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Bridging the Gap between Opinion Dynamics and Evolutionary Game Theory: Some Equivalence Results

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Abstract: Both opinion dynamics and evolutionary game theory have been receiving considerable attention in the control community in the last decade. In spite of this, they are generally regarded as two different fields and thus have been studied separately. It is proved in this paper that the voter model on the evolving network in the study of opinion dynamics and the Moran process on the complete network in evolutionary game theory are equivalent both in fixation probability and in limiting behavior, provided that the network evolves much faster than the opinions. These results bridge the gap between opinion dynamics and evolutionary game theory. With the aid of this bridge, we transform the in-group bias opinion dynamics on evolving networks, where individuals are likely to keep a steady relationship with those holding similar opinions, to the coordination game on the complete network; and the out-group bias opinion dynamics to the coexistence game. These equivalence results not only provide game-theoretical insights into opinion formation but also pave the way to understand fundamental concepts in game theory such as payoff and Nash equilibrium in the perspective of opinion dynamics.

Key Words: Evolutionary Game Theory, Opinion Dynamics, Dynamical Networks

1 Introduction

Opinion dynamics, which explore how opinions evolve, have attracted much attention from various disciplines, e.g., control theory, system science, artificial intelligence, applied mathematics and computer science [1–9]. In particular, opinion dynamics have a close relationship with multi-agent systems in control theory [2, 3], and can be insightful for crowd control [7–9, 11–14]. A basic assumption in opinion dynamics is that individuals would like to adopt popular opinions. The voter model is one of the classic models in opinion dynamics. Variants of the voter model are many with this basic assumption invariant [5–9]. As network science keeps developing [10, 15, 16], opinion dynamics on networks have attracted considerable attention [5–7, 9, 17–19]. Typically, individuals are assumed to be located on a network where their opinions are influenced by their neighbours'. Moreover, individuals are prone to adjusting their social ties. For example, individuals are likely to interact with those holding the same opinions [20, 21]. It is, thus, of great interest to investigate opinion dynamics on evolving social networks [17, 18].

Evolutionary game theory [22–24] is a mathematical tool to study how rational individuals adjust their strategies over time, when facing conflicts. It has been successfully applied in both biological and social systems [25, 26]. Recent decades also see an increasing interest of the control community in studying large-scale distributed systems via evolutionary game theory [27–34]. The basic assumption in evolutionary game theory is rationality, i.e., individuals are prone to optimizing their payoffs. The Moran process on the complete network is one of the classic evolutionary processes [25, 35–37], if there are a finite number of individuals in the system or population. Herein, individuals with higher payoff are more likely to be imitated. Variants of the Moran process are many with the assumption of rationality invariant [38–40]. If the population size is sufficiently large, the Moran process is approximated by the replicator equation [41]. It has been shown that the internal equilibrium of the replicator equation, if there is any, must be the Nash equilibrium for the underlying game [42], which is the key concept in game theory, depicting a state in which no player obtains more if one unilaterally alters its strategy [43]. Therefore, the Moran process as well as its infinitely large population size limit, i.e., the replicator equation, provides an algorithm to seek Nash equilibrium [24],

Opinion dynamics and evolutionary games are generally regarded as two distinct fields. In this paper we attempt to bridge the gap between the two fields by showing i) that for a population with a finite number of individuals, the consensus probability for the voter model on the evolving network is equivalent to the fixation probability for the Moran process on the complete network; ii) that for a population with infinitely many individuals, the voter model on the evolving network is captured by the replicator equation in evolutionary game theory.

The paper is organised as follows: Firstly, we propose a network update rule, in which only social ties are evolving. We term it as "the evolving network" in our manuscript. In this way, the voter model on the evolving network is introduced. Secondly, we introduce the Moran process on the complete network. Thirdly, we show that the voter model on the evolving network is equivalent to the Moran process on the complete network both in fixation probability and in limiting behavior. Fourthly, we give illustrative examples. Fifthly, we provide further discussions on these equivalence results. Finally, we end up with conclusions.

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2 Voter Model on the Evolving Network

Individuals on real networks would not only adjust their opinions but also adjust their social relationships. Here we propose the dynamics of the links, and study the voter model on the evolving network.

All the individuals are sitting on an undirected network, whose numbers of nodes and ties are N and L, respectively. Opinion dynamics happen with probability w, and link dynamics happen with probability $1-w$.

For opinion dynamics, we adopt the classic voter model: There are two opinions, say $+$ and $-$. An individual is randomly selected from the nodes. The focal individual adopts the opinion of $\pm$ with the probability proportional to the number of its neighbours with opinion $\pm$. In other words, the focal individual adopts opinion $\pm$ with probability $k_\pm/(k_+ + k_-)$, where $k_\pm$ refers to the number of its neighbours whose opinion is $\pm$. Noteworthily, this probability does not depend on the opinion of the focal individual. Thus it refers to a population where individuals have no insistency on their own opinion [8]. For the link dynamics, a link is randomly chosen. Let us denote the type of the selected link as XY, where $XY \in \{++,+-,--\}$. It breaks off with probability k_{XY}. If it breaks indeed, one of the two individuals at the extremes of the broken link is randomly selected. This individual reconnects randomly to another individual on the network, provided it is not its current neighbour.

When $w=1$, the social ties between individuals are invariant. The proposed model degenerates to the voter model on the static network, which has been intensively studied [5, 7, 9]. In this case, only the fraction of individuals adopting opinion $\pm$, i.e., $x_\pm$, evolves. And the social link is never broken. The system either ends up with the state, in which all the individuals adopt opinion $+$, or the state where all the individuals adopt opinion $-$. The consensus probability, with which the system ends up with all using $+$, indicates the likelihood of wining the voting for opinion $+$, and is the key quantity in opinion dynamics. The consensus probability is also termed as fixation probability in mathematics. Within this manuscript, we use the two terminologies simultaneously.

