Controllability of Switching Signed Networks

Yongqiang Guan, Lingling Tian, and Long Wang

Abstract—**This brief investigates the controllability of switching signed networks, in which the interconnection topologies are directed, weighted, and switched. First, we study the controllability from the union digraph perspective and derive a graph-theoretic sufficient condition. It is shown that a switching signed network can be controllable even if each of its subnetworks is uncontrollable. Second, we consider the controllability of simultaneously structurally balanced signed networks. It is found that the controllability of simultaneously structurally balanced networks is equivalent to the controllability of their corresponding unsigned networks. Finally, several numerical examples are given to illustrate the effectiveness of the theoretical results.**

Index Terms—**Controllability, signed networks, switching topologies, leader-follower framework.**

I. Introduction

Controllability is a fundamental issue in multiagent networks and has recently been extensively investigated [1]–[11]. In general, multiagent network controllability deals with the question whether the states of the whole network could be steered to any desired final state from any initial state in finite time by directly manipulating/controlling only a few nodes of the network. In [1], the concept of controllability for multi-agent networks under the leader-follower framework was first formulated, and necessary and sufficient condition for controllability in terms of Laplacian submatrix was proposed. In [2], controllability was studied from a graph-theoretic perspective, and equitable partition technique was introduced to investigate controllability. In [3] and [4], distance partition and almost equitable partition were employed to quantitatively study the controllable subspace of undirected networks and directed networks, respectively. Subsequently, an increasing number of scholars from different perspectives have made elaborate effort on multiagent network controllability. Recent results in this regard include controllability for simple topology structures such as multichain graphs [5], controllability of switching topologies [6], protocols design for controllability [7], controllability of heterogeneous dynamics networks [8], controllability destructive nodes [9], controllability of temporal networks [10], and target controllability [11].

Although great progress on multiagent network controllability has been made, most of the existing results are derived for the network with only cooperative interactions among agents. In both natural and man-made multi-agent networks, competitive (antagonistic) interactions among agents are ubiquitous, such as hostile individuals in social networks, population competition in ecological networks, and inhibition among cells in biochemical networks (see, e.g., [12], [13]). Recently, some complex group dynamics behaviors on signed networks have been studied, including bipartite consensus, bipartite tracking, bipartite containment [13]–[18]. In particular, the controllability of signed networks has also drawn much attention, such as controllability for undirected signed networks [19], controllability for directed and weighted signed networks [20], leader selection on signed multi-agent networks [21].

It is worth noting that the majority of the pervious papers only investigate the controllability of fixed signed networks. In this brief, we investigate the controllability of switching signed networks, in which the interconnection topologies are directed, weighted and switched. The main contributions are twofold. First, we study the controllability from the union digraph perspective and derive a graph-theoretic sufficient condition. It is shown that when the union digraph of all the possible topologies is controllable, the switching signed network would be controllable even if each of its subnetworks is not. Second, we explore the relationship between the controllability of simultaneously structurally balanced signed networks and the controllability of the corresponding unsigned networks. It is found that the controllability of simultaneously structurally balanced networks is equivalent to that of the corresponding unsigned networks. Such a graphical characterization of controllability provides a guide to design and analyze switching signed networks.

The rest of this brief is organized as follows. In Section II, some graph theory preliminaries and switching signed network model are presented. The controllability of switching signed networks is analyzed in Section III-A. In Section III-B, we consider the controllability of simultaneously structurally balanced signed networks. In Section IV, numerical simulations are given to illustrate the effectiveness of theoretical results. Finally, the conclusion is drawn in Section V.

Notation: Let $diag\{a_1, \dots, a_n\}$ be a diagonal matrix with $a_i, i = 1, \dots, n$, being the diagonal entries. Let $\mathbb{R}$ and $\mathbb{N}^+$ represent the set of real numbers and positive integers numbers, respectively. For $i \in \mathbb{N}^+$, e_i denotes the standard basis vectors. For any two sets X and Y, $X \backslash Y$ denotes the difference

Manuscript received May 6, 2019; revised June 13, 2019; accepted June 27, 2019. Date of publication July 1, 2019; date of current version June 4, 2020. This work was supported in part by the National Natural Science Foundation of China under Grant 61603288, Grant 61533001, Grant 61751301, and Grant 61773023, and in part by the Shaanxi Provincial Natural Science Foundation of China under Grant 2019JQ-265. This brief was recommended by Associate Editor J. Wu. *(Corresponding author: Long Wang.)*

Y. Guan is with the School of Mechano-Electronic Engineering, Xidian University, Xi'an 710071, China (e-mail: guan-jq@163.com).

