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RESEARCH ARTICLE

Controllability of heterogeneous multi-agent systems under directed and weighted topology

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This paper considers the controllability problem for both continuous- and discrete-time linear heterogeneous multi-agent systems with directed and weighted communication topology. First, two kinds of neighbor-based control protocols based on the distributed protocol of first-order and second-order multi-agent systems are proposed, under which it is shown that a heterogeneous multi-agent system is controllable if the underlying communication topology is controllable. Then, under special leader selection, the result shows that the controllability of a heterogeneous multi-agent system is solely decided by its communication topology graph. Furthermore, some necessary and/or sufficient conditions are derived for controllability of communication topology from algebraic and graphical perspectives. Finally, simulation examples are presented to demonstrate the effectiveness of the theoretical results.

Keywords: controllability; local interaction; leader-follower framework; heterogeneous multi-agent systems

1 Introduction

In the past decade, multi-agent distributed coordination has been intensively studied, and many results have been obtained and applied in science and engineering areas, such as formation control of unmanned air vehicles (UAVs) and unmanned ground vehicles (UGVs), data fusion of sensors, attitude alignment of satellite clusters, and multiple mobile robotic systems Olfati-Saber and Murray (2004), Xie and Wang (2007), Wang and Xiao (2007, 2010), Guan et al. (2013, 2014), Zheng et al. (2011), Zheng and Wang (2012a,b, 2014) etc.

It is known that controllability is a core concept of modern control, playing a fundamental role in analysis and synthesis of linear control systems. For multi-agent systems, the controllability problem aims at driving follower agents to achieve any configurations from any initial states only through controlling a few leaders externally. The study of the controllability of multi-agent systems remains to be a challenging task due to the fact that the system is decentralized rather than centralized, as well as that there are more factors, such as the design of protocols, leaders selection, interaction topologies, and system dimensions need to be considered. The concept of controllability of multi-agent systems was first formulated by Tanner, who established an algebraic necessary and sufficient condition in terms of eigenvalues and eigenvectors of system matrix corresponding to the follower nodes Tanner (2004). However, it is also pointed out that the lack of a graph-theoretic characterization of controllability prevents controllable topologies from being built. This motivated subsequent analysis of controllability from a graph-theoretic

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point of view. For instance, a necessary condition on controllability was proposed in terms of induced subgraphs, which involves a division of the whole graph Ji et al. (2008). The controllable subspaces were studied by employing weight-balanced partition Luo and Hong (2012). Equitable partition and relaxed equitable partition were proposed to check the controllability and observability of multi-agent systems O'Clery et al. (2013), Zhang et al. (2014). Uncontrollable topologies and graph-theoretic properties were given in Ji et al. (2009). In addition, the controllability of some special graphs was also investigated, such as paths and cycles Parlangeli and Notarstefano (2012), grid graphs Notarstefano and Parlangeli (2013). Specially, tree graph was studied in Ji et al. (2012), and necessary and sufficient conditions were proposed in terms of downer branch and subgraphs to characterize controllability. There are other relevant results, for instance, results on controllability were derived with respect to switching topology Liu et al. (2008, 2012), time-delay Ji et al. (2010), protocol design Ji et al. (2015), coupling weights Wang and Jiang (2009), leaders selection Commault and Dion (2013), structural controllability Zamani and Lin (2009), Liu et al. (2012), and complex network Liu et al. (2011).

Most aforementioned results mainly focus on homogeneous multi-agent systems, that is, all the agents have the identical dynamics or all the network nodes have the same properties. However, heterogeneity is common in multi-agent systems, the dynamics of the agents coupled with each others is not the same because of various restrictions or to reach the goals with lowest costs in the practical systems. For example, taking into account the dynamic environments and uncertainty external to the multi-robot system itself, heterogeneous systems with robots in different shapes and abilities are more applicable than the homogeneous systems in real world. In modern military field, the formation control of UAVs and UGVs can carry out complex tasks (such as battlefield surveillance, cooperative search) with only minimal command effort from humans. (see e.g. Zheng et al. (2011), Zheng and Wang (2012a,b, 2014)). The objective of the paper is to study controllability for heterogeneous multi-agent systems, in which the agents are governed by first-order and second-order integrators, respectively. In the controllability analysis of heterogeneous systems, the difficulty comes partly from how to cope with the different agent dynamics and control protocol. To overcome the difficulty, we developed an approach which consists of a interaction topology partition and a decomposition of the dynamics of the leaders and followers. With the partitioned interaction topology, the controllability of heterogeneous systems can be reduced to that of a pair of submatrices of Laplacian matrix by implementing decomposition of the agents dynamics. Since the matrix pair of Laplacian matrix is entirely decided by the interconnection topology, it is further exhibited that with the specified leaders, the controllability of heterogeneous systems is decided by the communication graph. The main contributions of this article are summarised as follows. (i) This is the first time that the controllability problem is studied for both continuous- and discrete-time linear heterogeneous multi-agent systems. Two kinds of models for controllability of heterogeneous multi-agent systems are proposed. As far as we know, these models are put forward here for the first time. (ii) We show that a heterogeneous multi-agent system is controllable if the graph of the underlying communication topology is controllable. Under special leaders selection, the result further shows that the controllability of a heterogeneous multi-agent system is solely decided by its interaction topology. (iii) We establish some necessary and/or sufficient conditions for controllability of communication topology from algebraic and graph theoretical perspectives, respectively.

The paper is organized as follows. In Section 2, the controllability problem is formulated, as well as graph preliminaries are included. Sections 3 and 4 present the main results of the paper. In Section 5, some numerical examples are presented. Finally, the concluding remarks are summarized in Section 6.

Notation: Throughout this paper, the following notations are used. X/Y denotes the difference of the sets X and Y, that is the set of those elements of X which do not belong to Y. $|X|$ represents the size of a set X. Let $\mathcal{I}_m$ and $\mathcal{I}_n/\mathcal{I}_m$ represent, respectively, the set of integers $\{1, 2, \ldots, m\}$ and $\{m+1, m+2, \ldots, n\}$. Let $\mathbf{0}(\mathbf{0}_{m\times n})$ denote an all-zero vector or matrix with compatible dimension (dimension $m \times n$). I_n denotes the $n \times n$ identity matrix, and

$diag\{a_1, a_2, \ldots, a_n\}$ represents the $n \times n$ diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_n$. Let $\mathbf{1}_n$ denote the all-1 vector with dimension n. $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and complex numbers, respectively. A permutation matrix $P \in \mathbb{R}^{n\times n}$ is a 0-1 matrix with a single nonzero element in each row and column.

2 Preliminaries

2.1 *Graph preliminaries*

A directed weighted graph is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represent, respectively, the vertex set and the edge set; $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix with $a_{ij} > 0$ representing the reliability of the interaction from vertex j to vertex i. An edge of $\mathcal{G}$ is denoted by $e_{ij} = (j, i)$, where j is called the parent vertex of i and i the child vertex of j. In this paper, we assume that there are no *self-loops*, i.e. $e_{ii} \notin \mathcal{E}$. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. For two graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$, we call $\mathcal{G}'$ a subgraph of $\mathcal{G}$, denoted by $\mathcal{G}' \subseteq \mathcal{G}$, if $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}'$. A subgraph $\mathcal{G}'$ is said to be induced from $\mathcal{G}$ if it is obtained by deleting a subset of nodes and all the edges connecting to those nodes. A directed path in a directed graph $\mathcal{G}$ is a sequence $i_1, \ldots, i_k$ of distinct vertices with $(i_s, i_{s+1}) \in \mathcal{E}$, for $s = 1, \ldots, k-1$. The vertex i_1 in the above path is the origin or root and the vertex i_k is called the end of this path. $\mathcal{G}$ is called strongly connected if there is a directed path that starts from i and ends at j between every pair of distinct vertices i, j in $\mathcal{G}$. A strongly connected component (SCC) of a directed graph is an induced subgraph that is maximal, and subject to being strongly connected. An independent strongly connected component (iSCC) of $\mathcal{G}$ is an induced subgraph $\mathcal{G}'$ such that it is a SCC and satisfies that $(j, i) \notin \mathcal{E}$ for any $j \in \mathcal{V}/\mathcal{V}'$ and $i \in \mathcal{V}'$. For a directed weighted graph $\mathcal{G}$, the in-degree of node i is defined as: $\deg_{in}(i) = \sum_{j\in\mathcal{N}_i} a_{ij}$. The degree matrix of $\mathcal{G}$ is a diagonal matrix defined as: $\triangle = [\triangle_{ij}] \in \mathbb{R}^{n\times n}$, where $\triangle_{ij} = \deg_{in}(i)$ for $i = j$, otherwise, $\triangle_{ij} = 0$. The Laplacian matrix $L(\mathcal{G}) = [l_{ij}] \in \mathbb{R}^{n\times n}$ of a graph $\mathcal{G}$, abbreviated as L, is defined by $l_{ij} = -a_{ij}$ if $i \neq j$ and $l_{ij} = \sum_{j\in\mathcal{N}_i} a_{ij}$ if $i = j$. It is obvious that $L = \triangle - \mathcal{A}$.

2.2 *Problem formulation*

The controllability problem is studied under leader-follower framework, where some agents are taken to play leaders' role and others are followers. The followers obey the neighbor-based law, but the leaders are free of such a constrain and are allowed to pick its control input arbitrarily. In this way the states of leader agents are treated as inputs, which can be utilized to control those of the follower agents. Given leader agents, the Laplacian of the network can be partitioned as $L = \begin{bmatrix} L_f & L_{fl} \\ L_{lf} & L_l \end{bmatrix}$, where L_f, L_l, L_{fl} and L_{lf} correspond to, respectively, the indices of followers, leaders, from leaders to the followers and from followers to the leaders. In what follows, we denote by $F \triangleq L_f, R \triangleq L_{fl}$. With respect to the selected leader agents, the matrix pair (F, R) is entirely decided by the interconnection topology graph.

Definition 2.1: A multi-agent system is said to be controllable if followers can be steered to proper positions to make up any desirable configuration, in a finite time, by regulating the moving of leaders.

