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Topological Characterizations of Leader-Follower Controllability on Signed Path and Cycle Networks

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***Abstract*—Leader-follower controllability of signed multi-agent networks is investigated in this paper, where the agents interact via neighbor-based Laplacian feedback and the interactions between agents admit positive and negative weights capturing cooperative and competitive interactions. To enable full control of the leader-follower signed network, graph-inspired topological characterizations of the controllability of signed networks are investigated. Specifically, sufficient conditions on the controllability of signed path and cycle networks are developed based on the investigation of the interaction between network topology and agent dynamics. Constructive examples are provided to illustrate how the developed controllability result on signed path and cycle networks can be potentially extended to general signed networks.**

I. Introduction

Leader-follower multi-agent systems have been increasingly applied in science and engineering. Typical applications of leader-follower systems include distributed coordination in robotic networks [1], formation and propagation of opinions in social networks [2], and analysis of biochemical reaction in biological networks [3]. The success of these applications relies on the capability of driving the network to a desired state by external controls via selected leaders, i.e., network controllability. Since signed networks that admit cooperative and antagonistic interactions can be potentially applied in engineering [4] and social networks [5] to capture both friendly and adversary relationships, the study of the controllability of signed networks has attracted increasing research attention recently. However, the interactions between network controllability and the underlying topological structure over signed graphs are still largely unexplored. Therefore, this paper is particularly motivated to investigate the controllability of signed networks from topological perspectives.

Leader-follower controllability of unsigned graphs was considered for the first time in [6], based on which graph theoretical approaches were then explored. For instance, equitable graph partition and graph symmetry were employed in [7] to characterize uncontrollability of unsigned graphs. Graph-distance based lower bounds on the rank of the controllability matrix were developed in [8]. Graphical and topological characterizations of network controllability were investigated in [9]. Besides the consideration of general unsigned networks, controllability of networks with particular topological structures also attracts significant research attention. In [10], controllability properties of circulant networks were developed. Necessary and sufficient conditions on unsigned path and cycle graph were developed via number theory in [11]. Characterizations of controllability were investigated on tree graphs [12], grid graphs [13], and lattice graphs [14]. However, the aforementioned results mainly focused on the characterization of network controllability on unsigned graphs. Few existing research considers signed networks.

Although controllability of signed graphs were partially studied via structural balance in recent works in [15] and [16], topological characterizations of controllability of signed graph are still largely unexplored. Since path and cycle graphs are basic building blocks for a variety of complex graphs, controllability of signed path and cycle graphs are particularly investigated in this work. Illustrative examples are provided showing how the controllability result developed for signed path and cycle graphs can be potentially extended for general signed graphs. The contributions of this work are multi-fold. First, in contrast to most existing matrix-theoretical approaches to characterize network controllability, graph-inspired understandings of network controllability are realized in this work. Specifically, we investigate the relationship between the network controllability and the underlying topology and characterize how leader-to-leader and leader-to-follower connections affect the controllability of a signed network with Laplacian dynamics. Second, topological sufficient conditions on the controllability of signed path, cycle, tree graphs are developed, which relaxes the constraint of requiring structural balance on the network topology as in previous research. Third, constructive examples are provided to illustrate how the developed controllability result on signed path and cycle networks can be potentially extended to general signed networks.

II. Problem Formulation

Consider a multi-agent network represented by an undirected signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, where the node set $\mathcal{V} = \{v_1, \dots, v_n\}$ and the edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ represents the agents and the interactions between pairs of agents, respectively. The potential interactions among agents are captured by the adjacency matrix $\mathcal{W} \in \mathbb{R}^{n \times n}$, where $w_{ij} \neq 0$ if $(v_i, v_j) \in \mathcal{E}$ and $w_{ij} = 0$ otherwise. No self-loop is considered, i.e., $w_{ii} = 0 \ \forall i = 1, \dots, n$. Different from classical unsigned graphs that contain non-

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negative adjacency matrix, $w_{ij} \colon \mathcal{E} \rightarrow \{\pm 1\}$ in this work admits positive or negative weight to capture collaborative or competitive relationships between agents, thus resulting in a signed graph $\mathcal{G}$. Specifically, v_i and v_j are called positive neighbors of each other if $w_{ij} = 1$ and negative neighbors if $w_{ij} = -1$. A path of length $k-1$ in $\mathcal{G}$ is a concatenation of edges $\{(v_1, v_2), (v_2, v_3), \cdots, (v_{k-1}, v_k)\} \subset \mathcal{E}$. A cycle is a path with identical starting and end node, i.e, $v_1 = v_k$. Graph $\mathcal{G}$ is connected if there exists a path between any pair of nodes in $\mathcal{V}$. The neighbor set of v_i is defined as $\mathcal{N}_i = \{v_j | (v_i, v_j) \in \mathcal{E}\}$, and the degree of v_i, denoted as $d_i \in \mathbb{Z}^+$, is defined as the number of its neighbors, i.e., $d_i = |\mathcal{N}_i| = \sum_{j \in \mathcal{N}_i} \operatorname{abs}(w_{ij})$, where $|\mathcal{N}_i|$ denotes the cardinality of $\mathcal{N}_i$ and $\operatorname{abs}(w_{ij})$ denotes the absolute value of w_{ij}. The signed graph Laplacian of $\mathcal{G}$ is defined as $\mathcal{L}(\mathcal{G}) \triangleq \mathcal{D} - \mathcal{W}$, where $\mathcal{D} \triangleq \operatorname{diag}\{d_1, \ldots, d_n\}$ is a diagonal matrix.

