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Professor Yukai Yang 9 September 2022
Time Series Econometrics

Problem 1

Let us define the following:

$$\xi_t = \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}, \quad v_t = \begin{pmatrix} w_t \\ 0 \end{pmatrix}. \tag{1}$$

Then for any τ future periods after t and r periods before t, we can write the second-order difference equation with a system of equations:

$$\xi_{t+\tau} = \mathbf{F}\xi_{t+\tau-1} + v_{t+\tau} \tag{2}$$
$$= \begin{pmatrix} y_{t+\tau} \\ y_{t+\tau-1} \end{pmatrix} = \mathbf{F}\begin{pmatrix} y_{t+\tau-1} \\ y_{t+\tau-2} \end{pmatrix} + \begin{pmatrix} w_{t+\tau} \\ 0 \end{pmatrix} \tag{3}$$
$$= \mathbf{F}^2 \left[\begin{pmatrix} y_{t+\tau-2} \\ y_{t+\tau-3} \end{pmatrix} + \begin{pmatrix} w_{t+\tau-1} \\ 0 \end{pmatrix} \right] + \begin{pmatrix} w_{t+\tau} \\ 0 \end{pmatrix} \tag{4}$$
$$\vdots \tag{5}$$
$$= \mathbf{F}^{\tau+r}\begin{pmatrix} y_{t-r-1} \\ y_{t-r-2} \end{pmatrix} + \sum_{s=-r}^{\tau} \mathbf{F}^{\tau-s}\begin{pmatrix} w_{t+s} \\ 0 \end{pmatrix}. \tag{6}$$

Q1.1. For $\tau = 3$, we have from (6) that

$$\xi_{t+3} = \begin{pmatrix} y_{t+3} \\ y_{t+2} \end{pmatrix} = \mathbf{F}^3\begin{pmatrix} y_{t-1} \\ y_{t-2} \end{pmatrix} + \begin{pmatrix} w_{t+3} \\ 0 \end{pmatrix} + \mathbf{F}\begin{pmatrix} w_{t+2} \\ 0 \end{pmatrix} \mathbf{F}^2 \begin{pmatrix} w_{t+1} \\ 0 \end{pmatrix} + \mathbf{F}^3\begin{pmatrix} w_t \\ 0 \end{pmatrix}. \tag{7}$$

With direct multiplication, we have

$$\mathbf{F}^3\begin{pmatrix} w_t \\ 0 \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}^3 \begin{pmatrix} w_t \\ 0 \end{pmatrix} = \begin{pmatrix} \phi_1^2+\phi_2 & \phi_1\phi_2 \\ \phi_1 & \phi_2 \end{pmatrix}\begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} w_t \\ 0 \end{pmatrix} \tag{8}$$
$$= \begin{pmatrix} \phi_1^3+2\phi_1\phi_2 & \phi_1^2\phi_2+\phi_2^2 \\ \phi_1^2+\phi_2 & \phi_1\phi_2 \end{pmatrix}\begin{pmatrix} w_t \\ 0 \end{pmatrix} \tag{9}$$
$$= w_t\begin{pmatrix} \phi_1^3+2\phi_1\phi_2 \\ \phi_1^2+\phi_2 \end{pmatrix}. \tag{10}$$

Then we have from the first row of system of equations (7) that

$$\frac{\partial y_{t+3}}{\partial w_t} = \frac{\partial w_t(\phi_1^3+2\phi_1\phi_2)}{\partial w_t} = \phi_1^3 + 2\phi_1\phi_2. \tag{11}$$

Q1.2. Similarly for $\tau = 4$, we have from (6) that

$$\begin{pmatrix} y_{t+4} \\ y_{t+3} \end{pmatrix} = \mathbf{F}^4\begin{pmatrix} y_{t-1} \\ y_{t-2} \end{pmatrix} + \begin{pmatrix} w_{t+4} \\ 0 \end{pmatrix} + \mathbf{F}\begin{pmatrix} w_{t+3} \\ 0 \end{pmatrix} + ... + \mathbf{F}^4\begin{pmatrix} w_t \\ 0 \end{pmatrix}. \tag{12}$$