Suppose that $\mathcal{G}_f$ and $\mathcal{G}_l$ represent the follower and leader subgraphs of $\mathcal{G}$, which are induced respectively by the follower and leader node sets. Let $\mathcal{G}_f^{c_1}, \ldots, \mathcal{G}_f^{c_\gamma}$ stand for the γ iSCCs in $\mathcal{G}_f$. The following definition is introduced to be a prerequisite for the investigation of controllability.

Definition 2.2: A directed graph $\mathcal{G}$ is said to be leader-follower connected if for each iSCC $\mathcal{G}_f^{c_i}, i = 1, \ldots, \gamma$, in the follower subgraph $\mathcal{G}_f$, there exists a leader in the leader subgraph $\mathcal{G}_l$, so that there is an edge from this leader to a node in $\mathcal{G}_f^{c_i}$.

An example of leader-follower connected digraph is displayed in Figure 1.

3 Controllability of heterogeneous multi-agent systems

For the simplicity of presentation, we assume that the states of all agents are in a one-dimensional space. However, the results of this paper are valid for the high-dimensional by the introduction of Kronecker product.

3.1 *Continuous-time multi-agent systems*

In this subsection, we consider the controllability of continuous-time heterogeneous multi-agent systems. Suppose that the system consists of first-order (single) and second-order (double) integrator agents. The number of agents is $n+l$, labeled from 1 to $n+l$, where the number of first-order integrator agents is $m(m<n+l)$. Each agent has the dynamics as follows:

$$\begin{cases} \dot{x}_i = u_i, & i \in \mathcal{I}_m, \\ \dot{x}_i = v_i, \dot{v}_i = u_i, & i \in \mathcal{I}_{n+l}/\mathcal{I}_m, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}, v_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the position-like, velocity-like and control input, respectively, of agent i. The interactions among agents are realized through the following protocol:

$$u_i = \begin{cases} \sum\limits_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i), & i \in \mathcal{I}_m, \\ \sum\limits_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i) + k_1 \sum\limits_{j\in\mathcal{N}_i} a_{ij}(v_j - v_i), & i \in \mathcal{I}_{n+l}/\mathcal{I}_m, \end{cases} \tag{2}$$

or

$$u_i = \begin{cases} \sum\limits_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i), & i \in \mathcal{I}_m, \\ \sum\limits_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i) + k_1 v_i, & i \in \mathcal{I}_{n+l}/\mathcal{I}_m, \end{cases} \tag{3}$$

where a_{ij} is the weight of the edge from agent j to agent i, k_1 is a nonzero feedback gain.

Remark 1 : Protocols (2) and (3) are put forward for the first time by Zheng et. al to solve consensus problem of heterogeneous multi-agent system (1) (see e.g. Zheng et al. (2011), Zheng and Wang (2012a,b)). In this paper, we employ them to investigate the controllability of system (1). In (2), the neighbor-based law between second-order agents is with the feedbacks of relative velocities, but the feedback of absolute velocity is used in (3). This difference will make two dynamic equations of followers with second-order integrator have different forms of expression (the details can be found in (5) and (6)), while it will not affect the results of the controllability of heterogeneous multi-agent systems.

Assume there are $l(= l_1 + l_2 \geq 1)$ leaders (l_1 and l_2 are, respectively, the number of leaders with first-order and second-order integrator) and n followers in system (1). The movements of these leaders are dominated by external control inputs $u_{\text{ext}} = [u_1^T, u_2^T]^T \in \mathbb{R}^{l_1+l_2}$, $u_1 \in \mathbb{R}^{l_1}$ and $u_2 \in \mathbb{R}^{l_2}$, which can drive the states of the leaders to arbitrary values.

Let $\mathcal{G}_1$ be a subgraph of $\mathcal{G}$, which consists of all the follower agents with first-order integrator dynamics. Similarly, let $\mathcal{G}_2$ be a subgraph of $\mathcal{G}$, which consists of all the follower agents with second-order integrator dynamics, $\mathcal{G}_3$ be a subgraph of $\mathcal{G}$, which consists of all the leader agents with first-order integrator dynamics, and $\mathcal{G}_4$ be a subgraph of $\mathcal{G}$, which consists of all the leader agents with second-order integrator dynamics. Then the associated Laplacian matrix of $\mathcal{G}$ can be conformably partitioned as

$$L = \begin{bmatrix} L_f & L_{fl} \\ L_{lf} & L_l \end{bmatrix} = \begin{bmatrix} F_1 & R_{12} & R_{13} & R_{14} \\ R_{21} & F_2 & R_{23} & R_{24} \\ R_{31} & R_{32} & F_3 & R_{34} \\ R_{41} & R_{42} & R_{43} & F_4 \end{bmatrix}, \tag{4}$$

where F_i corresponds to $\mathcal{G}_i$, $i = 1, 2, 3, 4$. $F \triangleq L_f = \begin{bmatrix} F_1 & R_{12} \\ R_{21} & F_2 \end{bmatrix}$ corresponds to the indices of the followers, and $R \triangleq L_{fl} = \begin{bmatrix} R_{13} & R_{14} \\ R_{23} & R_{24} \end{bmatrix}$ corresponds to the indices from leaders to the followers.

According to the partition of leaders and followers, the system (1) under protocol (2) can be written as

$$\begin{bmatrix} \dot{y}_1 \\ \dot{z}_1 \\ \dot{y}_2 \\ \dot{z}_2 \\ \dot{v}_{y_2} \\ \dot{v}_{z_2} \end{bmatrix} = \begin{bmatrix} -F_1 & -R_{13} & -R_{12} & -R_{14} & \mathbf{0} & \mathbf{0} \\ -R_{31} & -F_3 & -R_{32} & -R_{42} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I^{(2)} \\ -R_{21} & -R_{23} & -F_2 & -R_{24} & -k_1 F_2 & -k_1 R_{24} \\ -R_{41} & -R_{43} & -R_{42} & -F_4 & -k_1 R_{42} & -k_1 F_4 \end{bmatrix} \begin{bmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \\ v_{y_2} \\ v_{z_2} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ u_1 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ u_2 \end{bmatrix},$$

where $y_1 = [x_1 \ldots x_{m-l_1}]^T$ and $z_1 = [x_{m-l_1+1} \ldots x_m]^T$ are, respectively, the stacked vector of the first-order integrator followers' and leaders' positions. $y_2 = [x_{m+1} \ldots x_{n+l_1}]^T$, $z_2 = [x_{n+l_1+1} \ldots x_{n+l}]^T$, $v_{y_2} = [v_{m+1} \ldots v_{n+l_1}]^T$ and $v_{z_2} = [v_{n+l_1+1} \ldots v_{n+l}]^T$ are, respectively, the stacked vector of the second-order integrator followers' positions, leaders' positions, followers' velocities and leaders' velocities. Consequently, we can describe the dynamics of the followers to be the following system

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{v}_{y_2} \end{bmatrix} = \begin{bmatrix} -F_1 & -R_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I^{(1)} \\ -R_{21} & -F_2 & -k_1 F_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ v_{y_2} \end{bmatrix} + \begin{bmatrix} -R_{13} & -R_{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -R_{23} & -R_{24} & -k_1 R_{24} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ v_{z_2} \end{bmatrix}, \tag{5}$$

where the control inputs are the positions and velocities of the leaders.

If protocol (3) is employed, the closed-loop system of (1) is represented by

$$\begin{bmatrix} \dot{y}_1 \\ \dot{z}_1 \\ \dot{y}_2 \\ \dot{z}_2 \\ \dot{v}_{y_2} \\ \dot{v}_{z_2} \end{bmatrix} = \begin{bmatrix} -F_1 & -R_{13} & -R_{12} & -R_{14} & \mathbf{0} & \mathbf{0} \\ -R_{31} & -F_3 & -R_{32} & -R_{42} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I^{(2)} \\ -R_{21} & -R_{23} & -F_2 & -R_{24} & k_1 I^{(1)} & \mathbf{0} \\ -R_{41} & -R_{43} & -R_{42} & -F_4 & \mathbf{0} & k_1 I^{(2)} \end{bmatrix} \begin{bmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \\ v_{y_2} \\ v_{z_2} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ u_1 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ u_2 \end{bmatrix},$$

where $y_1, z_1, y_2, z_2, v_{y_2}$ and v_{z_2} are defined as above. Then the dynamics of the followers can be

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[1] $I^{(1)} \in \mathbb{R}^{(n-m+l_1)\times(n-m+l_1)}$

[2] $I^{(2)} \in \mathbb{R}^{l_2 \times l_2}$

written as the following system

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{v}_{y_2} \end{bmatrix} = \begin{bmatrix} -F_1 & -R_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I^{(1)} \\ -R_{21} & -F_2 & k_1 I^{(1)} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ v_{y_2} \end{bmatrix} + \begin{bmatrix} -R_{13} & -R_{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -R_{23} & -R_{24} & \mathbf{0} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ v_{z_2} \end{bmatrix}, \tag{6}$$

where the control inputs are the positions and velocities of the leaders.

Since the follower dynamics is described by (5) (or (6)), the controllability of system (1) comes down to that of system (5) (or (6)).

Next, two special cases are considered. One is that all the agents with first-order integrator dynamics are the leaders, that is, $l_1 = l, l_2 = 0$ (in this case, $l = m$). The other is that all the agents with second-order integrator dynamics are the leaders, that is $l_1 = 0, l_2 = l$ (in this case, $n = m$).

Case one ($l_1 = l, l_2 = 0$)

According to the partition of leaders and followers, the system (1) under protocol (2) is written as

$$\begin{bmatrix} \dot{y} \\ \dot{v}_y \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_n & \mathbf{0} \\ -F & -k_1 F & -R \\ -L_{lf} & \mathbf{0} & -L_l \end{bmatrix} \begin{bmatrix} y \\ v_y \\ z \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ u_{\text{ext}} \end{bmatrix}$$

where $y = [x_1 \dots x_n]^T$, $v_y = [v_1 \dots v_n]^T$ and $z = [x_{n+1} \dots x_{n+l}]^T$ are, respectively, the stacked vector of followers' positions, followers' velocities and leaders' positions. u_{ext} is the external control inputs. Then the dynamics of the followers can be written as

$$\begin{bmatrix} \dot{y} \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_n \\ -F & -k_1 F \end{bmatrix} \begin{bmatrix} y \\ v_y \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -R \end{bmatrix} z, \tag{7}$$

where the control inputs are the positions of the leaders.