L. Tian and L. Wang are with the Center for Systems and Control, College of Engineering, Peking University, Beijing 100871, China (e-mail: 1301111408@pku.edu.cn; longwang@pku.edu.cn).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCSII.2019.2926090

set of X and Y. Let $sign(x)$ denote the sign function of a scalar $x \in \mathbb{R}$.

II. Preliminaries and Network Model

A. Signed Graph

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ represent a directed signed graph (digraph), where $\mathcal{V} = \{1, \dots, n\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} = \{(i,j) : i,j \in \mathcal{V}\}$, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a node set, an edge set, and a weighted adjacency matrix, respectively. The weighted adjacency matrix $\mathcal{A} = [a_{ij}]$ is defined as: $a_{ij} \neq 0$ if $(j,i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise, where (j,i) denotes the edge from j to i. Note that a digraph $\mathcal{G}' = (\mathcal{V}, \mathcal{E}, |\mathcal{A}|)$ can be regarded as the unsigned version of the signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $|\mathcal{A}| = [|a_{ij}|] \in \mathbb{R}^{n\times n}$. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (j,i) \in \mathcal{E}\}$. Given a signed digraph $\mathcal{G}$, it is said to be structurally balanced if there exists a bipartition $\{\mathcal{V}_1, \mathcal{V}_2 : \mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}, \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset\}$ of $\mathcal{V}$ such that $a_{ij} \geq 0, \forall i,j \in \mathcal{V}_p$ $(p \in \{1,2\})$, $a_{ij} \leq 0, \forall i \in \mathcal{V}_p, j \in \mathcal{V}_q$ $(p,q \in \{1,2\}, p \neq q)$. It is said to be structurally unbalanced otherwise. For signed digraph $\mathcal{G}$, $c_i = \sum_{j\in\mathcal{N}_i} |a_{ij}|$ denotes the degree of node i. The Laplacian matrix of $\mathcal{G}$ is defined as $L = C - \mathcal{A}$, where $C = diag\{c_1, \dots, c_n\}$ is the degree matrix of $\mathcal{G}$. The union digraph of a collection of signed digraph $\mathcal{G}_k = \{\mathcal{V}, \mathcal{E}_k, \mathcal{A}_k\}$, $k = 1, \dots, N$, with the same node set $\mathcal{V} = \{1 \dots, n\}$, is denoted by $\bar{\mathcal{G}} = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathcal{A}}\}$, where $\bar{\mathcal{V}} = \mathcal{V}$, $\bar{\mathcal{E}} = \bigcup_{k=1}^{N} \mathcal{E}_k$, and $\bar{\mathcal{A}} = \sum_{k=1}^{N} \mathcal{A}_k$. A gauge transformation is a change of orthant order via a square matrix $D \in \mathcal{D}$, where $\mathcal{D} = \{D = diag\{d_1, \dots, d_n\} : d_i \in \{-1, +1\}\}$ represents the set of all gauge transformations in $\mathbb{R}^{n\times n}$.

B. Network Model

The controllability problem is usually considered under the leader-follower framework, in which the agents are partitioned into two sets: leaders and followers. The leaders can be manipulated by external inputs, and the followers obey network dynamics. Consider a signed network consisting of n agents, which are labeled by set $\mathcal{V}$, where $\mathcal{V}^L$ and $\mathcal{V}^F = \mathcal{V} \backslash \mathcal{V}^L$ denote the leader set and the follower set, respectively. Let $u(t) \in \mathbb{R}^m$ represent the control input. The dynamics of each agent satisfy

$$\begin{cases} \dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}[x_j(t) - sign(a_{ij})x_i(t)] + u_i(t), & i \in \mathcal{V}^L \\ \dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}[x_j(t) - sign(a_{ij})x_i(t)], & i \in \mathcal{V}^F \end{cases} \tag{1}$$

where $x_i(t) \in \mathbb{R}$ is the state of node i and $u_i(t) \in \mathbb{R}$ is the i-th entry of the control vector $u(t)$. Then, we can write the network model in a compact form

$$\dot{x}(t) = -Lx(t) + Bu(t), \tag{2}$$

where $x(t) = [x_1(t), \dots, x_n(t)]^T$, $u(t) = [u_1(t), \dots, u_m(t)]^T$. $L \in \mathbb{R}^{n\times n}$ is the Laplacian matrix of $\mathcal{G}$, and $B = [e_{i_1}, \dots, e_{i_m}] \in \mathbb{R}^{n\times m}$.

