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Controllability Improvement for Multi-agent Systems: Leader Selection and Weight Adjustment

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For an uncontrollable system, adding leaders and adjusting edge weights are two methods to improve controllability. In this paper, the controllability of multi-agent systems under directed topologies is studied, especially on the leader selection problem and the weight adjustment problem. For a multi-agent system, necessary and sufficient algebraic conditions for controllability with fewest leaders are proposed. To improve controllability by adjusting edge weights, the system is supposed to be structurally controllable, which holds if and only if the communication topology contains a spanning tree. It is also proved that the number of the fewest edges needed to be assigned on new weights equals the rank deficiency of controllability matrix. In addition, a leader selection algorithm and a weight adjustment algorithm are presented. Simulation examples are provided to illustrate the theoretical results.

Keywords: multi-agent systems; controllability improvement; leader selection; weight adjustment

1. Introduction

In the past few decades, due to the rapid development of computer science and communication technology, distributed cooperative control of multi-agent systems has become a hot topic in multidisciplinary research area. Many results have been obtained and applied in science and engineering areas, such as flocking in biology, formation of unmanned air vehicles, attitude alignment of satellite clusters and data fusion of sensors. Researches on multi-agent systems include several fundamental problems, such as consensus (Wang and Xiao., 2010; Xiao and Wang, 2007; Xiao et al., 2015), formation and containment control (Xiao et al., 2009; Zheng and Wang, 2014), flocking and swarming (Jing et al., 2014; Saber, 2006; Shi et al., 2004), stabilizability (Guan et al., 2013, 2014) and controllability (Tanner, 2004), etc.

Controllability is a significant issue on multi-agent systems and attracts increasing attentions. A multi-agent system is said to be controllable if when leaders get appropriate exogenous control inputs, the system will achieve any designed configuration from any given initial states within a finite time. The controllability problem of multi-agent systems was put forward by Tanner (Tanner, 2004), where an algebraic necessary and sufficient condition was presented under undirected communication topologies. Based on this, Ji et al. proposed a leader-follower connected structure and proved it to be a necessary condition for the controllability of a multi-agent system with multiple leaders (Ji et al., 2008, 2009). The models of the agents in the above are all with single-integrator dynamics. In (Wang et al., 2009), Wang et al. studied the systems whose agents are with high-order dynamics and general linear dynamics, and proved that the controllability is only determined by the communication topology, regardless of the agents' dynamics. Further researches presented necessary and sufficient conditions for the controllability of some special graphs, such as cycles and paths (Parlangeli and Notarstefano, 2012), stars and trees (Ji et al., 2012), grid graphs (Notarstefano and Parlangel, 2013) and regular graphs (Kibangou and Commault, 2014), to name a few. Since controllability of a multi-agent system is also closely related to the interaction rule, protocol design was studied in

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(Ji et al., 2015). Studies on directed topologies are only confined to strongly regular graphs and distance regular graphs (Kibangou and Commault, 2014), and graph partitions (Lou and Hong, 2012). With respect to switching topologies, Liu et al. obtained several achievements on controllability (Liu et al., 2008, 2012). In addition, the controllability of heterogeneous multi-agent systems is studied in (Guan et al., 2015).

A parallel research line in this field is structural controllability, which was proposed by Lin in (Lin, 1974) for linear time-invariant systems, and was brought into multi-agent systems in (Wang et al., 2009; Zamani, 2009). On the one hand, structural controllability was investigated under various models (Lin, 1974; Mayeda and Yamada, 1979; Zamani, 2009), whereas all results in (Zamani, 2009) ignored the zero row sum restriction of the Laplacian matrix. In other words, the interactions of the agents were not based on the distributed consensus protocol. Although the protocol in (Lou and Hong, 2012) and (Liu et al., 2013) is a distributed one, the structural controllability problem proposed in those papers either allows to cut off interactions between some agents, or is focused on undirected graphs. To date, the structural controllability problem of multi-agent systems with consensus algorithms has been solved under three connectedness assumptions in (Goldin, 2013). On the other hand, the existing results only qualitatively judged the structural controllability. However, if a system is not strongly structurally controllable (Mayeda and Yamada, 1979), how to arrange a set of feasible weights that could ensure controllability is critically important. Moreover, for an uncontrollable system, in the premise of not adding any leader, controllability could be also improved by adjusting edge weights. To the best of our knowledge, how to adjust the weight parameters in a multi-agent system, especially how to choose the fewest edges and endow proper new weights to improve the controllability has not been studied.

Inspired by the previous results, this paper studies the leader selection problem and the weight adjustment problem for multi-agent systems under directed communication topologies. The contributions in our research are threefold:

(1) All results in this paper are based on directed and weighted graphs. Since the Laplacian matrix of a directed graph, especially weighted graph, is not symmetric, therefore the eigenvalues may be complex numbers and the Laplacian matrix cannot always be diagonalizable. However, the investigations in this paper successfully overcome these difficulties.
(2) From the viewpoint of the left eigenvectors of the Laplacian matrix, necessary and sufficient conditions of single leader controllability and fewest leaders to control a system are established. The conditions provide a new method to investigate the leader selection problem for the controllability of multi-agent systems, and a leader selection algorithm is proposed.
(3) Necessary and sufficient graphic conditions for structural controllability are given under the distributed consensus protocol; a new problem named “weight adjustment” is put forward in order to assign proper new weights to the fewest edges to ensure controllability, along with the algorithm of proceeding the weight adjustment.

This paper is organized as follows: In Section 2, basic concepts and preliminaries are given. In Section 3, leader selection problem is investigated. In Section 4, weight adjustment problem is proposed and solved. An application on checking controllability of in-degree regular graphs is shown in Section 5. Two typical examples are shown in Section 6 to illustrate the theoretical results. Finally, the conclusions are summarized in Section 7.

Notations : Throughout this paper, the following notations are used. $\mathbf{1}_n$ is a vector with dimension n whose entries are all 1, and sometimes footprint n is omitted for convenience. If A is a square matrix. $diag(a_1, a_2, \cdots, a_n)$ and $\max\{b_1, b_2, \cdots, b_m\}$ represent the diagonal matrix with principal diagonals $a_1, a_2, \cdots, a_n$ and the maximum value in $b_1, b_2, \cdots, b_m$, respectively. The set of all real numbers is denoted by $\mathbb{R}^n$. $\Lambda(A)$ denotes the eigenvalue set of A. $|S|$ represents the cardinality of a set S. S/T means the set of all elements those in S but not in T.

