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Controllability of heterogeneous multiagent systems

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Summary

The existing results on controllability of multiagent systems (MASs) are mostly based on homogeneous nodes. This paper focuses on controllability of heterogeneous MASs, where the agents are modeled as two types. One type is that the agents have the same high-order dynamics, and the interconnection topologies of the information flow in different orders are supposed to be different; the other type is that the agents have generic linear dynamics, and the dynamics are supposed to be heterogeneous. For the first type, the necessary and sufficient condition for controllability of heterogeneous-topology system is derived via combination of Laplacian matrices. For the second type, the contribution also has two parts. The first part supposes that the agents have the same dimensional states and proves that controllability of this kind of MASs is equivalent to the controllability of each node and the whole interconnection topology, while the last parameter of the state feedback vector must not be 0. The second part supposes that the agents may have different dimensional states. For this kind of systems, the concept of β-controllability is proposed. The necessary and sufficient condition for β-controllability of heterogeneous-dynamic systems is also derived and it is also proved that the feedback gain vectors have the effect to improve controllability. Different illustrative examples are provided to demonstrate the effectiveness of the theoretical results in this paper.

KEYWORDS

controllability, feedback gain, heterogeneous multiagent systems, leader-follower connected

1 | INTRODUCTION

During the past few decades, control of networked systems has become a popular topic because of the broad applications of networks in areas such as dynamical systems, complex networks, and neural networks.[1-3] As a kind of common networked systems, multiagent systems (MASs) have attracted attentions from many researchers. The distributed coordination control of MASs has many practical applications, for example, flocking in biology groups, unmanned aerial vehicle cooperative formation, and attitude adjustment of spacecrafts, to name a few. Some basic and important issues were studied including consensus problem,[4-6] formation control,[7] and controllability and stabilizability.[8-13] In the following, this paper takes MASs as a representative to discuss controllability of networks.

Controllability of MASs was proposed by Tanner for the first time,[14] where a necessary and sufficient condition was presented using the Laplacian matrix and the corresponding eigenvalues. Afterwards, research studies on multiagent controllability divided into two directions, one of which is on the necessary and/or sufficient conditions for controllability, while the other is on the methods to achieve and maintain controllability. The investigations on controllability conditions varies in different models and topologies, eg, Wang et al studied controllability of MASs with high-order dynamic

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agents and generic linear dynamic agents in the work of Wang et al,[15] where they showed that controllability of these systems is congruously determined by the interconnection topology. Ji et al proposed a basic necessary condition named leader-follower connected topologies for controllability.[16] Following these works, Zhao et al generalized the results from the viewpoint of left eigenvectors of the graph Laplacians.[17] Rahmani et al provided some necessary conditions for controllability utilizing the equitable partition of the interconnection topology.[18] In addition, other useful necessary and sufficient conditions were obtained for controllability of some specific graphs, eg, paths, cycles, grid graphs, multichains, and signed networks,[19-22] to name a few. Besides, controllability of dynamic-edge MASs was proposed by Wang et al,[23] where PBH-like conditions were provided. Except for investigating the conditions of controllability, several new problems and properties were proposed for controllability. For example, to deal with controllability improvement problem, algorithms of selecting proper leaders and adjusting edge weights to improve controllability were provided by Zhao et al,[17] and protocol design to achieve controllability was studied by Ji et al.[24] In parallel, structural controllability was studied under various models in other works,[25-27] where some necessary and sufficient conditions were also established. Additionally, robustness of controllability and structural controllability were investigated subject to failure of agents and communication edges.[28,29]

However, the existing results on controllability mainly focus on homogeneous dynamic agents, where the agents have exactly the same kinetic models. In practice, for MASs, it is sometimes difficult to ensure that each agent has exactly the same dynamic. For example, the dynamics of marine vehicles on surface and underwater are quite different, and the robots of different generations act differently during the formation progress.[30] In the cooperative combat of the navy, army, and air forces, the battle units have different dynamic behaviors.[10] One calls these systems as heterogeneous-dynamic MASs. Furthermore, even if the agents share the same high-order dynamic, the interconnection topologies of the information flow in different orders may possibly be different,[31,32] which forms heterogeneous-topology MASs. These two kinds of systems are collectively called heterogeneous MASs. The study of heterogeneous MASs is just in its early stage,[33] especially on controllability problem.[10] For second-order heterogeneous-topology systems, the previous investigations only focused on consensus problems,[31,34,35] and to the best of our knowledge, controllability of high-order systems with heterogeneous topologies has not yet been discussed. Although consensus and output consensus problems have been studied for heterogeneous-dynamic MASs using various models,[30,33,36] controllability problem is yet to be explored.

Motivated by the above analysis, this paper studies controllability for heterogeneous MASs. The main contributions of this paper are summarized as follows.

1. Controllability protocols are proposed for different kinds of heterogeneous MASs, especially for heterogeneous-dynamic systems with nonidentical-dimension nodes, where feedback gain vectors of each agent are introduced. The feedback gains guarantee that all the agents can properly interact with their neighbors and also ensure that the whole system achieves controllability.
2. The necessary and sufficient conditions for controllability of heterogeneous MASs (including two situations where the networks are with high-order dynamical agents and with same-dimensional generic-linear dynamical agents) are provided for the first time. The conditions are described both in graphic and algebraic perspectives.
3. For the systems with different-dimensional generic-linear dynamical agents, a new concept of β-controllability is proposed and the necessary and sufficient condition for β-controllability is derived. It is also proved that the feedback gain vectors have the effect to improve controllability of MASs.

In summary, this paper generalizes the results of homogeneous MASs to heterogeneous MASs in several aspects.

This paper is organized as follows. Section 2 introduces and proposes some basic concepts and mathematic tools for this paper. Main results on controllability of heterogeneous-topology MASs and heterogeneous-dynamic MASs are obtained in Sections 3 and 4, respectively. Numerical examples are provided in Section 5 to illustrate the theoretical results. Conclusions are drawn in Section 6.

2 | PRELIMINARIES

2.1 | Notations

The set of n-dimensional real vectors is denoted by $\mathbb{R}^n$ and the set of $m \times n$ real matrices is denoted by $\mathbb{R}^{m\times n}$. Matrix $\mathrm{diag}(a_1, a_2, \dots, a_n)$ is the matrix with principal diagonals $a_1, a_2, \dots, a_n$, where a_i is a number or a matrix, $i = 1, 2, \dots, n$. Denote $(0, \dots, 0)^T, (1, \dots, 1)^T \in \mathbb{R}^n$ as 0_n and 1_n, respectively. Let $e_i(n)$ represent the ith column of the identity matrix I_n, and (n) is omitted without misunderstanding. Let $sp\{L_1, L_2, \dots, L_m\}$ denote the matrix space $\{L | L = \sum_{i=1}^{m} k_i L_i, k_i \in \mathbb{R}\}$. $\emptyset$

represents the empty set and $\otimes$ represents the Kronecker product. S/T represents the set of all the elements in S but not in T.

