

Properties of Derivatives

Definition The derivative of a function $f(x)$ is defined as the limit

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Derivative Rules

(i) Constant $f(x) = c \Rightarrow f'(x) = 0.$

(ii) Coefficient $f(x) = c g(x) \Rightarrow f'(x) = c g'(x).$

(iii) Addition $f(x) = g(x) + h(x) \Rightarrow f'(x) = g'(x) + h'(x).$

(iv) Power $f(x) = x^n, n \neq 0 \Rightarrow f'(x) = nx^{n-1}.$

(v) Product $f(x) = g(x)h(x) \Rightarrow f'(x) = g'(x)h(x) + h'(x)g(x).$

(vi) Chain $f(x) = g(h(x)) \Rightarrow f'(x) = g'(h(x))h'(x).$

(vii) Exponent $f(x) = e^x \Rightarrow f'(x) = e^x.$

(viii) Logarithm $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}.$

Proofs

(i) Constant $f(x) = c \Rightarrow f'(x) = 0.$

Proof $f(x) = c \Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$
 $= \lim_{h \rightarrow 0} \frac{c - c}{h}$
 $= \lim_{h \rightarrow 0} \frac{0}{h} = 0. \square$

(ii) Coefficient $f(x) = c g(x) \Rightarrow f'(x) = c g'(x).$

Proof $f(x) = c g(x) \Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$
 $= \lim_{h \rightarrow 0} \frac{c g(x+h) - c g(x)}{h}$
 $= \lim_{h \rightarrow 0} c \left[\frac{g(x+h) - g(x)}{h} \right]$
 $= c \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$
 $= c g'(x). \square$

(iii) Addition $f(x) = g(x) + h(x) \Rightarrow f'(x) = g'(x) + h'(x).$

Proof $f(x) = g(x) + h(x) \Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$
 $= \lim_{\Delta x \rightarrow 0} \frac{[g(x+\Delta x) + h(x+\Delta x)] - [g(x) + h(x)]}{\Delta x}$
 $= \lim_{\Delta x \rightarrow 0} \frac{[g(x+\Delta x) - g(x)] + [h(x+\Delta x) - h(x)]}{\Delta x}$
 $= \lim_{\Delta x \rightarrow 0} \left(\frac{g(x+\Delta x) - g(x)}{\Delta x} + \frac{h(x+\Delta x) - h(x)}{\Delta x} \right)$
 $= \lim_{\Delta x \rightarrow 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{h(x+\Delta x) - h(x)}{\Delta x}$
 $= g'(x) + h'(x). \square$

(IV) Power $f(x) = x^n, n \in \mathbb{N} \Rightarrow f'(x) = nx^{n-1}$.

Proof for $n \in \mathbb{N}$ $f(x) = x^n, n \in \mathbb{N} \Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$$

by binomial theorem

$$= \lim_{h \rightarrow 0} \frac{\sum_{k=0}^n \frac{n!}{k!(n-k)!} h^k x^{n-k} - x^n}{h}$$

$$= \lim_{h \rightarrow 0} \sum_{k=1}^n \frac{n!}{k!(n-k)!} h^{k-1} x^{n-k}$$

since $\lim_{h \rightarrow 0} h^{k-1} = \begin{cases} 1 & \text{if } k=1 \\ 0 & \text{if } k>1 \end{cases}$

$$= \frac{n!}{1!(n-1)!} x^{n-1}$$

$$= nx^{n-1}. \square$$

$$\begin{array}{c} | \\ (x+h)^0 = 1 \\ | \\ (x+h)^1 = 2x+1 \cdot h \\ | \quad 2 \quad | \\ (x+h)^2 = 2x^2 + 2xh + 1h^2 \\ | \quad 3 \quad 3 \quad | \\ (x+h)^3 = 2^3 + 3x^2h + 3xh^2 + h^3 \\ | \quad 4 \quad 6 \quad 4 \quad | \\ (x+h)^4 = 2^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 \end{array}$$

