



Randomized Transmission Protocols for Protection against Jamming Attacks in Multi-Agent Consensus[☆]

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ABSTRACT

Multi-agent consensus under jamming attacks is investigated. Specifically, inter-agent communications over a network are assumed to fail at certain times due to jamming of transmissions by a malicious attacker. A new stochastic communication protocol is proposed to achieve finite-time practical consensus between agents. In this protocol, communication attempt times of agents are randomized and unknown by the attacker until after the agents make their communication attempts. Through a probabilistic analysis, we show that the proposed communication protocol, when combined with a stochastic ternary control law, allows agents to achieve consensus regardless of the frequency of attacks. We demonstrate the efficacy of our results by considering two different strategies of the jamming attacker: a deterministic attack strategy and a more malicious communication-aware attack strategy.

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1. Introduction

Nowadays, control systems heavily utilize information and communication technologies. Especially, the Internet of Things is becoming widespread and remote sensing/control operations can now take place over wireless networks. With these new developments, the risk of cyber attacks against control systems is also increasing. Communication channels used in control systems are vulnerable to cyber attacks and ensuring cyber security of control systems has become a very important challenge (Sandberg, Amin, & Johansson, 2015).

Networked control systems are threatened by different types of cyber attacks. For instance, on a vulnerable network, measurement and control data can be altered by a malicious attacker (Fawzi, Tabuada, & Diggavi, 2014). In certain situations, attackers can even inject false data into the system without being noticed (Mo, Garone, Casavola, & Sinopoli, 2010). These attacks

require the attacker to be knowledgeable about the system dynamics. In the context of multi-agent systems, the presence of faulty or even malicious agents not following the given protocols may affect the global behavior of the overall system. There is a rich history in computer science on the development of resilient consensus algorithms (e.g., Lynch, 1996; Azadmanesh & Kieckhafer, 2002). Recently, this problem has gained interest in systems and control as well (Dibaji & Ishii, 2017; Dibaji, Ishii, & Tempo, 2018; LeBlanc, Zhang, Koutsoukos, & Sundaram, 2013; Tseng & Vaidya, 2015).

On the other hand, attackers who have limited information about the control system can resort to denial-of-service (DoS) attacks to prevent communication over networks. For instance, malicious routers in a network may intentionally drop measurement and control data (Awerbuch, Curtmola, Holmer, Nita-Rotaru, & Rubens, 2008; Mahmoud & Shen, 2014). Moreover, denial-of-service on wireless networks can also happen in the form of jamming attacks. A jamming attacker can block the transmissions on a wireless channel by emitting strong interference signals (Pelechrinis, Iliofotou, & Krishnamurthy, 2011; Xu, Trappe, Zhang, & Wood, 2005). Recently, researchers explored the effect of jamming and other types of denial-of-service attacks on networked control systems (Cetinkaya, Ishii, & Hayakawa, 2017, 2018; Cetinkaya, Ishii & Hayakawa, 2019; De Persis & Tesi, 2016; Feng & Tesi, 2017; Shisheh-Foroush & Martínez, 2016). Moreover, the effect of jamming on multi-agent consensus has also been explored (Senejohnny, Tesi, & De Persis, 2015, 2017).

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One of the main challenges in studying the multi-agent consensus problem under jamming attacks is that the attacker's actions cannot be known a priori. To account for the uncertainty in the generation of attacks, the works (Senejohnny et al., 2015, 2017) characterized jamming attacks through their average duration and frequency. It is shown there that multi-agent consensus can be achieved if the duration and the frequency of attacks satisfy certain conditions. Specifically, these works consider a self-triggered control framework, where each agent attempts to communicate with its neighbors and update its local control input only when a triggering condition is satisfied. For consensus, it is required that the ratio of the duration of the attacks to the total time is less than one. This ensures that the jamming does not span the entire time. Note that under the self-triggering framework, the communication attempt times for the agents are deterministic. Thus, an attacker who is knowledgeable on the multi-agent system can determine those time instants. This allows the attacker to block the communication by turning on the jamming attack very briefly at those instants without violating the duration condition. To avoid this issue, a restriction on the attack frequency becomes necessary. Specifically, the frequency of the attacks is required to be less than the frequency of the communication attempts by the agents.

Motivated by the discussion above, our goal in this paper is to investigate attack scenarios where the jamming is turned on and off very frequently. Our main contribution is a new stochastic consensus framework to deal with those attack scenarios. In our framework, we use the ternary control laws previously used in De Persis and Frasca (2013), Senejohnny et al. (2015, 2017). However, instead of the self-triggering method utilized in those works, we propose a stochastic communication protocol that can achieve consensus regardless of the frequency of the attacks. In this protocol, each agent attempts to communicate with its neighbors at random time instants. These time instants are hence unknown by the attacker.

We consider two attack strategies that are restricted by their average duration but *not* by their frequency. In the first strategy, the starting time and the duration of the jamming attacks are deterministic and do not depend on whether the agents try to communicate. On the other hand, in the second strategy the attacker is aware of the communication attempts of the agents and can preserve energy by turning off jamming right after a communication attempt is blocked. We show that in both strategies, our proposed stochastic communication protocol guarantees infinitely many successful communications in the long run. Furthermore, by using a probabilistic analysis, we show that almost-sure finite-time practical consensus is achieved regardless of attack frequency as long as the average ratio of attack durations is less than hundred-percent.

Our approach for analyzing the consensus under jamming differs largely from those in the literature. In particular, for the deterministic communication strategy proposed in Senejohnny et al. (2015, 2017), bounds on attack frequency can be used for establishing an upper-bound for the interval between two consecutive successful communication times of an agent. Here in this paper, such an upper-bound is not available and there is a positive probability that any finite number of consecutive communication attempts can be blocked by a jamming attacker. This difference is due to the fact that we do not consider a bound for attack frequencies and our communication protocol involves randomization of transmission times. We also note that although there are several works that deal with random connectivity issues and randomly switching graph topologies in multi-agent systems (e.g., Tahbaz-Salehi & Jadbabaie, 2010, Zhang & Tian, 2010, You, Li, & Xie, 2013), the analysis techniques in this paper are completely different from those works due to our approach of intentionally

randomizing the inter-agent communication times to mitigate jamming attacks which occur at uncertain times.

Our analysis for consensus relies on first establishing that under randomized transmissions, all agents can communicate with their neighbors infinitely many times in the long run. This is shown for the deterministic and the communication-aware attacks using different techniques. In the case of deterministic attacks, the independence of attacks and communication attempts plays an important role. Another big role is played by the uniform distribution of random communication attempt times. On the other hand, in the case of communication-aware attacks, the timing of attacks depends on all previous history of the communication times of agents. In the analysis of this case, we construct a filtration that represents the progression of the actions of the agents and those of the attacker. By utilizing this filtration, we show that our protocol can achieve a positive probability of at least one successful inter-agent transmission during carefully selected sufficiently long intervals spanning the time domain. We then utilize the monotone-convergence theorem for sets to show that even in communication-aware attacks, each agent can make infinitely many successful communications in the long run. This result allows us to show that with suitable choice of control parameters, each agent would be able to select appropriate control actions and apply them long enough to reach consensus in finite time.

In this paper, we show that randomization in inter-agent communications enables agents to reach consensus regardless of the frequency of jamming attacks. In recent works, randomization in communication has been exploited in different ways. For instance, randomized gossip algorithms is used in Boyd, Ghosh, Prabhakar, and Shah (2006) to allow networked operation under limited computation and communication resources. Furthermore, the work by Dibaji et al. (2018) introduced randomness in quantization as well as in communication times to increase resiliency against malicious nodes in multi-agent systems. Such advantages of using probabilistic methods have been found in resilient consensus in computer science and are often referred to as “impossibility results” (e.g., Lynch, 1996). In addition, random frequency hopping techniques are utilized by Navda, Bohra, Ganguly, and Rubenstein (2007) and Pöpper, Strasser, and Čapkun (2010) to mitigate jamming in wireless networks.

The paper is organized as follows. In Section 2, we explain the multi-agent consensus problem under jamming attacks. We propose a stochastic communication protocol and provide conditions for consensus under jamming attacks in Section 3. Then we discuss our protocol's efficacy under deterministic and communication-aware attacks in Section 4. In Section 5, we present numerical examples to demonstrate our results. Finally, we conclude the paper in Section 6.

We note that part of the results in Sections 3 and 4 appeared in our preliminary report (Kikuchi, Cetinkaya, Hayakawa, & Ishii, 2017) without proofs. In this paper, we provide complete proofs and more detailed discussions in Sections 3 and 4. Furthermore, new numerical examples are presented in Section 5.

The notation used in the paper is fairly standard. Specifically, we denote positive and nonnegative integers by \mathbb{N} and \mathbb{N}_0 , respectively. Furthermore, we use $(\cdot)^T$ to denote transpose, $|S|$ to denote the Lebesgue measure of a set $S \subset \mathbb{R}$, and $A \setminus B$ to denote the set of elements that belong to set A , but not to set B . The notations $\mathbb{P}[\cdot]$ and $\mathbb{E}[\cdot]$ respectively denote the probability and the expectation on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, we use $\mathbb{1}[E] : \Omega \rightarrow \{0, 1\}$ for the indicator of the event $E \in \mathcal{F}$, that is, $\mathbb{1}[E](\omega) = 1$, $\omega \in E$, and $\mathbb{1}[E](\omega) = 0$, $\omega \notin E$. To simplify the presentation, we omit the $\omega \in \Omega$ in the notation of random variables in certain equations.

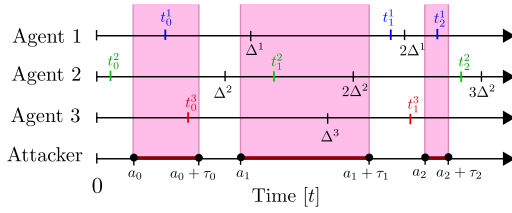


Fig. 1. Communication attempt times of 3 agents represented with $\{t_k^i\}_{k \in \mathbb{N}_0}$, $i \in \{1, 2, 3\}$, together with the jamming attacks represented with $\{a_k\}_{k \in \mathbb{N}_0}$ and $\{\tau_k\}_{k \in \mathbb{N}_0}$.

2. Multi-agent consensus under jamming attacks

In this paper, we investigate the consensus problem for a multi-agent system composed of n agents with scalar dynamics. The communication topology of the multi-agent system is represented by an undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ represents the set of nodes corresponding to the n agents, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges corresponding to the communication links between the agents. Let \mathcal{N}_i be the set of neighbors and d^i be the degree of node i . We use $L \in \mathbb{R}^{n \times n}$ to denote the Laplacian matrix associated with \mathcal{G} . Note that L is a symmetric matrix since \mathcal{G} is an undirected graph.

The evolution of the states of the multi-agent system is characterized through the scalar dynamics

$$\dot{x}^i(t) = u^i(t), \quad t \geq 0, \quad (1)$$

where $x^i(t)$ and $u^i(t)$ respectively denote the state and the local control input for agent i .

In this paper we design a piecewise-constant control input $u^i(t)$ for each agent i , as well as a protocol for the communication between the agents so that the agents achieve practical consensus, that is, $x(t) \triangleq [x^1(t) \ x^2(t) \ \dots \ x^n(t)]^T$ representing the agent states converges in finite time to a vector $x^* \in \mathbb{R}^n$ belonging to the approximate consensus set \mathcal{D}_ε with $\varepsilon > 0$ given by

$$\mathcal{D}_\varepsilon \triangleq \left\{ x \in \mathbb{R}^n : \left| \sum_{j \in \mathcal{N}_i} (x^j - x^i) \right| < \varepsilon, \quad i \in \mathcal{V} \right\}. \quad (2)$$

In what follows we first discuss the jamming attacks on the communication channels between the agents, and then we explain our proposed control and communication protocols for achieving consensus under jamming.

2.1. Jamming attacks

We consider scenarios where the communication channels between the agents are disabled by a jamming attacker. Specifically, we assume that when there is a jamming attack, communication on all links \mathcal{E} fail. This setup allows us to model attacks on a shared network that the agents use for communication. We note that the results presented in the paper can also be extended with small modifications to more general cases where the communication links are attacked individually by multiple attackers.

We follow the approach of De Persis and Tesi (2016), Senejohnny et al. (2015), Senejohnny et al. (2017) and use the sequences $\{a_k \geq 0\}_{k \in \mathbb{N}_0}$ and $\{\tau_k \geq 0\}_{k \in \mathbb{N}_0}$ to characterize the starting time instants and durations of the sequence of attacks, respectively. Specifically, a_k represents the starting time instant for the k th attack, and τ_k represents its duration. For each $k \in \mathbb{N}_0$, these scalars are assumed to satisfy $a_{k+1} > a_k + \tau_k$ (see the time sequence at the bottom of Fig. 1).