For the heterogeneous multi-agent system (1) under protocol (3), the dynamics of the closed-loop system is represented by

$$\begin{bmatrix} \dot{y} \\ \dot{v}_y \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_n & \mathbf{0} \\ -F & k_1 I_n & -R \\ -L_{lf} & \mathbf{0} & -L_l \end{bmatrix} \begin{bmatrix} y \\ v_y \\ z \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ u_{\text{ext}} \end{bmatrix}$$

where y, v_y and z are defined as above. Then the dynamics of the followers is described by

$$\begin{bmatrix} \dot{y} \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_n \\ -F & k_1 I_n \end{bmatrix} \begin{bmatrix} y \\ v_y \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -R \end{bmatrix} z, \tag{8}$$

where the control inputs are the positions of the leaders.

Case two ($l_1 = 0, l_2 = l$)

According to the partition of leaders and followers, the heterogeneous system (1) under protocol (2) can be written as follow

$$\begin{bmatrix} \dot{y} \\ \dot{z} \\ \dot{v}_z \end{bmatrix} = \begin{bmatrix} -F & -R & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_l \\ -L_{lf} & -L_l & -k_1 L_l \end{bmatrix} \begin{bmatrix} y \\ z \\ v_z \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ u_{\text{ext}} \end{bmatrix}$$

where $y = [x_1 \dots x_n]^T$, $z = [x_{n+1} \dots x_{n+l}]^T$ and $v_z = [v_{n+1} \dots v_{n+l}]^T$ are, respectively, the stacked vector of followers' positions, leaders' positions and leaders' velocities. Then the dynamics of the followers is given by

$$\dot{y} = -Fy - Rz, \tag{9}$$

where the control inputs are the positions of the leaders.

For the heterogenous system (1) under protocol (3), the dynamics of the whole closed-loop system is represented by

$$\begin{bmatrix} \dot{y} \\ \dot{z} \\ \dot{v}_z \end{bmatrix} = \begin{bmatrix} -F & -R & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_l \\ -L_{lf} & -L_l & k_1 I_n \end{bmatrix} \begin{bmatrix} y \\ z \\ v_z \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ u_{\text{ext}} \end{bmatrix}$$

where y, z and v_z are defined as above. Then the dynamics of the followers can be written as

$$\dot{y} = -Fy - Rz \tag{10}$$

where the control inputs are the positions of the leaders.

Theorem 3.1 : *For a given heterogeneous multi-agent system (1) with $l(= l_1 + l_2)$ leaders and n followers, the following assertions hold:*

i) System (1) is controllable under protocol (2) (or (3)) if the matrix pair (F, R) is controllable.

ii) Suppose $l_1 = l, l_2 = 0$ (or $l_1 = 0, l_2 = l$), the controllability of the system (1) under protocol (2) (or (3)) is equivalent to that of (F, R).

Proof Part i). Denote $A \triangleq \begin{bmatrix} -F_1 & -R_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \\ -R_{21} & -F_2 & -k_1F_2 \end{bmatrix}$ and $B \triangleq \begin{bmatrix} -R_{13} & -R_{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -R_{23} & -R_{24} & -k_1R_{24} \end{bmatrix}$, where $A \in \mathbb{R}^{(2n-m+l_1)\times(2n-m+l_1)}, B \in \mathbb{R}^{(l_1+2l_2)\times(l_1+2l_2)}$. Suppose that the matrix pair (F, R) is controllable. Note that (F, R) and $(-F, -R)$ have the same controllability, then $\text{rank}[sI + F \;\; -R] = n$, for $\forall s \in \mathbb{C}$, that is

$$\text{rank}\begin{bmatrix} sI + F_1 & R_{12} & -R_{13} & -R_{14} \\ R_{21} & sI + F_2 & -R_{23} & -R_{24} \end{bmatrix} = n. \tag{11}$$

On the other hand, it is obvious to see that

$$\begin{aligned}
\text{rank}[sI - A \;\; B] &= \text{rank}\begin{bmatrix} sI + F_1 & R_{12} & \mathbf{0} & -R_{13} & -R_{14} & \mathbf{0} \\ \mathbf{0} & sI & -I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ R_{21} & F_2 & sI + k_1F_2 & -R_{23} & -R_{24} & -k_1R_{24} \end{bmatrix} \\
&= \text{rank}\left[\begin{array}{cccc|cc} sI + F_1 & R_{12} & -R_{13} & -R_{14} & \mathbf{0} & \mathbf{0} \\ R_{21} & F_2 & -R_{23} & -R_{24} & sI + k_1F_2 & -k_1R_{24} \\ \hline \mathbf{0} & sI & \mathbf{0} & \mathbf{0} & -I & \mathbf{0} \end{array}\right] \\
&= \text{rank}\left[\begin{array}{cccc|cc} sI + F_1 & R_{12} & -R_{13} & -R_{14} & \mathbf{0} & \mathbf{0} \\ R_{21} & s^2I + (sk_1 + 1)F_2 & -R_{23} & -R_{24} & -k_1R_{24} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{array}\right] \\
&= \text{rank}\begin{bmatrix} sI + F_1 & R_{12} & -R_{13} & -R_{14} & \mathbf{0} \\ R_{21} & sI + F_2 + s(s-1)I + sk_1F_2 & -R_{23} & -R_{24} & -k_1R_{24} \end{bmatrix} \\
&\quad + (n - m + l_1),
\end{aligned} \tag{12}$$

Suppose that (F, R) is controllable, then (11) is satisfied. Therefore, by (11) and (12), one has $\text{rank}[sI - A \;\; B] = n + (n - m + l_1) = 2n - m + l_1$. It follows that (A, B) is controllable.

Part ii). If $l_1 = l, l_2 = 0$, then under protocol (2) the dynamics of the followers is given in (7). Denote $A \triangleq \begin{bmatrix} \mathbf{0} & I_n \\ -F & -k_1 F \end{bmatrix}$ and $B \triangleq \begin{bmatrix} \mathbf{0} \\ -R \end{bmatrix}$. We start the proof by observing the relations between the spectrum of A and that of F. Suppose λ is an eigenvalue of A and $[\beta_1^T \ \beta_2^T]$ is the associated left eigenvector with $\beta_k \in \mathbb{R}^n, k \in \{1, 2\}$, i.e. $[\beta_1^T \ \beta_2^T]A = \lambda[\beta_1^T \ \beta_2^T]$. It follows that

$$\begin{cases} -\beta_2^T F = \lambda \beta_1^T, \\ \beta_1^T - k_1 \beta_2^T F = \lambda \beta_2^T. \end{cases}$$

As a consequence

$$\beta_1^T = \lambda \beta_2^T + k_1 \beta_2^T F, \tag{13}$$

and

$$-(1 + \lambda k_1)\beta_2^T F = \lambda^2 \beta_2^T. \tag{14}$$

We say that $1+\lambda k_1 \neq 0$. Otherwise, (14) means that $\lambda = 0$ or $\beta_2^T = \mathbf{0}$. If $\lambda = 0$, then $1 = 0$. This is a contradiction. If $\beta_2^T = \mathbf{0}$, then (13) leads to $\beta_1^T = \mathbf{0}$ which contradicts with $[\beta_1^T \ \beta_2^T]$ being a left eigenvector of matrix A. As a result, (14) implies that $-\frac{\lambda^2}{1+\lambda k_1}$ is an eigenvalue of F, denoted by μ, and β_2^T is the corresponding left eigenvector. For now, it follows that for any eigenvalue λ of A with a left eigenvector $[\beta_1^T \ \beta_2^T]$, $\mu = -\frac{\lambda^2}{1+\lambda k_1}$ is an eigenvalue of F with the corresponding left eigenvector β_2^T. Conversely, for any given eigenvalue μ of F with a corresponding left eigenvector β^T, we can obtain that the roots of polynomial $s^2 + \mu k_1 s + \mu = 0$ with respect to s are the eigenvalues of A and $[\beta_1^T \ \beta^T]$ is the corresponding eigenvector with $\beta_1^T = (s + \mu k_1)\beta^T$.

Next, we prove that system (A, B) is controllable if and only if (F, R) is controllable.

(Sufficiency) By contradiction, if (A, B) is uncontrollable, then there exists an eigenvalue λ of A with an associated left eigenvector $[\beta_1^T \ \beta_2^T]$ and $\beta_k \in \mathbb{R}^n, k \in \{1, 2\}$ such that $[\beta_1^T \ \beta_2^T]B = \mathbf{0}$. It follows that $\mu = -\frac{\lambda^2}{1+\lambda k_1}$ is an eigenvalue of F with the corresponding left eigenvector β_2^T, and $\beta_2^T R = \mathbf{0}$. This contradicts that (F, R) is controllable.

(Necessary) If (F, R) is uncontrollable, then we can easily derive a contradiction to the assumption that the system (A, B) is completely controllable according to the relations between the spectrum of the two system matrices. Hence the details are omitted for brevity.

When $l_1 = 0, l_2 = l$, the conclusion is obviously. The proof is completed. □

Remark 2 : As seen from i) of Theorem 3.1, the controllability of (F, R) is only a sufficient condition for that of heterogeneous system (1). This means that system (1) is controllable even if (F, R) is not. This result different from homogeneous system whose controllability is equivalent to that of (F, R) and uniquely determined by the communication graph (see, for example Ji et al. (2015), Wang and Jiang (2009)). The result of ii) in Theorem 3.1 implies that for some special given leaders, the controllability of (F, R) is also a necessary condition for the controllability of heterogeneous multi-agent system (1). In this case, the controllability of all the above systems is uniquely determined by the communication graph, and accordingly is not affected by different agent dynamics. Therefore some criteria for the controllability of graph (or (F, R)) established in Ji et al. (2008, 2009, 2010, 2012) are suitable for determining the controllability.

