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Equitable Partitions in the Controllability of Undirected Signed Graphs*

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Abstract— **This paper studies the controllability of multi-agent networks with antagonistic interactions, where signed graphs are used to represent such networks. By analyzing and abstracting definitions of equitable partitions of unsigned graphs and signed graphs, new definitions of an equitable partition and an almost equitable partition in the controllability of undirected signed graphs are proposed, which are the natural extension of unsigned graphs. The reasonability of the definitions is discussed in detail. Under this new partition, a new general necessary condition is proposed for controllability of undirected signed graphs.**

I. INTRODUCTION

Recently, multi-agent networks have been studied extensively ([12]- [26]). Equitable partitions and almost equitable partitions have been proposed and used to study the controllability of cooperative multi-agent networks [1], [2], [3]. The controllability of multi-agent networks was first introduced in [3] and algebraic conditions were proposed in [3] and [4]. Various concepts and properties of graph partitions are used to study the controllability of multi-agent networks, such as equitable partition [5], relaxed equitable partition [6], [7], weight-balanced partition [8], etc.

While in the controllability study of antagonistic multi-agent networks whose information interactions among agents are usually described by signed graphs, there are only a few results basing on graph partitions [9]; Moreover, the connectivity matrix is defined as the sum of absolute values of weights in the diagonal terms for convergence purposes [10], [11]. It is unnecessary to demand the connectivity matrix defined as the above form in the study of controllability of graphs because there are no necessary relations between convergence and controllability. Thus we give a new form of connective matrix to discuss the controllability of undirected signed graphs without considering convergence. By analyzing and abstracting definitions of equitable partitions of unsigned graphs and signed graphs, we not only propose new definitions of equitable partitions but also discuss its reasonability in detail. The new definitions proposed in this paper are the natural extension of unsigned graphs. The definition of a generalized almost equitable partition in [9] is a special case of the definition proposed in this paper; and the most important advantage of new partitions is that it overcomes the shortcomings of the existing definitions which are only valid for certain class of graphs. Under the new partition proposed in this paper, a necessary condition of controllability is given.

The following discussion all assume that there is no self-loop in the graph and the weight is 1 or -1.

II. NOTATIONS

Throughout this paper, we consider a signed graph $\mathbb{G} = (\mathscr{G}, \theta)$ consisting of an unsigned graph $\mathscr{G} = (V, \varepsilon, A)$ and a signal mapping $\theta : \varepsilon \longrightarrow \{+,-\}$, where $\mathscr{G}$ is said to be the underlying graph of $\mathbb{G}$. The node set $V = \{1,2,...,n\}$ and the edge set $\varepsilon = \varepsilon^+ \bigcup \varepsilon^-$, where $\varepsilon^+ = \{(i,j)|a_{ij} = 1\}$ and $\varepsilon^- = \{(i,j)|a_{ij} = -1\}$ denote the sets of positive and negative edges, respectively. Given $\mathbb{G}$, denote adjacency matrix $A = [a_{ij}]$ and connective matrix $C = diag\{c_i\}$, where $c_i = \sum_{j=1}^{n} a_{ij}$. Then the Laplacian of $\mathbb{G}$ is defined as $L = C - A$.

For a k-partition $\pi = \{C_1, C_2, ..., C_k\}$, P is the characteristic matrix of π,

$$p_{ij} = \begin{cases} 1 & i \in C_j \\ 0 & \text{else} \end{cases}$$

The generalized Laplacian matrix L^π is denoted as the $k \times k$ matrix such that

$$(L^\pi)_{ij} = \begin{cases} -d_{ij} & i \neq j \\ \sum\limits_{i \neq j} d_{ij} & i = j \end{cases}$$

where $d_{ij} = \sum\limits_{r \in C_j} a_{sr}, s \in C_i$ (see [2]).

The vector space generated by the columns of the matrix P is denoted by im(P).

III. NEW DEFINITIONS OF EQUITABLE PARTITIONS IN THE CONTROLLABILITY OF UNDIRECTED SIGNED GRAPHS

A. Equitable Partitions of Undirected Unsigned Graph

Definition 1: [1] For a graph $\mathscr{G}$, A partition π of $V(\mathscr{G})$ with cells $C_1, ..., C_r$ is equitable if the number of neighbours in C_j of a vertex u in C_i is a constant b_{ij}, independent of u.