Let $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ denote the stacked system states, where each entry $x_i(t) \in \mathbb{R}$ represents the state of agent v_i. Suppose the system states evolve according to the following Laplacian dynamics,

$$\dot{x}(t) = -\mathcal{L}(\mathcal{G}) x(t), \tag{1}$$

where the graph Laplacian $\mathcal{L}(\mathcal{G})$ indicates that each agent v_i updates its state x_i only taking into account the states of its neighboring agents, i.e., $x_j \in \mathcal{N}_i$. Suppose the agent set $\mathcal{V}$ is classified into a leader set $\mathcal{V}_l \subset \mathcal{V}$ and a follower set $\mathcal{V}_f \subset \mathcal{V}$ with $\mathcal{V}_l \cup \mathcal{V}_f = \mathcal{V}$, thus forming a typical leader-follower network. Without loss of generality, assume that the first m agents form the follower set $\mathcal{V}_f = \{v_1, \ldots, v_m\}$, while the remaining agents form the leader set $\mathcal{V}_l = \{v_{m+1}, \ldots, v_n\}$. Let $x(t) = \left[x_f^T(t), x_l^T(t)\right]^T \in \mathbb{R}^n$ be the aggregated system states, where $x_f(t) \in \mathbb{R}^m$ and $x_l(t) \in \mathbb{R}^{n-m}$ represent the aggregated states of followers and leaders, respectively. Similar to [7], the graph Laplacian in (1) can be partitioned as

$$\mathcal{L}(\mathcal{G}) = \left[\begin{array}{c|c} \mathcal{L}_f(\mathcal{G}) & \mathcal{L}_{fl}(\mathcal{G}) \\ \hline \mathcal{L}_{lf}(\mathcal{G}) & \mathcal{L}_l(\mathcal{G}) \end{array}\right] \tag{2}$$

with $\mathcal{L}_f(\mathcal{G}) \in \mathbb{R}^{m \times m}$, $\mathcal{L}_{fl}(\mathcal{G}) = \mathcal{L}_{lf}^T(\mathcal{G}) \in \mathbb{R}^{m \times (n-m)}$, and $\mathcal{L}_l(\mathcal{G}) \in \mathbb{R}^{(n-m) \times (n-m)}$. Based on (1) and (2), the dynamics of the followers become

$$\dot{x}_f(t) = -\mathcal{L}_f(\mathcal{G}) x_f - \mathcal{L}_{fl}(\mathcal{G}) u(t), \tag{3}$$

where $u(t) \triangleq x_l(t)$ denotes the exogenous control signal dictated by the leaders.

Definition 1 (Leader-Follower Controllability)**.** Provided that the leaders are completely controllable and dictated by exogenous input $u(t)$, a leader-follower network with dynamics of (1) is called controllable, if the followers' state $x_f(t)$ in (3) can be driven to any target state by a proper design of $u(t)$. Mathematically, if the controllability matrix $\mathcal{C} = \left[\begin{array}{cccc} -\mathcal{L}_{fl} & \mathcal{L}_f \mathcal{L}_{fl} & \cdots & (-1)^m \mathcal{L}_f^{m-1} \mathcal{L}_{fl} \end{array}\right]$ has full row rank, the leader-follower system in (3) is controllable.

From Definition 1, the leader-follower controllability is dependent on the system matrices $\mathcal{L}_f$ and $\mathcal{L}_{fl}$ in (3). Since $\mathcal{L}(\mathcal{G})$ is determined by the topological structure of $\mathcal{G}$ and the roles of nodes, i.e., leaders or followers, $\mathcal{L}(\mathcal{G})$ can vary significantly with different leader set, resulting either a controllable or uncontrollable network. Therefore, the primary objective of this work is to characterize the relationship between leader-follower controllability and network topology and identify a subgroup of nodes (i.e., the leader set) in $\mathcal{G}$ such that leader-follower controllability in Definition 1 is ensured.

