Written Exam of the Course
Time Series Econometrics 2ST111 Fall 2015

Yukai Yang & Sebastian Andersson
Department of Statistics, Uppsala University
8:00 – 13:00, Thursday, 29 October 2015

UPPSALA UNIVERSITET

This is a five hours open-book exam. Please read carefully and answer all questions. The answers shall be clearly written, concise and relevant, and all steps shall be well explained. The total score is 100 points.

You can bring the textbook, the printed materials offered by the teachers (slides, notes), and any paper or printed articles that are relevant to the course. You are not allowed to use any calculator, computer, smart-phone or any devices with internet or bluetooth connection. You can bring paper dictionary, but the electronic dictionary is not allowed. You can use pen, pencil, eraser and ruler. You are not allowed to share any books, notes, papers, tools or devices with others during the exam.

Part 1: Moving Average Model (48 points)

Consider the following MA(2) model:

$$Y_t = \mu + \epsilon_t + \alpha_1 \epsilon_{t-1} + \alpha_2 \epsilon_{t-2} \tag{1}$$

where ϵ_t is a white noise process with zero mean and variance σ^2.

1. (16 points) Let Y_t be a stationary process. Derive explicitly (i) the unconditional mean $\mathrm{E}(Y_t)$, (ii) the variance $\gamma_0 = \mathrm{Var}(Y_t)$, (iii) the covariances $\gamma_i = \mathrm{Cov}(Y_t, Y_{t-i})$, as well as (iv) the autocorrelations ρ_i, for $i = 1, 2, ...$

2. (8 points) Explain the conditions on the parameters under which Y_t is a stationary process. Also explain the conditions under which the MA model is invertible.

Suppose now instead that you have the MA(1) process

$$Y_t = \mu + 0.5\epsilon_{t-1} + \epsilon_t, \tag{2}$$

where ϵ_t is some error term with $\mathrm{E}(\epsilon_t) = 0$ for all $t = 0, \pm1, \pm2, \dots$.

3. (8 points) Assume $\text{Cov}(\epsilon_t, \epsilon_\tau) = 0$ for all $t \neq \tau$. Is the process in (2) covariance-stationary? Explain why or why not.

4. (8 points) After some further investigation of the data, you find that it seems plausible that ϵ_t is a martingale difference sequence. Is the process in (2) covariance-stationary under this assumption? Explain why or why not.

5. (8 points) Another alternative, that you also find plausible, is that ϵ_t itself is an AR(1) process. More specifically, you now feel quite confident that the data-generating process for the error term can be described by

$$\epsilon_t = 0.4\epsilon_{t-1} + u_t$$

where u_t is a white noise term. Is the process in (2) covariance-stationary under this assumption? Explain why or why not.

Part 2: (12 points)

The weekly log US M2 money stock x_t (10 April 1995 – 29 January 2001, 304 observations, offered by Federal Reserve Bank of St. Louis) is plotted in levels in Figure 1. Its first order difference, plotted in Figure 2, shows that the time series is either trend-stationary I(0) or I(1). The log M2 seems to follow a linear trend, and fluctuation is somewhat small around the trend, which implies that it may be trend-stationary.

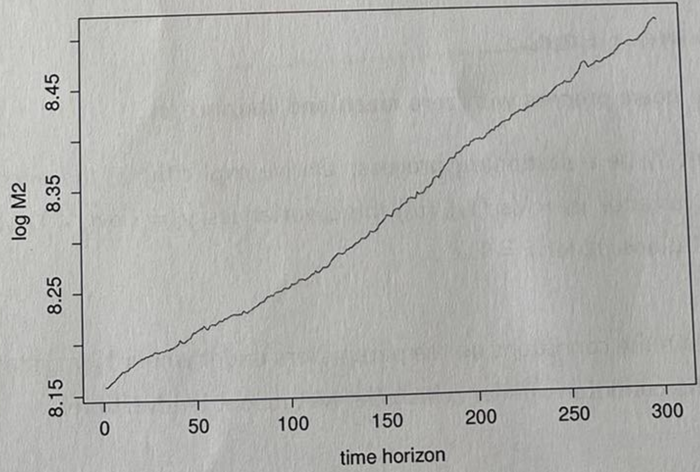

Figure 1: Weekly log US M2, 10 April 1995 – 29 January 2001, 304 obs.

