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Second-Order Controllability of Multi-Agent Systems with Multiple Leaders*

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Abstract *This paper proposes a new second-order continuous-time multi-agent model and analyzes the controllability of second-order multi-agent system with multiple leaders based on the asymmetric topology. This paper considers the more general case: velocity coupling topology is different from location coupling topology. Some sufficient and necessary conditions are presented for the controllability of the system with multiple leaders. In addition, the paper studies the controllability of the system with velocity damping gain. Simulation results are given to illustrate the correctness of theoretical results.*

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1 Introduction

The controllability of multi-agent systems is one of fundamental issues for coordinated control. Nowadays, the controllability problem of multi-agent systems is an important developing direction of control theory due to the wide application in communication, computation and cooperative control, etc.[1–22]

To date, study of the controllability of multi-agent systems remains to be a challenging task, which is due to the complexity of multi-agent systems such as dynamics of intelligent agents, information flows, and distributed control laws or protocols of networks. In 2004, Tanner[2] first studied the controllability of the first-order multi-agent system with a single leader based on nearest neighbor rules. In this model, the coupling weights are the same. A necessary and sufficient condition for the controllability was given. In Refs. [3–4], the authors studied the controllability of the system with switching topology and a single leader. Some necessary or sufficient controllable conditions for multi-agent systems are obtained. Furthermore, in Refs. [7, 11, 13] the controllability of multi-agent systems with multiple leaders based on fixed topology and switching topology was investigated, respectively. In Refs. [14–15], the controllability for multi-agent systems was studied from a graph-theoretic perspective. In Ref. [17], the authors studied the structural controllability of multi-agent systems. With the improvement of multi-agent systems' complexity, it is a key problem how to characterize the controllability-relevant topologies by designing appropriate protocols and algorithms, which can be described by single-integrator kinematics,[2] double-integrator dynamics,[19] and high-order-integrator dynamics,[10] respectively.

However, due to the complexity of multi-agent systems with high-order-integrator dynamics, the controllability of multi-agent dynamic systems was seldom studied. In fact, it is of physical interest and of theoretical interest to investigate the controllability for networks of high-order-integrator agents. For the controllability of the multi-agent system with different orders, the issue of controllability bears new features and difficulties, involving in how to define the controllability and to locate the leaders? Specially, as the dynamics of a network depends crucially on its interconnection topology, it is to regard that the controllability should rely on network connectivity or topology structure, which makes the controllability of the multi-agent system with high-order-integrator a nontrivial new problem. So there are very few results for the controllability of the multi-agent system with high-order[10] being available in the literature. From Theorem 3 and Theorem 7 in this paper, it can be found that the number of leaders has great influence on the controllability, as well as the positions and speeds of leaders have important effect on the

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controllability. Considering the physical and theoretical significance of high-order multi-agent systems, there exit some essential differences between the first-order, second-order and higher-order systems. In this paper, we focus on discussing the controllability of second-order continuous-time multi-agent systems with multiple leaders and general dynamic topology. Some sufficient and necessary conditions for controllability are presented. Compared to the existing works on the related problems,[2,10,19] the main contributions of the paper are threefold: (i) A novel model of continuous-time multi-agent system is second-order; (ii) The multiple leaders of continuous-time multi-agent systems are studied; (iii) The PBH test is introduced to justify the controllability of the system with general dynamics.

The rest of this paper is organized as follows. Section 2 states the model and introduces some preliminaries. In Sec. 3, we investigate the controllability of second-order continuous-time multi-agent systems and present the main results. Some simulations are presented in Sec. 4. Finally, the conclusion is given in Sec. 5.

2 Graph Theory Preliminaries and Model

An (undirected) graph $\mathscr{G}$ consists of a vertex set $V = \{1, 2, \ldots, n\}$ and an edge set $\mathscr{E} = \{(i, j) : i, j \in \mathscr{V}\}$, where an edge is an unordered pair of distinct vertices of $\mathscr{V}$. If $i, j \in \mathscr{V}$, and $(i, j) \in \mathscr{E}$, we say i and j are adjacent or j is a neighbor of i. The neighborhood set of node i is denoted by $\mathscr{N}_i = \{j \in \mathscr{V} : (i, j) \in \mathscr{E}\}$. $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{ij} > 0 \Leftrightarrow (j, i) \in \mathcal{E}$, and a_{ij} is called the coupling weight of edge (i, j). A diagonal matrix $D = \operatorname{diag}\{d_1, d_2, \ldots, d_n\} \in R^{n\times n}$ is a degree matrix of $\mathscr{G}$ with its diagonal elements $d_i = \sum_{j\in\mathcal{N}_i} a_{ij}$, $i = 1, 2, \ldots, n$. Then the Laplacian of the graph $\mathscr{G}$ (or matrix A) is defined as $L = D - A \in R^{n\times n}$.

