The controllability of weighted and directed networks with antagonistic interactions

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***Abstract*—This paper studies the controllability problem for weighted digraphs with antagonistic interactions. Based on the graph partition theory, the L-equitable partition(L-EP) is proposed. Using L-equitable partition, a graph-theoretic necessary condition is proposed for the controllability of the multi-agent system and an algorithm is given for the computation of the partition. Besides, the upper bound on the controllable subspace is derived by utilizing the coarsest L-EP under leaders V_l when the system is uncontrollable.**

***Index Terms*—Controllability; Multi-agent systems; Antagonistic interactions**

I. Introduction

Multi-agent systems have broad applications in science and engineering areas such as consensus [1]–[3], formation control [4], flocking [5], etc. As one of the fundamental problems of multi-agent systems, controllability has been extensively studied in recent decade. The controllability of multi-agent systems was first introduced in [6] by Tanner and followed by [7]–[22]. In this framework, the system is governed by the consensus algorithm. One or more agents are designated as leaders which can be influenced by external signals, while the remaining agents are followers which are indirectly controlled by the leaders via the nearest neighbor interconnections between them. Followers still obey the consensus algorithm but leaders update their state values by external inputs. The main idea of the controllability is that the followers can be led into desired locations to form anticipated configurations via the control inputs implemented by leaders. In this setting, the controllability of the networked system is determined by the number and position of the leaders. Some necessary and sufficient algebraic conditions were derived for the controllability [6]–[8]. With nodes(vertices) representing agents and edges indicating the interconnections, a multi-agent system can be denoted by a graph. Some previous work [6]–[8] has shown that the controllability of such systems relies closely on the structure and interconnection topology of the graph. However, it is difficult to completely answer the question about how the controllability is affected by the structure of networks. In the course of solving that problem, the graph partition is a powerful tool to graphically characterize the controllability for multi-agent systems [8], [25]. Equitable partitions have been used to provide necessary conditions for controllability with multiple leaders [8]. The more general notion of almost equitable partition was introduced in [26], [27] for the Laplacian matrix. It should be noted that the interaction between agents studied above is cooperative. However, in reality, some agents may be cooperative and others may be antagonistic. In nature, some animals are cooperative, and some animals are hostile. Similarly, in society, some companies cooperate with each other, some companies compete with each other. This is similar to the significance of the signed graph in which the weight of its edge may be equal to 1 or -1 [23]. The positive weighted edge represents the cooperative interaction and the negative weighted edge represents the antagonistic interaction between the two agents. In [24], the authors studied the controllability problem of multi-agent networks under an undirected signed graph, a necessary condition was proposed for the controllability of the network using the so-called generalized almost equitable partition. However, the relationship of two agents in reality will be more complicated than that in the signed graph. For instance, in the biological fields, lions could attack the antelope, but not vice versa. The harm of a lion to the antelope may be greater than that of a hyena. Therefore, the weighted digraph with antagonistic interactions should be investigated. In general, a directed graph is much more complicated than an undirected graph. The adjacency matrix and the Laplace matrix of the undirected graph are symmetric matrices, and the adjacency matrix and Laplacian matrix of the directed graph are not symmetric matrices. Hence the adjacency matrix and the Laplacian matrix of the directed graph will lose some good nature.

In this paper, we consider the controllability problem of multi-agent systems under weighted digraph. The contribution of this work is threefold. Firstly, the L-equitable partition is

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introduced for weighed digraph with antagonistic interactions. Necessary and sufficient conditions about the L-equitable partition are presented. Secondly, the coarsest L-EP under leaders and its algorithm are proposed. Thirdly, a necessary condition for the controllability of the network and tight bounds for the controllable subspace are provided.

The rest of this paper is organized as follows. In Section II, some preliminaries are provided and the controllability problem of antagonistic networks is formulated. In Section III, the main results are presented and several examples are provided to illustrate the main results. Finally, conclusions are given in Section IV.

II. Preliminaries and Problem Formulation

Throughout this paper, e_i is the identity vector whose i-th element is 1 while the other elements are 0. Given two sets X and Y, $X\backslash Y$ is the set whose elements belong to X but not to Y. The column space of a matrix M will be denoted by $im(M)$.

