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CONTROLLABILITY OF MULTI-AGENT SYSTEMS FROM A GRAPH-THEORETIC PERSPECTIVE*

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Abstract. In this work, we consider the controlled agreement problem for multi-agent networks, where a collection of agents take on leader roles while the remaining agents execute local, consensus-like protocols. Our aim is to identify reflections of graph-theoretic notions on system-theoretic properties of such systems. In particular, we show how the symmetry structure of the network, characterized in terms of its automorphism group, directly relates to the controllability of the corresponding multi-agent system. Moreover, we introduce network equitable partitions as a means by which such controllability characterizations can be extended to the multileader setting.

Key words. multi-agent systems, networked systems, controllability, automorphism group, equitable partitions, agreement dynamics, algebraic graph theory

AMS subject classifications. 93B05, 05C50, 05C25, 34B45, 94C15

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1. Introduction.

A networked system is a collection of dynamic units that interact over an information exchange network for its operation. Such systems are ubiquitous in diverse areas of science and engineering. Examples include physiological systems and gene networks [12]; large-scale energy systems; and multiple space, air, and land vehicles [1, 2, 20, 27, 37, 38]. There is an active research effort underway in the control and dynamical systems community to study these systems and lay out a foundation for their analysis and synthesis [6, 7, 9, 26]. As a result, over the past few years, a distinct area of research at the intersection of systems theory and graph theory has emerged. An important class of problems that lies at this intersection pertains to the *agreement* or the *consensus problem* [4, 15, 28, 30, 39]. The agreement problem concerns the development of processes by which a group of dynamic units, through local interactions, reach a common value of interest. As such, the agreement protocol is essentially an unforced dynamical system whose trajectory is governed by the interconnection geometry and the initial condition for each unit.

Our goal in this paper is to consider situations where network dynamics can be influenced by external signals and decisions. In particular, we postulate a case involving nodes in the network that do not abide by the agreement protocol; we refer to these agents as *leaders* or *anchors*.[1] The complement of the set of leaders in the network will be referred to as followers (respectively, floating nodes). The presence of these leader nodes generally alters the system behavior. The main topic under consideration in this paper is network controllability when leaders are agents of control. The controllability issue in leader-follower multi-agent systems was introduced in [36]

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[1]Depending on the context, we could equally consider them as the nonconformist agents.

by Tanner, who provided necessary and sufficient conditions for system controllability in terms of the eigenvectors of the graph Laplacian; we also refer to the related work of Olfati-Saber and Shamma in the context of consensus filters [31]. Subsequently, graph-theoretic characterizations of controllability for leader-follower multi-agent systems were examined by Ji, Muhammad, and Egerstedt [18] and Rahmani and Mesbahi [34]. In the present work, we further explore the ramifications of this graph-theoretic outlook on multi-agent systems controllability. First, we examine the roles of the graph Laplacian eigenvectors and the graph automorphism group for single-leader networks. We then extend these results to multileader setting via equitable partitions of the underlying graph.

This paper begins with the general form of the agreement dynamics over graphs. Next, we introduce transformations that, given the location of the leader nodes, produce the corresponding controlled linear time-invariant system. The study of the controllability for single-leader systems is then pursued via tools from algebraic graph theory. In this venue, we provide a sufficient graphical condition in terms of graph automorphisms for the system's uncontrollability. Furthermore, we introduce network equitable partitions as a means by which such controllability characterizations can be extended to the multileader setting.

2. Notation and preliminaries. In this section we recall some basic notions from graph theory, which is followed by the general setup of the agreement problem for multi-agent networks.

2.1. Graphs and their algebraic representation. Graphs are broadly adopted in the multi-agent literature to encode interactions in networked systems. An undirected graph $\mathcal{G}$ is defined by a set $\mathcal{V}_{\mathcal{G}} = \{1, \ldots, n\}$ of *nodes* and a set $\mathcal{E}_{\mathcal{G}} \subset \mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{G}}$ of *edges*. Two nodes i and j are *neighbors* if $(i, j) \in \mathcal{E}_{\mathcal{G}}$; the neighboring relation is indicated with $i \sim j$, while $\mathcal{P}(i) = \{j \in \mathcal{V}_{\mathcal{G}} : j \sim i\}$ collects all neighbors of node i. The *degree* of a node is given by the number of its neighbors; we say that a graph is *regular* if all nodes have the same degree. A *path* $i_0 i_1 \ldots i_L$ is a finite sequence of nodes such that $i_{k-1} \sim i_k$, $k = 1, \ldots, L$, and a graph $\mathcal{G}$ is *connected* if there is a path between any pair of distinct nodes. A subgraph $\mathcal{G}'$ is said to be *induced* from the original graph $\mathcal{G}$ if it can be obtained by deleting a subset of nodes and edges connecting to those nodes from $\mathcal{G}$.

The *adjacency* matrix of the graph $\mathcal{G}$, $A(\mathcal{G}) \in \mathbb{R}^{n \times n}$, with n denoting the number of nodes in the network, is defined by

$$[\,\mathcal{A}(\mathcal{G})\,]_{ij} := \begin{cases} 1 & \text{if} \quad (i,j) \in \mathcal{E}_{\mathcal{G}}, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{G}$ has m edges and is given an arbitrarily orientation, its node-edge *incidence* matrix $\mathcal{B}(\mathcal{G}) \in \mathbb{R}^{n \times m}$ is defined as

$$[\,\mathcal{B}(\mathcal{G})\,]_{kl} := \begin{cases} 1 & \text{if} \quad \text{node } k \text{ is the head of edge } l, \\ -1 & \text{if} \quad \text{node } k \text{ is the tail of edge } l, \\ 0 & \text{otherwise,} \end{cases}$$

where k and l are the indices running over the node and edge sets, respectively.

A matrix that plays a central role in many graph-theoretic treatments of multi-agent systems is the graph *Laplacian*, defined by

$$\mathcal{L}(\mathcal{G}) := \mathcal{B}(\mathcal{G})\,\mathcal{B}(\mathcal{G})^T; \tag{1}$$

thus the graph Laplacian is a (symmetric) positive semidefinite matrix. Let d_i be the degree of node i, and let $\mathcal{D}(\mathcal{G}) := \mathbf{Diag}([d_i]_{i=1}^n)$ be the corresponding diagonal degree matrix. It is easy to verify that $\mathcal{L}(\mathcal{G}) = \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$ [11]. As the Laplacian is positive semidefinite, its spectrum can be ordered as

$$0 = \lambda_1(\mathcal{L}(\mathcal{G})) \leq \lambda_2(\mathcal{L}(\mathcal{G})) \leq \cdots \leq \lambda_n(\mathcal{L}(\mathcal{G})),$$

with $\lambda_i(\mathcal{L}(\mathcal{G}))$ being the ith ordered eigenvalue of $\mathcal{L}(\mathcal{G})$. It turns out that the multiplicity of the zero eigenvalue of the graph Laplacian is equal to the number of connected components of the graph [14]. In fact the second smallest eigenvalue $\lambda_2(\mathcal{L}(\mathcal{G}))$ provides a judicious measure of the connectivity of $\mathcal{G}$. For more on the related matrix-theoretic and algebraic approaches to graph theory, we refer the reader to [5, 14, 24].

2.2. Agreement dynamics. Given a multi-agent system with n agents, we can model the network by a graph $\mathcal{G}$ where nodes represent agents and edges are interagent information exchange links.[2] Let $x_i(t) \in \mathbb{R}^d$ denote the state of node i at time t, whose dynamics is described by the single integrator

$$\dot{x}_i(t) = u_i(t), \qquad i = 1, \ldots, n,$$

with $u_i(t)$ being node i's control input. Next, we allow agent i to have access to the relative state information with respect to its neighbors and use it to compute its control. Hence, interagent coupling is realized through $u_i(t)$. For example, one can let

$$u_i(t) = -\sum_{i \sim j} (x_i(t) - x_j(t)). \tag{2}$$

The localized rule in (2) happens to lead to the solution of the rendezvous problem, which has attracted considerable attention in the literature [8, 17, 22]. Some other important networked system problems, e.g., formation control [3, 10, 13], consensus or agreement [25, 29, 30], and flocking [32, 35], share the same distributive flavor as the rendezvous problem.

The single integrator dynamics in conjunction with (2) can be represented as the Laplacian dynamics of the form

$$\dot{x}(t) = -\mathbb{L}(\mathcal{G})x(t), \tag{3}$$

where $x(t) = [x(t)_1^T,\ x(t)_2^T, \ldots, x(t)_n^T]^T$ denotes the aggregated state vector of the multi-agent system, $\mathbb{L}(\mathcal{G}) := \mathcal{L}(\mathcal{G}) \otimes I_d$, with I_d denoting the d-dimensional identity matrix, and $\otimes$ is the matrix Kronecker product [16]. In fact, if the dynamics of the agent's state is decoupled along each dimension, the behavior of the multi-agent system can be investigated one dimension at a time. Although our results can directly be extended to the case of (3), in what follows we will focus on the system

$$\dot{x}(t) = -\mathcal{L}(\mathcal{G})x(t), \tag{4}$$

capturing the multi-agent dynamics with individual agent states evolving in $\mathbb{R}$.

[2]Throughout this paper we assume that the network is static. As such, the movements of the agents will not cause edges to appear or disappear in the network.

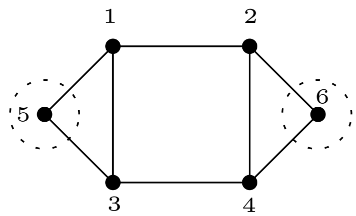

FIG. 1. *A leader-follower network with* $\mathcal{V}_f = \{1, 2, 3, 4\}$ *and* $\mathcal{V}_l = \{5, 6\}$.