Now, let $\mathcal{A}_k \triangleq [a_k, a_k + \tau_k]$ denote the k th attack interval (or instant if $\tau_k = 0$), during which the attacker prevents all transmissions on the communication channels between the agents. For any time interval $[\tau, t] \subset [0, \infty)$, we use $\overline{\mathcal{A}}(\tau, t)$ to denote the set of times under jamming attacks, that is,

$$\overline{\mathcal{A}}(\tau, t) \triangleq \bigcup_{k \in \mathbb{N}_0} \mathcal{A}_k \cap [\tau, t]. \quad (3)$$

Furthermore, for the same interval $[\tau, t]$, $\overline{\mathcal{A}}^c(\tau, t)$ denotes the complement of $\overline{\mathcal{A}}(\tau, t)$ in the sense that it is the set of times where there is no attack, that is,

$$\overline{\mathcal{A}}^c(\tau, t) \triangleq [\tau, t] \setminus \overline{\mathcal{A}}(\tau, t). \quad (4)$$

Conducting jamming attacks requires energy for transmitting interfering radio signals (Xu et al., 2005). Thus, an attacker with limited resources would not be able to continuously jam the communication channels for a long time. In such cases, the attacker may repeat cycles of jamming and idling to preserve energy. The following assumption from Senejohnny et al. (2015) provides a characterization of the duration of jamming for various attack scenarios.

Assumption 2.1. There exist $\kappa \geq 0$ and $\rho \in (0, 1)$ such that for each $\tau \geq 0$ and $t \geq \tau$,

$$|\overline{\mathcal{A}}(\tau, t)| \leq \kappa + \rho(t - \tau), \quad (5)$$

where $|\overline{\mathcal{A}}(\tau, t)|$ represents the total duration of the attacks in the interval $[\tau, t]$.

Notice that (5) implies $\limsup_{t \rightarrow \infty} |\overline{\mathcal{A}}(0, t)|/t \leq \rho$. As a consequence, the scalar $\rho \in (0, 1)$ can be considered as an upper-bound on the ratio of the duration of attacks in long intervals, and it is related to the average energy used by the attacker. Under Assumption 2.1, jamming attacks are allowed to start at arbitrary time instants as long as (5) holds. Note also that the longest duration for continuous jamming allowed under Assumption 2.1 can be obtained as $\kappa/(1 - \rho)$. Here, the scalar κ can be selected to model the attacker's capabilities for continuous jamming.

2.2. Stochastic ternary control

To achieve consensus we employ the ternary control approach previously used in De Persis and Frasca (2013), Senejohnny et al. (2015), Senejohnny et al. (2017). However, instead of the self-triggering method utilized in those studies, we propose a stochastic communication protocol. In what follows, we first explain the control framework. We then discuss our communication protocol in detail in Section 3.

Each agent $i \in \mathcal{V}$ attempts communicating with its neighbors \mathcal{N}_i at times $t_k^i \geq 0$, $k \in \mathbb{N}_0$. In particular, at a communication attempt time t_k^i , agent i sends an information request to all of its neighbors and asks for their states. If there is no jamming at time t_k^i , then the neighbors of agent i receive the request and send back their states, which will be used in the update of agent i 's local control input. In the case where there is a jamming attack at time t_k^i , agent i cannot send/receive information.

We use $\varphi_k^i \in \{0, 1\}$ to indicate whether the communication attempt at time t_k^i is successful or not. In particular, $\varphi_k^i = 0$ indicates a failure at time t_k^i due to a jamming attack, and $\varphi_k^i = 1$ implies that agent i successfully communicates with its neighbors at time t_k^i . Observe that

$$\varphi_k^i = \mathbb{1}[t_k^i \in \overline{\mathcal{A}}^c(0, t_k^i)], \quad k \in \mathbb{N}_0. \quad (6)$$

In this paper communication attempt times t_k^i are random variables, and consequently, φ_k^i are also random variables.

Now, let $\text{ave}^i(t) \triangleq \sum_{j \in \mathcal{N}_i} (x^j(t) - x^i(t))$ and $\text{sign}_\varepsilon: \mathbb{R} \rightarrow \{-1, 0, 1\}$ be defined by

$$\text{sign}_\varepsilon(z) \triangleq \begin{cases} \text{sign}(z), & \text{if } |z| \geq \varepsilon, \\ 0, & \text{otherwise,} \end{cases}$$

with $\varepsilon > 0$ given in (2).

In our framework, agent i attempts to communicate with its neighbors at time t_k^i to compute $\text{sign}_\varepsilon(\text{ave}^i(t_k^i))$. Notice that $\text{sign}_\varepsilon(\text{ave}^i(t_k^i))$ is a ternary indicator of the location of neighboring agents' states with respect to agent i 's own state. Let

$$\tilde{u}_k^i \triangleq \begin{cases} \text{sign}_\varepsilon(\text{ave}^i(t_k^i)), & \text{if } \varphi_k^i = 1, \\ 0, & \text{otherwise,} \end{cases} \quad k \in \mathbb{N}_0. \quad (7)$$

Notice that $\tilde{u}_k^i = \varphi_k^i \text{sign}_\varepsilon(\text{ave}^i(t_k^i))$. Hence, $\tilde{u}_k^i \neq 0$ implies $\varphi_k^i = 1$ (i.e., agent i successfully communicates with its neighbors at time t_k^i) and $\text{sign}_\varepsilon(\text{ave}^i(t_k^i)) \in \{-1, 1\}$ (i.e., $|\text{ave}^i(t_k^i)| \geq \varepsilon$). Now, let $\hat{u}_k^i \triangleq \tilde{u}_k^i$ for $k = 0$ and

$$\hat{u}_k^i \triangleq \begin{cases} \tilde{u}_k^i, & \text{if } \tilde{u}_k^i \neq 0, \\ \tilde{u}_{k-1}^i, & \text{if } \tilde{u}_k^i = 0 \text{ and } t_k^i < t_{k-1}^i + T^i, \\ 0, & \text{if } \tilde{u}_k^i = 0 \text{ and } t_k^i \geq t_{k-1}^i + T^i, \end{cases} \quad (8)$$

for $k \in \mathbb{N}$, where $T^i > 0$. The scalar \hat{u}_k^i denotes the control input to be applied after time t_k^i .

To understand how the control input value is decided, we consider a few cases. First, if the communication attempt at time t_k^i is successful, then $\tilde{u}_k^i = \text{sign}_\varepsilon(\text{ave}^i(t_k^i))$, otherwise $\tilde{u}_k^i = 0$. If $\tilde{u}_k^i \neq 0$ (or equivalently if $\tilde{u}_k^i \in \{-1, 1\}$), then the control input value is set as $\hat{u}_k^i = \tilde{u}_k^i$. On the other hand, if $\tilde{u}_k^i = 0$ and the times t_k^i and t_{k-1}^i are sufficiently close so that $t_k^i < t_{k-1}^i + T^i$, then \tilde{u}_{k-1}^i is applied as a control input (i.e., $\hat{u}_k^i = \tilde{u}_{k-1}^i$). Otherwise, if $\tilde{u}_k^i = 0$ and $t_k^i \geq t_{k-1}^i + T^i$, then the control input is set to $\hat{u}_k^i = 0$.

The control input \hat{u}_k^i is applied during the interval $[t_k^i, \hat{t}_k^i]$, where $\hat{t}_k^i \triangleq \min\{t_{k+1}^i, \tilde{t}_k^i\}$ with $\tilde{t}_k^i \triangleq t_k^i + T^i$ for $k = 0$, and

$$\tilde{t}_k^i \triangleq \begin{cases} t_k^i + T^i, & \text{if } \tilde{u}_k^i \neq 0, \\ t_{k-1}^i + T^i, & \text{if } \tilde{u}_k^i = 0 \text{ and } t_k^i < t_{k-1}^i + T^i, \\ t_k^i + T^i, & \text{if } \tilde{u}_k^i = 0 \text{ and } t_k^i \geq t_{k-1}^i + T^i, \end{cases} \quad (9)$$

for $k \in \mathbb{N}$. Thus, the continuous-time control input $u^i(t)$ for each agent i is given by $u^i(t) = 0$ for $t \in [0, t_0^i]$, and

$$u^i(t) \triangleq \begin{cases} \hat{u}_k^i, & t \in [t_k^i, \hat{t}_k^i], \\ 0, & t \in [\hat{t}_k^i, t_{k+1}^i], \end{cases} \quad k \in \mathbb{N}_0. \quad (10)$$

As we will discuss in the following sections, the scalar $T^i > 0$ and the communication attempt time instants t_0^i, t_1^i, \dots in our framework are selected to satisfy $t_k^i + T^i < t_{k+2}^i$ for every $k \in \mathbb{N}_0$. Notice that at time t_k^i , the control input is set to either one of the values $\text{sign}_\varepsilon(\text{ave}^i(t_k^i))$, $\text{sign}_\varepsilon(\text{ave}^i(t_{k-1}^i))$, or 0, depending on the status of the communication attempts at t_k^i and t_{k-1}^i . Moreover, this control input is applied until time $\hat{t}_k^i = \min\{t_{k+1}^i, \tilde{t}_k^i\}$. In the cases where $\tilde{t}_k^i < t_{k+1}^i$, the control input is set to zero at \tilde{t}_k^i until the next communication attempt time t_{k+1}^i . Furthermore, in the cases where $t_{k+1}^i < \tilde{t}_k^i$, agent i first attempts a communication with its neighbors at time t_{k+1}^i , and then updates the control input based on (10) with k replaced by $k + 1$.

In our approach, communication attempt time instants are random variables with certain distributions and their values are selected by the agents using random number generators. In particular, for each agent i , the first communication attempt time t_0^i can be selected at time 0. Moreover, for each k , the communication attempt time t_{k+1}^i can be selected at time t_k^i . Clearly, agent i can check if $t = t_{k+1}^i$ at each time $t > t_k^i$. The control input can thus be updated following the description in (10). Notice

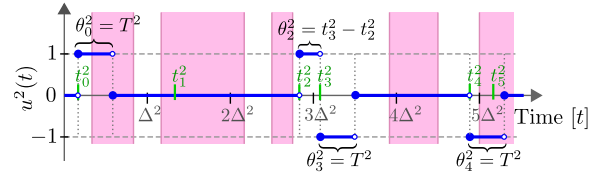


Fig. 2. Control input trajectory of agent 2 versus time. Jamming attack intervals are denoted with shaded (pink) regions.

that in certain cases, agent i keeps the same control input that is set at time t_k^i even after time t_{k+1}^i . This happens if the previous successful communication time t_k^i is sufficiently close to t_{k+1}^i and $\tilde{u}_{k+1}^i = 0$ (i.e., $\varphi_{k+1}^i \text{sign}_\varepsilon(\text{ave}^i(t_{k+1}^i)) = 0$). In such cases, we have $u(t) = \hat{u}_{k+1}^i = \hat{u}_k^i = \text{sign}_\varepsilon(\text{ave}^i(t_k^i))$ for $t \in [t_{k+1}^i, \hat{t}_{k+1}^i]$ with $\hat{t}_{k+1}^i = t_k^i + T^i$, which indicates that the control input $\text{sign}_\varepsilon(\text{ave}^i(t_k^i))$ is used until $t_k^i + T^i$ (which is after t_{k+1}^i).

Example 2.2. In Fig. 2, we show an example trajectory for the control input $u^2(t)$ of agent 2. In this example, we have $\varphi_0^2 = \varphi_2^2 = \varphi_3^2 = \varphi_4^2 = 1$ indicating that communication attempts at times $t_0^2, t_2^2, t_3^2, t_4^2$ are not attacked and thus successful. For this example scenario, at the successful communication time t_2^2 , agent 2 sets its control input to 1 based on the neighbor agents' states. This control input is then applied until time t_3^2 , at which another successful communication is made and the control input value is changed to -1 . The new control input value is kept constant until time $t_3^2 + T^2$ and set to zero at that time.

In our control approach, if agent i makes a successful communication at time t_k^i and sets a nonzero control input, then this input is kept constant at the latest until time $t_k^i + T^i$. If no new successful communication is made until that time, the control input is switched to zero at time $t_k^i + T^i$. If a successful communication is made before (i.e., $t_{k+1}^i < t_k^i + T^i$) and $|\text{ave}^i(t_{k+1}^i)| \geq \varepsilon$, then the control input value is set at time t_{k+1}^i to $\tilde{u}_{k+1}^i = \text{sign}_\varepsilon(\text{ave}^i(t_{k+1}^i))$. In the example scenario depicted in Fig. 2, the nonzero control inputs that are set at instants t_0^2, t_3^2 , and t_4^2 are applied for T^2 units of time. The control input is switched to zero at times $t_0^2 + T^2$, $t_3^2 + T^2$, and $t_4^2 + T^2$, since no new successful communication attempts yielding new nonzero control inputs are made during the intervals $[t_0^2, t_0^2 + T^2)$, $[t_3^2, t_3^2 + T^2)$, and $[t_4^2, t_4^2 + T^2)$.