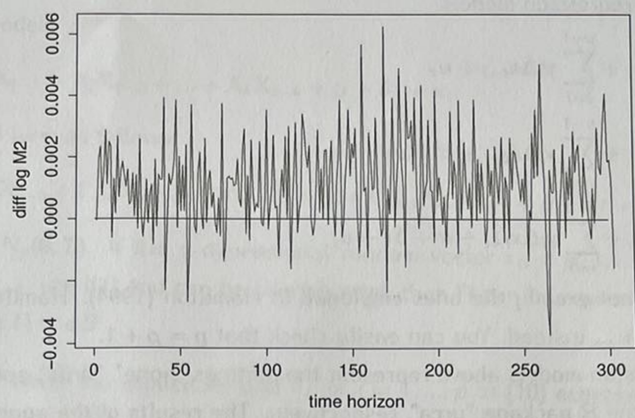

Figure 2: First order difference of log US M2

A common technique to check whether it is trend-stationary is to regress the following model

$$x_t = \alpha + \delta t + u_t$$

and plot the residuals $\hat{u}_t$. $\hat{u}_t$ is called the demeaned and detrended x_t, which is plotted in Figure 3. By doing so, we amplify the fluctuation around the trend by removing the trend. From Figure 3, we see that most probably x_t is not trend-stationary. Instead, it looks like a random walk.

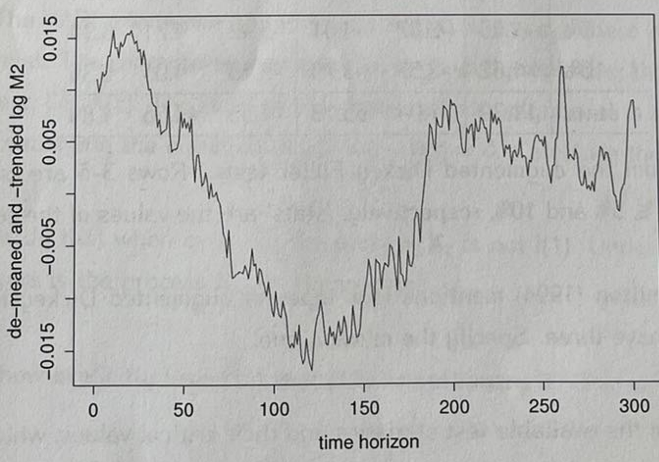

Figure 3: Demeaned and detrended log US M2

Now we need to give statistical evidence(s) and make inference about the truth. We run the following three regression models

$$\Delta x_t = \rho x_{t-1} + \sum_{i=1}^{p-1} \gamma_i \Delta x_{t-i} + u_t \tag{3}$$

$$\Delta x_t = \rho x_{t-1} + \sum_{i=1}^{p-1} \gamma_i \Delta x_{t-i} + \alpha + u_t \tag{4}$$

$$\Delta x_t = \rho x_{t-1} + \sum_{i=1}^{p-1} \gamma_i \Delta x_{t-i} + \alpha + \delta t + u_t \tag{5}$$

Note that they are not exactly the ones employed in Hamilton (1994). Hamilton (1994) uses the form $x_t = \varrho x_{t-1} + \ldots$ instead. You can easily check that $\varrho = \rho + 1$.

The three regression models above represent the settings "none", "drift", and "trend" in the function "ur.df()" in the R package "urca", respectively. The results of the augmented Dickey-Fuller test are summarized in Table 1. The null hypotheses are $\rho = 0$ for model (3); $\rho = 0$, $\alpha = 0$ for model (4); and $\rho = 0$, $\alpha = 0$, $\delta = 0$ for model (5). Note again that in Hamilton (1994), the unit root test has $H_0 : \varrho = 1$, but in our case (urca package), it is $H_0 : \rho = 0$.

The estimated models are given in (6)-(8), and the lag length $p = 2$ is chosen by AIC.

$$\Delta x_t = 0.00015\, x_{t-1} - 0.058\, \Delta x_{t-1} + u_t \tag{6}$$

$$\Delta x_t = 0.00085\, x_{t-1} - 0.062\, \Delta x_{t-1} - 0.0059 + u_t \tag{7}$$

$$\Delta x_t = -0.015\, x_{t-1} - 0.055\, \Delta x_{t-1} + 0.13 + 0.000019\, t + u_t \tag{8}$$

Model	(3)	(4)		(5)		
par.	ρ	ρ	α	ρ	α	δ
1%	-2.58	-3.44	6.47	-3.98	6.15	8.34
5%	-1.95	-2.87	4.61	-3.42	4.71	6.30
10%	-1.62	-2.57	3.79	-3.13	4.05	5.36
stats	11.39	1.04	65.13	-1.55	44.56	1.89

Table 1: Results from the augmented Dickey-Fuller tests. Rows 3-5 are critical values at significance levels 1%, 5% and 10%, respectively. 'stats' are the values of the test statistics.

