Time Series Econometrics, 2ST111

Lecture 2. Difference Equations and Lag Operators

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Outline of Today's Lecture

- Difference Equations (Hamilton, pp. 1-24)
  - first-order equations
  - pth-order equations
- Lag Operators (Hamilton, pp. 25-42)
  - first-order equations
  - pth-order equations
  - initial conditions

First-Order Difference Equations

Denote y_t the value of a variable at time t.

A linear first-order difference equation

$$y_t = \phi y_{t-1} + w_t \tag{1}$$

is an expression relating the variable y_t to its previous values.

- y_t as a linear function of y_{t-1} and w_t
- first-order due to that only y_{t-1} enters
- affine transformation

Example: Goldfeld's Model

Goldfeld's model (1973), estimated money demand function for US:

$$\begin{aligned} m_t &= 0.72 m_{t-1} + w_t \\ w_t &= 0.27 + 0.19 I_t - 0.045 r_{bt} - 0.019 r_{ct} \end{aligned} \tag{2}$$

- m_t the log of the real money holdings of the public
- I_t the log of aggregate real income
- r_{bt} the log of the interest rate on bank accounts
- r_{ct} the log of the interest rate on commercial paper

First-Order Difference Equations

We have seen that w_t is deterministic, which means that y_t is perfectly predictable.

Question:

If a dynamic system is described by $y_t = \phi y_{t-1} + w_t$, what are the effects on y of changes in the value of w?

Recursive Substitution

The answer is given by Recursive Substitution.

Let us expand y_2 in the following way:

$$\begin{aligned} y_2 &= \phi y_1 + w_2 = \phi(\phi y_0 + w_1) + w_2 \\ &= \phi^2 y_0 + \phi w_1 + w_2. \end{aligned} \tag{3}$$

Likewise, for y_3 we have

$$\begin{aligned} y_3 &= \phi y_2 + w_3 = \phi(\phi^2 y_0 + \phi w_1 + w_2) + w_3 \\ &= \phi^3 y_0 + \phi^2 w_1 + \phi w_2 + w_3. \end{aligned} \tag{4}$$

Recursive Substitution

By Recursive Substitution,

$$\begin{aligned} y_t &= \phi^t y_0 + \phi^{t-1} w_1 + \phi^{t-2} w_2 + \ldots + w_t \\ &= \phi^t y_0 + \sum_{i=1}^{t} \phi^{t-i} w_i. \end{aligned} \tag{5}$$

The effect on y_t of changing the value of w_i is, *ceteris paribus*,

$$\frac{\partial y_t}{\partial w_i} = \phi^{t-i}, \tag{6}$$

where $\partial y_t / \partial w_i$ denotes the partial derivative of y_t w.r.t. w_i.

Dynamic Multipliers

Let us expand $y_{t+\tau}$ instead of y_t recursively up to y_{-k}:

$$\begin{aligned} y_{t+\tau} &= \phi^{t+\tau+k} y_{-k} + \phi^{t+\tau+k-1} w_{-k+1} + \ldots + w_{t+\tau} \\ &= \phi^{t+\tau+k} y_{-k} + \sum_{i=-k+1}^{t+\tau} \phi^{t+\tau-i} w_i, \end{aligned} \tag{7}$$

with

$$\frac{\partial y_{t+\tau}}{\partial w_i} = \phi^{t+\tau-i}. \tag{8}$$

Note that k is not involved.

By setting $i = t$, we have the Dynamic Multiplier

$$\frac{\partial y_{t+\tau}}{\partial w_t} = \phi^{\tau}, \tag{9}$$

only depending on τ.

Remarks for the Dynamic Multiplier

$$\frac{\partial y_{t+\tau}}{\partial w_t} = \phi^\tau$$

- $0 < \phi < 1$, ϕ^τ decays geometrically.
- $-1 < \phi < 0$, ϕ^τ alternates in sign, $|\phi^\tau|$ decays geometrically.
- $1 < \phi$, ϕ^τ increases exponentially.
- $\phi < -1$, ϕ^τ alternates in sign, $|\phi^\tau|$ increases exponentially.