Proof for $n \in \mathbb{R} \setminus \{0\}$ let $y = f(x) = x^n, n \in \mathbb{R} \setminus \{0\}$.

$$y = x^n \Rightarrow \ln y = n \ln x.$$

$$\ln y = n \ln x \Rightarrow \frac{d \ln y}{dx} = \frac{d n \ln x}{dx} = nx^{-1}. (4.1)$$

2 by (i) coefficient and (iii) logarithm.

$$(4.1) \Rightarrow \frac{d \ln y}{dx} = \frac{d \ln y}{dy} \frac{dy}{dx} = y^{-1} \frac{dy}{dx} = nx^{-1}$$

by (vi) chain. $\Rightarrow \frac{dy}{dx} = f'(x) = nx^{-1}y$

$$= nx^{-1}x^n$$

$$= nx^n. \square$$

(V) Product $f(x) = g(x)h(x) \Rightarrow f'(x) = g'(x)h(x) + g(x)h'(x)$.

Proof $f(x) = g(x)h(x) \Rightarrow f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \rightarrow 0} \frac{g(x+\Delta x)h(x+\Delta x) - g(x)h(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{g(x+\Delta x)h(x+\Delta x) - g(x)h(x) + (g(x+\Delta x)h(x) - g(x+\Delta x)h(x))}{\Delta x}$$

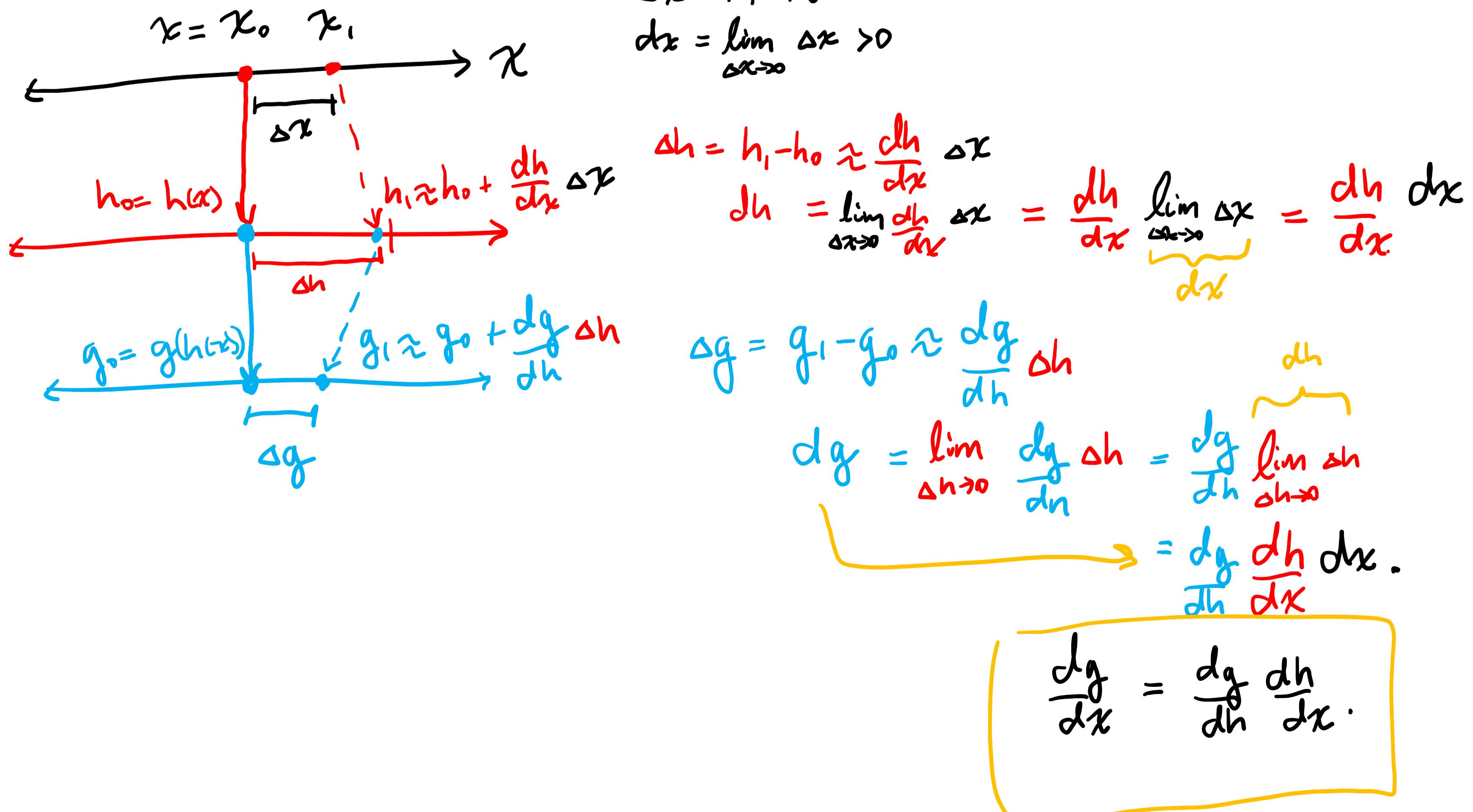
$$= \lim_{\Delta x \rightarrow 0} \frac{g(x+\Delta x)[h(x+\Delta x) - h(x)] + h(x)[g(x+\Delta x) - g(x)]}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} g(x+\Delta x) \frac{h(x+\Delta x) - h(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} h(x) \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= g(x)h'(x) + h(x)g'(x). \square$$