When $w=0$, individuals do not update their opinions, while the social ties evolve over time. In this case, we note that the link dynamics have two properties: i) The number of links is constant over time, leading to a constant average degree of the evolving network. ii) The smaller the breaking probability k_{XY} is, the longer the links of type XY endure. In fact, let us denote D_{XY} as the duration time of the social tie of type XY. D_{XY} is a random variable, with the distribution law given by $P(D_{XY}=j)=(1-k_{XY})^{j-1}k_{XY}$ $(j=1,2\cdots)$. In other words, D_{XY} follows the geometric distribution with successful probability k_{XY}. Thus the average duration time of the social tie of type XY, is $(k_{XY})^{-1}$. Concerning the limiting behavior of the social ties, we provide the following lemma.

Lemma 1 [20] *The link dynamics are a Markov chain with state space $S_2=\{++,+-,--\}$. If $k_{++}k_{+-}k_{--}x_+x_- \neq 0$, then there is a unique stationary distribution of the Markov chain and is given by $\pi_{ij} = \mathcal{N}(2-\delta_{ij})x_ix_j/k_{ij},\ \ ij\in S_2$, where δ_{ij} is Kronecker Delta, and $\mathcal{N}=\left(\sum_{ij\in S_n}(2-\delta_{ij})x_ix_j/k_{ij}\right)^{-1}>0$ is the normalisation factor.*

Remark 1 *If the undirected network is taken as the directed network, each undirected link XY, where $X,Y\in\{+,-\}$, can be taken as two directed links, i.e., $\overrightarrow{XY}$ and $\overrightarrow{YX}$, Lemma 1 can be rewritten as follows: The stationary distribution of the link dynamics is a Markov chain with state space $\tilde{S}_2=\{\overrightarrow{++},\overrightarrow{+-},\overrightarrow{-+},\overrightarrow{--}\}$. If $k_{++}k_{+-}k_{--}x_+x_-\neq 0$, then there is a unique stationary distribution of the Markov chain and is given by $\pi_{\overrightarrow{ij}}=\mathcal{N}x_ix_j/k_{ij},\ \ ij\in\tilde{S}_2$, where $\mathcal{N}$ is the normalisation factor.*

Remark 2 *Lemma 1 can be generalised to the case with more than two opinions. Denote S_n as the set consisting of all the link types, when there are n opinions in the population. There are $n(n+1)/2$ elements in set S_n. The link dynamics are proved to be a reversible Markov chain with state space S_n (see the appendix in [44] for the proof). With the aid of the detailed balance condition, we obtain that if $\Pi_{i=1}^n x_i \Pi_{XY\in S_n} k_{XY}\neq 0$, then the stationary distribution π_{ij} is proportional to $(2-\delta_{ij})x_ix_j/k_{ij},\ \ ij\in S_n$.*

3 Evolutionary Game Dynamics on the Complete Network

Evolutionary game dynamics are typically studied on the complete network [22–24]. Each individual plays the game with all the rest in the population. Let's concentrate on the two-player two-strategy games. In this case, each individual has two strategies, say strategies 1 and 2. When an individual taking strategy i meets the other individual taking the same strategy, each gets a_{ii} $(i=1,2)$; When an individual taking strategy 1 meets the other individual taking strategy 2, the individual with strategy 1 gets a_{12} and the other obtains a_{21}. This gives rise to the payoff matrix $A=(a_{ij})_{2\times 2}$. Based on the Darwinian's evolutionary theory, it is assumed that individuals with higher payoff are more likely to be imitated [25, 45].

For the deterministic evolutionary dynamics, the replicator equation on the complete network is the cornerstone. It is assumed that there are infinitely many individuals. The replicator equation is given by $\dot{x}_1 = x_1x_2(f_1-f_2)$, where x_i is the frequency of individuals taking strategy i, and $f_i=a_{i1}x_1+a_{i2}x_2$ is the average payoff $(i=1,2)$ [24, 42]. Based on the replicator equation, the frequency of the individuals taking strategy 1 increases, if the payoff of strategy 1 is greater than that of strategy 2.

For the stochastic evolutionary dynamics, the Moran process is typically adopted. We concentrate on the Moran process on a complete network of a finite size N. Each individual plays with a randomly selected individual (including itself), and gets its own payoffs. An individual with strategy i $(i=1,2)$ gets payoff $q_i(N_1)=a_{i1}\frac{N_1}{N}+a_{i2}\frac{N_2}{N}$, where N_i is the number of individuals taking strategy i. The payoff is then mapped to Darwinian fitness $f_i=f(q_i)$, measuring the number of offsprings an individual has on average. f should be positive and increasing [37]. An individual is selected to reproduce an identical offspring with a probability proportional to its fitness. The offspring then replaces another randomly selected individual (including the parent generation) in the system.

The Moran process is a Markov process. If all the individuals take strategy 1, then all the individuals would take strategy 1 for ever. The same applies to the case where all the individuals are taking strategy 2. Thus all the individuals taking strategy i ($i = 1, 2$) is an absorbing state for the Moran process. Let us denote the fixation probability ρ_k as the probability that the system ends up with a state in which all the individuals take strategy 1, conditioned on that there are k individuals take strategy 1 in the beginning. In particular, the fixation probability ρ_1 is taken as the key quantity to characterize the invasion of a mutant strategy [25, 36]. To obtain the fixation probability ρ_k, we introduce the following lemma.

Lemma 2 [46] *For a homogenous Markov chain with state space $S = \{0, 1, 2, \cdots, N\}$, if the following four conditions hold:*

i) both state 0 and state N are absorbing states;

ii) state i and state j intercommunicate for $i, j \in \{1, 2, \cdots, N-1\}$;

iii) state i is transient for $i \in \{1, 2, \cdots, N-1\}$;

iv) the transition matrix (p_{ij}) is tri-diagonal, i.e., if $|i-j| > 1$, $i, j \in S$, then $p_{i,j} = 0$,

then the fixation probability is given by

$$\rho_k = \frac{\sum_{i=0}^{k-1} \prod_{s=1}^{i} \frac{p_{s,s-1}}{p_{s,s+1}}}{\sum_{i=0}^{N-1} \prod_{s=1}^{i} \frac{p_{s,s-1}}{p_{s,s+1}}}, 0 \le k \le N. \tag{1}$$

Based on Lemma 2, the fixation probability for Moran process with identity payoff-to-fitness mapping $f(x) = x$ can be given in the following theorem.

