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Group consensus control for discrete-time heterogeneous multi-agent systems with time delays

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ABSTRACT

The problem of group consensus is considered for a class of discrete-time heterogeneous multi-agent systems(HMASs) with time delays in this paper. Due to network-induced time delays, the predictive scheme is proposed to compensate delays actively by using the received agents' outdate information, and a novel distributed protocol with prediction of states is designed. By getting rid of a conservation condition of in-degree balance which widely exists in relevant results, necessary and/or sufficient conditions are established for group consensus control of discrete-time HMASs. Finally, simulations are demonstrated that, under the proposed theoretical results, the performance of discrete-time HMASs with time delays is similar to that without delays.

1. Introduction

In recent years, as an important research branch of distributed system, multi-agent systems(MASs) have attracted the interest from researchers of biology, physics, mathematics, computer science, engineering and so on [1,2]. There are many studies on cooperative control of MASs, such as stable flocking [3], cooperative attack [4], controllability [5], formation control [6], consensus [7], etc. As we know, the consensus problem is a basic and important issue of MASs.

In many practical situations, agents must be able to adapt to the varying environment and then reach different consistent values. Therefore, it is an important issue that appropriate protocols are designed to make agents reach different consistent values. This problem is called group consensus problem. Different from consensus problem, the group consensus problem concerns a network which is divided into multiple sub-networks. All agents are divided into multiple groups consequently, and information exchange exists not only two agents in a group but also in different groups [8]. In addition, the phenomenon of group consensus is ubiquitous in nature and human society, such as multi-species foraging groups, the pattern formation of bacteria colonies, a school of fish and formation of opinion [9]. Research on group consensus problem not only contributes to a better understanding of natural collective behaviors but also helps to provide different perspectives for studying the distributed coordination of multi-agent systems [10]. The group consensus problem, which is suitable for addressing complex and flexible problems, has attracted the attention of scholars. In the past decade, group consensus analysis has been widely used in multi-robot manipulators, satellite clusters, unmanned aerial vehicles task assignment and so on [11–13].

Recently, great deals of research results about group consensus for MASs have emerged. For a given communication topology, the necessary and sufficient conditions are established to reach group consensus in [14,15]. Furthermore, a reverse group consensus problem for the dynamic agents with the inputs in the cooperation-competition network is investigated in [16], by defining the mirror graph. In addition, Ji et al. extend to study the group consensus for the multi-agent network with generally connected topology, which needs neither being strongly connected nor containing a directed spanning tree in [17]. Yu et al. present some necessary and/or sufficient conditions for a network achieving the group average consensus, and discuss how the information communication among the groups affects the group average consensus in [18]. The group consensus problem with hybrid protocol is investigated in [19], which is considered more consistent with practical situations. The necessary and sufficient conditions for containment control of MASs without and with time-delay are obtained in [20] through

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the iterative method and Z-transformation, and it is shown that analysis of the discrete-time MASs is compactly related to the step-size and the largest eigenvalue of the interaction topology. Shang derives sufficient conditions to guarantee L^1 group consensus of the discrete-time MASs with external stochastic inputs in [21]. The couple-group $\mathcal{L}_2-\mathcal{L}_\infty$ consensus problem of nonlinear MASs affected by external disturbances is discussed in [22], sufficient conditions are obtained by system transformation. Fang et al. propose conditions of the mean-square asymptotic stability and $\mathcal{H}_\infty$ performance by converting closed-loop systems into reduced-order systems in [23].

The above results focus on group consensus for homogeneous MASs. However, the actual application shows that agents may be governed by different dynamics each other due to various restrictions [24]. For example, multi-robot systems with different abilities and shapes are more applicable modeled by heterogeneous dynamics than the homogeneous systems due to restrictions and external uncertainty [25]. Therefore, the group consensus of heterogeneous MASs have more research significance. Group consensus of a class of heterogeneous MASs with input saturation via pinning scheme is investigated in [24]. For the discrete-time heterogeneous MASs composed of hybrid dynamic agents including first-order and second-order dynamic agents, the group consensus problems under fixed topology and dynamically changing interaction topologies are investigated in [26,27], respectively. Two partition coefficients are introduced to realize the group consensus of heterogeneous MASs, and the concrete convergence points are given in [28]. A novel consensus protocol is designed by utilizing cooperative and competitive interactions among agents in [29], and the group consensus problems are solved by getting rid of a conservation condition of in-degree balance. There are some recent results on the consensus control in the area of switched systems. For example, the novel asymptotic stability conditions with less conservatism are obtained for the induced switched piecewise affine (PWA) systems with dual switching mechanism and the design of PWA state-feedback controllers is implemented via the smooth approximation technique in [30]. The global asymptotic stability criteria for discrete-time Lure-type switched systems with parameter mismatches and transmission channel noises are obtained by virtue of the semi-time-varying Lyapunov function in [31]. Based on the average dwell-time approach and the multiple Lyapunov functions method, the passivity analysis for switched neural networks subject to stochastic disturbances and time-varying delays is concerned in [32].

Furthermore, due to pervasive networked environment, many phenomena different from traditional systems have arisen, such as time delays, quantization, missing measurements, saturations, etc. [33–35]. The occurrence of time delays is unavoidable due to the limited bandwidth of the communication channel and the limited transmission speed, when agents communicate each other [36,37]. Time delays can degrade the performance of MASs and even cause divergence, so good progress has also been made on group consensus problems of MASs with time delays in recent years. The group consensus of second-order discrete-time homogeneous MASs with time-varying delays under switching topologies is investigated in [38]. An et al. further investigate the group consensus of high-order homogeneous MASs with communication delays in [39]. Shang et al. study group consensus problems of homogeneous MASs under random noises and time delays in [40]. Hu et al. study event-triggered group consensus by the optimization methods aiming at maximizing the event-triggered parameter and the consensus region in [41]. Wen et al. investigate the dynamical group consensus problem of second-order heterogenous MASs with uniform constant input time delays in [42].

As far as I know, there are few results concerned group consensus of the high-order discrete-time heterogeneous MASs with time delays. Most of the above articles, to overcome the negative effects of communication delays on group consensus, have used directly theoretical results of time-delay systems, which is passive to accept for communication delays. For networked systems, Liu et al. have proposed a different strategy to overcome network delays, that is, the networked predictive scheme in [43–45]. The main idea of this scheme is to use the observed states to predict current and future states, and compensate the network delays actively. Inspired by the relevant works, the group consensus problem is investigated for a high-order discrete-time HMASs with time delays in this paper. The main contributions are summarized in the following three aspects. First, the novel distributed protocols are designed for the HMASs consisting of high-order agents, and the necessary and/or sufficient conditions to guarantee group consensus are derived. In addition, the result is extended to the case of multiple groups. Second, without the in-degree balance condition, the constraint of topological structure is relaxed and the group consensus of HMASs is achieved. Finally, based on the networked predictive scheme, the communication delays are compensated actively, the predictions of data at current time are obtained, which makes that the performance of discrete-time HMASs with time delays is similar to that without delays. Combining the above points of innovation will make the scope of our results more applicable.

The paper is organized as follows. The problem formulation is described and group consensus protocol is designed in Section 2. Some necessary and/or sufficient conditions of group consensus are presented, and the corresponding result is extended to multi-group consensus case in Section 3. Numerical examples are provided in Section 4. Conclusion is drawn in Section 5.

Notation. All notations used in this paper are fairly standard. Throughout the paper, $\mathcal{R}^n$ and $\mathcal{R}^{n\times n}$ serve as the n-dimensional Euclidean space and the space of $n\times n$ matrices. Superscript T stands for the transpose of a vector or a matrix. I_n is an $n\times n$ identity matrix, $\mathbf{0}_{m\times n}$ is an $m\times n$ zero matrix, and $\mathbf{1}_n$ is an n-dimensional column vector with all entries equal to 1. $\otimes$ represents the Kronecker product, which is defined if A is an $m\times n$ matrix and B is a $p\times q$ matrix, the Kronecker product of A and B is a block matrix of $mp\times nq$. Moreover, $\text{diag}(x_1, x_2, \ldots, x_n)$ is applied to express a diagonal matrix, whose diagonal entries are $x_1, x_2, \ldots, x_n$, successively, and $A\oplus B$ represents diag(A, B). Moreover, I and $\mathbf{0}$ represent an identity matrix and a zero matrix with appropriate dimensions, respectively, if their dimensions are not explicitly stated.

2. Problem formulation and group consensus control protocol

Let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A}(\mathcal{G}))$ be a weighted directed graph, where the node set $\mathcal{V}=\{v_1,v_2,\ldots,v_N\}$ represents $N(N\ge 2)$ agents, $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ stands for directed edge set, and a nonnegative weighted adjacency matrix $\mathcal{A}(\mathcal{G})=\left[a_{ij}\right]\in\mathcal{R}^{N\times N}$. The node indices belong to a finite index set $\ell=\{1,2,\ldots,N\}$. A directed edge $e_{ij}=\left(v_i,v_j\right)$ represents the information of the ith agent to the jth agent. The a_{ji}, which is nonzero, is the weight of the edge e_{ij}. Matrix composed of a_{ij} is the associated weighted adjacency matrix of graph $\mathcal{G}$. Moreover, it is assumed that $a_{ii}=0$ $(\forall i\in\ell)$ in this paper. The neighborhood set of node v_i is represented by $\mathcal{N}_i=\left\{v_j\in\mathcal{V}|\left(v_j,v_i\right)\in\mathcal{E}\right\}$. The in-degree of node v_i is defined as $d_{in}(i)=\sum_{j\in\mathcal{N}_i}a_{ij}$. Degree matrix is defined as $\mathcal{D}(\mathcal{G})=\text{diag}\left(d_{in}(1),d_{in}(2),\ldots,d_{in}(N)\right)$. Laplacian matrix $\mathcal{L}^{\mathcal{G}}=\left[l_{ij}\right]_{N\times N}$ of graph $\mathcal{G}$ is defined as $\mathcal{L}^{\mathcal{G}}=\mathcal{D}(\mathcal{G})-\mathcal{A}(\mathcal{G})$,

$$\text{where } l_{ij}=\begin{cases}\sum\limits_{j=1,j\neq i}^{N}a_{ij}, & \text{if } i=j\\ -a_{ij}, & \text{if } i\neq j\end{cases},\quad i,j=1,2,\ldots,N.$$

For a network $(\mathcal{G},x)$ composed of N agents, the state vector is defined as $x=\left[x_1^T\ x_2^T\ \cdots\ x_N^T\right]^T\in\mathcal{R}^{nN}$, and $x_i\in\mathcal{R}^n$ represents the state of the ith agent, such as position, velocity, liquid level, etc. A network $(\mathcal{G}_1, \chi_1)$ with topological graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1(\mathcal{G}))$ is said to be a sub-network of network $(\mathcal{G}, x)$ with a topological graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}(\mathcal{G}))$ if $\mathcal{V}_1 \subseteq \mathcal{V}$, $\mathcal{E}_1 \subseteq \mathcal{E}$, $\mathcal{A}_1(\mathcal{G})$ inherits $\mathcal{A}(\mathcal{G})$. In a similar way, $\mathcal{G}_1$ is called a sub-graph of $\mathcal{G}$. Moreover, if $\mathcal{V}_1 \subseteq \mathcal{V}$, $\mathcal{E}_1 \subseteq \mathcal{E}$ are strictly satisfied, and $\mathcal{E}_1 = \left\{(v_i, v_j) | (v_i, v_j) \in \mathcal{E}, v_i, v_j \in \mathcal{V}_1\right\}$, $(\mathcal{G}_1, \chi_1)$ is called a proper sub-network of $(\mathcal{G}, x)$ and $\mathcal{G}_1$ is said to be a proper sub-graph of $\mathcal{G}$.

Considering a complex network $(\mathcal{G}, x)$ composed of $N+M$ agents, where topological graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}(\mathcal{G}))$ is a weighted directed graph and composed of sub-graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1(\mathcal{G}))$ and sub-graph $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{A}_2(\mathcal{G}))$. Here, $\mathcal{V}_1 = \{v_1, v_2, \ldots, v_N\}$ and $\mathcal{V}_2 = \{v_{N+1}, v_{N+2}, \ldots, v_{N+M}\}$. For node v_i, the sets of its neighbor nodes in two sub-graphs are defined as $\mathcal{N}_{1i} = \left\{v_j \in \mathcal{V}_1 | (v_j, v_i) \in \mathcal{E}\right\}$ and $\mathcal{N}_{2i} = \left\{v_j \in \mathcal{V}_2 | (v_j, v_i) \in \mathcal{E}\right\}$. Obviously, $(\mathcal{G}, x)$ is composed of two sub-networks $(\mathcal{G}_1, \chi_1)$ and $(\mathcal{G}_2, \chi_2)$, where $\chi_1 = \left[x_1^T \; x_2^T \; \cdots \; x_N^T\right]^T \in \mathcal{R}^{nN}$, $\chi_2 = \left[x_{N+1}^T \; x_{N+2}^T \; \cdots \; x_{N+M}^T\right]^T \in \mathcal{R}^{nM}$, and $x = \left[\chi_1^T \; \chi_2^T\right]^T$. The communication between agents exists not only in the same group but also between the two groups. For simplify, the index sets of node sets $\mathcal{V}_1$ and $\mathcal{V}_2$ are denoted as $\ell_1 = \{1, 2, \ldots, N\}$ and $\ell_2 = \{N+1, N+2, \ldots, N+M\}$. Obviously, $\ell = \ell_1 \cup \ell_2$.

