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Graph partitions and the controllability of directed signed networks

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Abstract This paper studies the controllability problem of signed networks which is presented by weighted and directed signed graphs. Graph partitions such as structural balance and almost equitable partitions (AEPs) are studied. We generalize the definition of AEPs to any graphs, directed or undirected, signed or unsigned, with or without edge weights. Based on AEP theory, a graph-theoretic necessary condition is proposed for the controllability of directed signed networks and an algorithm is given for the computation of the coarsest partition. Besides, the upper bound on the controllable subspace is derived when the system is uncontrollable.

Keywords controllability, signed networks, graph partition, almost equitable partitions, structural balance

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1 Introduction

In recent two decades, the problems of consensus [1–13] and controllability [14–34] for networks have received significant attention. Tanner [15] first proposed the concept of controllability of networks interconnected via nearest neighbor rules. Since then, this problem has been studied by a lot of researchers. The study of the controllability of the networks mainly focuses on how to choose the appropriate nodes called leaders which can receive the external inputs so that the system can achieve the desired final state from any initial state within a finite time. The fundamental and challenging issue is how to quickly select the minimum number of leaders to make the network system controllable. Based on Kalman's controllability rank condition or Popov-Belevitch-Hautus (PBH) test, some necessary and sufficient algebraic conditions are obtained [15, 16]. To avoid a large amount of complicated computation relating to these algebraic conditions, the approach based on the graph theory deserves special attention [15]. Tanner [15] studied the relationship between connectivity and controllability, and concluded that connectivity seems to have an adverse effect on controllability. Ji et al. concluded that the star graph is uncontrollable for any choice of one single leader [19], and gave the concept of a destructive node [30]. Refs. [20, 32] investigated the controllability of path and cycle graphs. However, how to completely identify the relationship between the controllability and topological structures is still an open problem. Graph partitions play an important role in graphical characterization of controllability of the network [17, 35–40]. By using equitable partition, some necessary conditions were presented for the controllability of networks in the

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sense of graph theory [17, 35]. Under the undirected graph, Egerstedt et al. [36, 37] proposed the concept of external equitable partitions (EEPs) and gave a necessary condition for the controllability. Ref. [38] provided the definition of relaxed equitable partitions (REPs) and gave a necessary and sufficient condition for the controllability of a single leader network which is actually wrong [41]. Both EEPs and REPs are actually almost equitable partitions (AEPs). Under directed graph, Aguilar et al. [39] gave a necessary condition for the controllability of networks by making use of AEPs. With this approach, some uncontrollable cases can be quickly determined. Furthermore, the upper bound for the controllable subspace could be achieved [40]. It is noted that the definition of AEPs above is characterized by the fact that every vertex in the same cell has the same number of neighbors as in other cells.

It is also noted that the interactions of the networks studied above are cooperative. However, antagonistic interactions may exist in reality. Just like in nature, some animals are cooperative while some are hostile. Similarly, in society, some companies cooperate while some compete with each other. This is similar to the significance of the signed graph in which the weights of its edges may be equal to 1 or -1 [42]. The positive weighted edge represents the cooperative interaction while the negative weighted edge represents the antagonistic interaction. In this paper, the weights of the signed graph may be positive or negative numbers. The Laplacian of signed graph is different from that of unsigned graph, e.g., its row/column sum does not need to be zero and it can be positive definite. In [43], the authors studied the controllability problem of multi-agent networks under undirected signed graphs and a necessary condition was proposed for signed graphs' controllability using the so-called generalized almost equitable partition (GAEP). However, based on the number of positive and negative neighbors, Ref. [43] cannot give the sufficient condition for a GAEP of a graph. Therefore the definition of GAEP should be extended to handle more cases. Furthermore, the relationship between two individuals in reality will be more complicated. For instance, in nature, lions could attack the antelope, but not vice versa. The threat of a lion to the antelope may be greater than that of a hyena. In general, a directed graph is much more complicated than an undirected graph. The adjacency matrix and the Laplace matrix of the undirected graph are symmetric matrices, and the adjacency matrix and Laplacian matrix of the directed graph are not symmetric matrices. Hence the adjacency matrix and the Laplacian matrix of the directed graph will lose some good nature. In the paper, we attempt to investigate the graph partitions and provide a general definition of AEPs. The contribution of this work is threefold. First, a general definition of AEPs is introduced and then a necessary condition for the controllability is presented. Second, we addressed that the controllability of the structurally balanced graph is equivalent to that of the associated unsigned graph. Third, the coarsest AEP and its algorithm are presented. Besides, the upper bound for the controllable subspace is given.