Theorem 1 *For the Moran process on the complete network, if the following two conditions hold:*

i) all the payoff entries a_{ij} ($i, j \in \{1, 2\}$) are positive,

ii) the payoff-to-fitness mapping is given by the identity payoff-to-fitness mapping $f(x) = x$,

then the fixation probability ρ_k is given by

$$\rho_k = \frac{\sum_{i=0}^{k-1} \prod_{s=1}^{i} \frac{q_2(s)}{q_1(s)}}{\sum_{i=0}^{N-1} \prod_{s=1}^{i} \frac{q_2(s)}{q_1(s)}}, 0 \le k \le N. \tag{2}$$

Proof *Firstly, we denote the number of individuals taking strategy 1 as the state of the Markov chain. The state space of the Markov chain is given by $\{0, 1, 2, \cdots, N\}$. Secondly, both state 0 and N are absorbing states, as mentioned before. Thirdly, any state i, where $i \in \{1, 2, \cdots, N-1\}$ is transient. In fact, the system reaches state 0, provided an individual with strategy 2 is selected for reproduction and an individual with strategy 1 is replaced within the first i time steps, which is non-zero in probability. Fourthly, if $|i-j| > 1$, the transition probability $p_{i,j} = 0$, since the number of individuals taking strategy 1, i.e., i either increases by one, or decreases by one, or keeps the same within one time step. All of the four conditions in Lemma 2 are fulfilled, thus the fixation probabilities are obtained based on Eq.* (1).

On the other hand, $p_{i,i+1} = \frac{if(q_1(i))}{if(q_1(i))+(N-i)f(q_2(i))} \frac{N-i}{N}$, where the first term on the right hand side represents the probability with which an individual taking strategy 1 is selected for reproduction, whereas the second term on the right hand side indicates the probability with which an individual taking strategy 2 is replaced by the offspring. The similar argument yields that $p_{i,i-1} = \frac{(N-i)f(q_2(i))}{if(q_1(i))+(N-i)f(q_2(i))} \frac{i}{N}$. Thus the ratio $p_{i,i-1}/p_{i,i+1}$ is given by $f(q_2(i))/f(q_1(i))$. Taking $f(x) = x$ into account, the ratio $p_{i,i-1}/p_{i,i+1}$ is given by $q_2(i)/q_1(i)$. Based on Lemma 2, the fixation probability ρ_k is given by Eq. (2). ■

Theorem 1 indicates that the fixation probability for the Moran process is completely determined by the ratios between payoffs at a given state. Thus we have the following corollary.

Corollary 1 *If all the payoff entries in the payoff matrix A are positive and there exists a positive number r such that $B = rA$, then the fixation probability for the Moran process with the identity payoff-to-fitness mapping of payoff matrix B is identical to that of A.*

Lemma 3 [41] *For the Moran process with a payoff matrix A, if the population size is infinitely large, i.e., $N \to \infty$, then the deterministic evolution of the fraction of the individuals taking strategy 1, i.e., x_1, is given by the replicator equation $\dot{x}_1 = x_1 x_2 (f_1 - f_2)$, where $f_i = a_{i1}x_1 + a_{i2}x_2$ with $x_1 + x_2 = 1$.*

Lemma 3 shows a clear connection between the Moran process and the replicator equation. It is shown that the replicator equation can be a good approximation for the Moran process if the population size is sufficiently large.

4 Equivalence between the Voter Model on the Evolving Network and the Moran Process on the Complete Network

The voter model on the evolving network belongs to opinion dynamics, whereas the Moran process on the complete network belongs to evolutionary game theory. In other words, they belong to distinct fields. Here, we show that they are equivalent both in fixation probability and in limiting behavior.

Firstly, we show the equivalence in fixation probability.

Theorem 2 *If $w \to 0^+$ and $k_{++}k_{+-}k_{--} \neq 0$, then the fixation probability of the voter model on the evolving network is equivalent to that of the Moran process on the complete network with the payoff matrix*

$$M = \begin{array}{c c} & \begin{array}{cc} + & - \end{array} \\ \begin{array}{c} + \\ - \end{array} & \begin{pmatrix} \frac{1}{k_{++}} & \frac{1}{k_{+-}} \\ \frac{1}{k_{+-}} & \frac{1}{k_{--}} \end{pmatrix} \end{array}. \tag{3}$$

Proof *For the voter model on the evolving network, if $w \to 0^+$, the link dynamics evolve much faster than that of opinions. Since, $k_{++}k_{+-}k_{--} \neq 0$, based on Lemma 1, there exists a unique stationary distribution of the link dynamics, provided $x_+ x_- \neq 0$, i.e., there are both individuals with opinion $+$ and $-$. In this case, the opinion does not change until the link dynamics reach the stationary regime. In the stationary regime, based on Remark 1 of Lemma 1, the fraction of directed links $\overrightarrow{XY}$ is given by*

$$\pi_{\overrightarrow{XY}} = \mathcal{N} \frac{1}{k_{XY}} \frac{N_X N_Y}{N^2}, \quad X, Y \in \{+, -\}, \tag{4}$$

where N_X is the number of individuals whose opinion is X, and $\mathcal{N} > 0$ is the normalisation factor subject to $\pi_{\overrightarrow{++}} + \pi_{\overrightarrow{+-}} + \pi_{\overrightarrow{-+}} + \pi_{\overrightarrow{--}} = 1$.

Let us denote the number of individuals in opinion $\pm$ as $N_\pm$. We have that $N_+ + N_- = N$, and N_+ ranges from 0 to N. The voter model on the evolving network is a Markov chain with state N_+, giving rise to the state space $\{0, 1, \cdots, N\}$. Here state 0 and state N are absorbing states. In fact, both state 0 and state N represent that individuals reach consensus in opinion. Based on the voter model, the system keeps in the consensus state, provided that all the individuals reach consensus. Therefore, N_+ is a Markov chain with two absorbing states.