2.2 | Graph theory

An undirected graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ consists of a vertex set $\mathbb{V} = \{v_1, v_2, \dots, v_n\}$, and an edge set $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$. In graph $\mathbb{G}$, $e_{ij} \in \mathbb{E}$ if and only if $e_{ji} \in \mathbb{E}$, and v_i and v_j are said to be adjacent with each other. The neighbor set of v_j is denoted by $N_j = \{v_i \in \mathbb{V} \,|\, (v_i, v_j) \in \mathbb{E}\}$. For two different vertices $v_i \neq v_j$, we say they are in the same connected component if there exists a path between them; otherwise, they are in different connected components. The adjacency matrix of $\mathbb{G}$ is $A(\mathbb{G}) = (a_{ij}) \in \mathbb{R}^{n \times n}$, where $a_{ij} > 0$ is the weight of edge e_{ji} (as well as e_{ij}), and $a_{ij} = 0$ if $(v_j, v_i) \notin \mathbb{E}$.* The Laplacian matrix of $\mathbb{G}$ is $L(\mathbb{G}) = D - A$, $D = \text{diag}(d_1, d_2, \dots, d_n)$ where $d_k = \sum_{i=1, i \neq k}^{n} a_{ki}$, $k = 1, 2, \dots, n$. Suppose that $\mathbb{G}_1$ and $\mathbb{G}_2$ have the same number of nodes, then $\tilde{\mathbb{G}} = (\tilde{\mathbb{V}}, \tilde{\mathbb{E}})$ is said to be the union graph of $\mathbb{G}_1$ and $\mathbb{G}_2$ if $\tilde{\mathbb{E}} = \mathbb{E}_1 \cup \mathbb{E}_2$ with the adjacency matrix $A(\tilde{\mathbb{G}}) = A(\mathbb{G}_1) + A(\mathbb{G}_2)$.

2.3 | Controllability

Linear system $\dot{x} = Ax + Bu$, $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$ is said to be controllable if for any initial state $x(t_0) = x_0$ and target state x^*, there exist a control input u and a finite time $t_f > t_0$ such that $x(t_f) = x^*$. According to the PBH Test, the linear system is controllable if and only if no left eigenvector of A is orthogonal to B. If the system is controllable, one also says (A, B) is controllable for convenience. Especially, if A is a negative Laplacian matrix, ie, $A = -L$ with the corresponding graph $\mathbb{G}$, when $(-L, B)$ is controllable, one can also say that the topology graph $\mathbb{G}$ is controllable.

For MASs, information communication is based on distributed protocols and the interconnection topologies are modeled as undirected graphs in this paper, ie, if there is a connection between two agents, the interconnection couplings between them are the same.† The leader agents are supposed to receive external input(s) while the follower agents only receive information from neighbor agents. In the original study of multiagent controllability, the agent dynamics are modeled as single integrators $\dot{x}_i = u_i$, $i = 1, 2, \dots, n$ and the information communication between the agents obeys the protocol $u_i = \sum_{j \in N_i} a_{ij}(x_j - x_i) + u_{o,i}$.[14] Here, the communication protocol consists of two parts. One is the distributed part $\sum_{j \in N_i} a_{ij}(x_j - x_i)$, which represents the relative information interaction inside the system; the other is the external input(s) $u_{o,i}$ actuating the leader(s), which indirectly drives the whole system to the desired state. The MAS is said to be controllable if, for any initial state and any target state, the target state can be reached within a finite time by proper external input(s). If the MAS is controllable, one also says that the topology (of the system) is controllable. The most important concept for the interconnection topologies of MASs to be introduced in this paper, named "leader-follower connected," is also a basic necessary (not sufficient) condition for multiagent controllability.[16]

Definition 1. An interconnection graph $\mathbb{G}$ is said to be leader-follower connected if, for each connected component of $\mathbb{G}$, there exists at least one leader in the component.

3 | CONTROLLABILITY OF HETEROGENEOUS-TOPOLOGY MULTIAGENT SYSTEMS

Take a third-order dynamical agent as an example. Let x be the position of the agent, then $v = \dot{x}$ represents the velocity. The derivative of v is the acceleration $a = \dot{v}$, which is directly proportional to the driving force of the agent. If one takes control on the variation rate of the driving force, one actually controls the third-order derivative of x. Restricted by the communication ability, different agents may have different modes to receive position information, velocity information, and acceleration information from different neighbors. This leads to different patterns of information topologies,[34] which are called heterogeneous topologies in the following. From the viewpoint of graph theory, heterogeneous topologies have their own adjacency matrices for different orders of information.

Mathematically, this section discusses the MAS with mth-order dynamical agents. For a high-order MAS, the dynamics of the agents are modeled as $\dot{x}_i = x_i^{(1)}, \dot{x}_i^{(1)} = x_i^{(2)}, \dots, \dot{x}_i^{(m-1)} = x_i^{(m)}$, and $\dot{x}_i^{(m)} = u_i$, where u_i is the control input and $x_i^{(l)}$ is said to be the lth-order information, $l = 1, 2, \dots, m, i = 1, 2, \dots, n$. The control inputs are supposed to follow the

*The edge weight $a_{ij} > 0$ is allowed to be any positive number in this paper, and it is assumed that $a_{ii} = 0$, $i = 1, 2, \dots, n$.
†Undirected graph is the simple case. The study on asymmetrical topology is much more difficult, which will be discussed in our future work.

consensus-based protocol: $u_i = \sum_{l=1}^{m} \sum_{j \in N_i} k_i a_{ij}^{(l)} (x_j^{(l)} - x_i^{(l)}) + u_{oi}$, where $a_{ij}^{(l)}$ is the interconnection coupling of the lth-order information between $x_i^{(l)}$ and $x_j^{(l)}$, k_i is the feedback gain, $u_{oi} \in \mathbb{R}$ is the external control on the leader agent v_i, and $u_{oi} = 0$ when v_i is a follower; let $A_l = (a_{ij}^{(l)})_{n \times n}$ be the adjacency matrix of the lth-order information topology and L_l the corresponding Laplacian matrix, $l = 1, 2, \ldots, m$. Then, the state of each agent is

$$\begin{pmatrix} \dot{x}_i^{(1)} \\ \dot{x}_i^{(2)} \\ \vdots \\ \dot{x}_i^{(m)} \end{pmatrix} = \begin{pmatrix} 0_{m-1} & I_{m-1} \\ 0 & 0_{m-1}^T \end{pmatrix} \begin{pmatrix} x_i^{(1)} \\ x_i^{(2)} \\ \vdots \\ x_i^{(m)} \end{pmatrix} + \begin{pmatrix} 0_{m-1} \\ 1 \end{pmatrix} \left(\sum_{l=1}^{m} \sum_{j \in N_i} k_i a_{ij}^{(l)} \left(x_j^{(l)} - x_i^{(l)} \right) + u_{oi} \right). \tag{1}$$

Let $x = (x^{(1)T}, \ldots, x^{(m)T})^T$, where $x^{(i)} = (x_1^{(i)}, \ldots, x_n^{(i)})^T$. Without loss of generality, suppose that the leaders are $v_{p+1}, \ldots, v_n$, then summarize the compact form of the high-order MAS as

$$\dot{x} = \left(\begin{pmatrix} 0_{m-1} & I_{m-1} \\ 0 & 0_{m-1}^T \end{pmatrix} \otimes I_n - e_m \otimes e_1^T \otimes k_1 L_1 - \cdots - e_m \otimes e_m^T \otimes k_m L_m \right) x + e_m \otimes B u_o, \tag{2}$$

where e_i is the ith column of I_m, $B = (e_{p+1}, \ldots, e_n) \in \mathbb{R}^{n \times (n-p)}$, $u_o \in \mathbb{R}^{n-p}$.

System (2) is said to be controllable if, for any initial state $x(t_0) = x_0$ and target state x^*, there exist a finite time $T > t_0$, a control input u_o, and $k_1, \ldots, k_m \in \mathbb{R}$ such that $x(T) = x^*$. We conclude the controllability of system (2) as follows.

