Controllability of Multi-Agent Systems over Signed Graphs

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***Abstract*—This work studies the controllability problem of multi-agent system over directed signed topology. The model we adopted is leader-follower system and some agents are chosen as leaders to control the rest of all the agents. The definition of sign-weighted-balanced partition is proposed which can deal with controllability problem over signed graph and the upper bound of controllable subspace has been presented.**

***Index Terms*—Controllability, multi-agent systems, signed graphs**

I. Introduction

In the past two decades, multi-agent systems have received tremendous attention due to its broad applications in many areas, such as sensory networks, robotics systems, power systems and unmanned aerial vehicles [1]–[3]. In 2003, one of the seminal works [4] explained the classical nearest neighbor rules mathematically and initiated the study of distributed control of multi-agent systems. Reference [5] studied multi-agent systems under directed weighted and switching topology. Recently, [6] has investigated signed graph, especially structurally balanced graphs where both cooperation and competition interactions coexist, which is an extension of nonnegative graphs where all agents collaborate. In fact, the structurally balanced graphs constitute only a minority of signed graphs. Therefore, [7] studied a class of structurally unbalanced graphs, called eventually positive graphs and some distinct performance can be found in this kind of graphs. Reference [8] further introduced the definition of sign-consensus about states of agents, which means states of all agents will converge to a situation that has the same sign but different magnitudes in an entry-wise manner. For more recent works on collective behaviors of multi-agent systems over signed graphs, readers are referred to [9]–[13] and references therein.

Many works about multi-agent systems focus on consensus problems. The study of controllability of multi-agent systems is also important, yet has been insufficiently investigated so far. The definition of controllability for multi-agent systems has been presented in [14], which analyzed the controllability in a leader-follower framework and the controllable condition about topology has been proposed. Reference [15] studied controllability on leaders-followers system with multi-leaders from a graph-theoretic perspective. They gave the concept of equitable partition and demonstrated that there exists a connection between system controllability and this partition. They showed that some special structures, like leader symmetry, which characterized the network, play a crucial role on the controllability of multi-agents systems. Based on the concept of equitable partition, [16] extends this definition to a new partition, called relaxed equitable partition and provided a sufficient and necessary condition for controllability over leader-follower systems. Reference [17] used a concept weighted-balanced partition to analyze controllability over directed and weighted graphs. However, these above mentioned works only focus on nonnegative graph. The major difference between nonnegative graphs and signed graphs is that the row sum of conventional Laplacian matrix of signed graph may not be equal to zero and this property will increase analytical complexity when we study signed graphs. Hence, [18] proposed a generalized almost equitable partition (GAEP) over undirected signed graphs and provided an algorithm of GAEP and a necessary condition of system controllability.

In this paper, we analyze the controllability of leader-follower system over directed signed graphs and propose the concept of sign-weighted-balanced partition that can be used to deal with controllability problem over directed signed graphs.

The organization of this paper is as follows. Preliminaries and notations about graph are provided in Section II. In Section III, the controllability problem for multi-agent systems is formulated. In Section IV, the controllability for general signed graphs is analyzed. Section V concludes this paper.

II. Notations and Preliminaries

Notations: 1_m and 0_m are vectors with all elements being ones and zeros, respectively. The identity matrix is $I_N \in R^{N\times N}$. Matrix $A = [a_{ij}] \in R^{N\times M}$ means a matrix with N rows and M columns where a_{ij} represents its element. The empty set is $\emptyset$. Matrix A with all entries being positive can be denoted as $A > 0$. Similarly, a negative matrix A means that all its elements are negative. The diagonal matrix is $diag(\sigma_1, \sigma_2, \cdots, \sigma_n)$. The eigenvalue of $A \in R^{N\times N}$ is $\lambda_i(A)$ for $i = 1, 2, , \cdots, N$. The cardinality of set C is $|C|$. The sign function is

$$sgn(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

The communication network of multi-agent system can be represented by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the node set is $\mathcal{V} =$