To achieve consensus under the control law (10), it is important to design the times t_k^i , $k \in \mathbb{N}_0$, $i \in \mathcal{V}$, at which the agents attempt to communicate with each other. In this paper, we take a stochastic approach and design these times in Section 3 as random variables.

Remark 2.3. In Senejohnny et al. (2015), the communication attempt times t_k^i , $k \in \mathbb{N}_0$, $i \in \mathcal{V}$, are determined based on a deterministic self-triggering approach. There, the minimum interval between consecutive communication attempts for each agent is given by $\Delta^* > 0$. It is observed in Senejohnny et al. (2015) that if the attacker is allowed to attack at a frequency larger than the maximum frequency of communication attempts (given by $\frac{1}{\Delta^*}$), then all communication may be blocked even if Assumption 2.1 is satisfied. This is because as t_k^i are deterministic times, an attacker that is knowledgeable on how t_k^i are decided may be able to generate a strategy to pinpoint t_k^i with attacks of very short durations and preserve energy for the rest of the time. To avoid this problem, Senejohnny et al. (2015) considers the additional assumption that there exist $\eta \geq 0$ and $\sigma < \frac{1}{\Delta^*}$ such that

$$\mathcal{I}(\tau, t) \leq \eta + \sigma(t - \tau), \quad (11)$$

for all $\tau \geq 0$ and $t \geq \tau$, where $\mathcal{I}(\tau, t) \in \mathbb{N}_0$ denotes the number of attack intervals \mathcal{A}_n in the time frame $[\tau, t]$. The scalar $\sigma > 0$ in (11) represents an upper-bound on the attack frequency in the long run. Note that since $\sigma < \frac{1}{\Delta^*}$, the assumption (11) guarantees that the attack frequency in large time frames is smaller than the frequency of communication attempts. By utilizing ρ from (5) and σ from (11), the main result in Senejohnny et al. (2015) shows that consensus is achieved if

$$\rho + \sigma \Delta^* < 1. \quad (12)$$

In the following section, we propose a stochastic communication protocol, where communication attempt times t_k^i , $k \in \mathbb{N}_0$, $i \in \mathcal{V}$, are decided randomly. We show that in this case even if (11) and (12) are not satisfied due to high frequency attacks, consensus can still be achieved.

3. Stochastic communication protocol for consensus under jamming attacks

3.1. Stochastic communication protocol

We propose a communication protocol where each agent attempts to communicate with its neighbors at random times that are unknown to the attacker until the agents attempt communication at those times.

Definition 3.1 (Stochastic Communication Protocol). For each agent $i \in \mathcal{V}$, let $\Delta^i > 0$ be a fixed scalar, and set t_k^i , $k \in \mathbb{N}_0$, to be independent random variables such that t_k^i has uniform distribution on the interval $[k\Delta^i, (k+1)\Delta^i)$.

In this communication protocol, each agent i attempts to make transmission to its neighbors once in every Δ^i period. The communication attempt time t_k^i for the interval $[k\Delta^i, (k+1)\Delta^i)$ is selected randomly at time $k\Delta^i$ by agent i . Due to uniform distribution of t_k^i , $k \in \mathbb{N}_0$, we have $\mathbb{E}[t_{k+1}^i - t_k^i] = \Delta^i$, that is, the duration between consecutive communication attempts are Δ^i in expectation. However, note that $t_{k+1}^i - t_k^i \neq \Delta^i$, almost surely. We remark that the attacker is allowed to know how t_k^i is distributed, but not the value of t_k^i until the communication is attempted. In the example of Fig. 1 with 3 agents, the attacker is able to block the first communication attempts of agents 1 and 3. However, the first attempt of agent 2 is successful. Thus, for this example, $\varphi_0^1 = \varphi_0^3 = 0$ and $\varphi_0^2 = 1$.

3.2. Finite-time consensus analysis

In the ternary control approach, the speed of change in each agent's state is at most 1. Therefore, agents need to apply the ternary control input to their dynamics for sufficiently long durations to get closer to their neighbors. Notice that the duration of control input application is affected by the number of successful communication attempts. In particular, it is likely that an agent i would apply control input to its dynamics for a longer duration in total, if that agent makes many successful communications with its neighbors.

As we characterize in the control law (7)–(10), for each $k \in \mathbb{N}_0$ with $\varphi_k^i = 1$ and $|\text{ave}^i(t_k^i)| \geq \varepsilon$, the control input for agent i is set to a nonzero value at time t_k^i . As a preliminary discussion before our consensus analysis in Theorem 3.3, we first investigate the duration for which this nonzero control input is applied to agent i 's dynamics. To this end we define θ_k^i , $k \in \mathbb{N}_0$, by

$$\theta_k^i \triangleq \begin{cases} t_{k+1}^i - t_k^i, & \text{if } t_{k+1}^i - t_k^i < T^i \text{ and } \tilde{u}_{k+1}^i \neq 0, \\ T^i, & \text{otherwise.} \end{cases} \quad (13)$$

It follows from (7)–(10) that for each $k \in \mathbb{N}_0$ with $\varphi_k^i = 1$ and $|\text{ave}^i(t_k^i)| \geq \varepsilon$, the control input $\tilde{u}_k^i = \tilde{u}_k^i = \text{sign}_\varepsilon(\text{ave}^i(t_k^i))$ is applied for θ_k^i units of time after t_k^i . The applied control input is either 1 or -1 depending on the states of agent i and its neighbors at time t_k^i . Notice that the selected control input does not change during the interval $[t_k^i, t_k^i + \theta_k^i)$. We also note that θ_k^i is used for consensus analysis and its knowledge is not required for control purposes.

The scalar θ_k^i is always upper-bounded by the constant $T^i > 0$ in (13). In fact θ_k^i can either be $t_{k+1}^i - t_k^i$ or T^i , and the first case where $\theta_k^i = t_{k+1}^i - t_k^i$ happens only if the next communication attempt time t_{k+1}^i is close to t_k^i so that $t_{k+1}^i - t_k^i < T^i$. To explain this case, we first consider a successful communication time instant t_k^i with $|\text{ave}^i(t_k^i)| \geq \varepsilon$. Notice that $\varphi_k^i = 1$ and $\text{sign}_\varepsilon(\text{ave}^i(t_k^i)) \neq 0$, and therefore, the control input is set to a nonzero value at t_k^i (e.g., t_2^2 in Fig. 2). The next communication is attempted at time t_{k+1}^i . If the communication attempt at t_{k+1}^i is successful (i.e., $\varphi_{k+1}^i = 1$) and the neighboring agents' states are sufficiently far from agent i 's state such that $|\text{ave}^i(t_{k+1}^i)| \geq \varepsilon$, then $\tilde{u}_{k+1}^i \neq 0$, and thus the control input is updated to \tilde{u}_{k+1}^i at time t_{k+1}^i (e.g., t_3^2 in Fig. 2). If the communication attempt at time t_{k+1}^i fails (i.e., $\varphi_{k+1}^i = 0$) or $|\text{ave}^i(t_{k+1}^i)| < \varepsilon$, then $\theta_k^i = T^i$ and the control input is unchanged until time $t_k^i + T^i$. In the case where $t_{k+1}^i - t_k^i < T^i$, if $\varphi_{k+1}^i = 0$ or $|\text{ave}^i(t_{k+1}^i)| < \varepsilon$, then we have $\hat{u}_{k+1}^i = \tilde{u}_k^i = \text{sign}_\varepsilon(\text{ave}^i(t_k^i))$ by (7) and (8), and moreover, it follows from (9) that $\hat{t}_{k+1}^i = \min\{t_{k+2}^i, \tilde{t}_{k+1}^i\}$ with $\tilde{t}_{k+1}^i = t_k^i + T^i$.

By using properties of θ_k^i , we obtain the following key result, which establishes intervals in which the control input is guaranteed to be nonzero.

Lemma 3.2. Suppose $T^i \in (0, \Delta^i)$. Then, for every $k \in \mathbb{N}_0$ such that $\varphi_k^i = 1$ and $|\text{ave}^i(t_k^i)| \geq \varepsilon$, we have

$$|u^i(t)| = 1, \quad t \in [t_k^i, t_k^i + T^i). \quad (14)$$

Proof. By (7)–(10) and (13), we have

$$|u^i(t)| = 1, \quad t \in [t_k^i, t_k^i + \theta_k^i). \quad (15)$$

Notice that the value of θ_k^i can either be T^i or $t_{k+1}^i - t_k^i$. If $\theta_k^i = T^i$, then (15) implies (14).

On the other hand, if $\theta_k^i = t_{k+1}^i - t_k^i$, then we have

$$|u^i(t)| = 1, \quad t \in [t_k^i, t_{k+1}^i). \quad (16)$$

Moreover, (13) indicates that the case $\theta_k^i = t_{k+1}^i - t_k^i$ happens when $t_{k+1}^i - t_k^i \leq T^i$ and $\tilde{u}_{k+1}^i \neq 0$. In this case, since $\tilde{u}_{k+1}^i \neq 0$, it follows from (8)–(10) and (13) that

$$|u^i(t)| = 1, \quad t \in [t_{k+1}^i, t_{k+1}^i + \theta_{k+1}^i). \quad (17)$$

Now, as a consequence of (16) and (17), we get

$$|u^i(t)| = 1, \quad t \in [t_k^i, t_{k+1}^i + \theta_{k+1}^i). \quad (18)$$

Furthermore, since $t_{k+1}^i \geq t_k^i$ and $t_{k+2}^i - t_{k+1}^i \geq \Delta^i$,

$$\begin{aligned} t_{k+1}^i + \theta_{k+1}^i &\geq t_{k+1}^i + \min\{t_{k+2}^i - t_{k+1}^i, T^i\} \\ &= \min\{t_{k+2}^i, t_{k+1}^i + T^i\} \geq \min\{t_k^i + \Delta^i, t_k^i + T^i\}. \end{aligned}$$

Using this inequality together with $T^i < \Delta^i$, we obtain $t_{k+1}^i + \theta_{k+1}^i \geq t_k^i + T^i$. Hence, (14) follows from (18). \square

We are ready to present the first main result of this paper. The theorem below provides conditions under which the multi-agent system (1), (10) achieves consensus. In particular, they are given in terms of T^i of the control law, as well as Δ^i and φ_k^i , associated with the communication protocol.

Theorem 3.3. Consider the multi-agent system (1), (10) with $T^i \in (0, \min\{\frac{\varepsilon}{2d^i}, \Delta^i\})$ where $\varepsilon > 0$. Assume that under the stochastic communication protocol, it holds that

$$\mathbb{P}\left[\sum_{k=0}^{\infty} \varphi_k^i \geq M\right] = 1, \quad (19)$$

for all $M \in \mathbb{N}_0$ and $i \in \mathcal{V}$. Then $x(t)$ converges in finite time to a vector $x^* \in \mathbb{R}^n$ belonging to the set \mathcal{D}_ε given by (2), almost surely.

Proof. First, for each $i \in \mathcal{V}$, let $\mathcal{K}^i \triangleq \{k \in \mathbb{N}_0: \varphi_k^i = 1, |\text{ave}^i(t_k^i)| \geq \varepsilon\}$ and $\mathcal{T}^i \triangleq \bigcup_{k \in \mathcal{K}^i} [t_k^i, t_k^i + \theta_k^i)$. Notice that for a given $k \in \mathcal{K}^i$, the time t_k^i corresponds to a successful communication attempt time of agent i for which $|\text{ave}^i(t_k^i)| \geq \varepsilon$. Thus, \mathcal{T}^i represents the union of all nonzero control input application intervals after successful communication attempts of agent i .

Now, for $t \in \mathcal{T}^i$, let $\pi^i(t) \triangleq \max\{t_k^i: t_k^i \leq t, k \in \mathcal{K}^i\}$. Here, $\pi^i(t)$ corresponds to the agent i 's last successful communication attempt time t_k^i before time t such that $|\text{ave}^i(t_k^i)| \geq \varepsilon$. Furthermore, let

$$\mathcal{W}(t) \triangleq \{i \in \mathcal{V}: t \in \mathcal{T}^i\}, \quad t \geq 0. \quad (20)$$

Observe that $\mathcal{W}(t)$ corresponds to the set of agents that have a nonzero control input $u_i(t)$ at time t .

Now, let $V(x) \triangleq (1/2)x^T Lx$, $x \in \mathbb{R}^n$. By (1) and (10),

$$\begin{aligned} \dot{V}(x(t)) &= x^T(t) Lu(t) \\ &= - \sum_{i \in \mathcal{W}(t)} \text{ave}^i(t) \text{sign}_\varepsilon(\text{ave}^i(\pi^i(t))), \end{aligned} \quad (21)$$

for $t \geq 0$, where $u(t) \triangleq [u^1(t) \ u^2(t) \ \dots \ u^n(t)]^T$.

