

1 Limits

Definition: $\lim_{x \rightarrow a} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0$ where

$$\begin{aligned} a - \delta < x < a + \delta &\Rightarrow L - \varepsilon < f(x) < L + \varepsilon, \text{ i.e.} \\ |x - a| < \delta &\Rightarrow |f(x) - L| < \varepsilon \end{aligned}$$

Proposition: $f(x) = x \Rightarrow \lim_{x \rightarrow a} f(x) = a.$

Proof: $\forall \varepsilon > 0$, let $\delta = \varepsilon$. Then

$$|x - a| < \underbrace{\delta}_{=\varepsilon} \Rightarrow |x - a| < \varepsilon \Rightarrow \underbrace{|f(x) - a|}_{=f(x)} < \varepsilon. \square$$

Examples: $\lim_{x \rightarrow 1} x = 1.$ $\lim_{h \rightarrow 0} h = 0.$

Proposition: $\lim_{x \rightarrow a} cx = ca.$

Proof: $\forall \varepsilon > 0$, let $\delta = \frac{\varepsilon}{|c|}$. Then

$$|x - a| < \underbrace{\delta}_{=\frac{\varepsilon}{|c|}} \Rightarrow |c| \cdot |x - a| < \varepsilon \Rightarrow |cx - ca| < \varepsilon. \square$$

Examples: $\lim_{x \rightarrow 2} 8x = 8(2) = 16.$ $\lim_{y \rightarrow 1} cy = c \cdot (1) = c.$

Proposition: $\lim_{x \rightarrow a} f(x) = L \Rightarrow \lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot L = c \left[\lim_{x \rightarrow a} f(x) \right]$

Proof: $\lim_{x \rightarrow a} f(x) \Rightarrow \forall \varepsilon > 0, \exists \delta, \text{ where } (|x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon)$
 $\Rightarrow \forall \varepsilon > 0, \exists \delta_2 \text{ where } (|x-a| < \delta_2 \Rightarrow |f(x)-L| < \frac{\varepsilon}{|c|}).$

Now $\forall \varepsilon > 0$ let $\delta = \delta_2$. Then

$$|x-a| < \delta = \delta_2 \Rightarrow |f(x)-L| < \frac{\varepsilon}{|c|}$$

$$\Rightarrow |c| \cdot |f(x)-L| = |c \cdot f(x) - c \cdot L| < \varepsilon. \quad \square$$

Proposition: $\lim_{x \rightarrow a} f(x) = L, \lim_{x \rightarrow a} g(x) = K \Rightarrow \lim_{x \rightarrow a} [f(x) + g(x)] = L + K.$
 $\left[\lim_{x \rightarrow a} f(x) \right] + \left[\lim_{x \rightarrow a} g(x) \right]$

Proof: $\lim_{x \rightarrow a} f(x) = L \Rightarrow \forall \varepsilon > 0, \exists \delta_1 > 0 \text{ where } (|x-a| < \delta_1 \Rightarrow |f(x)-L| < \frac{\varepsilon}{2}).$
 $\Rightarrow \forall \varepsilon > 0, \exists \delta_1 > 0 \text{ where } (|x-a| < \delta_1 \Rightarrow |f(x)-L| < \frac{\varepsilon}{2}).$

$$\lim_{x \rightarrow a} g(x) = K \Rightarrow \forall \varepsilon > 0, \exists \delta_2 > 0 \text{ where } |x-a| < \delta_2 \Rightarrow |g(x)-K| < \frac{\varepsilon}{2}.$$

Now, $\forall \varepsilon$ let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$|x-a| < \delta \Rightarrow |f(x)-L| + |g(x)-K| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow |f(x)-L + g(x)-K| = |[f(x)+g(x)] - [L+K]| < \varepsilon. \quad \square$$

Let $f(x)-L = F, g(x)-K = G$, then

$$|f(x)-L + g(x)-K| = |F+G| = \begin{cases} |F|+|G| & \text{if } F, G > 0 \text{ or } F, G < 0 \\ |F|-|G| & \text{if } F > G \\ |G|-|F| & \text{if } G > F \end{cases} \leq \max\{|F|, |G|\} \\ \leq |F| + |G| = |f(x)-L| + |g(x)-K| < \varepsilon.$$

Proposition: $\lim_{x \rightarrow a} f(x) = L, \lim_{x \rightarrow a} g(x) = K \Rightarrow \lim_{x \rightarrow a} [f(x) \cdot g(x)] = \underbrace{L \cdot K}_{\left[\lim_{x \rightarrow a} f(x) \right] \left[\lim_{x \rightarrow a} g(x) \right]}$.

Proof: $\lim_{x \rightarrow a} f(x) = L \Rightarrow \forall \varepsilon > 0, \exists \delta'_1 > 0$ where $(|x - a| < \delta'_1 \Rightarrow |f(x) - L| < \varepsilon)$
 $\Rightarrow \forall \varepsilon > 0, \exists \delta'_1 > 0$ where $(|x - a| < \delta_1 \Rightarrow |f(x) - L| < \sqrt{\varepsilon})$

$\lim_{x \rightarrow a} g(x) = K \Rightarrow \forall \varepsilon > 0, \exists \delta_2 > 0$ where $|x - a| < \delta_2 \Rightarrow |g(x) - K| < \sqrt{\varepsilon}$.