A second-order continuous-time multi-agent network to be studied consists of $N + n_l$ dynamic agents interconnected through an information network, in which the agents indexed by $1, \ldots, N$ are assigned as the followers and the others indexed by $N+1, \ldots, N+n_l$ are referred to as leaders. A continuous-time model of the $N + n_l$ agents is described with double integrator dynamics

$$\dot{x}_i = v_i\,, \quad \dot{v}_i = u_i\,, \tag{1}$$

with

$$u_i = -\sum_{j\in\mathcal{N}_{i_j}} a_{ij}(v_i - v_j) - \sum_{p\in\mathcal{N}_{i_p}} \gamma_{ip} b_{ip}(v_i - v_p) - \sum_{j\in\mathcal{N}_{i_j}} c_{ij}(x_i - x_j) - \sum_{p\in\mathcal{N}_{i_p}} \varepsilon_{ip} d_{ip}(x_i - x_p) - k_i v_i\,, \tag{2}$$

where $x_i \in \mathbb{R}^m$ is the state of agent i ($i \in \underline{N} \triangleq \{1, 2, \ldots, N\}$), $x_p \in \mathbb{R}^m$ is the state of agent p ($p \in \underline{N+n_l} - \underline{N}$, $\underline{N+n_l} \triangleq \{1, \ldots, N+n_l\}$. $\mathscr{N}_i$ presents the neighbor set of agent i. $\mathscr{N}_{i_j} \bigcup \mathscr{N}_{i_p} = \mathscr{N}_i$, and $\mathscr{N}_{i_j} \bigcap \mathscr{N}_{i_p} = \emptyset$. The speed coupling matrix among the followers $\bar{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ with $a_{ij} \ge 0$ and $a_{ii} = 0$, the position coupling matrix among the followers $\bar{C} = [c_{ij}] \in \mathbb{R}^{N\times N}$ with $c_{ij} \ge 0$ and $c_{ii} = 0$; $\bar{B} = [b_{ip}] \in \mathbb{R}^{N\times n_l}$ with $b_{ip} > 0$ and $\bar{D} = [d_{ip}] \in \mathbb{R}^{N\times n_l}$ with $d_{ip} > 0$ represent the speed and position coupling strengths from the leaders to the followers. $\gamma_{ip} = 1$ and $\varepsilon_{ip} = 1$ if there has information from leader p to follower i; otherwise $\gamma_{ip} = 0$, $\varepsilon_{ip} = 0$. $-k_i v_i$ is the velocity damping term, if $k_i = 0$, then the system has no velocity damping gain.

Throughout this paper, it is assumed that the coupling weight of (i, j) is the same as that of (j, i), that is, $a_{ij} = a_{ji}$, $c_{ij} = c_{ji}$ for every pair of i, j.

Let $z = (v_1, \ldots, v_N, x_1, \ldots, x_N)^{\mathrm{T}}$, $y = (v_{N+1}, \ldots, v_{N+n_l}, x_{N+1}, \ldots, x_{N+n_l})^{\mathrm{T}}$ be the state vectors of all the followers and all the leaders, respectively. Then Eq. (1) can be rewritten as

$$\dot{z} = \hat{A} z + \hat{B} y\,, \tag{3}$$

where

$$\hat{A} \triangleq \begin{bmatrix} A - K & B \\ I_N & 0 \end{bmatrix}, \quad \hat{B} \triangleq \begin{bmatrix} P_v & P_x \\ 0 & 0 \end{bmatrix},$$

with $A = -L_v - R_v, B = -L_x - R_x$, where $L_v = [f_{ij}] \in \mathbb{R}^{N\times N}$ and $L_x = [h_{ij}] \in \mathbb{R}^{N\times N}$ with

$$f_{ij} = \begin{cases} -a_{ij}\,, & i \ne j \quad \text{and} \quad j \in \mathcal{N}_{i_j}\,, \\ \sum\limits_{j\in\mathcal{N}_{i_j}} a_{ij}\,, & i = j\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \qquad h_{ij} = \begin{cases} -c_{ij}\,, & i \ne j \quad \text{and} \quad j \in \mathcal{N}_{i_j}\,, \\ \sum\limits_{j\in\mathcal{N}_{i_j}} c_{ij}\,, & i = j\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

$$R_v = \operatorname{diag}\Big\{\sum_{p\in\mathcal{N}_{1_p}} \gamma_{1p} b_{1p}, \sum_{p\in\mathcal{N}_{2_p}} \gamma_{2p} b_{2p}, \ldots, \sum_{p\in\mathcal{N}_{N_p}} \gamma_{Np} b_{Np}\Big\} \in \mathbb{R}^{N\times N}\,,$$

$$R_x = \operatorname{diag}\Big\{\sum_{p\in\mathcal{N}_{1_p}} \varepsilon_{1p} d_{1p}, \sum_{p\in\mathcal{N}_{2_p}} \varepsilon_{2p} d_{2p}, \ldots, \sum_{p\in\mathcal{N}_{N_p}} \varepsilon_{Np} d_{Np}\Big\} \in \mathbb{R}^{N\times N}\,,$$