Theorem 1. *System (2) is controllable if and only if there exists an $\tilde{L} \in sp\{L_1, L_2, \ldots, L_m\}$ such that $(\tilde{L}, B)$ is controllable and the topology corresponding to L_1 is leader-follower connected.*

Proof. Let $Q = \begin{pmatrix} 0_{m-1} & I_{m-1} \\ 0 & 0_{m-1}^T \end{pmatrix} \otimes I_n - e_m \otimes e_1^T \otimes k_1 L_1 - \cdots - e_m \otimes e_m^T \otimes k_m L_m$. System (2) is controllable if and only if there exist $k_1, k_2, \ldots, k_m \neq 0$ such that $(Q, e_m \otimes B)$ is controllable, which is to say, no left eigenvector of Q is orthogonal to $e_m \otimes B$. Suppose that Q has an eigenvalue λ with the corresponding left eigenvector $\beta^T = [\beta_1^T, \beta_2^T, \ldots, \beta_m^T]$, where $\beta_i \in \mathbb{R}^n, i = 1, 2, \ldots, m$. Then, one has

$$\begin{cases} -k_1 \beta_m^T L_1 = \lambda \beta_1^T \\ \beta_1^T - k_2 \beta_m^T L_2 = \lambda \beta_2^T \\ \vdots \\ \beta_{m-1}^T - k_m \beta_m^T L_m = \lambda \beta_m^T. \end{cases} \tag{3}$$

With simple transformation, one obtains

$$\begin{cases} \beta_{m-1}^T = k_m \beta_m^T L_m + \lambda \beta_m^T \\ \beta_{m-2}^T = k_{m-1} \beta_m^T L_{m-1} + \lambda k_m \beta_m^T L_m + \lambda^2 \beta_m^T \\ \vdots \\ \beta_1^T = k_2 \beta_m^T L_2 + \lambda k_3 \beta_m^T L_3 + \cdots + \lambda^{m-2} k_m \beta_m^T L_m + \lambda^{m-1} \beta_m^T. \end{cases} \tag{4}$$

Substituting (4) into the first equation in (3) yields

$$\begin{aligned} -\lambda^m \beta_m^T &= k_1 \beta_m^T L_1 + \cdots + \lambda^{m-2} k_{m-1} \beta_m^T L_{m-1} + \lambda^{m-1} k_m \beta_m^T L_m \\ &= \beta_m^T \left(k_1 L_1 + \lambda k_2 L_2 + \cdots + \lambda^{m-2} k_{m-1} L_{m-1} + \lambda^{m-1} k_m L_m \right). \end{aligned} \tag{5}$$

According to the PBH Test, system (2) is controllable if and only if, for all $\lambda \in \Lambda(Q)$, the corresponding $\beta_m(\lambda) \neq 0$ satisfies $\beta_m^T B \neq 0$. On one hand, if system (2) is controllable, it means β_m^T is a left eigenvector of $\tilde{L} = k_1 L_1 + \lambda k_2 L_2 + \cdots + \lambda^{m-2} k_{m-1} L_{m-1} + \lambda^{m-1} k_m L_m$ for some λ and $k_1, k_2, \ldots, k_m$ not all being 0. Obviously, $\tilde{L} \in sp\{L_1, L_2, \ldots, L_m\}$ and $(\tilde{L}, B)$

is controllable, especially, when $\lambda = 0$, the left eigenvectors of L_1 corresponding to the eigenvalue 0 are not orthogonal to B, ie, the topology corresponding to L_1 is leader-follower connected. On the other hand, if there exist $k_1, k_2, \dots, k_m$ not all being 0 such that $(\tilde{L}, B)$ is controllable, ie, (5) holds for $\lambda = 1$ (here, $\lambda = 1$ is not required to be an eigenvalue of Q), then, for almost all[‡] $k_1, \dots, k_m \in \mathbb{R}$, $(k_1L_1 + \cdots + k_mL_m, B)$ are controllable. Therefore, there exist $k_1, k_2, \dots, k_m \neq 0$ such that, for finite choices of $\lambda \neq 0$, $(\tilde{L}, B)$ are all controllable, where $\tilde{L} = k_1L_1 + \lambda k_2L_2 + \cdots + \lambda^{m-2}k_{m-1}L_{m-1} + \lambda^{m-1}k_mL_m$. Meanwhile, when $\lambda = 0$, since the left eigenvectors of L_1 corresponding to the eigenvalue 0 are not orthogonal to B, it means all the left eigenvectors of $\tilde{L}$, each of which denoted as $\beta_m^T(\lambda)$, satisfy $\beta_m^T(\lambda)B \neq 0$. Since β^T can be calculated by (5), it also holds that $\beta^T e_m \otimes B \neq 0$, ie, system (2) is controllable. □

Remark 1. The necessary and sufficient condition for system (2) to be controllable has two parts, one is the existence of $\tilde{L}$ making $(\tilde{L}, B)$ controllable, and the other is the topology corresponding to L_1 being leader-follower connected. The second condition is not contained in the first one and is therefore important. As one can see, although the feedback gains $k_1, \dots, k_n$ affect the eigenvalues of Q, they do not affect the 0 eigenvalue. When $\lambda = 0$, the only condition regarding the topologies on controllability falls on the (left) eigenvectors of the eigenvalue 0 of L_1. Physically, the request of the second condition originates from the model $\dot{x}_i^{(m)} = u_i$, in which $x_i^{(1)}$ is the information of the lowest order. Corresponding to the eigenvalue 0, only the information of the lowest order affects the controllability of the system. If some of the agents cannot receive the lowest-order information from any leader, the lowest-order states of the agents can not be completely controlled, which makes the entire system uncontrollable. However, for the nonzero eigenvalues, the information in different orders can jointly affect the states of each order, and one can make the system controllable by adjusting proper feedback gains.

Theorem 1 provides a necessary and sufficient condition for system (2) from the graphic perspective. The corresponding algebraic expression can be stated as follows. Without loss of generality, suppose that $L_1 = \mathrm{diag}\{L_{11}, L_{12}, \dots, L_{1r}\}$, where $L_{1i} \in \mathbb{R}^{n_i \times n_i}$, $i = 1, 2, \dots, r$, are the diagonal Laplacian blocks each of whom has only a single 0 eigenvalue, and $n_1 + n_2 + \cdots + n_r = n$.

Proposition 1. *System (2) is controllable if and only if there exists an $\tilde{L} \in sp\{L_1, L_2, \dots, L_m\}$ such that not any eigenvector of $\tilde{L}$ is orthogonal to B, and for each $i = 0, 1, 2, \dots, r-1$, there exists a j_i satisfying $n_1 + \cdots + n_i + 1 \leq j_i \leq n_1 + \cdots + n_{i+1}$ (especially, $1 \leq j_0 \leq n_1$, if $i = 0$) such that B contains e_{j_i} as a column.*

Corollary 1 (See the work of Wang et al[15]). *For system (2), if $L_1 = L_2 = \cdots = L_m \triangleq L$, then the system is controllable if and only if (L, B) is controllable.*

Proof. For the trivial case of $L_1 = L_2 = \cdots = L_m \triangleq L$, one obtains $sp\{L_1, L_2, \dots, L_m\} = \{kL | k \in \mathbb{R}\}$. On one hand, if (L, B) is controllable, the left eigenvectors of L are not orthogonal to B. Since $L \in \{kL | k \in \mathbb{R}\}$, it can be declared that system (2) is controllable according to Theorem 1. On the other hand, if the system is controllable, there exists a $k \neq 0$ such that, when $k_1 = k_2 = \cdots = k_m = k$, $(k(1 + \lambda + \cdots + \lambda^{m-1})L, B)$ is controllable, ie, (L, B) is controllable. □

Theorem 1 implies that the system can be controllable even if none of the interconnection topologies of each order information is controllable. This is because the aggregate effect of the information in different orders may lead to the overall controllability of the system, although the interconnection topologies at individual orders are not controllable. Corollary 1 is a main result in the work of Wang et al,[15] and Theorem 1 generalizes it to the heterogeneous-topology situation. However, searching for the linear combinations of Laplacian matrices to verify the controllability of system (2) seems to be difficult, and thus, we provide a direct method via the union graph.