See Figure 1.1 on pp.4 in Hamilton.

Dynamic Multipliers

- The dynamic system is called stable if $|\phi| < 1$ and explosive if $|\phi| > 1$.
- The τth dynamic multiplier is the response of y τ-step ahead to a single impulse in w. It is also referred to as the impulse-response function.
- Think about what if $|\phi| = 1$.

Long Run Effect

Sometimes we are interested in the effect of a permanent change in w, i.e. the effect when $w_t, w_{t+1}, ... w_{t+\tau}$ all increase by one unit. Consider again

$$y_{t+\tau} = \phi^{t+\tau+k} y_{-k} + \sum_{i=-k+1}^{t+\tau} \phi^{t+\tau-i} w_i.$$

Let $k = 1 - t$, we have

$$y_{t+\tau} = \phi^{\tau+1} y_{t-1} + \sum_{i=t}^{t+\tau} \phi^{t+\tau-i} w_i.$$

Thus, if $w_i = 1$ for $i = t, ..., t+\tau$ (one unit), the Long-Run Effect

$$\sum_{i=t}^{t+\tau} \frac{\partial y_{t+\tau}}{\partial w_i} = \sum_{i=t}^{t+\tau} \phi^{t+\tau-i} = \phi^{\tau} + \phi^{\tau-1} + ... + 1.$$

When $\tau \to \infty$, it converges to $1/(1-\phi)$, if $|\phi| < 1$.

Cumulative Effect

We may be also interested in the Cumulative Effect of a one unit increase in w_t, that is

$$\sum_{\tau=0}^{\infty} \frac{\partial y_{t+\tau}}{\partial w_t}. \tag{10}$$

Provided that $|\phi| < 1$, it is the same as the long-run effect $1/(1-\phi)$.

pth-Order Difference Equations

The linear first-order difference equation

$$y_t = \phi y_{t-1} + w_t$$

is a special case ($p = 1$) of the linear pth-Order Difference Equation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \phi_p y_{t-p} + w_t \tag{11}$$

in which the value of y at time t depends on p of its own lags ($y_{t-1}, ..., y_{t-p}$) and the current value of w.

First-Order Vector Difference Equations

It is often convenient to rewrite the pth-order scalar difference equation as a First-Order Vector Difference Equation. Denote

$$\boldsymbol{\xi}_t = \begin{pmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{pmatrix}_p, \quad \mathbf{F} = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{p \times p}, \quad \mathbf{v}_t = \begin{pmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_p$$

Consider the following first-order vector difference equation:

$$\boldsymbol{\xi}_t = \mathbf{F}\boldsymbol{\xi}_{t-1} + \mathbf{v}_t. \tag{12}$$

In particular, when $p = 1$, $\boldsymbol{\xi}_t = y_t$, $\mathbf{F} = \phi_1$, and $\mathbf{v}_t = w_t$ (first-order difference equation).

First-Order Vector Difference Equations

More clearly, the system of equations are

$$\begin{pmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{pmatrix} + \begin{pmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{13}$$

Remarks

- The first-order vector system is equivalent to the pth-order scalar system (11).
- The advantage of rewriting the pth-order scalar system into a first-order vector system is that the latter one is often easier to handle.

First-Order Vector Difference Equations

Given the first-order vector difference equation (12), we expand $\boldsymbol{\xi}_{t+\tau}$ up to $t-1$ by recursive substitution as follows:

$$\boldsymbol{\xi}_{t+\tau} = \mathbf{F}^{\tau+1}\boldsymbol{\xi}_{t-1} + \mathbf{F}^{\tau}\mathbf{v}_t + \mathbf{F}^{\tau-1}\mathbf{v}_{t+1} + \ldots + \mathbf{v}_{t+\tau}. \tag{14}$$

The system of equations are

$$\begin{pmatrix} y_{t+\tau} \\ y_{t+\tau-1} \\ y_{t+\tau-2} \\ \vdots \\ y_{t+\tau-p+1} \end{pmatrix} = \mathbf{F}^{\tau+1} \begin{pmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{pmatrix} + \mathbf{F}^{\tau} \begin{pmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + \begin{pmatrix} w_{t+\tau} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{15}$$