Then with direct multiplication,

$$\mathbf{F}^4\begin{pmatrix} w_t \\ 0 \end{pmatrix} = \begin{pmatrix} \phi_1^3+2\phi_1\phi_2 & \phi_1^2\phi_2+\phi_2^2 \\ \phi_1^2+\phi_2 & \phi_1\phi_2 \end{pmatrix}\begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} w_t \\ 0 \end{pmatrix} \tag{13}$$
$$= \begin{pmatrix} \phi_1^4+3\phi_1^2\phi_2+\phi_2^2 & \phi_1^3\phi_2+2\phi_1\phi_2^2 \\ \phi_1^3+2\phi_1\phi_2 & \phi_1^2\phi_2+\phi_2^2 \end{pmatrix}\begin{pmatrix} w_t \\ 0 \end{pmatrix} \tag{14}$$
$$= w_t\begin{pmatrix} \phi_1^4+3\phi_1^2\phi_2+\phi_2^2 \\ \phi_1^3+2\phi_1\phi_2 \end{pmatrix}. \tag{15}$$

Then we have from the first row of system of equations (12) that

$$\frac{\partial y_{t+4}}{\partial w_t} = \phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2. \tag{16}$$

Problem 2.

Q2.1. For any eigenvector $v \in \mathbb{R}^p$, $v \neq 0$ and its eigenvalue $\lambda \in \mathbb{C}$,

$$\begin{aligned}
\mathbf{F}v = \lambda v \implies 0 &= |\mathbf{F} - \lambda \mathbf{I}| && (17)\\
&= \left| \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| && (18)\\
&= \left| \begin{pmatrix} \phi_1 - \lambda & \phi_2 \\ 1 & -\lambda \end{pmatrix} \right| && (19)\\
&= \lambda^2 - \phi_1\lambda - \phi_2. && (20)
\end{aligned}$$

Then with $\phi_1 = 3/4,\ \phi_1 = -1/8$, we get two values for λ from the quadratic formula,

$$\begin{aligned}
\lambda_1 &= \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} = \frac{3/4 + \sqrt{(3/4)^2 - 4(1/8)}}{2} = \frac{1}{2}, && (21)\\
\lambda_2 &= \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} = \frac{3/4 - \sqrt{(3/4)^2 - 4(1/8)}}{2} = \frac{1}{4}. && (22)
\end{aligned}$$

Then two eigenvectors with the eigenvalues are derived as follows.

$$\begin{aligned}
(\mathbf{F} - \lambda_1\mathbf{I})v_1 &= \begin{pmatrix} \frac{3}{4} - \frac{1}{2} & -\frac{1}{8} \\ 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}x_1 - \frac{1}{8}x_2 \\ x_1 - \frac{1}{2}x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ x_1 = 1 \implies v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. && (23)\\
(\mathbf{F} - \lambda_2\mathbf{I})v_2 &= \begin{pmatrix} \frac{3}{4} - \frac{1}{4} & -\frac{1}{8} \\ 1 & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}y_1 - \frac{1}{8}y_2 \\ y_1 - \frac{1}{4}y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ y_1 = 1 \implies v_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}. && (24)
\end{aligned}$$

Now define the matrices

$$\mathbf{T} = (v_1\ v_2) = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \tag{25}$$

so that for any $n \in \mathbb{N}$,

$$\mathbf{F}^n = \mathbf{T}\mathbf{\Lambda}^{\mathbf{n}}\mathbf{T}^{-\mathbf{1}}, \quad \mathbf{T}^{-\mathbf{1}} = \begin{pmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix}. \tag{26}$$