Suppose that the heterogeneous multi-agent systems(HMASs) are composed of $N+M$ agents with discrete-time dynamics. The dynamics of the ith agent are described by :

$$\begin{aligned} x_i(t+1) &= A_i x_i(t) + B_i u_i(t), \\ y_i(t) &= C_i x_i(t), \quad \forall\, i \in \ell, \end{aligned} \tag{1}$$

where $x_i(t) \in \mathcal{R}^n$ is the state vector, $u_i(t) \in \mathcal{R}^r$ is the control input vector, $y_i(t) \in \mathcal{R}^m$ is the measured output vector, and A_i, B_i, C_i are the constant matrices.

Remark 1. In the real world, the vast majority of control objects are continuous-time systems. However, information exchange between agents in networked MASs is achieved through the network. Therefore, it is necessary to control a continuous-time system using a discrete control device such as digital computers. It requires the discretization of systems. Common methods of discretization include differential transformation method, response invariant method, bilinear transformation method, zero-pole matching and so on [46]. Therefore, the discrete-time dynamic structure of agents will be described in this paper, and the group consensus problem of the discrete-time networked MASs will be studied based on the networked predictive control method.

The main purpose of this paper is to design a distributed protocol to make the states of the first N agents reach one consistent value, but states of the last M agents reach another consistent value. For the purpose of guaranteeing the feasibility of designing protocols, the following assumptions are reasonably made.

Assumption 1. The state of each agent is unavailable, but its output is detectable.

Assumption 2. The matrix pair (A_i, C_i) is detectable, $\forall\, i \in \ell$.

Assumption 3. Suppose that agent $i (\forall\, i \in \ell)$ receives the information from agent j ($\forall\, v_j \in \mathcal{N}_i$) with time delay $\tau_{ij}(t)$ at time t, where

$$0 < \tau_0 \le \check{\tau}_{ij}(t) \le \tau_{ij}(t) \le \hat{\tau}_{ij}(t) \le \tau, \quad \forall\, i, j \in \ell, \tag{2}$$

and τ is a known constant.

Assumption 4. The data packets transmitted through the network are with time stamps.

Assumption 5. Each agent in the first group can receive information of the $(N+1)$th agent, and each agent in the second group can receive information of the first agent.

Remark 2. The outputs of many systems in field of control engineering are measured at time instants due to the digital communication network or the applications of digital sensors [47]. However, in practice, due to economic costs and/or constraints on measurement, it is often difficult or even unavailable to get the information of all agent's states. So observers are used to estimate systems' states. The data packets transmitted through the network is very important for networked MASs, because the networked predictive scheme proposed in this paper is based on time. In addition, it is assumed that the components in the networked MASs have been synchronized. Furthermore, It can be seen as the first agent of each group has priority over other agents, or it can be regarded as a virtual leader of each group.

When the communication network has bounded time-varying delays at time t, agent i receives information of agent j with time-varying delays $\tau_{ij}(t)$, where τ_0 and τ are known lower and upper bounds. The dwell-time approach is used to deal with the time-varying delays problem of networked control systems in [48,49], that is, when the networked time delay $\tau_{ij}(t) < \tau$, the data in the network will be forced to wait until the time delay reaches the previous set. At this time, the time-varying delays are transformed into the constant time delay. Although the dwell-time approach is conservative. However, when it is very difficult to directly deal with time-varying delays, the dwell-time approach can be used as an effective method to study time-varying delays indirectly.

In order to overcome adverse effects caused by network delays, a network delay compensation scheme is proposed to predict the state of each agent. For the output of its neighboring agent j, a state observer of the jth agent is constructed as

$$\begin{aligned} \hat{x}_j(t-\tau+1|t-\tau) &= A_j \hat{x}_j(t-\tau|t-\tau-1) + B_j u_j(t-\tau) \\ &\quad + L_j\big(y_j(t-\tau) - \hat{y}_j(t-\tau)\big), \\ \hat{y}_j(t-\tau) &= C_j \hat{x}_j(t-\tau|t-\tau-1), \quad \forall\, j \in \mathcal{N}_i, \end{aligned} \tag{3}$$

where $\hat{x}_j(t-p|t-q) \in \mathcal{R}^n$ $(p < q)$ means state prediction of the jth agent at time $t-p$ by the information of the jth agent up to time $t-q$, $\hat{y}_j(t) \in \mathcal{R}^m$ is the observer output at time t. With (3), the state estimations at time t can be predicted forward by the outputs up to time $t-\tau$:

$$\begin{aligned} \hat{x}_j(t-\tau+d|t-\tau) &= A_j \hat{x}_j(t-\tau+d-1|t-\tau) \\ &\quad + B_j u_j(t-\tau+d-1), \quad d = 2, 3, \ldots, \tau. \end{aligned} \tag{4}$$

Assume that matrix $B_i (i \in \ell)$ has a right inverse, a group consensus control protocol using the above state estimation scheme is constructed as

$$u_i(t) = \begin{cases} B_{iR}^{-1}(K_a + \sum_{j=1, j\neq i}^{N} A_j)\hat{x}_i(t|t-\tau) \\ + K_i\big(\sum_{v_j \in \mathcal{N}_{1i}} a_{ij} \Delta\hat{x}_{i,j}(t|t-\tau) \\ + \sum_{v_j \in \mathcal{N}_{2i}} a_{ij} \Delta\hat{x}_{N+1,j}(t|t-\tau)\big), \forall\, i \in \ell_1, \\ B_{iR}^{-1}(K_b + \sum_{j=N+1, j\neq i}^{N+M} A_j)\hat{x}_i(t|t-\tau) \\ + K_i\big(\sum_{v_j \in \mathcal{N}_{2i}} a_{ij} \Delta\hat{x}_{i,j}(t|t-\tau) \\ + \sum_{v_j \in \mathcal{N}_{1i}} a_{ij} \Delta\hat{x}_{1,j}(t|t-\tau)\big), \forall\, i \in \ell_2, \end{cases} \tag{5}$$

where $\Delta\hat{x}_{i,j}(t|t-\tau) = \hat{x}_j(t|t-\tau) - \hat{x}_i(t|t-\tau), i, j \in \ell$, B_{iR}^{-1} is the right inverse of matrix B_i, K_a, K_b and K_i are the gain matrices to be designed, $i \in \ell$.

Remark 3. When time delays $\tau_{ij}(t) < \tau$, the measured output and control input of agent j from time $t-\tau$ to time $t-\tau_{ij}(t)$ can be obtained at time t under the networked predictive scheme. However, according to state estimations (4) and group consensus control protocol (5), the information of agent j from time $t-\tau+1$ to time $t-\tau_{ij}(t)$ needs not participate in operation, which increases computational cost, and thus has certain conservatism. However, it provides a unified prediction process for all agents, and overcomes the impact of time-varying delays on group consensus problem. In addition, the powerful operation ability of embedded microprocessors can ensure the execution of prediction algorithm. Therefore, group consensus control protocol (5) is effective in both theoretical derivation and engineering realization.

Definition 1. For discrete-time HMASs (1), it is called that protocol (5) can solve the group consensus problem or discrete-time HMASs (1) can achieve group consensus under protocol (5), if the following conditions hold:

(i) $\lim_{t\to+\infty}\left\|x_i(t)-x_j(t)\right\|=0, \quad \forall\, i,j\in\ell_1;$

(ii) $\lim_{t\to+\infty}\left\|x_i(t)-x_j(t)\right\|=0, \quad \forall\, i,j\in\ell_2;$

(iii) $\lim_{t\to+\infty}\left\|x_i(t)-\hat{x}_i(t|t-1)\right\|=0, \quad \forall\, i\in\ell,$

where $\hat{x}_i(t|t-1)$ means state prediction of the ith agent at time t by the information at time $t-1$.

For the convenience of analysis, the following new variables are introduced:

$$\begin{aligned}
\xi_i(t) &= x_i(t)-x_1(t), \quad \forall\, i\in\ell_1,\\
\eta_i(t) &= x_i(t)-x_{N+1}(t), \quad \forall\, i\in\ell_2,\\
\varepsilon_i(t) &= x_i(t)-\hat{x}_i(t|t-1), \quad \forall\, i\in\ell,\\
\xi(t) &= \begin{bmatrix}\xi_2^T(t) & \xi_3^T(t) & \cdots & \xi_N^T(t)\end{bmatrix}^T,\\
\eta(t) &= \begin{bmatrix}\eta_{N+2}^T(t) & \eta_{N+3}^T(t) & \cdots & \eta_{N+M}^T(t)\end{bmatrix}^T,\\
\delta(t) &= \begin{bmatrix}\xi^T(t) & \eta^T(t)\end{bmatrix}^T.
\end{aligned}$$

It is obvious that Definition 1 holds if and only if $\|\delta(t)\|\to 0$, and $\|\varepsilon_i(t)\|\to 0\,(i\in\ell)$, when $t\to\infty$.

Let $\mathcal{L}^{\mathcal{G}_i}$ $(i=1,2)$ be the Laplacian matrix of sub-graph $\mathcal{G}_i$ of graph $\mathcal{G}$. $\Omega^{\mathcal{G}_1}=\left[-a_{ij}\right]_{N\times M}$, $i=1,2,\ldots,N;\ j=N+1,N+2,\ldots,N+M$. $\Omega^{\mathcal{G}_2}=\left[-a_{ij}\right]_{M\times N}$, $i=N+1,N+2,\ldots,N+M;\ j=1,2,\ldots,N$. Before the presentation of main results, some relevant matrices are given as follows.

$$\mathcal{L}^{\mathcal{G}}=\begin{bmatrix}\mathcal{L}^{\mathcal{G}_1} & \Omega^{\mathcal{G}_1}\\ \Omega^{\mathcal{G}_2} & \mathcal{L}^{\mathcal{G}_2}\end{bmatrix},$$

$$\mathcal{L}^{\mathcal{G}_i}=\begin{bmatrix}\mathcal{L}_{11}^{\mathcal{G}_i} & \mathcal{L}_{12}^{\mathcal{G}_i}\\ \mathcal{L}_{21}^{\mathcal{G}_i} & \mathcal{L}_{22}^{\mathcal{G}_i}\end{bmatrix}=\begin{bmatrix}\mathcal{L}_{1}^{\mathcal{G}_i}\\ \mathcal{L}_{2}^{\mathcal{G}_i}\end{bmatrix}, \quad i=1,2,$$

$$\Omega^{\mathcal{G}_i}=\begin{bmatrix}\Omega_{11}^{\mathcal{G}_i} & \Omega_{12}^{\mathcal{G}_i}\\ \Omega_{21}^{\mathcal{G}_i} & \Omega_{22}^{\mathcal{G}_i}\end{bmatrix}=\begin{bmatrix}\Omega_{1}^{\mathcal{G}_i}\\ \Omega_{2}^{\mathcal{G}_i}\end{bmatrix}, \quad i=1,2,$$

where

$\mathcal{L}_{11}^{\mathcal{G}_1}\in\mathcal{R}$, $\mathcal{L}_{12}^{\mathcal{G}_1}\in\mathcal{R}^{1\times(N-1)}$, $\mathcal{L}_{21}^{\mathcal{G}_1}\in\mathcal{R}^{(N-1)\times 1}$, $\mathcal{L}_{22}^{\mathcal{G}_1}\in\mathcal{R}^{(N-1)\times(N-1)}$,
$\mathcal{L}_{11}^{\mathcal{G}_2}\in\mathcal{R}$, $\mathcal{L}_{12}^{\mathcal{G}_2}\in\mathcal{R}^{1\times(M-1)}$, $\mathcal{L}_{21}^{\mathcal{G}_2}\in\mathcal{R}^{(M-1)\times 1}$, $\mathcal{L}_{22}^{\mathcal{G}_2}\in\mathcal{R}^{(M-1)\times(M-1)}$,
$\mathcal{L}_{1}^{\mathcal{G}_1}\in\mathcal{R}^{1\times N}$, $\mathcal{L}_{2}^{\mathcal{G}_1}\in\mathcal{R}^{(N-1)\times N}$, $\mathcal{L}_{1}^{\mathcal{G}_2}\in\mathcal{R}^{1\times M}$, $\mathcal{L}_{2}^{\mathcal{G}_2}\in\mathcal{R}^{(M-1)\times M}$.

3. Main results

In this section, under group consensus control protocol (5), some necessary and/or sufficient conditions are established to achieve group consensus for discrete-time HMASs (1).

Theorem 1. *Under the premise of Assumptions 1–5, if there exists time delays (2) in the discrete-time HMASs (1) with two sub-networks $(\mathcal{G}_1,\chi_1)$ and $(\mathcal{G}_2,\chi_2)$, protocol (5) can solve the group consensus problem if and only if matrices Υ and $A_i-L_iC_i$ $(i\in\ell)$ are Schur, where*

$$\Upsilon=\begin{bmatrix}\Lambda_1 & \Pi_1\\ \Pi_2 & \Lambda_2\end{bmatrix},$$

$$\begin{aligned}
\Lambda_1 &= I_{N-1}\otimes(K_a+A_{s_1})-\widehat{BK}_2^N\left(\mathcal{L}_{22}^{\mathcal{G}_1}\otimes I_n\right)+\left(\mathbf{1}_{N-1}\mathcal{L}_{12}^{\mathcal{G}_1}\right)\otimes(B_1K_1),\\
\Pi_1 &= -\widehat{BK}_2^N\left(\Omega_{22}^{\mathcal{G}_1}\otimes I_n\right)+\left(\mathbf{1}_{N-1}\Omega_{12}^{\mathcal{G}_1}\right)\otimes(B_1K_1),\\
\Lambda_2 &= I_{M-1}\otimes(K_b+A_{s_2})-\widehat{BK}_N^{N+M}\left(\mathcal{L}_{22}^{\mathcal{G}_2}\otimes I_n\right)+\left(\mathbf{1}_{M-1}\mathcal{L}_{12}^{\mathcal{G}_2}\right)\otimes(B_{N+1}K_{N+1}),\\
\Pi_2 &= -\widehat{BK}_{N+2}^{N+M}\left(\Omega_{22}^{\mathcal{G}_2}\otimes I_n\right)+\left(\mathbf{1}_{M-1}\Omega_{12}^{\mathcal{G}_2}\right)\otimes(B_{N+1}K_{N+1}).
\end{aligned}$$

Here, $\widehat{BK}_2^N=\oplus\sum_{i=2}^N B_iK_i$, $\widehat{BK}_{N+2}^{N+M}=\oplus\sum_{i=N+2}^{N+M}B_iK_i$, $A_{s_1}=\sum_{j=1}^N A_j$, $A_{s_2}=\sum_{j=N+1}^{N+M}A_j$.

