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Consensus of signed networked multi-agent systems with nonlinear coupling and communication delays

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ABSTRACT

This paper investigates the consensus problem in networks of agents with antagonistic interactions and communication delays. For undirected signed networks, we respectively establish two dynamic models corresponding to linear and nonlinear coupling. Based on Lyapunov stability theory and some other mathematical analysis, it is proved that all agents on signed networks can reach agreement on a consensus value except for the sign. Further, a bipartite consensus solution is given for linear coupling networks, and an explicit expression associating with bipartite consensus solution is provided for nonlinear coupling networks. Finally, numerical simulations are given to demonstrate the effectiveness of our theoretical results.

1. Introduction

With the recent technological advance on communication resources, distributed control of multi-agent systems has been studied more and more widely [1–3]. Some significant results, such as consensus [4,5], state feedback control [6–8], formation control [9–16], and synchronization [17–24], have attracted attention of plentiful experts from diverse disciplines. Specially, the consensus of the multi-agent system is a focal research topic and has been intensively investigated in the past decade.

Consensus, which means to reach an agreement among all agents, is closely correlated with many practical applications, such as birds' migration [25,26], robots' coordination [27] and control of distributed sensor networks [28]. Some interesting works on consensus have been done in the past few years. In [29,30], consensus problem for nonlinearly coupled networks was investigated, and the results were generalized to directed networks. Aiming at detail-balanced multi-agent networks, [31] proposed a new protocol for finite-time consensus of multi-agent networks.

To achieve consensus, each node in a network has to transmit its state information to its neighbours via connections. However, because of physical and environmental limitations, communication constraints between connected nodes are unavoidable. As is well-known, the communication delay is one of the most universal communication constraints. Thus, researches on consensus of multi-agent systems containing communication delays received widespread attentions [32–41].

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The works in [39] focused on consensus problem of multi-agent networks with switching topology and time-delay. Lu et al. [36] studied the consensus over directed static networks with arbitrary finite communication delays, and found that the consensus of strongly connected networks can be asymptotically achieved for any limited communication delay.

A common feature of above literature [32–41] is the focus on cooperative system. The consensus of these systems is asymptotical achieved through collaboration, which is characterized by the nonnegative weights among agents. In many real-world cases, however, it is more reasonable to consider that some agents collaborate with each other, while others are competitive. Networks with antagonistic interactions are ubiquitous in real world [42,43], and it becomes a focus for studying in recent years. Proskurnikov et al. [44] investigated the opinion dynamics in social networks with hostile camps. Xia et al. [45] studied structural balance and opinion separation in trust-mistrust social networks. Moreover, [46] proved that bipartite consensus can be achieved over networks with antagonistic interactions. Furthermore, [47] gave the concept of interval bipartite consensus, and presented that interval bipartite consensus can be reached for directed networks with a spanning tree. Proskurnikov and Cao [48] studied the polarization in coopetitive networks of heterogeneous nonlinear agents. In [49], the pinning bipartite synchronization is studied for nonlinear systems with antagonistic interactions and switching topologies.

Motivated by the aforementioned discussions, we investigate the consensus problem of undirected signed networks with antagonistic interactions and communication delays. To the best of our knowledge, only a few work have been done about such problem. Due to this difficulty that antagonistic interactions and communication delays need to be simultaneously considered, new techniques are required to deal with this problem. According to matrix theory, Lyapunov theorem and some other mathematical analysis, we found that bipartite consensus can be achieved for those systems with communication delays. Furthermore, in order to obtain the final bipartite consensus solution, we construct an invariant function to study the relationship of the states of nodes and their initial states. Using some mathematical analysis skills, we provide the bipartite consensus solution with an explicit expression.

The reminder of this paper is organized as follows. In Section 2, we give some preliminaries containing some basic knowledge of graph theory, some necessary definitions and lemmas. In Section 3, efficient criteria are established for bipartite consensus of nonlinear coupling networks and linear coupling networks with communication delays. In addition, the final bipartite consensus value is analyzed in this part. In Section 4, numerical examples are given to demonstrate our main results. Finally, Section 5 shows the conclusion.

Notations: $\mathbb{R}^n$ denotes the n dimensional Euclidean space. $\mathbb{R}^{n\times n}$ is the set of $n\times n$ real matrices. The superscript "$\top$" represents the transpose. The signed function sign$(\cdot)$ is defined as follows: $sign(x)=1,\ x\geq 0$; $sign(x)=-1,\ x<0$.

2. Preliminaries

Let $G(\mathcal{V},\varepsilon,A)$ be an undirected (weighted) signed graph, where $\mathcal{V}=\{\nu_1,\nu_2,\dots,\nu_N\}$ is the set of finite nodes, $\varepsilon\in\mathcal{V}\times\mathcal{V}$ is the set of edges, and $A=[a_{ij}]\in R^{N\times N}$ is the adjacency matrix of G: $a_{ij}\neq 0\Longleftrightarrow(\nu_j,\ \nu_i)\in\varepsilon$. Since a_{ij} can be positive or negative, the adjacency matrix A uniquely corresponds to a signed graph. In this paper, the notation $G(A)$ is used to denote the signed graph corresponding to A for simplicity. A cycle of $G(A)$ is a sequence of edges $(\nu_{j_1},\nu_{j_2}),(\nu_{j_2},\nu_{j_3}),\dots,(\nu_{j_p},\nu_{j_1})$, which starts from ν_{j_1} and ends to ν_{j_1}. A cycle is *negative* if it contains an odd number of negative edge weights: $a_{j_1j_2}\cdot a_{j_2j_3}\dots a_{j_pj_1}<0$. It is *positive* if $a_{j_1j_2}\cdot a_{j_2j_3}\dots a_{j_pj_1}>0$.

