

Instruction: Type your answers to the following questions provided by LaTeX and submit a zipped file (included .pdf and .tex files) to E-learning (Sakai) by group (only 4-5 members in each group). Only team leader will submit it. One page per problem. Please use the solution template which is provided; summarize the work of each member in percentage (%).

GROUP 1984 — MEMBER LIST			
No.	Name	ID	Role
1	Quach Dang Giang	1952044	Leader, 50% (of this current work)
2	Huynh Phuoc Thien	1952463	Member, 20%
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The number of mandatory exercise: 12/12

The number of extra homework: 7

Problem 1. [10pts] Suppose that 8% of the patients tested in a clinic are infected with HIV. Furthermore, suppose that when a blood test for HIV is given, 98% of the patients infected with HIV test positive and that 3% of the patients not infected with HIV test positive. What is the probability that infected with it?

- (1) a patient testing positive for HIV with this test is infected with it?
- (2) a patient testing positive for HIV with this test is not infected with it?
- (3) a patient testing negative for HIV with this test is infected with it?
- (4) a patient testing negative for HIV with this test is not infected with it?

Solution:

We have:

- $P(A)$ is the probability of patients tested in a clinic are infected with HIV.
- $P(B)$ is the probability of patients test positive.
- $P(A) = 8\% \Rightarrow P(\bar{A}) = 92\%$
- $P(B|A) = 98\% \Rightarrow P(\bar{B}|A) = 2\%$
- $P(B|\bar{A}) = 3\% \Rightarrow P(\bar{B}|\bar{A}) = 97\%$

1. The probability that a patient testing positive for HIV with this test is infected with it:

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B|A) \times P(A) + P(B|\bar{A}) \times P(\bar{A})} = \frac{98 \times 8}{98 \times 8 + 3 \times 92} = \mathbf{0.7396}$$

2. The probability that a patient testing positive for HIV with this test is not infected with it:

$$P(\bar{A}|B) = \frac{P(B|\bar{A}) \times P(\bar{A})}{P(B|A) \times P(A) + P(B|\bar{A}) \times P(\bar{A})} = \frac{3 \times 92}{98 \times 8 + 3 \times 92} = \mathbf{0.2604}$$

3. The probability that a patient testing negative for HIV with this test is infected with it:

$$P(A|\bar{B}) = \frac{P(\bar{B}|A) \times P(A)}{P(\bar{B}|A) \times P(A) + P(\bar{B}|\bar{A}) \times P(\bar{A})} = \frac{2 \times 8}{2 \times 8 + 97 \times 92} = \mathbf{0.0018}$$

4. The probability that a patient testing negative for HIV with this test is not infected with it:

$$P(\bar{A}|\bar{B}) = \frac{P(\bar{B}|\bar{A}) \times P(\bar{A})}{P(\bar{B}|A) \times P(A) + P(\bar{B}|\bar{A}) \times P(\bar{A})} = \frac{97 \times 92}{2 \times 8 + 97 \times 92} = \mathbf{0.9982}$$

Problem 2. [5pts] The final exam of a discrete mathematics course consists of 50 true/false questions, each worth two points, and 25 multiple-choice questions, each worth four points. The probability that Linda answers a true/false question correctly is 0.9, and the probability that she answers a multiple-choice question correctly is 0.8. What is her expected score on the final?

Solution:

The expected score of Linda is:

$$50 \times 0.9 \times 2 + 25 \times 0.8 \times 4 = \mathbf{170}$$

Problem 3. [10pts] Please consider the poker rule via this link: [List of poker hands](#). Following this rule, what is the probability that a poker hand contains: Straight flush, Four of a kind, Full house, Flush, Straight, Three of a kind, Two pair, One pair, High card?

