Time Series Econometrics, 2ST111

Lecture 8. Nonstationarity & Deterministic Trends

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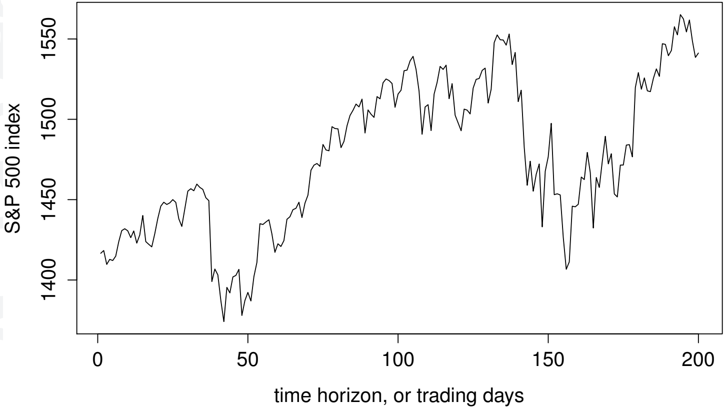
Outline of Today's Lecture

- Models of Nonstationary Time Series (pp.435-453 in Hamilton)
  - Deterministic Time Trend & Unit Root Approaches
  - Unit Root Process
  - ARIMA(p, d, q) Process
  - Linear vs. Exponential Time Trends
  - Comparison of Trend-Stationary & Unit Root Processes
- Processes with Deterministic Time Trends (pp.454-474 in Hamilton)
  - Asymptotic Results for OLS Estimators for the Simple Trend-Stationary Process
  - Order in Probability
  - Asymptotic Results for OLS Estimators for the Trend-Stationary AR(p) Process

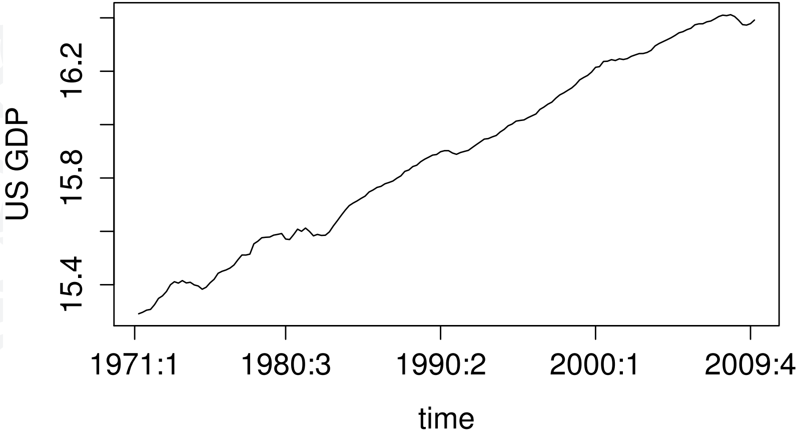
Why Nonstationarity instead of Stationary ARMA?

S&P daily closing price indices (time series plot), from 1 Jan to 17 Oct in 2007



Why Nonstationarity instead of Stationary ARMA?

Monthly log US GDP from Jan 1971 to April 2009



Models for Nonstationary Time Series

We consider two different approaches for modeling nonstationary time series

1. A deterministic time trend approach

$$y_t = \alpha + \delta t + \psi(L)\varepsilon_t, \tag{1}$$

   where $\alpha, \delta \in \mathbb{R}$ and $\alpha + \delta t$ is a deterministic time trend.

2. A unit root approach:

$$\Delta y_t = \delta + \psi(L)\varepsilon_t, \tag{2}$$

   where $\Delta = 1 - L$ and $\psi(1) \neq 0$.

From now on, the small letter y_t will be used for both random variables and the observations.

Remarks

- The stochastic process given by (1) is sometimes said to be trend-stationary, because if one subtracts the trend δt from it, the result is a stationary process.
- The condition that $\psi(1) \neq 0$ for the unit root process (2) ensures that y_t is nonstationary. $\varepsilon_t \overset{iid}{\sim} (0, \sigma^2)$
- The prototypical example of a unit root process (2) is obtained when $\psi(L) = 1$

$$y_t = y_{t-1} + \delta + \varepsilon_t, \tag{3}$$

which is known as a random walk with drift δ.
- There are several other approaches. For example, fractionally integrated processes and processes with occasional discrete shifts in trend. See pp.447-451 in Hamilton.

