} + 0.8\varepsilon_{t-j-2}) && (\dagger) \\ &+ 0.8Cov(\varepsilon_{t-2}, \varepsilon_{t-j} + 2.4\varepsilon_{t-j-1} + 0.8\varepsilon_{t-j-2}) && (\ddagger) \end{aligned}$$

Look at these three terms one by one. The thing to note here is that these covariances will only be non-zero if we have common error terms in the covariance, so what we can do is to look at for what j this is the case for each term. We then get

$$(*) = \begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$(\dagger) = \begin{cases} 2.4^2, & \text{if } j = 0 \\ 2.4, & \text{if } j = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$(\ddagger) = \begin{cases} 0.8^2, & \text{if } j = 0 \\ 0.8 * 2.4, & \text{if } j = 1 \\ 0.8, & \text{if } j = 2 \\ 0, & \text{otherwise} \end{cases}$$

Taking these cases together we get

$$\gamma_j = Cov(Y_t, Y_{t-j}) = \begin{cases} 1 + 2.4^2 + 0.8^2, & \text{if } j = 0 \\ 2.4 + 0.8 * 2.4, & \text{if } j = 1 \\ 0.8, & \text{if } j = 2 \\ 0, & \text{otherwise.} \end{cases}$$

These autocovariances do not depend on t, hence the process is stationary. But, this is an MA(2) process, so by identifying it as such we can immediately come to that conclusion, as this process is always stationary.

2 Exercise 3.2

Is the following $AR(2)$ process covariance-stationary?

$$(1 - 1.1L + 0.18L^2)Y_t = \varepsilon_t$$

$$E(\varepsilon_t \varepsilon_\tau) = \begin{cases} 1, & \text{for } t = \tau \\ 0, & \text{otherwise.} \end{cases}$$

If so, calculate its autocovariances.

Denote the lag polynomial by $\phi(L)$. To check stationarity, we can look at the roots of $\phi(z) = 0$ and if they are outside the unit circle. Similarly, we can premultiply it by $\lambda^2 = z^{-2}$ and solve for λ instead (which is sometimes easier), in which case the roots should be inside the unit circle for stationarity.

$$\begin{aligned}
\phi(z) &= 0 \\
(1 - 1.1z + 0.18z^2) &= 0 \\
z^{-2}(1 - 1.1z + 0.18z^2) &= 0 \\
(\lambda^2 - 1.1\lambda + 0.18) &= 0.
\end{aligned}$$

This is just a second-degree polynomial, for which we can use the quadratic formula:

$$x^2 + bx + c = 0 \Longrightarrow x = \frac{-b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}.$$

In our case, $x = \lambda$, $b = -1.1$ and $c = 0.18$, so we get

$$\begin{aligned}
\lambda &= \frac{1.1}{2} \pm \frac{\sqrt{1.1^2 - 4 \times 0.18}}{2} = 0.55 \pm \frac{\sqrt{1.21 - 0.72}}{2} = 0.55 \pm \frac{\sqrt{0.49}}{2} \\
&= 0.55 \pm 0.35 = \begin{cases} 0.9 \\ 0.2 \end{cases}
\end{aligned}$$

The two roots are 0.9 and 0.2, which are inside the unit circle. Similarly, since $z = \lambda^{-1}$, the roots of the characteristic equation are $0.9^{-1} = 10/9$ and $0.2^{-1} = 5$, which are outside of the unit circle. The process is therefore stationary.

For the autocovariances, we start with γ_0 (and we do this for a general AR(2) to keep the calculations cleaner):

$$\begin{aligned}
\gamma_0 = V(Y_t) &= V(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t) \\
&= \phi_1^2 V(Y_{t-1}) + \phi_2^2 V(Y_{t-2}) + V(\varepsilon_t) + 2\phi_1 Cov(Y_{t-1}, \varepsilon_t) \\
&+ 2\phi_2 Cov(Y_{t-2}, \varepsilon_t) + 2\phi_1\phi_2 Cov(Y_{t-1}, Y_{t-2}) \\
&= \phi_1^2 \gamma_0 + \phi_2^2 \gamma_0 + \sigma^2 + 2\phi_1\phi_2\gamma_1 \\
\gamma_0 &= \frac{\sigma^2 + 2\phi_1\phi_2\gamma_1}{1 - \phi_1^2 - \phi_2^2}
\end{aligned}$$