(vi) Chain $f(x) = g(h(x)) \Rightarrow f'(x) = g'(h)h'(x)$

Illustration



Corollary: Quotient $f(x) = \frac{g(x)}{h(x)} \Rightarrow f'(x) = \frac{g'(x)h(x) + h'(x)g(x)}{h(x)^2}$.

Proof $f(x) = \frac{g(x)}{h(x)} = g(x)[h(x)]^{-1} \Rightarrow f'(x) = \frac{dg(x)}{dx} \frac{1}{h(x)} + \frac{d(h(x)^{-1})}{dx} g(x)$ by (v) Product

$$\begin{aligned} &= \frac{g'(x)}{h(x)} + \frac{d(h(x))^{-1}}{dh} \frac{dh(x)}{dx} g(x) \\ &= \frac{g'(x)h(x)}{h(x)^2} + (-1)[h(x)]^{-2} h'(x) g(x) \\ &= \frac{g'(x)h(x) + h'(x)g(x)}{h(x)^2}. \quad \square \end{aligned}$$

Definitions $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{m \rightarrow 0} (1+m)^{\frac{1}{m}}$.

$$\log_a(y) = x \Leftrightarrow y = a^x.$$

Properties (LP1) $\log_a(a^x) = x$.

(LP2) $\log_a(x^n) = n \log_a(x)$.

(LP3) $a^{\log_a(x)} = x$.

(LP4) $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$

Continuous Compounding:

- r: period interest rate
- n: interest payment/period
- t: number of periods

$$\text{Compounded interest: } r_n = \left(1 + \frac{r}{n}\right)^{nt}$$

$$\text{Continuous compounding: } r_c = \lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^{nt} = \lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^{nrt} = \left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n\right]^{rt}$$

$$= e^{rt}.$$

(LP1) $\log_a(a^x) = x$.

Proof $y = a^x \Rightarrow \log(y) = x$

$$\Rightarrow \log(a^x) = x. \quad \square$$

Corollary: $\log_a(a) = 1, \log_a(1) = 0$.

Proof Set $x = 1$, then $\log(a^x) = x$

$$= \log(a^1) = \log(a) = 1. \quad \square$$

Set $x = 0$, then $\log(a^x) = x$

$$= \log(a^0) = \log(1) = 0. \quad \square$$

Other Properties

(LP5) $\log_b(xy) = \log_b(x) + \log_b(y)$

Proof Let $x = b^\alpha, y = b^\beta$. Then

$$\begin{aligned} \log_b(xy) &= z \Rightarrow b^z = xy = b^\alpha b^\beta \\ &= b^{\alpha+\beta} \\ &\Rightarrow \log_b(b^z) = \log_b(b^{\alpha+\beta}) \end{aligned}$$

(LP2) $\log_a(x^n) = n \log_a(x)$.