Corollary 2.

1. *System (2) is controllable if the union graph of $\mathbb{G}_1, \dots, \mathbb{G}_m$ is controllable and $\mathbb{G}_1$ is leader-follower connected.*
2. *System (2) is uncontrollable if the union graph of $\mathbb{G}_1, \dots, \mathbb{G}_m$ is not leader-follower connected, or $\mathbb{G}_1$ is not leader-follower connected.*
3. *If randomly selecting $k_1, k_2, \dots, k_m \neq 0$ makes $(k_1L_1 + k_2L_2 + \cdots + k_mL_m, B)$ uncontrollable, then the probability of system (2) being controllable is 0.*

‡ The choices of $k_1, \dots, k_m$ have Lebesgue 1 in $\mathbb{R}^m$.

Proof.

1. If the union graph of $\mathbb{G}_1, \dots, \mathbb{G}_m$ is controllable, it means $(\tilde{L}, B)$ is controllable, where $\tilde{L} = L_1 + \cdots + L_m$. Since $\mathbb{G}_1$ is leader-follower connected, by Theorem 1, system is controllable.
2. If the union graph of $\mathbb{G}_1, \dots, \mathbb{G}_m$ is not leader-follower connected, for all $k_1, \dots, k_m \in \mathbb{R}$, the corresponding graph of $\tilde{L} = k_1 L_1 + \cdots + k_m L_m$ is not leader-follower connected, ie, $(\tilde{L}, B)$ is uncontrollable, which makes system (2) uncontrollable. If $\mathbb{G}_1$ is not leader-follower connected, by Theorem 1, system (2) is not controllable (even if the union graph may be controllable).
3. This can be obtained directly from the proof of Theorem 1.

□

The first two statements of Corollary 2 provide either (only) necessary or (only) sufficient condition for system controllability, but it is much more efficient compared to Theorem 1. The third one is technically neither necessary nor sufficient. However, the meaning of this statement is that, if system (2) is controllable, then one can randomly select $k_1, k_2, \dots, k_m$ and the controllability will not be broken.

4 | CONTROLLABILITY OF HETEROGENEOUS-DYNAMIC MULTIAGENT SYSTEMS

In this section, the nodes in heterogeneous-dynamic MASs are supposed to be single input for convenience. In fact, all the results obtained in this section can be generalized to multiple-input situations. Due to space limitation, the multiple-input case is not specially discussed.

4.1 | Controllability of heterogeneous-dynamic MASs with identical-dimension nodes

Firstly, we discuss the MASs consisting of identical-dimension nodes, ie, the states of the agents are supposed to have the same dimension and each agent has its specific kinetic dynamic. The heterogeneous-dynamic MASs are modeled as

$$\dot{x}_i = A_i x_i + b_i u_i, \; i = 1, 2, \dots, n,$$

where $x_i \in \mathbb{R}^m, A_i \in \mathbb{R}^{m \times m}, b_i \in \mathbb{R}^m$. To discuss controllability, we should firstly assume that $(A_i, b_i), i = 1, 2, \dots, n$ are controllable pairs. Otherwise, we are not able to control the states of each single node, let alone to control the whole system. Therefore, there exist invertible matrices $T_1, \dots, T_n$ such that

$$T_i \begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \vdots \\ \dot{x}_{im} \end{pmatrix} = \begin{pmatrix} 0_{m-1} & I_{m-1} \\ \alpha_{i1} & \tilde{\alpha}_i \end{pmatrix} T_i \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{im} \end{pmatrix} + \begin{pmatrix} 0_{m-1} \\ 1 \end{pmatrix} u_i, \tag{6}$$

where $\begin{pmatrix} 0_{m-1} & I_{m_i-1} \\ \alpha_{i1} & \tilde{\alpha}_i \end{pmatrix} = T_i A_i T_i^{-1}$ and $\begin{pmatrix} 0_{m-1} \\ 1 \end{pmatrix} = T_i b_i$. The consensus-based protocol is designed as

$$u_i = -\alpha_i^T x_i + \sum_{j \in N_i} a_{ij} (\beta^T T_j x_j - \beta^T T_i x_i) + u_{oi}, \tag{7}$$

where $\alpha_i = (\alpha_{i1}, \tilde{\alpha}_i^T)^T, \; \beta = (c_1, c_2, \dots, c_m)^T \in \mathbb{R}^m, i = 1, 2, \dots, n.$

Remark 2. Mathematically, heterogeneous topology MAS is a special case of heterogeneous dynamic MAS, where $A_i = \begin{pmatrix} 0_{m-1} & I_{m-1} \\ 0 & 0_{m-1}^T \end{pmatrix}$. However, heterogeneous topology MAS has its own specific physical significance and unique results on the Laplacian matrices, and it is therefore studied in an independent section. In the heterogeneous dynamic situation, the MAS has only one interaction topology, and the controllability of the whole system is closely related to the controllability of each agent, which will be shown in the following.

To facilitate discussion, denote $u_o = (u_{o1}, u_{o2}, \dots, u_{on})^T$, $\tilde{x}_i = T_i x_i$ and $\tilde{x}^{(k)} = (\tilde{x}_{1,k}, \tilde{x}_{2,k}, \dots, \tilde{x}_{n,k})^T$, $k = 1, 2, \dots m$. Then, one yields the compact form of system (6) under the protocol (7)

$$\begin{pmatrix} \dot{\tilde{x}}^{(1)} \\ \dot{\tilde{x}}^{(2)} \\ \vdots \\ \dot{\tilde{x}}^{(m-1)} \\ \dot{\tilde{x}}^{(m)} \end{pmatrix} = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -c_1 L & -c_2 L & -c_3 L & \cdots & -c_m L \end{pmatrix} \begin{pmatrix} \tilde{x}^{(1)} \\ \tilde{x}^{(2)} \\ \vdots \\ \tilde{x}^{(m-1)} \\ \tilde{x}^{(m)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ B \end{pmatrix} u_o. \tag{8}$$

Remark 3. This protocol is derived from the works of Wang et al[15] and Ji et al.[24] The protocol has two parts. The absolute part $-\alpha_i^T x_i$ aims to neutralize the nonzero elements in the last row of $T_i A_i T_i^{-1}$, and the effect of the relative part is to transmit the information from the leader(s) to the followers in a distributed manner. The two parts have their respective importance. Without the absolute part, controllability of the whole system may not be achieved; without the relative part, the information from the leader(s) cannot be transmitted to the followers, which also definitely causes network uncontrollability.

