Written Exam of the Course Time Series Econometrics 2ST111 Fall 2021

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This is a four hours open-book exam. Please read carefully and answer all questions. The answers shall be clearly written, concise and relevant, and all steps shall be well explained. The total score is 100 points.

You can bring the textbook, the printed materials offered by the teachers (slides, notes), and any paper or printed articles that are relevant to the course. You are not allowed to use any calculator, computer, smart-phone or any devices with internet or bluetooth connection. You can bring paper dictionary, but the electronic dictionary is not allowed. You can use pen, pencil, eraser and ruler. You are not allowed to share any books, notes, papers, tools or devices with others during the exam.

Part 1: Vector AutoRegressive Model (50 points)

Consider the VAR(2) model:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} y_{1,t-2} \\ y_{2,t-2} \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} t + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}, \quad (1)$$

and its matrix version

$$\boldsymbol{y}_t = \boldsymbol{A}\boldsymbol{y}_{t-1} + \boldsymbol{B}\boldsymbol{y}_{t-2} + \boldsymbol{c} + \boldsymbol{d}t + \boldsymbol{u}_t, \quad (2)$$

where $\boldsymbol{u}_t$ is a white noise process with zero mean and positive definite covariance matrix $\boldsymbol{\Omega}$. The covariance matrix has the structure

$$\boldsymbol{\Omega} = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}. \quad (3)$$

Define the set of some parameters of the VAR(2) model $\theta = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{d}\}$ (note, not a vector).

Note that the our notations follow strictly the rule that, small letters in boldface stand for vectors, capitcal in boldface for matrices. However, when you write your answers, it is hard to write boldface, and therefore you do not need to do it. We understand that y_t means $\boldsymbol{y}_t$ but not any of y_{1t} and y_{2t} in your answer sheet.

1. (X points) Speficy the sufficient (trend) weak or covariance stationarity condition(s) for the VAR(2) model. The answer shall be in details in the sense that, if you mention for example the "companion matrix", then you have to speficy explicitly what the companion matrix is in this case.

   Answer: These questions takes approximately 10 min.

   - The sufficient condition for the VAR(2) model to be (weakly or covariance) stationary is that all the roots of the equation

     $$\boldsymbol{I}_2 - \boldsymbol{A}z - \boldsymbol{B}z^2 = 0$$

     implied by the lag polynomial of the model are all located outside of the unit circle or disk.

     The model can be rewritten in the companion matrix form

     $$\tilde{\boldsymbol{y}}_t = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{y}}_{t-1} + \ldots$$

     where $\tilde{\boldsymbol{y}}_t = (\boldsymbol{y}_t', \boldsymbol{y}_{t-1}')'$, and

     $$\tilde{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{I}_2 & \boldsymbol{0} \end{pmatrix}$$

     Then, equivalently, the sufficient condition can also be that all the eigenvalues of the companion matrix $\tilde{\boldsymbol{A}}$ are located inside the unit circle or disk.

     The student may answer either of them.

Now assume for simplicity that $\boldsymbol{d} = \boldsymbol{0}$ and that the VAR(2) model is weakly or covariance stationary.

2. (X points) Compute the unconditional expectation $\boldsymbol{\mu}$ of $\boldsymbol{y}_t$ in terms of the parameters θ.

   Answer: These questions takes approximately 15 min.

- Take expectation of (2), and denote

$$\begin{aligned} \mathrm{E}[\boldsymbol{y}_t] &= \boldsymbol{A}\mathrm{E}[\boldsymbol{y}_{t-1}] + \boldsymbol{B}\mathrm{E}[\boldsymbol{y}_{t-2}] + \boldsymbol{c} \implies \\ \boldsymbol{\mu} &= \boldsymbol{A}\boldsymbol{\mu} + \boldsymbol{B}\boldsymbol{\mu} + \boldsymbol{c} \implies \\ \boldsymbol{\mu} &= (\boldsymbol{I}_2 - \boldsymbol{A} - \boldsymbol{B})^{-1}\boldsymbol{c} \end{aligned}$$

3. (X points) Write the steady-state form the VAR(2) model. The steady-state form is sometimes called the process in deviations of the mean, that is, to rewrite (or reparameterize) the model in terms of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{\mu}$.