The following are some necessary definitions and lemmas for the derivation of the theoretical results.

Definition 1 [46]**.** A signed graph $G(A)$ is structurally balanced if it admits a bipartition of the nodes $\mathcal{V}_1,\ \mathcal{V}_2,\ \mathcal{V}_1\cup\mathcal{V}_2=\mathcal{V}$, $\mathcal{V}_1\cap\mathcal{V}_2=\emptyset$, such that $a_{ij}\geq 0,\ \forall\nu_i,\nu_j\in\mathcal{V}_q,(q\in\{1,2\})$, and $a_{ij}\leq 0,\ \forall\nu_i\in\mathcal{V}_q,\ \nu_j\in\mathcal{V}_r,\ q\neq r,\ (q,\ r\in\{1,\ 2\})$. It is said structurally unbalanced otherwise.

Definition 2. $\mathcal{D}=\{diag(\sigma)\mid\sigma=[\sigma_1,\sigma_2,\dots,\sigma_N],\sigma_i\in\{\pm 1\}\}$ is a set of diagonal matrices.

Lemma 1 [46]**.** *A connected signed graph $G(A)$ is structurally balanced if and only if any of the following equivalent conditions holds: (1) all cycles of $G(A)$ are positive; (2) $\exists D\in\mathcal{D}$ such that DAD has all nonnegative entries.*

Lemma 2 [46]**.** *A connected signed graph $G(A)$ is structurally unbalanced if and only if any of the following equivalent conditions holds: (1) one or more cycles of $G(A)$ are negative; (2) $\nexists D\in\mathcal{D}$ such that DAD has all nonnegative entries;*

3. Main results

Let $\mathcal{N}=\{1,2,\dots,N\}$ be a set of nodes. In this section, we will investigate the multi-agent systems with nonlinear coupling. Consider the following multi-agent systems:

$$\dot{x}_i(t)=\sum_{j=1}^{N}\mid a_{ij}\mid\left\{\text{sign}(a_{ij})h[x_j(t-\tau_{ij})]-h[x_i(t)]\right\},\ i\in\mathcal{N}. \tag{1}$$

where $x_i(t)\in\mathbb{R}$ is the state of node i at time t, and $\tau_{ij}>0$ denotes the communication delay from ν_j to ν_i for $i\neq j$ and $\tau_{ii}=0$. The function $h(\cdot):\mathbb{R}\to\mathbb{R}$ is assumed to be odd and strictly monotone increasing, which implies $h(0)=0$ and $h(-x)=-h(x)$. Further we assume that $h(\cdot)$ is unbounded. The adjacency matrix $A=[a_{ij}]$ is symmetric, which implies

$a_{ij} = a_{ji}, \forall i, j \in \mathcal{N}$, where a_{ij} denotes the weight between node i and node j in the coupled networks. Here it is assumed that there is no self-closed loop, which means that $a_{ii} = 0$.

Throughout this paper, the bipartite consensus of dynamical system (1) is said to be realized if $\lim_{t\to\infty} x_i(t) = \alpha$ for $i \in \mathcal{V}_1$ and $\lim_{t\to\infty} x_i(t) = -\alpha$ for $i \in \mathcal{V}_2$.

Theorem 1. *Consider the nonlinear coupled system (1) with a connected signed graph $G(A)$. The bipartite consensus can be asymptotically reached if $G(A)$ is structurally balanced. If instead $G(A)$ is structurally unbalanced, then $\lim_{t\to\infty} x(t) = 0$.*

Proof. Following Lemma 1, if $G(A)$ is structurally balanced, we can obtain that $\exists D \in \mathcal{D}$ such that DAD has all nonnegative entries. Let $Z(t) = Dx(t)$, i.e. $z_i(t) = \sigma_i x_i(t)$, one can easily get that

$$\dot{z}_i(t) = \sum_{j=1}^{N} | a_{ij} | \left\{ h[z_j(t-\tau_{ij})] - h[z_i(t)] \right\} \;\; i \in \mathcal{N}. \tag{2}$$

Following [36], we obtain that $\lim_{t\to\infty} z_i(t) \to \alpha \in \mathbb{R}$ for any $i \in \mathcal{N}$, which shows that $\lim_{t\to\infty} x_i(t) \to \sigma_i\alpha \in \mathbb{R}$ for $i \in \mathcal{N}$. Therefore, the bipartite consensus of system (1) can be reached if $G(A)$ is structurally balanced.

Next, we consider the case that $G(A)$ is structurally unbalanced. Following Lemma 2, we can conclude that $G(A)$ contains one or more negative cycles. For the sake of simplicity, let us first consider the simplest case of $G(A)$ with only one negative cycle. Without loss of generality, we assume that (ν_1, ν_2) belongs to the negative cycle and $a_{12} = a_{21} = a < 0$. According to Lemma 2, one can obtain that there is no $D \in \mathcal{D}$ such that DAD is a nonnegative matrix. However, for the subgraph $G(B)$, which denotes the rest part of $G(A)$ reducing the edge (ν_1, ν_2), it is structurally balanced and admits a bipartition of the nodes $\mathcal{V}_1$ and $\mathcal{V}_2$. Furthermore, one can find that $G(B)$ is connected and matrix B is irreducible. Then, we can conclude that nodes ν_1 and ν_2 simultaneously belong to $\mathcal{V}_1$ (or $\mathcal{V}_2$) and the rest nodes remain unchanged. Thus, we can choose $D_1 = diag(\sigma)$ with σ satisfying $\sigma_i = 1$ for $\nu_i \in \mathcal{V}_1$ and $\sigma_i = -1$ for $\nu_i \in \mathcal{V}_2$. Then $D_1AD_1 = A' = [a'_{ij}]_{N\times N}$ has exactly two negative elements, i.e. $a'_{12} = a'_{21} = a < 0$, and the rest elements are non-negative. The decomposition of the matrix A' shows as follows:

