(i) Linear function: $f(x)$ is a linear transformation $\iff$ it is invariant under

(1) scaling, i.e. $\forall a \in \mathbb{R}$, $a \cdot f(x) = f(ax)$, and

(2) addition, i.e. $f(x) + f(y) = f(x+y)$.

e.g.: $\cdot\ g(x) = m \cdot x$ is linear, since (1) $a \cdot g(x) = a(mx) \overset{\checkmark}{=} m(ax) = g(ax)$, and

(2) $g(x) + g(y) = mx + my \overset{\checkmark}{=} m(x+y) = g(x+y)$.

$\cdot\ f(x) = m \cdot x + b$ is not linear since (1) $a \cdot f(x) = a(mx+b) \neq amx + b = f(ax)$.

Q: Can you think of examples of other linear functions?

(vi) Functions: $f: \mathbb{R}^m \to \mathbb{R}^n$ denotes a function that takes length m vectors as inputs and outputs vectors of length n.

Examples: $\cdot$ A function $f: \mathbb{R}^n \to \mathbb{R}^n$ where $f(x) = f\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = [i x_{i, i=1 \dots n}] = \begin{bmatrix} 1x_1 \\ 2x_2 \\ \vdots \\ n x_n \end{bmatrix}$,

e.g. $f\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$, $f\left(\begin{bmatrix} 9 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 2 \\ 6 \end{bmatrix}$.

$\cdot$ A function $f: \mathbb{R}^n \to \mathbb{R}$ where $f(x) = \sum_{i=1}^{n} x_i$, e.g. $f\left(\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}\right) = 2+4+6 = 12$.

Q: Are these examples linear functions?