First-Order Vector Difference Equations

Denote

$$\mathbf{F}^s = \begin{pmatrix} f_{11}^{(s)} & f_{12}^{(s)} & \cdots & f_{1p}^{(s)} \\ f_{21}^{(s)} & f_{22}^{(s)} & \cdots & f_{2p}^{(s)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{p1}^{(s)} & f_{p2}^{(s)} & \cdots & f_{pp}^{(s)} \end{pmatrix} \tag{16}$$

For the first equation, we have

$$\begin{aligned} y_{t+\tau} &= f_{11}^{(\tau+1)} y_{t-1} + f_{12}^{(\tau+1)} y_{t-2} + \ldots + f_{1p}^{(\tau+1)} y_{t-p} + \\ &\quad \color{blue}{f_{11}^{(\tau)}} w_t + f_{11}^{(\tau-1)} w_{t+1} + \ldots + w_{t+\tau}. \end{aligned} \tag{17}$$

Thus, the dynamic multiplier (at time t for τ-step ahead) is given by

$$\frac{\partial y_{t+\tau}}{\partial w_t} = \color{blue}{f_{11}^{(\tau)}} \tag{18}$$

Dynamic Multiplier

- For $p = 1$, $f_{11}^{(\tau)} = \phi_1^{\tau}$.
- More generally, for any positive integer p,

$$\frac{\partial y_{t+1}}{\partial w_t} = f_{11}^{(1)} = \phi_1, \quad \frac{\partial y_{t+2}}{\partial w_t} = f_{11}^{(2)} = \phi_1^2 + \phi_2. \qquad (19)$$

- Recall the impulse-response function.

Dynamic Multiplier

- For larger values of τ, Hamilton suggests to compute $f_{11}^{(\tau)}$ by numerical simulation, see Hamilton pp.10.
- A simple anlytical characterization of the dynamic multiplier (18) can be obtained in terms of the eigenvalues of the matrix $\mathbf{F}$.
- The reason: it is related to the power of matrix $\mathbf{F}$.
- Recall that the eigenvalues of matrix $\mathbf{F}$ are those (complex) numbers λ who satisfy $|\mathbf{F} - \lambda \mathbf{I}_p| = 0$.
- For a general pth-order system, this determinant is a pth-order polynomial in λ whose p solutions are the eigenvalues of $\mathbf{F}$. See Proposition 1.1 on pp.10 and its proof in Appendix 1.A on pp.21 in Hamilton.

General Solution of a pth-Order Difference Equation

Distinct Eigenvalues

The matrix $\mathbf{F}$ with distinct eigenvalues can be decomposed (eigenvalue decomposition) as follows

$$\mathbf{F} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}.$$

Remarks:

- The columns of the $p \times p$ matrix $\mathbf{T}$ are the eigenvectors of $\mathbf{F}$.
- The elements on the main diagonal of the $p \times p$ diagonal matrix $\boldsymbol{\Lambda}$ are the eigenvalues.
- The decomposition is not unique. Different columns of $\mathbf{T}$ can be switched, but certain eigenvalue corresponds to its eigenvector at certain position.
- Most software functions keep the eigenvalues in decreasing order.

General Solution of a *p*th-Order Difference Equation

Distinct Eigenvalues

It is related to the power of the matrix. To see this, check for example $\tau = 2$

$$\begin{aligned}\mathbf{F}^2 &= \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1} \cdot \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1} = \mathbf{T}\boldsymbol{\Lambda}(\mathbf{T}^{-1}\mathbf{T})\boldsymbol{\Lambda}\mathbf{T}^{-1} \\ &= \mathbf{T}\boldsymbol{\Lambda}\boldsymbol{\Lambda}\mathbf{T}^{-1} = \mathbf{T}\boldsymbol{\Lambda}^2\mathbf{T}^{-1}.\end{aligned}$$