Remark 1: Definition 1 shows that every vertex in C_i has the same number of neighbours in C_j, which can be abstracted to the following mathematical relationship:

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$$\sum_{r\in C_j} a_{sr} = \sum_{w\in C_j} a_{tw}, \forall s,t \in C_i, i,j = 1,...,k. \tag{1}$$

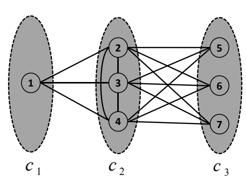
$(r,s),(w,t) \in \varepsilon$. If (1) is satisfied in the case of $i = j$ and $i \neq j$, then the partition is called an equitable partition (EP). If (1) is satisfied only when $i \neq j$, which is the following definition 2.

Definition 2: [2] For a graph $\mathscr{G}$, a partition $\pi = \{C_1, C_2, ..., C_k\}$ is said to be an almost equitable partition (AEP) if for any distinct cell C_i and C_j, every vertex in C_i has the same number of neighbors in C_j.

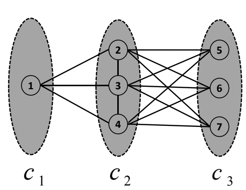
Remark 2: The relationship (1) plays a critical role in the equitable partition definition of undirected signed graph, which is the reason that we discuss EP/AEP of unsigned graphs in this section.

Remark 3: From the above definition we can see that the almost equitable partition doesn't consider the inner connection relations in the same cell. Results deriving from undirected graph that based on the almost equitable partition have not taken the inner connection relations in the same cell into consideration, which shows that the inner structures in the same cell have no influence on the discussion of the controllability.

Remark 4: To construct an almost equitable partition, a necessary condition must be satisfied for any two connected cells: $mk = nl, 1 \leq l \leq m$, where m, n is respectively the vertex number of the two cells, and k, l are positive integers.



(a) an equitable partition



(b) an almost equitable partition

Fig. 1. Two different partitions of undirected graphs.

Fig. 1 gives two different partitions of undirected graphs: (a) is an equitable partition, which studies the vertex relations not only between different cells but also in the same cell; (b) is an almost equitable partition, which only considers the vertex relations between different cells.

The following observation is straightforward.

Property 1: The controllability of the undirected unsigned graph basing on equitable partitions is the same as that under the restriction of (1).

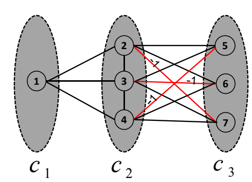
Proof: If Definition 1 and Definition 2 are satisfied, then (1) is fulfilled; vice versa, if (1) holds, Definition 1 is naturally satisfied, and when $i \neq j$, Definition 2 is also satisfied.

B. The Definition of a Generalized Almost Equitable Partition of Undirected Signed Graphs

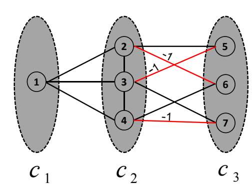
Definition 3: [9] $\pi = \{C_1..., C_k\}$ is called a generalized almost equitable partition (GAEP) if for every pair of distinct $i, j \in \{1,...,k\}$, there exists a non-negative number d_{ij+} such that any vertex in C_i has d_{ij+} positive neighbors in C_j and for every pair of $i, j \in \{1,...,k\}$(not necessarily distinct), there exists a non-negative number d_{ij-}such that any vertex in C_i has d_{ij-} negative neighbors in C_j.

Remark 5: Definition 3 shows that every vertex in C_i has the same positive edge number d_{ij+} in $C_j (i \neq j)$ and the same negative edge number d_{ij-} in C_j(i is not necessarily distinct from j), however, d_{ij+} is not necessarily equal to d_{ij-}.

Remark 6: Definition 3 also satisfies (1). Every vertex in C_i has the same positive edge numbers d_{ij+} and negative edge numbers d_{ij-} in $C_j (i \neq j)$, thus $d_{ij+} - d_{ij-}$ is equal for every vertex in C_i, i.e. $\sum_{r\in C_j} a_{sr} = \sum_{w\in C_j} a_{tw}, \forall s,t \in C_i, i,j = 1,...,k, i \neq j$.



