Written Exam of the Course
Time Series Econometrics 2ST111 2019

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This is a five hours open-book exam. Please read carefully and answer all questions. The answers shall be clearly written, concise and relevant, and all steps shall be well explained. The total score is 100 points.

You can bring the textbook, the printed materials offered by the teachers (slides, notes), and any paper or printed articles that are relevant to the course. You are not allowed to use any calculator, computer, smart-phone or any devices with internet or bluetooth connection. You can bring paper dictionary, but the electronic dictionary is not allowed. You can use pen, pencil, eraser and ruler. You are not allowed to share any books, notes, papers, tools or devices with others during the exam.

Part 1: Moving Average Model (48 points)

Consider the following MA(2) model:

$$Y_t = \mu + \epsilon_t + \alpha_1 \epsilon_{t-1} + \alpha_2 \epsilon_{t-2} \tag{1}$$

where ϵ_t is a white noise process with zero mean and variance σ^2.

1. (16 points) Let Y_t be a stationary process. Derive explicitly (i) the unconditional mean $\mathrm{E}(Y_t)$, (ii) the variance $\gamma_0 = \mathrm{Var}(Y_t)$, (iii) the covariances $\gamma_i = \mathrm{Cov}(Y_t, Y_{t-i})$, as well as (iv) the autocorrelations ρ_i, for $i = 1, 2, ...$
2. (8 points) Explain the conditions on the parameters under which Y_t is a stationary process. Also explain the conditions under which the MA model is invertible.

Suppose now instead that you have the MA(1) process

$$Y_t = \mu + 0.5\epsilon_{t-1} + \epsilon_t, \tag{2}$$

where ϵ_t is some error term with $\mathrm{E}(\epsilon_t) = 0$ for all $t = 0, \pm 1, \pm 2, \ldots$.

3. (8 points) Assume $\text{Cov}(\epsilon_t, \epsilon_\tau) = 0$ for all $t \neq \tau$. Is the process in (2) covariance-stationary? Explain why or why not.

4. (8 points) After some further investigation of the data, you find that it seems plausible that ϵ_t is a martingale difference sequence. Is the process in (2) covariance-stationary under this assumption? Explain why or why not.

5. (8 points) Another alternative, that you also find plausible, is that ϵ_t itself is an AR(1) process. More specifically, you now feel quite confident that the data-generating process for the error term can be described by

$$\epsilon_t = 0.4\epsilon_{t-1} + u_t$$

where u_t is a white noise term. Is the process in (2) covariance-stationary under this assumption? Explain why or why not.

Part 2: (12 points)

The weekly log US M2 money stock x_t (10 April 1995 – 29 January 2001, 304 observations, offered by Federal Reserve Bank of St. Louis) is plotted in levels in Figure 1. Its first order difference, plotted in Figure 2, shows that the time series is either trend-stationary I(0) or I(1). The log M2 seems to follow a linear trend, and fluctuation is somewhat small around the trend, which implies that it may be trend-stationary.

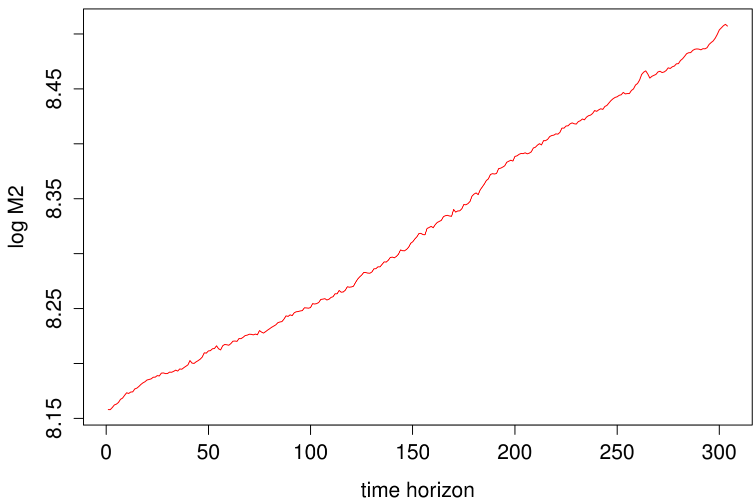

Figure 1: Weekly log US M2, 10 April 1995 – 29 January 2001, 304 obs.