3. Controlled agreement. We now endow leadership roles to a subset of agents in the Laplacian dynamics (4); the other agents in the network, the followers, continue to abide by the agreement protocol. In this paper, we use subscripts l and f to denote affiliations with leaders and followers, respectively. For example, a *follower graph* $\mathcal{G}_f$ is the subgraph induced by the follower node set $\mathcal{V}_f \subset \mathcal{V}_{\mathcal{G}}$. Leadership designations induce a partition of incidence matrix $\mathcal{B}(\mathcal{G})$ as

$$\mathcal{B}(\mathcal{G}) = \begin{bmatrix} \mathcal{B}_f(\mathcal{G}) \\ \mathcal{B}_l(\mathcal{G}) \end{bmatrix}, \tag{5}$$

where $\mathcal{B}_f(\mathcal{G}) \in \mathbb{R}^{n_f \times m}$, and $\mathcal{B}_l(\mathcal{G}) \in \mathbb{R}^{n_l \times m}$. Here n_f and n_l are the cardinalities of the follower group and the leader group, respectively, and m is the number of edges. The underlying assumption of this partition, without loss of generality, is that leaders are indexed last in the original graph $\mathcal{G}$. As a result of (1) and (5), the graph Laplacian $\mathcal{L}(\mathcal{G})$ is given by

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} \mathcal{L}_f(\mathcal{G}) & l_{fl}(\mathcal{G}) \\ l_{fl}(\mathcal{G})^T & \mathcal{L}_l(\mathcal{G}) \end{bmatrix}, \tag{6}$$

where

$$\mathcal{L}_f(\mathcal{G}) = \mathcal{B}_f\mathcal{B}_f^T, \quad \mathcal{L}_l(\mathcal{G}) = \mathcal{B}_l\mathcal{B}_l^T, \quad \text{and} \;\; l_{fl}(\mathcal{G}) = \mathcal{B}_f\mathcal{B}_l^T.$$

Here we omitted the dependency of $\mathcal{B}, \mathcal{B}_f$, and $\mathcal{B}_l$ on $\mathcal{G}$, which we will continue to do whenever this dependency is clear from the context. As an example, Figure 1 shows a leader-follower network with $\mathcal{V}_l = \{5, 6\}$ and $\mathcal{V}_f = \{1, 2, 3, 4\}$. This gives

$$\mathcal{B}_f = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}, \; \mathcal{B}_l = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

and

$$\mathcal{L}_f(\mathcal{G}) = \begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix}, \qquad l_{fl}(\mathcal{G}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

The control system we now consider is the controlled agreement dynamics (or leader-follower system), where followers evolve through the Laplacian-based dynamics

$$\dot{x}_f(t) = -\mathcal{L}_f\, x_f(t) - l_{fl}\, u(t), \tag{7}$$

where u denotes the exogenous control signal dictated by the leaders' states.

DEFINITION 3.1. *Let l be a leader node in $\mathcal{G}$, i.e., $l \in \mathcal{V}_l(\mathcal{G})$. The indicator vector with respect to l,*

$$\delta_l : \mathcal{V}_f \to \{0,1\}^{n_f},$$

is such that

$$\delta_l(i) := \begin{cases} 1 & \textit{if} \quad i \sim l, \\ 0 & \textit{otherwise.} \end{cases}$$

We note that each column of $-l_{fl}$ is an indicator vector, i.e., $l_{fl} = [-\delta_{n_f+1}, \ldots, -\delta_n]$.

Let d_{il}, with $i \in \mathcal{V}_f$, denote the number of leaders adjacent to follower i, and define the follower-leader degree matrix

$$\mathcal{D}_{fl}(\mathcal{G}) := \mathbf{Diag}([d_{il}]_{i=1}^{n_f}), \tag{8}$$

which leads to the relationship

$$\mathcal{L}_f(\mathcal{G}) = \mathcal{L}(\mathcal{G}_f) + \mathcal{D}_{fl}(\mathcal{G}), \tag{9}$$

where $\mathcal{L}(\mathcal{G}_f)$ is the Laplacian matrix of the follower graph $\mathcal{G}_f$.

Remark 3.2. We should emphasize the difference between $\mathcal{L}_f(\mathcal{G})$ and $\mathcal{L}(\mathcal{G}_f)$. The matrix $\mathcal{L}_f(\mathcal{G})$ is the principle diagonal submatrix of the original Laplacian matrix $\mathcal{L}(\mathcal{G})$ related to the followers, while $\mathcal{L}(\mathcal{G}_f)$ is the Laplacian matrix of the subgraph $\mathcal{G}_f$ induced by the followers. For simplicity, we will write $\mathcal{L}_f$ and l_{fl} to represent $\mathcal{L}_f(\mathcal{G})$ and $l_{fl}(\mathcal{G})$, respectively, when their dependency on $\mathcal{G}$ is clear from the context.

Since the row sum of the Laplacian matrix is zero, the sum of the ith row of $\mathcal{L}_f(\mathcal{G})$ and that of $-l_{fl}(\mathcal{G})$ are both equal to d_{il}, i.e.,

$$\mathcal{L}_f(\mathcal{G})\,\mathbf{1}_{n_f} = \mathcal{D}_{fl}(\mathcal{G})\,\mathbf{1}_{n_f} = -l_{fl}(\mathcal{G})\,\mathbf{1}_{n_l}, \tag{10}$$

where $\mathbf{1}$ is a vector with ones at each component.

If there is only one leader in the network, then according to the indexing convention, $\mathcal{V}_l = \{n\}$. In this case, we have $l_{fl}(\mathcal{G}) = -\delta_n$ and $\mathcal{D}_{fl}(\mathcal{G}) = \mathbf{Diag}(\delta_n)$. For instance, the indicator vector for the node set $\mathcal{V}_f = \{1,2,3\}$ in the graph shown in Figure 2 with respect to the leader $\{4\}$ is $\delta_4 = [\,1,\,1,\,0\,]^T$.

PROPOSITION 3.3. *If a single node is chosen to be the leader, the original Laplacian $\mathcal{L}(\mathcal{G})$ is related to the Laplacian of the follower graph $\mathcal{L}(\mathcal{G}_f)$ via*

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} \mathcal{L}(\mathcal{G}_f) + \mathcal{D}_{fl}(\mathcal{G}) & -\delta_n \\ -\delta_n^T & d_n \end{bmatrix}, \tag{11}$$

where d_n denotes the degree of agent n.

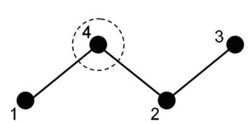

FIG. 2. *Path graph with node "4" being the leader.*

Another way to construct the system matrices $\mathcal{L}_f(\mathcal{G})$ and $l_{fl}(\mathcal{G})$ is from the Laplacian of the original graph via

$$\mathcal{L}_f = P_f^T\,\mathcal{L}(\mathcal{G})\,P_f \quad \text{and} \quad l_{fl} = P_f^T\,\mathcal{L}(\mathcal{G})\,T_{fl}, \tag{12}$$

where $P_f \in \mathbb{R}^{n \times n_f}$ is constructed by eliminating the columns of the $n \times n$ identity matrix that correspond to the leaders, and $T_{fl} \in \mathbb{R}^{n \times n_l}$ is formed by grouping these eliminated columns into a new matrix. For example, in Figure 1 these matrices assume the form

$$P_f = \begin{bmatrix} I_4 \\ \mathbf{0}_{2\times 4} \end{bmatrix} \quad \text{and} \quad T_{fl} = \begin{bmatrix} \mathbf{0}_{4\times 2} \\ I_2 \end{bmatrix}.$$

PROPOSITION 3.4. *If a single node is chosen to be the leader, one has*

$$T_{fl} = (I_n - \tilde{P})\mathbf{1}_n \quad \textit{and} \quad l_{fl} = -\mathcal{L}_f \mathbf{1}_{n_f}$$

in (12), *where* $\tilde{P} = [P_f \;\; \mathbf{0}_{n\times n_l}]$ *is the* $n \times n$ *square matrix obtained by expanding* P_f *with zero block of proper dimensions.*

Proof. The first equality directly follows from the definition of P_f and T_{fl}. Without loss of generality, assume that the last node is the leader; then $[\, P_f \;\; T_{fl}\,] = I_n$. Multiplying both sides by $\mathbf{1}_n$ and noting that $\tilde{P}\,\mathbf{1}_n = P_f \mathbf{1}_{n_f}$, one has $T_{fl} = (I_n - \tilde{P})\mathbf{1}_n$. Moreover,

$$l_{fl} = P_f^T \mathcal{L}(\mathcal{G})\{(I - \tilde{P})\mathbf{1}_n\} = P_f^T \mathcal{L}(\mathcal{G})\mathbf{1}_n - P_f^T \mathcal{L}(\mathcal{G}) P_f \mathbf{1}_{n_f}.$$

The first term on the right-hand side of the equality is zero, as $\mathbf{1}$ belongs to the null space of $\mathcal{L}(\mathcal{G})$; the second term, on the other hand, is simply $\mathcal{L}_f \mathbf{1}$. ∎

Alternatively, for the case when the exogenous signal is constant, the dynamics (7) can be rewritten as

$$\begin{bmatrix} \dot{x}_f(t) \\ \dot{u}(t) \end{bmatrix} = - \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_f(t) \\ u(t) \end{bmatrix}. \tag{13}$$

This corresponds to zeroing-out the rows of the original graph Laplacian associated with the leader. Zeroing-out a row of a matrix can be accomplished via a reduced identity matrix Q_r, with zeros at the diagonal elements that correspond to the leaders, with all other diagonal elements being kept as one. In this case,

$$\begin{bmatrix} \mathcal{L}_f & l_{fl} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = Q_r \mathcal{L}(\mathcal{G}), \tag{14}$$

where

$$Q_r = \begin{bmatrix} I_{n_f} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

and all the zero matrices are of proper dimensions.