To see that the condition $\psi(1) \neq 0$ ensures that y_t is nonstationary, suppose that y_t is stationary with $MA(\infty)$ representation

$$y_t = \mu + \chi(L)\varepsilon_t. \tag{4}$$

By taking the first-order difference, we have

$$(1-L)y_t = \underbrace{(1-L)\mu}_{=0} + \underbrace{(1-L)\chi(L)}_{=\psi(L)}\varepsilon_t, \tag{5}$$

where $\psi(1) = (1-1)\chi(1) = 0$.

Claim: Let $Y_t = \mu + \chi(L)\varepsilon_t$.

Then $\psi(1) \neq 0 \Rightarrow Y_t$ not stationary.

Proof: Can show Y_t stationary $\Rightarrow \psi(1) = 0$.

Let $(1-L)Y$

Unit Root Process

It is sometimes convenient to work with a slightly different representation of the unit root process in (2). Let

$$y_t = \alpha + \delta t + u_t, \tag{6}$$

where u_t is a zero-mean ARMA(p, q) process

$$\phi(L)u_t = \theta(L)\varepsilon_t. \tag{7}$$

Assume that ε_t is white noise, the MA lag polynomial $\theta(L)$ is invertible, and the AR lag polynomial is stable.

Unit Root Process

If the lag polynomial in the AR part is stable, (7) can be written as

$$u_t = \frac{1 + \theta_1 L + \theta_2 L^2 + ... + \theta_q L^q}{(1 - \lambda_1 L)(1 - \lambda_2 L)...(1 - \lambda_p L)} \varepsilon_t = \psi(L)\varepsilon_t \qquad (8)$$

where $\sum_{i=0}^{\infty} |\psi_i| < \infty$. It is exactly the form in (1) (trend stationary)!

Unit Root Process

Now suppose that one $\lambda_i = 1$ and $|\lambda_j| < 1$ for $j \neq i$. Without loss of generality, let $\lambda_1 = 1$

$$(1-L)u_t = \frac{1+\theta_1 L + \theta_2 L^2 + ... + \theta_q L^q}{(1-\lambda_2 L)(1-\lambda_3 L)...(1-\lambda_p L)}\varepsilon_t = \psi^*(L)\varepsilon_t \tag{9}$$

with $\sum_{i=0}^{\infty} |\psi_i^*| < \infty$.

By taking the first-order difference of y_t, we obtain

$$(1-L)y_t = \underbrace{(1-L)\alpha}_{=0} + \underbrace{(1-L)\delta t}_{=\delta} + (1-L)u_t = \delta + \psi^*(L)\varepsilon_t, \tag{10}$$

which is exactly the unit root process in (2).

Unit Root Process

$u_t = I(1) \Rightarrow \Delta u_t = I(0)$

$\varepsilon_t = I(0) \Rightarrow \sum_t \varepsilon_t = I(1)$

Remarks:

- (6) together with (9) and (10) explain why (2) is called a unit root process. One of the roots of the lag polynomial in the AR part of u_t equals one, and all other roots lie outside the unit disk.
- The unit root process (2) with only one unit root is also called integrated of order 1, or simply I(1).
- $\psi^*(L)\varepsilon_t$ in (9) and (10) with $\psi^*(1) \neq 0$ is called I(0) process.
- If, unfortunately, two roots equal one, and the other roots lie outside the unit disk, then y_t has be to be differenced twice to reach I(0).

$$\Delta^2 y_t = \kappa + \psi^*(L)\varepsilon_t. \tag{11}$$

  y_t in this case is called I(2).
- Think about in which case $\kappa \neq 0$ (quadratic trend).

