# Time Series Econometrics
# Supplementary Lecture 3

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## 1 Exercise 5.1

*Show that the value of [5.4.16] at $\theta = \tilde{\theta}$, $\sigma^2 = \tilde{\sigma}^2$ is identical to its value at $\theta = \tilde{\theta}^{-1}$, $\sigma^2 = \tilde{\theta}^2\tilde{\sigma}^2$.*

The likelihood is

$$\mathcal{L}(\boldsymbol{\theta}) = -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\log(d_{tt}) - \frac{1}{2}\sum_{t=1}^{T}\frac{\tilde{y}_t^2}{d_{tt}}$$

where

$$d_{tt}(\theta, \sigma^2) = \sigma^2\frac{\sum_{j=0}^{t}\theta^{2j}}{\sum_{j=0}^{t-1}\theta^{2j}}, \quad \tilde{y}_t(\theta) = y_t - \mu - \frac{\theta\sum_{j=0}^{t-2}\theta^{2j}}{\sum_{j=0}^{t-1}\theta^{2j}}\tilde{y}_{t-1}$$

where the terms are written as functions of parameters to explicitly express which parameter set we are considering. We also have that $\tilde{y}_1 = y_1 - \mu$.

The likelihood will be the same if the individual terms can be shown to be the same. In other words, if $d_{tt}(\tilde{\theta}, \tilde{\sigma}^2) = d_{tt}(\tilde{\theta}^{-1}, \tilde{\theta}^2\tilde{\sigma}^2)$ and $\tilde{y}_t^2(\tilde{\theta}) = \tilde{y}_t^2(\tilde{\theta}^{-1})$, then the likelihoods are the same.

We start with $d_{tt}$:

$$d_{tt}(\tilde{\theta}, \tilde{\sigma}^2) = \tilde{\sigma}^2\frac{\sum_{j=0}^{t}\tilde{\theta}^{2j}}{\sum_{j=0}^{t-1}\tilde{\theta}^{2j}}$$

and for the other set

$$
\begin{aligned}
d_{tt}(\tilde{\theta}^{-1}, \tilde{\theta}^2\tilde{\sigma}^2) &= \tilde{\theta}^2\tilde{\sigma}^2 \frac{\sum_{j=0}^{t} \tilde{\theta}^{-2j}}{\sum_{j=0}^{t-1} \tilde{\theta}^{-2j}} = \cancel{\tilde{\theta}^2}\tilde{\sigma}^2 \frac{\tilde{\theta}^{2(t\cancel{-1})}}{\tilde{\theta}^{2(t-1)}} \frac{\sum_{j=0}^{t} \tilde{\theta}^{-2j}}{\sum_{j=0}^{t-1} \tilde{\theta}^{-2j}} \\
&= \tilde{\sigma}^2 \frac{\sum_{j=0}^{t} \tilde{\theta}^{-2j}}{\sum_{j=0}^{t-1} \tilde{\theta}^{-2j}} \frac{\tilde{\theta}^{2t}}{\tilde{\theta}^{2(t-1)}} = \tilde{\sigma}^2 \frac{\sum_{j=0}^{t} \tilde{\theta}^{2(t-j)}}{\sum_{j=0}^{t-1} \tilde{\theta}^{2(t-1-j)}} \\
&= \tilde{\sigma}^2 \frac{\sum_{j=0}^{t} \tilde{\theta}^{2(t-j)}}{\sum_{j=0}^{t-1} \tilde{\theta}^{2(t-1-j)}} = \tilde{\sigma}^2 \frac{\sum_{j=0}^{t} \tilde{\theta}^{2j}}{\sum_{j=0}^{t-1} \tilde{\theta}^{2(j-1)}} \\
&= d_{tt}(\tilde{\theta}, \tilde{\sigma}^2)
\end{aligned}
$$

The second to last equality is obvious if writing the sums out; it's the same sum, just summing in opposite directions.