1. (4 points) Hamilton (1994) mentions four types of augmented Dickey-Fuller tests, but here we only have three. Specify the missing one.

2. (8 points) From the available test statistics and their critical values, which model do you think is the best one among the others (models (3)-(5))? Motivate your choice.

Exam 2015-10-29 Solutions

Part 1

Moving average model $Y_t = \mu + \epsilon_t + \alpha_1\epsilon_{t-1} + \alpha_2\epsilon_{t-2}$

Where ϵ_t is a white noise process (0,σ^2).

1. Let Y_t be stationary
    a. The unconditional mean $E(Y_t)$
    b. Variance $\gamma_0 = Var(Y_t)$
    c. $\gamma_i = Cov(Y_t, Y_{t+j})$
    d. $\rho_i, i = 1,2, \ldots$

1.a.

$$E(Y_t) = \mu + E(\epsilon_t) + \alpha_1 E(\epsilon_{t-1}) + \alpha_2 E(\epsilon_{t-2}) = \mu$$

1.b.

$$\begin{aligned} Var(Y_t) = E[(Y_t-\mu)^2] &= E[(\epsilon_t + \alpha_1\epsilon_{t-1} + \alpha_2\epsilon_{t-2})^2] \\ &= E[\epsilon_t^2 + 2\alpha_1\epsilon_t\epsilon_{t-1} + \alpha_2\epsilon_t\epsilon_{t-2} + \alpha_1^2\epsilon_{t-1}^2 + 2\alpha_1\alpha_2\epsilon_{t-1}\epsilon_{t-2} + \alpha_2\epsilon_t\epsilon_{t-2} \\ &+ \alpha_2^2\epsilon_{t-2}^2] =_{Assume\ E(\epsilon_t\epsilon_{t-j})=0\ for\ j\neq 0, and\ E(\epsilon_t^2)=\sigma^2} \sigma^2 + \alpha_1^2\sigma^2 + \alpha_2^2\sigma^2 \\ &= (1+\alpha_1^2+\alpha_2^2)\sigma^2 \end{aligned}$$

1.c.

$$\begin{aligned} \gamma_1 = E[(Y_t-\mu)(Y_{t-1}-\mu)] &= E[(\epsilon_t + \alpha_1\epsilon_{t-1} + \alpha_2\epsilon_{t-2})(\epsilon_{t-1} + \alpha_1\epsilon_{t-2} + \alpha_2\epsilon_{t-3})] \\ &= (\alpha_1 + \alpha_1\alpha_2)\sigma^2 \end{aligned}$$

$$\gamma_2 = E[(Y_t-\mu)(Y_{t-2}-\mu)] = E[(\epsilon_t + \alpha_1\epsilon_{t-1} + \alpha_2\epsilon_{t-2})(\epsilon_{t-2} + \alpha_1\epsilon_{t-3} + \alpha_2\epsilon_{t-4})] = \alpha_2\sigma^2$$

$$\gamma_k = 0\ \forall\ k > 2$$

1.d.

$$\rho_i = \frac{\gamma_1}{\gamma_0} = \frac{\alpha_1 + \alpha_1\alpha_2}{1+\alpha_1^2+\alpha_2^2}$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\alpha_2}{1+\alpha_1^2+\alpha_2^2}$$

$$\rho_k = 0\ \forall\ k > 2$$

2. Explain conditions of the parameters under which Y_t is a stationary process.

- A finite order MA process is stationary for all combinations of the parameters.
- The finite lag polynomial $1 + \alpha_1 L + \alpha_2 L^2$ is a special case of the absolute summable infinite lag polynomial $1 + \sum_{i=1}^{\infty}\Psi_i L^i$ satisfying $\sum|\Psi_i| < \infty$. The moving average process with absolute summable lag polynomial is stationary.