2. Preliminaries and problem formulation

2.1 *Graph theory*

A directed graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ consists of two parts, $\mathbb{V} = \{v_1, v_2, \cdots, v_n\}$ is the set of the nodes in the graph, and $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ represents the edge set. An edge e_{ij} in $\mathbb{E}$ is denoted by (v_i, v_j) if the edge points at v_j from v_i. v_i is called the parent node while v_j is called the child node and we say v_i is a neighbor of v_j. The neighbor set of v_j is denoted by $N_j = \{v_i \in \mathbb{V} | (v_i, v_j) \in \mathbb{E}\}$. The in-degree of node v is the total number of its neighbors, denoted as $\deg_{in}(v)$. Assume that there is no self-loop at any node, i.e. $(v_i, v_i) \notin \mathbb{E}$. A directed path $\mathbb{P}_n$ is a graph with n nodes and the edges are only $(v_1, v_2), (v_2, v_3), \cdots, (v_{n-1}, v_n)$. A tree graph $\mathbb{T}_v$ with root v is a graph that for each node other than v, there exists one and only one path from v to this node. In a tree graph, a node is called a leaf if it has no child, and two nodes are said to be in different branches when there is no path from any one of them to the another. A graph $\mathbb{G}$ is said to contain a spanning tree if there exists a tree whose nodes are all those in $\mathbb{V}$ and the edges in the tree are also in $\mathbb{E}$. A spanning forest of $\mathbb{G}$ is a set of trees covering $\mathbb{V}$ with no common nodes, and edges are all in $\mathbb{E}$. The minimal spanning forest is a spanning forest with fewest trees. Length of the shortest path from v_i to v_j is called the distance from v_i to v_j, denoted by $d(v_i \to v_j)$. Especially, if $v_i = v_j$, $d(v_i \to v_j) = 0$. $d(v_i \to v_j) = \infty$ when there is no path from v_i to v_j. The distance partition is defined as follows.

Definition 1: The distance partition of graph $\mathbb{G}$ relative to node v consists of a series of sets $D_0, D_1, D_2, \cdots, D_l$ and D_∞, where $D_0 = \{v\}$, $D_i = \{w \in \mathbb{V} | d(v \to w) = i\}$ and $D_\infty = \{w \in \mathbb{V} | d(v \to w) = \infty\}$. $\bigcup_i D_i = \mathbb{V}, i = 0, 1, 2, \cdots, l, \infty$.

In this paper, $\mathbb{G}$ is fixed. The adjacency matrix of $\mathbb{G}$ is $A(\mathbb{G}) = [a_{ij}] \in \mathbb{R}^{n \times n}$, where a_{ij} is the weight of edge e_{ji}, and $a_{ij} = 0$ if $(v_j, v_i) \notin \mathbb{E}$. The Laplacian matrix of $\mathbb{G}$ is $L = D - A$, $D = diag(d_1, d_2, \cdots, d_n)$ where $d_k = \deg_{in}(k)$ is the in-degree of node $k, k = 1, 2, \cdots, n$. A matrix M is said to be **cyclic** if no two blocks in the Jordan canonical form share the same eigenvalue (or equivalently, the eigenpolynomial of M equals its minimal polynomial).

Since the mapping between the communication topology of a system and the corresponding graph is a bijection, "node" and "agent" are not distinguished in this paper for convenience.

2.2 *Problem formulation*

Consider a multi-agent system with n single-integrator dynamic agents:

$$\dot{x}_i = u_i, \ i = 1, 2, \cdots, n. \tag{1}$$

Here x_i and u_i represent the state and the control input on agent i, respectively. For simplicity, only one dimensional states are considered in the following, i.e. $x_i \in \mathbb{R}$. However, the results obtained from this paper can be extended to arbitrary dimensional systems via Kronecker products. Agents that can be driven by external inputs are called **leaders**. The set of leaders are denoted by $\mathbb{V}_l = \{i_1, i_2, \cdots, i_m\}$. The rest agents are called **followers**, denoted by $\mathbb{V}_f = \mathbb{V}/\mathbb{V}_l$. The control inputs on the agents obey the distributed consensus-based protocol:

$$u_i = \begin{cases} \sum\limits_{j \in N_i} (x_j - x_i) + u_{o,i}, & i \in \mathbb{V}_l, \\ \sum\limits_{j \in N_i} (x_j - x_i), & i \in \mathbb{V}_f, \end{cases} \tag{2}$$

where $u_{o,i}$ is the external control on the leader agent i.

The compact form of system (1) with protocol (2) is summarized as follows.

$$\dot{x} = -Lx + Bu, \tag{3}$$

where $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ and $u = (u_1, u_2, \cdots, u_m)^T \in \mathbb{R}^m$ represent the states and the control inputs, respectively. L is the Laplacian matrix and $B = (e_{i_1}, e_{i_2}, \cdots, e_{i_m}) \in \mathbb{R}^{n\times m}$. $e_i \in \mathbb{R}^n$ is a vector with the i-th entry 1 and the rest 0.

Definition 2: Multi-agent system (3) (or the corresponding communication graph) is said to be controllable if for any initial state $x(t_0)$ and target state x^*, $x(t_0)$ can be actuated to $x(t_1) = x^*$ in finite time $t_1 > t_0$ with external controls u on the leaders.

Especially, if all the edges can be weighted freely, or more specifically, 0 entries in the adjacency matrix of the communication graph remain to be 0, and all other entries can be weighted positive numbers freely, then, the concept of structural controllability is proposed.

Definition 3: Multi-agent system (3) (or the corresponding communication graph) is said to be structurally controllable if there exists a group of weights to make the system controllable.

If multi-agent system (3) is not controllable, there are two methods to improve controllability, i.e. adding leaders or adjusting edge weights. The former one derives the leader selection problem, which will be introduced in the next section, and the latter method derives the weight adjustment problem, which will be discussed in Section 4.

3. Leader selection problem

Problem 1: (**Leader selection problem**) For multi-agent system (3), find a set of nodes $\mathbb{V}_l \subseteq \mathbb{V}$ with the minimum $|\mathbb{V}_l|$, such that when all nodes in $\mathbb{V}_l$ are chosen as leaders, the system is controllable.

The investigation of the leader selection problem begins with a basic concept.

Definition 4: (**r Leaders Controllable System**) Multi-agent system (3) is said to be r leaders controllable if the minimum $|\mathbb{V}_l| = r$. Especially, if $r = 1$, system (3) is called single leader controllable (SLC).

3.1 *Single leader controllability*

Let us start by single leader controllability problem. As a matter of fact, controllability of system (3) is invariant under any labeling of the nodes in communication graph $\mathbb{G}$. Suppose that $B = e \triangleq e_i \in \mathbb{R}^n$. Since the controllability of system (3) is same to that of system $\dot{x} = Lx + Bu$, the latter system is studied for simplicity. Consider the controllability matrix $C = (e, Le, L^2e, \cdots, L^{n-1}e)$, system (3) is controllable if and only if $rank(C) = n$.