In practice, the signed networks may be time-variable due to various reasons, such as link failure, creation, and external disturbance. This case is usually described more reasonably by switching topologies, in which the adjacency weight $a_{ij}(t)$, the neighbour set $\mathcal{N}_i(t)$, and the Laplacian matrix $L(t)$ are time varying. Moreover, the leader set $\mathcal{V}^L(t)$ is also time varying. Let $\{\mathcal{G}_k\}_{k=1}^{N}$ be the set of signed digraphs with all possible topologies, and $\Lambda = \{1, \dots, N\}$ as its index set. Let $\sigma(t) : [0, \infty) \to \Lambda$ represent a piecewise constant switching signal to be designed.

Now one can view (2) as a system in the form of a switching network

$$\dot{x}(t) = -L_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \tag{3}$$

where $L_{\sigma(t)} \in \mathbb{R}^{n\times n}$ is the Laplacian matrix of $\mathcal{G}_{\sigma(t)}$, and $B_{\sigma(t)} = [e_{i_1}, \dots, e_{i_{m_{\sigma(t)}}}] \in \mathbb{R}^{n\times m_{\sigma(t)}}$. Note that system (3) contains N subsystems $(-L_k, B_k), k = 1, \dots, N$, and $\sigma(t) = i$ implies that the subsystem $(-L_i, B_i)$ is activated at time instant t.

Definition 1: The system (3) is said to be controllable, if for any initial state $x_0 \in \mathbb{R}^n$ and any terminal state $x_f \in \mathbb{R}^n$, there exists a switching signal $\sigma(t)$ and a piecewise continuous input function $u(t), t \in [t_0, t_T]$, such that the system can be driven from $x(t_0) = x_0$ to $x(t_T) = x_f$.

III. Main Results

A. Controllability Analysis of Switching Signed Networks

In this section, we give some graph-theoretic characterization on controllability of switching signed networks. In particular, the controllability is studied from the union digraph perspective.

Lemma 1: System (3) is controllable if and only if $rank(\mathcal{Q}_s(-L_i, B_i)) = n$, where

$$\begin{aligned} \mathcal{Q}_s(-L_i, B_i) \triangleq [&B_1, \dots, B_N, -L_1B_1, \dots, -L_NB_N, \\ &L_1^2B_1, \dots, L_NL_1B_1, \dots, L_N^2B_N, \dots, \\ &(-1)^{n-1}L_1^{n-1}B_1, \dots, (-1)^{n-1}L_N^{n-1}B_N]. \end{aligned}$$

Proof: The result is a direct consequence of [22, Th. 1]. ■

Remark 1: The matrix $\mathcal{Q}_s(-L_i, B_i)$ is called the controllability matrix of the system (3). If we use $Im(Q)$ to represent the range space of a matrix Q, then $Im(\mathcal{Q}_s(-L_i, B_i))$ is the controllable subspace of the system (3) [22]. Lemma 1 implies that the system (3) is controllable if and only if $Im(\mathcal{Q}_s(-L_i, B_i)) = \mathbb{R}^n$. In fact, Lemma 1 provides a sufficient and necessary algebraic condition for checking the controllability of system (3). However, this condition may fail as the network size increases. When the network size increases, $\mathcal{Q}_s(-L_i, B_i)$ becomes a high-dimensional matrix. In this case, the computation result of checking the rank of $\mathcal{Q}_s(-L_i, B_i)$ may suffer from numerical inaccuracy.

Next, we will propose some graph-theoretic conditions for controllability of the switched system even if each subsystem is not controllable.

Theorem 1: System (3) is controllable if and only if the system

$$\dot{x}(t) = -L_{\sigma(t)}x(t) + \bar{B}\bar{u}(t) \tag{4}$$

is controllable, where $\bar{u}(t) = [u_1(t), \dots, u_M(t)]^T \in \mathbb{R}^M$, $\bar{B} = [B_1, \dots, B_N] \in \mathbb{R}^{n\times M}$, $M \triangleq \sum_{i=1}^{N} m_i$.

Proof: By Lemma 1, system (4) is controllable if and only if the matrix

$$\mathcal{Q}_s(-L_i,\bar{B}) \triangleq [\bar{B}, -L_1\bar{B}, \dots, -L_N\bar{B}, L_1^2\bar{B}, \dots, L_NL_1\bar{B}, \dots, L_N^2\bar{B}, \dots, (-1)^{n-1}L_1^{n-1}\bar{B}, \dots, (-1)^{n-1}L_N^{n-1}\bar{B}] \quad (5)$$

is full row rank. It is easy to check that the rank of $\mathcal{Q}_s(-L_i,\bar{B})$ is the same as that of $\mathcal{Q}_s(-L_i, B_i)$. ■

Remark 2: As seen from Theorem 1, when the leaders are known, the switching behaviors of leaders between different interconnection topologies have no effect on controllability of system (3). As a special case, if the interconnection topologies associated with system (3) are fixed, i.e., $\mathcal{G}^k = \mathcal{G}, k = 1, \dots, N$, the controllability of system $(-L, B_{\sigma(t)})$ is equivalent to that of $(-L,\bar{B})$.