Solution:

Getting 5 random cards from 52-card deck. The number of outcomes in the sample space is :

$$\binom{52}{5}$$

The Situations:

- **Straight Flush:** All 5 cards are from the same suit and they form a straight (they may also be a royal flush). The number of such hands is

$$\binom{4}{1} \times \binom{10}{1} = 40$$

IF YOU MEAN TO EXCLUDE ROYAL FLUSHES, SUBTRACT 4 (the number of royal flush hand), the number of hands would then be $4 \times 10 - 4 = 36$, and the probability is

$$36 \div \binom{52}{5} \approx \mathbf{0.001385\%}$$

- **Four of a kind:** This hand has the pattern AAAAB where A and B are from distinct kinds. The number of such hands is

$$\binom{13}{1} \times \binom{12}{1} \times \binom{4}{1} = 624$$

The probability is approximately **0.0240%** ($624 : 52C5$).

- **Full house:** This hand has the pattern AAABB where A and B are from distinct kinds. The number of such hands is

$$\binom{13}{1} \times \binom{4}{3} \times \binom{12}{1} \times \binom{4}{2} = 3744$$

The probability is approximately **0.1441%**.

- **Flush:** Here all 5 cards are from the same suit (they may also be a straight). The probability of such hands is

$$\binom{4}{1} \times \binom{13}{5} \div \binom{52}{5} \approx 0.19807\%$$

IF YOU MEAN TO EXCLUDE STRAIGHT FLUSHES AND ROYAL FLUSH, SUBTRACT 40: the probability would then be

$$\left(\binom{4}{1} \times \binom{13}{5} - 40 \right) \div \binom{52}{5} \approx 0.19654\%$$

The probability is approximately **0.19654%**.

- **Straight:** This is five cards in a sequence (e.g., 4,5,6,7,8), with aces allowed to be either 1 or 13 (low or high) and with the cards allowed to be of the same suit (e.g., all hearts) or from some different suits. The number of such hands is $10 \times \binom{4}{1}^5$. The probability is 0.3940%. *IF YOU MEAN TO EXCLUDE STRAIGHT FLUSHES AND ROYAL FLUSHES* the number of such hands is

$$\binom{10}{1} \times \binom{4}{1}^5 - 36 - 4 = 10200$$

The probability is approximately **0.392465%**.

- **Three of a kind:** This hand has the pattern AAABC where A, B, and C are from distinct kinds. The number of such hands is

$$\binom{13}{1} \times \binom{4}{3} \times \binom{12}{2} \times \binom{4}{1}^2 = 54912$$

The probability is approximately **2.112845%**.

- **Two pairs:** This hand has the pattern AABBC where A, B, and C are from distinct kinds. The number of such hands is

$$\binom{13}{2} \times \binom{4}{2}^2 \times \binom{11}{1} \times \binom{4}{1} = 123552$$

The probability is approximately **4.7539%**.

- **One pair:** This the hand with the pattern AABCD, where A, B, C and D are from the distinct "kinds" of cards: aces, twos, threes, tens, jacks, queens, and kings (there are 13 kinds, and four of each kind, in the standard 52 card deck). The number of such hands is

$$\binom{13}{1} \times \binom{4}{2} \times \binom{12}{4} \times \binom{4}{1}^3 = 1098240$$

The probability is approximately **42.2569%**.

- **High card:** We have to choose 5 distinct kinds (13-choose-5) but exclude any straights (subtract 10). We can have any pattern of suits except the 4 patterns where all 5 cards have the same suit:

$$4^5 - 4$$

The total number of such hands is

$$\left[\binom{13}{5} - 10 \right] \times (4^5 - 4) = 1302540$$

The probability is approximately **50.1177%**

Problem 4. [10pts]

- (1) A group contains 20 people (5 men and 15 women) is randomly assigned to 4 different groups (each group has 5 people). Calculate the probability that females will be in the same group?
- (2) A phone number consisting of 9 random numbers, a man forgets two last digits of the phone number. However, he knows that the two digits are different. He chooses two these numbers randomly, find the probability that he call the correct number?
- (3) Arrange 7 people consist of four sons, two daughters, and a child sitting in seven chairs around a round-table. What is the probability of a child sitting between two girls?
- (4) Rolling three dice at the same time, knowing that at least one of them is five dots. Calculate the probability that the total number of dots of 3 dice is 8?
- (5) A soccer tournament has 12 teams, 9 teams are foreign and 3 teams are domestic. The organizers want to divide 12 teams into 3 groups A, B, C, (4 teams per group). Calculate the probability that three domestic teams are in three different tables.
- (6) Let A,B,C be any events. Prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

