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Distributed Controller Design and Analysis of Second-Order Signed Networks With Communication Delays

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***Abstract*— This article concentrates on dealing with distributed control problems for second-order signed networks subject to not only cooperative but also antagonistic interactions. A distributed control protocol is proposed based on the nearest neighbor rules, with which necessary and sufficient conditions are developed for consensus of second-order signed networks whose communication topologies are described by strongly connected signed digraphs. Besides, another distributed control protocol in the presence of a communication delay is designed, for which a time margin of the delay can be determined simultaneously. It is shown that under the delay margin condition, necessary and sufficient consensus results can be derived even though second-order signed networks with a communication delay are considered. Simulation examples are included to illustrate the validity of our established consensus results of second-order signed networks.**

***Index Terms*— Antagonistic interaction, communication delay, consensus, distributed control, second-order signed network.**

I. Introduction

TRADITIONAL networked systems (TNSs) are constituted by a group of agents, involving pure cooperative interactions among agents. Unsigned digraphs are usually used to describe the communication topologies of TNSs, in which nodes and edges with positive weights denote agents and cooperative interactions among agents, respectively. Practically, there may exist not only cooperative but also antagonistic interactions in networked systems, especially in the area of social networks. This class of networked systems is named signed networked systems (SNSs), where the communication description should resort to signed digraphs whose edges with negative weights can denote antagonistic interactions among agents. SNSs have received more and more attention due to its wide applications in, e.g., opinion dynamics, economic analysis, and political science (see [1] for more details).

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Over the past decade, distributed control problems of TNSs have become one of the hottest topics in the field of automation control [2]–[27]. Consensus means that the states of all nodes converge to a common value, which is a fundamental problem in investigating the distributed control of TNSs. In [2]–[6], distributed protocols are designed to guarantee the consensus of TNSs with single integrator dynamics, where topological conditions play an important role in the convergence analysis. When the dynamics of TNSs are double integrators, distributed control problems are investigated in [7]–[14], from which we can realize that the consensus of TNSs with double integrator dynamics is not just a simple extension of that in the presence of single integrator dynamics. In order to ensure the consensus, some limitations need to be imposed on the control gains of protocols other than topological conditions. In [15] and [16], given second-order TNSs without and with the communication delay, necessary and sufficient conditions are proposed to make sure the consensus of all nodes. It is worth noting that second-order TNSs include TNSs with double integrator as a particular case. For TNSs with general linear dynamics, consensus issues are addressed in [17]–[21]. When TNSs consist of the unknown nonlinear dynamics, distributed control protocols are designed such that the consensus objective can be achieved [22]–[27].

Recently, distributed control of SNSs has become an interesting focus on the domain of networked systems. Different from TNSs, SNSs may exhibit plentifully collective behaviors because there exist antagonistic interactions among nodes. In [28], a fundamental framework is established for investigating distributed control problems of SNSs. A distributed protocol is given to ensure the bipartite consensus when the corresponding communication topologies are strongly connected. When the strong connectivity is relaxed to quasi-strong connectivity (or containing a spanning tree), all rooted nodes can still achieve the bipartite consensus and all nonrooted nodes terminally lie in the interval that is constructed by two polarized values of rooted nodes, which is called interval bipartite consensus [29]–[31]. When signed digraphs are not connected, all nodes spread in the convex hull that is formed by the convergence values of leaders [32]. In addition, many other kinds of collective behaviors are discovered, such as modulus consensus [33]–[35], bipartite formation [36], finite-time bipartite consensus [37]–[41], bipartite tracking consensus [42], and bipartite swarming behavior [43].

The abovementioned literature [28]–[43] only centers on the SNSs with single integrator dynamics. Meanwhile, the finite-time bipartite consensus problems are studied for SNSs subject to double integrator dynamics in [44] and [45], based on which distributed protocols are designed to ensure all nodes in face of bounded external disturbances reaching the bipartite consensus within a finite time. Given SNSs with general linear dynamics, the condition for bipartite consensus is offered in [46]–[51]. Furthermore, bipartite consensus issues are discussed for SNSs suffering from the nonlinear coupling in [52] and [53]. Until now, there have been no studies on the distributed control problems of second-order SNSs with oscillating dynamics [54], [55].

In this article, we focus on second-order SNSs under strongly connected signed digraphs and explore the convergence behaviors of them. A distributed protocol is provided for SNSs based on the nearest neighbor rule. We can derive the necessary and sufficient conditions for bipartite consensus and state stability, in which both topological structures and control gains of the distributed protocol are essential. Besides, we can calculate the mathematical expressions for the steady states of second-order SNSs subject to different system parameters and disclose new collective behaviors. In contrast to [28]–[32], the steady states of second-order SNSs may be not convergent any longer, where they can accomplish a time-varying trajectory with the constant velocity or a sine/cosine function. When considering SNSs in the presence of a communication delay, we can induce a time margin for the communication delay. Benefitting from the time margin, we can develop the necessary and sufficient consensus results for second-order SNSs. It is shown that the bipartite consensus (respectively, state stability) objective can be achieved if and only if the corresponding signed digraphs are structurally balanced (respectively, unbalanced) and the communication delay is less than the time margin. In addition, four simulation examples are given to demonstrate our derived theoretical results.

The rest of this article is organized as follows. In Section II, some preliminaries for signed digraphs and problem descriptions are provided. The bipartite consensus and state stability issues for second-order SNSs without communication delays are discussed in Section III, based on which consensus results are established for second-order SNSs with a communication delay in Section IV. Simulation examples and conclusions are given in Sections V and VI, respectively.

Notations: For a positive integer n, we denote $\mathcal{F}_n = \{1, 2, \ldots, n\}$, $1_n = [1, 1, \ldots, 1]^T \in \mathbb{R}^n$, $0_n = [0, 0, \ldots, 0]^T \in \mathbb{R}^n$, $\operatorname{diag}\{d_1, d_2, \ldots, d_n\}$ as a diagonal matrix whose diagonal elements are $d_1, d_2, \ldots, d_n$ and nondiagonal elements are zero, and $|a|$ as the absolute value of a real number $a \in \mathbb{R}$. For a complex number $c \in \mathbb{C}$, let $\operatorname{Re}(c)$ and $\operatorname{Im}(c)$ be the real part and imaginary part of c, respectively. The complex number c can be written as $c = \operatorname{Re}(c) + \operatorname{Im}(c)\imath$, where $\imath^2 = -1$. We denote $|c|$ as the modulus of c, i.e., $|c| = (\operatorname{Re}^2(c) + \operatorname{Im}^2(c))^{1/2}$. In addition, let $\operatorname{sgn}(a)$ represent the sign function of any real scalar $a \in \mathbb{R}$.