$$A' = A_{12} + A_{21} + B',$$

where $A_{ij}, i, j \in \mathcal{N}$ denotes a matrix in which the element lied in the intersection of ith row and jth column is $a_{ij} \neq 0$ and others all are 0. $B' = [b'_{ij}]_{N\times N}$ is a nonnegative adjacency matrix. In order to clearly express the matrix B', we define a function as follows:

$$c(i, j) = \begin{cases} 0, & (i, j) = (1, 2) \text{ or } (2, 1); \\ 1, & \text{otherwise}. \end{cases}$$

Hence, we get $b'_{ij} = c(i, j)|a_{ij}|$. It is easy to find that B' is irreducible. Let $\bar{B} = [\bar{b}_{ij}]_{N\times N}$ be the Laplacian matrix of $G(B')$, and its elements are defined as : $\bar{b}_{ij} = b'_{ij}$ $(i \neq j)$, $\bar{b}_{ii} = -\sum_{j=1}^{N} b'_{ij}$. Therefore, $\xi = (1, 1, \dots, 1)^T$ is the left eigenvector of $\bar{B}$ corresponding to the zero eigenvalue, i.e. $\xi^\top \bar{B} = 0$, which implies that $\bar{b}_{ii} = -\sum_{j=1, j\neq i}^{N} \bar{b}_{ji}$. Further because $\bar{b}_{ii} = -\sum_{j=1}^{N} b'_{ij}$, one can obtain that

$$\sum_{j=1}^{N} b'_{ij} = \sum_{j=1}^{N} b'_{ji} \;\; \text{and} \;\; \sum_{i=1}^{N} b'_{ji} = \sum_{i=1}^{N} b'_{ij}, \tag{3}$$

Let $Z(t) = Dx(t)$, i.e. $z_i(t) = \sigma_i x_i(t)$ for any $i \in \mathcal{N}$, we have

$$\dot{z}_i(t) = \sum_{j=1}^{N} | a_{ij} | \left\{ \sigma_i\sigma_j \text{sign}(a_{ij}) h[z_j(t-\tau_{ij})] - h[z_i(t)] \right\}. \tag{4}$$

Consider the following Lyapunov–Krasovskii functional for system (1)

$$V(x(t)) = V_1(x(t)) + V_2(x(t)), \tag{5}$$

where

$$V_1(x(t)) = \sum_{i=1}^{N} \int_0^{x_i(t)} h(s)ds, \tag{6}$$

and

$$V_2(x(t)) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{t-\tau_{ij}}^{t} |a_{ij}| h^2[x_j(s)]ds. \tag{7}$$

Calculating the time derivative of $V_i(t) (i = 1, 2)$ along the trajectories of system (1), we have

$$\begin{aligned}
\dot{V}_1(x(t)) &= \sum_{i=1}^{N} h[x_i(t)]\dot{x}_i(t) \\
&= \sum_{i=1}^{N} h[x_i(t)] \sum_{j=1}^{N} |a_{ij}| \Big\{ \mathrm{sgn}(a_{ij}) h[x_j(t-\tau_{ij})] - h[x_i(t)] \Big\} \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}| \Big\{ \sigma_i\sigma_j \mathrm{sgn}(a_{ij}) h[z_i(t)] h[z_j(t-\tau_{ij})] - h^2[z_i(t)] \Big\} \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} b'_{ij} \Big\{ h[z_i(t)] h[z_j(t-\tau_{ij})] - h^2[z_i(t)] \Big\} + a_{12} \Big\{ h[z_1(t)] h[z_2(t-\tau_{12})] \\
&\quad + h^2[z_1(t)] \Big\} + a_{21} \Big\{ h[z_2(t)] h[z_1(t-\tau_{21})] + h^2[z_2(t)] \Big\} \\
&= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} b'_{ij} \Big\{ 2h[z_i(t)] h[z_j(t-\tau_{ij})] - 2h^2[z_i(t)] \Big\} \\
&\quad + a \Big\{ h[z_1(t)] h[z_2(t-\tau_{12})] + h^2[z_1(t)] + h[z_2(t)] h[z_1(t-\tau_{21})] + h^2[z_2(t)] \Big\},
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\dot{V}_2(x(t)) &= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}| \Big\{ h^2[x_j(t)] - h^2[x_j(t-\tau_{ij})] \Big\} \\
&= \frac{1}{2} \sum_{i=1}^{N} b_{ji} \cdot \sum_{j=1}^{N} h^2[x_j(t)] - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} \cdot h^2[x_j(t-\tau_{ij})] \\
&\quad + \frac{1}{2} a_{12} \Big\{ h^2[x_2(t-\tau_{12})] - h^2[x_2(t)] \Big\} \\
&\quad + \frac{1}{2} a_{21} \Big\{ h^2[x_1(t-\tau_{21})] - h^2[x_1(t)] \Big\} \\
&= \frac{1}{2} \sum_{i=1}^{N} b'_{ij} \sum_{j=1}^{N} h^2[z_i(t)] - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} \cdot h^2[z_j(t-\tau_{ij})] \\
&\quad + \frac{1}{2} a \Big\{ h^2[z_2(t-\tau_{12})] - h^2[z_2(t)] + h^2[z_1(t-\tau_{21})] - h^2[z_1(t)] \Big\}.
\end{aligned} \tag{9}$$

