Controllability and stabilizability of networks of linear systems

Jochen Trumpf, *Senior Member, IEEE*, and Harry L. Trentelman, *Fellow, IEEE*

Abstract—We provide necessary and sufficient conditions on the node systems, the graph adjacency matrix, and the input matrix such that a heterogeneous network of multi-input multi-output (MIMO) linear time-invariant (LTI) node systems with constant linear couplings is controllable or stabilizable through the external input. We also provide specializations of these general conditions for homogeneous networks. Finally, we give a very simple necessary and sufficient condition under which a homogeneous network of single-input single-output (SISO) LTI node systems is stable in the absence of the external input.

Index Terms—behavioral approach, controllability and stabilizability, linear systems, network analysis

I. INTRODUCTION

In this paper we study system theoretic properties of interconnections of linear systems. Given is a collection of linear time invariant input/state/output systems, together with an interconnection topology represented by a weighted directed graph. The systems are interconnected through their inputs and outputs as prescribed by the given graph, and at the same time new external input and output channels are specified for the interconnected system. The systems that are being interconnected are called *the node systems*, the system resulting after interconnection will be called *the network*. In this paper we will deal with finding conditions on the node systems, the adjacency matrix of the graph, and the new input matrix such that the network is controllable or stabilizable through the new external input. We also discuss stability in the absence of the external input. Of course, similar questions arise concerning observability and detectability through the new external output. By dualization however, results on controllability and stabilizability immediately lead to results on observability and detectability. Therefore, in this paper we will not discuss the latter issues.

The problem of finding conditions under which these basic system theoretic properties hold for interconnected systems has been studied before, and dates back to work by Gilbert [1], who studied controllability and observability of systems in parallel, series and feedback interconnections. Other earlier references on this topic are work of Callier and Nahum [2], and of Fuhrmann [3]. Obviously, interconnection structures in general are more complex than those treated in these references, and therefore need to be described by means of weighted directed graphs. This has resulted in more recent contributions on system theoretic properties of interconnections. Among these, we mention the work of Hara et al. [4], the recent textbook by Fuhrmann and Helmke [5], as well as recent works by Zhou [6] and Wang et al. [7].

In Hara et al. [4], controllability and observability was studied for networks in which every node system is a copy of the same single-input single-output system (such networks are called *homogeneous*). Using an argument based on the control canonical form for controllable single input systems, it was shown that such a network is controllable if and only if the node system is controllable and observable, and the interconnection structure is represented by a controllable pair. In Fuhrmann and Helmke [5], the more general

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framework of heterogeneous networks consisting of possibly distinct multi-input multi-output node systems was studied. In this more general framework a necessary and sufficient condition was established for controllability of the network, reminiscent of the classical Popov-Belevitch-Hautus (PBH) test. This result was however established under the restrictive condition that all node systems are observable. Zhou [6] provides a necessary and sufficient condition for controllability of heterogeneous networks where certain transfer matrices associated with the node systems have full column normal rank. This can only occur if each node system has at least as many external inputs as internal outputs. Wang et al. [7] provide a necessary and sufficient condition for the controllability of a homogeneous network of MIMO LTI node systems where the interconnection structure is given at the level of node systems rather than at the level of individual node system inputs and outputs.

The main contributions of the present paper are the following:

1) We will generalize the result of Fuhrmann and Helmke in [5] to the case that the node systems are not necessarily observable and obtain a PBH-like test for network controllability applicable to general linear networks.
2) The result of Hara et al. in [4] on homogeneous networks of single-input single-output systems will be generalized to multi-input multi-output systems.
3) We will extend both of the above results to *stabilizability* of networks.

Although this paper deals with systems in input-state-output form, an important role in our analysis will be played by elements and ideas from the behavioral approach [8]. This approach will provide us with flexibility in using the most suitable system representations for the problems at hand. In the end however, our final results will be formulated in terms of classical concepts involving state space representations and polynomial matrices.

This paper deals with controllability and stabilizability of networks of linear systems. It does *not* deal with problems of weak or strong structural controllability of systems on graphs as studied for example in [9], [10], [11], [12] and the references therein. Note that, in the literature on weak or strong structural controllability, conditions are usually given in terms of graph theoretical concepts (cactus graphs, matchings, color change rules), whereas in our work conditions involve system theoretic properties of the coupling matrices and the node systems. A further related but significantly different area of work concerns the study of controllability of networks of single integrator systems. Here, controllability conditions are given in terms of graph theoretical concepts (almost equitable partitions) but the node systems have trivial dynamics. See Aguilar et al. [13] for a recent representative contribution to that line of work, and also Camlibel et al. [14] for a good explanation of a common misconception in the related classical literature.

The outline of this paper is as follows. In Section II we introduce the problem formulation of this paper together with some notation that we will use. Section III will deal with finding a convenient kernel representation of the network that will reveal which properties should be satisfied for controllability. Then, in Section IV we will formulate our results on network controllability. The general scheme is that we start with the most general heterogeneous multi-input multi-output case, and then gradually specialize to the homogeneous single-
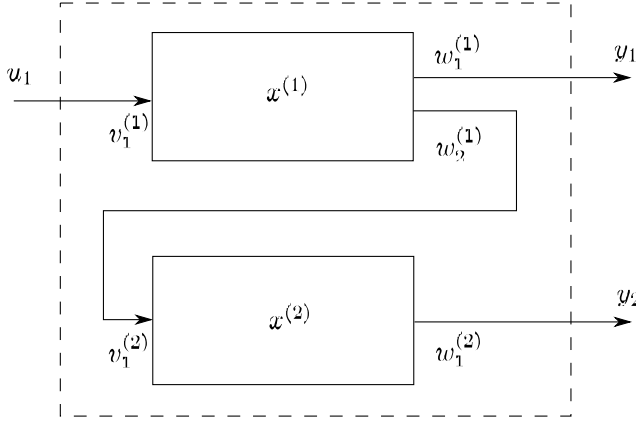
Fig. 1. An example network with two node systems with internal states $x^{(1)}$ resp. $x^{(2)}$. The network has one external input u_1 and two external outputs y_1 and y_2. The external inputs are connected to node inputs $v_l^{(k)}$, while the node outputs $w_j^{(i)}$ can be connected to external outputs or to node inputs.

input single-output case. In Section V we will extend our results to stabilizability and stability. Concluding remarks can be found in Section VI. In order to enhance readability, some technicalities are deferred to the Appendix in Section VII.