   Answer: These questions takes approximately 10 min.

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   $$\boldsymbol{y}_t - \boldsymbol{\mu} = \boldsymbol{A}(\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{B}(\boldsymbol{y}_{t-2} - \boldsymbol{\mu}) + \boldsymbol{u}_t$$

4. (X points) Suppose that you have the sample $\{\boldsymbol{y}_t\}_{t=1}^T$. Propose a feasible scheme to test the hypothesis $H_0 : \boldsymbol{\mu} = \boldsymbol{m}$ for some known values $\boldsymbol{m}$. Describe the scheme in details.

   Answer: These questions takes approximately 15 min.

   There are many ways. For example, estimate the covariances of $\boldsymbol{y}_t$, and then compute the long run covariance. Or estimate the steady-state model, and then do the test, for example LR test.

5. (X points) The VAR(2) model has an infinite VMA representation with the lag polynomial $\boldsymbol{\Psi}(L)$ in the moving average part. Derive the close forms for $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$ and $\boldsymbol{\Psi}_4$ given $\boldsymbol{A}$ and $\boldsymbol{B}$.

   Answer: These questions takes approximately 20 min.

   Due to

   $$(\boldsymbol{I}_2 - \boldsymbol{A}L - \boldsymbol{B}L^2)^{-1} = \boldsymbol{I}_2 + \boldsymbol{\Psi}_1 L + \boldsymbol{\Psi}_2 L^2 + \ldots,$$

   or

   $$(\boldsymbol{I}_2 - \boldsymbol{A}L - \boldsymbol{B}L^2)(\boldsymbol{I}_2 + \boldsymbol{\Psi}_1 L + \boldsymbol{\Psi}_2 L^2 + \ldots) = \boldsymbol{I}_2.$$

- $$\Psi_1 = \boldsymbol{A}$$

- $$\Psi_2 = \boldsymbol{A}^2 + \boldsymbol{B}$$

- $$\Psi_3 = \boldsymbol{A}^3 + \boldsymbol{AB} + \boldsymbol{BA}$$

- $$\Psi_4 = \boldsymbol{A}^4 + \boldsymbol{A}^2\boldsymbol{B} + \boldsymbol{ABA} + \boldsymbol{BA}^2 + \boldsymbol{B}^2$$

Assume now that the vector system (2) contains unit root(s) and that $\boldsymbol{d} \neq 0$. Furthermore, $\boldsymbol{y}_t \sim I(1)$.

6. (X points) The VAR(2) model (2) has the VECM form

$$\Delta \boldsymbol{y}_t = \boldsymbol{\Pi} \boldsymbol{y}_{t-1} + \boldsymbol{\Gamma} \Delta \boldsymbol{y}_{t-1} + \boldsymbol{c} + \boldsymbol{d}t + \boldsymbol{u}_t. \tag{4}$$

Find $\boldsymbol{\Pi}$ and $\boldsymbol{\Gamma}$ in terms of the elements in θ.

Answer: These questions takes approximately 10 min.

- $\boldsymbol{\Pi} = \boldsymbol{A} + \boldsymbol{B} - \boldsymbol{I}$, $\boldsymbol{\Gamma} = -\boldsymbol{B}$.

7. (X points) The rank of $\boldsymbol{\Pi}$, say r, has special meaning under the cointegration framework. Briefly describe the meaning of r, and the meaning of $n - r$, where n is the dimension of the vector system (number of the dependent variables).

Answer: These questions takes approximately 10 min.

- r is the number of cointegrating vectors, or better say, the dimension number of the subspace spanned by the cointegrating vectors.