Note that this depends on γ_1. Next, calculate γ_1:

$$\begin{aligned}
\gamma_1 &= Cov(Y_t, Y_{t-1}) = Cov(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t, Y_{t-1}) \\
&= \phi_1 Cov(Y_{t-1}, Y_{t-1}) + \phi_2 Cov(Y_{t-2}, Y_{t-1}) + Cov(\varepsilon_t, Y_{t-1}) \\
&= \phi_1\gamma_0 + \phi_2\gamma_1 \\
\gamma_1 &= \frac{\phi_1\gamma_0}{1 - \phi_2}
\end{aligned}$$

Note that this, in turn, depends on γ_0. To get rid of this circular dependence, substitute γ_1 into γ_0:

$$
\begin{aligned}
\gamma_0 = \frac{\sigma^2 + 2\phi_1\phi_2\gamma_1}{1-\phi_1^2-\phi_2^2} &= \frac{\sigma^2 + 2\phi_1\phi_2\frac{\phi_1\gamma_0}{1-\phi_2}}{1-\phi_1^2-\phi_2^2} = \frac{\sigma^2(1-\phi_2) + 2\phi_1^2\phi_2\gamma_0}{(1-\phi_1^2-\phi_2^2)(1-\phi_2)} \\
\gamma_0\left(1 - \frac{2\phi_1^2\phi_2}{(1-\phi_1^2-\phi_2^2)(1-\phi_2)}\right) &= \frac{\sigma^2(1-\phi_2)}{(1-\phi_1^2-\phi_2^2)(1-\phi_2)} \\
\gamma_0\left(\frac{(1-\phi_1^2-\phi_2^2)(1-\phi_2) - 2\phi_1^2\phi_2}{(1-\phi_1^2-\phi_2^2)(1-\phi_2)}\right) &= \frac{\sigma^2(1-\phi_2)}{(1-\phi_1^2-\phi_2^2)(1-\phi_2)} \\
\gamma_0 &= \frac{\sigma^2(1-\phi_2)}{(1-\phi_1^2-\phi_2^2)(1-\phi_2) - 2\phi_1^2\phi_2}
\end{aligned}
$$

The kth autocovariance, for a general k, we can get as:

$$
\begin{aligned}
&\gamma_k = Cov(Y_{t-k}, Y_t) = Cov(\phi_1 Y_{t-k-1} + \phi_2 Y_{t-k-2} + \varepsilon_{t-k}, Y_t) \\
&\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2}
\end{aligned}
$$

In our case, $\phi_1 = 1.1$, $\phi_2 = -0.18$ and $\sigma^2 = 1$ so

$$
\begin{aligned}
\gamma_0 &= 7.89 \\
\gamma_1 &= 7.35 \\
\gamma_2 &= 6.66
\end{aligned}
$$

and so on.

3 Exercise 3.3

A covariance stationary AR(p) process,

$$(1 - \phi_2 L - \phi_2 L^2 - \cdots - \phi_p L^p)(Y_t - \mu) = \varepsilon_t,$$

has an $MA(\infty)$ representation given by

$$(Y_t - \mu) = \psi(L)\varepsilon_t,$$

with

$$\psi(L) = (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)^{-1}$$

or

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)(\psi_0 + \psi_1 L + \psi_2 L^2 + \cdots) = 1.$$

In order for this equation to be true, the implied coefficient on L^0 must be unity and the coefficients on L^1, L^2, L^3, ... must be zero. Write out these conditions explicitly and show that they imply a recursive algorithm for generating the $MA(\infty)$ weights $\psi_0, \psi_1, \ldots$. Show that this recursion is algebraically equivalent to setting ψ_j equal to the $(1,1)$ *element of the matrix* **F** *raised to the jth power as in equation [1.2.28].*