Additionally, N_+, either increases by one or decreases by one or stays the same within one time step. For example, N_+ increases by one if an individual in opinion $-$ is selected. This happens with probability $\frac{N_-}{N}$. Then the focal individual in opinion $-$ adopts opinion $+$, or an individual in opinion $-$ finds a neighbour in opinion $+$. This occurs with probability $h\pi_{\overrightarrow{-+}}/(h\pi_{\overrightarrow{-+}} + h\pi_{\overrightarrow{--}})$, where h is the average degree of the evolving network. In our model, h is constant. Therefore the probability with which the number of individuals in opinion $+$, i.e., N_+, increases by one is

$$T^+_{N_+} = \frac{N_-}{N} \frac{\pi_{\overrightarrow{-+}}}{\pi_{\overrightarrow{-+}} + \pi_{\overrightarrow{--}}}. \tag{5}$$

Similarly, the transition probability with which the number of individuals in opinion $+$ decreases by one is

$$T^-_{N_+} = \frac{N_+}{N} \frac{\pi_{\overrightarrow{+-}}}{\pi_{\overrightarrow{+-}} + \pi_{\overrightarrow{++}}}. \tag{6}$$

By the normalisation condition, the transition probability with which the number of individuals in opinion $+$ stays the same is $T^0_{N_+} = 1 - T^+_{N_+} - T^-_{N_+}$.

Now, we calculate the likelihood with which the entire population ends up with consensus in opinion $+$ if there are k individuals $(k = 0, 1, 2, \cdots, N)$ in opinion $+$ in the beginning. That is the consensus probability or fixation probability in $+$, i.e., $\tilde{\rho}_k$, which is defined in Section 2. We notice that the voter model on the evolving network fulfills the following conditions: i) the state space of the Markov chain is $\{0, 1, \cdots, N\}$; ii) the Markov chain has two absorbing states, i.e., state 0 and state N; iii) state i and state j intercommunicate for $i, j \in \{1, 2, \cdots, N-1\}$; iv) state i is transient for $i \in \{1, 2, \cdots, N-1\}$; v) the transition probability matrix is tri-diagonal. Based on Lemma 2, the fixation probability in $+$ is given by

$$\tilde{\rho}_k = \frac{\sum_{i=0}^{k-1} \prod_{N_+=1}^{i} \frac{T^-_{N_+}}{T^+_{N_+}}}{\sum_{i=0}^{N-1} \prod_{N_+=1}^{i} \frac{T^-_{N_+}}{T^+_{N_+}}}. \tag{7}$$

Based on Corollary 1, the fixation probability $\tilde{\rho}_k$ is determined by the ratios between transition probabilities, i.e., $\frac{T^-_{N_+}}{T^+_{N_+}}$. Taking Eq. (4) into Eq. (7), we obtain that the ratios are given by

$$\begin{aligned} \frac{T^-_{N_+}}{T^+_{N_+}} &= \frac{N_+}{N_-} \frac{\pi_{\overrightarrow{-+}} + \pi_{\overrightarrow{--}}}{\pi_{\overrightarrow{+-}} + \pi_{\overrightarrow{++}}} \\ &= \frac{\frac{1}{k_{-+}} \frac{N_+}{N} + \frac{1}{k_{--}} \frac{N_-}{N}}{\frac{1}{k_{++}} \frac{N_+}{N} + \frac{1}{k_{+-}} \frac{N_-}{N}}. \end{aligned} \tag{8}$$

The numerator of Eq. (8) is exactly the payoff of an individual taking strategy $-$ when there are N_+ individuals taking strategy $+$ with payoff matrix Eq. (3) on the complete network, i.e., $f_-(N_+) = k^{-1}_{-+}\frac{N_+}{N} + k^{-1}_{--}\frac{N_-}{N}$. Similarly, the denominator of Eq. (8) is exactly the payoff of an individual taking strategy $+$ when there are N_+ individuals taking strategy $+$ in the population, i.e., $f_+(N_+) = k^{-1}_{++}\frac{N_+}{N} + k^{-1}_{+-}\frac{N_-}{N}$. Thus it holds true that

$$\frac{f_-(N_+)}{f_+(N_+)} = \frac{T^-_{N_+}}{T^+_{N_+}}, (1 \le N_+ \le N-1). \tag{9}$$

Based on Theorem 1, the fixation probability of strategy $+$ for the Moran process on the complete network with game matrix Eq. (3) when there are k strategy $+$ individuals in the beginning is given by

$$\rho_k = \frac{\sum_{i=0}^{k-1} \prod_{N_+=1}^{i} \frac{f_-(N_+)}{f_+(N_+)}}{\sum_{i=0}^{N-1} \prod_{N_+=1}^{i} \frac{f_-(N_+)}{f_+(N_+)}}, \tag{10}$$

which is exactly the fixation probability on the evolving network, i.e., Eq. (7).

■

Next, we show the equivalence in limiting behavior for large population size.

Theorem 3 *If $w \to 0^+$, $k_{++}k_{+-}k_{--} \neq 0$ and the population size is infinitely large, i.e., $N \to \infty$, then the deterministic evolution of the fraction of the individuals adopting opinion $+$ for the voter model on the evolving network is equivalent to that of the Moran process on the complete network with payoff matrix (3).*

Proof *On the one hand, since $w \to 0^+$ and $k_{++}k_{+-}k_{--} \neq 0$ hold, following the same argument in the proof of Theorem 2, the voter model on the evolving network is proven to be a Markov process, whose transition matrix is tri-diagonal. In addition, Eqs. (4)(5)(6) still hold.*

If the population size is infinitely large, i.e., $N \to \infty$, the deterministic evolution of the fraction of individuals adopting opinion $+$, i.e., x_+, is given by

$$\dot{x}_+ = T^+_{N_+} - T^-_{N_+}. \tag{11}$$

Taking Eqs. (4)(5)(6) into Eq. (11) yields that

$$\dot{x}_+ = \frac{x_+ x_-}{k_{+-}} \left(\frac{1}{q_-(x_+)} - \frac{1}{q_+(x_+)} \right), \tag{12}$$

where $q_i(x_+) = k^{-1}_{i+}x_+ + k^{-1}_{i-}x_- > 0$ is the average payoff of strategy i ($i \in \{+, -\}$) with game matrix Eq. (3). Multiplying $k_{+-}q_-(x_+)q_+(x_+) > 0$ on the right hand side of

Eq. (12) *only alters the evolutionary speed with the asymptotical behavior invariant. Therefore, the limiting behavior of voting is captured by*

$$\dot{x}_+ = x_+ x_- (q_+(x_+) - q_-(x_+)), \tag{13}$$

i.e., the replicator equation with payoff matrix (3).