Using Eqs (8) and (9), we get that

$$\begin{aligned}
\dot{V}(x(t)) = &- \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} b'_{ij} \Big\{ h^2[z_i(t)] - 2h[z_i(t)] h[z_j(t-\tau_{ij})] + h^2[z_j(t-\tau_{ij})] \Big\} \\
&+ \frac{1}{2} a \Big\{ 2h[z_1(t)] h[z_2(t-\tau_{12})] + h^2[z_1(t)] + h^2[z_2(t-\tau_{12})] \\
&+ 2h[z_2(t)] h[z_1(t-\tau_{21})] + h^2[z_2(t)] + h^2[z_1(t-\tau_{21})] \Big\} \\
= &- \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} b'_{ij} \Big\{ h[z_i(t)] - h[z_j(t-\tau_{ij})] \Big\}^2 \\
&+ \frac{1}{2} a \Big\{ h[z_1(t)] + h[z_2(t-\tau_{12})] \Big\}^2 \\
&+ \frac{1}{2} a \Big\{ h[z_2(t)] + h[z_1(t-\tau_{21})] \Big\}^2 \\
\leq &\, 0.
\end{aligned} \tag{10}$$

Let $S = \{x(t) \mid \dot{V}(x(t)) = 0\}$. Then it follows from Eq. (10) that $S = \{x \in C([t-\tau, t], R^N) \mid b'_{ij}\{h[z_i(t)] - h[z_j(t-\tau_{ij})]\} = 0,\ h[z_1(t)] + h[z_2(t-\tau_{12})] = 0$, and $h[z_2(t)] + h[z_1(t-\tau_{21})] = 0\}$. Combining with the property of $h(\cdot)$, we can get that the set S is an invariant set with respect to system (4). According to the LaSalle's invariance principle [50], one can easily show that $x(t) \to S$ as $t \to \infty$. Thus, we have $\lim_{t\to\infty}\{h[z_i(t)] - h[z_j(t-\tau_{ij})]\} = 0$ for $b'_{ij} > 0$, $\lim_{t\to\infty}\{h[z_1(t)] + h[z_2(t-\tau_{12})]\} = 0$, and $\lim_{t\to\infty}\{h[z_2(t)] + h[z_1(t-\tau_{21})]\} = 0$. Hence, we have $\lim_{t\to\infty} \dot{z}_i(t) = 0$. In addition, since $h(\cdot)$ is unbounded and strictly increasing with $h(0) = 0$, we get that $\lim_{t\to\infty}[z_i(t) - z_j(t-\tau_{ij})] = 0$ when $b'_{ij} > 0$ and $\lim_{t\to\infty}[z_1(t) + z_2(t-\tau_{12})] = 0$. According to the fact that B is irreducible, we conclude that $z_1(t) = z_2(t) = \cdots = z_N(t)$ as $t \to \infty$. It follows from $\lim_{t\to\infty}[z_1(t) +$

$z_2(t-\tau_{12})]=0$ that $z_1(t)=-z_2(t)$ as $t\to\infty$. Hence, the following equality can be concluded: $\lim_{t\to\infty}z_1(t)=\lim_{t\to\infty}z_2(t)=\cdots=\lim_{t\to\infty}z_N(t)=0$. Therefore, we obtain that $\lim_{t\to\infty}x_i(t)=\lim_{t\to\infty}\sigma_i z_i(t)=0$ for $i\in\mathcal{N}$.

Now consider the case of $G(A)$ with $m(m\geq 2)$ negative cycles. Referring to above approach, in order to obtain a structurally balanced subgraph $G(B'')$, we respectively select m negative edges from m negative cycles as follows: $(\nu_{i_1},\nu_{j_1}),\ldots,(\nu_{i_m},\nu_{j_m})$. Hence, Similarly there exists an diagonal matrix $D_2\in\mathcal{D}$ such that

$$D_2AD_2=\sum_{k=1}^{k=m}(A_{i_kj_k}+A_{j_ki_k})+B'',$$

where $B''=[b''_{ij}]_{N\times N}$ is a nonnegative irreducible matrix. Further, it follows from Lemma 2 that there at least exist a $r\in\{1,2,\ldots,m\}$ to render $a_{i_rj_r}=a_{j_ri_r}=a_r<0$. Hence, according to (6) and (8), we have

$$\begin{aligned}\dot V_1(t)=&\sum_{i=1}^{N}\sum_{j=1}^{N}b''_{ij}\Big\{\sigma_i\sigma_j\mathrm{sign}(a_{ij})h[z_i(t)]h[z_j(t-\tau_{ij})]-h^2[z_i(t)]\Big\}\\&+\sum_{k=1,k\neq r}^{m}|a_{i_kj_k}|\Big\{\sigma_{i_k}\sigma_{j_k}\mathrm{sign}(a_{i_kj_k})h[z_{i_k}(t)]h[z_{j_k}(t-\tau_{i_kj_k})]-h^2[z_{i_k}(t)]\Big\}\\&+\sum_{k=1,k\neq r}^{m}|a_{j_ki_k}|\Big\{\sigma_{j_k}\sigma_{i_k}\mathrm{sign}(a_{j_ki_k})h[z_{j_k}(t)]h[z_{i_k}(t-\tau_{j_ki_k})]-h^2[z_{j_k}(t)]\Big\}\\&+a_{i_rj_r}\Big\{h[z_{i_r}(t)]h[z_{j_r}(t-\tau_{i_rj_r})]+h^2[z_{i_r}(t)]\Big\}\\&+a_{j_ri_r}\Big\{h[z_{j_r}(t)]h[z_{i_r}(t-\tau_{j_ri_r})]+h^2[z_{j_r}(t)]\Big\},\end{aligned}\tag{11}$$

