Properties of Derivatives

Definition The derivative of a function $f(x)$ is defined as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Derivative Rules

(i) Constant $f(x) = c \Rightarrow f'(x) = 0.$

(ii) Coefficient $f(x) = c\,g(x) \Rightarrow f'(x) = c\,g'(x).$

(iii) Addition $f(x) = g(x) + h(x) \Rightarrow f'(x) = g'(x) + h'(x).$

(iv) Power $f(x) = x^n, n \neq 0 \Rightarrow f'(x) = n x^{n-1}.$

(v) Product $f(x) = g(x)h(x) \Rightarrow f'(x) = g'(x)h(x) + h'(x)g(x).$

(vi) Chain $f(x) = g(h(x)) \Rightarrow f'(x) = g'(h)\,h'(x).$

(vii) Exponent $f(x) = e^x \Rightarrow f'(x) = e^x.$

(viii) Logarithm $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}.$

Proofs

(i) Constant $f(x) = c \Rightarrow f'(x) = 0.$

Proof $f(x) = c$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = 0. \square$$

(ii) Coefficient $f(x) = c\,g(x) \Rightarrow f'(x) = c\,g'(x).$

Proof

$$f(x) = c\,g(x) \Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{c\,g(x+h) - c\,g(x)}{h}$$
$$= \lim_{h \to 0} c \left[\frac{g(x+h) - g(x)}{h}\right]$$
$$= c \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$
$$= c\,g'(x). \square$$

(iii) Addition $f(x) = g(x) + h(x) \Rightarrow f'(x) = g'(x) + h'(x).$

Proof

$$f(x) = g(x) + h(x) \Rightarrow f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{[g(x+\Delta x) + h(x+\Delta x)] - [g(x) + h(x)]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{[g(x+\Delta x) - g(x)] + [h(x+\Delta x) - h(x)]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left(\frac{g(x+\Delta x) - g(x)}{\Delta x} + \frac{h(x+\Delta x) - h(x)}{\Delta x} \right)$$
$$= \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x}$$
$$= g'(x) + h'(x). \square$$

(iv) Power $f(x) = x^n, n \neq 0 \Rightarrow f'(x) = nx^{n-1}$.

Proof for $n \in \mathbb{N}$ $\quad f(x) = x^n, n \in \mathbb{N} \Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

by binomial theorem $\quad = \lim_{h \to 0} \dfrac{\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} h^k x^{n-k} - x^n}{h}$

$$= \lim_{h\to 0} \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} h^{k-1} x^{n-k}$$

since $\lim_{h\to 0} h^{k-1} = \begin{cases} 1 & \text{if } k=1 \\ 0 & \text{if } k>1 \end{cases}$ $\quad = \dfrac{n}{1!(n-1)!} x^{n-1}$

$$= nx^{n-1}. \square$$

Proof for $n \in \mathbb{R} \setminus \{0\}$ Set $y = f(x) = x^n, n \in \mathbb{R}/\{0\}$.

$$y = x^n \Rightarrow \ln y = n \ln(x).$$

$$\ln y = n \ln x \Rightarrow \frac{d \ln y}{dx} = \frac{d\, n \ln x}{dx} = nx^{-1}. \quad (4.1)$$

by (ii) coefficient and (viii) logarithm.

$$(4.1) \Rightarrow \frac{d \ln y}{dx} = \frac{d \ln y}{dy} \frac{dy}{dx} = y^{-1} \frac{dy}{dx} = nx^{-1}$$

by (vi) Chain. $\quad \Rightarrow \dfrac{dy}{dx} = f'(x) = nx^{-1} y$

$$= nx^{-1} x^n$$

$$= nx^{-1}. \square$$

(v) Product $\quad f(x) = g(x) h(x) \Rightarrow f'(x) = g'(x) h(x) + g(x) h'(x)$.

Proof $\quad f(x) = g(x) h(x) \Rightarrow f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{g(x+\Delta x) h(x+\Delta x) - g(x) h(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{g(x+\Delta x) h(x+\Delta x) - g(x) h(x) + \overbrace{(g(x+\Delta x) h(x) - g(x+\Delta x) h(x))}^{=0}}{\Delta x}$$