Proof. By state observer (3) and state estimations (4), it is easy to get the following state error equation:

$$\varepsilon_i(t+1)=(A_i-L_iC_i)\varepsilon_i(t), \quad \forall\, i\in\ell, \tag{6}$$

$$\hat{x}_i(t|t-\tau)=x_i(t)-\theta_i(\tau)E_i(t), \quad \forall\, i\in\ell, \tag{7}$$

where

$$\begin{aligned}
\theta_i(\tau) &= \begin{bmatrix}I & L_iC_i & A_iL_iC_i & \cdots & A_i^{\tau-2}L_iC_i\end{bmatrix}\in R^{n\times n\tau},\\
E_i(t) &= \begin{bmatrix}\varepsilon_i^T(t) & \varepsilon_i^T(t-1) & \cdots & \varepsilon_i^T(t-\tau+1)\end{bmatrix}^T\in R^{n\tau\times 1}.
\end{aligned}$$

Let

$$\begin{aligned}
\hat{\xi}_i(t|t-\tau) &= \hat{x}_i(t|t-\tau)-\hat{x}_1(t|t-\tau), \quad \forall\, i\in\ell_1,\\
\hat{\eta}_i(t|t-\tau) &= \hat{x}_i(t|t-\tau)-\hat{x}_{N+1}(t|t-\tau), \quad \forall\, i\in\ell_2.
\end{aligned}$$

Based on the above equations, it is obtained that

$$\begin{aligned}
\hat{\xi}_i(t|t-\tau) &= \xi_i(t)-\theta_i(\tau)E_i(t)+\theta_1(\tau)E_1(t), \quad \forall\, i\in\ell_1;\\
\hat{\eta}_i(t|t-\tau) &= \eta_i(t)-\theta_i(\tau)E_i(t)+\theta_{N+1}(\tau)E_{N+1}(t), \quad \forall\, i\in\ell_2.
\end{aligned}\tag{8}$$

Combining (5) and (8) results in

$$\begin{aligned}
\xi_i(t+1) = {} & (K_a+A_{s_1})\xi_i(t)-(K_a+A_{s_1}-A_i)\theta_i(\tau)E_i(t)\\
& +(K_a+A_{s_1}-A_1)\theta_1(\tau)E_1(t)\\
& +B_iK_i\left(\sum_{j=1}^N a_{ij}\left(\xi_j(t)-\xi_i(t)\right)\right)-B_1K_1\left(\sum_{j=1}^N a_{1j}\xi_j(t)\right)\\
& +B_iK_i\left(\sum_{j=N+1}^{N+M} a_{ij}\eta_j(t)\right)-B_1K_1\left(\sum_{j=N+1}^{N+M} a_{1j}\eta_j(t)\right)\\
& +B_iK_i\left(\sum_{j=1}^N a_{ij}\left(\left(\theta_i(\tau)E_i(t)-\theta_j(\tau)E_j(t)\right)\right)\right)\\
& +B_1K_1\left(\sum_{j=1}^N a_{1j}\left(\theta_j(\tau)E_j(t)-\theta_1(\tau)E_1(t)\right)\right)\\
& -B_iK_i\left(\sum_{j=N+1}^{N+M} a_{ij}\left(\theta_j(\tau)E_j(t)-\theta_{N+1}(\tau)E_{N+1}(t)\right)\right)\\
& +B_1K_1\left(\sum_{j=N+1}^{N+M} a_{1j}\left(\theta_j(\tau)E_j(t)-\theta_{N+1}(\tau)E_{N+1}(t)\right)\right),\\
& \forall\, i\in\ell_1.
\end{aligned}\tag{9}$$

Similarly,

$$\begin{aligned}
\eta_i(t+1) = {} & (K_b+A_{s_2})\eta_i(t)-(K_b+A_{s_2}-A_i)\theta_i(\tau)E_i(t)\\
& +(K_b+A_{s_2}-A_{N+1})\theta_{N+1}(\tau)E_{N+1}(t)\\
& +B_iK_i\left(\sum_{j=N+1}^{N+M} a_{ij}\left(\eta_j(t)-\eta_i(t)\right)\right)\\
& -B_{N+1}K_{N+1}\left(\sum_{j=N+1}^{N+M} a_{(N+1)j}\eta_j(t)\right)
\end{aligned}$$

$$+ B_iK_i\left(\sum_{j=1}^{N} a_{ij}\xi_j(t)\right) - B_{N+1}K_{N+1}\left(\sum_{j=1}^{N} a_{(N+1)j}\xi_j(t)\right)$$
$$+ B_iK_i\left(\sum_{j=N+1}^{N+M} a_{ij}\big(\theta_i(\tau)E_i(t) - \theta_j(\tau)E_j(t)\big)\right)$$
$$+ B_{N+1}K_{N+1}\left(\sum_{j=N+1}^{N+M} a_{(N+1)j}\big(\theta_j(\tau)E_j(t) - \theta_{N+1}(\tau)E_{N+1}(t)\big)\right)$$
$$- B_iK_i\left((\sum_{j=1}^{N} a_{ij}\big(\theta_j(\tau)E_j(t) - \theta_1(\tau)E_1(t)\big)\right)$$
$$+ B_{N+1}K_{N+1}\left(\sum_{j=1}^{N} a_{(N+1)j}\big(\theta_j(\tau)E_j(t) - \theta_1(\tau)E_1(t)\big)\right),$$
$$\forall\, i \in \ell_2. \tag{10}$$

Let

$$E(t) = \begin{bmatrix} E_1^T(t) & E_2^T(t) & \cdots & E_N^T(t) & E_{N+1}^T(t) & \cdots & E_{N+M}^T(t) \end{bmatrix}^T. \tag{11}$$

In the light of (5), (9), (10) and (11), it is obtained that a compact representation contains intragroup tracking error vectors and estimation error vectors as

$$\begin{bmatrix} \xi(t+1) \\ \eta(t+1) \\ E(t+1) \end{bmatrix} = \begin{bmatrix} \Lambda_1 & \Pi_1 & \Gamma_1 \\ \Pi_2 & \Lambda_2 & \Gamma_2 \\ \mathbf{0} & \mathbf{0} & \Xi \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \\ E(t) \end{bmatrix}, \tag{12}$$

where

$$\Gamma_1 = \begin{bmatrix} P_1 & P_2 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} P_3 & P_4 \end{bmatrix}, \ \Xi = \oplus\sum_{i=1}^{N+M} (I_\tau \otimes (A_i - L_iC_i)),$$

$$P_1 = \left(-\left(I_{N-1} \otimes (K_a + A_{s_1}) - \left(\oplus\sum_{i=2}^{N} A_i\right)\right)\begin{bmatrix} \mathbf{0}_{(N-1)n\times n} & I_{(N-1)n} \end{bmatrix}\right.$$
$$+ (I_{N-1} \otimes (K_a + A_{s_1} - A_1))\begin{bmatrix} \mathbf{1}_{N-1} \otimes I_n & \mathbf{0}_{(N-1)n} \end{bmatrix}$$
$$\left. + (\widehat{BK}_2^N(\mathcal{L}_2^{\mathcal{G}_1} \otimes I_n) - (\mathbf{1}_{N-1} \otimes \mathcal{L}_1^{\mathcal{G}_1}) \otimes (B_1K_1))\right)\theta_{s_1}(\tau),$$

$$P_2 = ((\widehat{BK}_2^N(\Omega_2^{\mathcal{G}_1} \otimes I_n) - (\mathbf{1}_{N-1}\Omega_1^{\mathcal{G}_1}) \otimes (B_1K_1)) - (\widehat{BK}_2^N(\gamma_{s_1}^{\mathcal{G}_2} \otimes I_n)$$
$$- ((\mathbf{1}_{N-1}\gamma_1^{\mathcal{G}_2}) \otimes (B_1K_1)))\begin{bmatrix} I_n & \mathbf{0}_{n\times(M-1)n} \end{bmatrix})\theta_{s_2}(\tau),$$

$$P_3 = ((\widehat{BK}_{N+2}^{N+M}(\Omega_2^{\mathcal{G}_2}\otimes I_n) - (\mathbf{1}_{M-1}\Omega_1^{\mathcal{G}_2})\otimes(B_{N+1}K_{N+1}) - (\widehat{BK}_{N+2}^{N+M}(\gamma_{s_2}^{\mathcal{G}_1} \otimes I_n)$$
$$- ((\mathbf{1}_{M-1}\gamma_{N+1}^{\mathcal{G}_2}) \otimes (B_{N+1}K_{N+1})))\begin{bmatrix} I_n & \mathbf{0}_{n\times(N-1)n} \end{bmatrix}\theta_{s_1}(\tau),$$

$$P_4 = \left(-\left(I_{M-1} \otimes (K_b + A_{s_2}) - \left(\oplus\sum_{i=N+2}^{N+M} A_i\right)\right)\begin{bmatrix} \mathbf{0}_{(M-1)n\times n} & I_{(M-1)n} \end{bmatrix}\right.$$
$$+ (I_{M-1} \otimes (K_b + A_{s_2} - A_{N+1}))\begin{bmatrix} \mathbf{1}_{M-1} \otimes I_n & \mathbf{0}_{(M-1)n} \end{bmatrix}$$
$$\left. + (\widehat{BK}_{N+2}^{N+M}(\mathcal{L}_2^{\mathcal{G}_2} \otimes I_n) - (\mathbf{1}_{M-1} \otimes \mathcal{L}_1^{\mathcal{G}_2}) \otimes (B_{N+1}K_{N+1}))\right)\theta_{s_2}(\tau),$$

$$\gamma_i^{\mathcal{G}_2} = \sum_{j=N+1}^{N+M} -a_{ij}, \quad i = 1, 2, \ldots, N,$$

$$\gamma_i^{\mathcal{G}_1} = \sum_{j=1}^{N} -a_{ij}, \quad i = N+1, N+2, \ldots, N+M,$$

$$\gamma_{s_1}^{\mathcal{G}_2} = \left[\left(\gamma_2^{\mathcal{G}_2}\right)^T \quad \left(\gamma_3^{\mathcal{G}_2}\right)^T \quad \cdots \quad \left(\gamma_N^{\mathcal{G}_2}\right)^T\right]^T,$$

$$\gamma_{s_2}^{\mathcal{G}_1} = \left[\left(\gamma_{N+2}^{\mathcal{G}_1}\right)^T \quad \left(\gamma_{N+3}^{\mathcal{G}_1}\right)^T \quad \cdots \quad \left(\gamma_{N+M}^{\mathcal{G}_1}\right)^T\right]^T,$$

$$\theta_{s_1}(\tau) = \oplus\sum_{i=1}^{N} \theta_i(\tau), \quad \theta_{s_2}(\tau) = \oplus\sum_{i=N+1}^{N+M} \theta_i(\tau).$$

Let

$$\Psi(t) = \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}, \Upsilon = \begin{bmatrix} \Lambda_1 & \Pi_1 \\ \Pi_2 & \Lambda_2 \end{bmatrix}, \Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}.$$

The above formula can be simplified to

$$\begin{bmatrix} \Psi(t+1) \\ E(t+1) \end{bmatrix} = \begin{bmatrix} \Upsilon & \Gamma \\ \mathbf{0} & \Xi \end{bmatrix} \begin{bmatrix} \Psi(t) \\ E(t) \end{bmatrix}. \tag{13}$$

Clearly, the system matrix of (13) is an upper block triangular matrix. It is well-known that a block triangular matrix is Schur if and only if its sub-matrices on the diagonal line are Schur. Based on Definition 1, protocol (5) can solve the group consensus problem for discrete-time HMASs (1) if and only if system (13) is asymptotically stable, which implies that matrices Υ and $A_i - L_iC_i$ are required to have their eigenvalues with magnitude less than one. The proof is completed. □

Remark 4. Theorem 1 shows that when there are communication delays between agents, under the role of the protocol (5) based on the networked predictive scheme, the group consensus of HMASs is related to the structure of each agent and the communication topology between them, and it has nothing to do with communication delays. It can be seen that the networked predictive scheme can effectively compensate the impact of communication delays.

Remark 5. In the existing articles referred to solving group consensus problem, the weights of communication between different subgroups are often limited. For example, adjacency element a_{ij} associated with edge e_{ji} between two different subgroups satisfies that

$$\sum_{v_j\in\mathcal{N}_{2i}} a_{ij} = \alpha, \quad \forall i \in \ell_1,$$
$$\sum_{v_j\in\mathcal{N}_{1i}} a_{ij} = \beta, \quad \forall i \in \ell_2,$$

where $\alpha,\ \beta \in R$ are constant numbers in [39]. The constants $\alpha,\ \beta$ are required to be zero in [40,42,50], which is called in-degree balance. In reality, there are many limitations in in-degree balance, because it will result in no actual communication between subsystems. In other words, it will cause the interaction between agents in different subsystems to be offset [51,52]. Similarly, the weights of communication between the agents coming from different subnetworks in [39] also impose restrictions. It is that the influence from every agent in one group to all agents in the other group is always equal at any time. Obviously, all these assumptions are too special to be applied to investigate the problem of group consensus of multi-agent networks. It's worth nothing that there is no restriction on adjacency elements associated with edges between two different subgroups in Theorem 1, that is, $a_{ij}\,(i,\, j \in \ell_1 \text{ or } i,\, j \in \ell_2)$ can take any nonnegative real number and $a_{ij}\,(i \in \ell_1,\, j \in \ell_2 \text{ or } i \in \ell_2,\, j \in \ell_1)$ can take any real number. So the constraint of in-degree balance in [40,42,50] is relaxed, moreover, the conditions in [39] is also relaxed, which makes the communication topology more flexible.

