Linear Algebra

Simultaneous Equations

Example

$$1x + 1y + 2z = -1,$$
$$3x + 1y + 7z = -7,$$
$$1x + 7y + 1z = 7.$$

Step 1. Rewrite as augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 3 & 1 & 7 & -7 \\ 1 & 7 & 1 & 7 \end{array}\right].$$

Step 2. Row elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 3 & 1 & 7 & -7 \\ 1 & 7 & 1 & 7 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -2 & 1 & -4 \\ 0 & 6 & -1 & 8 \end{array}\right] \xrightarrow[R_2 = \frac{1}{-2}R_2]{R_3 = R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 2 & -4 \end{array}\right]$$

$$\xrightarrow[R_3 = \frac{1}{2}R_3]{R_2 = R_2 + \frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[R_1 = R_1 - 2R_2]{R_1 = R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \Rightarrow \begin{array}{l} x = 2, \\ y = 1, \\ z = -2. \end{array}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 7 \\ 1 & 7 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \\ 7 \end{bmatrix} \begin{array}{l} = 2 + 1 - 4 \\ = 6 + 1 - 14 \\ = 2 + 7 - 2 \end{array} \iff \begin{array}{l} x = 2, \\ y = 1, \\ z = -2. \end{array}$$

Example

$$-2x_0 + x_1 + 2x_2 = 0,$$
$$4x_0 - x_1 - 5x_2 = 4,$$
$$2x_0 - 3x_1 - x_2 = -6.$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 4 & -1 & -5 & 4 \\ 2 & -3 & -1 & -6 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & -2 & 1 & -6 \end{array}\right] \xrightarrow[R_3 = R_3 + 2R_2]{R_1 = -\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 = -R_3]{R_2 = R_2 - R_3} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[R_1 = R_1 + R_3]{R_1 = R_1 + \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \Rightarrow \begin{array}{l} x_0 = -1 \\ x_1 = 2 \\ x_2 = -2 \end{array}$$

$$\begin{bmatrix} -2 & 1 & 2 \\ 4 & -1 & -5 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -6 \end{bmatrix} \iff \begin{array}{l} x_0 = -1 \\ x_1 = 2 \\ x_2 = -2 \end{array}$$

Example Let $f(n) = \sum_{i=0}^{n-1} i$. Show that $f(n) = y_0 + y_1 \cdot n + y_2 \cdot n^2$ for some coefficients y_j, $j = 0, 1, 2$.

$$f(n) = \sum_{j=0}^{2} y_j n^j \Rightarrow f(0) = \sum_{i=0}^{-1} i = 0 = y_0 + y_1(0) + y_2(0)^2 \Rightarrow y_0 = 0$$

$$f(1) = \sum_{i=0}^{0} i = 0 = y_1 + y_2 \Rightarrow y_1 = -y_2$$

$$f(2) = \sum_{i=0}^{1} i = 1 = y_1 \cdot 2 + y_2 \cdot 4 = -y_2 \cdot 2 + y_2 \cdot 4 \Rightarrow y_2 = \frac{1}{2},\ y_1 = -\frac{1}{2}.$$

$$\therefore\ y_0 = 0,\ y_1 = -\frac{1}{2},\ y_2 = \frac{1}{2} \Rightarrow \sum_{i=0}^{n} i = \frac{n^2 - n}{2}.$$

Now we want to show that $\sum_{i=0}^{n-1} i = \frac{n^2 - n}{2}$, $n \geq 1$.

Proof: (by induction).

(i) **Base case.** $n = 1 \Rightarrow \left(\sum_{i=0}^{n-1} i = \sum_{i=0}^{-1} i = 0\right)$ and $\left(\frac{n^2 - n}{2} = \frac{1^2}{2} - 0 = 0\right) \Rightarrow \sum_{i=0}^{n-1} i = \frac{n^2 - n}{2}$, $n = 0$ ✓

(ii) **Inductive step.** Show $\sum_{i=1}^{n-1} i = \frac{n^2 - n}{2} \Rightarrow \sum_{i=1}^{n} i = \frac{(n+1)^2 - (n+1)}{2} = \frac{n^2 + n}{2}$.

$$\sum_{i=1}^{n-1} i = \frac{n^2 - n}{2} \Rightarrow \sum_{i=1}^{n} i = \left(\sum_{i=1}^{n-1} i\right) + n = \frac{n^2 - n}{2} + n = \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2}.\ ✓\ \square$$

__Example__ Let $f(n) = 6\sum_{i=0}^{n-1} i^2 = y_0 + y_1 n + y_2 n^2 + y_3 n^3$. Find coefficients y_j, $j = 0, 1, 2, 3$.