II. PROBLEM FORMULATION

We study networks of (finite-dimensional) linear time-invariant node systems of the form

$$
\begin{aligned}
\dot{x}^{(i)} &= \alpha^{(i)} x^{(i)} + \beta^{(i)} v^{(i)}, \\
w^{(i)} &= \gamma^{(i)} x^{(i)},
\end{aligned} \tag{1}
$$

where $x^{(i)}(t) \in \mathbb{R}^{n_i}$ denotes the internal state of the i-th node system at time t, $v^{(i)}(t) \in \mathbb{R}^{m_i}$ its input and $w^{(i)}(t) \in \mathbb{R}^{p_i}$ its output. We assume that there are $N > 0$ nodes, i.e. $i = 1, \ldots, N$. Our notation mostly follows [5].

The special case where $\alpha^{(i)} = \alpha^{(0)}$, $\beta^{(i)} = \beta^{(0)}$ and $\gamma^{(i)} = \gamma^{(0)}$ for all $i = 1, \ldots, N$, i.e. where all node system dynamics are the same, is called a *homogeneous* network, while the general case of different node system dynamics is referred to as a *heterogeneous* network. A *single input single output (SISO)* network is a network where $m_i = p_i = 1$ for all $i = 1, \ldots, N$, i.e. a network consisting of SISO node systems. We will see that homogeneous SISO networks have very special properties.

We consider static linear couplings between the node systems as well as between the external inputs and outputs and the nodes (see Fig. 1). In general, such couplings can be modelled as

$$
v^{(i)} = \sum_{j=1}^{N} A_{ij} w^{(j)} + B_i u, \tag{2}
$$

where $u(t) \in \mathbb{R}^m$ is the external network input, and

$$
y = \sum_{i=1}^{N} C_i w^{(i)}, \tag{3}
$$

where $y(t) \in \mathbb{R}^p$ is the external network output. We will not consider external network outputs for the remainder of this paper.

By introducing the block diagonal matrices

$$
\alpha = \begin{pmatrix} \alpha^{(1)} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha^{(N)} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta^{(1)} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta^{(N)} \end{pmatrix}
$$

and $\gamma = \begin{pmatrix} \gamma^{(1)} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma^{(N)} \end{pmatrix}$

and the block matrices

$$
A = \begin{pmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix},
$$

the network (without the external network output) can be compactly represented as

$$
\begin{aligned}
\dot{x} &= \alpha x + \beta v, \\
w &= \gamma x, \\
v &= Aw + Bu.
\end{aligned} \tag{4}
$$

Here we have stacked the individual node system states, inputs and outputs in the obvious way to obtain $x(t) \in \mathbb{R}^{\sum_{i=1}^{N} n_i}$, $v(t) \in \mathbb{R}^{\sum_{i=1}^{N} m_i}$ and $w(t) \in \mathbb{R}^{\sum_{i=1}^{N} p_i}$.

The transfer matrix $\gamma^{(i)}(sI - \alpha^{(i)})^{-1}\beta^{(i)}$ of node system i from $v^{(i)}$ to $w^{(i)}$ will be denoted by $g^{(i)}(s)$. The transfer matrix $\gamma(sI - \alpha)^{-1}\beta$ from the joint input vector v to the joint output vector w is then equal to

$$
G(s) = \begin{pmatrix} g^{(1)}(s) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g^{(N)}(s) \end{pmatrix}.
$$

Our goal in this paper is to study controllability and stabilizability of the state x of the network (4) through the external control input u, and to give characterizations of controllability and stabilizability in terms of properties of the node systems (1) and the coupling matrices (A, B). Considering the special case $B = 0$ (or, equivalently, $u = 0$) we will also obtain characterizations of stability of the network.

III. SYSTEM REPRESENTATIONS OF NETWORKS

We will use elements of the behavioral approach [8] in our analysis. The first step in a behavioral analysis is always the choice of a convenient system representation. To this end, it is imperative to think clearly about the selection of variables in the representation. The full behavior $\mathcal{B}$ of the network (4), i.e. the linear space of C^∞ solutions of Eqs. (4), has variables (x, v, w, u). However, controllability of the network through the external control input u obviously only depends on the behavior of the variables (x, u), i.e. formally, on the projected behavior $\mathcal{B}_{(x,u)}$ [8]. This behavior can be obtained by elimination of the variables (v, w) and, by inspection, is given by

$$
\dot{x} = (\alpha + \beta A \gamma) x + \beta B u. \tag{5}
$$

Together with the output equation

$$
w = \gamma x, \tag{6}
$$

this is nothing but a standard linear time-invariant input/state/output system with behavior $\mathcal{B}_{(x,u,w)}$ where the system matrices $(\alpha + \beta A\gamma, \beta B, \gamma)$ have special structure.

The following proposition shows that controllability of the input/state behavior $\mathcal{B}_{(x,u)}$ is equivalent to controllability of the input/output behavior $\mathcal{B}_{(u,w)}$ provided that the state x of system (5) and (6) is observable from (u,w), i.e. if system (5) and (6) is observable in the classical sense. Here, controllability of $\mathcal{B}_{(x,u)}$ and $\mathcal{B}_{(u,w)}$ is to be understood in the behavioral sense, but note that behavioral controllability of $\mathcal{B}_{(x,u)}$ is the same as classical controllability [8]. Similarly, the input/state behavior $\mathcal{B}_{(x,u)}$ is stabilizable in the behavioral sense if and only if the input/output behavior $\mathcal{B}_{(u,w)}$ is stabilizable in the behavioral sense, provided that system (5) and (6) is observable. Also behavioral stabilizability of $\mathcal{B}_{(x,u)}$ is the same as classical stabilizability [8]. Note, however, that controllability of $\mathcal{B}_{(u,w)}$ is *not* the same as classical output controllability.

Proposition 1. *[8, Ex. 6.25] Let $\mathcal{B}$ be the full behavior of the input/state/output system*

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx \end{aligned} \tag{7}$$

with variables (x,u,y). Then the following holds.

1) *If $\mathcal{B}_{(x,u)}$ is controllable, equivalently (A,B) is a controllable pair, then $\mathcal{B}_{(u,y)}$ is controllable.*
2) *If $\mathcal{B}_{(x,u)}$ is stabilizable, equivalently (A,B) is a stabilizable pair, then $\mathcal{B}_{(u,y)}$ is stabilizable.*

Moreover, if (7) is observable, then both in 1. and 2. also the converse holds.

Proof. A proof can be found in the Appendix. □

The following lemma shows that observability of the node systems implies observability of system (5) and (6).

