Uppsala universitet
Statistiska institutionen

Intyg av skrivningsvakt att legitimationskontroll samt kontroll av kod är utförd:
Signatur

TENTAMEN

Code (for the exam): AO-0006-DST

Course: Time series analysis

Date: 5/11

Task number	1	2	3	4	Bonus	Sum	Grade
Number of submitted sheets per task (for students to state	4	1	2				
Points (for examiner to state)	32	6	17			55	G

1 Write the code on every sheet submitted.

2 Make sure your solutions are easy to read and easy to understand. For each task you solve, please start with a new sheet. State the number of submitted sheets in the table above.

3 You may present your solutions in Swedish or English.

Part 1 white noise: $E(\varepsilon_i^2)=\sigma^2$ $E(\varepsilon_i\varepsilon_j)=0$ if $i\neq j$
$E(\varepsilon_i)=0$

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1) i) $E(Y_t)=E(\mu+\varepsilon_t+\alpha_1\varepsilon_{t-1}+\alpha_2\varepsilon_{t-2})$

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$$=E(\mu)+E(\varepsilon_t)+\alpha_1E(\varepsilon_{t-1})+\alpha_2E(\varepsilon_{t-2})$$
$$=\mu+0+\alpha_1\cdot 0+\alpha_2\cdot 0$$
$$=\mu.$$

ii)

$$\gamma_0=Var(Y_t)=E[(Y_t-\mu)^2]=E(Y_t^2-2\mu Y_t+\mu^2)$$
$$=E(Y_t^2)-2\mu E(Y_t)+\mu^2=E(Y_t^2)-\mu^2$$
$$=E(\mu^2+\alpha_1^2\varepsilon_{t-1}^2+\alpha_2^2\varepsilon_{t-2}^2+\varepsilon_t^2+2\mu\varepsilon_t+2\mu\alpha_1\varepsilon_{t-1}$$
$$+2\mu\alpha_2\varepsilon_{t-2}+2\alpha_1\varepsilon_t\varepsilon_{t-1}+2\alpha_2\varepsilon_t\varepsilon_{t-2}$$
$$+2\alpha_1\alpha_2\varepsilon_{t-1}\varepsilon_{t-2})-\mu^2$$
$$=\mu^2+\alpha_1^2E(\varepsilon_{t-1}^2)+\alpha_2^2E(\varepsilon_{t-2}^2)+E(\varepsilon_t^2)+2\mu E(\varepsilon_t)$$
$$+2\mu\alpha_1E(\varepsilon_{t-1})+2\mu\alpha_2E(\varepsilon_{t-2})+2\alpha_1E(\varepsilon_t\varepsilon_{t-1})+2\alpha_2E(\varepsilon_t\varepsilon_{t-2})$$
$$+2\alpha_1\alpha_2E(\varepsilon_{t-1}\varepsilon_{t-2})-\mu^2$$
$$=\alpha_1^2\sigma^2+\alpha_2^2\sigma^2+\sigma^2+2\mu\cdot 0+2\mu\alpha_1\cdot 0+2\mu\alpha_2\cdot 0$$
$$+2\alpha_1\cdot 0+2\alpha_2\cdot 0+2\alpha_1\alpha_2\cdot 0$$
$$=\sigma^2+\alpha_1^2\sigma^2+\alpha_2^2\sigma^2=\sigma^2(1+\alpha_1^2+\alpha_2^2)$$
$$=\sigma^2\sum_{i=0}^{2}\alpha_i^2 \quad \text{where } \alpha_0=1$$

iii) $\gamma_1=E(Y_t-\mu)(Y_{t-1}-\mu)=E(Y_tY_{t-1}-\mu Y_t-\mu Y_{t-1}+\mu^2)=E(Y_tY_{t-1})-\mu^2$
$$=E[(\mu+\varepsilon_t+\alpha_1\varepsilon_{t-1}+\alpha_2\varepsilon_{t-2})(Y_{t-1})]-\mu^2$$
$$=E[\mu Y_{t-1}+\varepsilon_tY_{t-1}+\alpha_1\varepsilon_{t-1}Y_{t-1}+\alpha_2\varepsilon_{t-2}Y_{t-1}]-\mu^2$$
$$=\mu^2+0+\alpha_1\sigma^2+\alpha_1\alpha_2\sigma^2-\mu^2=\sigma^2(\alpha_1+\alpha_1\alpha_2)$$

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iii) cont.