If weighted adjacency matrix $\mathcal{A}(\mathcal{G})$ of discrete-time HMASs (1) satisfies in-degree balance, that is, the sum of adjacent weights from every agent in one subgroup to all agents in the other subgroup should be equal to zero, the following conclusions can be obtained.

Corollary 1. *When Assumptions 1–5 hold, if discrete-time HMASs (1) with time delays (2) satisfies that* $\sum_{v_j\in\mathcal{N}_{2i}} a_{ij} = 0\ (\forall\, i \in \ell_1)$ *and* $\sum_{v_j\in\mathcal{N}_{1i}} a_{ij} = 0\ (\forall\, i \in \ell_2)$*, the following conclusions are equivalent:*

(i) Protocol (5) can solve the group consensus problem.

(ii) For arbitrarily chosen pair of matrices $U_{12} \in \mathcal{R}^{N\times(N-1)}$ *and* $U_{22} \in \mathcal{R}^{M\times(M-1)}$*, satisfying that* $U_1 = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & U_{12} \end{bmatrix}$*,* $U_2 = \begin{bmatrix} \frac{1}{\sqrt{M}}\mathbf{1}_M & U_{22} \end{bmatrix}$

are orthogonal matrices, there exist $B_{i_0} \in \{B_1, B_2, \ldots, B_N\}$ *and* $K_{i_0} \in R^{m\times n}$ *so that* $B_iK_i = B_{i_0}K_{i_0} (i \in \mathcal{V})$, Ψ *and* $A_i - L_iC_i$ $(i \in \mathcal{V})$ *are Schur, where* $\Psi = \digamma_3 - \Phi \otimes \left(B_{i_0}K_{i_0}\right)$, $\Phi = \hat{U}^T\mathcal{L}^g\hat{U}$, $\hat{U} = diag(U_{12}, U_{22})$, $\digamma_3 = \left(I_{N-1} \otimes \left(K_a + A_{s_1}\right)\right) \oplus \left(I_{M-1} \otimes \left(K_b + A_{s_2}\right)\right)$.

Proof. Let

$$\hat{x}(t) = \begin{bmatrix}\hat{x}_1^T(t) & \hat{x}_2^T(t) & \cdots & \hat{x}_{N+M}^T(t)\end{bmatrix}^T,$$
$$u(t) = \begin{bmatrix}u_1^T(t) & u_2^T(t) & \cdots & u_{N+M}^T(t)\end{bmatrix}^T. \tag{14}$$

Combining (11) and (14) results in

$$\hat{x}(t) = x(t) - \theta_s(\tau)E(t). \tag{15}$$

Combining (14) and (15) results in

$$\begin{aligned} u(t) = \Bigg(&\oplus\sum_{i=1}^{N+M} B_{iR}^{-1}((K_{s_1} - \oplus\sum_{i=1}^{N} A_i) \oplus (K_{s_2} - \oplus\sum_{i=N+1}^{N+M} A_i)) \\ &= -(\oplus\sum_{i=1}^{N+M} K_i)(\mathcal{L}^g \otimes I_n)\Bigg)(x(t) - \theta_s(\tau)E(t)), \end{aligned}$$

where

$$K_{s_1} = (I_N \otimes (K_a + A_{s_1})), \quad K_{s_2} = (I_M \otimes (K_b + A_{s_2})),$$
$$\widehat{BK}_1^{N+M} = \oplus\sum_{i=1}^{N+M} B_iK_i, \quad \theta_s(\tau) = \oplus\sum_{i=1}^{N+M}\theta_i(\tau).$$

The closed-loop system is as follows:

$$\begin{aligned} x(t+1) &= \left((K_{s_1} \oplus K_{s_2} - \widehat{BK}_1^{N+M}(\mathcal{L}^g \otimes I_n)\right)(x(t) - \theta_s(\tau)E(t)) \\ &= \digamma_1 x(t) - \digamma_2 E(t), \end{aligned} \tag{16}$$

where

$$\digamma_1 = K_{s_1} \oplus K_{s_2} - \widehat{BK}_1^{N+M}(\mathcal{L}^g \otimes I_n),$$
$$\digamma_2 = K_{s_1} \oplus K_{s_2} - \widehat{BK}_1^{N+M}(\mathcal{L}^g \otimes I_n)\theta_s(\tau).$$

Let

$$R = diag(R_1, R_2), R_1 = \begin{bmatrix}-\mathbf{1}_{N-1} & I_{N-1}\end{bmatrix}, R_2 = \begin{bmatrix}-\mathbf{1}_{M-1} & I_{M-1}\end{bmatrix}.$$

The error system is as follows:

$$\delta(t+1) = \Upsilon\delta(t) - (R \otimes I_n)\digamma_2 E(t), \tag{17}$$

where

$$\Upsilon = \digamma_3 - \digamma_4((\mathcal{L}^g R^T(RR^T)^{-1}) \otimes I_n),$$
$$\digamma_4 = \begin{bmatrix}-\mathbf{1}_{N-1} \otimes (B_1K_1) & \widehat{BK}_2^N\end{bmatrix} \oplus \begin{bmatrix}-\mathbf{1}_{M-1} \otimes (B_{N+1}K_{N+1}) & \widehat{BK}_{N+2}^{N+M}\end{bmatrix}.$$

According to Corollary 4.2 in [53], it is obtained easily that

$$x(t+1) = \left((K_{s_1} \oplus K_{s_1}) - \mathcal{L}^g \otimes \left(B_{i_0}K_{i_0}\right)\right)x(t) - \digamma_2 E(t), \tag{18}$$

$$\delta(t+1) = \Upsilon\delta(t) - (R \otimes I_n)\digamma_2 E(t), \tag{19}$$

where $\Upsilon = \digamma_3 - ((R\mathcal{L}^g R^T\left(RR^T\right)^{-1}) \otimes \left(B_{i_0}K_{i_0}\right))$.

It is easy to prove that

$$R\mathcal{L}^g R^T(RR^T)^{-1} = T\Phi T^{-1}, \tag{20}$$

where $T = \mathrm{diag}(R_1U_{12}, R_2U_{22})$.

Note that

$$\begin{aligned} &(T \otimes I_n)^{-1}\Upsilon(T \otimes I_n) \\ &= (T \otimes I_n)^{-1}\left(\digamma_3 - ((R\mathcal{L}^g R^T(RR^T)^{-1}) \otimes \left(B_{i_0}K_{i_0}\right))\right)(T \otimes I_n) \\ &= \digamma_3 - \left((T^{-1}R\mathcal{L}^g R^T(RR^T)^{-1}T) \otimes \left(B_{i_0}K_{i_0}\right)\right) \\ &= \digamma_3 - \Phi \otimes \left(B_{i_0}K_{i_0}\right) \\ &= \Psi. \end{aligned}$$

It means that Υ is similar to Ψ. Since similar matrices have the same eigenvalues, Υ is Schur if and only if Ψ is Schur. Based on Theorem 1, it is obvious that the discrete-time HMASs (1) subjected to control protocol (5) can achieve group consensus if and only if Ψ and $A_i - L_iC_i$ $(i \in \ell)$ are Schur. This completes the proof. □

When the communication weights between agents are unconstrained or constrained, the necessary and sufficient conditions of protocol (5) solving the group consensus problem are given in Theorem 1 and Corollary 1, respectively. However, the solution of gain matrices in protocol (5) is not given. For convenience of solution, a solution method of gain matrices is given in the following conclusion.

Corollary 2. *Consider the discrete-time HMASs (1) with time delays (2). If Assumptions 1–5 and the following conditions hold, protocol (5) can solve the group consensus problem.*

(i) The matrices $A_i - L_iC_i$ $(i \in \mathcal{V})$ *are Schur.*

(ii) There exist matrices $X = X^T > 0, Y, Z \in R^{w\times n}$ *such that*

$$YX^{-1} = I_{N+M-2} \otimes Z, \tag{21}$$

$$\begin{bmatrix} X & -\left(\hat{A}_sX + TY\right)^T \\ -\left(\hat{A}_sX + TY\right) & X \end{bmatrix} > 0. \tag{22}$$

Besides, the feedback gain matrices in protocol (5) can be taken as $\overline{K} = Z$, *where*

$$\overline{K} = \begin{bmatrix}K_a^T & K_1^T & K_2^T & \ldots & K_N^T & K_b^T & K_{N+1}^T & K_{N+2}^T & \cdots & K_{N+M}^T\end{bmatrix}^T,$$
$$\hat{A}_s = (I_{N-1} \otimes A_{s_1}) \oplus (I_{M-1} \otimes A_{s_2}),$$
$$T = \begin{bmatrix}\left[T_{ij}\right] & \left[T_{iq}\right] \\ \left[T_{pj}\right] & \left[T_{pq}\right]\end{bmatrix} \in R^{(N+M-2)n\times(N+M-2)w},$$

$$T_{ij} = \begin{cases} [0_{n\times n}\ l_{1,j+1}B_1\ \overbrace{0_{n\times r}}^{i-1}\ -l_{i+1,j+1}B_{i+1}\ \overbrace{0_{n\times r}}^{N-i-1}\ 0_{n\times n}\ \overbrace{0_{n\times r}}^{M}\ ]', & i \neq j \\ [I_n\ l_{1,i+1}B_1\ \overbrace{0_{n\times r}}^{i-1}\ -l_{i+1,i+1}B_{i+1}\ \overbrace{0_{n\times r}}^{N-i-1}\ 0_{n\times n}\ \overbrace{0_{n\times r}}^{M}\ ]', & i = j \end{cases}$$

$$T_{iq} = [0_{n\times n}\ -a_{1,q+1}B_1\ \overbrace{0_{n\times r}}^{i-1}\ a_{i+1,q+1}B_{i+1}\ \overbrace{0_{n\times r}}^{N-i-1}\ 0_{n\times n}\ \overbrace{0_{n\times r}}^{M}\ ]'$$

$$T_{pj} = [0_{n\times n}\ \overbrace{0_{n\times r}}^{N}\ 0_{n\times n}\ -a_{N+1,j+1}B_{N+1}\ \overbrace{0_{n\times r}}^{p-N-1}\ a_{p+1,j+1}B_{p+1}\ \overbrace{0_{n\times r}}^{N+M-p-1}\ ]'$$

$$T_{pq} = \begin{cases} [0_{n\times n}\ \overbrace{0_{n\times r}}^{N}\ 0_{n\times n}\ l_{N+1,q+1}B_{N+1}\ \overbrace{0_{n\times r}}^{p-N-1}\ -l_{p+1,q+1}B_{p+1}\ \overbrace{0_{n\times r}}^{N+M-p-1}\ ]', & p \neq q \\ [0_{n\times n}\ \overbrace{0_{n\times r}}^{N}\ I_n\ l_{N+1,p+1}B_{N+1}\ \overbrace{0_{n\times r}}^{p-N-1}\ -l_{p+1,p+1}B_{p+1}\ \overbrace{0_{n\times r}}^{N+M-p-1}\ ]', & p = q \end{cases}$$

$i, j = 1, 2, \ldots, N-1; \quad p, q = N+1, N+2, \ldots, N+M-1; \quad w = 2n + (N+M)r.$

Proof. From Theorem 1,

$$\Upsilon = \hat{A}_s + T\hat{K}, \tag{23}$$

where $\hat{K} = I_{N+M-2} \otimes \overline{K}$.

Combining (22) and Schur complement lemma results in

$$\left(\hat{A}_sX + TY\right)^T X^{-1}\left(\hat{A}_sX + TY\right) - X < 0,$$

and

$$\left(\hat{A}_s + TYX^{-1}\right)^T X^{-1}\left(\hat{A}_s + TYX^{-1}\right) - X^{-1} < 0. \tag{24}$$

Set $\hat{K} = YX^{-1}$, $P = X^{-1}$. Combining (21) and (24) results in

$$\begin{aligned} \left(\hat{A}_s + T\hat{K}\right)^T &P\left(\hat{A}_s + T\hat{K}\right) - P < 0, \\ &\overline{K} = Z. \end{aligned} \tag{25}$$

From Lyapunov stability theory, $\hat{A}_s + T\hat{K}$ is Schur. As can be seen from (23), Υ is Schur, and then protocol (5) can solve the group consensus problem for the discrete-time HMASs (1) with time delays (2). □

Remark 6. When the discrete-time HMASs (1) has not time delays and the state of each agent is unavailable (i.e., Assumption 1), a distributed control protocol can be designed as following:

$$u_i(t) = \begin{cases} B_{iR}^{-1}(K_a + \sum_{j=1,j\neq i}^{N} A_j)\hat{x}_i(t|t-1) \\ + K_i\big(\sum_{v_j\in\mathcal{N}_{1i}} a_{ij}\Delta\hat{x}_{i,j}(t|t-1) \\ + \sum_{v_j\in\mathcal{N}_{2i}} a_{ij}\Delta\hat{x}_{N+1,j}(t|t-1)\big), \forall\, i\in\ell_1, \\ B_{iR}^{-1}(K_b + \sum_{j=N+1,j\neq i}^{N+M} A_j)\hat{x}_i(t|t-1) \\ + K_i\big(\sum_{v_j\in\mathcal{N}_{2i}} a_{ij}\Delta\hat{x}_{i,j}(t|t-1) \\ + \sum_{v_j\in\mathcal{N}_{1i}} a_{ij}\Delta\hat{x}_{1,j}(t|t-1)\big), \forall\, i\in\ell_2. \end{cases} \tag{26}$$

That is, protocol (26) is as same as the case of $\tau = 1$ in protocol (5). Because Theorem 1 and Corollaries 1–2 are independent of time delays, the results are still applicable to the discrete-time HMASs (1) without time delays under Assumptions 1–2 and Assumptions 4–5.