Lemma 2. (γ, α) *observable implies* $(\gamma, \alpha + \beta A\gamma)$ *observable.*

Proof. This follows from the dual of the well known result that controllability is invariant under constant state feedback, see e.g. [15, Ex. 3.9]. □

In order to apply Proposition 1 via Lemma 2, we start with an unobservable/observable Kalman decomposition of each node system

$$\alpha^{(i)} = \begin{pmatrix} \alpha_{11}^{(i)} & \alpha_{12}^{(i)} \\ 0 & \alpha_{22}^{(i)} \end{pmatrix}, \ \beta^{(i)} = \begin{pmatrix} \beta_1^{(i)} \\ \beta_2^{(i)} \end{pmatrix},$$
$$\gamma^{(i)} = \begin{pmatrix} 0 & \gamma_2^{(i)} \end{pmatrix},$$

such that $\left(\gamma_2^{(i)}, \alpha_{22}^{(i)}\right)$ is observable. Recall that $\alpha^{(i)} \in \mathbb{R}^{n_i \times n_i}$. Let $n_{i,1}$ and $n_{i,2}$ be the sizes of $\alpha_{11}^{(i)}$ and $\alpha_{22}^{(i)}$, respectively, so that $n_{i,1} + n_{i,2} = n_i$. By stacking all the unobservable node system states into a vector x_1 and separately stacking all the observable node system states into a vector x_2 we can rewrite the collection of the node system dynamics (the uncoupled network dynamics) as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} v,$$
$$w = \gamma_2 x_2,$$

where the block system matrices are reordered versions of the block matrices (α, β, γ) in Eq. (4) and (γ_2, α_{22}) is observable. Note that α_{11} has size $\sum_{i=1}^N n_{i,1}$. The joint node transfer matrix is then $G(s) = \gamma(sI - \alpha)^{-1}\beta = \gamma_2(sI - \alpha_{22})^{-1}\beta_2$.

The following trick allows us to apply Proposition 1 via Lemma 2. We introduce additional "virtual" outputs

$$w' = x_1 \tag{8}$$

of the node systems to obtain the observable augmented uncoupled network dynamics

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} v, \\ \begin{pmatrix} w' \\ w \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned} \tag{9}$$

We now augment the network coupling matrices to

$$\overline{A} = \begin{pmatrix} 0 & A \end{pmatrix} \text{ and } \overline{B} = B \tag{10}$$

and obtain the new coupling equation

$$v = \overline{A} \begin{pmatrix} w' \\ w \end{pmatrix} + \overline{B}u = Aw + Bu. \tag{11}$$

Observe that the (x,v,w,u) behavior $\overline{\mathcal{B}}_{(x,v,w,u)}$ of the augmented network (9) and (11) is equal to the full behavior $\mathcal{B}$ of the original network as the "virtual" outputs w' are not connected to any node system inputs. Here, $x^\top = (x_1^\top, x_2^\top)$. But then $\overline{\mathcal{B}}_{(x,u)} = \mathcal{B}_{(x,u)}$.

In the following, we derive a kernel representation of $\overline{\mathcal{B}}_{(w',w,u)}$. By definition, $\overline{\mathcal{B}} = \overline{\mathcal{B}}_{(x_1,x_2,v,w',w,u)} = \operatorname{Ker}R(\frac{\mathrm{d}}{\mathrm{d}t})$ where

$$R(s) = \begin{pmatrix} sI - \alpha_{11} & -\alpha_{12} & -\beta_1 & 0 & 0 & 0 \\ 0 & sI - \alpha_{22} & -\beta_2 & 0 & 0 & 0 \\ -I & 0 & 0 & I & 0 & 0 \\ 0 & -\gamma_2 & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & -A & -B \end{pmatrix}.$$

By observability of (γ_2, α_{22}), there exist polynomial matrices $X(s)$ and $Y(s)$ such that

$$\begin{pmatrix} X(s) & Y(s) \end{pmatrix} \begin{pmatrix} sI - \alpha_{22} \\ \gamma_2 \end{pmatrix} = I. \tag{12}$$

Choose a left coprime factorization

$$D_{so}(s)^{-1} N_{so}(s) = \gamma_2 (sI - \alpha_{22})^{-1},$$

where the subscript 'so' refers to the fact that it is a factorization of the transfer matrix from the (observable parts of) the states to the outputs of all node systems. Compute

$$\begin{pmatrix} * & * & * & * & 0 \\ * & * & * & * & 0 \\ I & \alpha_{12}X(s) & sI - \alpha_{11} & -\alpha_{12}Y(s) & 0 \\ 0 & N_{so}(s) & 0 & D_{so}(s) & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} R(s) =$$

$$\begin{pmatrix} \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} & * & * & * & * \\ \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix}} & \begin{matrix} Z(s) \\ -N_{so}(s)\beta_2 \\ I \end{matrix} & \begin{matrix} sI - \alpha_{11} \\ 0 \\ 0 \end{matrix} & \begin{matrix} -\alpha_{12}Y(s) \\ D_{so}(s) \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ -B \end{matrix} \end{pmatrix},$$

where $Z(s) = -\beta_1 - \alpha_{12}X(s)\beta_2$ and the entries marked with '*' in the first two rows of the first matrix have been chosen such that the resulting matrix is unimodular; this is possible since $\begin{pmatrix} N_{so}(s) & D_{so}(s) \end{pmatrix}$ is left coprime. Observe that the matrix formed from the first two block columns of $R(s)$ has full column rank and hence the upper left block of the last matrix in the preceding equation has full column rank; it hence also has full row rank as it is a square matrix. The elimination theorem [8, Theorem 6.2.6] now implies that the lower right block in the last matrix provides a kernel

representation of $\overline{B}_{(v,w',w,u)}$. We continue by also eliminating v. Compute

$$
\begin{pmatrix} I & 0 & -Z(s) \\ 0 & I & N_{so}(s)\beta_2 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} Z(s) & sI - \alpha_{11} & -\alpha_{12}Y(s) & 0 \\ -N_{so}(s)\beta_2 & 0 & D_{so}(s) & 0 \\ I & 0 & -A & -B \end{pmatrix}
$$
$$
= \begin{pmatrix} 0 & sI - \alpha_{11} & -\alpha_{12}Y(s) + Z(s)A & Z(s)B \\ 0 & 0 & D_{so}(s) - N_{so}(s)\beta_2 A & -N_{so}(s)\beta_2 B \\ I & 0 & -A & -B \end{pmatrix}
$$

then the elimination theorem yields $\overline{\mathcal{B}}_{(w',w,u)} = \operatorname{Ker} P(\frac{\mathrm{d}}{\mathrm{d}t})$ where

$$
P(s) = \begin{pmatrix} sI - \alpha_{11} & -\alpha_{12}Y(s) + Z(s)A & Z(s)B \\ 0 & D_{so}(s) - N_{so}(s)\beta_2 A & -N_{so}(s)\beta_2 B \end{pmatrix}.
$$

Recall that here $Z(s) = -\beta_1 - \alpha_{12}X(s)\beta_2$, the polynomial matrices $X(s)$ and $Y(s)$ are given by (12) and $D_{so}(s)^{-1}N_{so}(s) = \gamma_2 (sI - \alpha_{22})^{-1}$ is a left coprime factorization.