Then we have from (6) that for any $j \in \mathbb{N}$,

$$\begin{pmatrix} y_{t+j} \\ y_{t+j-1} \end{pmatrix} = \mathbf{F}^j \begin{pmatrix} y_{t-1} \\ y_{t-2} \end{pmatrix} + \sum_{i=0}^{j} \mathbf{F}^{j-i} \begin{pmatrix} w_{t+i} \\ 0 \end{pmatrix}, \tag{27}$$

where the coefficient on $\begin{pmatrix} w_t \\ 0 \end{pmatrix}$ is $\mathbf{F}^j$, and

$$\begin{aligned}
\mathbf{F}^j = \mathbf{T}\mathbf{\Lambda}^j\mathbf{T}^{-\mathbf{1}} &= \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}^j \begin{pmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix} && (28)\\
&= \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \left(\frac{1}{2}\right)^j & 0 \\ 0 & \left(\frac{1}{4}\right)^j \end{pmatrix} \begin{pmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix} && (29)\\
&= w_t \begin{pmatrix} \left(\frac{1}{2}\right)^{j-1} - \left(\frac{1}{4}\right)^j \\ \left(\frac{1}{2}\right)^{j-2} - \left(\frac{1}{4}\right)^{j-1} \end{pmatrix}. && (30)
\end{aligned}$$

Then from the first row of (27) we have that

$$\frac{\partial y_{t+j}}{\partial w_t} = \left(\frac{1}{2}\right)^{j-1} - \left(\frac{1}{4}\right)^{j}. \tag{31}$$

<u>**Q2.2.**</u> From equation (31) we can see that

$$\begin{aligned}
\frac{\partial y_{t+3}}{\partial w_t} &= \left(\frac{1}{2}\right)^{3-1} - \left(\frac{1}{4}\right)^{3} = \frac{15}{64} && (32)\\
&= \left(\frac{3}{4}\right)^{3} + 2\left(\frac{3}{4}\right)\left(-\frac{1}{8}\right) = \phi_1^3 + 2\phi_1\phi_2, && (33)\\
\frac{\partial y_{t+4}}{\partial w_t} &= \left(\frac{1}{2}\right)^{4-1} - \left(\frac{1}{4}\right)^{4} = \frac{31}{256} && (34)\\
&= \left(\frac{3}{4}\right)^{4} + 3\left(\frac{3}{4}\right)^{2}\left(-\frac{1}{8}\right) + \left(-\frac{1}{8}\right)^{2} = \phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2, && (35)
\end{aligned}$$

which are the same results as in problem 1.

<u>**Q2.2.**</u> Since $|\lambda_1| = \frac{1}{2} < 1$ and $|\lambda_2| = \frac{1}{4} < 1$, we have that

$$\begin{aligned}
\lim_{j\to\infty} \mathbf{F}^j = \lim_{j\to\infty} \mathbf{T}\mathbf{\Lambda}^j\mathbf{T^{-1}} &= \lim_{j\to\infty} \mathbf{T}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}^j \mathbf{T^{-1}} && (36)\\
&= \lim_{j\to\infty} \mathbf{T}\begin{pmatrix}\left(\frac{1}{2}\right)^j & 0\\ 0 & \left(\frac{1}{4}\right)^j\end{pmatrix} \mathbf{T^{-1}} && (37)\\
&= \mathbf{T}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\mathbf{T^{-1}} && (38)\\
&= \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}. && (39)
\end{aligned}$$

Therefore

$$\lim_{j\to\infty} \frac{\partial y_{t+j}}{\partial w_t} = \lim_{j\to\infty}\left(\left(\frac{1}{2}\right)^{j-1} - \left(\frac{1}{4}\right)^{j}\right) = 0, \tag{40}$$

and the system is stable.

Problem 3.