For system (3), assume that all possible interconnection topologies are digraph $\mathcal{G}_k = \{\mathcal{V}, \mathcal{E}_k, A_k\}$, $k = 1, \dots, N$, where the corresponding leader sets are $\mathcal{V}_k^L \subseteq \mathcal{V}$. The union digraph $\bar{\mathcal{G}}$ of all the possible interconnection topologies is the representation of linear time-invariant system $(-\bar{L},\bar{B})$, where $\bar{L} = \sum_{k=1}^N L_k$, and the corresponding leader set is $\bar{\mathcal{V}}^L = \bigcup_{k=1}^N \mathcal{V}_k^L$.

Theorem 2: If the union digraph $\bar{\mathcal{G}}$ is controllable (i.e., $(-\bar{L},\bar{B})$ is controllable), then system (3) is controllable.

Proof: Assume that the union digraph $\bar{\mathcal{G}}$ is controllable, this implies that the controllability matrix $\mathcal{Q}_c(-\bar{L},\bar{B}) \triangleq [\bar{B}, (-\bar{L})\bar{B}, \dots, (-\bar{L})^{n-1}\bar{B}]$ is full row rank. Expanding the matrix $\mathcal{Q}_c(-\bar{L},\bar{B})$ yields,

$$\begin{aligned}[B_1, \dots, B_N,\\ &(-L_1)B_1 + \cdots + (-L_N)B_1, \dots,\\ &(-L_1)B_N + \cdots + (-L_N)B_N, \dots,\\ &(-L_1)^{n-1}B_1 + \cdots + (-L_N)^{n-1}B_1, \dots,\\ &(-L_1)^{n-1}B_N + \cdots + (-L_N)^{n-1}B_N]. \end{aligned} \quad (6)$$

One has that the matrix (6) is full row rank. Next, we add some column vectors to the matrix (6) and obtain that

$$\begin{aligned}[B_1, \dots, B_N,\\ &(-L_1)B_1 + \cdots + (-L_N)B_1, \dots,\\ &(-L_1)B_N + \cdots + (-L_N)B_N,\\ &(-L_2)B_1, \dots, (-L_N)B_1, \dots, (-L_N)B_N, \dots,\\ &(-L_1)^{n-1}B_1 + \cdots + (-L_N)^{n-1}B_1, \dots,\\ &(-L_1)^{n-1}B_N + \cdots + (-L_N)^{n-1}B_N,\\ &(-L_2)(-L_1)^{n-2}B_1, \dots, (-L_N)^{n-1}B_1, \dots, (-L_N)^{n-1}B_N]. \end{aligned} \quad (7)$$

This matrix still has n linear independent column vectors, so it has full row rank. As some column elementary transformations do not change the matrix rank, we subtract $(-L_2)B_1, \dots, (-L_N)B_1$ from $(-L_1)B_1 + \cdots + (-L_N)B_1$; subtract $(-L_2)B_N, \dots, (-L_N)B_N$ from $(-L_1)B_N + \cdots + (-L_N)B_N$; subtract $(-L_2)(-L_1)^{n-2}B_1, \dots, (-L_N)^{n-1}B_1$ from $(-L_1)^{n-1}B_1 + \cdots + (-L_N)^{n-1}B_1$ and subtract $(-L_2)(-L_1)^{n-2}B_N, \dots, (-L_N)^{n-1}B_N$ from $(-L_1)^{n-1}B_N + \cdots + (-L_N)^{n-1}B_N$. The matrix (7) becomes

$$\begin{aligned}[B_1, \dots, B_N, -L_1B_1, \dots, -L_NB_1, \dots, -L_NB_N, \dots,\\ &(-1)^{n-1}L_1^{n-1}B_1, \dots, (-1)^{n-1}L_N^{n-1}\\ &B_1, \dots, (-1)^{n-1}L_N^{n-1}B_N], \end{aligned} \quad (8)$$

which is full row rank. By Lemma 1, system (3) is controllable. ■

Remark 3: Theorem 2 provides a graph-theoretic sufficient condition for controllability of switching signed networks in terms of the union digraph of all the possible topologies. On one hand, the result indicates that, to some extent, we can judge the controllability of switching networks by checking the controllability of the union digraph. This merit can greatly reduce the computational cost and significantly improve the computational accuracy for checking controllability. On the other hand, Theorem 2 implies that when the union digraph is controllable, the switching signed network would be controllable even if each of its subnetworks is not (see Example 1 in Section V). However, it is noted that the condition in Theorem 2 is sufficient rather than necessary. That is to say, even if the union digraph is uncontrollable, a switching signed network could still be controllable (see Example 2 in Section V).