(a)



(b)

Fig. 2. Generalized almost equitable partitions.

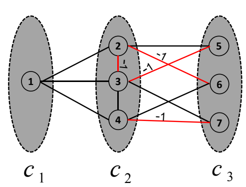
Fig. 2 gives two different GAEP of seven agents with identical cells: In figure (a), every vertex in C_2 has 2 positive edges and 1 negative edge in C_3 ; In figure (b), every vertex in C_2 has 1 positive edge and 1 negative edge in C_3. We take vertices of C_2 in figure (a) as an example: $d_{23+} - d_{23-} = 2 - 1 = 1, d_{33+} - d_{33-} = 2 - 1 = 1, d_{43+} - d_{43-} = 2 - 1 = 1$, i.e. $\sum_{r\in C_3} a_{sr} = \sum_{w\in C_3} a_{tw}, \forall s,t \in C_2$.

Remark 7: The constraint conditions to be GAEP are too strong, which brings about that only a class of signed graphs could be studied.

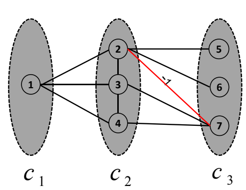
C. The Discussion of Controllability Definition of Undirected Signed Graphs

In the study of consensus problem of signed graphs, the connectivity matrix C is defined as the sum of absolute values of weights in the diagonal terms for convergence purposes [10], [11], i.e. $c_i = \sum_{j=1}^{n} |a_{ij}|$.

While in the study of controllability of a graph, if we define c_i as above, which would result in contradictions: the discussion of controllability has certain limitations and only a few signed graphs can be involved. For example, in [9], the discussion of controllability asks to add a restrictive condition that every vertex in C_i has the same negative edge number d_{ij-} in C_j (i is not necessarily distinct from j). If this restrictive condition is not satisfied, then the discussion of controllability cannot carry on.



(a) an example that cannot satisfy the above restrictive condition



(b) an example that does not satisfy Definition 3

Fig. 3. Two examples.

Fig. 3 (a) gives an example that cannot satisfy the above restrictive condition: every vertex in C_2 has different negative edge numbers in figure (a). Fig. 3 (b) gives an example that does not satisfy Definition 3: every vertex in C_2 has different numbers of positive edges and negative edges in C_3 in figure (b).

While in the study of controllability of a graph, it is unnecessary to demand the connectivity matrix defined as the above form. Thus we manage to consider the following definition of c_i in the connectivity matrix: $c_i = \sum_{j=1}^{n} a_{ij}$. How does the change of connection relation in the same cell affect the controllability of signed graph? Enlightened by relationship (1), we firstly consider the following definition:

Given a signed graph G and a k-partition $\pi = \{C_1, C_2, ..., C_k\}$, $\forall s,t \in C_i, i, j = 1, ..., k$

$$\sum_{r\in C_j} a_{sr} = \sum_{w\in C_j} a_{tw}, \forall s,t \in C_i, i, j = 1, ..., k. \tag{1}$$

$$c_s - \sum_{f\in C_i} a_{sf} = c_t - \sum_{h\in C_i} a_{th}, i = j \tag{2}$$

$$c_i = \sum_{j=1}^{n} a_{ij} \tag{3}$$

$$\sum_{r\in C_j} a_{sr} = \sum_{w\in C_j} a_{tw}, i \neq j \tag{4}$$

where $(r,s), (w,t), (f,s), (h,t) \in \varepsilon$.

If (1) is satisfied, the partition is said to be an equitable partition. If (1) and (2) are satisfied for any $s,t \in C_i, i, j = 1, ..., n, i \neq j$, the partition is said to ba an almost equitable partition, i.e., the partition satisfying (2) and (4) is said to ba an almost equitable partition.

If c_i is defined as (3), then (2) is equal to $\sum_{r\in C_j} a_{sr} = \sum_{w\in C_j} a_{tw}$. Thus, in this situation, (1) and (2) can be written into one equation (4) in the case of $i \neq j$. The controllability discussion of signed graphs can be carried on as that of unsigned graphs(see the next section).