Explain conditions for invertibility

$Y_t - \mu = \alpha(L)\epsilon_t$ where $\alpha(L) = 1 + \alpha_1 L + \alpha_2 L^2$

$\alpha(L) = (1-\lambda_1 L)(1-\lambda_2 L)$ where $\lambda_1 \& \lambda_2$ are the inverse roots of $1 + \alpha_1 z + \alpha_2 z^2 = 0$

The MA process is invertible **iff**

- The roots of $1 + \alpha_1 z + \alpha_2 z^2 = 0$ are outside the unit circle
- The (complex) numbers …… missed that.

3. Assume $Cov(\epsilon_t, \epsilon_\tau) = 0 \; \forall \; t = \tau$ Is the process cov.stationary?
   $Y_t = \mu + 0.5\epsilon_{t-1} + \epsilon_t$, where $E(\epsilon_t) = 0 \; \forall \; t \in integers$
- Not necessary. Since $Var(\epsilon_t)$ is not specified.
- Cov. Stationary process requires that the variance of Y_t is finite and constant over time.

4. ϵ_t is a martingale-sequence (MGS), is the process cov.-stat.?
- No, not necessarily.
- The MDS assumption does not restrict the variance of ϵ_t being constant over time.

5. ϵ_t is an AR(1)
   $\epsilon_t = 0.4\epsilon_{t-1} + u_t$, where u_t is white noise.
   Is it cov. Stat. now?
- ϵ_t is stable since $|\phi| < 1$ and stationary since u_t is white noise ($E(u_t) = 0, V(u_t) = \sigma^2$
  $Cov(u_t, u_\tau) = 0 \; \forall \; t \neq \tau \Rightarrow V(\epsilon_t) = \gamma_0 = \sigma^2$
- The variance of Y_t does not depend on t, so it is cov.stationary, since mean, variance & covariance independent of t.
- The process is an ARMA(1,1) with white noise errors and stable & invertible lag polynomials.

Part 2

A DF-test is done by these kinds of regressions.

$\Delta x_t = \rho x_{t-1} + \sum_{i=1}^{p} \gamma_i \Delta x_{t-i} + u_t$ ("none") (urca::ur.df() in R) $H_0: \rho = 0$

Added constant: $\Delta x_t = \rho x_{t-1} + \sum_{i=1}^{p} \gamma_i \Delta x_{t-i} + \alpha + u_t$ ("drift") $H_0: \rho = 0, \alpha = 0$

Added trend: $\Delta x_t = \rho x_{t-1} + \sum_{i=1}^{p} \gamma_i \Delta x_{t-i} + \alpha + \delta t + u_t$ ("trend") $H_0: \rho = 0, \alpha = 0, \delta = 0$

Results of an augmented DF test

	(3)	(4)		(5)			
	ρ	ρ	α	ρ	α	δ	
5%	-1.95	-2.87	4.61	-3.48	4.71	6.3	
Stats	11.39	1.04	65.13	-1.55	44.56	1.89	

1. Specify the missing aug. DF-test.
   Case 3, in which $\alpha \neq 0$ is presumed. (the corresponding ρ converges at rate $T^{\frac{3}{2}}$)
2. From the test statistics, which model do you think is the best one?
   In any of these three we accept that $\rho = 0$.
   ➔ We choose model (4)
   $\delta = 0$ can be accepted in (5). $\alpha = 0$ is rejected in (4 & 5). $\rho = 0$ is accepted in all three.
   Start with largest model, and work your way down.

Part 3

The VAR(k) $X_t = A_1 X_{t-1} + \cdots + A_k X_{t-k} + \mu + \delta t + \epsilon_t$

Has the VECM

$$\Delta X_t = \Pi X_{t-1} + \Gamma_1 \Delta X_{t-1} + \cdots + \Gamma_{k-1} \Delta X_{t-k+1} + \mu + \delta t + \epsilon_t$$

$$\epsilon_i \overset{i.i.d}{\sim} N_p(0, Z)$$

If the p-dimensional vector x_t are $I(1)$ and can be cointegrated, then Π can be represented as $\Pi = \alpha\beta^T$.

