Solutions to TSE Exam, 141128

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1 Task 1

Consider an MA(2) process:

$$y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2},$$

where $\epsilon_t \sim N(0,1)$ and $E(\epsilon_t \epsilon_\tau) = 0$ if $t \neq \tau$.

1. (x p) The state space representation is given by:

$$\begin{aligned} \boldsymbol{\xi}_{t+1} &= \mathbf{F}\boldsymbol{\xi}_t + \mathbf{v}_{t+1}, & E(\mathbf{v}_t\mathbf{v}_t') &= \mathbf{Q} \\ \mathbf{y}_t &= \mathbf{A}'\mathbf{x}_t + \mathbf{H}'\boldsymbol{\xi}_t + \mathbf{w}_t, & E(\mathbf{w}_t\mathbf{w}_t') &= \mathbf{R} \end{aligned}$$

where $E(\mathbf{v}_t\mathbf{v}_\tau') = E(\mathbf{w}_t\mathbf{w}_\tau') = \mathbf{0}$ if $t \neq \tau$. Write the MA(2) process on state space form, and state explicitly the values of all matrices in the representation for this process.

2. (x p) Suppose that $y_1 = 2$, $\theta_1 = 0.5$, $\theta_2 = 0$ and $\epsilon_0 = 0$. Find the starting values of the Kalman filter:

$$\begin{aligned} \hat{\boldsymbol{\xi}}_{1|0} &= E(\boldsymbol{\xi}_1) \\ \mathbf{P}_{1|0} &= E\{[\boldsymbol{\xi}_1 - E(\boldsymbol{\xi}_1)][\boldsymbol{\xi}_1 - E(\boldsymbol{\xi}_1)]\} \end{aligned}$$

and compute the filter estimate $\hat{\epsilon}_{1|1}$.

2 Task 2

Consider the VAR model defined by the equations

$$\begin{aligned} y_{1,t} &= y_{1,t-1} - 0.3y_{2,t-1} + u_{1,t} \\ y_{2,t} &= 0.5y_{1,t-1} + 0.15y_{2,t-1} + u_{2,t} \end{aligned}$$

where $E(u_{1,t}u_{1,\tau}) = 2$ if $t = \tau$ and 0 otherwise, $E(u_{2,t}u_{2,\tau}) = 3$ if $t = \tau$ and 0 otherwise, and $E(u_{1,t}u_{2,\tau}) = 0$ for all t and τ.

$\Omega \quad \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

1. (x p) Is this system covariance-stationary?
2. (x p) Calculate the moving average weights $\boldsymbol{\Psi}_s$ for $s = 1, 2, 3$.
3. (x p) Suppose that $y_{1,t} = 2$ and $y_{2,t} = -1$. Calculate the forecasts for time $t+1$ and $t+2$ given this information. What is the mean squared error of $\hat{y}_{1,t+1|t}$ and $\hat{y}_{1,t+2|t}$?

3 Solution to Task 1

Part 1

State and measurement equations:

$$\begin{pmatrix} \epsilon_{t+1} \\ \epsilon_t \\ \epsilon_{t-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \epsilon_t \\ \epsilon_{t-1} \\ \epsilon_{t-2} \end{pmatrix} + \begin{pmatrix} \epsilon_{t+1} \\ 0 \\ 0 \end{pmatrix}.$$

$$y_t = \begin{pmatrix} 1 & \theta_1 & \theta_2 \end{pmatrix} \begin{pmatrix} \epsilon_t \\ \epsilon_{t-1} \\ \epsilon_{t-2} \end{pmatrix}$$

In terms of the general state-space representation:

$$\begin{aligned} \boldsymbol{\xi}_{t+1} &= \mathbf{F}\boldsymbol{\xi}_t + \mathbf{v}_{t+1}, & E(\mathbf{v}_t\mathbf{v}_t') &= \mathbf{Q} \\ \mathbf{y}_t &= \mathbf{A}'\mathbf{x}_t + \mathbf{H}'\boldsymbol{\xi}_t + \mathbf{w}_t, & E(\mathbf{w}_t\mathbf{w}_t') &= \mathbf{R} \end{aligned}$$

we have that

$$\boldsymbol{\xi}_{t+1} = \begin{pmatrix} \epsilon_{t+1} \\ \epsilon_t \\ \epsilon_{t-1} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{v}_{t+1} = \begin{pmatrix} \epsilon_{t+1} \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{y}_t = y_t, \quad \mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} 1 & \theta_1 & \theta_2 \end{pmatrix}, \quad \mathbf{A}' = \mathbf{x}_t = \mathbf{w}_t = \mathbf{R} = 0.$$