4. Reachability.

First, we examine whether we can steer the system (7) into the agreement subspace, $\mathbf{span}\{\mathbf{1}\}$, when the exogenous signal is constant, i.e., $x_i = c$, for all $i \in \mathcal{V}_l$ and $c \in \mathbb{R}$ is a constant. As shown in (14), in this case the controlled agreement can be represented as

$$\dot{x}(t) = -Q_r \mathcal{L}(\mathcal{G})\, x(t) = -\mathcal{L}_r(\mathcal{G})\, x(t), \tag{15}$$

where Q_r is the reduced identity matrix and $\mathcal{L}_r(\mathcal{G}) := Q_r \mathcal{L}(\mathcal{G})$ is the reduced Laplacian matrix. Let us now examine the convergence properties of (15) with respect to

$\mathbf{span}\{\mathbf{1}\}$. Define $\zeta(t)$ as the projection of the followers' state $x_f(t)$ onto the subspace orthogonal to the agreement subspace $\mathbf{span}\{\mathbf{1}\}$. This subspace is denoted by $\mathbf{1}^\perp$; in [30] it is referred to as the *disagreement* subspace. One can then model the disagreement dynamics as

$$\dot{\zeta}(t) = -\mathcal{L}_r(\mathcal{G})\,\zeta(t). \tag{16}$$

Choosing a standard quadratic Lyapunov function for (16),

$$V(\zeta(t)) = \frac{1}{2}\,\zeta(t)^T\zeta(t),$$

reveals that its time rate of change assumes the form

$$\dot{V}(\zeta(t)) = -\zeta(t)^T\,\overline{\mathcal{L}}_r(\mathcal{G})\,\zeta(t),$$

where $\overline{\mathcal{L}}_r(\mathcal{G}) = (1/2)\,[\,\mathcal{L}_r(\mathcal{G}) + \mathcal{L}_r(\mathcal{G})^T\,]$.

PROPOSITION 4.1. *The agreement subspace is reachable for the controlled agreement protocol* (7).

Proof. Since $\dot{V}(\zeta) < 0$ for all $\zeta \neq 0$ and $Q_r\mathcal{L}(\mathcal{G})\,\mathbf{1} = 0$, for any leader nodes, the agreement subspace remains a globally attractive subspace of (15). ∎

PROPOSITION 4.2. *In the case of one leader, the matrix $\mathcal{L}_r(\mathcal{G})$ has a real spectrum and the same inertia as $\mathcal{L}(\mathcal{G})$.*

Proof. Let $E = \mathbf{1}\mathbf{1}^T$ denote the matrix of all ones. Since $E\mathcal{L}(\mathcal{G}) = 0$ and $Q_r\mathcal{L}(\mathcal{G}) = \mathcal{L}_r(\mathcal{G})$, $(Q_r + E)\,\mathcal{L}(\mathcal{G}) = \mathcal{L}_r(\mathcal{G})$. Hence $\mathcal{L}_r(\mathcal{G})$ is a product of a positive definite matrix, namely $Q_r + E$, and the symmetric matrix $\mathcal{L}(\mathcal{G})$. By Theorem 7.6.3 of [16], $\mathcal{L}_r(\mathcal{G})$ is diagonalizable and has a real spectrum. In fact, it has the same inertia as $\mathcal{L}(\mathcal{G})$. ∎

5. Controllability analysis of single-leader networks. In this section, we investigate the controllability properties of single-leader networks. Following our previously mentioned indexing convention, the index of the leader is assumed to be n. For notational convenience in this section, we will equate x_f with x and x_l with u. Moreover, we identify matrices A and B with $-\mathcal{L}_f$ and $-l_{fl}$, respectively. Thus, the system (7) is specified by

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{17}$$

The controllability of the controlled agreement (17) can be investigated using the Popov–Hautus–Belevitch (PHB) test [19, 33]. Specifically, (17) is uncontrollable if and only if there exists a left eigenvector ν of A, i.e., $\nu^T A = \lambda\nu^T$ for some λ, such that

$$\nu^T B = 0.$$

Since A is symmetric, its left and right eigenvectors are the same. Hence, the necessary and sufficient condition for controllability of (17) is that none of the eigenvectors of A should be simultaneously orthogonal to all columns of B. Additionally, in order to investigate the controllability of (17), one can form the controllability matrix as

$$\mathcal{C} = [\,B \;\; AB \;\; \cdots \;\; A^{n_f-1}B\,]. \tag{18}$$

As A is symmetric, it can be written in the form $U\Lambda U^T$, where Λ is the diagonal matrix of eigenvalues of A; U, on the other hand, is the unitary matrix comprised of

A's pairwise orthogonal unit eigenvectors. Since $B = UU^T B$, by factoring the matrix U from the left in (18), the controllability matrix assumes the form

$$\mathcal{C} = U\ [\, U^T B \ \ \Lambda U^T B \ \ \ldots \ \ \Lambda^{n_f - 1} U^T B\,]. \tag{19}$$

In this case, U is full rank and its presence does not alter the rank of the matrix product in (19). If one of the columns of U is perpendicular to all the columns of B, then $\mathcal{C}$ will have a row equal to zero, and hence the matrix $\mathcal{C}$ is rank deficient [36]. On the other hand, in the case of one leader, if any two eigenvalues of A are equal, then $\mathcal{C}$ will have two linear dependent rows, and again, the controllability matrix becomes rank deficient. Assume that ν_1 and ν_2 are two eigenvectors corresponding to the same eigenvalue and that none of them is orthogonal to B. Then $\nu = \nu_1 + c\nu_2$ is also an eigenvector of A for that eigenvalue. This will then allow us to choose $c = -\nu_1^T B / \nu_2^T B$, which renders $\nu^T B = 0$. In other words, we are able to find an eigenvector that is orthogonal to B. Hence, we arrive at the following observation.

Proposition 5.1. *Consider a leader-follower network whose evolution is described by* (17). *This system is controllable if and only if none of the eigenvectors of A is (simultaneously) orthogonal to (all columns of) B. Moreover, if A does not have distinct eigenvalues, then* (17) *is not controllable.*

Proposition 5.1 is also valid for the case with more than one leader and implies that in any finite time interval, the floating dynamic units can be independently steered from their initial states to an arbitrary final one based on local interactions with their neighbors. This controllability results is of course valid when the states of the leader nodes are assumed to be unconstrained.

Corollary 5.2. *The networked system* (17) *with a single leader is controllable if and only if none of the eigenvectors of A is orthogonal to* $\mathbf{1}$.

Proof. As shown in Proposition 3.4, the elements of B correspond to row-sums of A, i.e., $B = -A\,\mathbf{1}$. Thus, $\nu^T B = -\nu^T A\,\mathbf{1} = -\lambda\,(\nu^T\,\mathbf{1})$. By Proposition 4.2 one has $\lambda \neq 0$. Thereby, $\nu^T B = 0$ if and only if $\mathbf{1}^T \nu = 0$. ∎

Proposition 5.3. *If the networked system* (17) *is uncontrollable, there exists an eigenvector ν of A such that $\sum_{i \sim n} \nu(i) = 0$.*

Proof. Using Corollary 5.2, when the system is uncontrollable, there exists an eigenvector of A that is orthogonal to $\mathbf{1}$. As $A\nu = \lambda\nu$, we deduce that $\mathbf{1}^T(\,A\,\nu\,) = 0$. Moreover, using Proposition 3.3, we obtain

$$\nu^T \left\{\, \mathcal{L}(\mathcal{G}_f) + \mathcal{D}_{fl}(\mathcal{G}) \,\right\} \mathbf{1} = 0.$$

But $\mathcal{L}(\mathcal{G}_f)\mathbf{1} = 0$, and thereby

$$\nu^T\, \mathcal{D}_{fl}(\mathcal{G})\mathbf{1} = \nu^T \delta_n = 0,$$

which implies that $\sum_{i \sim n} \nu(i) = 0$. ∎

Proposition 5.4. *Suppose that the leader-follower system* (17) *is uncontrollable. Then there exists an eigenvector of $\mathcal{L}(\mathcal{G})$ that has a zero component on the index that corresponds to the leader.*

Proof. Let ν be an eigenvector of A that is orthogonal to $\mathbf{1}$ (by Corollary 5.2, such an eigenvector exists). Attach a zero to ν; using the partitioning (11), we then have

$$\mathcal{L}(\mathcal{G}) \begin{bmatrix} \nu \\ 0 \end{bmatrix} = \begin{bmatrix} A & -\delta_n \\ -\delta_n^T & d_n \end{bmatrix} \begin{bmatrix} \nu \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda\nu \\ -\delta_n^T \nu \end{bmatrix},$$

where δ_n is the indicator vector of the leader's neighbors. From Proposition 5.3 we know that $\delta_n^T \nu = 0$. Thus,

$$\mathcal{L}(\mathcal{G}) \begin{bmatrix} \nu \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} \nu \\ 0 \end{bmatrix}.$$

In the other words, $\mathcal{L}(\mathcal{G})$ has an eigenvector with a zero on the index that corresponds to the leader. ◻

A direct consequence of Proposition 5.4 is the following.

COROLLARY 5.5. *Suppose that none of the eigenvectors of $\mathcal{L}(\mathcal{G})$ has a zero component. Then the leader-follower system* (7) *is controllable for any choice of the leader.*

5.1. Controllability and graph symmetry. The controllability of the interconnected system depends not only on the geometry of the interunit information exchange but also on the position of the leader with respect to the graph topology. In this section, we examine the controllability of the system in terms of graph-theoretic properties of the network. In particular, we will show that there is an intricate relation between the controllability of (17) and the symmetry structure of the graph, as captured by its automorphism group. We first need to introduce a few useful constructs.

DEFINITION 5.6. *A permutation matrix is a $\{0, 1\}$-matrix with a single nonzero element in each row and column.*

DEFINITION 5.7. *The system* (17) *is anchor symmetric with respect to anchor a if there exists a nonidentity permutation J such that*

$$JA = AJ, \tag{20}$$

where $A = -\mathcal{L}_f = -P_f^T\, \mathcal{L}(\mathcal{G})\, P_f$ is constructed as in (12). *We call the system asymmetric if it does not admit such a permutation for any anchor.*

As an example, the graph represented in Figure 3(a) is leader symmetric with respect to $\{6\}$ but asymmetric with respect to any other leader node set. On the other hand, the graph of Figure 3(b) is leader symmetric with respect to a single leader located at every node. The utility of the notion of leader symmetry is now established through its relevance to the system-theoretic concept of controllability.