$$P_v = \begin{bmatrix} \gamma_{1(N+1)} b_{1(N+1)} & \gamma_{1(N+2)} b_{1(N+2)} & \cdots & \gamma_{1(N+n_l)} b_{1(N+n_l)} \\ \gamma_{2(N+1)} b_{2(N+1)} & \gamma_{2(N+2)} b_{2(N+2)} & \cdots & \gamma_{2(N+n_l)} b_{2(N+n_l)} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{N(N+1)} b_{N(N+1)} & \gamma_{N(N+2)} b_{N(N+2)} & \cdots & \gamma_{N(N+n_l)} b_{N(N+n_l)} \end{bmatrix} \in \mathbb{R}^{N\times n_l}\,,$$

$$P_x = \begin{bmatrix} \varepsilon_{1(N+1)}d_{1(N+1)} & \varepsilon_{1(N+2)}d_{1(N+2)} & \cdots & \varepsilon_{1(N+n_l)}d_{1(N+n_l)} \\ \varepsilon_{2(N+1)}d_{2(N+1)} & \varepsilon_{2(N+2)}d_{2(N+2)} & \cdots & \varepsilon_{2(N+n_l)}d_{2(N+n_l)} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{N(N+1)}d_{N(N+1)} & \varepsilon_{N(N+2)}d_{N(N+2)} & \cdots & \varepsilon_{N(N+n_l)}d_{N(N+n_l)} \end{bmatrix} \in \mathbb{R}^{N\times n_l},$$

and $K = \mathrm{diag}\,[k_1, \ldots, k_N]$. Since $a_{ij} = a_{ji}$, $c_{ij} = c_{ji}$, we can know that A and B are both symmetric matrices.

3 Main Results

3.1 *Ideal Case: $K = 0$*

In this section, we first give the definition of the controllability of second-order continuous-time multi-agent systems with multiple leaders.

Definition 1 A nonzero state z_0 of system (3) is controllable at the initial time t_0 if there exists a finite time t_f and a control input $y(t)$, such that $z(t_0) = z_0$ and $z(t_f) = 0$. If any nonzero state z_0 of system (3) is controllable, then system (3) is said to be controllable. If $x(t_0) = x_0$ and $x(t_f) = 0$, then system (3) is position controllable, and if $v(t_0) = v_0$ and $v(t_f) = 0$, then system (3) is speed controllable.

Definition 2[5] (Controllability matrix) The controllability matrix of system (3) is

$$Q = [\hat{B} \vdots \hat{A}\hat{B} \vdots \hat{A}^2\hat{B} \vdots \cdots \vdots \hat{A}^{(2N-1)}\hat{B}]\,.$$

Lemma 1[5] (Rank criterion for controllability) system (3) is controllable if $\mathrm{rank}\,(Q) = 2N$.

Lemma 2[5] (PBH rank criterion) system (3) is controllable if (3) satisfies one of the following conditions

(i) $\mathrm{rank}\,[\lambda_i I - \hat{A}, \hat{B}] = 2N$, where λ_i is eigenvalue of $\hat{A}$ for $\forall i = 1, 2, \ldots, 2N$.

(ii) $\mathrm{rank}\,[sI - \hat{A}, \hat{B}] = 2N$, $\forall s \in \mathbb{C}$.

From lemmas 1–2, we can derive the following main result for system (3).

Theorem 1 System (3) is controllable if the following conditions are satisfied:

(i) The eigenvalues of $\hat{A}$ are all distinct;

(ii) All the row vectors of U^{-1} are not orthogonal to at least one column in $\hat{B}$ simultaneously, where U is composed of the eigenvectors of $\hat{A}$.

Proof If the eigenvalues of $\hat{A}$ are all distinct, which can be expressed as $\hat{A} = U\hat{D}U^{-1}$, where $\hat{D} = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_{2N})$ and U composed of the eigenvectors of $\hat{A}$. Thus controllability matrix Q is

$$\begin{aligned} Q &= [\hat{B} \vdots (U\hat{D}U^{-1})\hat{B} \vdots (U\hat{D}U^{-1})^2\hat{B} \vdots \cdots \vdots (U\hat{D}U^{-1})^{(2N-1)}\hat{B}] \\ &= [\hat{B} \vdots U\hat{D}U^{-1}\hat{B} \vdots U\hat{D}^2U^{-1}\hat{B} \vdots \cdots \vdots U\hat{D}^{(2N-1)}U^{-1}\hat{B}] \\ &= U[U^{-1}\hat{B} \vdots \hat{D}U^{-1}\hat{B} \vdots \hat{D}^2U^{-1}\hat{B} \vdots \cdots \vdots \hat{D}^{(2N-1)}U^{-1}\hat{B}]\,. \end{aligned} \tag{4}$$

