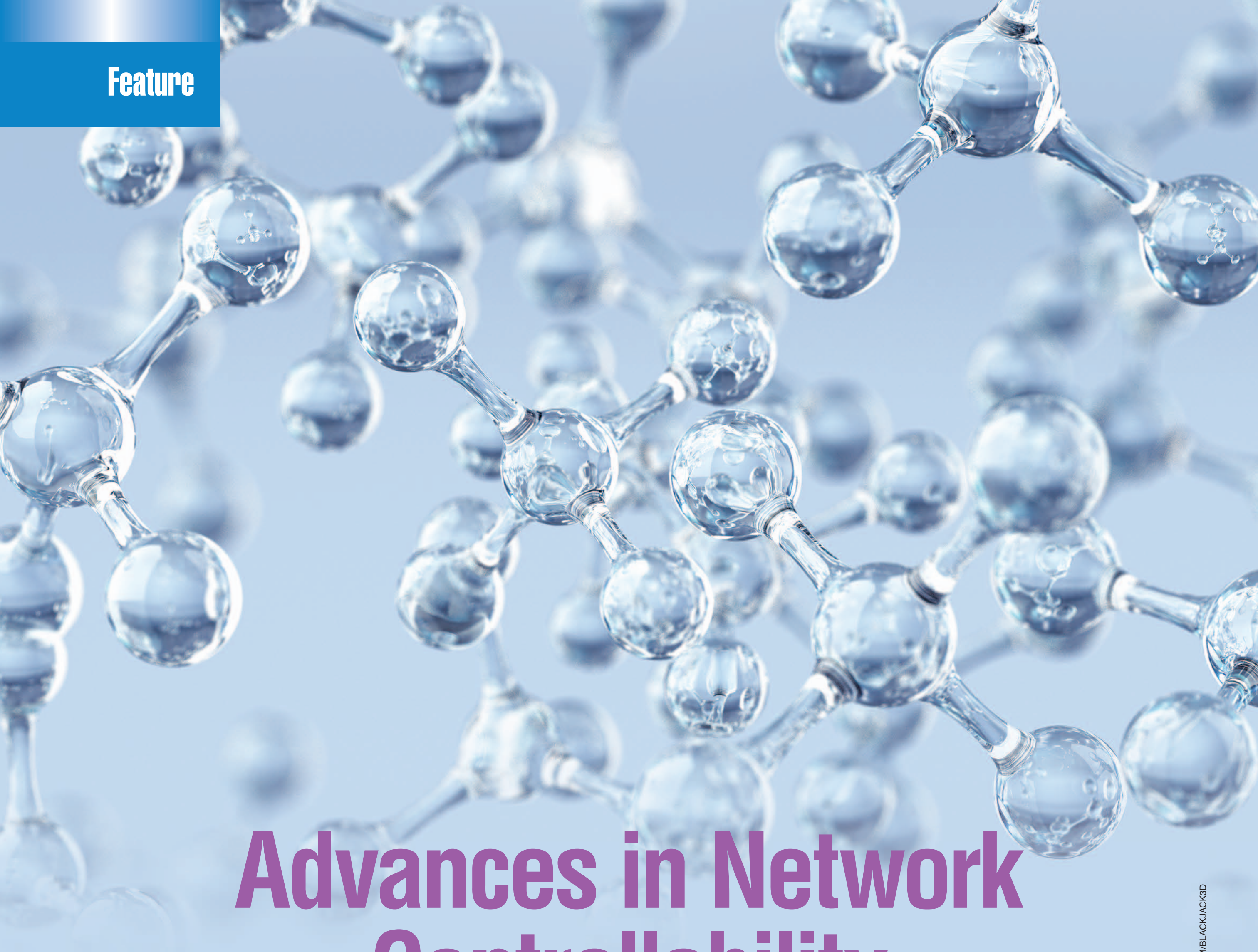
Feature



©ISTOCKPHOTO.COM/BLACKJACK3D

Advances in Network Controllability

Linying Xiang, Fei Chen, Wei Ren, and Guanrong Chen

Abstract

The last decade has seen an explosion of research in network controllability. The present article reviews some basic concepts, significant progress, important results and recent advances in the studies of the controllability of networked linear dynamical systems, regarding the relationship of the network topology, node-system dynamics, external control inputs and inner dynamical interactions with the controllability of such complex networked dynamical systems. Different approaches to analyzing the network controllability are evaluated. Some advanced topics on the selection of driver nodes, optimization of network controllability and control energy are discussed. Potential applications to real-world networked systems are also described. Finally, a near-future research outlook is highlighted.

Digital Object Identifier 10.1109/MCAS.2019.2909446
Date of publication: 20 May 2019

1. Introduction

Controllability, as one of the fundamental concepts in control theory, quantifies the ability to steer a dynamical system from an arbitrary initial state to an arbitrary terminal state in finite time [1], [2]. In general, controllability is a prerequisite of control actions. Many important applications have been found, not only in systems engineering and control technology, but also in such areas as chemical and logistics processes, nuclear reactors, power systems, aerospace engineering and recently quantum systems and nanotechnology.

The classical notion of controllability in control theory focuses on the inherent dynamics of a single albeit higher-dimensional system. In the big-data era and omni-networking world today, the traditional control theory is encountered more and more large-scale complex networks [3]–[7], where nodes are higher-dimensional dynamical systems and edges represent the interactions among them. Typical examples include the Internet, WWW, wireless communication networks, transportation networks, power grids, sensor networks, brain neural networks, metabolic networks, gene regulatory networks, social networks, and many others. Developments in engineering, physics and biology have recently extended the classic concepts and notions on systems control to networks control. It has been noted that perturbations on one node in a network can influence and alter the states of many other nodes through their local interactions. This interconnectedness can be exploited to effectively control a complex network by manipulating the states of only a small fraction of nodes, in which the underlying network structure plays a crucial role. Therefore, it is of theoretical and practical importance to explore the controllability of complex networked systems from a network-theoretic perspective. This can help better understand, predict and optimize the collective behaviors of various networked dynamical systems in practical applications.

In the past decade, research on network controllability has attracted increasing attention and, in effect, become an exciting and rapidly developing research direction. The goal of this article is to survey on the current flourishing advances in the studies of the controllability of networked linear dynamical systems. Fundamental concepts and selected theoretical results on both state controllability and structural controllability for different types of complex networked systems are reviewed and discussed. Several specific applications of network controllability are described. Finally, a near-future research outlook will be highlighted.

2. Notions of Network Controllability

In the present literature, there are several notions of network controllability, which strongly depend on the types of the networked control systems and the forms of admissible control inputs, but they can be classified into two essential types of state controllability and structural controllability in general.

2.1 State Controllability

The basic concept of (complete) state controllability was introduced by Kalman in the 1960s [1], for a linear time-invariant (LTI) dynamical system of the form

$$\dot{x}(t) = A_0 x(t) + B_0 u(t) \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the internal state vector of the system at time t, $u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T \in \mathbb{R}^m$ is the input vector at time t, $A_0 \in \mathbb{R}^{n\times n}$ is the system matrix and $B_0 \in \mathbb{R}^{n\times m}$ is the input matrix.

Definition 1
The LTI system (1) is said to be (completely) *state controllable* if, for any initial state $x(t_0) \in \mathbb{R}^n$ and any final state $x(t_f) \in \mathbb{R}^n$, there exist a finite time t_1 and an input $u(t) \in \mathbb{R}^m, t \in [t_0, t_1]$, such that $x(t_1; x(t_0), u) = x(t_f)$.

This definition implies that any initial state $x(t_0)$ can be steered to any final state $x(t_f)$ in finite time. Here, the finite time t_1 is not fixed, the trajectory of the dynamical system (1) between t_0 and t_1 is not specified, and there is no constraint on the input vector $u(t)$.

The classic algebraic controllability criteria are given as follows [2].

Theorem 1 (State Controllability Theorem)
The LTI system (1) is completely state controllable (state controllable, or simply, controllable) if and only if one of the following conditions is satisfied:

i) the controllability matrix

$$Q = [B_0, A_0 B_0, \ldots, A_0^{n-1} B_0] \tag{2}$$

has full row rank; that is,

$$rank(Q) = n. \tag{3}$$

ii) $rank[sI_n - A_0, B_0] = n, \quad \forall s \in \mathbb{C}$.

iii) the relationship $\upsilon^T A_0 = \lambda \upsilon^T$ implies $\upsilon^T B_0 \neq 0$, where υ is the nonzero left eigenvector of A_0 associated with the eigenvalue λ.

iv) the Gramian matrix

$$W_c = \int_{t_0}^{t_1} e^{A_0 t} B_0 B_0^T e^{A_0^T t} dt \tag{4}$$

is nonsingular.

Conditions (i), (ii), (iii) and (iv) in Theorem 1 are referred to as the Kalman rank criterion, Popov-Belevitch-Hautus (PBH) rank criterion, PBH eigenvector test, and Gramian matrix criterion, respectively. They are equivalent for the LTI system (1).

Today, strongly stimulated by the rapid and promising development of network science and engineering,

Linying Xiang and Fei Chen are with the School of Control Engineering, Northeastern University at Qinhuangdao, Qinhuangdao 066004, P. R. China, (e-mails: xianglinying@neuq.edu.cn; xianglyhk@gmail.com). Fei Chen is also with the State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, Shenyang 110004, P. R. China. Wei Ren is with the Department of Electrical and Computer Engineering, University of California, Riverside, CA 92521, USA, Guanrong Chen is with the Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong SAR, P. R. China.

the concept of state controllability of an LTI system has been extended to a complex network of many such systems interconnected together. The latter focuses on the effects of the interactions among the multiple systems on the collective dynamics of the whole network.

To introduce this notion, consider a set of one-dimensional LTI control systems, $\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij} x_j(t) + \sum_{m=1}^{M} b_{im} u_m(t)$, $i = 1, 2, \ldots, N$, and form a network by these systems as nodes, altogether described by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T \in \mathbb{R}^N$ is the state vector of all nodes at time t, in which $x_i(t)$ can represent the opinion of a person in a social network or the position of a sensor in a mobile sensor network; $u(t) = [u_1(t), u_2(t), \ldots, u_M(t)]^T \in \mathbb{R}^M$ is the input vector, in which $u_i(t)$ can represent the command of a leader in a social network or the message signal of a device in a sensor network; $B = (b_{im}) \in \mathbb{R}^{N \times M}$ is the input matrix identifying the nodes that are being directly controlled, in which b_{im} represents the strength of an external control signal $u_m(t)$ imposed on node i; $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ is the adjacency matrix of the underlying network, in which the element $a_{ij} \neq 0$ (usually, $a_{ij} > 0$) denotes the weight of a directed edge (j, i) from node j to node i (for an undirected network, $a_{ij} = a_{ji}$), for all i, $j = 1, 2, \ldots, N$. The overall networked system described by (5) can be denoted by the matrix pair (A, B).

The concept of state controllability for the above-described linear network can be analogously defined, as follows.

Definition 2

The linear network (5) is said to be *state controllable* if, for any initial state $x(t_0) \in \mathbb{R}^N$ and any final state $x(t_f) \in \mathbb{R}^N$, there exist a finite time t_1 and an input $u(t) \in \mathbb{R}^M$, $t \in [t_0, t_1]$, such that $x(t_1; x(t_0), u) = x(t_f)$.

The aforementioned Kalman rank criterion suggests that the networked system (5) is state controllable if and only if the controllability matrix $Q = [B, AB, \ldots, A^{N-1}B]$ is of full row rank, i.e., $rank(Q) = N$. This can be easily tested for lower-dimensional systems, namely with a small N, but is very difficult to verify for large-scale and complex-structured networks. Moreover, to numerically check this rank condition, one has to know the exact parameter values in matrices A and B. In practice, system parameter values may vary or never be known precisely due to noise or measuring errors. For example, for a gene regulatory network, one does not have any method today to estimate the edge weights, only knowing whether or not there is an edge. Hence, it is hard to numerically verify the Kalman rank condition assuming known and fixed edge weights. In addition, this rank criterion does not show how to find an appropriate B for a given A so as to satisfy the required full-rank condition. These situations had led to very few applications in large-sized complex networked systems, motivating the introduction of the following concept of structural controllability for networked systems.

2.2 Structural Controllability

Structural control theory, firstly introduced by Lin in 1974 [8], offers a more comprehensive framework to avoid the above-mentioned limitations of the Kalman rank criterion. This concept emphasizes the crucial role of the underlying network structure in determining the state controllability, where the matrices A and B are both parameterized but preserving the system structure. An LTI system (A, B) is a *structured system* if the entries in A and B are either fixed zeros, which reflect the fixed un-connecting sub-structure of the network, or independent nonzero parameters, which reflect the variable edge weights of the network. More precisely, a fixed zero indicates the absence of a relation between some state variables of nodes, while a nonzero parameter characterizes the relationship between two corresponding state variables. The two matrices A and B are called *structured matrices*. A structured system can represent a large class of linear systems since its nonzero parameters can vary, but the connecting structure of the system has been determined by the locations of those fixed zeros, which cannot be changed.

To a structured system, one can associate it with a digraph whose nodes denote the (state and input) variables and edges indicate the connections between some variables [8]. Using this type of representation, one can investigate system properties from a graph-theoretic perspective. More precisely, a structured system (A, B) can be represented by a digraph $G(A, B) = (V, E)$ with $V = V_A \cup V_B$ being the node set and $E = E_A \cup E_B$ being the edge set. Here, $V_A = \{x_1, x_2, \ldots, x_N\} = \{v_1, v_2, \ldots, v_N\}$ is the set of *state nodes*, corresponding to the N nodes in the original network $G(A)$ (e.g. the nodes in Fig. 1(a) marked by red); $V_B = \{u_1, u_2, \ldots, u_M\} = \{v_{N+1}, v_{N+2}, \ldots, v_{N+M}\}$ is the set of *input nodes* corresponding to the M inputs (e.g. the nodes in Fig. 1(b) marked by blue); $E_A = \{(x_j, x_i) \mid a_{ij} \neq 0\}$ is the set of edges between state nodes (e.g. the edges in Fig. 1(b) marked by black); $E_B = \{(u_m, x_i) \mid b_{im} \neq 0\}$ is the set of edges between input nodes and state nodes (e.g. the edges in Fig. 1(b) marked by green). Note that the system matrix A is the weighted adjacency matrix of the original network. The state nodes connected to some input nodes are called *controlled nodes* (e.g. x_1 and x_2 in Figs. 1(b) and (c)). Denoting the number of controlled nodes as M', one has $M' \geq M$ because one input node can be connected to multiple state nodes (see Fig. 1(c), where a single input node u_1 is simultaneously connected to two state nodes x_1 and x_2). Those controlled nodes that do not share input nodes are called *driver nodes* (e.g., x_1 and x_2 in Fig. 1(b)). Obviously, the number of driver nodes is equal to the number of independent inputs.

A state node x_i is *inaccessible* if there are no directed paths reaching x_i from the input nodes (see Fig. 2(a)). The digraph $G(A, B)$ contains a *dilation* if there is a subset of nodes $S \subset V_A$ such that their common-neighbor set of S, denoted as $T(S)$, has fewer nodes than S itself (see Fig. 2(b)). Here, $T(S)$ is the set of nodes v_j, in which there is a directed edge from v_j to some other node in S. Note that the input nodes are not allowed to belong to S but may belong to $T(S)$. For a digraph, a sequence of oriented edges (v_j, v_{j+1}), $j = 1, 2, \ldots, k-1$, where the nodes $\{v_1, v_2, \ldots, v_k\}$ are distinct, is called a simple directed path. When v_k coincides with v_1, the sequence of edges is called a simple directed cycle. For a digraph $G(A, B)$, one can define the following subgraphs (see Fig. 2(c)):

i) a *stem* is a simple directed path originating from an input node;
ii) a *bud* is a simple directed cycle with an additional edge that ends at, but does not begin from, a node of the directed cycle;
iii) a *cactus* is defined recursively: Start with a stem, which is a cactus. Let C_a, O and e be respectively a cactus, a simple directed cycle that is disjoint with C_a, and an additional directed edge that connects C_a to O in $G(A, B)$. Then, $C_a \cup \{e\} \cup O$ is also a cactus. After all, $G(A, B)$ is spanned by cacti if there exists a set of disjoint cacti that cover all state nodes.