B. Controllability of Simultaneously Structurally Balanced Signed Networks

Structural balance is an important and basic concept in signed networks. The concept and properties of structural balance are widely used to study distributed control problem of multi-agent networks with antagonistic interactions, see, e.g., [13]–[18]. For fixed signed networks, some research results show that, to some extent, the controllability of a structurally balanced network is equivalent to the controllability of its corresponding unsigned network, see, e.g., [19]–[21]. In this section, we use the structural balance theory to study the controllability of switching signed networks.

Definition 2: All signed digraphs $\mathcal{G}_k = (\mathcal{V}, \mathcal{E}_k, \mathcal{A}_k), k = 1, \dots, N$, are said to be simultaneously structurally balanced if there exists a fixed bipartition $\{\mathcal{V}_1, \mathcal{V}_2 : \mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}, \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset\}$ of $\mathcal{V}$ such that $a_{ij}^k \geq 0, \forall i,j \in \mathcal{V}_p$ $(p \in \{1,2\})$, $a_{ij}^k \leq 0, \forall i \in \mathcal{V}_p, j \in \mathcal{V}_q$ $(p,q \in \{1,2\}, p \neq q)$.

Lemma 2 [13]: A weighted and signed digraph $\mathcal{G}$ is structurally balanced if and only if there exists $D \in \mathcal{D}$ such that the entries of $\mathcal{A}^D = D\mathcal{A}D$ are all non-negative and the off-diagonal entries of $L^D = DLD$ are all non-positive.

Theorem 3: Assume that the interconnection topologies associated with system (3) are simultaneously structurally balanced. If the leaders are chosen from the same subset (i.e., either $\mathcal{V}_k^L \subseteq \mathcal{V}_1$ or $\mathcal{V}_k^L \subseteq \mathcal{V}_2, k = 1, \dots, N$), then the controllability of system $(-L_{\sigma(t)}, B_{\sigma(t)})$ is equivalent to that of $(-L_{\sigma(t)}^D, B_{\sigma(t)})$.

Proof: Since all signed digraphs $\mathcal{G}_k, k = 1, \dots, N$, are simultaneously structurally balanced, one can obtain by Lemma 2 that there exists a common signature matrix $D \in \mathcal{D}$ such that $L_k^D = DL_kD, k = 1, \dots, N$, where L_k^D is a Laplacian matrix of the unsigned digraph $\mathcal{G}_k' = (\mathcal{V}, \mathcal{E}_k, |\mathcal{A}_k|)$. By

Lemma 1 and the fact that $D = D^{-1}$, one has

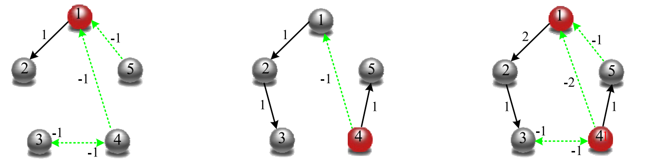
$$\begin{aligned}
&rank(\mathcal{Q}_s(-L_i, B_i)) \\
&\quad = rank(\mathcal{Q}_s(-L_i, \bar{B})) \\
&\quad = rank([\bar{B}, -DL_1^D D\bar{B}, \ldots, -DL_N^D D\bar{B}, \\
&D(L_1^D)^2 D\bar{B}, \ldots, DL_N^D L_1^D D\bar{B}, \ldots, D(L_N^D)^2 D\bar{B}, \ldots, \\
&(-1)^{n-1} D(L_1^D)^{n-1} D\bar{B}, \ldots, (-1)^{n-1} D(L_N^D)^{n-1} D\bar{B}])
\end{aligned} \tag{9}$$