Our next goal is to show $\text{ave}^i(t) \text{sign}_\varepsilon(\text{ave}^i(\pi^i(t))) = |\text{ave}^i(t)|$, $i \in \mathcal{W}(t)$. To this end, we need to show that $\text{sign}_\varepsilon(\text{ave}^i(\pi^i(t))) = \text{sign}(\text{ave}^i(t))$, $i \in \mathcal{W}(t)$. To show this equivalence, we utilize an important property of ternary control law. Specifically, in the interval $[\pi^i(t), t]$, the state of an agent $j \in \mathcal{V}$ can change by at most $t - \pi^i$, that is, $x^j(t) \in [x^j(\pi^i(t)) - (t - \pi^i), x^j(\pi^i(t)) + (t - \pi^i)]$. By using this property and noting that agent i has d^i number of neighbors, we obtain that if $\text{ave}^i(\pi^i(t)) \geq \varepsilon$, then

$$\begin{aligned} &\sum_{j \in \mathcal{N}_i} (x^j(t) - x^j(t)) \\ &\geq \sum_{j \in \mathcal{N}_i} (x^j(\pi^i(t)) - (t - \pi^i) - (x^j(\pi^i(t)) + (t - \pi^i))) \\ &= \text{ave}^i(\pi^i(t)) - 2d^i(t - \pi^i(t)) \geq \varepsilon - 2d^i(t - \pi^i(t)). \end{aligned} \quad (22)$$

Since $t \in \mathcal{T}^i$ for $i \in \mathcal{W}(t)$ and $\theta_k^i \leq T^i$ for $k \in \mathbb{N}_0$, we have $t - \pi^i(t) < T^i$. Furthermore, since $T^i < \varepsilon/2d^i$, we have $t - \pi^i(t) < \varepsilon/2d^i$. Hence, it follows from (22) that if $\text{ave}^i(\pi^i(t)) \geq \varepsilon$, then

$$\sum_{j \in \mathcal{N}_i} (x^j(t) - x^j(t)) > \varepsilon - 2d^i \frac{\varepsilon}{2d^i} = 0. \quad (23)$$

Similarly, we can show that if $\text{ave}^i(\pi^i(t)) \leq -\varepsilon$, then

$$\sum_{j \in \mathcal{N}_i} (x^j(t) - x^j(t)) < -(\varepsilon - 2d^i \frac{\varepsilon}{2d^i}) = 0. \quad (24)$$

Noting that $|\text{ave}^i(\pi^i(t))| \geq \varepsilon$ for $i \in \mathcal{W}(t)$, we obtain from (23) and (24) that $\text{sign}_\varepsilon(\text{ave}^i(\pi^i(t))) = \text{sign}(\text{ave}^i(t))$, $i \in \mathcal{W}(t)$. Consequently, we have $\text{ave}^i(t) \text{sign}_\varepsilon(\text{ave}^i(\pi^i(t))) = |\text{ave}^i(t)|$, $i \in \mathcal{W}(t)$.

It then follows from (21) that

$$\dot{V}(x(t)) = - \sum_{i \in \mathcal{W}(t)} |\text{ave}^i(t)|, \quad t \geq 0.$$

Since $|\text{ave}^i(t)| \geq \varepsilon - 2d^i(t - \pi^i(t)) \geq \varepsilon - 2d^i T^i$, $i \in \mathcal{W}(t)$, we have

$$\begin{aligned} \dot{V}(x(t)) &\leq - \sum_{i \in \mathcal{W}(t)} (\varepsilon - 2d^i T^i) \leq -\alpha \sum_{i \in \mathcal{W}(t)} 1 \\ &= -\alpha \sum_{i \in \mathcal{V}} g^i(t), \quad t \geq 0, \end{aligned} \quad (25)$$

where $\alpha \triangleq \min_{i \in \mathcal{V}} (\varepsilon - 2d^i T^i)$ and $g^i(t) \triangleq \mathbb{1}[t \in \mathcal{T}^i]$. By integrating both sides of (25),

$$V(x(t)) \leq V(x(0)) - \alpha \int_0^t \sum_{i \in \mathcal{V}} g^i(s) ds, \quad t \geq 0. \quad (26)$$

Since $V(x(t)) \geq 0$, it follows from (26) that

$$\int_0^t \sum_{i \in \mathcal{V}} g^i(s) ds \leq \frac{V(x(0))}{\alpha}, \quad t \geq 0. \quad (27)$$

Now, let $H^i \triangleq \{\omega \in \Omega: \sum_{k=0}^{\infty} \varphi_k^i = \infty\}$ and $H \triangleq \bigcap_{i \in \mathcal{V}} H^i$. By (19), we have $\mathbb{P}[H^i] = 1$, $i \in \mathcal{V}$, and thus, $\mathbb{P}[H] = 1$. In what follows, we show that finite time approximate consensus is achieved for every $\omega \in H$. Define

$$\underline{k}^i(t) \triangleq \inf\{k \in \mathbb{N}_0: t_k^i > t, \varphi_k^i = 1\},$$

and $\underline{t}^i(t) \triangleq t_{\underline{k}^i(t)}^i$, $i \in \mathcal{V}$. Here, $\underline{t}^i(t)$ denotes the first successful communication time instant of agent i after time t . Notice that for every $\omega \in H$, we have $\underline{t}^i(t) < \infty$, $t \geq 0$.

Let $\underline{T} \triangleq \min_{i \in \mathcal{V}} T^i$ and $\bar{T} \triangleq \max_{i \in \mathcal{V}} T^i$. Moreover, let $\vartheta_0 \triangleq \bar{T}$, and

$$\vartheta_{k+1} \triangleq \max_{i \in \mathcal{V}} \underline{t}^i(\vartheta_k) + 2\bar{T}, \quad k \in \mathbb{N}_0.$$

First, we show that for every $\omega \in H$,

$$\int_{\vartheta_k - \bar{T}}^{\vartheta_{k+1} - \bar{T}} \sum_{i \in \mathcal{V}} g^i(s) ds \geq (\max_{i \in \mathcal{V}} \mathbb{1}[|\text{ave}^i(\vartheta_k)| \geq \varepsilon]) \underline{T}. \quad (28)$$

Notice that $\int_{\vartheta_k - \bar{T}}^{\vartheta_{k+1} - \bar{T}} \sum_{i \in \mathcal{V}} g^i(s) ds \geq 0$, since $g^i(s) \geq 0$ for $i \in \mathcal{V}$. Therefore, in the case where $\max_{i \in \mathcal{V}} \mathbb{1}[|\text{ave}^i(\vartheta_k)| \geq \varepsilon] = 0$, we have (28). Now, consider the case where $\max_{i \in \mathcal{V}} \mathbb{1}[|\text{ave}^i(\vartheta_k)| \geq \varepsilon] = 1$. In this case, there exists an agent $i^* \in \mathcal{V}$ such that $|\text{ave}^{i^*}(\vartheta_k)| \geq \varepsilon$. This agent makes a successful communication with its neighbors at time $\underline{t}^{i^*}(\vartheta_k) \in (\vartheta_k, \vartheta_{k+1})$. If $|\text{ave}^{i^*}(\underline{t}^{i^*}(\vartheta_k))| \geq \varepsilon$, then since $\varphi_{\underline{k}^{i^*}(\vartheta_k)}^{i^*} = 1$ and $T^{i^*} \in (0, \Delta^{i^*})$, by Lemma 3.2, we have $|u^{i^*}(t)| = 1$, $t \in [\underline{t}^{i^*}(\vartheta_k), \underline{t}^{i^*}(\vartheta_k) + T^{i^*}]$. Since $|u^{i^*}(t)| = 1$ implies $g^{i^*}(t) = 1$, we have $g^{i^*}(t) = 1$, $t \in [\underline{t}^{i^*}(\vartheta_k), \underline{t}^{i^*}(\vartheta_k) + T^{i^*}]$. As a result,

$$\begin{aligned} \int_{\vartheta_k - \bar{T}}^{\vartheta_{k+1} - \bar{T}} \sum_{i \in \mathcal{V}} g^i(s) ds &\geq \int_{\underline{t}^{i^*}(\vartheta_k)}^{\underline{t}^{i^*}(\vartheta_k) + T^{i^*}} \sum_{i \in \mathcal{V}} g^i(s) ds \\ &\geq \int_{\underline{t}^{i^*}(\vartheta_k)}^{\underline{t}^{i^*}(\vartheta_k) + T^{i^*}} g^{i^*}(s) ds = T^{i^*} \geq \underline{T}. \end{aligned} \quad (29)$$

On the other hand, if $|\text{ave}^{i^*}(\underline{t}^{i^*}(\vartheta_k))| < \varepsilon$, then there may be two cases: (1) agent i^* was in the process of changing its state at time ϑ_k and (2) the state of another agent $j^* \in \mathcal{N}_{i^*}$ became closer to the state of agent i^* in the interval between the times ϑ_k and $\underline{t}^{i^*}(\vartheta_k)$, since $|\text{ave}^{i^*}(\vartheta_k)| \geq \varepsilon$. Case 1 implies that for some $\hat{k} \in \mathbb{N}_0$, we have $\varphi_{\hat{k}}^{i^*} = 1$ and $|\text{ave}^{i^*}(t_{\hat{k}}^{i^*})| \geq \varepsilon$ with $t_{\hat{k}}^{i^*} \in [\vartheta_k - \bar{T}, \vartheta_k]$. Thus, by Lemma 3.2, we get $|u^{i^*}(t)| = 1$ for $t \in [t_{\hat{k}}^{i^*}, t_{\hat{k}}^{i^*} + T^{i^*}]$ implying $g^{i^*}(t) = 1$, $t \in [t_{\hat{k}}^{i^*}, t_{\hat{k}}^{i^*} + T^{i^*}]$. Consequently,

$$\int_{\vartheta_k - \bar{T}}^{\vartheta_{k+1} - \bar{T}} \sum_{i \in \mathcal{V}} g^i(s) ds \geq \int_{t_{\hat{k}}^{i^*}}^{t_{\hat{k}}^{i^*} + T^{i^*}} g^{i^*}(s) ds = T^{i^*} \geq \underline{T}. \quad (30)$$

Case 2 implies that agent j^* updated its state at least once in the interval $[\vartheta_k - \bar{T}, \underline{t}^{j^*}(\vartheta_k) + T^{j^*}]$. In other words, for some $\tilde{k} \in \mathbb{N}_0$, we have $\varphi_k^{j^*} = 1$ and $|\text{ave}^{j^*}(t_k^*)| \geq \varepsilon$ with $t_k^* \in [\vartheta_k - \bar{T}, \underline{t}^{j^*}(\vartheta_k) + T^{j^*}]$. It then follows by Lemma 3.2 that $|u^{j^*}(t)| = 1$ for $t \in [\underline{t}_k^{j^*}, \underline{t}_k^{j^*} + T^{j^*}]$. Therefore, $g^{j^*}(t) = 1$ for $t \in [\underline{t}_k^{j^*}, \underline{t}_k^{j^*} + T^{j^*}]$, and hence,

$$\int_{\vartheta_k - \bar{T}}^{\vartheta_{k+1} - \bar{T}} \sum_{i \in \mathcal{V}} g^i(s) ds \geq \int_{\underline{t}_k^{j^*}}^{\underline{t}_k^{j^*} + T^{j^*}} g^{j^*}(s) ds = T^{j^*} \geq \underline{T}. \quad (31)$$

The inequalities (29)–(31) that are obtained for different cases imply (28).