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$\{v_1, v_2, \ldots, v_N\}$ and the edge set is $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Each node in the graph represents an agent and the edge in graph means there is a path between two agents. In general, we employ the adjacency matrix to describe the structure of the topology. The adjacency matrix related to signed graph is a real matrix. In nonnegative graphs the adjacency matrices are nonnegative, which means all its entries are positive or zero. In general, the nonnegative graph is a subset of signed graph and the research of signed graph is more challenging.. The a_{ij} represents the weight of communication between agent i and j. $a_{ij} \neq 0$ if and only if there exists an edge from node j to node i; otherwise $a_{ij} = 0$. Let $\mathcal{N}_i = \{j | a_{ij} \neq 0,\ j = 1, 2, \ldots, n\}$ be the set of neighbors of node i. In the sequel, the graph associated with adjacency matrix $\mathcal{A}$ is written as $\mathcal{G}(\mathcal{A})$. The graph $\mathcal{G}(\mathcal{A})$ is undirected if and only if $\mathcal{A} = \mathcal{A}^T$. The graph $\mathcal{G}$ is strongly connected if and only if there exists a path from node i to node j for any pairs of nodes (i, j) in graph $\mathcal{G}$ and it is strongly connected when $\mathcal{A}$ is irreducible. A graph has a spanning tree if and only if there exists a root node such that there is a path from root node to any other nodes. The Laplacian matrix in this paper is

$$L = diag(\sum_{j \in \mathcal{N}_1} |a_{1j}|, \cdots, \sum_{j \in \mathcal{N}_n} |a_{nj}|) - \mathcal{A}.$$

Lemma 1: [6] If the signed graph $\mathcal{G}(\mathcal{A})$ is structurally balanced, then the following conditions hold.

a) The edges pairs are all digon sign-symmetry, i.e., $a_{ij}a_{ji} \geq 0$ and the corresponding undirected graph $\mathcal{G}(\mathcal{A}_u)$ is structurally balanced.

b) There exists a signature matrix D such that the following equation holds

$$\overline{\mathcal{A}} = D\mathcal{A}D$$

where $\overline{\mathcal{A}}$ is a nonnegative matrix and the matrix D can be denoted as

$$D = diag\{\sigma_1, \sigma_2, \cdots, \sigma_N\}, \quad \sigma_i \in \{-1, 1\}$$

∎

Definition 1: [17] A weight-balanced partition $\mathcal{P} = \{C_1, C_2, \cdots, C_r\}$ on follower agents set is defined according to i, j belong to $C_k (1 \leq k \leq r)$ if and only if $b_{ik} = b_{jk}$ and $\sum_{p \in C_l} a_{ip} = \sum_{p \in C_l} a_{jp}$ for all $l \neq k$, with $C_1, C_2, \cdots, C_r$ are cells of $\mathcal{P}$. In addition, if there exists at least one cell in partition, which contains more than two elements then we call this partition nontrivial. ∎

Lemma 2: [17] Consider system (1). It is uncontrollable if $\mathcal{G}$ has a nontrivial weighted-balanced partition $\mathcal{P}$. And, the dimension of its controllable subspace is not bigger than the cardinality of partition $\mathcal{P}$. ∎

III. Problem formulation

In this paper, the agents over a signed graph $\mathcal{G}$ are divided into two groups, i.e., the follower group $\mathcal{F} = \{1, 2, \cdots, n\}$ and the leader group $\mathcal{L} = \{n+1, n+2, \cdots, n+k\}$. Each agent is modeled as follows

$$\begin{aligned} \dot{x}_i(t) &= -\sum_{j \in \mathcal{N}_i} a_{ij}(sgn(a_{ij})x_i(t) - x_j(t)), \quad && i \in \mathcal{F} \\ \dot{x}_k(t) &= v_k, && k \in \mathcal{L} \end{aligned} \tag{1}$$

where $\mathcal{N}_i$ is the neighbour set of agent i and the v_k is the control input of the leader node k. In other words, each follower node is influenced by its neighbors, but each leader node is only driven by an external force. System (1) can be written as

$$\dot{x}(t) = -Lx(t) + \mathcal{V} \tag{2}$$

where $x(t) = [x_1, x_2, ..., x_{n+k}]^T$, matrix L is the Laplacian matrix of graph $\mathcal{G}$ and $\mathcal{V} = [0_n^T, v]^T$ where $v = [v_1, v_2, \cdots, v_k]$. The Laplacian matrix L can be written as

$$L = \begin{bmatrix} \mathcal{L}_f(g) & \mathcal{L}_{fl}(g) \\ 0_{k \times n} & 0_{k \times k} \end{bmatrix} \in \mathbb{R}^{(n+k) \times (n+k)} \tag{3}$$

where $\mathcal{L}_f(g) \in R^{n \times n}$, $\mathcal{L}_{fl}(g) \in R^{n \times k}$. Notations $0_{k \times n} \in \mathbb{R}^{k \times n}$ and $0_{k \times k} \in \mathbb{R}^{k \times k}$ are matrices with all elements being zeros. Then, the system (2) can be putted into the following form