Since U is nonsingular. Thus, we only need to know the rank of the following matrix

$$\tilde{Q} = [U^{-1}\hat{B} \vdots \hat{D}U^{-1}\hat{B} \vdots \hat{D}^2U^{-1}\hat{B} \vdots \cdots \vdots \hat{D}^{(2N-1)}U^{-1}\hat{B}]\,,$$

which is the controllability matrix of the decoupled system

$$\dot{w} = \hat{D}w + U^{-1}\hat{B}u\,. \tag{5}$$

Now, we analyze the controllability of system (5). Let

$$U^{-1} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{2N} \end{bmatrix} \in \mathbb{R}^{2N\times 2N}\,,$$

where η_i $(i = 1, 2, \ldots, 2N)$ are all the row vectors of U^{-1}, and

$$\hat{B} = [\beta_{N+1}, \beta_{N+2}, \ldots, \beta_{N+n_l}, \delta_{N+1}, \delta_{N+2}, \ldots, \delta_{N+n_l}] \in \mathbb{R}^{2N\times 2n_l}\,,$$

where

$$\beta_{N+j} = \begin{bmatrix} \gamma_{1(N+j)}b_{1(N+j)} \\ \gamma_{2(N+j)}b_{2(N+j)} \\ \vdots \\ \gamma_{N(N+j)}b_{N(N+j)} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{2N\times 1}, \quad \delta_{N+j} = \begin{bmatrix} \varepsilon_{1(N+j)}d_{1(N+j)} \\ \varepsilon_{2(N+j)}d_{2(N+j)} \\ \vdots \\ \varepsilon_{N(N+j)}d_{N(N+j)} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{2N\times 1} \quad \text{for} \quad \forall j = 1, 2, \ldots, n_l\,.$$

Then,

$$\begin{aligned}\tilde{Q} &= [U^{-1}\hat{B} \vdots \hat{D}U^{-1}\hat{B} \vdots \hat{D}^2U^{-1}\hat{B} \vdots \cdots \vdots \hat{D}^{(2N-1)}U^{-1}\hat{B}] \\ &= \left[\begin{bmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_{2N}\end{bmatrix}[\beta_{N+1},\beta_{N+2},\ldots,\beta_{N+n_l},\delta_{N+1},\delta_{N+2},\ldots,\delta_{N+n_l}] \vdots\right. \\ &\quad \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_{2N}\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_{2N}\end{bmatrix}[\beta_{N+1},\beta_{N+2},\ldots,\beta_{N+n_l},\delta_{N+1},\delta_{N+2},\ldots,\delta_{N+n_l}] \vdots \cdots \\ &\quad \left.\cdots \vdots \begin{bmatrix}\lambda_1^{2N-1} & 0 & \cdots & 0\\ 0 & \lambda_2^{2N-1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_{2N}^{2N-1}\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_{2N}\end{bmatrix}[\beta_{N+1},\beta_{N+2},\ldots,\beta_{N+n_l},\delta_{N+1},\delta_{N+2},\ldots,\delta_{N+n_l}]\right],\end{aligned}$$

where λ_i ($\forall i = 1,2,\ldots,2N$) is the eigenvalues of $\hat{A}$, and

$$\hat{D} = \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_{2N}\end{bmatrix}.$$

By elementary column transformation of matrix, we can get

$$\begin{aligned}\tilde{Q} \longrightarrow &\left[\begin{bmatrix}(\eta_1,\beta_{N+1}) & & & \\ & (\eta_2,\beta_{N+1}) & & \\ & & \ddots & \\ & & & (\eta_{2N},\beta_{N+1})\end{bmatrix} M \vdots \begin{bmatrix}(\eta_1,\beta_{N+2}) & & & \\ & (\eta_2,\beta_{N+2}) & & \\ & & \ddots & \\ & & & (\eta_{2N},\beta_{N+2})\end{bmatrix} M \vdots \cdots\right. \\ &\cdots \vdots \begin{bmatrix}(\eta_1,\beta_{N+n_l}) & & & \\ & (\eta_2,\beta_{N+n_l}) & & \\ & & \ddots & \\ & & & (\eta_{2N},\beta_{N+n_l})\end{bmatrix} M \vdots \begin{bmatrix}(\eta_1,\delta_{N+1}) & & & \\ & (\eta_2,\delta_{N+1}) & & \\ & & \ddots & \\ & & & (\eta_{2N},\delta_{N+1})\end{bmatrix} M \vdots \\ &\left.\begin{bmatrix}(\eta_1,\delta_{N+2}) & & & \\ & (\eta_2,\delta_{N+2}) & & \\ & & \ddots & \\ & & & (\eta_{2N},\delta_{N+2})\end{bmatrix} M \vdots \cdots \vdots \begin{bmatrix}(\eta_1,\delta_{N+n_l}) & & & \\ & (\eta_2,\delta_{N+n_l}) & & \\ & & \ddots & \\ & & & (\eta_{2N},\delta_{N+n_l})\end{bmatrix} M\right],\end{aligned}$$

where (η_i,β_{N+j}) and (η_i,δ_{N+j}) are vector inner product for $i = 1,2,\ldots,2N$, $j = 1,2,\ldots,n_l$, and M is the transposed matrix of Vandermonde matrix

$$M = \begin{bmatrix}1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{2N-1}\\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{2N-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & \lambda_{2N} & \lambda_{2N}^2 & \cdots & \lambda_{2N}^{2N-1}\end{bmatrix}.$$