Using Eqs. (5), (9) and (11) gives that

$$\begin{aligned}\dot V(t)=&-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}b''_{ij}\Big\{h[z_i(t)]-h[z_j(t-\tau_{ij})]\Big\}^2\\&+\frac{1}{2}\sum_{k=1,k\neq r}^{m}a_k\Big\{h[z_{i_k}(t)]-\sigma_{i_k}\sigma_{j_k}\mathrm{sign}(a_{i_kj_k})h[z_{j_k}(t-\tau_{i_kj_k})]\Big\}^2\\&+\frac{1}{2}\sum_{k=1,k\neq r}^{m}a_k\Big\{h[z_{j_k}(t)]-\sigma_{j_k}\sigma_{i_k}\mathrm{sign}(a_{j_ki_k})h[z_{i_k}(t-\tau_{j_ki_k})]\Big\}^2\\&+\frac{1}{2}a_r\{[h[z_{i_r}(t)]+h[z_{j_r}(t-\tau_{i_rj_r})]\}^2+\frac{1}{2}a_r\{h[z_{j_r}(t)]+h[z_{i_r}(t-\tau_{j_ri_r})]\}^2\\&\leq 0.\end{aligned}$$

where $a_k=a_{i_kj_k}=a_{j_ki_k}<0$. Therefore, similarly we can get that $\lim_{t\to\infty}x_i(t)=0$ for any $i\in\mathcal{N}$. □

Remark 1. There is a situation needing to be considered. If a negative edge simultaneously belong to two or more negative cycles, we should admit that the edge only belongs to one of those cycles. Then we can still make a hypothesis that this negative edge belongs to $\mathcal{V}_1$ or $\mathcal{V}_2$. According to the proof progress of Theorem 1, it is obvious that the results of Theorem 1 still hold.

Specially, we consider a multi-agent system formed by N linearly coupled identical nodes, where each node's dynamic is described as follows:

$$\dot x_i(t)=\sum_{j=1}^{N}|\,a_{ij}\,|\,[\mathrm{sign}(a_{ij})x_j(t-\tau_{ij})-x_i(t)],\ \ i\in\mathcal{N},\tag{12}$$

where $x_i(t)\in\mathbb{R}^n$ is the state of node i at time t. Then, one can obtain the following corollary.

Corollary 1. *Consider the networked multi-agent system (12) with a connected signed graph $G(A)$. The bipartite consensus can be asymptotically reached if $G(A)$ is structurally balanced. If instead $G(A)$ is structurally unbalanced, then $\lim_{t\to\infty}x(t)=0$.*

Remark 2. Different from the result in Ref. [46], our result shows that the bipartite consensus can be realized even in the presence of communication delays.

According to Theorem 1, if $G(A)$ is structurally balanced, the bipartite consensus can be asymptotically reached and we have $\lim_{t\to\infty}x_i(t)=\alpha$ for $i\in\mathcal{V}_1$, $\lim_{t\to\infty}x_i(t)=-\alpha$ for $i\in\mathcal{V}_2$. Calculating the bipartite consensus value of α is not an easy task due to the delays. Here, the bipartite solution of (1) when the initial conditions of system (1) are given. The initial conditions about system (1) are provided as $x_i(s) = \sigma_i\psi_i(s) \in C([-\tau, 0], \mathbb{R})$, where $\tau = \max_{i,j}(\tau_{ij})$. Hence, we have $z_i(s) = \psi_i(s) \in C([-\tau, 0], \mathbb{R})$. Let $\zeta(0) = (1/N)\big[\sum_{i=1}^N \psi_i(0) + \sum_{i=1}^N \sum_{j=1}^N |a_{ij}| \int_{-\tau_{ij}}^0 h(\psi_j(s))ds\big]$ and $\mathbf{1} = (1, 1, \dots, 1)^\top_{1\times N}$.

Theorem 2. *Consider a connected signed graph $G(A)$ which is structurally balanced. If $D \in \mathcal{D}$ renders DAD nonnegative, then the bipartite solution of (1) is $\lim_{t\to\infty} x(t) = \alpha D\mathbf{1}$, where $\alpha \in \mathbb{R}$ meets a relational expression as follows:*

$$N\alpha + h(\alpha)\sum_{i=1}^N\sum_{j=1}^N(|a_{ij}|\tau_{ij}) - N\times\zeta(0) = 0. \tag{13}$$

Proof. Referring to the proof of Theorem 1, one can obtain that $\lim_{t\to\infty} z_i(t) = \alpha$ and $\lim_{t\to\infty} x(t) = \alpha D\mathbf{1}$. Let

$$\zeta(t) = \frac{1}{N}\Big[\sum_{i=1}^N z_i(t) + \sum_{i=1}^N\sum_{j=1}^N |a_{ij}| \int_{t-\tau_{ij}}^t h(z_j(s))ds\Big]. \tag{14}$$

Using (2), we can obtain

$$\begin{aligned}\dot{\zeta}(t) =& \frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N |a_{ij}|\big\{h[z_j(t-\tau_{ij})] - h[z_i(t)]\big\}\\ &+ \frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N |a_{ij}|\big\{h[z_j(t)] - h[z_j(t-\tau_{ij})]\big\}\\ =& -\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N |a_{ij}|h[z_i(t)] + \frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N |a_{ij}|h[z_j(t)]\\ =&0.\end{aligned} \tag{15}$$