Similar to Theorem 1, the group consensus problem with two groups can be extended to a more general one, i.e., the agents in a network reach multiple consistent states asymptotically.

Considering a complex network $(\mathcal{G}, x)$ composed of agents with discrete-time dynamics (1) is divided into P sub-networks $(\mathcal{G}_1, \chi_1), (\mathcal{G}_2, \chi_2), \ldots, (\mathcal{G}_P, \chi_P)$, and there are $\mathcal{N}_i$ agents in the ith sub-group, $i = 1, 2, \ldots, P$. Denote $\vartheta_0 = 0, \vartheta_i = \mathcal{N}_1 + \cdots + \mathcal{N}_2 + \cdots + \mathcal{N}_i, i = 1, 2, \ldots, P$. Obviously, index sets $\ell_1 = \{1, 2, \ldots, \vartheta_1\}$, $\ell_2 = \{\vartheta_1 + 1, \vartheta_1 + 2, \ldots, \vartheta_2\}, \ldots, \ell_P = \{\vartheta_{P-1} + 1, \vartheta_{P-1} + 2, \ldots, \vartheta_P\}$, and $\ell = \bigcup_{i=1}^{P} \ell_i$. Similarly, the following relevant matrices are given

$$\mathcal{L}^{\mathcal{G}} = \begin{bmatrix} \mathcal{L}^{\mathcal{G}_1} & \Omega^{\mathcal{G}_{12}} & \cdots & \Omega^{\mathcal{G}_{1P}} \\ \Omega^{\mathcal{G}_{21}} & \mathcal{L}^{\mathcal{G}_2} & \cdots & \Omega^{\mathcal{G}_{2P}} \\ & & \ddots & \\ \Omega^{\mathcal{G}_{P1}} & \Omega^{\mathcal{G}_{P2}} & \cdots & \mathcal{L}^{\mathcal{G}_P} \end{bmatrix},$$

$$\mathcal{L}^{\mathcal{G}_i} = \begin{bmatrix} \mathcal{L}_{11}^{\mathcal{G}_i} & \mathcal{L}_{12}^{\mathcal{G}_i} \\ \mathcal{L}_{21}^{\mathcal{G}_i} & \mathcal{L}_{22}^{\mathcal{G}_i} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_1^{\mathcal{G}_i} \\ \mathcal{L}_2^{\mathcal{G}_i} \end{bmatrix}, \quad i = 1, 2, \ldots, P,$$

$$\Omega^{\mathcal{G}_{ij}} = \begin{bmatrix} \Omega_{11}^{\mathcal{G}_{ij}} & \Omega_{12}^{\mathcal{G}_{ij}} \\ \Omega_{21}^{\mathcal{G}_{ij}} & \Omega_{22}^{\mathcal{G}_{ij}} \end{bmatrix} = \begin{bmatrix} \Omega_1^{\mathcal{G}_{ij}} \\ \Omega_2^{\mathcal{G}_{ij}} \end{bmatrix}, \quad i = 1, 2, \ldots, P, \quad j = 1, 2, \ldots, P.$$

For the complex network $(\mathcal{G}, x)$, the following protocol can be designed

$$u_i(t) = \begin{cases} B_{iR}^{-1}(K_a + \sum_{j=1,j\neq i}^{\vartheta_1} A_j)\hat{x}_i(t|t-\tau) \\ +K_i\begin{pmatrix} \sum_{v_j\in\mathcal{N}_{1i}} a_{ij}\Delta\hat{x}_{i,j}(t|t-\tau) \\ +\sum_{v_j\in\mathcal{N}_{2i}} a_{ij}\Delta\hat{x}_{\vartheta_1+1,j}(t|t-\tau) \\ +\cdots+\sum_{v_j\in\mathcal{N}_{Pi}} a_{ij}\Delta\hat{x}_{\vartheta_{P-1}+1,j}(t|t-\tau) \end{pmatrix}, & \forall\, i\in\ell_1, \\ B_{iR}^{-1}(K_b + \sum_{j=\vartheta_1+1,j\neq i}^{\vartheta_2} A_j)\hat{x}_i(t|t-\tau) \\ +K_i\begin{pmatrix} \sum_{v_j\in\mathcal{N}_{1i}} a_{ij}\Delta\hat{x}_{1,j}(t|t-\tau) \\ +\sum_{v_j\in\mathcal{N}_{2i}} a_{ij}\Delta\hat{x}_{i,j}(t|t-\tau) \\ +\cdots+\sum_{v_j\in\mathcal{N}_{Pi}} a_{ij}\Delta\hat{x}_{\vartheta_{P-1}+1,j}(t|t-\tau) \end{pmatrix}, & \forall\, i\in\ell_2, \\ \vdots \\ B_{iR}^{-1}\Big(K_P + \sum_{j=\vartheta_{P-1}+1,j\neq i}^{\vartheta_P} A_j\Big)\hat{x}_i(t|t-\tau) \\ +K_i\begin{pmatrix} \sum_{v_j\in\mathcal{N}_{1i}} a_{ij}\Delta\hat{x}_{1,j}(t|t-\tau) \\ +\sum_{v_j\in\mathcal{N}_{2i}} a_{ij}\Delta\hat{x}_{\vartheta_1+1,j}(t|t-\tau) \\ +\cdots+\sum_{v_j\in\mathcal{N}_{Pi}} a_{ij}\Delta\hat{x}_{i,j}(t|t-\tau) \end{pmatrix}, & \forall\, i\in\ell_P. \end{cases} \tag{27}$$

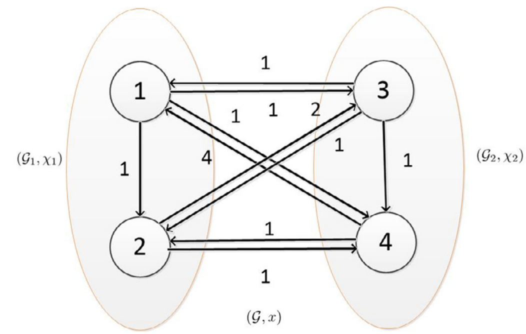

Fig. 1. Directed fixed topology $\mathcal{G}$ in Example 1.

Definition 2. For discrete-time HMASs (1), it is called that protocol (27) can solve the multi-group consensus problem or discrete-time HMASs (1) can achieve multi-group consensus under protocol (27), if the following conditions hold:

$(i)\ \lim_{t\to+\infty} \|x_i(t) - x_j(t)\| = 0, \quad \forall\, i, j \in \ell_\sigma,\ \sigma = 1, 2, \ldots, P;$

$(ii)\ \lim_{t\to+\infty} \|x_i(t) - \hat{x}_i(t|t-1)\| = 0, \quad \forall\, i \in \ell.$

where $\hat{x}_i(t|t-1)$ means state prediction of the ith agent at time t by the information at time $t-1$.

Corollary 3. *Under the premise of Assumptions 1–5, protocol (27) can solve the multi-group consensus problem, if and only if matrices Υ and $A_i - L_iC_i (i \in \ell)$ are Schur, where $\Upsilon = [\Upsilon_{ij}] \in R^{(\vartheta_P - P)n\times(\vartheta_P - P)n}$, $i, j = 1, 2, \ldots, P$, where*

$$\Upsilon_{ij} = \begin{cases} I_{(\mathcal{N}_i-1)}\otimes(K_i + A_{s_i}) - \widehat{BK}_{\vartheta_{i-1}+2}^{\vartheta_i}(\mathcal{L}_{22}^{\mathcal{G}_i}\otimes I_n) + (\mathbf{1}_{\mathcal{N}_i-1}\mathcal{L}_{12}^{\mathcal{G}_i})\otimes\tilde{B}_i, & i = j, \\ -\widehat{BK}_{\vartheta_{i-1}+2}^{\vartheta_i}(\Omega_{22}^{\mathcal{G}_{ij}}\otimes I_n) + (\mathbf{1}_{\mathcal{N}_i-1}\Omega_{12}^{\mathcal{G}_{ij}})\otimes\tilde{B}_i, & i \neq j. \end{cases}$$

Here,

$$\widehat{BK}_{\vartheta_{i-1}+2}^{\vartheta_i} = \oplus\sum_{i=\vartheta_{i-1}+2}^{\vartheta_i} B_iK_i, \quad i = 1, 2, \ldots, P,$$

$$A_{s_i} = \sum_{j=\vartheta_{i-1}+1}^{\vartheta_i} A_j, \quad i = 1, 2, \ldots, P,$$

$$\tilde{B}_i = B_{\vartheta_{i-1}+1}K_{\vartheta_{i-1}+1}, \quad i = 1, 2, \ldots, P.$$

4. Numerical simulations

In this section, some numerical examples are presented to describe the effectiveness of the above theoretical results.

Example 1. Given a network $(\mathcal{G}, x)$ composed of $N + M$ agents, where $N = 2, M = 2, \ell_1 = \{1, 2\}, \ell_2 = \{3, 4\}$. The dynamics of agents are described as (1), where

$$A_1 = \begin{bmatrix} 1.25 & 0 \\ 0.1 & 0.6 \end{bmatrix}, B_1 = \begin{bmatrix} -0.01 & 0.03 & -0.02 \\ 0 & -0.05 & 0.03 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & 0 \\ 0.3 & -0.1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.01 & 0 & 0.05 \\ 0 & 0.25 & -0.02 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1.3 & 0.2 \\ 0.2 & 0.6 \end{bmatrix}, B_3 = \begin{bmatrix} -0.02 & 0 & -0.01 \\ -0.01 & 0.05 & -0.01 \end{bmatrix}, C_3 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 0.1 \\ 0 & -1.2 \end{bmatrix}, B_4 = \begin{bmatrix} -0.2 & 0.04 & -0.1 \\ -0.10 & 0.05 & 0.01 \end{bmatrix}, C_4 = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

The topological graph of HMASs is shown in Fig. 1.

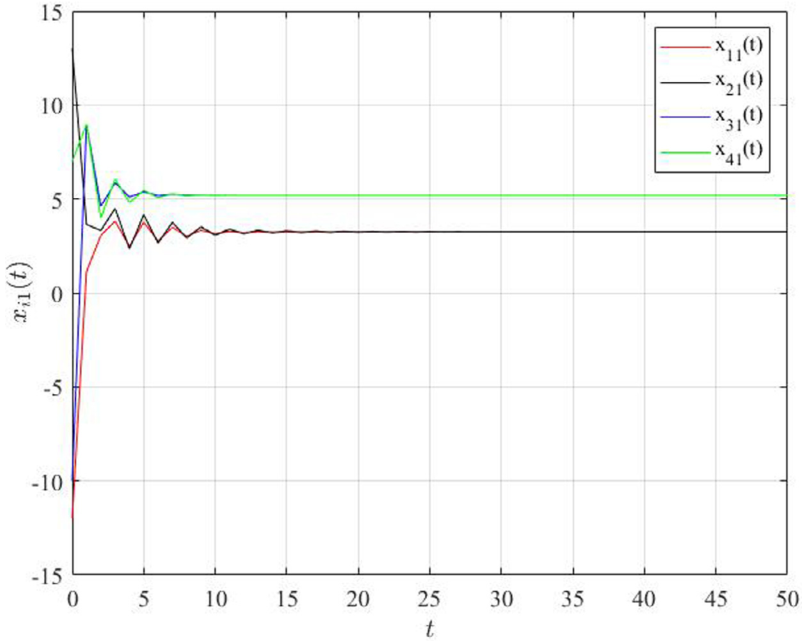

Fig. 2. State trajectory $x_{i1}(t)$ in Example 1, $i = 1, 2, 3, 4$ $(\tau = 3)$.

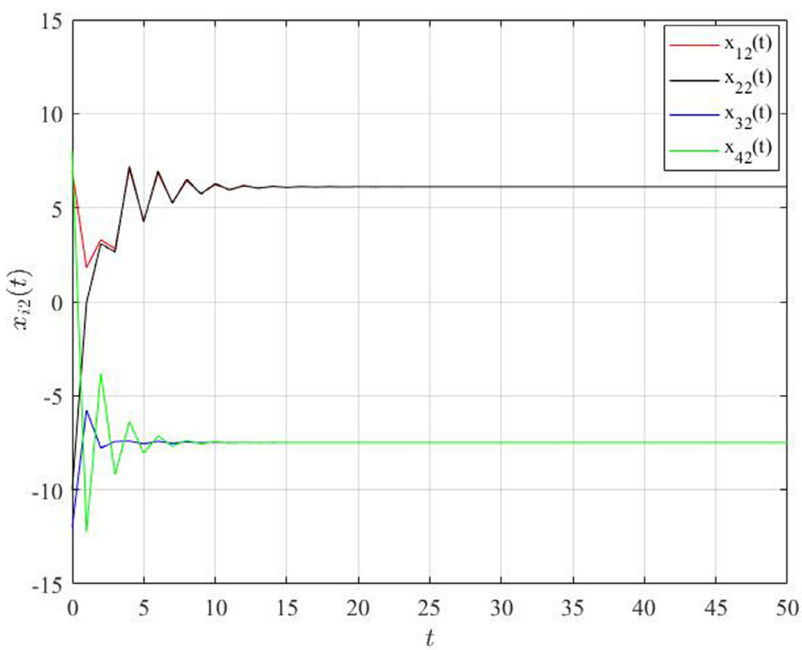

Fig. 3. State trajectory $x_{i2}(t)$ in Example 1, $i = 1, 2, 3, 4$ $(\tau = 3)$.

