Pinning Control of Boolean Networks via Injection Mode

Zequn Liu, Daizhan Cheng, *Fellow, IEEE*, and Jiangbo Liu

***Abstract*—The pinning control of Boolean networks with both controls and injection modes is proposed. The system concerned is converted into a classical Boolean network with mix-valued logical controls. Under this framework three other fundamental control problems, including controllability, observability and stabilization, are investigated.**

***Index Terms*—Boolean network, pinning control, injection mode, controllability, observability, stabilization.**

I. Introduction

BOOLEAN network was firstly proposed by Kauffman for modeling genetic regulatory network(GRN) [10]. Since then it has received considerable attention from both the biologists and system scientists, and developed rapidly. It is, nowadays, a commonly used prominent tool in analyzing and manipulating GRN.

Since the mathematical model of Boolean networks is a logical dynamic system, it is not easy to analyse the dynamics of Boolean networks. Recently, a new matrix product, called semi-tensor product (STP), was proposed by Cheng and his colleagues [1]. Using STP, the logical model of Boolean networks can be converted into an algebraic form, called the algebraic state space representation (ASSR) of Boolean networks. In the light of ASSR, an algebraic framework of Boolean (control) networks has been proposed [1]. Following [1], many standard control problems have been investigated, for instance, the controllability and observability [5], [13], stability and stabilization [2], [26], structure and disturbance decoupling [3], [9], [23], optimization [6], just to mention a few.

Nowadays, pinning control of Boolean networks becomes a hot topic. Pinning control is concerned with a selected set of nodes that controls are applied to while the rest transform without control [7]. There are mainly two reasons for this. (1) For a Boolean network, particularly for large scale Boolean networks in biology etc., only a few particular nodes are allowed (or convenient) to inject inputs [20]. (2) Recent researches discovered in the cellular reprogramming field that full control and reprogramming of biological systems may be achieved by controlling only a few key factors [21].Thus pinning control is more effective and economic than the conventional control method that all the nodes need to be designed. Several fundamental control problems for pinning control Boolean networks have been discussed. [14]firstly studied the stabilization of Boolean networks by pinning control. Globally stable Boolean networks were obtained by changing the columns of the corresponding transition matrices and algorithms to design the required pinning controllers were given. This work was then generalized to the case of Boolean networks with time delays [15]. Pinning controllability was also investigated in [19], [8]. [17]studied the disturbance decoupling problem of Boolean Networks by pinning control. In [16], pinning control design for the robust output tracking of k-valued logical networks with disturbances was discussed. Pinning control was also applied to output robustness [18].

Unlike all existing results, where the fixed injection mode is used, this paper proposes using time-varying injection mode. This is reasonable, because if a node is allowed to add any control, the injection mode should also be manipulated by control designer. The main idea of this paper is to convert a pinning control Boolean network with time-varying injection mode (PCBNTVIM) into an (almost) standard control Boolean network. Then all control problems of pinning control Boolean networks can be handled by classical control techniques of Boolean networks with some mild modifications.

The problems considered in this paper include controllability, stabilization, and observability. The paper is organized as follows: Section II provides a mathematical formulation for PCBNTVIM. The controllability, stabilization, and observability of PCBNTVIM are considered in Sections III,IV,V respectively. Section VI studies the pinning control networks with time-invariant injection mode. Section VII is a brief conclusion.

Before ending this introduction we give a list for notations, which are used in the sequel:

1) $\mathbb{R}$: Field of real numbers;
2) $\mathcal{M}_{m\times n}$: set of $m\times n$ dimensional real matrices.
3) $\mathrm{Col}(A)$ ($\mathrm{Row}(A)$): the set of columns (rows) of A; $\mathrm{Col}_i(A)$ ($\mathrm{Row}_i(A)$): the i-th column (row) of A.
4) δ_n^i: i-th column of the identity matrix I_n.
5) $\mathcal{D}_k$: $\mathcal{D}_k=\{1,2,\cdots,k\}$; $\mathcal{D}=\mathcal{D}_2$.
6) Δ_k: $\Delta_k=\mathrm{Col}(I_k)$.

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7) $\mathbf{1}_k$: $\mathbf{1}_k = (\underbrace{1, 1, \cdots, 1}_{k})^T \in \mathbb{R}^k$.
8) $\mathbf{1}_{m\times n}$: $\mathbf{1}_{m\times n} \in \mathcal{M}_{m\times n}$ with all entries equal 1.
9) $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\bar{\vee}$ (logical operators): conjunction, disjunction, conditional, biconditional, exclusive or, respectively [22].
10) $\mathcal{L}_{m\times n}$: The set of logical matrices. (A matrix $L \in \mathcal{M}_{m\times n}$ is called a logical matrix, if $\mathrm{Col}(L) \subset \Delta_m$.)
11) $\delta_m[i_1, \cdots, i_n]$: A brief notation for logical matrices, that is,
$$\delta_m[i_1, \cdots, i_n] := \left[\delta_m^{i_1}, \cdots, \delta_m^{i_n}\right]$$

II. Problem Formulation

A. STP of Matrices

This subsection gives a brief review on STP of matrices. STP is the main tool used in this paper. We refer to [1] for details.