$$\dot{x}_f(t) = -\mathcal{L}_f(g)x_f(t) - \mathcal{L}_{fl}(g)x_l(t) \tag{4}$$

$$\dot{x}_l(t) = v^T \tag{5}$$

where $x_f = [x_1, x_2, \cdots, x_n]^T \in R^n$ and $x_l = [x_{n+1}, x_{n+2}, \cdots, x_{n+k}]^T \in R^k$ are states of followers and leaders, respectively. Further, let $x_l(t) = u(t)$. Then (4) can be treated as a linear control system

$$\dot{x}_f(t) = Ax_f(t) + Bu(t) \tag{6}$$

and the state of the leader node is now acting as the control input of the follower nodes. The matrix A is

$$A = \mathcal{L}_f(g) = \begin{bmatrix} H_1 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & H_2 & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & H_n \end{bmatrix} \in R^{n \times n}$$

where $H_i = \sum_{j=1}^{n} |a_{ij}|$ and $B = -\mathcal{L}_{fl}(g) = [b_{ij}] \in R^{n \times k}$ $(i = 1, 2, \cdots, n)$. This paper studies the controllability of followers agents, which is described by the following definition.

Definition 2: [20] System (6) is said to be controllable if for any initial states $x_{initial}$ and any finial states x_{final} of followers, there exists $u(t)$ that transfers $x_{initial}$ to x_{final} in a finite time. ∎

Lemma 3: [20] The system (6) is controllable if and only if any of the following conditions hold

(a) The controllability matrix

$$C \triangleq \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is full rank.

(b) For each eigenvalue λ_i of A, $i = \{1, 2, \cdots, n\}$. The matrix $[A - \lambda_i I, B]$ is full rank. ∎

The objective of this paper is to analyze the effect of the graph on the controllability of the multi-agent system.

IV. Controllability for General Signed Graphs

In this section, we propose a new partition which is called sign-weighted-balanced partition to analyze the controllability of signed graphs. The definition of sign-weighted-balanced partition is as follows.

Definition 3: A partition $\pi = \{C_1, C_2, \ldots, C_r\}$ on followers agents $\mathcal{F}$ is called sign-weighted-balanced partition, if for any $i, j \in C_l$ $(1 \le l \le r)$, we have $b_{is} = b_{js}$ $(1 \le s \le k)$ and $\sum_{m \in \mathcal{C}_p} a_{im} = \sum_{m \in \mathcal{C}_p} a_{jm} (1 \le p \le r)$, where $a_{im}a_{jm} \ge 0$. If there exists at least one cell C_k in partition which contains more than one elements, then this partition is nontrivial. ∎

Then we can obtain a characteristic matrix $P(\pi) = [P_{ij}]$ of sign-weighted-balanced partition π, where

$$P_{ij} = \begin{cases} 1, i \in C_j \\ 0, i \notin C_j \end{cases}$$

Algorithm 1: *The way to find a sign-weighted-balanced partition.*

Step 1: Let the initial partition be $\pi = \{\{1\}, \{2\}, \cdots, \{n\}\}$.

Step 2: Let $l = 1$. Agents $i, j, \cdots, m \in (1, 2, \cdots, n)$ should be grouped into one cell if $b_{il} = b_{jl} = \cdots = b_{ml}$. Based on this rule, we can generate new partition $\pi = \{C_1, C_2, \cdots, C_f\}$.

Step 3: Let $l = l + 1$. The C_h cannot be split if and only if $b_{il} = b_{jl} = \cdots = b_{ml}, \forall (i, j, \cdots, m) \in C_h (h = 1, 2, \cdots, f)$ holds, otherwise the cell should be further split according to the rule in Step 2. Repeat the above procedures for each cell in π and update the partition. Repeat Step 3 until $l = k + 1$.

Step 4: Suppose the partition is $\pi = \{C_1, \cdots, C_h\}, h \in R$ now. Each cell in π cannot be split if the following condition holds

$$a_{im}a_{jm} \ge 0$$

$\forall i, j \in C_p$, $m \in C_q$ and $(C_p, C_q) \in \pi$. Otherwise, agents i, j cannot be grouped into one cell and C_p now is split into two cells, i.e., one cell contains agent i and the other is a singleton containing agent j. Repeat the above procedures until no cell can be split in π and and replace the old partition with the new one.