In the following of this paper, denote the Jordan form of L as $J = diag(J_0, J_1, J_2, \cdots, J_s)$, and the corresponding similarity transformation matrix is $P^{-1} = (\xi_1, \xi_2, \xi_3, \cdots, \xi_n)^T$, $P^{-1}LP = J$. Here

$$J_j = \begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix}_{n_j \times n_j},$$

$j = 0, 1, \cdots, s, n_0 + n_1 + n_2 + \cdots + n_s = n$. Denote $m_j = n_0 + n_1 + n_2 + \cdots + n_j$, $j = 1, 2, \cdots, s$, obviously $m_s = n$. $\xi_1^T, \xi_{m_0+1}^T, \xi_{m_1+1}^T, \cdots, \xi_{m_{s-1}+1}^T$ are the linearly independent left eigenvectors of L. Correspondingly,

$\xi^T_{m_j+2}, \xi^T_{m_j+3}, \cdots, \xi^T_{m_{j+1}}$ are the linearly independent generalized eigenvectors of λ_j.

Theorem 1: *Multi-agent system (3) is SLC if and only if the following two conditions are satisfied simultaneously:*
1. The Laplacian matrix L is cyclic;
2. There exists an integer i, $1 \le i \le n$*, such that the i-th entries of all the left eigenvectors of L are not* 0.
In this circumstance, system (3) is controllable with the single leader agent i.

Proof. According to the PBH Test, system (3) is controllable if and only if $rank(\lambda I + L, B) = n$ holds for all $\lambda \in \Lambda(-L)$. Considering that $rank(\lambda I + L, B) = rank(\lambda I + J, P^{-1}e)$, next we shall prove that $rank(\lambda I + J, P^{-1}e) = n$ if and only if the two conditions are satisfied simultaneously.
(Necessity) If there are two different Jordan blocks in J sharing the same eigenvalue λ^*, the two rows in $(\lambda I + J, P^{-1}e)$ corresponding to the last rows of the two Jordan blocks will always be linearly dependent when $\lambda = \lambda^*$, which means $rank(\lambda^* I + J, P^{-1}e) < n$, therefore condition 1 is necessary. When the Laplacian matrix L is cyclic, then, all the Jordan blocks in J have different eigenvalues, thus $rank(\lambda I + J, P^{-1}e) = n$ holds for all $\lambda \in \Lambda(-L)$ if and only if the m_0-th, m_1-th, m_2-th, $\cdots$, m_s-th entries of $P^{-1}e$ all not be 0. If $e = e_i$ could satisfy this condition, i.e. the i-th entries of all the left eigenvectors of L are not 0, then the system is controllable, meanwhile, only the dynamic of the i-th agent is affected by the external input u, i.e. agent i is the leader.
(Sufficiency) Since all the Jordan blocks have different eigenvalues, suppose that $\lambda = \lambda_i$ is the eigenvalue of $-J_i$, then, $\lambda I + J_j$ are all of full row rank when $j \neq i$, and thus all the other rows are of full row rank except for the n_i rows corresponding to J_i. Considering that the first $n_i - 1$ rows are also of full row rank due to the 1 entries, and condition 2 ensures that the last entry of the n_i-th row is not 0. Therefore, $\lambda I + J_j$ is of full row rank, and the system is controllable. □

The second condition in Theorem 1 can be equivalently described as "There exists a column η^i in $P^{-1} = (\eta^1, \eta^2, \cdots, \eta^n)$ such that $\eta^i_{m_j} \neq 0$ for all $j = 0, 1, 2, \cdots, s$", which will benefit proving some results in the following.

Corollary 1: *The following two assertions hold:*
1. For multi-agent system (3), suppose that the eigenvalues of the Laplacian matrix satisfy condition 1 of Theorem 1. Agent i can be selected as the single leader to ensure the controllability if and only if the i-th entries of all the left eigenvectors of L are not 0.
2. If multi-agent system (3) is SLC, there must be a spanning tree in the communication graph with the root being the leader.

Proof. Assertion 1 is an equivalent expression of Theorem 1 and the proof is omitted. For assertion 2, if the graph doesn't contain a spanning tree, then $rank(L) < n - 1$ (see Lemma 5 in (Liu et al., 2012)), which leads to that the eigenvalue 0 corresponds to more than one Jordan blocks, and thus L is not cyclic, i.e. system (3) is not controllable. □

Remark 1: The existing results are mainly focused on the judgment of the controllability of multi-agent systems, while this paper investigates the leader selection problem to realize the controllability. Although it is intuitive to judge the controllability of multi-agent system (3) from the perspective of graph theory, to find a graphic necessary and sufficient condition for controllability is rather difficult and is still an opening problem. Ji et al. proposed a necessary and sufficient condition via the eigenvalues of the Laplacian matrix (Ji et al., 2009), but the results are only applicable to judge the controllability of some specific systems with fixed leaders, and the interaction topology should be undirected. However, Theorem 1 and Corollary 1 showed necessary and sufficient conditions based on the left eigenvectors of L, which contribute to searching for the fewest leaders and designing leader selection algorithms. The investigations on the left eigenvectors of the Laplacian matrix provide a new method to study the controllability of multi-agent systems.

3.2 *r leaders controllability*

Based on the SLC problem, a question arises that if system (3) is not SLC, whether how many leaders are needed to control the system? For a general directed topology, the next theorem shows how to check the controllability of system (3) with multiple leaders, as well as whether $|\mathbb{V}_l|$ is minimal.

For multi-agent system (3), $J = diag(J_0, J_1, \cdots, J_s)$ is the Jordan form of Laplacian matrix L. Distinct eigenvalues of L are denoted as $\lambda_0, \lambda_1, \cdots, \lambda_t, t \leq s$. $P^{-1}LP = J$.

Theorem 2: *System (3) is r leaders controllable if and only if there exist r integers, denoted as* $1 \leq c_1 < c_2 < \cdots < c_r \leq n$, *satisfying the following two conditions simultaneously:*
1. If $\xi_1^T, \xi_2^T, \cdots, \xi_s^T$ *are linearly independent left eigenvectors of the same eigenvalue* λ *of L, then, the matrix* $\Omega_\lambda = [\omega_{ij}] \in \mathbb{C}^{s \times r}$ *is of full row rank, where* ω_{ij} *represents the* c_j*-th entry of* ξ_i, $i = 1, 2, \cdots, s$, $j = 1, 2, \cdots, r$.
2. Any combination of less than r integers couldn't satisfy condition 1 for all the eigenvalues of L.
In this circumstance, system (3) is controllable with the leader agents $c_1, c_2, \cdots, c_r$.