Definition 2.1: Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$ be two real matrices, and the least common multiple of n and p be $t = \mathrm{lcm}\{n, p\}$. The STP of A and B, denoted by $A \ltimes B$, is defined as
$$\left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right). \tag{1}$$
where I_k is the $k \times k$ identity matrix, and $\otimes$ is the Kronecker product.

STP has some "commutative" properties, which will be used in the sequel:

Proposition 2.2: Given $A \in \mathcal{M}_{m\times n}$.

1. Let $Z \in \mathbb{R}^t$ be a column vector. Then
$$ZA = (I_t \otimes A)Z. \tag{2}$$
2. Let $Z \in \mathbb{R}^t$ be a row vector. Then
$$AZ = Z(I_t \otimes A). \tag{3}$$

Define
$$\begin{array}{l} M_f = I_p \otimes \mathbf{1}_q^T \\ M_s = \mathbf{1}_p^T \otimes I_q. \end{array} \tag{4}$$

Then we have the following result:

Proposition 2.3: Given $x \in \Delta_p$ and $y \in \Delta_q$, then
$$\begin{array}{l} M_f xy = x \\ M_s xy = y. \end{array} \tag{5}$$

Next, we define the swap matrix, which is useful for commutativity.

Definition 2.4: An (m, n) order swap matrix $W_{[m,n]} \in \mathcal{M}_{mn\times mn}$ is defined as
$$\begin{array}{rcl} W_{[m,n]} & = & \delta_{mn}[1, m+1, \cdots, (n-1)m+1; \\ & & 2, m+2, \cdots, (n-1)m+2; \\ & & \cdots; m, 2m, \cdots, nm]. \end{array} \tag{6}$$

The main function of a swap matrix is to exchange the order of two vector factors.

Proposition 2.5: 1. Given two column vectors $X \in \mathbb{R}^m$, $Y \in \mathbb{R}^n$, then
$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X. \tag{7}$$
2. Given two row vectors $X \in \mathbb{R}^m$, $Y \in \mathbb{R}^n$, then
$$X \ltimes Y \ltimes W_{[m,n]} = Y \ltimes X. \tag{8}$$

Definition 2.6: 1) Let $x_i \in \mathcal{D}_{k_i}$, $i = 1, \cdots, n$. A mapping $f : \prod_{i=1}^n \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$ is called a mix-valued logical function. A 2-valued logical function is also called a Boolean function.

2) Let $x_i(t) \in \mathcal{D}_{k_i}$, $i = 1, \cdots, n$. The system
$$\begin{cases} x_1(t+1) & = f_1(x_1(t), \cdots, x_n(t)) \\ x_2(t+1) & = f_2(x_1(t), \cdots, x_n(t)) \\ \vdots & \\ x_n(t+1) & = f_n(x_1(t), \cdots, x_n(t)), \end{cases} \tag{9}$$
where $f_i : \prod_{i=1}^n \mathcal{D}_{k_i} \to \mathcal{D}_{k_i}$, $i = 1, \cdots, n$, is called a mix-valued logical dynamic system. 2-valued logical dynamic system is also called Boolean network.

Identifying
$$i \sim \delta_k^i, \quad i = 1, \cdots, k,$$
then $x \in \mathcal{D}_k$ can be expressed as $x \in \Delta_k$. The latter is called the vector expression of logical variable.

Theorem 2.7: Given mix-valued logical function $f : \prod_{i=1}^n \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$ denoted as
$$y = f(x_1, \cdots, x_n), \tag{10}$$
then there exists a unique $M_f \in \mathcal{L}_{k_0\times k}$, where $k = \prod_{i=1}^n k_i$, such that in vector expression we have
$$y = M_f \ltimes_{i=1}^n x_i. \tag{11}$$
M_f is called the structure matrix of f.

Using Theorem 2.7 to each equation of (9), we have
$$\begin{cases} x_1(t+1) & = M_1 \ltimes_{i=1}^n x_i(t) \\ x_2(t+1) & = M_2 \ltimes_{i=1}^n x_i(t) \\ \vdots & \\ x_n(t+1) & = M_n \ltimes_{i=1}^n x_i(t), \end{cases} \tag{12}$$
where M_i is the structure matrix of f_i, $i = 1, \cdots, n$.

Definition 2.8: Let $A \in \mathcal{M}_{m\times r}$ and $B \in \mathcal{M}_{n\times r}$. The Khatri-Rao product of A and B, denoted by $A * B$, is defined as follows: [11]
$$\begin{array}{rcl} A * B & = & [\mathrm{Col}_1(A) \ltimes \mathrm{Col}_1(B), \mathrm{Col}_2(A) \ltimes \mathrm{Col}_2(B), \\ & & \cdots, \mathrm{Col}_r(A) \ltimes \mathrm{Col}_r(B)] \in \mathcal{M}_{mn\times r}. \end{array} \tag{13}$$

Theorem 2.9: Denote by $x(t) = \ltimes_{i=1}^n x_i$. Then (12) can be expressed as
$$x(t+1) = Lx(t), \tag{14}$$
where
$$L = M_1 * M_2 * \cdots * M_n.$$
(14) is called the algebraic state space expression of the mix-valued logical dynamic system (9).