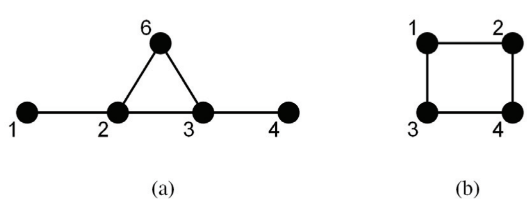

FIG. 3. *Interconnected topologies that are leader symmetric:* (a) *only with respect to node* $\{6\}$*;* (b) *with respect to a leader at any node.*

PROPOSITION 5.8. *The system* (17) *is uncontrollable if it is leader symmetric.*

Proof. If the system is leader symmetric, then there is a nonidentity permutation J such that

$$JA = AJ. \tag{21}$$

Recall that, by Proposition 5.1, if the eigenvalues of A are not distinct, then (17) is not controllable. We thus consider the case where all eigenvalues λ are distinct and satisfy

$A\nu = \lambda\nu$; thereby, for all eigenvalue/eigenvector pairs (λ, ν) one has $JA\nu = J(\lambda\nu)$. Using (21), however, we see that $A\,(J\nu) = \lambda\,(J\nu)$, and $J\nu$ is also an eigenvector of A corresponding to the eigenvalue λ. Given that λ is distinct and A admits a set of orthonormal eigenvectors, we conclude that for one such eigenvector ν, $\nu - J\nu$ is also an eigenvector of A. Moreover, $J\,B = J^T B = B$, as the elements of B correspond to the row-sums of the matrix A, i.e., $B = -A\,\mathbf{1}$. Thereby,

$$(\nu - J\nu)^T B = \nu^T B - \nu^T J^T\, B = \nu^T B - \nu^T B = 0. \tag{22}$$

This, on the other hand, translates into having one of the eigenvectors of A, namely $\nu - J\nu$, be orthogonal to B. Proposition 5.1 now implies that the system (17) is uncontrollable. □

Proposition 5.8 states that leader symmetry is a sufficient condition for uncontrollability of the system. It is instructive to examine whether leader asymmetry leads to a controllable system.

PROPOSITION 5.9. *Leader symmetry is not a necessary condition for system uncontrollability.*

Proof. In Figure 4, the subgraph shown by solid lines, $\mathcal{G}_f$, is the smallest asymmetric graph [21], in the sense that it does not admit any nonidentity automorphism. Let us augment this graph with the node "a" and connect it to all vertices of $\mathcal{G}_f$. Constructing the corresponding system matrix A (i.e., setting it equal to $-\mathcal{L}_f(\mathcal{G})$), we have

$$-A = \mathcal{L}(\mathcal{G}_f) + \mathcal{D}_{fl}(\mathcal{G}) = \mathcal{L}(\mathcal{G}_f) + I,$$

where I is the identity matrix of proper dimensions. Consequently, A has the same set of eigenvectors as $\mathcal{L}(\mathcal{G}_f)$. Since $\mathcal{L}(\mathcal{G}_f)$ has an eigenvector orthogonal to $\mathbf{1}$, A also has an eigenvector that is orthogonal to $\mathbf{1}$. Hence, the leader-follower system is not controllable. Yet, the system is not symmetric with respect to a; more on this will appear in section 5.2. □

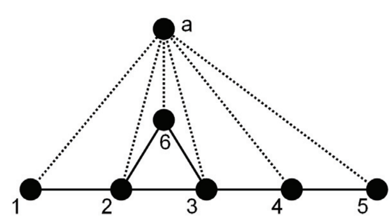

FIG. 4. *Asymmetric information topology with respect to the leader $\{a\}$. The subgraph shown by solid lines is the smallest asymmetric graph.*

It is intuitive that a highly connected leader will result in faster convergence to the agreement subspace. However, a highly connected leader also increases the chances that a symmetric graph, with respect to leader, emerges. A limiting case for this latter scenario is the complete graph. In such a graph, $n-1$ leaders are needed to make the corresponding controlled system controllable. This requirement is of course not generally desirable, as it means that the leader group includes all nodes except for one node! The complete graph is "the worse" case configuration as far as its controllability properties. Generally at most $n-1$ leaders are needed to make any information exchange network controllable. In the meantime, a path graph with a leader at one end is controllable. Thus it is possible to make a complete graph controllable by keeping the links on the longest path between a leader and all other nodes and deleting the unnecessary information exchange links to break its inherent symmetry. This procedure is not always feasible; for example, a star graph is not amenable to such graphical alterations.

5.2. Leader symmetry and graph automorphism. In section 5.1 we discussed the relationship between leader symmetry and controllability. In this section we will further explore the notion of leader symmetry with respect to graph automorphisms.

DEFINITION 5.10. *An automorphism of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation ψ of its node set such that*

$$(\psi(i)\,, \psi(j)) \in \mathcal{E}_{\mathcal{G}} \iff (i, j) \in \mathcal{E}_{\mathcal{G}}.$$

The set of all automorphisms of $\mathcal{G}$, equipped with the composition operator, constitutes the automorphism group of $\mathcal{G}$; note that this is a "finite" group. It is clear that the degree of a node remains unchanged under the action of the automorphism group; i.e., if ψ is an automorphism of $\mathcal{G}$, then $d_v = d_{\psi(v)}$ for all $v \in \mathcal{V}_{\mathcal{G}}$.

PROPOSITION 5.11 (see [5]). *Let $\mathcal{A}(\mathcal{G})$ be the adjacency matrix of the graph $\mathcal{G}$ and ψ a permutation on its node set $\mathcal{V}$. Associate with this permutation the permutation matrix Ψ such that*

$$\Psi_{ij} := \begin{cases} 1 & \textit{if} \quad \psi(i) = j, \\ 0 & \textit{otherwise.} \end{cases}$$

Then ψ is an automorphism of $\mathcal{G}$ if and only if

$$\Psi\,\mathcal{A}(\mathcal{G}) \,=\, \mathcal{A}(\mathcal{G})\,\Psi.$$

In this case, the least positive integer z for which $\Psi^z = I$ is called the order of the automorphism.

Recall that from Definition 5.7 leader symmetry for (17) corresponds to having

$$JA = AJ,$$

where J is a nonidentity permutation. From Proposition 3.3, however,

$$A = -(\mathcal{L}(\mathcal{G}_f) + \mathcal{D}_{fl}(\mathcal{G})).$$

Thus using the identity $\mathcal{L}(\mathcal{G}_f) = \mathcal{D}(\mathcal{G}_f) - \mathcal{A}(\mathcal{G}_f)$, one has

$$J\,\{\mathcal{D}(\mathcal{G}_f) - \mathcal{A}(\mathcal{G}_f) + \mathcal{D}_{fl}(\mathcal{G})\} = \{\mathcal{D}(\mathcal{G}_f) - \mathcal{A}(\mathcal{G}_f) + \mathcal{D}_{fl}(\mathcal{G})\}\;J. \tag{23}$$

Pre- and postmultiplication of (a permutation matrix) J does not change the structure of diagonal matrices. Also, all diagonal elements of $\mathcal{A}(\mathcal{G})$ are zero. We can thereby rewrite (23) as two separate conditions,

$$J\mathcal{D}_f(\mathcal{G}) = \mathcal{D}_f(\mathcal{G})\,J \quad \text{and} \quad J\mathcal{A}(\mathcal{G}_f) = \mathcal{A}(\mathcal{G}_f)J, \tag{24}$$

with $\mathcal{D}_f(\mathcal{G}) := \mathcal{D}(\mathcal{G}_f) + \mathcal{D}_{fl}(\mathcal{G})$. The second equality in (24) states that sought after J in (20) is in fact an automorphism of $\mathcal{G}_f$.

PROPOSITION 5.12. *Let Ψ be the permutation matrix associated with ψ. Then $\Psi\,\mathcal{D}_f(\mathcal{G}) = \mathcal{D}_f(\mathcal{G})\,\Psi$ if and only if*

$$d_i + \delta_n(i) = d_{\psi(i)} + \delta_n(\psi(i)).$$

In the case where ψ is an automorphism of $\mathcal{G}_f$, this condition simplifies to

$$\delta_n(i) = \delta_n(\psi(i)).$$

Proof. Using the properties of permutation matrices, one has

$$[\Psi\mathcal{D}_f(\mathcal{G})]_{ik} = \sum_t \Psi_{it}\mathcal{D}_{tk} = \begin{cases} d_k + \delta_n(k) & \text{if} \quad i \to k, \\ 0 & \text{otherwise} \end{cases}$$

and

$$[\mathcal{D}_f(\mathcal{G})\Psi]_{ik} = \sum_t \mathcal{D}_{it}\,\Psi_{tk} = \begin{cases} d_i + \delta_n(i) & \text{if} \quad i \to k, \\ 0 & \text{otherwise.} \end{cases}$$

For these matrices to be equal elementwise, one needs to have $d_i + \delta_n(i) = d_k + \delta_n(k)$ when $\psi(i) = k$. The second statement in the proposition follows from the fact that the degree of a node remains invariant under the action of the automorphism group. ∎

The next two results follow immediately from the above discussion.