On the other hand, if $N \to \infty$, based on Lemma 3, for the Moran process on the complete network with payoff matrix (3)*, the deterministic evolution of the fraction of the individuals taking strategy + is given by the replicator equation Eq.* (13).

Therefore, the replicator equation Eq. (13) *with payoff matrix* (3) *captures both the limiting behavior of the voter model on the evolving network and that of the Moran process on the complete network.* ■

Remark 3 *Theorem 3 not only shows that the voter model on the evolving network and the Moran process on the complete network are equivalent in their limiting behaviors, but also indicates that both of the limiting behaviors are captured by the replicator equation with the same payoff matrix.*

Remark 4 *The replicator equation is diffeomorphic to the Lotka-Voltrera equation, the basic equation in ecology with interacting species [24]. Therefore, Theorem 3 also bridges opinion dynamics and ecological theory.*

We have found that both the fixation probability (Theorem 2) and the limiting behavior (Theorem 3) of the voter model on the evolving network are captured by evolutionary processes with the payoff matrix Eq. (3). Concerning the payoff matrix, it is not assumed a priori, as it is typically done in evolutionary game theory. In fact, it is an emergent payoff matrix arising from the microscopic dynamics of the voter model on the evolving network, in which no game interaction is assumed. Furthermore, the payoff entry M_{ij} is k_{ij}^{-1}, i.e., the expected duration time of social link of type ij. Therefore, the payoff in evolutionary games on the complete network can be taken as the duration time in the voter model on the evolving network (Fig. 1).

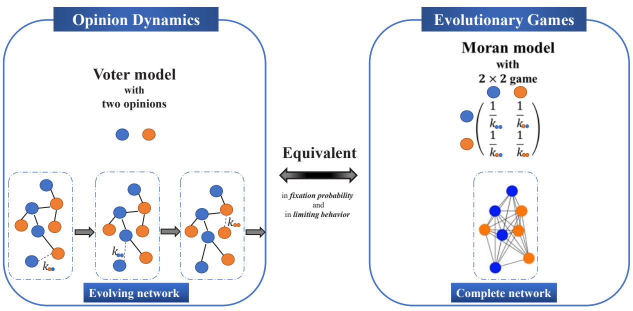

Fig. 1: Bridging the gap between opinion dynamics and evolutionary games.

5 Illustrative Examples

The equivalence results, i.e., Theorem 2 and Theorem 3, help us explore opinion dynamics from evolutionary game theory. The emergent payoff matrix is the key component of the bridge. In the following, based on Theorem 3, we give three examples to show how to tackle limiting behavior of the opinion dynamics on the evolving network with an evolutionary game perspective. In fact, analogous results can be given to obtain the consensus probability for the opinion dynamics on the evolving network via the existing literature in evolutionary games [36, 46].

Example 1 *In opinion dynamics, the in-group bias refers to the case where individuals interact more often with those holding the same opinion. If all the individuals are in-group bias, it refers to $k_{++} < k_{+-}$ and $k_{--} < k_{+-}$ in the context of our model. Based on Theorem 3, we obtain that for large population size, the voting behavior is captured by the replicator equation with payoff matrix Eq.* (3)*. Noteworthily, the payoff matrix represents a coordination game. Standard dynamical analysis of the replicator equation [24] shows that the system ends up with the consensus state in opinion + if the initial fraction of individuals adopting opinion + exceeds $x_+^* = (k_{--}^{-1} - k_{+-}^{-1})/(k_{++}^{-1} - 2k_{+-}^{-1} + k_{--}^{-1})$. Otherwise, the system ends up with consensus state in opinion −. Simulations are performed to validate this result in Fig. 2. In addition, (x_+^*, x_+^*) is exactly the Nash equilibrium of the payoff matrix Eq.* (3).

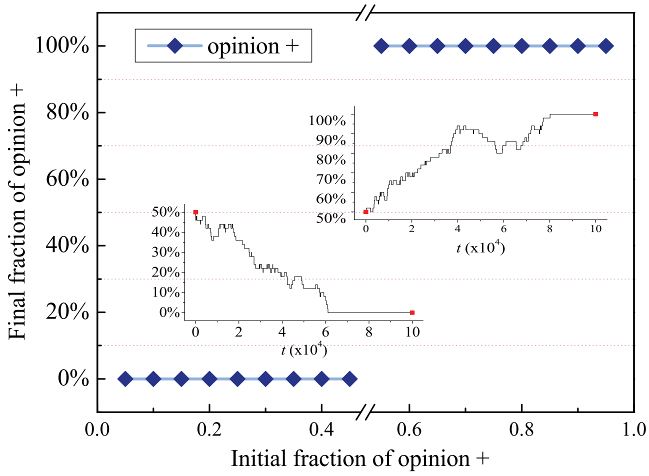

Fig. 2: If both opinions are in-group bias, the opinion dynamics are described by the replicator equation of a coordination game. Herein $k_{++} = 0.1$, $k_{--} = 0.1$, $k_{+-} = 0.9$. Other parameters: $N = 50$, $w = 0.01$.