B. ASSR of Pinning Boolean Network

A node is called a pinning node, if a control is allowed to inject the network through this node. Assume a Boolean Network has n nodes $x_1, \cdots, x_n$, where m nodes are pinning ones. The dynamics of the network is described as

$$\begin{array}{l} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)) \\ \vdots \\ x_m(t+1) = f_m(x_1(t), \cdots, x_n(t)) \\ x_{m+1}(t+1) = f_{m+1}(x_1(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)). \end{array} \tag{15}$$

Without loss of generality, we can assume $x_1, \cdots, x_m$ are pinning nodes. Then we can inject inputs through them, and convert system (15) into the following form:

$$\begin{array}{l} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)) \diamondsuit_1(t) u_1(t) \\ \vdots \\ x_m(t+1) = f_m(x_1(t), \cdots, x_n(t)) \diamondsuit_m(t) u_m(t) \\ x_{m+1}(t+1) = f_{m+1}(x_1(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)). \end{array} \tag{16}$$

where $\diamondsuit_i$, $i = 1, \cdots, m$ are used to represent the injection modes, which are binary logical operators, $u_i \in \{0, 1\}$, $i = 1, \cdots, m$ are pinning controls. Denote by Θ the set of binary logical operators with $\emptyset$, where $\emptyset$ stands for "no injection control". Denote by $\Xi_i \subset \Theta$ the feasible set of injection modes for pinning nod i, then

$$\diamondsuit_i \in \Xi_i \subset \Theta, \quad i = 1, \cdots, m.$$

Note that the total number of binary logical operators is $2^{2^2} = 16$, hence $|\Theta| = 17$, because we allow $\emptyset$ to represent "no injection". Assume

$$|\Xi_i| = k_i, \quad i = 1, 2, \cdots, m.$$

Then we can formally consider $\diamondsuit_i$ as a k_i-valued logical variable. Precisely speaking, let $\Xi = \{O_1, O_2, \cdots, O_s\} \subset \Theta$, and $\diamondsuit \in \Xi$. Then we consider $\diamondsuit$ as an s-valued logical variable, and denote it by ξ. In vector form, we have

$$\diamondsuit = O_i \Leftrightarrow \xi = \delta_s^i, \quad i = 1, 2, \cdots, s.$$

In this way, the ASSR of (16) can also be obtained as

$$x(t+1) = M\xi(t)u(t)x(t), \tag{17}$$

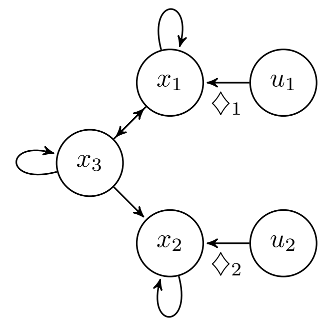
where $x(t) = \ltimes_{i=1}^n x_i(t) \in \Delta_{2^n}$, $u(t) = \ltimes_{j=1}^m u_j(t) \in \Delta_{2^m}$, $\xi(t) = \ltimes_{j=1}^m \xi_j(t) \in \Delta_{d_\xi}$, and

$$d_\xi = \prod_{j=1}^m |\Xi_j|,$$

is the dimension of $\xi(t)$.

We use an example to depict this.

Example 2.10: Consider a pinning control system



$$\begin{array}{l} x_1(t+1) = (x_1(t) \vee x_3(t)) \diamondsuit_1(t) u_1(t) \\ x_2(t+1) = (x_2(t) \wedge x_3(t)) \diamondsuit_2(t) u_2(t) \\ x_3(t+1) = x_1(t) \leftrightarrow x_3(t). \end{array} \tag{18}$$

where

$$\begin{array}{l} \diamondsuit_1 \in \Xi_1 = \{\bar{\vee}, \rightarrow\} \\ \diamondsuit_2 \in \Xi_2 = \{\emptyset, \rightarrow, \vee\}. \end{array}$$

(i) If $\diamondsuit_1(t) = \bar{\vee}$, that is, $\xi_1 = \delta_2^1$, then we have

$$x_1(t+1) = (x_1(t) \vee x_3(t)) \bar{\vee} u_1(t). \tag{19}$$

In vector form, we have

$$\begin{array}{rcl} x_1(t+1) & = & P_1^1 x_1(t) x_3(t) u_1(t) \\ & = & Q_1^1 u(t) x(t), \end{array} \tag{20}$$

where $x(t) = \ltimes_{i=1}^2 x_i(t)$, $u(t) = u_1(t)u_2(t)$ and

$$\begin{array}{rcl} P_1^1 & = & \delta_2[2, 1, 2, 1, 2, 1, 1, 2], \\ Q_1^1 & = & \delta_2[2, 2, 2, 2, 2, 1, 2, 1, 2, 2, 2, 2, 2, 1, 2, 1, \\ & & 1, 1, 1, 1, 1, 2, 1, 2, 1, 1, 1, 1, 1, 2, 1, 2]. \end{array}$$