A. L-invariant subpace

Definition 1. *[28] Let V be a vector space and $M : V \to V$ be a linear operator over V. A vector space $\mathcal{W} \subseteq V$ is said to be an invariant subspace of M if for every $w \in \mathcal{W}$, $Mw \in \mathcal{W}$ (we also write $M\mathcal{W} \subseteq \mathcal{W}$).*

Lemma 1. *[30] For matrices $M \in R^{n\times n}$ and $P \in R^{n\times r}$, $im(P)$ is an invariant subspace of M if and only if there exists a matrix $Q \in R^{r\times r}$ such that $MP = PQ$.*

Lemma 2. *[30] If $MP = PQ$ and the columns of P are linearly independent, then every eigenvalue of Q is an eigenvalue of M.*

B. Graphs

A directed graph is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, \cdots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represent the node set and the edge set, respectively. $A = [a_{ij}]_n$ is the adjacency matrix of $\mathcal{G}$. $a_{ij} \neq 0$ represents $(j, i) \in \mathcal{E}$. If the edge points at i from j. j is called the parent node while i is called the child node and we say j is a neighbor of i. The neighbor set of i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}, j \neq i\}$. The degree of node i can be denoted by $d_i = \sum_{j\in\mathcal{N}_i} |a_{ij}|$. The degree matrix of a graph $\mathcal{G}$ is a diagonal matrix $D = diag\{d_1, \cdots, d_n\}$. The Laplacian matrix L of a graph $\mathcal{G}$ can be defined as $L = D - A$. Thus, the entries of the matrix L can be written as

$$l_{ij} = \begin{cases} d_i, & i = j, \\ -a_{ij}, & i \neq j. \end{cases}$$

C. Graph partitions

The vertices $\mathcal{V}$ can be partitioned into several subsets with specific properties.

Definition 2. *A partition π of $\mathcal{V}$ is to partition $\mathcal{V}$ into r cells $C_1, C_2, \cdots, C_r$, where $r > 1$ and $C_i \subset \mathcal{V}, \mathcal{V} = \cup_{i=1}^{r} C_i, C_i \cap C_j = \emptyset, i \neq j$. C_i is nontrivial if $1 < |C_i| < n$, otherwise, it is trivial. The partition π is nontrivial if it contains at least one nontrivial cell, otherwise it is trivial.*

Definition 3. *Let π_1, π_2 be two partitions of the same $\mathcal{V}$. Then we say that π_1 is coarser than π_2 if each cell in π_1 is a union of cells in π_2.*

Definition 4. *[30] A characteristic matrix $P \in R^{n\times r}$ of a partition π of $V(G)$ is a matrix with the characteristic vectors of the cells as its columns. The entries of the matrix P are*

$$p_{ij} = \begin{cases} 1, & if \quad i \in C_j, \\ 0, & otherwise. \end{cases}$$

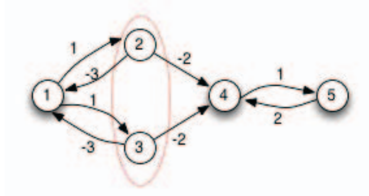

Fig. 1: π: $C_1 = \{1\}$, $C_2 = \{2, 3\}$, $C_3 = \{4\}$, $C_4 = \{5\}$.

For example, the characteristic matrix of the partition of the graph in Fig. 1 is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is clear that

$$P^T P = \begin{bmatrix} |C_1| & & & \\ & |C_2| & & \\ & & \ddots & \\ & & & |C_r| \end{bmatrix}.$$

Since each cell at least has one node, then $P^T P$ is nonsingular.

D. Problem formulation

Consider a multi-agent system with n linear agents described by the graph $\mathcal{G}$. Let x_i denote the state of agent i, whose dynamics is described by the protocol

$$\dot{x}_i = -\sum_{i\in\mathcal{N}_i} (|a_{ij}|x_i - a_{ij}x_j), \quad i = 1, \cdots, n \tag{1}$$

For simplicity, only one dimensional case is considered. The compact dynamics can be written as $\dot{x}(t) = -Lx(t)$, where x is the vector of the agents' states and L is the graph Laplacian. Let $V_l = \{i_1, \cdots, i_m\}$ be the set of leaders controlled by external inputs. Then the dynamical system is

$$\dot{x}(t) = -Lx(t) + Bu(t) \tag{2}$$

where $B = [e_{i_1}, e_{i_2}, \cdots, e_{i_m}]$ is the control input matrix, and $u(t) \in R^m$ is the input vector. The system described by equation (2) is said to be controllable if it can be driven from any initial state to any desired final state in finite time.