ARIMA(p, d, q)

A general stochastic process is called an autoregressive integrated moving average process, or simply an ARIMA(p, d, q) process. It takes the form as follows

$$\begin{aligned} y_t = \alpha + \delta t + u_t \quad &\sim \quad \mathrm{I}(d) \\ \Delta^d \phi(L) u_t = \theta(L)\varepsilon_t \quad &\sim \quad \mathrm{I}(0) \end{aligned} \tag{12}$$

with $\phi(L) = 1 - \phi_1 L - \ldots - \phi_p L^p$ stable, and $\theta(L) = 1 + \theta_1 L + \ldots + \theta_q L^q$ invertible. We only consider the case when the integration order d is a non-negative integer, or it's gonna be fractionally integrated.

Taking dth difference produces a stationary ARMA(p, q) process $\Delta^d y_t$.

If $d = 0$ but $\delta \neq 0$, it is trend stationary.

ARIMA(p, d, q)

The ARIMA(p, d, q) can also be rewritten in the following form:

- First, take the dth difference on y_t process

$$\Delta^d y_t = \Delta^d \alpha + \Delta^d \delta t + \Delta^d u_t.$$

- Since $\Delta^d \phi(L) u_t = \theta(L)\varepsilon_t$, then $\Delta^d u_t = \phi(L)^{-1}\theta(L)\varepsilon_t = \psi(L)\varepsilon_t$

$$\Delta^d y_t = \Delta^d \alpha + \Delta^d \delta t + \psi(L)\varepsilon_t.$$

- Multiply both sides by $\phi(L)$

$$\Delta^d \phi(L) y_t = \Delta^d \phi(1)\alpha + \Delta^d \phi(L)\delta t + \theta(L)\varepsilon_t.$$

Note that $\phi(L)\delta t = \gamma t + \eta$ and $\phi(1)\alpha = \zeta$. Then

$$\Delta^d \phi(L) y_t = \Delta^d \tilde{\alpha} + \Delta^d \tilde{\delta} t + \theta(L)\varepsilon_t$$

where $\tilde{\alpha} = \eta + \zeta$ and $\tilde{\delta} = \gamma$. This ARIMA form is more often employed.

Linear vs. Exponential Time Trends

- In practice, many economic time series exhibit an exponential trend rather than a linear trend
- Because of this, it is common to take logs of economic time series before attempting to model them with the trend-stationary or unit root process, respectively.
- Note that

$$\begin{aligned}
\Delta \log y_t &= \log y_t - \log y_{t-1} = \log \frac{y_t}{y_{t-1}} \\
&= \log \frac{y_{t-1} + y_t - y_{t-1}}{y_{t-1}} = \log \left(1 + \frac{y_t - y_{t-1}}{y_{t-1}}\right) \\
&\approx \frac{y_t - y_{t-1}}{y_{t-1}} \quad \text{provided that } \left|\frac{y_t - y_{t-1}}{y_{t-1}}\right| \text{very small.}
\end{aligned}$$

This is referred to as the growth rate in discrete time.

Linear vs. Exponential Time Trends

- In finance, the continuous compound growth rate is widely used.
- Suppose that the value P goes to F after one year. The annual growth rate R_y is computed from the identity

$$F = P(1 + R_y).$$

- If the corresponding interest is compounded every $1/n$ period, then the compound annual growth rate R_n is computed from

$$F = P\left(1 + \frac{R_n}{n}\right)^n.$$

- Suppose that the compound time period can be infinitely small, then the continuous compound annual growth rate r is obtained from

$$\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n = \exp r = \frac{F}{P}, \quad \text{or} \quad r = \log F - \log P.$$

Linear vs. Exponential Time Trends

- The continuous compound growth rate has the very nice feature that

$$\exp(r(t_1 - t_0)) \exp(r(t_2 - t_1)) = \exp(r(t_2 - t_0))$$

  where t_0, t_1, t_2 are time points.
- If the growth rate r_t is a time-varying from time point t_0 to point t_1, the discrete growth rate for that period is

$$R = \exp \int_{t_0}^{t_1} r_t \mathrm{d}t,$$

- It is reasonable to assume that some growth rates are I(0).

Comparison of Trend-Stationary & Unit Root Processes

Let us compare the forecasts of a trend-stationary and unit root process.