It follows from (27) and (28) that for every $\omega \in H$,

$$\begin{aligned} \sum_{k=0}^{\infty} \max_{i \in \mathcal{V}} \mathbb{1}[|\text{ave}^i(\vartheta_k)| \geq \varepsilon] &\leq \frac{1}{\underline{T}} \sum_{k=0}^{\infty} \int_{\vartheta_k - \bar{T}}^{\vartheta_{k+1} - \bar{T}} \sum_{i \in \mathcal{V}} g^i(s) ds \\ &= \int_0^{\infty} \sum_{i \in \mathcal{V}} g^i(s) ds \leq \frac{1}{\underline{T}} \frac{V(x(0))}{\alpha}, \end{aligned}$$

which implies that there exists $k^* \in \mathbb{N}_0$ such that for every $k \in \{k^*, k^* + 1, \dots\}$, $\max_{i \in \mathcal{V}} \mathbb{1}[|\text{ave}^i(\vartheta_k)| \geq \varepsilon] = 0$. Thus, for each agent $i \in \mathcal{V}$, we have $|\text{ave}^i(\vartheta_k)| < \varepsilon$ and $|\text{ave}^i(\vartheta_{k+1})| < \varepsilon$ for $k \in \{k^*, k^* + 1, \dots\}$. Therefore, $\max_{t \in [\vartheta_k, \vartheta_{k+1})} \max_{i \in \mathcal{V}} \mathbb{1}[|\text{ave}^i(t)| \geq \varepsilon] = 1$ implies that at least one agent $i^* \in \mathcal{V}$ started changing its state at some time $\tilde{t}_k^{i^*} \in (\vartheta_k, \vartheta_{k+1})$, and thus by Lemma 3.2, we have $|u^{i^*}(t)| = 1$ for $t \in [\tilde{t}_k^{i^*}, \tilde{t}_k^{i^*} + T^{i^*}]$ implying $g^{i^*}(t) = 1$, $t \in [\tilde{t}_k^{i^*}, \tilde{t}_k^{i^*} + T^{i^*}]$. Consequently, by using an argument similar to the one that we used for obtaining (28), we obtain

$$\begin{aligned} \max_{t \in [\vartheta_k, \vartheta_{k+1})} \max_{i \in \mathcal{V}} \mathbb{1}[|\text{ave}^i(t)| \geq \varepsilon] &\leq \frac{1}{\underline{T}} \int_{\vartheta_k}^{\vartheta_{k+1} + \bar{T}} \sum_{i \in \mathcal{V}} g^i(s) ds \leq \frac{1}{\underline{T}} \int_{\vartheta_k - \bar{T}}^{\vartheta_{k+2} - \bar{T}} \sum_{i \in \mathcal{V}} g^i(s) ds \\ &= \frac{1}{\underline{T}} \int_{\vartheta_k - \bar{T}}^{\vartheta_{k+1} - \bar{T}} \sum_{i \in \mathcal{V}} g^i(s) ds + \frac{1}{\underline{T}} \int_{\vartheta_{k+1} - \bar{T}}^{\vartheta_{k+2} - \bar{T}} \sum_{i \in \mathcal{V}} g^i(s) ds, \end{aligned}$$

for every $k \in \{k^*, k^* + 1, \dots\}$. Now, by using this inequality together with (27),

$$\begin{aligned} \sum_{k=k^*}^{\infty} \max_{t \in [\vartheta_k, \vartheta_{k+1})} \max_{i \in \mathcal{V}} \mathbb{1}[|\text{ave}^i(t)| \geq \varepsilon] &\leq \frac{1}{\underline{T}} \sum_{k=k^*}^{\infty} \int_{\vartheta_k - \bar{T}}^{\vartheta_{k+1} - \bar{T}} \sum_{i \in \mathcal{V}} g^i(s) ds \\ &\quad + \frac{1}{\underline{T}} \sum_{k=k^*+1}^{\infty} \int_{\vartheta_k - \bar{T}}^{\vartheta_{k+1} - \bar{T}} \sum_{i \in \mathcal{V}} g^i(s) ds \\ &\leq \frac{2}{\underline{T}} \int_0^{\infty} \sum_{i \in \mathcal{V}} g^i(s) ds \leq \frac{2}{\underline{T}} \frac{V(x(0))}{\alpha}, \end{aligned}$$

which implies that for every $\omega \in H$, there exists $\tilde{k}^* \in \{k^*, k^* + 1, \dots\}$ such that

$$\max_{t \in [\vartheta_k, \vartheta_{k+1})} \max_{i \in \mathcal{V}} \mathbb{1}[|\text{ave}^i(t)| \geq \varepsilon] = 0, \quad k \in \{\tilde{k}^*, \tilde{k}^* + 1, \dots\}.$$

Thus, for every $\omega \in H$, we have $|\text{ave}^i(t)| < \varepsilon$ for $i \in \mathcal{V}$ and $t \geq \vartheta_{\tilde{k}^*}$, implying almost sure finite time approximate consensus, since $\mathbb{P}[H] = 1$. \square

Condition (19) in Theorem 3.3 is concerned with the long run statistics of inter-agent communication attempts. Specifically, $\sum_{k=0}^{\infty} \varphi_k^i$ in (19) represents the total number of successful communication attempts of agent i . Under the condition (19), this number exceeds all nonnegative integers almost surely, indicating that each agent i can achieve infinitely many successful

communications with its neighbors in the long run. In Section 4, we will show that (19) holds and consensus can be achieved under different attack strategies that satisfy Assumption 2.1 on the average duration of jamming.

For the proof of Theorem 3.3, we utilize the function $(1/2)x^T(t)Lx(t)$ and explore its evolution. This approach is also utilized by De Persis and Frasca (2013), Senejohnny et al. (2015), and Senejohnny et al. (2017). However, there are some key differences. An important role is played by Lemma 3.2. Notice also that the analysis is facilitated by the choice of parameters $T^i \in (0, \min\{\frac{\varepsilon}{2d^i}, \Delta^i\})$, $i \in \mathcal{V}$. In particular, $T^i < \Delta^i$ allows us to utilize Lemma 3.2 to characterize the durations for which agent i has a nonzero control input. Furthermore, $T^i < \frac{\varepsilon}{2d^i}$ ensures a certain decay for the Lyapunov-like function $(1/2)x^T(t)Lx(t)$ after a successful communication by agent i .

In our ternary control approach, after each successful transmission at t_k^i , agent i may only apply a constant and bounded control input for a maximum duration of T^i . Thus the worst-case change in relative state positions $(\sum_{j \in \mathcal{N}_i} (x^j(t) - x^i(t)))$ during a control input application by agent i can be calculated, since all agents can only change their states with speed 1. This worst case is taken into account in the design of T^i for agent i . If instead of the ternary input, real-valued $\text{ave}^i(t_k^i)$ is used, then agent i would not be able to know how its neighbors may move. This is because the control input of a neighboring agent $j \in \mathcal{N}^i$ depends on other agents' states that agent i does not have access to. As a result, agent i may not be able to choose an appropriate control input application duration to guarantee a decrease in the Lyapunov-like function.

We note that the notion of *finite-time approximate consensus* that we utilize in Theorem 3.3 is similar to the notion of *finite-time contractive stability* discussed in Dorato (2006) for nonlinear dynamical systems. In particular, agent states that are initially outside the approximate consensus set \mathcal{D}_ε enter \mathcal{D}_ε in finite time and stay there, with probability one. This approximate consensus time is a random variable that depends on the initial agent states, the network topology, the communication attempt times, as well as the jamming times and durations.

Our control approach and hence the consensus time achieved with it have different characteristics from the ones in De Persis and Frasca (2013), Senejohnny et al. (2015), and Senejohnny et al. (2017). In those works, the control input values are changed at communication attempt time instants and at each communication attempt time t_k^i , agent i decides the next communication time based on the average distance $|\text{ave}^i(t_k^i)|$ from its neighbors. If this distance is large, then the duration until the next communication time becomes large. On the other hand, in our approach, when agent i makes a successful communication at time t_k^i and sets a nonzero control input, this input is not necessarily kept constant until the next communication time t_{k+1}^i . As we discuss above, the time t_{k+1}^i may be far from t_k^i . In such cases, the control input is set to zero at time $t_k^i + T^i$ and zero input is used until time t_{k+1}^i . With this approach, we are allowed to pick large Δ^i values in our randomized communication approach to increase the expected interval length between consecutive communication attempts and reduce the number of transmissions. Consensus is guaranteed as long as $T^i \in (0, \min\{\frac{\varepsilon}{2d^i}, \Delta^i\})$, $i \in \mathcal{V}$. In the absence of jamming attacks, the approximate consensus time achieved with our control approach can be larger than the time achieved with the approach of De Persis and Frasca (2013), since in our approach, control inputs of agents may be set to zero between consecutive communication times. We note that to reduce the expected total duration for which the control input is zero, T^i can be selected close to Δ^i , which also reduces the time when consensus is achieved.

4. Deterministic jamming and communication-aware jamming

In this section, we consider two different attack strategies that a jamming attacker may follow. We show that consensus can be achieved in both cases.

4.1. Consensus under deterministic attacks

First, we consider the attack strategy where the starting time and the duration of the jamming attacks do not depend on the time instants at which the agents try to communicate. In particular, concerning the sequences $\{a_k\}_{k \in \mathbb{N}_0}$ and $\{\tau_k\}_{k \in \mathbb{N}_0}$, we assume the following.

Assumption 4.1. The sequences $\{a_k\}_{k \in \mathbb{N}_0}$ and $\{\tau_k\}_{k \in \mathbb{N}_0}$, which characterize the jamming attacks, are decided deterministically, that is, for every $\omega \in \Omega$ and $k \in \mathbb{N}_0$,

$$a_k(\omega) = \bar{a}_k, \quad \tau_k(\omega) = \bar{\tau}_k, \quad (32)$$

where $\bar{a}_k \geq 0$ and $\bar{\tau}_k \geq 0$ for $k \in \mathbb{N}_0$ are fixed scalars.

Assumption 4.1 is useful to model scenarios where the attacker cannot detect the transmissions on the communication channels. Note that the attacker may still be knowledgeable on certain properties of the multi-agent system such as the number of agents, communication topology, as well as the scalars Δ^i , $i \in \mathcal{V}$, used in the communication protocol.

Our analysis relies on a few key definitions. First, let

$$\gamma^i \triangleq \min\{k \in \mathbb{N} : k\Delta^i > \kappa/(1-\rho)\}, \quad i \in \mathcal{V}. \quad (33)$$

Now define $\hat{\Delta}^i > 0$ and $\hat{\varphi}_k^i \in \{0, 1\}$, $k \in \mathbb{N}_0$, by

$$\hat{\Delta}^i \triangleq \gamma^i \Delta^i, \quad (34)$$

$$\hat{\varphi}_k^i \triangleq \begin{cases} 0, & \text{if } \varphi_{k\gamma^i}^i = 0, \dots, \varphi_{(k+1)\gamma^i-1}^i = 0, \\ 1, & \text{otherwise.} \end{cases} \quad (35)$$

With these definitions, $\hat{\Delta}^i$ is an integer multiple of Δ^i that is selected to be larger than $\kappa/(1-\rho)$. In the interval $[k\hat{\Delta}^i, (k+1)\hat{\Delta}^i)$, agent i makes γ^i number of communication attempts with its neighbors, and moreover, $\hat{\varphi}_k^i$ takes the value 0 if all of these attempts fail and 1 if one or more of these attempts are successful. We emphasize that γ^i , $\hat{\Delta}^i$, and $\hat{\varphi}_k^i$ are used only for the purpose of analysis, and their values are not needed in our stochastic communication protocol.

We now show that under Assumptions 2.1 and 4.1, agents can successfully communicate with their neighbors infinitely many times in the long run, almost surely.

Proposition 4.2. For any jamming attacks described by sequences $\{a_k\}_{k \in \mathbb{N}_0}$ and $\{\tau_k\}_{k \in \mathbb{N}_0}$ that satisfy Assumptions 2.1 and 4.1, the equality in (19) holds.

Proof. It follows from (35) that for every $i \in \mathcal{V}$,

$$\mathbb{P}\left[\sum_{k=0}^{\infty} \varphi_k^i \geq M\right] \geq \mathbb{P}\left[\sum_{k=0}^{\infty} \hat{\varphi}_k^i \geq M\right], \quad M \in \mathbb{N}_0. \quad (36)$$

In what follows, we show (19) by proving that

$$\mathbb{P}\left[\sum_{k=0}^{\infty} \hat{\varphi}_k^i \geq M\right] = 1, \quad M \in \mathbb{N}_0, \quad i \in \mathcal{V}. \quad (37)$$

First, let $B_k^i \triangleq \{\omega \in \Omega : \hat{\varphi}_k^i(\omega) = 1\}$, $k \in \mathbb{N}_0$, and $E \triangleq \bigcap_{l=0}^{\infty} \left(\bigcup_{k \geq l} B_k^i\right)$. Furthermore, for each $k \in \mathbb{N}_0$, let $\beta_k^i: \Omega \rightarrow \{0, 1, \dots, \gamma^i - 1\}$ be a random variable distributed according to

$\mathbb{P}[\beta_k^i = l] = 1/\gamma^i$ for each $l \in \{0, 1, \dots, \gamma^i - 1\}$, and define $\hat{t}_k^i \triangleq t_{k\gamma^i + \beta_k^i}^i$, $k \in \mathbb{N}_0$.