Therefore, $\zeta(t)$ in (15) is a constant. That is,

$$\zeta(t) = \zeta(0) = \frac{1}{N}\left[\sum_{i=1}^N \psi_i(0) + \sum_{i=1}^N\sum_{j=1}^N |a_{ij}| \int_{-\tau_{ij}}^0 h(\psi_j(s))ds\right]. \tag{16}$$

Then, we can get

$$\zeta(0) = \lim_{t\to\infty}\zeta(t) = \frac{1}{N}\left[N\alpha + \sum_{i=1}^N\sum_{j=1}^N |a_{ij}|\tau_{ij}h(\alpha)\right]. \tag{17}$$

Hence, we have

$$N\alpha + h(\alpha)\sum_{i=1}^N\sum_{j=1}^N(|a_{ij}|\tau_{ij}) - N\times\zeta(0) = 0. \tag{18}$$

□

Remark 3. In this theorem, although the value of α cannot be given by an explicit expression, we can get a numerical solution by iterative algorithm from (18). In Numerical example, Example 1 gives a numerical solution to (18) for $h(x) = x + 0.5sin(x)$, which illustrates the computational feasibility.

In particular, according to Corollary 1, if $G(A)$ is structurally balanced, the bipartite consensus can be asymptotically reached and we have $\lim_{t\to\infty} x_i(t) = \beta$ for $i \in \mathcal{V}_1$, $\lim_{t\to\infty} x_i(t) = -\beta$ for $i \in \mathcal{V}_2$, where $\beta \in \mathbb{R}^n$. The bipartite consensus value $\beta = (\beta_1, \beta_2, \dots, \beta_n)^\top$ will be obtained by an exact expression when the initial conditions of system (12) are given. The initial conditions about system (12) are provided as $x_i(s) = \sigma_i\varphi_i(s) \in C([-\tau, 0], \mathbb{R}^n)$. Let $Z(t) = Dx(t)$, i.e. $z_i(t) = \sigma_i x_i(t)$, then one gets $z_i(s) = \varphi_i(s) \in C([-\tau, 0], \mathbb{R}^n)$.

Corollary 2. *Consider a connected signed graph $G(A)$ which is structurally balanced. If $D \in \mathcal{D}$ renders DAD nonnegative, then the bipartite solution of (12) is $\lim_{t\to\infty} x(t) = (D\mathbf{1}) \otimes \big[N\xi(0)/\sum_{i=1}^N(1 + \sum_{j=1}^N |a_{ij}|\tau_{ij})\big]$.*

Proof. According to Corollary 1, one get that $\lim_{t\to\infty} z_i(t) = \beta$ and $\lim_{t\to\infty} x(t) = (D\mathbf{1}) \otimes \beta$. Let $\xi(t) = (\xi_1(t), \xi_2(t), \dots, \xi_n(t))^\top$, where $\xi_r(t) = (1/N)[\sum_{i=1}^N z_{ir}(t) + \sum_{i=1}^N\sum_{j=1}^N |a_{ij}| \int_{t-\tau_{ij}}^t z_{jr}(s)ds]$, $r \in \{1, 2, \dots, n\}$. Referring to the proof of Theorem 2, one can obtain $\dot{\xi}_r(t) = 0$, which implies $\xi_r(t) = \xi_r(0)$. From $\xi_r(0) = \lim_{t\to\infty}\xi_r(t)$, we have

$$\beta = \frac{N\xi(0)}{\sum_{i=1}^N(1 + \sum_{j=1}^N |a_{ij}|\tau_{ij})}, \tag{19}$$

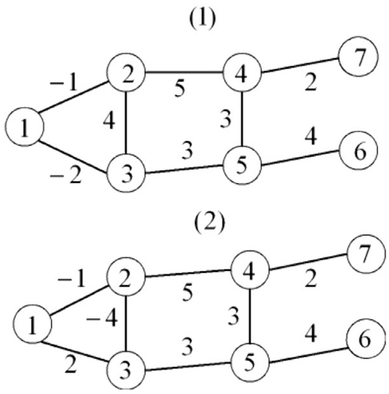

Fig. 1. Two signed undirected connectivity graphs with seven nodes. (1) structurally balanced. (2) structurally unbalanced.

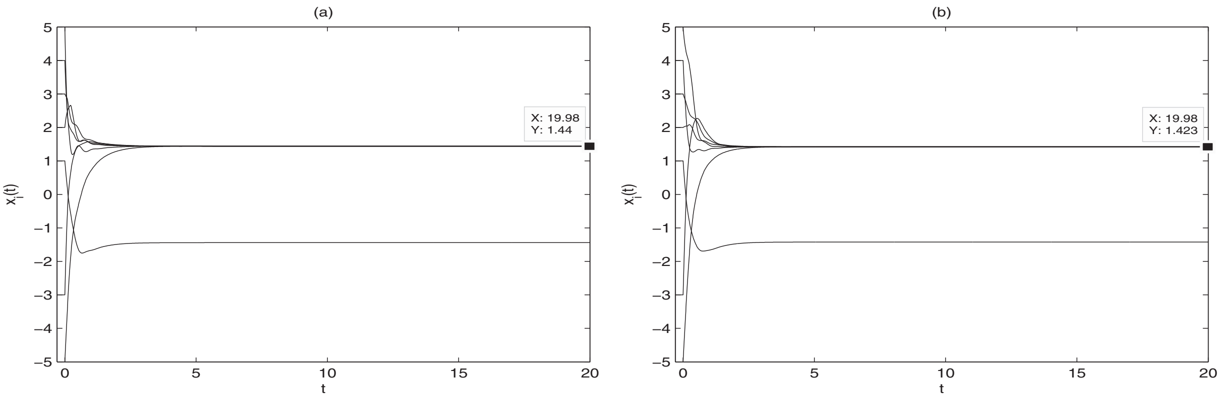

Fig. 2. The left subgraph: the bipartite consensus on multi-agent system (12) with structurally balanced graph and linear coupling in Example 1. The right subgraph: the bipartite consensus on multi-agent system (1) with structurally balanced graph and nonlinear coupling in Example 1.