Proof $\log_a(x) = y \Rightarrow x = a^y$

$$\Rightarrow x^n = (a^y)^n = a^{ny}$$

$$\Rightarrow \log_a(x^n) = \log_a(a^{ny}) = ny$$

$$\Rightarrow \log_a(x^n) = ny$$

$$\begin{aligned} \log_a(x) = y &\Rightarrow ny = n \log_a(x) \\ \Rightarrow \log_a(x^n) &= n \log_a(x). \quad \square \end{aligned}$$

(LP3) $a^{\log_a(x)} = x$.

Proof $\log_a(x) = y \Rightarrow a^y = x$

$$\Rightarrow a^{\log_a(x)} = x. \quad \square$$

(LP4) $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$.

Proof $\log_a(x) = y \Rightarrow a^y = x \Rightarrow \log_b(x) = y \log_b(a)$

$$\Rightarrow y = \log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

(vii) Exponential $f(x) = e^x \Rightarrow f'(x) = e^x.$

Proof $f(x) = e^x \Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^x (e^h - 1)}{h}$$

$$= e^x \lim_{h \rightarrow 0} \frac{(e^h - 1)}{h}$$

$$= e^x \lim_{y \rightarrow \infty} \left\{ \begin{array}{l} y \\ \ln(1+y) \end{array} \right\}$$

$$= e^x \lim_{y \rightarrow \infty} \frac{1}{\frac{1}{y} \ln(1+y)}$$

$$= e^x \lim_{y \rightarrow \infty} \frac{1}{\ln((1+y)^{\frac{1}{y}})}$$

$$= e^x \frac{1}{\lim_{y \rightarrow \infty} \ln((1+y)^{\frac{1}{y}})}$$

$$= e^x \frac{1}{\ln(e)} \stackrel{\text{definition of } e}{=} 1$$

$$= e^x (1) = e^x. \square$$

Let $y = e^h - 1$
 $\Rightarrow h = \ln(1+y)$
 $\lim_{h \rightarrow 0} y = 0 = \lim_{h \rightarrow 0} h = \lim_{y \rightarrow 0} y$

(viii) Logarithm $f(x) = \ln(x) \Rightarrow f'(x) = \frac{1}{x}.$

Proof $f(x) = y = \ln(x) \Rightarrow e^y = x$

$$\Rightarrow \frac{de^y}{dx} = \frac{dx}{dx} = 1$$

$$\Rightarrow \frac{de^y}{dy} \frac{dy}{dx} = e^y \frac{dy}{dx} = 1$$

(vi) Chain Rule $\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}. \square$$

Corollary: $f(x) = a^x \Rightarrow f'(x) = a^x \ln(a).$

Proof 1 $f(x) = y = a^x \Rightarrow \ln(y) = x \ln(a)$

$$\Rightarrow \frac{d \ln(y)}{dx} = \frac{d}{dx} x \ln(a) = \ln(a)$$

$$\Rightarrow \frac{d \ln(y)}{dy} \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx} = \ln(a)$$

$$\Rightarrow \frac{dy}{dx} = y \ln(a) = a^x \ln(a).$$

Proof 2 $f(x) = a^x = e^{\ln(a^x)} = e^{x \ln(a)}$

$$f'(x) = e^{x \ln(a)} \cdot \ln(a) = a^x \ln(a). \square$$

Corollary: $f(x) = \log_a(x) \Rightarrow f'(x) = \frac{1}{x \ln(a)}.$

Proof $f(x) = \log_a(x) = \frac{\ln(x)}{\ln(a)} \Rightarrow f'(x) = \frac{1}{\ln(a)} \frac{d \ln(x)}{dx}$

$$= \frac{1}{x \ln(a)}. \square$$