Proof. Consider the matrix $(\lambda I + J, P^{-1}B)$, denote $\tilde{B} = P^{-1}B = (\tilde{B}_0^T, \tilde{B}_1^T, \cdots, \tilde{B}_s^T)^T$, where $\tilde{B}_i^T$ is of the same row size as J_i, $i = 0, 1, 2, \cdots, s$.

$$
\begin{aligned}
&(\lambda I + J, \tilde{B}) \\
&= \left(\begin{pmatrix} \lambda I + J_0 & & & \\ & \lambda I + J_1 & & \\ & & \ddots & \\ & & & \lambda I + J_s \end{pmatrix}, \begin{pmatrix} \tilde{B}_0 \\ \tilde{B}_1 \\ \vdots \\ \tilde{B}_s \end{pmatrix} \right).
\end{aligned} \tag{4}
$$

According to the PBH Test, system (3) is controllable if and only if the rank of (4) is n, i.e. all $(\lambda I + J_l, \tilde{B}_l), l = 0, 1, 2, \cdots, s$ are of full row rank for all $\lambda \in \Lambda(-L)$. Consider the submatrix

$$
\left(\begin{pmatrix} \lambda I + J_{i_1} & & & \\ & \lambda I + J_{i_2} & & \\ & & \ddots & \\ & & & \lambda I + J_{i_{k_i}} \end{pmatrix}, \begin{pmatrix} \tilde{B}_{i_1} \\ \tilde{B}_{i_2} \\ \vdots \\ \tilde{B}_{i_{k_i}} \end{pmatrix} \right),
$$

it is always of full row rank if and only if $rank(\Omega_{\lambda_i}) = k_i$, therefore condition 1 is necessary and sufficient. Condition 2 ensures the minimality of $\mathbb{V}_l$. □

If only condition 1 of Theorem 2 is satisfied, the system is also controllable, whereas $|\mathbb{V}_l|$ may not be minimal.

Theorem 2 is just a theoretical result, not a direct method to search for the fewest leaders. For a general directed graph, this problem is extremely similar to the minimal controllability problem discussed in (Olshevsky, 2014), which appears to be NP-hard. How to put forward an effective algorithm to search for the fewest leaders is a problem worthy of study. Here we propose a leader selection algorithm based on the left eigenvectors of the Laplacian matrix L.

4. Weight adjustment problem

As introduced, there are two methods to improve the controllability of a multi-agent system, one is by adding leaders, and the other is by adjusting edge weights. When the leaders' positions in the interaction topology are fixed, the controllability could be achieved only by assigning new weights to some proper edges. This yields the weight adjustment problem, which will be discussed in this section.

Leader Selection Algorithm

For multi-agent system (3), get the Laplacian matrix L; Get all the linearly independent left eigenvectors of L, and denote them as $\xi_1^T, \xi_2^T, \cdots, \xi_s^T$;
if all these eigenvectors correspond to different eigenvalues
  if $\xi_1(i) \neq 0, \cdots, \xi_s(i) \neq 0$ holds for some $1 \leq i \leq n$
    Output "The system is SLC. Agent i could be the leader."
  else
    for $r = 2 : n$
      if there exist r integers $1 \leq c_1 < \cdots < c_r \leq n$ such that $\sum_{j=1}^{r} \xi_i(c_j)\bar{\xi}_i(c_j) \neq 0$ holds for $i = 1 : s$
        Output "The system is r leaders controllable. Agents $c_1, \cdots, c_r$ could be the leaders."
      end if
    end for
else
  Get the different eigenvalues $\lambda_1, \cdots, \lambda_s$ of L,
  for $i = 1 : s$
    get the linearly independent left eigenvectors of λ_i, denoted as $\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_{k_i}}$;
  end for
  for $r = 2 : n$
    if there exist r integers $1 \leq c_1 < \cdots < c_r \leq n$ such that $rank(\Omega_{\lambda_i}) = k_i$ holds for $i = 1 : s$
      Output "The system is r leaders controllable. Agents $c_1, \cdots, c_r$ could be the leaders."
    end if
  end for
end if

4.1 *Structural controllability*

In order to control a system by adjusting edge weights, the system should be structurally controllable. The structural controllability of multi-agent systems under distributed protocols is studied in (Lou and Hong, 2012) and (Goldin, 2013). The former investigation allows to remove edges in the communication graph, and the latter one shows a necessary and sufficient condition for structural controllability under three connectedness assumptions. In (Goldin, 2013), it shows that the structural controllability can be achieved under three connectedness assumptions if and only if the interaction topology is leader-follower connected. However, it can be proved, the three assumptions are redundant since all of them can be derived from the leader-follower connectedness. Next we shall simplify the expressions of the conditions for structural controllability, and give the proofs in another method. The investigation starts from a property of tree graphs.

Proposition 1: *$\mathbb{T}_v$ with root v being the single leader is controllable if and only if the edges in different branches in $\mathbb{T}_v$ share no equal weight.*

Proof. See Appendix A. □

The next lemma and corollary are in another description of the main results in (Goldin, 2013). To facilitate comprehension, they will be described in a graphic perspective, and the proofs will be shown in an alternative approach.

Lemma 1: *System (3) is structurally controllable with one leader if and only if the communication graph contains a spanning tree with the root being the leader.*

Proof. See Appendix B. □

Corollary 2: *System (3) is structurally controllable with at least r leaders if and only if the minimal* *spanning forest of the communication graph contains r trees with the roots being the leaders.*

Proof. (Necessity) Refer to the necessity proof of Theorem 1.
(Sufficiency) When $r = 2$, i.e. the spanning forest $\mathbb{F} = \{\mathbb{T}_{v_1}, \mathbb{T}_{v_2}\}$. Since the graph is not structurally controllable, at least two leaders are needed. Select v_1 and v_2 as leaders. With proper weights, $\mathbb{T}_{v_1}$ and $\mathbb{T}_{v_2}$ could be controlled by v_1 and v_2 respectively. For each edge whose parent node lies in $\mathbb{T}_{v_1}$ (or $\mathbb{T}_{v_2}$) and the child node lies in $\mathbb{T}_{v_2}$ (or $\mathbb{T}_{v_1}$), assign a weight small enough to neglect the effect of it, then the whole graph remains controllable. Hence the conclusion holds for $r = 2$. Suppose that the conclusion holds for $r = n$. When $r = n+1$, i.e. $\mathbb{F} = \{\mathbb{T}_{v_1}, \mathbb{T}_{v_2}, \cdots, \mathbb{T}_{v_{n+1}}\}$, by the induction hypothesis, any n trees in $\mathbb{F}$ are structurally controllable with their roots being the leaders. Without loss of generality, suppose $\mathbb{T}_{v_1}, \mathbb{T}_{v_2}, \cdots, \mathbb{T}_{v_n}$ are controllable. Select v_{n+1} as a new leader, assign proper weights to $\mathbb{T}_{n+1}$, $\mathbb{F}$ is controllable. Assign small weights to the connections among the trees could make the whole graph controllable. According to mathematical induction, the conclusion holds for any positive integer r. □

4.2 *The fewest edges to be assigned on new weights*

For a structurally controllable system, how to adjust the weights on the edges to ensure controllability is concerned. The next theorem will solve this problem. To describe more explicitly, the problem of weight adjustment problem is defined mathematically as follows.