II. Problem Descriptions

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a signed digraph including a node set $\mathcal{V} = \{\upsilon_1, \upsilon_2, \ldots, \upsilon_n\}$, an edge set $\mathcal{E} \subseteq \{(\upsilon_i, \upsilon_j) : \upsilon_i, \upsilon_j \in \mathcal{V}\}$, and an adjacency weight matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ whose entries satisfy $a_{ij} \neq 0 \Leftrightarrow (\upsilon_j, \upsilon_i) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. Let $\mathcal{G}$ have no self-loops, i.e., $a_{ii} = 0$, $\forall i \in \mathcal{F}_n$. An edge (υ_i, υ_j) denotes that υ_j can receive the information from υ_i, and υ_i is called a neighbor of υ_j. The index set $N(j) = \{i : (\upsilon_i, \upsilon_j) \in \mathcal{E}\}$ represents all neighbors of υ_j. $\mathcal{P} = \{(\upsilon_{m_0}, \upsilon_{m_1}), (\upsilon_{m_1}, \upsilon_{m_2}), \ldots, (\upsilon_{m_k-1}, \upsilon_{m_k})\}$ denotes a directed path of length k, in which $\upsilon_{m_0}, \upsilon_{m_1}, \ldots, \upsilon_{m_k}$ are distinct nodes. The signed digraph $\mathcal{G}$ is strongly connected if there exists a directed path from every node to every other node. The Laplacian matrix of $\mathcal{G}$, denoted by L, satisfies $L = \operatorname{diag}\{\sum_{j=1}^n |a_{1j}|, \sum_{j=1}^n |a_{2j}|, \ldots, \sum_{j=1}^n |a_{nj}|\} - \mathcal{A}$. The signed digraph $\mathcal{G}$ is said to be structurally balanced if its node set $\mathcal{V}$ can be divided into two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ that fulfill $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ such that $a_{ij} \geq 0$ if $\upsilon_i, \upsilon_j \in \mathcal{V}_1$ or $\upsilon_i, \upsilon_j \in \mathcal{V}_2$, and $a_{ij} \leq 0$ if $\upsilon_i \in \mathcal{V}_1, \upsilon_j \in \mathcal{V}_2$ or $\upsilon_j \in \mathcal{V}_1, \upsilon_i \in \mathcal{V}_2$. Otherwise, we say that $\mathcal{G}$ is structurally unbalanced.

We give an induced unsigned digraph $\overline{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \overline{\mathcal{A}})$ from $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, in which $\overline{\mathcal{A}} = [|a_{ij}|] \in \mathbb{R}^{n\times n}$. Besides, the Laplacian matrix of $\overline{\mathcal{G}}$ is $\overline{L} = \operatorname{diag}\{\sum_{j=1}^n |a_{1j}|, \sum_{j=1}^n |a_{2j}|, \ldots, \sum_{j=1}^n |a_{nj}|\} - \overline{\mathcal{A}}$. A set of all gauge transformations is

$$\mathcal{D}_n = \{D_n = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} : \sigma_i \in \{-1, 1\}, i \in \mathcal{F}_n\}.$$

From [28], if $\mathcal{G}$ is structurally balanced, then there exists a gauge transformation $D_n \in \mathcal{D}_n$ such that $\overline{L} = D_n L D_n$ holds.

For every node υ_i, $\forall i \in \mathcal{F}_n$, its dynamics fulfills

$$\begin{cases} \dot{x}_i(t) = \nu_i(t) \\ \dot{\nu}_i(t) = \alpha x_i(t) + \beta \nu_i(t) + u_i(t) \end{cases} \tag{1}$$

where $x_i(t) \in \mathbb{R}$, $\nu_i(t) \in \mathbb{R}$, and $u_i(t) \in \mathbb{R}$ denote the position state, velocity state, and control protocol to be designed of υ_i, respectively, and $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ are the system parameters. It is worthwhile noticing that the practical systems described by (1) are common in our daily life, such as pendulum system, mass-spring system, negative-resistance oscillator system, and Euler–Lagrange mechanical system.

For arbitrary initial states $x_i(0)$ and $\nu_i(0)$, $\forall i \in \mathcal{F}_n$, we say that the system (1) achieves:

1) bipartite consensus if

$$\lim_{t\to\infty} (|x_i(t)| - |x_j(t)|) = 0$$
$$\lim_{t\to\infty} (|\nu_i(t)| - |\nu_j(t)|) = 0 \quad \forall i, j \in \mathcal{F}_n;$$

2) state stability if

$$\lim_{t\to\infty} x_i(t) = 0, \quad \lim_{t\to\infty} \nu_i(t) = 0 \quad \forall i \in \mathcal{F}_n.$$

In the following, we are interested in studying how to ensure that the system (1) without or with a communication delay can accomplish the bipartite consensus and state stability under a strongly connected signed digraph $\mathcal{G}$.

III. Consensus Results of Second-Order SNSs

A distributed control protocol is given by

$$u_i(t) = -k_1 \sum_{j\in N(i)} |a_{ij}|[x_i(t) - \mathrm{sgn}(a_{ij})x_j(t)] - k_2 \sum_{j\in N(i)} |a_{ij}|[\nu_i(t) - \mathrm{sgn}(a_{ij})\nu_j(t)], \quad i \in \mathcal{F}_n \tag{2}$$

where k_1 and k_2 are control gains. Let $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ and $\nu(t) = [\nu_1(t), \nu_2(t), \ldots, \nu_n(t)]^T$ denote the position state vector and the velocity state vector, respectively. With L, we can rewrite (1) and (2) as a compact form of

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\nu}(t) \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ \alpha I_n - k_1 L & \beta I_n - k_2 L \end{bmatrix} \begin{bmatrix} x(t) \\ \nu(t) \end{bmatrix} \triangleq \Psi \begin{bmatrix} x(t) \\ \nu(t) \end{bmatrix}. \tag{3}$$

Next, we explore the behavior analysis of the system (3). Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ represent the eigenvalues of L. Without loss of generality, it follows from [28] that the following statements hold.

1) $\lambda_1 = 0$ and $\lambda_2, \lambda_3, \ldots, \lambda_n$ have positive real parts when $\mathcal{G}$ is strongly connected and structurally balanced.
2) All eigenvalues of L have positive real parts when $\mathcal{G}$ is strongly connected and structurally unbalanced.

When $\mathcal{G}$ is structurally balanced, we will introduce a tree-type transformation for second-order SNSs. Based on this tree-type transformation, a reduced-order system can be derived, in which bipartite consensus problems of second-order SNSs are equivalently converted into stability problems of the reduced-order system. To be specific, since $\mathcal{G}$ is structurally balanced, a matrix $D_n = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \in \mathcal{D}_n$ satisfying $\overline{L} = D_n L D_n$ exists. Define $\hat{x} = Qx$ and $\hat{\nu} = Q\nu$, where

$$Q = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ \sigma_1 & -\sigma_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_1 & 0 & 0 & \cdots & -\sigma_n \end{bmatrix} \triangleq \begin{bmatrix} C \in \mathbb{R}^{1\times n} \\ E \in \mathbb{R}^{(n-1)\times n} \end{bmatrix}.$$

Obviously, Q is invertible and

$$Q^{-1} = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ \sigma_2 & -\sigma_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_n & 0 & 0 & \cdots & -\sigma_n \end{bmatrix} \triangleq [\, D_n 1_n \quad F \in \mathbb{R}^{n\times(n-1)} \,].$$

Substituting $\hat{x} = Qx$ and $\hat{\nu} = Q\nu$ into (3) can lead to

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\hat{\nu}}(t) \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ \alpha I_n - k_1 QLQ^{-1} & \beta I_n - k_2 QLQ^{-1} \end{bmatrix} \times \begin{bmatrix} \hat{x}(t) \\ \hat{\nu}(t) \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ \alpha I_n - k_1 H & \beta I_n - k_2 H \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{\nu}(t) \end{bmatrix} \tag{4}$$

where $H = \left[\begin{smallmatrix} 0 & CLF \\ 0_{n-1} & ELF \end{smallmatrix}\right]$, and $LD_n 1_n = 0_n$ is inserted. Denote $\hat{x}^1(t) = Cx(t) = \sigma_1 x_1(t)$, $\hat{\nu}^1(t) = C\nu(t) = \sigma_1 \nu_1(t)$, $\hat{x}^2(t) = Ex(t) \in \mathbb{R}^{n-1}$, and $\hat{\nu}^2(t) = E\nu(t) \in \mathbb{R}^{n-1}$. Thus, we revisit (4) and can derive two subsystems as

$$\begin{bmatrix} \dot{\hat{x}}^1(t) \\ \dot{\hat{\nu}}^1(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} \hat{x}^1(t) \\ \hat{\nu}^1(t) \end{bmatrix} - \begin{bmatrix} 0_{n-1}^T & 0_{n-1}^T \\ k_1 CLF & k_2 CLF \end{bmatrix} \begin{bmatrix} \hat{x}^2(t) \\ \hat{\nu}^2(t) \end{bmatrix} \tag{5}$$

and

$$\begin{bmatrix} \dot{\hat{x}}^2(t) \\ \dot{\hat{\nu}}^2(t) \end{bmatrix} = \begin{bmatrix} 0_{(n-1)\times(n-1)} & I_{n-1} \\ \alpha I_{n-1} - k_1 ELF & \beta I_{n-1} - k_2 ELF \end{bmatrix} \times \begin{bmatrix} \hat{x}^2(t) \\ \hat{\nu}^2(t) \end{bmatrix} \triangleq \Phi \begin{bmatrix} \hat{x}^2(t) \\ \hat{\nu}^2(t) \end{bmatrix}. \tag{6}$$