Thus $\tilde{Q}$ has full row rank if any one block matrix has row full rank. Without loss of generality, we analyze the first block

$$\begin{bmatrix}(\eta_1,\beta_{N+1}) & & & \\ & (\eta_2,\beta_{N+1}) & & \\ & & \ddots & \\ & & & (\eta_{2N},\beta_{N+1})\end{bmatrix} M.$$

M is row full rank if and only if $\lambda_i \neq \lambda_j$ when $i \neq j$ for $i,j = 1,2,\ldots,2N$. By condition (i), we know that M has row full rank. At the same time, it is obvious that matrix

$$\begin{bmatrix}(\eta_1,\beta_{N+1}) & & & \\ & (\eta_2,\beta_{N+1}) & & \\ & & \ddots & \\ & & & (\eta_{2N},\beta_{N+1})\end{bmatrix},$$

has row full rank when all elements on the diagonal are nonzero, i.e., η_i (the row vectors of U^{-1}, $i=1,2,\ldots,2N$) are not orthogonal to β_{N+1} simultaneously. By condition (ii), we can know the above matrix has full row rank. Therefore, $\mathrm{rank}\,(\tilde{Q}) = 2N$, i.e., $\mathrm{rank}\,(Q) = 2N$. Then system (5) is controllable, and then system (3) is controllable.

Remark 1 Notice that although it is in general a sufficient condition, Theorem 1 is simpler and more easily checkable, and it does not require the system matrix $\hat{A}$ is symmetric. In most recent references such as Refs. [2–3, 11] require the system matrix $\hat{A}$ is symmetric. This merit is very desirable in applications as it can provide more freedom in the design of a network.

From Theorem 1, we can have the following result immediately.

Theorem 2 System (3) is controllable if the following conditions are satisfied:

(i) The eigenvalues of $\hat{A}$ are all distinct;

(ii) Every row of $U^{-1}\hat{B}$ has at least one nonzero element, where U is composed of the eigenvectors of $\hat{A}$.

Proof For system (5), we can have

$$[\lambda_i I_{2N} - \hat{D}, U^{-1}\hat{B}] = \begin{bmatrix} \lambda_i-\lambda_1 & 0 & \cdots & 0 & \hat{b}_{1,1} & \hat{b}_{1,2} & \cdots & \hat{b}_{1,2n_l} \\ 0 & \lambda_i-\lambda_2 & \cdots & 0 & \hat{b}_{2,1} & \hat{b}_{2,2} & \cdots & \hat{b}_{2,2n_l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i-\lambda_{2N} & \hat{b}_{2N,1} & \hat{b}_{2N,2} & \cdots & \hat{b}_{2N,2n_l} \end{bmatrix}, \tag{6}$$

where λ_i $(i=1,2,\ldots,2N)$ are the eigenvalues of $\hat{A}$, and

$$\hat{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{2N} \end{bmatrix},$$

$$U^{-1}\hat{B} = \begin{bmatrix} \hat{b}_{1,1} & \hat{b}_{1,2} & \cdots & \hat{b}_{1,2n_l} \\ \hat{b}_{2,1} & \hat{b}_{2,2} & \cdots & \hat{b}_{2,2n_l} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{b}_{2N,1} & \hat{b}_{2N,2} & \cdots & \hat{b}_{2N,2n_l} \end{bmatrix}.$$

Because $\lambda_i, (i=1,\ldots,2N)$ are all distinct, then i-th element of $\lambda_i I_{2N} - \hat{D}$ is 0, but the others are not equal to 0. And by condition (ii), we know every row of $U^{-1}\hat{B}$ has at least one nonzero element. According to lemma 2, system (5) is controllable. Hence, system (3) is controllable. ■

Remark 2 Notice that Theorem 2 is simpler and more easily checkable than Theorem 1 by computing.

For the convenience of calculation, we can also have the eigenpolynomial of $\hat{A}$ as follows:

$$\begin{aligned} f(\lambda) &= |\lambda I_{2N} - \hat{A}| = \begin{vmatrix} \lambda I_N - A & -B \\ -I_N & \lambda I_N \end{vmatrix} \\ &= (-1)^N \begin{vmatrix} -I_N & \lambda I_N \\ \lambda I_N - A & -B \end{vmatrix} \\ &= (-1)^N \begin{vmatrix} -I_N & 0 \\ \lambda I_N - A & \lambda^2 I_N - \lambda A - B \end{vmatrix} \\ &= (-1)^{2N} |\lambda^2 I_N - \lambda A - B| = |\lambda^2 I_N - \lambda A - B|\,. \end{aligned}$$

In the following, we can obtain simpler and more easily checkable results.

Theorem 3 System (3) is controllable if system (3) satisfies one of the following conditions

(i) $\mathrm{rank}\,[P_x] = N$;

(ii) $\mathrm{rank}\,[P_v] = N$;

(iii) $\mathrm{rank}\,[P_v, P_x] = N$;

(iv) $\mathrm{rank}\,[\lambda_i^2 I_N - \lambda_i A - B, P_v] = N$;

(v) $\mathrm{rank}\,[\lambda_i^2 I_N - \lambda_i A - B, P_x] = N$.