Problem 2: (**Weight adjustment problem**) For multi-agent system (3), when it is not controllable whereas all the roots in the spanning forest of the interaction topology are chosen as leaders, find a set of edges $\mathbb{E}_m \subseteq \mathbb{E}$ with minimum $|\mathbb{E}_m|$, such that when the weights are properly adjusted on the edges in $\mathbb{E}_m$, the system is controllable without changing any leader.

Investigating the weight adjustment problem first requires that, when taking all the leaders as roots, there must be a forest with these roots covering all nodes in the interaction graph. Especially, when there is only one leader, the graph should contain a spanning tree.

Theorem 3: *Suppose that the communication graph of multi-agent system (3) contains a directed spanning tree, and the root is the single leader. If the rank of the controllability matrix is $n-r$, then, there exist r edges such that the system could be controllable by adjusting the weights on them, and any adjustment on less than r edges cannot make system (3) controllable.*

Proof. When $r = 0$, the statement is obvious. Without loss of generality, assume the agents are labeled as follows: Label the root as 1, get the distance partition of the interaction graph $\{D_0, D_1, \cdots, D_p\}$ where D_0 is the root, and there is no D_∞ due to the existence of spanning tree in the graph. Label the nodes from those in D_1 to those in D_p successively. With this method, for agent i in $D_q, q = 2, 3, \cdots, p$, the first $q-1$ entries of the i-th row in L are 0.

For $1 \le r < n$, we prove that there exists one edge whose weight if be adjusted properly, could increase the rank of the controllability matrix C by 1. Mathematically, this equals to prove that if $k_i \neq 0, i = 1, 2, \cdots, s$,

$$(k_1 e_{i_1} + k_2 e_{i_2} + \cdots + k_s e_{i_s})^T L^m e_1 = 0, \tag{5}$$

for $m = 0, 1, 2, \cdots, n-1$, then, there exist ΔL and m_0 such that $(k_1 e_{i_1} + k_2 e_{i_2} + \cdots + k_s e_{i_s})^T (L + \Delta L)^{m_0} e_1 \neq 0$ where ΔL contains only two opposite nonzero elements who lie in the same row. The positive one is in the principal diagonal and the other is in front of it, $m_0 \le n-1$.
Suppose that the two nonzero elements are in the j-th row, $j \neq i_t, t = 1, 2, \cdots, s$, it can be verified,

$$(k_1 e_{i_1} + k_2 e_{i_2} + \cdots + k_s e_{i_s})^T (L + \Delta L)^m e_1 = 0 \tag{6}$$

holds for all $m = 0, 1, 2, \cdots, n-1$. If we intend to increase the rank of C, the revised edge must be selected from one of the $i_1, i_2, \cdots, i_s$-th rows in L. Take the i_1-th row as an example.

Next we show the existence of ΔL and m_0. Suppose that the equation (6) holds for all agents j connected to agent i_1, all $\varepsilon > 0$, and $m = 0,1,2,\cdots,n-1$. Let $\Delta L_{i_1,j} = -\varepsilon, \Delta L_{i_1,i_1} = \varepsilon$ and the other entries in ΔL are all 0. Under the assumption (5), combined with (6), we get

$$(k_1 e_{i_1} + k_2 e_{i_2} + \cdots + k_s e_{i_s})^T ((L+\Delta L)^m - L^m) e_1 = 0 \tag{7}$$

holds for $m = 0,1,2,\cdots,n-1$. When $m = 1$, $(k_1 e_{i_1} + k_2 e_{i_2} + \cdots + k_s e_{i_s})^T \Delta L e_1 = -k_1 \Delta L_{i_1,1} = 0$ yields $\Delta L_{i_1,1} = 0$, which means $L_{i_1,1} = 0$ holds for all j, and thus the root is not a parent node of agent i_1. Denote $D_m = (L+\Delta L)^m - L^m$, therefore $D_{m+1} = D_m L + D_m \Delta L + L^m \Delta L$. Since the $(i_1,1)$ entry in $D_m \Delta L$ and $L^m \Delta L$ are both 0, the $(i_1,1)$ entry in $D_m L$ must also be 0 to satisfy (7). Considering that the $(i_1,1)$ entry in $D_1 L$ is $\varepsilon(L_{i_1,1} - L_{j,1})$ and $L_{i_1,1} = 0$, this will lead to $L_{j,1} = 0$, which means all the parent nodes of agent i_1 are not connected from the root. Similarly, let $m = 2$, it can be proved that all the nodes connected to the parent nodes of agent i_1, are not connected from the root. Therefore, continue this procedure to $m = n-1$, we get that agent i_1 couldn't get information from the root of the tree, which contradicts the assumption of the spanning tree. Here we get that (7) doesn't hold for all agents j, $\varepsilon > 0$, and $m = 0,1,2,\cdots,n-1$, i.e. there exist ΔL and m_0 such that $(k_1 e_{i_1} + k_2 e_{i_2} + \cdots + k_s e_{i_s})^T (L+\Delta L)^{m_0} e_1 \neq 0$.

Apparently, ΔL only changes the i_1-th row of C. Without loss of generality, suppose no $s-1$ vectors of $C_{r_{i_1}}, C_{r_{i_2}}, \cdots, C_{r_{i_s}}$ are linearly dependent. Next we prove that there exists a proper $\varepsilon > 0$ such that $C_{r_{i_1}} + \delta C_{r_{i_1}}, C_{r_{i_2}}, \cdots, C_{r_{i_s}}$ are linearly independent. Consider the equation $k_1^{'}(C_{r_{i_1}} + \delta C_{r_{i_1}}) + k_2^{'} C_{r_{i_2}} + \cdots + k_s^{'} C_{r_{i_s}} = 0$, it is equal to $(k_1^{'} - k_1) C_{r_{i_1}} + (k_2^{'} - k_2) C_{r_{i_2}} + \cdots + (k_s^{'} - k_s) C_{r_{i_s}} + k_1^{'} \delta C_{r_{i_1}} = 0$. With the discussion above, we can get that if $\delta C_{r_{i_1}} \neq 0$, $\delta C_{r_{i_1}}$ will change with ε nonlinearly and thus a proper ε will ensure that $\delta C_{r_{i_1}}$ is linearly independent to $C_{r_{i_1}} + \delta C_{r_{i_1}}, C_{r_{i_2}}, \cdots, C_{r_{i_s}}$. If $k_1^{'} \neq 0$, then, $(k_1^{'} - k_1) C_{r_{i_1}} + (k_2^{'} - k_2) C_{r_{i_2}} + \cdots + (k_s^{'} - k_s) C_{r_{i_s}} + k_1^{'} \delta C_{r_{i_1}}$ will never be 0, therefore $k_1^{'} = 0$. Since $C_{r_{i_1}} + \delta C_{r_{i_1}}, C_{r_{i_2}}, \cdots, C_{r_{i_s}}$ are linearly independent, once $k_1^{'} = 0$, $k_i^{'} - k_i = -k_i$ for all $i = 2,3,\cdots,s$. Finally, $k_1^{'} = k_2^{'} = \cdots = k_s^{'} = 0$. This means that, there exist proper ΔL and m_0 to eliminate one of the linearly dependent rows in the controllability matrix. Based on this, assigning a proper weight to one proper edge, the rank of C will increase 1 and only 1. This implies that exactly r different edges (should be from different rows of L) are needed to adjust weights to fulfill the decreased rank of C. □