Note that \hat{t}_k^i is a random variable distributed uniformly in $[k\hat{\Delta}^i, (k+1)\hat{\Delta}^i)$. Since $B_k^i = \bigcup_{l=0}^{\gamma^i-1} \{t_{k\gamma^i+l}^i \in \mathcal{A}^c(k\hat{\Delta}^i, (k+1)\hat{\Delta}^i)\}$, we have $B_k^i \supseteq \{\hat{t}_k^i \in \mathcal{A}^c(k\hat{\Delta}^i, (k+1)\hat{\Delta}^i)\}$. Hence,

$$\begin{aligned} \mathbb{P}[B_k^i] &\geq \mathbb{P}[\hat{t}_k^i \in \mathcal{A}^c(k\hat{\Delta}^i, (k+1)\hat{\Delta}^i)] \\ &= \mathbb{P}[\hat{t}_k^i \notin \mathcal{A}(k\hat{\Delta}^i, (k+1)\hat{\Delta}^i)] \\ &= \frac{\hat{\Delta}^i - |\mathcal{A}(k\hat{\Delta}^i, (k+1)\hat{\Delta}^i)|}{\hat{\Delta}^i}, \quad k \in \mathbb{N}_0. \end{aligned}$$

By Assumption 2.1 and $\hat{\Delta}^i > \kappa/(1-\rho)$, we obtain

$$\mathbb{P}[B_k^i] \geq \frac{\hat{\Delta}^i - \kappa - \rho\hat{\Delta}^i}{\hat{\Delta}^i} = 1 - \rho - \frac{\kappa}{\hat{\Delta}^i} > 0, \quad k \in \mathbb{N}_0.$$

As a consequence, $\sum_{k=0}^{\infty} \mathbb{P}[B_k^i] = \infty$. Now, since t_0^i, t_1^i, \dots are independent and the sequences $\{a_k\}_{k \in \mathbb{N}_0}$ and $\{\tau_k\}_{k \in \mathbb{N}_0}$ are deterministic (by Assumption 4.1), the events B_0^i, B_1^i, \dots are independent. Therefore, it follows from $\sum_{k=0}^{\infty} \mathbb{P}[B_k^i] = \infty$ and the Borel–Cantelli Lemma (see Theorem 3.22 of Karr, 1993) that $\mathbb{P}[E] = 1$. Consequently, noting that $\{\omega \in \Omega : \sum_{k=0}^{\infty} \hat{\varphi}_k^i(\omega) \geq M\} \supseteq E$, we obtain $\mathbb{P}[\sum_{k=0}^{\infty} \hat{\varphi}_k^i \geq M] \geq \mathbb{P}[E]$. Hence, (37) holds. Finally, (19) follows from (36) and (37). \square

Proposition 4.2 implies that agents can achieve infinitely many successful communications with their neighbors in the long run under any deterministic attack strategy satisfying (5) in Assumption 2.1.

The proof of Proposition 4.2 relies on a few essential principles. First of all, we do not directly compute the successful communications in each Δ^i -length interval. Instead, we look at the longer $\hat{\Delta}^i$ -length intervals and compute how many of these intervals include successful communications. This is useful due to the fact that $\hat{\Delta}^i$ is chosen for the analysis to be larger than the longest possible duration $\kappa/(1-\rho)$ of a continuous jamming attack. Regardless of how large $\kappa \geq 0$ and $\rho \in (0, 1)$ can be, there always exists such a $\hat{\Delta}^i$ as given in (34). We remark again that since $\hat{\Delta}^i$ is needed only for the analysis, its value is not necessary for the multi-agent operation.

In the proof of Proposition 4.2, we also take advantage of the uniform distribution of the communication attempt times in each Δ^i -length interval. The probability of at least one successful communication in a $\hat{\Delta}^i$ -length interval $[k\hat{\Delta}^i, (k+1)\hat{\Delta}^i)$ is lower-bounded by the probability of an event that we construct in the proof. This is the event that one of communication attempt times that is selected uniformly randomly from the γ^i number of attempt times in the interval $[k\hat{\Delta}^i, (k+1)\hat{\Delta}^i)$ does not face a jamming attack. The uniform distribution property of the attempt times over the Δ^i -length intervals and thus the $\hat{\Delta}^i$ -length intervals allow derivation of the probability bound.

The proof of Proposition 4.2 also relies on the fact that the attacks are deterministic, and hence the attack times do not depend on the communication attempt times.

Next, by using Proposition 4.2 and Theorem 3.3, we show that under deterministic attacks, the multi-agent system (1), (10) achieves consensus in finite time, almost surely.

Theorem 4.3. Consider the multi-agent system (1), (10) with $T^i \in (0, \min\{\frac{\varepsilon}{2d^i}, \Delta^i\})$ where $\varepsilon > 0$. For any jamming attacks described by sequences $\{a_k\}_{k \in \mathbb{N}_0}$ and $\{\tau_k\}_{k \in \mathbb{N}_0}$ that satisfy Assumptions 2.1 and 4.1, $x(t)$ converges in finite time to a vector $x^* \in \mathbb{R}^n$ belonging to the set \mathcal{D}_ε given by (2), almost surely.

Proof. By Proposition 4.2, we have (19). Thus, the result follows from Theorem 3.3. \square

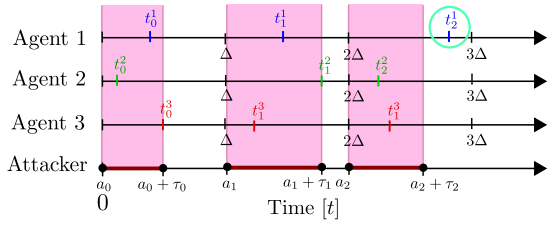


Fig. 3. Attack times for the communication-aware attack strategy.

We emphasize again that the result does not depend on the frequency of attacks. In particular, the proposed stochastic communication protocol allows us to deal with attack scenarios where the jamming is turned on and off very frequently.

4.2. Consensus under communication-aware attacks

We now explore an attack strategy where the attacker can sense communication attempts on the channel and turns the jamming on and off based on the activity of the agents. Here, we consider a simpler setup where $\Delta^i = \Delta$, $i \in \mathcal{V}$, with $\Delta > 0$. In this setup, k th communication attempt time of each agent is in the interval $[k\Delta, (k+1)\Delta)$, i.e., $t_k^i \in [k\Delta, (k+1)\Delta)$, $i \in \mathcal{V}$. Let $\bar{t}_k \triangleq \max_{i \in \mathcal{V}} \{t_k^i\}$.

We consider an attack strategy where the attacker knows about the communication protocol as well as Δ . The attacker generates an attack so that for each interval $[k\Delta, (k+1)\Delta)$,

- (i) the jamming attack starts from $t = k\Delta$,
- (ii) the jamming attack continues until time \bar{t}_k as long as [Assumption 2.1](#) is satisfied.

Under this strategy, for the interval $[k\Delta, (k+1)\Delta)$, the attacker turns off jamming right after all communication attempts are blocked. This allows the attacker to preserve energy to be used later. Furthermore, the attacker can block transmissions among the agents in a more thorough way than the deterministic strategy. Clearly, if the attacker wants to interfere at the randomly chosen communication attempt times for sure, he has to keep jamming until \bar{t}_k in each interval $[k\Delta, (k+1)\Delta)$.

To characterize $a_k, \tau_k, k \in \mathbb{N}_0$, for this strategy, first let

$$s_k \triangleq \max \left\{ s \in [0, \Delta] : |\bar{\mathcal{A}}(\tau, k\Delta)| + s \leq \kappa + \rho(k\Delta + s - \tau), \tau \in [0, k\Delta] \right\},$$

for $k \in \mathbb{N}_0$. Note that $s_k \in [0, \Delta]$ denotes the largest duration that a jamming attack starting at $k\Delta$ can last without violating the condition (5) in [Assumption 2.1](#). Now, let

$$\underline{s}_k \triangleq \min\{\bar{t}_k - k\Delta, s_k\}, \quad k \in \mathbb{N}_0.$$

Observe that \underline{s}_k gives the duration of the attack in the interval $[k\Delta, (k+1)\Delta)$ for this strategy. In particular, the jamming attack is turned on for $t \in [k\Delta, k\Delta + \underline{s}_k]$, and turned off for $t \in (k\Delta + \underline{s}_k, (k+1)\Delta)$. Hence, a_k, τ_k can be given by

$$a_k = k\Delta, \quad \tau_k = \underline{s}_k, \quad k \in \mathbb{N}_0. \quad (38)$$

Consequently, the set of time instants where communication is not possible in the interval $[k\Delta, (k+1)\Delta)$ is then given by the set $\mathcal{A}_k \triangleq [a_k, a_k + \tau_k] = [k\Delta, k\Delta + \underline{s}_k]$. We remark that the communication-aware jamming described by (38) satisfies (5) in [Assumption 2.1](#) by construction.

To illustrate the properties of the communication-aware attack strategy, we show an example attack scenario in [Fig. 3](#). Here, the attacker is able to block all communications that are

attempted in the interval $[0, \Delta)$ by jamming the network between times 0 and $\bar{t}_0 = \max_{i \in \mathcal{V}} t_0^i$. After blocking, the attacker turns off jamming and waits until the next interval $[\Delta, 2\Delta)$. In the interval $[\Delta, 2\Delta)$, the duration of the attack is relatively large, because agent 2 attempts communication towards the end of the interval. The attacker has to use large energy resources for this interval. As a result, the attacker cannot conduct an attack with very long duration in the interval $[2\Delta, 3\Delta)$, since [Assumption 2.1](#) holds only for a short duration of length τ_2 . And after that, the attacker has to turn off jamming to save resources. Therefore, in the interval $[2\Delta, 3\Delta)$, it happens that agent 1 can successfully communicate after the jamming is turned off. As a result, we have $\varphi_0^1 = 0, \varphi_1^1 = 0, \varphi_2^1 = 1$.

We show in the following that the agents achieve consensus under the communication-aware attack strategy given in (38) by using our proposed stochastic communication protocol.

Now, consider $\gamma^i \in \mathbb{N}$, $\hat{\Delta}^i > 0$, and $\hat{\varphi}_k^i \in \{0, 1\}$, $k \in \mathbb{N}_0$, $i \in \mathcal{V}$, given by (33)–(35) with $\Delta^i = \Delta$. Note that in this subsection, we have $\gamma^i = \gamma^j$, $\hat{\Delta}^i = \hat{\Delta}^j$ for all $i, j \in \mathcal{V}$, since $\Delta^i = \Delta^j$, $i, j \in \mathcal{V}$. We can thus simplify the notation by setting

$$\gamma \triangleq \gamma^1, \quad \hat{\Delta} \triangleq \hat{\Delta}^1. \quad (39)$$

The analysis of consensus under communication-aware jamming attacks is quite different from the case with deterministic jamming attacks. Here, we utilize a filtration representing the timing of the attacks and communication attempt instants. In particular, we consider the filtration $\{\mathcal{H}_k^i\}_{k \in \mathbb{N}_0}$, where \mathcal{H}_k^i denotes the σ -algebra generated by the random variables $a_0, a_1, \dots, a_{(k+1)\gamma-1}, \tau_0, \tau_1, \dots, \tau_{(k+1)\gamma-1}$, and $t_0^i, t_1^i, \dots, t_{(k+1)\gamma-1}^i$. Notice that $\varphi_j^i, j \in \{0, \dots, (k+1)\gamma-1\}$, are \mathcal{H}_k^i -measurable random variables, because φ_j^i is determined by a_j, τ_j , and t_j^i . Consequently, $\hat{\varphi}_j^i, j \in \{0, \dots, k\}$, are also \mathcal{H}_k^i -measurable. In the statement of the results below, in addition to $\{\mathcal{H}_k^i\}_{k \in \mathbb{N}_0}$, we also use the σ -algebra \mathcal{H}_{-1}^i , which we define as $\mathcal{H}_{-1}^i \triangleq \{\emptyset, \Omega\}$.

In what follows our main objective is to show that the agents can communicate with their neighbors infinitely many times in the long run satisfying (19), even though the network faces communication-aware jamming attacks described in (38). We show this by establishing several key results. First, we investigate the probability of successful communications in the intervals $[k\hat{\Delta}, (k+1)\hat{\Delta})$, $k \in \mathbb{N}_0$. The following result provides a positive lower-bound for the conditional probability of a successful communication in $[k\hat{\Delta}, (k+1)\hat{\Delta})$ given \mathcal{H}_{k-1}^i (i.e., the information on all previous intervals).

Lemma 4.4. Consider the stochastic communication protocol in [Definition 3.1](#). For the attacks given by (38), we have

$$\mathbb{P}[\hat{\varphi}_k^i = 1 \mid \mathcal{H}_{k-1}^i] \geq 2q^\gamma, \quad k \in \mathbb{N}_0, \quad i \in \mathcal{V}, \quad (40)$$

where

$$q \triangleq \frac{\tilde{\Delta}}{\Delta}, \quad \tilde{\Delta} \triangleq \frac{(1-\rho)(\hat{\Delta}-\Delta)}{\gamma+1}, \quad \Delta \triangleq \frac{\kappa}{1-\rho}. \quad (41)$$

Proof. See [Cetinkaya, Kikuchi, Hayakawa and Ishii \(2019\)](#). \square

In (40), the conditional probability term $\mathbb{P}[\hat{\varphi}_k^i = 1 \mid \mathcal{H}_{k-1}^i]$ is an \mathcal{H}_{k-1}^i -measurable random variable. Furthermore, its expectation gives the probability of having $\hat{\varphi}_k^i = 1$, i.e., $\mathbb{P}[\hat{\varphi}_k^i = 1] = \mathbb{E}[\mathbb{P}[\hat{\varphi}_k^i = 1 \mid \mathcal{H}_{k-1}^i]]$. Hence, [Lemma 4.4](#) implies $\mathbb{P}[\hat{\varphi}_k^i] > 0$, $k \in \mathbb{N}_0$. In other words, for each interval $[k\hat{\Delta}, (k+1)\hat{\Delta})$, our stochastic communication protocol guarantees a positive probability for a successful communication.