3.2 *Discrete-time multi-agent systems*

In this subsection, we consider the controllability of discrete-time heterogeneous multi-agent system. The dynamics of the ith agent is described as

$$\begin{cases} x_i((k+1)T) = x_i(kT) + Tu_i(kT), & i \in \mathcal{I}_m, \\ x_i((k+1)T) = x_i(kT) + Tv_i(kT), v_i((k+1)T) = v_i(kT) + Tu_i(kT), & i \in \mathcal{I}_{n+l}/\mathcal{I}_m, \end{cases} \tag{15}$$

where $x_i(kT) \in \mathbb{R}, v_i(kT) \in \mathbb{R}$ and $u_i(kT) \in \mathbb{R}$ are the position-like, velocity-like and control input, respectively, of agent i at time kT, where $T > 0$ is the sampling period. In what follows, kT will be replace by k for simplicity.

The interactions among agents are realized through the following protocol:

$$u_i(k) = \begin{cases} \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)), & i \in \mathcal{I}_m, \\ \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)) + k_1 \sum_{j\in\mathcal{N}_i} a_{ij}(v_j(k) - v_i(k)), & i \in \mathcal{I}_{n+l}/\mathcal{I}_m, \end{cases} \tag{16}$$

or

$$u_i(k) = \begin{cases} \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)), & i \in \mathcal{I}_m, \\ \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)) + k_1 v_i(k), & i \in \mathcal{I}_{n+l}/\mathcal{I}_m, \end{cases} \tag{17}$$

where a_{ij} is the weight of the edge from agent j to agent i, k_1 is a nonzero feedback gain.

We still assume that there are $l(= l_1 + l_2 \geq 1)$ leaders (l_1 and l_2 are, respectively, the number of leaders with first-order and second-order dynamics) and n followers in system (15). According to the partition of leaders and followers, the system (15) under protocol (16) can be written as

$$\begin{bmatrix} y_1(k+1) \\ z_1(k+1) \\ y_2(k+1) \\ z_2(k+1) \\ v_{y_2}(k+1) \\ v_{z_2}(k+1) \end{bmatrix} = \begin{bmatrix} I^{(3)} - TF_1 & -TR_{13} & -TR_{12} & -TR_{14} & \mathbf{0} & \mathbf{0} \\ -TR_{31} & I^{(4)} - TF_3 & -TR_{32} & -TR_{42} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I^{(1)} & \mathbf{0} & TI^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I^{(2)} & \mathbf{0} & TI^{(2)} \\ -TR_{21} & -TR_{23} & -TF_2 & -TR_{24} & I^{(1)} - k_1TF_2 & -k_1TR_{24} \\ -TR_{41} & -TR_{43} & -TR_{42} & -TF_4 & -k_1TR_{42} & I^{(2)} - k_1TF_4 \end{bmatrix} \begin{bmatrix} y_1(k) \\ z_1(k) \\ y_2(k) \\ z_2(k) \\ v_{y_2}(k) \\ v_{z_2}(k) \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbf{0} \\ u_1(k) \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ u_2(k) \end{bmatrix},$$

where $y_1(k) = [x_1(k) \ldots x_{m-l_1}(k)]^T$ and $z_1(k) = [x_{m-l_1+1}(k) \ldots x_m(k)]^T$ are, respectively, the stacked vector of the first-order followers' and leaders' positions. $y_2(k) = [x_{m+1}(k) \ldots x_{n+l_1}(k)]^T$, $z_2(k) = [x_{n+l_1+1}(k) \ldots x_{n+l}(k)]^T$, $v_{y_2}(k) = [v_{m+1}(k) \ldots v_{n+l_1}(k)]^T$ and $v_{z_2}(k) = [v_{n+l_1+1}(k) \ \ldots$

[3] $I^{(3)} \in \mathbb{R}^{(m-l_1)\times(m-l_1)}$
[4] $I^{(4)} \in \mathbb{R}^{l_1\times l_1}$

$v_{n+l}(k)]^T$ are, respectively, the stacked vector of the second-order followers' positions, leaders' positions, followers' velocities and leaders' velocities. Consequently, we can describe the dynamics of the followers to be the following system

$$\begin{bmatrix} y_1(k+1) \\ y_2(k+1) \\ v_{y_2}(k+1) \end{bmatrix} = \begin{bmatrix} I^{(3)} - TF_1 & -TR_{12} & \mathbf{0} \\ \mathbf{0} & I^{(1)} & TI^{(1)} \\ -TR_{21} & -TF_2 & I^{(1)} - k_1TF_2 \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \\ v_{y_2}(k) \end{bmatrix} + \begin{bmatrix} -TR_{13} & -TR_{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -TR_{23} & -TR_{24} & -k_1TR_{24} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \\ v_{z_2}(k) \end{bmatrix}, \tag{18}$$

where the control inputs are the positions and velocities of the leaders.

If protocol (17) is employed, the closed-loop system of (15) is represented by

$$\begin{bmatrix} y_1(k+1) \\ z_1(k+1) \\ y_2(k+1) \\ z_2(k+1) \\ v_{y_2}(k+1) \\ v_{z_2}(k+1) \end{bmatrix} = \begin{bmatrix} I^{(3)} - TF_1 & -TR_{13} & -TR_{12} & -TR_{14} & \mathbf{0} & \mathbf{0} \\ -TR_{31} & I^{(4)} - TF_3 & -TR_{32} & -TR_{42} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I^{(1)} & \mathbf{0} & TI^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I^{(2)} & \mathbf{0} & TI^{(2)} \\ -TR_{21} & -TR_{23} & -TF_2 & -TR_{24} & (1+k_1T)I^{(1)} & \mathbf{0} \\ -TR_{41} & -TR_{43} & -TR_{42} & -TF_4 & \mathbf{0} & (1+k_1T)I^{(2)} \end{bmatrix} \begin{bmatrix} y_1(k) \\ z_1(k) \\ y_2(k) \\ z_2(k) \\ v_{y_2}(k) \\ v_{z_2}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ u_1(k) \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ u_2(k) \end{bmatrix},$$

where $y_1, z_1, y_2, z_2, v_{y_2}$ and v_{z_2} are defined as above. Then the dynamics of the followers is presented as the following system

$$\begin{bmatrix} y_1(k+1) \\ y_2(k+1) \\ v_{y_2}(k+1) \end{bmatrix} = \begin{bmatrix} I^{(3)} - TF_1 & -TR_{12} & \mathbf{0} \\ \mathbf{0} & I^{(1)} & TI^{(1)} \\ -TR_{21} & -TF_2 & (1+k_1T)I^{(1)} \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \\ v_{y_2}(k) \end{bmatrix} + \begin{bmatrix} -TR_{13} & -TR_{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -TR_{23} & -TR_{24} & \mathbf{0} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \\ v_{z_2}(k) \end{bmatrix}, \tag{19}$$

where the control inputs are the positions and velocities of the leaders.

Similarly, two special cases are also considered. One is that all the first-order agents are the leaders, that is, $l_1 = l, l_2 = 0$ (in this case, $l = m$). The other is that all the second-order agents are the leaders, that is, $l_1 = 0, l_2 = l$ (in this case, $n = m$).

Case one ($l_1 = l, l_2 = 0$)

According to the partition of leaders and followers, the system (15) under protocol (16) is written as

$$\begin{bmatrix} y(k+1) \\ v_y(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} I_n & TI_n & \mathbf{0} \\ -TF & I_n - k_1F & -TR \\ -TL_{lf} & \mathbf{0} & I_l - TL_l \end{bmatrix} \begin{bmatrix} y(k) \\ v_y(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ u_{\text{ext}}(k) \end{bmatrix}$$

where $y(k) = [x_1(k) \dots x_n(k)]^T$, $v_y(k) = [v_1(k) \dots v_n(k)]^T$ and $z(k) = [x_{n+1}(k) \dots x_{n+l}(k)]^T$ are, respectively, the stacked vector of followers' positions, followers' velocities and leaders' positions. $u_{\text{ext}}(k)$ is the external control inputs. Then the dynamics of the followers is proposed as follows:

$$\begin{bmatrix} y(k+1) \\ v_y(k+1) \end{bmatrix} = \begin{bmatrix} I_n & TI_n \\ -TF & I_n - k_1TF \end{bmatrix} \begin{bmatrix} y(k) \\ v_y(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -TR \end{bmatrix} z(k), \tag{20}$$

where the control inputs are the positions of the leaders.

For the heterogeneous multi-agent system (15) under protocol (17), the dynamics of the whole closed-loop system is represented by

$$\begin{bmatrix} y(k+1) \\ v_y(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} I_n & TI_n & \mathbf{0} \\ -TF & (1+k_1T)I_n & -TR \\ -TL_{lf} & \mathbf{0} & I_l - TL_l \end{bmatrix} \begin{bmatrix} y(k) \\ v_y(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ u_{\text{ext}}(k) \end{bmatrix}$$

where y, v_y and z are defined as above. Then the dynamics of the followers can be represented by

$$\begin{bmatrix} y(k+1) \\ v_y(k+1) \end{bmatrix} = \begin{bmatrix} I_n & TI_n \\ -TF & (1+k_1T)I_n \end{bmatrix} \begin{bmatrix} y(k) \\ v_y(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -TR \end{bmatrix} z(k), \tag{21}$$

where the control inputs are the positions of the leaders.

Case two ($l_1 = 0, l_2 = l$)

According to the partition of leaders and followers, the heterogeneous system (15) under protocol (16) can be written as

$$\begin{bmatrix} y(k+1) \\ z(k+1) \\ v_z(k+1) \end{bmatrix} = \begin{bmatrix} I_n - TF & -TR & \mathbf{0} \\ \mathbf{0} & I_l & TI_l \\ -TL_{lf} & -TL_l & I_l - k_1TL_l \end{bmatrix} \begin{bmatrix} y(k) \\ z(k) \\ v_z(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ u_{\text{ext}}(k) \end{bmatrix},$$

where $y(k) = [x_1(k) \dots x_n(k)]^T$, $z = [x_{n+1}(k) \dots x_{n+l}(k)]^T$ and $v_z(k) = [v_{n+1}(k) \dots v_{n+l}(k)]^T$ are, respectively, the stacked vector of followers' positions, leaders' positions and leaders' velocities. Then the dynamics of the followers is given as the following form

$$\dot{y} = (I_n - TF)y - TRz, \tag{22}$$

where the control inputs are the positions of the leaders.