To forecast a trend-stationary process, the known deterministic component

$$\alpha + \delta t$$

is simply added to the forecast of the stationary component

$$\psi(L)\varepsilon_t$$

Comparison of Trend-Stationary & Unit Root Processes

Hence, for the trend-stationary process

$$y_t = \alpha + \delta t + \psi(L)\varepsilon_t,$$

the s-step ahead forecast of y_{t+s} at time t is

$$\begin{aligned} \hat{y}_{t+s|t} &= \hat{E}(y_{t+s}|y_t, y_{t-1}, \dots) \\ &= \alpha + \delta(t+s) + \psi_s\varepsilon_t + \psi_{s+1}\varepsilon_{t-1} + \psi_{s+2}\varepsilon_{t-2} + \dots \qquad (13) \end{aligned}$$

The absolute summability of $\{\psi_i\}_{i=0}^{\infty}$ implies that this s-step ahead forecast converges in mean square to the time trend. That is

$$\lim_{s\to\infty} \mathrm{E}[\hat{y}_{t+s|t} - \alpha - \delta(t+s)]^2 = 0$$

Comparison of Trend-Stationary & Unit Root Processes

By contrast, it can be shown that the s-step-ahead forecast of y_{t+s} at time t for the unit process

$$\Delta y_t = \delta + \psi(L)\varepsilon_t$$

is given by

$$\hat{y}_{t+s|t} = s\delta + y_t + (\psi_s + \psi_{s-1} + ... + \psi_1)\varepsilon_t + (\psi_{s+1} + \psi_s + ... + \psi_2)\varepsilon_{t-1} + ... \quad (14)$$

In particular, if $\psi(L)\varepsilon_t = \varepsilon_t$, then

$$\hat{y}_{t+s|t} = s\delta + y_t \quad (15)$$

Comparison of Trend-Stationary & Unit Root Processes

It can be shown that the mean squared error (MSE)

$$\mathrm{MSE} = \mathsf{E}(y_{t+s} - \hat{y}_{t+s|t})^2 \tag{16}$$

for the trend-stationary process converges to a constant as $s \to \infty$.

By contrast, the MSE for the unit root process diverges as $s \to \infty$.

Comparison of Trend-Stationary & Unit Root Processes

It can also be shown that, for the trend-stationary process, the dynamic multiplier is $\partial y_{t+s}/\partial \varepsilon_t = \psi_s$, and hence

$$\lim_{s\to\infty} \frac{\partial y_{t+s}}{\partial \varepsilon_t} = 0. \tag{17}$$

By contrast, for the unit root process, $\partial y_{t+s}/\partial \varepsilon_t = \sum_{i=0}^{s} \psi_i$, and hence

$$\lim_{s\to\infty} \frac{\partial y_{t+s}}{\partial \varepsilon_t} = \sum_{i=0}^{\infty} \psi_i. \tag{18}$$

Thus, the effect of (or a shock occuring to) ε_t dies out eventually in trend stationary process, but is permanent in unit root process.

Trend-Stationary Process

The trend-stationary process

$$y_t = \alpha + \delta t + \psi(L)\varepsilon_t \tag{19}$$

with $\sum_{i=0}^{\infty} |\psi_i| < \infty$ and ε_t is white noise.

Consider the special case when $\psi(L) = 1$, simply

$$y_t = \alpha + \delta t + \varepsilon_t \tag{20}$$

where α and β are unknown. Alternatively, we write it as

$$y_t = \mathbf{x}_t'\boldsymbol{\beta} + \varepsilon_t \tag{21}$$

where $\mathbf{x}_t = (1, t)'$ and $\boldsymbol{\beta} = (\alpha, \delta)'$.