III. Topological Characterization of Leader-Follower Controllability

Lemma 1. *[17] Consider a linear time-invariant system*

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{4}$$

where $A \in \mathbb{R}^{n \times n}$ is the system matrix, $B \in \mathbb{R}^{n \times m}$ is the input matrix, and $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ represent the system states and the control input, respectively. Let $E(\cdot)$ denote the set of left eigenvectors of a matrix. Per the well known Popov-Belevitch-Hautus test, the system in (4) is uncontrollable if and only if there exists a left eigenvector $\nu \in E(A)$ (i.e., $\nu^T A = \lambda \nu^T$ for some eigenvalue λ) such that $\nu^T B = 0_m$, where 0_m is an m-dimensional vector of all zeros. In other words, the system (4) is controllable if and only if $\nu \notin \ker\left(B^T\right)$, $\forall \nu \in E(A)$, where $\ker(\cdot)$ indicates the kernel space.

Proposition 1. *Consider a signed leader-follower network $\mathcal{G}$ where the followers evolve according to (3) and the leaders take exogenous input, i.e., $x_l(t) = u(t)$. Provided that the follower-to-follower and leader-to-follower connections are intact, the leader-follower controllability is invariant to alterations of leader-to-leader connections, e.g., creating new edges, removing existing edges, or changing edge weights between leaders.*

As an immediate consequence of Lemma 1, Proposition 1 provides a topological characterization of network controllability, which is instructive in constructing a controllable graph from a set of controllable sub-graphs and paves a way to develop controllable conditions in the subsequent section. To show how a controllable graph can be constructed, the case of two sub-graphs is first considered in Proposition 2, which is then extended to the case of multiple sub-graphs in Proposition 3.

First, consider two leader-follower signed graphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{W}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{W}_2)$, where $\mathcal{G}_i$, $i = \{1, 2\}$, has a follower set $\mathcal{V}_{fi} = \{1, \ldots, m_i\}$ and a leader set $\mathcal{V}_{li} = \{m_i + 1, \ldots, n_i\}$ with $\mathcal{V}_{fi} \cup \mathcal{V}_{li} = \mathcal{V}_i$ and $\mathcal{V}_{fi} \cap \mathcal{V}_{li} = \emptyset$, where m_i and n_i denote the cardinality of the follower set and its node set, respectively. The edge set $\mathcal{E}_i$ and weight matrix $\mathcal{W}_i$ indicate the underlying connections among leaders and followers within $\mathcal{G}_i$, $i = \{1, 2\}$.

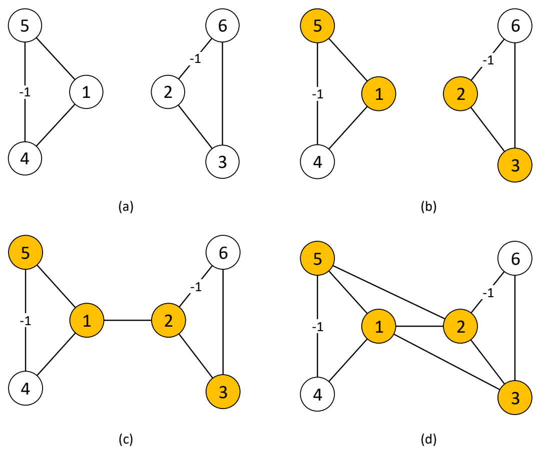

Figure 1. Examples of constructing a controllable graph from controllable sub-graphs. (a) Two sub-graphs. (b) Each sub-graph is controllable provided that the nodes $\{1, 5\}$ and $\{2, 3\}$ are selected as leaders, respectively. (c) The combined graph is controllable if a new leader-to-leader connection (i.e., the edge $(1, 2)$) is created. (d) The combined graph remains controllable if new leader-to-leader connections (i.e., the edges $(1, 2)$, $(1, 3)$, and $(5, 2)$) are created.

Proposition 2. *Provided that the two signed graphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{W}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{W}_2)$ are controllable and evolve according to (1), $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0, \mathcal{W}_0)$ remains controllable if $\mathcal{G}_0$ is constructed such that 1) $\mathcal{V}_0 = \mathcal{V}_1 \cup \mathcal{V}_2$ with the follower set $\mathcal{V}_{f0} = \{1, \ldots, m_1 + m_2\}$ and the leader set $\mathcal{V}_{l0} = \{m_1 + m_2 + 1, \ldots, n_1 + n_2\}$, where the nodes are re-indexed, without loss of generality, as the first $m_1 + m_2$ nodes are followers and the rest nodes are leaders, 2) $\mathcal{E}_0 = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}'$ where $\mathcal{E}' \subset \mathcal{V}_{l1} \times \mathcal{V}_{l2}$, and 3) $\mathcal{W}_0 = \begin{bmatrix} \mathcal{W}_1 & \bar{\mathcal{W}} \\ \bar{\mathcal{W}}^T & \mathcal{W}_2 \end{bmatrix} \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$ where $\bar{\mathcal{W}}$ indicates the weights associated with the edges in $\mathcal{E}'$.*

To extend Proposition 2 to the case of multiple sub-graphs, consider a set of signed graphs $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i, \mathcal{W}_i)$, $i = \{1, \ldots, n\}$, where each has a follower set $\mathcal{V}_{fi} = \{1, \ldots, m_i\}$ and a leader set $\mathcal{V}_{li} = \{m_i + 1, \ldots, n_i\}$ with m_i and n_i indicating cardinality of the follower set and its node set, respectively.