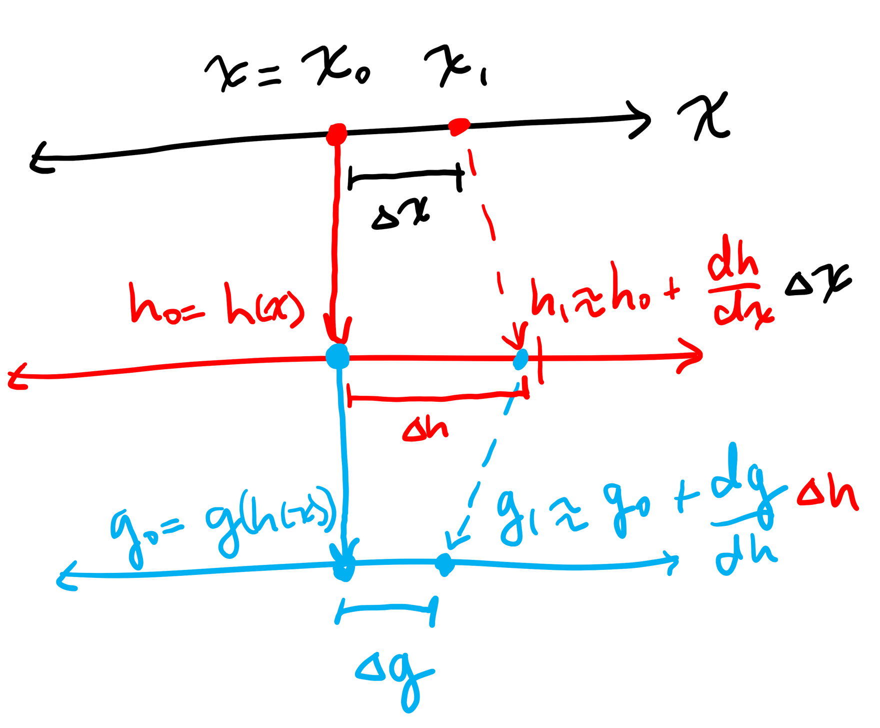
$$= \lim_{\Delta x \to 0} \frac{g(x+\Delta x)[h(x+\Delta x) - h(x)] + h(x)[g(x+\Delta x) - g(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} g(x+\Delta x) \frac{h(x+\Delta x) - h(x)}{\Delta x} + \lim_{\Delta x \to 0} h(x) \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= g(x) h'(x) + h(x) g'(x). \square$$

(vi) Chain $\quad f(x) = g(h(x)) \Rightarrow f'(x) = g'(h) h'(x)$

Illustration



$\Delta x = x_1 - x_0$

$dx = \lim_{\Delta x \to 0} \Delta x > 0$

$\Delta h = h_1 - h_0 \approx \dfrac{dh}{dx} \Delta x$

$$dh = \lim_{\Delta x \to 0} \frac{dh}{dx} \Delta x = \frac{dh}{dx} \underbrace{\lim_{\Delta x \to 0} \Delta x}_{dx} = \frac{dh}{dx} dx.$$

$\Delta g = g_1 - g_0 \approx \dfrac{dg}{dh} \Delta h$

$$dg = \lim_{\Delta h \to 0} \frac{dg}{dh} \Delta h = \frac{dg}{dh} \overbrace{\lim_{\Delta h \to 0} \Delta h}^{dh}$$

$$= \frac{dg}{dh} \frac{dh}{dx} dx.$$

$$\boxed{\frac{dg}{dx} = \frac{dg}{dh} \frac{dh}{dx}.}$$

Corollary: Quotient $f(x) = \frac{g(x)}{h(x)} \Rightarrow f'(x) = \frac{g'(x)h(x) + h'(x)g(x)}{h(x)^2}$.

Proof $f(x) = \frac{g(x)}{h(x)} = g(x)[h(x)]^{-1} \Rightarrow f'(x) = \frac{dg(x)}{dx}\frac{1}{h(x)} + \frac{d(h(x)^{-1})}{dx} g(x)$ by (v) Product

$$= \frac{g'(x)}{h(x)} + \overbrace{\frac{d[h(x)]^{-1}}{dh}\frac{dh(x)}{dx}}^{\text{by (vi) Chain}} g(x)$$

$$= \frac{g'(x)h(x)}{h(x)^2} + (-1)[h(x)]^{-2} h'(x) g(x)$$

$$= \frac{g'(x)h(x) + h'(x)g(x)}{h(x)^2}. \quad \square$$

Definition $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{m\to 0}(1+m)^{\frac{1}{m}}$.

$\log_a(x) = y \Leftrightarrow y = a^x$.