OLS for the Simple Trend-Stationary Process

Denote $\hat{\boldsymbol{\beta}}_T$ the OLS estimator for the parameter vector $\boldsymbol{\beta}$, given the sample $y_1, ..., y_T$ of size T

$$\hat{\boldsymbol{\beta}}_T = \begin{pmatrix} \hat{\alpha}_T \\ \hat{\delta}_T \end{pmatrix} = \left(\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' \right)^{-1} \left(\sum_{t=1}^{T} \mathbf{x}_t y_t \right). \tag{22}$$

It can be readily shown that

$$\hat{\boldsymbol{\beta}}_T = \boldsymbol{\beta} + \left(\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' \right)^{-1} \left(\sum_{t=1}^{T} \mathbf{x}_t \varepsilon_t \right). \tag{23}$$

Hence

$$\hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta} = \left(\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' \right)^{-1} \left(\sum_{t=1}^{T} \mathbf{x}_t \varepsilon_t \right),$$

or equivalently

$$\begin{pmatrix} \hat{\alpha}_T - \alpha \\ \hat{\delta}_T - \delta \end{pmatrix} = \left[\sum_{t=1}^{T} \begin{pmatrix} 1 & t \\ t & t^2 \end{pmatrix} \right]^{-1} \left[\sum_{t=1}^{T} \begin{pmatrix} \varepsilon_t \\ t\varepsilon_t \end{pmatrix} \right] = \begin{pmatrix} \sum 1 & \sum t \\ \sum t & \sum t^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum \varepsilon_t \\ \sum t\varepsilon_t \end{pmatrix},$$

where $\sum$ denotes $\sum_{t=1}^{T}$.

OLS for the Simple Trend-Stationary Process

In order to find a non-degenerate limiting distribution (chapter 8 in Hamilton), typically we consider the statistic

$$\sqrt{T}(\hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta}) = \left(\frac{1}{T} \sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' \right)^{-1} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \mathbf{x}_t \varepsilon_t \right).$$

Recall in maximum likelihood that $\sqrt{T}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \mathcal{I}(\theta)^{-1})$ under certain conditions.

OLS for the Simple Trend-Stationary Process

Usually one assumes that

$$\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t' \xrightarrow{p} \mathbf{Q} \tag{24}$$

for some nonsingular matrix $\mathbf{Q}$, and that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbf{x}_t\varepsilon_t \xrightarrow{d} N(\mathbf{0},\sigma^2\mathbf{Q})$$

which implies that

$$\sqrt{T}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}) \xrightarrow{d} N(\mathbf{0},\sigma^2\mathbf{Q}^{-1}).$$

OLS for the Simple Trend-Stationary Process

However, this procedure does not work for the trend stationary process.

To see this, let us check the assumptions. First,

$$\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' = \begin{pmatrix} \sum 1 & \sum t \\ \sum t & \sum t^2 \end{pmatrix} = \begin{pmatrix} T & T(T+1)/2 \\ T(T+1)/2 & T(T+1)(2T+1)/6 \end{pmatrix} \quad (25)$$

Then, $T^{-1} \sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'$ definitely diverges as $T \to \infty$.

What about ... try $T^{-3} \sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'$ instead of $T^{-1} \sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'$?

$$\lim_{T \to \infty} \frac{1}{T^3} \sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' = \begin{pmatrix} 0 & 0 \\ 0 & 1/3 \end{pmatrix}$$

which is singular! Usch!

OLS for the Simple Trend-Stationary Process

- It turns out that the OLS estimators $\hat{\alpha}_T$ and $\hat{\delta}_T$ have different asymptotic rates of convergence.
- In order to arrive at a non-degenerate limiting distribution, $\hat{\alpha}_T$ must be multiplied by $T^{1/2}$, while $\hat{\delta}_T$ by $T^{3/2}$.
- Then define the matrix below

$$\mathbf{S}_T = \begin{pmatrix} T^{1/2} & 0 \\ 0 & T^{3/2} \end{pmatrix}.$$

If we left-multiply $\mathbf{S}_T$ to $\hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta}$, what will happen?