By Kalman's controllability rank condition [29], system (2) is controllable if and only if the $n \times nm$ controllability matrix $C = [B, LB, ..., L^{n-1}B]$ has full row rank, namely, $rank(C) = n$. Another well-known controllability test is Popov-Belevitch-Hautus (PBH) test. By the PBH test and the fact that L is symmetric, the following useful lemma can be proven.

Lemma 3. *System (2) is controllable if and only if there does not exist a nonzero vector v such that the following two equations are met simultaneously*

$$\begin{aligned} Lv &= \lambda v \\ B^T v &= 0 \end{aligned} \tag{3}$$

The controllable subspace of system (2) is $\mathcal{C} := im(C)$ denoted by $\langle L, B \rangle$.

Lemma 4. *[28] The following statements are equivalent:*
(i) The system (2) *is controllable,*
(ii) $\langle L, B \rangle = R^n$,
(iii) $rank(C) = n$.

However, the rank criterion requires a large number of calculations on the matrix C, and the rank of the matrix is very sensitive to the change of the entries of the matrix. Since the controllability of the multi-agent systems relies closely on the structure and interconnection topology of the graph. We should explore how the controllability is affected by the structure of networks in order to avoid the shortcomings of algebraic conditions.

III. Main rusults

A. L-equitable partition

Definition 5. *A partition π of $\mathcal{V}$, $\{C_1, C_2, \cdots, C_r\}$, is said to be a L-equitable partition(L-EP) if for all $s, t \in C_i, i, j = 1, \cdots, r$, then*

$$\sum_{k\in C_j} l_{sk} = \sum_{k\in C_j} l_{tk}. \tag{4}$$

Remark 1. *The L-EP is not the same as the generalized almost equitable partition (GAEP) of [24]. For instance, the graph in Fig. 2, π: $C_1 = \{1\}$, $C_2 = \{2, 3, 4\}$, $C_3 = \{5, 6, 7\}$ is a L-EP but not a GAEP. In fact, a GAEP must be a L-EP but not vice versa.*

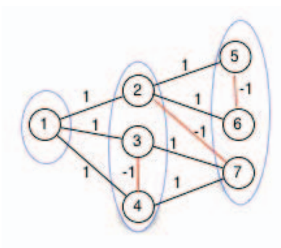

Fig. 2: π: $C_1 = \{1\}$, $C_2 = \{2, 3, 4\}$, $C_3 = \{5, 6, 7\}$.

Definition 6. *The directed graph with the r cells of π as its vertices and $\alpha_{ij} := \sum_{k\in C_j} l_{sk}, s \in C_i$ arcs from the jth to the ith cell of π is called the quotient of $\mathcal{G}$ over π, which is denoted by G/π. Let A_π and L_π denote the adjacency and Laplacian matrix of G/π, respectively. Therefore, the entries of the adjacency matrix of this quotient are given by $A(G/\pi) = \alpha_{ij}$.*

Theorem 1. *Let $\mathcal{G}$ be a graph, L its Laplacian matrix, $\pi = (C_1, \cdots, C_r)$ a partition of $\mathcal{V}$ and P the characteristic matrix of π. Then π is a L-EP if and only if there is a $r \times r$ matrix Q such that*

$$LP = PQ$$

$Q = P^+LP$, where $P^+ = (P^TP)^{-1}P^T$ is the pseudo-inverse of P. If π is a L-EP then Q is the Laplacian matrix L_π of G/π