- (7) Suppose that E and F are events such that $p(E) = 0.7$ and $p(F) = 0.5$. Show that

$$p(E \cup F) \geq 0.7 \quad \text{and} \quad p(E \cap F) \geq 0.2$$

Solution:

1. Choosing sample space:

$$\binom{20}{5} \times \binom{15}{5} \times \binom{10}{5} \times \binom{5}{5}$$

Choosing 3 groups for woman:

$$\binom{4}{3}$$

Arranging woman into groups:

$$\binom{15}{5} \times \binom{10}{5} \times \binom{5}{5}$$

The probability that females will be in the same group: 2. Last 2 digits of a number $\Rightarrow 10^2$ possible combination. Since the two digits are different, we eliminated 10 cases from 00 to 99.

The probability that he call the correct number: $\frac{1}{90}$

3. The probability of a child sitting between two girls:

$$\frac{4! \times 2}{6!} = \frac{1}{15}$$

4. The probability that the total number of dots of 3 dice is 8:

$$3 \times \frac{2}{36} = \frac{1}{6}$$

5. The probability that three domestic teams are in three different tables:

$$\frac{3 \times \binom{9}{3} \times 2 \times \binom{6}{3}}{\binom{12}{4} \times \binom{8}{4}} = \frac{16}{55}$$

6. Considering A,B, C are sets of cases:

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

$$p(B \cup C) = p(B) + p(C) - p(B \cap C)$$

$$p(A \cup C) = p(A) + p(C) - p(A \cap C)$$

$$\begin{aligned}
 P(A \cup B \cup C) &= P(A \cup B) + P(C) - P((A \cup B) \cap C) \\
 &= P(A) + P(B) - P(A \cap B) + P(C) - P((A \cup B - A \cap B) \cap C) \\
 &= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)
 \end{aligned}$$

\Rightarrow Correct

7. We have: $P(E) = 0.7$; $P(F) = 0.5 \Rightarrow P(E \cap F) \leq 0.5$ if F is a subset of E .

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.12 \Rightarrow P(E \cap F) \geq 0.7$$

$$P(E \cap F) = P(E) + P(F) - P(E \cup F) = 1.2 - P(E \cup F). \text{ Because } P(E \cup F) \leq 1 \Rightarrow P(E \cap F) \leq 0.2.$$

Problem 5. [5pts] To go to school, a student must cross a railroad track. The time trains run through this rail are a little bit different every day. However, this student estimates she have to wait for the train approximate 15% of the school days. During one week of schooling there are 5 days, the probability will be how much if this student

- (1) See a train on Monday and do it again on Tuesday??
- (2) The first time she meet a train is Thursday?
- (3) See a train everyday?
- (4) See a train at least once a week?

Solution:

1. If this student see a train on Monday and do it again on Tuesday, the probability will be:

$$0.15^2 = \mathbf{2.25\%}$$

2. If the first time she meet a train is on Thursday, the probability will be:

$$0.85^3 \times 0.15 \approx \mathbf{9.21\%}$$

3. If this student see a train everyday, the probability will be:

$$0.15^5 \approx \mathbf{0.0076\%}$$

4. The probability of the student seeing a train at least once a week is:

$$1 - 0.85^5 \approx \mathbf{55.63\%}$$

Problem 6. [5pts] A straight iron rod of length l (unit length) is cut randomly into three pieces. Find the probability that these three iron pieces form a triangle.

Solution:

Let the lengths of the three part of the rod be x, y and $l-(x+y)$

Obviously we have (1) $x > 0; y > 0$ and $x + y < l \Rightarrow y < l - x \dots$

So in order that these three parts form the sides of a triangle, we should have : (2)

$$x + y > l - (x + y) \Rightarrow y > \frac{l}{2} - x$$

$$x + l - (x + y) > y \Rightarrow y < \frac{l}{2}$$

$$y + l - (x + y) > x \Rightarrow x < \frac{l}{2}$$

Since in a triangle, the sum of any 2 sides is greater than the third. We have (3)