The rest of this paper is organized as follows. In Section 2, some preliminaries are provided and the controllability problem of signed networks is formulated. In Section 3, graph partitions including structural balance and AEP are introduced. In Section 4, the controllability of weighted and directed signed networks is presented and several examples are provided to illustrate the main results. Finally, the conclusion is given in Section 5.

2 Preliminaries and problem formulation

Throughout this paper, e_i is the identity vector whose i-th element is 1 while the other elements are 0. Given two sets X and Y, $X\backslash Y$ is the set whose elements belong to X but not to Y. For a matrix $M \in \mathbb{R}^{n\times n}$, the column space of a matrix M is denoted by $\mathrm{im}(M)$ and $|M|$ can be represented as a matrix where $[|M|]_{ij} = |[M]_{ij}|$.

2.1 Signed graphs

A signed digraph is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, \ldots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represent the vertex set and the edge set, respectively. $A = [a_{ij}]_n$ is the adjacency matrix of $\mathcal{G}$. $a_{ij} \neq 0$ represents $(j, i) \in \mathcal{E}$, where j is called the parent vertex of i and i the child of j. If the edge points at i from j, j

is called the parent vertex while i is called the child vertex and we say j is a neighbor of i. We assume that there are no self-loops, i.e., $a_{ii} = 0$. A directed path in a signed digraph $\mathcal{G}$ is a sequence $i_1, \ldots, i_k$ of distinct vertices with $(i_s, i_{s+1}) \in \mathcal{E}$, for $s = 1, \ldots, k-1$ and a weak path, with either (i_s, i_{s+1}) or $(i_{s+1}, i_s) \in \mathcal{E}$. A digraph $\mathcal{G}$ is strongly connected if there is a directed path that starts from i and ends at j between every pair of distinct vertices i, j in $\mathcal{G}$, and is weakly connected if any two vertices can be jointed by a weak path. A cycle in a graph G is a sequence $\{v_1, \ldots, v_r\}$ of r distinct vertices, $r > 1$, such that $(v_1, v_2), \ldots, (v_{r-1}, v_r), (v_r, v_1) \in \mathcal{E}$. A weak cycle is a sequence $\{v_1, \ldots, v_r\}$ of r distinct vertices, $r > 1$, such that (v_i, v_{i+1}), or $(v_{i+1}, v_i) \in \mathcal{E}, i = 1, \ldots, r$, where $v_{r+1} = v_1$. A positive cycle of a signed graph is a cycle in which the number of negative edges is even. A negative cycle is not positive [42]. The graph is said an unsigned graph if $a_{ij} \geqslant 0$ for all i and j. The neighbor set of i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (j,i) \in \mathcal{E}, j \neq i\}$. The indegree of the vertex i can be denoted by $c_i = \sum_{j \in \mathcal{N}_i} |a_{ij}|$. The indegree matrix of a graph $\mathcal{G}$ is a diagonal matrix $C = \mathrm{diag}(c_1, \ldots, c_n)$. The Laplacian matrix L of a graph $\mathcal{G}$ can be defined as $L = C - A$. Thus, the entries of the matrix L can be written as

$$l_{ij} = \begin{cases} c_i, & i = j, \\ -a_{ij}, & i \neq j. \end{cases}$$

2.2 Invariant subspace and linear controllability

Definition 1 ([44]). Given $M : \mathbb{X} \to \mathbb{X}$ and a subspace $\mathbb{W}$ of $\mathbb{X}$, we say that $\mathbb{W}$ is M-invariant (or, if the map M is supposed to be obvious, simply invariant) if for all $x \in \mathbb{W}$ we have $Mx \in \mathbb{W}$, which can be written as $M\mathbb{W} \subset \mathbb{W}$.

Lemma 1 ([45]). For matrices $M \in \mathbb{R}^{n \times n}$ and $P \in \mathbb{R}^{n \times r}$, $\mathrm{im}(P)$ is an M-invariant subspace if and only if there exists a matrix $Q \in \mathbb{R}^{r \times r}$ such that $MP = PQ$.

Lemma 2 ([44]). Given matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times r}$, we use $\langle A, B \rangle$ to denote the smallest A-invariant subspace containing $\mathrm{im}(B)$. The pair (A, B) is called controllable if $\dim(\langle A, B \rangle) = n$.