Proposition 3. *Provided a set of controllable signed graphs $\mathcal{G}_i$, $i = \{1, \ldots, n\}$, evolving according to (1), $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0, \mathcal{W}_0)$ remains controllable if $\mathcal{G}_0$ is constructed such that 1) $\mathcal{V}_0 = \bigcup_{i=1}^n \mathcal{V}_i$, 2) $\mathcal{E}_0 = \bigcup_{i=1}^n \mathcal{E}_i \cup \mathcal{E}'$ where $\mathcal{E}' \subset \prod_{i=1}^n \mathcal{V}_{li}$, indicating the additional edges $\mathcal{E}'$ are restricted to leader-to-leader connections within the leader sets $\mathcal{V}_{li}$, and 3) $\mathcal{W}_0 = \bigcup_{i=1}^n \mathcal{W}_i \cup \mathcal{W}'$ where $\mathcal{W}'$ indicates weights associated with the edge set $\mathcal{E}'$.*

Example 1. Fig. 1 shows how a controllable graph can be constructed from a set of controllable sub-graphs. Fig. 1 (a) contains two sub-graphs, which become controllable if the nodes $\{1, 5\}$ and $\{2, 3\}$ are selected as leaders, respectively, as shown in Fig. 1 (b). The combined graph in Fig. 1 (c) is constructed by connecting the two leaders $\{1, 2\}$. It can be verified that the combined graphs in Fig. 1 (c) remains controllable, since its construction follows the rules in Proposition 2. It is worth pointing out that only sufficient conditions to preserve network controllability are developed in Proposition 2. There might exist different ways in connecting leaders to preserve network controllability. As a different construction, the combined graph in Fig. 1 (d) is constructed by including three new edges (i.e., $(1, 2)$, $(1, 3)$, and $(5, 2)$), which is also controllable according to Proposition 2.

IV. Controllability for Signed Path and Cycle Graphs

This section focuses on developing sufficient conditions on the controllability of signed path and cycle graph. A path graph is a graph where all internal nodes have degree two except that two end nodes have degree one. A cycle graph is a graph where all nodes have degree two. As a key tool to study the controllability of signed networks, structural balance is introduced.

Definition 2 (Structural Balance)**.** A signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ is structurally balanced if the node set $\mathcal{V}$ can be partitioned into $\mathcal{V}_1$ and $\mathcal{V}_2$ with $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, where $w_{ij} > 0$ if $v_i, v_j \in \mathcal{V}_q$, $q \in \{1, 2\}$, and $w_{ij} < 0$ if $v_i \in \mathcal{V}_q$ and $v_j \in \mathcal{V}_r$, $q \neq r$, and $q, r \in \{1, 2\}$.

Definition 2 indicates that v_i and v_j are positive neighbors if they are from the same subset, i.e., either $\mathcal{V}_1$ or $\mathcal{V}_2$, and negative neighbors if v_i and v_j are from different subset. To characterize structural balance, necessary and sufficient conditions are provided.

Lemma 2. *[4] A connected signed graph $\mathcal{G}$ is structurally balanced if and only if any of the following equivalent conditions holds: 1) all cycles of $\mathcal{G}$ are positive, i.e., the product of edge weights on any cycle is positive; 2) there exists a diagonal matrix $\Xi = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\}$ with $\sigma_i \in \{\pm 1\}$ such that $\Xi \mathcal{W} \Xi$ has non-negative entries; 3) 0 is an eigenvalue of graph Laplacian $\mathcal{L}(\mathcal{G})$.*

Lemma 3. *Consider a structurally balanced signed graph $\mathcal{G}$ with nodes partitioned into $\mathcal{V}_1$ and $\mathcal{V}_2$. If leaders are selected from the same subset (i.e., either $\mathcal{V}_1$ or $\mathcal{V}_2$) and followers evolve according to (3), the leader-follower controllability of $\mathcal{G}$ remains the same as its corresponding unsigned graph $\mathcal{G}'$, where $\mathcal{G}' = (\mathcal{V}, \mathcal{E}, \mathcal{W}')$ has the same node and edge set as $\mathcal{G}$ except that $\mathcal{W}' = \Xi \mathcal{W} \Xi = \operatorname{abs}(\mathcal{W})$, where Ξ is defined in Lemma 2 and $\operatorname{abs}(\mathcal{W})$ denotes the entry-wise absolute value of $\mathcal{W}$.*

This lemma is a variant of Theorem 3 in [15] that accounts for the follower dynamics in (3).