Lemma 1. *System (8) is controllable if and only if (L, B) is controllable and $c_m \neq 0$.*

Proof. Denote $\Omega = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -c_1 L & -c_2 L & -c_3 L & \cdots & -c_m L \end{pmatrix}$ and $\tilde{B} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ B \end{pmatrix}$. By the Rank Test, system (8) is controllable if and only if $(\tilde{B}, \Omega\tilde{B}, \dots, \Omega^{mn-1}\tilde{B})$ has full row rank. Considering that $\operatorname{rank}(\tilde{B}, \Omega\tilde{B}, \dots, \Omega^{mn-1}\tilde{B}) = \operatorname{rank}\begin{pmatrix} 0 & 0 & 0 & \cdots & * \\ 0 & 0 & 0 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & B & -c_m LB & \cdots & * \\ B & -c_m LB & -c_{m-1}LB + c_m^2 L^2 B & \cdots & * \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & 0 & 0 & \cdots & (-c_m)^{nm-n} L^{nm-n} B \\ 0 & 0 & 0 & \cdots & (-c_m)^{nm-n+1} L^{nm-n+1} B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & B & -c_m LB & \cdots & (-c_m)^{nm-2} L^{nm-2} B \\ B & -c_m LB & (-c_m)^2 L^2 B & \cdots & (-c_m)^{nm-1} L^{nm-1} B \end{pmatrix}$, one has that $(\tilde{B}, \Omega\tilde{B}, \dots, \Omega^{mn-1}\tilde{B})$ has full row rank if and only if $(B, -c_m LB, (-c_m)^2 L^2 B, \dots, (-c_m)^{nm-1} L^{nm-1} B)$ has full row rank, which means (L, B) is controllable and $c_m \neq 0$. □

Lemma 1 provides the necessary and sufficient condition for controllability of system (8). This condition implies that, for heterogeneous-dynamic MASs, controllability not only relies on the graph controllability of the interconnection topology but also on the gain vector β, especially the last parameter of β. The condition is based on the assumption that (A_i, b_i), $i = 1, 2, \dots, n$ are all controllable, and thus, we derive the entire necessary and sufficient condition for controllability of heterogeneous-dynamic MASs.

Theorem 2. *System (6) with the communication protocol (7) is controllable if and only if the following conditions hold simultaneously.*

(1) Matrix pairs (A_i, b_i), $i = 1, 2, \dots, n$, are controllable.
(2) The interconnection topology is controllable.
(3) In the protocol, $c_m \neq 0$.

Proof. Sufficiency: If (A_i, b_i), $i = 1, 2, \dots, n$, are controllable, then system (6) with the communication protocol (7) has the compact form (8). By Lemma 1, the system is controllable.
Necessity: If there exists a matrix pair (A_i, b_i) that is not controllable, then the state of agent v_i is not controllable, which leads to the overall uncontrollability of the system. Thus condition (1) is necessary, and the system has the compact form (8). By Lemma 1, conditions (2) and (3) also hold. □

4.2 | Controllability of heterogeneous-dynamic MASs with nonidentical-dimension nodes

For the most general case, we suppose that the states of each agent have their own dimensions. The heterogeneous-dynamic MASs are modeled as

$$\dot{x}_i = A_i x_i + b_i u_i, \ i = 1, 2, \ldots, n,$$

where $x_i \in \mathbb{R}^{m_i}, A_i \in \mathbb{R}^{m_i \times m_i}, b_i \in \mathbb{R}^{m_i}$. Similar to the last subsection, we assume that $(A_i, b_i), i = 1, 2, \ldots, n$, are controllable pairs, ie, there exist invertible matrices $T_1, \ldots, T_n$ such that

$$T_i \begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \vdots \\ \dot{x}_{im_i} \end{pmatrix} = \begin{pmatrix} 0_{m_i-1} & I_{m_i-1} \\ \alpha_{i1} & \tilde{\alpha}_i \end{pmatrix} T_i \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{im_i} \end{pmatrix} + \begin{pmatrix} 0_{m_i-1} \\ 1 \end{pmatrix} u_i, \tag{9}$$

where $\begin{pmatrix} 0_{m_i-1} & I_{m_i-1} \\ \alpha_{i1} & \tilde{\alpha}_i \end{pmatrix} = T_i A_i T_i^{-1}$ and $\begin{pmatrix} 0_{m_i-1} \\ 1 \end{pmatrix} = T_i b_i$. As one can see, when $i \neq j$, x_i and x_j may have different dimensions. To ensure successful interaction between agents, different feedback gains are required for each agent to unify the dimensions of information. Thus, the consensus-based protocol is designed as $u_i = -\alpha_i^T x_i + \sum_{j \in N_i} a_{ij} (\beta_j^T T_j x_j - \beta_i^T T_i x_i) + u_{oi}$, where $\alpha_i = (\alpha_{i1}, \tilde{\alpha}_i^T)^T, \beta_i \in \mathbb{R}^{m_i}, i = 1, 2, \ldots, n$.

The compact form of system (9) under the protocol is

$$\dot{\tilde{x}} = (\Omega - \tilde{L})\tilde{x} + \left(e_{\rho_1}, \ldots, e_{\rho_r}\right) u_o, \tag{10}$$

where $\rho_1 = \sum_{i=1}^{i_1} m_i, \ldots, \rho_r = \sum_{i=1}^{i_r} m_i, \Omega = (\omega_1^T, \omega_2^T, \ldots, \omega_n^T)^T, \tilde{L} = (\tilde{L}_1^T, \tilde{L}_2^T, \ldots, \tilde{L}_n^T)^T, \quad \omega_i = e_i^T \otimes \begin{pmatrix} 0_{m-1} & I_{m_i-1} \\ 0 & 0_{m-1}^T \end{pmatrix}, \tilde{L}_i = e_{m_i} \otimes (l_{i1}, \ldots, l_{in}) \otimes \beta_i^T, i = 1, 2, \ldots, n$.

For some specific $\beta_i \in \mathbb{R}^{m_i}, i = 1, 2, \ldots, n$, system (10) is said to be controllable if for any initial state $x(t_0) = x_0$ and target state x^*, there exist a finite time $T > t_0$, a control input u_o, such that $x(T) = x^*$. To investigate controllability of system (10), the simplest single-leader case is firstly considered. The Laplacian matrix of the interconnection topology is denoted to be L in the following.

Lemma 2. *For MAS (10), if there is only one leader (ie, $r = 1$) and the interconnection topology is connected, then for any selection of the single leader, there exists a $K = \mathrm{diag}(k_1, \ldots, k_n), k_i \in \mathbb{R}, i = 1, 2, \ldots, n$ such that (LK, B) is controllable.*

Proof. According to the PBH Test, (LK, B) is controllable if and only if for any $\mu \in \mathbb{C}$, $\mathrm{rank}(\mu I - LK, B) = n$. Since the interconnection topology is connected, the Laplacian matrix of the topology contains only a single 0 eigenvalue. Suppose that $LT = T\Lambda = T \begin{pmatrix} 0 & 0_{m-1}^T \\ 0_{m-1} & \bar{\Lambda} \end{pmatrix}$, where $\bar{\Lambda} = \mathrm{diag}\{\lambda_2, \ldots, \lambda_n\}$, $T = (\xi_1, \ldots, \xi_n)$. Denote $0 = \lambda_1$, obviously, $\lambda_2, \ldots, \lambda_n$ are all positive real numbers, and $\xi_1, \ldots, \xi_n$ are the (left) eigenvectors corresponding to $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$ satisfying $\xi_i^T \xi_i = 1$ and $\xi_i^T \xi_j = 0$ if $i \neq j$, which makes $\xi_1 = \frac{1}{\sqrt{n}} 1_n$. Considering that $(\mu I - LK, B) = (\mu TT^{-1} - T\Lambda T^{-1}K, TT^{-1}B) = T(\mu T^{-1} - \Lambda T^{-1}K, T^{-1}B)$, one obtains $\mathrm{rank}(\mu I - LK, B) = \mathrm{rank}(\mu T^{-1} - \Lambda T^{-1}K, T^{-1}B)$, where $T^{-1} = T^T$.

