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From Matrix-weighted Consensus to Multipartite Average Consensus

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***Abstract*—This paper studies a consensus protocol, which is termed multipartite average consensus, of a multi-agent system over signed undirected networks in which all agents could be partitioned into multiple subgroups and all agents in the same group converge to the average of their initial values. The proposed protocol is based on results from a matrix-weighted consensus. It is shown that a matrix-weighted consensus can be transformed into a multipartite average consensus. We then discuss a method to combine some subgroups or all subgroups into a union group in order that all agents in the union group reach an average consensus. Finally, we provide examples and simulations to validate the statement.**

***Index Terms*—Multi-agent systems; signed networks; group consensus; multipartite average consensus.**

I. INTRODUCTION

THE consensus problems in networked multi-agent systems have widely been studied in the literature [1]–[6], where consensus algorithms have been applied to formation control [7]–[10], social networks [11], [12], coordination for power generation in a microgrid [13], etc. In particular, complex networks characterized by a graph including signed scalar-weighted edges or matrix-weighted edges have recently received much attention for a consensus in multi-agent systems [14]–[22].

The complex network corresponding to a graph with signed scalar-weighted edges is called signed network, and this network is generally studied for group consensus, cluster consensus and bipartite consensus [5]. Roughly speaking, if agents in a group over a signed network are partitioned into multiple subgroups such that all agents in each partitioned subgroup converge to a common value regardless of the other subgroups, then the process is sometimes called group consensus [14], [23]. On the other hand, if all agents in each partitioned subgroup converge to a common value different from common values of the other subgroups, then the scenario is called cluster consensus [15], [24]. That is, in a cluster consensus, no consensus between any two distinct partitioned subgroups can occur, whereas in a group consensus, there can happen a consensus between two different partitioned subgroups. Moreover, if agents in a signed network are split into two antagonistic subgroups such that all agents in each partitioned subgroup converge to a common value with the opposite sign of the convergent value of the other subgroup, then the scenario is called bipartite consensus [16]–[18], [21], [22]. Another complex network is called matrix-weighted network corresponding to a graph with matrix-weighted edges. In such a network, agents' states are represented by vectors and a relationship between two different agents is coupled with a matrix weight, which can be used to describe interconnections between agents and reach a consensus of a group for coupled systems [19], [20]. Such a consensus process is called matrix-weighted consensus.

In this paper, we study how to design a group average consensus protocol over signed undirected networks. In the literature [14], [23], [25]–[27], it has been studied to find conditions for achieving a group consensus with multiple subgroups over directed networks or with two partitioned subgroups over undirected networks. To the best of our knowledge, a group consensus protocol with multiple subgroups has not been studied over signed undirected networks, which motivates our study to focus on that consensus protocol. In particular, we make use of the concept of matrix-weighted consensus studied in [19] to explore a group average consensus over signed undirected networks. Moreover, we consider not only a concept of achieving a group average consensus but also a concept of reaching a single average consensus, where the single average consensus means a consensus such that all agents' states converge to their initial average. To distinguish our protocol from other group consensus protocols, the consensus protocol studied in this paper is termed multipartite average consensus.

Consequently, the main objective of our work is to design a consensus protocol for exponentially achieving a multipartite average consensus over signed undirected networks. To achieve this objective, we make two main contributions as follows. The first main contribution is to derive conditions for reaching a multipartite average consensus over a signed undirected network from the perspective of a novel approach, i.e., from the concept of a matrix-weighted consensus. A matrix-weighted consensus problem can be interpreted as a scalar-weighted consensus problem, which provides insights on studying group consensus problems over signed undirected networks. We establish a relationship between the two consensus problems such that if a matrix-weighted consensus system achieves an average consensus then the transformed system from the matrix-weighted consensus system achieves a

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multipartite average consensus, details of which are presented in Section II and Section III. We then extend the result of the first contribution to another group consensus problems by unifying different clusters in a consensus system, which is regarded as our second contribution. That is, the second main contribution is that we provide a method to combine some or all subgroups into a union group such that all agents in the union group converge to their initial average, which is studied in Section IV.

II. Background and Problem Formulation

We first introduce several notations. We denote the null space of a matrix by $\mathcal{N}(\cdot)$, and the identity matrix by $I_k \in \mathbb{R}^{k\times k}$. Also, $\mathbb{1}_n \in \mathbb{R}^n$ denotes the vector whose all entries are 1, and the symbol $\otimes$ denotes the Kronecker product.

In what follows, we briefly introduce matrix- and scalar-weighted consensus, and formulate the main problems to be studied in the paper.

A. Matrix-weighted consensus

Consider an undirected graph $\mathcal{G}$ denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{A}$ respectively denote the vertex set defined as $\mathcal{V} = \{1, 2, \cdots, n\}$, the edge set $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ with $m = |\mathcal{E}|$ and the set of matrix weights defined as $\mathcal{A} = \{A_{ij} \in \mathbb{R}^{k\times k} \mid A_{ij}$ is positive semidefinite, $\{i, j\} \in \mathcal{E}\}$. Also, let us order all m matrix weights as $A_{l_{ij}}, l \in \{1, 2, \cdots, m\}$, where A_l is sometimes used instead of $A_{l_{ij}}$ for notational convenience if we expect no confusion. Then, we can have the matrix-weighted Laplacian $L \in \mathbb{R}^{kn\times kn}$ as follows

$$L = [L_{ij}] = \bar{H}^\top \text{blkdiag}(A_l)\bar{H}, \tag{1}$$

where $\bar{H} = H \otimes I_k$, and $H \in \mathbb{R}^{m\times n}$ and blkdiag$(\cdot)$ denote the incidence matrix (refer to [19, Lemma 1]) and a block diagonal matrix of matrices in $(\cdot)$, respectively. Furthermore, we can express the matrix-weighted consensus system as follows

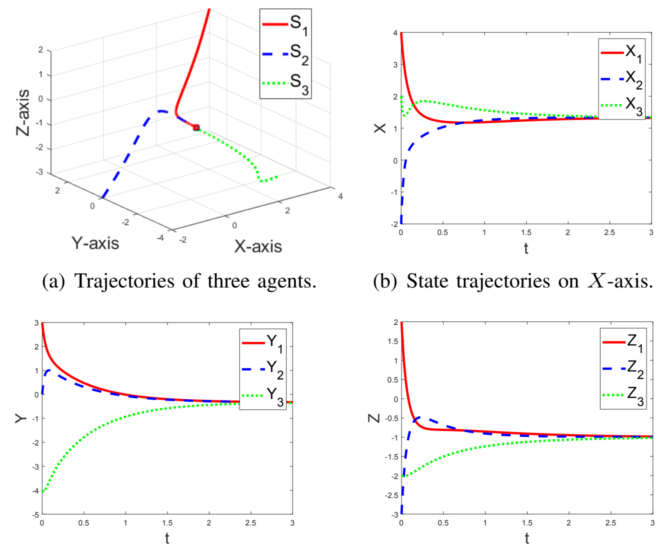
$$\dot{x} = -Lx, \tag{2}$$

where the symbol x denotes a state vector such that $x = [x_1^\top, x_2^\top, \cdots, x_n^\top]^\top \in \mathbb{R}^{kn}$ in which $x_u = [x_{u1}, x_{u2}, \cdots, x_{uk}]^\top \in \mathbb{R}^k, u \in \{1, 2, \cdots, n\}$. Through the system (2), the matrix-weighted consensus is defined as follows.

Definition 1 (Matrix-weighted consensus [19]). *The system (2) is said to achieve a matrix-weighted consensus if $x_i = x_j, \forall i, j \in \mathcal{V}$.*

A matrix-weighted consensus discussed in this paper is confined to an average consensus over matrix-weighted networks such that $\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} x_j(t) = \frac{1}{n}\sum_{u=1}^n x_u(0), \forall i, j \in \mathcal{V}$. Here, we introduce a result to reach a matrix-weighted consensus with an algebraic approach derived in [19].

Theorem 1. *[19, Theorem 2] The system (2) converges to its initial average $x^* = \mathbb{1}_n \otimes \bar{x}$ exponentially fast, where $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$ for all initial values, if and only if $\mathcal{N}(L) = \text{span}\{\mathbb{1}_n \otimes I_k\}$.*



(a) Trajectories of three agents.

(b) State trajectories on X-axis.

(c) State trajectories on Y-axis.

(d) State trajectories on Z-axis.

Fig. 1: Simulation: a matrix-weighted consensus of three agents in $\mathbb{R}^3$ under the consensus protocol (2), where three agents are denoted by $S_1 = [X_1, Y_1, Z_1]$, $S_2 = [X_2, Y_2, Z_2]$ and $S_3 = [X_3, Y_3, Z_3]$.