Proof. Given a graph $\mathcal{G}$, π is a partition of $\mathcal{V}$, and P the characteristic matrix of π. Then

$$L * P = \begin{bmatrix} \sum l_{1j}p_{j1} & \sum l_{1j}p_{j2} & \cdots & \sum l_{1j}p_{jr} \\ \sum l_{2j}p_{j1} & \sum l_{2j}p_{j2} & \cdots & \sum l_{2j}p_{jr} \\ \vdots & \vdots & \ddots & \vdots \\ \sum l_{nj}p_{j1} & \sum l_{nj}p_{j2} & \cdots & \sum l_{nj}p_{jr} \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{j\in C_1} l_{1j} & \sum_{j\in C_2} l_{1j} & \cdots & \sum_{j\in C_r} l_{1j} \\ \sum_{j\in C_1} l_{2j} & \sum_{j\in C_2} l_{2j} & \cdots & \sum_{j\in C_r} l_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j\in C_1} l_{nj} & \sum_{j\in C_2} l_{nj} & \cdots & \sum_{j\in C_r} l_{nj} \end{bmatrix}$$

Suppose that a matrix $Q \in R^{r\times r}$ is

$$Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1r} \\ q_{21} & q_{22} & \cdots & q_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ q_{r1} & q_{r2} & \cdots & q_{rr} \end{bmatrix},$$

then

$$P * Q = \begin{bmatrix} q_{r_1 1} & q_{r_1 2} & \cdots & q_{r_1 r} \\ q_{r_2 1} & q_{r_2 2} & \cdots & q_{r_2 r} \\ \vdots & \vdots & \ddots & \vdots \\ q_{r_n 1} & q_{r_n 2} & \cdots & q_{r_n r} \end{bmatrix}$$

where r_i indicates that i belongs to the cell C_{r_i}. If $s, t \in C_i$, then

$$(P * Q)_{sj} = (P * Q)_{tj}.$$

(Sufficiency)

If $L * P = P * Q$, then $(L * P)_{sj} = (L * P)_{tj}$, for $s, t \in C_i$. It follows that

$$\sum_{k\in C_j} l_{sk} = \sum_{k\in C_j} l_{tk},$$

that is to say, π is a L-EP.

(Necessity)

If π is a L-EP, then (4) is satisfied. Let $Q = L_\pi$, then $L * P = P * Q$. □

The trivial cells of a nontrivial L-EP play an important role in the study of the controllability of the weighed digraph with antagonistic interactions.

Definition 7. *Suppose that $\pi = (C_1, \cdots, C_r)$ is a nontrivial L-EP and all its trivial cells are $C_{r_1}, \cdots, C_{r_m}$. If the leaders set V_l satisfies that $V_l \subseteq \cup_{k=1}^{m} C_{r_k}$, then π is said to be a nontrivial L-EP under leaders V_l.*

B. The coarsest L-EP under leaders

Definition 8. *If π is a nontrivial L-EP under leaders V_l and π is coarser than any other nontrivial L-EP under leaders V_l, then π is the coarsest L-EP under leaders V_l.*

Inspired by [24], an algorithm to compute the coarsest L-EP under leaders V_l for a given graph with antagonistic interactions is proposed. The algorithm is described as follows.

1) Let $\pi_0 = \{i_1, \cdots, i_m, V_f\}$ be the initial partition.
2) Relabel the cells in the current partition: $C_1, \cdots, C_r, C_f$, if C_f is a nontrivial cell, for every node s of C_f, compute $q_{sj} = \sum_{k \in C_j} l_{sk}, j = 1, \cdots, r, f$. Suppose that there exists a node t such that $q_{sj} = q_{tj}$, then let s and t group into one cell. Replace the old cell with the newly created cells.
3) Repeat Step 2) until no cell can be split.

Lemma 5. *For a directed weighted graph $\mathcal{G}$ with antagonistic interactions, the partition obtained via Steps 1)-3) is the coarsest L-EP under leaders V_l.*

Proof. The proof is similar to the proof of Theorem 2 of [24], and hence is omitted. □

C. Controllability of a weighed digraph with antagonistic interactions

In this subsection, the controllability of a weighed digraph with antagonistic interactions is investigated by utilizing L-EPs. The following result characterizes the relation between the controllability and the L-EP.

Theorem 2. *Let $\mathcal{G}$ be a weighted digraph with antagonistic interactions and suppose that $\pi = (C_1, \cdots, C_r)$ is a nontrivial L-EP under leaders V_l and P is the characteristic matrix of π, then*

i) the system is uncontrollable;
ii) $\langle L, B \rangle \subseteq im(P)$;
iii) $dim\langle L, B \rangle \leq r$.