PROPOSITION 5.13. *The interconnected system* (17) *is leader symmetric if and only if there is a nonidentity automorphism for $\mathcal{G}_f$ such that the indicator function remains invariant under its action.*

COROLLARY 5.14. *The interconnected system* (17) *is leader asymmetric if the automorphism group of the floating (or follower) subgraph contains only the trivial (identity) permutation.*

5.3. Controllability of special graphs. In this section we investigate the controllability of ring and path graphs.

PROPOSITION 5.15. *A ring graph, with only one leader, is never controllable.*

Proof. With only one leader in the ring graph, the follower graph $\mathcal{G}_f$ becomes the path graph with one nontrivial automorphism, i.e., its mirror image. Without loss of generality, choose the first node as the leader and index the remaining follower nodes by a clockwise traversing of the ring. Then the permutation $i \to n - i + 2$ for $i = 2, \ldots, n$ is an automorphism of $\mathcal{G}_f$. In the meantime, the leader "1" is connected to both node 2 and node n; hence $\delta_n = [\,1, 0, \ldots, 0, 1\,]^T$ remains invariant under the permutation. Using Proposition 5.13, we conclude that the corresponding system (17) is leader symmetric and thus uncontrollable. ∎

PROPOSITION 5.16. *A path graph is controllable for all choices of the leader node if and only if it is of even order.*

Proof. Suppose that the path graph is of odd order; then choose the middle node $\frac{n+1}{2}$ as the leader. Note that $\psi(k) = n - k + 1$ is an automorphism for the floating subgraph. Moreover, the leader is connected to nodes $\frac{n+1}{2} - 1$ and $\frac{n+1}{2} + 1$, and $\psi(\frac{n+1}{2} - 1) = \frac{n+1}{2} + 1$. Thus

$$\delta_n = [\,0, \ldots, 0, 1, 1, 0, \ldots, 0\,]^T$$

remains invariant under the permutation ψ and the system is uncontrollable. The converse statement follows analogously. ∎

Hence although in general leader symmetry is a sufficient—yet not necessary—condition for uncontrollability of (17), it is necessary and sufficient for uncontrollability of the path graph.

COROLLARY 5.17. *A path graph with a single leader is controllable if and only if it is leader asymmetric.*

6. Rate of convergence. In previous sections, we discussed controllability properties of controlled agreement dynamics in terms of the symmetry structure of the network. When the resulting system is controllable, the nodes can reach agreement arbitrarily fast.

PROPOSITION 6.1. *A controllable agreement dynamics* (17) *can reach the agreement subspace arbitrarily fast.*

Proof. The (invertible) controllability Grammian for (17) is defined as

$$W_a(t_0, t_f) = \int_{t_0}^{t_f} e^{sA} B B^T e^{sA^T} \, ds. \tag{25}$$

For any $t_f > t_0$, the leader can then transmit the signal

$$u(t) = B^T e^{A^T(t_f - t_0)} W_a(t_0, t_f)^{-1} \left(x_f - e^{A(t_f - t_0)} x_0 \right) \tag{26}$$

to its neighbors; in (26) x_0 and x_f are the initial and final states for the follower nodes, and t_0 and t_f are prespecified initial and final maneuver times. ∎

Next let us examine the convergence properties of the leader-follower network with a leader that transmits a constant signal (15). In this venue, define the quantity

$$\mu_2(\mathcal{L}_r(\mathcal{G})) := \min_{\substack{\zeta \neq 0 \\ \zeta \perp \mathbf{1}}} \frac{\zeta^T \, \overline{\mathcal{L}}_r(\mathcal{G}) \, \zeta}{\zeta^T \zeta}. \tag{27}$$

PROPOSITION 6.2. *The rate of convergence of the disagreement dynamics* (16) *is bounded by* $\mu_2(\mathcal{L}_r(\mathcal{G}))$ *and* $\lambda_2(\mathcal{L}(\mathcal{G}))$*, when the leader transmits a constant signal.*

Proof. Using the variational characterization of the second smallest eigenvalue of the graph Laplacian [14, 16], we have

$$\begin{aligned}
\lambda_2(\mathcal{L}(\mathcal{G})) = \min_{\substack{\zeta \neq 0 \\ \zeta \perp \mathbf{1}}} \frac{\zeta^T \mathcal{L}(\mathcal{G}) \zeta}{\zeta^T \zeta} &\leq \min_{\substack{\zeta \neq 0 \\ \zeta \perp \mathbf{1} \\ \zeta = Q\beta}} \frac{\zeta^T \mathcal{L}(\mathcal{G}) \zeta}{\zeta^T \zeta} \\
&= \min_{\substack{Q\beta \neq 0 \\ Q\beta \perp \mathbf{1}}} \frac{\beta^T Q \mathcal{L}(\mathcal{G}) Q \beta}{\beta^T Q \beta} \\
&= \min_{\substack{Q\beta \neq 0 \\ Q\beta \perp \mathbf{1}}} \frac{\beta^T Q \left\{ \frac{1}{2}(Q\mathcal{L}(\mathcal{G}) + \mathcal{L}(\mathcal{G})Q) \right\} Q\beta}{\beta^T Q \beta} \\
&= \min_{\substack{Q\beta \neq 0 \\ Q\beta \perp \mathbf{1}}} \frac{\beta^T Q \left(\frac{1}{2}(\mathcal{L}_r(\mathcal{G}) + \mathcal{L}_r(\mathcal{G})^T) \right) Q\beta}{\beta^T Q \beta} \\
&= \min_{\substack{\zeta \neq 0 \\ \zeta \perp \mathbf{1}}} \frac{\zeta^T \overline{\mathcal{L}}_r(\mathcal{G}) \zeta}{\zeta^T \zeta} = \mu_2(\overline{\mathcal{L}}_r(\mathcal{G})),
\end{aligned}$$

where β is an arbitrary vector with the appropriate dimension, Q is the matrix introduced in (14), and $Q^2 = Q$. In the last variational statement, we observe that ζ

should have a special structure, i.e., $\zeta = Q\beta$ (a zero at the row corresponding to the leader). An examination of the error dynamics suggests that such a structure always exists. As the leader does not update its value in the static leader case, the difference between the leader's state and the agreement value is always zero. Thus with respect to the disagreement dynamics (16),

$$\begin{aligned}\dot{V}(\zeta) &= -\zeta^T\,\overline{\mathcal{L}}_r(\mathcal{G})\,\zeta \le -\mu_2(\mathcal{L}_r(\mathcal{G}))\zeta^T\zeta \\ &\le -\lambda_2(\mathcal{L}(\mathcal{G}))\,\zeta^T\zeta. \qquad \square\end{aligned}$$

7. Controllability of multiple-leader networks.

Some applications of multi-agent systems may require multiple leaders. As our subsequent discussion shows, in this case, one needs an additional set of graph-theoretic tools to analyze the network controllability. In this venue, we first introduce equitable partitions and interlacing theory that play important roles in our analysis. We then present the main theorem of this section, providing a graph-theoretic characterization of controllability for multiple-leader networks.

7.1. Interlacing and equitable partitions.

A *cell* $C \subset \mathcal{V}_\mathcal{G}$ is a subset of the node set. A *partition* of the graph is then a grouping of its node set into different cells.

DEFINITION 7.1. *An r-partition π of $\mathcal{V}_\mathcal{G}$, with cells $C_1, \ldots, C_r$, is said to be* equitable *if each node in C_j has the same number of neighbors in C_i for all i, j. We denote the cardinality of the partition π by $r = |\pi|$.*

Let b_{ij} be the number of neighbors in C_j of a node in C_i. The directed graph with the cells of an equitable r-partition π as its nodes, and with b_{ij} edges from the ith to the jth cells of π, is called the quotient of $\mathcal{G}$ over π and is denoted by $\mathcal{G}/\pi$. An obvious trivial partition is the n-partition, $\pi = \{\{1\}, \{2\}, \ldots, \{n\}\}$. If an equitable partition contains at least one cell with more than one node, we call it a nontrivial equitable partition (NEP), and the adjacency matrix of a quotient is given by

$$\mathcal{A}(\mathcal{G}/\pi)_{ij} = b_{ij}.$$

Equitable partitions of a graph can be obtained from its automorphisms. For example, in the Peterson graph shown in Figure 5(a), one equitable partition π_1 (Figure 5(b)) is given by the two orbit of the automorphism groups, namely the 5 inner vertices and the 5 outer vertices. The adjacency matrix of the quotient is then given by

$$\mathcal{A}(\mathcal{G}/\pi_1) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The equitable partition can also be introduced by the equal distance partition. Let $C_1 \subset \mathcal{V}_\mathcal{G}$ be a given cell, and let $C_i \subset \mathcal{V}_\mathcal{G}$ be the set of vertices at distance $i-1$ from C_1. C_1 is said to be *completely regular* if its distance partition is equitable. For instance, every node in the Peterson graph is completely regular and introduces the partition π_2 as shown in Figure 5(c). The adjacency matrix of this quotient is given by

$$\mathcal{A}(\mathcal{G}/\pi_2) = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

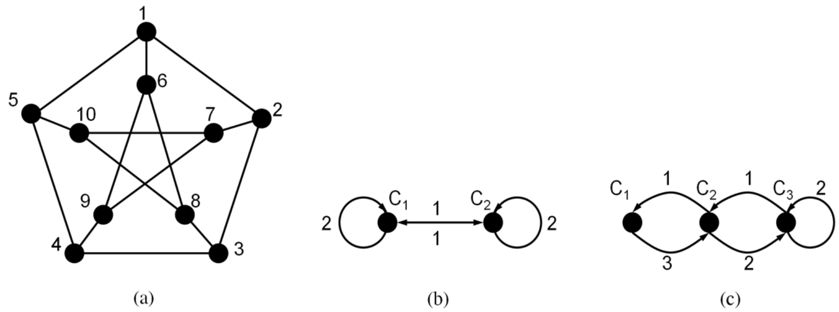

FIG. 5. (a) *Example of equitable partitions on the Peterson graph* $\mathcal{G} = J(5, 2, 0)$ *and the quotients;* (b) *the NEP introduced by the automorphism is* $\pi_1 = \{C_1^1, C_2^1\}$, $C_1^1 = \{1, 2, 3, 4, 5\}$, $C_2^1 = \{6, 7, 8, 9, 10\}$; *and* (c) *the NEP introduced by an equal-distance partition is* $\pi_2 = \{C_1^2, C_2^2, C_3^2\}$, $C_1^2 = \{1\}$, $C_2^2 = \{2, 5, 6\}$, $C_3^2 = \{3, 4, 7, 8, 9, 10\}$.