Proof From Lemma 2, we can have

$$\begin{aligned} &\mathrm{rank}\,[\lambda_i I_{2N} - \hat{A}, \hat{B}] \\ &= \mathrm{rank}\begin{bmatrix} \lambda_i I_N - A & -B & P_v & P_x \\ -I_N & \lambda_i I_N & 0 & 0 \end{bmatrix} \\ &= \mathrm{rank}\begin{bmatrix} \lambda_i I_N - A & \lambda_i^2 I_N - \lambda_i A - B & P_v & P_x \\ -I_N & 0 & 0 & 0 \end{bmatrix} \\ &= \mathrm{rank}\begin{bmatrix} 0 & \lambda_i^2 I_N - \lambda_i A - B & P_v & P_x \\ -I_N & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

where λ_i $(\forall i = 1,2,\ldots,2N)$ are eigenvalues of $\hat{A}$.

Obviously, if system (3) satisfies one of the conditions (i), (ii), (iii), (iv), (v), according to Lemma 2, system (3) is controllable.

Specially, if $a_{ij} = c_{ij}$, $b_{ip} = d_{ip}$, $\gamma_{ip} = \varepsilon_{ip}$ for $i = 1,2,\ldots,N, j \in \mathcal{N}_{i_j}$, $p \in \mathcal{N}_{i_p}$, it is easy to get $A = B$ and $P_v = P_x$. We can have more special results in the following.

Theorem 4 System (3) with $A = B$ is controllable if the following conditions are satisfied:

(i) $(a_i \pm \sqrt{a_i^2 + 4a_i})/2$ $(\forall i = 1,\ldots,N)$ are all distinct, where a_i are the eigenvalues of A;

(ii) Every row of $U^{-1}\hat{B}$ has at least one nonzero element, where U is composed of the eigenvectors of $\hat{A}$.

Proof Since $A^{\mathrm{T}} = A$, then there must exist an orthogonal matrix $T \in \mathbb{R}^{N\times N}$, whose columns are composed of the eigenvectors of A, such that

$$T^{\mathrm{T}}AT = \mathrm{diag}\,(a_1, a_2, \ldots, a_N)\,,$$

where $T^{\mathrm{T}} = T^{-1}$. Because $A = B$, we have

$$T^{\mathrm{T}}BT = \mathrm{diag}\,(a_1, a_2, \ldots, a_N)\,.$$

Considering $f(\lambda)$, we can have

$$\begin{aligned} f(\lambda) &= |\lambda^2 I_N - \lambda A - B| \\ &= \begin{vmatrix} \lambda^2 - a_1\lambda - a_1 & & \\ & \ddots & \\ & & \lambda^2 - a_N\lambda - a_N \end{vmatrix} \end{aligned}$$

$$= \prod_{i=1}^{N}(\lambda^2 - a_i\lambda - a_i)$$
$$= \prod_{i=1}^{N}\left(\lambda - \frac{a_i + \sqrt{a_i^2 + 4a_i}}{2}\right) \times \left(\lambda - \frac{a_i - \sqrt{a_i^2 + 4a_i}}{2}\right).$$

Obviously, $(a_i \pm \sqrt{a_i^2 + 4a_i})/2$ $(\forall i = 1, \ldots, N)$ are the eigenvalues of $\hat{A}$. From Theorem 1, system (3) is controllable if conditions (i) and (ii) hold. ■

3.2 *General Case: $K \neq 0$*

In this section, we consider system (3) has the velocity damping term, i.e., $K \neq 0$. Similar to the ideal case, we also have the following results.

Theorem 5 System (3) is controllable if the following conditions are satisfied:

(i) The eigenvalues of $\hat{A}$ are all distinct;

(ii) All the row vectors of U^{-1} are not orthogonal to at least one column in $\hat{B}$ simultaneously, where U is composed of the eigenvectors of $\hat{A}$.

Theorem 6 System (3) is controllable if the following conditions are satisfied:

(i) The eigenvalues of $\hat{A}$ are all distinct;

(ii) Every row of $U^{-1}\hat{B}$ has at least one nonzero element, where U is composed of the eigenvectors of $\hat{A}$.

Theorem 7 System (3) is controllable if system (3) satisfies one of the following conditions

(i) $\text{rank}\,[P_x] = N$;

(ii) $\text{rank}\,[P_v] = N$;

(iii) $\text{rank}\,[P_v, P_x] = N$;

(iv) $\text{rank}\,[\lambda_i^2 I_N - \lambda_i(A - K) - B, P_v] = N$;

(v) $\text{rank}\,[\lambda_i^2 I_N - \lambda_i(A - K) - B, P_x] = N$.