(ii) If $\diamondsuit_1(t) = \rightarrow$, that is, $\xi_1 = \delta_2^2$, then we have

$$x_1(t+1) = (x_1(t) \vee x_3(t)) \rightarrow u_1(t). \tag{21}$$

In vector form, we have

$$\begin{array}{rcl} x_1(t+1) & = & P_1^2 x_1(t) x_3(t) u_1(t) \\ & = & Q_1^2 u(t) x(t), \end{array} \tag{22}$$

where

$$\begin{array}{rcl} P_1^2 & = & \delta_2[1, 2, 1, 2, 1, 2, 1, 1], \\ Q_1^2 & = & \delta_2[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \\ & & 2, 2, 2, 2, 2, 1, 2, 1, 2, 2, 2, 2, 2, 1, 2, 1]. \end{array}$$

Setting

$$Q_1 = [Q_1^1, Q_1^2],$$

it follows that

$$x_1(t+1) = Q_1 \xi_1(t) u(t) x(t). \tag{23}$$

Next, we consider $x_2(t+1)$. Similar argument shows the following:

(i) If $\diamondsuit_2(t) = \emptyset$, that is, $\xi_2 = \delta_3^1$, then we have

$$x_2(t+1) = x_2(t) \wedge x_3(t). \tag{24}$$

In vector form, we have

$$\begin{array}{rcl} x_2(t+1) & = & P_2^1 x_2(t) x_3(t) \\ & = & Q_2^1 u(t) x(t), \end{array} \tag{25}$$

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where

$$\begin{aligned} P_2^1 &= \delta_2[1,2,2,2], \\ Q_2^1 &= \delta_2[1,2,2,2,1,2,2,2,1,2,2,2,1,2,2,2, \\ &\quad 1,2,2,2,1,2,2,2,1,2,2,2,1,2,2,2]. \end{aligned}$$

(ii) If $\diamondsuit_2(t) = \rightarrow$, that is, $\xi_2 = \delta_3^2$, then we have

$$x_2(t+1) = (x_2(t) \wedge x_3(t)) \rightarrow u_2(t). \tag{26}$$

In vector form, we have

$$\begin{aligned} x_2(t+1) &= P_2^2 x_2(t) x_3(t) u_2(t) \\ &= Q_2^2 u(t) x(t), \end{aligned} \tag{27}$$

where

$$\begin{aligned} P_2^2 &= \delta_2[1,2,1,1,1,1,1,1] \\ Q_2^2 &= \delta_2[1,1,1,1,1,1,1,1,2,1,1,1,2,1,1,1, \\ &\quad 1,1,1,1,1,1,1,1,2,1,1,1,2,1,1,1]. \end{aligned}$$

(iii) If $\diamondsuit_2(t) = \vee$, that is, $\xi_2 = \delta_3^3$, then we have

$$x_2(t+1) = (x_2(t) \wedge x_3(t)) \vee u_2(t). \tag{28}$$

In vector form, we have

$$\begin{aligned} x_2(t+1) &= P_2^3 x_2(t) x_3(t) u_2(t) \\ &= Q_2^3 u(t) x(t), \end{aligned} \tag{29}$$

where

$$\begin{aligned} P_2^3 &= \delta_2[1,1,1,2,1,2,1,2] \\ Q_2^3 &= \delta_2[1,1,1,1,1,1,1,1,1,2,2,2,1,2,2,2, \\ &\quad 1,1,1,1,1,1,1,1,1,2,2,2,1,2,2,2]. \end{aligned}$$

Setting

$$Q_2 = [Q_2^1, Q_2^2, Q_2^3],$$

it follows that

$$x_2(t+1) = Q_2 \xi_2(t) u(t) x(t). \tag{30}$$

We also know that

$$x_3(t+1) = M_e x_1(t) x_3(t). \tag{31}$$

Using Proposition 2.3, we have the following expressions:

(i) System (23) can be expressed as

$$x_1(t+1) = M_1 \xi(t) u(t) x(t), \tag{32}$$

where $\xi(t) = \xi_1(t)\xi_2(t)$, $u(t) = u_1(t)u_2(t)$, $x(t) = \ltimes_{i=1}^3 x_i(t)$, and

$$M_1 = Q_1 \left(I_2 \otimes \mathbf{1}_3^T \right). \tag{33}$$

(ii) System (30) can be expressed as

$$x_2(t+1) = M_2 \xi(t) u(t) x(t), \tag{34}$$

where

$$M_2 = Q_2 \left(\mathbf{1}_2^T \otimes I_3 \right). \tag{35}$$

(iii) System (31) can be expressed as

$$x_3(t+1) = M_3 \xi(t) u(t) x(t), \tag{36}$$

where

$$M_3 = M_e \left(I_2 \otimes \mathbf{1}_2^T \right) \left(\mathbf{1}_4^T \otimes I_2 \right) \left(\mathbf{1}_6^T \otimes I_2 \right). \tag{37}$$