A. Signed Path Graph

Let $\mathcal{G}_p = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ denote a signed path graph, where $\mathcal{V} = \{1, \ldots, n\}$ and $\mathcal{E} = \{(i, i+1) | \, i \in \{1, \ldots, n-1\}\}$ represent the node and edge set, respectively, and the weight matrix $\mathcal{W} \in \mathbb{R}^{n\times n}$ indicates associated positive or negative weights in $\mathcal{E}$.

Theorem 1. *A signed path graph $\mathcal{G}_p = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ with followers evolving according to (3) is controllable if one of the end nodes (i.e., v_1 or v_n) is selected as leader.*

Proof: As indicated in Corollary 1 in [4], a spanning tree is always structurally balanced. As a particular case of the spanning tree, the path graph $\mathcal{G}_p$ is thus structurally balanced. Since $\mathcal{G}_p$ is structurally balanced, its node set can be partitioned into $\mathcal{V}_1$ and $\mathcal{V}_2$ as in Definition 2. If one leader is considered, the leader must be selected from either $\mathcal{V}_1$ and $\mathcal{V}_2$, i.e., the same subset. Hence, Lemma 3 indicates that the controllability of $\mathcal{G}_p$ is equivalent to its corresponding unsigned graph $\mathcal{G}_p^{'} = (\mathcal{V}, \mathcal{E}, \mathcal{W}')$, where $\mathcal{W}'$ consists of non-negative edge weights. Given that an unsigned path graph is controllable if an end node is selected as leader in [6] and [7], it can be concluded that the signed path graph $\mathcal{G}_p$ is also controllable if an end node is selected as leader from Lemma 3.

Since $\mathcal{G}_p$ is structurally balanced, its node set can be partitioned into $\mathcal{V}_1$ and $\mathcal{V}_2$ as in Definition 2. If one leader is considered, the leader must be selected from either $\mathcal{V}_1$ and $\mathcal{V}_2$, i.e., the same subset. Hence, Lemma 3 indicates that the controllability of $\mathcal{G}_p$ is equivalent to its corresponding unsigned graph $\mathcal{G}_p^{'} = (\mathcal{V}, \mathcal{E}, \mathcal{W}')$, where $\mathcal{W}'$ consists of non-negative edge weights. Given that an unsigned path graph is controllable if an end node is selected as leader in [7], it can be concluded that the signed path graph $\mathcal{G}_p$ is also controllable if an end node is selected as leader from Lemma 3. ■

The following theorem extends the result in Theorem 1 to multi-leader selection.

Theorem 2. *A signed path graph $\mathcal{G}_p$ with followers evolving according to (3) is controllable if multiple adjacent nodes in $\mathcal{G}_p$ are selected as leaders.*

Proof: The case of two leaders is first considered in this proof, which will then be extended to include multiple leaders. Suppose that an arbitrary pair of adjacent nodes v_k and v_{k+1}, $\forall k \in \{1, \dots, n-1\}$, in $\mathcal{G}_p$ are selected as leaders. Let $\mathcal{G}_{p1} = \{\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{k-1}, v_k\}\}$ denote the sub-graph of $\mathcal{G}_p$ with the leader node v_k and $\mathcal{G}_{p2} = \{\{v_{k+1}, v_{k+2}\}, \cdots, \{v_{n-1}, v_n\}\}$ denote the sub-graph of $\mathcal{G}_p$ with the leader node v_{k+1}, respectively. Based on Theorem 1, $\mathcal{G}_{p1}$ and $\mathcal{G}_{p2}$ are both controllable, since v_k and v_{k+1} are the end nodes of the sup-graph $\mathcal{G}_{p1}$ and $\mathcal{G}_{p2}$, respectively. Since $\mathcal{G}_p$ can be constructed by connecting the two leaders in $\mathcal{G}_{p1}$ and $\mathcal{G}_{p2}$ while keeping the rest graph intact, Proposition 1 indicates that $\mathcal{G}_p$ will remain controllable.