and

$$\lim_{t\to\infty} x(t) = (D\mathbf{1}) \otimes \frac{N\xi(0)}{\sum_{i=1}^{N}(1+\sum_{j=1}^{N}|a_{ij}|\tau_{ij})}. \tag{20}$$

□

Remark 4. Referring to expression (19), for the case of network without communication delays, one can obtain that $\beta = (1/N)\sum_{i=1}^{N}\varphi_i(0)$. This result is consistent with the one obtained in [46]. This shows that our results are more general. Moreover, we conclude that a bipartite consensus solution is not only associated with initial values of $x_i(t)$, but also closely related with communication delays and networked structure.

Remark 5. According to Corollary 2, it is obvious that $\alpha \neq 0$ if $G(A)$ is structurally balanced unless $\xi_r(0) = 0$.

4. Numerical examples

In this section, numerical examples will be provided to demonstrate the effectiveness of our theoretical results.

Example 1. Consider the structurally balanced graph of Fig. 1. Now, we will give an example to illustrate the correctness of our main results.

For systems (12) and (1), all nonzero communication delays are listed as follows: $\tau_{12} = 0.1,\ \tau_{13} = 0.3,\ \tau_{21} = 0.15,\ \tau_{23} = 0.2,\ \tau_{24} = 0.1,\ \tau_{31} = 0.11,\ \tau_{32} = 0.16,\ \tau_{35} = 0.23,\ \tau_{42} = 0.1,\ \tau_{45} = 0.2,\ \tau_{47} = 0.12,\ \tau_{53} = 0.1,\ \tau_{54} = 0.15,\ \tau_{56} = 0.24,\ \tau_{65} = 0.25,\ \tau_{74} = 0.15$, and $x_1(s) = 1,\ x_2(s) = 2,\ x_3(s) = 3,\ x_4(s) = 4,\ x_5(s) = 5,\ x_6(s) = -3,\ x_7(s) = -5,\ \forall s \in [-0.3, 0]$. Let $D = diag\{-1, 1, 1, 1, 1, 1, 1\}$. Then, we have $z(t) = [-1, 2, 3, 4, 5, -3, -5],\ \forall s \in [-0.3, 0]$. Further, we define that $h(s) = s + 0.5 * sin(s)$. According to Corollary 2, one can easily conclude that $\alpha = 1.44$. Following Theorem 2, one can get the numerical solution $\beta = 1.42$ by iterative algorithm. Numerical results are depicted in the left and right of Fig. 2, which verify our theoretical results very well.

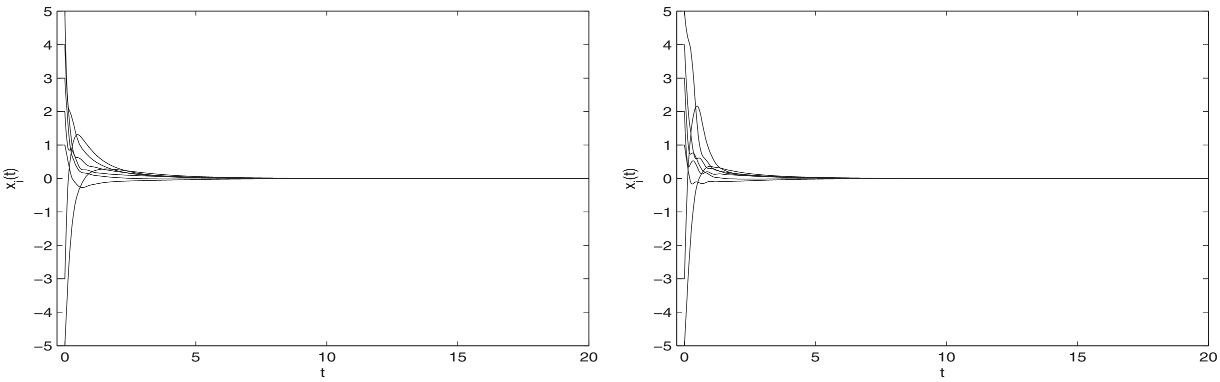

Fig. 3. The left subgraph: state trajectories of system (12) with structurally unbalanced graph and linear coupling in Example 2. The right subgraph: state trajectories of system (1) with structurally unbalanced graph and nonlinear coupling in Example 2.