IV. Network controllability

The following is our key result on network controllability. Please refer to Section III for the notation used in the theorem statement.

Theorem 3. *The network* (4) *is controllable if and only if the polynomial matrix*

$$
P(s) = \begin{pmatrix} sI - \alpha_{11} & -\alpha_{12}Y(s) + Z(s)A & Z(s)B \\ 0 & D_{so}(s) - N_{so}(s)\beta_2 A & -N_{so}(s)\beta_2 B \end{pmatrix}
$$

is left prime.

Proof. The input/state/output system obtained by plugging Eq. (11) into Eq. (9) has behavior $\overline{\mathcal{B}}_{(x,w',w,u)}$, and is observable by Lemma 2. Hence, controllability of $\mathcal{B}_{(x,u)} = \overline{\mathcal{B}}_{(x,u)}$ is equivalent to controllability of $\overline{\mathcal{B}}_{(w',w,u)}$ by Proposition 1. A full row rank kernel representation of $\overline{\mathcal{B}}_{(w',w,u)}$ is given by the matrix $P(s)$ and the result now follows from [8, Theorem 5.2.10]. □

Note that the matrix $P(s)$ in the above theorem is in general significantly smaller than the matrix in a classical, directly applied PBH-test would be. In the sequel we will obtain several consequences of the key result of Theorem 3. However, before embarking on this, we first formulate a basic result that establishes a relation between our special factorization $D_{so}^{-1}(s)N_{so}(s)\beta_2$ of the joint node transfer matrix $\gamma(sI - \alpha)^{-1}\beta = \gamma_2(sI - \alpha_{22})^{-1}\beta_2$ and any arbitrary left coprime factorization of this transfer matrix.

Lemma 4. *Let $D(s)^{-1}N(s)$ be any left coprime factorization of $\gamma(sI - \alpha)^{-1}\beta$. Assume that (α, β) is a controllable pair. Then $D_{so}^{-1}(s)N_{so}(s)\beta_2$ is also a left coprime factorization and there exists a unimodular polynomial matrix $U(s)$ such that*

$$
\begin{pmatrix} D_{so}(s) & -N_{so}(s)\beta_2 \end{pmatrix} = U(s) \begin{pmatrix} D(s) & -N(s) \end{pmatrix}.
$$

Proof. This follows immediately from Lemma 22 and Lemma 23 in the Appendix. □

Then, as a first consequence of Theorem 3 we recover a result by Fuhrmann and Helmke [5, Theorem 9.8] that is reminiscent of the classical Popov-Belevitch-Hautus test (PBH test) for controllability. The result deals with the special case where all node systems are controllable and observable.

Corollary 5. [Fuhrmann-Helmke test] *Assume that all node systems are controllable and observable. Let $D(s)^{-1}N(s)$ be any left coprime factorization of $\gamma (sI - \alpha)^{-1} \beta$. Then the network (4) is controllable if and only if the polynomial matrix*

$$
\begin{pmatrix} D(s) - N(s)A & -N(s)B \end{pmatrix} \tag{13}
$$

is left prime.

Proof. By observability, the first block row in $P(s)$ is not present. The result then follows immediately from Theorem 3 and Lemma 4. □

Before proceeding, we make the trivial observation that controllability of all node systems is a necessary condition for controllability of the network (4).

Lemma 6. *If the network* (5) *is controllable then all pairs $(\alpha^{(i)}, \beta^{(i)})$, $i = 1, 2, \ldots, N$ are controllable.*

Proof. Let $\eta \in \mathbb{C}^{\sum_{i=1}^{N} n_i}$ and $\lambda \in \mathbb{C}$ be such that $\eta^*\beta = 0$ and $\eta^*\alpha = \lambda\eta^*$. Then $\eta^*(\alpha + \beta A\gamma) = \eta^*\alpha = \lambda\eta^*$ and controllability of the network implies $\eta = 0$. It follows that the pair (α, β) is controllable and hence all pairs $(\alpha^{(i)}, \beta^{(i)})$, $i = 1, 2, \ldots, N$ are controllable. □

Returning now to the general case that not all node systems are required to be observable, the following result gives a necessary condition on the rank of the external input matrix B for the network to be controllable. The result gives a necessary lower bound on the rank of B in terms of the number of unobservable agents that share a common unobservable eigenvalue.

Theorem 7. *Assume that among the N node systems (1) there are k unobservable ones, indexed by, say $i_1, i_2, \ldots i_k$. Moreover, assume these have at least one unobservable eigenvalue in common, i.e., there exists $\lambda \in \mathbb{C}$ such that λ is an unobservable eigenvalue of the pair $(\gamma^{(j)}, \alpha^{(j)})$ for $j = i_1, i_2, \ldots, i_k$ with geometric multiplicities $\ell_1, \ldots, \ell_k$, respectively. Then the network is controllable only if $\operatorname{rank}(B) \geq \sum_{i=1}^{k} \ell_i \geq k$.*

Proof. Let $P(s)$ be given by (3). Obviously, for any $\lambda \in \mathbb{C}$ we must have $\operatorname{rank}(P(\lambda)) \leq \operatorname{rowdim}(P)$, where 'rowdim' denotes the number of rows. Counting this number of rows we find

$$
\operatorname{rowdim}(P) = \sum_{i=1}^{N} n_{i,1} + \sum_{i=1}^{N} p_i. \tag{14}
$$

Now let λ be a joint unobservable eigenvalue of k node systems with respective multiplicities $\ell_1, \ldots, \ell_k$. Then λ must be at least a $\sum_{i=1}^{k} \ell_i$-fold eigenvalue of α_{11} and hence $\operatorname{rank}(\lambda I - \alpha_{11}) \leq \sum_{i=1}^{N} n_{i,1} - \sum_{i=1}^{k} \ell_i$. Now looking at the column space of $P(\lambda)$ we estimate:

$$
\operatorname{rank}(P(\lambda)) \leq \sum_{i=1}^{N} n_{i,1} - \sum_{i=1}^{k} \ell_i + \sum_{i=1}^{N} p_i + \operatorname{rank}(B). \tag{15}
$$