If n adjacent nodes are selected as leaders, following similar argument above, $\mathcal{G}_p$ can always be partitioned into controllable sub-path graphs. Iteratively invoking Proposition 1 indicates $\mathcal{G}_p$ is always controllable if adjacent nodes are selected as leaders. ■

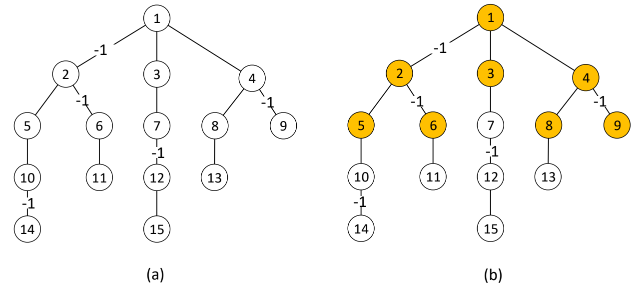

Figure 2. (a) A signed tree graph. (b) A controllable signed tree graph with the marked nodes $\{1, 2, 3, 4, 5, 6, 8, 9\}$ selected as leaders.

Theorem 2 relaxes the constraint in Lemma 3 that leaders have to be selected from the same partitioned subset. Specifically, Theorem 2 indicates that, for signed path graphs, leaders are allowed to be selected from different partitioned subsets, as long as they are adjacent nodes in $\mathcal{G}_p$.

A signed tree is a particular topology where any two vertices are connected by exactly one simple path and the edges admit negative weights. Networks with tree topology have been broadly applied to model multi-agent networks, cyber-physical systems, smart grid, and power networks (cf. [18] and [19] for more applications). Since a tree graph can be naturally partitioned into connected path graphs, the subsequent theorem extends leader selection rules developed in Theorem 1 and Theorem 2 to signed trees.

Let $\mathcal{G}_t = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ denote a signed tree, where $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{W}$ represent the node set, edge set, and weight matrix, respectively. Since $\mathcal{G}_t$ is a tree with no cycles, it is well known that $|\mathcal{E}| = |\mathcal{V}| - 1$.

Theorem 3. *A signed tree is controllable if node with $d_i \geq 3$ and nodes are from $\mathcal{N}_i$ are selected as leaders.*

The proof is omitted here, since Theorem 3 is an immediate consequence of Theorem 1, Theorem 2, and Proposition 1. Example 2 is provided to illustrate Theorem 3.

Example 2. Consider a signed tree shown in Fig. 2 (a), where the roles (i.e., leaders or followers) are not assigned yet. To show our results, the tree can be first partitioned into 5 paths, i.e., $\{5, 10, 14\}$, $\{6, 11\}$, $\{3, 7, 12, 15\}$, $\{8, 13\}$, and $\{2, 1, 4, 9\}$. If the leaders are selected as $\{1, 2, 3, 4, 5, 6, 8, 9\}$, then each of the path graphs is controllable by Theorem 1. Since the path graphs are connected in a way that only leaders from each path are connected to form the tree, the tree is controllable based on Theorem 2 and Proposition 1. Therefore, it can be verified that the selected leaders ensure leader-follower controllability of the tree graph.

B. Signed Cycle Graph

A cycle graph $\mathcal{G}_c = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ is defined similar to $\mathcal{G}_p$, except that the edge set in $\mathcal{G}_c$ is defined as $\mathcal{E} = \{(i,\ i \mod (n) + 1) | \, i \in \{1, \dots, n\}\}$. If $\mathcal{G}_c$ is structurally balanced, based on Lemma 2, there exists a diagonal matrix

Ξ such that $\Xi\mathcal{W}\Xi = \mathrm{abs}(W)$ is non-negative. The graph $\mathcal{G}_c^{'} = (\mathcal{V}, \mathcal{E}, \mathrm{abs}(W))$ is then called the unsigned graph of $\mathcal{G}_c$, since $\mathcal{G}_c^{'}$ no longer has negative edge weights.

Lemma 4. *[11] An unsigned cycle graph is controllable if two adjacent nodes are chosen as leaders.*

This section focuses on extending the controllability conditions on unsigned cycles graphs developed in Lemma 4 to signed cycle graphs, where the weight matrix $\mathcal{W}$ can take negative entries.

Lemma 5. *Consider a signed cycle graph $\mathcal{G}_c = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, where the followers evolve according to (3). The leader-follower controllability of $\mathcal{G}_c$ is equivalent to that of its sign-reversed graph $\bar{\mathcal{G}}_c = (\mathcal{V}, \mathcal{E}, -\mathcal{W})$, where the weight matrix $\mathcal{W}$ in $\mathcal{G}_c$ is replaced by $-\mathcal{W}$ in $\bar{\mathcal{G}}_c$.*