Definition 3
An LTI system (A, B) is *structurally controllable* if one can set some values to the nonzero parameters in A and B such that the resulting system is state controllable in the sense of Kalman defined in Definition 2 above.

Note that cactus is the minimum structure containing no inaccessible nodes and no dilations. In other words, for a given cactus, the removal of any edge will lead to either inaccessibility or dilation.

A simple and elegant necessary and sufficient condition for the structural controllability of a system can be given by simply inspecting its topology [8], [9].

Theorem 2 (Structural Controllability Theorem)
An LTI system (A, B) is structurally controllable if and only if

i) the digraph $G(A, B)$ contains no inaccessible nodes and no dilations; or
ii) the digraph $G(A, B)$ is spanned by cacti.

Note that, differing from the Kalman algebraic (rank) condition, this theorem offers a geometric (graphical) condition. Note also that if a system is structurally controllable then it is state controllable for most possible parameter realizations, except perhaps for some improperly related ones in the parameter space. Therefore, structural controllability is a generic property of

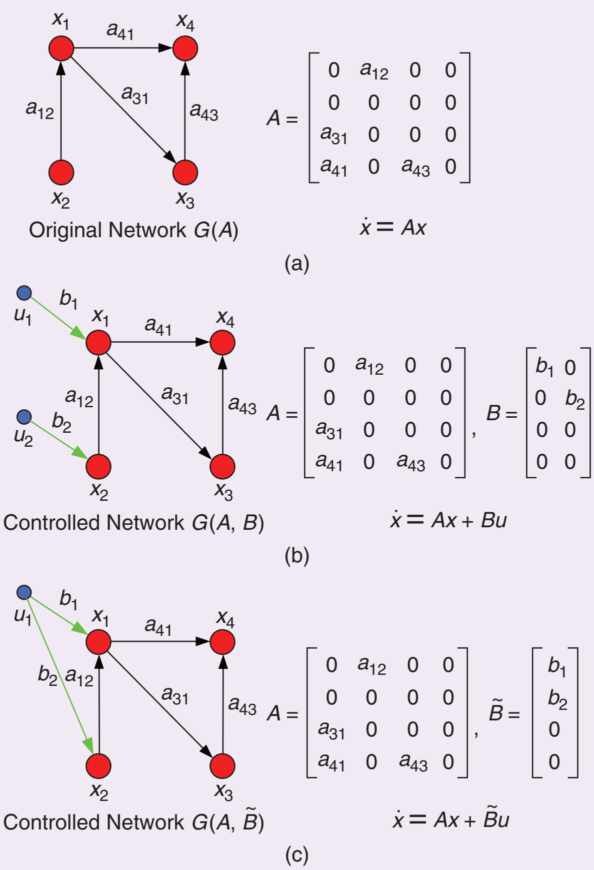


Figure 1. Graphical representation of a linear networked system (A, B). (a) The original network without external control input. All nodes in the network are called state nodes, where the state node set is $V_A = \{x_1, x_2, x_3, x_4\}$. The state matrix A denotes the weighted wiring diagram of the digraph $G(A)$. (b) The network is controlled by two input nodes, where the input node set is $V_B = \{u_1, u_2\}$. The input matrix B identifies the state nodes $\{x_1, x_2\}$ that are controlled by the input $u(t) = [u_1(t), u_2(t)]^T$ with two independent signals $u_1(t)$ and $u_2(t)$, respectively. Therefore, both x_1 and x_2 are driver nodes. (c) The network is controlled by a single input node, where the input node set is $V_{\tilde{B}} = \{u_1\}$. The input matrix $\tilde{B}$ identifies the state nodes $\{x_1, x_2\}$ that are controlled by the same input signal $u(t) = u_1(t)$ with different control gains b_1 and b_2. Here, if x_2 is chosen to be a driver node then x_1 cannot be so, because they share the same input node u_1.

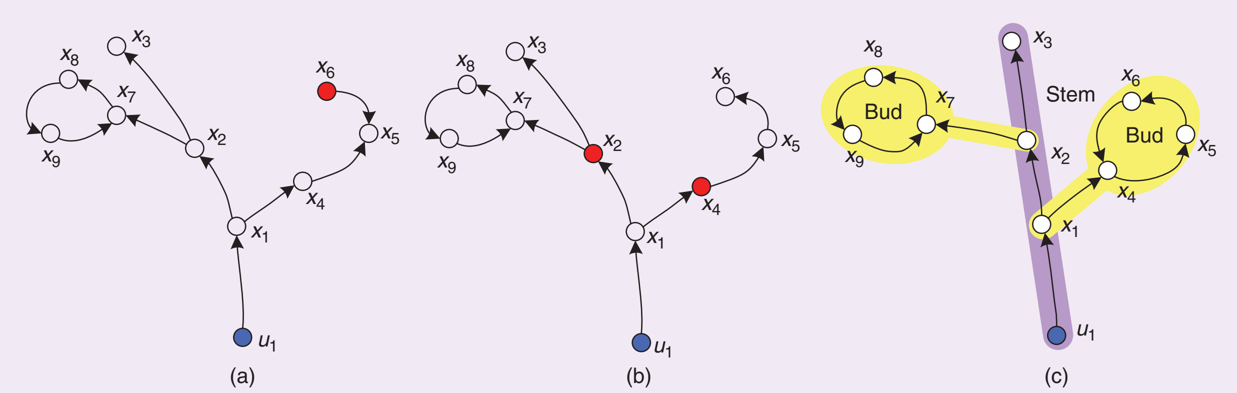


Figure 2. Inaccessibility, dilation and cactus. (a) The red node x_6 is inaccessible from the input node u_1 (blue node). (b) The red nodes in the set $S = \{x_2, x_4\}$ lead to a dilation. Their common-neighbor set $T(S) = \{x_1\}$ contains one node (less than 2). (c) A cactus contains neither inaccessible nodes nor dilations. There are one stem (purple) and two buds (yellow).

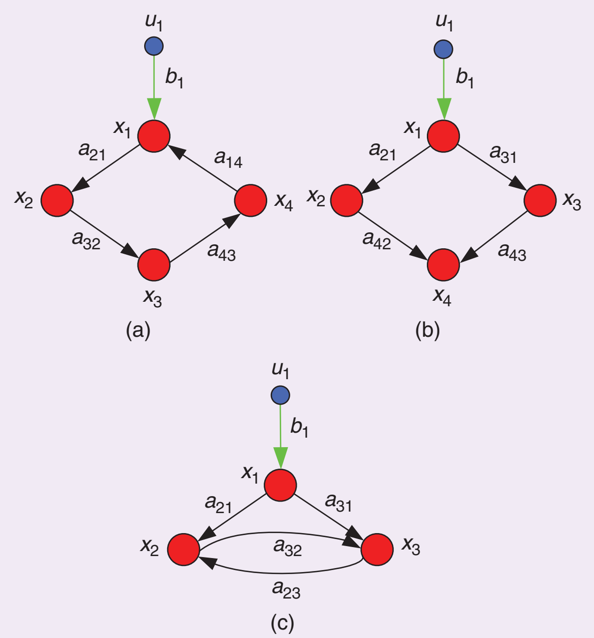


Figure 3. Control diagrams of a directed network. (a) A directed cycle can be completely controlled by controlling only one node therein. The controllability is independent of any specific (nonzero) values of b_1, a_{21}, a_{32} and a_{43}, so the networked system is strongly structurally controllable. (b) This directed network can never be completely controlled by only controlling the node x_1. (c) This directed network is completely state controllable for almost all nonzero parameter realizations, which will be uncontrollable only for some parameters satisfying the constraint of $a_{23}a_{31}^2 = a_{32}a_{21}^2$, thus this networked system is structurally controllable.

a practical (parametrized) control system. It highlights the importance of the system structures. In this setting, furthermore, when the system is state controllable for all nonzero parameter realizations, it is called *strongly structurally controllable* [10]. Conversely, if a system is state controllable then it is certainly structurally controllable (considering the given set of matrix values as a particular parametric realization).

The advantage of structural controllability is due to the fact that one can determine a network's controllability even if the exact weight values of some or all edges are unknown. As will be demonstrated below, this framework considerably expands the practical applicability of the classical control theory and techniques to real-world networked systems with incomplete modeling or uncertainties.

It should be stressed that the two concepts and notions of "state controllability" and "structural controllability" are obviously closely related, but not equivalent. It is even possible that a networked system is not state controllable but is structurally controllable. Here, some illustrative examples are given to show the differences between state controllability, structural controllability, and strong structural controllability for networked systems.

Example 1

The linear system shown in Fig. 3(a) is described by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & a_{14} \\ a_{21} & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & a_{43} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1(t).$$

The controllability matrix of this networked system is given by

$$Q = [B, AB, A^2B, A^3B] = b_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{21} & 0 & 0 \\ 0 & 0 & a_{32}a_{21} & 0 \\ 0 & 0 & 0 & a_{43}a_{32}a_{21} \end{bmatrix}.$$

This system is state controllable because $rank(Q) = 4 = N$ as long as the parameters b_1, a_{21}, a_{32} and a_{43} are all nonzero. That is, its controllability is independent of the specific values of b_1, a_{21}, a_{32} and a_{43}. Therefore, this system is also strongly structurally controllable.

The linear system shown in Fig. 3(b) is described by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ a_{31} & 0 & 0 & 0 \\ 0 & a_{42} & a_{43} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1(t).$$

Its controllability matrix is given by

$$Q = [B, AB, A^2B, A^3B] = b_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{21} & 0 & 0 \\ 0 & a_{31} & 0 & 0 \\ 0 & 0 & a_{42}a_{21} + a_{43}a_{31} & 0 \end{bmatrix}.$$

This system is state uncontrollable because $rank(Q) = 3 < N$. Further note that this system is always state and structurally uncontrollable no matter how the parameters b_1, a_{21}, a_{31}, a_{42} and a_{43} are chosen.

The linear system shown in Fig. 3(c) is described by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} u_1(t).$$

Its controllability matrix is given by

$$Q = [B, AB, A^2B] = b_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23}a_{31} \\ 0 & a_{31} & a_{32}a_{21} \end{bmatrix}.$$

This system will be state controllable if one choose the detailed parameters such that $rank(Q) = 3 = N$. In the special case of $a_{23}a_{31}^2 = a_{32}a_{21}^2$, one has $rank(Q) = 2 < N$, so the system becomes state uncontrollable. Therefore, this system is structurally controllable. Clearly, it is not strongly structurally controllable. Figure 3(c) illustrates the relationship between state controllability and structural controllability. ■

3. Analysis of Network Controllability

In this section, various approaches to evaluating the controllability of a complex networked dynamical system are presented. Fundamental concepts and selected theoretical results on both state controllability and structural controllability for different types of complex networked systems are summarized. In particular, how the network topology, node-system dynamics, external control inputs and inner dynamical interactions, altogether or respectively affect the controllability of a general network of linear dynamical systems are discussed.

3.1 Kalman Rank Criterion

In a networked dynamical system, agents as nodes are endowed with state variables and dynamics, and are interconnected via a communication network in a certain topology. The controllability issue in such a networked multi-agent system under the so-called leader-follower framework was first considered in [11], where the problem is formulated as the classical state controllability of a single-input linear system.

Specifically, consider a networked multi-agent system described by

$$\dot{x}_i(t) = \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, 2, \ldots, N, \tag{6}$$

where $x_i(t) \in \mathbb{R}$ is the state vector of the ith agent at time t, a_{ij} is the (i, j)th element of the adjacency matrix of an undirected graph G that denotes the information flows among the agents, $a_{ij} = 1$ if agents i and j $(j \neq i)$ are neighbors (i.e., connected to each other), and $a_{ij} = 0$ otherwise (herein, self-loops and multi-connected edges are excluded), and N_i is the neighbor set of agent i, which is assumed fixed, therefore the interconnection graph G is time invariant.

System (6) can be written in the Laplacian dynamics form

$$\dot{x}(t) = -Lx(t), \tag{7}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T \in \mathbb{R}^N$ denotes the aggregated state vector of the agents and $L = (L_{ij}) \in \mathbb{R}^{N \times N}$ is the Laplacian matrix defined by

$$L_{ij} = \begin{cases} -a_{ij}, & j \neq i, \\ \sum_{j \in N_i} a_{ij}, & j = i. \end{cases} \tag{8}$$

The agents in the network can be divided into two groups: *leaders* and *followers*, where external control inputs are injected only to the leaders. Denote the set of controlled agents as the leader set, V_l, and the remaining agents as the follower set, V_f, where the subscripts l and f denote the leaders and followers, respectively. It is clear that $V_l \bigcup V_f = V$ (the whole network), and $V_l \bigcap V_f = \varnothing$. Define the *follower graph* G_f to be the subgraph induced by V_f and the *leader graph* G_l the subgraph induced by V_l. Obviously, G_l and G_f are disjoint.

Without loss of generality, one can reorganize the indices of the agents in such a way that the first N_f $(1 < N_f < N)$ agents are followers; that is, one can label the followers from 1 to N_f and the leaders from $N_f + 1$ to N. The associated Laplacian matrix L is thereby partitioned as

$$L = \begin{bmatrix} L_f & L_{fl} \\ L_{fl}^T & L_l \end{bmatrix}, \tag{9}$$

where L_f and L_l are $N_f \times N_f$ and $(N - N_f) \times (N - N_f)$ matrices, respectively. However, these two sub-matrices generally do not have the Laplacian matrix properties. The matrix L_f is the principle diagonal submatrix of the original Laplacian matrix L related to the followers, while L_l is the one related to the leaders, while L_{fl} denotes the connections between the followers and the leaders.

Note that the motion of each leader in this networked multi-agent system is free; that is, its state variables can change without being influenced by followers. The behavior of each follower obeys the local nearest-neighbor rule and is dominated by the leaders, directly or indirectly. In this case, the trajectories of the leaders can be viewed as exogenous control inputs to the followers. Therefore, the followers evolve through the quasi-Laplacian-based dynamics, described by

$$\dot{x}_f(t) = -L_f x_f(t) - L_{fl} x_l(t), \tag{10}$$

where $x_f(t) = [x_1(t), x_2(t), \ldots, x_{N_f}(t)]^T \in \mathbb{R}^{N_f}$ represents the state vector of the followers and $x_l(t) = [x_{N_f+1}(t), x_{N_f+2}(t), \ldots, x_N(t)]^T \in \mathbb{R}^{N-N_f}$ is that of the leaders.

The controllability of a networked multi-agent system can be defined as follows: *the followers can be steered by the leaders from any initial states to arbitrary final states in finite time*.

Remark 1

There are two issues to be clarified. First, the notion of controllability for networked multi-agent systems bears some new physical features, depending on the application background. In essence, it is a kind of formation control [23]. It deals with multi-agent systems under the leader-follower framework, where the agents are divided into two groups, leaders and followers. The leaders are considered as exogenous control inputs to the followers, while the followers can be guided to pre-desired locations so as to form some anticipated configurations due to the control actions implemented by the leaders, while the leaders do not participate in any network configuration changes and updates. Second, if one equates x_f with x and x_l with u, and identifies matrices A and B with $-L_f$ and $-L_{fl}$, respectively, the leader-follower network (10) can be reformulated as (5). Note that the system matrix A in the LTI system (5) is merely the adjacency matrix of the network, while in the leader-follower setting described by (10) it is a quasi-Laplacian matrix. The subtle difference between these two matrices will lead to very different and even contrary controllability results.

Based on the Kalman rank criterion, some necessary and sufficient algebraic conditions are derived in [11], in terms of the eigenvalues and eigenvectors of a submatrix of the Laplacian matrix corresponding to the followers.

Theorem 3 [11]

The multi-agent system (10) with a single leader is state controllable if and only if the following conditions are satisfied simultaneously:

i) all the eigenvalues of L_f are distinct;
ii) the eigenvectors of L_f are not orthogonal to L_{fl}.

Example 2

As examples, Fig. 4 shows two leader-follower networks with $V_f = \{1, 2, 3\}$ and $V_l = \{4\}$.