Now, $\forall \varepsilon$ let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$|x - a| < \delta \Rightarrow |[f(x) - L][g(x) - K] - 0| = |f(x) - L| \cdot |g(x) - K| < (\sqrt{\varepsilon})(\sqrt{\varepsilon}) = \varepsilon$$

$$\Rightarrow \lim_{x \rightarrow a} [f(x) - L][g(x) - K] = 0$$

$$\Rightarrow \lim_{x \rightarrow a} [f(x)g(x) - Kg(x) - Lf(x) + LK] = 0$$

$$\Rightarrow \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} [f(x)K + Lg(x) - LK]$$

$$= LK + \underbrace{LK - LK}_{=0} = LK$$

Proposition: $\lim_{x \rightarrow a} [\log_b(x)] = \log_b(a), b > 1.$ $= \ln \left[\lim_{x \rightarrow a} f(x) \right]$

Proof: $\forall \varepsilon$ let $\delta_1 = |a(1-b^\varepsilon)|$ $\delta_2 = |a(b^\varepsilon-1)|$, and

$\delta = \min \{\delta_1, \delta_2\}$. Then

$$|x-a| < \delta \Rightarrow a-\delta < x < a+\delta$$

$$\Rightarrow \min\{a-a(1-b^\varepsilon), a-a(b^\varepsilon-1)\} < x < \min\{a+a(1-b^\varepsilon), a+a(b^\varepsilon-1)\}$$

$$\Rightarrow \min\{ab^{-\varepsilon}, a(2-b^\varepsilon)\} < x < \min\{a(2-b^{-\varepsilon}), ab^\varepsilon\}$$

$x < \min\{y, z\} \Rightarrow x < y, z.$

$$\Rightarrow \min\{\log_b(a)-\varepsilon, \log_b(a[2-b^\varepsilon])\} < \log_b(x) < \log_b(ab^\varepsilon) = \log_b(a) + \varepsilon$$

$$\Rightarrow \min\{-\varepsilon, \underbrace{\log_b(2-b^\varepsilon)}_{>0 \text{ or undefined}}\} < \log_b(x) - \log_b(a) < \varepsilon$$

$\Rightarrow \min\{-\varepsilon, \dots\} = -\varepsilon$

$$\Rightarrow -\varepsilon < \log_b(x) - \log_b(a) < \varepsilon$$

$$\Rightarrow |\log_b(x) - \log_b(a)| < \varepsilon. \quad \square$$

Proposition: $\lim_{x \rightarrow a} f(x) = L \Rightarrow \lim_{x \rightarrow a} [f(x)]^n = L^n, (L \neq 0) \wedge (n \leq 0).$ $= \left[\lim_{x \rightarrow a} f(x) \right]^n$

Proof: $\lim_{x \rightarrow a} f(x) = L \Rightarrow \lim_{x \rightarrow a} \ln(f(x)) = \ln(L)$

$$\Rightarrow \lim_{x \rightarrow a} n \cdot \ln(f(x)) = n \cdot \ln(L)$$

$$= \lim_{x \rightarrow a} \ln([f(x)]^n) = \ln\left(\lim_{x \rightarrow a} [f(x)]^n\right) = \ln(L^n)$$

$$\Rightarrow \lim_{x \rightarrow a} [f(x)]^n = L^n. \quad \square$$

Proposition: $\lim_{x \rightarrow a} f(x) = L > 0 \Rightarrow \lim_{x \rightarrow a} \log_b f(x) = \log_b L, b > 0.$

Proof: $\lim_{x \rightarrow a} f(x) = L \Rightarrow \forall \varepsilon > 0, \exists \delta_1$ where $(|x-a| < \delta_1 \Rightarrow |f(x)-L| < |L(b^\varepsilon-1)|)$
and $\exists \delta_2$ where $(|x-a| < \delta_2 \Rightarrow |f(x)-L| < |L(b^{-\varepsilon}-1)|)$

Next, $\forall \varepsilon > 0$ let $\delta = \min\{\delta_1, \delta_2\}$. Then,

Case 1: $\delta = \delta_1 = \min\{\delta_1, \delta_2\} \Rightarrow \delta_1 < \delta_2 \Rightarrow b^{-\varepsilon} < b^\varepsilon.$

$$|x-a| < \delta = \delta_1 \Rightarrow |f(x)-L| < |L(b^{-\varepsilon}-1)| = L \cdot \max\{b^{-\varepsilon}-1, 1-b^{-\varepsilon}\}$$

$$\Rightarrow 0 < L \cdot \min\{b^{-\varepsilon}-1, 1-b^{-\varepsilon}\} < f(x)-L < L \cdot \max\{b^{-\varepsilon}-1, 1-b^{-\varepsilon}\}$$

$$\Rightarrow 0 < L \cdot \min\{b^{-\varepsilon}, 2-b^{-\varepsilon}\} < f(x) < L \cdot \max\{b^{-\varepsilon}, 2-b^{-\varepsilon}\}$$

$$\Rightarrow 0 < \quad < f(x) < L \cdot b^{-\varepsilon} < L \cdot b^\varepsilon$$

$$\Rightarrow \log_b(L) + \log \quad < \log_b[f(x)] <$$