If we define

$$c_i = \sum_{j=1}^{n} |a_{ij}|, \tag{5}$$

then according to the above definition, there are three contradictions:

Firstly, the property that an equitable partition must be an almost equitable partition cannot be guaranteed. Fig. 4 gives an example satisfying (1), which is an equitable partition. We take vertices in C_2 as an example. Since $c_2 - \sum_{f\in C_2} a_{2f} = 6 - 2 = 4$, $c_3 - \sum_{h\in C_2} a_{3h} = 4 - 2 = 2$, $c_4 - \sum_{h\in C_2} a_{4h} = 4 - 2 = 2$, (2) is not satisfied. Thus this partition isn't an almost equitable partition. The property that an equitable partition must be an almost equitable partition cannot be satisfied.

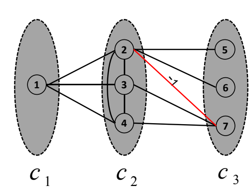


Fig. 4. A counter-example

Secondly, even if the above property is satisfied, the restriction condition (2) must be added when inner connective relations in the cell are considered. Otherwise, the discussion of controllability cannot carry on;

Thirdly, in the study of controllability under the above almost equitable partition, there are only a few graphs satisfying (2) and (5), which are included in the graphs satisfying (4). Thus the controllability of this case that can be discussed is in accordance with that of (4).

Taking Fig. 4 as an example, we just change the inner structure in the cell C_2 without changing the connective relationship between cells: there are 27 graphs satisfying (4), while there are only 4 graphs satisfying (2) and (5), which are included in these 24 graphs. For brevity, we only give 2 graphs satisfying (2) and (5) in Fig. 5.

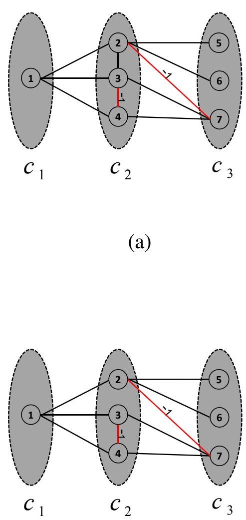


Fig. 5. Two graphs satisfying (2) and (5).

We take vertices of C_2 in Fig. 5 (a) as an example.

If c_i is defined as (5), then $c_2 - \sum_{f \in C_2} a_{2f} = 5 - 1 = 4$, $c_3 - \sum_{h \in C_2} a_{3h} = 4 - (1 - 1) = 4$, $c_4 - \sum_{h \in C_2} a_{4h} = 3 - (-1) = 4$, (2) is satisfied.

From the above analysis we see that although (5) is a necessary condition in the discussion of convergence for consensus, which has limitations in the discussion of controllability. Thus we adopt connective matrix C defined as (3).

Through the above analysis, we propose new definitions of equitable partitions in the controllability of undirected signed graphs as follows.

Definition 4: Given a signed graph $\mathscr{G}$ and a k-partition $\pi = \{C_1, C_2, ..., C_k\}$, if

$$\sum_{r \in C_j} a_{sr} = \sum_{w \in C_j} a_{tw}, \forall s, t \in C_i, i, j = 1, ..., k. \qquad (1)$$

is satisfied, where$(r,s), (w,t) \in \varepsilon$, the partition π is said to be an equitable partition (EP). If for any $s, t \in C_i, i \neq j, i, j = 1, ..., n$, (1) is satisfied, the partition π is said to be an almost equitable partition (AEP).

Remark 8: For Definition 4, there are three aspects to be explained.

1. Every vertex in C_i has the same sum of weights in $C_j, i \neq j$.

2. As is shown in Definition 4, an EP must be an AEP but not vice versa. Definition 4 is a natural extension of the EP/AEP from unsigned graphs to general signed graphs.

3. The definitions we give can discuss not only problems in [9] (see Remark 6) and the above special case but also other problems which cannot be discussed by [9] and the above special case, which is more extensive.

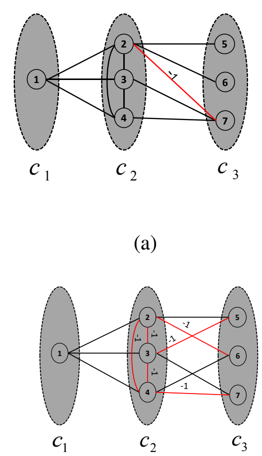


Fig. 6. Equitable partitions

Fig. 6 gives two different equitable partitions: (a) hasn't been discussed before; (b) is an EP under new definitions, while it is an AEP in [9]. Fig. 7 gives two different almost equitable partitions under our definitions: (a) cannot be involved in the AEP discussed in [9], and (b) is the special case satisfying relationship (2) and (5). Fig. 6 and Fig. 7 show that new definitions may discuss more kinds of signed graphs.