<u>**Q3.1.**</u> We have that $\forall s \in \mathbb{Z},\ z_s$ are i.i.d. and

$$z_s = \begin{pmatrix} y_{2s-1}\\ y_{2s}\end{pmatrix} \sim N(\mathbf{0}, \mathbf{\Sigma}), \quad \mathbf{\Sigma} = \begin{pmatrix}1 & \gamma\\ \gamma & 1\end{pmatrix}. \tag{41}$$

By definition of the multivariate Gaussian, all linear combinations of the elements of z_s must be normally distributed. In other words, z_s is Gaussian if and only if

$$\forall v \in \mathbb{R}^2,\ v \neq \mathbf{0}, \quad \exists \mu, \sigma \in \mathbb{R}, \quad v' z_s \sim N(\mu, \sigma^2). \tag{42}$$

For period t, let $s = \frac{t}{2}$ and $v = \begin{pmatrix}0\\1\end{pmatrix}$ if t is even and $s = \frac{t+1}{2}$ and $v = \begin{pmatrix}1\\0\end{pmatrix}$ if t is odd. Then for some μ, σ,

$$v' z_s = y_t \sim N(\mu, \sigma^2). \tag{43}$$

From the definition of expectation of a random vector, we have that

$$E(z_s) = \begin{pmatrix} E(y_{2s-1}) \\ E(y_{2s}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies E(y_t) = \mu = 0. \tag{44}$$

From the definition of the covariance matrix of a random vector, we have that

$$\begin{aligned} Var(z_s) &= E[(z_s - E(z_s))(z_s - E(z_s))'] & (45) \\ &= E\begin{pmatrix} (y_{2s-1} - E(y_{2s-1}))^2 & (y_{2s-1} - E(y_{2s-1}))(y_{2s} - E(y_{2s})) \\ (y_{2s-1} - E(y_{2s-1}))(y_{2s} - E(y_{2s})) & (y_{2s} - E(y_{2s}))^2 \end{pmatrix} & (46) \\ &= \begin{pmatrix} E(y_{2s-1} - E(y_{2s-1}))^2 & E[(y_{2s-1} - E(y_{2s-1}))(y_{2s} - E(y_{2s}))] \\ E[(y_{2s-1} - E(y_{2s-1}))(y_{2s} - E(y_{2s}))] & E(y_{2s} - E(y_{2s}))^2 \end{pmatrix} & (47) \\ &= \begin{pmatrix} Var(y_{2s-1}) & Cov(y_{2s-1}, y_{2s}) \\ Cov(y_{2s-1}, y_{2s}) & Var(y_{2s}) \end{pmatrix} = \mathbf{\Sigma} = \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}. & (48) \end{aligned}$$

Since $Var(y_{2s-1}) = Var(y_{2s}) = 1$, in any case $Var(y_t) = \sigma^2 = 1$. Then $y_t \sim N(0, 1)$.

Q3.2. The jth autocovariance of y_t is given by

$$\gamma_{jt} = Cov(y_t, y_{t-j}) = E(y_t - \mu_{y_t})(y_{t-j} - \mu_{y_{t-j}}). \tag{49}$$

Let $x_t = (y_{t-1}, y_t)'$ be a random variable where

$$x_t = \begin{pmatrix} y_{t-1} \\ y_t \end{pmatrix} \sim N(\mathbf{0}, \mathbf{\Sigma_x}), \quad \mathbf{\Sigma_x} = \begin{pmatrix} 1 & \gamma_{1t} \\ \gamma_{1t} & 1 \end{pmatrix} \tag{50}$$

since $y_t \sim N(0, 1)$ for all $t \in \mathbb{Z}$. Then when t is even, we have that $x_t = z_{t/2}$ and $\gamma_{1t} = \gamma$.

Since z_s's are i.i.d., we have that

$$\begin{aligned} Cov(z_s, z_{s+1}) = \mathbf{0} &= E[(z_s - E(z_s))(z_{s+1} - E(z_{s+1}))'] & (51) \\ &= E\left[\begin{pmatrix} y_{2s-1} - E(y_{2s-1}) \\ y_{2s} - E(y_{2s}) \end{pmatrix} \begin{pmatrix} y_{2s+1} - E(y_{2s+1}) & y_{2s+2} - E(y_{2s+2}) \end{pmatrix}\right] & (52) \\ &= \begin{pmatrix} Cov(y_{2s-1}, y_{2s+1}) & Cov(y_{2s-1}, y_{2s+2}) \\ Cov(y_{2s}, y_{2s+1}) & Cov(y_{2s}, y_{2s+2}) \end{pmatrix}, & (53) \end{aligned}$$

so $Cov(y_{2s}, y_{2s+1}) = 0$ for all $s \in \mathbb{Z}$. Then when t is odd, choosing $s = \frac{t-1}{2}$ gives us that $\gamma_{1t} = Cov(y_{t-1}, y_t) = Cov(y_{2s}, y_{2s+1}) = 0$.