From $\hat{x}^2(t) = Ex(t)$ and $\hat{\nu}^2(t) = E\nu(t)$, it is immediate to develop that the system (3) can achieve the bipartite consensus if and only if the reduced-order system (6) converges to zero. By the fact of

$$QLQ^{-1} = \begin{bmatrix} 0 & CLF \\ 0_{n-1} & ELF \end{bmatrix}$$

we realize that the matrix ELF is positive stable (namely, all the eigenvalues of it have positive real parts). To proceed, we calculate the characteristic polynomial of Φ as

$$\begin{aligned} \det(sI_{2n-2} - \Phi) &= \begin{vmatrix} sI_{n-1} & -I_{n-1} \\ -\alpha I_{n-1} + k_1 ELF & sI_{n-1} - \beta I_{n-1} + k_2 ELF \end{vmatrix} \\ &= |s^2 I_{n-1} - \beta I_{n-1} s + k_2 ELFs - \alpha I_{n-1} + k_1 ELF|. \end{aligned}$$

Clearly, there exist two invertible matrices $P \in \mathbb{R}^{(n-1)\times(n-1)}$ and $P^{-1} \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$P^{-1} ELFP \triangleq \Lambda = \begin{bmatrix} \lambda_2 & * & & \\ & \lambda_3 & \ddots & \\ & & \ddots & * \\ & & & \lambda_n \end{bmatrix}$$

where the element $*$ may be 0 or 1. We can further deduce

$$\begin{aligned} \det(sI_{2n-2} - \Phi) &= |s^2 I_{n-1} - \beta I_{n-1} s + k_2 \Lambda s - \alpha I_{n-1} + k_1 \Lambda| \\ &= \prod_{i=2}^{n} f_i(s) \end{aligned}$$

where $f_i(s) = s^2 - (\beta - k_2\lambda_i)s + k_1\lambda_i - \alpha$. Therefore, $2n-2$ eigenvalues of the matrix Φ can be developed by solving the following $n-1$ characteristic equations:

$$s^2 - (\beta - k_2\lambda_i)s + k_1\lambda_i - \alpha = 0 \quad \forall i \in \{2, 3, \ldots, n\}. \tag{7}$$

When $\mathcal{G}$ is structurally unbalanced, all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of L have positive real parts. Similarly, one can derive the characteristic polynomial of Ψ as follows:

$$\det(sI_{2n} - \Psi) = \prod_{i=1}^{n} f_i(s)$$

where $f_i(s) = s^2 - (\beta - k_2\lambda_i)s + k_1\lambda_i - \alpha$.

The following theorem can establish the bipartite consensus and state stability results for second-order SNSs.

Theorem 1: Consider a strongly connected signed digraph $\mathcal{G}$, and let the gains k_1 and k_2 satisfy

$$\begin{aligned} k_2^2\mathrm{Im}^2(\lambda_i)+4k_1\mathrm{Re}(\lambda_i)-4\alpha &> 0\\ \beta-k_2\mathrm{Re}(\lambda_i) &\neq 0\\ \frac{k_1^2\mathrm{Im}^2(\lambda_i)}{(\beta-k_2\mathrm{Re}(\lambda_i))^2}+\frac{k_1k_2\mathrm{Im}^2(\lambda_i)}{\beta-k_2\mathrm{Re}(\lambda_i)}-k_1\mathrm{Re}(\lambda_i)+\alpha &< 0\\ k_1k_2|\lambda_i|^2+\alpha\beta-(k_1\beta+k_2\alpha)\mathrm{Re}(\lambda_i) &> 0 \end{aligned} \tag{8}$$

for each nonzero eigenvalue λ_i, $\forall i\in\mathcal{F}_n$. Then, for any initial state conditions $x(0)$ and $\nu(0)$, the following two results hold for the system (3).

1) The bipartite consensus can be achieved if and only if $\mathcal{G}$ is structurally balanced. Besides, the steady state of the system (3) can be determined in two cases.

   a) If $\beta^2+4\alpha=0$, then

   $$\lim_{t\to\infty}\left\{\begin{bmatrix}x(t)\\ \nu(t)\end{bmatrix}-e^{\frac{\beta}{2}t}\left(\xi_1\zeta_1^T+\xi_1\zeta_2^Tt+\xi_2\zeta_2^T\right)\times\begin{bmatrix}x(0)\\ \nu(0)\end{bmatrix}\right\}=0_{2n} \tag{9}$$

   where ξ_i (respectively, ζ_i) is the right (respectively, left) eigenvector or generalized eigenvector of Ψ for the eigenvalue $(\beta/2)$ for each $i\in\{1,2\}$ such that

   $$\Psi\xi_1=\frac{\beta}{2}\xi_1,\quad \zeta_1^T\Psi=\frac{\beta}{2}\zeta_1^T+\zeta_2^T,\ \zeta_1^T\xi_1=1$$

   and

   $$\Psi\xi_2=\xi_1+\frac{\beta}{2}\xi_2,\quad \zeta_2^T\Psi=\frac{\beta}{2}\zeta_2^T,\ \zeta_2^T\xi_2=1.$$

   b) Otherwise, if $\beta^2+4\alpha\neq 0$, then

   $$\lim_{t\to\infty}\left\{\begin{bmatrix}x(t)\\ \nu(t)\end{bmatrix}-\left(e^{\lambda_{11}t}\xi_1\zeta_1^T+e^{\lambda_{12}t}\xi_2\zeta_2^T\right)\times\begin{bmatrix}x(0)\\ \nu(0)\end{bmatrix}\right\}=0_{2n} \tag{10}$$

   where λ_{11} and λ_{12} are the roots of the characteristic equation $s^2-\beta s-\alpha=0$, and ξ_i (respectively, ζ_i) is the right (respectively, left) eigenvector of Ψ associated with the eigenvalue λ_{1i}, $i\in\{1,2\}$ such that

   $$\Psi\xi_1=\lambda_{11}\xi_1,\quad \zeta_1^T\Psi=\lambda_{11}\zeta_1^T,\ \zeta_1^T\xi_1=1$$

   and

   $$\Psi\xi_2=\lambda_{12}\xi_2,\quad \zeta_2^T\Psi=\lambda_{12}\zeta_2^T,\ \zeta_2^T\xi_2=1.$$

2) The state stability can be accomplished if and only if $\mathcal{G}$ is structurally unbalanced.

To prove Theorem 1, we need the following lemma that can build necessary and sufficient conditions for Hurwitz stability of $f_i(s)$ corresponding to the nonzero eigenvalue λ_i, $i\in\mathcal{F}_n$.

Lemma 1: For the nonzero eigenvalue $\lambda_i\neq 0$, $i\in\mathcal{F}_n$, $f_i(s)$ is Hurwitz stable if and only if k_1 and k_2 satisfy (8).

Proof: See Appendix A. ■

As a benefit of Lemma 1, Theorem 1 can be presented in an alternative description as follows.

Theorem 2: Consider the system (3) under a strongly connected signed digraph $\mathcal{G}$. For any initial state conditions $x(0)$ and $\nu(0)$, the following results hold.