IV. CONTROLLABILITY OF UNDIRECTED SIGNED GRAPHS UNDER THE NEW AEP

A. System Description

Consider a multi-agent network with node set $V = \{1, .., n\}$. Without loss of generality, we assume that the first $m (m \leq n)$ agents in the network are selected as leaders and each leader is assigned with an external input. The other nodes are followers. Let $V_l = \{1, .., m\}$ and $V_f = V / V_l$ be the sets of leaders and followers, respectively. Let $u \in \mathbb{R}^m$ represent the control input. The dynamics of each agent are given by

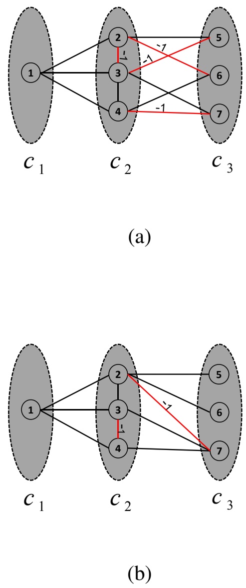


Fig. 7. Almost equitable partitions

$$\begin{cases} \dot{x}_i = -\sum\limits_{j\in N_i} a_{ij}(x_i - x_j) + u_i, & i \in V_l \\ \dot{x}_i = -\sum\limits_{j\in N_i} a_{ij}(x_i - x_j), & i \in V_f \end{cases} \tag{6}$$

where $x_i \in \mathbb{R}$ is the state of agent i and u_i is the i-th entry of the control vector u. The above dynamics of the agents can be expressed as

$$\dot{x} = -Lx + Bu \tag{7}$$

where $x = [x_1, ..., x_n]^T, u = [u_1, ..., u_n]^T$, $B = [e_1, ..., e_m] \in \mathbb{R}^{n\times m}$ and $L \in \mathbb{R}^{n\times n}$ is the Laplacian of $\mathbb{G}$. $L = C - A$, where $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the adjacency matrix of $\mathbb{G}$ and $C = diag\{c_i\} \in \mathbb{R}^{n\times n}$ is the connective matrix of $\mathbb{G}$, $c_i = \sum\limits_{j=1}^{n} a_{ij}$.

The sum of every row of Laplacian L is equal to zero, thus 0 is one of the eigenvalues of L and $[1, ..., 1]^T \in \mathbb{R}^n$ is one of the right eigenvectors of L corresponding to eigenvalue 0.

B. Controllability of Antagonistic Networks

Next, we employ AEP of Definition 4 to investigate how to characterize the controllability of an antagonistic network.

Definition 5: A partition π_l is said to be a leader-isolated AEP if π_l is a AEP and each leader is in a singleton class.

A partition π_l^* is called the coarsest leader-isolated AEP if for any leader-isolated π_l, each class in π_l is a subclass of some class in π_l^*.

Property 2: For each signed graph $\mathscr{G}$, there exists a unique coarsest leader-isolated AEP.

The proof is similar to that of Lemma 1 of [9], and hence is omitted. Fig. 8 gives an example of the coarsest leader-isolated AEP.

Next, we prove that im(P) is L-invariant.

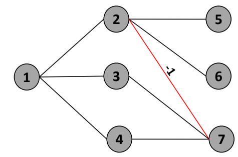


Fig. 8. A signal graph with a single leader 1. The coarest leader-isolated AEP is $\pi_l^* = \{\{1\}, \{2, 3, 4\}, \{5, 6, 7\}\}$

Lemma 1: Let $\mathscr{G}$ be a undirected signed graph, L is Laplacian matrix. Let $\pi = (C_1, \ldots, C_k)$ be a k-partition of $V(\mathscr{G})$ and P the characteristic matrix of π. Then π is an almost equitable k-partition if and only if there is a $k \times k$ matrix B such that

$$LP = PB.$$

If π is an almost equitable k-partition, then B is the generalized Laplacian matrix L^π.