Now assume that the network is controllable. Then by Theorem 3, $P(\lambda)$ has full row rank, i.e., $\operatorname{rank}(P(\lambda)) = \operatorname{rowdim}(P)$. By combining (14) and (15) this implies $\operatorname{rank}(B) \geq \sum_{i=1}^{k} \ell_i \geq k$. □

A. Controllability of homogeneous networks

Note that the previous results were all concerned with the heterogeneous case. We will now specialize to the case that the network is homogeneous, i.e., all node systems are identical, and given by the triple $(\alpha^{(0)}, \beta^{(0)}, \gamma^{(0)})$. If the node system is unobservable, then a necessary condition for controllability of the network is that the rank of B is at least equal to the number of nodes.

Corollary 8. *Assume the network is homogeneous. Assume that the node system is unobservable with an unobservable eigenvalue of geometric multiplicity ℓ. Then the network is controllable only if $\operatorname{rank}(B) \geq N\ell \geq N$.*

Proof. This follows immediately from Theorem 7. □

By collecting the above results we arrive at the following consequence of Theorem 3, again concerning the homogeneous case.

Theorem 9. *Assume the network is homogeneous. Let $D(s)^{-1}N(s)$ be a left coprime factorization of $\gamma(sI-\alpha)^{-1}\beta$. Assume that $\operatorname{rank}(B) < N$. Then the network is controllable if and only if the node system is controllable and observable, and*

$$\begin{pmatrix} D(s) - N(s)A & -N(s)B \end{pmatrix} \qquad (16)$$

is left prime.

Proof. If $\operatorname{rank}(B) < N$ then by Corollary 8 the node system is observable. Controllability of the node system follows from Lemma 6. By applying Corollary 5 we then obtain that (16) is left prime. The converse follows immediately from Corollary 5. $\square$

Specializing even further, we now consider the case that the network is homogeneous and the node system is a single input single output system. Note that in that case the input matrix B has N rows. Thus, either $\operatorname{rank}(B) = N$ or $\operatorname{rank}(B) < N$. In case that B has full row rank N, the corresponding network turns out to be controllable if and only if the node system is controllable, irrespective of the coupling matrix A. The case that $\operatorname{rank}(B) < N$ is the main result of Hara et al. [4, Proposition 3.1]. We recover this result as a special case of our Theorem 9.

Corollary 10. *Assume the network is homogeneous and the node system is a single-input single-output system. Then the following holds.*

1) *If $\operatorname{rank}(B) = N$ then the network is controllable if and only if the node system is controllable.*
2) *If $\operatorname{rank}(B) < N$ then the network is controllable if and only if the node system is controllable and observable, and the pair (A, B) is controllable.*

Proof. A proof of 1. can be given using the PBH test. Assume the node system is controllable and let $\eta \in \mathbb{C}^{\sum_{i=1}^{N} n_i}$ and $\lambda \in \mathbb{C}$ be such that $\eta^*\beta B = 0$ and $\eta^*(\alpha + \beta A\gamma) = \lambda\eta^*$. Since B has full row rank this implies $\eta^*\beta = 0$ and hence $\eta^*\alpha = \lambda\eta^*$. Controllability of the node system then implies $\eta = 0$, so the network is controllable. The converse implication follows from Lemma 6.

We will now prove 2. First note that a left coprime factorization $D(s)^{-1}N(s)$ of $\gamma(sI-\alpha)^{-1}\beta$ is obtained by taking a left coprime factorization $d(s)^{-1}n(s)$ of $\gamma^{(0)}\left(sI-\alpha^{(0)}\right)^{-1}\beta^{(0)}$, and then putting $D(s) := d(s)I$ and $N(s) = n(s)I$, with I the $N \times N$ identity matrix. Then (16) reduces to

$$\begin{pmatrix} d(s)I - n(s)A & -n(s)B \end{pmatrix} \qquad (17)$$

First assume that the network is controllable. Lemma 6 then yields controllability of the node system. Moreover, by Theorem 9, the node system is observable and (17) is left prime. We want to prove that for each $\mu \in \mathbb{C}$ the complex matrix $(\mu I - A \quad -B)$ has full row rank. To prove this, let μ be given. Consider the polynomial equation $d(s) - \mu n(s) = 0$ in the unknown s. Clearly it has a solution, say $\lambda \in \mathbb{C}$. Note that $n(\lambda) \neq 0$ for otherwise we would also have $d(\lambda) = 0$ which would contradict coprimeness of $d(s)$ and $n(s)$. Thus we obtain $\mu = \frac{d(\lambda)}{n(\lambda)}$. Since

$$\begin{pmatrix} d(\lambda)I - n(\lambda)A & -n(\lambda)B \end{pmatrix} \qquad (18)$$

has full row rank, the same now holds for $(\mu I - A \quad -B)$.

We will now prove the converse implication. Assume that the node system is controllable and assume that (A, B) is controllable. Our aim is to show that (18) has full row rank for all $\lambda \in \mathbb{C}$. Take any λ. If $n(\lambda) \neq 0$ then define $\mu := \frac{d(\lambda)}{n(\lambda)}$. Since $(\mu I - A \quad -B)$ has

full row rank the same now holds for (18). On the other hand, if $n(\lambda) = 0$ then necessarily $d(\lambda) \neq 0$. This again follows from the fact that $n(s)$ and $d(s)$ are coprime. Also in that case (18) has full row rank and hence (17) is left prime. Together with observability of the node system, an application of Theorem 9 then completes the proof. $\square$

Remark 11. For the homogeneous SISO case, Corollary 10 gives a complete picture of how to express controllability of the network in terms of conditions on the node system and the coupling matrices. In particular, for the case $\operatorname{rank}(B) < N$ these involve two conditions on the node system only (controllablity and observability) and a condition on the coupling matrices only (controllability). For the case $\operatorname{rank}(B) = N$ there is only a condition on the node system (controllability). This is in contrast with the homogeneous MIMO case: for the case $\operatorname{rank}(B) < N$ we have conditions in terms of the node system only (again controllablity and observability), but the left primeness condition on (16) involves both the node system and the coupling matrices. In addition, it not clear how to obtain similar necessary and sufficient conditions for the case that $\operatorname{rank}(B) \geq N$. Note that in the MIMO case, B has Nm_0 rows, where m_0 is the number of inputs of the node system. Of course, if $\operatorname{rank}(B) = Nm_0$ (full row rank) then, like in the SISO case, controllability of the network is equivalent to controllability of the node system. A complete picture for the case that $N \leq \operatorname{rank}(B) < Nm_0$ remains unclear beyond the general condition given in Theorem 3.