8. (X points) If the two variables y_{1t} and y_{2t} can be cointegrated, give the possible value for r, and briefly explain why.

   Answer: These questions takes approximately 5 min.

   - $r = 1$. $r = 0$ means I(1) but no cointegration; $r = 2$ full rank and then no unit root.

9. (X points) In order to test and estimate the cointegration relation between the two variables y_{1t} and y_{2t}, the Engle-Granger's Procedure can be applied. Briefly describe the procedure.

   Answer: This question takes approximately 20 min. In the slides.

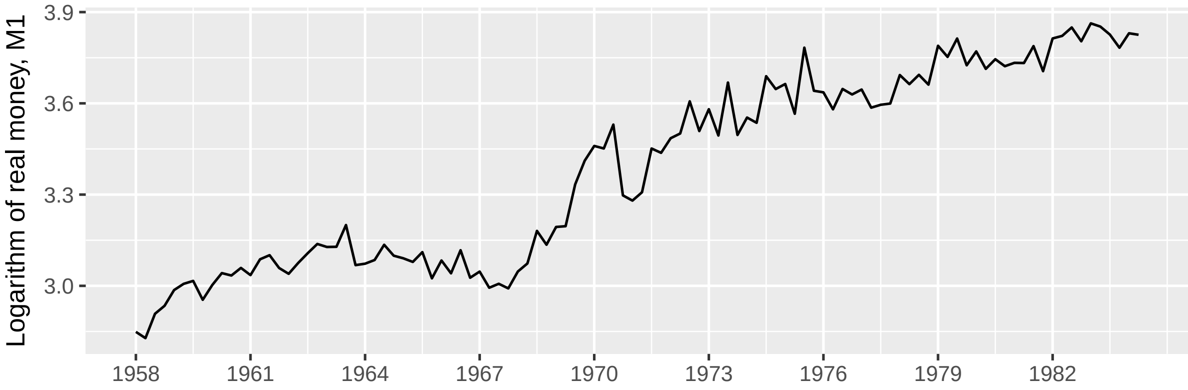
10. (8 points) The Johansen's procedure can also be applied to test and estimate the cointegration relation. Briefly show the difference between the two procedures in this case (two dependent variables).

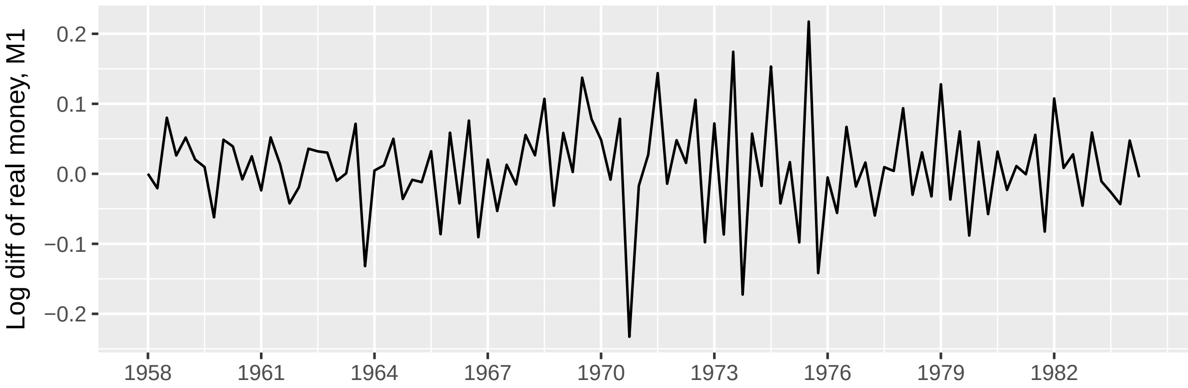
    Answer: This question takes approximately 15 min. The student should at least mention that Johansen's procedure allows for the case that not both of the variables must be I(1), while the E-G procedure requires. If one variable is I(1) and the other I(0), the Johansen's procedure will find one cointegration relation $(0, 1)$.