By induction, we have the general result

$$\mathbf{F}^\tau = \mathbf{T}\boldsymbol{\Lambda}^\tau\mathbf{T}^{-1}. \tag{20}$$

where

$$\boldsymbol{\Lambda}^\tau = \begin{pmatrix} \lambda_1^\tau & 0 & \cdots & 0 \\ 0 & \lambda_2^\tau & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p^\tau \end{pmatrix}$$

General Solution of a *p*th-Order Difference Equation

Distinct Eigenvalues

Proposition 1.2 on pp.12 in Hamilton says that the dynamic multiplier has the close form

$$\frac{\partial y_{t+\tau}}{\partial w_t} = f_{11}^{(\tau)} = c_1\lambda_1^\tau + c_2\lambda_2^\tau + \dots + c_p\lambda_p^\tau \tag{21}$$

where

$$c_i = \frac{\lambda_i^{p-1}}{\prod_{k\neq i}(\lambda_i - \lambda_k)}.$$

Remarks:

- It can be shown that $\sum_{i=1}^{p} c_i = 1$. The dynamic multiplier is a weighted average of λ_i^τ.
- Some of the eigenvalues may be complex. They will appear as complex conjugates.

General Solution of a Second-Order Difference Equation

Distinct Eigenvalues

A summary of the dynamics for a Second-Order Difference Equation with a nice graph are given on pp.17-18 in Hamilton.

What if $\mathbf{F}$ has repeated eigenvalues? Note that some c_i does not exist.

Solution: The previous result for the dynamic multiplier can be generalized using the Jordan decomposition.

$$\mathbf{F} = \mathbf{M}\mathbf{J}\mathbf{M}^{-1}. \tag{22}$$

See pp.18-19 in Hamilton for details.

Infinite History

If the modulus (absolute value) of the eigenvalues of $\mathbf{F}$ are all less than one, that is, $|\lambda_i| < 1$, $\mathbf{F}^\tau$ goes to zero as $\tau \to \infty$.

If all values of w and y are taken to be bounded, we can think of a 'solution' of y_t in terms of the infinite history of w

$$y_t = w_t + \psi_1 w_{t-1} + \psi_2 w_{t-2} + \psi_3 w_{t-3} + \ldots \tag{23}$$

where, likewise, $\psi_\tau = \partial y_{t+\tau} / \partial w_t = f_{11}^{(\tau)}$ is the row 1 column 1 element of $\mathbf{F}^\tau$.

Cumulative Effect

Again, if all the eigenvalues of **F** are less than one in modulus, it can be shown that the cumulative effect of a one-time change in w on y is

$$\sum_{\tau=0}^{\infty} \frac{\partial y_{t+\tau}}{w_t} = \frac{1}{1 - \phi_1 - \phi_2 - ... - \phi_p} \tag{24}$$

Sample, Window and Time Series

We think of the sample

$$\{y_t\}_{t=1}^{n} = y_1,\ y_2,\ \dots,\ y_n$$

as a 'window' out of an infinite past and infinite future

$$\{y_t\}_{t=-\infty}^{\infty} = \dots, y_{-1},\ y_0,\ \underbrace{y_1,\ y_2,\ \dots,\ y_n}_{sample},\ y_{n+1},\ y_{n+2},\ \cdots$$

The time series $\{y_t\}_{t=-\infty}^{\infty}$ is typically identified by its tth element.

Time Series Operators

A time series operator transforms one or more time series into a new time series.

Example (multiplication operator)

$$y_t = \beta x_t$$

Example (addition operator)

$$y_t = x_t + w_t$$

Note that they are transformations from $\{x_t\}_{t=-\infty}^{\infty}$ and $\{w_t\}_{t=-\infty}^{\infty}$ to $\{y_t\}_{t=-\infty}^{\infty}$, not just one observation at t.

Since the multiplication or addition operators amount to element-by-element multiplication or addition, they obey the fundamental laws of algebra (the commutative, associate and distributive laws).

For example (distributive),

$$\beta x_t + \beta w_t = \beta(x_t + w_t)$$

The Lag Operator

A highly useful time series operator is the Lag Operator, L.