1) When $\mathcal{G}$ is structurally balanced, the system (3) achieves the bipartite consensus if and only if the two gains k_1 and k_2 satisfy the condition (8) for all nonzero eigenvalues $\lambda_i\neq 0$, $\forall i\in\mathcal{F}_n$.
2) When $\mathcal{G}$ is structurally unbalanced, the system (3) reaches the state stability if and only if the two gains k_1 and k_2 satisfy the condition (8) for all eigenvalues λ_i, $\forall i\in\mathcal{F}_n$.

In comparison with Theorem 1, Theorem 2 reveals that we can also develop necessary and sufficient selection conditions on the gains k_1 and k_2 to establish consensus results of SNSs. It is shown that the consensus objective can be accomplished by the system (3) if k_1 and k_2 satisfy the condition (8) and the system (3) can not realize the consensus objective, otherwise.

With the development of Lemma 1, we will exhibit the proof of Theorem 1.

Proof of Theorem 1: Sufficiency: When the signed digraph $\mathcal{G}$ is structurally balanced, we can verify from the relationship between the system (3) and its subsystem (6) that the following four statements are equivalent.

a) The system (3) achieves the bipartite consensus.
b) The subsystem (6) is asymptotically stable.
c) All eigenvalues of Φ have negative real parts.
d) $f_i(s)$ is Hurwitz stable, $\forall i\in\{2,3,\ldots,n\}$.

When $\mathcal{G}$ is structurally unbalanced, we can demonstrate that the following three statements are equivalent.

a) The system (3) reaches the state stability.
b) All eigenvalues of Ψ have negative real parts.
c) $f_i(s)$ is Hurwitz stable, $\forall i\in\mathcal{F}_n$.

Because k_1 and k_2 satisfy (8) for each nonzero eigenvalue λ_i, it follows from Lemma 1 that the system (3) can achieve the bipartite consensus (respectively, state stability) when $\mathcal{G}$ is structurally balanced (respectively, unbalanced).

Necessity: Actually, the necessity of them can be derived by the mutually exclusive relationship between the structural balance and unbalance of the signed digraph $\mathcal{G}$.

Next, we calculate the steady states of $x(t)$ and $\nu(t)$ when $\mathcal{G}$ is structurally balanced. The solution of the system (3) is

$$\begin{bmatrix}x(t)\\ \nu(t)\end{bmatrix}=e^{\Psi t}\begin{bmatrix}x(0)\\ \nu(0)\end{bmatrix}. \tag{11}$$

The system (3) reaching the bipartite consensus implies that the subsystem (6) is asymptotically stable. Hence, the steady states of $x(t)$ and $\nu(t)$ are dependent on the subsystem (5) whose solution is provided by

$$\begin{bmatrix}\hat{x}^1(t)\\ \hat{\nu}^1(t)\end{bmatrix}=e^{At}\begin{bmatrix}\hat{x}^1(0)\\ \hat{\nu}^1(0)\end{bmatrix}-\int_0^t e^{A(t-\tau)}B\begin{bmatrix}\hat{x}^2(\tau)\\ \hat{\nu}^2(\tau)\end{bmatrix}d\tau \tag{12}$$

where $A=\left[\begin{smallmatrix}0&1\\ \alpha&\beta\end{smallmatrix}\right]$ and $B=\left[\begin{smallmatrix}0_{n-1}^T&0_{n-1}^T\\ k_1CLF&k_2CLF\end{smallmatrix}\right]$. It is obvious from (12) that two eigenvalues of A play an important role in the terminal state of the subsystem (5). Two eigenvalues

λ_{11} and λ_{12} of A can be induced by solving the characteristic equation of A as follows:

$$s^2 - \beta s - \alpha = 0. \tag{13}$$

Based on (13), we have

$$\lambda_{11} = \frac{\beta + \sqrt{\beta^2 + 4\alpha}}{2} \text{ and } \lambda_{12} = \frac{\beta - \sqrt{\beta^2 + 4\alpha}}{2} \tag{14}$$

where λ_{11} and λ_{12} are also the eigenvalues of Ψ due to

$$\begin{aligned}\det(sI_{2n} - \Psi) &= \det\left(sI_{2n} - \begin{bmatrix} 0_{n\times n} & I_n \\ \alpha I_n - k_1 H & \beta I_n - k_2 H \end{bmatrix}\right) \\ &= \det(s^2 I_n - \beta I_n s + k_2 H s - \alpha I_n + k_1 H) \\ &= \det(s^2 - \beta s - \alpha)\det(sI_{2n-2} - \Phi).\end{aligned}$$

From (14), we consider the following two cases separately.

Case a): $\beta^2 + 4\alpha = 0$. In this case, we can deduce $\lambda_{11} = \lambda_{12} = (\beta/2)$, which implies that the algebraic multiplicity of $(\beta/2)$ is two. To proceed, we calculate the geometric multiplicity of $(\beta/2)$. We introduce a matrix

$$\Theta = \frac{\beta}{2} I_{2n} - \Psi = \begin{bmatrix} \frac{\beta}{2} I_n & -I_n \\ \frac{\beta^2}{4} I_n + k_1 L & -\frac{\beta}{2} I_n + k_2 L \end{bmatrix}.$$

If $\beta = 0$ holds, then the matrix Θ becomes

$$\Theta = \begin{bmatrix} 0_n & -I_n \\ k_1 L & k_2 L \end{bmatrix}. \tag{15}$$

Because $\mathcal{G}$ is structurally balanced, we have $\operatorname{rank}(L) = n - 1$. This, together with (15), guarantees $\operatorname{rank}(\Theta) = 2n - 1$.

If $\beta \neq 0$ holds, then with the elementary column operations, we can obtain

$$\Theta \begin{bmatrix} I_n & \frac{2}{\beta} I_n \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} \frac{\beta}{2} I_n & 0_{n\times n} \\ \frac{\beta^2}{4} I_n + k_1 L & \left(\frac{2}{\beta} k_1 + k_2\right) L \end{bmatrix}. \tag{16}$$

It can be easily seen from (16) that $\operatorname{rank}(\Theta) = 2n - 1$ holds. Therefore, we know $\operatorname{rank}(\Theta) = 2n-1$ regardless of whether β is zero or not, which indicates that the geometric multiplicity of the eigenvalue $(\beta/2)$ is one. Therefore, we can obtain the Jordan canonical form of Ψ as

$$J = \begin{bmatrix} \frac{\beta}{2} & 1 & 0_{1\times(2n-2)} \\ 0 & \frac{\beta}{2} & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & J' \end{bmatrix}$$

where $J' \in \mathbb{C}^{(2n-2)\times(2n-2)}$ is an upper diagonal block matrix whose diagonal entries are the rest $2n-2$ nonzero eigenvalues of Ψ. The condition (8) can ensure that all diagonal entries of J' have negative real parts. We can employ J to deduce

$$\begin{aligned} e^{\Psi t} &= \hat{Q} e^{Jt} \hat{Q}^{-1} = [\, \xi_1 \quad \xi_2 \quad \cdots \quad \xi_{2n} \,] \\ &\quad \times \begin{bmatrix} e^{\frac{\beta}{2}t} & t e^{\frac{\beta}{2}t} & 0_{1\times(2n-2)} \\ 0 & e^{\frac{\beta}{2}t} & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & e^{J't} \end{bmatrix} \begin{bmatrix} \zeta_1^T \\ \zeta_2^T \\ \vdots \\ \zeta_{2n}^T \end{bmatrix} \end{aligned} \tag{17}$$

where $\xi_i \in \mathbb{R}^{2n}$ (respectively, $\zeta_i \in \mathbb{R}^{2n}$), $\forall i \in \{1, 2, \ldots, 2n\}$, is the right eigenvector or right generalized eigenvector (respectively, left eigenvector or left generalized eigenvector) for $2n$ eigenvalues of Ψ. With $\Psi\hat{Q} = \hat{Q}J$ and $\hat{Q}^{-1}\Psi = J\hat{Q}^{-1}$, we can verify

$$\begin{aligned} \Psi\xi_1 &= \frac{\beta}{2}\xi_1, \quad \zeta_1^T\Psi = \frac{\beta}{2}\zeta_1^T + \zeta_2^T \\ \Psi\xi_2 &= \xi_1 + \frac{\beta}{2}\xi_2, \quad \zeta_2^T\Psi = \frac{\beta}{2}\zeta_2^T \end{aligned}$$

where the vectors ξ_1 and ξ_2 (respectively, ζ_2 and ζ_1) represent the right (respectively, left) eigenvector and right (respectively, left) generalized eigenvector for the eigenvalue $(\beta/2)$ of Ψ. Since all diagonal elements of the matrix J' have negative real parts, it is immediate to derive the steady state (9) of the system (3) from (11) and (17).