OLS for the Simple Trend-Stationary Process

$$
\begin{aligned}
\mathbf{S}_T(\hat{\boldsymbol{\beta}}_T - \boldsymbol{\beta}) &= \mathbf{S}_T \left(\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' \right)^{-1} \left(\sum_{t=1}^{T} \mathbf{x}_t \varepsilon_t \right) \\
&= \mathbf{S}_T \left(\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' \right)^{-1} \mathbf{S}_T \mathbf{S}_T^{-1} \left(\sum_{t=1}^{T} \mathbf{x}_t \varepsilon_t \right) \\
&= \left[\mathbf{S}_T^{-1} \left(\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' \right) \mathbf{S}_T^{-1} \right]^{-1} \left(\mathbf{S}_T^{-1} \sum_{t=1}^{T} \mathbf{x}_t \varepsilon_t \right) \\
&= \mathbf{Q}_T^{-1} \mathbf{u}_T
\end{aligned}
$$

where

$$
\mathbf{Q}_T = \mathbf{S}_T^{-1} \left(\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' \right) \mathbf{S}_T^{-1} \quad \text{and} \quad \mathbf{u}_T = \mathbf{S}_T^{-1} \sum_{t=1}^{T} \mathbf{x}_t \varepsilon_t .
$$

OLS for the Simple Trend-Stationary Process

- First, we see that $\mathbf{Q}_T \to \mathbf{Q}$ as $T \to \infty$, where, from (25),
$$\mathbf{Q} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}$$
is nonsingular.
- Second,
$$\mathbf{u}_T = \begin{pmatrix} T^{-1/2} & 0 \\ 0 & T^{-3/2} \end{pmatrix} \begin{pmatrix} \sum \varepsilon_t \\ \sum t\varepsilon_t \end{pmatrix}.$$
The first element of $\mathbf{u}_T$
$$u_{1T} = \sqrt{T} \times \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t \quad \text{(Familiar? Yes! CLT)},$$
and the second element
$$u_{2T} = \frac{1}{T^{3/2}} \sum_{t=1}^{T} t\varepsilon_t = \sqrt{T} \times \frac{1}{T} \sum_{t=1}^{T} \left(\frac{t}{T}\right) \varepsilon_t,$$

OLS for the Simple Trend-Stationary Process

- Suppose that sequence ε_t is independently identical distributed with zero mean, finite constant variance σ^2, and $\mathrm{E}(\varepsilon_t^4) < \infty$.
- Consider first the limiting distribution of u_{1T}.

Theorem (Classical CLT)

Let $\bar{y}_T = T^{-1} \sum_{t=1}^{T} y_t$, where $y_1, ..., y_T$ is a sequence of i.i.d. random variables with finite mean μ and finite variance σ^2. Then

$$\sqrt{T}(\bar{y}_T - \mu) \xrightarrow{d} N(0, \sigma^2)$$

- Thus, it follows that

$$u_{1T} \xrightarrow{d} N(0, \sigma^2) \tag{26}$$

OLS for the Simple Trend-Stationary Process

- Next we consider the limiting distribution of u_{2T}.
- Define $v_t = (\frac{t}{T})\varepsilon_t$. Since

$$E(v_t) = E(v_t | v_{t-1}, v_{t-2}, ...) = \left(\frac{t}{T}\right) E(\varepsilon_t) = 0,$$

for all t, v_t is a martingale difference sequence (MDS).

Theorem (MDS CLT, Proposition 7.8 in Hamilton)

Let $\{y_t\}_{t=1}^{\infty}$ be a MDS with $\bar{y}_T = T^{-1}\sum_{t=1}^{T} y_t$. Suppose that

1. $E(y_t^2) = \sigma_t^2$ *with* $\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}\sigma_t^2 = \sigma^2 < \infty$,
2. $E|y_t|^r < \infty$ *for some* $r > 2$ *and all* t, *and*
3. $T^{-1}\sum_{t=1}^{T} y_t^2 \xrightarrow{p} \sigma^2$.

Then

$$\sqrt{T}\bar{y}_T \xrightarrow{d} N(0, \sigma^2)$$

OLS for the Simple Trend-Stationary Process

- Let us verify that conditions 1-3 of the MDS CLT are satisfied for v_t.
- Condition 1,

$$\sigma_t^2 = \mathrm{E}(v_t^2) = \left(\frac{t}{T}\right)^2 \mathrm{E}(\varepsilon_t^2) = \left(\frac{t}{T}\right)^2 \sigma^2$$

with

$$\begin{aligned} \frac{1}{T}\sum_{t=1}^{T}\sigma_t^2 &= \frac{1}{T}\sum_{t=1}^{T}\left(\frac{t}{T}\right)^2 \sigma^2 = \frac{\sigma^2}{T^3}\sum_{t=1}^{T} t^2 \\ &= \frac{\sigma^2}{T^3} \times \frac{T(T+1)(2T+1)}{6} \to \frac{\sigma^2}{3} \end{aligned}$$

as $T \to \infty$. Therefore, condition 1 is satisfied.