Part 2

Since $\theta_2 = 0$, we have an MA(1). We can therefore use a state space representation of smaller dimension (as for an MA(1), see p.375), where $\boldsymbol{\xi}_t = (\epsilon_t, \epsilon_{t-1})'$. Then the initial value is

$$E(\boldsymbol{\xi}_1) = E\begin{pmatrix} \epsilon_1 \\ \epsilon_0 \end{pmatrix} = \mathbf{0}$$

since all error terms have expectation 0. The MSE matrix:

$$\mathbf{P}_{1|0} = \begin{pmatrix} E(\epsilon_1^2) & E(\epsilon_1\epsilon_0) \\ E(\epsilon_1\epsilon_0) & E(\epsilon_0^2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The term $\hat{\epsilon}_{1|1}$ is the second element of $\hat{\boldsymbol{\xi}}_{2|1}$. We can calculate this by:

$$\hat{\boldsymbol{\xi}}_{2|1} = \mathbf{F}\hat{\boldsymbol{\xi}}_{1|0} + \mathbf{F}\mathbf{P}_{1|0}\mathbf{H}(\mathbf{H}'\mathbf{P}_{1|0}\mathbf{H} + \mathbf{R})^{-1}(\mathbf{y}_1 - \mathbf{A}'\mathbf{x}_1 - \mathbf{H}'\hat{\boldsymbol{\xi}}_{1|0})$$

which can be greatly simplified using that some terms are zero, and also that $\mathbf{P}_{1|0}$ is the identity matrix:

$$\begin{aligned}
\hat{\boldsymbol{\xi}}_{2|1} &= \mathbf{F}\hat{\boldsymbol{\xi}}_{1|0} + \mathbf{F}\mathbf{P}_{1|0}\mathbf{H}(\mathbf{H}'\mathbf{P}_{1|0}\mathbf{H} + \mathbf{R})^{-1}(\mathbf{y}_1 - \mathbf{A}'\mathbf{x}_t - \mathbf{H}'\hat{\boldsymbol{\xi}}_{1|0}) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \theta_1 \end{pmatrix}\left[\begin{pmatrix} 1 & \theta_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \theta_1 \end{pmatrix}\right]^{-1} y_1 \\
&= \begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{y_1}{1+\theta_1^2}.
\end{aligned}$$

Thus, $\hat{\epsilon}_{1|1} = y_1/(1+\theta_1^2) = 8/5$.

4 Solution to Task 2

Part 1

Rewrite the system as

$$\mathbf{y}_t = \boldsymbol{\Phi}\mathbf{y}_{t-1} + \mathbf{u}_t$$

where $\mathbf{y}_t = (y_{1,t}, y_{2,t})'$, $\mathbf{u}_t = (u_{1,t}, u_{2,t})'$ and

$$\boldsymbol{\Phi} = \begin{pmatrix} 1 & -0.3 \\ 0.5 & 0.15 \end{pmatrix}.$$

The eigenvalues are given as the solutions to

$$\begin{aligned}
|\lambda \mathbf{I} - \boldsymbol{\Phi}| &= 0 \\
\begin{vmatrix} \lambda - 1 & 0.3 \\ -0.5 & \lambda - 0.15 \end{vmatrix} &= 0 \\
(\lambda - 1)(\lambda - 0.15) + 0.15 &= 0 \\
\lambda^2 - 1.15\lambda + 0.3 &= 0,
\end{aligned}$$

which are the roots of a quadratic equation. We can solve this by using the quadratic formula, and we then get

$$\begin{aligned}
\lambda &= \frac{1.15 \pm \sqrt{1.15^2 - 4 \times 0.3}}{2} \\
\lambda &= \frac{1.15}{2} \pm \frac{0.35}{2}.
\end{aligned}$$

The eigenvalues are $\lambda_1 = 0.75$ and $\lambda_2 = 0.4$, both of which are inside the unit circle. Hence, the system is stationary by p. 259.