Case b): $\beta^2 + 4\alpha \neq 0$. In this case, λ_{11} and λ_{12} are different. Thus, the Jordan canonical form of Ψ can be written as

$$J = \begin{bmatrix} \lambda_{11} & 0 & 0_{1\times(2n-2)} \\ 0 & \lambda_{12} & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & J' \end{bmatrix}$$

which leads to

$$\begin{aligned} e^{\Psi t} &= \hat{Q} e^{Jt} \hat{Q}^{-1} = [\, \xi_1 \quad \xi_2 \quad \cdots \quad \xi_{2n} \,] \\ &\quad \times \begin{bmatrix} e^{\lambda_{11}t} & 0 & 0_{1\times(2n-2)} \\ 0 & e^{\lambda_{12}t} & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & e^{J't} \end{bmatrix} \begin{bmatrix} \zeta_1^T \\ \zeta_2^T \\ \vdots \\ \zeta_{2n}^T \end{bmatrix}. \end{aligned} \tag{18}$$

Because of all diagonal elements of J' with negative real parts, we can induce the steady state (10) of the system (3) from (11) and (18). We complete the proof of Theorem 1. ■

Remark 1: From Theorem 1, we can deal with the consensus problems of second-order SNSs, in which necessary and sufficient conditions for the bipartite consensus and state stability of the system (3) can be developed. This significantly extends the existing results on signed networks that only consider the first-order integrator dynamics (see [28]–[32]). It is worth noticing that Theorem 1 can be extended to the system (1) with a vector state space $\mathbb{R}^N$ ($N \geq 1$) by introducing the Kronecker product ($\otimes$).

For structurally unbalanced signed networks, it can be found from Theorem 1 that the control gains k_1 and k_2 play a central role in the distribution of all eigenvalues of Ψ. It is shown that no matter what the values of α and β, we can pick out k_1 and k_2 such that all eigenvalues of Ψ consist of negative real parts, which leads to that the system (3) achieves the state stability. However, when signed networks are structurally balanced, it follows from (14) that the two eigenvalues λ_{11} and λ_{12} of Ψ have no relationship with k_1 and k_2 and can be determined by the system parameters α and β. Therefore, the system (3) may exhibit diverse dynamic behaviors via selecting different values of α and β. Next, we further explore specific dynamic behaviors of the system (3) under different values of α and β. It is immediate to derive from (14) that there exist the following cases for the eigenvalues λ_{11} and λ_{12}.

C1) $\beta > 0$ and $\alpha \in \mathbb{R}$: $\mathrm{Re}(\lambda_{11}) > 0$.
C2) $\beta = 0$ and $\alpha > 0$: $\lambda_{11} > 0$ and $\lambda_{12} < 0$.
C3) $\beta < 0$ and $\alpha > 0$: $\lambda_{11} > 0$ and $\lambda_{12} < 0$.
C4) $\beta = 0$ and $\alpha = 0$: $\lambda_{11} = \lambda_{12} = 0$.
C5) $\beta = 0$ and $\alpha < 0$: $\lambda_{11} = \sqrt{-\alpha}\imath$ and $\lambda_{12} = -\sqrt{-\alpha}\imath$.
C6) $\beta < 0$ and $\alpha = 0$: $\lambda_{11} = 0$ and $\lambda_{12} < 0$.
C7) $\beta < 0$ and $\alpha < 0$: $\mathrm{Re}(\lambda_{11}) < 0$ and $\mathrm{Re}(\lambda_{12}) < 0$.

Through the abovementioned analyses, when the system parameters α and β meet one of the cases C1)–C3), the eigenvalue λ_{11} of Ψ has positive real part. From (9) and (10), we can realize that the system (3) is exponentially divergent. Here, we only study the steady states of $x(t)$ and $v(t)$ when α and β satisfy any of the cases C4)–C7), in which two eigenvalues λ_{11} and λ_{12} of Ψ do not contain positive real parts.

When α and β satisfy the case C4) (i.e., $\alpha = \beta = 0$), the system (1) transforms into

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = u_i(t) \end{cases} \quad \forall i \in \mathcal{F}_n \tag{19}$$

that is a double integrator dynamic model (see [7]–[14], [44], [45]). Using the protocol (2) to the system (19) yields

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ -k_1 L & -k_2 L \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}. \tag{20}$$

Based on Theorem 1, the dynamic behaviors of the system (20) can be exhibited in the following corollary.

Corollary 1: Consider the system (20) under a signed digraph $\mathcal{G}$ that is strongly connected, and let its gains k_1 and k_2 satisfy

$$k_1 > 0, \quad k_2 > 0 \text{ and } \frac{k_2^2}{k_1} > \frac{\mathrm{Im}^2(\lambda_i)}{\mathrm{Re}(\lambda_i)|\lambda_i|^2} \tag{21}$$

for each nonzero eigenvalue λ_i, $i \in \mathcal{F}_n$. Then, the following two results are established for the system (20).

1) The bipartite consensus can be achieved if and only if $\mathcal{G}$ is structurally balanced. In addition, the steady state of the system (20) is given by

$$\lim_{t\to\infty} \left\{ \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} - \begin{bmatrix} D_n 1_n p^T & t D_n 1_n p^T \\ 0_{n\times n} & D_n 1_n p^T \end{bmatrix} \times \begin{bmatrix} x(0) \\ v(0) \end{bmatrix} \right\} = 0_{2n} \tag{22}$$

where $D_n 1_n$ and p are the right and left eigenvectors for the zero eigenvalue of L, respectively, and $p^T D_n 1_n = 1$.

2) The state stability can be accomplished if and only if $\mathcal{G}$ is structurally unbalanced.

Proof: With $\alpha = \beta = 0$, the condition (8) becomes

$$\begin{aligned} k_2^2 \mathrm{Im}^2(\lambda_i) + 4k_1 \mathrm{Re}(\lambda_i) &> 0 \\ -k_2 \mathrm{Re}(\lambda_i) &\neq 0 \\ k_1^2 \mathrm{Im}^2(\lambda_i) &< k_1 k_2^2 \mathrm{Re}(\lambda_i)|\lambda_i|^2 \\ k_1 k_2 |\lambda_i|^2 &> 0 \end{aligned} \tag{23}$$

for each nonzero eigenvalue λ_i, $i \in \mathcal{F}_n$. From $k_1 k_2 |\lambda_i|^2 > 0$, it follows that $k_1 > 0$ and $k_2 > 0$ (or $k_1 < 0$ and $k_2 < 0$). If $k_1 > 0$ and $k_2 > 0$, then (23) is rewritten as

$$\frac{k_2^2}{k_1} > \frac{\mathrm{Im}^2(\lambda_i)}{\mathrm{Re}(\lambda_i)|\lambda_i|^2}.$$

If $k_1 < 0$ and $k_2 < 0$, then it makes an error on $k_1^2 \mathrm{Im}^2(\lambda_i) < k_1 k_2^2 \mathrm{Re}(\lambda_i)|\lambda_i|^2$. Hence, the condition (23) is identical to the condition (21). As shown in the proof of Theorem 1, the rest of proof can be derived.