$$\frac{l}{2} - x < y \cap 0 < x < \frac{l}{2}$$

Hence, on using (1) and (3), the probability is given by

$$\frac{\int_0^{\frac{l}{2}} \int_{\frac{l}{2}-x}^{\frac{l}{2}} dy dx}{\int_0^a \int_0^{a-x} dy dx} = \frac{\int_0^{\frac{l}{2}} \left(\frac{l}{2} - \left(\frac{l}{2} - x \right) \right) dx}{\int_0^a (a - x) dx} = \frac{\left[\frac{x^2}{2} \right]_0^{\frac{l}{2}}}{\left[-\frac{(a-x)^2}{2} \right]_0^a} = \frac{\frac{a^2}{8}}{\frac{a^2}{2}} = \frac{1}{4}$$

\Rightarrow The probability that these three iron pieces form a triangle is $\frac{1}{4}$

Problem 7. [10pts] A player in an archery Olympic game has 80% probability to shoot an arrow hit the target. Assume that every shot is independent, and this archer shoot 6 arrows,

- (1) Find the mean and standard deviation of the number of the shoot that the arrow hit the target?
- (2) Find the probability the first hit target is the third shoot?
- (3) Find the probability at least one arrow hit the target?
- (4) Find the probability the first arrow hit the target is the fourth or fifth shoot?
- (5) Find the probability there are exactly 4 arrows hit the target?
- (6) Find the probability there are at least 4 arrows hit the target?
- (7) Find the probability there are at most 4 arrows hit the target?

Solution:

$$p = 0.8; q = 0.2$$

$$1. \text{ Mean value } \mu = \frac{1}{p} = \mathbf{1.25}, \text{ Standard Deviation } \sigma = \sqrt{\frac{q}{p^2}} = \mathbf{0.559}$$

2. The probability the first hit target is the third shoot:

$$q^2 \times p = \mathbf{0.032}$$

3. The probability at least one arrow hit the target:

$$1 - q^5 = \mathbf{0.99968}$$

4.

- The probability the first arrow hit the target is the fourth shoot

$$q^3 \times p = \frac{\mathbf{4}}{\mathbf{625}}$$

- The probability the first arrow hit the target is the fifth shoot:

$$q^4 \times p = \frac{\mathbf{4}}{\mathbf{3125}}$$

- The probability the first arrow hit the target is the fourth or fifth shoot:

$$\frac{4}{625} + \frac{4}{3125} = \frac{\mathbf{24}}{\mathbf{3125}}$$

5. The probability there are exactly 4 arrows hit the target:

$$\binom{6}{4} \times p^4 \times q^2 = \mathbf{0.24576}$$

6.

- The probability there are exactly 4 arrows hit the target: 0.24576.

- The probability there are exactly 5 arrows hit the target: $\binom{6}{5} \times p^5 \times q = 0.393216$

- The probability there are exactly 6 arrows hit the target: $\binom{6}{6} \times p^6 \times q^0 = 0.262144$

\Rightarrow The probability there are at least 4 arrows hit the target:

$$0.24576 + 0.393216 + 0.262144 = \mathbf{0.90112}$$

7. The probability there are at most 4 arrows hit the target: $1 - 0.393216 - 0.262144 = \mathbf{0.34464}$

Problem 8. A student who wants to go to school must go through five intersections with traffic lights, and must stop if there is a red light. This student estimates the probability model for the number of red lights that this person encounters, as follows.

$X = \text{number of red light}$	0	1	2	3	4	5
$P(X = x)$	0.05	0.25	0.35	0.15	0.15	0.05

- (1) This student should expect to meet how many red lights a day??
- (2) Calculate the standard deviation.