Step 5: Now the partition can be written as $\pi = \{C_1, \cdots, C_r\}, r \in R$. Let $W_i = \sum_{m \in C_k} a_{im}$ where i means agent i which belongs to C_p, $(C_p, C_k) \in \pi$. Agents i, j in C_p can be grouped into one cell if following condition holds

$$W_i = W_j$$

, otherwise the cell should be split. Repeat the above procedure until no cell can be split. ∎

Example 1: Consider the multi-agent system (6) over a graph shown in Fig. 1, where the agent 6 is chosen as leader.

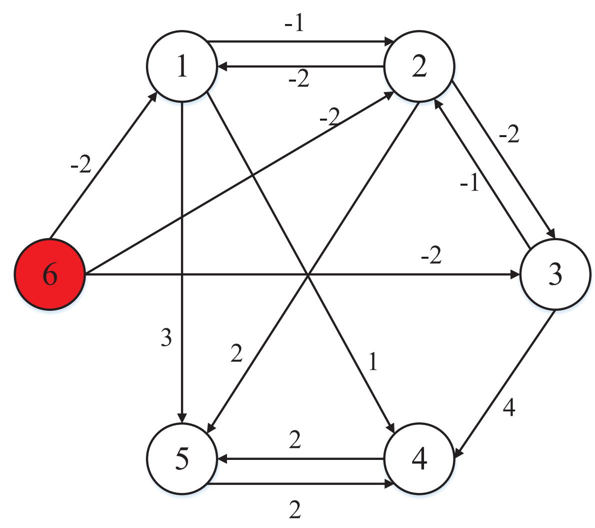

Fig. 1: A signed graph $\mathcal{G}_1$.

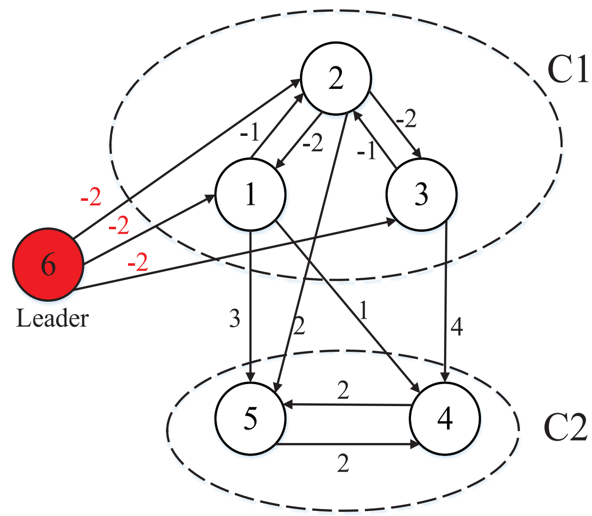

Fig. 2: The sign-weighted-balanced partition over followers set of $\mathcal{G}_1$.

The sign-weighted-balanced partition of Fig. 1 is shown in Fig. 2. The characteristic matrix of above partition is

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

∎

The controllable subspace of system (6) can be denoted as

$$K = Im(B) + A \times Im(B) + \cdots + A^{n-1} \times Im(B) \quad (7)$$

where $Im(B)$ is vector space spanned by columns of B, $A \times Im(B)$ means vector space $\{Ax : x \in Im(B)\}$ and the $+$

denotes the union of spaces. Then, we introduce the definition of quotient of $\mathcal{G}$ over partition π.

Definition 4: [15] The directed graph $\mathcal{G}/\pi = (\bar{\nu}, \bar{\varepsilon})$ is called the quotient of partition $\pi = \{C_1, C_2, \ldots, C_r\}$, if the cells in partition π are nodes in directed graph $\mathcal{G}/\pi$. In other words, $(1, 2, \cdots, r) \in \bar{\nu}$ and there exists an edge which belongs to edge set $\bar{\varepsilon}$ if and only if the following equation holds

$$\sum_{m \in \mathcal{C}_j} a_{pm} = \sum_{m \in \mathcal{C}_j} a_{qm}, \forall (p, q) \in C_i, \{C_i, C_j\} \in \pi$$

where a_{pm} is the weight of edge from node m to node p. Let $d_{ij} = \sum_{m \in \mathcal{C}_j} a_{pm} = \sum_{m \in \mathcal{C}_j} a_{qm}$ be the weight of an edge from cells C_j to C_i. Then, the adjacency matrix of graph $\mathcal{G}/\pi$ is

$$\mathcal{A}(\mathcal{G}/\pi) = [d_{ij}] \tag{8}$$

∎

Figure 3 is the quotient of partition of the Example 1.