Properties
(LP1) $\log_a(a^x) = x$.
(LP2) $\log_a(x^n) = n\log_a(x)$.
(LP3) $a^{\log_a(x)} = x$.
(LP4) $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$

Continuous Compounding:
- r: period interest rate
- n: interest payment/period
- t: number of periods

Compounded interest: $r_n = \left(1+\frac{r}{n}\right)^{nt}$

Continuous compounding: $r_c = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = \lim_{x\to\infty}\underbrace{\left(1+\frac{1}{x}\right)^{xrt}}_{x = \frac{n}{r}\ (n = xr)} = \left[\lim_{x\to}\left(1+\frac{1}{x}\right)^{x}\right]^{rt}$ $\boxed{= e^{rt}}$.

Corollary: $\log_a(a) = 1$, $\log_a(1) = 0$.

Proof Set $x = 1$, then $\log_a(a^x) = x$
$= \log_a(a^1) = \log_a(a) = 1$. $\square$

Set $x=0$, then $\log(a^x) = x$
$= \log(a^0) = \log(1) = 0$. $\square$

(LP1) $\log_a(a^x) = x$.

Proof $y = a^x \Rightarrow \log(y) = x$
$\Rightarrow \log(a^x) = x$. $\square$

(LP2) $\log_a(x^n) = n\log_a(x)$.

Proof $\log_a(x) = y \Rightarrow x = a^y$
$\Rightarrow x^n = (a^y)^n = a^{ny}$
$\Rightarrow \log_a(x^n) = \log_a(a^{ny}) = ny$
$\Rightarrow \log_a(x^n) = ny$

$\log_a(x) = y \Rightarrow ny = n\log_a(x)$
$\Rightarrow \log_a(x^n) = n\log_a(x)$. $\square$

(LP3) $a^{\log_a(x)} = x$.

Proof $\log_a(x) = y \Rightarrow a^y = x$
$\Rightarrow a^{\log_a(x)} = x$. $\square$

(LP4) $\log_a(x) = \frac{\log_b(x)}{\log_b(x)}$.

Proof $\log_a(x) = y \Rightarrow a^y = x \Rightarrow \log_b(x) = y\log_b(a)$
$\Rightarrow y = \log_a(x) = \frac{\log_b(x)}{\log_b(a)}$

(vii) Exponential $f(x) = e^x \Rightarrow f'(x) = e^x$.

Proof $f(x) = e^x \Rightarrow f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}$$

$$= e^x \lim_{h\to 0} \frac{(e^h - 1)}{h}$$

$$= e^x \lim_{y\to 0} \frac{y}{\ln(1+y)}$$

Let $y = e^h - 1$

$h = \ln(1+y)$

$\lim\limits_{h\to 0} y = 0 = \lim\limits_{h\to 0} h = \lim\limits_{y\to 0} y$

$$= e^x \lim_{y\to\infty} \frac{1}{\frac{1}{y}\ln(1+y)}$$

$$= e^x \lim_{y\to\infty} \frac{1}{\ln([1+y]^{\frac{1}{y}})}$$

$$= e^x \frac{1}{\lim\limits_{y\to\infty} \ln([1+y]^{\frac{1}{y}})}$$

$$= e^x \frac{1}{\ln(\underbrace{\lim\limits_{y\to 0} [1+y]^{\frac{1}{y}}}_{\text{definition of } e})}$$

$$= e^x \frac{1}{\underbrace{\ln(e)}_{=1}}$$

$$= e^x (1) = e^x. \ \square$$

(viii) Logarithm $f(x) = \ln(x) \Rightarrow f'(x) = \dfrac{1}{x}$.

Proof $f(x) = y = \ln(x) \Rightarrow e^y = x$

$$\Rightarrow \frac{de^y}{dx} = \frac{dx}{dx} = 1$$

$$\Rightarrow \underbrace{\frac{de^y}{dy}\frac{dy}{dx}}_{\text{(vi) Chain}} = e^y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}. \ \square$$

Corollary: $f(x) = a^x \Rightarrow f'(x) = a^x \ln(a)$.

Proof $f(x) = y = a^x \Rightarrow \ln(y) = x\ln(a)$

$$\Rightarrow \frac{d\ln(y)}{dx} = \frac{d\, x\ln(a)}{dx} = \ln(a)$$

$$\Rightarrow \frac{d\ln(y)}{dy}\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx} = \ln(a)$$

$$\Rightarrow \frac{dy}{dx} = y\ln(a) = a^x \ln(a).$$

Corollary: $f(x) = \log_a(x) \Rightarrow f'(x) = \dfrac{1}{x\ln(a)}$.

Proof $f(x) = \log_a(x) = \dfrac{\ln(x)}{\ln(a)} \Rightarrow f'(x) = \dfrac{1}{\ln(a)}\dfrac{d\ln(x)}{dx}$

$$= \frac{1}{x\ln(a)}. \ \square$$