Linear Algebra

1. Simultaneous Equations

Example

$$1x + 1y + 2z = -1,$$
$$3x + 1y + 7z = -7,$$
$$1x + 7y + 1z = 7.$$

Step 1. Rewrite as augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 3 & 1 & 7 & -7 \\ 1 & 7 & 1 & 7 \end{array}\right].$$

Step 2. Row elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 3 & 1 & 7 & -7 \\ 1 & 7 & 1 & 7 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -2 & 1 & -4 \\ 0 & 6 & -1 & 8 \end{array}\right] \xrightarrow[R_2 = \frac{1}{-2} R_2]{R_3 = R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 2 & -4 \end{array}\right]$$

$$\xrightarrow[R_3 = \frac{1}{2} R_3]{R_2 = R_2 + \frac{1}{4} R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[R_1 = R_1 - 2R_2]{R_1 = R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\Rightarrow x = 2, \quad y = 1, \quad z = -2.$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 7 \\ 1 & 7 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \\ 7 \end{bmatrix} \begin{matrix} = 2 + 1 - 4 \\ = 6 + 1 - 14 \\ = 2 + 7 - 2 \end{matrix} \iff \begin{matrix} x = 2, \\ y = 1, \\ z = -2. \end{matrix}$$

Example

$$-2x_0 + x_1 + 2x_2 = 0,$$
$$4x_0 - x_1 - 5x_2 = 4,$$
$$2x_0 - 3x_1 - x_2 = -6.$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 4 & -1 & -5 & 4 \\ 2 & -3 & -1 & -6 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & -2 & 1 & -6 \end{array}\right] \xrightarrow[R_3 = R_3 + 2R_2]{R_1 = -\frac{1}{2} R_1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 = -R_3]{R_2 = R_2 - R_3} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[R_1 = R_1 + R_3]{R_1 = R_1 + \frac{1}{2} R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \Rightarrow \begin{matrix} x_0 = -1 \\ x_1 = 2 \\ x_2 = -2 \end{matrix}$$

$$\begin{bmatrix} -2 & 1 & 2 \\ 4 & -1 & -5 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -6 \end{bmatrix} \iff \begin{matrix} x_0 = -1 \\ x_1 = 2 \\ x_2 = -2 \end{matrix}$$

2. Vectors

(i) **Column vector**: $x \in \mathbb{R}^{n \times 1}$ or $x \in \mathbb{R}^n$ denotes a column vector of length n containing real numbers.

The i^{th} element of x is denoted by x_i or $x_{i,1}$.

i.e.: $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, $x_i \in \mathbb{R}$ for $i = 1, 2, \dots n$.

(ii) **Row vector**: $x^T \in \mathbb{R}^{1 \times n}$ or $x^T \in \mathbb{R}^n$ denotes a row vector of length n (also the transpose of column vector $\underline{x}$). x^T_i or $x_{1,i}$ denotes the i^{th} element of x'.

i.e.: $x' = [x_1 \ x_2 \ \dots \ x_n] \in \mathbb{R}^n$, $x^T_i = x_i \in \mathbb{R}$.

(iii) **Unit basis vector**: a unit basis vector $e_i \in \mathbb{R}^n$ is a vector where the i^{th} element of the vector is 1 and all other elements are 0.

Example: $e_1 \in \mathbb{R}^3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. $e_3 \in \mathbb{R}^5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Geometric Representations of vectors

Let $e_1 = \rightarrow$ and $e_2 = \uparrow$.

Then $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \nearrow$ and $y = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \nwarrow$

(iv) Scaling: For any scalar $a \in \mathbb{R}$ and vector $x \in \mathbb{R}^n$, $ax = a \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a x_1 \\ \vdots \\ a x_n \end{bmatrix}$.

Examples: $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $5x = \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix}$, $\sin(\theta) x = \begin{bmatrix} \sin(\theta) \\ 2\sin(\theta) \\ 3\sin(\theta) \end{bmatrix}$, $\ln(a) x = \begin{bmatrix} \ln(a) \\ 2\ln(a) \\ 3\ln(a) \end{bmatrix} = \begin{bmatrix} \ln(a) \\ \ln(a^2) \\ \ln(a^3) \end{bmatrix}$.

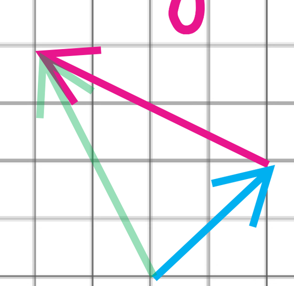
Let $y = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \nwarrow$, then $\frac{1}{2} \cdot y = \begin{bmatrix} -1 \\ 1/2 \end{bmatrix} = \nwarrow$.

(v) Vector addition: For any $x, y \in \mathbb{R}^n$,

$$x + y = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}, \quad x^T + y^T = [x_1 \cdots x_n] + [y_1 \cdots y_n] = [x_1 + y_1 \ \cdots \ x_n + y_n].$$

Example: Let $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \nearrow$ and $y = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \nwarrow$.

Then $x + y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \nwarrow$



(vi) Inner dot product: For any $x \in \mathbb{R}^n$ and a given row vector $y^T \in \mathbb{R}^n$, their inner product is

$$y^T \cdot x = [y_1 \cdots y_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} y_i^T x_i = y_1 \cdot x_1 + y_2 \cdot x_2 \cdots + y_n \cdot x_n.$$

Examples: Let $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $y = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $y^T x = [4 \ 5 \ 6] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 = 32$.

$$x^T x = [1 \ 2 \ 3] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 = 14.$$