In the following, we explore the steady state of the system (20) when $\mathcal{G}$ is structurally balanced. In this case, there exists a matrix $D_n \in \mathcal{D}_n$ fulfilling $\overline{L} = D_n L D_n$ and $L D_n 1_n = 0_n$. Denote $p \in \mathbb{R}^n$ as the left eigenvector of L for the zero eigenvalue that satisfies $p^T L = 0_n^T$ and $p^T D_n 1_n = 1$. Without loss of generality, we can select $\xi_1 = [\,(D_n 1_n)^T \;\; 0_n^T\,]^T$ and $\xi_2 = [\,0_n^T \;\; (D_n 1_n)^T\,]^T$. It can be easily verified that $\zeta_1 = [\,p^T \;\; 0_n^T\,]^T$ and $\zeta_2 = [\,0^T \;\; p^T\,]^T$ hold, where $\zeta_1^T \xi_1 = 1$ and $\zeta_2^T \xi_2 = 1$. It is immediate to obtain from (9) that the steady state of the system (20) is provided by (22). ■

Remark 2: From Corollary 1, we can figure out the distributed control problems of SNSs with a double integrator dynamics, where necessary and sufficient conditions are given to ensure the bipartite consensus and state stability. We should point out that topological structures are not sufficient to make sure the consensus results of SNSs, and selecting the appropriate gains k_1 and k_2 is also an essential element. It significantly enhances the existing results of networked systems that can only admit cooperative interactions among nodes (see [9]).

When α and β satisfy the case C5) (i.e., $\alpha < 0$ and $\beta = 0$), the system (1) turns into

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \alpha x_i(t) + u_i(t) \quad \forall i \in \mathcal{F}_n. \end{cases} \tag{24}$$

Actually, the system (24) is the dynamic model of harmonic oscillators (see [56]–[58] for more discussions). Based on L, substituting the protocol (2) into the system (24) results in

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ \alpha I_n - k_1 L & -k_2 L \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}. \tag{25}$$

The dynamic behaviors of the system (25) can be expressed in the following corollary.

Corollary 2: Consider the system (24) under a signed digraph $\mathcal{G}$ that is strongly connected. Let the protocol (2) be applied to the system (24) and its gains k_1 and k_2 satisfy

$$\begin{aligned} k_2^2 \mathrm{Im}^2(\lambda_i) + 4k_1 \mathrm{Re}(\lambda_i) - 4\alpha &> 0 \\ -k_2 \mathrm{Re}(\lambda_i) &\neq 0 \\ \frac{k_1^2 \mathrm{Im}^2(\lambda_i)}{(k_2 \mathrm{Re}(\lambda_i))^2} - \frac{k_1 \mathrm{Im}^2(\lambda_i)}{\mathrm{Re}(\lambda_i)} - k_1 \mathrm{Re}(\lambda_i) + \alpha &< 0 \\ k_1 k_2 |\lambda_i|^2 - k_2 \alpha \mathrm{Re}(\lambda_i) &> 0 \end{aligned} \tag{26}$$

for each nonzero eigenvalue λ_i, $i \in \mathcal{F}_n$. Then, the following results hold.

1) The system (25) can reach the bipartite consensus if and only if $\mathcal{G}$ is structurally balanced. Besides, the steady state of the system (25) is given by

$$\lim_{t\to\infty} \left\{ \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} - \left[\cos(\sqrt{|\alpha|}t) \times \begin{bmatrix} D_n 1_n p^T & 0_{n\times n} \\ 0_{n\times n} & D_n 1_n p^T \end{bmatrix} \right.\right.$$
$$+ \sin(\sqrt{|\alpha|}t)$$

$$\times \begin{bmatrix} 0_{n\times n} & \frac{1}{\sqrt{|\alpha|}} D_n 1_n p^T \\ -\sqrt{|\alpha|} D_n 1_n p^T & 0_{n\times n} \end{bmatrix}\Bigg] \times \begin{bmatrix} x(0) \\ v(0) \end{bmatrix} \Bigg\} = 0_{2n} \tag{27}$$

where $D_n 1_n$ and p are the right and left eigenvectors of L corresponding to the zero eigenvalue, respectively, and $p^T D_n 1_n = 1$.

2) The system (25) can achieve the state stability if and only if $\mathcal{G}$ is structurally unbalanced.

Proof: Based on the proof of Theorem 1, we can directly develop the proof of Corollary 2.

Next, we study the steady state of the system (25) when $\mathcal{G}$ is structurally balanced. Due to $\alpha < 0$ and $\beta = 0$, we realize that the matrix Ψ has a pair of conjugated imaginary eigenvalues: $\sqrt{|\alpha|}\iota$ and $-\sqrt{|\alpha|}\iota$. The Jordan canonical form of Ψ is

$$J = \hat{Q}^{-1}\Psi\hat{Q} = \begin{bmatrix} \sqrt{|\alpha|}\iota & 0 & 0_{1\times(2n-2)} \\ 0 & -\sqrt{|\alpha|}\iota & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & J' \end{bmatrix}.$$

The condition (26) can ensure all diagonal elements of J' with negative real parts. We can further induce

$$e^{\Psi t} = \hat{Q}e^{Jt}\hat{Q}^{-1} = [\,\xi_1 \quad \xi_2 \quad \cdots \quad \xi_{2n}\,] \times \begin{bmatrix} e^{\sqrt{|\alpha|}\iota t} & 0 & 0_{1\times(2n-2)} \\ 0 & e^{-\sqrt{|\alpha|}\iota t} & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & e^{J't} \end{bmatrix} \begin{bmatrix} \zeta_1^T \\ \zeta_2^T \\ \vdots \\ \zeta_{2n}^T \end{bmatrix}.$$

Since $\mathcal{G}$ is structurally balanced, it is immediate to know that there exist two vectors $D_n 1_n$ and p satisfying $LD_n 1_n = 0_n$, $p^T L = 0_n^T$ and $p^T D_n 1_n = 1$. Without loss of generality, we can select

$$\xi_1 = \begin{bmatrix} -\frac{1}{2\sqrt{|\alpha|}} D_n 1_n \iota \\ \frac{1}{2} D_n 1_n \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} \frac{1}{2\sqrt{|\alpha|}} D_n 1_n \iota \\ \frac{1}{2} D_n 1_n \end{bmatrix}$$

and

$$\zeta_1 = \begin{bmatrix} \sqrt{|\alpha|} p\iota \\ p \end{bmatrix}, \quad \zeta_2 = \begin{bmatrix} -\sqrt{|\alpha|} p\iota \\ p \end{bmatrix}$$

that satisfy $\zeta_1^T \xi_1 = 1$ and $\zeta_2^T \xi_2 = 1$. Due to all diagonal elements of J' with negative real parts, we can obtain

$$\begin{aligned}
\lim_{t\to\infty} e^{\Psi t} &= \lim_{t\to\infty} \left(e^{\sqrt{|\alpha|}\iota t}\xi_1\zeta_1^T + e^{-\sqrt{|\alpha|}\iota t}\xi_2\zeta_2^T\right) \\
&= \lim_{t\to\infty} \Bigg\{ e^{\sqrt{|\alpha|}\iota t} \begin{bmatrix} \frac{1}{2} D_n 1_n p^T & -\frac{\iota}{2\sqrt{|\alpha|}} D_n 1_n p^T \\ \frac{\sqrt{|\alpha|}\iota}{2} D_n 1_n p^T & \frac{1}{2} D_n 1_n p^T \end{bmatrix} \\
&\quad + e^{-\sqrt{|\alpha|}\iota t} \begin{bmatrix} \frac{1}{2} D_n 1_n p^T & \frac{\iota}{2\sqrt{|\alpha|}} D_n 1_n p^T \\ -\frac{\sqrt{|\alpha|}\iota}{2} D_n 1_n p^T & \frac{1}{2} D_n 1_n p^T \end{bmatrix} \Bigg\}.
\end{aligned}$$