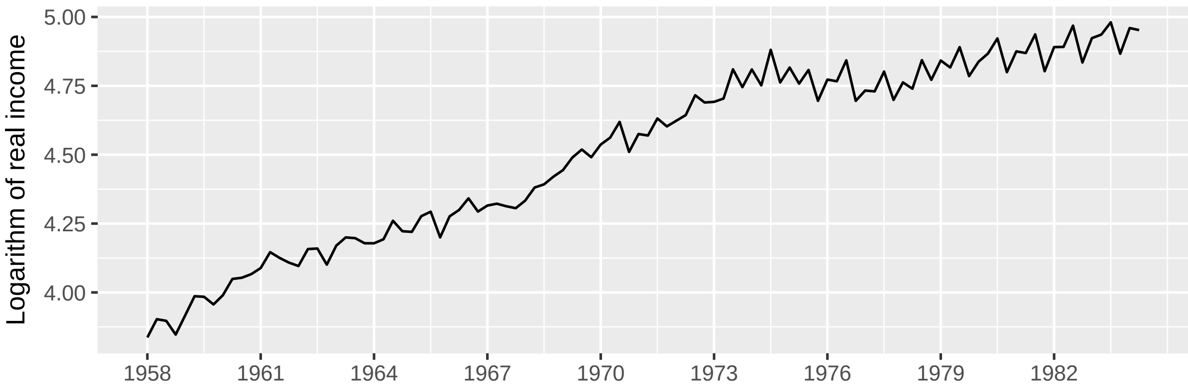
Part 2: Finnish Economy on the Demand for Money(12 points)

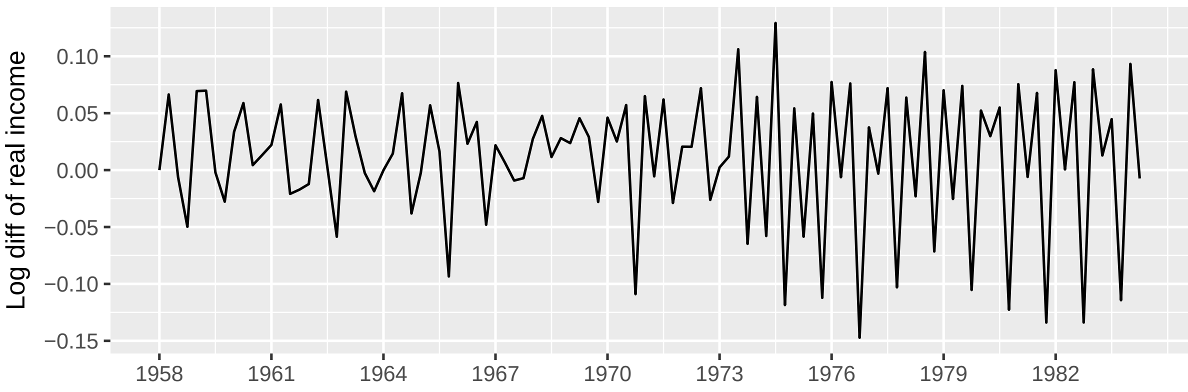
We chose the example and the corresponding data set given by Johansen and Juselius (1990) about the Finnish economy on the demand for money. The economic theory suggests that the money demand m can be represented by a function $m = f(y, p, c)$, where y is the real income, p the price level and c the cost of holding money.