Denote $Q = (\mu T^{-1} - \Lambda T^{-1}K, T^{-1}B)$, since B contains only one column, it follows that $q_{ij} = \xi_{ij}(\mu - \lambda_i k_j)$ if $j \leq n$ and $q_{i,n+1} = \xi_{il}$ (here, ξ_{ij} represents the jth element of ξ_i). Next, we prove that there exist $k_1, k_2, \ldots, k_n$ such that, for any $\mu \in \mathbb{C}$, Q is of full row rank. Denote $\bar{Q} = (\mu(\xi_2, \ldots, \xi_n)^T - \bar{\Lambda}(\xi_2, \ldots, \xi_n)^T K)$, if there exist $i, \ldots, i+s$ such that $\lambda_i = \cdots = \lambda_{i+s} = \tilde{\lambda}$, then for any selection of $k_1 \neq k_2 \neq \cdots \neq k_n \neq 0$, one concludes that the ith, $\ldots$, $(i+s)$th rows of $\bar{Q}$ are linearly independent for all μ. Otherwise, obtain some $(s+1)$ linearly independent columns of $(\xi_i, \ldots, \xi_{i+s})^T$, denoted as $M = (\eta_1, \ldots, \eta_{s+1})$, and $M = (m_{hj})_{(s+1)\times(s+1)}$ is invertible. Consider $M_\mu = (m_{hj}(\mu - \tilde{\lambda} k_{i+j-1}))_{(s+1)\times(s+1)}$, M_μ is not of full rank if and only if $\mu = \tilde{\lambda} k_{i+j-1}$ for some $1 \leq j \leq s+1$. However, when $\mu = \tilde{\lambda} k_{i+j-1}$, consider the other columns of $(\xi_i, \ldots, \xi_{i+s})^T$, if each of them is linearly dependent with the columns of M_μ, since $(\xi_i, \ldots, \xi_{i+s})^T 1_n = 0$, the $(j-1)$th column is also linearly dependent with the columns of M_μ, which makes a contradiction. Therefore, for any $\mu = \tilde{\lambda} k_{i+j-1}$, there exists another column (denoted to be the i_0-th column) in $(\xi_i, \ldots, \xi_{i+s})^T$ such that, when the $(j-1)$th column of M is replaced by this column, M is also invertible. Selecting $k_{i_0} \neq k_{i+j-1}$ makes $\bar{Q}$ full row rank. This implies that, for any selection of $k_1 \neq k_2 \neq \cdots \neq k_n \neq 0$, the ith, $\ldots$, $(i+s)$th rows of $\bar{Q}$ are linearly independent for all μ.

Similarly, it is straightforward to prove that, for different eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_s}$, for any selection of $k_1, k_2, \ldots, k_n \neq 0$ satisfying $\lambda_p k_i \neq \lambda_q k_j, i, j = 1, 2, \ldots, n, i \neq j, p, q \in \{i_1, \ldots, i_s\}$, the $i_1, \ldots, i_s$-th rows of $\bar{Q}$ are linearly independent for all μ. Furthermore, one can prove that, for any Λ, almost all selections of $k_1, k_2, \ldots, k_n \neq 0$ make $\bar{Q}$ be always of full row rank for all μ. Since $\xi_1^T B \neq 0$, one concludes that there exists a $K = \mathrm{diag}(k_1, \ldots, k_n)$ such that $\mathrm{rank}(\mu I - LK, B) = n$ holds for all $\mu \in \mathbb{C}$, ie, (LK, B) is controllable. □

Corollary 3. *If the interconnection topology of MAS (10) is connected, then almost all selections of* $K = \mathrm{diag}(k_1, \ldots, k_n),\ k_i \in \mathbb{R},\ i = 1, 2, \ldots, n$ *make* (LK, B) *be controllable for any leader(s).*

Proof. According to the proof of Lemma 2, if the interconnection topology is connected, for any single leader, almost all selections of K could make (LK, B) be controllable. Since the number of leaders is finite, it can be also obtained that almost all selections of K could make (LK, B) be controllable for any selection of a single leader. Since adding more leaders will never break the controllability, one concludes that almost all selections of $K = \mathrm{diag}(k_1, \ldots, k_n)$ guarantee that, for any selection of leaders, (LK, B) is controllable. □

This corollary implies that, if the interconnection topology is connected, one can randomly select K to successfully make system (10) controllable with probability 1. The feedback gain selection proceeds as follows. First, randomly select $k_1, k_2, \ldots, k_n \in \mathbb{R}$, retaining four decimal places. Then, check $\mathrm{rank}(B, LKB, (LK)^2 B, \ldots, ((LK)^{n-1} B))$, if it equals n, the (LK, B) is controllable; otherwise, reselect $k_1, k_2, \ldots, k_n \in \mathbb{R}$ until $\mathrm{rank}(B, LKB, (LK)^2 B, \ldots, ((LK)^{n-1} B)) = n$. Finally, output "Selecting $K = \mathrm{diag}(k_1, \ldots, k_n)$ makes (LK, B) controllable." According to Corollary 3, the procedure will usually pass without circulation.

Example 1. Consider an interconnection topology depicted in Figure 1, if agent v_1 is the single leader, (L, e_1) is obviously uncontrollable. However, let $K = \mathrm{diag}(0.3575, 0.5155, 0.4863, -0.2155)$, one can see that (LK, e_1) is controllable, and the value assignment proceeds only once without circulation.

Definition 2. System (10) is said to be β-controllable if there exist $\beta_i \in \mathbb{R}^{m_i}$, $i = 1, 2, \ldots, n$, such that, if the communication protocol is $u_i = -\alpha_i^T x_i + \sum_{j \in N_i} a_{ij}(\beta_j^T T_j x_j - \beta_i^T T_i x_i) + u_{oi}$, then the system is controllable.

Remark 4. In the traditional study of multiagent controllability, the feedback gains for communication of each agent are usually all equal.[15,24] Considering of the heterogeneity of the agent dynamics, $\beta_1, \beta_2, \ldots, \beta_n$ may have different dimensions, which means they may not be all equal. From this viewpoint, β-controllability is a generalization of controllability due to the freely designable parameters, ie, system (10) is said to be β-controllable if one can find proper $\beta_1, \beta_2, \ldots, \beta_n$ to make the system controllable. In summary, controllability of system (10) is the specific case of β-controllability with the equally pre-given $\beta_1, \ldots, \beta_n$; while β-controllability of system (10) is the general case of controllability where designing $\beta_1, \beta_2, \ldots, \beta_n$ is a part of the approaches to achieve system controllability.

By Lemma 3, if system (10) is β-controllable, then it is controllable for almost all $\beta_1, \ldots, \beta_n$. In the following, we shall show the necessary and sufficient conditions for β-controllability.