Remark 3 In Ref. [2], Tanner studied a first-order model of the multi-agent system with a single leader acting as the input. In Refs. [10] and [19], the authors considered the simple second-order multi-agent system with a single leader, while this paper has investigated a novel model of continuous-time multi-agent system with multiple leaders which is not only second-order, but also has the speed damping. Therefore, the controllability matrix in this paper is more complex than those of Refs. [10] and [19], and can be transformed into a controllability matrix of the decoupled system, which is a hard point in analyzing. In addition, by computing the eigenpolynomial of the system $\hat{A}$, the influences of speeds and the speed damping for the controllability are studied. Furthermore, it is to analyze the number, the positions and speeds of leaders influencing on the controllability in Theorem 3 and Theorem 7. All of these are seldom involved to the existing literature Refs. [2], [10], and [19]. It is evident that networks with first-order and a single leader studied in Ref. [2], and networks with second-order and a single leader studied in Refs. [10] and [19] belong to the special cases of the model in this paper, which brings the two kinds of networks into a unified framework.

4 Numerical Examples and Simulations

In this section, we give some examples to illustrate the effective of the proposed theoretical results.

4.1 *Example 1*

Consider a five-agent network with agents 4 and 5 as the leaders and with a fixed topology described by Fig. 1, where the solid lines show the information link of position between two agents and the dotted lines show the information link of velocity between two agents.

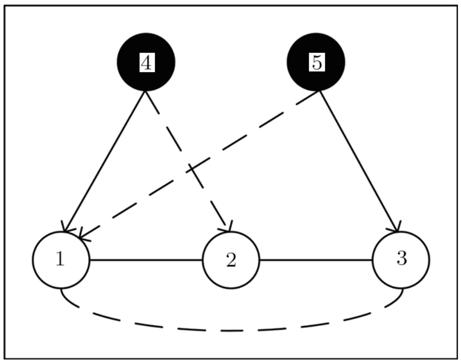

Fig. 1 The topology $\mathscr{G}$.

The system (3) is defined by

$$\hat{A} = \begin{bmatrix} -7 & 0 & 3 & -7 & 1 & 0 \\ 0 & -5 & 0 & 1 & -3 & 2 \\ 3 & 0 & -3 & 0 & 2 & -9 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 & 4 & 6 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

By computing using *Matlab*, the eigenvalues of $\hat{A}$ are

$$\lambda = \{-7.6316, -0.7527 + 2.8854\mathrm{i}, -0.7527 - 2.8854\mathrm{i}, -4.3660, -0.9658, -0.5312\},$$

$$U^{-1}\hat{B} = \begin{bmatrix} 0.0073 - 0.0000\mathrm{i} & -4.1080 + 0.0000\mathrm{i} & -6.1620 + 0.0000\mathrm{i} & 3.7119 + 0.0000\mathrm{i} \\ -0.2126 - 0.4104\mathrm{i} & 0.9982 + 0.1530\mathrm{i} & 1.4972 + 0.2295\mathrm{i} & 3.2915 + 0.8139\mathrm{i} \\ -0.2126 + 0.4104\mathrm{i} & 0.9982 - 0.1530\mathrm{i} & 1.4972 - 0.2295\mathrm{i} & 3.2915 - 0.8139\mathrm{i} \\ -5.9081 - 0.0000\mathrm{i} & -0.2222 - 0.0000\mathrm{i} & -0.3333 - 0.0000\mathrm{i} & -0.7651 - 0.0000\mathrm{i} \\ -0.1152 + 0.0000\mathrm{i} & 0.7656 - 0.0000\mathrm{i} & 1.1483 - 0.0000\mathrm{i} & -0.5976 - 0.0000\mathrm{i} \\ -1.4775 + 0.0000\mathrm{i} & -0.2140 - 0.0000\mathrm{i} & -0.3210 - 0.0000\mathrm{i} & -0.4604 - 0.0000\mathrm{i} \end{bmatrix}.$$

It is easy to find that the eigenvalues of $\hat{A}$ are all distinct, and every row of $U^{-1}\hat{B}$ has at least one nonzero element. According to Theorem 2, system 3 is controllable.

Figures 2–3 show the simulation results of formation control of the second-order continuous-time network. The follower agents (the black star dots) move from a random initial configuration to desired ones: aligning in a straight line (the circles) with controllable velocity as shown in Figs. 2 and 3, respectively.

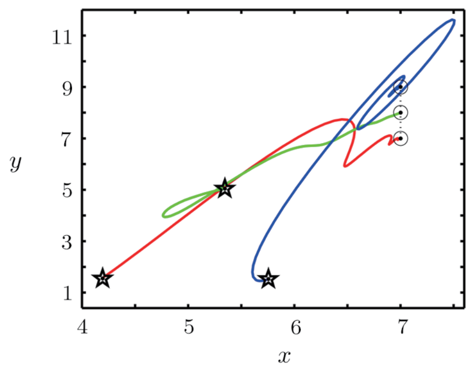

Fig. 2 Align in a straight line.