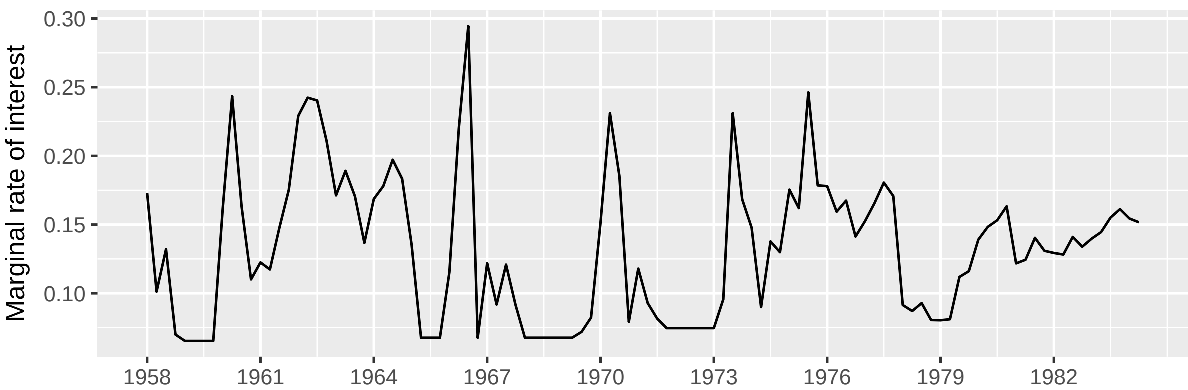
The Finnish data spans from 1958Q1 to 1984Q3. It uses the M1 (mainly the total cash in circulation excluding the bank reserves) as the measure of the money demand, the inflation rate Δp, and the marginal rate of interest i^m of the Bank of Finland as a proxy for the actual costs of holding money. By presumming multiplicative effects, money, income and prices are measured in logatithms. All the time series are plotted in the following:

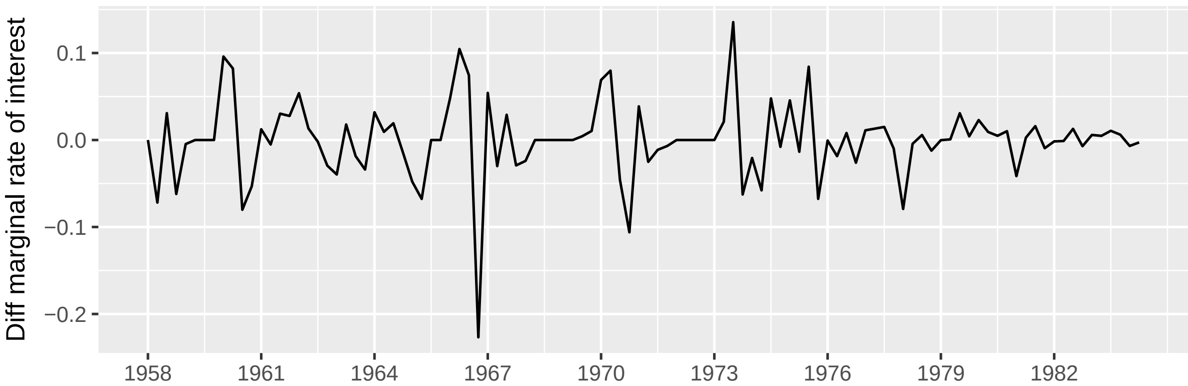


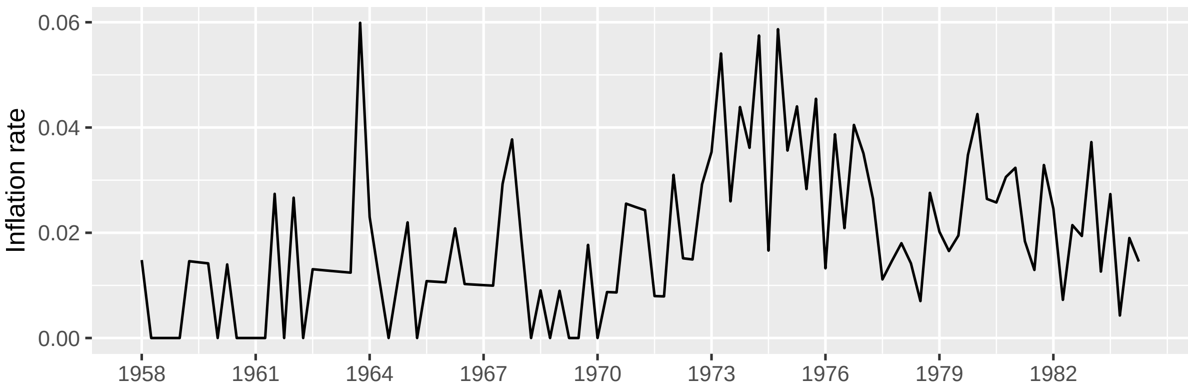


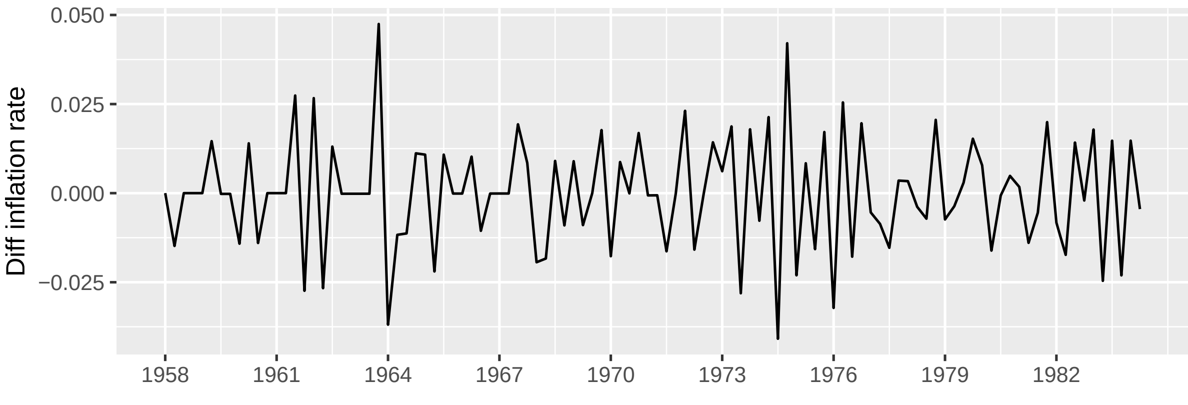












1. (X points) The first step of the analysis is to determine the integration orders for the variables. Hamilton (1994) gives four types of augmented Dickey-Fuller

tests. Let us assume that the differenced variables are stationary. Propose ADF test(s) for each of the four variables. Briefly explain why. Note that for certain variable, perhaps more than one ADF test will be applied.

Answer: This question takes approximately 30 min. But before doing this, it takes around 10 min to read and understand the question.

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2. (X points) In the following, you find the results from the trace tests based on different hypotheses. Are the variables cointegrated? Propose a suitable cointegration rank.

```
##           trace 10pct  5pct  1pct
## r <= 3 |  2.25  6.50  8.18 11.65
## r <= 2 | 10.04 15.66 17.95 23.52
## r <= 1 | 39.27 28.71 31.52 37.22
## r = 0  | 79.21 45.23 48.28 55.43
```

Answer: This question takes approximately 10 min.

- Yes.
- 2. 2 or 3 is also correct.

3. (X points) Consider that in this case, if we use the Engle-Granger's procedure, can we get the same result about the cointegration? Briefly explain why or why not.

Answer: This question takes approximately 10 min.

- No.
- The E-G procedure can only find one cointegration relation. And E-G procedure requires that the I(0) variables shall be removed.

References

Hamilton, J. D.: 1994, *Time Series Econometrics*, Princeton University Press.

Johansen, S. and Juselius, K.: 1990, Maximum likelihood estimation and inference on cointegration – with applications to the demand for money, *Oxford Bulletin of Economics and Statistics* **52**, 169–210.