In Fig. 4(a), one has

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad L_f = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad L_{fl} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

The eigenvalues of L_f are $\lambda(L_f) = \{0.2679, 1, 3.7321\}$, which are distinct. Denote by $\upsilon(L_f)$ their corresponding eigenvectors. Then, $\upsilon^T(L_f)L_{fl} = \{0.4597, 0, 0.8881\}$. That is, one of the eigenvectors is orthogonal to L_{fl}. It follows from Theorem 3 that this single-leader network is uncontrollable.

In Fig. 4(b), one has

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad L_f = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad L_{fl} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenvalues of L_f are $\lambda(L_f) = \{0.1981, 1.5550, 3.2470\}$, which are distinct. As for the eigenvectors of L_f, one has $\upsilon^T(L_f)L_{fl} = \{0.328, -0.737, 0.591\}$. That is, none of the eigenvectors is orthogonal to L_{fl}. Therefore, according to Theorem 3, this single leader network is state controllable. ■

Note that, although the conditions in Theorem 3 are derived for a single-leader system, they can be easily extended to multi-leader cases (see Proposition 1 in [13]). In addition, in [12] a sufficient condition is given for multi-leader controllability based on the algebraic characteristics of a submatrix of the incidence matrix of the network. The controllability of such undirected networked systems depends on both the network topology and the number

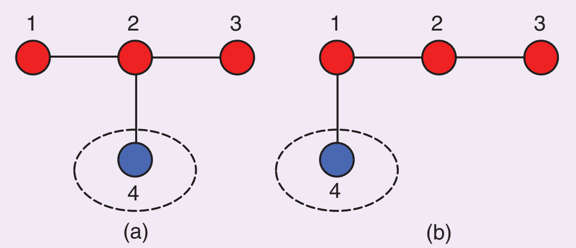


Figure 4. Two leader-follower networks with $V_f = \{1, 2, 3\}$ (red nodes) and $V_l = \{4\}$ (blue node). (a) The network is leader-symmetric with respect to $\{4\}$. (b) The network is leader-asymmetric with respect to $\{4\}$.

of leaders. This result is constructive in that it allows selecting leaders for the purpose of making a networked multi-agent system to be state controllable.

3.2 Network Equitable Partitions

Although elegant, the conditions given in Theorem 3 are not graph-theoretic in that the controllability could not be directly determined based on the network topology. Rahmani and Mesbahi [13] discussed an intricate relationship between the state controllability and the graph symmetry, and then gave a sufficient graph-theoretic condition for determining the uncontrollability. Subsequently, they explored the notion of leader symmetry with respect to the graph automorphism [14].

The notion of graph symmetry is introduced as follows.

Definition 4 [14]

The system (10) is *leader symmetric* with respect to a single leader if there exists a non-identical permutation P such that

$$PL_f = L_f P. \tag{11}$$

A sufficient graph-theoretic condition for the uncontrollability is given below.

Theorem 4 [13], [14]

The multi-agent system (10) with a single leader is uncontrollable if it is leader symmetric.

Example 3

The network depicted in Fig. 4(a) is leader symmetric with respect to $\{4\}$ but asymmetric with respect to any other leader node set (e.g., Fig. 4(b)). According to Theorem 4, the leader-follower network shown in Fig. 4(a) is uncontrollable, which is consistent with the conclusion given by Theorem 3. ■

Considering the case that some multi-agent systems may require multiple leaders, Ji et al. used nontrivial equitable partitions and interlacing theory to identify controllable networks, generalizing some existing results to the multi-leader case [15], [16]. They provided a more precise interpretation of the intrinsic relationship between the controllability and the network topology. The concept of equitable partition is rooted in the graph symmetry with respect to a leader [13], [14].

To introduce the graph-theoretic characterization of the controllability for multi-leader networks, some more definitions are needed.

Definition 5 [15], [16]

A *cell* $C \subset V_G$ is a subset of the node set. A *nontrivial cell* is a cell with more than one node. A *partition* of the graph is a grouping of its node set into different cells.

An r-partition π of V_G, with cells $C_1, C_2, \ldots, C_r$, is said to be *equitable* if each node in C_i has the same number of neighbors in C_j for all i, j. The cardinality of the partition π is denoted by $r = |\pi|$.

Let b_{ij} be the number of neighbors in C_j of a node in C_i. The directed graph with the cells of an equitable r-partition π as its nodes, and with b_{ij} edges from the i th cell to the j th cell of π, is called the *quotient* of G over π and is denoted by G/π. A trivial partition is the N-partition, $\pi = \{\{1\}, \{2\}, \ldots, \{N\}\}$. If an equitable partition contains at least one cell with more than one node, it is called a *nontrivial equitable partition* (NEP), and the adjacency matrix of a quotient is given by $A(G/\pi)_{ij} = b_{ij}$.

Example 4

In the undirected network shown in Fig. 5(a), one nontrivial equitable partition is $\pi = \{\{1\}, \{2,3\}, \{4\}, \{5\}, \{6\}\}$. The adjacency matrix of its quotient (see Fig. 5(b)) is given by

$$A(G/\pi) = \begin{bmatrix} 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

■

A necessary condition for a multi-leader networked system to be controllable was established in [15], [16], as follows.

Theorem 5 [15], [16]

Given a connected graph G with the induced follower graph G_f, a necessary condition for system (10) to be state controllable is that no NEPs π and π_f on G and G_f, respectively, share a nontrivial cell.

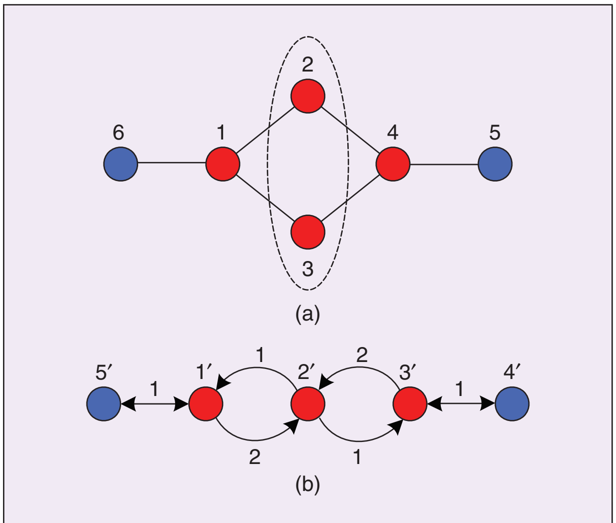


Figure 5. (a) The equitable partition $\pi = \{\{1\}, \{2,3\}, \{4\}, \{5\}, \{6\}\}$ on a 2-leader network with the leader set $\{5, 6\}$, which are marked in blue. (b) The quotient of the network over π.

Theorem 5 is established based on the following algebraic criterion.

Theorem 6 [16]
The multi-agent system (10) with multiple leaders is state controllable if and only if L and L_f do not share any common eigenvalues.

Example 5
Figure 5(a) shows a 2-leader network with $V_f = \{1, 2, 3, 4\}$ and $V_l = \{5, 6\}$. One has

$$L = \begin{bmatrix} 3 & -1 & -1 & 0 & 0 & -1 \\ -1 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$
$$L_f = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}, \quad L_{fl} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

An NEP of the follower graph G_f is introduced by $\pi_f = \{\{1\}, \{2, 3\}, \{4\}\}$. It is apparent that π and π_f share a nontrivial cell $\{2, 3\}$. According to Theorem 5, it is uncontrollable. One can also determine its uncontrollability from an algebraic perspective using Theorem 6. The eigenvalues of L and L_f are $\lambda(L) = \{0, 0.5858, 1.2679, 2, 3.4142, 4.7321\}$ and $\lambda(L_f) = \{0.4384, 2, 3, 4.5616\}$, respectively. Obviously, there exists one common eigenvalue, 2, between L and L_f. It then follows from Theorem 6 that this network is uncontrollable. ■

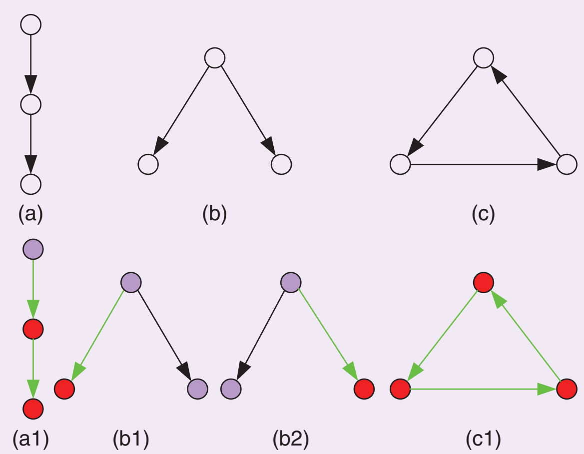


Figure 6. Matching. (a), (b) and (c) are original digraphs. (a1), (b1), (b2) and (c1) are maximum matchings of (a)–(c), respectively. Edges in the matching are colored in green. Matched (or unmatched) nodes are shown as shaded red (or purple), respectively. These notations will be used for Figs. 9 and 10 below.

Since Ji et al. [15], [16] employed the equitable partition to analyze the controllability of multi-agent systems, various improvements of the network partition have been made. Martini et al. [17] provided necessary and sufficient graphical conditions for the controllability of multi-agent systems using relaxed equitable partitions. Zhang et al. [18] provided lower and upper bounds for the controllable subspaces in terms of distance partitions and maximal almost equitable partitions, respectively. They systematically constructed a class of uncontrollable diffusively coupled multi-agent systems with a single leader and multi-chain topologies based on the minimal leader-invariant relaxed equitable partition (LEP) [19]. By providing counter examples to the results obtained in [17], Camlibel et al. showed that LEP cannot fully characterize the controllable subspace [20]. Observing that the existing techniques, e.g. equitable partition [16] and relaxed equitable partition [17], are not appropriate to deal with weighted systems, Lou and Hong [21] employed weight-balanced partition to classify the interconnection graphs and to characterize the controllable subspaces with a given nontrivial weight-balanced partition. They also provided necessary and sufficient graph-theoretic conditions on structural controllability and strong structural controllability. Moreover, they considered the effect of the zero row-sum restriction on the system matrices regarding the structural controllability. Recall that they were the first to consider the strong structural controllability [10] for multi-agent systems. Recently, Ji et al. introduced a new concept of controllability-destructive nodes and showed that the complete graphical characterization of multi-agent controllability can be reduced to the identification of the topological structures of the controllability-destructive nodes [22].

3.3 Structural Controllability Theory

3.3.1 Nodal dynamics

The controllability of networked multi-agent systems with a leader-follower structure is by nature a kind of formation control problem [23]. The methodologies and their corresponding results in the current literature are applicable only to small-sized and simple-structured networked systems. The controllability issue becomes more complicated and challenging when being applied to large-scale and complex-structured dynamical networks. Liu et al. [24], based on Lin's structural controllability theory, considered large-scale directed and weighted networks (5) with structured matrices A and B. They demonstrated that the ability to steer a complex network toward any desired states can be measured by the minimum number of driver nodes that need to be driven by external signals in order to achieve full control of the whole network. They called it the *minimum inputs theorem*. To introduce this theorem, some concepts and notations are introduced first.

In a directed graph, an edge subset $\tilde{M}$ is a *matching* if no two edges in $\tilde{M}$ share a common starting node or a common ending node. A node is *matched* if it is the ending node of an edge in the matching; otherwise, it is *unmatched*. A matching of maximum size is called a *maximum matching*. A maximum matching is *perfect* if all nodes are matched.

Example 6

In Fig. 6(a), all but the starting node are matched (see Fig. 6(a1)). In Fig. 6(b), one can find multiple maximum matchings (Figs. 6(b1) and (b2)). In a directed cycle, the maximum matching is perfect (Fig. 6(c1)). ■

Theorem 7 (Minimum Inputs Theorem) [24]

To fully control a directed network $G(A)$, the minimum number N_{I} of independent inputs, or equivalently the minimum number N_{D} of driver nodes, is given by

$$N_{\mathrm{I}} = N_{\mathrm{D}} = \max\{N - |M^*|, 1\}, \tag{12}$$

where $|M^*|$ is the size of a maximum matching in $G(A)$; that is, the driver nodes correspond to the unmatched nodes. If all nodes in the network are matched, i.e., $|M^*| = N$, a single input is sufficient to fully control the network, hence $N_{\mathrm{I}} = N_{\mathrm{D}} = 1$. In this case, any node can be chosen as the driver node.

The minimum inputs theorem maps the structurally controllability of a large-scale directed network into a purely graph-theoretical problem of finding a maximum matching of a diagraph. More importantly, it avoids the trouble in exhaustively searching for all node combinations in a minimum driver node set, since the driver nodes can be obtained from a solution of the matching problem.

Combining techniques from network science, control theory and statistical physics, Liu et al. [24] showed that the minimum number of driver nodes is mainly determined by the degree distribution of the network, and that the driver nodes tend to avoid high-degree nodes. They also found that sparse and heterogeneous networks are the most difficult ones to control, but dense and homogeneous networks can be controlled using only a few driver nodes.

Note that degree distribution alone is not sufficient to characterize many basic properties of a complex network, including the controllability. Also, the effects of different structural properties cannot be separated from each other in order to determine the network controllability. Thus, the effects of other network characteristics on the controllability were investigated from various perspectives. Pósfai et al. [25] found that clustering and modularity have no discernible impacts on the controllability, but the symmetries of the underlying matching problem can produce linear, quadratic or no dependence on the degree correlation coefficients, depending on the nature of the underlying correlations. Menichetti et al. [26] showed that the density of nodes with in-degree and out-degree equal to one or two already determines the number of driver nodes needed for a network. They also showed that random networks with minimum in-degree and out-degree greater than two are always fully controllable by an infinitesimal fraction of driver nodes, regardless of many other properties of the degree distribution.

3.3.2 Edge Dynamics

Most existing works focused on linear time-invariant (LTI) nodal dynamics, for which the nodal controllability is investigated based on the state variables of individual node systems. However, in many real-world networks, edge dynamics could also be important [27]–[30]. For example, in social networks, a node (person) constantly processes the information received from its upstream neighbors and makes decisions that are communicated to its downstream neighbors. In this process, nodes are not only passive components (receiving information) but also active components with information processing capabilities (identifying and transmitting information). One node can receive different information from different upstream neighbors through different inbound edges and then transmit the information or decisions to different downstream neighbors through outbound edges. Regarding the internal information of the node being mutually mixed and even being eliminated, it is more reasonable and accurate to assume that the information received and passed by a node is represented by the state variables on its incoming and outgoing edges, respectively. The node itself acts like a switchboard, mapping the signals from the incoming edges onto the outgoing edges. This is also the case in an urban transport network, where the edges represent road connections and the state variables on the edges represent the amount of traffic flow along a particular conjunction in a given direction. To model such systems, Nepusz and Vicsek [31] developed a switchboard dynamics (SBD) model by introducing a dynamical process on the edges of a directed network. The equation governing the LTI edge dynamics of a directed complex network is

$$\dot{y}_i^+(t) = M_i y_i^-(t) - \tau_i \otimes y_i^+(t) + \delta_i u_i(t), \quad i = 1, 2, \ldots, N, \tag{13}$$

where y_i^+ and y_i^- are vectors of outbound and inbound edges of node i, respectively; M_i is the mixing (or switching) matrix with the number of rows equaling the out-degree and the number of columns equaling the in-degree of node i; τ_i is a vector of damping terms corresponding to the edges in $y_i^+(t)$; $\delta_i = 1$ if node i is a driver node, but $\delta_i = 0$ otherwise. In this model, each node i acts as a small switchboard-like device mapping the signals of the inbound edges to the outbound edges via a linear operator M_i.

Reformulating the SBD in terms of the edge variables yields an LTI system described by

$$\dot{x}(t) = (W - T)x(t) + Hu(t), \tag{14}$$

where $W \in \mathbb{R}^{M \times M}$ is the transpose of the adjacency matrix of the line digraph $L(G)$ of the original digraph G; T is a diagonal matrix composing of damping terms of each edge, and H is a diagonal matrix where the ith element is δ_v if node v is the head of edge i. Note that the nodes of $L(G)$ correspond to the edges in the original digraph G, and each edge of $L(G)$ represents a length-two directed path of G. For illustration, a directed network and its corresponding line digraph are shown in Figs. 7(a) and (b).