2.3 Problem formulation

Consider a network described by the signed digraph $\mathcal{G}$. Let x_i denote the state of node i, whose dynamics is described by the protocol

$$\dot{x}_i = -\sum_{i \in \mathcal{N}_i} (|a_{ij}| x_i - a_{ij} x_j), \quad i = 1, \ldots, n. \tag{1}$$

For simplicity, only one dimensional case is considered. The compact dynamics can be written as $\dot{x}(t) = -Lx(t)$, where x is the vector of the states and L is the Laplacian of $\mathcal{G}$. Let $\mathcal{V}_l = \{i_1, \ldots, i_m\}$ be the set of leaders controlled by external inputs. Then the dynamical system is

$$\dot{x}(t) = -Lx(t) + Bu(t), \tag{2}$$

where $B = [e_{i_1}, e_{i_2}, \ldots, e_{i_m}]$ is the control input matrix, and $u(t) \in \mathbb{R}^m$ is the input vector.

By Kalman's controllability rank condition [46], system (2) is controllable if and only if the $n \times nm$ controllability matrix $Q = [B, LB, \ldots, L^{n-1}B]$ has full row rank, that is, $\mathrm{rank}(Q) = n$. The controllable subspace of system (2) is $\langle L, B \rangle := \mathrm{im}(Q)$.

Lemma 3 ([44]). The following statements are equivalent:

(i) The system (2) is controllable;

(ii) $\langle L, B \rangle = \mathbb{R}^n$;

(iii) $\mathrm{rank}(Q) = n$.

3 Graph partitions

The vertices $\mathcal{V}$ of $\mathcal{G}$ can be partitioned into several subsets with specific properties.

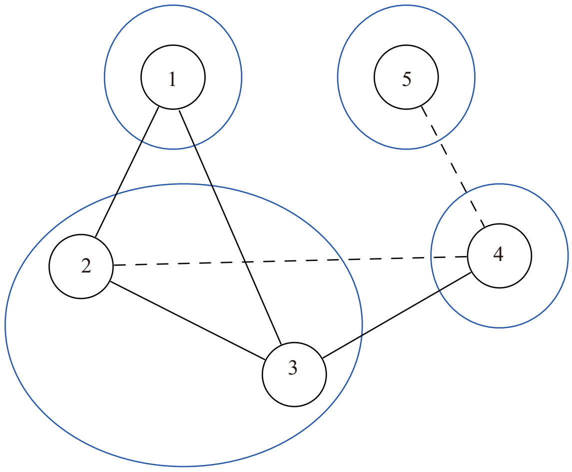

Figure 1 (Color online) π: $C_1 = \{1\}$, $C_2 = \{2, 3\}$, $C_3 = \{4\}$, $C_4 = \{5\}$. The solid and dashed lines represent the positive and negative edges, respectively.

Definition 2. A partition π of $\mathcal{V}$ is to partition $\mathcal{V}$ into r cells $C_1, C_2, \ldots, C_r$, where $r > 1$ and $C_i \subset \mathcal{V}$, $\mathcal{V} = \cup_{i=1}^{r} C_i$, $C_i \cap C_j = \emptyset, i \neq j$. C_i is nontrivial if $1 < |C_i| < n$, otherwise, it is trivial. The partition π is nontrivial if it contains at least one nontrivial cell, otherwise it is trivial.

Definition 3. Let π_1, π_2 be two partitions of the same $\mathcal{V}$. Then we say that π_1 is coarser than π_2 if each cell in π_1 is a union of cells in π_2.

Definition 4 ([45])**.** A characteristic matrix $P \in \mathbb{R}^{n\times r}$ of a partition π of $\mathcal{V}$ is a matrix with the characteristic vectors of the cells as its columns. The entries of the matrix P are

$$p_{ij} = \begin{cases} 1, & \text{if } i \in C_j, \\ 0, & \text{otherwise.} \end{cases}$$

For example, the characteristic matrix of the partition of the graph in Figure 1 is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is clear that

$$P^{\mathrm{T}}P = \begin{bmatrix} |C_1| & & & \\ & |C_2| & & \\ & & \ddots & \\ & & & |C_r| \end{bmatrix}.$$

Because each cell at least has one node, $P^{\mathrm{T}}P$ is nonsingular.

3.1 Almost equitable partitions

Unlike the case of unsigned graphs, L of signed graphs can be positive definite and its row/column sum does not need to be zero. Therefore, the definition of AEPs in unsigned graphs cannot be directly applied to the signed graphs. For example, according to the definition of AEPs of [39], in Figure 2, π: $C_1 = \{1, 2\}$, $C_2 = \{3, 4, 5\}$, is an AEP. In Figure 3, π satisfies this definition but π does not satisfy Theorem 2 of [39]. Therefore, we need to investigate the definition of AEPs in signed graphs.