Remark 2: Refer to the proof of Theorem 3, the next two claims can be achieved.
1. Suppose that the controllability of system (3) can be improved by adjusting the weight on edge e^* and the weight increment is ε, i.e. $\Delta L_{e^*} = -\varepsilon$, then there exists an $n-1$ order polynomial of ε, say $f(\varepsilon)$, such that ΔL_{e^*} fails to increase the rank of L if and only if $f(\varepsilon) = 0$. Hence, there are no more than $n-1$ values of ε that would fail to improve controllability. Therefore, if we randomly endue a new weight to e^*, the probability of successfully increase the rank of C is 1.
2. If e^* should be selected from the i-th row of C, denoted as L_{r_i}, then the first nonzero entry in L_{r_i} could be the edge to adjust weight. This means we could redesign the weight of the edge that connects to agent i from the agent with the minimum identifer in N_i.

Here we show an algorithm on how to perform weight adjustment on proper edges. To express more explicitly, the algorithm here is designed for the graph that contains a spanning tree. However, it can be generalized to all directed graphs.

5. An application

In this subsection, an application of controllability and structural controllability will be shown for a kind of special graphs named in-degree regular graphs.

Definition 5: (**In − degree Regular Graph**) A directed graph is called in-degree regular if the in-degrees of each node are equal, i.e. $\deg_{in}(i) = \deg_{in}(j)$ for all $1 \leq i, j \leq n$.

Weight Adjustment Algorithm

Get all the nodes that could be a root of a spanning tree, identify them as $v_1, v_2, \cdots, v_m$, $C = 0_{n\times n}$;
Select an weight increment $\theta > 0$. $r = n$.
For $k = 1 : m$
  Label the root v_k as 1, get the distance partition $\{D_0 = v_k, D_1, \cdots, D_p\}$, label the whole system from nodes in D_1 to nodes in D_p successively, get the Laplacian matrix L and the controllability matrix C;
  if $rank(C) == n$
    The system is controllable with leader agent k, exit;
  else if $rank(C) < r$
    $r = rank(C), s = k$;
  end if
end for
Use row elimination to get the all-0 rows, get their row identifers $i_1, \cdots, i_s$;
while $rank(C) < n$
  for $1 \leq j \leq s$
    Add weight $j\theta$ to the edge corresponding to the first nonzero element in the i_j-th row of L;
  end for
  Get $\tilde{L}, L = \tilde{L}$; Calculate C; $\theta = 1.1\theta$;
end while
The number of fewest edges to be assigned new weights is s, and an available graph Laplacian is L.

Theorem 4: *An in-degree regular graph can be controlled by agent* 1 *if and only if matrix* $M = [m_{ij}] \in \mathbb{R}^{(n-1)\times(n-1)}$ *is invertible, where* m_{ij} *is the number of different paths from agent* 1 *to agent* $i+1$ *with length* j.

Proof. According to Definition 5, the Laplacian matrix $L = D - A = dI - A$, $L^k = \sum_{i=0}^{k} C_k^i d^i (-A)^{n-i}$. The controllability matrix $C = (e, Le, L^2 e, \cdots, L^{n-1}e)$, and

$$\begin{aligned} & rank(C) \\ = & rank(e, (dI - A)e, \cdots, \sum_{i=0}^{n-1} C_k^i d^i (-A)^{n-i} e) \\ = & rank(e, Ae, \cdots, A^{n-1}e). \end{aligned}$$

As assumed, $e = (1, 0, \cdots, 0)^T$ and the i-th entry of $A^j e$ is the number of paths from agent 1 to agent i with length j. M is the submatrix of C by deleting the first row and the first column. When M is invertible, C is of full rank and thus the system is controllable, vice versa. □

For in-degree regular graphs, controllability can be validated more intuitively.

Corollary 3: *The next two assertions on in-degree regular graphs hold:*
1. For an in-degree regular graph with n-nodes, whose adjacency matrix is A. If each column in $\sum_{i=1}^{n-1} A^i$ *contains at least one* 0*, the system is not SLC.*
2. For an in-degree regular graph, denote $S = \sum_{k=1}^{n-1} A^k$*, if at least m columns of S are needed to ensure the sum of them contains no* 0 *entry, then it should be* $|\mathbb{V}_l| \geq m$ *to make system (3) controllable.*

Proof. For assertion 1, if each column of $\sum_{i=1}^{n-1} A^i$ contains at least one 0, no matter which agent is selected

as the leader, there will be at least one agent that couldn't get information from the leader, and this makes an in-degree regular graph uncontrollable.
For assertion 2, choosing m columns of S whose sum contains no 0 entry is to ensure a leader-follower connected structure. Therefore, at least m leaders are needed for controllability. □

Proposition 2: *An in-degree regular graph is structurally controllable if and only if there exists one column of* $\sum_{k=1}^{n-1} A^k$ *that contains no* 0 *entry except for the principal diagonal elements, where* A *is the adjacency matrix of the graph.*

Proof. Without loss of generality, consider the first column of $\sum_{k=1}^{n-1} A^k$, delete the first entry and denote the remained vector as η. $\eta_i \neq 0$ if and only if there exists a path from agent 1 to agent $i+1$. Therefore, η contains no 0 entry if and only if the graph contains a spanning tree, which is a necessary and sufficient condition of structural controllability. □

6. Simulation

Three numerical examples are presented in this section to illustrate the Theorems 1 and 2, Proposition 1 and Theorem 3, respectively.

Example 1: Figure 1 shows a system with four agents. The Laplacian matrix is shown as follows:

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & -1 & 3 \end{pmatrix},$$

and a set of linearly independent left eigenvectors of L is $\{\xi_1^T = (1,0,0,0), \xi_2^T = (-1,1,0,0), \xi_3^T = (0,-1,1,0), \xi_4^T = (0,0,-1,1)\}$. Since no matter which agent is selected as the single leader, there always exists a left eigenvector ξ_i, $i = 1,2,3,4$ such that the corresponding entry is 0, therefore system (3) is not SLC under the interaction topology depicted in Figure 1. If agent 1 and agent 3 are selected as leaders, the graph is controllable, but once agent 2 is selected as a leader, two more leaders are needed. Moreover, it also demonstrates that even if the eigenvalues of the Laplacian matrix are distinct, the system may also not be SLC.