Set up problem in matrix form:

$$f\left(\begin{bmatrix}0\\1\\2\\3\end{bmatrix}\right) = 6\begin{bmatrix}\sum_{i=0}^{-1} i^2\\ \sum_{i=0}^{0} i^2\\ \sum_{i=0}^{1} i^2\\ \sum_{i=0}^{2} i^2\end{bmatrix} = \begin{bmatrix}0\\0\\6\\30\end{bmatrix} = \begin{bmatrix}1&0&0&0\\1&1&1&1\\1&2&4&8\\1&3&9&27\end{bmatrix}\begin{bmatrix}y_0\\y_1\\y_2\\y_3\end{bmatrix} \Rightarrow \begin{aligned} 0 &= y_0\\ 0 &= y_0 + y_1 + y_2 + y_3\\ 6 &= y_0 + 2y_1 + 4y_2 + 8y_3\\ 30 &= y_0 + 3y_1 + 9y_2 + 27y_3\end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{cccc|c}1&0&0&0&0\\1&1&1&1&0\\1&2&4&8&6\\1&3&9&27&30\end{array}\right] \xrightarrow[R_i = R_i - R_1]{i \neq 1,} \left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&1&1&0\\0&2&4&8&6\\0&3&9&27&30\end{array}\right] \xrightarrow[R_4 = R_4 - 3R_2]{R_3 = R_3 - 2R_2} \left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&1&1&0\\0&0&2&6&6\\0&0&6&24&30\end{array}\right] \xrightarrow[R_3 = \frac{1}{2}R_3]{R_4 = R_4 - 3R_3} \left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&1&1&0\\0&0&1&3&3\\0&0&0&6&12\end{array}\right]$$

$$\xrightarrow[R_4 = \frac{1}{6}R_4]{R_3 = R_3 - \frac{1}{2}R_4} \left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&1&1&0\\0&0&1&0&-3\\0&0&0&1&2\end{array}\right] \xrightarrow[R_2 = R_2 - R_4]{R_2 = R_2 - R_3} \left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&1\\0&0&1&0&-3\\0&0&0&1&2\end{array}\right]$$

$$\Leftarrow \quad \begin{bmatrix}1&0&0&0\\1&1&1&1\\1&2&4&8\\1&3&9&27\end{bmatrix}\begin{bmatrix}0\\1\\-3\\2\end{bmatrix} = \begin{bmatrix}0\\0\\6\\30\end{bmatrix}, \quad \begin{aligned} y_0 &= 0\\ y_1 &= 1\\ y_2 &= -3\\ y_3 &= 2\end{aligned}$$

$\Rightarrow 6\sum_{i=0}^{n-1} i^2 = n - 3n^2 + 2n^3$, which we will show is true.

__Proof__: by induction.

__Base case__ $n = 0$, $6\sum_{i=0}^{n-1} i^2 = 0 = n - 3n^2 + 2n^3 = 0$. ✓

__Inductive step__ Show $6\sum_{i=0}^{n-1} i^2 = n - 3n^2 + 2n^3 \Rightarrow 6\sum_{i=0}^{n} i^2 = (n+1) - 3(n+1)^2 + 2(n+1)^3$
$= n + 3n^2 + 2n^3$.

$$6\sum_{i=0}^{n-1} i^2 = n - 3n^2 + 2n^3 \Rightarrow 6\sum_{i=0}^{n} i^2 = \sum_{i=0}^{n-1} i^2 + 6n^2 = n - 3n^2 + 2n^3 + 6n^2$$
$$= n + 3n^2 + 2n^3. \checkmark \; \square$$