Lemma 4.5. Consider the attack strategy described by (38). Under the stochastic communication protocol in Definition 3.1, we have

$$\mathbb{P}[\bigcap_{k=0}^{N-1} \{\hat{\phi}_k^i = \bar{\varphi}_{k+1}\}] \leq \prod_{j=1}^N (1 - 2q^\gamma(1 - \bar{\varphi}_j)), \quad (42)$$

for $\bar{\varphi}_1, \bar{\varphi}_2, \dots, \bar{\varphi}_N \in \{0, 1\}$ and $N \in \mathbb{N}$.

Proof. See Cetinkaya, Kikuchi et al. (2019). \square

Lemma 4.5 provides an upper-bound for the probability of the event that the random variables $\hat{\phi}_0^i, \hat{\phi}_1^i, \dots, \hat{\phi}_{N-1}^i$ take the particular values $\bar{\varphi}_1, \bar{\varphi}_2, \dots, \bar{\varphi}_N \in \{0, 1\}$, respectively. This result is important because the upper-bound can be given in terms of the scalar q , which depends on ρ and κ characterizing the attacker's capabilities as well as the parameter Δ of the communication protocol. Notice that if the sequence $\bar{\varphi}_1, \bar{\varphi}_2, \dots, \bar{\varphi}_N$ is formed of m number of 1s and $N - m$ number of 0s, then the probability bound in (42) is given by $(1 - 2q^\gamma)^{N-m}$. The following result is built upon this observation.

Proposition 4.6. Consider the attack strategy described by (38). Under the stochastic communication protocol in Definition 3.1, we have

$$\mathbb{P}[\sum_{k=0}^{N-1} \hat{\phi}_k^i \geq M] \geq 1 - \sum_{m=0}^{M-1} \frac{N!}{m!(N-m)!} (1 - 2q^\gamma)^{N-m}, \quad (43)$$

for all $M \in \{0, 1, \dots, N\}$ and $N \in \mathbb{N}$.

Proof. First, we obtain

$$\begin{aligned} \mathbb{P}[\sum_{k=0}^{N-1} \hat{\phi}_k^i \geq M] &= 1 - \mathbb{P}[\bigcup_{m=0}^{M-1} \{\sum_{k=0}^{N-1} \hat{\phi}_k^i = m\}] \\ &\geq 1 - \sum_{m=0}^{M-1} \mathbb{P}[\sum_{k=0}^{N-1} \hat{\phi}_k^i = m]. \end{aligned} \quad (44)$$

Now, let $\Pi_{N,m}^i \triangleq \{\bar{\varphi} \in \{0, 1\}^N : \bar{\varphi}^T \bar{\varphi} = m\}$ for $m \in \{0, 1, \dots, M\}$ and $N \in \{M, M+1, \dots\}$. Notice that

$$\begin{aligned} \mathbb{P}[\sum_{k=0}^{N-1} \hat{\phi}_k^i = m] &= \mathbb{P}[\bigcup_{\bar{\varphi} \in \Pi_{N,m}^i} \bigcap_{k=0}^{N-1} \{\hat{\phi}_k^i = \bar{\varphi}_{k+1}\}] \\ &\leq \sum_{\bar{\varphi} \in \Pi_{N,m}^i} \mathbb{P}[\bigcap_{k=0}^{N-1} \{\hat{\phi}_k^i = \bar{\varphi}_{k+1}\}]. \end{aligned} \quad (45)$$

By using Lemma 4.5 we obtain from (45) that

$$\mathbb{P}[\sum_{k=0}^{N-1} \hat{\phi}_k^i = m] \leq \sum_{\bar{\varphi} \in \Pi_{N,m}^i} \prod_{j=1}^N (1 - 2q^\gamma(1 - \bar{\varphi}_j)). \quad (46)$$

Note that $\prod_{j=1}^N (1 - 2q^\gamma(1 - \bar{\varphi}_j)) = (1 - 2q^\gamma)^{N-m}$ for $\bar{\varphi} \in \Pi_{N,m}^i$. Furthermore, the set $\Pi_{N,m}^i$ has $\frac{N!}{m!(N-m)!}$ elements. Therefore, it follows from (46) that

$$\begin{aligned} \mathbb{P}[\sum_{k=0}^{N-1} \hat{\phi}_k^i = m] &\leq \sum_{\bar{\varphi} \in \Pi_{N,m}^i} (1 - 2q^\gamma)^{N-m} \\ &= \frac{N!}{m!(N-m)!} (1 - 2q^\gamma)^{N-m}. \end{aligned} \quad (47)$$

Finally, by using (44) and (47), we arrive at (43). \square

Proposition 4.6 provides a lower bound of the probability that agent i can communicate with its neighbors at least M times during the interval $[0, N\Delta]$. Notice that as N approaches ∞ , this lower bound approaches 1.

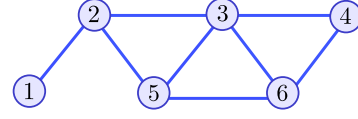


Fig. 4. Communication topology of the multi-agent system.

Theorem 4.7. Consider the attack strategy described by (38). Under the stochastic communication protocol in Definition 3.1, the equality in (19) holds.

Proof. Our initial goal is to show

$$\mathbb{P}[\sum_{k=0}^{\infty} \hat{\phi}_k^i \geq M] = 1, \quad M \in \mathbb{N}_0, \quad i \in \mathcal{V}. \quad (48)$$

To this end, first let $A_N \triangleq \{\omega \in \Omega : \sum_{k=0}^{N-1} \hat{\phi}_k^i \geq M\}$, $N \in \mathbb{N}$. Notice that $\mathbb{P}[A_N] = 0$ for $N < M$. For $N \geq M$, Proposition 4.6 implies

$$\mathbb{P}[A_N] \geq 1 - \sum_{m=0}^{M-1} \frac{N!}{m!(N-m)!} (1 - 2q^\gamma)^{N-m}. \quad (49)$$

Since $1 - 2q^\gamma < 1$, it follows from (49) that $\lim_{N \rightarrow \infty} \mathbb{P}[A_N] = 1$. The events A_N , $N \in \mathbb{N}$, satisfy $A_N \subseteq A_{N+1}$. Hence, by the monotone-convergence theorem for sets (see Section 1.10 in Williams, 1991),

$$\mathbb{P}[\sum_{k=0}^{\infty} \hat{\phi}_k^i \geq M] = \mathbb{P}[\lim_{N \rightarrow \infty} A_N] = \lim_{N \rightarrow \infty} \mathbb{P}[A_N] = 1. \quad (50)$$

Finally, since $\mathbb{P}[\sum_{k=0}^{\infty} \hat{\phi}_k^i \geq M] \geq \mathbb{P}[\sum_{k=0}^{\infty} \hat{\phi}_k^i \geq M]$, it follows from (50) that (19) holds. \square

Theorem 4.7 shows that the agents can communicate with their neighbors infinitely many times in the long run, even though the network is attacked by an attacker that follows the communication-aware attack strategy described in (38). The next theorem is the main result for the multi-agent system under communication-aware attacks.

Theorem 4.8. Consider the multi-agent system (1), (10) with $T^i \in (0, \min\{\frac{\varepsilon}{2d^i}, \Delta^i\})$ where $\varepsilon > 0$. For the attack strategy described by (38), $x(t)$ converges in finite time to a vector $x^* \in \mathbb{R}^n$ belonging to the set \mathcal{D}_ε given by (2), almost surely.

Proof. By Theorem 4.7, we have (19). Consequently, the result follows from Theorem 3.3. \square

So far we considered the consensus problem under both deterministic attacks and communication-aware attacks. In both cases, the randomness in the communication attempt times is the key property that enables consensus regardless of the frequency of jamming. A difference is that the attacker following the communication-aware attack strategy can sense the network activity and switch off the jamming attack right after blocking a communication attempt. This allows the attacker to preserve energy. This is further illustrated through numerical examples in the next section.

5. Numerical examples

In this section, we illustrate our results for the multi-agent system with $n = 6$ agents whose topology is shown in Fig. 4.

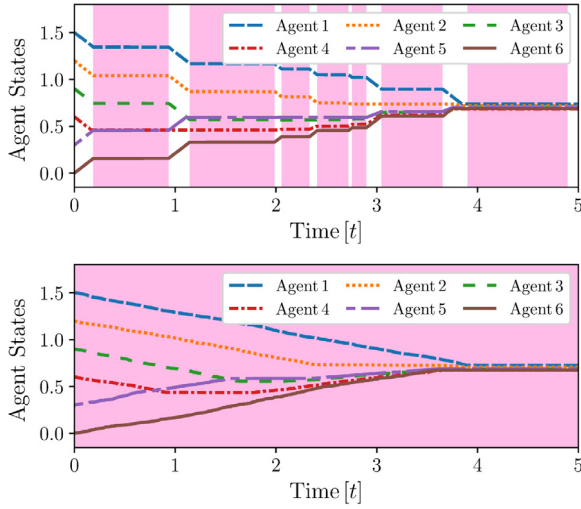


Fig. 5. Evolution of agent states under deterministic attacks with low frequency (top) and high frequency (bottom) settings.

5.1. Deterministic attacks

We first consider a deterministic attack scenario where the strategy of the attacker satisfies [Assumptions 2.1](#) and [4.1](#) with $\kappa = 0.2$, $\rho = 0.8$. We utilize our proposed stochastic communication protocol with $\Delta^i = 0.001 + 0.0001(i + 1)$, $i \in \mathcal{V}$. For the control laws [\(10\)](#), we choose $T^i = \Delta^i/1.01$, which satisfy $T^i \in (0, \min\{\frac{\varepsilon}{2d^i}, \Delta^i\})$ with $\varepsilon = 0.02$. Since [Proposition 4.2](#) implies [\(19\)](#), it follows from [Theorem 4.3](#) that the multi-agent system achieves consensus.

In the top part of [Fig. 5](#), we show sample paths of agent states under jamming attacks with low frequency. We see that consensus is achieved around the time $t = 3.85$. Each agent i attempts to communicate once at a random time instant at every Δ^i units of time. The agents keep their states constant during long jamming intervals.

The attack depicted in the top part of [Fig. 5](#) is of low frequency, as the jamming is turned on and off only 7 times during the interval $[0, 5]$. We also consider a high frequency case in the bottom plot of [Fig. 5](#), where jamming is turned on and off 7935 times during the interval $[0, 5]$, but the agent communication attempt times are the same as those in the top plot. Also in this case, the agents reach the consensus set \mathcal{D}_ε around the time $t = 3.91$. Both the low and the high frequency attacks in [Fig. 5](#) are generated randomly and independently of the communication attempt times of the agents. Through repeated simulations, we also observe that consensus is reached around the same time.

Next, we consider periodically generated jamming attacks with

$$a_k \triangleq \frac{k}{\sigma} + \frac{(1-\rho)}{\sigma}, \quad \tau_k \triangleq \frac{\rho}{\sigma}, \quad k \in \mathbb{N}_0, \quad (51)$$

where $\sigma > 0$ denotes the frequency of attacks (i.e., the number of attack intervals in 1 unit of time). Moreover, $\rho > 0$ indicates the ratio of the duration of attacks in each period. For each $\rho \in \{0.2, 0.5, 0.8\}$ and $\sigma \in \{10^1, 10^3, 10^5\}$ we repeat the simulation 50 times. For each simulation $j \in \{1, \dots, 50\}$, we calculate $t_c(j) \triangleq \inf\{t: x^i(t) \in \mathcal{D}_\varepsilon, i \in \mathcal{V}\}$, which is the time agents reach consensus. Then we obtain their mean $m_c > 0$ and standard deviation $s_c > 0$.

[Table 1](#) indicates that increasing the ratio ρ of the attack duration allows the attacker to delay the consensus. On the other

Table 1

Mean m_c and standard deviation s_c of consensus times for different values of ρ and σ in deterministic attacks [\(51\)](#).