Then we have that for $j = 1$,

$$\gamma_{jt} = \gamma_{1t} = \begin{cases} \gamma, & t \text{ is even}, \\ 0, & t \text{ is odd}. \end{cases} \tag{54}$$

If $\gamma \neq 0$, then the $j = 1$ autocovariance is not independent of time t, and the sequence $\{y_t\}$ is non-stationary (neither covariance-stationary nor strictly stationary).

Problem 4.

The white noise process $\{\varepsilon_t\}$ has the property that for all $t \in \mathbb{Z}$,

$$E(\varepsilon_t) = 0, \quad Var(\varepsilon_t) = E(\varepsilon^2) = \sigma^2, \quad E(\varepsilon_t \varepsilon_\tau) = \begin{cases} \sigma^2, & t = \tau, \\ 0, & t \neq \tau. \end{cases} \tag{55}$$

For the first moments of the MA(1) processes, we have that

$$\begin{aligned} E(y_t) &= E(\mu + \varepsilon_t + \theta \varepsilon_{t-1}) & (56) \\ &= E(\mu) + E(\varepsilon_t) + \theta E(\varepsilon_{t-1}) & (57) \\ &= \mu, & (58) \end{aligned}$$

$$
\begin{aligned}
E(\tilde{y}_t) &= E(\mu + \tilde{\varepsilon}_t + \tilde{\theta}\tilde{\varepsilon}_{t-1}), \quad \tilde{\varepsilon}_t = \theta\varepsilon_t,\ \tilde{\theta} = 1/\theta && (59)\\
&= E\left(\mu + \theta\varepsilon_t + \frac{1}{\theta}\theta\varepsilon_{t-1}\right) && (60)\\
&= E(\mu) + \theta E(\varepsilon_t) + E(\varepsilon_{t-1}) && (61)\\
&= \mu. && (62)
\end{aligned}
$$

Then $E(y_t) = E(\tilde{y}_t) = \mu$. For the auto-covariances, we have that for any $j \in \mathbb{N}$,

$$
\begin{aligned}
\gamma_{jt} = Cov(y_t, y_{t-j}) &= E(y_t - \mu)(y_{t-j} - \mu) && (63)\\
&= E(\mu + \varepsilon_t + \theta\varepsilon_{t-1} - \mu)(\mu + \varepsilon_{t-j} + \theta\varepsilon_{t-j-1} - \mu) && (64)\\
&= E(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-j} + \theta\varepsilon_{t-j-1}) && (65)\\
&= E(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-j} + \theta\varepsilon_{t-j-1}) && (66)\\
&= E(\varepsilon_t\varepsilon_{t-j} + \theta\varepsilon_t\varepsilon_{t-j-1} + \theta\varepsilon_{t-1}\varepsilon_{t-j} + \theta^2\varepsilon_{t-1}\varepsilon_{t-j-1}) && (67)\\
&= E(\varepsilon_t\varepsilon_{t-j}) + \theta E(\varepsilon_t\varepsilon_{t-j-1}) + \theta E(\varepsilon_{t-1}\varepsilon_{t-j}) + \theta^2 E(\varepsilon_{t-1}\varepsilon_{t-j-1}) && (68)\\
&= \begin{cases} (1+\theta^2)\sigma^2, & j = 0, \\ \theta\sigma^2, & j = 1, \\ 0, & j \geq 2, \end{cases} && (69)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\gamma}_{jt} = Cov(\tilde{y}_t, \tilde{y}_{t-j}) &= E(\tilde{y}_t - \mu)(\tilde{y}_{t-j} - \mu) && (70)\\
&= E(\mu + \tilde{\varepsilon}_t + \tilde{\theta}\tilde{\varepsilon}_{t-1} - \mu)(\mu + \tilde{\varepsilon}_{t-j} + \tilde{\theta}\tilde{\varepsilon}_{t-j-1} - \mu) && (71)\\
&= E(\theta\varepsilon_t + \frac{1}{\theta}\theta\varepsilon_{t-1})(\theta\varepsilon_{t-j} + \frac{1}{\theta}\theta\varepsilon_{t-j-1}) && (72)\\
&= E(\theta^2\varepsilon_t\varepsilon_{t-j} + \theta\varepsilon_t\varepsilon_{t-j-1} + \theta\varepsilon_{t-1}\varepsilon_{t-j} + \varepsilon_{t-1}\varepsilon_{t-j-1}) && (73)\\
&= \theta^2 E(\varepsilon_t\varepsilon_{t-j}) + \theta E(\varepsilon_t\varepsilon_{t-j-1}) + \theta E(\varepsilon_{t-1}\varepsilon_{t-j}) + E(\varepsilon_{t-1}\varepsilon_{t-j-1}) && (74)\\
&= \begin{cases} (\theta^2+1)\sigma^2, & j = 0, \\ \theta\sigma^2, & j = 1, \\ 0, & j \geq 2. \end{cases} && (75)
\end{aligned}
$$