We provide a simulation example on a matrix-weighted consensus in Fig. 1, where matrix weights are chosen as $A_{12} = \begin{bmatrix} 4 & 0 & 3 \\ 0 & 10 & 1 \\ 3 & 1 & 5 \end{bmatrix}$ and $A_{23} = \begin{bmatrix} 15 & 5 & 0 \\ 5 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, that is, Laplacian matrix L is given as

$$L = \begin{bmatrix} 4 & 0 & 3 & -4 & 0 & -3 & 0 & 0 & 0 \\ 0 & 10 & 1 & 0 & -10 & -1 & 0 & 0 & 0 \\ 3 & 1 & 5 & -3 & -1 & -5 & 0 & 0 & 0 \\ -4 & 0 & -3 & 19 & 5 & 3 & -15 & -5 & 0 \\ 0 & -10 & -1 & 5 & 13 & 1 & -5 & -3 & 0 \\ -3 & -1 & -5 & 3 & 1 & 6 & 0 & 0 & -1 \\ 0 & 0 & 0 & -15 & -5 & 0 & 15 & 5 & 0 \\ 0 & 0 & 0 & -5 & -3 & 0 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

and it holds that $\mathcal{N}(L) = \text{span}\{\mathbb{1}_3 \otimes I_3\}$. It is observed as in Fig. 1 that the state values of the three agents on the same axis converge to the same value. We make use of this property in the following subsection to interpret a scalar-weighted consensus over a signed network, where the scalar-weighted consensus means the classic concept of consensus with scalar-weighted edges.

B. Scalar-weighted consensus

This subsection introduces a concept of transforming the matrix-weighted graph $\mathcal{G}$ to a scalar-weighted graph, and shows how the matrix-weighted consensus system (2) is expressed in terms of a scalar-weighted consensus system.

Let us first consider the scalar-weighted graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{W})$, where $\mathcal{G}'$ is induced by $\mathcal{G}$ as follows: the vertex set $\mathcal{V}'$ is denoted by $\mathcal{V}' = \{1, 2, \cdots, kn\}$ where the kn vertices can be partitioned into k subgroups according to the state x defined in (2) such that $\mathcal{V}' = \mathcal{V}'_1 \cup \mathcal{V}'_2 \cup \cdots \cup \mathcal{V}'_k$ and $\mathcal{V}'_1, \mathcal{V}'_2, \cdots, \mathcal{V}'_k$ are disjoint sets with $|\mathcal{V}'_s| = n, \forall s \in \{1, 2, \cdots, k\}$; the edge set $\mathcal{E}'$ is defined as $\mathcal{E}' = \{\{i, j\} \mid L_{ij} \neq 0, i, j \in \mathcal{V}', i \neq j\}$ with $m' = |\mathcal{E}'|$; the weight set $\mathcal{W}$ is defined as $\mathcal{W} = \{w_{ij} \in$

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$\mathbb{R} \mid w_{ij} = -L_{ij}$ for $L_{ij} \neq 0, i, j \in \mathcal{V}', i \neq j\}$. Note that in this paper, we only consider undirected graphs without self-loops; we call vertex subsets, i.e., $\mathcal{V}'_1, \mathcal{V}'_2, \cdots, \mathcal{V}'_k$, clusters; we use a term 'disjoint clusters' to say that all agents in a cluster of disjoint clusters achieve an average consensus regardless of the other disjoint clusters. With the ordered weight set expressed as $\mathcal{W} = \{w_1, w_2, \cdots, w_{m'}\}$, the scalar-weighted Laplacian $L' \in \mathbb{R}^{kn \times kn}$ is given by

$$L' = H'^{\top} W H', \tag{4}$$

where $W \in \mathbb{R}^{m' \times m'}$ denotes a diagonal weight matrix such that $W = \text{diag}\{w_1, w_2, \cdots, w_{m'}\}$ and $H' \in \mathbb{R}^{m' \times kn}$ denotes the oriented incidence matrix of $\mathcal{G}'$. It is remarkable that if a negative weight in $\mathcal{W}$ is included then the definition of the scalar-weighted Laplacian L' equals the definition of the signed Laplacian discussed in [28]; for example, the scalar-weighted Laplacian (3) induced from the concept of matrix-weighted Laplacian has the same form as the signed Laplacian introduced in [28]. We then have the following scalar-weighted consensus system transformed from the matrix weighted system (2).

$$\dot{x} = -L'x. \tag{5}$$

This system is also a distributed system since each agent only needs weighted values of its neighbors in $\mathcal{G}'$, where the neighbors of agent i is defined as $\{j \in \mathcal{V} \mid \{i, j\} \in \mathcal{E}'\}$.

Remark 1. *According to the definitions, the matrix forms of L and L' are the same, i.e., $L = L'$, and thus the both consensus systems (2) and (5) have the same mathematical form. Moreover, in this case, L' is positive semidefinite since L is positive semidefinite [19, Lemma 2].*

Fig. 2 shows an example of how we make a scalar-weighted graph over a signed network from a matrix-weighted graph, where A_{12} and A_{23} are chosen as

$$A_{12} = \begin{bmatrix} 4 & 0 & 3 & 2 \\ 0 & 10 & 1 & 7 \\ 3 & 1 & 5 & 0 \\ 2 & 7 & 0 & 15 \end{bmatrix}, \; A_{23} = \begin{bmatrix} 15 & 5 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

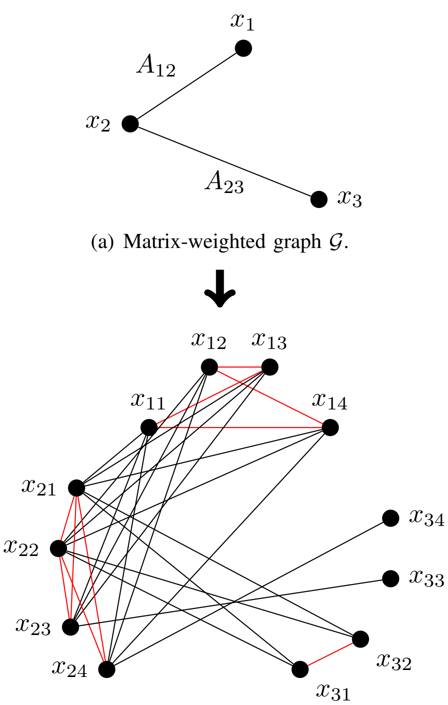
and L is given by

$$L = \begin{bmatrix} A_{12} & -A_{12} & 0 \\ -A_{12} & A_{12} + A_{23} & -A_{23} \\ 0 & -A_{23} & A_{23} \end{bmatrix}.$$

With this idea, we can have the main concept for a scalar-weighted consensus defined as follows.

Definition 2 (Multipartite average consensus)**.** *A multi-agent system is said to achieve a k-partite average consensus if there exist k disjoint clusters such that all agents in the same cluster converge to their average for all clusters.*

In particular, if a system achieves a 1-partite average consensus, then the consensus is termed single average consensus.



(a) Matrix-weighted graph $\mathcal{G}$.

(b) Scalar-weighted graph $\mathcal{G}'$ where $\mathcal{V}'_1 = \{x_{11}, x_{21}, x_{31}\}$, $\mathcal{V}'_2 = \{x_{12}, x_{22}, x_{32}\}$, $\mathcal{V}'_3 = \{x_{13}, x_{23}, x_{33}\}$ and $\mathcal{V}'_4 = \{x_{14}, x_{24}, x_{34}\}$.

Fig. 2: Example: scalar-weighted graph in Fig. 2(b) transformed from the matrix-weighted graph depicted in Fig. 2(a), where $x_1 = [x_{11}, x_{12}, x_{13}, x_{14}]^{\top}$, $x_2 = [x_{21}, x_{22}, x_{23}, x_{24}]^{\top}$ and $x_3 = [x_{31}, x_{32}, x_{33}, x_{34}]^{\top}$, and these states denote vertices. Moreover, for Fig. 2(b), we denote edges with positive and negative weights by black and red lines, respectively.

III. Multipartite average consensus over signed networks

In this section, we analyze the scalar-weighted consensus system (5) generated from the matrix-weighted consensus system (2), and then study more general multipartite average consensus problems over a signed network induced by transformed graph of $\mathcal{G}'$. To this end, we first explore a multipartite average consensus problem such that all states of the system (5) satisfy $\lim_{t\to\infty} x_{1g}(t) = \lim_{t\to\infty} x_{2g}(t) = \cdots = \lim_{t\to\infty} x_{ng}(t) = \frac{1}{n}\sum_{i=1}^{n} x_{ig}(t), \forall g \in \{1, 2, \cdots, k\}$. We then study a problem of reaching a multipartite average consensus in a transformed system from the system (5) with similarity transformation.