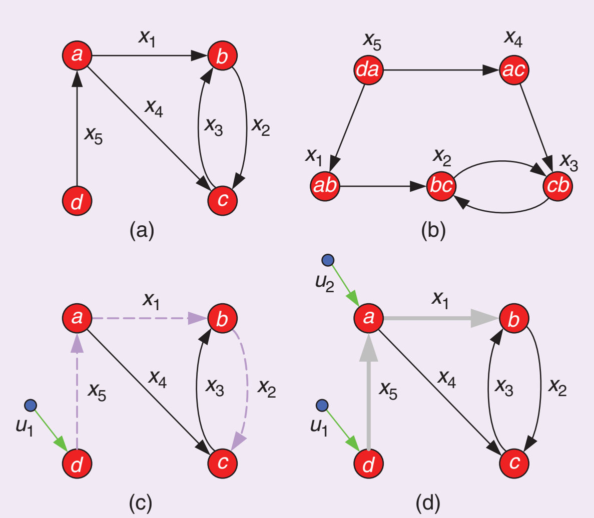


Figure 7. Control of an SBD. (a) A directed network G with four nodes: a, b, c, and d, and five edges: x_i $(i = 1, 2, \ldots, 5)$. (b) The line digraph $L(G)$ of the original directed network G. (c) Applying the minimum inputs theorem to G yields a control path denoted by dashed (purple) lines, where only the node d can be chosen as the driver node; that is, the input signal u_1 should be applied to the node d in order to ensure structurally controllability. (d) Applying the minimum inputs theorem to $L(G)$ yields two driver nodes in the line digraph, which correspond to two driven edges (marked in grey) in the original digraph G. To control the edges x_1 and x_5, two independent input signals u_1 and u_2 should be added to the nodes d and a in the original digraph G. That is, $N_D = 1$ is required for the network based on nodal dynamics but $N_D = 2$ for that based on edge dynamics.

Note that Eq. (14) essentially describes an LTI system in the form of (5) with the substitutions of $A = W - T$ and $B = H$. The controllability of system (14), which will be referred to as the controllability of edge dynamics below, can be addressed by employing the structural control theory if $W - T$ and H are structured matrices. Applying the minimum inputs theorem (Theorem 7) to $L(G)$ gives a set of control paths and driver nodes in the line digraph, or equivalently, a set of driven edges (see Fig. 7(d), marked in grey) in the original digraph G. It is important to stress that the minimum inputs theorem only ensures the number of driver nodes in $L(G)$ be minimum, which does not imply that the obtained set of driver nodes in G is also minimum. Comparing the results from Figs. 7(c) and (d), it is clear that the minimum number of driver nodes in G (to ensure structural controllability for nodal dynamics) is less than those in $L(G)$ (to ensure structural controllability based on edge dynamics).

Based on the structural controllability theory and line digraph theory, Nepusz and Vicsek [31] investigated the controllability of directed networks with edge dynamics, and derived several results from nodal dynamics: (i) driver nodes prefer hubs with large out-degrees; (ii) heterogeneous and sparse networks are more controllable than homogeneous and dense ones; (iii) positive correlation between the in-degree and out-degree of a node enhances the controllability of edge dynamics, without affecting the controllability of the nodal systems; (iv) adding self-loops to individual nodes enhances the controllability of nodal dynamics, but leaves the controllability of edge dynamics unchanged.

In the original switchboard dynamics (SBD), the switching matrix must be a structural matrix. In [32], Pang et al. relaxed this restriction of a structural matrix and introduced a general switchboard dynamics (GSBD) with any type of switching matrices, and found that the interaction strength plays an even more significant role in the controllability of edge dynamics than the network structure does.

3.4 Popov-Belevitch-Hautus Rank Criterion

The classical Popov-Belevitch-Hautus (PBH) rank criterion (Sec. 2.1) connects the state controllability of (A, B) to the maximum multiplicity of the eigenvalues of the system matrix A. This criterion can be used to exactly solve the minimum inputs problem, as summarized below.

Theorem 8 (Maximum Multiplicity Theorem) [33]

For arbitrary network structures and edge weights, the minimum number of driver nodes is equal to the maximum geometric multiplicity of the eigenvalues of A, i.e.,

$$N_D = \max_i \{\mu(\lambda_i)\}, \tag{15}$$

where λ_i $(i=1,2,\ldots,l)$ denotes the distinct eigenvalues of the matrix A, and $\mu(\lambda_i)=N-rank(\lambda_i I_N-A)$ is the geometric multiplicity of λ_i. For undirected networks with arbitrary edge weights, N_{D} is determined by the maximum algebraic multiplicity of the eigenvalues λ_i of A, i.e.,

$$N_{\mathrm{D}}=\max_i\{\rho(\lambda_i)\}, \tag{16}$$

where $\rho(\lambda_i)$ is the algebraic multiplicity of λ_i.

Note that condition (15) is rigorous and generally valid without any limitation on the matrix A. Also, condition (16) is valid for general undirected networks with diagonalizable matrix A.

3.4.1 Exact Controllability

Structural controllability provides a basic framework for determining the controllability of directed networks characterized by their structured matrices (Sec. 2.2). This framework is ideal for many complex networked systems for which only the underlying wiring diagram is known (i.e., zero or nonzero values, indicating the absence or presence of physical connections) but not the edge characteristics (e.g., their weights). Yet, the assumption of having independent free parameters is quite strong, usually not satisfied by many real-world systems, since system parameters are often related to each other. For example, in undirected networks the matrix A in system (5) is symmetric, and in unweighted networks all the edge weights are the same, both reflecting certain mutual dependence among parameters. In these cases, the structural control theory could yield misleading results on the minimum number of driver nodes. Based on the PBH rank criterion in control theory, Yuan et al. [33] developed a so-called *exact controllability* paradigm as an alternative to the conventional structural controllability framework, which offers a useful tool to study the controllability of complex dynamical networks with arbitrary structures and edge-weight distributions, including directed, undirected, weighted and unweighted networks, even with self-loops. The notion of exact controllability of complex networks is essentially the state controllability of complex systems, but for the purpose of readability and uniformity, sometimes more convenient to use.

The exact controllability theory was applied to fractal networks [34], bipartite networks [35] and multiplex networks [36]. Nie et al. [37] further investigated the effects of degree correlation on the exact controllability of multiplex networks. They found that the minimal number of driver nodes decreases with the correlation for networks with lower density of interconnections, but conversely for networks with higher density of interconnections.

3.4.2 Node Self-Dynamics

It is obvious that, other than the network topology and edge dynamics, the node system (nodal dynamics) is also a crucial factor affecting the controllability. Cowan et al. [38] pointed out that the main results in [24] depend strongly on a critical assumption about the model: each node has an infinite time constant (i.e., each node is treated as a pure integrator). However, several real networks considered therein, including food webs, power grids, electronic circuits, regulatory networks, and neuronal networks, typically manifest more general dynamics at each node, for example with finite time constants. In [38], the structural controllability of directed networks with linear nodal dynamics is analyzed, assuming that each node has a self-loop. They found that only one single time-varying input is needed to guarantee the network controllability. Zhao et al. [39] investigated the synergistic effect of network topology and the dth-order individual dynamics on the exact controllability. Interestingly, they found a global symmetry accounting for the invariance of the controllability with respect to the exchange of the densities between any two different types of node dynamics, irrespective of the network topology. Recently, some more general and higher-dimensional node systems were investigated [40]–[42].

Consider a general directed and weighted network consisting of identical multi-input multi-output (MIMO) LTI node-systems [40],

$$\begin{aligned}\dot{x}_i(t)&=Wx_i(t)+\sum_{j=1}^{N}a_{ij}Hy_j(t)+\delta_i Bu_i(t),\\ y_i(t)&=Cx_i(t),\quad i=1,2,\ldots,N,\end{aligned} \tag{17}$$

in which $x_i(t)\in\mathbb{R}^n$ is the state vector, $u_i(t)\in\mathbb{R}^p$ the input vector and $y_i(t)\in\mathbb{R}^m$ the output vector of node i; $H\in\mathbb{R}^{n\times m}$ is the inner-coupling matrix and $A=(a_{ij})\in\mathbb{R}^{N\times N}$ the outer-coupling matrix, i.e., the adjacency matrix of the network; $C\in\mathbb{R}^{m\times n}$ is the output matrix; $B\in\mathbb{R}^{n\times p}$ is the input matrix; $\delta_i=1$ if node i is under control, but $\delta_i=0$ otherwise. This networked system (17) can be rewritten in a compact form as

$$\dot{x}(t)=(I_N\otimes W+A\otimes HC)x(t)+(\Delta\otimes B)u(t), \tag{18}$$

with $\Delta=diag\{\delta_1,\delta_2,\ldots,\delta_N\}$.

The following result was obtained by Wang et al. [40], based on the classic PBH rank condition.

Theorem 9 [40]

The networked system (18) with identical MIMO node systems is state controllable if and only if the following system of two matrix equations has a unique zero solution $X=0$:

$$\Delta^T XB = 0 \quad \text{and} \quad A^T XHC = X(\lambda I_n - W) \quad \text{for all} \quad \lambda \in \mathbb{C}. \tag{19}$$

Lately, Hao et al. [41] further investigated the controllability of networked MIMO systems and derived a more convenient necessary and sufficient condition, which is effective and can be easily verified. Xiang et al. [42] extended the above results to heterogeneous networked MIMO systems and found that the controllability of heterogeneous networks is very different in some aspects from that of homogeneous networks.

Considering the situation where less transmitted information is more economical, the controllability of networked systems with one-dimensional communication was investigated by Wang et al. [43]. In this single-input single-output (SISO) setting, for a directed network of identical node systems with higher-dimensional states, described by (17) with $m = p = 1$ therein, a precise and easily verified necessary and sufficient condition was derived. To state the result, some more notations are needed. Let

$$\Omega = \{i \mid \delta_i = 1, \quad i = 1, 2, \ldots, N\},$$

and define

$$\Lambda(s) = \left\{ \left[[\alpha_1^T, \alpha_2^T, \ldots, \alpha_N^T] \middle| \begin{matrix} \alpha_i \in \Lambda_1(s) \ \text{ for } \ i \notin \Omega \\ \alpha_i \in \Lambda_2(s) \ \text{ for } \ i \in \Omega \end{matrix} \right] \right\},$$

where $s \in \sigma(W)$, which is the spectrum of matrix W, and

$$\Lambda_1(s) = \{\xi \in \mathbb{C}^{1\times n} \mid \xi(sI - W) = 0\},$$

$$\Lambda_2(s) = \{\xi \in \mathbb{C}^{1\times n} \mid \xi \in \Lambda_1(s), \ \ \xi B = 0\}.$$

Theorem 10 [43]

The networked system (18) with identical SISO higher-dimensional node systems is state controllable if and only if the following conditions are satisfied simultaneously:

i) (W, H) is controllable;
ii) (W, C) is observable;
iii) $\kappa A \neq 0$ for $\kappa \in \Lambda(s)$, $\kappa \neq 0$, $\forall s \in \sigma(W)$;
iv) $rank\,[I_N - \gamma A, \eta\Delta] = N$, where $\gamma = C(sI_n - W)^{-1}H$, $\eta = C(sI_n - W)^{-1}B$, $\forall s \notin \sigma(W)$.

When Theorem 9 is applied to some special types of directed networks, such as directed trees and cycles, more precise conditions can be obtained [40]. In addition, Theorem 10 can also be derived from Theorem 9, but the former is easier to check. For example, condition (iii) in Theorem 10 are automatically satisfied for cycles and condition (iv) holds automatically for chains [43].

3.5 Popov-Belevitch-Hautus Eigenvector Test

Consider a directed and weighted networked system with linear or linearized non-identical dynamics, described by [44]

$$\dot{x}_i(t) = w_i \Gamma x_i(t) + \sum_{j=1}^{N} L_{ij} \Gamma(x_i(t) - x_j(t)) + \delta_i B u_i(t), \tag{20}$$

where $x_i(t) \in \mathbb{R}^n$ is the state vector of the ith node, $u_i(t) \in \mathbb{R}^p$ the control input of node i, and $B \in \mathbb{R}^{n\times p}$ the input matrix; $\delta_i = 1$ if node i is subject to an exogenous control signal and $\delta_i = 0$ otherwise; $w_i \Gamma x_i(t)$ ($w_i \in \mathbb{R}$ and $w_i \neq 0$) describes the intrinsic dynamics of node i; $\Gamma = (\gamma_{ij}) \in \mathbb{R}^{n\times n}$ is a constant matrix indicating the inner-coupling between different components; $L = (L_{ij}) \in \mathbb{R}^{N\times N}$ is the Laplacian matrix of the network.

Without loss of generality, the indices of the nodes can be reorganized in such a way that the first q $(1 \le q < N)$ nodes (leaders) are chosen to be controlled. Defining $x(t) = [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T \in \mathbb{R}^{nN}$ and $u(t) = [u_1^T(t), \ldots, u_q^T(t), 0^T, \ldots, 0^T]^T \in \mathbb{R}^{pN}$, system (20) can be rewritten in a matrix form as

$$\dot{x}(t) = [(W - L) \otimes \Gamma] x(t) + (\Delta \otimes B) u(t), \tag{21}$$

where $W = diag\{w_1, w_2, \ldots, w_N\}$ and $\Delta = diag\{\overbrace{1, \ldots, 1}^{q}, \overbrace{0, \ldots, 0}^{N-q}\}$.

The following result was established by Xiang et al. [44], based on the PBH eigenvector test.

Theorem 11 [44]

The leader-followers networked system (21) with non-identical node-systems is state controllable if and only if the following two conditions are satisfied simultaneously:

i) (Γ, B) is controllable;
ii) there exists no left-eigenvector of matrix $L - W$ with the first q entries being all zeroes, where q is the number of leaders.

Example 7

Consider Fig. 8, with

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

It can be easily verified that the pair (Γ, B) is controllable. Also, one has

$$L - W = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 3 & 0 \\ -1 & 0 & -2 \end{bmatrix}.$$

The eigenvalues of $L - W$ are $\lambda(L - W) = \{-2, -1, 3\}$ and the corresponding left-eigenvectors are $\{[1, 0, 1]^T, [1, 0, 0]^T, [1, -4, 0]^T\}$. Here, the first node is the leader. According to Theorem 11, the integrated networked system (Fig. 8(c)) is controllable. It should be noted that the digraph (Fig. 8(a)) is uncontrollable if only node 1 is controlled, although each node system (Fig. 8(b)) is controllable. ■

Based on Theorem 11, several specific results ($w_1 = w_2 = \ldots = w_N = w$) can be obtained, as follows:

i) A directed path is controllable if the beginning node is selected to be the only leader.
ii) A directed cycle with a single leader is controllable.
iii) A complete digraph with a single leader is uncontrollable.
iv) A star digraph (with more than 2 nodes) is uncontrollable if the center node is the only leader.

3.6 Switched Systems Theory

In the real world, the interaction topologies of networked systems may be varying with time [45]. For example, in p2p communication networks, contact patterns among individuals such as telephoning and emailing, are typically temporal [46]. Recently, attention was attracted to the controllability of temporal networks [47]–[49]. Temporal networks are networks in which nodes and edges may appear and disappear at various time instants. Although temporal networks are essentially time-varying systems, the irreversibility of time and the deterministic chronological order in these networks distinguish them from general time-varying systems. In the framework of conventional switching networks [50], [51], the edges can periodically switch to repeat some topological forms, but this is not the case in temporally networks due to time causality, rendering the existing methodologies of switched systems theory inapplicable to temporal networks in general.

A temporally switching network G is characterized by a node set $\{1, 2, \ldots, N\}$, a static (time-invariant) network topology set represented by $G_1, G_2, \ldots, G_m$ in a specified order, where G_k (and G_m) exists only on the time interval $[t_{k-1}, t_k)$, $k = 1, 2, \ldots, m-1$ (and $[t_{m-1}, t_m]$). The adjacency matrix $A_k = [a_{ij}(k)] \in \mathbb{R}^{N \times N}$ of G_k, is defined by

$$a_{ij}(k) \begin{cases} \neq 0, & \text{edge } (j, i, [t_{k-1}, t_k)) \neq \varnothing, \\ = 0, & \text{otherwise}, \end{cases}$$

where $a_{ij}(k)$ denotes the directed edge from node j to node i on the time interval $[t_{k-1}, t_k)$.