If $\mathcal{V}_k^L \subseteq \mathcal{V}_1, k = 1, \ldots, N$, $D\bar{B} = \bar{B}$. If $\mathcal{V}_k^L \subseteq \mathcal{V}_2, k = 1, \ldots, N$, $D\bar{B} = -\bar{B}$. It follows from (9) that

$$\begin{aligned}
&= rank([\bar{B}, -DL_1^D D\bar{B}, \ldots, -DL_N^D D\bar{B}, \\
&\quad D(L_1^D)^2 D\bar{B}, \ldots, DL_N^D L_1^D D\bar{B}, \ldots, D(L_N^D)^2 D\bar{B}, \ldots, \\
&\quad (-1)^{n-1} D(L_1^D)^{n-1} D\bar{B}, \ldots, (-1)^{n-1} D(L_N^D)^{n-1} D\bar{B}]) \\
&= rank(D[D\bar{B}, -L_1^D D\bar{B}, \ldots, -L_N^D D\bar{B}, \\
&\quad (L_1^D)^2 D\bar{B}, \ldots, L_N^D L_1^D D\bar{B}, \ldots, (L_N^D)^2 D\bar{B}, \ldots, \\
&\quad (-1)^{n-1} (L_1^D)^{n-1} D\bar{B}, \ldots, (-1)^{n-1} (L_N^D)^{n-1} D\bar{B}]) \\
&= rank(\pm D[\bar{B}, -L_1^D \bar{B}, \ldots, -L_N^D \bar{B}, \\
&\quad (L_1^D)^2 \bar{B}, \ldots, L_N^D L_1^D \bar{B}, \ldots, (L_N^D)^2 \bar{B}, \ldots, \\
&\quad (-1)^{n-1} (L_1^D)^{n-1} \bar{B}, \ldots, (-1)^{n-1} (L_N^D)^{n-1} \bar{B}]) \\
&= rank(\mathcal{Q}_s(-L_i^D, \bar{B})) \\
&= rank(\mathcal{Q}_s(-L_i^D, B_i)),
\end{aligned}$$

which indicates that the controllability of $(-L_{\sigma(t)}^D, B_{\sigma(t)})$ remains the same as the controllability of $(-L_{\sigma(t)}, B_{\sigma(t)})$. ■

Remark 4: As seen from Theorem 3, if the leaders are chosen from the same subset, the controllability of simultaneously structurally balanced networks is equivalent to that of corresponding unsigned networks. The result bridges the gap between the controllability of switching signed networks and switching unsigned networks, and brings significant convenience for controllability analysis and design of signed networks. However, it is noted that the simultaneously structurally balanced condition is a fundamental assumption in Theorem 3, which usually requires that the interconnection topology of each subsystem is structurally balanced. Therefore, Theorem 3 is conservative to some extent. In future work, we will consider further relaxing the simultaneously structurally balanced assumption for switching signed networks.



(a) $\mathcal{G}_1$ (b) $\mathcal{G}_2$ (c) $\bar{\mathcal{G}} = \mathcal{G}_1 \cup \mathcal{G}_2$

Fig. 1. Two possible interconnection topologies $\mathcal{G}_1$, $\mathcal{G}_2$, and the corresponding union digraph $\bar{\mathcal{G}}$, where $\mathcal{V}_1^L = \{1\}$, $\mathcal{V}_2^L = \{4\}$ and $\bar{\mathcal{V}}^L = \{1, 4\}$. The black solid and green dotted lines represent the positive and negative edges, respectively.

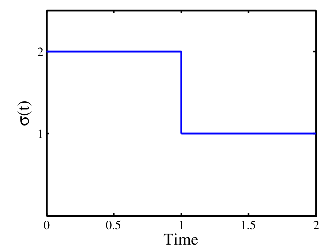


Fig. 2. The switching signal $\sigma(t)$ of Example 1.

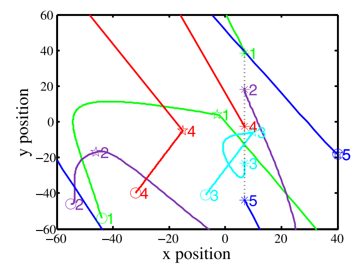
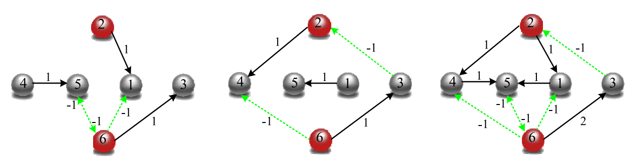


Fig. 3. The trajectories of the agents, where "∘", "∗" and "☆" denote the initial states, the final states and the states at the switching time instant $t = 1$, respectively.



(a) $\mathcal{G}_1$ (b) $\mathcal{G}_2$ (c) $\bar{\mathcal{G}} = \mathcal{G}_1 \cup \mathcal{G}_2$

Fig. 4. Two possible interconnection topologies $\mathcal{G}_1$, $\mathcal{G}_2$, and the corresponding union digraph $\bar{\mathcal{G}}$, where $\mathcal{V}_1^L = \mathcal{V}_2^L = \bar{\mathcal{V}}^L = \{2, 6\}$. The black solid and green dotted lines represent the positive and negative edges, respectively.