V. NETWORK STABILIZABILITY

We now turn to necessary and sufficient conditions for stabilizability of the network. While doing this, we will also address the issue of stability. In particular, we aim at finding conditions, preferably in terms of conditions on the node systems and conditions on the coupling matrices, under which the network (5) is stabilizable by means of the external input u. The development will follow that of the controllability case and therefore some of the details will be omitted.

Again, we will use results from the behavioral approach [8] whenever convenient. We refer to Section III for the notation used in the sequel. Recall that $D_{so}(s)^{-1}N_{so}(s) = \gamma_2(sI-\alpha_{22})^{-1}$ is a left coprime factorization and that $X(s)$ and $Y(s)$ are polynomial matrices such that (12) holds; these exist by observability. We now state necessary and sufficient conditions for network stabilizability.

Theorem 12. *The network* (4) *is stabilizable if and only if the complex matrix*

$$P(\lambda) = \begin{pmatrix} \lambda I - \alpha_{11} & -\alpha_{12}Y(\lambda) + Z(\lambda)A & Z(\lambda)B \\ 0 & D_{so}(\lambda) - N_{so}(\lambda)\beta_2 A & -N_{so}(\lambda)\beta_2 B \end{pmatrix}$$

has full row rank for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) \geq 0$.

Proof. Again, the input/state/output system obtained by plugging Eq. (11) into Eq. (9) has behavior $\overline{\mathcal{B}}_{(x,w',w,u)}$ and is observable by Lemma 2. Hence, stabilizability of $\mathcal{B}_{(x,u)} = \overline{\mathcal{B}}_{(x,u)}$ is equivalent to stabilizability of $\overline{\mathcal{B}}_{(w',w,u)}$ by Proposition 1. Since a full row rank kernel representation of $\overline{\mathcal{B}}_{(w',w,u)}$ is given by the matrix $P(s)$, the result then follows from [8, Theorem 5.2.30]. $\square$

We now turn to studying the consequences of this theorem. Before doing this we formulate the following lemma on coprime factorizations of $G(s) = \gamma(sI-\alpha)^{-1}\beta = \gamma_2(sI-\alpha_{22})^{-1}\beta_2$.

Lemma 13. *Let $D(s)^{-1}N(s)$ be any left coprime factorization of $\gamma(sI-\alpha)^{-1}\beta$. Assume that (α, β) is stabilizable. Then $(D_{so}(\lambda) \quad -N_{so}(\lambda)\beta_2)$ has full row rank for all $\lambda \in \mathbb{C}$ with*

$\text{Re}(\lambda) \geq 0$ *and there exists a polynomial matrix* $W(s)$ *such that* $W(\lambda)$ *is nonsingular for all* $\lambda \in \mathbb{C}$ *with* $\text{Re}(\lambda) \geq 0$ *and*

$$\begin{pmatrix} D_{so}(s) & -N_{so}(s)\beta_2 \end{pmatrix} = W(s) \begin{pmatrix} D(s) & -N(s) \end{pmatrix}.$$

Proof. This follows immediately from Lemma 22 and Lemma 23 in the Appendix. □

For the special case that all node systems are stabilizable and detectable we can now formulate an obvious Fuhrmann-Helmke test for stabilizability, analogous to the test for controllability in Corollary 5.

Corollary 14. *Assume that all node systems are stabilizable and detectable. Let* $D(s)^{-1}N(s)$ *be a left coprime factorization of* $\gamma (sI - \alpha)^{-1} \beta$. *Then the network (4) is stabilizable if and only if*

$$\begin{pmatrix} D(\lambda) - N(\lambda)A & -N(\lambda)B \end{pmatrix} \qquad (19)$$

has full row rank for all $\lambda \in \mathbb{C}$ *with* $\text{Re}(\lambda) \geq 0$.

Proof. If the network is stabilizable, then by Theorem 12 the matrix formed by the lower right blocks in $P(\lambda)$ must have full row rank for all $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$. By Lemma 13 the same then holds for (19). The converse follows similarly, with the additional observation that, by detectability, $\lambda I - \alpha_{11}$ is nonsingular for all $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$. □

We now first make the observation that stabilizability of all node systems is necessary for stabilizability of the network. Similar to the controllability case in Lemma 6, a proof can be given using the PBH test and will be omitted here.

Lemma 15. *If the network (5) is stabilizable then all pairs* $(\alpha^{(i)}, \beta^{(i)})$ *(* $i = 1, 2, \ldots, N$*) are stabilizable.*

In the sequel we will return to the general situation that not all node systems are detectable. Then we have the following lower bound on the rank of the input matrix B for network stabilizability.

Theorem 16. *Assume that among the N node systems (1) there are k nondetectable ones, indexed by, say $i_1, i_2, \ldots i_k$. Moreover, assume that there exists a common $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$ such that λ is an unobservable eigenvalue of the pair $(\gamma^{(j)}, \alpha^{(j)})$ for $j = i_1, i_2, \ldots, i_k$ with geometric multiplicities $\ell_1, \ldots, \ell_k$, respectively. Then the network is stabilizable only if $\text{rank}(B) \geq \sum_{i=1}^{k} \ell_i \geq k$.*

Proof. A proof runs along the same lines as the proof of Theorem 7, where now $\lambda \in \mathbb{C}$ is a joint unobservable eigenvalue with $\text{Re}(\lambda) \geq 0$. The details are omitted. □

A. Stabilizability of homogeneous networks

Again, all previous results deal with the heterogeneous multi-input multi-output (MIMO) case. We will now consider the case that the network is homogeneous. In that case we have the following result.

Corollary 17. *Assume the network is homogeneous. Assume that the node system is nondetectable with an unobservable eigenvalue of geometric multiplicity ℓ. Then the network is stabilizable only if $\text{rank}(B) \geq N\ell \geq N$.*

The following analogue of Theorem 9 now follows.