A. Multipartite average consensus based on matrix-weighted consensus

Consider a multipartite average consensus such that the system (5) achieves $\lim_{t\to\infty} x_{1g}(t) = \lim_{t\to\infty} x_{2g}(t) = \cdots = \lim_{t\to\infty} x_{ng}(t) = \frac{1}{n}\sum_{i=1}^{n} x_{ig}(t), \forall g \in \{1, 2, \cdots, k\}$, which means that the system (5) converges to $\mathbb{1}_n \otimes \bar{x} = (\mathbb{1}_n \otimes I_k)\,\bar{x} \in \text{span}\{\mathbb{1}_n \otimes I_k\}$. Then, in the light of $L' = L$, i.e., (2)=(5), we can reach the following results directly from Theorem 1.

Theorem 2. *The scalar-weighted consensus system (5) globally exponentially achieves a k-partite average consensus*

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such that $\lim_{t\to\infty} x_{1g}(t) = \lim_{t\to\infty} x_{2g}(t) = \cdots = \lim_{t\to\infty} x_{ng}(t) = \frac{1}{n}\sum_{i=1}^{n} x_{ig}(t), \forall g \in \{1, 2, \cdots, k\}$ *if and only if* $\mathcal{N}(L) = \operatorname{span}\{\mathbb{1}_n \otimes I_k\}$.

Corollary 1. *The matrix-weighted consensus system* (2) *globally exponentially converges to the system's average if and only if the scalar-weighted consensus system* (5) *globally exponentially achieves a* k*-partite average consensus such that* $\lim_{t\to\infty} x_{1g}(t) = \lim_{t\to\infty} x_{2g}(t) = \cdots = \lim_{t\to\infty} x_{ng}(t) = \frac{1}{n}\sum_{i=1}^{n} x_{ig}(t), \forall g \in \{1, 2, \cdots, k\}$.

Proof. This proof is directly proved by Theorem 1 and Theorem 2. □

B. Multipartite average consensus with similarity transformation

In this subsection, we explore a multipartite average consensus problem with using a similarity transformation of L' based on the fact of Theorem 2. That is, with the assumption that the system (2) achieves a matrix-weighted consensus, we can achieve various results on a multipartite average consensus by reordering the states and disjoint clusters. Note that any problem in the paper is based on the matrix-weighted consensus, and thus is always accompanied with the following assumption.

Assumption 1. *It is assumed that* $\mathcal{N}(L) = \mathcal{N}(L') = \operatorname{span}\{\mathbb{1}_n \otimes I_k\}$.

Let us define a new graph $\mathcal{G}'_S$ as $\mathcal{G}'_S = (\mathcal{V}'_S, \mathcal{E}'_S, \mathcal{W}'_S)$ where $\mathcal{V}'_S$ is composed of k disjoint clusters such that $\mathcal{V}'_S = \mathcal{V}'_{S1} \cup \mathcal{V}'_{S2} \cup \cdots \cup \mathcal{V}'_{Sk}$ with $|\mathcal{V}'_{S\bar{s}}| = n, \forall \bar{s} \in \{1, 2, \cdots, k\}$. Further, we define a new incidence matrix H'_S as follows

$$H'_S = H'P, \tag{6}$$

where $P \in \mathbb{R}^{kn\times kn}$ denotes a permutation matrix satisfying $PP^\top = I_{kn}$. The permutation matrix leads to a reordering of agents' states and disjoint clusters, which makes a k-partite average consensus distinct from that generated from Theorem 2. Then, a new scalar-weighted Laplacian L'_S corresponding to the graph $\mathcal{G}'_S$ is defined as $L'_S = H'^\top_S W H'_S$. The new sets for edges and weights, i.e., $\mathcal{E}'_S$ and $\mathcal{W}'_S$, are defined according to the definition of H'_S. Moreover, the following new scalar-weighted consensus system is given by

$$\dot{x} = -L'_S x \tag{7}$$

We provide an example of $\mathcal{G}'_S$ transformed from $\mathcal{G}'$ as illustrated in Fig. 3, where we choose a permutation matrix as

$$P = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \end{bmatrix}.$$

Based on the system (7), we have the following result.

Lemma 1. *The consensus system* (7) *has the following properties.*

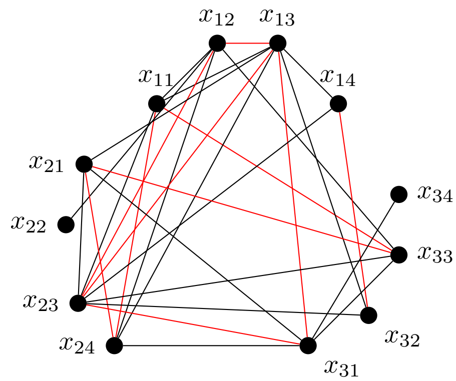

Fig. 3: Example: transformed graph $\mathcal{G}'_S$ from $\mathcal{G}'$ in Fig. 2(b), where the black and red lines denote positively and negatively weighted edges, respectively.

(i) L'_S *has the same spectrum as* L'*, and thus it has* k *zero eigenvalues,*

(ii) $\mathcal{N}(L'_S) = \{x \mid x = P^{-1}v, v \in \mathcal{N}(L')\}$,

(iii) the consensus system (7) *has a unique equilibrium point* $\hat{x}^* = P^{-1}(\mathbb{1}_n \otimes \bar{x}_S)$*, where* $\bar{x}_S = \frac{1}{n}(\mathbb{1}_n^\top \otimes I_k) Px$,

(iv) $\bar{x}_S$ *is invariant for all* $t \geq 0$.

Proof. (i) With respect to the definition of L'_S, we have

$$L'_S = H'^\top_S W H'_S = P^{-1}H'^\top W H' P = P^{-1}L'P. \tag{8}$$

We then see that L'_S and L' are similar and eigenvalues of them are the same. Thus, since L' is positive semidefinite with k zero eigenvalues (refer to [19]), L'_S is also.

(ii) It directly follows from (8) that $\mathcal{N}(L'_S) = \mathcal{N}(P^{-1}L'P) = \{x \mid x = P^{-1}v, v \in \mathcal{N}(L') = \operatorname{span}\{\mathbb{1}_n \otimes I_k\}\}$.

(iii) We use a contradiction for this proof. First, it is clear that $\hat{x}^* = P^{-1}(\mathbb{1}_n \otimes \bar{x}_S) = P^{-1}(\mathbb{1}_n \otimes I_k)\bar{x}_S \in \mathcal{N}(L'_S)$ from Lemma 1-(ii). Next, assume that there exists $x \in \mathcal{N}(L'_S)$ other than $x = \hat{x}^*$, i.e., $x \neq \hat{x}^*$. Then, we have $x = P^{-1}(\mathbb{1}_n \otimes I_k)z \neq \hat{x}^*$, where $z \in \mathbb{R}^k$ is an arbitrary vector. Observe the following result.

$$\begin{aligned} \bar{x}_S = \frac{1}{n}(\mathbb{1}_n^\top \otimes I_k) Px &= \frac{1}{n}(\mathbb{1}_n^\top \otimes I_k) PP^{-1}(\mathbb{1}_n \otimes I_k)z \\ &= \frac{1}{n}(\mathbb{1}_n^\top \mathbb{1}_n \otimes I_k) z \\ &= \frac{1}{n}(n \otimes I_k) z = z. \end{aligned} \tag{9}$$

Substituting (9) into $\hat{x}^*$ gives $\hat{x}^* = P^{-1}(\mathbb{1}_n \otimes z) = P^{-1}(\mathbb{1}_n \otimes I_k)z = x$. However, this is a contradiction. Therefore, the scalar-weighted consensus system (7) has a unique equilibrium point $\hat{x}^* = P^{-1}(\mathbb{1}_n \otimes \bar{x}_S)$.

(iv) Let us consider the following time derivative of $\bar{x}_S$.

$$\dot{\bar{x}}_S = \frac{1}{n}(\mathbb{1}_n^\top \otimes I_k) P\dot{x} = -\frac{1}{n}(\mathbb{1}_n^\top \otimes I_k) PL'_S x. \tag{10}$$

L'_S is symmetric and satisfies $\mathcal{N}(L'_S) = \{x \mid x = P^{-1}v, v \in \mathcal{N}(L')\}$ by Lemma 1-(ii). Thus, $(\mathbb{1}_n^\top \otimes I_k) PL'_S = 0$ and $\dot{\bar{x}}_S = 0$. Hence, we have the statement. □

Then, we can reach the following main result in this subsection.