$$\gamma_2 = E(y_t-\mu)(y_{t-2}-\mu) = E(y_t y_{t-2} - \mu y_t - \mu y_{t-2} + \mu^2)$$
$$= E(y_t y_{t-2}) - \mu^2 = E((\mu + \varepsilon_t + \alpha_1 \varepsilon_{t-1} + \alpha_2 \varepsilon_{t-2}) y_{t-2}) - \mu^2$$
$$= \mu^2 + 0 + 0 + \alpha_2 E(\varepsilon_{t-2} y_{t-2}) - \mu^2 = \alpha_2 E(\varepsilon_{t-2} y_{t-2})$$
$$= \alpha_2 E(\varepsilon_{t-2}(\mu + \varepsilon_{t-2} + \alpha_1 \varepsilon_{t-3} + \alpha_2 \varepsilon_{t-4}))$$
$$= \alpha_2 \sigma^2$$

$$\gamma_3 = E(y_t-\mu)(y_{t-3}-\mu) = E(y_t y_{t-3} - \mu y_t - \mu y_{t-3} + \mu^2)$$
$$= E((\mu + \varepsilon_t + \alpha_1 \varepsilon_{t-1} + \alpha_2 \varepsilon_{t-2}) y_{t-3}) - \mu^2$$
$$= \mu^2 + 0 + 0 + 0 - \mu^2 = 0$$

$$\gamma_j = \begin{cases} 0 & j > 2 \\ \sigma^2 \sum_{i=0}^{2-j} \alpha_i \alpha_{i+j} & j = 0,1,2 \\ \gamma_{-j} & j < 0 \end{cases} \qquad \alpha_0 = 1$$

iv)

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\sigma^2(\alpha_1 + \alpha_1\alpha_2)}{\sigma^2(1+\alpha_1^2+\alpha_2^2)} = \frac{\alpha_1 + \alpha_1\alpha_2}{1 + \alpha_1^2 + \alpha_2^2}$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\sigma^2(\alpha_2)}{\sigma^2(1+\alpha_1^2+\alpha_2^2)} = \frac{\alpha_2}{1+\alpha_1^2+\alpha_2^2}$$

$$\rho_3 = \frac{\gamma_3}{\gamma_0} = \frac{0}{\gamma_0} = 0$$

$$\rho_i = 0 \quad \text{if} \quad i > 2$$

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2) y_t is always covariance stationary as long as μ doesn't depend on time as well as the covariance doesn't depend on time. One can add that α_1 and α_2 must be $< \infty$, but thats always the case and due to this the coefficient α_i is absolute summable and y_t is covariance stationary.

If we can find values of α_1 and α_2 so that the roots of the lag polynomial $(1 + a_1 L + a_2 L^2 = 0)$ or $(1 + a_1 z + a_2 z^2 = 0)$ is outside the unit disc, and at the same time have the same first and second moments as the original model, then it is invertible part 1 8/8

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3) it depend, if the variance of ε_t is $< \infty$ and don't change over time it is. if its not $< \infty$ and is changing over time it is not covariance-stationary.

4) 0/8

5) the ~~v~~variable is: $Y_t = \mu +$

0/8

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1) 0/4

~~Fo~~ The missing case is where we test

H_0: $\alpha \neq 0$ No! and $p=0$ $\alpha=0$ pressumed

and estimate it the same way as

$$\Delta x_t = p x_{t-1} \sum_{i=1}^{p-1} \gamma_i \Delta x_{t-i} + \alpha + u_t \quad (4)$$

2) 6/8

I think model 4 is the best.

If we look at test 5: we can only reject $\alpha=0$ X i.e. we have an intercept in the differenced X data but no drift beacuse $p=0$ and no linear trend because $\delta=0$. X which seems to be correct based on a look on figure 2.

if we look at test (4), after we assume $\delta=0$ on previous discussion, we get to the same conclusion: that $p=0$ and $\alpha \neq 0$.

So model (4) seems to be the best which indicate that X_t is a radom walk with drift rather then trend stationary as we also saw in the discussion seems more aperopriate than assume trend stationarity.