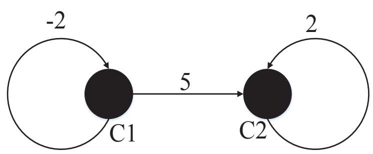

Fig. 3: The quotient of Fig. 2 with sign-weighted-balanced partition π.

Let $\Delta = diag(H_1, H2, \cdots, H_n)$. Define a matrix

$$E = \Delta - \mathcal{A}(\mathcal{G}/\pi) \tag{9}$$

The adjacency matrix of Fig. 3 and the matrix E are

$$\mathcal{A}(\mathcal{G}/\pi) = \begin{bmatrix} -2 & 0 \\ 5 & 2 \end{bmatrix}, E = \begin{bmatrix} 6 & 0 \\ -5 & 5 \end{bmatrix}$$

Next, we propose the following theorem.

Theorem 1: Consider a signed directed and weighted graph $\mathcal{G}$ with sign-weighted-balanced partition $\pi = \{C_1, \cdots, C_r\}$. There exists a matrix E defined as (9) such that $Im(P)$ is A-invariant, where P and A are characteristic matrix of partition π and system matrix of (6), respectively. And the controllable subspace K is bounded by $K \subseteq Im(P)$.

proof

Without loss of generality, we suppose that agents $(1, 2, \cdots, |C_1|) \in C_1$ and $(|C_1| + 1, \cdots, |C_2|) \in C_2, \cdots$. Then the characteristic matrix can be denoted as

$$P = \begin{bmatrix} 1_{|C_1|} & & & \\ & 1_{|C_2|} & & \\ & & \ddots & \\ & & & 1_{|C_r|} \end{bmatrix} \in R^{n \times r}$$

Then, we have

$$(AP)_{ij} = \begin{cases} -\sum\limits_{j \in C_j} a_{ij}, i \neq j \\ H_i - \sum\limits_{j \in C_i} a_{ij}, i = j \end{cases}$$

and

$$(PE)_{ij} = \begin{cases} -d_{ij}, i \neq j \\ H_i - d_{ii}, i = j \end{cases}$$

According to the definition of matrix E, we have

$$AP = PE \tag{10}$$

The vectors space spanned by P is A-invariant if and only if there exists a matrix E such equation (10) holds and the space can be denoted as $Im(P)$ mathematically. By the definition of sign-weighted-balanced partition, we know that matrix B in (6) is

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{bmatrix} \tag{11}$$

where $b_{ij} = b_{zj}, j = 1, 2, \cdots, k$, when $i, z \in C_p, p = 1, 2, \cdots, r$. It is trivial to show that the columns of B can be generated by linear combinations of columns of P. Therefore, $Im(B) \subseteq Im(P)$. As in [18], we can obtain

$$\begin{aligned} K &= Im(B) + A \times Im(B) + \cdots + A^{n-1} \times Im(B) \\ &\subseteq Im(P) + A \times Im(P) + \cdots + A^{n-1} \times Im(P) \\ &= Im(P) \end{aligned} \tag{12}$$

This completes the proof.

Based on Theorem 1, we can conclude the following result.

Corollary 1: System (6) over graph $\mathcal{G}$ is uncontrollable if $\mathcal{G}$ has a nontrivial sign-weighted-balanced partition $\pi = \{C_1, C_2, \ldots, C_r\}$.

proof

System (6) is controllable if and only if controllability matrix C has full rank. Once the sign-weighted-balanced partition π is nontrivial, the dimension of characteristic matrix of partition π must be less than n. This means that the dimension of control subspace is less than n. Therefore, the controllability matrix C is not full rank and system (6) is uncontrollable . ∎

Example 2: The Fig. 2 is a signed graph with nontrivial Sign-weighted-balanced partition, and we can easily derive the system matrix A and input matrix B as follows.

$$A = \begin{bmatrix} 4 & 2 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \\ -1 & 0 & -4 & 7 & -2 \\ -3 & -2 & 0 & -2 & 7 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

With straightforward calculation, we can show that the rank of its controllability matrix is 2 in the above example and system (6) is uncontrollable over graph $\mathcal{G}_1$. ∎

V. Conclusion

In this paper, we have investigated the controllability problem for linear multi-agent system over directed weighted signed graphs. The relationships between agents consist of both competition and cooperation. The definition of Sign-weighted-balanced partition was proposed to analyze the controllability of signed graphs. The upper bound of controllable subspace based on Sign-weighted-balanced partition has been proposed. It is our future work to study the controllability when topology is switching.

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