Example 2 *The out-group bias in opinion dynamics indicates that individuals interact more often with those holding different opinions. If all the individuals are out-group bias, it refers to $k_{++} > k_{+-}$ and $k_{--} > k_{+-}$ in our context. Theorem 3 transforms the voting behavior on the evolving network to the replicator equation of a coexistence game. In this case, standard analysis shows that there is only one internal equilibrium of the replicator equation and it is stable. In other words, opinion + and opinion − coexist, provided that they coexist in the beginning. Furthermore, the replicator equation shows that the stable internal equilibrium is given by $y_+^* = (k_{--}^{-1} - k_{+-}^{-1})/(k_{++}^{-1} - 2k_{+-}^{-1} + k_{--}^{-1})$, which quantitatively captures the popularity of opinion + in the long run. Simulations are performed to validate this result in Fig. 3. In addition, (y_+^*, y_+^*) is exactly the strict Nash equilibrium of the payoff matrix Eq.* (3).

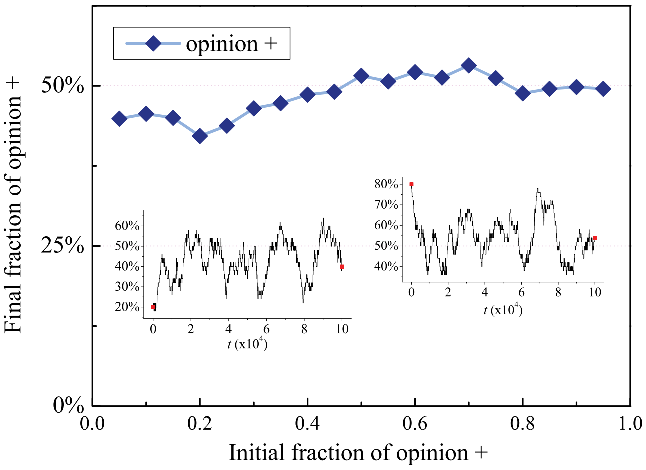

Fig. 3: If both opinions are out-group bias, the opinion dynamics are described by the replicator equation of a coexistence game. Herein $k_{++} = 0.9$, $k_{--} = 0.9$, $k_{+-} = 0.1$. Other parameters: $N = 50$, $w = 0.01$.

Example 3 *If one of the opinions, say $+$ without loss of generality, is in-group bias, and the other opinion, that is $-$ in this case, is out-group bias, it refers to $k_{++} < k_{+-} < k_{--}$ in our context. Theorem 3 transforms the voting behavior on the evolving network to the replicator equation of a dominant game. Strategy $+$ is the Nash equilibrium. Noteworthily, it is also the Pareto optimum, in which no one can be better off by making the other worse off. In other words, there is no dilemma between group interest and the individual interest in the emergent game. In this case, standard analysis shows the system would end up with the consensus state $+$, provided there are individuals adopting opinion $+$ in the beginning. Simulations are performed to validate this result in Fig. 4.*

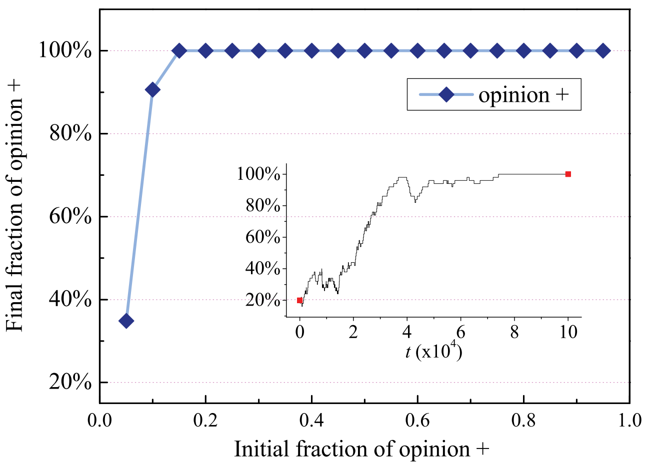

Fig. 4: If opinion $+$ is in-group bias and opinion $-$ is out-group bias, the opinion dynamics are described by the replicator equation of a dominant game. Herein $k_{++} = 0.1$, $k_{--} = 0.9$, $k_{+-} = 0.5$. Other parameters: $N = 50$, $w = 0.01$.

6 Some Further Discussions

Voter model. The adopted voter model in our paper is widely used in statistical physics. The voter model apparently differs from the French-DeGroot model, which is widely used in the control community. The voter model is stochastic whereas the French-DeGroot model is deterministic. On the other hand, based on the duality between the French-DeGroot model and the Markov chain [2], the voter model can be interpreted as the French-DeGroot model with two constraints: i) all the individuals are not stubborn at all; ii) all the neighbours have equal weights in influencing the decision making of the focal individual. Therefore, Example 2 also shows that out-group bias in opinion dynamics could lead to the coexistence of opinions, without introducing stubborn individuals as in [9]. It turns out that the adjustment of social ties alone could solve the diversity puzzle, which calls for the explanation why consensus in opinion is rare in societies [2].

The evolving network. The proposed evolving network captures the dynamical nature of the social network, which is also present in evolutionary games [47]. The driving force of the evolving network is social-bias, or preferential attachment, which also motivates the bounded confidence model [48, 49] and biased opinion formation process [50, 51] in opinion dynamics. In contrast with the biased opinion formation process, in which the social bias is at work in the opinion update rule, our model is the same as the bounded confidence model in the sense that the social bias is at work in adjusting the neighbours. In spite of this similarity, our model not only captures the homophily which is the basic assumption in the bounded confidence model (see Example 1) but also captures other social preferential attachments beyond homophily (see Example 2 and Example 3).

Intuitions of the equivalence results. To obtain the intuitions of the equivalence results, we review some previous results in evolutionary game theory. It has been shown that the Moran process of a pairwise game on the evolving network is captured by the replicator equation with a transformed matrix. The entries of the transformed matrix turns out to be the product of the original payoff entries and the corresponding duration time of the social link. The intuition is that the evolving social network alters the interaction rate between individuals with different strategies. And the emergent non-uniform interaction rates result in a payoff rescaling [20]. In fact, the voter model is the Moran process with a payoff matrix, in which all the payoff entries are 1. Therefore, the voter model on the evolving network is captured by the replicator equation of the transformed matrix based on [20]. In this case, the entry of the transformed payoff matrix is the duration time of the corresponding social link, which is exactly the emergent payoff matrix Eq. (3). In other words, the non-uniform interaction rates give rise to the emergent payoff matrix, which can be seen as the product of 1 and the duration time.