IV. Simulation

In this section, we provide several numerical examples to verify the effectiveness of the derived theoretical results.

Example 1: Consider a switching signed network with five agents. The interconnection topologies are depicted in Figure 1 (a)-(b), where $\mathcal{G}_k, k = 1, 2$, are two possible topologies. The union digraph $\bar{\mathcal{G}}$ is shown in Figure 1 (c). It is easy to check that $\mathcal{G}_k$ (i.e., subsystem $(-L_i, B_i)$), $i = 1, 2$ is uncontrollable. However, the union digraph $\bar{\mathcal{G}}$ (i.e., system $(-\bar{L}, \bar{B})$) is controllable. By Theorem 2, system (3) is controllable. In fact, by calculating, we obtain $rank([B_1, B_2, -L_1B_1, -L_1B_2, -L_2B_2]) = 5$, and thus system (3) is controllable. Design a switching signal $\sigma(t)$ as shown in Figure 2. Figure 3 depicts the trajectories of the controllable agents, where the initial states, the final states and the states at the switching time instant $t = 1$ are denoted by circles, asterisks and pentagram, respectively. The initial states are randomly selected, and the final states are designed so as to form a straight line configuration.

Example 2: Suppose that the interconnection topologies of system (3) are displayed in Figure 4 (a)-(b), where $\mathcal{G}_k, k = 1, 2$, are two possible topologies, which are also simultaneously structurally balanced (i.e., $\mathcal{V}_1 = \{1, 2, 4, 5\}$ and $\mathcal{V}_2 = \{3, 6\}$). The union digraph $\bar{\mathcal{G}}$ is shown in Figure 4 (c). It is easy to check that $\mathcal{G}_k$ (i.e., subsystem $(-L_i, B_i)$), $k = 1, 2$, and $\bar{\mathcal{G}}$ (i.e., system $(-\bar{L}, \bar{B})$) are uncontrollable. However, system (3) is controllable. In fact, $rank([B, -L_1B, -L_2B]) = 6$, and thus system (3) is controllable. Design a switching signal $\sigma(t)$ as shown in Figure 5. Figure 6 depicts the trajectories of the six

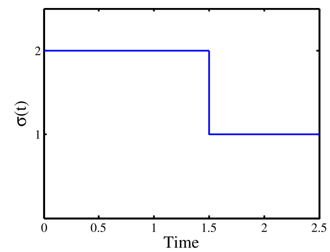


Fig. 5. The switching signal $\sigma(t)$ of Example 2.

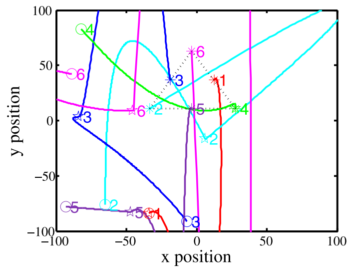


Fig. 6. The trajectories of the agents, where "○", "∗" and "☆" denote the initial states, the final states and the states at the switching time instant $t = 1.5$, respectively.

controllable agents, where the initial states, the final states and the states at the switching time instant $t = 1.5$ are denoted by circles, asterisks and pentagram, respectively. The initial states are randomly selected, and the final states are designed so as to form a triangle configuration.

V. Conclusion

In this brief, we consider the controllability of switching signed networks. It has been shown that when the union digraph of all the possible interconnection topologies is controllable, the switching signed network can be controllable even if each of its subnetworks is uncontrollable. It has also been pointed out the controllability of simultaneously structurally balanced networks is equivalent to the controllability of their corresponding unsigned networks if the leaders are chosen from the same vertex set. One future direction may include investigations of the structural controllability [23] and the stabilizability of switching signed networks [24], [25].

References

[1] H. G. Tanner, "On the controllability of nearest neighbor interconnections," in *Proc. IEEE Conf. Decis. Control*, vol. 3, 2004, pp. 2467–2472.

[2] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, "Controllability of multi-agent systems from a graph-theoretic perspective," *SIAM J. Control Optim.*, vol. 48, no. 1, pp. 162–186, Jan. 2009.

[3] S. Zhang, M. Cao, and M. K. Camlibel, "Upper and lower bounds for controllable subspaces of networks of diffusively coupled agents," *IEEE Trans. Autom. Control*, vol. 59, no. 3, pp. 745–750, Mar. 2014.

[4] Y. Guan, Z. Ji, L. Zhang, and L. Wang, "Controllability of multi-agent systems under directed topology," *Int. J. Robust Nonlin.*, vol. 27, no. 18, pp. 4333–4347, Dec. 2017.