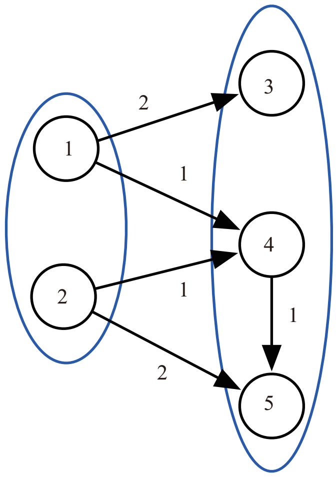

Figure 2 (Color online) A weighted directed unsigned graph.

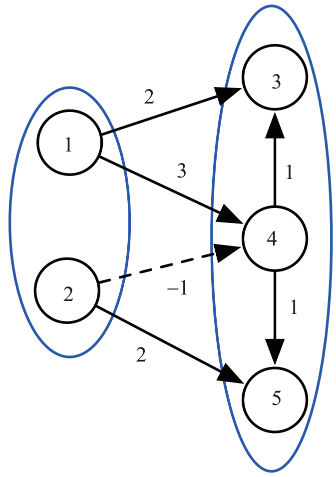

Figure 3 (Color online) A weighted directed signed graph.

Definition 5. Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a weighted and directed signed graph and L is its Laplacian. A partition π of $\mathcal{V}$, $\{C_1, C_2, \ldots, C_r\}$ is said to be an AEP if for all $s, t \in C_i$, $i, j = 1, \ldots, r$, the following equality holds:

$$\sum_{k\in C_j} l_{sk} = \sum_{k\in C_j} l_{tk}. \tag{3}$$

Remark 1. This definition is a mathematical definition which can be applicable to any graphs, directed or undirected, signed or unsigned, with or without edge weights.

Definition 6. The directed graph with the r cells of π as its vertices and $\alpha_{ij} := \sum_{k\in C_j} l_{sk}$, $s \in C_i$ arcs from the j-th to the i-th cell of π is called the quotient of $\mathcal{G}$ over π, which is denoted by G/π. Let A_π and L_π denote the adjacency and Laplacian matrix of G/π, respectively. Then, the entries of the adjacency matrix of this quotient are given by $A(G/\pi) = \alpha_{ij}$.

Theorem 1. Let $\mathcal{G}$ be a signed graph, L its Laplacian, $\pi = \{C_1, \ldots, C_r\}$ a partition of $\mathcal{V}$, and P the characteristic matrix of π. Then π is an AEP if and only if there is a matrix $Q \in \mathbb{R}^{r\times r}$ such that

$$LP = PQ,$$

$Q = P^+LP$, where $P^+ = (P^{\mathrm{T}}P)^{-1}P^{\mathrm{T}}$ is the pseudo-inverse of P. If π is an AEP then Q is the Laplacian L_π of G/π.

Proof. Given a graph $\mathcal{G}$, π is a partition of $\mathcal{V}$, and P is the characteristic matrix of π. Then

$$LP = \begin{bmatrix} \sum l_{1j}p_{j1} & \sum l_{1j}p_{j2} & \cdots & \sum l_{1j}p_{jr} \\ \sum l_{2j}p_{j1} & \sum l_{2j}p_{j2} & \cdots & \sum l_{2j}p_{jr} \\ \vdots & \vdots & & \vdots \\ \sum l_{nj}p_{j1} & \sum l_{nj}p_{j2} & \cdots & \sum l_{nj}p_{jr} \end{bmatrix} = \begin{bmatrix} \sum_{j\in C_1} l_{1j} & \sum_{j\in C_2} l_{1j} & \cdots & \sum_{j\in C_r} l_{1j} \\ \sum_{j\in C_1} l_{2j} & \sum_{j\in C_2} l_{2j} & \cdots & \sum_{j\in C_r} l_{2j} \\ \vdots & \vdots & & \vdots \\ \sum_{j\in C_1} l_{nj} & \sum_{j\in C_2} l_{nj} & \cdots & \sum_{j\in C_r} l_{nj} \end{bmatrix}.$$

Suppose that a matrix $Q \in \mathbb{R}^{r\times r}$ is

$$Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1r} \\ q_{21} & q_{22} & \cdots & q_{2r} \\ \vdots & \vdots & & \vdots \\ q_{r1} & q_{r2} & \cdots & q_{rr} \end{bmatrix}.$$