For the heterogenous system (15) under protocol (17), the dynamics of the whole closed-loop system is represented by

$$\begin{bmatrix} y(k+1) \\ z(k+1) \\ v_z(k+1) \end{bmatrix} = \begin{bmatrix} I_n - TF & -TR & \mathbf{0} \\ \mathbf{0} & I_l & TI_l \\ -TL_{lf} & -TL_l & (1+k_1T)I_l \end{bmatrix} \begin{bmatrix} y(k) \\ z(k) \\ v_z(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ u_{\text{ext}}(k) \end{bmatrix}$$

where $y(k), z(k)$ and $v_z(k)$ are defined as above. Then the dynamics of the followers can be written as

$$\dot{y} = (I_n - TF)y - TRz, \tag{23}$$

where the control inputs are the positions of the leaders.

Denote Φ and Γ as

$$\Phi \triangleq \begin{bmatrix} I_n & TI_n \\ -TF & I_n - k_1TF \end{bmatrix}, \Gamma \triangleq \begin{bmatrix} \mathbf{0} \\ -TR \end{bmatrix}.$$

Theorem 3.2 : *For a given heterogeneous multi-agent system (15) with $l(= l_1 + l_2)$ leaders and n followers, let λ denotes an eigenvalue of matrix Φ, then the following assertions hold:*

i) System (15) is controllable under protocol (16) (or (17)) if the matrix pair (F, R) is controllable.

ii) Suppose $l_1 = l, l_2 = 0$, and $T \neq (1 - \lambda)k_1$, then the controllability of the system (15) under protocol (16) (or (17)) is equivalent to that of (F, R).

iii) Suppose $l_1 = 0, l_2 = l$, the controllability of the system (15) under protocol (16) (or (17)) is equivalent to that of (F, R).

Proof Part i). Denote $A \triangleq \begin{bmatrix} I - TF_1 & -TR_{12} & \mathbf{0} \\ \mathbf{0} & I & TI \\ -TR_{21} & -TF_2 & I - k_1TF_2 \end{bmatrix}$ and $B \triangleq \begin{bmatrix} -TR_{13} & -TR_{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -TR_{23} & -TR_{24} & -k_1TR_{24} \end{bmatrix}$, where $A \in \mathbb{R}^{(2n-m+l_1)\times(2n-m+l_1)}, B \in \mathbb{R}^{(l_1+2l_2)\times(l_1+2l_2)}$. Suppose that the matrix pair (F, R) is controllable. Note that (F, R) and $(-TF, -TR)$ have the same controllability, then $\mathrm{rank}[sI + TF \;\; -TR] = n$, for $\forall s \in \mathbb{C}$, that is

$$\mathrm{rank} \begin{bmatrix} sI + TF_1 & TR_{12} & -TR_{13} & -TR_{14} \\ TR_{21} & sI + TF_2 & -TR_{23} & -TR_{24} \end{bmatrix} = n. \tag{24}$$

On the other hand, it is obvious to see that

$$\begin{aligned}
&\mathrm{rank}[sI - A \;\; B] \\
&= \mathrm{rank} \begin{bmatrix} (s-1)I + TF_1 & TR_{12} & \mathbf{0} & -TR_{13} & -TR_{14} & \mathbf{0} \\ \mathbf{0} & (s-1)I & -TI & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ TR_{21} & TF_2 & (s-1)I + k_1TF_2 & -TR_{23} & -TR_{24} & -k_1TR_{24} \end{bmatrix} \\
&= \mathrm{rank} \left[\begin{array}{cccc:cc} (s-1)I + TF_1 & TR_{12} & -TR_{13} & -TR_{14} & \mathbf{0} & \mathbf{0} \\ TR_{21} & TF_2 & -TR_{23} & -TR_{24} & -k_1TR_{24} & (s-1)I + k_1TF_2 \\ \hdashline \mathbf{0} & (s-1)I & \mathbf{0} & \mathbf{0} & \mathbf{0} & -TI \end{array}\right] \\
&= \mathrm{rank} \left[\begin{array}{cccc:cc} (s-1)I + TF_1 & TR_{12} & -TR_{13} & -TR_{14} & \mathbf{0} & \mathbf{0} \\ TR_{21} & \frac{(s-1)^2}{T}I + (sk_1 - k_1 + T)F_2 & -TR_{23} & -TR_{24} & -k_1TR_{24} & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -TI \end{array}\right] \\
&= \mathrm{rank} \begin{bmatrix} (s-1)I + TF_1 & TR_{12} & -TR_{13} & -TR_{14} & \mathbf{0} \\ TR_{21} & (s-1)I + TF_2 + \frac{(s-1)^2 - T(s-1)}{T}I + (s-1)k_1F_2 & -TR_{23} & -TR_{24} & -k_1TR_{24} \end{bmatrix} \\
&\quad + (n - m + l_1),
\end{aligned} \tag{25}$$

Suppose that (F, R) is controllable, then (24) is satisfied. Therefore, by (24) and (25), one has $\mathrm{rank}[sI - A \;\; B] = n + (n - m + l_1) = 2n - m + l_1$. It follows that (A, B) is controllable.

Part ii). If $l_1 = l, l_2 = 0$, then under protocol (16) the dynamics of the followers is given in (20). We start the proof by observing the relations between the spectrum of Φ and that of F. Suppose λ is an eigenvalue of Φ and $[\beta_1^T \;\; \beta_2^T]$ is the associated left eigenvector with $\beta_j \in \mathbb{R}^n, j \in \{1, 2\}$, i.e. $[\beta_1^T \ \beta_2^T]\Phi = \lambda[\beta_1^T \ \beta_2^T]$. It follows that

$$\begin{cases} \beta_1^T - T\beta_2^T F = \lambda\beta_1^T, \\ T\beta_1^T + \beta_2^T - k_1 T\beta_2^T F = \lambda\beta_2^T. \end{cases}$$

As a consequence

$$\beta_1^T = \frac{\lambda - 1}{T}\beta_2^T + k_1\beta_2^T F, \tag{26}$$

and

$$-(T^2 + (\lambda - 1)k_1 T)\beta_2^T F = (\lambda - 1)^2\beta_2^T. \tag{27}$$

Note that $T \neq (1-\lambda)k_1$, which means that $T^2 + (\lambda - 1)k_1 T \neq 0$. As a result, (27) implies that $-\frac{(\lambda-1)^2}{T^2+(\lambda-1)k_1 T}$ is an eigenvalue of F, denoted by μ, and β_2^T is the corresponding left eigenvector. For now, it follows that for any eigenvalue λ of Φ with a left eigenvector $[\beta_1^T \ \beta_2^T]$, $\mu = -\frac{(\lambda-1)^2}{T^2+(\lambda-1)k_1 T}$ is an eigenvalue of F with the corresponding left eigenvector β_2^T. Conversely, for any given eigenvalue μ of F with a corresponding left eigenvector β^T, we can obtain that the roots of polynomial $s^2 + (\mu k_1 T - 2)s + \mu T^2 - \mu k_1 T + 1 = 0$ with respect to s are the eigenvalues of Φ and $[\beta_1^T \ \beta^T]$ is the corresponding eigenvector with $\beta_1^T = (\frac{s-1}{T} + \mu k_1)\beta^T$.

Next, we prove that system (Φ, Γ) is controllable if and only if (F, R) is controllable.

(Sufficiency) By contradiction, if (Φ, Γ) is uncontrollable, then there exists an eigenvalue λ of Φ with an associated left eigenvector $[\beta_1^T \ \beta_2^T]$ and $\beta_j \in \mathbb{R}^n, j \in \{1, 2\}$ such that $[\beta_1^T \ \beta_2^T]\Gamma = \mathbf{0}$. It follows that $\mu = -\frac{(\lambda-1)^2}{T^2+(\lambda-1)k_1 T}$ is an eigenvalue of F with the corresponding left eigenvector β_2^T, and $\beta_2^T R = \mathbf{0}$. This contradicts that (F, R) is controllable.

(Necessary) If (F, R) is uncontrollable, then we can easily derive a contradiction to the assumption that the system (Φ, Γ) is completely controllable according to the relations between the spectrum of the two system matrices. Hence the details are omitted for brevity.

Part iii). When $l_1 = 0, l_2 = l$, the controllability of system (15) under protocol (16) (or (17) is equivalent to that of $(I_n - TF, -TR)$. Therefore, for $\forall s \in \mathbb{C}$, one has

$$\begin{aligned} &\mathrm{rank}[(s-1)I_n + TF \ \ -TR] = n, \\ \Leftrightarrow\ &\mathrm{rank}[\frac{(s-1)}{T}I_n + TF \ \ -R] = n, \\ \Leftrightarrow\ &\mathrm{rank}[wI_n + TF \ \ -R] = n, \text{for } \forall w = \frac{s-1}{T} \in \mathbb{C}. \end{aligned}$$

This implies that the controllability of the system (22) (or (23)) is equivalent to that of (F, R). The proof is completed. □

4 Controllability analysis

With the given leader agents, the above arguments show that the controllability of several systems can uniformly point to the interaction topology associated with (F, R). In this section, the controllability of (F, R) is analyzed in detail. Some algebraic and graphical criteria are proposed for controllability. Suppose that there are n followers and l leaders over a network with leader-following structure. We will label the followers from 1 to n and the leaders from $n+1$ to $n+l$.

Definition 4.1: A directed and weighted graph $\mathcal{G}$ is said to be leader symmetric with respect to the leaders if there exists a nonidentity permutation matrix P such that

$$PF = FP,\ PR = R. \tag{28}$$

An example of leader symmetric digraph is depicted in Figure 2.

Theorem 4.2: *For a given communication topology $\mathcal{G}$ with l leaders and n followers, the following assertions hold:*

i) (F, R) is controllable if F and L have no common eigenvalues.

ii) (F, R) is controllable if and only if there is no left eigenvector of L with the last l entries being all zeroes.

iii) (F, R) is controllable, only if $\mathcal{G}$ is leader-follower connected and leader asymmetric.