Solution:

X	p(X=x)	$X \times p(X = x)$	$(X - E(X))^2$	$(X - E(X))^2 \times p(X = x)$
0	0.05	0	5.0625	0.253125
1	0.25	0.25	1.5625	0.390625
2	0.35	0.7	0.0625	0.021875
3	0.15	0.45	0.5625	0.084375
4	0.15	0.6	3.0625	0.459375
5	0.05	0.25	7.5625	0.378125

$$E(X) = \sum x.p(X = x) = 0 \times 0.05 + 1 \times 0.25 + 2 \times 0.35 + 3 \times 0.15 + 4 \times 0.15 + 5 \times 0.05 = 2.25$$

$$V(X) = 0.253125 + 0.390625 + 0.021875 + 0.084375 + 0.459375 + 0.378125 = 1.5875$$

$$SD : 1.259960317$$

1. This student should expect to meet **2.25** red lights a day.
2. The standard deviation: **1.259960317**

Problem 9. [5pts] Suppose that Frida selects a ball by first picking one of two boxes at random and then selecting a ball from this box at random. The first box contains two white balls and three blue balls, and the second box contains four white balls and one blue ball. What is the probability that Frida picked a ball from the first box if she has selected a blue ball?

Solution:

$P(F) = 0.5$: probability of choosing the first box.

$P(\neg F) = 0.5$: probability of choosing the second box.

$P(B) = 0.4$: probability of choosing the blue ball.

$P(F \cap B) = 0.3$: probability of choosing the blue ball in the first box. This is because there are 3 blue balls in the first box of the total 10 balls.

The probability that Frida picked a ball from the first box if she has selected a blue ball:

$$P(F|B) = P(F \cap B)/P(B) = 0.3/0.4 = \mathbf{0.75}$$

Problem 10. [5 pts] We throw a coin until a head turns up for the second time, where p is the probability that a throw results in a head and we assume that the outcome of each throw is independent of the previous outcomes. Let X be the number of times we have thrown the coin.

- (1) Determine $P(X = 2)$; $P(X = 3)$; and $P(X = 4)$:
- (2) Show that $P(X = n) = (n - 1) \times p^2 \times (1 - p)^{n-2}$ for $n \geq 2$

Solution:

1.

$$P(X = 2) = p^2$$

$$P(X = 3) = \binom{2}{1} \times p^2 \times (1 - p) = 2p^2(1 - p)$$

$$P(X = 4) = \binom{3}{1} \times p^2 \times (1 - p)^2 = 3p^2(1 - p)^2$$

2.

For $P(X=n)$, the throw will result in a head exactly 2 times, one at the n -th throw, and the other from 1 of the previous $n-1$ throws. We have $\binom{n-1}{1}$ ways to pick 1 from the mentioned $n-1$ throws, and the probability of getting 2 heads out of n throws is $p^2 \times (1 - p)^{n-2}$. So

$$P(X = n) = \binom{n-1}{1} \times p^2 \times (1 - p)^{n-2} = (n - 1)p^2(1 - p)^{n-2}$$

Problem 11. [5 pts] A computer virus is trying to corrupt two files. The first file will be corrupted with probability 0.4. Independently of it, the second file will be corrupted with probability 0.3.

- (1) Compute the probability mass function (pmf) of X , the number of corrupted files.
- (2) Draw a graph of its cumulative distribution function (cdf).

Solution:

1.

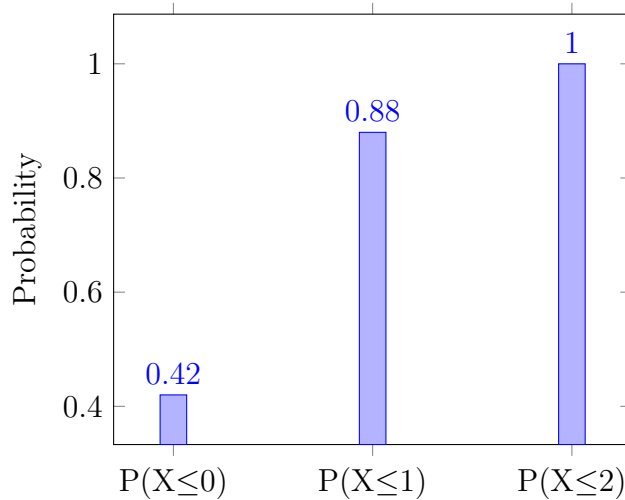
$$P(0) = (1 - 0.4) \times (1 - 0.3) = 0.42$$

$$P(1) = 0.4 \times (1 - 0.3) + 0.3 \times (1 - 0.4) = 0.46$$

$$P(2) = 0.4 \times 0.3 = 0.12$$

X	0	1	2
P(X=x)	0.42	0.46	0.12

2.