Example 2: Figure 2 shows a system with five agents, and the interaction topology is a tree graph. Take agent 1 as the single leader, the controllability matrix is:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -a_{21} & -a_{21}^2 & -a_{21}^3 & -a_{21}^4 \\ 0 & 0 & a_{21}a_{32} & a_{21}a_{32}(a_{21}+a_{32}) & a_{21}a_{32}(a_{21}^2+a_{21}a_{32}+a_{32}^2) \\ 0 & 0 & a_{21}a_{42} & a_{21}a_{42}(a_{21}+a_{42}) & a_{21}a_{42}(a_{21}^2+a_{21}a_{42}+a_{42}^2) \\ 0 & 0 & a_{21}a_{52} & a_{21}a_{52}(a_{21}+a_{52}) & a_{21}a_{52}(a_{21}^2+a_{21}a_{52}+a_{52}^2) \end{pmatrix}.$$

Apparently, the matrix is of full rank if and only if $a_{32} \neq a_{42} \neq a_{52}$. System (3) is structurally controllable and it is controllable for almost all sets of weights due to the leader-follower connected structure. However, if the system is not controllable, it holds that $a_{32} = a_{42}$ or $a_{32} = a_{52}$ or $a_{42} = a_{52}$. For whichever situation, adjusting the edge weight a_{21} will never improve the controllability, which means the system could become controllable by adjusting the weights on some specific edges, rather than any single edge, i.e. the weight adjustment problem is nontrivial.

Example 3: Figure 3 shows a directed communication graph of system (3). The Laplacian matrix is

$$L=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

with eigenvalues $0, 2, 0.2451, 1.8774 \pm 0.7449i$, and a set of linearly independent left eigenvectors is $\{\xi_1^T = (1,0,0,0,0), \xi_2^T = (0,0,0,1,-1), \xi_3^T = (-1.2672, 0.3106, 0.5451, 0.2345, 0.1770), \xi_4^T = (0.1336 + 0.1283i, -0.1553 - 0.3404i, -0.2726 + 0.0740i, -0.1172 + 0.4143i, 0.4115 - 0.2762i), \xi_4^T = (0.1336 - 0.1283i, -0.1553 + 0.3404i, -0.2726 - 0.0740i, -0.1172 - 0.4143i, 0.4115 + 0.2762i)\}$.

Apparently the system is not SLC. Actually it can be controlled by two leaders, one of which must be agent 1 and the other could be agent 4 or 5. Meanwhile, the graph contains a spanning tree, and thus system (3) is structurally controllable with one leader. Select agent 1 as the leader, we can get that the rank of the controllability matrix is 4, thus the system can be controlled only adjusting the weight on one edge. Here adjust the weight a_{53} from the original 1 to 1.1, and the Laplacian matrix turns to be

$$\tilde{L}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1.1 & -1 & 2.1 \end{pmatrix},$$

with the eigenvalues $0, 0.2493, 1.8930, 1.9788 \pm 0.7305i$, and the first column of the corresponding $\tilde{P}^{-1}$ is $[1, -1.2639, 0.0105, 0.1267 + 0.1412i, 0.1267 - 0.1412i]^T$, which means the system becomes controllable. Similarly, taking any adjustment on the weight a_{21} will never influent the controllability of the graph. This illustrates the results in Section 4.

7. Conclusions

This paper has studied the controllability of multi-agent systems with directed communication topologies. The concept of leader selection was explicitly introduced. Algebraic necessary and sufficient conditions on how to select the fewest leaders were presented based on the left eigenvectors of the Laplacian matrix. Considering that the controllability may also be achieved by adjusting edge weights, this paper also studied the weight adjustment problem, which aims to determine the fewest edges to be assigned on new weights to ensure controllability, as well as the weight values. The result showed that a multi-agent system is structurally controllable if and only if the communication graph contains a spanning forest with the roots of the trees being the leaders. The number of fewest edges equals the rank deficiency of controllability matrix. Algorithms on leader selection and weight adjustment were also provided.

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Appendix A. Proof of Proposition 1

Proof. (Necessity) If two edges of $\mathbb{T}_v$ in different branches share the same weight, there must be two equal elements $L_{ii} = L_{jj} = \lambda$ in the principal diagonal of L, and $L_{ij} = L_{ji} = 0$. Since λ is an eigenvalue of L, there must be two linearly independent eigenvectors of λ. Therefore, in the Jordan form of L, two Jordan blocks share a common eigenvalue λ. According to Theorem 1, the system is not SLC.

(Sufficiency) For clarity, the proof is stated in four steps. First assume all the edges have different weights, hence $P^{-1}LP = D$ where D is the diagonal form of L, and we prove that $p_{ij} = 0$ for all $i < j$. Then we show the expressions of p_{ij} for all $1 \le i, j \le n$. Next we prove that none of the elements in the first column of P^{-1} is 0. Finally we prove that controllability will not be broken when adding one leaf to a controllable tree with any weight different from the weights in other branches.

Part 1: Denote $R = \lambda_j I - L$, obviously when $j \ge 1$, the only nonzero element in the first row of R is r_{11}, thus $p_{1j} = 0$. Suppose when $i < i^*$, where $i^* \le j-1$, $p_{ij} = 0$, since $0 = \sum_{k=1}^{n} l_{i^*k} p_{kj} - \lambda_j p_{jj} = \sum_{k=1}^{i^*} l_{i^*k} p_{kj} = l_{i^*i^*} p_{i^*j}$ and $l_{i^*i^*} \neq 0$, so $p_{i^*j} = 0$. According to mathematical induction, $p_{ij} = 0$ for all $i < j$.

Part 2: Since $p_i = (p_{1i}, p_{2i}, \cdots, p_{ni})^T$ is an eigenvector of L, $(L - \lambda_i I) p_i = 0$, which means $\sum_{k=1}^{n} l_{ik} p_{ki} - \lambda_i p_{ii} = 0$. From part 1 we know $p_{1i} = p_{2i} = \cdots = p_{i-1,i} = 0$, and for $\mathbb{T}_v$, $l_{i,i+1} = l_{i,i+2} = \cdots = l_{i,n} = 0$, this yields $(l_{ii} - \lambda_i) p_{ii} = 0$. Owing to $l_{ii} - \lambda_i = 0$, p_{ii} could be any number, and $p_{ii} = 1$ is chosen here without loss of generality. For each $i > j$, we get $0 = \sum_{k=1}^{n} l_{ik} p_{ki} - \lambda_j p_{ij} = \sum_{k=1}^{i} l_{ik} p_{ki} - \lambda_j p_{ij}$. As mentioned before, there exists one and only one $k_i < i$ for each i such that $l_{ik} \neq 0$. Correspondingly, $p_{ij} = \frac{l_{ik_i} p_{k_i j}}{\lambda_j - l_{ii}}$. Combine the results afore yields

$$p_{ij} = \begin{cases} 0, & i < j \\ 1, & i = j \\ \frac{l_{ik_i} p_{k_i j}}{\lambda_j - l_{ii}}, & i > j \end{cases}.$$

