1. Defn Let $v \in \mathbb{R}^n$ be eigenvector of $A \in \mathbb{R}^{n \times n}$.

Then 1) $Av = \lambda v$ where $\lambda \in \mathbb{R}$

2) Eigenvalue $\lambda \in \mathbb{R}$ has nontrivial solution.

$$Av = \lambda v, \; v \neq 0 \;\Rightarrow\; \lambda \text{ is eigenvalue of } v.$$

Then $Ax = \lambda x, \; x \neq \vec{0} \;\Rightarrow\; \vec{0} = Ax - \lambda I_n x$

$$= (A - \lambda I_n) x$$

$$\Rightarrow 0 = |A - \lambda I_n|$$

2.
$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t$$
$$\Rightarrow \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_{t-2} \end{pmatrix} + \begin{pmatrix} w_t \\ 0 \end{pmatrix}$$

System of p equations:

$$\begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{bmatrix}_{p\times 1} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_p \\ 1 & 0 & & 0 \\ 0 & 1 & & \\ & 0 & \ddots & \\ & & & 1 \end{bmatrix}_{p\times p} \begin{bmatrix} y_{t-1} \\ \vdots \\ y_{t-p} \end{bmatrix}_{p\times 1} + \begin{bmatrix} w_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{p\times 1}$$

$$= \quad \xi_t = F\xi_{t-1} + V_t$$

Eigenvalues
$$0 = |F - \lambda I|$$
$$= \left| \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right|$$
$$= \left| \begin{pmatrix} \phi_1 - \lambda & \phi_2 \\ 1 & -\lambda \end{pmatrix} \right| = -\lambda(\phi_1 - \lambda) - \phi_2 = \lambda^2 - \lambda\phi_1 - \phi_2$$

$$\Rightarrow \quad \lambda = \frac{\phi_1}{2} \pm \frac{\sqrt{\phi_1^2 + 4\phi_2}}{2},$$

$$\lambda_1 = a + ib$$

$$\lambda_2 = a - ib$$

$$\left(1 - \phi_1 L - \phi_2 L^2\right) = (1 + \lambda_1 L)(1 + \lambda_2 L) \qquad \text{eq. (35)}$$

$$= 1 + (\lambda_1 + \lambda_2)L + \lambda_1 \lambda_2 L^2$$

$$\phi_1 = -(\lambda_1 + \lambda_2) = -2a.$$

$$\phi_2 = -\lambda_1 \lambda_2 = (a + ib)(a - ib)$$

$$= (a^2 - i^2 b^2)$$

$$= (a^2 + b^2).$$

3. Assume eigenvalues are distinct.

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_p \end{pmatrix}_{p\times p}$$

$p\times 1$ Eigenvectors $x_1, \dots, x_p$

$$T = (x_1 \ \cdots \ x_p)_{p\times p}$$

$$FT = F(x_1 \ \cdots \ x_p) = (Fx_1 \ \cdots \ Fx_p)$$

From defn know

$$Fx_1 = \lambda_1 x_1, \ \dots, \ Fx_p = \lambda_p x_p, \quad Fx_i \in \mathbb{R}^{p\times 1}$$

$$FT = (\lambda_1 x_1 \ \ \lambda_2 x_2 \ \dots \ \lambda_p x_p)_{p\times p}$$

$$= (x_1 \ \dots \ x_p) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{pmatrix}$$

$$= T\Lambda$$

$$FTT^{-1} = F = T\Lambda T^{-1}$$

$$\xi_t = F\xi_{t-1} + V_t$$

$$= F^t \xi_0 + \sum_{s=0}^{t-1} F^s V_{t-s}$$

$$= (T\Lambda T^{-1})^t \xi_0 + \sum_{s=0}^{t-1} (T\Lambda T^{-1})^s V_{t-s}$$

$$(T\Lambda T^{-1})^k = (T\Lambda \underbrace{T^{-1})(T}_{I}\Lambda T^{-1}) \cdots (T\Lambda T^{-1})$$

$$= T\Lambda^k T^{-1}$$

$$\Rightarrow \xi_t = (T\Lambda^k T^{-1})\xi_0 + \sum_{s=0}^{t-1} (T\Lambda^s T^{-1}) V_{t-s}$$

$$\Lambda^t = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{pmatrix}^t = \begin{pmatrix} \lambda_1^{(t)} & & 0 \\ & \ddots & \\ 0 & & \lambda_p^{(t)} \end{pmatrix}$$

If $|\lambda_i| < 1$, $i = 1, \dots, p$, $\quad \Lambda^t \to 0$.