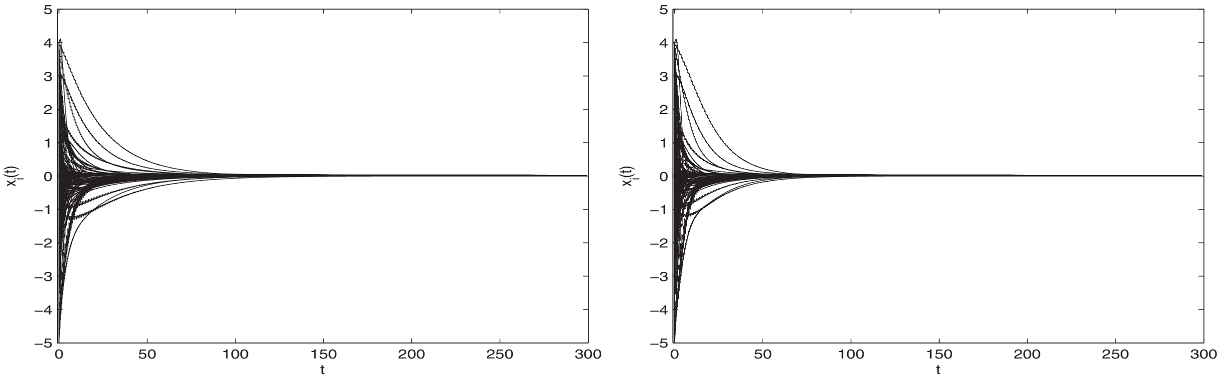

Fig. 4. The left subgraph: state trajectories of system (12) with structurally unbalanced graph and linear coupling in Example 3. The right subgraph: state trajectories of system (1) with structurally unbalanced graph and nonlinear coupling in Example 3.

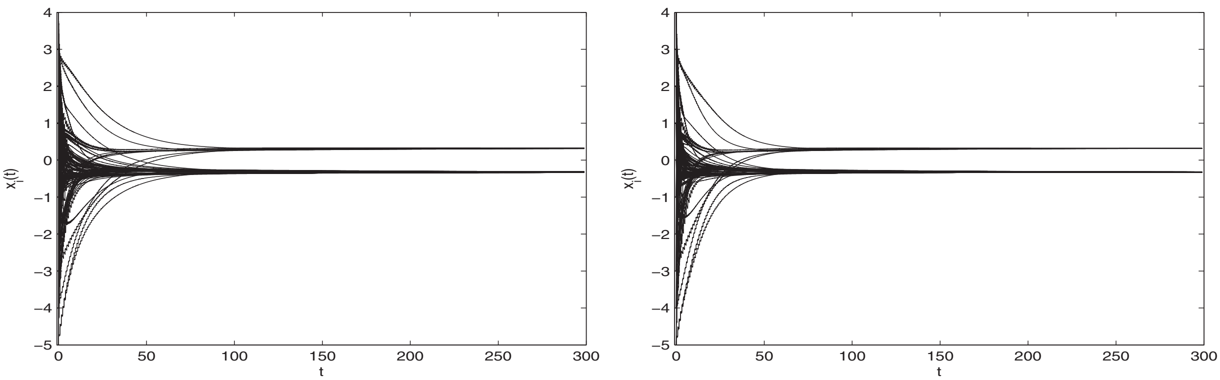

Fig. 5. The left subgraph: the bipartite consensus on multi-agent system (12) with structurally balanced graph and linear coupling in Example 3. The right subgraph: the bipartite consensus on multi-agent system (1) with structurally balanced graph and nonlinear coupling in Example 3.

Example 2. Consider the structurally unbalanced graph of Fig. 1(2). All communication delays and initial conditions are the same as those mentioned in example 1. Fig. 3 show that the consensus solutions of systems (12) and (1) are $\lim_{t\to\infty} x(t) = 0$.

Example 3. Now let us consider a larger network topology with 100 nodes and signed weight edges. Here two simple signed networks with 100 nodes are constructed, where one is structurally balanced and another one is structurally unbalanced. The network with structurally balanced coupling is constructed as follows: we present 20 identical circular networks with 5 nodes, whose 5 nodes are numbered 1, 2, 3, 4, 5 and their adjacency matrix are $A_3 = [a_{ij}]_{5\times5}$, where a_{ij} is chosen randomly from (-10,0) or (0,10). Now the first two edges of the circular network are defined as negative edges and others are positive edges, ie. $a_{12} < 0$, $a_{23} < 0$ $a_{34} > 0$, $a_{45} > 0$, $a_{15} > 0$. Here these circular networks are arranged in a sequence. A connected graph with 100 nodes and structurally balanced coupling can be obtained by stochastic interconnections among the 3rd, 4th, and 5th nodes of adjacent circular networks. Similarly the network with structurally unbalanced coupling can be obtained according to the above method when the first three edges of pentagon are defined as negative edges and other steps are the same. All communication delays of systems (12) and (1) are uniformly distributed in (0,1). The left of Fig. 4 shows that the consensus of system (12) can be achieved for $a_{ij} \in (-10, 10)$ and $\tau_{ij} \in (0, 1)$. The right of Fig. 4 shows that the consensus of system (1) also can be achieved for above conditions. Throughout this example, the nonlinear function $h(x) = x + 0.5sin(x)$ is not changed. Fig. 5 show that the bipartite consensus of systems (12) and (1) can be asymptotically reached, respectively.

5. Conclusion

In this paper, we study the consensus problem in undirected networks of agents with antagonistic interactions and communication delays. For nonlinear coupling signed network $G(A)$, we conclude that bipartite consensus can be achieved if $G(A)$ is structurally balanced, and if instead $G(A)$ is structurally unbalanced then $\lim_{t\to\infty} x(t) = 0$. Further, an expression associating with bipartite consensus solution is provided for nonlinear coupling networks. In particular, similar bipartite consensus conclusions are obtained for the linear coupling signed network. Further, for the structurally balanced network, a bipartite consensus solution is given for linear coupling. Three numerical examples are provided to demonstrate the effectiveness of our proposed results. In our near future work, we will study the consensus problem over directed networks with antagonistic interactions and communication delays.

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