Finally, we have

$$x(t+1) = M \xi(t) u(t) x(t), \tag{38}$$

where

$$\begin{aligned} M &= M_1 * M_2 * M_3 \\ &= \delta_8[5,8,7,8,6,3,8,3,5,8,7,8,6,3,8,3, \\ &\quad 1,4,3,4,2,7,4,7,1,4,3,4,2,7,4,7, \\ &\quad 5,6,5,6,6,1,6,1,7,6,5,6,8,1,6,1, \\ &\quad 1,2,1,2,2,5,2,5,3,2,1,2,4,5,2,5, \\ &\quad 5,6,5,6,6,1,6,1,5,8,7,8,6,3,8,3, \\ &\quad 1,2,1,2,2,5,2,5,1,4,3,4,2,7,4,7, \\ &\quad 1,4,3,4,2,3,4,3,1,4,3,4,2,3,4,3, \\ &\quad 5,8,7,8,6,3,8,3,5,8,7,8,6,3,8,3, \\ &\quad 1,2,1,2,2,1,2,1,3,2,1,2,4,1,2,1, \\ &\quad 5,6,5,6,6,1,6,1,7,6,5,6,8,1,6,1, \\ &\quad 1,2,1,2,2,1,2,1,1,4,3,4,2,3,4,3, \\ &\quad 5,6,5,6,6,1,6,1,5,8,7,8,6,3,8,3]. \end{aligned} \tag{39}$$

III. Controllability

Define a control-transition matrix of (16) (or equivalently, (17)) as

$$T = \sum_{\mathcal{B}\, i=1}^{d_\xi} \sum_{\mathcal{B}\, j=1}^{2^m} M \delta_{d_\xi}^i \delta_{2^m}^j, \tag{40}$$

where $\sum_{\mathcal{B}}$ is the Boolean addition of Boolean matrices, that is, $1 +_{\mathcal{B}} 1 = 1$ [12].

Using T, we construct the controllability matrix as

$$\mathcal{C} := \sum_{\mathcal{B}\, i=1}^{2^n} T^{(i)}, \tag{41}$$

where $T^{(i)}$ is performed by Boolean product [12].

Similar to classical case [24], it is easy to prove the following result:

Theorem 3.1: Consider the pinning control system (16) (with its ASSR (17)).

(i) The state $\delta_{2^n}^i$ is reachable from $\delta_{2^n}^j$, if and only if, the (i,j)-th entries of $\mathcal{C}$, $c_{i,j} = 1$.

(ii) $\delta_{2^n}^i$ is reachable (from any initial state), if and only if, $\mathrm{Row}_i(\mathcal{C}) = \mathbf{1}_{2^n}^T$.

(iii) $\delta_{2^n}^j$ is controllable (to any destination state), if and only if, $\mathrm{Col}_j(\mathcal{C}) = \mathbf{1}_{2^n}$.

(iv) The pinning control system is controllable (from any initial state to any destination state), if and only if,

$$\mathcal{C} := \mathbf{1}_{2^n \times 2^n}. \tag{42}$$

Example 3.2: Recall Example 2.10. Using (38), it is easy to calculate that

$$T = \begin{bmatrix} 1&0&1&0&0&1&0&1 \\ 0&1&0&1&1&0&1&0 \\ 0&0&1&0&0&1&0&1 \\ 0&1&0&1&0&0&1&0 \\ 1&0&1&0&0&0&0&0 \\ 0&1&0&1&1&0&1&0 \\ 0&0&1&0&0&0&0&0 \\ 0&1&0&1&0&0&1&0 \end{bmatrix} \tag{43}$$

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Then a straightforward computation shows that

$$\mathcal{C} = \mathbf{1}_{8\times 8}. \tag{44}$$

According to Theorem 3.1, the pinning control system (18) is controllable.

IV. Stabilization

Definition 4.1: Consider pinning control system (16). A state x_0 is called a control-fixed state, if there exist pinning controls u_i, $i = 1, \cdots, m$ and injection modes $\diamondsuit_i$, $i = 1, \cdots, m$, such that

$$x_0 = M\xi u x_0.$$

Consider a pinning control system Σ, denote its set of control-fixed states by

$$F_c(\Sigma),$$

(or briefly, F_c). The following results are obvious:

Proposition 4.2: Consider system (16) with its control transition matrix $T = (t_{i,j})$. Then $\delta_{2^n}^i \in F_c$, if and only if, $t_{i,i} = 1$. Hence,

$$|F_c| = \mathrm{trace}(T). \tag{45}$$

Proposition 4.3: Consider system (16) with its control transition matrix $T = (t_{i,j})$ and controllability matrix $\mathcal{C}$. Then the system is stabilizable to $x_0 = \delta_{2^n}^i$, if and only if,

(1) $x_0 \in F_c$;
(2) x_0 is reachable. That is.

$$\mathrm{Row}_i(\mathcal{C}) = \mathbf{1}_{2^n}^T.$$

Example 4.4: Recall Example 2.10 again. Using (43), one sees that

$$F_c = \{\delta_8^1, \delta_8^2, \delta_8^3, \delta_8^4\}.$$

Since the system is completely controllable, it follows that the system is stabilizable by time-varying pinning controls and injection modes to any state $x_0 \in F_c$.