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Theorem 3. *Based on a signed network corresponding to $\mathcal{G}'_S$, the scalar-weighted consensus system (7) converges to the point $\hat{x}^*$ exponentially fast for any initial point. Equivalently, the consensus system globally exponentially achieves a k-partite average consensus.*

Proof. We prove that the system (7) converges to $\hat{x}^* = P^{-1}(\mathbb{1}_n \otimes \bar{x}_S)$ if $\mathcal{N}(L') = \text{span}\{\mathbb{1}_n \otimes I_k\}$. First, let us define the differentiable Lyapunov function candidate V as $V = \frac{1}{2}\mathbf{e}^\top \mathbf{e}$, where $\mathbf{e} = x - \hat{x}^* = x - P^{-1}(\mathbb{1}_n \otimes \bar{x}_S)$ and V is radially unbounded. Then, since $\bar{x}_S$ is invariant from Lemma 1-(iv), the time derivative of V is given by

$$\dot{V} = \mathbf{e}^\top \dot{\mathbf{e}} = -\mathbf{e}^\top L'_S x = -\mathbf{e}^\top L'_S \mathbf{e} = -\|MP\mathbf{e}\|^2 \leq 0, \quad (11)$$

where, with the fact of Lemma 1-(ii), it holds that

$$\begin{aligned} L'_S \mathbf{e} &= L'_S\left(x - P^{-1}(\mathbb{1}_n \otimes \bar{x}_S)\right) \\ &= L'_S\left(x - P^{-1}(\mathbb{1}_n \otimes I_k)\bar{x}_S\right) = L'_S x, \end{aligned}$$

and $L' = L = \bar{H}^\top \text{blkdiag}(A_k)^{\frac{1}{2}} \text{blkdiag}(A_k)^{\frac{1}{2}} \bar{H} = M^\top M$ from (1). Next, consider the following result.

$$\begin{aligned} &\mathbf{e}^\top P^{-1}(\mathbb{1}_n \otimes I_k) \\ &= x^\top P^{-1}(\mathbb{1}_n \otimes I_k) - \left(\mathbb{1}_n^\top \otimes \bar{x}_S^\top\right) P P^{-1}(\mathbb{1}_n \otimes I_k) \\ &= x^\top P^{-1}(\mathbb{1}_n \otimes I_k) - \bar{x}_S^\top\left(\mathbb{1}_n^\top \otimes I_k\right)(\mathbb{1}_n \otimes I_k) \\ &= x^\top P^{-1}(\mathbb{1}_n \otimes I_k) - \bar{x}_S^\top\left(\mathbb{1}_n^\top \mathbb{1}_n \otimes I_k\right) \\ &= x^\top P^{-1}(\mathbb{1}_n \otimes I_k) - n\bar{x}_S^\top \\ &= 0, \end{aligned} \quad (12)$$

which implies that $\mathbf{e}$ is orthogonal to $\{x \mid x = P^{-1}v, v \in \text{span}\{\mathbb{1}_n \otimes I_k\}\} = \mathcal{N}(L'_S)$. Thus, we have

$$\dot{V} = -\mathbf{e}^\top L'_S \mathbf{e} \leq -\lambda_{k+1}(L'_S)\mathbf{e}^\top \mathbf{e} = -2\lambda_{k+1}(L'_S)V, \quad (13)$$

where $\lambda_{k+1}(L'_S)$ denotes the smallest nonzero eigenvalue of L'_S and $\lambda_{k+1}(L'_S) > 0$ since L'_S is positive semidefinite from Lemma 1-(i). Hence, since $\hat{x}^* = P^{-1}(\mathbb{1}_n \otimes \bar{x}_S)$ is the unique equilibrium point from Lemma 1-(iii), $\hat{x}^*$ is globally exponentially stable.

Furthermore, the above fact implies that all agents in any cluster out of k disjoint clusters achieve an average consensus. Consequently, we can say that the scalar-weighted consensus system (7) globally exponentially achieves a k-partite average consensus. □

Although this subsection assumes that the system (2) achieves a matrix-weighted consensus as in Assumption 1, we can also obtain a k-partite average consensus without the assumption as follows.

Proposition 1. *Assume that L'_S is positive semidefinite. Then, the scalar-weighted consensus system (7) globally exponentially achieves a k-partite average consensus such that the system (7) converges to a unique equilibrium point $\hat{x}^*$ if and only if $\mathcal{N}(L'_S) = \{x \mid x = P^{-1}v, v \in \text{span}\{\mathbb{1}_n \otimes I_k\}\}$.*

Proof. The sufficiency was already proved in Theorem 3, and the necessity is proved in the same way as Theorem 1. □

IV. Union of Disjoint Clusters

In this section, we extend the result of a k-partite average consensus problem studied in Subsection III-A to a $\bar{k}$-partite average consensus problem by unifying disjoint clusters, where $1 \leq \bar{k} < k$. We specifically study how we combine disjoint clusters to make them a union cluster with additional edges, where the weights of the additional edges can be either positive or negative. The additional edges can be regarded as indices for new neighbors to exchange information or to measure relative displacement, such as relative position, between an agent and its new neighbor. Once they are combined to a union cluster, then all agents in the union cluster converge to a common value. From this work, we can relax the condition of the fixed number of agents in a cluster, e.g., $|\mathcal{V}'_s| = n, \forall s \in \{1, 2, \cdots, k\}$ discussed in Section III; that is, we can make the number of agents in a cluster different from that in another clusters. Consequently, we can have results on a $\bar{k}$-partite average consensus by the work of combining disjoint clusters.

Consider a union graph $\mathcal{G}_U$ defined as $\mathcal{G}_U = \mathcal{G}' \cup \tilde{\mathcal{G}}$, where $\tilde{\mathcal{G}}$ is a graph generated by additional new edge set $\tilde{\mathcal{E}}$ and weight set $\tilde{\mathcal{W}}$. The augmenting graph $\tilde{\mathcal{G}}$ is defined as $\tilde{\mathcal{G}} = \left(\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{\mathcal{W}}\right)$ with corresponding new scalar-weighted Laplacian $\tilde{L} \in \mathbb{R}^{kn \times kn}$, where $\tilde{\mathcal{V}} = \mathcal{V}'$ and $\tilde{L}$ is defined in the same way as L'; for example, see Fig. 4 where $\tilde{\mathcal{V}} = \mathcal{V}'$, $\tilde{\mathcal{E}} = \{\{x_{12}, x_{31}\}\}$ and $\tilde{\mathcal{W}} = \{2\}$. Then, the scalar-weighted Laplacian L_U corresponding to $\mathcal{G}_U$ is given by $L_U = L' + \tilde{L}$. Also, we have the following consensus system.

$$\dot{x} = -L_U x \quad (14)$$

This consensus system is of the same form as the conventional scalar-weighted system, and thus the system (14) is also a distributed system with new neighbors $\{j \in \mathcal{V}_U \mid \{i, j\} \in \mathcal{E}_U\}$ of agent i, where $\mathcal{V}_U$ and $\mathcal{E}_U$ denote the vertex set and edge set in $\mathcal{G}_U$, respectively. One can say that although the consensus system (14) is a distributed system, it could be hard for each agent in real situations to recognize what edges should be added or how many edges should be to achieve a $\bar{k}$-partite average consensus. However, this paper aims to characterize the effect of adding edges (links) to a graph (network); thus further studies in real cases are left for future work. It is remarkable that if $\tilde{\mathcal{G}}$ includes at least one negative edge then it is not guaranteed that $\tilde{L}$ is positive semidefinite[1]. In Subsection IV-C, we are going to discuss what occurs if negative edges in $\tilde{\mathcal{G}}$ are included. We first explore how we combine two disjoint clusters to make $(k-1)$-partite average consensus in the following subsection.

A. $(k-1)$-partite average consensus over signed networks

First, we can express the identity matrix I_k as $I_k = [u_1, u_2, \cdots, u_k]$ where $u_a \in \mathbb{R}^k, a \in \{1, 2, \cdots, k\}$ denotes

[1]The structure of signed Laplacian as shown in (4) leads to this property while the conventional Laplacian matrix is always positive semidefinite. As is well known, if a symmetric matrix M satisfies $v^\top M v \geq 0$ for a nonzero vector v then M is said to be positive semidefinite. However, due to the structure of signed Laplacian, the condition $v^\top M v \geq 0$ is not guaranteed. Readers may also refer to [28].