Proof Part i). Suppose by contradiction that system (F, R) is not controllable, that is, there exists a vector β^T such that $\beta^T F = \lambda\beta^T$ for some $\lambda \in \mathbb{C}$, with $\beta^T R = \mathbf{0}$. Moreover,

$$[\beta^T\ \mathbf{0}^T]\begin{bmatrix} F & R \\ L_{lf} & L_l \end{bmatrix} = [\beta^T F\ \beta^T R] = \lambda[\beta^T\ \mathbf{0}^T],$$

which implies that λ is also an eigenvalue of L with left eigenvector $[\beta^T\ \mathbf{0}^T]$. This proof is thus completed.

Part ii). The proof is similar to Theorem 4 in Ji et al. (2009), and is thus omitted due to space limitations.

Part iii). We first prove that if (F, R) is controllable, then $\mathcal{G}$ is leader-follower connected. The proof is conducted by contradiction. Let $\mathcal{G}_f^{c_1}, \ldots, \mathcal{G}_f^{c_t}, \ldots, \mathcal{G}_f^{c_\gamma}$ represent the γ iSCCs of $\mathcal{G}_f$, where $1 \le t \le \gamma$. Denote by $\{v_{n_{t-1}+1}, \ldots, v_{n_t}\}$ the node set of $\mathcal{G}_f^{c_t}$, where $n_0 = 0, n_{\gamma+1} = n$. Through rearranging the indices of the n agents in $\mathcal{G}_f$, the matrix $[F, R]$ has the form

$$[F, R] = \begin{bmatrix} F_{f_1} & \cdots & \mathbf{0} & \mathbf{0} & R_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \cdots & F_{f_\gamma} & \mathbf{0} & R_\gamma \\ * & \cdots & * & F_{f_{\gamma+1}} & R_{\gamma+1} \end{bmatrix} \tag{29}$$

where $*$ represents zero or nonzero block. The indices of $F_{f_i} \in \mathbb{R}^{(n_i - n_{i-1})\times(n_i - n_{i-1})}$ are associated with those of agents in $\mathcal{G}_f^{c_i}, i = 1, \ldots, \gamma$. $R = [R_1, \ldots, R_{\gamma+1}]^T$ with $R_i \in \mathbb{R}^{(n_i - n_{i-1})\times l}, i = 1, \ldots, \gamma + 1$. Then the controllability matric $\mathbb{Q}$ of the system (F, R) has the form

$$\mathbb{Q} = [R,\ FR,\ \ldots\ , F^{n-1}R] = \begin{bmatrix} R_1 & F_{f_1}R_1 & \cdots & F_{f_1}^{n-1}R_1 \\ \vdots & \vdots & & \vdots \\ R_\gamma & F_{f_\gamma}R_\gamma & \cdots & F_{f_\gamma}^{n-1}R_\gamma \\ * & * & \cdots & * \end{bmatrix}. \tag{30}$$

If $\mathcal{G}$ is not leader-follower connected, there must exist at least one iSCC $\mathcal{G}_f^{c_i}, i \in \{1, \ldots, \gamma\}$ of $\mathcal{G}_f$ such that there is no path from any leader to any followers in $\mathcal{G}_f^{c_i}$, then $R_i = \mathbf{0}$. This results in the ith block row of $\mathbb{Q}$ is zero, which contradicts the assumption that (F, R) is controllable.

Next, we prove that if (F, R) is controllable, then $\mathcal{G}$ is leader asymmetric. Suppose that $\mathcal{G}$ is leader symmetric, then there exists a nonidentity permutation matrix P such that $PF = FP$. Let λ and β be the corresponding eigenvalue and left eigenvector of F, respectively, satisfying $\beta^T F = \lambda\beta^T$. Using (28), one has $\beta^T PF = \beta^T FP = \lambda\beta^T P$. This implies that $\beta^T P$ is also a left eigenvector of F corresponding the eigenvalue λ. Furthermore, one has $\beta^T - \beta^T P$ is also a left eigenvector of F. In fact, $(\beta^T - \beta^T P)F = \beta^T F - \beta^T PF = \lambda(\beta^T - \beta^T P)$. On the other hand, one has $(\beta^T - \beta^T P)R = \beta^T R - \beta^T PR = \beta^T R - \beta^T R = \mathbf{0}$. This suggests that the eigenvector $\beta^T - \beta^T P$ of F is orthogonal to R. Therefore, (F, R) is uncontrollable. This is a contradiction. The proof is completed. □

Remark 3 : Claim i) characterizes the controllability from the viewpoint of eigenvalue of Laplacian matrix, while claim ii) from the viewpoint of eigenvector. Claim i) is an extension of Lemma 7.9 in Ji et al. (2009) from undirected graphs to weighted directed graphs. However, the condition of claim i) is only sufficient. That is, system (F, R) might be controllable even if F and L have common eigenvalues. This indicates that the controllability of directed graph is different from controllability of undirected graphs. Claim ii) implies that the verification of controllability and the selection of leaders can be facilitated by checking the left eigenvectors of Laplacian matrix. In particular, when the graph $\mathcal{G}$ is undirected and unweighted, the result of ii) becomes Theorem 4 in Ji et al. (2009). Claim iii) provides a graph-theoretic result for the controllability of system (F, R). Note that leader-follower connected and leader asymmetric is only a necessary condition rather than a sufficient one. For instance, the topology graph $\mathcal{G}$ depicted in Figure 1 is leader-follower connected and leader asymmetric about the leader $\{9, 10\}$ but is is not controllable. In fact, after simple calculation, we can find rank $\mathbb{Q} = rank[R, FR, \ldots, F^7R] = 7 < 8$. Thus, (F, R) is not controllable.

It is interesting to note that the neighbor-based controllability depends not only on the interaction topology but also on the leaders' role Ji et al. (2012). Here, we study how the controllability is affected by leaders's role and the associated interconnection topology, and discuss the relationship between leaders' role and controllability. First, we will show that the topology graph can be further simplified to a subgraph under which the controllability still remains unchanged.

As illustrated in Figure 3, let $\mathcal{Z}$ be a subgraph of $\mathcal{G}$, which is assumed to be induced on the leader node set of system. Let $\mathcal{C}_1, \ldots, \mathcal{C}_d$ be the disjoint components of $\mathcal{G}/\mathcal{Z}$. The vertices of $\mathcal{Z}$ can be divided into two categories. Those not incident to any $\mathcal{C}_i, i = 1, \ldots, d$, constitute the vertex set of $\mathcal{Z}_0$, and the others are the vertices of $\mathcal{Z}_1 = \mathcal{Z}/\mathcal{Z}_0$. We see that $\mathcal{C}_1, \ldots, \mathcal{C}_d, \mathcal{Z}_0$ and $\mathcal{Z}_1$ constitute a partition of the information communication graph $\mathcal{G}$. Hereafter, we call it a $\mathcal{Z}$-partition. The partition has two extreme cases depicted in Figure 4. Note that given $\mathcal{G}$, the $\mathcal{Z}$-partition is not unique, which relies on the selected leader node set. With respect to the partition, the Laplacian can written as

$$L = \begin{bmatrix} L_1 & & & R_1 & \mathbf{0} \\ & \ddots & & \vdots & \vdots \\ & & L_d & R_d & \mathbf{0} \\ \bar{R}_1 & \cdots & \bar{R}_d & L_{\mathcal{Z}_1} & R_{d+1} \\ B_1 & \cdots & B_d & \bar{R}_{d+1} & L_{\mathcal{Z}_0} \end{bmatrix}.$$

Here, L_i and $L_{\mathcal{Z}_j}$ correspond to $\mathcal{C}_i$ and $\mathcal{Z}_j$, respectively, $i = 1, \ldots, d; j = 0, 1$. The subgraph $\mathcal{G}/\mathcal{Z}_0$ consists of $\mathcal{C}_1, \ldots, \mathcal{C}_d$ and $\mathcal{Z}_1$. We call it a $\mathcal{Z}_1$-leader subgraph in the sense that the leader set corresponds to $\mathcal{Z}_1$.

Theorem 4.3 : *(F, R) with leaders corresponding to the node set $\mathcal{Z}$ is controllable if and only if the $\mathcal{Z}_1$-leader subgraph is controllable.*

Proof (Necessity) Suppose the system (F, R) is uncontrollable. By ii) of Theorem 4.2, there is an left eigenvector β^T of an eigenvalue λ of L, with some zero coordinates corresponding to the leader agents. Let $\mathcal{Z}$ be the subgraph on the vertex set of all the zero coordinates of β^T. Then, one can make a $\mathcal{Z}$-partition for the communication graph $\mathcal{G}$, and β^T can be conformably written as

$$\beta^T = [\beta_1^T, \ldots, \beta_d^T, \mathbf{0}_{\mathcal{Z}_1}^T, \mathbf{0}_{\mathcal{Z}_0}^T], \tag{31}$$

where $\mathbf{0}_{\mathcal{Z}_j}$ are zero vectors corresponding to $L_{\mathcal{Z}_j}, j = 0, 1$. Note that there are no zero coordinates in β_i since all the zero coordinates are contained in $\mathbf{0}_{\mathcal{Z}_1}$ and $\mathbf{0}_{\mathcal{Z}_0}$. Denote by $\tilde{L} \triangleq diag\{L_1, \dots, L_d\}, \tilde{R} \triangleq [R_1^T, \dots, R_d^T]^T$ $\bar{R} \triangleq [\bar{R}_1, \dots, \bar{R}_d]$ and $\tilde{\beta}^T \triangleq [\beta_1^T, \dots, \beta_d^T]$. It follows from $\beta^T L = \lambda \beta^T$ that

$$[\tilde{\beta}^T \ \mathbf{0}_{\mathcal{Z}_1}^T] \begin{bmatrix} \tilde{L} & \tilde{R} \\ \bar{R} & L_{\mathcal{Z}_1} \end{bmatrix} = \lambda [\tilde{\beta}^T \ \mathbf{0}_{\mathcal{Z}_1}^T]. \tag{32}$$

Since $\begin{bmatrix} \tilde{L} & \tilde{R} \\ \bar{R} & L_{\mathcal{Z}_1} \end{bmatrix}$ is the generalized Laplacian matrix associated with the $\mathcal{Z}_1$-leader subgraph, ii) of Theorem 4.2 and (32) imply that the claim holds.