4. Let $t_i = \begin{pmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ \lambda_i \\ 1 \end{pmatrix}$

So that

$$F t_i = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_p \\ 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ \lambda_i \\ 1 \end{pmatrix}$$

Time Series Econometrics
Supplementary Lecture 1

Yukai Yang

Department of Statistics
Uppsala University

1 Eigenvalues and eigenvectors

In many of the coming chapters, eigenvalues, eigenvectors and certain related decompositions will appear from time to time. We will therefore start by repeating some results.

The starting point for eigenvalues is the equation

$$\underbrace{\mathbf{A}}_{p\times p}\underbrace{\mathbf{x}}_{p\times 1} = \lambda \underbrace{\mathbf{x}}_{p\times 1}$$

where $\mathbf{x} \neq \mathbf{0}$. Rewrite:

$$\begin{aligned} \mathbf{A}\mathbf{x} - \lambda\mathbf{I}_p\mathbf{x} &= \mathbf{0} \\ (\mathbf{A} - \lambda\mathbf{I}_p)\mathbf{x} &= \mathbf{0}. \end{aligned}$$

Assume that $(\mathbf{A} - \lambda\mathbf{I}_p)$ is invertible. This would imply that $\mathbf{x}$ is the $\mathbf{0}$ vector, which we have already assumed it is not. Thus, if $\mathbf{x} \neq \mathbf{0}$, then $(\mathbf{A} - \lambda\mathbf{I}_p)$ is singular. Singularity is equivalent to the determinant being zero. For this reason,

$$|\mathbf{A} - \lambda\mathbf{I}_p| = \mathbf{0},$$

and this equation we use to solve for λ.

Why are eigenvalues important to us in this course?

1. Stability (stationarity)
2. Decomposition and powers of matrices
3. Dynamic multipliers (impulse responses)

2 Stability

To put this into the time series context, we can use a second-order difference equation as an example:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t.$$

Write this as:

$$\begin{aligned}
\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} &= \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_{t-2} \end{pmatrix} + \begin{pmatrix} w_t \\ 0 \end{pmatrix} \\
\boldsymbol{\xi}_t &= \mathbf{F}\boldsymbol{\xi}_{t-1} + \mathbf{v}_t.
\end{aligned}$$

To see if this is a stable process (or stationary, if w_t were stochastic), we study the eigenvalues of the matrix $\mathbf{F}$. From the equation above:

$$\begin{aligned}
|\mathbf{F} - \lambda \mathbf{I}_2| &= 0 \\
\left| \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| &= \begin{vmatrix} \phi_1 - \lambda & \phi_2 \\ 1 & -\lambda \end{vmatrix} \\
&= -\phi_2 - \lambda(\phi_1 - \lambda) = \lambda^2 - \lambda\phi_1 - \phi_2 = 0
\end{aligned}$$

Completing the square gives us:

$$\begin{aligned}
\lambda^2 - \frac{2\lambda\phi_1}{2} + \frac{\phi_1^2}{4} &= \frac{\phi_1^2}{4} + \phi_2 \\
\left(\lambda - \frac{\phi_1}{2}\right)^2 &= \frac{\phi_1^2}{4} + \phi_2 \\
\lambda - \frac{\phi_1}{2} &= \pm\sqrt{\frac{\phi_1^2}{4} + \phi_2} \\
\lambda &= \frac{\phi_1}{2} \pm \sqrt{\frac{\phi_1^2}{4} + \phi_2} \\
&= \frac{\phi_1}{2} \pm \frac{\sqrt{\phi_1^2 + 4\phi_2}}{2}.
\end{aligned}$$

The two roots are therefore:

$$\lambda_1 = \frac{\phi_1}{2} + \frac{\sqrt{\phi_1^2 + 4\phi_2}}{2}, \quad \lambda_2 = \frac{\phi_1}{2} - \frac{\sqrt{\phi_1^2 + 4\phi_2}}{2}.$$

Because of this, it is possible to have complex roots. This happens when $\phi_1^2 + 4\phi_2 < 0$. This is possible even when we have only real-valued parameters. The complex roots always come in pairs as complex conjugates, meaning that if $\lambda_1 = a + bi$, then $\lambda_2 = a - bi$. The complex parts disappear, as we can see by factorizing the lag polynomial:

$$\begin{aligned}
(1 - \phi_1 L - \phi_2 L^2) &= (1 - \lambda_1 L)(1 - \lambda_2 L) \\
(1 - \phi_1 L - \phi_2 L^2) &= (1 - [\lambda_1 + \lambda_2]L + \lambda_1\lambda_2 L^2).
\end{aligned}$$

