1 Limits

Definition: $\lim\limits_{x\to a} f(x) = L$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ where

$$a-\delta < x < a+\delta \Rightarrow L-\varepsilon < f(x) < L+\varepsilon \text{, i.e.}$$
$$|x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$$

Proposition: $f(x) = x \Rightarrow \lim\limits_{x\to a} f(x) = a$.

Proof: $\forall \varepsilon > 0$, let $\delta = \varepsilon$. Then

$$|x-a| < \underbrace{\delta}_{=\varepsilon} \Rightarrow |\underbrace{x}_{=f(x)}-a| < \varepsilon \Rightarrow |f(x)-a| < \varepsilon. \square$$

Examples: $\lim\limits_{x\to 1} x = 1$. $\lim\limits_{h\to 0} h = 0$.

Proposition: $\lim\limits_{x\to a} cx = ca$.

Proof: $\forall \varepsilon > 0$, let $\delta = \frac{\varepsilon}{|c|}$. Then

$$|x-a| < \delta = \frac{\varepsilon}{|c|} \Rightarrow |c|\cdot|x-a| < \varepsilon \Rightarrow |cx-ca| < \varepsilon. \square$$

Examples: $\lim\limits_{x\to 2} 8x = 8(2) = 16$. $\lim\limits_{y\to 1} cy = c\cdot(1) = c$.

Proposition: $\lim_{x\to a} f(x) = L \Rightarrow \lim_{x\to a}[c\cdot f(x)] = c\cdot L = c\left[\lim_{x\to a} f(x)\right]$

Proof: $\lim_{x\to a} f(x) \Rightarrow \forall \varepsilon>0, \exists \delta_1$ where $(|x-a|<\delta_1 \Rightarrow |f(x)-L|<\varepsilon)$

$\Rightarrow \forall \varepsilon>0, \exists \delta_2$ where $\left(|x-a|<\delta_2 \Rightarrow |f(x)-L| < \frac{\varepsilon}{|c|}\right)$.

Now $\forall \varepsilon>0$ let $\delta=\delta_2$. Then

$$|x-a| < \delta = \delta_2 \Rightarrow |f(x)-L| < \frac{\varepsilon}{|c|}$$

$$\Rightarrow |c|\cdot|f(x)-L| = |c\cdot f(x) - c\cdot L| < \varepsilon. \square$$

Proposition: $\lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = K \Rightarrow \lim_{x\to a}[f(x)+g(x)] = \underbrace{L+K}_{=\left[\lim_{x\to a} f(x)\right] + \left[\lim_{x\to a} g(x)\right]}.$

Proof: $\lim_{x\to a} f(x)=L \Rightarrow \forall\varepsilon>0, \exists\delta_1'>0$ where $(|x-a|<\delta_1' \Rightarrow |f(x)-L|<\varepsilon.)$

$\Rightarrow \forall\varepsilon>0, \exists\delta_1>0$ where $\left(|x-a|<\delta_1 \Rightarrow |f(x)-L|<\frac{\varepsilon}{2}\right)$

$\lim_{x\to a} g(x)=K \Rightarrow \forall\varepsilon>0, \exists\delta_2>0$ where $|x-a|<\delta_2 \Rightarrow |g(x)-K|<\frac{\varepsilon}{2}$.

Now, $\forall\varepsilon$ let $\delta=\min\{\delta_1,\delta_2\}$. Then

$$|x-a|<\delta \Rightarrow |f(x)-L| + |g(x)-K| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \underbrace{|f(x)-L+g(x)-K|} = |[f(x)+g(x)] - [L+K]| < \varepsilon. \square$$

Let $f(x)-L=F$, $g(x)-K=G$, then

$$|f(x)-L+g(x)-K| = |F+G| = \begin{cases} |F|+|G| & \text{if } F,G>0 \text{ or } F,G<0 \\ |F|-|G| & \text{if } F>G \\ |G|-|F| & \text{if } G>F \end{cases} \Bigg\} \le \max\{|F|,|G|\}$$

$$\le |F|+|G| = |f(x)-L| + |g(x)-K| < \varepsilon.$$

Proposition: $\lim\limits_{x\to a} f(x) = L$, $\lim\limits_{x\to a} g(x) = K \Rightarrow \lim\limits_{x\to a}[f(x)\cdot g(x)] = \underbrace{L\cdot k}_{=\left[\lim\limits_{x\to a} f(x)\right]\left[\lim\limits_{x\to a} g(x)\right]}.$

Proof: $\lim\limits_{x\to a} f(x) = L \Rightarrow \forall \varepsilon > 0, \exists \delta_1' > 0$ where $\left(|x-a| < \delta_1' \Rightarrow |f(x) - L| < \varepsilon\right).$

$\Rightarrow \forall \varepsilon > 0, \exists \delta_1' > 0$ where $\left(|x-a| < \delta_1 \Rightarrow |f(x) - L| < \sqrt{\varepsilon}\right)$

$\lim\limits_{x\to a} g(x) = K \Rightarrow \forall \varepsilon > 0, \exists \delta_2 > 0$ where $|x-a| < \delta_2 \Rightarrow |g(x) - k| < \sqrt{\varepsilon}.$