Consider a linear temporally switching system of the form [48]

$$\dot{x}(t) = A_o(t)x(t) + Bu(t), \quad x(t_0) = x_0 \qquad (22)$$

where $x(t) \in \mathbb{R}^N$ is the state vector, $u(t) \in \mathbb{R}^M$ is the input vector, $B \in \mathbb{R}^{N \times M}$ is the constant input matrix, and the elements of the state matrix $A_o(t) : \mathbb{R} \to \mathbb{R}^{N \times N}$ are piecewise constant functions of $t \in [t_0, t_1) \cup [t_1, t_2) \cup \cdots \cup [t_{m-1}, t_m]$. This temporally switching system with fixed external inputs can be simply represented by the matrix pair $(A_o(t), B)$.

Definition 6

The linear temporally switching system (22) is state controllable on the time interval $[t_0, t_m]$ if, for any initial state $x_0 \in \mathbb{R}^N$ at $t_0 \geq 0$, there exists an input $u(t) \in \mathbb{R}^M$, $t \in [t_0, t_m]$ such that $x(t_m) = 0$.

The following result was obtained by Hou et al. [48].

Theorem 12 [48]

The linear temporally switching system (22) is state controllable on the time interval $[t_0, t_m]$ if and only if its controllability matrix

$$[e^{(t_m - t_{m-1})A_{om}} \cdots e^{(t_2 - t_1)A_{o2}} Q_1, \ldots, e^{(t_m - t_{m-1})A_{om}} Q_{m-1}, Q_m]$$

has full row rank, where $Q_i = [B, A_{oi}B, \ldots, A_{oi}^{N-1}B]$ is the controllability matrix of the static network on the ith time interval $[t_{i-1}, t_i)$, $i = 1, 2, \ldots, m-1$, and $[t_{m-1}, t_m]$.

For notational simplicity, a temporally switching network with fixed external inputs is denoted by $(\tilde{A}, B)$. Note that $(\tilde{A}, B)$ can be described by the matrix pairs (A_k, B), $k = 1, 2, \ldots, m$, in a time sequence. A linear temporally switching network $(\tilde{A}, B)$ and a linear temporally switching system $(A_o(t), B)$ have the *same structure* if and only if they have the same number of system matrices, $A_1, A_2, \ldots, A_m$ and $A_{o1}, A_{o2}, \ldots, A_{om}$, and moreover they have the same (fixed) zero and (parametric)

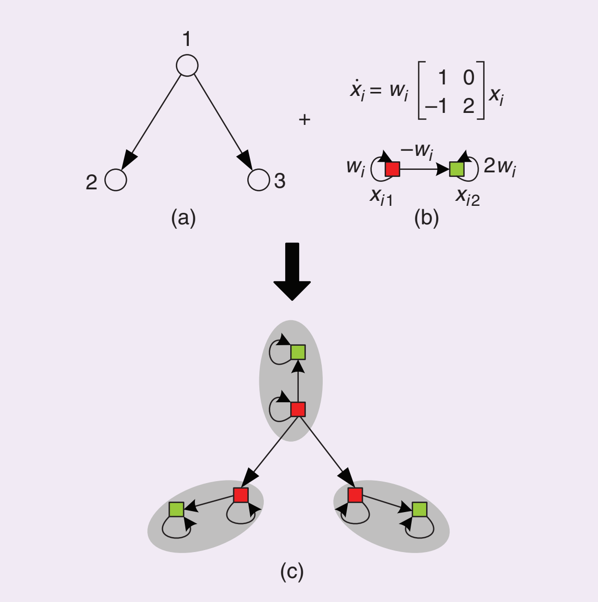


Figure 8. Integration of network topology and node self-dynamics. (a) Network Topology (b) Dynamic Unit (c) Integrated Network.

nonzero patterns in their corresponding system matrices $A_1, A_2, \ldots, A_m$ and $A_{o1}, A_{o2}, \ldots, A_{om}$.

Definition 7
A linear temporally switching network G represented by $(\tilde{A}, B)$ with fixed external inputs is structurally controllable if and only if there exists a state controllable linear temporally switching system (22) with the same structure as $(\tilde{A}, B)$.

Note that the temporally switching network G is structurally controllable if and only if for each admissible realization of the independent nonzero parameters on the time intervals $[t_{k-1}, t_k)$, $k = 1, 2, \ldots, m-1$, and $[t_{m-1}, t_m]$, the corresponding system (A_k, B) is state controllable for all $k = 1, 2, \ldots, m$.

With a new temporal interpretation of the dilation and intersection concepts, a graph-theoretic criterion was derived by Hou et al. [48], via the concept of n-walk, as follows.

Theorem 13 [48]
A linear temporally switching network G of size N with fixed r $(1 < r \leq N)$ external inputs is structurally controllable if and only if there exists an n-walk in the graph of $(\tilde{A}, B)$.

For discrete-time networked systems, an alternative characterization can be derived from the algebraic systems theory [52].

Consider the state-space form of a general discrete-time linear network,

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{23}$$

with appropriate dimensions.
The concept of *state controllability-from-0* (or *reachability*) for network (23) is defined as follows.

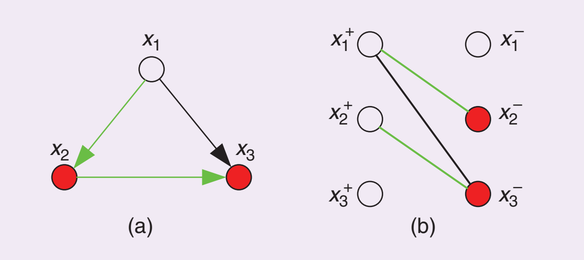


Figure 9. Determination of maximum matching. The maximum matching of the digraph (a) can be determined from its bipartite representation (b), which is obtained by splitting each node x_i into two nodes x_i^+ and x_i^-, and then placing an edge (x_j^+, x_i^-) in the bipartite graph if there is a directed edge from x_j to x_i in the original digraph.

Definition 8
The discrete-time linear network (23) is said to be *state controllable-from-0* (or *reachable*) if, for the zero initial state $x(0) = 0$ and any final state $x(t_f) \in \mathbb{R}^N$, there exist a positive integer T and a finite input sequence $u(0), u(1), \ldots, u(T)$ such that $x(T) = x(t_f)$.

Recall [52] that the transfer function (matrix) of the network (23), $G(z) = C(zI - A)^{-1}B$, always has a right-coprime factorization $G(z) = N_r(z)D_r^{-1}(z)$ and a left-coprime factorization $G(z) = D_l^{-1}(z)N_l(z)$.

A characterization of the reachability for the network (23) is given as follows [52].

Theorem 14
The discrete-time linear network (23) is reachable if and only if

i) $rank\,[B, AB, \ldots, A^{N-1}B] = N$.
ii) $zI - A$ and B are left coprime for all $z \in \mathbb{C}$.
iii) $rank\,[zI - A, B] = N$ for all $z \in \mathbb{C}$.

4. Synthesis of Network Controllability

In order to fully control a complex networked system, the first important step is to identify the driver nodes that can ensure the network controllability. Mathematically, it is the problem of how to design an appropriate input matrix B to achieve this goal. For system (5), there are many possible selections of B that can satisfy the controllability conditions. The objective here is first to find a set of B corresponding to the minimum number N_D of driver nodes required to control the whole network. The second step is to design an appropriate input $u(t) = [u_1(t), u_2(t), \ldots, u_{N_D}(t)]^T$ through which one can steer the networked system from any initial state to any desired state in finite time. For system (5), there are many possible inputs that can achieve this goal of control. The task, then, is to use an optimal control input to minimize the required energy cost.

In this section, the issues on the selection of driver nodes, optimization of network controllability and control energy are discussed.

4.1 Selection of Driver Nodes

4.1.1 Approaches Based on Maximum Matching

The minimum inputs theorem (Theorem 7) provides a guideline to identify the driver nodes in a directed network using maximum matching. From the structural controllability theorem (Theorem 2), the cactus is the most economical topological pattern to propagate control influence, because it is a minimal structure such that the removal of any edge will render the network uncontrollable. A maximum matching exhibits the important edges using which one can construct the cactus structures efficiently in a network. Therefore, the maximum matching not only reveals the minimum controllability structure but also consists of a backbone of the key control influence routes.

The maximum matching of a digraph can be identified by mapping the digraph to its bipartite representation. Consider a directed network $G(A)$, whose bipartite representation can be described by $Y(A) \equiv (V_A^+ \cup V_A^-, E_A)$, where $V_A^+ = \{x_1^+, x_2^+, \ldots, x_N^+\}$ and $V_A^- = \{x_1^-, x_2^-, \ldots, x_N^-\}$ are the sets of nodes corresponding to the columns and rows of the matrix A, respectively, and $E_A = \{(x_j^+, x_i^-) | a_{ij} > 0\}$ is the edge set of the bipartite graph. More precisely, each node x_i of the original digraph is split into two nodes x_i^+ and x_i^-. If there is a directed edge from node x_j to node x_i in the original digraph, then an edge between x_j^+ and x_i^- is established in its bipartite version. For example, the maximum matching of the original digraph, e.g., (x_1, x_2), (x_2, x_3) in Fig. 9(a), can be determined from its bipartite representation, e.g., (x_1^+, x_2^-), (x_2^+, x_3^-) in Fig. 9(b). The maximum matching of any bipartite graph can be obtained using the Hopcropt-Karp algorithm [53].

One can determine the set of driver nodes to ensure the structural controllability, in three steps: (i) for the given directed network, obtain its bipartite representation; (ii) identify a maximum matching on the underlying bipartite graph based on the Hopcropt-Karp algorithm; (iii) add a unique control input to each unmatched node (driver node).

The minimum set of driver nodes is usually not unique [24], but one can often obtain multiple potential control configurations with the same number of driver nodes. Take a general directed network for example. One can easily find two different maximum matchings, as shown in Figs. 10(b) and (c), respectively for the original digraph (Fig. 10(a)). As a result, there are two minimum driver node sets, with the same number of $N_D = 2$. Adding a unique input to each driver node, the structural controllability of the underlying network can be guaranteed.

Since some nodes may appear in some minimum driver node set but not in the other, it is natural to ask: (i) what is the role of each individual node in controlling a complex networked system? (ii) which nodes play most crucial roles in controlling the complex networked system? These two intrigued questions prompt the investigations of nodes classification and the measure of network controllability, which can provide another guideline for efficiently selecting the driver nodes.

Wang et al. [54] proposed a structural index, i.e. control range, to quantify the size of the subnetwork that a node can effectively control. Later, they [55] introduced a direction-specific index, namely a domination centrality, to assess the intervention capabilities of nodes in a directed network. The concept of control centrality used to quantify the ability of a single node in controlling a directed weighted network was introduced in [56]. The distribution of the control centrality was calculated for several real-world networks, revealing that it is mainly determined by the network's degree distribution. Jia and Barabási [57] introduced the concept of control capacity to quantify the likelihood that a node is a driver node. They developed a random sampling algorithm to efficiently measure this quantity with a limited number of minimum driver nodes. Ding et al. [58] introduced the concept of control backbone to quantify a node's importance for maintaining structural controllability of a directed network and they developed a random sampling algorithm to effectively compute it. On the other hand, Jia et al. [59] classified the nodes in a network based on their roles in control. Accordingly, a node is critical, intermittent or redundant, if it acts as a driver node in all, in some or in none of the control configurations. They also developed an analytical framework to identify the category that each node belongs, leading to the discovery of two distinct control modes in complex networked systems: centralized versus distributed controls. In addition, they confirmed that the emergence of the bifurcation in control coincides with the formation of the core and that the structure of the core determines the control mode of the network [60]. Ruths et al. [61] decomposed the driver nodes into three groups, i.e., source nodes, external dilations and internal dilations, and then depicted the control profiles of the network according to this classification.

4.1.2 Approaches Based on Elementary Column Transformation

Yuan et al. [33] offered a general method for identifying a minimum set of driver nodes in a network, based on the PBH rank criterion and the elementary column transformation. Recall that the PBH rank criterion states that the

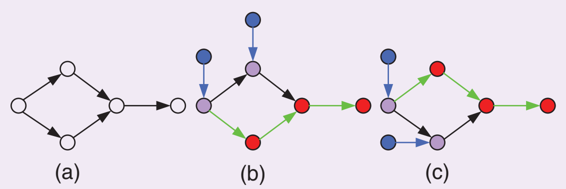


Figure 10. Identifying the driver nodes based on maximum matching. For a general directed network, as shown in (a), there could be multiple maximum matchings (green edges), as shown in (b) and (c). Therefore, one can identify multiple minimum driver node sets (purple nodes). To each driver node (i.e., unmatched node) one must add a unique control signal (blue), necessarily to ensure structural controllability.

networked system (5) is state controllable if and only if $rank\,[sI_N - A, B] = N, \forall s \in \mathbb{C}$. According to the maximum multiplicity theorem (Theorem 8), N_D is determined by the maximum geometric multiplicity $\mu(\lambda^M)$ of the eigenvalue λ^M. Therefore, the input matrix B used to ensure the network controllability must satisfy the condition that $rank\,[\lambda^M I_N - A, B] = N$. Since the rank of the matrix $[\lambda^M I_N - A, B]$ is contributed by the number of linearly independent rows, employing the elementary column transformation on the matrix $\lambda^M I_N - A$ (or $A - \lambda^M I_N$) will yield a set of linearly dependent rows that violate the full-rank condition. The control signals located via B should be imposed on the identical rows, so as to eliminate all linear correlations and to ensure the full-rank condition. The steps to identify the driver nodes to ensure exact controllability are as follows: (i) for the given network, compute the eigenvalues λ of the matrix A and their geometric multiplicities $\mu(\lambda)$, to find the eigenvalue λ^M corresponding to the maximum geometric multiplicity $\mu(\lambda^M)$; (ii) obtain the matrix $A - \lambda^M I_N$; (iii) perform an elementary column transformation on $A - \lambda^M I_N$ to get its column canonical form, which reveals the linear dependence among all rows. The rows that are linearly dependent on the others correspond to the driver nodes with number $N - rank\,(\lambda^M I_N - A)$, which is the maximum geometric multiplicity $\mu(\lambda^M)$.

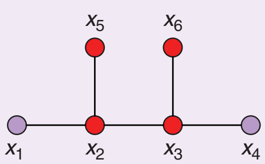
Figure 11 shows a schematic example on the determination of driver nodes using the elementary column transformation. Note that the configuration of driver nodes is not unique because it is determined by the order in operating the elementary column transformation and by the different choices of the linearly dependent rows. However, the minimum number of driver nodes is unique, which depends on the value of $\mu(\lambda^M)$. This approach to identifying driver nodes is applicable to any network, including particularly networks characterized by structured matrices.

4.2 Optimization Methods

Optimization of the network controllability is of great importance in most real applications. To date, some representative methods of optimizing the network controllability can be classified as structure perturbations [62], [63] and edge orientations [64].

Based on the minimum inputs theorem (Theorem 7), Wang et al. [62] proposed a general approach to optimizing the structural controllability of complex networks by minimum structure perturbations. The optimization process involves three steps: (i) finding the minimum number of independent matching paths using the maximum matching algorithm; (ii) randomly ordering all found matching paths; (iii) linking the ending points of each matching path to the starting nodes of the matching paths next to it in the above order. In operation, the minimum number N_D of driver nodes can be reduced to one, maintaining the network connectivity. The principle of the perturbation strategy is validated theoretically and demonstrated numerically for both homogeneous and heterogeneous random networks and for a number of real networks from nature and society. It is found that, for most real networks, about 5% of additional edges are sufficient for optimizing the controllability. In [63], an adding-edge strategy and a turning-edge strategy are proposed to optimize the controllability of switchboard edge dynamics via minimum structural perturbations. Both simulations and analysis indicate that the minimum number of adding-edges required for the optimal controllability is equal to the minimum number of turning-edges, and networks with positively correlated in- and out-degrees are easier to achieve optimal edge controllability.