By definition,

$$L\, x_t = x_{t-1}. \tag{25}$$

Furthermore,

$$L\,(L\, x_t) = L\, x_{t-1} = x_{t-2}.$$

The associate law holds, and then we have $L\,(L\, x_t) = (L\, L) x_t$. And we define the power of the lag operator $L^2 = L\, L$.

By induction, we have the general form

$$L^k\, x_t = x_{t-k}, \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{26}$$

and the special case $L^0\, x_t = x_t$.

The inverse of the lag operator can also be defined, $L^{-k}\, x_t = x_{t+k}$, and in general we have $L^{-j}\, L^k = L^{k-j}$.

Remarks:

- The lag operator is a unary operator, which only requires one operand. So it belongs to the family of the minus sign ($-$) or the factorial (!), but totally different from the multiplication ($\times$) and the addition ($+$) operators who are binary and require two operands.
- The lag operator is commutative with some other operators, and therefore, the lag operator is distributive over those operators.

$$L(x_t + w_t) = Lx_t + Lw_t, \quad \text{(distributive over } +\text{)}$$

Applying $+$ first (LHS) or L first (RHS) produces the same result (it commutes $+$).

$$L(x_t \cdot w_t) = Lx_t \cdot Lw_t, \quad \text{(distributive over } \cdot\text{)}$$

- The special case, the lag of a constant

$$L\beta = \beta$$

For better understanding, consider the lag operator L implies a function $lag(x_t) = x_{t-1}$ with only one argument (unary), the addition operator implies a function $add(x, y) = x + y$ with two arguments (binary).

The lag operator commutes the addition operator implies that

$$lag(add(x_t, y)) = add(lag(x_t), lag(y_t)). \tag{27}$$

The same result holds for the multiplication operator, and division, but not all (because you can define any kind of operator as you wish).

The Lag Operator

We think of the lag operator as a third operator in addition to the addition and the multiplication, and then we apply the fundamental laws of algebra carefully.

For example, you can do

$$y_t = (\alpha + \beta L)Lx_t \iff y_t = (\alpha L + \beta L^2)x_t$$

or

$$y_t = (1 - \lambda_1 L)(1 - \lambda_2 L)x_t \iff y_t = (1 - \lambda_2 L - \lambda_1 L - \lambda_1 \lambda_2 L^2)x_t$$

The expressions such as $\alpha L + \beta L^2$ and $1 - \lambda_2 L - \lambda_1 L - \lambda_1 \lambda_2 L^2$ without time varying terms x_t are referred to as polynomials in the lag operator or simply lag polynomials.

First-Order Difference Equations (revisited)

The first-order difference equation can be written in terms of the lag operators

$$y_t = \phi L y_t + w_t \iff (1 - \phi L) y_t = w_t. \tag{28}$$

Consider 'multiplying' both sides of (28) by the lag polynomial

$$1 + \phi L + \phi^2 L^2 + ... + \phi^{t-1} L^{t-1}.$$

This yields

$$(1 - \phi^t L^t) y_t = (1 + \phi L + \phi^2 L^2 + ... + \phi^{t-1} L^{t-1}) w_t \tag{29}$$

or equivalently (same as that from recursive substitution),

$$y_t = \phi^t y_0 + w_t + \phi w_{t-1} + \phi^2 w_{t-2} + ... + \phi^{t-1} w_1 \tag{30}$$

First-Order Difference Equations (revisited)

Since $(1 + \phi L + \phi^2 L^2 + \ldots + \phi^{t-1} L^{t-1})(1 - \phi L) = 1 - \phi^t L^t$, we have

$$1 + \phi L + \phi^2 L^2 + \ldots + \phi^{t-1} L^{t-1} = \frac{1 - \phi^t L^t}{1 - \phi L}. \tag{31}$$

If $|\phi| < 1$, ϕ^t converges to zero as $t \to \infty$, and

$$1 + \phi L + \phi^2 L^2 + \ldots = \lim_{t \to \infty} \frac{1 - \phi^t L^t}{1 - \phi L} = (1 - \phi L)^{-1}. \tag{32}$$

We find the inverse of $1 - \phi L$, such that $(1 - \phi L)^{-1}(1 - \phi L) = 1$.