1. Write the explicit form of $\Gamma_j, i = 1, \ldots, p$ in … expressed in $A_i, i = 1, .,., p-1$ [19.1.38]

$$\Gamma_i = -\sum_{j=i+1}^{p} A_j$$

Viktigt ☆

$$\Gamma_1 = -\sum_{j=2}^{p} A_j$$

2. $\Delta X_{1t} = \alpha_1 (X_{1,t-1} - \beta_2 X_{2,t-1}) + \mu_1 + \epsilon_{1t}$

$$\Delta X_{2t} = \mu_2 + \epsilon_{2t}$$

Find $\alpha, \beta, \Gamma, \mu, A = A_1$ as expressed in terms of the not. In (11)

$$\alpha = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ -\beta_2 \end{pmatrix}, \Pi = \alpha\beta^T = \begin{pmatrix} \alpha_1 & -\alpha_1\beta_2 \\ 0 & 0 \end{pmatrix}$$

[19.1.40] $A_1 = I + \Pi = \begin{pmatrix} 1+\alpha_1 & -\alpha_1\beta_2 \\ 0 & 1 \end{pmatrix}, \Phi = \rho + I$

$rank(\Pi) = 0 \Rightarrow Y_t$ is not cointegrated (I(1)) $0 < rank(\Pi) = r < n$

Y_t is $I(1)$ with r linearly independent cointegrating vectors.

$$\Pi = \alpha_{n \times r} \beta_{r \times n}{}^T, rank(\alpha) = rank(\beta^T) = r$$

3. Model (11) has the VAR(1) form as (9)

Find the repr. Of X_t: $\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}$ in terms of X_0 & $\epsilon_t = \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix}$

+ find $E(X_t|X_0)$ & $Var(X_t|X_0)$

$$X_t = AX_{t-1} + \mu + \epsilon_t = A^T X_0 + \sum_{i=0}^{t-1} A^i (\mu + \epsilon_t - 1)$$

$$E(X_t|X_0) = A^t X_0 + \sum_{i=0}^{t-1} A^i \mu$$

$$Var(X_t|X_0) = Var\left(\sum_{i=0}^{t-1} A^i \epsilon_{t-i} \,|X_0\right) = \sum_{i=0}^{t-1} A^i \Sigma A^{i^T}$$

4. $VAR(1)$ can be written as $A(L)X_t = \mu + \epsilon_t$. Characteristic equation $A(z)$

$$|A(z)| = |1 - Az| = 0$$

Calculate roots

$$\begin{vmatrix} 1-(1-\alpha_1)z & \alpha_1\beta_2 z \\ 0 & 1-z \end{vmatrix} = (1-(1+\alpha_1)z)(1-z) = 0$$

Two roots: $z_1 = \frac{1}{1+\alpha_1}$ & $z_2 = 1$

5. Verify that when $\alpha_1 = -2, X_t\ is\ not\ I(1)$
   Under what condition on parameters is X_t an $I(1)$?

   We need to check that when $\alpha_1 = \rho(?)$

   $$\Delta X_{1t} = -2X_{1,t-1} + 2\beta_2 X_{2,t-1} + \mu + \epsilon_{1t}$$

   If X_{1t} is $I(1)$, then ΔX_{1t} must be $I(0)$
   → We cannot say(?) that ΔX_t is $I(0)$, which implies that X_{1t} is not $I(1)$. This example shows that unit root does not imply $I(1)$.

   → The conditions is simply that $-2 < \alpha_1 \leq 0$. When $\alpha_1 = 0$ the system is I(1), but the two variables cannot be cointegrated.

6. Show that from (11) $\beta^t X_t$ is stationary if $-2 < \alpha_1 < 0$.
   Make use of lecture slides (lecture 10)

   $$\beta^T X_t = (\beta^T \alpha + I)\mathrm{B}^T X_{t-1} + \beta^T \mu + \beta^T \epsilon_t$$

   Denote $Y_t = \beta^T X_t$

   $$Y_t = (\beta^t \alpha + 1)Y_{t-1} + \beta^T \mu + \beta^T \epsilon_t$$

   Need to check if $\beta^T \alpha + 1 = 1 + \alpha_1$

   $$|1+\alpha_1| < 1 \rightarrow -2 < \alpha_1 < 0$$

7. Suppose that $-2 < \alpha < 0$, under what conditions on parameters is there no linear trend?
   Slides from lecture 10 (slide no. 38,39,41)

   Granger VMA representation for a cointegrated VAR(1) with intercept & trend terms. By setting trend zero we end up with $X_t = CX_0 + C\mu t + C\sum_{i=0}^{t-1} \epsilon_{t-i} + \alpha(\beta^T\alpha)^{-1}\sum_{(i=0)}^{\infty} \mathrm{B}^i\, \beta^T(\mu + \epsilon_{t-i})$. The linear trend term is $C\mu t$. Thus we have $C\mu = 0$.