Considering the difficulty and impracticality of changing the original network topology for a large-scale real system, Xiao et al. [64] introduced an effective method for enhancing the structural controllability of a complex



$$A - \lambda^M I_N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Column Canonical Form} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \lambda = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \quad \begin{matrix} \lambda^M = 0 \\ \mu(\lambda^M) = 2 \end{matrix}$$

Figure 11. Identifying the driver nodes based on elementary column transformation. For a general undirected network, the matrix $A - \lambda^M I_N$, the column canonical form of $A - \lambda^M I_N$ after an elementary column transformation, the eigenvalues λ and the eigenvalue λ^M corresponding to the maximum geometric multiplicity $\mu(\lambda^M)$ of A are given, respectively. The rows that are linearly dependent on the others in the column canonical form are marked in red. The nodes corresponding to them are the driver nodes, which are marked in purple in the original network. The set of driver nodes is not unique since it depends on the elementary column transformation, while the number of driver nodes is unique.

network by assigning edge directions while keeping the network topology unchanged. They showed that the disassortativity pattern could weaken the orientation for optimal controllability, while the assortativity pattern has no impact on the edge orientation regarding the optimal controllability. In [65], a method was developed for enhancing the structural controllability of a directed network by changing the directions of a small potion of edges, while keeping the total number of edges unchanged. The basic idea is to find candidate edges based on matching paths. It was found that the nodes linked to candidate edges have a distinct character, which provides a strategy for improving the controllability based on the network local structures. Since the whole topology of a real network is usually not visible and one can only get some local structure information, this strategy is more practical compared to those that demand complete information of the whole network topology.

In [62]–[65], the optimal controllability is characterized by the minimum number of driver nodes. Various effective and efficient strategies were developed for minimizing the number of driver nodes to ensure the network controllability in different scenarios. Ding et al. [66] proposed a design strategy for optimal control configurations of a directed network that can maximize its controllability index (i.e., the maximum dimension of the controllable subnetwork) with a fixed number of driver nodes.

4.3 Control Energy

When controlling a real networked system, an important and unavoidable issue is the energy cost of the actual control. Much effort has been devoted to the network controllability study from an energy perspective [67]–[72]. Some works revealed that the energy required for controlling a complex network depends on the number of driver nodes, and that the energy distribution follows an algebraic scaling law [67], [68].

Assume that the weighted network (5) can be driven from any initial state $x_0 \in \mathbb{R}^N$ to any desired state $x_d \in \mathbb{R}^N$ over a finite time interval $t \in [0, \tau]$ by injecting an appropriate external input $u(t)$. The associated control energy is defined as $\varepsilon(\tau) = \int_0^\tau \|u(t)\| dt$. For a fixed number of driver nodes, the external input $u(t)$ has many different choices to steer the system from x_0 to x_d. Among all the possible inputs, the optimal control input that minimizes the control energy is given by

$$u(t) = B^T e^{A^T(\tau - t)} W^{-1}(\tau) \nu_\tau, \tag{24}$$

where $W(\tau) = \int_0^\tau e^{At} BB^T e^{A^T t} dt$ is the positive definite (PD) and symmetric Gramian matrix, which serves for determining quantitatively if a system is controllable. It follows from the classical Gramian matrix criterion that the system is controllable if and only if $W(\tau)$ is nonsingular. The required energy corresponding to the optimal control input (24) is given by $\varepsilon(\tau) = \nu_\tau^T W^{-1}(\tau) \nu_\tau$, where $\nu_\tau = x_d - e^{A\tau} x_0$ denotes the difference vector between the desired state under control and the final state after a free evolution without control. Without loss of generality, one can set the desired state at $x_d = 0$. Then, the minimal energy can be rewritten as

$$\varepsilon(\tau) = x_0^T H^{-1}(\tau) x_0,$$

where $H(\tau) = e^{-A\tau} W(\tau) e^{-A^T\tau}$. When the system (5) is controllable, $H(\tau)$ is positive definite. Thus, the normalized control energy can be obtained as follows:

$$E(\tau) = \frac{\varepsilon(\tau)}{\|x_0\|^2} = \frac{x_0^T H^{-1}(\tau) x_0}{x_0^T x_0}. \tag{25}$$

When x_0 is parallel to the direction of one of the eigenvectors of $H(\tau)$, the inverse of the corresponding eigenvalue denotes the normalized energy associated with the action of controlling the system along the particular eigen-direction. Using the classical Rayleigh-Ritz theorem, the normalized control energy has the bounds

$$\frac{1}{\eta_{\max}} = E_{\min} \le E(\tau) \le E_{\max} = \frac{1}{\eta_{\min}}, \tag{26}$$

where $\eta_{\max}$ and $\eta_{\min}$ are the maximum and minimum eigenvalues of the positive definite matrix $H(\tau)$, respectively [67].

Now, consider the undirected weighted networked system (5) with linear node self-dynamics, characterized by $a_{ii} = -a - \sum_{j=1}^N a_{ij}$, where a is a tunable parameter that can make the symmetric matrix A either positive definite or negative definite (ND). Assume that one single node with index c is controlled, i.e., $N_D = 1$. The lower and upper bounds of the control energy can be estimated as follows [67]:

$$E_{\min} \sim \begin{cases} \tau^{-1} & \text{small } \tau, \\ \dfrac{1}{[(A+A^T)^{-1}]_{cc}} & \text{large } \tau,\ A \textit{ is } \text{PD}, \\ \tau^{-1} \to 0 & \text{large } \tau, A \textit{ is } \text{semi PD}, \\ e^{2\lambda_N \tau} \to 0 & \text{large } \tau,\ A \text{ is not PD}, \end{cases} \tag{27}$$

$$E_{\max} \sim \begin{cases} \tau^{-\theta}(\theta \gg 1) & \text{small } \tau, \\ \varepsilon(A,\ c) & \text{large } \tau, A \text{ is not ND}, \\ \tau^{-1} \to 0 & \text{large } \tau, A \text{ is semi ND}, \\ e^{2\lambda_1 \tau} \to 0 & \text{large } \tau, A \text{ is ND}. \end{cases} \tag{28}$$

Here, $\lambda_1 > \lambda_2 > \cdots > \lambda_N$ are the eigenvalues of A, and $\varepsilon(A, c)$ denotes a positive energy that depends on both the matrix A and the controlled node c. The scaling laws in (27) and (28) can be easily extended to directed weighted networks (if A is diagonalizable), where the decay exponents λ_1 and λ_N are replaced by $\mathrm{Re}(\lambda_1)$ and $\mathrm{Re}(\lambda_N)$, respectively.

Herein, the complexity of the fundamental problem of control energy has been reduced from the complicated and intractable Gramian matrix $H(\tau)$ to the simple system matrix A, which is directly related to the network topology. The results in (27) and (28) suggest that the scaling of the energy cost for controlling complex networks is sensitive to the control time τ. For small τ, setting a relatively longer time for control can lead to less energy consumption. For large τ, there exists a situation in which the control energy converges to a steady-state value and cannot be further reduced even if much longer time is allowed to spend.

The control energy is also sensitive to the direction in the state space along which one wants to move the whole system. Moving a network state to various directions in the state space requires different amounts of energy. Yan et al. [68] showed that the variability of control energy along different directions in the state space depends strongly on the number of driver nodes. In particular, for a scale-free network with degree exponent γ, if all nodes are directly driven ($N_{\mathrm{D}} = N$), control is energetically feasible, as the maximum energy increases sublinearly with the system size $\varepsilon_{\max} \sim N^{1/(\gamma-1)}$. However, if only one single node is controlled ($N_{\mathrm{D}} = 1$), control in some directions is energetically prohibitive, which increases exponentially with the system size $\varepsilon_{\max} \sim e^{N}$. For the cases with $1 < N_{\mathrm{D}} < N$, the maximum energy decays exponentially as the number of driver nodes increases, $\varepsilon_{\max} \sim e^{N/N_{\mathrm{D}}}$.

Subsequently, Chen et al. [69] uncovered the general mechanism for the above energy scaling behavior, by identifying the longest control chains (LCC), which are the fundamental structures embedded in a network that dominate the required control energy. Based on the LCC, they developed an optimization strategy to drastically reduce the control energy, with applications to some real-world networks. Notice that the LCC are conceptually different from the control signal paths (CSP), or the stems [8], [24]. Finding the LCC in a network can be done in a few steps: (i) use the maximum matching algorithm to find all driver nodes; (ii) identify the shortest paths from each of the driver nodes to each of the non-driver nodes; (iii) pick out the longest such shortest paths as LCC. Actually, control energy flows through the control chains, while the control signal is propagated along the CSP. While control signals travel through the CSP, the energy may not flow along the same paths due to the interactions among the nearest neighbors via their physical connections. For example, the state change of a node can lead to the state changes of all its nearest neighbors through the energy exchange among them; that is, control energy flows through the LCC, not CSP.

Quite recently, Wang et al. [70] developed some strategies to turn a physically uncontrollable network into a physically controllable one based on the idea of LCC [69]. Their findings indicate that, although full control can be theoretically guaranteed by the prevailing the structural controllability theory, it is necessary to balance the number of driver nodes and the required energy cost in practical applications of the control theory.

For the discrete version of system (5) described by $x(k+1) = Ax(k) + Bu(k)$, Pasqualetti et al. [71] investigated the joint problem of selecting a set of driver nodes and of designing a control input to steer the network to a target state. By combining the state controllability with graph theory, they utilized the smallest eigenvalue of the T-step controllability Gramian matrix $W_T = \sum_{\tau=0}^{T-1} A^{\tau} BB^{T} (A^{T})^{\tau}$ as the measure of the network controllability, which quantifies the worst-case control energy, and characterized the tradeoff between the control energy and the number of driver nodes, thereby offering an open-loop distributed control strategy with guaranteed performance and computational complexity.

5. Applications of Network Controllability

In parallel, the recent advances in network controllability theories had stimulated the investigation of practical applications of complex networked systems, including protein interaction networks [73], human liver metabolic networks [75], gene regulatory networks [76], interbank networks [77], brain networks [78]–[80], power systems [81], and multi-link planar robots [82].

5.1 Protein Interaction Networks

In view of the maximum matching principle and approach to determining the driver nodes in a directed network, Nacher and Akutsu [74] proposed an equivalent optimization procedure for identifying the minimum dominating sets (MDSet) of nodes for the control of undirected networks. They showed a link between the MDSet and the structural controllability: if every edge in a network is bi-directional and every node in the MDSet can control all of its outgoing edges separately, then the network is structurally controllable by selecting the nodes in the MDSet as the driver nodes. For illustration, consider an example (see Fig. 12). A set $S \subseteq V$ of nodes in an undirected network $G(V, E)$

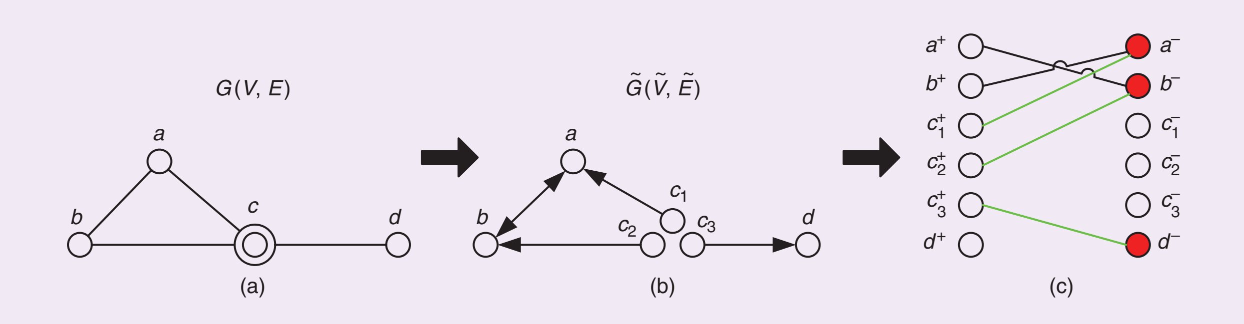


Figure 12. Illustration of the relation between the MDSet and the structural controllability. (a) The original network and its MDSet, where the MDSet is denoted by double circle. (b) The transformed network. (c) The constructed bipartite network, where green lines correspond to maximum matching.

is an MDSet if every node $v \in V$ is either an element of S or adjacent to an element of S. For a given network $G(V, E)$ and an MDSet S, construct a directed network as follows. For each node $v \in S$, split this node into k nodes, $v_1, v_2, \ldots, v_k$, where k is the degree of node v, and create a directed edge from each v_i to a node adjacent to v such that the out-degree of each v_i is 1. For each pair of nodes (u, v) such that $u, v \notin S$ and $(u, v) \in E$, create directed edges (u, v) and (v, u). Let $\tilde{G}(\tilde{V}, \tilde{E})$ denote the resulting network (Fig. 12(b)). Next, construct a bipartite graph from $\tilde{G}(\tilde{V}, \tilde{E})$ as follows (Fig. 12(c)). For each node $v \in \tilde{V}$, create two nodes v^+ and v^-. For each edge $(u, v) \in \tilde{E}$, create an edge between u^+ and v^-. Because S is a dominating set, there exists a matching $\tilde{M}$ satisfying (i) for each node $v \notin S$, v^- appears in $\tilde{M}$; (ii) for each node $v \in S$, none of v_i^- appears in $\tilde{M}$. Here, $\tilde{M}$ is maximum because no v_i^- with $v \in S$ has an edge in the bipartite graph. Therefore, by choosing $\{v_i | v \in S\}$ as the set of driver nodes, the above conclusion follows from the minimum inputs theorem (Theorem 7).

Motivated by the work of Nacher and Akutsu [74], Wuchty [73] identified MDSet in human and yeast protein interaction networks, where an MDSet is defined as an optimized subset of proteins from where each remaining (i.e., non-MDSet) protein can be immediately reached through one interaction. Such proteins play a key role in the control of the underlying networks. The findings suggest that MDSet proteins are enriched with cancer-related and virus-targeted genes. Furthermore, MDSet proteins have a higher impact on the network resilience than hub proteins. Indicating their relevance to the controllability of biological networks, a strong involvement was found in bottleneck interactions, regulatory and phosphorylation events, as well as genetic interactions. Specifically, Wuchty [73] determined the MDSet by solving a binary integer programming problem, where each protein $v \in V$ in a protein interaction network $G(V, E)$ is assigned a binary variable, x_v. If $v \in S$, then $x_v = 1$, and $x_v = 0$ otherwise. The computation of the MDSet is performed on $\min \sum_{v \in V} x_v$, subject to the constraint of $x_v + \sum_{\omega \in N_v} x_\omega \geq 1$, where N_v is the set of neighbors of protein v. In Fig. 13, the concept of binary integer programming problems is illustrated.

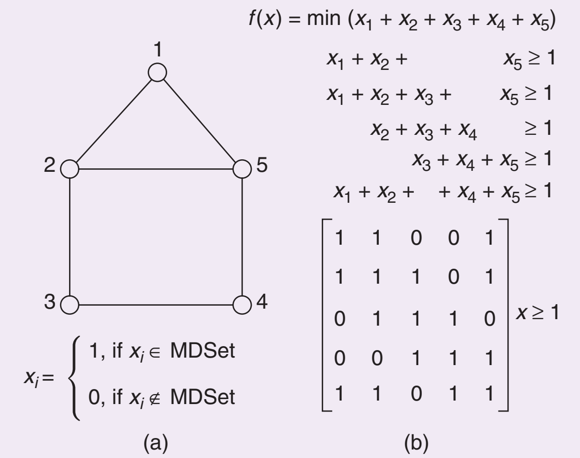


Figure 13. Concept of binary integer programming problems. (a) An undirected network. (b) Formulation of the binary integer programming problem for identifying MDSet.

5.2 Interbank Networks

The network of interbank loans represents a building block of modern financial economies, which also represents a crucial gear for the transmission of monetary policies to the real economy. It is known that targeted intervention on individual banks could be more effective in guaranteeing and restoring the efficient allocation of credits. This suggests the need for monitoring the system and keeping track of the banks that are systemically relevant, from a control perspective.