V. Observability

Consider a PCBNTVIM as (16) with outputs

$$y_j(t) = h_j(x_1(t), \cdots, x_n(t)), \quad j = 1, \cdots, p. \tag{46}$$

The ASSR of (46) is assumed to be

$$y(t) = Hx(t), \tag{47}$$

where $y(t) = \ltimes_{j=1}^p y_j(t)$, $H \in \mathcal{L}_{2^p\times 2^n}$ is the structure matrix of $h = (h_1, \cdots, h_p)$.

We consider the observability of system (16) and (46). Since there are several different definitions for observability of Boolean networks, we need to specify the observability concerned in this paper.

Definition 5.1: Consider a pinning control Boolean network with time-varying injection mode (PCBNTVIM) (16) and (46). It is observable, if for any two distinct initial states $x_0 \neq \tilde{x}_0$, there exist a control sequence $\{u(t),\ t = 0, 1, \cdots\}$ and a sequence of injection modes $\{\xi(t),\ t = 0, 1, \cdots\}$ such that the corresponding output sequences are distinct. That is

$$\{y(t),\ t = 0, 1, \cdots\}\big|_{x_0} \neq \{y(t),\ t = 0, 1, \cdots\}\big|_{\tilde{x}_0}.$$

Similar to standard Boolean control network, we can define any other observability corresponding to those for standard one, and prove that the above one is the most sensitive one, which means any other observability implies the above one [25].

We first briefly review the technique proposed by [4] to verify observability of PCBNTVIM. Though Definition 5.1 is a little bit different from standard one for Boolean control networks, it is easy to verify that the technique developed by [4] remains available.

Let $N = \{1, 2, \cdots, 2^n\}$ be the set of states. A set $s \subset 2^N$ is a subset of N. Its index vector, denoted by $V(s) \in \mathbb{R}^{2^n}$, is defined as

$$(V(s))_i = \begin{cases} 1, & i \in s \\ 0, & i \notin s. \end{cases}$$

Let $P_0 \in 2^{2^N}$ be the collection of initial sets of states and $P_d \in 2^{2^N}$ be the collection of destination sets of states and

$$\begin{array}{lll} P^0 & := & \left\{s_1^0, s_2^0, \cdots, s_\alpha^0\right\} \subset 2^N, \\ P^d & := & \left\{s_1^d, s_2^d, \cdots, s_\beta^d\right\} \subset 2^N. \end{array} \tag{48}$$

The index matrices of P^0 and P^d can be defined as

$$\begin{array}{lll} J_0 & := & \left[V(s_1^0) \quad V(s_2^0) \quad \cdots \quad V(s_\alpha^0)\right] \in \mathcal{B}_{2^n\times\alpha}; \\ J_d & := & \left[V(s_1^d) \quad V(s_2^d) \quad \cdots \quad V(s_\beta^d)\right] \in \mathcal{B}_{2^n\times\beta}. \end{array} \tag{49}$$

Then we have the following result.

Theorem 5.2: Consider PCBNTVIM (16) with the set of initial sets P^0 and the set of destination sets P^d as defined in (48), and the set controllability matrix is

$$\mathcal{C}_S := J_d^T \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{B}} J_0 \in \mathcal{B}_{\beta\times\alpha}. \tag{50}$$

Then

1) system (16) is set controllable from s_j^0 to s_i^d, if and only if, $c_{i,j} = 1$;
2) system (16) is set controllable at s_j^0, if and only if $\mathrm{Col}_j(\mathcal{C}_S) = \mathbf{1}_\beta$;
3) system (16) is completely set controllable, if and only if, $\mathcal{C}_S = \mathbf{1}_{\beta\times\alpha}$.

We give a corollary about set controllability, which is straightforward verifiable:

Corollary 5.3: If the original Boolean control system is completely controllable, then it is also set controllable.

Proof: Since $\mathcal{C} = \mathbf{1}_{n\times n}$, using formula (50), it is easy to see that

$$\mathcal{C}_S = \mathbf{1}_{\beta\times\alpha}.$$

The conclusion follows. ■

Using set controllability, [4] developed a technique to verify observability. Following [4], we can get the following result for verifying observability of PCBNTVIM.

