Local discontinuous Galerkin methods with novel basis for fractional diffusion equations with non-smooth solutions

Liyao Lyu [a], Zheng Chen [b]

[a] Department of Computational Mathematics, Science and Engineering, Michigan State University
[b] Department of Mathematics, University of Massachusetts

MICHIGAN STATE UNIVERSITY

Introduction

Consider the equations in the form

$$\frac{\partial u(x,t)}{\partial t} = d\frac{\partial^\beta u(x,t)}{\partial x^\beta} + f(x,t), \quad x\in[a,b], \quad \beta\in(1,2), \quad d>0 \tag{1}$$

The fractional derivative here is a Caputo derivative

$${}_a^C D_x^\alpha v(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-\xi)^{n-\alpha-1}\frac{d^n v(\xi)}{d\xi^n}d\xi, \quad x>a, \quad \alpha\in[n-1,n) \tag{2}$$

The equation (1) can be rewritten into the following system

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} - \sqrt{d}\frac{\partial q(x,t)}{\partial x} = f(x,t) && \text{in } \Omega_T,\\ &q - {}_a D_x^{\beta-2} p(x,t) = 0 && \text{in } \Omega_T,\\ &p - \sqrt{d}\frac{\partial u(x,t)}{\partial x} = 0 && \text{in } \Omega_T, \end{aligned} \tag{3}$$

Local discontinuous Galerkin methods were developed for fractional diffusion equations, and demonstrated to achieve optimal order of accuracy both theoretically and computationally for smooth enough underlying solutions. **However, the order degeneracy is observed when applied to problems with non-smooth solutions.** Consider solving a non-smooth problem with a solution $u(\cdot,t)\in H^\alpha$, the LDG method using finite element space V^k of piecewise polynomials with degree up to k, will achieve an error with a theoretical order $\min\{k+1,\alpha\}$, and the numerical order is observed to be like $\min\{k+1,\alpha+0.5\}$

Novel approximation space and mesh

Consider that the solution to problem (1) has an weak singularity at the left end of the domain, and is of form

$$w(x)(x-a)^\beta \tag{4}$$

with some unknown but smooth function $w(x)$.
Near the end $x=a$, we adopt the mapped polynomial functions $P^k((x-a)^\gamma)$ as basis functions to approximate the solution, with $\frac{1}{\gamma}$ being the smallest positive integer makes $\frac{\beta}{\gamma}\in\mathbb{N}$. Since $\beta\in(1,2)$, it is clear that $\gamma\in(0,1)$.
We now define the finite element space $V_\gamma^k$for both trial functions and test functions:

$$V_\gamma^k = \left\{v : v\big|_{I_j}\in P^k((x-a)^\gamma), \text{ if } x_j\le\hat{x};\ v\big|_{I_j}\in P^k(x), \text{ otherwise}\right\}. \tag{5}$$

In the domain near starting point the exact solution, being considered as a function of the mapped variable $y=(x-a)^\gamma$, is a regular function. In order to deal with ill-conditioned mass matrix, we adopted graded mesh for the cells with irregular mesh.

$$x_j = a + \left(\frac{b-a}{N}j\right)^{1/\gamma}; \tag{6}$$

for $n<j\le M$,

$$x_j = b-(M-j)\frac{b-a}{N}.s \tag{7}$$

Semi-defined LDG scheme

The semi-discrete LDG scheme to solve systemx is defined as follows. Find $u_h, q_h, p_h\in V_\gamma^k$ such that, for all test functions $v,w,z\in V_\gamma^k$ and all $j=1,2,\dots,M$, we have

$$\begin{aligned} \left(\frac{\partial u_h(x,t)}{\partial t}, v(x)\right)_{I_j} + \sqrt{d}\left(q_h(x,t), \frac{\partial v(x)}{\partial x}\right)_{I_j} - \sqrt{d}\hat{q}(x,t)v(x)\Big|_{x_{j-1}^+}^{x_j^-} &= (f(x,t), v(x))_{I_j},\\ (q_h(x,t), w(x))_{I_j} - ({}_a D_x^{\beta-2}p_h(x,t), w(x))_{I_j} &= 0,\\ (p_h(x,t), z(x))_{I_j} + \sqrt{d}\left(u_h(x,t), \frac{\partial z(x)}{\partial x}\right)_{I_j} - \sqrt{d}\hat{u}(x,t)z(x)\Big|_{x_{j-1}^+}^{x_j^-} &= 0, \end{aligned} \tag{8}$$

Alternating fluxes

Here, we use the so-called "alternating fluxes", which is a popular and attractive choice and defined as

$$\hat{q}(x_j,t) = q_h^+(x_j,t),\ \hat{u}(x_j,t) = u_h^-(x_j,t); \tag{9}$$

or

$$\hat{q}(x_j,t) = q_h^-(x_j,t),\ \hat{u}(x_j,t) = u_h^+(x_j,t) \tag{10}$$

at any interior cell interfaces; at the domain boundaries,

$$\hat{u}(a,t)=0,\ \hat{u}(b,t)=g(t), \tag{11}$$

and

$$\hat{q}(a,t) = q_h^+(a,t),\ \hat{q}(b,t) = q_h^-(b,t), \tag{12}$$

which reflect the Dirichlet boundary conditions.