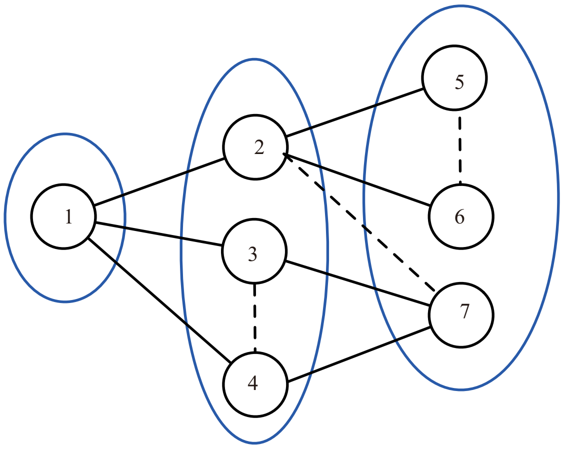

Figure 4 (Color online) π: $C_1 = \{1\}$, $C_2 = \{2,3,4\}$, $C_3 = \{5,6,7\}$. The solid and dashed lines represent the positive and negative edges, respectively.

Then

$$PQ = \begin{bmatrix} q_{r_1 1} & q_{r_1 2} & \cdots & q_{r_1 r} \\ q_{r_2 1} & q_{r_2 2} & \cdots & q_{r_2 r} \\ \vdots & \vdots & & \vdots \\ q_{r_n 1} & q_{r_n 2} & \cdots & q_{r_n r} \end{bmatrix},$$

where r_i indicates that i belongs to the cell C_{r_i}. If $s, t \in C_i$, then

$$(PQ)_{sj} = (PQ)_{tj}.$$

(Sufficiency) If $LP = PQ$, then $(LP)_{sj} = (LP)_{tj}$, for $s, t \in C_i$. It follows that

$$\sum_{k \in C_j} l_{sk} = \sum_{k \in C_j} l_{tk},$$

that is to say, π is an AEP.

(Necessity) If π is an AEP, then (3) is satisfied. Let $Q = L_\pi$. Then $LP = PQ$.

Remark 2. Because the definition of GAEP in [43] only uses the number of positive and negative neighbors, it can neither be extended to the weight graph nor make the necessary and sufficient conditions for the Lemma 2 of [43] to be obtained. The AEP here is not the same as the GAEP. In fact, the definition of AEP in the paper is more general than that of GAEP in [43]. A GAEP must be an AEP but not vice versa. For instance, the graph in Figure 4, π: $C_1 = \{1\}$, $C_2 = \{2,3,4\}$, $C_3 = \{5,6,7\}$ is an AEP but not a GAEP.

3.2 Structural balance

Definition 7 ([42])**.** A signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is said to be structurally balanced if its vertices set can be partitioned into two disjoint subsets, such that $a_{ij} \geqslant 0$ for two vertices in the same subset, and $a_{ij} \leqslant 0$ for two vertices in different subsets. It is said structurally unbalanced otherwise.

Remark 3. For a structurally balanced signed graph, one of the two subsets may be empty. For example, a connected unsigned graph is structurally balanced, however, it contains only positive edges. Another definition utilizes the positive cycles (for undirected graph) or positive weak cycles (for digraph). In fact, for a signed digraph, it is structurally balanced if and only if all of its weak cycles are positive [42].

In Figure 5, the two graphs are structurally balanced. In Figure 6, the two graphs are structurally unbalanced. Lemma 4 would be used.

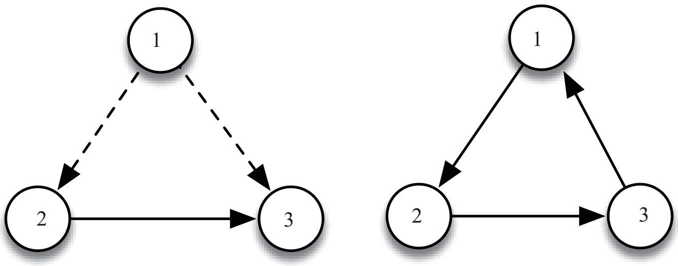

Figure 5 Two structurally balanced signed graphs. The dashed lines denote the negative weighted edges.

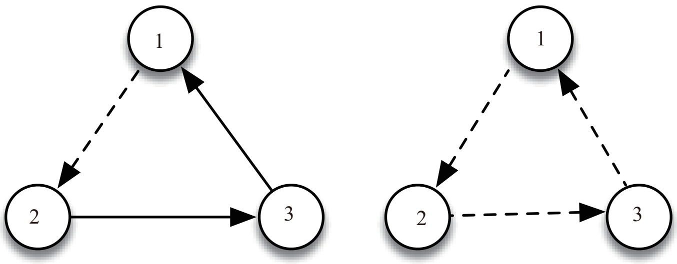

Figure 6 Two structurally unbalanced signed graphs. The dashed lines denote the negative weighted edges.