ρ	$\sigma = 10^1$		$\sigma = 10^3$		$\sigma = 10^5$	
	m_c	s_c	m_c	s_c	m_c	s_c
0.2	1.143	0.005	1.108	0.015	1.110	0.013
0.5	1.822	0.006	1.663	0.029	1.682	0.035
0.8	4.532	0.035	3.962	0.086	4.010	0.107

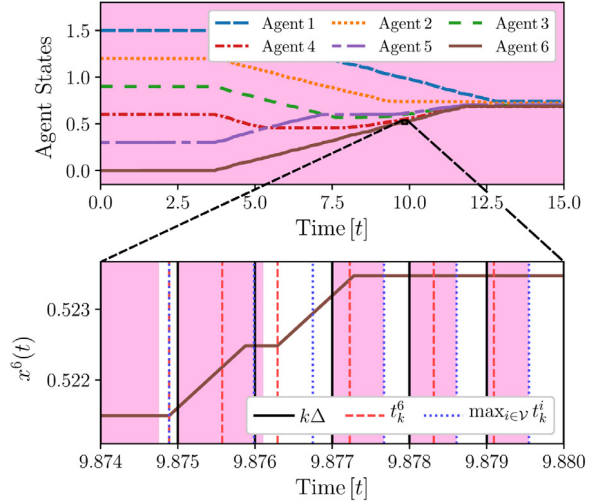


Fig. 6. Evolution of agent states under communication-aware attacks.

hand, consensus time is not influenced much by how frequent the attacks are. For each value of ρ , mean consensus time m_c is similar under all attack frequency settings $\sigma = 10^1, 10^3, 10^5$. Furthermore, consensus times are finite in all simulations and they do not show large deviation (i.e., s_c is small) in all cases. The cases with $\rho = 0.8$ indicate that periodic attacks and the attack timings shown in [Fig. 5](#) do not differ much in their effects on consensus times.

5.2. Communication-aware attacks

Next, we consider the scenario where the attacker follows the communication-aware attack strategy of [\(38\)](#) with the same parameters $\kappa = 0.2$ and $\rho = 0.8$ as in [Section 5.1](#).

In this scenario, the intervals for the communication are selected as $\Delta^i = \Delta = 0.001$, $i \in \mathcal{V}$. Similar to the deterministic case discussed above, for the control law [\(10\)](#), we choose $T^i = \Delta/1.01$, $i \in \mathcal{V}$, which satisfy $T^i \in (0, \min\{\frac{\varepsilon}{2d^i}, \Delta^i\})$ with $\varepsilon = 0.02$. Furthermore, [Theorem 4.7](#) implies that [\(19\)](#) holds. Therefore, it follows from [Theorem 4.8](#) that the multi-agent system with the stochastic communication protocol achieves consensus.

We show the evolution of the agent states in [Fig. 6](#). Notice that every communication attempt in the interval $[0, 3.18]$ is blocked by the attacker. However, the attacker's energy resources eventually become not sufficient. We observe in the enlarged plot in the bottom part of [Fig. 6](#) that some of the communication attempts cannot be blocked by the attacker and the agents eventually achieve consensus.

Even though the value of ρ is the same with $\rho = 0.8$ as in the deterministic attacks case of the previous example, the communication-aware attacks can be more malicious in the sense

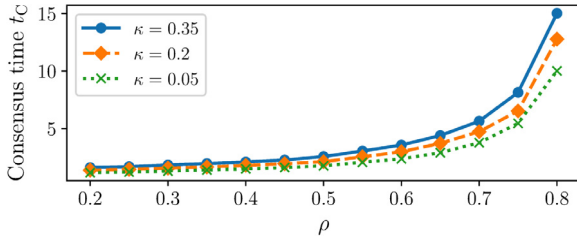


Fig. 7. Effect of the bound ρ of average attack duration ratio to consensus time t_c for different values of κ .

that they can delay consensus (compare Figs. 5 and 6). To further investigate how ρ and κ affect the consensus time, we run simulations with different values of ρ and κ but with the same communication attempt times used for constructing Fig. 6. We observe in Fig. 7 that consensus time t_c increases as ρ increases. The scalar $\kappa \geq 0$ also has an effect on the consensus time. In particular, increasing κ delays the consensus, since the duration for continuous jamming becomes larger.

We remark that in communication-aware attacks, the attacker turns jamming on and off once in every Δ -length intervals. Hence, the frequency of attacks is equal to the frequency of communication attempts. This case is outside the class of attacks considered previously in Senejohnny et al. (2015). On the other hand, the class of attacks under which our communication protocol allows consensus is not restricted by the frequency of attacks. Specifically, as long as the average ratio of the duration of attacks in the long run is bounded by $\rho < 1$, consensus can be achieved.

Remark 5.1. Although the randomized transmission approach expands the class of attacks under which consensus can be achieved, the number of communication attempts made by agents can be large compared to that in the self-triggered communication schemes of De Persis and Frasca (2013), Senejohnny et al. (2015), and Senejohnny et al. (2017). In the self-triggered communication schemes, each agent i computes its next communication time t_{k+1}^i based on neighboring agents' states at time t_k^i . If the states of neighboring agents are far from agent i 's state, then t_{k+1}^i takes a large value. This approach reduces the number of transmissions and keeps the communication costs low. In the randomized transmission scheme of this paper, the number of communication attempts made by agent i in a fixed interval $[0, t]$ can be reduced by increasing Δ^i in Definition 3.1. However, setting a large value to Δ^i results in slow convergence towards the consensus set, since each agent i applies control inputs only for at most T^i units of time between each communication attempt. The approximate consensus time achieved with our control approach can be larger than the time achieved with the approach of above-mentioned works, since in our approach, control inputs of agents may be set to zero between consecutive communication times.

To reduce the expected total time where the control inputs have zero value before reaching consensus, T^i can be selected close to Δ^i . This also reduces the time to reach consensus. We observe through simulations that with smaller T^i (given by $T^i = \Delta^i/2.1$) consensus is achieved at time $t_c = 22.92$. On the other hand, when T^i is larger ($T^i = \Delta^i/1.01$ as in the setting of this section) consensus is achieved for the same attack parameters at an earlier time $t_c = 12.79$ as shown in Fig. 6.

We note that a new hybrid self-triggered and randomized approach can be useful in reducing the communication loads while protecting against a large class of attacks and achieving faster consensus. In the hybrid approach, the neighboring agents'

states obtained at time t_k^i can be used by agent i for choosing the length Δ_{k+1}^i of the interval in which the next transmission attempt time t_{k+1}^i is randomly selected.

6. Conclusion

We proposed a stochastic communication protocol for multi-agent consensus under jamming attacks. In this protocol, agents attempt to exchange information with their neighbors at uniformly distributed random time instants. We showed that our proposed communication protocol guarantees consensus as long as the jamming attacks satisfy a certain condition on the average ratio of their duration. We demonstrated our results both for a deterministic attack strategy and a communication-aware attack strategy.

The analysis in this paper enables a natural extension to the case with multiple jamming attackers that can attack different links at different times. In such a problem setting, if the deterministic or the communication-aware attack for each communication link satisfies Assumption 2.1, then two agents can communicate over that link infinitely many times in the long run. This allows agents to achieve consensus through a modified control law where each agent can communicate with different neighbors at different times.

References

- Awerbuch, B., Curtmola, R., Holmer, D., Nita-Rotaru, C., & Rubens, H. (2008). ODSBR: An on-demand secure Byzantine resilient routing protocol for wireless ad hoc networks. *ACM Transactions on Information and System Security*, 10(4), 6.
- Azadmanesh, M. H., & Kieckhafer, R. M. (2002). Asynchronous approximate agreement in partially connected networks. *International Journal of Distributed and Parallel Systems*, 5(1), 26–34.
- Boyd, S., Ghosh, A., Prabhakar, B., & Shah, D. (2006). Randomized gossip algorithms. *IEEE/ACM Transactions on Networking*, 14, 2508–2530.
- Cetinkaya, A., Ishii, H., & Hayakawa, T. (2017). Networked control under random and malicious packet losses. *IEEE Transactions on Automatic Control*, 62(5), 2434–2449.
- Cetinkaya, A., Ishii, H., & Hayakawa, T. (2018). The effect of time-varying jamming interference on networked stabilization. *SIAM Journal on Control and Optimization*, 56, 2398–2435.
- Cetinkaya, A., Ishii, H., & Hayakawa, T. (2019). Analysis of stochastic switched systems with application to networked control under jamming attacks. *IEEE Trans. Autom. Control*, 64, 2013–2028.
- Cetinkaya, A., Kikuchi, K., Hayakawa, T., & Ishii, H. (2019). Randomized transmission protocols for protection against jamming attacks in multi-agent consensus. Online, <http://arxiv.org/abs/1802.01281>.
- De Persis, C., & Frasca, P. (2013). Robust self-triggered coordination with ternary controllers. *IEEE Transactions on Automatic Control*, 58(12), 3024–3038.
- De Persis, C., & Tesi, P. (2016). Networked control of nonlinear systems under denial-of-service. *Systems & Control Letters*, 96, 124–131.
- Dibaji, S. M., & Ishii, H. (2017). Resilient consensus of second-order agent networks: Asynchronous update rules with delays. *Automatica*, 81, 123–132.
- Dibaji, S. M., Ishii, H., & Tempo, R. (2018). Resilient randomized quantized consensus. *IEEE Transactions on Automatic Control*, 63(8), 2508–2522.
- Dorato, P. (2006). An overview of finite-time stability. In *Current Trends in Nonlinear Systems and Control* (pp. 185–194). Birkhäuser.
- Fawzi, H., Tabuada, P., & Diggavi, S. (2014). Secure estimation and control for cyber-physical systems under adversarial attacks. *IEEE Transactions on Automatic Control*, 59(6), 1454–1467.
- Feng, S., & Tesi, P. (2017). Resilient control under denial-of-service: Robust design. *Automatica*, 79, 42–51.
- Karr, A. (1993). *Probability*. Springer.
- Kikuchi, K., Cetinkaya, A., Hayakawa, T., & Ishii, H. (2017). Stochastic communication protocols for multi-agent consensus under jamming attacks. In *Proc. IEEE Conf. Dec. Control* (pp. 1657–1662).
- LeBlanc, H. J., Zhang, H., Koutsoukos, X., & Sundaram, S. (2013). Resilient asymptotic consensus in robust networks. *IEEE Journal on Selected Areas in Communications*, 31(4), 766–781.
- Lynch, N. A. (1996). *Distributed Algorithms*. Morgan Kaufmann.
- Mahmoud, M. M. E. A., & Shen, X. S. (2014). *Security for Multi-Hop Wireless Networks*. Springer.

- Mo, Y., Garone, W., Casavola, A., & Sinopoli, B. (2010). False data injection attacks against state estimation in wireless sensor networks. In *Proc. IEEE Conf. Dec. Control* (pp. 5967–5972).
- Navda, V., Bohra, A., Ganguly, S., & Rubenstein, D. (2007). Using channel hopping to increase 802.11 resilience to jamming attacks. In *Proc. IEEE INFOCOM* (pp. 2526–2530).
- Pelechrinis, K., Iliofotou, M., & Krishnamurthy, S. V. (2011). Denial of service attacks in wireless networks: The case of jammers. *IEEE Communications Surveys and Tutorials*, 13(2), 245–257.
- Pöpper, C., Strasser, M., & Čapkun, S. (2010). Anti-jamming broadcast communication using uncoordinated spread spectrum techniques. *IEEE Journal on Selected Areas in Communications*, 28(5), 703–715.
- Sandberg, H., Amin, S., & Johansson, K. H. (2015). Cyberphysical security in networked control systems: An introduction to the issue. *IEEE Control Systems Magazine*, 35(1), 20–23.
- Senejohnny, D., Tesi, P., & De Persis, C. (2015). Self-triggered coordination over a shared network under denial-of-service. In *Proc. IEEE Conf. Dec. Control* (pp. 3469–3474).
- Senejohnny, D., Tesi, P., & De Persis, C. (2017). A jamming-resilient algorithm for self-triggered network coordination. *IEEE Transactions on Control of Network Systems*, 5(3), 981–990.
- Shisheh-Foroush, H., & Martínez, S. (2016). On triggering control of single-input linear systems under pulse-width modulated DoS signals. *SIAM Journal on Control and Optimization*, 54(6), 3084–3105.
- Tahbaz-Salehi, A., & Jadbabaie, A. (2010). Consensus over ergodic stationary graph processes. *IEEE Transactions on Automatic Control*, 55(1), 225–230.
- Tseng, L., & Vaidya, N. H. (2015). Fault-tolerant consensus in directed graphs. *Proc. ACM Symp. Princ. Distrib. Computing* (pp. 451–460).
- Williams, D. (1991). *Probability with Martingales*. Cambridge University Press.
- Xu, W., Trappe, W., Zhang, Y., & Wood, T. (2005). The feasibility of launching and detecting jamming attacks in wireless networks. In *Proc. ACM Int. Symp. Mobile Ad Hoc Network. Computing* (pp. 46–57).
- You, K., Li, Z., & Xie, L. (2013). Consensus condition for linear multi-agent systems over randomly switching topologies. *Automatica*, 49(10), 3125–3132.
- Zhang, Y., & Tian, Y.-P. (2010). Consensus of data-sampled multi-agent systems with random communication delay and packet loss. *IEEE Transactions on Automatic Control*, 55(4), 939–943.



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