Thus we have that $\gamma_{jt} = Cov(y_t, y_{t-j}) = Cov(\tilde{y}_t, \tilde{y}_{t-j}) = \tilde{\gamma}_{jt}$.

Problem 5.

Q5.1. By recursive substitution, we have that

$$
\begin{aligned}
\varepsilon_t &= (1 - \phi L)y_t && (76)\\
&= y_t - \phi L y_t && (77)\\
&= y_t - \phi y_{t-1} && (78)\\
\implies y_t &= \phi y_{t-1} + \varepsilon_t && (79)\\
\implies y_{t+1} &= \phi y_t + \varepsilon_{t+1} && (80)\\
\implies y_{t+2} &= \phi y_{t+1} + \varepsilon_{t+2} && (81)\\
&= \phi(\phi y_t + \varepsilon_{t+1}) + \varepsilon_{t+2} && (82)\\
&= \phi^2 y_t + \phi\varepsilon_{t+1} + \varepsilon_{t+2} && (83)\\
\implies y_{t+3} &= \phi y_{t+2} + \varepsilon_{t+3} && (84)\\
&= \phi(\phi^2 y_t + \phi\varepsilon_{t+1} + \varepsilon_{t+2}) + \varepsilon_{t+3} && (85)\\
&= \phi^3 y_t + \phi^2\varepsilon_{t+1} + \phi\varepsilon_{t+2} + \varepsilon_{t+3} && (86)\\
&\vdots && (87)\\
\implies y_{t+s} &= \phi^s y_t + \phi^{s-1}\varepsilon_{t+1} + \phi^{s-2}\varepsilon_{t+2} + ... + \phi\varepsilon_{t+s-1} + \varepsilon_{t+s} && (88)\\
&= \phi^s y_t + \sum_{i=0}^{s-1}\phi^i\varepsilon_{t+s-i}. && (89)
\end{aligned}
$$

Q5.2. Given information set I_t at time t, we have that

$$E(y_t|I_t) = y_t, \tag{90}$$
$$E(\varepsilon_\tau|I_t) = E(\varepsilon_\tau) = 0, \quad \tau > t. \tag{91}$$

Then for $E(y_{t+s}|I_t)$, we have

$$\begin{aligned}
E(y_{t+s}|I_t) &= E\left(\phi^s y_t + \sum_{i=0}^{s-1} \phi^i \varepsilon_{t+s-i} \,\middle|\, I_t\right) && (92)\\
&= \phi^s E(y_t|I_t) + \sum_{i=0}^{s-1} \phi^i E\left(\varepsilon_{t+s-i}|I_t\right) && (93)\\
&= \phi^s y_t + E(\varepsilon_{t+s}|I_t) + ... + \phi^{s-2} E(\varepsilon_{t+2}|I_t) + \phi^{s-1} E(\varepsilon_{t+1}|I_t) && (94)\\
&= \phi^s y_t. && (95)
\end{aligned}$$

Time Series Econometrics: Home work assignment 1

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Abstract

Please write your report in LaTeX. The report should be clearly written such that it is easy to understand what is done and why. Please attach any computer code in an appendix.