Proof: Since every node in a cycle graph has degree two, the degree matrix $\mathcal{D}$ of a cycle graph is $2I_n$, where I_n represents the n-dimensional identity matrix. Without loss of generality, suppose the first m nodes are followers and the rest $n-m$ nodes are leaders. Graph Laplacian of $\mathcal{G}_c$ can then be partitioned as

$$\mathcal{L} = \mathcal{D} - \mathcal{W} = \left[\begin{array}{c|c} 2I_m - \mathcal{W}_f & -\mathcal{W}_{fl} \\ \hline -\mathcal{W}_{fl}^T & 2I_{n-m} - \mathcal{W}_l \end{array}\right], \quad (5)$$

where $\mathcal{W}_f \in \mathbb{R}^{m\times m}$, $\mathcal{W}_{fl} \in \mathbb{R}^{m\times(n-m)}$, and $\mathcal{W}_l \in \mathbb{R}^{(n-m)\times(n-m)}$. By the dynamics (1) and (5), the followers evolve according to

$$\dot{x}_f(t) = -(2I_m - \mathcal{W}_f)\, x_f - (-\mathcal{W}_{fl})\, u(t)$$

where $x_f \in \mathbb{R}^m$ and $u(t) \in \mathbb{R}^{n-m}$ denote the followers' states and leaders' input, respectively. Hence, based on Def. 1, the leader-follower controllability of $\mathcal{G}_c$ is completely determined by $2I_n - \mathcal{W}_f$ and $-\mathcal{W}_{fl}$.

Consider the sign reversed cycle graph $\bar{\mathcal{G}}_c$. Similarly, the leader-follower controllability of $\bar{\mathcal{G}}_c$ is determined by $2I_n + \mathcal{W}_f$ and $\mathcal{W}_{fl}$. Note that the set of eigenvectors of $2I_n - \mathcal{W}_f$ is identical to that of $2I_n + \mathcal{W}_f$. In addition, since $2I_n - \mathcal{W}_f$ and $2I_n + \mathcal{W}_f$ are symmetric matrices, their right and left eigenvectors are equal. Let μ be an eigenvector of $2I_n - \mathcal{W}_f$ (or $2I_n + \mathcal{W}_f$). If $\mu \in \ker\left(-\mathcal{W}_{fl}^T\right)$, it is always true that $\mu \in \ker\left(\mathcal{W}_{fl}^T\right)$. Similarly, if $\mu \notin \ker\left(-\mathcal{W}_{fl}^T\right)$, it will also be true that $\mu \notin \ker\left(\mathcal{W}_{fl}^T\right)$. Therefore, based on Lemma 1, the controllability of $\mathcal{G}_c$ is equivalent to that of $\bar{\mathcal{G}}_c$. ∎

Theorem 4. *Provided that all followers evolve according to (3), a signed cycle graph $\mathcal{G}_c$ is controllable if any two adjacent nodes in $\mathcal{G}_c$ are selected as leaders.*

Proof: Different from path graphs that are inherently structurally balanced, cycle graphs may not be structurally balanced. In addition, when considering two adjacent nodes as leaders, the sign of inter-leader edge introduces additional challenge to the analysis of leader-follower controllability. Therefore, based on the topological structure of $\mathcal{G}_c$ and the sign of inter-leader edge, four cases are discussed.

Case 1: Structurally balanced $\mathcal{G}_c$ with positive inter-leader edge. Since $\mathcal{G}_c$ is structurally balanced, its node set $\mathcal{V}$ can be partitioned into two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$. Based on Lemma 2, the positive inter-leader edge indicates that the two leaders are from the same subset, i.e., either $\mathcal{V}_1$ or $\mathcal{V}_2$. In addition, there exists an unsigned cycle graph $\mathcal{G}_c^{'}$, which has the same edge and node set as $\mathcal{G}_c$ except that the edge weights in $\mathcal{G}_c^{'}$ are all positive. Given that leaders are selected from the same subset, Lemma 3 indicates that the controllability of $\mathcal{G}_c$ is equivalent to that of $\mathcal{G}_c^{'}$. Since $\mathcal{G}_c^{'}$ is controllable from Lemma 4 if two adjacent nodes are selected as leaders, it concludes that $\mathcal{G}_c$ is also controllable with similar leader selection approach from Lemma 3.

Case 2: Structurally unbalanced $\mathcal{G}_c$ with negative inter-leader edge. Since $\mathcal{G}_c$ is structurally unbalanced cycle graph, Lemma 2 indicates that $\mathcal{G}_c$ is a negative cycle, which implies there exists an odd number of negative edges in $\mathcal{G}_c$. Since graph controllability is invariant to changes of inter-leader edges from Proposition 1, the controllability of $\mathcal{G}_c$ with negative inter-leader edge will remain the same if the sign of inter-leader edge flips from negative to positive. After flipping the sign to be positive, $\mathcal{G}_c$ will now contain an even number of negative edges and becomes structurally balanced from Lemma 2, which implies that $\mathcal{G}_c$ is controllable due to the equivalent to Case 1.