Proof. Because $\pi = (C_1, \cdots, C_r)$ is a L-EP of $\mathcal{V}$ and P is the characteristic matrix of π, according to Theorem 1 and Lemma 1, $im(P)$ is L-invariant. Since π is a nontrivial L-EP, then $r < n$. If π is a nontrivial L-EP under leaders V_l, then $im(B) \subseteq im(P)$. Therefore, $\langle L, B \rangle \subseteq im(P)$ and $dim\langle L, B \rangle \leq r$. According to Lemma 4, the system is uncontrollable. □

Example 1. *Consider the weighted digraph shown in Fig. 1. $\pi = (\{1\}, \{2, 3\}, \{4\}, \{5\})$ is a nontrivial L-EP. Every of cells: {1}, {4} and {5} is a trivial cell. By Theorem 2, if the leaders belong to the set {1,4,5}, the system would be uncontrollable. For instance, if the agents 1,4,5 are taken as the leaders, the system is uncontrollable. In this case, the Laplacian matrix L and the control matrix B can be written as follows*

$$L = \begin{bmatrix} 6 & 3 & 3 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 6 & -2 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The rank of the controllability matrix of system is 4. Thus, the system is uncontrollable. If agents 4 and 5 are taken as the leaders, the Laplacian matrix L and the control matrix B can be written as follows

$$L = \begin{bmatrix} 6 & 3 & 3 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 6 & -2 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The rank of the controllability matrix of system is 2. Thus, the system is uncontrollable.

Theorem 2 provides an upper bound on the controllable subspace if there exists a nontrivial L-EP under leaders V_l. In the following we utilize the coarsest L-EP under leaders V_l to get a tighter upper bound.

Theorem 3. *Suppose that $P*$ is the characteristic matrix of the coarsest L-EP under leaders V_l and π a nontrivial L-EP under leaders V_l, then the controllable subspace of the system* (2) *satisfies $\langle L, B \rangle \subseteq im(P*) \subseteq im(P)$.*

Proof. According to Theorem 2, $\pi = (C_1, \cdots, C_r)$ is a nontrivial L-EP under leaders V_l and P is the characteristic matrix of π. Then $\langle L, B \rangle \subseteq im(P)$. Assume that $P*$ is the characteristic matrix of the coarsest L-EP under leaders V_l, then $\langle L, B \rangle \subseteq im(P*)$ and $im(P*) \subseteq im(P)$. Therefore, $\langle L, B \rangle \subseteq im(P*) \subseteq im(P)$. □

Example 2. *Consider the graph shown in Fig. 2. π_1 and π_2 are two nontrivial L-EPs. π_1: $C_1 = \{1\}$, $C_2 = \{2, 3, 4\}$, $C_3 = \{5, 6, 7\}$. π_2: $C_1 = \{1\}$, $C_2 = \{2, \}$, $C_3 = \{3, 4\}$,$C_4 = \{5, 6\}$, $C_5 = \{7\}$. π_1 is coarser than π_2. Suppose that P_1 and P_2 are the characteristic matrices of π_1 and π_2, respectively.*

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

If the agent 1 is taken as the leader, according to Theorem 2, the system is uncontrollable. In this case, the Laplacian *matrix L and the control matrix B can be written as follows*

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & -1 & -1 & 1 \\ -1 & 0 & 3 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The rank of the L-EP matrix is 3. $\langle L, B\rangle = im(P_1) \subset im(P_2)$.

Remark 2. *It is noted that the condition in Theorem 2 is necessary but not sufficient for the controllability. That is to say, if the leaders belong to a nontrivial cell, the controllability of the system is inconclusive. For the system in Fig. 1, agent 2 belongs to the nontrivial cell $\{2, 3\}$. If agent 2 is taken as the leader, the rank of the controllability matrix is 5, and accordingly the system is controllable. For the system in Fig. 2, if agent 3 is taken as the leader, the rank of the controllability matrix is 4. Thus, the system is uncontrollable. How to get the sufficient and necessary conditions is still an open question.*

IV. Conclusion

In this paper, we considered the controllability problem for weighted digraphs with antagonistic interactions. The main results of the paper are the L-equitable partition and a necessary condition for the controllability involving L-EP. We generalized the known results on the role of uncontrollability to weighted digraphs with antagonistic interactions. The results were illustrated by several examples in the paper.

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