First, rewrite the condition of multiplication of the polynomials being equal to 1 using sums and then expand it:

$$\begin{aligned}(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)(\psi_0 + \psi_1 L + \psi_2 L^2 + \cdots) &= 1 \\ \left(1 - \sum_{i=1}^{p} \phi_i L^i\right)\left(\psi_0 + \sum_{k=1}^{\infty} \psi_k L^k\right) &= 1 \\ \psi_0 + \sum_{k=1}^{\infty} \psi_k L^k - \psi_0 \sum_{i=1}^{p} \phi_i L^i - \sum_{i=1}^{p} \phi_i L^i \sum_{k=1}^{\infty} \psi_k L^k &= 1\end{aligned}$$

Think of this is a regular polynomial with some coefficients, in which the idea might be easier to see. What we have is essentially

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1.$$

However, the left hand side has x terms, but the right hand side does not. That means that the coefficients on the x terms should all be 0, as we otherwise would have x terms on the right hand side - which we don't. So for the left hand side to be equal to the right hand side, we must have that $a_0 = 1$ and $a_1 = a_2 = \cdots = 0$. What we do in order to find the MA weights is to use these restrictions.

In our case, we have our polynomial in L instead of x, but the principle is the same. What we can do then is to find all terms that are multiplied by L^1, since we know that these should sum to zero. Similarly, we find the terms that are multiplied by L^2, etc. First, however, we can note that all terms are multiplied by L except ψ_0. This is our a_0, and for the RHS to equal the LHS this must be 1. So with $\psi_0 = 1$, we have

$$\begin{aligned}1 + \sum_{k=1}^{\infty} \psi_k L^k - \sum_{i=1}^{p} \phi_i L^i - \sum_{i=1}^{p} \phi_i L^i \sum_{k=1}^{\infty} \psi_k L^k &= 1 \\ \sum_{k=1}^{\infty} \psi_k L^k - \sum_{i=1}^{p} \phi_i L^i - \sum_{i=1}^{p} \phi_i L^i \sum_{k=1}^{\infty} \psi_k L^k &= 0\end{aligned}$$

For the coefficient on L^1, we get one term from the first sum ($k = 1$) and from the second ($i = 1$). The third sum's L term of smallest degree is L^2, so this is not relevant here. What we have is thus

$$\begin{aligned}(\psi_1 - \phi_1)L &= 0\\ \psi_1 - \phi_1 &= 0\\ \psi_1 &= \phi_1.\end{aligned}$$

For L^2, we get one term from the first sum ($k = 2$), one from the second ($i = 2$) and one from the third (when the indices in these two sums are $i = 1$ and $k = 1$, respectively, because then we get an L^2 term). Hence,

$$\begin{aligned}(\psi_2 - \phi_2 - \phi_1\psi_1)L^2 &= 0\\ \psi_2 - \phi_2 - \phi_1\psi_1 &= 0\\ \psi_2 &= \phi_2 + \phi_1\psi_1\end{aligned}$$

For L^3, we get one term from the first sum ($k = 3$) and one from the second ($i = 3$) and then two from the third sum: one when $k = 1$ and $i = 2$ in these sums, and one when $k = 2$ and $i = 1$. This gives us

$$\begin{aligned}(\psi_3 - \phi_3 - \phi_2\psi_1 - \phi_1\psi_2)L^3 &= 0\\ \psi_3 - \phi_3 - \phi_2\psi_1 - \phi_1\psi_2 &= 0\\ \psi_3 &= \phi_3 + \phi_2\psi_1 + \phi_1\psi_2\end{aligned}$$

In order to calculate the weights like this for ψ_s, we need $s \leq p$, or otherwise the second sum does not contribute as it doesn't have an L^s term in this case. So what happens when calculating ψ_s for $s > p$? In this case, we still get one term from the first sum. The third sum also gives us something, and more specifically this will be the terms in the two sums that correspond to the cases when the summation indices sum to s, i.e. when $k + i = s$. The first part of the third sum starts at $i = 1$. Thus, then the relevant term in the second sum is the one corresponding to $s - 1$, since then we will have $\phi_1 L^1 \times \psi_{s-1}L^{s-1} = \phi_1\psi_{s-1}L^s$ and we have an L^s term as desired. Similarly, for $i = 2$ we get $s - 2$ from the second term, and so on. Lastly, we will have $i = p$, and then the relevant term in the second part of the sum is $s - p$, so that we then get $\phi_p L^p \times \psi_{s-p}L^{s-p} = \phi_p\psi_{s-p}L^s$.