Now, $\forall \varepsilon$ let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$|x-a| < \delta \Rightarrow \left|[f(x) - L][g(x) - k] - 0\right| = |f(x) - L|\cdot|g(x) - k| < (\sqrt{\varepsilon})(\sqrt{\varepsilon}) = \varepsilon$$

$$\Rightarrow \lim_{x\to a}[f(x) - L][g(x) - k] = 0$$

$$\Rightarrow \lim_{x\to a}\left[f(x)g(x) - kf(x) - Lg(x) + Lk\right] = 0$$

$$\Rightarrow \lim_{x\to a}[f(x)g(x)] = \lim_{x\to a}[f(x)k + Lg(x) - Lk]$$

$$= Lk + \underbrace{Lk - Lk}_{=0} = Lk$$

Proposition: $\lim_{x\to a}\left[\log_b(x)\right] = \log_b(a),\ b>1.$ $= \ln\left[\lim_{x\to a} f(x)\right]$

Proof: $\forall \varepsilon$ let $\delta_1 = \left|a(1-b^{-\varepsilon})\right|$ $\delta_2 = \left|a(b^{\varepsilon}-1)\right|$, and

$\delta = \min\{\delta_1, \delta_2\}$. Then

$$|x-a|<\delta \Rightarrow a-\delta < x < a+\delta$$

$$\Rightarrow \min\{a-a(1-b^{-\varepsilon}),\ a-a(b^{\varepsilon}-1)\} < x < \min\{a+a(1-b^{-\varepsilon}),\ a+a(b^{\varepsilon}-1)\}$$

$$\Rightarrow \min\{ab^{-\varepsilon},\ a(2-b^{\varepsilon})\} < x \underbrace{< \min}_{x<\min\{y,z\}\ \Rightarrow\ x<y,z.}\{a(2-b^{-\varepsilon}),\ ab^{\varepsilon}\}$$

$$\Rightarrow \min\{\log_b(a)-\varepsilon,\ \log_b(a[2-b^{\varepsilon}])\} < \log_b(x) < \log_b(ab^{\varepsilon}) = \log_b(a)+\varepsilon$$

$$\Rightarrow \min\{-\varepsilon,\ \underbrace{\log([2-b^{\varepsilon}])\}}_{\substack{>0 \text{ or undefined} \\ \Rightarrow \min\{-\varepsilon,\ldots\}=-\varepsilon}} < \log_b(x)-\log_b(a) < \varepsilon$$

$$\Rightarrow -\varepsilon < \log_b(x)-\log_b(a) < \varepsilon$$

$$\Rightarrow \left|\log(x)-\log b(a)\right| < \varepsilon.\ \square$$

Proposition: $\lim_{x\to a} f(x) = L \Rightarrow \lim_{x\to a}[f(x)]^n = L^n,\ (L\neq 0)\wedge(n\leq 0).$ $= \left[\lim_{x\to a} f(x)\right]^n$

Proof: $\lim_{x\to a} f(x) = L \Rightarrow \lim_{x\to a} \ln(f(x)) = \ln(L)$

$$\Rightarrow \lim_{x\to a} n\cdot\ln(f(x)) = n\cdot\ln(L)$$

$$= \lim_{x\to a} \ln([f(x)]^n) = \ln\left(\lim_{x\to a}[f(x)]^n\right) = \ln(L^n)$$

$$\Rightarrow \lim_{x\to a}\left[f(x)\right]^n = L^n.\ \square$$

Proposition: $\lim_{x \to a} f(x) = L > 0 \Rightarrow \lim_{x \to a} \log_b f(x) = \log_b L$, $b > 0$.

Proof: $\lim_{x \to a} f(x) = L \Rightarrow \forall \varepsilon > 0$, $\exists \delta_1$ where $\left(|x-a| < \delta_1 \Rightarrow |f(x) - L| < |L(b^{-\varepsilon} - 1)| \right)$
and $\exists \delta_2$ where $\left(|x-a| < \delta_2 \Rightarrow |f(x) - L| < |L(b^{\varepsilon} - 1)| \right)$

Next, $\forall \varepsilon > 0$ let $\delta = \min\{\delta_1, \delta_2\}$. Then,

Case 1: $\delta = \delta_1 = \min\{\delta_1, \delta_2\} \Rightarrow \delta_1 < \delta_2 \Rightarrow b^{-\varepsilon} < b^{\varepsilon}$.

$$|x-a| < \delta = \delta_1 \Rightarrow |f(x) - L| < |L(b^{-\varepsilon} - 1)| = L \cdot \max\{b^{-\varepsilon} - 1, 1 - b^{-\varepsilon}\}$$

$$\Rightarrow 0 < L \cdot \min\{b^{-\varepsilon} - 1, 1 - b^{-\varepsilon}\} < f(x) - L < L \cdot \max\{b^{-\varepsilon} - 1, 1 - b^{-\varepsilon}\}$$

$$\Rightarrow 0 < L \cdot \min\{b^{-\varepsilon}, 2 - b^{-\varepsilon}\} < f(x) < L \cdot \max\{b^{-\varepsilon}, 2 - b^{-\varepsilon}\}$$

$$\Rightarrow 0 < \qquad < f(x) < L \cdot b^{-\varepsilon} < L \cdot b^{\varepsilon}$$

$$\Rightarrow \log_b(L) + \log \qquad < \log_b[f(x)] <$$