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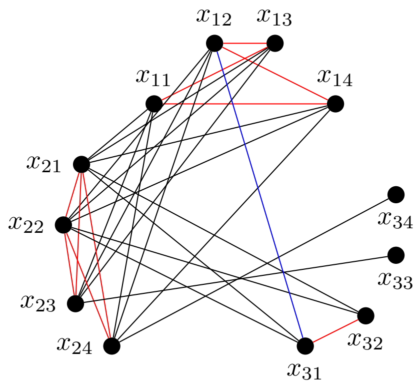

Fig. 4: Example: new graph $\mathcal{G}_U$ from $\mathcal{G}'$ in Fig. 2(b) with additional edge $\{x_{12}, x_{31}\}$ and weight 2, where the black and red lines denote positively and negatively weighted edges, respectively. We denote the additional edge by a blue line.

a unit vector whose all elements are 0 except that the a-th element is 1. Then, we can make additional connection between two disjoint clusters i and j with additional weighted edges. In particular, the aim for the connection is to make $\mathcal{N}(L_U) = \text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\}$, where $\bar{I}_{ij}$ denotes a symmetric matrix transformed from I_k such that u_i and u_j in I_k are substituted with $u_i + u_j$, i.e., $\bar{I}_{ij} = [u_1, \cdots, u_i + u_j, \cdots, u_i + u_j, \cdots, u_k] \in \mathbb{R}^{k\times k}$. For example, in Fig. 4, there is an edge between two vertices in two disjoint clusters in $\mathcal{G}'$, and it holds that $\mathcal{N}(L_U) = \text{span}\{\mathbb{1}_3 \otimes \bar{I}_{12}\}$ with an additional positive weighted edge; see Example 1 in Appendix for the detail. In brief, the condition $\mathcal{N}(L_U) = \text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\}$ means that all agents in clusters i and j converge to a common value under an assumption that the system (14) achieves a $(k-1)$-partite average consensus. Let us first define special edges becoming bridges between disjoint clusters.

Definition 3 (Union edge)**.** *Consider connections between disjoint clusters i and j in $\mathcal{G}'$ by weighted edges in $\tilde{\mathcal{G}}$ such that the sum of all weights from a vertex in the cluster i to all vertices in the cluster j is not equal to zero, which implies that $\tilde{L}(\mathbb{1}_n \otimes u_j) \neq 0$. We call these edges union edges in $\tilde{\mathcal{G}}$.*

Note that if there exist only union edges in $\tilde{\mathcal{G}}$, then it also holds that $\tilde{L}(\mathbb{1}_n \otimes u_i) \neq 0$ and $\tilde{L}(\mathbb{1}_n \otimes (u_i + u_j)) = 0$ due to the property of the Laplacian; for example, see Example 2 in Appendix. Let us then consider the following result.

Lemma 2. *Suppose $\tilde{\mathcal{G}}$ includes only union edges to connect disjoint clusters i and j in $\mathcal{G}'$. Then, it holds that $\text{span}\{\mathbb{1}_n \otimes I_k\} \setminus \text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\} \not\subset \mathcal{N}(\tilde{L})$.*

Proof. This proof is omitted due to the space limitation. □

In the paper, we only consider union edges in $\tilde{\mathcal{G}}$ to generate a union graph $\mathcal{G}_U$ from $\mathcal{G}'$. Thus, we reach the following fact with Lemma 2.

Lemma 3. *Suppose $\mathcal{G}_U$ is induced with only union edges to connect disjoint clusters i and j in $\mathcal{G}'$, and $\tilde{L}$ is positive semidefinite. Then, $\mathcal{N}(L_U) = \text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\}$ holds and L_U is positive semidefinite.*

Proof. First, it is clear that if both L' and $\tilde{L}$ are positive semidefinite, then L_U is also positive semidefinite. Next, consider the graph $\tilde{\mathcal{G}}$ with only edges between i and j clusters. Then, $\text{span}\{\mathbb{1}_n \otimes (u_i + u_j)\} \subset \mathcal{N}(\tilde{L})$ holds, where it also holds that $\text{span}\{\mathbb{1}_n \otimes (u_i + u_j)\} \subset \text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\} \subset \mathcal{N}(\tilde{L})$. Let us consider the fact that $\mathcal{N}(L') \cap \mathcal{N}(\tilde{L}) \subset \mathcal{N}(L' + \tilde{L})$. Then, since both L' and $\tilde{L}$ are positive semidefinite, we have

$$\mathcal{N}(L') \cap \mathcal{N}(\tilde{L}) = \mathcal{N}(L' + \tilde{L}).$$

This fact gives the following result:

$$\mathcal{N}(L_U) = \mathcal{N}(L') \cap \mathcal{N}(\tilde{L}) = \text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\}$$

with the fact of $\text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\} \subset \text{span}\{\mathbb{1}_n \otimes I_k\}$, the assumption of $\mathcal{N}(L') = \text{span}\{\mathbb{1}_n \otimes I_k\}$ and the result of $\text{span}\{\mathbb{1}_n \otimes I_k\} \setminus \text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\} \not\subset \mathcal{N}(\tilde{L})$ from Lemma 2. Therefore, we have the statement. □

Next, to reach the main theorem, we explore the following two properties.

Lemma 4. *Suppose $\mathcal{N}(L_U) = \text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\}$. Then, the scalar-weighted consensus system (14) has a unique equilibrium point as follows*

$$x_U^* = (\mathbb{1}_n \otimes I_k)\,\bar{x}_U,$$

where $\bar{x}_U = \text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\left(\mathbb{1}_n^\top \otimes \bar{I}_{ij}\right)x \in \mathbb{R}^k$ and $\text{diag}^{-1}(\cdot)$ denotes the inverse of $\text{diag}(\cdot)$.

Proof. This proof is proved in a similar way to Lemma 1-(iii). First, let us check whether x_U^* is in $\mathcal{N}(L_U)$ or not.

$$\begin{aligned}
&L_U x_U^* \\
&= L_U\,(\mathbb{1}_n \otimes I_k)\,\text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\left(\mathbb{1}_n^\top \otimes \bar{I}_{ij}\right)x \\
&= L_U\left(\mathbb{1}_n \otimes \text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\right)\left(\mathbb{1}_n^\top \otimes \bar{I}_{ij}\right)x \\
&= L_U\left(\mathbb{1}_n\mathbb{1}_n^\top \otimes \text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\bar{I}_{ij}\right)x \\
&= L_U\left(\mathbb{1}_n \otimes \text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\bar{I}_{ij}\right)\left(\mathbb{1}_n^\top \otimes I_k\right)x \\
&= 0,
\end{aligned}$$

where, due to the structure of $\text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\bar{I}_{ij}$, it holds that $\text{span}\left\{\mathbb{1}_n \otimes \text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\bar{I}_{ij}\right\} \subseteq \text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\} = \mathcal{N}(L_U)$. Thus, x_U^* lies in $\mathcal{N}(L_U)$.

Next, assume that there exists $x \in \mathcal{N}(L_U) = \text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\}$ except for $x = x_U^*$. Then, we have $x = \left(\mathbb{1}_n \otimes \bar{I}_{ij}\right)z = (\mathbb{1}_n \otimes I_k)\,\bar{I}_{ij}z \neq x_U^*$, where $z \in \mathbb{R}^k$ is an arbitrary vector. Moreover, the following equation holds.

$$\begin{aligned}
\bar{x}_U &= \text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\left(\mathbb{1}_n^\top \otimes \bar{I}_{ij}\right)\left(\mathbb{1}_n \otimes \bar{I}_{ij}\right)z \\
&= \text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\left(\mathbb{1}_n^\top\mathbb{1}_n \otimes \bar{I}_{ij}\bar{I}_{ij}\right)z \\
&= n\text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\bar{I}_{ij}\bar{I}_{ij}z \\
&= \bar{I}_{ij}z,
\end{aligned} \tag{15}$$

where it is remarkable that $n\text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\bar{I}_{ij} \neq I_k$. By substituting (15) into x_U^*, we have $x_U^* = (\mathbb{1}_n \otimes I_k)\,\bar{I}_{ij}z = x$, which contradicts the fact that $x \neq x_U^*$. Hence, $x_U^* = (\mathbb{1}_n \otimes I_k)\,\bar{x}_U$ is a unique equilibrium point of the system. □

Lemma 5. *Suppose $\mathcal{N}(L_U) = \text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\}$. Under the system (14), the point x_U^* is invariant for all $t \geq 0$.*

This lemma is proved in a similar way as Lemma 1-(iv). Now, we can finally have the following main theorem.