Graph of cumulative distribution function

Problem 12. [5 pts] Suppose that a Bayesian spam filter is trained on a set of 500 spam messages and 200 messages that are not spam. The word "exciting" appears in 40 spam messages and in 25 messages that are not spam. Would an incoming message be rejected as spam if it contains the word "exciting" and the threshold for rejecting spam is 0.9?

Solution:

E: Spam message; F: the word "exciting" appears

$p(E) = 5/7$: the probability of spam emails.

$p(\neg E) = 2/7$: the probability of not spam emails.

$$p(F|E) = \frac{p(F \cap E)}{p(E)} = \frac{40}{500} = \frac{2}{25}$$

$$p(F|\neg E) = \frac{p(F \cap \neg E)}{p(\neg E)} = \frac{25}{200} = \frac{1}{8}$$

$$p(E|F) = \frac{p(F|E) \times p(E)}{p(F|E) \times p(E) + p(F \cap \neg E) \times p(\neg E)} = \frac{(2/25) \times (5/7)}{(2/25) \times (5/7) + (1/8) \times (2/7)} = \frac{8}{13} < 0.9$$

So the message containing the word "exciting" will not be rejected as spam.

Bonus Exercises:

7.1.2: The probability that a fair die comes up six when it is rolled is $\frac{1}{6}$

7.1.6: What is the probability that a card selected at random from a standard deck of 52 cards is an ace or a heart?

There are 12 cards which are a heart and not an ace, 3 cards which are an ace and not a heart, and 1 card which is a heart and an ace.

\Rightarrow The probability is $\frac{12+3+1}{52} = \frac{4}{13}$

7.2.5

With two dices, the combinations of the first and second dice to produce a sum of 7 are

(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)

\Rightarrow The probability is $\frac{1}{7} \times \frac{1}{7} + \frac{1}{7} \times \frac{1}{7} + \frac{1}{7} \times \frac{1}{7} + \frac{2}{7} \times \frac{2}{7} + \frac{1}{7} \times \frac{1}{7} + \frac{1}{7} \times \frac{1}{7} = \frac{9}{49}$

7.3.2

$$p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{p(F|E) * p(E)}{p(F)} = \frac{\frac{5}{8} \times \frac{2}{3}}{\frac{3}{4}} = \frac{5}{9}$$

7.3.8

Let A: use opium, B: test positive .Given:

$P(B|\neg A) = 0.02, P(\neg B|A) = 0.05, P(A) = 0.01$

a. The probability that someone who tests negative for opium use does not use opium:

$$P(\neg A|\neg B) = \frac{P(\neg B|\neg A) \times P(\neg A)}{P(\neg B|\neg A) \times P(\neg A) + P(\neg B|A) \times P(A)} = \frac{(1 - 0.02) \times (1 - 0.01)}{(1 - 0.02) \times (1 - 0.01) + 0.05 \times 0.01} \approx \mathbf{0.9995}$$

b. The probability that someone who tests positive for opium use actually uses opium:

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B|A) \times P(A) + P(B|\neg A) \times P(\neg A)} = \frac{(1 - 0.05) \times 0.01}{(1 - 0.05) \times 0.01 + 0.02 \times (1 - 0.01)} \approx \mathbf{0.3242}$$

7.4.1

Since the coin is unbiased, $P(H) = 0.5$

$$E(H) = (\text{number of trials}) \times P(H) = 5 \times \frac{1}{2} = \mathbf{2.5}$$

7.4.5

$P(X = 3) = \frac{2}{7}, P(X = 1) = P(X = 2) = \dots = P(X = 6) = \frac{1}{7}$

X is the number appear on the first die, Y is the number appear on the second die

$E(X+Y) = E(X)+E(Y)$

$$E(X) = E(Y) = 1 \times \frac{1}{7} + 2 \times \frac{1}{7} + 3 \times \frac{2}{7} + \dots + 6 \times \frac{1}{7} = \frac{24}{7}$$

$$E(X + Y) = \frac{24}{7} + \frac{24}{7} = \frac{48}{7}$$