A common technique to check whether it is trend-stationary is to regress the following model

$$x_t = \alpha + \delta t + u_t$$

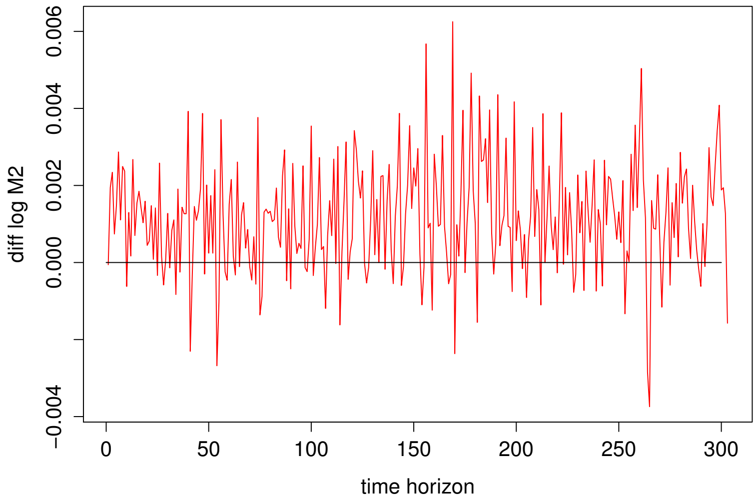

Figure 2: First order difference of log US M2

and plot the residuals $\hat{u}_t$. $\hat{u}_t$ is called the demeaned and detrended x_t, which is plotted in Figure 3. By doing so, we amplify the fluctuation around the trend by removing the trend. From Figure 3, we see that most probably x_t is not trend-stationary. Instead, it looks like a random walk.

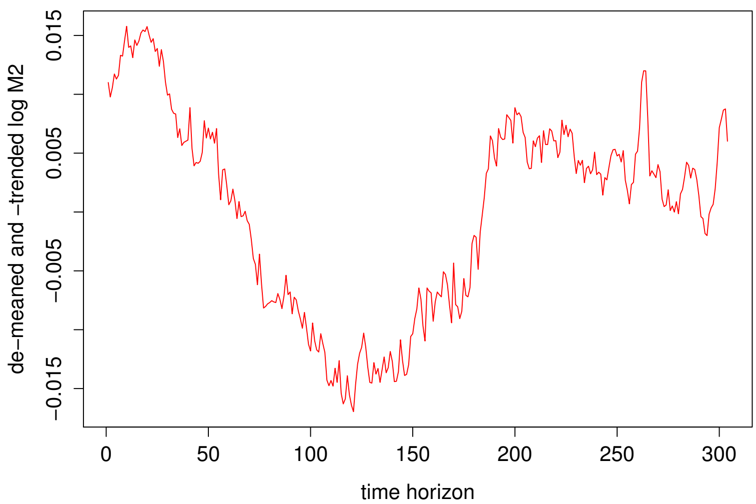

Figure 3: Demeaned and detrended log US M2

Now we need to give statistical evidence(s) and make inference about the truth. We run the following three regression models

$$\Delta x_t = \rho x_{t-1} + \sum_{i=1}^{p-1} \gamma_i \Delta x_{t-i} + u_t \tag{3}$$

$$\Delta x_t = \rho x_{t-1} + \sum_{i=1}^{p-1} \gamma_i \Delta x_{t-i} + \alpha + u_t \tag{4}$$

$$\Delta x_t = \rho x_{t-1} + \sum_{i=1}^{p-1} \gamma_i \Delta x_{t-i} + \alpha + \delta t + u_t \tag{5}$$

Note that they are not exactly the ones employed in Hamilton (1994). Hamilton (1994) uses the form $x_t = \varrho x_{t-1} + \ldots$ instead. You can easily check that $\varrho = \rho + 1$.

The three regression models above represent the settings "none", "drift", and "trend" in the function "ur.df()" in the R package "urca", respectively. The results of the augmented Dickey-Fuller test are summarized in Table 1. The null hypotheses are $\rho = 0$ for model (3); $\rho = 0$, $\alpha = 0$ for model (4); and $\rho = 0$, $\alpha = 0$, $\delta = 0$ for model (5). Note again that in Hamilton (1994), the unit root test has $H_0 : \varrho = 1$, but in our case (urca package), it is $H_0 : \rho = 0$.

The estimated models are given in (6)-(8), and the lag length $p = 2$ is chosen by AIC.

$$\Delta x_t = 0.00015\, x_{t-1} - 0.058\, \Delta x_{t-1} + u_t \tag{6}$$

$$\Delta x_t = 0.00085\, x_{t-1} - 0.062\, \Delta x_{t-1} - 0.0059 + u_t \tag{7}$$

$$\Delta x_t = -0.015\, x_{t-1} - 0.055\, \Delta x_{t-1} + 0.13 + 0.000019\, t + u_t \tag{8}$$

Model	(3)	(4)		(5)		
par.	ρ	ρ	α	ρ	α	δ
1%	-2.58	-3.44	6.47	-3.98	6.15	8.34
5%	-1.95	-2.87	4.61	-3.42	4.71	6.30
10%	-1.62	-2.57	3.79	-3.13	4.05	5.36
stats	11.39	1.04	65.13	-1.55	44.56	1.89

Table 1: Results from the augmented Dickey-Fuller tests. Rows 3-5 are critical values at significance levels 1%, 5% and 10%, respectively. 'stats' are the values of the test statistics.