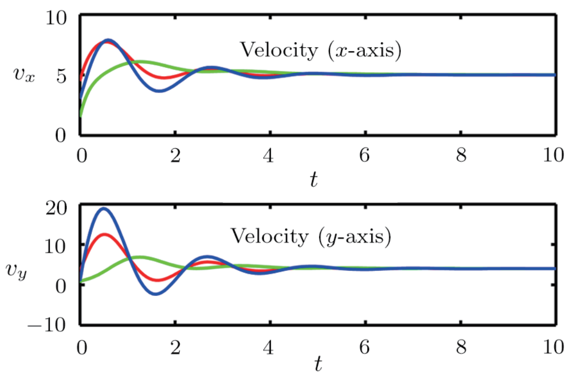

Fig. 3 Velocity controllability.

4.2 *Example 2*

Consider a five-agent network with agents 4 and 5 as the leaders and with a fixed topology described by Fig. 1, if the system has velocity damping term. Let $K = \mathrm{diag}\,(1, 1, 2)$, and the system is defined by

$$\hat{A} = \begin{bmatrix} -8 & 0 & 3 & -7 & 1 & 0 \\ 0 & -6 & 0 & 1 & -3 & 2 \\ 3 & 0 & -5 & 0 & 2 & -9 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 & 4 & 6 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similar to example 1, by computing using *Matlab*, the eigenvalues of $\hat{A}$ are

$$\lambda = \{-9.0181, -5.4963, -1.6234 + 2.5157\mathrm{i}, -1.6234 - 2.5157\mathrm{i}, -0.8234, -0.4155\},$$

$$\hat{U}^{-1}\hat{B} = \begin{bmatrix} -0.0310 + 0.0000\mathrm{i} & 3.8056 - 0.0000\mathrm{i} & 5.7084 - 0.0000\mathrm{i} & -3.9849 - 0.0000\mathrm{i} \\ -5.5304 - 0.0000\mathrm{i} & -0.2677 - 0.0000\mathrm{i} & -0.4016 - 0.0000\mathrm{i} & -0.6617 - 0.0000\mathrm{i} \\ 0.2299 + 0.5012\mathrm{i} & -1.1029 - 0.5871\mathrm{i} & -1.6543 - 0.8806\mathrm{i} & -3.1443 - 2.0317\mathrm{i} \\ 0.2299 - 0.5012\mathrm{i} & -1.1029 + 0.5871\mathrm{i} & -1.6543 + 0.8806\mathrm{i} & -3.1443 + 2.0317\mathrm{i} \\ -0.0408 - 0.0000\mathrm{i} & 0.5792 + 0.0000\mathrm{i} & 0.8688 + 0.0000\mathrm{i} & -0.4707 + 0.0000\mathrm{i} \\ -1.0508 + 0.0000\mathrm{i} & -0.1502 + 0.0000\mathrm{i} & -0.2253 + 0.0000\mathrm{i} & -0.3685 \end{bmatrix}.$$

we can also find the eigenvalues of $\hat{A}$ are all distinct, and every row of $U^{-1}\hat{B}$ has at least one nonzero element. According to Theorem 6, system (3) is controllable.

Figures 4–5 show the simulation results of formation control of the second-order continuous-time network with velocity damping term. The follower agents (the black star dots) move from a random initial configuration to desired ones: forming a regular triangle (the circles) with controllable velocity as shown in Figs. 4 and 5, respectively.

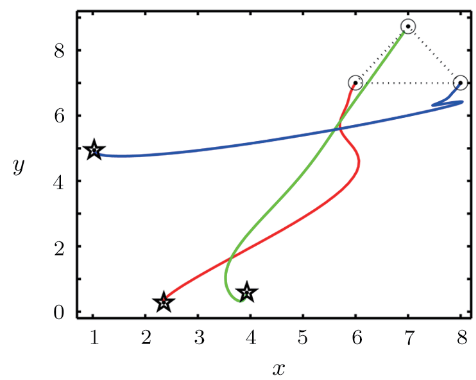

Fig. 4 Form a regular triangle with velocity damping term.

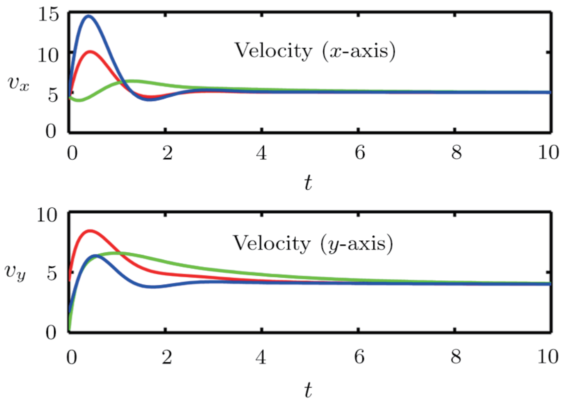

Fig. 5 Velocity controllability with velocity damping term.

5 Conclusion

In this paper, we have investigated the controllability of second-order continuous-time multi-agent systems with multiple leaders and velocity damping term based on the asymmetric topology. By the PBH rank test, some sufficient and necessary conditions for the controllability of such system are obtained. The results in this paper give some simpler and more easily checkable methods to prove the controllability and provide further insight into the effect of the mutual interaction patterns on the collective motion of a multi-agent system. Future work will consider the controllability of higher-order multi-agent system.

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