Here, we can see that

$$\begin{aligned}
\phi_1 &= \lambda_1 + \lambda_2 = a + ib + a - ib = 2a \\
\phi_2 &= \lambda_1 \lambda_2 = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2
\end{aligned}$$

A dynamic system is stable if $|\lambda_i| < 1$, i.e. the largest (in modulus) eigenvalue is less than 1. What is the implication of stability? To see this, we will do what we call recursive substitution for the same model as before:

$$\begin{aligned}
\boldsymbol{\xi}_t &= \mathbf{F}\boldsymbol{\xi}_{t-1} + \mathbf{v}_t \\
\boldsymbol{\xi}_t &= \mathbf{F}(\mathbf{F}\boldsymbol{\xi}_{t-2} + \mathbf{v}_{t-1}) + \mathbf{v}_t \\
\boldsymbol{\xi}_t &= \mathbf{F}(\mathbf{F}(\mathbf{F}\boldsymbol{\xi}_{t-3} + \mathbf{v}_{t-2}) + \mathbf{v}_{t-1}) + \mathbf{v}_t \\
\boldsymbol{\xi}_t &= \mathbf{F}^3\boldsymbol{\xi}_{t-3} + \mathbf{F}^2\mathbf{v}_{t-2} + \mathbf{F}\mathbf{v}_{t-1} + \mathbf{v}_t \\
&\vdots \\
\boldsymbol{\xi}_t &= \mathbf{F}^t\boldsymbol{\xi}_0 + \mathbf{F}^{t-1}\mathbf{v}_1 + \mathbf{F}^{t-2}\mathbf{v}_2 + \cdots + \mathbf{F}\mathbf{v}_{t-1} + \mathbf{v}_t \\
&= \mathbf{F}^t\boldsymbol{\xi}_0 + \sum_{s=0}^{t-1} \mathbf{F}^s \mathbf{v}_{t-s}.
\end{aligned}$$

3 Eigenvalue decomposition

To check stability, the object of study is $\mathbf{F}$. To see this, we first simplify things and assume its eigenvalues are distinct (the generalization if not is straight-forward). Collect them on the diagonal in the diagonal $p \times p$ matrix $\boldsymbol{\Lambda}$:

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_p \end{pmatrix}$$

We denote the corresponding $p \times 1$ eigenvectors by $\mathbf{x}_1, \ldots, \mathbf{x}_p$ such that we get the $p \times p$ matrix $\mathbf{T}$ with the eigenvectors:

$$\mathbf{T} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_p \end{pmatrix}.$$

Consider now premultiplying $\mathbf{T}$ by $\mathbf{F}$:

$$\mathbf{FT} = \mathbf{F}\begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_p \end{pmatrix} = \begin{pmatrix} \mathbf{F}\mathbf{x}_1 & \mathbf{F}\mathbf{x}_2 & \cdots & \mathbf{F}\mathbf{x}_p \end{pmatrix}$$

From the definition of eigenvalues, we know that $\mathbf{F}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$, $\mathbf{F}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$, etc. This then means that

$$\mathbf{FT} = \begin{pmatrix} \mathbf{F}\mathbf{x}_1 & \mathbf{F}\mathbf{x}_2 & \cdots & \mathbf{F}\mathbf{x}_p \end{pmatrix} = \begin{pmatrix} \lambda_1\mathbf{x}_1 & \lambda_2\mathbf{x}_2 & \cdots & \lambda_p\mathbf{x}_p \end{pmatrix}.$$

The right hand side we can rewrite as

$$\begin{pmatrix}\lambda_1\mathbf{x}_1 & \lambda_2\mathbf{x}_2 & \cdots & \lambda_p\mathbf{x}_p\end{pmatrix} = \begin{pmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_p\end{pmatrix}\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_p\end{pmatrix} = \mathbf{T\Lambda}.$$

So connecting these equations, we have established that $\mathbf{FT} = \mathbf{T\Lambda}$. Under the assumption of distinct eigenvalues, the eigenvectors are linearly independent. Linearly independent columns in a (square) matrix means it is invertible – hence, $\mathbf{T}$ is invertible. Thus, finally, we can say that

$$\mathbf{F} = \mathbf{T\Lambda T}^{-1}.$$

This is something that will be used many times. We will see why if we go back to the dynamic process. We expressed it as:

$$\boldsymbol{\xi}_t = \mathbf{F}^t\boldsymbol{\xi}_0 + \sum_{s=0}^{t-1}\mathbf{F}^s\mathbf{v}_{t-s}.$$