Theorem (L^2 stability)

The scheme (8) is L^2 stable, and the solutions satisfies, for all $t\in[0,T]$,

$$\|e_{u_h}(\cdot,t)\|_{L^2}^2 + 2\cos((\beta/2-1)\pi)\int_0^t \|{}_a D_x^{\beta/2-1}e_{p_h}(\cdot,t)\|_{L^2}^2\,dt = \|e_{u_h}(\cdot,0)\|_{L^2}^2. \tag{13}$$

Theorem (Error Estimation)

The error for the scheme (8) with flux (9) or (10) and (11)-(12) satisfies

$$\|u-u_h\|_{L^2}\le Ch^{k+1}. \tag{14}$$

$\gamma = 0.5$

Consider equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{2}{3\Gamma(1.5)}\frac{\partial^{1.5}u(x,t)}{\partial x^{1.5}} - e^{-t}(x^{1.5}+1), \quad x\in(0,1), \tag{15}$$

on the computational domain $x\in\Omega=(0,1)$. Given initial condition

$$u_0(x) = x^{1.5}, \tag{16}$$

and Dirichlet boundary conditions

$$u(0,t)=0, \quad u(1,t)=e^{-t}, \tag{17}$$

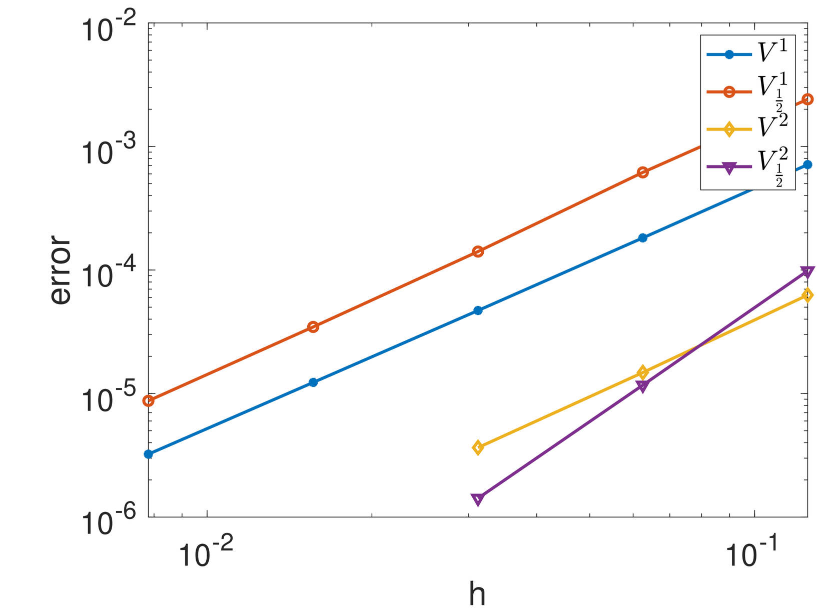
the exact solution is $u(x,t)=e^{-t}x^{1.5}$.

Table 1: The error and order of convergence with space V^1 on uniform mesh and $V^1_{1/2}$ at $T=1$.

h	V^1 error	order	h	$V^1_{1/2}$ error	order
1/8	7.13e-04		1/4	2.41e-03	
1/16	1.82e-04	1.97	1/8	6.17e-04	1.96
1/32	4.70e-05	1.95	1/16	1.41e-04	2.12
1/64	1.23e-05	1.92	1/32	3.46e-05	2.03
1/128	3.23e-06	1.93	1/64	8.73e-06	1.98

Table 2: The error and order of convergence for LDG methods with space V^2 on uniform mesh and $V^2_{1/2}$ at $T=1$.

h	V^2 error	order	h	$V^2_{1/2}$ error	order
1/8	6.27e-05		1/4	9.85e-05	
1/16	1.48e-05	2.07	1/8	1.17e-05	3.06
1/32	3.65e-06	2.02	1/16	1.42e-06	3.03



$\gamma = \frac{1}{3}$

Consider

$$\frac{\partial u(x,t)}{\partial t} = \frac{9\sqrt{3}\Gamma(\frac{2}{3})}{8\pi}\frac{\partial^{\frac{4}{3}}u(x,t)}{\partial x^{\frac{4}{3}}} - e^{-t}(x^{\frac{4}{3}}+1) \tag{18}$$

in $\Omega=(0,1)$ with exact solution $u(x)=e^{-t}x^{\frac{4}{3}}$.

Table 3: The error and order of convergence for LDG methods solving problem with V^1 on uniform mesh and $V^1_{1/3}$ on the same mesh as before, at $T=0.1$ with cfl $=0.0001$.

h	V^1 error	order	h	$V^1_{1/3}$ error	order
1/8	1.44e-3		1/4	6.23e-03	
1/16	4.14e-4	1.80	1/8	1.76e-03	1.82
1/24	1.98e-4	1.81	1/12	7.47e-04	2.12
1/32	1.18e-4	1.79	1/16	4.22e-04	1.98
1/40	7.89e-5	1.81	1/20	2.64e-04	2.09
1/48	5.65e-5	1.82	1/24	1.85e-04	1.96

Table 4: The error and order of convergence for LDG methods solving problem with V^1 on uniform mesh and $V^1_{1/3}$ on the same mesh as before, at $T=0.1$ with cfl $=0.0001$.

h	V^2 error	order	h	$V^2_{1/3}$ error	order
1/8	2.10e-04		1/4	5.19e-04	
1/16	5.82e-05	1.85	1/8	5.55e-05	3.21
1/24	2.78e-05	1.82	1/12	1.58e-05	3.07
1/32	1.65e-05	1.81	1/16	7.25e-06	2.82
1/40	1.11e-05	1.74	1/20	2.15e-06	2.49