Lemma 4. A signed digraph $\mathcal{G}$ is structurally balanced if and only if $\exists\ D = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, $\sigma_i \in \{\pm 1\}$ such that DAD has all nonnegative entries.

Proof. (Necessity) Suppose that $\mathcal{G}$ is structurally balanced. Then its vertices set can be partitioned into two disjoint subsets $\mathcal{V}_1$ and $\mathcal{V}_2$, such that $a_{ij} \geqslant 0$ for two vertices in the same subset, and $a_{ij} \leqslant 0$ for two vertices in different subsets.

Let $D = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, $\sigma_i = 1$ when $i \in \mathcal{V}_1$, otherwise, $\sigma_i = -1$. Let $\bar{A} = DAD$. Then $[\bar{A}]_{ij} = \sigma_i \sigma_j a_{ij}$.

For $\forall v_i, v_j \in \mathcal{V}_p$, $p \in \{1, 2\}$, $a_{ij} \geqslant 0$, thus $\sigma_i \sigma_j = 1$. Therefore $[\bar{A}]_{ij} = \sigma_i \sigma_j a_{ij} \geqslant 0$.

For $\forall v_i \in \mathcal{V}_q$, $\forall v_j \in \mathcal{V}_r$, $q \neq r$, $q, r \in \{1, 2\}$, $a_{ij} \leqslant 0$, thus $\sigma_i \sigma_j = -1$. Therefore $[\bar{A}]_{ij} = \sigma_i \sigma_j a_{ij} \geqslant 0$.

Subsequently, DAD has all nonnegative entries.

(Sufficiency) Suppose that there exists D such that DAD has all nonnegative entries. If $\sigma_i = 1$ then let $v_i \in \mathcal{V}_1$. Similarly, if $\sigma_i = -1$ then let $v_i \in \mathcal{V}_2$. Assume that $\bar{A} = DAD$. It follows that $A = D\bar{A}D$ and $a_{ij} = \sigma_i \sigma_j [\bar{A}]_{ij}$, where $[\bar{A}]_{ij} \geqslant 0$. For $\forall v_i, v_j \in \mathcal{V}_p$, $p \in \{1, 2\}$, it is easily seen that $\sigma_i \sigma_j = 1$. Consequently, $a_{ij} \geqslant 0$. Similarly, for $v_i \in \mathcal{V}_q$, $v_j \in \mathcal{V}_r$, $q \neq r$, $q, r \in \{1, 2\}$, we obtain $\sigma_i \sigma_j = -1$. Thus $a_{ij} = \sigma_i \sigma_j [\bar{A}]_{ij} \leqslant 0$. Therefore, $\mathcal{G}$ is structurally balanced.

4 Controllability of directed signed networks

According to Lemma 4, if a signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is structurally balanced then $\exists\ D = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, $\sigma_i \in \{\pm 1\}$ such that DAD has all nonnegative entries. By the proof of Lemma 4, $DAD = |A|$. That is to say, $\mathcal{G}^U(\mathcal{V}, \mathcal{E}, |A|)$ is an unsigned graph which is called the corresponding unsigned graph of $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. Let $\bar{L}$ denote the Laplacian matrix of $\mathcal{G}^U$. We take into account the controllability of (L, B) and that of $(\bar{L}, B)$. In Theorem 3 of [43], the controllability of (L, B) is equivalent to that of $(\bar{L}, B)$, however, the leaders need to be chosen from the same subset, i.e., $\mathcal{V}_l \subset \mathcal{V}_1$ or $\mathcal{V}_l \subset \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are two disjoint subsets of $\mathcal{V}$ of $\mathcal{G}$, such that $a_{ij} \geqslant 0$ for two vertices in the same subset, and $a_{ij} \leqslant 0$ for two vertices in the different subsets. In this paper, this premise is removed and the improved result is provided by Theorem 2.

Theorem 2. Suppose that the signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is structurally balanced and its corresponding unsigned graph is $\mathcal{G}^U = (\mathcal{V}, \mathcal{E}, |A|)$. Then the controllability of $\mathcal{G}$ under leaders $\mathcal{V}_l$ is equivalent to that of $\mathcal{G}^U$ under leaders $\mathcal{V}_l$.