Replace $\mathbf{F}$ by $\mathbf{T\Lambda T}^{-1}$:

$$\boldsymbol{\xi}_t = \left(\mathbf{T\Lambda T}^{-1}\right)^t\boldsymbol{\xi}_0 + \sum_{s=0}^{t-1}\left(\mathbf{T\Lambda T}^{-1}\right)^s\mathbf{v}_{t-s}.$$

The advantage of the decomposition is this:

$$\left(\mathbf{T\Lambda T}^{-1}\right)^k = \underbrace{\left(\mathbf{T\Lambda T}^{-1}\right)\left(\mathbf{T\Lambda T}^{-1}\right)\cdots\left(\mathbf{T\Lambda T}^{-1}\right)}_{k\text{ times}} = \mathbf{T}\underbrace{\mathbf{\Lambda\Lambda}\cdots\mathbf{\Lambda}}_{k\text{ times}}\mathbf{T}^{-1} = \mathbf{T\Lambda}^k\mathbf{T}^{-1}$$

giving us

$$\boldsymbol{\xi}_t = \mathbf{T\Lambda}^t\mathbf{T}^{-1}\boldsymbol{\xi}_0 + \sum_{s=0}^{t-1}\mathbf{T\Lambda}^s\mathbf{T}^{-1}\mathbf{v}_{t-s}.$$

Now, the connection between the eigenvalues and stability becomes imminent. Since $\mathbf{\Lambda}$ is diagonal, it follows that

$$\mathbf{\Lambda}^t = \begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_p\end{pmatrix}^t = \begin{pmatrix}\lambda_1^t & & & \\ & \lambda_2^t & & \\ & & \ddots & \\ & & & \lambda_p^t\end{pmatrix} \longrightarrow \begin{pmatrix}0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0\end{pmatrix}$$

if $|\lambda_i| < 1$ for $i = 1, \ldots, p$. If this is violated, the matrix will eventually "explode".

4 Finding T

To find $\mathbf{T}$, one does not need to compute the eigenvectors directly. First, consider Proposition 1.1: *The eigenvalues of* $\mathbf{F}$ *are the values of* λ *that satisfy*

$$\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_{p-1}\lambda - \phi_p = 0.$$

Define $\mathbf{t}_i$ as

$$\mathbf{t}_i = \begin{pmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ \lambda_i \\ 1 \end{pmatrix}$$

and consider $\mathbf{F}\mathbf{t}_i$:

$$\begin{aligned} \mathbf{F}\mathbf{t}_i &= \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ \lambda_i \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_i^{p-1}\phi_1 + \lambda_i^{p-2}\phi_2 + \cdots + \lambda_i\phi_{p-1} + \phi_p \\ \lambda_i^{p-1} \\ \vdots \\ \lambda_i \end{pmatrix} \end{aligned}$$

From Proposition 1.1, however, we can see that $\lambda^p - (\lambda_i^{p-1}\phi_1 + \cdots + \phi_p) = 0$, so the first row of $\mathbf{F}\mathbf{t}_i$ must be equal to λ_i^p. This means

$$\mathbf{F}\mathbf{t}_i = \begin{pmatrix} \lambda_i^p \\ \lambda_i^{p-1} \\ \vdots \\ \lambda_i \end{pmatrix} = \lambda_i \begin{pmatrix} \lambda_i^{p-1} \\ \lambda_i^{p-2} \\ \vdots \\ 1 \end{pmatrix} = \lambda_i\mathbf{t}_i$$

This means that $\mathbf{t}_i$ is the eigenvector associated with the eigenvalue λ_i. This leads us to an expression of $\mathbf{T}$ which only involves the eigenvalues:

$$\mathbf{T} = \begin{pmatrix} \lambda_1^{p-1} & \lambda_2^{p-1} & \cdots & \lambda_p^{p-1} \\ \lambda_1^{p-2} & \lambda_2^{p-2} & \cdots & \lambda_p^{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_p \end{pmatrix}$$

Example: consider the second-order difference equation

$$y_t = 0.4y_{t-1} + 0.05y_{t-2} + w_t.$$

The $\mathbf{F}$ matrix:

$$\mathbf{F} = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0.4 & 0.05 \\ 1 & 0 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 0.5$ and $\lambda_2 = -0.1$. Thus,

$$\mathbf{T} = \begin{pmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0.5 & -0.1 \\ 1 & 1 \end{pmatrix}.$$