Part 2

For a VAR(1), the moving average weights are simply given by

$$\begin{aligned}\mathbf{\Psi}_1 &= \mathbf{\Phi}\\ \mathbf{\Psi}_2 &= \mathbf{\Phi}^2\\ \mathbf{\Psi}_3 &= \mathbf{\Phi}^3.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{\Psi}_1 &= \begin{pmatrix}1 & -0.3\\ 0.5 & 0.15\end{pmatrix}\\ \mathbf{\Psi}_2 &= \begin{pmatrix}1 & -0.3\\ 0.5 & 0.15\end{pmatrix}\begin{pmatrix}1 & -0.3\\ 0.5 & 0.15\end{pmatrix} = \begin{pmatrix}0.85 & -0.345\\ 0.575 & -0.1275\end{pmatrix}\\ \mathbf{\Psi}_3 &= \begin{pmatrix}0.85 & -0.345\\ 0.575 & -0.1275\end{pmatrix}\begin{pmatrix}1 & -0.3\\ 0.5 & 0.15\end{pmatrix} = \begin{pmatrix}0.68 & -0.31\\ 0.51 & -0.19\end{pmatrix}.\end{aligned}$$

Part 3

The forecasts for a VAR(1) without an intercept are given by

$$\hat{\mathbf{y}}_{t+s|t} = \mathbf{\Phi}^s\mathbf{y}_t,$$

where in our case $s = 1, 2$ and $\mathbf{y}_t = (2, -1)'$. We get:

$$\begin{aligned}\hat{\mathbf{y}}_{t+1|t} &= \mathbf{\Phi}\mathbf{y}_t = \begin{pmatrix}1 & -0.3\\ 0.5 & 0.15\end{pmatrix}\begin{pmatrix}2\\ -1\end{pmatrix} = \begin{pmatrix}2.3\\ 0.85\end{pmatrix}\\ \hat{\mathbf{y}}_{t+2|t} &= \mathbf{\Phi}^2\mathbf{y}_t = \mathbf{\Phi}\hat{\mathbf{y}}_{t+1|t} = \begin{pmatrix}1 & -0.3\\ 0.5 & 0.15\end{pmatrix}\begin{pmatrix}2.3\\ 0.85\end{pmatrix} = \begin{pmatrix}2.045\\ 1.2775\end{pmatrix}.\end{aligned}$$

The mean squared errors are defined as:

$$MSE(\hat{\mathbf{y}}_{t+s|t}) = E[(\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t})(\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t})']$$

where

$$(\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t}) = \left(\mathbf{\Phi}^s\mathbf{y}_t + \sum_{j=0}^{s-1}\mathbf{\Psi}_j\mathbf{u}_{t+s-j}\right) - \mathbf{\Phi}^s\mathbf{y}_t = \sum_{j=0}^{s-1}\mathbf{\Psi}_j\mathbf{u}_{t+s-j}.$$

Since $\mathbf{u}_t$ and $\mathbf{u}_\tau$ are independent for $t \neq \tau$, it follows that the MSE is

$$\begin{aligned}MSE(\hat{\mathbf{y}}_{t+s|t}) &= E\left(\sum_{j=0}^{s-1}\mathbf{\Psi}_j\mathbf{u}_{t+s-j}\sum_{k=0}^{s-1}\mathbf{u}'_{t+s-k}\mathbf{\Psi}'_k\right)\\ &= \mathbf{\Omega} + \mathbf{\Psi}_1\mathbf{\Omega}\mathbf{\Psi}'_1 + \cdots + \mathbf{\Psi}_{s-1}\mathbf{\Omega}\mathbf{\Psi}'_{s-1}\end{aligned}$$

where $\mathbf{\Omega}$ is the covariance matrix of $\mathbf{u}_t$, i.e.

$$\mathbf{\Omega} = E(\mathbf{u}_t\mathbf{u}_t') = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

We are still only interested in $s = 1$ and $s = 2$, which give us

$$MSE(\hat{\mathbf{y}}_{t+1|t}) = \mathbf{\Omega} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$MSE(\hat{\mathbf{y}}_{t+2|t}) = \mathbf{\Omega} + \mathbf{\Psi}_1\mathbf{\Omega}\mathbf{\Psi}_1' = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & -0.3 \\ 0.5 & 0.15 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0.5 \\ -0.3 & 0.15 \end{pmatrix}$$
$$= \begin{pmatrix} 6 & -1.8 \\ 3 & 4.35 \end{pmatrix}.$$

Thus, the MSE of $\hat{y}_{1,t+1|t}$ is 2 and the MSE of $\hat{y}_{1,t+2|t}$ is 6.