In summary, for $s > p$, we get

$$\begin{aligned}(\psi_s - \phi_1\psi_{s-1} - \phi_2\psi_{s-2} - \cdots - \phi_p\psi_{s-p})L^p &= 0\\ \psi_s - \phi_1\psi_{s-1} - \phi_2\psi_{s-2} - \cdots - \phi_p\psi_{s-p} &= 0\\ \psi_s &= \phi_1\psi_{s-1} + \phi_2\psi_{s-2} + \cdots + \phi_p\psi_{s-p}\end{aligned}$$

However, since we have defined ψ_0 to be 1, this is actually true even for $s = p$. In this case, the last term is $\phi_p\psi_{p-p} = \phi_p$ and we get the same as if we had done it like we did for ψ_1, ψ_2, ψ_3. Here we now have the recursive algorithm for calculating these weights: given the p previous weights, we can compute the next weight. Next is to show that it is the same as the $(1,1)$ element of the matrix $\mathbf{F}^j$.

To do this, we will first need to define $\psi_{-1} = \psi_{-2} = \cdots = \psi_{-p+1} = 0$. If we do this, then

$$\psi_s = \phi_1\psi_{s-1} + \phi_2\psi_{s-2} + \cdots + \phi_p\psi_{s-p}$$

is true for all $s = 1, 2, \ldots$. But, what we can note is that this is just a difference equation as in Chapter 1. We can write this as a first order process:

$$\underbrace{\begin{pmatrix} \psi_s \\ \psi_{s-1} \\ \vdots \\ \psi_{s-p} \end{pmatrix}}_{\boldsymbol{\xi}_s} = \underbrace{\begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_p \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}}_{\mathbf{F}} \underbrace{\begin{pmatrix} \psi_{s-1} \\ \psi_{s-2} \\ \vdots \\ \psi_{s-p-1} \end{pmatrix}}_{\boldsymbol{\xi}_{s-1}}$$

We can use recursive substitution of this to write it as a function of the initial values:

$$\begin{aligned} \boldsymbol{\xi}_s &= \mathbf{F}\boldsymbol{\xi}_{s-1} = \mathbf{F}^2\boldsymbol{\xi}_{s-2} = \mathbf{F}^3\boldsymbol{\xi}_{s-3} = \cdots \\ &= \mathbf{F}^s\boldsymbol{\xi}_0 \\ &= \mathbf{F}^s \begin{pmatrix} \psi_0 \\ \psi_{-1} \\ \vdots \\ \psi_{-p+1} \end{pmatrix} = \mathbf{F}^s \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{aligned}$$

by the definitions of $\psi_0, \psi_{-1}, \ldots$. In more detail, we have

$$\begin{aligned} \begin{pmatrix} \psi_s \\ \psi_{s-1} \\ \vdots \\ \psi_{s-p} \end{pmatrix} &= \begin{pmatrix} f_{1,1}^{(s)} & f_{1,2}^{(s)} & \cdots & f_{1,p}^{(s)} \\ f_{2,1}^{(s)} & f_{2,2}^{(s)} & \cdots & f_{2,p}^{(s)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{p,1}^{(s)} & f_{p,2}^{(s)} & \cdots & f_{p,p}^{(s)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ \begin{pmatrix} \psi_s \\ \psi_{s-1} \\ \vdots \\ \psi_{s-p} \end{pmatrix} &= \begin{pmatrix} f_{1,1}^{(s)} \\ f_{2,1}^{(s)} \\ \vdots \\ f_{p,1}^{(s)} \end{pmatrix} \end{aligned}$$

Hence, $\psi_s = f_{1,1}^{(s)}$ as in [1.2.28].

4 Exercise 3.8

Show that the MA(2) process in Exercise 3.1 is not invertible. Find the invertible representation for the process. Calculate the autocovariances of the invertible representation using equation [3.3.12] and verify that these are the same as obtained in Exercise 3.1.