Theorem 18. *Assume the network is homogeneous. Let $D(s)^{-1}N(s)$ be a left coprime factorization of $\gamma (sI - \alpha)^{-1} \beta$. Assume that $\text{rank}(B) < N$. Then the network is stabilizable if and only if the node system is stabilizable and detectable, and*

$$\begin{pmatrix} D(\lambda) - N(\lambda)A & -N(\lambda)B \end{pmatrix} \qquad (20)$$

has full row rank for all $\lambda \in \mathbb{C}$ *with* $\text{Re}(\lambda) \geq 0$.

Proof. Let $\text{rank}(B) < N$. If the network is stabilizable then by Corollary 17 the node system is detectable. By Lemma 15 the node system is stabilizable. By Corollary 14, (20) then has full row rank for all $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$. The converse follows immediately from Corollary 14. □

Finally, we specialize to the homogeneous SISO case to obtain an extension of the controllability result of Hara et al. [4] dealing with stabilizability. In the following, let

$$g^{(0)}(s) = \gamma^{(0)} \left(sI - \alpha^{(0)} \right)^{-1} \beta^{(0)} \qquad (21)$$

be the transfer function of the node system. We have the following characterization of stabilizability of the network in terms of the node system and the coupling matrices.

Theorem 19. *Assume the network is homogeneous and the node system is a single-input single-output system with transfer function $g^{(0)}(s)$, cf. (21). Then the following holds.*

  1) *If $\text{rank}(B) = N$ then the network is stabilizable if and only if the node system is stabilizable.*
  2) *If $\text{rank}(B) < N$ then the network is stabilizable if and only if the node system is stabilizable and detectable, and*

$$\left(\frac{1}{g^{(0)}(\lambda)} I - A \quad -B \right) \qquad (22)$$

  has full row rank for all $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$ and $g^{(0)}(\lambda) \neq 0$.

Proof. As in the proof of Corollary 10, let the polynomials $d(s)$ and $n(s)$ be obtained by taking a coprime factorization $d(s)^{-1}n(s)$ of $g^{(0)}(s)$. Assume that the network is stabilizable. By Lemma 15 and Theorem 18 we have that the node system is stabilizable and detectable, and

$$\begin{pmatrix} d(\lambda)I - n(\lambda)A & -n(\lambda)B \end{pmatrix} \qquad (23)$$

has full row rank for all $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$. Take any such λ with $g^{(0)}(\lambda) = d(\lambda)^{-1}n(\lambda) \neq 0$. Then $n(\lambda) \neq 0$. Noting that $\frac{1}{g^{(0)}(\lambda)} = \frac{d(\lambda)}{n(\lambda)}$, the condition (22) then follows.

We now prove the converse. Assume the node system is stabilizable and assume that (22) holds for all $\text{Re}(\lambda) \geq 0$ with $g^{(0)}(\lambda) \neq 0$. For such λ we clearly have $n(\lambda) \neq 0$ so (23) has full row rank. It remains to show that this also holds for $\text{Re}(\lambda) \geq 0$ with $g^{(0)}(\lambda) = 0$. For such λ we have $n(\lambda) = 0$. By coprimeness of $n(s)$ and $d(s)$ we then have that $d(\lambda) \neq 0$ so also in this case we have that (23) has full row rank. Thus, (20) has full row rank for all $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$. Together with detectability of the node system the result then follows from Theorem 18. □

Unlike in the case of controllability, the full row rank condition on (22) does not reduce to ordinary stabilizability of the pair (A, B). The following example illustrates this.

Example 20. As node system take the stabilizable and detectable system $\dot{x}^{(0)} = ax^{(0)} + v^{(0)}, w^{(0)} = x^{(0)}$ with $a \in \mathbb{R}$. Its transfer function is $g^{(0)}(s) = \frac{1}{s-a}$. For given coupling matrices (A, B), network stabilizability then holds if and only if $\begin{pmatrix} (\lambda - a)I - A & -B \end{pmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$. This is equivalent with the condition that $\begin{pmatrix} \mu I - A & -B \end{pmatrix}$ has full row rank for all $\mu \in \mathbb{C}$ with $\text{Re}(\mu) \geq -a$, in other words: stabilizability with respect to the stability domain $\{\mu \in \mathbb{C} \,|\, \text{Re}(\mu) < -a\}$.

B. Stability of autonomous networks

Finally, we turn to the question under which conditions on the node systems and coupling matrix A the autonomous network

$$\dot{x} = (\alpha + \beta A \gamma) x \qquad (24)$$

is stable. Obviously, the network is stable if and only if it is stabilizable with $B = 0$. Clearly then, in the (most general) heterogeneous MIMO case, we obtain from Theorem 12 that the network is stable if and only if

$$P(s) = \begin{pmatrix} sI - \alpha_{11} & -\alpha_{12}Y(s) + Z(s)A \\ 0 & D_{so}(s) - N_{so}(\lambda)\beta_2 A \end{pmatrix} \qquad (25)$$

is Hurwitz. It follows that all eigenvalues of α_{11} must have negative real part, so all node systems must be detectable. From Lemma 15 also all node systems must be stabilizable.

In case the network is MIMO but homogeneous, it follows immediately from this that the network is stable if and only if the node system is stabilizable and detectable and $(D(s) - N(s)A)$ is Hurwitz, where $D(s)^{-1}N(s)$ is a left coprime factorization of $\gamma(sI - \alpha)^{-1}\beta$.

We finally look at the homogeneous SISO case to obtain the following immediate consequence of Theorem 19.

Corollary 21. *Assume the network is homogeneous and the node system is a single-input single-output system with transfer function $g^{(0)}(s)$, cf. (21). Then the network is stable if and only if the node system is stabilizable and detectable, and*

$$\frac{1}{g^{(0)}(\lambda)} I - A \qquad (26)$$

is nonsingular for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$ and $g^{(0)}(\lambda) \neq 0$.

VI. Conclusions

In this paper we have studied controllability, stabilizability and stability of networks of linear systems. In particular, we have established necessary and sufficient conditions for controllability of heterogeneous networks of MIMO node systems, generalizing a controllability condition of Fuhrmann and Helmke [5] to the case that the node system is not necessarily observable. For the case of homogeneous networks of MIMO nodes we have shown that if the rank of the external input matrix is strictly less than the number of nodes, then observability of the node system is even a necessary condition for controllability. This result can also be considered as a MIMO extension of a similar result of Hara et al. [4] on SISO node systems. Finally, we have extended our results on controllability to conditions for stabilizability and stability of networks. In particular, we have extended the celebrated result of Hara et al. [4] to conditions for stabilizability of homogeneous networks of SISO node systems.

VII. Appendix

In this appendix we collect some technicalities needed in this paper.