First-Order Difference Equations (revisited)

Suppose that $|\phi| < 1$. We divide both sides of $(1 - \phi L)y_t = w_t$ by $1 - \phi L$:

$$(1 - \phi L)^{-1}(1 - \phi L)y_t = (1 - \phi L)^{-1}w_t.$$

Then we obtain

$$y_t = w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 w_{t-3} + \dots. \tag{33}$$

pth-Order Difference Equations (revisited)

The general pth-order difference equation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + w_t \tag{34}$$

can be written in terms of the lag operator as well

$$(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p) y_t = w_t, \tag{35}$$

where the lag polynomial in (35) can be factorized as

$$1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p = (1 - \lambda_1 L)(1 - \lambda_2 L) \times \ldots \times (1 - \lambda_p L) \tag{36}$$

Why the lag polynomial can be factorized and how?

Consider the equation with complex number z

$$1 - \phi_1 z - \phi_2 z^2 - ... - \phi_p z^p = (1 - \lambda_1 z)(1 - \lambda_2 z) \times ... \times (1 - \lambda_p z). \quad (37)$$

Is that possible to find $\lambda_1, ..., \lambda_p$ such that, for any value of z, the equation holds? The answer is yes!

Immediately we find that the equation holds when $z = 0$. For $z \neq 0$, turn to the next page.

Why the lag polynomial can be factorized and how?

When $z \neq 0$, first define $\lambda = 1/z$, then divide both sides of the equation by z^p, and we obtain:

$$\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - ... - \phi_p = (\lambda - \lambda_1)(\lambda - \lambda_2) \times ... \times (\lambda - \lambda_p). \quad (38)$$

Now looks familiar? If so, you get it!

$\lambda_1, ..., \lambda_p$ are actually the roots of the equation

$$\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - ... - \phi_p = 0. \quad (39)$$

There must be p complex roots which can be repeated. If complex, then conjugates.

Why the lag polynomial can be factorized and how?

Recall the matrix $\mathbf{F}$ in the corresponding first-order vector difference equation. The eigenvalue problem $|\mathbf{F} - \lambda \mathbf{I}_p| = 0$ or $|\lambda \mathbf{I}_p - \mathbf{F}| = 0$ is actually equivalent to the root-finding problem in (41).

To see this, consider the eigenvalue decomposition $\mathbf{F} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}$. We have $|\lambda \mathbf{I}_p - \mathbf{F}| = |\lambda \mathbf{I}_p - \boldsymbol{\Lambda}| = (\lambda - \lambda_1)(\lambda - \lambda_2) \times ... \times (\lambda - \lambda_p) = 0$.

If you think that it is beautiful, you get it!

If all the p roots are found, the polynomial $\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - ... - \phi_p$ can be factorized like the RHS of (38). Thus, we have (37).

$$\lambda \in \mathbb{C}, \quad F = \begin{bmatrix} \phi_1 & \cdots & \phi_p \\ 1 & \ddots & 0 \\ 0 & \ddots & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_p \end{bmatrix}, \quad T = \Big[$$

$$F = T \Lambda T^{-1}$$

$$|\lambda I_p - F| = |\lambda I_p - T \Lambda T^{-1}|$$

$$= |\lambda I_p -$$

Given $\lambda = 1/z$ and $z \neq 0$, we have two equivalent root-finding problems

$$1 - \phi_1 z - \phi_2 z^2 - ... - \phi_p z^p = 0,$$

and

$$\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - ... - \phi_p = 0.$$

The latter one is equivalent to the eigenvalue problem $|\lambda \mathbf{I}_p - \mathbf{F}| = 0$.