Theorem 4. *If $\mathcal{G}_U$ is induced with only union edges to connect disjoint clusters i and j in $\mathcal{G}'$ and $\tilde{L}$ is positive semidefinite, then the system (14) globally exponentially achieves a $(k-1)$-partite average consensus.*

Proof. First, from Lemma 3, we have the fact that $\mathcal{N}(L_U) = \text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\}$ and L_U is positive semidefinite. Define the differentiable Lyapunov function candidate $V = \frac{1}{2}\mathbf{e}^\top \mathbf{e}$, where $\mathbf{e} = x - x_U^*$. Then, with the fact that x_U^* is invariant from Lemma 5, the time derivative of V is given by

$$\dot{V} = \mathbf{e}^\top \dot{\mathbf{e}} = -\mathbf{e}^\top L_U x = -\mathbf{e}^\top L_U \mathbf{e}. \tag{16}$$

where $L_U\mathbf{e} = L_U x$ since $L_U x_U^* = 0$ holds from Lemma 4.

Moreover, we have the following result.

$$\begin{aligned}
&\text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\left(\mathbb{1}_n \otimes \bar{I}_{ij}\right)^\top \mathbf{e} \\
&= \text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\left(\mathbb{1}_n^\top \otimes \bar{I}_{ij}\right) x \\
&\quad - \text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\left(\mathbb{1}_n^\top \otimes \bar{I}_{ij}\right)\left(\mathbb{1}_n \otimes I_k\right)\bar{x}_U \\
&= \bar{x}_U - \text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\left(\mathbb{1}_n^\top\mathbb{1}_n \otimes \bar{I}_{ij}\right)\bar{x}_U \\
&= \bar{x}_U - n\text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\bar{I}_{ij}\bar{x}_U \\
&= \bar{x}_U - \bar{x}_U = 0,
\end{aligned} \tag{17}$$

where it is remarkable that $n\text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)\bar{I}_{ij} \neq I_k$. Thus, since $\text{diag}^{-1}\left(n\bar{I}_{ij}\mathbb{1}_k\right)$ is a diagonal matrix and is of full rank, it holds that

$$\left(\mathbb{1}_n \otimes \bar{I}_{ij}\right)^\top \mathbf{e} = 0,$$

which implies that $\mathbf{e} \perp \mathcal{N}(L_U)$. With this fact, we have

$$\dot{V} = -\mathbf{e}^\top L_U \mathbf{e} \leq -\lambda(L_U)\mathbf{e}^\top\mathbf{e} = -2\lambda(L_U)V, \tag{18}$$

where $\lambda(L_U)$ denotes the smallest positive eigenvalue of L_U. Hence, the unique equilibrium point x_U^* is globally exponentially stable, which implies that all agents in any cluster of $(k-1)$ disjoint clusters achieve an average consensus. Therefore, we can have the statement. □

In the same manner as Proposition 1, Theorem 4 can be rewritten in a general form without Assumption 1 as follows.

Proposition 2. *Assume that L_U is positive semidefinite. Then, the scalar-weighted consensus system (14) globally exponentially achieves a $(k-1)$-partite average consensus such that the system (14) converges to a unique equilibrium point $\bar{x}_U$ if and only if $\mathcal{N}(L_U) = \text{span}\{\mathbb{1}_n \otimes \bar{I}_{ij}\}$.*

Proof. The sufficiency was already proved in Theorem 4, and the necessity is proved in the same way as Theorem 1. □

A $(k-1)$-partite average consensus can be extended to a reduced-multipartite average consensus with new union of disjoint clusters in the same way as discussed in Subsection IV-A; we briefly discuss how we derive general conditions to achieve a $(k-p)$-partite average consensus, where $2 \leq p \leq k-1$, in Appendix. In the following subsection, we particularly address a problem on a union of all disjoint clusters for a single average consensus.

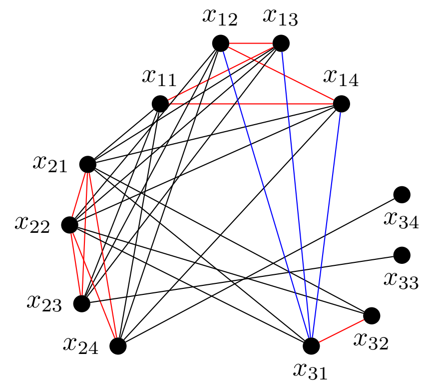

Fig. 5: Example: union graph $\mathcal{G}_U$ from $\mathcal{G}'$ in Fig. 2(b) and $\tilde{\mathcal{G}}$ with a edge set $\tilde{\mathcal{E}} = \{\{x_{12}, x_{31}\}, \{x_{13}, x_{31}\}, \{x_{14}, x_{31}\}\}$ and a weight set $\tilde{\mathcal{W}} = \{2, 3, 3\}$, where the black and red lines denote positively and negatively weighted edges, respectively. The blue lines denote the additional weighted edges.

B. Single average consensus over signed networks

In this subsection, we study how to make a multipartite average consensus a single average consensus by augmenting additional positive and negative edges among disjoint clusters. Fig. 5 shows an example of union graph with augmenting graph with 3 positive weighted edges. Let us first consider the following result.

Lemma 6. *Suppose $\tilde{\mathcal{G}}$ includes only union edges among every pair of disjoint clusters in $\mathcal{G}'$. Then, it holds that* $\text{span}\{\mathbb{1}_n \otimes I_k\} \setminus \text{span}\{\mathbb{1}_{kn}\} \not\subset \mathcal{N}(\tilde{L})$.

Proof. This proof is omitted due to the space limitation. □

We then explore a condition that L_U has a simple zero eigenvalue and invariance of the system's average in the following two lemmas.

Lemma 7. *If $\tilde{\mathcal{G}}$ includes only union edges among every pair of disjoint clusters in $\mathcal{G}'$ and $\tilde{L}$ is positive semidefinite, then $\mathcal{N}(L_U) = \text{span}\{\mathbb{1}_{kn}\}$ holds and L_U is positive semidefinite.*

Proof. This proof is proved by contradiction. We first assume that there exists a nonzero vector x such that $x \in \mathcal{N}(L_U)$ other than $x \in \text{span}\{\mathbb{1}_{kn}\}$. Then, we have $L_U x = 0, x \neq c\mathbb{1}_{kn}$, where c is a constant. Thus, we also have

$$x^\top L_U x = x^\top L' x + x^\top \tilde{L} x = 0. \tag{19}$$

Now, we can observe whether or not there exists x satisfying (19) by the following two cases.

1) When $x \in \mathcal{N}(L')$, the equation (19) yields $x^\top \tilde{L} x = 0$. However, this contradicts the condition $\text{span}\{\mathbb{1}_n \otimes I_k\} \setminus \text{span}\{\mathbb{1}_{kn}\} \not\subset \mathcal{N}(\tilde{L})$ from Lemma 6.
2) When $x \notin \mathcal{N}(L')$, the both equations $x^\top L' x = 0$ and $x^\top \tilde{L} x = 0$ must hold since L' and $\tilde{L}$ are positive semidefinite. However, $x \notin \mathcal{N}(L')$ implies $x^\top L' x \neq 0$. This is also a contradiction.

Hence, $\mathcal{N}(L_U) = \text{span}\{\mathbb{1}_{kn}\}$ holds. Moreover, since L' and $\tilde{L}$ are positive semidefinite, L_U is also positive semidefinite. □

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Lemma 8. *The average $\hat{x} = \frac{1}{kn}\sum_{i=1}^{n}\sum_{j=1}^{k} x_{ij} \in \mathbb{R}$ is invariant for all $t \geq 0$.*

This lemma is proved in a similar manner as Lemma 1-(iv). With the fact of Lemma 7 and Lemma 8, we can finally have the main result for a single average consensus.

Theorem 5. *The consensus system (14) globally exponentially converges to $\hat{x}\mathbb{1}_{kn}$ if $\tilde{\mathcal{G}}$ includes only union edges among every pair of disjoint clusters in $\mathcal{G}'$ and $\tilde{L}$ is positive semidefinite.*

Proof. This theorem is proved in the same way as Theorem 4 or the conventional consensus problem [29]. □

In the same way as Proposition 1 and Proposition 2, Theorem 5 can also be expressed in a general form without Assumption 1; however, since the general form shows the same result as the traditional average consensus [29], that form is omitted in this paper.