Assumed that there exist time-varying delays $\tau_{ij}(t)$ when agents transmit data over network, and the upper bound $\tau = 3$. Using the pole-assignment technique, the observer gain matrix L_i is determined as

$$L_1 = \begin{bmatrix} -4.8630 \\ -0.8717 \end{bmatrix}, L_2 = \begin{bmatrix} 0.5277 \\ -0.2305 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} -0.8931 \\ 0.1689 \end{bmatrix}, L_4 = \begin{bmatrix} 0.3782 \\ -0.6684 \end{bmatrix}.$$

According to Corollary 2, control gain matrix K_i is chosen as

$$K_a = \begin{bmatrix} -0.7850 & -0.0014 \\ 1.4986 & -0.1834 \end{bmatrix}, K_b = \begin{bmatrix} 0.2601 & -0.0867 \\ 0.0758 & -0.0839 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} -10.7989 & 0.9057 \\ -7.1538 & 0.5184 \\ 2.1325 & -0.1299 \end{bmatrix}, K_2 = \begin{bmatrix} -3.1463 & 0.2264 \\ 5.8416 & -0.4225 \\ -11.3129 & 0.6822 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -1.0245 & -1.1224 \\ 0.1234 & -0.3946 \\ -0.2075 & 0.0775 \end{bmatrix}, K_4 = \begin{bmatrix} -0.2603 & -0.0018 \\ 0.1898 & -0.2675 \\ 0.1453 & -0.5368 \end{bmatrix}.$$

The eigenvalues of Υ, i.e. $\lambda_1 = 0.0849, \lambda_2 = 0.4130, \lambda_3 = -0.5485, \lambda_4 = -0.0208$, are obtained by computations. Obviously, all eigenvalues of Υ lie in the unit circle. Therefore, according to Theorem 1, protocol (5) can solve the group consensus problem. Given the initial states as follows:

$$x_1(0) = \begin{bmatrix} -12 & 7 \end{bmatrix}^T, x_2(0) = \begin{bmatrix} 13 & -10 \end{bmatrix}^T,$$

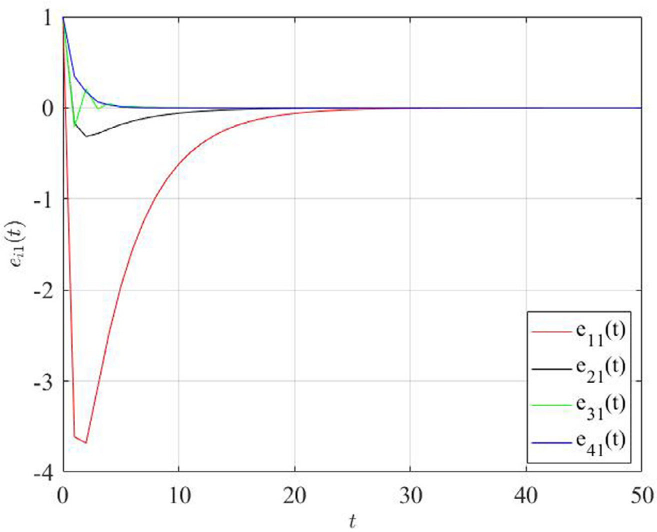

Fig. 4. Error trajectory $e_{i1}(t)$ in Example 1, $i = 1, 2, 3, 4$ ($\tau = 3$).

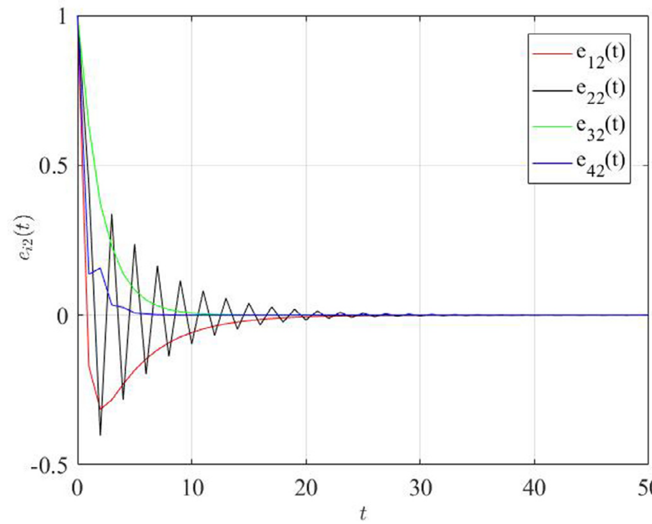

Fig. 5. Error trajectory $e_{i2}(t)$ in Example 1, $i = 1, 2, 3, 4$ ($\tau = 3$).

$$\begin{aligned}
x_3(0) &= \begin{bmatrix}-10 & -12\end{bmatrix}^T, x_4(0) = \begin{bmatrix}7 & 8\end{bmatrix}^T. \\
e_1(0) &= \begin{bmatrix}0.1 & -0.1\end{bmatrix}^T, e_2(0) = \begin{bmatrix}0.1 & 0.1\end{bmatrix}^T, \\
e_3(0) &= \begin{bmatrix}0.3 & 0\end{bmatrix}^T, e_4(0) = \begin{bmatrix}0.2 & -0.2\end{bmatrix}^T, \\
e_1(-1) &= e_2(-1) = e_3(-1) = e_4(-1) = -\begin{bmatrix}1 & 1\end{bmatrix}^T, \\
e_1(-2) &= e_2(-2) = e_3(-2) = e_4(-2) = \begin{bmatrix}1 & 1\end{bmatrix}^T.
\end{aligned} \tag{28}$$

Fig. 2 and Fig. 3 show that system can achieve group consensus under the proposed group consensus protocol (5) and Fig. 4 and Fig. 5 show the error trajectory of each agent.

Example 2. Consider the discrete-time HMASs in Example 1, the topological graph is still shown in Fig. 1. Assume that there exist no time delays (i.e. $\tau = 0$), when agents communicate information each other. According to Theorem 1 and Remark 6, conditions that the states of discrete-time HMASs (1) achieve group consensus are identical, whether there exist time delays or not. Therefore, the gain matrices K_i and L_i in Example 1 is still applicable to the discrete-time HMASs (1) without time delays. The initial conditions of HMASs and observer are chosen as (28), Fig. 6 and 7 show that system can achieve group consensus under protocol (26). Fig. 8 and 9 show the error trajectory of each agent.

The performance of discrete-time HMASs with time-varying delays and without time delays is compared in Example 1 and 2. It could be displayed more intuitively by plotting the states of HMASs without time delays and with time-varying delays in the same figure, as shown in Fig. 10 and 11. Time-delay case is indicated by solid lines and no time-delay case is indicated by dashed lines. Fig. 10 and Fig. 11 demonstrate that, based on the networked predictive scheme, the performance of discrete-time HMASs with time-varying delays is similar to that without delays. Simulation results further show that the networked predictive scheme can

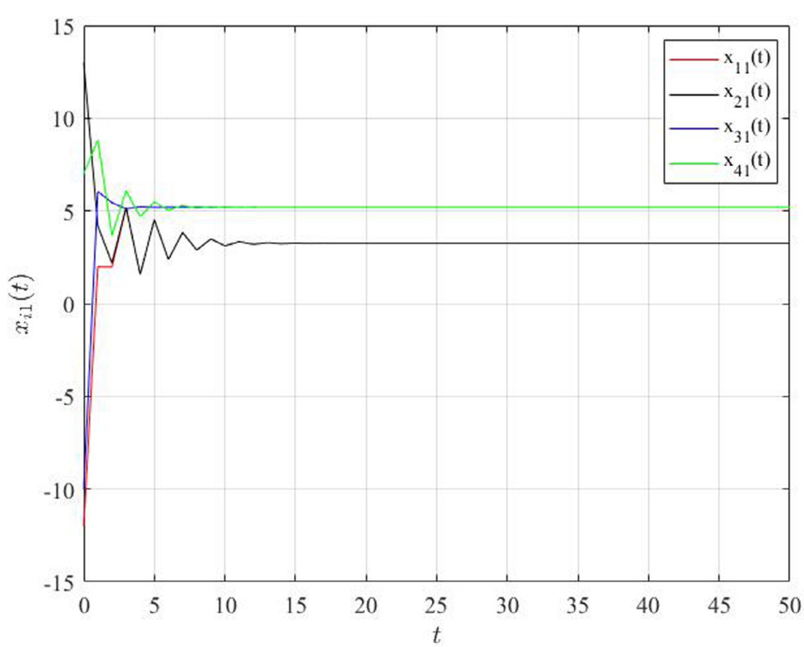

Fig. 6. State trajectory $x_{i1}(t)$ in Example 2, $i = 1, 2, 3, 4$ ($\tau = 0$).

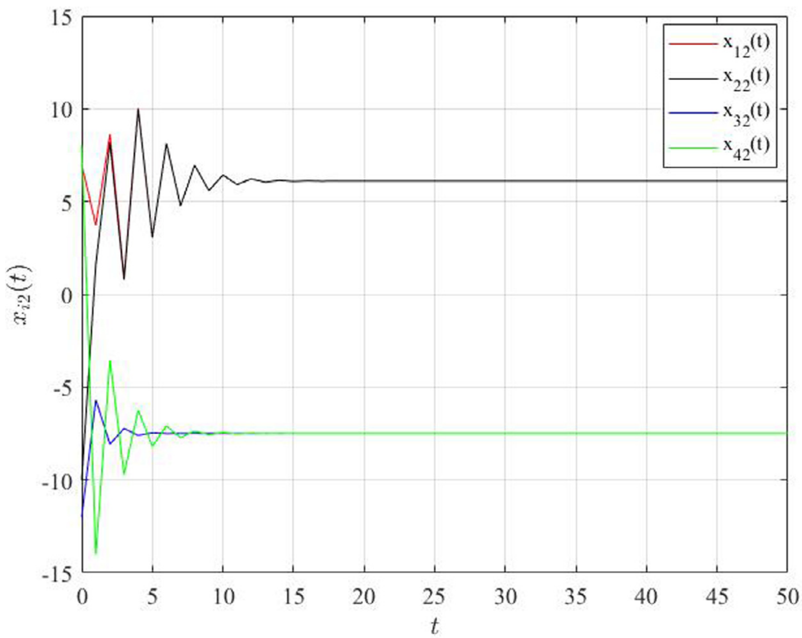

Fig. 7. State trajectory $x_{i_2}(t)$ in Example 2, $i = 1, 2, 3, 4$ ($\tau = 0$).

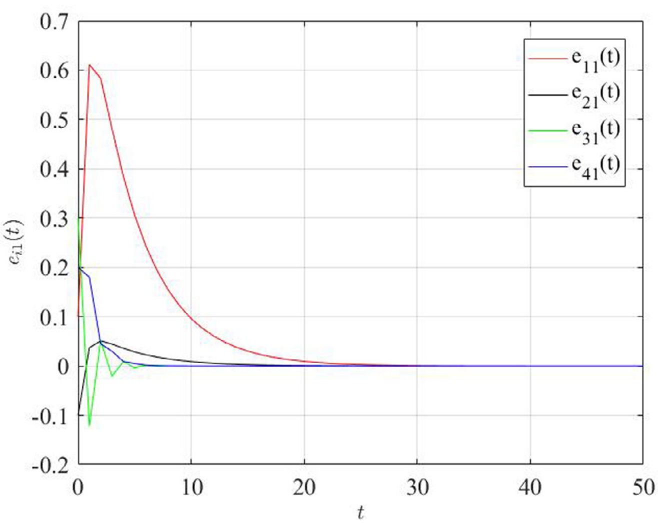

Fig. 8. Error trajectory $e_{i_1}(t)$ in Example 2, $i = 1, 2, 3, 4$ ($\tau = 0$).

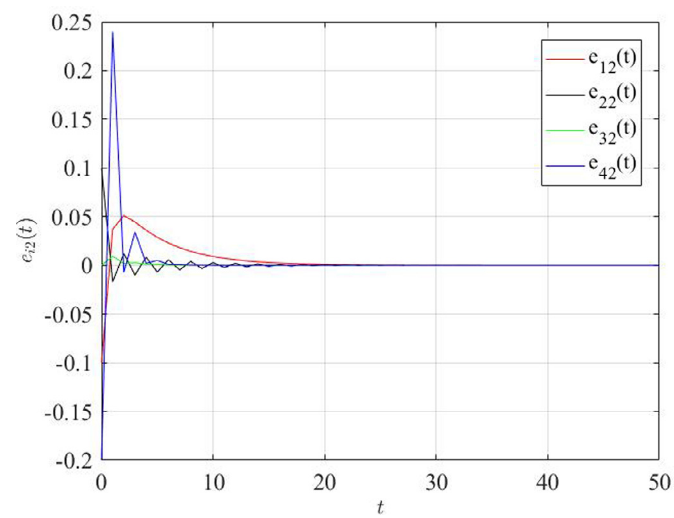

Fig. 9. Error trajectory $e_{i_2}(t)$ in Example 2, $i = 1, 2, 3, 4$ ($\tau = 0$).

compensate the network delays actively and improve system performance.