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Part 3 AO-0006-DST

1) 4/4 $\Gamma_1 = -(A_2 + A_3 + \cdots + A_k)$

$\Gamma_2 = -(A_3 + A_4 + \cdots + A_k)$

$\Gamma_i = -\left(\sum_{j=i}^{p-1} A_{j+1}\right)$ Where $i = 1, \ldots, p$

$j = i, \ldots, p-1$

and $p = K$

2) 4/4 $\alpha = \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}$ $\beta = \begin{bmatrix} 1 \\ -\beta_2 \end{bmatrix}$ $\Pi = \begin{bmatrix} \alpha_1 & -\beta_2\alpha_1 \\ 0 & 0 \end{bmatrix}$

$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ $A = \begin{bmatrix} 1+\alpha_1 & -\alpha_1\beta_2 \\ 0 & 1 \end{bmatrix}$

3) 2/12 $X_t = X_0 + \mu \cdot t + \sum_{s=1}^{t} U_s$

~~4) $\begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = \begin{bmatrix} X_{10} \\ X_{20} \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \cdot t + \sum_{s=1}^{t} \begin{bmatrix} \varepsilon_{1s} \\ \varepsilon_{2s} \end{bmatrix}$~~

4) 4/4

$A(L) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} (1+\alpha_1)L & -\alpha_1\beta_2 L \\ 0 & L \end{bmatrix} = \begin{bmatrix} 1-(1+\alpha_1)L & -\alpha_1\beta_2 L \\ 0 & 1-L \end{bmatrix}$

$|A(z)| = 0 \Rightarrow \begin{vmatrix} 1-(1+\alpha_1)z & -\alpha_1\beta_2 z \\ 0 & 1-z \end{vmatrix} = (1-(1+\alpha_1)z)(1-z)$ see this!

$\frac{1}{1+\alpha_1}$ and 1

$\Rightarrow z^2(1+\alpha_1) = 0$

$= 1 - z - (z + z\alpha_1)(1-z) = 1 - z - z + z^2 - z\alpha_1 + z^2\alpha = 1 - 2z - z\alpha_1 + z^2 + z^2\alpha_1$

$= 1 + z(-2-\alpha_1) + z^2(1+\alpha_1) \ldots$ then pq-formel ....

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Part 3

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5) $\begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = \begin{bmatrix} (\alpha_1+1) X_{1t-1} - \alpha_1 \beta_2 X_{2t-1} + \mu_1 + \varepsilon_{1t} \\ X_{2t-1} + \mu_2 + \varepsilon_{2t} \end{bmatrix}$ $\alpha_1 = -2$

2/8

$$\begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = \begin{bmatrix} -X_{1t-1} + 2\beta_2 X_{2t-1} + \mu_1 + \varepsilon_{1t} \\ X_{2t-1} + \mu_2 + \varepsilon_{2t} \end{bmatrix}$$

$$= \begin{bmatrix} -(-X_{1t-2} + 2\beta_2 X_{2t-2} + \mu_1 + \varepsilon_{1t-1}) + 2\beta_2 X_{2t-1} + \mu_1 + \varepsilon_{1t} \\ X_{2t-1} + \mu_2 + \varepsilon_{2t} \end{bmatrix}$$

$$= \begin{bmatrix} X_{1t-2} + 2\beta_2 \Delta X_{2t-1} + \Delta \varepsilon_{1t} \\ X_{2t-1} + \mu_2 + \varepsilon_{2t} \end{bmatrix}$$

$$= \begin{bmatrix} X_{1t-2} + 2\beta_2(\mu_2 + \varepsilon_{2t-1}) + \Delta \varepsilon_{1t} \\ X_{2t-1} + \mu_2 + \varepsilon_{2t} \end{bmatrix}$$

$=$

X

1/4

6) $\beta X_t = [1 \; -\beta_2] \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = X_{1t} - \beta_2 X_{2t}$

$$= (1+\alpha_1) X_{1t-1} - \alpha_1 \beta_2 X_{2t-1} + \mu_1 + \varepsilon_{1t} - \beta_2 X_{2t}$$

$$= (1+\alpha_1) X_{1t-1} - \beta_2(\alpha_1 X_{2t-1} + X_{2t}) + \mu_1 + \varepsilon_{1t} - \beta_2 X_{2t}$$

$$= (1+\alpha_1) X_{1t-1} - \beta_2(\alpha X_{2t-1} + X_{2t-1} + \mu_2 + \varepsilon_{2t}) + \mu_1 + \varepsilon_{1t}$$

$$= (1+\alpha_1) X_{1t-1} - \beta_2(1+\alpha) X_{2t-1} - \beta_2\mu_2 - \beta_2 \varepsilon_{2t} + \mu_1 + \varepsilon_{1t}$$

? If $-2 < \alpha < 0$ then $|(1+\alpha)| < 1$

and the prosess can be seen as a stable ARMA process.

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