(Sufficiency) By ii) of Theorem 4.2, the uncontrollability of the $\mathcal{Z}_1$-leader subgraph means that (32) holds. Since the Laplacian matrix of $\mathcal{G}$ is

$$\begin{bmatrix} \tilde{L} & \tilde{R} & \mathbf{0} \\ \bar{R} & L_{\mathcal{Z}_1} & R_{d+1} \\ \tilde{B} & \bar{R}_{d+1} & L_{Z_0} \end{bmatrix},$$

where $\tilde{B} \triangleq [B_1, \dots, B_d]$. (32) and direct computation show that $\beta^T L = \lambda \beta^T$ holds for the eigenvector β^T given by (31). Accordingly, the system (F, R) is uncontrollable if leaders come from $\mathcal{Z}_1$ and/or $\mathcal{Z}_0$ by ii) of Theorem 4.2. □

Theorem 4.3 yields the following proposition, which means that for a multi-agent system all we need is to consider the controllability for the simplified $\mathcal{Z}_1$-leader subgraph.

Proposition 4.4: *The controllability is not affected by the leaders not directly incident to the follower subgraph. These leaders correspond to the node set of $\mathcal{Z}_0$.*

Lemma 4.5: *If (F, R) with graph $\mathcal{G}$ is controllable, then each iSCC $\mathcal{G}^{c_j}$ of $\mathcal{G}, j = 1, \dots, d$, contains at least one leader.*

Proof To clarify the presentation, the proof is carried out by taking two steps.

Step 1: Suppose $\tilde{\beta}^T = [\tilde{\beta}_1 \ \tilde{\beta}_2 \ \dots \ \tilde{\beta}_n], \tilde{\beta}_i \in \mathbb{R}, i \in \mathcal{I}_n$ is a left eigenvector of L corresponding to the eigenvector $\mathbf{0}$, i.e., $\tilde{\beta}^T L = \mathbf{0}$. By Theorem 2.13 of Wieland (2011), one has $\tilde{\beta} \in span\{\beta_1, \beta_2, \dots, \beta_d\}$, i.e.,

$$\begin{matrix} \tilde{\beta}_i > 0 \ \ if \ \ i \in \mathcal{V}^{c_j}, \\ \tilde{\beta}_i = 0 \ \ if \ \ i \notin \mathcal{V}^{c_j}, \end{matrix} \quad j \in \mathcal{I}_d$$

According to ii) of Theorem 4.2, leaders must be selected from the iSCC.

Step 2: Suppose that there exists an iSCC $\mathcal{G}_j$ such that there is no leader located in it. By Definition 2.2 we know that $\mathcal{G}$ is not leader-follower connected. It follows from iii) of Theorem 4.2 that (F, R) is uncontrollable. This is a contradiction.

Combining Step 1 with Step 2, we get the conclusion. The proof is completed. □

Proposition 4.6: *For a given communication topology $\mathcal{G}$, the number of leaders selected to ensure controllability of (F, R) is at least d, where d is the number of iSCCs in graph $\mathcal{G}$.*

Remark 4: Proposition 4.6 relates leaders selection to a constructive partition of the topology. This interprets the leaders' role to a certain extent from the viewpoint of the communication topology. It provides a lower bound on the smallest number of leaders to make (F, R) controllable. In fact, in special cases, the single leader could control the entire multi-agent system, for example, a path graph with the root node as leader is controllability (see, e.g., Ji et al. (2009), Parlangeli and Notarstefano (2012) and the references therein). On the other hand, the upper bound on the number of leaders is n, which seems trivial, that is all agents are leaders. Proposition 4.6 indicates that not only the location of leaders but also the number of leaders have to be considered in controllability of multi-agent system.

5 Illustrative examples

In this section, some illustrative examples are provided to verify the derived theoretical results. Without loss of generality, we assume that the weighted adjacency matrix $\mathcal{A}$ is 0-1 matrix, the feedback gain $k_1 = 1$ and the sampling period is taken as $T = 1s$.

Example 5.1 The example is used to illustrate claim i) of Theorem 3.1. Figure 5 depicts a directed leader-follower connected topology with $\mathcal{V}_f = \{1, 2, 3\}$, and $\mathcal{V}_l = \{4, 5\}$. Suppose that the vertices 1 and 4 denote the first-order integrator agents and the rest of vertices denote the second-order agents. It can be verified that

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

After simple calculation, we can find rank $\mathbb{Q}(F, R) = rank\ [R,\ FR,\ F^2R] = 3$. Therefore system (F, R) is controllable. The associated system matrices of (5) are

$$A \triangleq \begin{bmatrix} -F_1 & -R_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \\ -R_{21} & -F_2 & -k_1F_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & -1 \end{bmatrix}, \quad B \triangleq \begin{bmatrix} -R_{13} & -R_{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -R_{23} & -R_{24} & -k_1R_{24} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Computing the rank of the controllability matrix yields: rank $\mathbb{Q}(A, B) =$ rank$[B,\ AB,\ A^2B,\ A^3B,\ A^4B] = 5$. It implies that the heterogeneous multi-agent system (5) is controllable. Figure 6 depicts the trajectories of the three controllable followers in the plane.

Example 5.2 In this example, we still consider the controllability of heterogeneous multi-agent system (5). Figure 7 depicts a directed leader-follower topology with $\mathcal{V}_f = \{1, 2, 3\}$, and $\mathcal{V}_l = \{4, 5, 6\}$. Suppose that the vertices 1 and 4 denote the first-order integrator agents and the rest of vertices denote the second-order agents. It can be verified that

$$F = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}, \quad R = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}.$$

After simple calculation, we can find rank $\mathbb{Q}(F, R) = rank\ [R,\ FR,\ F^2R] = 2 < 3$. Thus, system (F, R) is uncontrollable. The system matrices of (5) are

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -3 & 1 & -3 & 1 \\ 0 & 0 & -2 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

It can be easily computed that rank $\mathbb{Q}(A, B) = $ rank$[B,\ AB,\ A^2B,\ A^3B,\ A^4B] = 5$. Therefore, the heterogeneous multi-agent system (5) is controllable. Figure 8 depicts the trajectories of the three controllable followers in the plane.

Example 5.3 This example is employed to verify claim ii) of Theorem 3.1. Consider the interacting topology depicted in Figure 9, where $\mathcal{V}_f = \{1, 2, 3, 4\}$ and $\mathcal{V}_l = \{5, 6\}$. Suppose that the vertices 5 and 6 denote the first-order integrator agents and the rest of vertices denote the second-order agents. The matrices F and R according to $\mathcal{G}$ are given as follows:

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 2 \end{bmatrix}, \quad R = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

After simple calculation, we can find rank $\mathbb{Q}(F, R) = \text{rank}\ [R,\ FR,\ F^2R,\ F^3R] = 4$. Thus, system (F, R) is controllable. The system matrices of (7) are

$$A \triangleq \begin{bmatrix} \mathbf{0} & I_4 \\ -F & -k_1F \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & -2 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 & 1 & -2 \end{bmatrix}, \quad B \triangleq \begin{bmatrix} \mathbf{0} \\ -R \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

One has rank $\mathbb{Q}(A, B) = \text{rank}\ [B,\ AB,\ \ldots,\ A^7B] = 8$. Thus, the second-order multi-agent system (7) is controllable. Figure 10 depicts the trajectories of the four controllable followers in the plane.

Example 5.4 The example is employed to validate claim i) of Theorem 3.2. Consider the interacting topology of heterogeneous multi-agent system (15) depicted by Figure 5. Suppose that the vertices 1 and 4 denote the first-order agents and the rest of vertices denote the second-order agents. It follows from Example 5.1 that system (F, R) is controllable. The associated system matrices of (19) are

$$A \triangleq \begin{bmatrix} I - TF_1 & -TR_{12} & \mathbf{0} \\ \mathbf{0} & TI & I \\ -TR_{21} & -TF_2 & I - k_1TF_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{bmatrix}, B \triangleq \begin{bmatrix} -TR_{13} & -TR_{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -TR_{23} & -TR_{24} & -k_1TR_{24} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

After simple calculation, we obtain that rank $\mathbb{Q}(A, B) = \text{rank}[B,\ AB,\ A^2B,\ A^3B,\ A^4B] = 5$. It implies that the heterogeneous multi-agent system (19) is controllable. Figure 11 depicts the trajectories of the three controllable followers in the plane.

Example 5.5 In the fifth example we still consider the controllability of heterogeneous multi-agent system (15). Consider the interacting topology of heterogeneous multi-agent system (15) depicted by Figure 7. Suppose that the vertices 1 and 4 denote the first-order agents and the rest of vertices denote the second-order agents. It follows from Example 5.2 that system (F, R)

is uncontrollable. The associated system matrices of (19) are

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & -3 & 1 & -2 & 1 \\ 0 & 0 & -2 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1&0&1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&1&0&1&0 \\ 1&0&1&0&1 \end{bmatrix}.$$

Computing the rank of the controllability matrix yields: rank $\mathbb{Q}(A, B) = \text{rank}[B,\ AB,\ A^2B,\ A^3B,\ A^4B] = 5$. Therefore, the heterogeneous multi-agent system (19) is controllable. Figure 12 depicts the trajectories of the three controllable followers in the plane.

Example 5.6 This example is employed to verify claim ii) of Theorem 3.2. Figure 9 depicts a directed leader-follower topology with $\mathcal{V}_f = \{1, 2, 3, 4\}$ and $\mathcal{V}_l = \{5, 6\}$. Suppose that the vertices 5 and 6 denote the first-order agents and the rest of vertices denote the second-order agents. It follows from Example 5.3 that system (F, R) is controllable. The associated system matrices of (20) are

$$\Phi \triangleq \begin{bmatrix} I_4 & TI_4 \\ -TF & I_4 - k_1TF \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 & 1 & -1 \end{bmatrix}, \quad \Gamma \triangleq \begin{bmatrix} \mathbf{0} \\ -TR \end{bmatrix} = \begin{bmatrix} 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 1&0 \\ 0&1 \\ 0&0 \\ 0&0 \end{bmatrix}.$$

After simple calculation, one has rank $\mathbb{Q}(\Phi, \Gamma) = \text{rank}\ [\Gamma,\ \Phi\Gamma,\ \ldots\ , \Phi^7\Gamma] = 8$. Thus, the second-order multi-agent system (7) is controllable. Figure 13 depicts the trajectories of the four controllable followers in the plane.