Case 3: Structurally balanced $\mathcal{G}_c$ with negative inter-leader edge. Based on Lemma 5, the controllability of $\mathcal{G}_c$ is invariant if all of its signs are flipped. Based on the number of negative edges in $\mathcal{G}_c$, two sub-cases are further discussed. 1) If $\mathcal{G}_c$ in this case has an even number of edges, it must have an even number of negative edges and an even number of positive edges from Lemma 2. After flipping all of its signs, $\mathcal{G}_c$ remains structurally balanced but with positive inter-leader edge, which implies that $\mathcal{G}_c$ in this case is controllable due to the equivalence to Case 1. 2) If $\mathcal{G}_c$ has an odd number of edges, it must have an even number of negative edges and an odd number of positive edges, due to Lemma 2. After flipping all of its signs, $\mathcal{G}_c$ becomes $\mathcal{G}_c^*$, which is structurally unbalanced with positive inter-leader edge and equivalent to Case 4 below. If the sign of any leader-follower edge in $\mathcal{G}_c^*$ is flipped again, $\mathcal{G}_c^*$ becomes $\mathcal{G}_c^\star$, which is structurally balanced with positive inter-leader edge due to the change of negative edges from an odd number of to an even number. Following similar procedure as in the proof of Lemma 5, it can be trivially verified that the change of sign of any leader-follower edge will not affect the controllability of the system. In other words, the controllability of $\mathcal{G}_c^\star$ remains the same as $\mathcal{G}_c^*$, which is then the same as $\mathcal{G}_c$ due to Lemma 5. Since $\mathcal{G}_c^\star$ is structurally balanced with positive inter-leader edge, which is the same as Case 1, $\mathcal{G}_c^\star$ is controllable, and hence $\mathcal{G}_c$ is also controllable.

Case 4: Structurally unbalanced $\mathcal{G}_c$ with positive inter-leader edge. Since this case has been discussed in Case 3, $\mathcal{G}_c$ is controllable.

Based on Cases 1-4, $\mathcal{G}_c$ is controllable if adjacent nodes

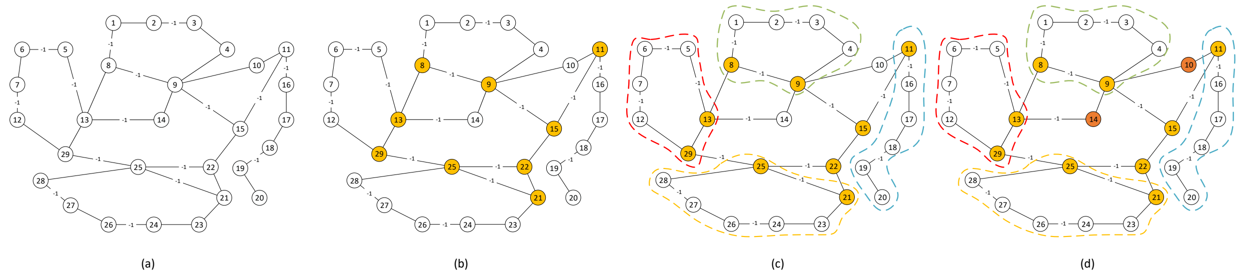

Figure 3. (a) A signed graph with 29 nodes. (b) The initial selection of leader nodes (i.e., nodes with degree more than two) are marked. (c) Based on the initially selected leader node, the signed graph is partitioned into a set of path and cycle graphs. (d) Update the leader nodes based on the rules in Theorem 1-4 to ensure the resulting leader-follower network is controllable.

are selected as leaders. ∎

C. Extensions

This section shows how the conditions developed for signed path and cycle graphs can be potentially extended to more general signed networks. Since path and cycle graphs are basic building blocks for general signed graphs, the idea of leader selection in this section is to first partition the general signed graphs into a set of path and cycle graphs, where the results developed in Theorems 1-4 can be applied.

Example 3. Consider the signed graph in Fig. 3 (a) where the objective is to select a set of leaders such that the leader-follower network is controllable. The leader nodes $\{8, 9, 11, 13, 15, 21, 22, 25, 29\}$ are first identified in Fig. 3 (b) to partition the graph. Based on the selected leader nodes, the signed graph is partitioned into a set of path and cycle graphs, which are shown in dashed lines. Checking each part with sufficient conditions developed in previous sections to ensure that each path and cycle graph are controllable. For instance, the selected nodes $\{8, 9\}$ ensure the controllability of the cycle graph formed by $\{1, 2, 3, 4, 8, 9\}$. The leader set is then updated to include more nodes (i.e., addition leaders $\{10, 14\}$) whenever necessary such that the individual path and cycle graphs are connected satisfying Propositions 1-3, which ensues the controllability of the original signed graph.

V. Conclusion

Topological characterizations of leader-follower controllability on signed path and cycle networks are developed in this work. Future research will consider reducing the size of the leader group such that the network is controllable by a small number of leaders.

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