Proof of Proposition 1.
We prove 2. Let (u, y) be a trajectory in $\mathcal{B}_{(u,y)}$. There exists x such that $y = Cx$, and $(x, u) \in \mathcal{B}_{(x,u)}$. Since $\mathcal{B}_{(x,u)}$ is stabilizable, there exists $(x', u') \in \mathcal{B}_{(x,u)}$ such that $(x'(t), u'(t)) \to 0$ as $t \to \infty$ and $(x(t), u(t)) = (x'(t), u'(t))$ for all $t \leq 0$. Define $y' = Cx'$. Then $(u', y') \in \mathcal{B}_{(u,y)}$, $(u'(t), y'(t)) \to 0$ as $t \to \infty$ and $(u(t), y(t)) = (u'(t), y'(t))$ for all $t \leq 0$. This shows that $\mathcal{B}_{(u,y)}$ is stabilizable.

Assume now that (7) is observable. Then in particular it is detectable. Let (x, u) be a trajectory in $\mathcal{B}_{(x,u)}$. Define $y = Cx$. Then (u, y) in $\mathcal{B}_{(u,y)}$. By stabilizability there exists $(u', y') \in \mathcal{B}_{(u,y)}$ with $(u'(t), y'(t)) \to 0$ as $t \to \infty$ and $(u(t), y(t)) = (u'(t), y'(t))$

for all $t \leq 0$. Clearly, there exists x' such that $y' = Cx'$ and $(x', u') \in \mathcal{B}_{(x,u)}$. By observability, since (u, y) and (u', y') coincide on the negative halve line, this implies $x(t) = x'(t)$ for $t \leq 0$. Finally, since $(u'(t), y'(t)) \to 0$ as $t \to \infty$, by detectability we have $x'(t) \to 0$ as well. Also, $(x(t), u(t)) = (x'(t), u'(t))$ for all $t \leq 0$. This proves stabilizability of $\mathcal{B}_{(x,u)}$.

A proof of 1. uses similar ideas as above, and will be omitted here.

Lemma 22. *Consider the system (7). Let $D(s)^{-1}N(s) = C(sI - A)^{-1}$ be a left coprime factorization. Then the following statements hold.*

1) *If (A, B) is a controllable pair then $(D(s) \quad - N(s)B)$ is left prime.*
2) *If (A, B) is a stabilizable pair then $(D(\lambda) \quad - N(\lambda)B)$ has full row rank for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$.*

Proof. Let $\mathcal{B}$ be the full behavior of (7). The full behavior has a kernel representation $\mathcal{B} = \mathrm{Ker} R(\frac{d}{dt})$, with

$$R(s) = \begin{pmatrix} sI - A & -B & 0 \\ -C & 0 & I \end{pmatrix}.$$

Note that if (A, B) is controllable, then the projected behavior $\mathcal{B}_{(u,y)}$ is controllable by Proposition 1. If (A, B) is stabilizable, then the projected behavior $\mathcal{B}_{(u,y)}$ is stabilizable by Proposition 1. We derive a kernel representation for this projected behavior. Compute

$$\begin{pmatrix} * & * \\ N & D \end{pmatrix} R(s) = \begin{pmatrix} * & * & * \\ 0 & -N(s)B & D(s) \end{pmatrix}.$$

Here the *'s in the first row of the first matrix have been chosen such that the resulting matrix is unimodular. This is possible due to coprimeness of $D(s)$ and $N(s)$. Since the first block column of $R(s)$ has full column rank, the first block column of the matrix on the right hand side must have full column rank as well. Thus, the '*' in the upper left corner of this matrix (being square) must have full row rank. Hence, the projected behavior $\mathcal{B}_{(u,y)}$ has a kernel representation $-N(\frac{d}{dt})Bu + D(\frac{d}{dt})y = 0$. If this behavior is controllable we must have that $(D(\lambda) \quad - N(\lambda)B)$ has full row rank for all $\lambda \in \mathbb{C}$. If it is stabilizable then the same holds for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$ (see [8]). $\square$

Lemma 23. *Let $G(s)$ be a proper real rational matrix. Assume that $G(s) = D_1(s)^{-1}N_1(s) = D_2(s)^{-1}N_2(s)$ are two factorizations with $D_i(s)$ and $N_i(s)$ $(i = 1, 2)$ polynomial matrices. Then the following holds.*

1) *If both factorizations are left coprime, then there exists a unimodular polynomial matrix $U(s)$ such that $(D_2(s) \, -N_2(s)) = U(s)(D_1(s) \quad - N_1(s))$.*
2) *If $(D_1(s) \quad - N_1(s))$ is left prime and $(D_2(\lambda) \quad - N_2(\lambda))$ has full row rank for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$, then there exists a polynomial matrix $W(s)$ such that $(D_2(s) \, - N_2(s)) = W(s)(D_1(s) \, -N_1(s))$ with the property that $W(\lambda)$ is nonsingular for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$.*

Proof. For a proof of 1., consider the behaviors $\mathcal{B}_1$ and $\mathcal{B}_2$ represented, respectively, by the kernel representations $D_1(\frac{d}{dt})y - N_1(\frac{d}{dt})u = 0$ and $D_2(\frac{d}{dt})y - N_2(\frac{d}{dt})u = 0$. Since both are a factorization of the same transfer matrix, by [8, Theorem 8.2.7] the controllable parts of $\mathcal{B}_1$ and $\mathcal{B}_2$ are equal. However, since $(D_1(\lambda) - N_1(\lambda))$ and $(D_2(\lambda) - N_2(\lambda))$ both have full row rank for all $\lambda \in \mathbb{C}$, both $\mathcal{B}_1$ and $\mathcal{B}_2$ are in fact controllable. We conclude that $\mathcal{B}_1 = \mathcal{B}_2$. It then follows that a unimodular $U(s)$ exists as claimed [8, Theorem 3.6.2].

To prove 2., again consider the behaviors $\mathcal{B}_1$ and $\mathcal{B}_2$. In this case, only $\mathcal{B}_1$ is controllable, so $\mathcal{B}_1$ is equal to the controllable part of $\mathcal{B}_2$.

Clearly this implies that $\mathcal{B}_1 \subset \mathcal{B}_2$. Then there exists a polynomial matrix $W(s)$ such that $(D_2(s) - N_2(s)) = W(s)(D_1(s) - N_1(s))$. Since for $i = 1, 2$, $(D_i(\lambda) - N_i(\lambda))$ has full row rank for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$, we must have that $W(\lambda)$ is nonsingular for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$. $\qquad\square$

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