Part 3: Now consider the first column of P^{-1}, denoted as $q = (q_1, q_2, \cdots, q_n)^T$. Obviously, $q_1 \neq 0$, otherwise, $Pq \neq e$ where $e = (1, 0, 0, \cdots, 0)^T$. Suppose $q_i \neq 0$ with $i < i^*$, and $\sum_{k=1}^{i} p_{ik} q_k = e_i$. Since $e_{i^*} = \sum_{k=1}^{n} p_{i^*k} q_k = \sum_{k=1}^{i^*} p_{i^*k} q_k$, $q_{i^*} = -\sum_{k=1}^{i^*-1} p_{i^*k} q_k = -l_{i^*k_{i^*}} \left(\frac{p_{k_{i^*}1}}{\lambda_1 - l_{i^*i^*}}, \frac{p_{k_{i^*}2}}{\lambda_2 - l_{i^*i^*}}, \cdots, \frac{p_{k_{i^*}, i^*-1}}{\lambda_{i^*-1} - l_{i^*i^*}}\right)(q_1, q_2, \cdots, q_{i^*-1})^T$. It follows from part 2 that the only nonzero element in the i^* row of P except for $p_{i^*1} = p_{i^*i^*} = 1$ is $p_{i^*k_{i^*}}$. Denote $\xi_{i^*} = -l_{i^*k_{i^*}} \left(\frac{p_{k_{i^*}1}}{\lambda_1 - l_{i^*i^*}}, \frac{p_{k_{i^*}2}}{\lambda_2 - l_{i^*i^*}}, \cdots, \frac{p_{k_{i^*}, i^*-1}}{\lambda_{i^*-1} - l_{i^*i^*}}\right)^T$, and $\eta_j = (p_{j1}, p_{j2}, \cdots, p_{j,i^*-1})^T$. Obviously, η_j are linearly independent, $i = 1, 2, \cdots, i^*-1$. Thus, there exist $c_1, c_2, \cdots, c_{i^*-1}$ such that $\sum_{k=1}^{i^*-1} c_k \eta_k^T = \xi_i^T$. Consider $\eta_2, \cdots, \eta_{i^*-1}, \xi$, which are also linearly independent, so that $c_1 \neq 0$. According to the induction hypothesis, $q_{i^*} = (q_1, q_2, \cdots, q_{i^*-1}) \sum_{k=1}^{i^*-1} c_k \eta_k = \sum_{k=1}^{i^*-1} c_k e_k = c_1 \neq 0$. Thus, each element in q is not 0, and in this case, condition 2 in Theorem 1 is satisfied.

Part 4: Suppose the diagonal form of L is J, $P^{-1}LP = J$. If weight μ of the new edge is different from every other weight, the Jordan form of the new graph is

$$\hat{J} = \begin{pmatrix} J & \\ & \mu \end{pmatrix} = \begin{pmatrix} P^{-1} & 0 \\ \alpha^T & 1 \end{pmatrix} \begin{pmatrix} L & 0 \\ \gamma^T & \mu \end{pmatrix} \begin{pmatrix} P & 0 \\ \beta^T & 1 \end{pmatrix}.$$

This means $\alpha^T P + \beta^T = 0$ and $\alpha^T LP + \gamma^T P + \mu\beta^T = \beta^T(\mu I - J) + \gamma^T P = 0$. It follows that

$$\gamma^T = \alpha^T(\mu I - L). \tag{A1}$$

According to the proper of γ and L in (A1), we can easily prove that all entries in α that correspond to the path from the root to the new leaf are not 0. It follows from Theorem 1 that the system is controllable. On the other hand, if μ equals to the weight of an edge in the path from the root to the new edge, μ is different from any weights in other branches due to the necessity condition. This guarantees that there is only one eigenvector $c(0,0,\cdots,0,1)^T$ correspond to eigenvalue μ, where c is a nonzero coefficient, and thus condition 1 in Theorem 1 is satisfied. Condition 2 can be proved similarly. □

Appendix B. Proof of Lemma 1

(Necessity) If the communication graph doesn't contain a spanning tree, there must be at least two agents that couldn't get information from each other, and thus the minimum spanning forest contains more than one trees, denoted as $\mathbb{T}_1, \mathbb{T}_2, \cdots, \mathbb{T}_r$. Once the leader is selected in some $\mathbb{T}_i$, there always be agents in other trees that couldn't get information from the leader and apparently the system is not controllable.

(Sufficiency) Suppose that the Laplacian matrix of the spanning tree $\mathbb{T}$ is L_T, and the corresponding similarity transformation matrix is P_T. Consider the communication topology $\mathbb{G}$ with the Laplacian matrix $L = L_T + \varepsilon L_R$, where L_R is the Laplacian matrix of the subgraph of $\mathbb{G}$ by deleting the edges in $\mathbb{T}$. Denote the similarity transformation matrix of L as P, $\Delta P \triangleq P^{-1} - P_T^{-1}$. Apparently, when $\varepsilon \to 0$, $\Delta P \to 0$. There exists a set of weights that makes all the eigenvalues of L_T distinct and $\mathbb{T}$ is controllable by Proposition 1. Therefore, all entries in the first column of P_T^{-1} are not 0. When ε is small enough, all eigenvalues of L remain distinct and all entries in ΔP will be small enough such that the first column of P^{-1} contains no 0. Therefore, system (3) is structurally controllable with the root of $\mathbb{T}$ being the single leader.

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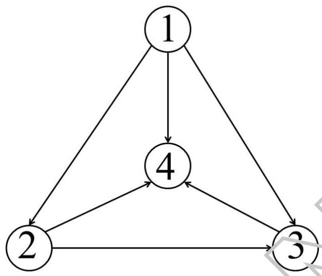

Figure 1.: Interaction topology of Example 1

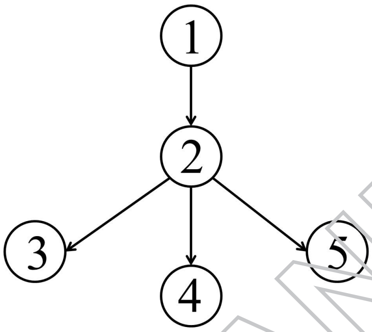

Figure 2.: Interaction topology of Example 2

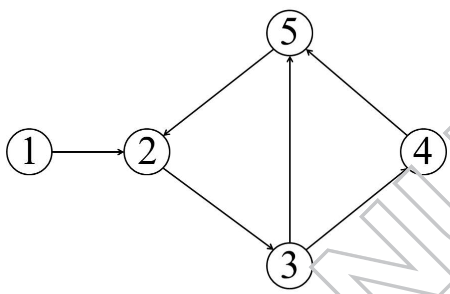

Figure 3.: Interaction topology of Example 3