'Traditionally', or in the literature,

- we call the roots of the former one 'the roots of the lag polynomial',
- and we call the roots of the latter one 'the eigenvalues of the companion matrix', as **F** is termed the companion matrix of the pth-order difference equation.

pth-Order Difference Equations (revisited)

Provided the factorization

$$1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p = (1 - \lambda_1 L)(1 - \lambda_2 L) \times ... \times (1 - \lambda_p L),$$

if

- all the roots of the corresponding lag polynomial are greater than one in modulus (lie outside the unit circle or unit disk), or equivalently,
- all the eigenvalues of the corresponding companion matrix are less than one in modulus (lie inside the unit circle or unit disk),

then we call the pth-order difference equation stable, and each $1 - \lambda_i L$, $i = 1, .., p$, can be inverted, that is

$$(1 - \lambda_i L)^{-1} = 1 + \lambda_i L + + \lambda_i^2 L^2 + + \lambda_i^3 L^3 + ... \qquad (40)$$

Thus, the pth-order difference equation for y_t

$$(1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p) y_t = w_t$$

can be transformed into

$$y_t = (1 - \lambda_1 L)^{-1} (1 - \lambda_2 L)^{-1} ... (1 - \lambda_p L)^{-1} w_t, \tag{41}$$

by multiplying $(1 - \lambda_i L)^{-1}$, $i = 1, ..., p$, on both sides.

pth-Order Difference Equations (revisited)

Recall the dynamic multiplier in (21). Likewise, if the eigenvalues λ_i are distinct, we have first

$$(1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}...(1-\lambda_p L)^{-1} = \sum_{i=1}^{p} \frac{c_i}{1-\lambda_i L}. \tag{42}$$

See [2.4.8] on pp.34 in Hamilton. c_is are defined in (21). Combined with (40), we have

$$\begin{aligned} y_t &= \sum_{i=1}^{p} \frac{c_i}{1-\lambda_i L} \cdot w_t = \sum_{i=1}^{p} c_i(1+\lambda_i L + +\lambda_i^2 L^2 + +\lambda_i^3 L^3 + ...) \cdot w_t \\ &= w_t \sum_{i=1}^{p} c_i + w_{t-1} \sum_{i=1}^{p} c_i \lambda_i + w_{t-2} \sum_{i=1}^{p} c_i \lambda_i^2 + ... \\ &= \sum_{j=0}^{\infty} \left(w_{t-j} \sum_{i=1}^{p} c_i \lambda_i^j \right) \quad \text{(get used to it)} \end{aligned} \tag{43}$$

pth-Order Difference Equations (revisited)

From (43), we can obtain the dynamic multiplier

$$\frac{\partial y_{t+\tau}}{\partial w_t} = \sum_{i=1}^{p} c_i \lambda_i^{\tau}$$

(41) is often written as

$$\begin{aligned} y_t &= (1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}...(1-\lambda_p L)^{-1} w_t, \\ &= \prod_{i=1}^{p}(1+\lambda_i L + +\lambda_i^2 L^2 + +\lambda_i^3 L^3 + ...) w_t \\ &= \psi_0 w_t + \psi_1 w_{t-1} + \psi_2 w_{t-2} + ... \\ &= \psi(L) w_t \end{aligned} \tag{44}$$

where $\psi(L) = \psi_0 + \psi_1 L + ...$ represents the lag polynomial, when the difference equation is stable.

Whether the eigenvalues are distinct is irrelevant for this form.

However, when they are distinct, then $\psi_j = \frac{\partial y_{t+j}}{\partial w_t} = \sum_{i=1}^{p} c_i \lambda_i^j$, as c_i exit.

Initial Conditions

Given the pth-order difference equation

$$y_t = \sum_{i=1}^{p} \phi_i y_{t-i} + w_t,$$

p initial values of y

$$y_0,\ y_{-1},\ \dots,\ y_{1-p},\ ,$$

and a sequence of w

$$w_1,\ w_2,\ \dots,\ w_t,\ ,$$

we can calculate the sequence of y from time 1 to t

$$y_1,\ y_2,\ \dots,\ y_t,\ ,$$

Initial Conditions

However, there are many examples in economics and finance in which a theory does not specify the initial values y_0, y_{-1}, ..., y_{1-p}. See the example and discussion on pp.36-42 in Hamilton.

To be continued! Thank you!