Example 3. Given a network $(\mathcal{G}, x)$ composed of $N + M$ agents, where $N = 2, M = 2, \ell_1 = \{1, 2\}, \ell_2 = \{3, 4\}$. The dynamics of agents are described as (1), where

$$A_1 = \begin{bmatrix} 0 & 0.65 & 0.22 \\ -0.65 & 0.65 & 0.23 \\ 0 & 0.03 & 1.02 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0.03 & -0.02 \\ 0 & -0.05 & 0.03 \\ 0.01 & 0 & 0.05 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.3 & 0 & 1 \\ -0.6 & 1 & 0.7 \\ 0.8 & 0.3 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.02 & 0.01 & 0.03 \\ -0.04 & 0.01 & 0 \\ -0.01 & 0.05 & -0.01 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0.5 & 3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & -0.5 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} -0.02 & 0 & -0.01 \\ 0.02 & 0.01 & 0.01 \\ 0 & 0 & 0.03 \end{bmatrix}, C_3 = \begin{bmatrix} -0.2 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, B_4 = \begin{bmatrix} 0 & 0 & -0.01 \\ 0 & 0.05 & 0.01 \\ 0.02 & 0.01 & 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 0.25 & -2 \end{bmatrix}.$$

The topological graph of HMASs is shown in Fig. 12.

The upper bound of time-varying delays $\tau_{ij}(t)$ is $\tau = 2$. And the observer gain matrix L_i is determined as

$$L_1 = \begin{bmatrix} -0.0876 & 0.3856 \\ -0.2815 & 0.4973 \\ 0.4231 & 0.3328 \end{bmatrix}, L_2 = \begin{bmatrix} -0.1745 & 0.1766 \\ 0.7179 & 0.8663 \\ 0.2229 & 0.4705 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 0.4109 & 0.4882 \\ -0.2908 & 0.1339 \\ -0.0151 & -0.0183 \end{bmatrix}, L_4 = \begin{bmatrix} -0.7048 & 0.2998 \\ 0.3124 & -0.3139 \\ 0.5379 & 0.6241 \end{bmatrix}.$$

The control gain matrix K_i is chosen as

$$K_a = \begin{bmatrix} -0.2593 & -0.7046 & -1.551 \\ 1.0180 & -1.4923 & -0.5129 \\ -0.6486 & -0.5188 & -1.3094 \end{bmatrix},$$

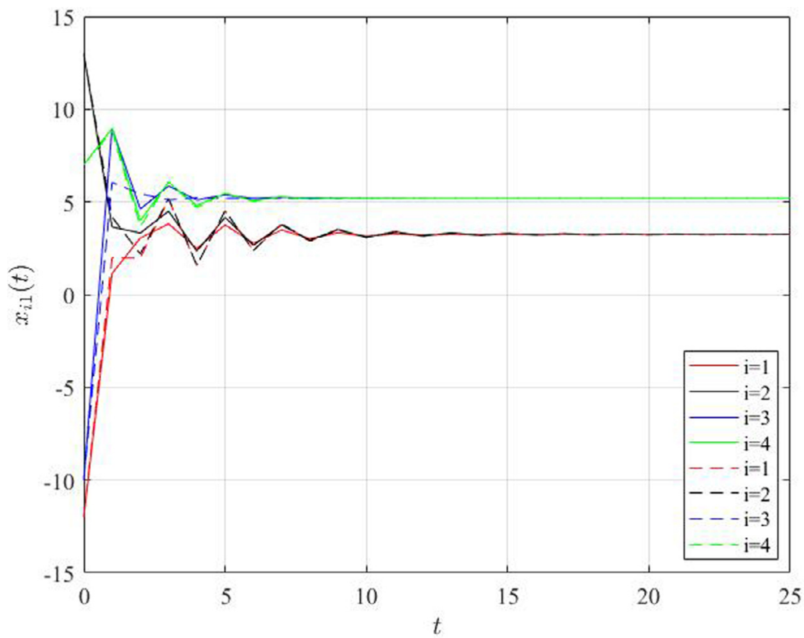

Fig. 10. State trajectory $x_{i1}(t)$ in Example 2, $i = 1, 2, 3, 4$.

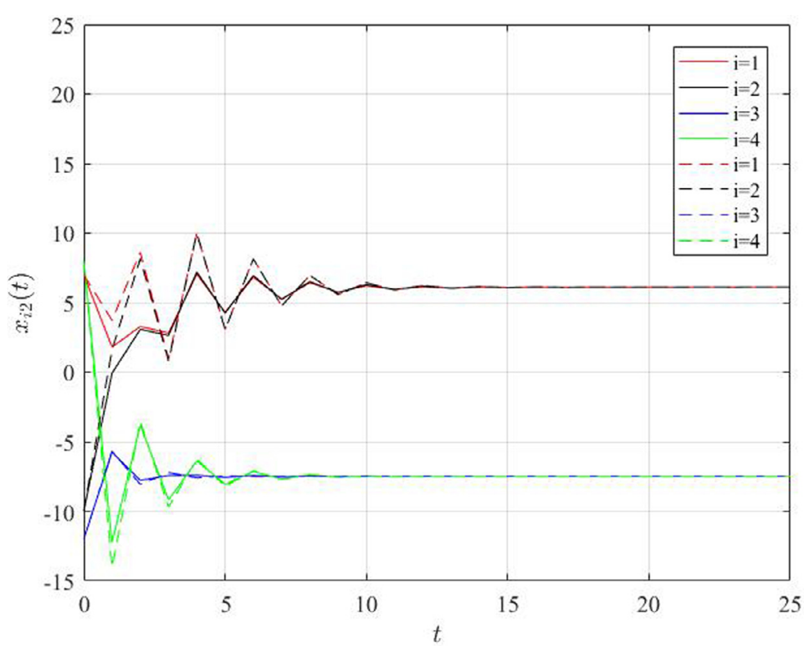

Fig. 11. State trajectory $x_{i2}(t)$ in Example 2, $i = 1, 2, 3, 4$.

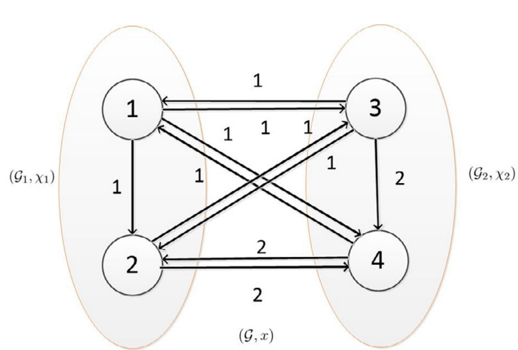

Fig. 12. Directed fixed topology $\mathcal{G}$ in Example 3.

$$K_b = \begin{bmatrix} 2.1934 & 0.0030 & 1.0294 \\ -0.9591 & 0.7282 & -1.311 \\ -1.1867 & 0.3786 & 0.6936 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 1.9445 & -2.0396 & 3.0311 \\ 9.0972 & -2.6799 & 0.6219 \\ 4.0208 & -8.5465 & 14.975 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 4.9220 & 0.5601 & -3.6779 \\ 2.9762 & -3.8795 & 6.1963 \\ -1.0651 & 2.4286 & -4.2942 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -1.5766 & -4.0137 & -3.0309 \\ 0.4657 & -1.5735 & -0.4906 \\ -8.8321 & -1.7657 & -4.6418 \end{bmatrix},$$

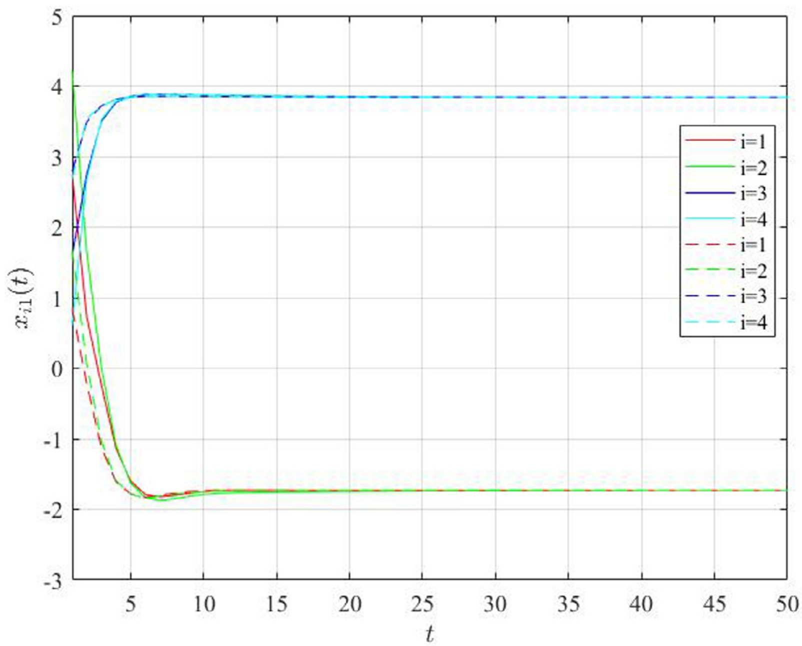

Fig. 13. State trajectory $x_{i_1}(t)$ in Example 3, $i = 1, 2, 3, 4$.

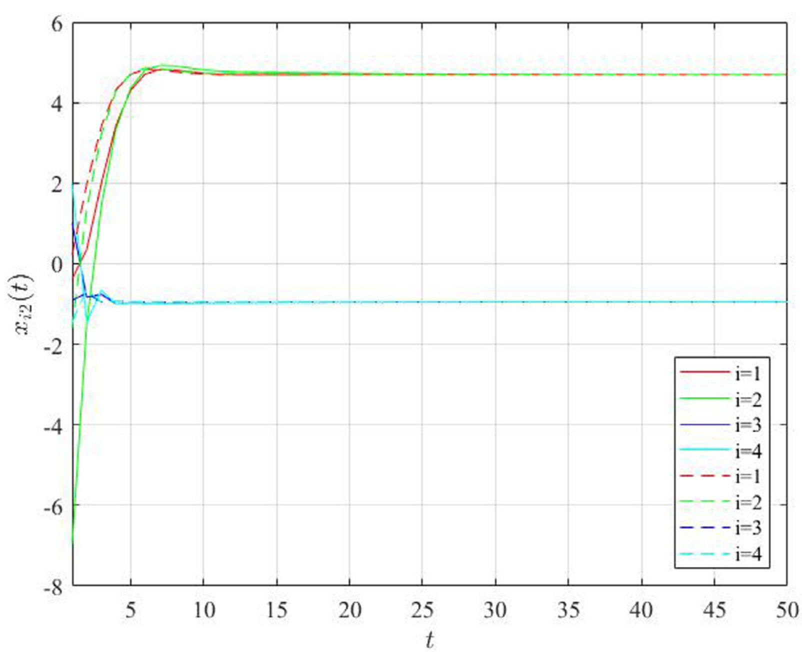

Fig. 14. State trajectory $x_{i_2}(t)$ in Example 3, $i = 1, 2, 3, 4$.

$$K_4 = \begin{bmatrix} -6.6474 & -1.2456 & -3.4568 \\ 0.04657 & -0.1574 & -0.0491 \\ -5.9926 & -4.8966 & -5.3517 \end{bmatrix}.$$

The eigenvalues of Υ, i.e. $\lambda_1 = 0.3525, \lambda_2 = 0.1944, \lambda_3 = -0.06246, \lambda_4 = -0.1064 + 0.1562i, \lambda_5 = -0.1064 - 0.1562i, \lambda_6 = -0.0389$, are obtained by computations. Obviously, all eigenvalues of Υ lie in the unit circle. Therefore, according to Theorem 1, protocol (5) can solve the group consensus problem.

The performance of discrete-time HMASs with upper bound of time-varying delays $\tau = 2$ and without time delays is compared in Figs. 13–15. Time-delay case is indicated by solid lines and no time-delay case is indicated by dashed lines. Figs. 16–18 show the error trajectory when $\tau = 2$.

Feasibility and effectiveness of Theorem 1 are illustrated by Examples 1–3. Next, the following simulation example is to illustrate the feasibility of Corollary 1.

Example 4. Given a network $(\mathcal{G}, x)$ composed of $N + M$ agents, where $N = 2, M = 2, \ell_1 = \{1, 2\}, \ell_2 = \{3, 4\}$. The dynamics of agents are described as (1), where

$$A_1 = \begin{bmatrix} 1.25 & 0 \\ 1 & 2 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 3 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 & 5 \\ 1 & 2.5 & 2 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

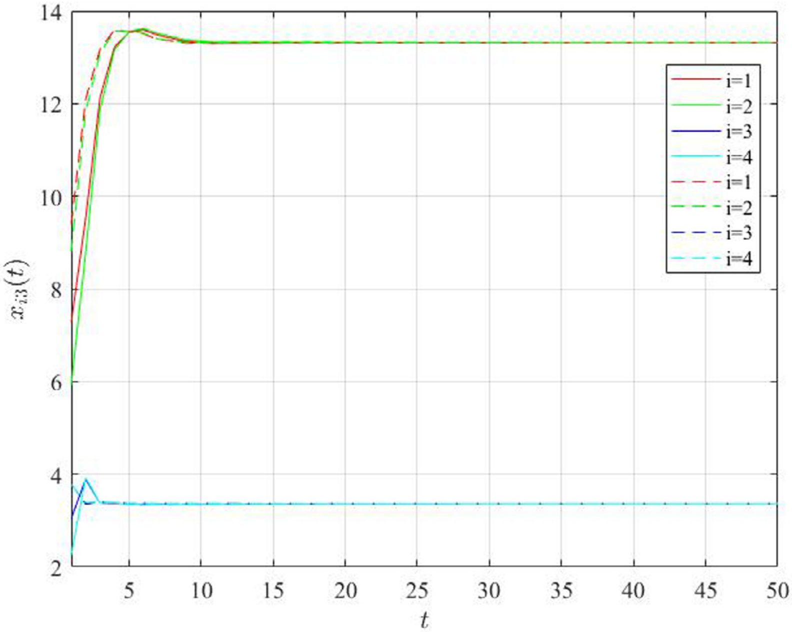

Fig. 15. State trajectory $x_{i3}(t)$ in Example 3, $i = 1, 2, 3, 4$.