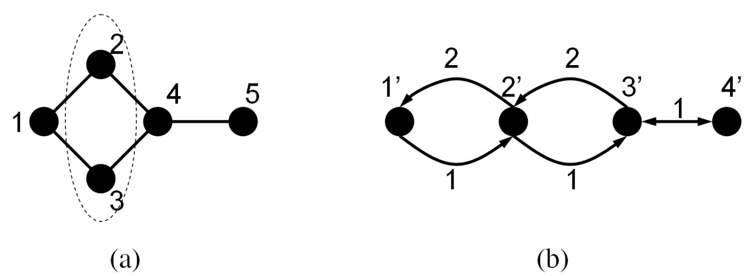

FIG. 6. (a) *The equitable partition and* (b) *the quotient of a graph.*

The adjacency matrix of the original graph and the quotient are closely related through the interlacing theorem. First, let us introduce the notion of the characteristic matrix of an equitable partition.

DEFINITION 7.2. *A* characteristic vector $p_i \in \mathbb{R}^n$ *of a nontrivial cell* C_i *has* 1*'s in components associated with* C_i *and* 0*'s elsewhere.*[3] *A* characteristic matrix $P \in \mathbb{R}^{n\times r}$ *of a partition* π *of* $\mathcal{V}_\mathcal{G}$ *is a matrix with characteristic vectors of the cells as its columns. For example, the characteristic matrix of the equitable partition of the graph in Figure* 6(a) *is given by*

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{28}$$

LEMMA 7.3 (see [14, Lemma 9.3.1]). *Let* P *be the characteristic matrix of an equitable partition* π *of the graph* $\mathcal{G}$, *and let* $\hat{\mathcal{A}} = \mathcal{A}(\mathcal{G}/\pi)$. *Then* $\mathcal{A}P = P\hat{\mathcal{A}}$ *and* $\hat{\mathcal{A}} = P^+\mathcal{A}P$, *where* $P^+ = (P^TP)^{-1}P^T$ *is the pseudo-inverse of* P.

[3] A nontrivial cell is a cell with more than one node.

As an example, the graph in Figure 6 has a nontrivial cell $(2,3)$. The adjacency matrix of the original graph is

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The adjacency matrix of the quotient, on the other hand, is

$$\hat{\mathcal{A}} = P^{+}\mathcal{A}P = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

LEMMA 7.4 (see [14, Lemma 9.3.2]). *Let $\mathcal{G}$ be a graph with adjacency matrix $\mathcal{A}$, and let π be a partition of $\mathcal{V}_\mathcal{G}$ with characteristic matrix P. Then π is equitable if and only if the column space of P is $\mathcal{A}$-invariant.*

LEMMA 7.5 (see [23]). *Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$, let S be a subspace of $\mathbb{R}^n$. Then $S^\perp$ is A-invariant if and only if S is A-invariant.*

The proof of this lemma is well known and can be found, for example, in [23].

Remark 7.6. Let $\mathcal{R}(\cdot)$ denote the range space. Suppose $|\mathcal{V}_\mathcal{G}| = n$, $|C_i| = n_i$, and $|\pi| = r$. Then we can find an orthogonal decomposition of $\mathbb{R}^n$ as

$$\mathbb{R}^n = \mathcal{R}(P) \oplus \mathcal{R}(Q). \tag{29}$$

In this case the matrix Q satisfies $\mathcal{R}(Q) = \mathcal{R}(P)^\perp$, and its columns, together with those of P, form a basis for $\mathbb{R}^n$. Note that by Lemma 7.5, $\mathcal{R}(Q)$ is also $\mathcal{A}$-invariant.

One way of obtaining the Q matrix is via the orthonormal basis of $\mathcal{R}(P)^\perp$. Let us denote the normalized matrix (each column of which is a norm one vector) by $\bar{Q}$. Next, define

$$\bar{P} = P(P^TP)^{-\frac{1}{2}} \tag{30}$$

as the normalized P matrix.[4] Since $\bar{P}$ and $\bar{Q}$ have the same column space as P and Q, respectively, they satisfy $\bar{P}^T\bar{Q} = \mathbf{0}$ and $\bar{Q}^T\bar{Q} = I_{n-r}$. In other words,

$$T = [\bar{P} \mid \bar{Q}] \tag{31}$$

is a matrix, constructed based on the equitable partition π, whose columns constitute an orthonormal basis for $\mathbb{R}^n$.

THEOREM 7.7 (see [14, Theorem 9.3.3]). *If π is an equitable partition of a graph $\mathcal{G}$, then the characteristic polynomial of $\hat{\mathcal{A}} = \mathcal{A}(\mathcal{G}/\pi)$ divides the characteristic polynomial of $\mathcal{A}(\mathcal{G})$.*

LEMMA 7.8 (see [14, Theorem 9.5.1]). *Let $\Phi \in \mathbb{R}^{n\times n}$ be a real symmetric matrix, and let $R \in \mathbb{R}^{n\times m}$ be such that $R^TR = I_m$. Set $\Theta = R^T\Phi R$ and let $\nu_1, \nu_2, \ldots, \nu_m$ be an orthogonal set of eigenvectors for Θ such that $\Theta\nu_i = \lambda_i(\Theta)\nu_i$, where $\lambda_i(\Theta) \in \mathbb{R}$ is*

[4]Note that the invertibility of P^TP follows from the fact that the cells of the partition are nonempty. In fact, P^TP is a diagonal matrix with $(P^TP)_{ii} = |C_i|$.

an eigenvalue of Θ. *Then*

1. *the eigenvalues of* Θ *interlace the eigenvalues of* Φ.
2. *if* $\lambda_i(\Theta) = \lambda_i(\Phi)$, *then there is an eigenvector* v *of* Θ *with eigenvalue* $\lambda_i(\Theta)$ *such that* $R\nu$ *is an eigenvector of* Φ *with eigenvalue* $\lambda_i(\Phi)$.
3. *if* $\lambda_i(\Theta) = \lambda_i(\Phi)$ *for* $i = 1, \ldots, l$, *then* $R\nu_i$ *is an eigenvector for* Φ *with eigenvalue* $\lambda_i(\Phi)$ *for* $i = 1, \ldots, l$.
4. *if the interlacing is tight, then* $\Phi R = R\Theta$.

Based on the controllability results introduced in section 5, together with some basic properties of the graph Laplacian, we first derive the following lemma.

LEMMA 7.9. *Given a connected graph, the system* (7) *is controllable if and only if* $\mathcal{L}$ *and* $\mathcal{L}_f$ *do not share any common eigenvalues.*

Proof. We can reformulate the lemma as stating that the system is uncontrollable if and only if there exists at least one common eigenvalue between $\mathcal{L}$ and $\mathcal{L}_f$.

Necessity. Suppose that the system is uncontrollable. Then by Proposition 5.1 there exists a vector $\nu_i \in \mathbb{R}^{n_f}$ such that $\mathcal{L}_f\nu_i = \lambda\nu_i$ for some $\lambda \in \mathbb{R}$, with $l_{fl}^T\nu_i = \mathbf{0}$. Now, since

$$\begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \mathcal{L}_l \end{bmatrix} \begin{bmatrix} \nu_i \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_f\nu_i \\ l_{fl}^T\nu_i \end{bmatrix} = \lambda \begin{bmatrix} \nu_i \\ \mathbf{0} \end{bmatrix},$$

λ is also an eigenvalue of $\mathcal{L}$, with eigenvector $[\nu_i^T, \mathbf{0}]^T$. The necessary condition thus follows.

Sufficiency. It suffices to show that if $\mathcal{L}$ and $\mathcal{L}_f$ share a common eigenvalue, then the system $(\mathcal{L}, l_{fl})$ is not completely controllable. Since $\mathcal{L}_f$ is a principal submatrix of $\mathcal{L}$, it can be given by

$$\mathcal{L}_f = P_f^T \mathcal{L} P_f,$$

where $P_f = [I_{n_f}, 0]^T$ is the $n \times n_f$ matrix defined in (12). Following the fourth statement of Lemma 7.8,[5] if $\mathcal{L}_f$ and $\mathcal{L}$ share a common eigenvalue, say λ, then the corresponding eigenvector satisfies

$$\nu = P_f\nu_f = \begin{bmatrix} \nu_f \\ \mathbf{0} \end{bmatrix},$$

where ν is λ's eigenvector of $\mathcal{L}$ and ν_f is that of $\mathcal{L}_f$. Moreover, we know that

$$\mathcal{L}\nu = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \mathcal{L}_l \end{bmatrix} \begin{bmatrix} \nu_f \\ \mathbf{0} \end{bmatrix} = \lambda \begin{bmatrix} \nu_f \\ \mathbf{0} \end{bmatrix},$$

which gives us $l_{fl}^T\nu_f = \mathbf{0}$; thus the system is uncontrollable. $\square$

Remark 7.10. Lemma 7.9 is an extension of Corollary 5.2, Propositions 5.3, and Proposition 5.4 to multileader settings.

7.2. Controllability analysis based on equitable partitions. In this section, we will utilize a graph-theoretic approach to characterize the necessary condition for a multiple-leader networked system to be controllable. The way we approach this necessary condition is through Lemma 7.9. In what follows we will show first that matrices $\mathcal{L}$ and $\mathcal{L}_f$ are both similar to some block diagonal matrices. Furthermore,

[5]Here the matrix P_f plays the same role as the matrix R in the fourth statement of Lemma 7.8.

we show that under certain assumptions, the diagonal block matrices obtained from the diagonalization of $\mathcal{L}$ and $\mathcal{L}_f$ have common diagonal block(s).

LEMMA 7.11. *If a graph $\mathcal{G}$ has an NEP π with characteristic matrix P, then the corresponding adjacency matrix $\mathcal{A}(\mathcal{G})$ is similar to a block diagonal matrix*

$$\bar{\mathcal{A}} = \begin{bmatrix} \mathcal{A}_P & \mathbf{0} \\ \mathbf{0} & \mathcal{A}_Q \end{bmatrix},$$

where $\mathcal{A}_P$ is similar to the adjacency matrix $\hat{\mathcal{A}} = \mathcal{A}(\mathcal{G}/\pi)$ of the quotient graph.