Comparison between our equivalence results and other equivalence results. Efforts have been made to establish the equivalence between game theory and Friedkin-Johnson model in opinion dynamics [52]. In Friedkin-Johnson model, each individual not only i) takes the neighbours' opinions but also ii) takes its initial opinion into account. In fact, each individual is assigned a utility function. It consists of two parts, one is for penalising disagreement which refers to i), and the other is for rewarding consistency in its own opinion which refers to ii). The two parts are weighted by, respectively, individuals' susceptibilities to social influence and stubbornness to their initial opinions. And every individual optimises its individual utility function. In this way, the Friedkin-Johnson model can be interpreted in a game theoretical way. The game perspective shows that there is a conflict between individual interest and global interest. In short, it is a conflict between i) and ii). In our work, no one takes its own opinion into account in opinion updating. In other words, there is no ii). Therefore, there is no conflict between i) and ii). Our equivalence results, however, show that the conflict can also arise between taking opinion $+$ and opinion $-$. In fact, if both opinions are in-group bias, both consensus states are stable Nash equilibria, as shown in Example 1. The problem is to figure out which opinion is dominant eventually in the population. Noteworthily, it is typically assumed that there is a unique Nash equilibrium, when problems in such multi-agent systems are addressed in a game-theoretical way. Our equivalence results, however, facilitate us to figure out the likelihood for opinion $+$ to win the vote based on existing results in evolutionary game theory.

Potential generalisations. The dyadic opinions can be violated. In fact Remark 2 can help us extend the equivalence results to the case with more than two opinions. Works along this line are in progress.

Limitations. The fast link rewiring process is the key technical assumption to ensure that the equivalence results hold, which is the limitation of our work.

7 Conclusions

Opinion dynamics and evolutionary game theory are two distinct fields. To the best of our knowledge, they have been studied separately up till now. In this paper, we have studied the classic models in the two fields, i.e., the voter model in opinion dynamics and the Moran process in evolutionary game theory. It is proven that the voter model on the evolving network is equivalent to the Moran process on the complete network both in fixation probability (Theorem 2) and in limiting behavior (Theorem 3), given that the link rewiring evolves much faster than the opinion updates. These two equivalence results bridge the gap between opinion dynamics and evolutionary game theory.

On the one hand, the bridge transforms the in-group bias [53] of the voter model into the coordination game, which is widely used in addressing the emergence and maintenance of the social convention [54]. On the other hand, the bridge deepens the understanding of the basic concepts in game theory, i.e., payoff as well as Nash equilibrium with the aid of the voter model on the evolving network.

References

[1] N. Wiener, *The Human Use of Human Beings: Cybernetics and Society*. Boston: Houghton Mifflin, 1950.

[2] A. V. Proskurnikov and R. Tempo. A tutorial on modeling and analysis of dynamic social networks. Part I. *Annu. Rev. Control*, 43:65-79, 2017.

[3] A. V. Proskurnikov and R. Tempo. A tutorial on modeling and analysis of dynamic social networks. Part II. *Annu. Rev. Control*, 45:166-190, 2018.

[4] D. Acemoglu and A. Ozdaglar, Opinion dynamics and learning in social networks, *Dyn. Games Appl.*, 1(1): 3–49, 2011.

[5] A. Mirtabatabaei and F. Bullo, Opinion dynamics in heterogeneous networks: Convergence conjectures and theorems, *SIAM J. Control Optim.*, 50: 2763–2785, 2012.

[6] X. Lin, Q. Jiao, and L. Wang, Opinion propagation over signed networks: Models and convergence analysis, *IEEE Trans. Autom. Control*, 64(8): 3431–3438, 2019.

[7] A. V. Proskurnikov, A. S. Matveev, and M. Cao, Opinion dynamics in social networks with hostile camps: Consensus vs. polarization, *IEEE Trans. Autom. Control*, 61(6): 1524–1536, 2016.

[8] J. Ghaderi and R. Srikant, Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate, *Automatica*, 50(12): 3209–3215, 2014.

[9] Y. Tian and L. Wang, Opinion dynamics in social networks with stubborn agents: An issue-based perspective, *Automatica*, 96: 213–223, 2018.

[10] A.-L. Barabási, Scale-free networks: A decade and beyond, *Science*, 325(5939): 412–413, 2009.

[11] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, Controllability of complex networks, *Nature*, 473: 167–173, 2011.

[12] F. Xiao and L. Wang, Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays, *IEEE Trans. Autom. Control*, 53(8): 1804–1816, 2008.

[13] G. Palla, I. Derényi, I. Farkas, and T. Vicsek, Uncovering the overlapping community structure of complex networks in nature and society, *Nature*, 435: 814–818, 2005.

[14] P. Michelucci and J. L. Dickinson, The power of crowds, *Science*, 351(6268): 32–33, 2016.

[15] A.-L. Barabási, *Network Science*. Cambridge: Cambridge University Press, 2016.

[16] A. Li, S. P. Cornelius, Y.-Y. Liu, L. Wang, and A.-L. Barabási, The fundamental advantages of temporal networks, *Science*, 358(6366): 1042–1046, 2017.

[17] M. Perc and A. Szolnoki, Coevolutionary games - A mini review, *BioSystems*, 99(2): 109–125, 2010.

[18] T. Gross and B. Blasius, Adaptive coevolutionary networks: A review, *J. R. Soc. Interface*, 5(20): 259–271, 2008.

[19] L. Wang and F. Xiao, Finite-time consensus problems for networks of dynamic agents, *IEEE Trans. Autom. Control*, 55(4): 950–955, 2010.

[20] B. Wu, D. Zhou, F. Fu, Q. Luo, L. Wang, and A. Traulsen, Evolution of cooperation on stochastical dynamical networks, *PLoS ONE*, 5(6): e11187, 2010.