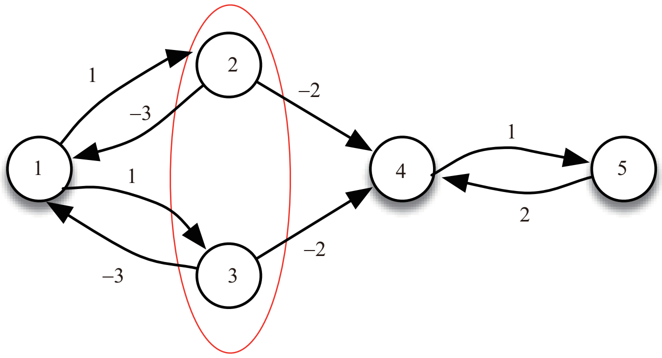

Figure 7 (Color online) π: $C_1 = \{1\}$, $C_2 = \{2, 3\}$, $C_3 = \{4\}$, $C_4 = \{5\}$.

Proof. The controllability matrix of (L, B) is $Q_1 = [B, LB, \ldots, L^{n-1}B]$ and the controllability matrix of $(\bar{L}, B)$ is

$$Q_2 = [B, \bar{L}B, \ldots, \bar{L}^{n-1}B] = [B, DLDB, \ldots, DL^{n-1}DB] = D[DB, LDB, \ldots, L^{n-1}DB],$$

where D is a nonsingular diagonal matrix and $B = [e_{i_1}, e_{i_2}, \ldots, e_{i_m}]$. Hence, the rank of Q_1 is equal to that of Q_2, and $\mathrm{im}(B)$ is equal to $\mathrm{im}(DB)$. Thus, the controllability of (L, B) is equivalent to that of (L, DB). Therefore, the controllability of (L, B) is equivalent to that of $(\bar{L}, B)$. The trivial cells of a nontrivial AEP play an important role in the study of the controllability of the weighed signed digraph.

Definition 8. Suppose that $\pi = \{C_1, \ldots, C_r\}$ is a nontrivial AEP and all its trivial cells are $C_{r_1}, \ldots, C_{r_m}$. If the leaders set $\mathcal{V}_l$ satisfies that $\mathcal{V}_l \subseteq \cup_{k=1}^{m} C_{r_k}$, then π is said to be a nontrivial AEP under leaders $\mathcal{V}_l$.

If π is a nontrivial AEP under leaders $\mathcal{V}_l$ and π is coarser than any other nontrivial AEP under leaders $\mathcal{V}_l$, then π is the coarsest AEP under leaders $\mathcal{V}_l$.

Inspired by [43], an algorithm to compute the coarsest AEP under leaders $\mathcal{V}_l$ for a given signed network is proposed. The algorithm is described as follows.

(1) Let $\pi_0 = \{i_1, \ldots, i_m, V_f\}$ be the initial partition.

(2) Relabel the cells in the current partition: $C_1, \ldots, C_r, C_f$. If C_f is a nontrivial cell, for every node s of C_f, compute $q_{sj} = \sum_{k \in C_j} l_{sk}$, $j = 1, \ldots, r, f$. Suppose that there exists a node t such that $q_{sj} = q_{tj}$. Then let s and t group into one cell. Replace the old cell with the newly created cells.

(3) Repeat step (2) until no cell can be split.

Lemma 5. For a signed digraph $\mathcal{G}$, the partition obtained via steps (1)–(3) is the coarsest AEP under leaders $\mathcal{V}_l$.

Proof. The proof is similar to that of Theorem 2 of [43], and hence is omitted. The following result characterizes the relationship between the controllability and the AEP.

Theorem 3. Let $\mathcal{G}$ be a signed digraph and suppose that $\pi = \{C_1, \ldots, C_r\}$ is a nontrivial AEP under leaders $\mathcal{V}_l$ and P is the characteristic matrix of π. Then

(i) The system is uncontrollable;

(ii) $\langle L, B \rangle \subseteq \mathrm{im}(P)$;

(iii) $\dim\langle L, B \rangle \leqslant r$.

Proof. Because $\pi = \{C_1, \ldots, C_r\}$ is an AEP of $\mathcal{V}$ and P is the characteristic matrix of π, according to Theorem 1 and Lemma 1, $\mathrm{im}(P)$ is L-invariant. Because π is a nontrivial AEP, $r < n$. If π is a nontrivial AEP under leaders $\mathcal{V}_l$, then $\mathrm{im}(B) \subseteq \mathrm{im}(P)$. Therefore, $\langle L, B \rangle \subseteq \mathrm{im}(P)$ and $\dim\langle L, B \rangle \leqslant r$. According to Lemma 3, the system is uncontrollable.