OLS for the Simple Trend-Stationary Process

- Condition 2, check $r = 4$,

$$\mathrm{E}|v_t^4| = \mathrm{E}(v_t^4) = \left(\frac{t}{T}\right)^4 \mathrm{E}(\varepsilon_t^4) < \infty,$$

  which is true as it has been presumed.

- Condition 3,

$$\frac{1}{T}\sum_{t=1}^{T} v_t^2 = \frac{1}{T}\sum_{t=1}^{T}\left[\left(\frac{t}{T}\right)\varepsilon_t\right]^2 \xrightarrow{p} \frac{\sigma^2}{3}$$

  is verified by checking $T^{-1}\sum_{t=1}^{T} v_t^2 - \sigma^2/3 \xrightarrow{m.s.} 0$ on pp.459 in Hamilton.

- Thus, it follows that

$$u_{2T} \xrightarrow{d} N(0, \sigma^2/3) \tag{27}$$

- Finally, the vector version is shown on pp.459 in Hamilton

$$\begin{pmatrix} T^{-1/2}\sum \varepsilon_t \\ T^{-1/2}\sum(\frac{t}{T})\varepsilon_t \end{pmatrix} \xrightarrow{d} N_2(\mathbf{0}, \sigma^2\mathbf{Q}) \tag{28}$$

where

$$\mathbf{Q} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}$$

- Then we have

$$\begin{pmatrix} T^{1/2}(\hat{\alpha}_T - \alpha) \\ T^{3/2}(\hat{\delta}_T - \delta) \end{pmatrix} \xrightarrow{d} N_2(\mathbf{0}, \mathbf{Q}^{-1} \times \sigma^2\mathbf{Q} \times \mathbf{Q}^{-1}) = N_2(\mathbf{0}, \sigma^2\mathbf{Q}^{-1}) \tag{29}$$

OLS for the Simple Trend-Stationary Process

- Look at the model again

$$y_t = \alpha + \delta t + \varepsilon_t$$

- We see that the estimators $\hat{\alpha}_T$ and $\hat{\delta}_T$ are both consistent. In particular, $\hat{\delta}$ is called super consistent, as $\hat{\delta}_T$ is consistent and $\hat{\delta}_T = \delta + O_p(T^{-3/2})$ with $-3/2 < -1/2$.
- The super consistency means that the estimator converges faster in terms of the order in probability than the square root convergence.
- Square root convergence $\sqrt{T} X_T \xrightarrow{d} X$.
- Super consistency: $\hat{\delta}_T - \delta \xrightarrow{p} 0$, and $\sqrt{T}(\hat{\delta}_T - \delta) \xrightarrow{p} 0$ and even $T(\hat{\delta}_T - \delta) \xrightarrow{p} 0$, while $\hat{\alpha}_T$ is not so super.

OLS for the Trend-Stationary AR(p) Process

- Chapter 16.3 in Hamilton considers a more general trend stationary process generated by

$$y_t = \alpha + \delta t + \phi_1 y_{t-1} + ... + \phi_p y_{t-p} + \varepsilon_t,$$

where ε_t is an independent white noise process with $\mathrm{E}(\varepsilon_t^4) < \infty$, and $\phi(L)$ is stable.

- The same approach used to establish the limiting distribution of the OLS estimators $\hat{\alpha}_T$ and $\hat{\delta}_T$ for the simple trend-stationary process are used to establish the asymptotic distribution of the OLS estimators

$$\hat{\alpha}_T,\ \hat{\delta}_T,\ \hat{\phi}_{1,T}, ...\hat{\phi}_{p,T}$$

for the trend-stationary AR(p) process.

To be continued! Thank you!