5 Dynamic multipliers

The dynamic multiplier is the change in y at a future time period induced by a change in w, i.e. $\frac{\partial y_{t+j}}{\partial w_t}$. In our case:

$$\begin{aligned}
\frac{\partial \boldsymbol{\xi}_{t+j}}{\partial w_t} &= \frac{\partial}{\partial w_t}\left(\mathbf{F}^{t+j}\boldsymbol{\xi}_0 + \sum_{s=0}^{t+j-1} \mathbf{F}^s \mathbf{v}_{t+j-s}\right) \\
&= 0 + \frac{\partial}{\partial w_t}\left(\sum_{s=0}^{t+j-1} \mathbf{F}^s \mathbf{v}_{t+j-s}\right) \\
&= 0 + \frac{\partial}{\partial w_t}\left(\mathbf{v}_{t+j} + \mathbf{F}\mathbf{v}_{t+j-1} + \cdots + \mathbf{F}^j\mathbf{v}_t + \cdots + \mathbf{F}^{t+j-1}\mathbf{v}_1\right) \\
&= \mathbf{F}^j \frac{\partial}{\partial w_t}\mathbf{v}_t = \mathbf{F}^j \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\end{aligned}$$

The first row of $\boldsymbol{\xi}_{t+j}$ contains y_{t+j}, so $\frac{\partial y_{t+j}}{\partial w_t} = f_{11}^{(j)}$, where $f_{11}^{(j)}$ is element $(1,1)$ of $\mathbf{F}^j$.

This is not very convenient to calculate by hand, especially if j is large. But look more closely at $\mathbf{F}^j$:

$$\mathbf{F}^j = \mathbf{T}\boldsymbol{\Lambda}^j\mathbf{T}^{-1} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1p} \\ t_{21} & t_{22} & \cdots & t_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ t_{p1} & t_{p2} & \cdots & t_{pp} \end{pmatrix} \begin{pmatrix} \lambda_1^j & & & \\ & \lambda_2^j & & \\ & & \ddots & \\ & & & \lambda_p^j \end{pmatrix} \begin{pmatrix} t^{11} & t^{12} & \cdots & t^{1p} \\ t^{21} & t^{22} & \cdots & t^{2p} \\ \vdots & \vdots & \ddots & \vdots \\ t^{p1} & t^{p2} & \cdots & t^{pp} \end{pmatrix}$$

$$= \begin{pmatrix} t_{11}\lambda_1^j & t_{12}\lambda_2^j & \cdots & t_{1p}\lambda_p^j \\ t_{21}\lambda_1^j & t_{22}\lambda_2^j & \cdots & t_{2p}\lambda_p^j \\ \vdots & \vdots & \ddots & \vdots \\ t_{p1}\lambda_1^j & t_{p2}\lambda_2^j & \cdots & t_{pp}\lambda_p^j \end{pmatrix} \begin{pmatrix} t^{11} & t^{12} & \cdots & t^{1p} \\ t^{21} & t^{22} & \cdots & t^{2p} \\ \vdots & \vdots & \ddots & \vdots \\ t^{p1} & t^{p2} & \cdots & t^{pp} \end{pmatrix}$$

Element (1,1) after making this last multiplication gives us:

$$f_{11}^{(j)} = t_{11}t^{11}\lambda_1^j + t_{12}t^{21}\lambda_2^j + \cdots + t_{1p}t^{p1}\lambda_p^j = \sum_{i=1}^{p} t_{1i}t^{i1}\lambda_i^j.$$

We let $c_i = t_{1i}t^{i1}$, so that $f_{11}^{(j)} = \sum_{i=1}^{p} c_i\lambda_i^j$. This might not seem a whole lot easier, but thanks to Proposition 1.2 we know that

$$c_i = \frac{\lambda_i^{p-1}}{\prod_{\substack{k=1\\k\neq i}}^{p}(\lambda_i - \lambda_k)}.$$

Pg. 8-12.

Thus, knowing the eigenvalues makes it very easy to find the dynamic multipliers, even if j is large. For the second order difference equation, this means that it is:

$$f_{11}^{(j)} = c_1\lambda_1^j + c_2\lambda_2^j = \frac{\lambda_1^{j+1}}{(\lambda_1 - \lambda_2)} + \frac{\lambda_2^{j+1}}{(\lambda_2 - \lambda_1)}.$$