First, we split the product state space $\Delta_{2^n} \times \Delta_{2^n}$ into a partition of three components as

$$D = \{zx \mid z = x\}, \tag{51}$$

$$\Theta = \{zx \mid z \neq x \text{ and } Hz = Hx\}, \tag{52}$$

$$\Xi = \{zx \mid Hz \neq Hx\}. \tag{53}$$

Then we construct an auxiliary network as

$$\begin{cases} z(t+1) = M\xi(t)u(t)z(t) \\ x(t+1) = M\xi(t)u(t)x(t), \end{cases} \tag{54}$$

and setting initial set and destination set as

$$P^0 := \bigcup_{zx\in\Theta} \{\{zx\}\} \tag{55}$$

and

$$P^d := \{\Xi\}. \tag{56}$$

Then we have the following result:

Theorem 5.4: System (16) and (46) is observable, if and only if, the auxiliary system (54) is set controllable from P^0 to P^d, which are defined in (55) and (56) respectively.

To calculate the transition matrix of auxiliary system (54), we have

$$\begin{array}{rcl} x(t+1) & = & M\xi(t)u(t)x(t) \\ & = & M\xi(t)u(t)(I_{2^n}\otimes \mathbf{1}^T_{2^n})x(t)z(t) \\ & = & M[(I_{d_\xi 2^m}\otimes (I_{2^n}\otimes \mathbf{1}^T_{2^n})]\xi(t)u(t)x(t)z(t) \\ & := & M_1\xi(t)u(t)x(t)z(t). \end{array}$$

Similarly,

$$\begin{array}{rcl} z(t+1) & = & M\xi(t)u(t)z(t) \\ & = & M\xi(t)u(t)(\mathbf{1}^T_{2^n}\otimes I_{2^n})x(t)z(t) \\ & = & M[(I_{d_\xi 2^m}\otimes (\mathbf{1}^T_{2^n}\otimes I_{2^n})]\xi(t)u(t)x(t)z(t) \\ & := & M_2\xi(t)u(t)x(t)z(t). \end{array}$$

Setting $\eta(t) = x(t)z(t)$, then (54) can be converted into its ASSR form as

$$\eta(t+1) = L\xi(t)u(t)\eta(t), \tag{57}$$

where $L = M_1 * M_2$.

Finally, using this L we can calculate the control-transition matrix T as

$$T = \sum_{\mathcal{B}\, i=1}^{d_\xi} \sum_{\mathcal{B}\, j=1}^{2^m} L\delta^i_{d_\xi}\delta^j_{2^m}. \tag{58}$$

In above standard calculation, the sizes of some inter median matrices will be large. We propose the following algorithm:

Algorithm 5.5:
- Step 1: Splitting M into $\ell = d_\xi 2^m$ blocks as

$$M = [M_{11},\cdots,M_{1,2^m},\cdots,M_{d_\xi,1},\cdots,,M_{d_\xi,2^m}], \tag{59}$$

where each $M_{i,j}\in\mathcal{L}_{2^m\times 2^m}$, which is the transition matrix for $\xi=\delta^i_{d_\xi}$, and $u=\delta^j_{2^m}$.
- Step 2: Calculate

$$L_0 := \sum_{\mathcal{B}\, i=1}^{d_\xi} \sum_{\mathcal{B}\, j=1}^{2^m} M_{i,j}. \tag{60}$$

- Step 3: Calculate

$$\begin{array}{l} T_1 := L_0(I_n\otimes \mathbf{1}^T_n) \\ T_2 := L_0(\mathbf{1}^T_n\otimes I_n) \end{array} \tag{61}$$

- Step 4: Construct the transition matrix L as:

$$L = T_1 * T_2. \tag{62}$$

Example 5.6: Recall Example 2.10 again. Using Algorithm 5.5, it is easy to calculate that the auxiliary system of (18) in the form of (54), has its controllability matrix as $\mathcal{C} = \mathbf{1}_{64\times 64}$. That is, the auxiliary system of (18) is completely controllable. Using Theorem 5.2 and Corollary 5.3, one sees that (18) with any state-depending output is observable.

VI. Time-invariant Injection Mode

Although the time-varying injection mode is easy to be modeled in the theoretic analysis, it might be difficult to be realized in engineering. Here we consider pinning control networks with time-invariant injection mode. Now the injection mode can be designed, but it will not be changed once it has been decided.

Consider the pinning control system (16) (with its ASSR (17)) . We consider the controllability with time-invariant injection mode. The injection mode can be regarded as a networked input, generated by the following input network function as

$$v(t+1) = v(t). \tag{63}$$

Then (63) has its normal form as

$$\begin{array}{rcl} v(t+1) & = & \left(\mathbf{1}^T_{2^m}\otimes I_\ell\otimes \mathbf{1}^T_{2^n}\right)u(t)v(t)x(t) \\ & := & M_1u(t)v(t)x(t), \end{array} \tag{64}$$

where

$$M_1 = \left(\mathbf{1}^T_{2^m}\otimes I_\ell\otimes \mathbf{1}^T_{2^n}\right).$$

Then we consider (16), which has the normal form of mixed input as

$$\begin{array}{rcl} x(t+1) & = & Pv(t)u(t)x(t) \\ & = & PW_{[2^m,\ell]}u(t)v(t)x(t) \\ & := & M_2u(t)v(t)x(t), \end{array} \tag{65}$$

where

$$M_2 = PW_{[2^m,\ell]}.$$

Putting (64) and (65) together, we can obtain the auxiliary network as

$$w(t+1) = Mu(t)w(t), \tag{66}$$

where $w(t) = v(t)x(t)$,

$$M = M_1 * M_2.$$

The index matrices of P^0 and P^d can be defined by equivalent class as

$$J_0 = J_d := J = \mathbf{1}_\ell \otimes I_{2^n}. \tag{67}$$

Then the set controllability matrix of (66) is

$$\mathcal{C}_S = J^T\mathcal{C}J. \tag{68}$$

where $\mathcal{C}$ is the controllability matrix of (16).