[5] M. Cao, S. Zhang, and M. K. Camlibel, "A class of uncontrollable diffusively coupled multiagent systems with multichain topologies," *IEEE Trans. Autom. Control*, vol. 58, no. 2, pp. 465–469, Feb. 2013.

[6] B. Liu, T. Chu, L. Wang, and G. Xie, "Controllability of a leader–follower dynamic network with switching topology," *IEEE Trans. Autom. Control*, vol. 53, no. 4, pp. 1009–1013, May 2008.

[7] Z. Ji, H. Lin, and H. Yu, "Protocols design and uncontrollable topologies construction for multi-agent networks," *IEEE Trans. Autom. Control*, vol. 60, no. 3, pp. 781–786, Mar. 2015.

[8] Y. Guan, Z. Ji, L. Zhang, and L. Wang, "Controllability of heterogeneous multi-agent systems under directed and weighted topology," *Int. J. Control*, vol. 89, no. 5, pp. 1009–1024, May 2016.

[9] Z. Ji and H. Yu, "A new perspective to graphical characterization of multiagent controllability," *IEEE Trans. Cybern.*, vol. 47, no. 6, pp. 1471–1483, Jun. 2017.

[10] A. Li, S. P. Corneliue, Y. Y. Liu, L. Wang, and A. L. Barabási, "The fundamental advantages of temporal networks," *Science*, vol. 358, no. 6366, pp. 1042–1046, Nov. 2017.

[11] Y. Guan and L. Wang, "Target controllability of multiagent systems under fixed and switching topologies," *Int. J. Robust Nonlin.*, vol. 29, no. 9, pp. 2725–2741, Jun. 2019.

[12] G. Facchetti, G. Iacono, and C. Altafini, "Computing global structural balance in large-scale signed social networks," *Proc. Nat. Acad. Sci. USA*, vol. 108, no. 52, pp. 20953–20958, Dec. 2011.

[13] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 935–946, Apr. 2013.

[14] J. Qin, W. Fu, W. X. Zheng, and H. Gao, "On the bipartite consensus for generic linear multiagent systems with input saturation," *IEEE Trans. Cybern.*, vol. 47, no. 8, pp. 1948–1958, Aug. 2017.

[15] Q. Jiao, H. Zhang, S. Xu, F. L. Lewis, and L. Xie, "Bipartite tracking of homogeneous and heterogeneous linear multi-agent systems," *Int. J. Control*, to be published. doi: 10.1080/00207179.2018.1467044.

[16] J. Qu, Z. Ji, C. Lin, and H. Yu, "Fast consensus seeking on networks with antagonistic interactions," *Complexity*, vol. 2018, Dec. 2018, Art. no. 7831317.

[17] S. Bhowmick and S. Panja, "Leader–follower bipartite consensus of linear multiagent systems over a signed directed graph," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, to be published. doi: 10.1109/TCSII.2018.2884006.

[18] L. Pan, H. Shao, M. Mesbahi, Y. Xi, and D. Li, "Bipartite consensus on matrix-valued weighted networks," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, to be published. doi: 10.1109/TCSII.2018.2884483.

[19] C. Sun, G. Hu, and L. Xie, "Controllability of multiagent networks with antagonistic interactions," *IEEE Trans. Autom. Control*, vol. 62, no. 10, pp. 5457–5462, Oct. 2017.

[20] Y. Guan and L. Wang, "Controllability of multi-agent systems with directed and weighted signed networks," *Syst. Control Lett.*, vol. 116, pp. 47–55, Jun. 2018.

[21] B. She, S. Mehta, C. Ton, and Z. Kan, "Controllability ensured leader group selection on signed multiagent networks," *IEEE Trans. Cybern.*, to be published. doi: 10.1109/TCYB.2018.2868470.

[22] G. Xie and L. Wang, "Controllability and stabilizability of switched linear-systems," *Syst. Control Lett.*, vol. 48, no. 2, pp. 135–155, Feb. 2003.

[23] Y. Guan and L. Wang, "Structural controllability of multi-agent systems with absolute protocol under fixed and switching topologies," *Sci. China Inf. Sci.*, vol. 60, no. 9, Sep. 2017, Art. no. 092203.

[24] Y. Guan, Z. Ji, L. Zhang, and L. Wang, "Decentralized stabilizability of multi-agent systems under fixed and switching topologies," *Syst. Control Lett.*, vol. 62, no. 5, pp. 438–446, May 2013.

[25] Y. Guan and X. Kong, "Stabilizability of discrete-time multi-agent systems under fixed and switching topologies," *Int. J. Syst. Sci.*, vol. 50, no. 2, pp. 294–306, Feb. 2019.