For $\tilde{y}_t$, they will be the same if its term depending on $\theta$ is the same for both cases. For the first one, it is

$$
\frac{\tilde{\theta} \sum_{j=0}^{t-2} \tilde{\theta}^{2j}}{\sum_{j=0}^{t-1} \tilde{\theta}^{2j}}
$$

and in the second case

$$
\begin{aligned}
\frac{\theta^{-1} \sum_{j=0}^{t-2} \tilde{\theta}^{-2j}}{\sum_{j=0}^{t-1} \tilde{\theta}^{-2j}} = \frac{\tilde{\theta}^{2(t-1)}}{\tilde{\theta}^{2(t-1)}} \frac{\tilde{\theta}^{-1} \sum_{j=0}^{t-2} \tilde{\theta}^{-2j}}{\sum_{j=0}^{t-1} \tilde{\theta}^{-2j}} &= \frac{\tilde{\theta}^{2(t-2)}\tilde{\theta}^2\tilde{\theta}^{-1}}{\tilde{\theta}^{2(t-1)}} \frac{\sum_{j=0}^{t-2} \tilde{\theta}^{-2j}}{\sum_{j=0}^{t-1} \tilde{\theta}^{-2j}} \\
&= \tilde{\theta} \frac{\sum_{j=0}^{t-2} \tilde{\theta}^{2(t-2-j)}}{\sum_{j=0}^{t-1} \tilde{\theta}^{2(t-1-j)}} = \frac{\tilde{\theta} \sum_{j=0}^{t-2} \tilde{\theta}^{2j}}{\sum_{j=0}^{t-1} \tilde{\theta}^{2j}}
\end{aligned}
$$

and they are the same (last equality is easily verified by writing the terms out). Hence, what we have shown is that

$$
\begin{aligned}
&\mathcal{L}(\tilde{\theta}, \tilde{\sigma}^2) - \mathcal{L}(\tilde{\theta}^{-1}, \tilde{\theta}^2\tilde{\sigma}^2) \\
&= \left( -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T} \log\left( d_{tt}(\tilde{\theta}, \tilde{\sigma}^2) \right) - \frac{1}{2}\sum_{t=1}^{T} \frac{\tilde{y}_t^2(\tilde{\theta})}{d_{tt}(\tilde{\theta}, \tilde{\sigma}^2)} \right) \\
&- \left( -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T} \log\left( d_{tt}(\tilde{\theta}^{-1}, \tilde{\theta}^2\tilde{\sigma}^2) \right) - \frac{1}{2}\sum_{t=1}^{T} \frac{\tilde{y}_t^2(\tilde{\theta}^{-1})}{d_{tt}(\tilde{\theta}^{-1}, \tilde{\theta}^2\tilde{\sigma}^2)} \right) \\
&= \frac{1}{2}\sum_{t=1}^{T} \left[ \underbrace{\log\left( d_{tt}(\tilde{\theta}^{-1}, \tilde{\theta}^2\tilde{\sigma}^2) \right) - \log\left( d_{tt}(\tilde{\theta}, \tilde{\sigma}^2) \right)}_{=0} + \underbrace{\frac{\tilde{y}_t^2(\tilde{\theta}^{-1})}{d_{tt}(\tilde{\theta}^{-1}, \tilde{\theta}^2\tilde{\sigma}^2)} - \frac{\tilde{y}_t^2(\tilde{\theta})}{d_{tt}(\tilde{\theta}, \tilde{\sigma}^2)}}_{=0} \right] \\
&= 0
\end{aligned}
$$

## 2 Exercise 7.3

*Does a martingale difference sequence have to be covariance-stationary?*

No. For $Y_t$ to be an MDS, it is required that $E(Y_t) = 0$ and that $E(Y_t|Y_{t-1}, Y_{t-2}, \dots) = 0$. If $Y_t \sim N(0, t^2)$, it will not be covariance stationary, but it will satisfy the MDS conditions.

## 3 Exercise 7.4

*Let $Y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$, where $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and $\{\varepsilon_t\}$ is a martingale difference sequence with $E(\varepsilon_t^2) = \sigma^2$. Is $Y_t$ covariance stationary?*

Since the error term is a martingale difference sequence, this means that it is serially uncorrelated. An MDS condition is stronger than no serial correlation, but weaker than serial independence. So it is just a regular white noise error term, and hence what we have is an $MA(\infty)$, which is stationary.