1. (4 points) Hamilton (1994) mentions four types of augmented Dickey-Fuller tests, but here we only have three. Specify the missing one.

2. (8 points) From the available test statistics and their critical values, which model do you think is the best one among the others (models (3)-(5))? Motivate your choice.

Part 3: VAR and Cointegration (40 points)

The VAR(k) model

$$\mathbf{X}_t = \mathbf{A}_1\mathbf{X}_{t-1} + \mathbf{A}_2\mathbf{X}_{t-2} + \ldots + \mathbf{A}_k\mathbf{X}_{t-k} + \boldsymbol{\mu} + \boldsymbol{\delta} t + \boldsymbol{\varepsilon}_t \tag{9}$$

has the VECM form as follows

$$\Delta\mathbf{X}_t = \boldsymbol{\Pi}\mathbf{X}_{t-1} + \boldsymbol{\Gamma}_1\Delta\mathbf{X}_{t-1} + \boldsymbol{\Gamma}_2\Delta\mathbf{X}_{t-2} + \ldots + \boldsymbol{\Gamma}_{k-1}\Delta\mathbf{X}_{t-k+1} + \boldsymbol{\mu} + \boldsymbol{\delta} t + \boldsymbol{\varepsilon}_t, \tag{10}$$

where $\boldsymbol{\varepsilon}_t \overset{i.i.d.}{\sim} N_p(\mathbf{0}, \boldsymbol{\Sigma})$. If the p-dimensional random vector $\mathbf{x}_t$, $t = 1, \ldots, T$, given the initial values $\mathbf{x}_0, \ldots, \mathbf{x}_{1-k}$, are I(1) and can be cointegrated, then $\boldsymbol{\Pi}$ can be represented by the product of two matrices $\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$.

1. (4 points) Write the explicit form of $\boldsymbol{\Gamma}_i$ for $i = 1, \ldots, p$ in (10) expressed in terms of $\mathbf{A}_i$ for $i = 1, \ldots, p-1$ in (9).

Then consider the vector system

$$\begin{aligned} \Delta X_{1t} &= \alpha_1(X_{1,t-1} - \beta_2 X_{2,t-1}) + \mu_1 + \varepsilon_{1t} \\ \Delta X_{2t} &= \mu_2 + \varepsilon_{2t}. \end{aligned} \tag{11}$$

2. (4 points) Find the matrices $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\Pi}$, $\boldsymbol{\mu}$, and $\mathbf{A} = \mathbf{A}_1$ (as $k = 1$) expressed in terms of the notations in (11).
3. (12 points) The model (11) has the VAR(1) form as (9). Find the representation of $\mathbf{X}_t = (X_{1t}, X_{2t})'$ in terms of $\mathbf{X}_0$ and $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \varepsilon_{2t})'$, $t = 1, \ldots, T$ given all the paramters, and find $\mathrm{E}(\mathbf{X}_t|\mathbf{X}_0)$ and $\mathrm{Var}(\mathbf{X}_t|\mathbf{X}_0)$.
4. (4 points) The VAR(1) form can be written as $\mathbf{A}(L)\mathbf{X}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t$, where $\mathbf{A}(L) = \mathbf{I} - \mathbf{A}L$ is the lag polynomial. The characteristic polynomial is given by replacing the lag operator L by the variable z, i.e. $\mathbf{A}(z)$. The roots of the characteristic polynomial is the all the possible values of z satisfying the equation $|\mathbf{A}(z)| = |\mathbf{I} - \mathbf{A}z| = 0$. Calculate the roots.
5. (8 points) Verify that when $\alpha_1 = -2$, the process $\mathbf{X}_t$ is not I(1). Under what condition on the parameters is the process $\mathbf{X}_t$ an I(1) process?
6. (4 points) Show explicitly from (11) that $\boldsymbol{\beta}'\mathbf{X}_t$ is stationary if $-2 < \alpha_1 < 0$.
7. (4 points) Suppose that $-2 < \alpha_1 < 0$. Under what condition on the parameters is there no linear trend?

References

Hamilton, J. D.: 1994, *Time Series Econometrics*, Princeton University Press.