The λ roots are 0.4 and 2. It is thus non-invertible since one of the roots is greater than 1. On p. 68, the invertible representation is discussed. In short, what this means for us is that if

$$Y_t = (1 + \lambda_1 L)(1 + \lambda_2 L)\tilde{\varepsilon}_t$$

is non-invertible such that $|\lambda_1| < 1$ and $|\lambda_2| > 1$ and $E(\tilde{\varepsilon}_t \tilde{\varepsilon}_\tau) = \tilde{\sigma}^2$ for $t = \tau$ and 0 otherwise, then the invertible representation is given by

$$Y_t = (1 + \lambda_1 L)(1 + \lambda_2^{-1} L)\varepsilon_t,$$

where $E(\varepsilon_t \varepsilon_\tau) = \lambda_2^2 \tilde{\sigma}^2$ for $t = \tau$ and 0 otherwise. For us, $\lambda_1 = 0.4$, $\lambda_2 = 2$ and $\tilde{\sigma}^2 = 1$, so the invertible representation is

$$\begin{aligned} Y_t &= (1 + 0.4L)(1 + 2^{-1} L)\varepsilon_t \\ Y_t &= (1 + 0.9L + 0.2L^2)\varepsilon_t, \end{aligned}$$

where $E(\varepsilon_t \varepsilon_\tau) = 4$ for $t = \tau$ and 0 otherwise.

5 Exercise 4.1c

Use formula [4.3.6] to show that for a covariance-stationary process, the projection of Y_{t+1} on a constant and Y_t is given by

$$\hat{E}(Y_{t+1}|Y_t) = (1 - \rho_1)\mu + \rho_1 Y_t$$

where $\mu = E(Y_t)$ and $\rho_1 = \gamma_1/\gamma_0$.

c) Show that for an $AR(2)$ process, the implied forecast is

$$\mu + [\phi_1/(1 - \phi_2)](Y_t - \mu).$$

Is the error associated with this forecast correlated with Y_t? Is it correlated with Y_{t-1}?

From [4.3.5], a linear forecast of Y_{t+1} on a constant and Y_t is given by $\boldsymbol{\alpha}^{(m)\prime}\mathbf{X}_t$, where $\mathbf{X}_t = (1, Y_t, \ldots, Y_{t-m+1})'$ and

$$\boldsymbol{\alpha}^{(m)\prime} = \begin{pmatrix} \mu, & \gamma_1 + \mu^2, & \ldots, & \gamma_m + \mu^2 \end{pmatrix} \begin{pmatrix} 1 & \mu & \cdots & \mu \\ \mu & \gamma_0 + \mu^2 & \cdots & \gamma_{m-1} + \mu^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mu & \gamma_{m-1} + \mu^2 & \cdots & \gamma_0 + \mu^2 \end{pmatrix}^{-1}.$$

With $m = 1$, $\mathbf{X}_t = \begin{pmatrix} 1 & Y_t \end{pmatrix}'$ and

$$\begin{aligned}
\boldsymbol{\alpha}^{(1)'} &= \begin{pmatrix} \mu, & \gamma_1 + \mu^2 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ \mu & \gamma_0 + \mu^2 \end{pmatrix}^{-1} \\
&= \frac{1}{\gamma_0 + \mu^2 - \mu^2} \begin{pmatrix} \mu, & \gamma_1 + \mu^2 \end{pmatrix} \begin{pmatrix} \gamma_0 + \mu^2 & -\mu \\ -\mu & 1 \end{pmatrix} \\
&= \frac{1}{\gamma_0} \begin{pmatrix} \mu(\gamma_0 + \mu^2) - \mu(\gamma_1 + \mu^2), & \gamma_1 + \mu^2 - \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu - \frac{\gamma_1}{\gamma_0}\mu, & \frac{\gamma_1}{\gamma_0} \end{pmatrix} = \begin{pmatrix} \mu(1 - \rho_1), & \rho_1 \end{pmatrix}.
\end{aligned}$$

So the linear forecast is given by

$$\hat{E}(Y_{t+1}|Y_t) = \begin{pmatrix} \mu(1 - \rho_1), & \rho_1 \end{pmatrix} \begin{pmatrix} 1 \\ Y_t \end{pmatrix} = \mu(1 - \rho_1) + \rho_1 Y_t.$$