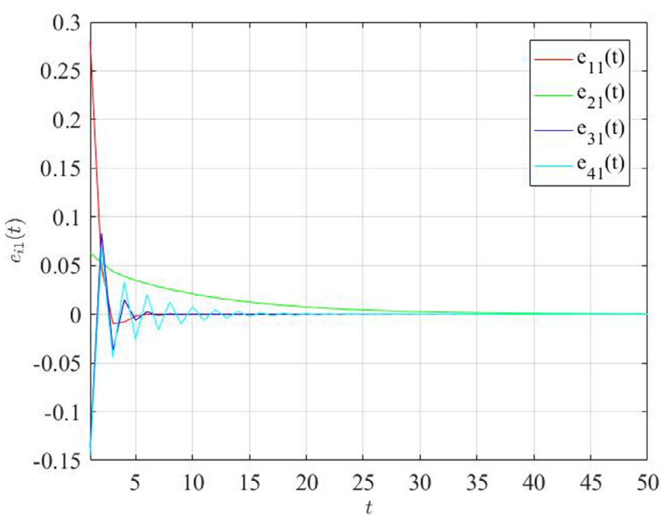

Fig. 16. Error trajectory $e_{i1}(t)$ in Example 3, $i = 1, 2, 3, 4$ ($\tau = 2$).

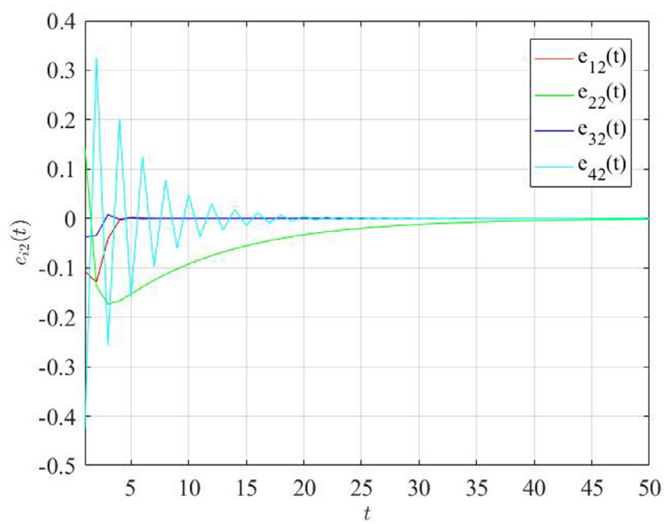

Fig. 17. Error trajectory $e_{i2}(t)$ in Example 3, $i = 1, 2, 3, 4$ ($\tau = 2$).

$$A_3 = \begin{bmatrix} -3 & 1 \\ 0 & -1.5 \end{bmatrix}, B_3 = \begin{bmatrix} -0.02 & 0.01 & -0.04 \\ -0.01 & 0.05 & -0.02 \end{bmatrix}, C_3 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 1 \\ 0 & -1.2 \end{bmatrix}, B_4 = \begin{bmatrix} 2 & 4 & 1 \\ 1 & -5 & 1 \end{bmatrix}, C_4 = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

The topological graph of HMASs is shown in Fig. 19.

The upper bound of time-varying delays $\tau_{ij}(t)$ is $\tau = 3$. And the observer gain matrix L_i is determined as

$$L_1 = \begin{bmatrix} 0.6412 \\ 2.3602 \end{bmatrix}, L_2 = \begin{bmatrix} 1.3028 \\ -2 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} -3.6801 \\ 1.1837 \end{bmatrix}, L_4 = \begin{bmatrix} 0.8431 \\ 0.8189 \end{bmatrix}.$$

Choose $B_{i0} = B_3$, according to Corollary 1, The control gain matrix K_i is determined as

$$K_a = \begin{bmatrix} -0.2712 & -0.9099 \\ -3.8483 & -1.0817 \end{bmatrix}, K_b = \begin{bmatrix} 1.9661 & -1.8559 \\ 0.2427 & -2.5693 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.0223 & 0.0273 \\ -0.0064 & 0.0048 \\ 0 & -0.0026 \end{bmatrix}, K_2 = \begin{bmatrix} -0.0026 & 0.0014 \\ -0.0122 & 0.0091 \\ 0.0014 & -0.0039 \end{bmatrix},$$

$$K_3 = K_{i0} = \begin{bmatrix} -0.1145 & 0.2365 \\ -0.7213 & 0.5633 \\ 0.2289 & 0.4730 \end{bmatrix}, K_4 = \begin{bmatrix} -0.0049 & -0.0014 \\ 0.0044 & -0.0037 \\ 0.0035 & 0 \end{bmatrix}.$$

The eigenvalues of Υ, i.e. $\lambda_1 = 0.1386, \lambda_2 = -0.3444, \lambda_3 = 0.2218, \lambda_4 = 0.5510$, are obtained by computations. Obviously, all eigenvalues of Υ lie in the unit circle. Therefore, according to Theorem 1, protocol (5) can solve the group consensus problem.

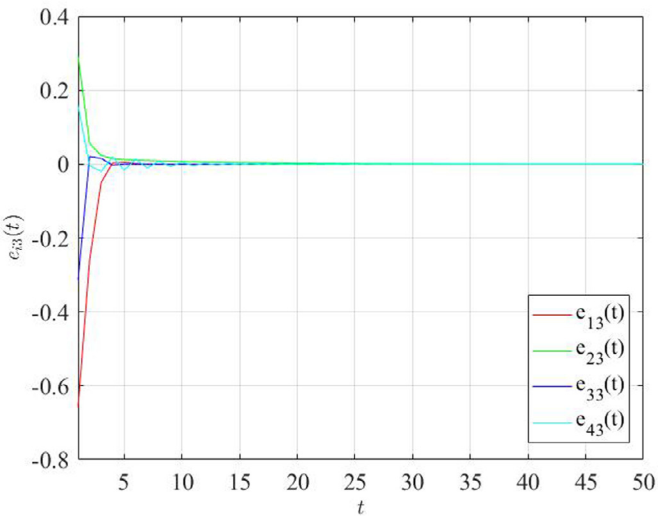

Fig. 18. Error trajectory $e_{i_3}(t)$ in Example 3, $i = 1, 2, 3, 4$ ($\tau = 2$).

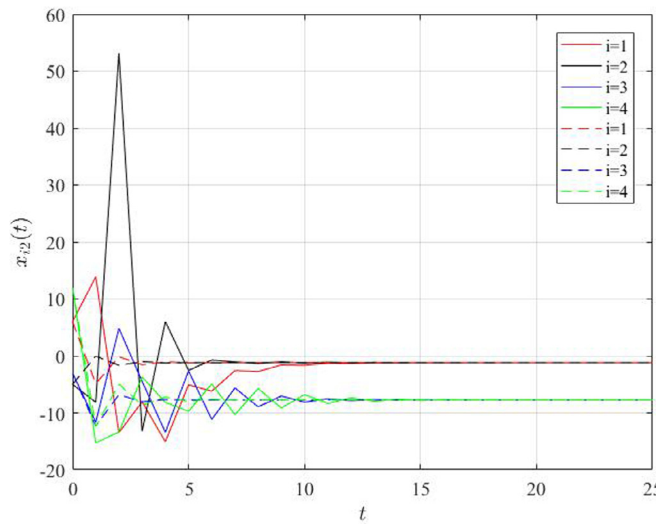

Fig. 21. State trajectory $x_{i_2}(t)$ in Example 4, $i = 1, 2, 3, 4$.

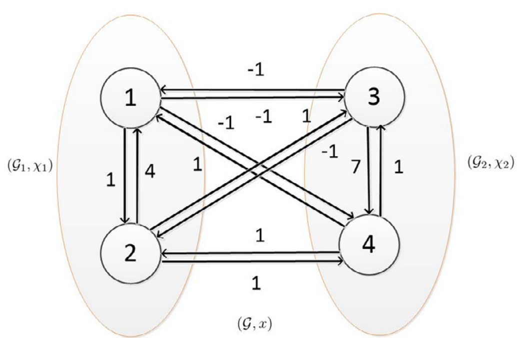

Fig. 19. Directed fixed topology $\mathcal{G}$ in Example 4.

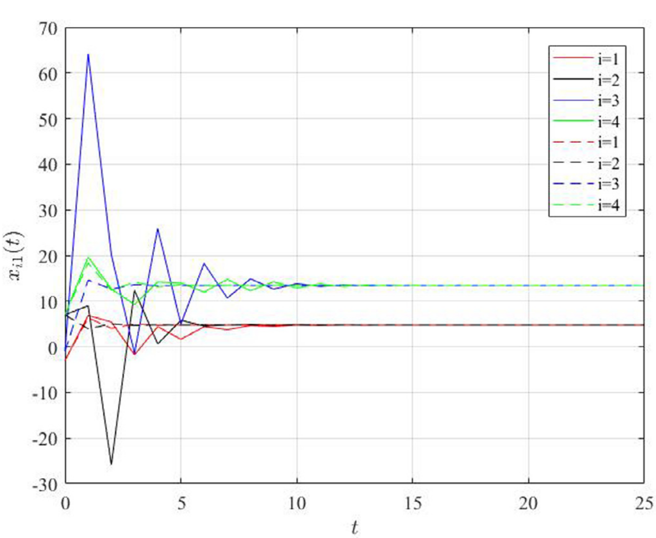

Fig. 20. State trajectory $x_{i_1}(t)$ in Example 4, $i = 1, 2, 3, 4$.

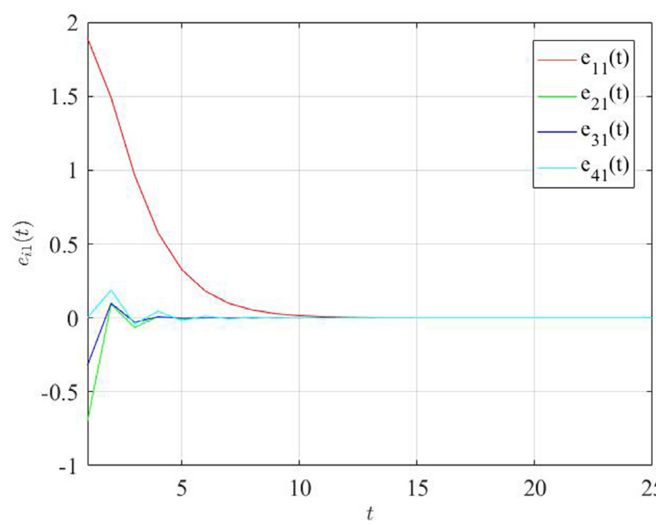

Fig. 22. Error trajectory $e_{i_1}(t)$ in Example 4, $i = 1, 2, 3, 4$ ($\tau = 3$).

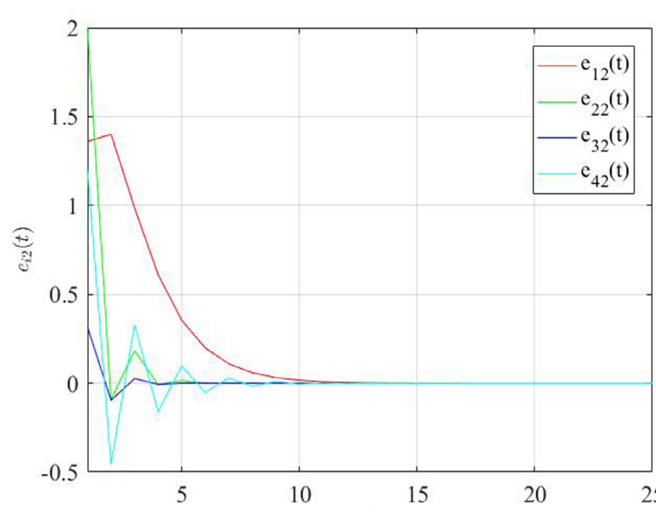

Fig. 23. Error trajectory $e_{i_2}(t)$ in Example 4, $i = 1, 2, 3, 4$ ($\tau = 3$).

The performance of discrete-time HMASs with upper bound of time-varying delays $\tau = 3$ and without time delays is compared in Fig. 20 and 21. Time-delay case is indicated by solid lines and no time-delay case is indicated by dashed lines. Fig. 22 and 23 show the error trajectory when $\tau = 3$.

5. Conclusion

The group consensus control for high-order discrete-time heterogeneous multi-agent systems with time delays is studied in this paper. Based on algebraic graph theory and Lyapunov stability theory, sufficient and/or necessary conditions for the multi-agent system to reach group consensus are derived. The group consensus of HMASs without or with the in-degree balance condition are achieved, respectively. In addition, the corresponding result is extended to multi-group consensus case. Numerical simulations are given to illustrate that each group can reach different consistent values under the proposed protocols. For future work, it would be interesting to extend our technique of reaching group consensus in the scenarios of switching topologies and uncertain systems.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Chong Tan: Conceptualization, Methodology, Software, Funding acquisition. **Liang Yue:** Writing - original draft, Writing - review & editing, Software. **Yanjiang Li:** Software, Project administration, Funding acquisition. **Guo-Ping Liu:** Methodology, Supervision.

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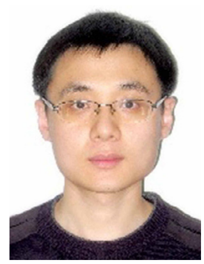

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