It follows from Euler's theorem that $e^{\sqrt{|\alpha|}t\iota} = \cos(\sqrt{|\alpha|}t) + \iota\sin(\sqrt{|\alpha|}t)$ and $e^{-\sqrt{|\alpha|}t\iota} = \cos(\sqrt{|\alpha|}t) - \iota\sin(\sqrt{|\alpha|}t)$. Thus, we can deduce

$$\begin{aligned}
\lim_{t\to\infty} e^{\Psi t} &= \lim_{t\to\infty} \Bigg\{ (\cos(\sqrt{|\alpha|}t) + \iota\sin(\sqrt{|\alpha|}t)) \\
&\quad \times \begin{bmatrix} \frac{1}{2} D_n 1_n p^T & -\frac{\iota}{2\sqrt{|\alpha|}} D_n 1_n p^T \\ \frac{\sqrt{|\alpha|}\iota}{2} D_n 1_n p^T & \frac{1}{2} D_n 1_n p^T \end{bmatrix} \\
&\quad + (\cos(\sqrt{|\alpha|}t) - \iota\sin(\sqrt{|\alpha|}t)) \\
&\quad \times \begin{bmatrix} \frac{1}{2} D_n 1_n p^T & \frac{\iota}{2\sqrt{|\alpha|}} D_n 1_n p^T \\ -\frac{\sqrt{|\alpha|}\iota}{2} D_n 1_n p^T & \frac{1}{2} D_n 1_n p^T \end{bmatrix} \Bigg\} \\
&= \lim_{t\to\infty} \Bigg\{ \cos(\sqrt{|\alpha|}t) \begin{bmatrix} D_n 1_n p^T & 0_{n\times n} \\ 0_{n\times n} & D_n 1_n p^T \end{bmatrix} + \sin(\sqrt{|\alpha|}t) \\
&\quad \times \begin{bmatrix} 0_{n\times n} & \frac{1}{\sqrt{|\alpha|}} D_n 1_n p^T \\ -\sqrt{|\alpha|} D_n 1_n p^T & 0_{n\times n} \end{bmatrix} \Bigg\}.
\end{aligned}$$

This, together with (11), guarantees the steady state of system (25) given by (27). We complete this proof. ■

Remark 3: Based on Corollary 2, we can realize that the second-order SNSs can accomplish the bipartite consensus although the system matrix Ψ has no zero eigenvalues. New collective behavior of SNSs can be disclosed. It is shown that all nodes can terminally converge to a time-varying trajectory that is sine and cosine function, which includes the potential application values in the area of harmonic oscillators.

When α and β satisfy the case C6) (i.e., $\alpha = 0$ and $\beta < 0$), the system (1) becomes

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \beta v_i(t) + u_i(t) \quad \forall i \in \mathcal{F}_n. \end{cases} \tag{28}$$

Substituting the protocol (2) into the system (28) leads to

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ -k_1 L & \beta I_n - k_2 L \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}. \tag{29}$$

The system (29) is common in the actual applications, such as unmanned aerial vehicles, underwater vehicles, and mobile robots (see [59]–[61] for more details). The dynamic behaviors of the system (29) can be presented in the following corollary.

Corollary 3: Consider the system (29) under a signed digraph $\mathcal{G}$ that is strongly connected, and the gains k_1 and k_2 satisfy

$$\begin{aligned}
k_2^2\mathrm{Im}^2(\lambda_i) + 4k_1\mathrm{Re}(\lambda_i) &> 0 \\
\beta - k_2\mathrm{Re}(\lambda_i) &\neq 0 \\
\frac{k_1^2\mathrm{Im}^2(\lambda_i)}{(\beta - k_2\mathrm{Re}(\lambda_i))^2} + \frac{k_1k_2\mathrm{Im}^2(\lambda_i)}{\beta - k_2\mathrm{Re}(\lambda_i)} - k_1\mathrm{Re}(\lambda_i) &< 0 \\
k_1k_2|\lambda_i|^2 - k_1\beta\mathrm{Re}(\lambda_i) &> 0
\end{aligned} \tag{30}$$

for each nonzero eigenvalue λ_i, $i \in \mathcal{F}_n$. Then, the following results can be established for the system (29).

1) The bipartite consensus is achieved if and only if $\mathcal{G}$ is structurally balanced. Moreover, the steady state of the system (29) is given by

$$\begin{bmatrix} x(\infty) \\ v(\infty) \end{bmatrix} = \begin{bmatrix} D_n 1_n p^T & -\frac{1}{\beta} D_n 1_n p^T \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} \begin{bmatrix} x(0) \\ v(0) \end{bmatrix} \tag{31}$$

where the vectors $D_n 1_n$ and p are the right and left eigenvectors of L associated with zero eigenvalue, respectively, and $p^T D_n 1_n = 1$.

2) The system (29) can reach the state stability if and only if $\mathcal{G}$ is structurally unbalanced.

Proof: Following the proof of Theorem 1, it is immediate to develop the proof of Corollary 3.

To proceed, we calculate the steady state of the system (29) under the structurally balanced signed digraph $\mathcal{G}$. Due to $\alpha = 0$ and $\beta < 0$, we obtain that $\lambda_{11} = 0$ and λ_{12} is a negative real number. The condition (30) ensures the rest $2n-2$ eigenvalues of Ψ with negative real parts. It thus can deduce

$$e^{\Psi t} = \hat{Q} e^{Jt} \hat{Q}^{-1} = [\, \xi_1 \quad \xi_2 \quad \cdots \quad \xi_{2n} \,] \times \begin{bmatrix} e^{\lambda_{11} t} & 0 & 0_{1\times(2n-2)} \\ 0 & e^{\lambda_{12} t} & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & e^{J' t} \end{bmatrix} \begin{bmatrix} \zeta_1^T \\ \zeta_2^T \\ \vdots \\ \zeta_{2n}^T \end{bmatrix}$$

where $J' \in \mathbb{C}^{(2n-2)\times(2n-2)}$ is an upper diagonal block matrix whose all diagonal elements contain negative real parts. With the structural balance of $\mathcal{G}$, we can select a matrix $D_n \in \mathcal{D}_n$ satisfying $LD_n 1_n = 0_n$. Let $p \in \mathbb{R}^n$ be the left eigenvector of L for the zero eigenvalue that fulfills $p^T L = 0_n^T$ and $p^T D_n 1_n = 1$. Furthermore, we select $\xi_1 = [\, (D_n 1_n)^T \; 0_n^T \,]^T$ and $\zeta_1 = [\, p^T \; -p^T/\beta \,]^T$ as the right eigenvector and the left eigenvector of Ψ for the sole zero eigenvalue, respectively. Hence, we can derive that the steady state of the system (29) is provided by (31). We complete this proof. ■

Remark 4: Because TNSs can be viewed as a particular case of SNSs when there exist no antagonistic interactions among nodes, the convergence results of Corollary 3 can also apply to TNSs, from which the necessary and sufficient conditions for consensus of TNSs are developed. This significantly improves the existing results that only consider sufficient conditions for consensus of TNSs (see [59, Th. 3.2] for more details).

When α and β satisfy the case C7) (i.e., $\alpha < 0$ and $\beta < 0$), two eigenvalues λ_{11} and λ_{12} have negative real parts. Based on Theorem 1, we can deduce the following convergence results for the system (3).

Corollary 4: Consider the system (3) with parameters $\alpha < 0$ and $\beta < 0$ under a strongly connected signed digraph $\mathcal{G}$. The system (3) can reach the state stability if and only if the gains k_1 and k_2 satisfy the condition (8) regardless of whether $\mathcal{G}$ is structurally balanced or unbalanced.