Proof: Necessary: suppose π is a k-partition of $V(\mathscr{G})$ and satisfies 1, then each column of LP is the linear combination of the columns of P. Consequently, each column of LP is constant on the indices corresponding vertices in the same subsets of π. Suppose that $i \in C_j$, for $i \neq j$,

$$(LP)_{ij} = \sum_{r=1}^{n} L_{ir}P_{rj} = -\sum_{r\in V_j} a_{ir}P_{rj} = -\sum_{r\in V_j} a_{ir},$$
$$-\sum_{r\in V_j} a_{tr} = -\sum_{r\in V_j} a_{tr}P_{rj} = (LP)_{tj},$$

for every t in the same subset as i, leading to the conclusion that π is an almost equitable k-partition.

Sufficient: suppose π is an almost equitable k-partition. Let us start by considering the matrix product PL^π, and suppose that $i \in C_j$, then the entries of the line $(PL^\pi)_i$ are given by

$$\begin{aligned} row_i(PL^\pi) &= row_j(L^\pi) \\ &= (-d_{j1}, \cdots, -d_{j,j-1}, \sum_{r\neq j} d_{jr}, -d_{j,j+1}, \cdots, -d_{jk}) \end{aligned}$$

$$\begin{aligned} row_i(LP) = (&-\sum_{r\in V_1} a_{ir}, \cdots, -\sum_{r\in V_{j-1}} a_{ir}, -\sum_{r=1}^{n} a_{ir} - \sum_{r\in V_j} a_{ir}, \\ &-\sum_{r\in V_{j+1}} a_{ir}, \cdots, -\sum_{r\in V_k} a_{ir}) \\ = (&-d_{j1}, \cdots, -d_{j,j-1}, \sum_{r\neq j} d_{jr}, -d_{j,j+1}, \cdots, -d_{jk}) \end{aligned}$$

and thus

$$LP = PL^\pi.$$

■

Proposition 1: Given a signed graph $\mathscr{G}$, let L, P and L^π as defined in Lemma 1, then (λ, u) is an eigenpair of L^π if and only if (λ, Pu) is an eigenpair of L. Moreover, im(P) is L-invariant.

Lemma 2: For any leader-isolated π_l, the controllable subspace K satisfies $K \subseteq im(P_{\pi_l})$.

Proof: From the system (7), each column of the matrix B is a column of P, thus $im(B) \subseteq im(P)$. For any leader-isolated π_l, $im(B) \subseteq im(P_{\pi_l})$. From lemma 1 we know that im(P) is L-invariant, then the controllable subspace K satisfies

$$\begin{aligned} K &= im(B) + L \times im(B) + \cdots + L^{n-1} \times im(B) \\ &\subseteq im(P_{\pi_l}) + L \times im(P_{\pi_l}) + \cdots + L^{n-1} \times im(P_{\pi_l}) \\ &= im(P_{\pi_l}). \end{aligned}$$

■

Here, the operator "+" represents the union of two spaces. Lemma 2 gives an upper bound on the controllable subspace based on the leader-isolated AEP.

From lemma 2, we can directly get the following theorem:

Theorem 1: The controllable subspace K of the system (7) satisfies $K \subseteq im(P_{\pi_l^*})$, where π_l^* is the coarsest leader-isolated AEP.

Based on Theorem 1, we propose a necessary condition for the controllability of the system (7).

Proposition 2: If the system (7) is controllable, then each class in the coarsest leader-isolated AEP π_l^* is a singleton class.

Proof: If there exists a non-singleton class in the coarsest leader-isolated AEP, then the dimension of $im(P_{\pi_l^*})$ is less than n. From Theorem 1 we know that the dimension of the controllable subspace K of the system (7) is less than n, thus the system (7) is uncontrollable, which contradicts with the condition that the system (7) is controllable. ■

V. CONCLUSION

In this paper, we only consider the controllability of antagonistic networks without considering its convergence. Under this premise, a new connective matrix is given and new definitions of EP/AEP are proposed reasonably. A new necessary condition for controllability of antagonistic networks is proposed under the new partitions. The future work is to investigate the the controllability of antagonistic networks using spectral analysis of the system matrix under the new partitions.

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