C. Discussions on negative union edges

This subsection provides brief discussions on using negative union edges to combine disjoint clusters. First, as proved in Theorem 4 and Theorem 5, we need to prove that $\tilde{L}$ is positive semidefinite. Moreover, as is well known, $\tilde{L}$ is always positive semidefinite when only positive edges are considered in $\tilde{\mathcal{G}}$. Thus, positive union edges always satisfy Theorem 4 and Theorem 5. However, in this paper, we deal with not only positive edges but also negative edges as union edges, and it is not guaranteed that $\tilde{L}$ is positive semidefinite. Results on positive semidefiniteness of Laplacian matrices with negative edges for connected graphs are introduced in the literature [28], [30]–[32]. For disconnected graphs, $\tilde{L}$ is positive semidefinite if each Laplacian corresponding to each connected subgraph of $\tilde{\mathcal{G}}$ is positive semidefinite.

We can observe another cases achieving a single average consensus. When we combine disjoint clusters with union edges, we can achieve a single average consensus even if $\tilde{L}$ is not positive semidefinite. For example, as shown in Fig. 6, $\tilde{L}$ is indefinite but L_U corresponding to $\mathcal{G}_U$ is positive semidefinite with a simple zero eigenvalue, which implies that the system (14) with L_U achieves a single average consensus as discussed in this section. This phenomenon could be explained with the following theorem.

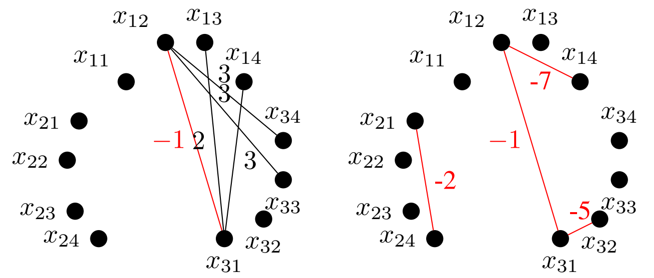
Theorem 6. *[28, Theorem 1] L_U is positive semidefinite and has a simple zero eigenvalue if and only if $\mathcal{G}_U$ is connected and satisfies $\Gamma_{\mathcal{F}_U^-} > 0$, where $\Gamma_{\mathcal{F}_U^-}$ is an effective resistance matrix of a spanning forest[2] $\mathcal{F}_U^-$ of $\mathcal{G}_U^-$ and $\mathcal{G}_U^-$ denotes a spanning subgraph of $\mathcal{G}_U$ including only negative edges.*

This theorem shows that, even when $\tilde{L}$ is not positive semidefinite, we can reach a single average consensus if $\Gamma_{\mathcal{F}_U^-} > 0$ as shown in Fig. 6(b).

V. Simulation results

We provide four simulations: the first one is a result of a 4-partite average consensus based on a graph in Fig. 2(b);

[2]A spanning forest is a subgraph that is composed of trees including all vertices in a graph.



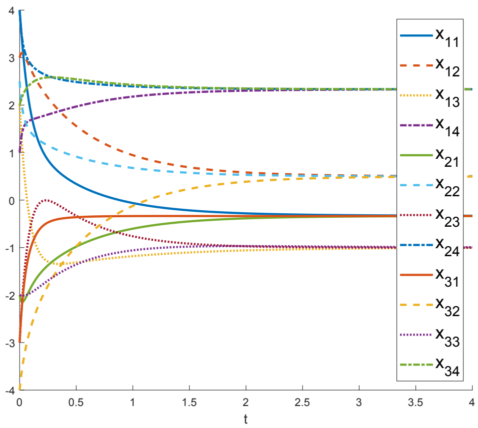
(a) Union edges in $\tilde{\mathcal{G}}$. (b) Spanning forest of $\mathcal{G}_U^-$.

Fig. 6: Example: Union edges to characterize a union graph and a spanning forest of $\mathcal{G}_U^-$, where $\tilde{L}$ is indefinite and the black and red lines denote positively and negatively weighted edges, respectively.

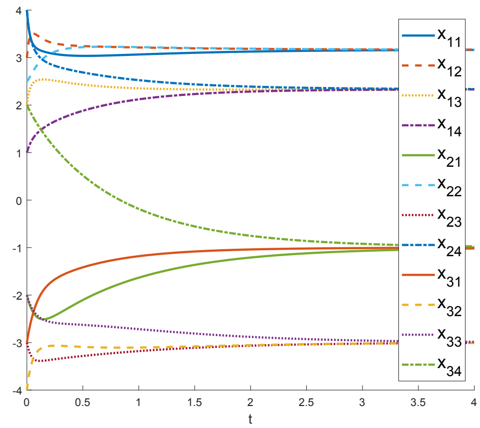
another simulation is also a result of a 4-partite average consensus with respect to a transformed graph in Fig. 3 from the graph in Fig. 2(b); another in Fig. 7(c) shows a result of a 3-partite average consensus under a union graph with a connection between two disjoint clusters; the other simulation is to show a single average consensus based on a union graph in Fig. 5 with some additional edges to the graph in Fig. 2(b). First, we choose the initial states for all simulations as $x = [x_1^\top, x_2^\top, x_3^\top]^\top = [4, 3, 2, 1, -2, 2.5, -3, 4, -3, -4, -2, 2]^\top$, where $x_1 = [x_{11}, x_{12}, x_{13}, x_{14}]^\top$, $x_2 = [x_{21}, x_{22}, x_{23}, x_{24}]^\top$ and $x_3 = [x_{31}, x_{32}, x_{33}, x_{34}]^\top$. As shown in Fig. 7(a), the consensus system (5) induced by the matrix-weighted consensus system (2) can achieve a multipartite average consensus based on the graph $\mathcal{G}'$ in Fig 2(b), where the multiple joint clusters are $\mathcal{V}_1' = \{x_{11}, x_{21}, x_{31}\}$, $\mathcal{V}_2' = \{x_{12}, x_{22}, x_{32}\}$, $\mathcal{V}_3' = \{x_{13}, x_{23}, x_{33}\}$ and $\mathcal{V}_4' = \{x_{14}, x_{24}, x_{34}\}$, and agents in each cluster converge to their average. Fig. 7(b) shows that agents under the consensus system (7) converge to the unique equilibrium point as discussed in Subsection III-B. Also, as shown in Fig. 7(b), the disjoint clusters are different from the clusters in Fig. 7(a), and we can see that agents in each cluster converge to their average. Fig. 7(c) shows a 3-partite average consensus under the consensus system (14), where we can particularly see that the agents in two clusters $\mathcal{V}_1'$ and $\mathcal{V}_2'$ introduced in Fig. 7(a) converge to the same value, i.e., average of agents. Finally, Fig. 7(d) illustrates the trajectories of all agents under the consensus system (14), where all agents converge to their average.

VI. Conclusion

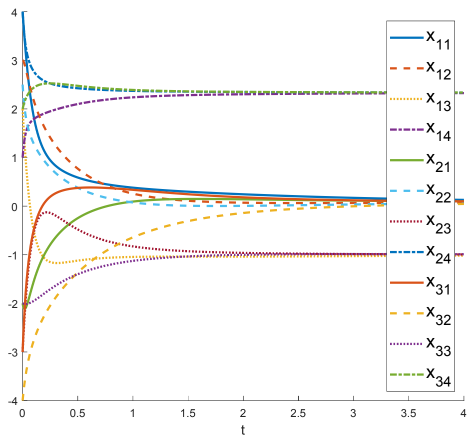
In this paper, we proposed the novel approach to reach a multipartite average consensus over signed networks from the perspective of a matrix-weighted consensus. This work particularly provides the result that a system over signed networks achieves a multipartite average consensus if and only if the system over matrix-weighted networks achieves a matrix-weighted consensus. From this approach, we can interpret a consensus system with multiple clusters over signed network. Moreover, we extended the result of a multipartite average consensus (k-partite average consensus) with k dis-



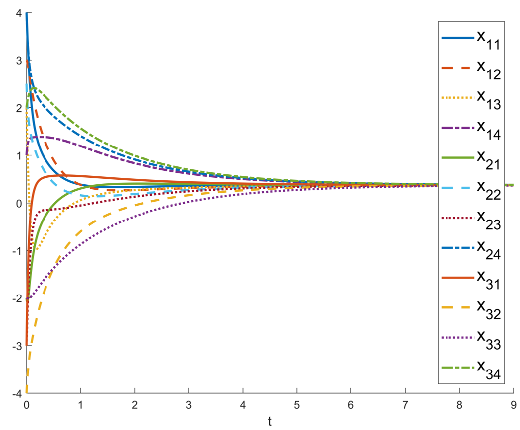
(a) 4-partite average consensus under the consensus system (5) with the graph $\mathcal{G}'$ in Fig 2(b).



(b) 4-partite average consensus under the consensus system (7) with the transformed graph $\mathcal{G}'_S$ as in Fig 3.



(c) 3-partite average consensus under the consensus system (14) with the graph $\mathcal{G}_U$ as depicted in Fig. 4.