Within this context, Delpini et al. [77] considered the structural controllability of the interbank lending network and investigated its time evolution behavior. Specifically, they empirically analyzed the controllability of the interbank money markets, concentrating on the especial case of the Italian electronic trading system, which is open to European banking players. The data set therein is composed of 2750 daily snapshots of the Italian interbank money market from 1999 to 2009. On a daily scale, let the state variable x_i represent the level of funding bank i that provides to the others and assume that x_i depends on the funding it gets from its neighbors. Note that banks influence each other through "funding contagion" and the influence of bank i on j is proportional to the funding provided by i to j. Therefore, all these daily transactions form a directed and weighted network, described by graphs $G(V_t, E_t)$, with $t = 1, 2, \ldots, 2750$, whose nodes are the banks participating in the deals with edges denoting the amounts of the corresponding cash flows on day t. The concept of external control is implemented by liquidity interventions of the central banks in individual institutions of the network. During the time span (≈ 10 years) of the data, the interbank lending system underwent dramatic changes in its functioning. The resulting system is thereby modeled as a temporal network, a time-ordered sequence of realizations of the same system is represented as a graph.

In general, daily transactions are highly volatile in an interbank lending network, which can be aggregated over different time horizons Δ to reduce the level of fluctuations. It is shown that on a monthly basis the system is quite efficient from the control perspective. At every aggregation scale Δ and for every available network instance, the maximum matching of the digraph can be computed, thereby the set of driver nodes can be determined. A scale-free decay of the fraction of driver nodes is observed, with increasing time resolution, implying that policies have to be adjusted to the time scales in order to be effective. The findings also reveal that banks that drive the lending system are not hubs, nor do they correspond to the top lenders as would be expected. This strongly suggests the necessity to rethink of the policies based exclusively on the too-big-too-fail specification of a systemically important institution.

5.3 Structural Brain Networks

A brain network can be represented by a graph, where nodes denote units of the brain that perform a specific function such as vision or audition. At a large scale these units may be several centimeters of tissues, while at a small scale these may be individual neurons. In a structural brain network, the edges can represent structural links such as fiber bundles at a large scale or synapses at a small scale. The ability to efficiently control brain dynamics holds great promise for the enhancement of cognitive functions in humans, and the betterment of their quality of life. Recently, Gu et al. [78] applied network control theory to a whole-brain structural network in humans and investigated how the human brain moves between diverse cognitive states on the basis of white matter microstructure.

In [78], the structural brain network is constructed by subdividing the whole brain into anatomically distinct brain areas (network nodes), over five levels of spatial resolution, from 83 regions to 1015 regions, where the nodes are connected by the number of white matter streamlines. This procedure results in a sparse, weighted, and undirected structural brain network. With the dynamics of neural processing, a linear discrete-time and time-invariant network model is constructed, as follows:

$$x(k+1) = Ax(k) + B_\kappa u_\kappa(k), \tag{29}$$

where $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^N$ describes the state (i.e., the magnitude of neurophysiological activity) of brain regions over time; $A \in \mathbb{R}^{N \times N}$ is a symmetric and weighted adjacency matrix, whose non-diagonal elements denote the numbers of white matter streamlines connecting two different brain regions, and the diagonal elements are assumed zeros; the input matrix $B_\kappa = [e_{k_1}\ e_{k_2}\ \ldots\ e_{k_M}]$ identifies the set of driver nodes $\kappa = \{k_1, k_2, \ldots, k_M\}$ in the brain, where $e_i \in \mathbb{R}^N$ denotes the ith canonical vector; the input $u_\kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}^M$ corresponds to the stimuli of the controlled regions κ.

The classic criteria in the control theory ensure that the controllability of the network (29) from the set of nodes κ is equivalent to the invertibility of controllability Gramian W_κ, where

$$W_\kappa = \sum_{\tau=0}^{\infty} A^\tau B_\kappa B_\kappa^T A^\tau. \tag{30}$$

Gu et al. [78] examined three metrics of controllability: *average controllability*, *modal controllability*, and *boundary controllability*. The average controllability of a network quantifies the role of a node in moving the system to many easily reachable states. It equals the average input energy from a set of driver nodes and over all possible target states. Here, $\mathrm{Trace}(W_\kappa)$ is adopted as a measure of the average controllability for the brain network. The modal controllability refers to the ability of a node to control each evolutionary mode of a dynamical network, which can be used to identify the states that are difficult to control by using a set of driver nodes. It can be computed from the eigenvector matrix of the adjacency matrix A, based on the PBH eigenvector test. The boundary controllability measures the ability of a set of driver nodes to decouple the trajectories of the disjoint brain regions. The results suggest that densely connected areas, particularly in the default mode system, facilitate the movement of the brain to easy-to-reach states. Weakly connected areas, particularly in cognitive control systems, facilitate the movement of the brain to difficult-to-reach states. Areas located on the boundary between network communities, particularly in attentional control systems, facilitate the integration or segregation of diverse cognitive systems. All these results imply that the structural network differences between cognitive circuits dictate their distinct roles in brain network functions.

Subsequently, Medaglia et al. [79] examined the relationship between the brain network controllability and the cognitive control performance. They calculated the modal controllability and the boundary controllability on a structural brain network derived from diffusion tensor imaging data of 125 healthy adults, and tested the performance of individuals in traditional cognitive control tasks. It was observed that individual differences in these two control features are significantly correlated with individual differences in cognitive control performances. In particular, increases in the controllability in some regions are advantageous for the cognitive control performance, whereas increases in the others are disadvantageous.

To address the issue whether controllability metrics for the brain network change with ages, Tang et al. [80] examined two types of (average and modal) controllability in a cohort of 882 healthy youths from ages 8 to 22. They calculated the whole-brain average (modal) controllability as the mean average (modal) controllability value across all brain regions in a single individual. They found that individuals whose brains display high mean average controllability also display high mean modal controllability. Moreover, mean average controllability significantly increases with aging, as does the mean modal controllability. These suggest that the brain network is highly optimized to support a diverse range of possible dynamics and this range of supporting dynamics increases with aging.

6. Research Outlook

This article has offered a brief overview on the network controllability of networked linear dynamical systems. Two types of network controllability, namely state controllability and structural controllability, have been introduced and illustrated. Different approaches to evaluating the network controllability have been described and discussed. Some advanced topics on the selection of driver nodes, optimization of network controllability and control energy have also been reviewed. Finally, some potential applications to real-world networked systems have been highlighted.

Although significant progress has been made in this exciting field recently, there are still many relevant theoretical and applied research problems awaiting for further exploration under the unified framework of complex dynamical networks [83]–[85]. Future work could be devoted to, for instance, the following issues.

- *Nonlinear Dynamics*: In practical applications, most networked systems are highly nonlinear; therefore, controlling networked systems with nonlinear dynamics is necessary and important. Although some efforts have been made on this task [86]–[89], there is still no efficient framework for investigating the controllability of networked nonlinear dynamical systems. The main difficulty lies in the extremely diverse nonlinear dynamical functions and behaviors, which make it practically impossible to formulate a universal mathematic framework for control theory and practice. While the controllability of nonlinear systems can be assessed based on Lie brackets, it seems difficult to implement the abstract framework for large-scale complex networks, especially for the networks where nonlinear dynamical behaviors take place on the edges.
- *Heterogeneous Node Dynamics*: Most of the results in network controllability studies were obtained based on the assumption of identical node self-dynamics [40], [41], [43], or even the same scalar node dynamics, for all nodes [24]–[26], [33]–[37]. The introduction of higher-dimensional and heterogeneous node dynamics makes the controllability of the entire networked system very complicated since, apart from the complexity of the network topology, the intrinsic dynamics of the non-identical nodes bring in extra dynamical complexity [42], [44]. At present, the network controllability of heterogeneous node systems remains an outstanding and challenging problem.
- *Robustness of Network Controllability*: In general, complex networked systems are always confronting with random or intentional edges/nodes attacks, or subject to failures in communicating edges/nodes. These attacks or failures will lead to cascading reactions and then to global failures, significantly affecting the global environment such as human daily life. In such scenarios, robustness analysis of the network controllability is extremely important for reliable control of real-world complex systems. Some previous works analyzed the robustness of network

controllability for complex systems with nodal dynamics [90]–[93]. Yan et al. recently constructed a multiplex congruence network of natural numbers, which is a scale-free network but has strong robustness of network controllability, revealing the importance of chain structures in such networks [94]. Most recently, Lou et al. constructed a snapback network model, showing that it has even stronger robustness of controllability against both targeted and random node-degree or node-betweenness attacks, due to its prominent chain-loop structure [95]. The topic of robust controllability deserves further investigation especially with regard to edge dynamics [96].

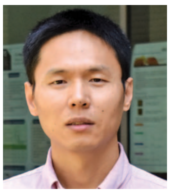
- *Multiplex Networks*: Almost all studies on the network controllability are performed on single-layer networks. In most natural and engineered systems, however, a set of entities interact with each other in complicated patterns that can encompass different kinds of relationships, change in time and in space, and involve different types of components [97]. Such systems can be modeled as multiplex networks [98] (or multilayer networks [99], see also [94], [95]), which consist of multiple networks at different levels, or with multiple types of edges or properties, and so on. In the emerge of Internet of Things today, it has become increasingly important to develop more effective and also more realistic frameworks for future studies of multiplex network controllability as well as its robustness.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under grant 61473240, the Natural Science Foundation of Fujian Province of China under grant 2017J01119, the Fundamental Research Funds for the Central Universities under grant N182304013, the National Science Foundation of USA under grant ECCS-1611423, and the Hong Kong Research Grants Council under the GRF Grant CityU11200317.

Linying Xiang (M'12) is currently an Associate Professor with the School of Control Engineering, Northeastern University at Qinhuangdao, Qinhuangdao, China. She received the Ph.D. degree in Control Theory and Control Engineering from the Department of Automation at Nankai University, Tianjin, China in 2008. From November 2008 to October 2010, she was a Postdoctoral Research Fellow and from March 2013 to June 2013 she was a Senior Research Assistant, both with the Department of Electronic Engineering, City University of Hong Kong, Hong Kong. From July 2017 to July 2018, she was a Visiting Associate Researcher with the Department of Electrical and Computer Engineering, University of California, Riverside. She was an Assistant Professor (2010–2014) and an Associate Professor (2014–2018) with the Department of Automation at Xiamen University, Xiamen, China. Dr. Xiang's research interests are in the areas of synchronization, control and optimization of complex networks, and coordinated control of multi-agent systems.

Fei Chen (M'12, SM'15) is currently a Professor with the State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, China and the School of Control Engineering, Northeastern University at Qinhuangdao, China. He received the Ph.D. degree in Control Theory and Control Engineering from Nankai University, Tianjin, China, in 2009. Before joining Northeastern University, he was an Associate Professor at Xiamen University, Xiamen, China and a Postdoctoral Researcher at Utah State University. He held visiting positions at University of California, Riverside and City University of Hong Kong. His current research focuses on cooperative control of multi-agent systems and modeling and control of complex networks. Dr. Chen received the 2009 Distinguished Ph.D. Dissertation Award from Nankai University, China. He was a 2014 Outstanding Reviewer for *IEEE Transactions on Control of Network Systems*.



Wei Ren (M'04, F'16) is currently a Professor with the Department of Electrical and Computer Engineering, University of California, Riverside, CA. He received the Ph.D. degree in Electrical Engineering from Brigham Young University, Provo, UT, in 2004. Prior to joining UC Riverside, he was a faculty member at Utah State University and a postdoctoral research associate at the University of Maryland, College Park. His research focuses on distributed control of multi-agent systems and autonomous control of unmanned vehicles.

Guanrong Chen (M'89, SM'92, F'97) received the MSc degree in Computer Science from Sun Yat-sen University, Guangzhou, China in 1981 and the PhD degree in Applied Mathematics from Texas A&M University, College Station, Texas in 1987. He has been a Chair Professor and the

Founding Director of the Centre for Chaos and Complex Networks at the City University of Hong Kong since year 2000, prior to that he was a tenured Full Professor at the University of Houston, Texas. He was awarded the 2011 Euler Gold Medal, Russia, and conferred Honorary Doctorate by the Saint Petersburg State University, Russia in 2011 and by the University of Le Havre, Normandy, France in 2014. He is a Member of the Academia of Europe and a Fellow of The World Academy of Sciences, and is a Highly Cited Researcher in Engineering according to Thomson Reuters.

References

[1] R. E. Kalman, "Mathematical description of linear dynamical systems," *J. Soc. Ind. Appl. Math. Ser. A*, vol. 1, no. 2, pp. 152–192, 1963.

[2] C. K. Chui and G. Chen, *Linear Systems and Optimal Control*. New York: Springer-Verlag, 1989.

[3] C. W. Wu, "Synchronization and convergence of linear dynamics in random directed networks," *IEEE Trans. Autom. Control*, vol. 51, no. 7, pp. 1207–1210, 2006.

[4] X. F. Wang, X. Li, and J. Lü, "Control and flocking of networked systems via pinning," *IEEE Circuits Syst. Mag.*, vol. 10, no. 3, pp. 83–91, 2010.

[5] G. Chen, "Pinning control and controllability of complex dynamical networks," *Int. J. Autom. Comput.*, vol. 14, no. 1, pp. 1–9, 2017.

[6] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Syst. Mag.*, vol. 27, no. 2, pp. 71–82, 2007.

[7] F. Chen and W. Ren, "Distributed consensus in networks," *Wiley Encycl. Elect. Electron. Eng.*, pp. 1–15, 2016. doi: 10.1002/047134608X.W8314.

[8] C.-T. Lin, "Structural controllability," *IEEE Trans. Autom. Control*, vol. 19, no. 3, pp. 201–208, 1974.

[9] C.-T. Lin, "System structure and minimal structure controllability," *IEEE Trans. Autom. Control*, vol. 22, no. 5, pp. 855–862, 1977.

[10] H. Mayeda and T. Yamada, "Strong structural controllability," *SIAM J. Control Optim.*, vol. 17, no. 1, pp. 123–138, 1979.

[11] H. G. Tanner, "On the controllability of nearest neighbor interconnections," in *Proc. 43rd IEEE Conf. Decision Control*, Dec. 2004, pp. 2467–2472.

[12] M. Ji, A. Muhammad, and M. Egerstedt, "Leader-based multi-agent coordination: controllability and optimal control," in *Proc. American Control Conf.*, June 2006, pp. 1358–1363.

[13] A. Rahmani and M. Mesbahi, "On the controlled agreement problem," in *Proc. American Control Conf.*, June 2006, pp. 1376–1381.

[14] A. Rahmani and M. Mesbahi, "Pulling the strings on agreement: anchoring, controllability, and graph automorphisms," in *Proc. American Control Conf.*, July 2007, pp. 2738–2743.

[15] M. Ji and M. Egerstedt, "A graph-theoretic characterization of controllability for multi-agent systems," in *Proc. American Control Conf.*, July 2007, pp. 4588–4593.

[16] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, "Controllability of multi-agent systems from a graph-theoretic perspective," *SIAM J. Control Optim.*, vol. 48, no. 1, pp. 162–186, 2009.

[17] S. Martini, M. Egerstedt, and A. Bicchi, "Controllability analysis of multi-agent systems using relaxed equitable partitions," *Int. J. Syst. Control Commun.*, vol. 2, no. 1/2/3, pp. 100–121, 2010.

[18] S. Zhang, M. Cao, and M. K. Camlibel, "Upper and lower bounds for controllable subspaces of networks of diffusively coupled agents," *IEEE Trans. Autom. Control*, vol. 59, no. 3, pp. 745–750, 2014.

[19] M. Cao, S. Zhang, and M. K. Camlibel, "A class of uncontrollable diffusively coupled multiagent systems with multichain topologies," *IEEE Trans. Autom. Control*, vol. 58, no. 2, pp. 465–469, 2013.

[20] M. K. Camlibel, S. Zhang, and M. Cao, "Comments on 'Controllability analysis of multi-agent systems using relaxed equitable partitions," *Int. J. Syst. Control Commun.*, vol. 4, no. 1/2, pp. 72–75, 2012.