Proof. Let the matrix $T = [\bar{P} \mid \bar{Q}]$ be the orthonormal matrix with respect to π, as defined in (31). Let

$$\bar{\mathcal{A}} = T^T \mathcal{A} T = \begin{bmatrix} \bar{P}^T \mathcal{A} \bar{P} & \bar{P}^T \mathcal{A} \bar{Q} \\ \bar{Q}^T \mathcal{A} \bar{P} & \bar{Q}^T \mathcal{A} \bar{Q} \end{bmatrix}. \tag{32}$$

Since $\bar{P}$ and $\bar{Q}$ have the same column spaces as P and Q, respectively, they inherit their $\mathcal{A}$-invariance property, i.e., there exist matrices B and C such that

$$\mathcal{A}\bar{P} = \bar{P}B \quad \text{and} \quad \mathcal{A}\bar{Q} = \bar{Q}C.$$

Moreover, since the column spaces of $\bar{P}$ and $\bar{Q}$ are orthogonal complements of each other, one has

$$\bar{P}^T \mathcal{A} \bar{Q} = \bar{P}^T \bar{Q} C = \mathbf{0}$$

and

$$\bar{Q}^T \mathcal{A} \bar{P} = \bar{Q}^T \bar{P} B = \mathbf{0}.$$

In addition, by letting $D_p^2 = P^T P$, we obtain

$$\bar{P}^T \mathcal{A} \bar{P} = D_P^{-1} P^T \mathcal{A} P D_P^{-1} = D_P (D_P^{-2} P^T \mathcal{A} P) D_P^{-1} = D_P \hat{\mathcal{A}} D_P^{-1}, \tag{33}$$

and therefore the first diagonal block is similar to $\hat{\mathcal{A}}$. ∎

LEMMA 7.12. *Let P be the characteristic matrix of an NEP in $\mathcal{G}$. Then $\mathcal{R}(P)$ is K-invariant, where K is any diagonal block matrix of the form*

$$K = \mathbf{Diag}([\underbrace{k_1, \dots, k_1}_{n_1}, \underbrace{k_2, \dots, k_2}_{n_2}, \dots, \underbrace{k_r, \dots, k_r}_{n_r}]^T) = \mathbf{Diag}([k_i \mathbf{1}_{n_i}]_{i=1}^r),$$

$k_i \in \mathbb{R}$, $n_i = |C_i|$ is the cardinality of the cell, and $r = |\pi|$ is the cardinality of the partition. Consequently,

$$\bar{Q}^T K \bar{P} = \mathbf{0},$$

where $\bar{P} = P(P^T P)^{-\frac{1}{2}}$ and $\bar{Q}$ is chosen in such a way that $T = [\bar{P} \mid \bar{Q}]$ is an orthonormal matrix.

Proof. We note that

$$P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_r \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & \dots & p_r \end{bmatrix},$$

where $P_i \in \mathbb{R}^{n_i \times r}$ is a row block which has 1's in column i and 0's elsewhere. On the other hand, p_i is a characteristic vector representing C_i, which has 1's in the positions associated with C_i and zeros otherwise. Recall the example given in (28) with

$$P = \left[\begin{array}{c|c|cc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]; \tag{34}$$

we can then find

$$P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

while $p_2 = [0\ 1\ 1\ 0\ 0]^T$. A little algebra reveals that

$$KP = \begin{bmatrix} k_1 P_1 \\ k_2 P_2 \\ \vdots \\ k_r P_r \end{bmatrix} = \begin{bmatrix} k_1 p_1 & k_2 p_2 & \dots & k_r p_r \end{bmatrix} = P\hat{K},$$

where $\hat{K} = \mathbf{Diag}([k_1, k_2, \dots, k_r]^T) = \mathbf{Diag}([k_i]_{i=1}^r)$; hence $\mathcal{R}(P)$ is K-invariant. Since $\mathcal{R}(\bar{Q}) = \mathcal{R}(P)^\perp$, by Lemma 7.5 it is K-invariant as well and

$$\bar{Q}^T K \bar{P} = \bar{Q}^T \bar{P} \hat{K} = \mathbf{0}. \qquad \square$$

By the definition of equitable partitions, the subgraph induced by a cell is regular and every node in the same cell has the same number of neighbors outside the cell. Therefore, the nodes belonging to the same cell have the same degree, and thus by Lemma 7.12, $\mathcal{R}(\bar{Q})$ and $\mathcal{R}(P)$ are $\mathcal{D}$-invariant, where $\mathcal{D}$ is the degree matrix given by

$$\mathcal{D} = \mathbf{Diag}([d_i \mathbf{1}_{n_i}]_{i=1}^r),$$

with $d_i \in \mathbb{R}$ denoting the degree of each node in the cell. Since the graph Laplacian satisfies $\mathcal{L}(\mathcal{G}) = \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$, Lemmas 7.11 and 7.12 imply that $\mathcal{R}(\bar{Q})$ and $\mathcal{R}(P)$ are $\mathcal{L}$-invariant. Thereby, we have following corollary.

COROLLARY 7.13. *Given the same condition as in Lemma* 7.11, $\mathcal{L}$ *is similar to a diagonal block matrix*

$$\bar{\mathcal{L}} = T^T \mathcal{L} T = \begin{bmatrix} \mathcal{L}_P & \mathbf{0} \\ \mathbf{0} & \mathcal{L}_Q \end{bmatrix}, \tag{35}$$

where $\mathcal{L}_P = \bar{P}^T \mathcal{L} \bar{P}$ *and* $\mathcal{L}_Q = \bar{Q}^T \mathcal{L} \bar{Q}$, *and* $T = [\bar{P} \mid \bar{Q}]$ *defines an orthonormal basis for* $\mathbb{R}^n$ *with respect to* π.

As (35) defines a similarity transformation, it follows that $\mathcal{L}_P$ and $\mathcal{L}_Q$ carry all the spectral information of $\mathcal{L}$, i.e., they share the same eigenvalues as $\mathcal{L}$.

As we have shown in section 2, in a leader-follower network, the graph Laplacian can be partitioned as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \mathcal{L}_l \end{bmatrix}.$$

Transformations similar to (35) can also be found for $\mathcal{L}_f$ in the presence of NEPs in the follower graph $\mathcal{G}_f$.

COROLLARY 7.14. *Let $\mathcal{G}_f$ be a follower graph, and let $\mathcal{L}_f$ be the diagonal submatrix of $\mathcal{L}$ related to $\mathcal{G}_f$. If there is an NEP π_f in $\mathcal{G}_f$ and a π in $\mathcal{G}$ such that all the nontrivial cells in π_f are also cells in π, then there exists an orthonormal matrix T_f such that*

$$\bar{\mathcal{L}}_f = T_f^T \mathcal{L}_f T_f = \begin{bmatrix} \mathcal{L}_{fP} & \mathbf{0} \\ \mathbf{0} & \mathcal{L}_{fQ} \end{bmatrix}. \tag{36}$$

Proof. Let $\bar{P}_f = P_f(P_f^T P_f)^{\frac{1}{2}}$, where P_f is the characteristic matrix for π_f. Moreover, let $\bar{Q}_f$ be defined on an orthonormal basis of $\mathcal{R}(P_f)^{\perp}$. In this way, we obtain an orthonormal basis for $\mathbb{R}^{n_f}$ with respect to π_f. Moreover, by (9), $\mathcal{L}_f(\mathcal{G}) = \mathcal{D}_f^l(\mathcal{G}) + \mathcal{L}(\mathcal{G}_f)$, where $\mathcal{L}(\mathcal{G}_f)$ denotes the Laplacian matrix of $\mathcal{G}_f$ while $\mathcal{D}_f^l$ is the diagonal follower-leader degree matrix defined in (8). Since all the nontrivial cells in π_f are also cells in π, $\mathcal{D}_f$ satisfies the condition in Lemma 7.12, i.e., nodes from an identical cell in π_f have the same degree. Hence by Lemma 7.11 and Lemma 7.12, $\mathcal{R}(\bar{P}_f)$ and $\mathcal{R}(\bar{Q}_f)$ are $\mathcal{L}_f$-invariant and consequently,

$$\bar{\mathcal{L}}_f = T_f^T \mathcal{L}_f T_f = \begin{bmatrix} \mathcal{L}_{fP} & \mathbf{0} \\ \mathbf{0} & \mathcal{L}_{fQ} \end{bmatrix}, \tag{37}$$

where $T_f = [\bar{P}_f \mid \bar{Q}_f]$, $\mathcal{L}_{fP} = \bar{P}_f^T \mathcal{L}_f \bar{P}_f$, and $\mathcal{L}_{fQ} = \bar{Q}_f^T \mathcal{L}_f \bar{Q}_f$. ☐

Again, the diagonal blocks of $\bar{\mathcal{L}}_f$ contain the entire spectral information of $\mathcal{L}_f$. We are now in the position to prove the main result of this section.

THEOREM 7.15. *Given a connected graph $\mathcal{G}$ and the induced follower graph $\mathcal{G}_f$, the system* (7) *is not controllable if there exist NEPs on $\mathcal{G}$ and $\mathcal{G}_f$, say π and π_f, such that all nontrivial cells of π are contained in π_f; i.e., for all $C_i \in \pi \backslash \pi_f$, one has $|C_i| = 1$.*

Proof. In Corollaries 7.13 and 7.14, we have shown that $\mathcal{L}$ and $\mathcal{L}_f$ are similar to some block diagonal matrices. Here we further expand on the relationship between such matrices.