Lemma 3. *If* $(A_i, b_i), i = 1, \ldots, n$, *are controllable, then system (10) is* β*-controllable if and only if the interconnection topology is leader-follower connected.*

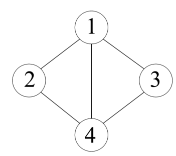

FIGURE 1 A graph with four nodes

Proof. Suppose that $\Omega - \tilde{L}$ has an eigenvalue λ with the corresponding left eigenvector $\theta^T = [\theta_1^T, \theta_2^T, \dots, \theta_m^T]$, where $\theta_i = (\theta_{i1}, \theta_{i2}, \dots, \theta_{im_i-1}, \theta_{im_i})^T \in \mathbb{R}^{m_i}, i = 1, 2, \dots, m$. Specially, denote $\theta_{im_i} = \eta_i$ and $\eta = (\eta_1, \dots, \eta_n)^T$, one obtains that $\theta^T(\Omega - \tilde{L}) = \lambda\theta^T$. Furthermore, $\lambda\theta_{i,m_i-j} = \theta_{i,m_i-j-1} + \eta^T\beta_{i,m_i-j}l_i$ yields

$$\begin{cases} \theta_{i,m_i-1} = \lambda\eta_i - \eta^T\beta_{i,m_i}l_i \\ \theta_{i,m_i-2} = \lambda^2\eta_i - \lambda\eta^T\beta_{i,m_i}l_i^T - \eta^T\beta_{i,m_i-1}l_i^T \\ \quad\vdots \\ \theta_{i,1} \quad = \lambda^{m_i-1}\eta_i - \lambda^{m_i-2}\eta^T\beta_{i,m_i}l_i^T - \cdots - \eta^T\beta_{i,2}l_i^T \\ \lambda\theta_{i,1} \quad = \eta^T\beta_{11}l_1. \end{cases}$$

Therefore, $\eta^T(\rho_1 l_1, \rho_2 l_2, \dots, \rho_n l_n) = \lambda^{m_j}\eta^T, j = 1, 2, \dots, n$, where $\rho_i = \beta_{i1} + \lambda\beta_{i2} + \cdots + \lambda^{m_i-1}\beta_{im_i}, i = 1, 2, \dots, n$. This means system (10) is controllable if and only if there exist $\rho_1, \dots, \rho_n$ such that $(L \cdot \mathrm{diag}(\rho_1, \dots, \rho_n), B)$ is controllable.
Sufficiency: First prove that, if the interconnection topology of an MAS is leader-follower connected, then there exists a $K = \mathrm{diag}(k_1, \dots, k_n)$ such that (LK, B) is controllable.
If the interconnection topology is connected, this is exactly Lemma 2. If the interconnection topology contains $r > 1$ connected components, then one can properly reorder the identifiers of the agents such that $L = \mathrm{diag}(L_1, \dots, L_r), B = \mathrm{diag}(B_1, \dots, B_r)$. Referring to the blocks of L and dividing K into blocks $K = \mathrm{diag}(K_1, \dots, K_r)$, one obtains that (LK, B) is controllable if and only if $(\mathrm{diag}(L_1K_1, \dots, L_rK_r), \mathrm{diag}(B_1, \dots, B_r))$ is controllable, which holds if and only if $(L_iK_i, B_i), i = 1, 2, \dots, r$, are controllable. However, according to Lemma 2, since L_i corresponds to a connected subgraph and $B_i \neq 0$, there exists a K_i such that (L_iK_i, B_i) is controllable, $i = 1, 2, \dots, r$. Selecting K_i making (L_iK_i, B_i) be controllable ensures that (LK, B) is controllable.

If the interconnection topology of system (10) is leader-follower connected, there exists a K such that (LK, B) is controllable. Selecting $\beta_{i1} = k_i$ and $\beta_{ij} = 0, i = 1, \dots, n, j = 2, \dots, m_i$ makes $(L \cdot \mathrm{diag}(\rho_1, \dots, \rho_n), B)$ be controllable.

Necessity: If there exist $\rho_1, \dots, \rho_n$ such that $(L \cdot \mathrm{diag}(\rho_1, \dots, \rho_n), B)$ is controllable, it concludes that L corresponds to a leader-follower connected graph. Otherwise, $L = \mathrm{diag}(L_1, \dots, L_r)$, $L_i \in \mathbb{R}^{n_i}$, $i = 1, 2, \dots, r$, $n_1 + \cdots + n_r = n$. Without loss of generality, suppose that the rth connected component of the interconnection topology contains no leader, ie, the last r rows of B are all 0. Apparently, for any K, LK contains a left eigenvector $\zeta^T = (0_{n_1}^T, \dots, 0_{n_{r-1}}^T, \tilde{\zeta}^T)$, which satisfies $\zeta^T B = 0$. This contradicts that there exist $\rho_1, \dots, \rho_n$ such that $(L \cdot \mathrm{diag}(\rho_1, \dots, \rho_n), B)$ is controllable.□

Theorem 3. *System (10) is β-controllable if and only if the following two conditions are satisfied simultaneously.*

(i) Matrix pairs $(A_i, b_i), i = 1, 2, \dots, n$, are all controllable.
(ii) The interconnection topology of the system is leader-follower connected.

Proof. Sufficiency: Since matrix pairs $(A_i, b_i), i = 1, 2, \dots, n$ are all controllable, by Lemma 3, the interconnection topology being leader-follower connected makes system (10) controllable.

Necessity: If (A_i, b_i) is not controllable for some i, the state of agent i is uncontrollable, which makes the whole system uncontrollable. Therefore, $(A_i, b_i), i = 1, \dots, n$ must be controllable. According to Lemma 3, since system (10) is controllable, the interconnection topology must be leader-follower connected. □

Theorem 3 shows that system (10) is β-controllable if and only if the agents' dynamics are all controllable and the interconnection topology is leader-follower connected. This means system (10) can be controllable even if (L, B) is not controllable, ie, the graph controllability is not absolutely necessary, the controllability can be achieved via adjusting the gain parameters in the communication protocol. Theorem 3 also has an equivalent algebraic expression. Without loss of generality, suppose that $L_1 = \mathrm{diag}\{L_{11}, L_{12}, \dots, L_{1r}\}$ as in Section 3.

Proposition 2. *System (10) is β-controllable if and only if for each $k = 1, 2, \dots, n$, not any eigenvector of $\tilde{A}_k$ is orthogonal to b_k, and for each $i = 0, 1, 2, \dots, r-1$, there exists a j_i satisfying $n_1 + \cdots + n_i + 1 \le j_i \le n_1 = \cdots + n_{i+1}$ (especially, $1 \le j_0 \le n_1$, if $i = 0$) such that B contains e_{j_i} as a column.*

The following corollary has appeared in the work of Wang et al,[15] which can be directly obtained from Theorem 3 as a special case.