6 Conclusion

In this paper, controllability of the continuous- and discrete-time linear heterogeneous multi-agent systems with agents modeled by first-order and second-order dynamics was considered. We proposed two kinds of models for controllability of heterogeneous multi-agent systems and proved that the heterogeneous multi-agent system is controllable if the underlying communication topology is controllable. The result indicates that the controllability of heterogeneous multi-agent system cannot be uniquely determined by the communication topology, which is different from that of homogeneous multi-agent system. For any special given leader set, it was shown that the controllability of a heterogeneous multi-agent system is is equivalent to that of (F, R). Finally, some necessary and/or sufficient conditions for controllability of communication topology from algebraic and graph theoretical perspectives were established. Consideration of the issuer of time delays, switching topologies, or random communication topology graph leaves some interesting topics for future research.

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Figure 1. A leader-follower connected graph. The vertices $9, 10$ and the corresponding edges between them constitute a leader subgraph $\mathcal{G}_l$. The remaining nodes and the corresponding edges among them form the follower subgraph $\mathcal{G}_f$. $\mathcal{G}_f$ contains two distinct iSCCs $\mathcal{G}_f^{c_i}, i = 1, 2$, where $\mathcal{V}_f^{c1} = \{1, 2, 3\}$ and $\mathcal{V}_f^{c2} = \{6\}$ (Note that $\{4, 5\}$ is a SCC rather that an iSCC).

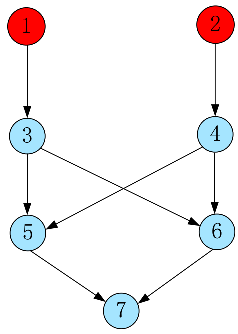

Figure 2. A leader symmetric communication graph, where the vertices 1 and 2 are leaders and all the link wights are taken as 1.

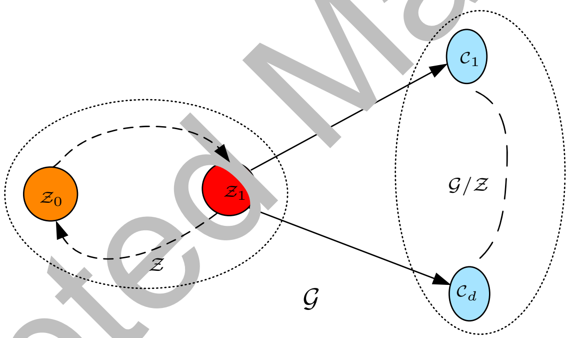

Figure 3. A $\mathcal{Z}$-partition of communication graph $\mathcal{G}$.

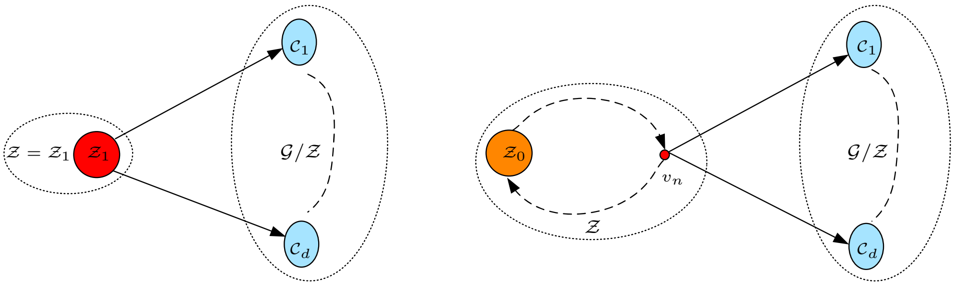

Figure 4. Two extreme cases of the partition: (a) indicates that $\mathcal{Z}_1$ is full of $\mathcal{Z}$; while (b) exhibits that $\mathcal{Z}_1$ shrinks to a single vertex.

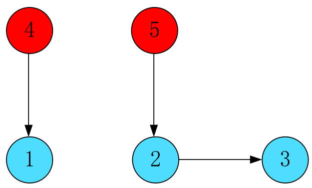

Figure 5. A communication topology, which is controllable.

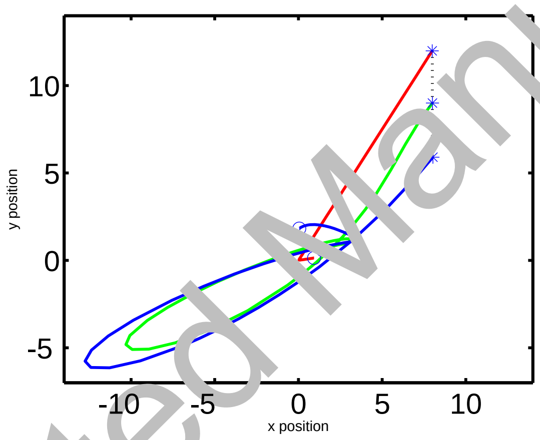

Figure 6. A straight line configuration. The circle and asterisk denotes the initial state and the final desired configuration, respectively.

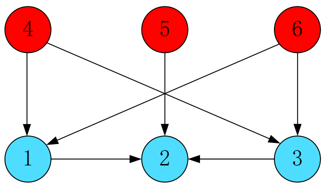

Figure 7. A communication topology, which is uncontrollable.

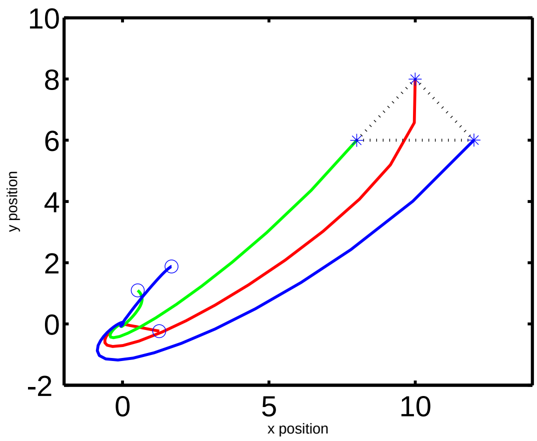

Figure 8. A triangle configuration. The circle and asterisk denotes the initial state and the final desired configuration, respectively.

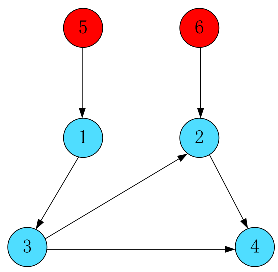

Figure 9. A communication topology, which is controllable.

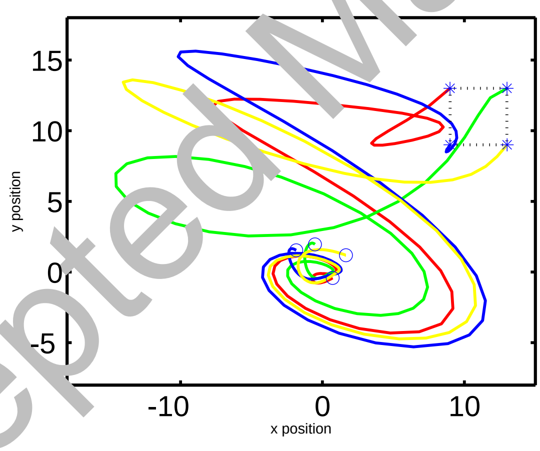

Figure 10. A rectangle configuration. The circle and asterisk denotes the initial state and the final desired configuration, respectively.

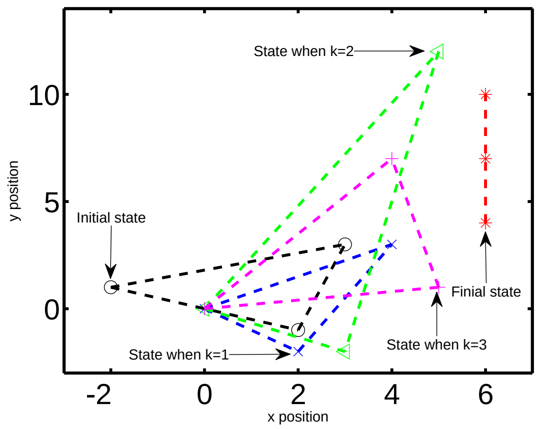

Figure 11. A straight line configuration. The circle and asterisk denotes the initial state and the final desired configuration, respectively.

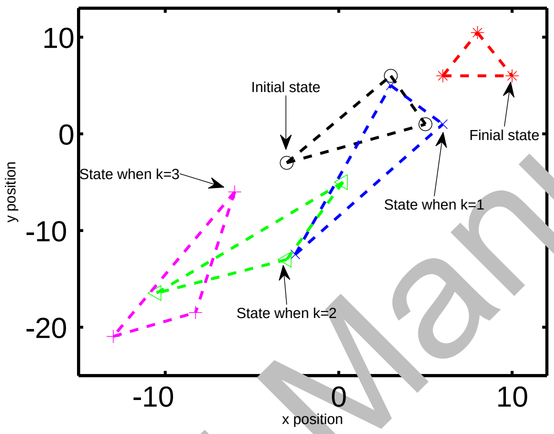

Figure 12. A regular triangle configuration. The circle and asterisk denotes the initial state and the final desired configuration, respectively.

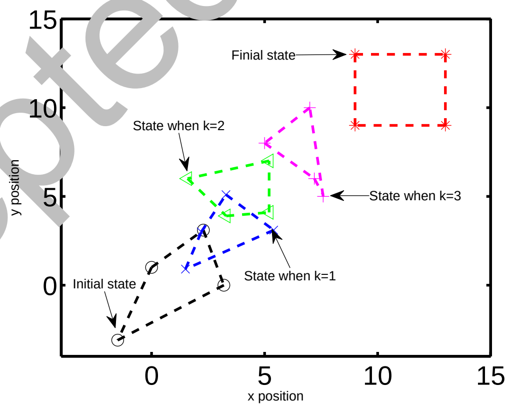

Figure 13. A square configuration. The circle and asterisk denotes the initial state and the final desired configuration, respectively.