From the construction, we have the following result.

Theorem 6.1: Consider a pinning control Boolean network with time-invariant injection mode (16), the controllability of (16) can be converted into the set controllability of the auxiliary system (66), which is determined by its set controllability matrix $\mathcal{C}_S = (c_{ij})$. Particularly, we have the following itemized results:

1) The pinning control system (16) with time-invariant injection mode is controllable from state $\delta_{2^n}^i$ to $\delta_{2^n}^j$, if and only if, $c_{i,j}=1$.
2) $\delta_{2^n}^i$ is reachable (from any initial state), if and only if, $\mathrm{Row}_i(\mathcal{C}_S)=\mathbf{1}_{2^n}^T$.
3) $\delta_{2^n}^j$ is controllable (to any destination state), if and only if, $\mathrm{Col}_j(\mathcal{C}_S)=\mathbf{1}_{2^n}$.
4) The pinning control system is controllable (16) with time-invariant injection mode (from any initial state to any destination state), if and only if, $\mathcal{C}_S=\mathbf{1}_{2^n\times 2^n}$.

Example 6.2: Recall Example2.10. Assuming the injection mode is time-invariant. We construct the auxiliary network:

$$\begin{array}{l} v(t+1)=M_1u(t)v(t)x(t)\\ x(t+1)=M_2u(t)v(t)x(t), \end{array} \tag{69}$$

where

$$M_1=\mathbf{1}_{2^m}^T\otimes I_6\otimes \mathbf{1}_8^T.$$

Using(39),

$$\begin{array}{rcl} M_2 &=& PW_{[4,6]}\\ &=& \delta_8[5,8,7,8,6,3,8,3,5,6,5,6,6,1,6,1,\\ && 5,6,5,6,6,1,6,1,1,4,3,4,2,3,4,3,\\ && 1,2,1,2,2,1,2,1,1,2,1,2,2,1,2,1,\\ && 5,8,7,8,6,3,8,3,7,6,5,6,8,1,6,1,\\ && 5,8,7,8,6,3,8,3,1,4,3,4,2,3,4,3,\\ && 3,2,1,2,4,1,2,1,1,4,3,4,2,3,4,3,\\ && 1,4,3,4,2,7,4,7,1,2,1,2,2,5,2,5,\\ && 1,2,1,2,2,5,2,5,5,8,7,8,6,3,8,3,\\ && 5,6,5,6,6,1,6,1,5,6,5,6,6,1,6,1,\\ && 1,4,3,4,2,7,4,7,3,2,1,2,4,5,2,5,\\ && 1,4,3,4,2,7,4,7,5,8,7,8,6,3,8,3,\\ && 7,6,5,6,8,1,6,1,5,8,7,8,6,3,8,3] \end{array}$$

Using $M_1,\ M_2$, it can be calculated in turn that

$$M=M_1*M_2,$$

$$\mathcal{C}=\sum_{\mathcal{B}\, i=1}^{6}\sum_{\mathcal{B}\, j=1}^{4} M\delta_6^i\delta_4^j,$$

$$J=\mathbf{1}_6\otimes I_8.$$

Finally

$$\mathcal{C}_S=J^T\mathcal{C}J=\mathbf{1}_{8\times 8}.$$

According to Theorem6.1, the pinning control network with time-invariant injection mode(18) is controllable.

All other problems such as observability, stabilization etc. via time-invariant injection mode can also be handled in a similar way.

VII. Conclusion

In this paper some fundamental control problems, including controllability, stabilization, observability, of pinning control Boolean network were investigated. Unlike existing results, the time-varying injection mode was firstly proposed and investigated. A fundamental contribution is proposing a method to convert a pinning control system into a (almost) standard Boolean control system, to which the standard STP approach is applicable. The controllability of pinning control networks with time-invariant injection mode is then also studied, which shows the relationship between existing results and our work. The technique developed is essentially to provide a new framework for formulating pinning control Boolean networks. Hence, it can also be used for investigating other control problems of pinning control systems.

Choosing pinning modes efficiently is a challenging problem. Properly choosing injection mode might reduce the number of pinning nodes. These problems are interesting and fundamental. We leave them for further study.

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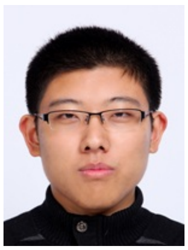

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