(d) Single average consensus under the consensus system (14) with the graph $\mathcal{G}_U$ in Fig 5.

Fig. 7: Simulations: multipartite average consensus over signed networks.

joint clusters to the result of a $\bar{k}$-partite average consensus by combining the disjoint clusters, where $1 \leq \bar{k} < k$.

For future work, we would like to study a multipartite average consensus problem in a discrete-time system or based on a matrix-weighted consensus protocol over directed networks. To achieve this aim, it should firstly be preceded to explore a matrix-weighted consensus problem in a discrete-time system or over directed networks since such a consensus problem has not been studied yet. Although, in [33], a matrix-weighted consensus problem over directed networks has been studied, the problem is confined to directed networks corresponding to leader-following graphs. For example, to explore a matrix-weighted consensus protocol in a discrete-time system, for the given system

$$x(t+1) = Ax(t), \tag{20}$$

where $A = I_{kn} - \epsilon L, 0 < \epsilon < 1$, we need to find ϵ such that it holds that $\mathcal{N}(\epsilon L) = \text{span}\{\mathbb{1}_n \otimes I_k\}$ and $\rho(A) = 1$ in which $\rho(A)$ denotes the spectral radius of A. In the case of a matrix-weighted consensus protocol over directed networks, it is not guaranteed anymore that Laplacian matrix is a symmetric matrix, and thus we need to study a new approach to analyze Laplacian matrix corresponding to a directed graph and a new consensus protocol as in [33].

APPENDIX

Example 1

In Fig. 4, $\tilde{L}$ is given by

$$\tilde{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

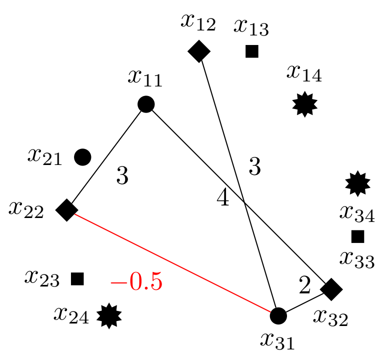


Fig. 8: Example: $\tilde{\mathcal{G}}$ with additional edges and weights, where the vertices in the same group are denoted by the same shape. We denote edges with positive and negative weights by black and red lines, respectively.

and the union graph $\mathcal{G}_U$ has the Laplacian L_U as follows

$$L_U = L' + \tilde{L}$$
$$= \begin{bmatrix} 4 & 0 & 3 & 2 & -4 & 0 & -3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 12 & 1 & 7 & 0 & -10 & -1 & -7 & -2 & 0 & 0 & 0 \\ 3 & 1 & 5 & 0 & -3 & -1 & -5 & 0 & 0 & 0 & 0 & 0 \\ 2 & 7 & 0 & 15 & -2 & -7 & 0 & -15 & 0 & 0 & 0 & 0 \\ -4 & 0 & -3 & -2 & 19 & 5 & 3 & 2 & -15 & -5 & 0 & 0 \\ 0 & -10 & -1 & -7 & 5 & 13 & 1 & 7 & -5 & -3 & 0 & 0 \\ -3 & -1 & -5 & 0 & 3 & 1 & 6 & 0 & 0 & 0 & -1 & 0 \\ -2 & -7 & 0 & -15 & 2 & 7 & 0 & 20 & 0 & 0 & 0 & -5 \\ 0 & -2 & 0 & 0 & -15 & -5 & 0 & 0 & 17 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & -3 & 0 & 0 & 5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Thus, it holds that $\mathcal{N}(L_U) = \mathrm{span}\{\mathbb{1}_3 \otimes \bar{I}_{12}\}$ where $\bar{I}_{12} = [u_1 + u_2, u_1 + u_2, u_3, u_4] \in \mathbb{R}^{4\times 4}$.

Example 2

Fig. 8 shows a graph $\tilde{\mathcal{G}}$ with union edges between cluster 1 to cluster 2, where we assume that clusters are defined as $\mathcal{V}_1' = \{x_{11}, x_{21}, x_{31}\}$, $\mathcal{V}_2' = \{x_{12}, x_{22}, x_{32}\}$, $\mathcal{V}_3' = \{x_{13}, x_{23}, x_{33}\}$ and $\mathcal{V}_4' = \{x_{14}, x_{24}, x_{34}\}$. As shown in the figure, the sum of all weights from vertex x_{31} in cluster 1 to all vertices in cluster 2 is not equal to zero. Similarly, the sum of all weights from vertex x_{11} to all vertices in cluster 2 is also not equal to zero. Therefore, the edges in $\tilde{\mathcal{G}}$ are union edges. Consider $\tilde{L}$ being of the following form:

$$\tilde{L} = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 2.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0.5 & 0 & 0 & 4.5 & -2 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, it holds that $\tilde{L}(\mathbb{1}_n \otimes u_2) \neq 0$. Moreover, when $\tilde{L}$ is denoted by $\tilde{L} = [\tilde{l}_{ij}]$, we have $\tilde{l}_{ij} = \sum_{j=1, j\neq i}^{kn} -\tilde{l}_{ij}$ for $i = j$. Thus, if there exist only union edges, then it intuitively holds that $\tilde{L}(\mathbb{1}_n \otimes u_1) \neq 0$, and further $\tilde{L}(\mathbb{1}_n \otimes (u_1 + u_2)) = 0$.

Extension of a $(k-1)$-partite average consensus to a general multipartite average consensus with additional edges

In a similar way as the discussion for a $(k-1)$-partite average consensus in Section IV, we can extend the result of a $(k-1)$-partite average consensus to that of a general $(k-p)$-partite average consensus, where $2 \leq p \leq k-1$, by combining disjoint clusters with additional edges. Let us first consider $(k-2)$ disjoint clusters generated from $(k-1)$ disjoint clusters with additional edges such that $\mathcal{N}(L_U) = \mathrm{span}\{\mathbb{1}_n \otimes \bar{I}'_{(k-2)}\}$ holds, where $\bar{I}'_{(k-2)}$ denotes a symmetric matrix as $\bar{I}'_{(k-2)} = [u_1, \cdots, u_i + u_j + u_{i'}, \cdots, u_i + u_j + u_{i'}, \cdots, u_i + u_j + u_{i'}, \cdots, u_k] \in \mathbb{R}^{k\times k}$ or $\bar{I}'_{(k-2)} = [u_1, \cdots, u_i + u_j, \cdots, u_i + u_j, \cdots, u_{i'} + u_{j'}, \cdots, u_{i'} + u_{j'}, \cdots, u_k] \in \mathbb{R}^{k\times k}, i < j < i' < j'$. We can observe from the definition of $\bar{I}'_{(k-2)}$ that $\mathrm{rank}\left\{\bar{I}'_{(k-2)}\right\} = k-2$. Then, the system (14) has a unique equilibrium point as

$$x_U^* = (\mathbb{1}_n \otimes I_k)\,\bar{x}_{U_{(k-2)}},$$

where $\bar{x}_{U_{(k-2)}} = \mathrm{diag}^{-1}\left(n\bar{I}'_{(k-2)}\mathbb{1}_k\right)\left(\mathbb{1}_n^\top \otimes \bar{I}'_{(k-2)}\right)x \in \mathbb{R}^k$. This is proved in a similar way as Lemma 4.

To make this process more general, let us consider $(k-p)$ disjoint clusters generated from $(k-(p-1))$ disjoint clusters with additional edges such that $\mathcal{N}(L_U) = \mathrm{span}\{\mathbb{1}_n \otimes \bar{I}'_{(k-p)}\}$ holds, where $\bar{I}'_{(k-p)} \in \mathbb{R}^{k\times k}$ is a symmetric matrix whose column vectors are composed of $\mathrm{span}\{I_k\}$ and it holds that $\mathrm{rank}\left\{\bar{I}'_{(k-p)}\right\} = k-p$. As a result, the system (14) has a unique equilibrium point as

$$x_U^* = (\mathbb{1}_n \otimes I_k)\,\bar{x}_{U_{(k-p)}},$$

where $\bar{x}_{U_{(k-p)}} = \mathrm{diag}^{-1}\left(n\bar{I}'_{(k-p)}\mathbb{1}_k\right)\left(\mathbb{1}_n^\top \otimes \bar{I}'_{(k-p)}\right)x \in \mathbb{R}^k$. This is also proved in a similar way as Lemma 4. Moreover, conditions to achieve a $(k-p)$-partite average consensus are derived in the same manner as Theorem 4.

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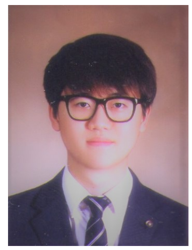

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