1 Problem 1

Consider the second-order difference equation ($p = 2$)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t.$$

Using direct multiplication, show that

1. the effect on y_{t+3} of a one-unit increase in w_t is
$$\phi_1^3 + 2\phi_1\phi_2 \tag{1}$$
2. the effect on y_{t+4} of a one-unit increase in w_t is
$$\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2 \tag{2}$$

2 Problem 2

Consider the same difference equation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t,$$

where $\phi_1 = 3/4$ and $\phi_2 = -1/8$.

1. Using the eigenvalues of the matrix $\mathbf{F}$, show that

$$\frac{\partial y_{t+j}}{\partial w_t} = \left(\frac{1}{2}\right)^{j-1} - \left(\frac{1}{4}\right)^{j}. \tag{3}$$

2. For $j = 3$ and $j = 4$, verify that (1) and (2) produce the same results as (3).

3. Is the system stable? Motivate your answer.

3 Problem 3

Let $\{y_t\}_{t=-\infty}^{\infty}$ be given by

$$\mathbf{z}_s = \begin{pmatrix} y_{2s-1} \\ y_{2s} \end{pmatrix}, \quad s = 0, \pm 1, \pm 2, \ldots$$

where $\mathbf{z}_s$ is iid $N(\mathbf{0}, \mathbf{\Sigma})$, with

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}.$$

Using standard results for the multivariate normal distribution,

1. verify that $y_t \sim N(0,1)$ for all $t = 0, \pm 1, \pm 2, \ldots$

2. show that if $\gamma \neq 0$, then $\{y_t\}_{t=-\infty}^{\infty}$ is neither strictly stationary nor covariance stationary

Hint: In (ii), compare the distribution of $\mathbf{z}_1$ with that of $(y_2, y_3)'$.

4 Problem 4

Let $\{\epsilon_t\}_{t=-\infty}^{\infty}$ be a white noise process and $\theta \neq 0$. Consider the two $MA(1)$ processes $\{y_t\}_{t=-\infty}^{\infty}$ and $\{\tilde{y}_t\}_{t=-\infty}^{\infty}$ given by

$$y_t = \mu + \epsilon_t + \theta \epsilon_{t-1}$$

and

$$\tilde{y}_t = \mu + \tilde{\epsilon}_t + \tilde{\theta} \tilde{\epsilon}_{t-1}$$

respectively, where $\tilde{\epsilon}_t = \theta\epsilon_t$ and $\tilde{\theta} = 1/\theta$.

Verify that $E(y_t) = E(\tilde{y}_t) = \mu$ and

$$E(y_t - \mu)(y_{t-j} - \mu) = E(\tilde{y}_t - \mu)(\tilde{y}_{t-j} - \mu),$$

for $j = 0, 1, 2, \ldots$.[1]

5 Problem 5

Consider the simple $AR(1)$ process

$$(1 - \phi L)y_t = \epsilon_t$$

where $\{\epsilon_t\}_{t=-\infty}^{\infty}$ is a white noise process.

1. Show, by recursive substitution, that

$$y_{t+s} = \phi^s y_t + \sum_{i=0}^{s-1} \phi^i \epsilon_{t+s-i}.$$

2. Use the above formula to compute the conditional expectation $E(y_{t+s}|I_t)$, where I_t is the information set available at time t.

[1]That is, verify that for any invertible $MA(1)$ representation, there is a noninvertible $MA(1)$ representation with the same first and second moments as the invertible representation.