Example 1. Consider the signed digraph shown in Figure 7. $\pi = (\{1\}, \{2, 3\}, \{4\}, \{5\})$ is a nontrivial AEP. Each of cells: $\{1\}$, $\{4\}$ and $\{5\}$ is a trivial cell. By Theorem 3, if leaders belong to the set $\{1,4,5\}$, the system would be uncontrollable. For instance, if nodes 1,4,5 are taken as the leaders, the system is uncontrollable.

In this case, the Laplacian matrix L and the input matrix B can be written as follows:

$$L=\begin{bmatrix} 6 & 3 & 3 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 6 & -2 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad B=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The rank of the controllability matrix is 4. Thus, the system is uncontrollable. If nodes 4 and 5 are taken as leaders, the Laplacian matrix L and the input matrix B can be written as follows:

$$L=\begin{bmatrix} 6 & 3 & 3 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 6 & -2 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad B=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The rank of the controllability matrix is 2. Thus, the system is uncontrollable.

Theorem 3 provides an upper bound on the controllable subspace if there exists a nontrivial AEP under leaders $\mathcal{V}_l$. In the following we utilize the coarsest AEP under leaders $\mathcal{V}_l$ to get a tighter upper bound.

Theorem 4. Suppose that $\hat{P}$ is the characteristic matrix of the coarsest AEP under leaders $\mathcal{V}_l$ and P is the characteristic matrix of a nontrivial AEP π under leaders $\mathcal{V}_l$. Then the controllable subspace of the system (2) satisfies $\langle L, B\rangle \subseteq \mathrm{im}(\hat{P}) \subseteq \mathrm{im}(P)$.

Proof. According to Theorem 3, $\pi = \{C_1, \ldots, C_r\}$ is a nontrivial AEP under leaders $\mathcal{V}_l$ and P is the characteristic matrix of π. Then $\langle L, B\rangle \subseteq \mathrm{im}(P)$. Assume that $\hat{P}$ is the characteristic matrix of the coarsest AEP under leaders $\mathcal{V}_l$. Then $\langle L, B\rangle \subseteq \mathrm{im}(\hat{P})$ and $\mathrm{im}(\hat{P}) \subseteq \mathrm{im}(P)$. Therefore, $\langle L, B\rangle \subseteq \mathrm{im}(\hat{P}) \subseteq \mathrm{im}(P)$.

Example 2. Consider the graph shown in Figure 4. π_1 and π_2 are two nontrivial AEPs. π_1: $C_1 = \{1\}$, $C_2 = \{2,3,4\}$, $C_3 = \{5,6,7\}$. π_2: $C_1 = \{1\}$, $C_2 = \{2,\}$, $C_3 = \{3,4\}$, $C_4 = \{5,6\}$, $C_5 = \{7\}$. π_1 is coarser than π_2. Suppose that P_1 and P_2 are the characteristic matrices of π_1 and π_2, respectively.

$$P_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_2=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

If node 1 is taken as the leader, according to Theorem 3, the system is uncontrollable. In this case, the Laplacian matrix L and the input matrix B can be written as follows:

$$L=\begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & -1 & -1 & 1 \\ -1 & 0 & 3 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 3 \end{bmatrix}, \quad B=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The rank of the controllability matrix is 3. $\langle L, B\rangle = \mathrm{im}(P_1) \subset \mathrm{im}(P_2)$.

Remark 4. It is noted that the condition in Theorem 3 is necessary but not sufficient for the controllability. That is to say, if the leaders belong to a nontrivial cell, the controllability of the system is inconclusive. For the system in Figure 1, node 2 belongs to the nontrivial cell $\{2, 3\}$. If node 2 is taken as the leader, the rank of the controllability matrix is 5, and accordingly the system is controllable. For the system in Figure 4, if node 3 is taken as the leader, the rank of the controllability matrix is 4. Thus, the system is uncontrollable. How to get the sufficient and necessary conditions by graph partitions is still an open question.

5 Conclusion

In this paper, we developed a general definition of AEPs which offers a universal tool to characterize the controllability of multi-agent systems with arbitrary structures and link weights, including directed and undirected, weighted and unweighted, signed and unsigned networks. Our approach that transforms the network controllability problem into a graph problem greatly facilitates computation and offers a graph theoretic characterization for the controllability of multi-agent systems. We presented preliminary results for the characterization of the controllability of weighted and directed signed networks by graph partitions. By quickly ignoring those cases that cannot be controlled, we just need to judge the remaining cases although the main result is a necessary condition. We also provided the upper bound on the controllable subspace and addressed that the controllability of the structurally balanced graph is equivalent to that of the associated unsigned graph.

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