Corollary 4. *For system (10), when* $A_1 = A_2 = \cdots = A_n \triangleq A \in \mathbb{R}^{m\times m}$*, and* $b_1 = b_2 = \cdots = b_n \triangleq b \in \mathbb{R}^m$*,*

(1) The system is β*-controllable if and only if* (A, b) *is controllable and the interconnection topology is leader-follower connected.*
(2) If it is required that $\beta_1 = \cdots = \beta_n = (c_1, c_2, \ldots, c_m)^T$*, then the system is controllable if and only if* (A, b) *is controllable and* (L, B) *is controllable and* $c_m \neq 0$*.*

Proof. If $A_1 = A_2 = \cdots = A_n \triangleq A$, and $b_1 = b_2 = \cdots = b_n \triangleq b$, then $(A_i, b_i), i = 1, 2, \ldots, n$ all being controllable is equivalent to (A, b) being controllable. By Theorem 3, the system is controllable if and only if (A, b) is controllable and the interconnection topology is leader-follower connected; if $\beta_1 = \cdots = \beta_n$, this is another expression of Theorem 2. □

Remark 5. Counterintuitively, for the simple case that the agents are all of the first-order dynamic, leader-follower connection is only a necessary condition, but it is both necessary and sufficient for β-controllability of the general case, ie, agents of heterogenous generic-linear dynamics. The reason of this difference relies in the feedback gains $\beta_1, \ldots, \beta_n$. Actually, the effect of $\beta_1, \ldots, \beta_n$ is to match the state information of each agent. Degenerate it to the simple case $\dot{x} = u$, if one gives each agent an independent feedback gain k_i, respectively, the protocol turns to be $u_i = \sum_{j\in N_i} a_{ij}(k_j x_j - k_i x_i) + u_o$, and the compact form is summarized as $\dot{x} = -LKx + Bu_o$. By Lemma 3, the system (with first-order dynamic agents) is controllable (for almost all selections of K) if and only if the interconnection topology is leader-follower connected.

Corollaries 1 and 4 are the main results obtained in the work of Wang et al,[15] which appear to be the special homogeneous cases of this paper. Actually, this work is a generalization in the work of Wang et al[15] in both modeling and mathematical derivation. Compared to the existing results, there are some difficulties in dealing with heterogeneous MASs. For example, there is no common eigenvector of the Laplacian matrices for high-order MASs and communication difficulty in generic-linear dynamic MASs due to the different dimensions of the nodes. Besides, proving (LK, B) being controllable in Lemma 2 is much harder than proving structural controllability of the MAS. To deal with the difficulties, this paper brings in the feedback gains for the agents in heterogeneous MASs to ensure the agents successfully communicate with others, which also benefit the whole system achieving controllability. Moreover, the properties of the eigenvectors of the Laplacian matrices are revealed more profoundly.

5 | EXAMPLES

In this section, two examples are provided to illustrate the main results in this paper.

Example 2. Consider system (2) with five third-order agents, ie, $\dot{x}_i^{(3)} = u_i, i = 1, 2, 3, 4, 5$. Suppose that the interconnection topologies of the first-order, second-order, and third-order information are depicted in Figures 2, 3, and 4, respectively. Obviously, if agent 1 is the single leader, none of the graphs is controllable, even if Figure 2 is leader-follower connected. However, the union graph of them is given in Figure 5, which is controllable. Actually, if one selects $k_1 = k_2 = k_3 = 1$, the high-order system becomes controllable.

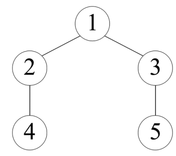

FIGURE 2 The first-order information topology of system (2)

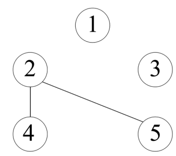

FIGURE 3 The second-order information topology of system (2)

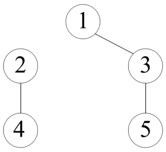

FIGURE 4 The third-order information topology of system (2)

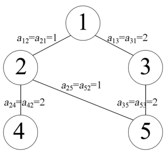

FIGURE 5 The union graph of Figures 2, 3, and 4

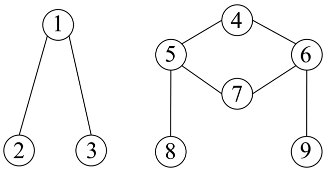

FIGURE 6 A graph with nine nodes

Example 3. For system (10) with heterogeneous dynamic agents, the interconnection topology is depicted in Figure 6. Suppose that the dynamics of the agents are $\dot{x}_1 = u_1$, $\dot{x}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u_2$, $\dot{x}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} u_3$, $\dot{x}_4 = \begin{pmatrix} 0 & 1 & 1.5 \\ 2 & 0 & -0.5 \\ 0 & -1 & 1 \end{pmatrix} x_4 + \begin{pmatrix} 0.5 \\ 0 \\ 1.5 \end{pmatrix} u_4$, $\dot{x}_5 = -x_5 + u_5$, $\dot{x}_6 = \begin{pmatrix} 0 & -0.5 & 1 \\ 1 & -1 & 2 \\ 0 & -1 & 1 \end{pmatrix} x_6 + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} u_6$, $\dot{x}_7 = \begin{pmatrix} 1 & 0 & 1.5 & 0 & 1 \\ 0.5 & 0 & 2 & -1 & 0 \\ 0 & -2 & 0.5 & 1 & 0 \\ 2 & 0 & 1 & 2 & 0.5 \\ 1 & 0 & -1.5 & 0.5 & 0 \end{pmatrix} x_7 + \begin{pmatrix} 1 \\ 0 \\ 0.5 \\ 0 \\ 2 \end{pmatrix} u_7$, $\dot{x}_8 = \begin{pmatrix} 1 & 2 & -0.5 & 2 \\ -1 & 0 & 1.5 & 2 \\ -2 & 1 & 0.5 & 1 \\ 1.5 & 0.5 & 2 & -1 \end{pmatrix} x_8 + \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} u_8$, $\dot{x}_9 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x_9 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} u_9$, respectively. Apparently, if agents 1 and 4 are selected as the leaders, the interconnection topology is leader-follower connected, but the matrix pair

$$\left(\begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 3 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

is not controllable. However, if the feedback gains are (randomly) selected as $\beta_1 = 0.6324, \beta_2 = (0.8147, 0.9058)^T, \beta_3 = (0.9575, 0.9649, 0.1576)^T \beta_4 = (0.1270, 0.9134, 0.8003)^T, \beta_5 = 0.2348, \beta_6 = (0.0975, 0.2785, 05469)^T \beta_7 = (0.4218, 0.0357, 0.7431, 0.0318, 0.6948)^T, \beta_8 = (0.7922, 0.9340, 0.6555, 0.0462)^T, \beta_9 = (0.9595, 0.6787, 0.1712, 0.0971)^T$, the system becomes controllable. Therefore, the effect of $\beta_1, \ldots, \beta_9$ has two aspects. One is to make the agents able to communicate with each other, and the other is actually to contribute to achieving controllability.

Remark 6. The three examples (including Example 1 in Section 4.2) have their respective roles. Example 1 demonstrates that the controllability can be achieved via randomly assigning K for only one selection step without circulation. Example 2 provides a simulation of heterogeneous-topology MAS to illustrate Theorem 1; Example 3 provides a simulation of heterogeneous-dynamic MAS to illustrate Theorem 3.

6 | CONCLUSIONS AND FUTURE WORK

In conclusion, this paper considered controllability of heterogeneous MASs. The main results in this paper provided necessary and sufficient conditions on controllability of heterogeneous-topology systems and heterogeneous-dynamic systems. For an MAS with high-order dynamical agents, it is controllable if and only if there exists a Laplacian matrix, which is a linear combination of the Laplacian matrices of each order information, whose corresponding topology is controllable, and the topology corresponding to the first-order information should be leader-follower connected. For an MAS with same-dimension generic-linear dynamical agents, the necessary and sufficient condition for system controllability is that each agent contains a controllable dynamic and the interconnection topology of the system is controllable, while the last parameter of the state feedback vector must not be 0. For an MAS with different-dimension generic-linear dynamical agents, the concept of β-controllability is proposed and the necessary and sufficient condition for β-controllability is that, except for each agent that contains a controllable dynamic, the interconnection topology only needs to be leader-follower connected. In our future work, controllability of heterogeneous MASs will be discussed with directed interconnection topologies, especially for the heterogeneous-dynamic systems.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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