[21] Y. C. Lou and Y. G. Hong, "Controllability analysis of multi-agent systems with directed and weighted interconnection," *Int. J. Control*, vol. 85, no. 10, pp. 1486–1496, 2012.

[22] Z. J. Ji and H. S. Yu, "A new perspective to graphical characterization of multiagent controllability," *IEEE Trans. Cybern.*, vol. 47, no. 6, pp. 1471–1483, 2017.

[23] N. Cai and Y. S. Zhong, "Formation controllability of high-order linear time-invariant swarm systems," *IET Control Theory Appl.*, vol. 4, no. 4, pp. 646–654, 2010.

[24] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, "Controllability of complex networks," *Nature*, vol. 473, pp. 167–173, 2011.

[25] M. Pósfai, Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, "Effect of correlations on network controllability," *Sci. Rep.*, vol. 3, no. 1067, p. 1067, 2013.

[26] G. Menichetti, L. Dall'Asta, and G. Bianconi, "Network controllability is determined by the density of low in-degree and out-degree nodes," *Phys. Rev. Lett.*, vol. 113, no. 7, p. 078701, 2014.

[27] Y. P. Wang, J. Xiang, Y. J. Li, and M. Z. Q. Chen, "Controllability of dynamic-edge multi-agent systems," *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 3, pp. 857–567, 2017. doi: 10.1109/TCNS.2017.2648513.

[28] Z. C. Hong, F. Chen, L. Y. Xiang, and W. Lan, "A study on the relationship between consensus of edge dynamics and node dynamics," in *Proc. Youth Academic Annu. Conf. Chinese Association Automation*, May 2017, pp. 1183–1187.

[29] X. L. Wang, H. S. Su, L. Wang, and X. F. Wang, "Edge consensus on complex networks: a structural analysis," *Int. J. Control*, vol. 90, no. 8, pp. 1584–1596, 2017.

[30] X. L. Wang, H. S. Su, X. F. Wang, and G. Chen, "Nonnegative edge quasi-consensus of networked dynamical systems," *IEEE Trans. Circuits Syst. II*, vol. 64, no. 3, pp. 304–308, 2017.

[31] T. Nepusz and T. Vicsek, "Controlling edge dynamics in complex networks," *Nat. Phys.*, vol. 8, no. 7, pp. 568–573, 2012.

[32] S.-P. Pang, W.-X. Wang, F. Hao, and Y.-C. Lai, "Universal framework for edge controllability of complex networks," *Sci. Rep.*, vol. 7, p. 4224, 2017.

[33] Z. Z. Yuan, C. Zhao, Z. R. Di, W.-X. Wang, and Y.-C. Lai, "Exact controllability of complex networks," *Nat. Commun.*, vol. 4, p. 2447, 2013.

[34] J. W. Li, Z. Z. Yuan, Y. Fan, W.-X. Wang, and Z. R. Di, "Controllability of fractal networks: an analytical approach," *Europhys. Lett.*, vol. 105, no. 5, p. 58001, 2014.

[35] M. Xu, C.-Y. Xu, H. Wang, C.-Z. Deng, and K.-F. Cao, "Analytical controllability of deterministic scale-free networks and Cayley trees," *Eur. Phys. J. B*, vol. 88, p. 168, 2015.

[36] Z. Z. Yuan, C. Zhao, W.-X. Wang, Z. R. Di, and Y.-C. Lai, "Exact controllability of multiplex networks," *New J. Phys.*, vol. 16, p. 103036, 2014.

[37] S. Nie, X. W. Wang, and B. H. Wang, "Effect of degree correlation on exact controllability of multiplex networks," *Physica A*, vol. 436, pp. 98–102, 2015.

[38] N. J. Cowan, E. J. Chastain, D. A. Vilhena, J. S. Freudenberg, and C. T. Bergstrom, "Nodal dynamics, not degree distributions, determine the structural controllability of complex networks," *PLoS One*, vol. 7, no. 6, p. e38398, 2012.

[39] C. Zhao, W.-X. Wang, Y.-Y. Liu, and J.-J. Slotine, "Intrinsic dynamics induce global symmetry in network controllability," *Sci. Rep.*, vol. 5, p. 8422, 2015.

[40] L. Wang, G. Chen, X. F. Wang, and W. K. S. Tang, "Controllability of networked MIMO systems," *Automatica*, vol. 69, pp. 405–409, 2016.

[41] Y. Q. Hao, Z. S. Duan, and G. Chen, "Further on the controllability of networked MIMO LTI systems," *Int. J. Robust Nonlinear Control*, vol. 28, no. 5, pp. 1778–1788, 2018.

[42] L. Y. Xiang, P. R. Wang, F. Chen, and G. Chen, "Controllability of heterogeneous directed networked MIMO systems," arXiv:1812.03302v2, 2018.

[43] L. Wang, X. F. Wang, and G. Chen, "Controllability of networked higher-dimensional systems with one-dimensional communication," *Phil. Trans. R. Soc. A*, vol. 375, p. 20160215, 2017.

[44] L. Y. Xiang, J. J. H. Zhu, F. Chen, and G. Chen, "Controllability of weighted and directed networks with nonidentical node dynamics," *Math. Probl. Eng.*, vol. 2013, p. 405034, 2013.

[45] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, 2005.

[46] P. Holme and J. Saramäki, "Temporal networks," *Phys. Rep.*, vol. 519, no. 3, pp. 97–125, 2012.

[47] Y. J. Pan and X. Li, "Structural controllability and controlling centrality of temporal networks," *PLoS One*, vol. 9, no. 4, p. e94998, 2014.

[48] B. Y. Hou, X. Li, and G. Chen, "Structural controllability of temporally switching networks," *IEEE Trans. Circuits Syst. I*, vol. 63, no. 10, pp. 1771–1781, 2016.

[49] P. Yao, B. Y. Hou, Y. J. Pan, and X. Li, "Structural controllability of temporal networks with a single switching controller," *PLoS One*, vol. 12, no. 1, p. e0170584, 2017.
[50] B. Liu, T. G. Chu, L. Wang, and G. M. Xie, "Controllability of a leader-follower dynamic network with switching topology," *IEEE Trans. Autom. Control*, vol. 53, no. 4, pp. 1009–1013, 2008.
[51] X. M. Liu, H. Lin, and B. M. Chen, "Graph-theoretic characterisations of structural controllability for multi-agent system with switching topology," *Int. J. Control*, vol. 86, no. 2, pp. 222–231, 2013.
[52] P. A. Fuhrmann and U. Helmke, *The Mathematics of Networks of Linear Systems*. New York: Springer, 2015.
[53] J. E. Hopcroft and R. M. Karp, "An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs," *SIAM J. Comput.*, vol. 2, no. 4, pp. 225–231, 1973.
[54] B. B. Wang, L. Gao, and Y. Gao, "Control range: a controllability-based index for node significance in directed networks," *J. Stat. Mech.*, vol. 4, p. P04011, 2012.
[55] B. B. Wang, L. Gao, Y. Gao, Y. Deng, and Y. Wang, "Controllability and observability analysis for vertex domination centrality in directed networks," *Sci. Rep.*, vol. 4, p. 5399, 2014.
[56] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, "Control centrality and hierarchical structure in complex networks," *PLoS One*, vol. 7, no. 9, p. e44459, 2012.
[57] T. Jia and A.-L. Barabási, "Control capacity and a random sampling method in exploring controllability of complex networks," *Sci. Rep.*, vol. 3, p. 2354, 2013.
[58] J. Ding and Y.-Z. Lu, "Control backbone: an index for quantifying a node's importance for the network controllability," *Neurocomputing*, vol. 153, pp. 309–318. 2015.
[59] T. Jia, Y.-Y. Liu, E. Csóka, M. Pósfai, J.-J. Slotine, and A.-L. Barabási, "Emergence of bimodality in controlling complex networks," *Nat. Commun.*, vol. 4, p. 2002, 2013.
[60] T. Jia and M. Pósfai, "Connecting core percolation and controllability of complex networks," *Sci. Rep.*, vol. 4, p. 5379, 2014.
[61] J. Ruths and D. Ruths, "Control profiles of complex networks," *Science*, vol. 343, no. 6177, pp. 1373–1376, 2014.
[62] W.-X. Wang, X. Ni, Y.-C. Lai, and C. Grebogi, "Optimizing controllability of complex networks by minimum structural perturbations," *Phys. Rev. E*, vol. 85, no. 2, p. 026115, 2012.
[63] S.-P. Pang and F. Hao, "Optimizing controllability of edge dynamics in complex networks by perturbing network structure," *Physica A*, vol. 470, pp. 217–227, 2017.
[64] Y. D. Xiao, S. Y. Lao, L. L. Hou, and L. Bai, "Edge orientation for optimizing controllability of complex networks," *Phys. Rev. E*, vol. 90, no. 4, p. 042804, 2014.
[65] L. L. Hou, S. Y. Lao, M. Small, and Y. D. Xiao, "Enhancing complex network controllability by minimum link direction reversal," *Phys. Lett. A*, vol. 379, no. 20–21, pp. 1321–1325, 2015.
[66] J. Ding, P. Tan, and Y.-Z. Lu, "Optimizing the controllability index of directed networks with the fixed number of control nodes," *Neurocomputing*, vol. 171, pp. 1524–1532, 2016.
[67] G. Yan, J. Ren, Y.-C. Lai, C.-H. Lai, and B. Li, "Controlling complex networks: How much energy is needed?" *Phys. Rev. Lett.*, vol. 108, no. 21, p. 218703, 2012.
[68] G. Yan, G. Tsekenis, B. Barzel, J.-J. Slotine, Y.-Y. Liu, and A.-L. Barabási, "Spectrum of controlling and observing complex networks," *Nat. Phys.*, vol. 11, pp. 779–786, 2015.
[69] Y.-Z. Chen, L.-Z. Wang, W.-X. Wang, and Y.-C. Lai, "Energy scaling and reduction in controlling complex networks," *R. Soc. Open Sci.*, vol. 3, p. 160064, 2016.
[70] L.-Z. Wang, Y.-Z. Chen, W.-X. Wang, and Y.-C. Lai, "Physical controllability of complex networks," *Sci. Rep.*, vol. 7, p. 40198, 2017.
[71] F. Pasqualetti, S. Zampieri, and F. Bullo, "Controllability metrics, limitations and algorithms for complex networks," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 1, pp. 40–52, 2014.
[72] J. Sun and A. E. Motter, "Controllability transition and nonlocality in network control," *Phys. Rev. Lett.*, vol. 110, no. 20, p. 208701, 2013.
[73] S. Wuchty, "Controllability in protein interaction networks," *Proc. Natl. Acad. Sci. USA*, vol. 111, no. 19, pp. 7156–7160, 2014.
[74] J. C. Nacher and T. Akutsu, "Dominating scale-free networks with variable scaling exponent: Heterogeneous networks are not difficult to control," *New J. Phys.*, vol. 14, p. 073005, 2012.
[75] X. M. Liu and L. Q. Pan, "Detection of driver metabolites in the human liver metabolic network using structural controllability analysis," *BMC Syst. Biol.*, vol. 8, p. 51, 2014.
[76] C. Zhao, A. Bin, W. M. Ye, Y. Fan, and Z. R. Di, "Motif for controllable toggle switch in gene regulatory networks," *Physica A*, vol. 419, pp. 498–505, 2015.
[77] D. Delpini, S. Battiston, M. Riccaboni, G. Gabbi, F. Pammolli, and G. Caldarelli, "Evolution of controllability in interbank networks," *Sci. Rep.*, vol. 3, p. 1626, 2013.
[78] S. Gu et al., "Controllability of structural brain networks," *Nat. Commun.*, vol. 6, p. 8414, 2015.
[79] J. D. Medaglia et al., "Cognitive control in the controllable connectome," arXiv: Preprint, arXiv:1606.09185, 2016.
[80] E. Tang et al., "Developmental increases in white matter network controllability support a growing diversity of brain dynamics," *Nat. Commun.*, vol. 8, p. 1252, 2017.
[81] Y.-S. Li, D.-Z. Ma, H.-G. Zhang, and Q.-Y. Sun, "Critical nodes identification of power systems based on controllability of complex networks," *Appl. Sci.*, vol. 5, no. 3, pp. 622–636, 2015.
[82] Y. Q. Hao, Z. S. Duan, and G. H. Wen, "Controllability and observability of an n-link robot with multiple active links," *Int. J. Robust Nonlinear Control*, vol. 27, no. 18, pp. 4633–4647, 2017.
[83] X. F. Wang and G. Chen, "Complex networks: small-world, scale-free and beyond," *IEEE Circuits Syst. Mag.*, vol. 3, no. 1, pp. 6–20, 2003.
[84] L. Y. Cui, S. Kumara, and R. Albert, "Complex networks: an engineering view," *IEEE Circuits Syst. Mag.*, vol. 10, no. 3, pp. 10–25, 2010.
[85] S. L. Tan, J. Lü, G. Chen, and D. J. Hill, "When structure meets function in evolutionary dynamics on complex networks," *IEEE Circuits Syst. Mag.*, vol. 14, no. 4, pp. 36–50, 2014.
[86] A. J. Whalen, S. N. Brennan, T. D. Sauer, and S. J. Schiff, "Observability and controllability of nonlinear networks: the role of symmetry," *Phys. Rev. X*, vol. 5, p. 011005, 2015.
[87] L.-Z. Wang et al., "A geometrical approach to control and controllability of nonlinear dynamical networks," *Nat. Commun.*, vol. 7, p. 11323, 2016.
[88] J. G. T. Aanudo, G. Yang, and R. Albert, "Structure-based control of complex networks with nonlinear dynamics," *Proc. Natl. Acad. Sci. USA*, vol. 114, no. 28, pp. 7234–7239, 2017.
[89] Y.-Z. Sun, S.-Y. Leng, Y.-C. Lai, C. Grebogi, and W. Lin, "Closed-loop control of complex networks: a trade-off between time and energy," *Phys. Rev. Lett.*, vol. 119, no. 19, p. 198301, 2017.
[90] C.-L. Pu, W.-J. Pei, and A. Michaelson, "Robustness analysis of network controllability," *Physica A*, vol. 391, no. 18, pp. 4420–4425, 2012.
[91] S. Nie, X.-W. Wang, H. F. Zhang, Q. L. Li, and B. H. Wang, "Robustness of controllability for networks based on edge-attack," *PLoS One*, vol. 9, no. 2, p. e89066, 2014.
[92] S. Jafari, A. Ajorlou, and A. G. Aghdam, "Leader localization in multi-agent systems subject to failure: a graph-theoretic approach," *Automatica*, vol. 47, no. 8, pp. 1744–1750, 2011.
[93] M. A. Rahimian and A. G. Aghdam, "Structural controllability of multi-agent networks: robustness against simultaneous failures," *Automatica*, vol. 49, no. 11, pp. 3149–3157, 2013.
[94] X. Y. Yan, W.-X. Wang, G. Chen, and D. H. Shi, "Multiplex congruence network of natural numbers," *Sci. Rep.*, vol. 6, p. 23714, 2016.
[95] Y. Lou, L. Wang, and G. Chen, "Toward stronger robustness of network controllability: a snapback network model," *IEEE Trans. Circuits Syst. I*, vol. 65, no. 9, pp. 2983–2991, 2018.
[96] S.-P. Pang, F. Hao, and W.-X. Wang, "Robustness of controlling edge dynamics in complex networks against node failure," *Phys. Rev. E*, vol. 94, p. 052310, 2016.
[97] M. Kivelä, A. Arenas, M. Barthelemy, J. P. Gleeson, Y. Moreno, and M. A. Porter, "Multilayer networks," *J. Compl. Netw.*, vol. 2, pp. 203–271, 2014.
[98] M. Pósfai, J. X. Gao, S. P. Cornelius, A.-L. Barabási, and R. M. D'Souza, "Controllability of multiplex, multi-time-scale networks," *Phys. Rev. E*, vol. 94, p. 032316, 2016.
[99] S. Boccaletti et al., "The structure and dynamics of multilayer networks," *Phys. Rep.*, vol. 544, pp. 1–122, 2014.