In an AR(2), we have

$$\gamma_1 = \frac{\phi_1}{1 - \phi_2}\gamma_0, \quad \rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\phi_1}{1 - \phi_2}$$

so the forecast is given by

$$\begin{aligned}
\hat{E}(Y_{t+1}|Y_t) &= \mu(1 - \rho_1) + \rho_1 Y_t \\
&= \mu + \rho_1(Y_t - \mu) \\
&= \mu + \frac{\phi_1}{1 - \phi_2}(Y_t - \mu).
\end{aligned}$$

Let the forecast error be $\hat{\varepsilon}_{t+1} = Y_{t+1} - \hat{E}(Y_{t+1}|Y_t)$. The forecast error is:

$$\begin{aligned}
\hat{\varepsilon}_{t+1} &= Y_{t+1} - \hat{E}(Y_{t+1}|Y_t) \\
&= (\mu + \phi_1 Y_t + \phi_2 Y_{t-1} + \varepsilon_{t+1}) - (\mu + \rho_1(Y_t - \mu)) \\
&= \mu\rho_1 + (\phi_1 - \rho_1)Y_t + \phi_2 Y_{t-1} + \varepsilon_{t+1}.
\end{aligned}$$

The covariance with Y_t:

$$\begin{aligned}
Cov(\hat{\varepsilon}_{t+1}, Y_t) &= Cov(\mu\rho_1 + (\phi_1 - \rho_1)Y_t + \phi_2 Y_{t-1} + \varepsilon_{t+1}, Y_t) \\
&= (\phi_1 - \rho_1)V(Y_t) + \phi_2 Cov(Y_{t-1}, Y_t) \\
&= (\phi_1 - \rho_1)\gamma_0 + \phi_2\gamma_1 \\
&= \left(\phi_1 - \frac{\phi_1}{1 - \phi_2}\right)\gamma_0 + \phi_2 \frac{\phi_1}{1 - \phi_2}\gamma_0 \\
&= -\frac{\phi_1\phi_2}{1 - \phi_2}\gamma_0 + \frac{\phi_1\phi_2}{1 - \phi_2}\gamma_0 \\
&= 0.
\end{aligned}$$

Covariance with Y_{t-1}:

$$
\begin{aligned}
Cov(\hat{\varepsilon}_{t+1}, Y_{t-1}) &= Cov(\mu\rho_1 + (\phi_1 - \rho_1)Y_t + \phi_2 Y_{t-1} + \varepsilon_{t+1}, Y_{t-1}) \\
&= (\phi_1 - \rho_1)Cov(Y_t, Y_{t-1}) + \phi_2 V(Y_{t-1}) \\
&= \left(\phi_1 - \frac{\phi_1}{1-\phi_2}\right)\gamma_1 + \phi_2\gamma_0 \\
&= \left(\phi_1 - \frac{\phi_1}{1-\phi_2}\right)\frac{\phi_1}{1-\phi_2}\gamma_0 + \phi_2\gamma_0 \\
&= -\frac{\phi_1^2\phi_2}{(1-\phi_2)^2}\gamma_0 + \phi_2\gamma_0 \\
&= \left(\phi_2 - \frac{\phi_1^2\phi_2}{(1-\phi_2)^2}\right)\gamma_0 \\
&= \left(\frac{\phi_2[(1-\phi_2)^2 - \phi_1^2]}{(1-\phi_2)^2}\right)\gamma_0
\end{aligned}
$$

For a stationary process, $\phi_2 \neq 1$ and $\gamma_0 \neq 0$, so this covariance will only be zero if either $\phi_2 = 0$ (i.e. an AR(1)) or if $\phi_1 = \pm(\phi_2 - 1)$. But remembering the triangle region of stationarity for an AR(2) (p. 17), this would mean that we are on the border of the stationarity region (the upper left and right sides of the triangle). So for a stationary AR(2), the forecast error $\hat{\varepsilon}_{t+1}$ with $m = 1$ is only uncorrelated with Y_{t-1} if the AR(2) is really an AR(1).