[21] T. Wu, F. Fu, and L. Wang, Moving away from nasty encounters enhances cooperation in ecological prisoner's dilemma game, *PLoS ONE*, 6(11): e27669, 2011.

[22] J. Maynard Smith, *Evolution and the Theory of Games*. Cambridge: Cambridge University Press, 1982.

[23] J. W. Weibull, *Evolutionary Game Theory*. Cambridge: The MIT Press, 1995.

[24] J. Hofbauer and K. Sigmund, *Evolutionary Games and Population Dynamics*. Cambridge: Cambridge University Press, 1998.

[25] M. A. Nowak, A. Sasaki, C. Taylor, and D. Fudenberg, Emergence of cooperation and evolutionary stability in finite populations, *Nature*, 428(6983): 646–650, 2004.

[26] A. Traulsen, D. Semmann, R. D. Sommerfeld, et al., Human strategy updating in evolutionary games, *Proc. Natl. Acad. Sci. USA*, 107(7): 2962–2966, 2010.

[27] K. G. Vamvoudakis, H. Modares, B. Kiumarsi, and F. L. Lewis, Game theory-based control system algorithms with real-time reinforcement learning: how to solve multiplayer games online. *IEEE Control Syst. Mag.*, 37(1), 33-52, 2017.

[28] Q. Nicanor, O. M. Carlos, B. G. Julian et al. The role of population games and evolutionary dynamics in distributed control

systems: the advantages of evolutionary game theory. *IEEE Control Syst. Mag.*, 37(1), 70-97, 2017.
[29] Y. Huang, J. Chen, L. Huang, and Q. Zhu, Dynamic games for secure and resilient control system design. accepted by *Natl. Sci. Rev.*
[30] D. Cheng, F. He, H. Qi, and T. Xu, Modeling, analysis and control of networked evolutionary games, *IEEE Trans. Autom. Control*, 60(9): 2402–2415, 2015.
[31] D. Cheng, T. Xu, F. He, and H. Qi, On dynamics and Nash equilibriums of networked games, *IEEE/CAA J. Autom. Sin.*, 1(1): 10–18, 2014.
[32] D. Cheng, T. Xu, and H. Qi, Evolutionarily stable strategy of networked evolutionary games, *IEEE Trans. Neural Netw. Learn. Syst.*, 25(7): 1335–1345, 2014.
[33] R.-R. Zhang and L. Guo, Controllability of Nash equilibrium in game-based control systems, *IEEE Trans. Autom. Control*, 64(10): 4180–4187, 2019.
[34] R.-R. Zhang and L. Guo, Controllability of stochastic game-based control systems, *SIAM J. Control Optim.*, 57(6): 3799–3826, 2019.
[35] P. A. P. Moran, *The Statistical Processes of Evolutionary Theory*. Oxford: Clarendon, 1962.
[36] A. Traulsen, M. A. Nowak, and J. M. Pacheco, Stochastic dynamics of invasion and fixation, *Phys. Rev. E*, 74(1): 011909, 2006.
[37] B. Wu, P. M. Altrock, L. Wang, and A. Traulsen, Universality of weak selection, *Phys. Rev. E*, 82(4): 046106, 2010.
[38] L. A. Imhof and M. A. Nowak, Evolutionary game dynamics in a wright-fisher process, *J. Math. Biol.*, 52(5): 667–681, 2006.
[39] T. Antal, A. Traulsen, H. Ohtsuki, C. E. Tarnita, and M. A. Nowak, Mutation-selection equilibrium in games with multiple strategies, *J. Theor. Biol.*, 258(4): 614–622, 2009.
[40] J. Du, B. Wu, P. M. Altrock, and L. Wang, Aspiration dynamics of multi-player games in finite populations, *J. R. Soc. Interface*, 11(94): 20140077, 2014.
[41] A. Traulsen, J. C. Claussen, and C. Hauert, Coevolutionary dynamics: From finite to infinite populations, *Phys. Rev. Lett.*, 95(23): 238701, 2005.
[42] P. D. Taylor and L. B. Jonker, Evolutionarily stable strategies and game dynamics, *Math. Biosci.*, 40(1-2): 145–156, 1978.
[43] J. F. Nash, Equilibrium points in n-person games, *Proc. Natl. Acad. Sci. USA*, 36(1): 48–49, 1950.
[44] B. Wu, D. Zhou, and L. Wang, Evolutionary dynamics on stochastic evolving networks for multiple-strategy games, *Phys. Rev. E*, 84(4): 046111, 2011.
[45] L. E. Blume, The statistical mechanics of strategic interaction, *Games Econ. Behav.*, 5(3): 387–424, 1993.
[46] S. Karlin and H. M. Taylor, *A First Course in Stochastic Processes*. New York: Academic Press, 1975.
[47] X. Chen, F. Fu and L. Wang, Social tolerance allows cooperation to prevail in an adaptive environment. *Phy. Rev. E*, 80: 051104, 2009.
[48] R. Hegselmann and U. Krause, Opinion dynamics and bounded confidence: Models, analysis and simulation, *J. Artif. Soc. S.*, 5(3), 2002.
[49] A. Mirtabatabaei, P. Jia and F. Bullo, Eulerian opinion dynamics with bounded confidence and exogenous inputs, *SIAM J. Appl. Dyn. Syst.*, 13(1): 425–446, 2014.
[50] P. Dandekar, A. Goel and D. T. Lee, Biased assimilation, homophily, and the dynamics of polarization, *Proc. Natl. Acad. Sci. USA*, 110(15): 5791–5796, 2013.
[51] Z. Chen, J. Qin, B. Li, H. Qi, P. Buchhorn and G. Shi, Dynamics of opinions with social biases. *Automatica*, 106: 374–383, 2019.
[52] D. Bindel, J. Kleinberg and S. Oren, How bad is forming your own opinion? *Games Econ. Behav.*, 92: 248–265, 2015.
[53] F. Fu, C. E. Tarnita, N. A. Christakis, et al., Evolution of in-group favoritism, *Sci. Rep.*, 2: 460, 2012.
[54] T. C. Schelling, *Micromotives and Macrobehavior*. New York: W. W. Norton, 1978.