Assume that $\pi \cap \pi_f = \{C_1, C_2, \ldots, C_{r_1}\}$. According to the underlying condition, one has $|C_i| \geq 2,\ i = 1, 2, \ldots, r_1$. Without loss of generality, we can index the nodes in such a way that the nontrivial cells comprise the first n_1 nodes, where[6]

$$n_1 = \sum_{i=1}^{r_1} |C_i| \leq n_f < n.$$

As all the nontrivial cells of π are in π_f, their characteristic matrices have similar structures,

$$P = \begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & I_{n-n_1} \end{bmatrix}_{n \times r} \quad \text{and} \quad P_f = \begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & I_{n_f - n_1} \end{bmatrix}_{n_f \times r_f},$$

where P_1 is an $n_1 \times r_1$ matrix containing the nontrivial part of the characteristic matrices. Since $\bar{P}$ and $\bar{P}_f$ are the normalizations of P and P_f, respectively, they

[6]We have introduced n_1 for notational convenience. It is easy to verify that $n_1 - r_1 = n - r = n_f - r_f$.

have the same block structures. Consequently $\bar{Q}$ and $\bar{Q}_f$, the matrices containing the orthonormal bases of $\mathcal{R}(P)$ and $\mathcal{R}(P_f)$, have the following structures:

$$\bar{Q} = \begin{bmatrix} Q_1 \\ \mathbf{0} \end{bmatrix}_{n\times(n_1-r_1)} \quad \text{and} \quad \bar{Q}_f = \begin{bmatrix} Q_1 \\ \mathbf{0} \end{bmatrix}_{n_f\times(n_1-r_1)},$$

where Q_1 is an $n_1 \times (n_1 - r_1)$ matrix that satisfies $Q_1^T P_1 = \mathbf{0}$. We observe that $\bar{Q}_f$ is different from $\bar{Q}$ only by $n - n_f$ rows of zeros. In other words, the special structures of $\bar{Q}$ and $\bar{Q}_f$ lead to the relationship

$$Q_f = R^T Q,$$

where $R = [I_{n_f}, 0]^T$. Now, recall the definition of $\mathcal{L}_Q$ and $\mathcal{L}_{Qf}$ from (35) and (36), leading us to

$$\mathcal{L}_Q = \bar{Q}^T \mathcal{L} \bar{Q} = \bar{Q}_f^T R^T \mathcal{L} R \bar{Q}_f = \bar{Q}_f^T \mathcal{L}_f \bar{Q}_f = \mathcal{L}_{fQ}. \tag{38}$$

Therefore $\mathcal{L}_f$ and $\mathcal{L}$ share the same eigenvalues associated with $\mathcal{L}_Q$; hence by Lemma 7.9, the system is not controllable. □

Theorem 7.15 provides a method to identify uncontrollable multi-agent systems in the presence of multiple leaders. In an uncontrollable multi-agent system, vertices in the same cell of an NEP, satisfying the condition in Theorem 7.15, are not distinguishable from the leaders' point of view. In other words, agents belonging to a shared cell among π and π_f, when identically initialized, remain undistinguished to the leaders throughout the system evolution. Moreover, the controllable subspace for this multi-agent system can be obtained by collapsing all the nodes in the same cell into a single "meta-agent." However, since the NEPs may not be unique, as we have seen in the case of the Peterson graph, more work is required before a complete understanding of the intricate interplay between controllability and NEPs is obtained.

Two immediate ramifications of the above theorem are as follows.

Corollary 7.16. *Given a connected graph $\mathcal{G}$ with the induced follower graph $\mathcal{G}_f$, a necessary condition for* (7) *to be controllable is that no NEPs π and π_f, on $\mathcal{G}$ and $\mathcal{G}_f$, respectively, share a nontrivial cell.*

Corollary 7.17. *If $\mathcal{G}$ is disconnected, a necessary condition for* (7) *to be controllable is that all of its connected components are controllable.*

8. Simulation and discussions.

In this section we will explore controllable and uncontrollable leader-follower networks that are amenable to analysis via methods proposed in this paper.

Example 1 (single leader with symmetric followers). In Figure 6, if we choose node 5 as the leader, the symmetric pair $(2, 3)$ in the follower graph renders the network uncontrollable, as stated in [34]. The dimension of the controllable subspace is three, while there are four nodes in the follower group. This result can also be interpreted via Theorem 7.15, since the corresponding automorphisms introduce equitable partitions.

Example 2 (single leader with equal distance partitions). We have shown in Figure 5 that the Peterson graph has two NEPs. One is introduced by the automorphism group and the other (π_2) is introduced by the equal-distance partition. Based on π_2, if we choose node 1 as the leader, the leader-follower network ends up with a controllable subspace of dimension two. Since there are four orbits in the automorphism group,[7] this dimension pertains to the two-cell equal-distance partitions.[8]

[7] They are $\{2, 5, 6\}, \{7, 10\}, \{8, 9\}$, and $\{3, 4\}$.
[8] They are $\{2, 5, 6\}$ and $\{3, 4, 7, 8, 9, 10\}$.

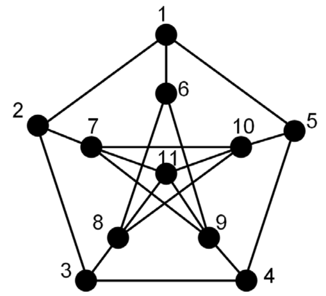

Fig. 7. *A 2-leader network based on the Peterson graph.*

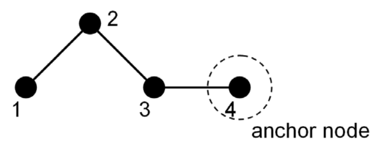

Fig. 8. *A path-like information exchange network.*

Example 3 (multiple leaders). This example is a modified leader graph based on the Peterson graph. In Figure 7, we add another node (11) connected to $\{3, 4, 7, 8, 9, 10\}$ as the second leader in addition to node 1. In this network, there is an equal-distance partition with four cells $\{1\}$, $\{2, 5, 6\}$, $\{3, 4, 7, 8, 9, 10\}$, and $\{11\}$. In this case, the dimension of the controllable subspace is still two, which is consistent with the second example above.

Example 4 (single-leader controllability). To demonstrate the controllability notion for the leader-follower system (7), consider a path-like information network, as shown in Figure 8. In this figure, the last node is chosen as the leader. By Proposition 5.17, this system is controllable. The system matrices in (7) assume the form

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Using (26), one can find the controller that drives the leader-follower system from any initial state to an arbitrary final state. For this purpose, we chose to re-orient the planar triangle on the node set $\{1, 2, 3\}$. The maneuver time is set to be five seconds. Figure 9 shows the initial and the final positions of the nodes along with their respective trajectories.

Figure 10, on the other hand, depicts the leader node state trajectory as needed to perform the required maneuver. This trajectory corresponds to the speed of node 4 in the xy-plane. We note that as there are no restrictions on the leader's state trajectory, the actual implementation of this control law can become infeasible, especially when the maneuver time is arbitrarily short. This observation is apparent in the previous example, in this scenario, the speed of node 4 changes rather rapidly from 20 [m/s] to -50 [m/s]. To further explore the relationship between the location of the leader node and the convergence time to the agreement subspace, an extensive set of simulations was also carried out. In these simulations, at each step, a random connected graph

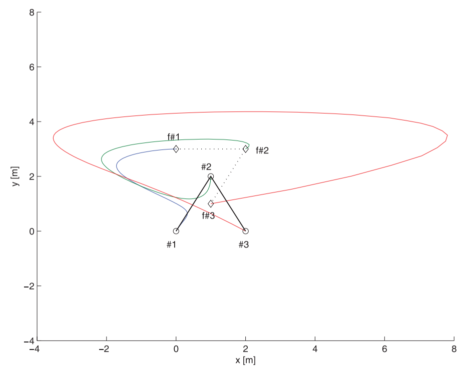

FIG. 9. *Initial and final positions of dynamic units and their respective state trajectories; $f\#i$ denotes the final position for agent i, $i = 1, 2, 3$.*

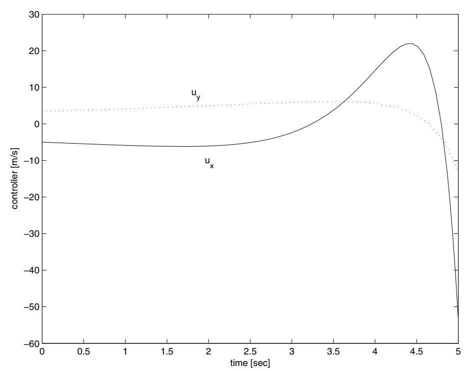

FIG. 10. *The leader node's velocity acts as a controller for the networked system.*

with 12 nodes and an edge probability of 0.3 was constructed. We then monitored the dynamics of the agreement protocol for the case when the center point of the graph was chosen to be the leader, as well as for the cases when the an arbitrary noncentral node is chosen.[9] These simulations were performed with 10 sets of randomly chosen initial conditions; the overall convergence time for each system was chosen to be the average of the total convergence times for all initial conditions. Figure 11 shows the result for 50 such iterations. We note that the convergence time is improved for the cases where the center of the graph is chosen as the leader.

9. Conclusions. In this paper, we considered the controlled agreement dynamics over a network. We first derived a set of transformations that can be employed to derive the system matrices for scenarios where one or more of the nodes (leader nodes) update their state values based on an external command. The other nodes in the graph (floating vertices) are assumed to update their states according to their relative states with their neighbors. In such a setting, we studied the controllability of the resulting dynamic system. It was shown that there is an intricate relationship between the uncontrollability of the corresponding multi-agent system and various graph-theoretic properties of the network. In particular, we pointed out the

[9]The center of the graph is a node with the following property: Its maximum distance to other nodes in the graph is minimum. We note that the center does not have to be unique.

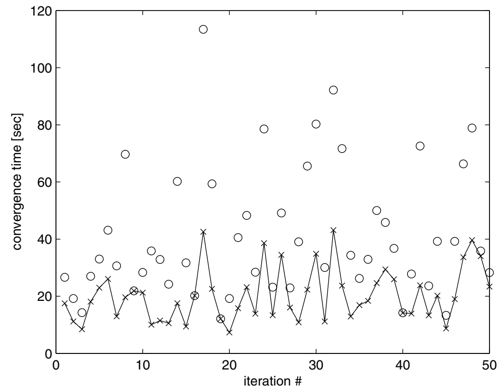

FIG. 11. *Convergence time comparison* (x: *center node is the leader.* o: *an arbitrary noncentral point is the leader*).

importance of the network automorphism group and its nontrivial equitable partitions in the controllability properties of the interconnected system. Some of the ramifications of this correspondence were then explored. The results of the present work point to a promising research direction at the intersection of graph theory and control theory that aims to study system-theoretic attributes from a purely graph-theoretic outlook.

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