Proof: When $\mathcal{G}$ is structurally balanced, because of $\alpha < 0$ and $\beta < 0$, we can obtain from (14) that two eigenvalues λ_{11} and λ_{12} of the matrix Ψ have negative real parts. It follows from Lemma 1 that the rest $2n-2$ eigenvalues of Ψ contain negative real parts if and only if the condition (8) holds for k_1 and k_2. Therefore, the system (3) can reach the state stability if and only if k_1 and k_2 satisfy the condition (8).

When $\mathcal{G}$ is structurally unbalanced, this consequence can be derived from Theorem 2. We complete this proof. ■

Remark 5: From Corollary 4, it follows that the second-order SNSs can reach the state stability even if the associated signed digraphs are structurally balanced, which is different from the existing results (see [28]–[49]). By Corollaries 1–4, we can realize that the system parameters α and β play a key role in determining the dynamic behaviors of second-order SNSs, in which second-order SNSs with different system parameters can generate the diverse dynamic behaviors.

IV. Consensus Results of Second-Order SNSs With A Communication Delay

In this section, we explore how to ensure the consensus of second-order SNSs in the presence of a communication delay. A distributed protocol is given by

$$\begin{aligned} u_i(t) = &-k_1 \sum_{j\in N(i)} |a_{ij}|[x_i(t-\tau) - \operatorname{sgn}(a_{ij})x_j(t-\tau)] \\ &- k_2 \sum_{j\in N(i)} |a_{ij}|[v_i(t-\tau) - \operatorname{sgn}(a_{ij})v_j(t-\tau)] \\ &\forall i \in \mathcal{F}_n \end{aligned} \tag{32}$$

where $\tau > 0$ is a fixed communication delay. Benefitting from L, we can rewrite (1) and (32) as a compact form of

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ \alpha I_n & \beta I_n \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} - \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ k_1 L & k_2 L \end{bmatrix} \begin{bmatrix} x(t-\tau) \\ v(t-\tau) \end{bmatrix}. \tag{33}$$

When $\mathcal{G}$ is structurally balanced, we will provide a reduced-order system, with which the bipartite consensus problems of the system (33) can be transformed into the stability problems. To be specific, substituting $\widetilde{x}(t) = Qx(t)$, $\widetilde{v}(t) = Qv(t)$, $\widetilde{x}(t-\tau) = Qx(t-\tau)$, and $\widetilde{v}(t-\tau) = Qv(t-\tau)$ into (33) yields

$$\begin{bmatrix} \dot{\widetilde{x}}(t) \\ \dot{\widetilde{v}}(t) \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ \alpha I_n & \beta I_n \end{bmatrix} \begin{bmatrix} \widetilde{x}(t) \\ \widetilde{v}(t) \end{bmatrix} - \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ k_1 QLQ^{-1} & k_2 QLQ^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{x}(t-\tau) \\ \widetilde{v}(t-\tau) \end{bmatrix}. \tag{34}$$

Denote $\widetilde{x}^1(t-\tau) = Cx(t-\tau)$, $\widetilde{x}^2(t-\tau) = Ex(t-\tau)$, $\widetilde{v}^1(t-\tau) = Cv(t-\tau)$, and $\widetilde{v}^2(t-\tau) = Ev(t-\tau)$. The system (34) can be divided into two subsystems

$$\begin{bmatrix} \dot{\widetilde{x}}^1(t) \\ \dot{\widetilde{v}}^1(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} \widetilde{x}^1(t) \\ \widetilde{v}^1(t) \end{bmatrix} - \begin{bmatrix} 0_{n-1}^T & 0_{n-1}^T \\ k_1 CLF & k_2 CLF \end{bmatrix} \begin{bmatrix} \widetilde{x}^2(t-\tau) \\ \widetilde{v}^2(t-\tau) \end{bmatrix} \tag{35}$$

and

$$\begin{bmatrix} \dot{\widetilde{x}}^2(t) \\ \dot{\widetilde{v}}^2(t) \end{bmatrix} = \begin{bmatrix} 0_{(n-1)\times(n-1)} & I_{n-1} \\ \alpha I_{n-1} & \beta I_{n-1} \end{bmatrix} \begin{bmatrix} \widetilde{x}^2(t) \\ \widetilde{v}^2(t) \end{bmatrix} - \begin{bmatrix} 0_{(n-1)\times(n-1)} & 0_{(n-1)\times(n-1)} \\ k_1 ELF & k_2 ELF \end{bmatrix} \times \begin{bmatrix} \widetilde{x}^2(t-\tau) \\ \widetilde{v}^2(t-\tau) \end{bmatrix}. \tag{36}$$

We realize that the system (33) can reach bipartite consensus if and only if the reduced-order system (36) is asymptotically stable. Let

$$z(t) = \begin{bmatrix} \widetilde{x}^2(t) \\ \widetilde{v}^2(t) \end{bmatrix}$$

and the subsystem (36) can be rewritten as

$$\dot{z}(t) = \begin{bmatrix} 0_{(n-1)\times(n-1)} & I_{n-1} \\ \alpha I_{n-1} & \beta I_{n-1} \end{bmatrix} z(t) - \begin{bmatrix} 0_{(n-1)\times(n-1)} & 0_{(n-1)\times(n-1)} \\ k_1 ELF & k_2 ELF \end{bmatrix} z(t-\tau). \tag{37}$$

Using the Laplacian transform to (37) leads to

$$\begin{aligned} sZ(s) - z(0) &= \begin{bmatrix} 0_{(n-1)\times(n-1)} & I_{n-1} \\ \alpha I_{n-1} & \beta I_{n-1} \end{bmatrix} Z(s) - \begin{bmatrix} 0_{(n-1)\times(n-1)} & 0_{(n-1)\times(n-1)} \\ k_1 ELFe^{-\tau s} & k_2 ELFe^{-\tau s} \end{bmatrix} Z(s) \\ &= \begin{bmatrix} 0_{(n-1)\times(n-1)} & I_{n-1} \\ \alpha I_{n-1} - k_1 ELFe^{-\tau s} & \beta I_{n-1} - k_2 ELFe^{-\tau s} \end{bmatrix} Z(s). \end{aligned}$$

We can further validate

$$Z(s) = \Gamma^{-1}(s) z(0)$$

where

$$\Gamma(s) = \begin{bmatrix} sI_{n-1} & -I_{n-1} \\ -\alpha I_{n-1} + k_1 ELFe^{-\tau s} & (s-\beta)I_{n-1} + k_2 ELFe^{-\tau s} \end{bmatrix}.$$

As a result, the stability analysis of subsystem (36) can reduce to the studies for a multiple-input–multiple-output (MIMO) transfer function $\Gamma^{-1}(s)$. Furthermore, it can derive

$$\det(\Gamma(s)) = |(s^2 - \beta s - \alpha)I_{n-1} + e^{-\tau s}(k_2 s + k_1)\Lambda| = \prod_{j=2}^{n} f_j(s,\tau)$$

where $f_j(s,\tau) = s^2 + e^{-\tau s}(k_2 s + k_1)\lambda_j - \beta s - \alpha$.

When $\mathcal{G}$ is structurally unbalanced, all eigenvalues of L have positive real parts. Let $w(t) = [x^T(t), v^T(t)]^T$ and the system (33) turns into

$$\dot{w}(t) = \begin{bmatrix} 0_{n\times n} & I_n \\ \alpha I_n & \beta I_n \end{bmatrix} w(t) - \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ k_1 L & k_2 L \end{bmatrix} w(t-\tau). \tag{38}$$

Employing Laplacian transformation to the system (38) yields

$$\begin{aligned} sW(s) - w(0) &= \begin{bmatrix} 0_{n\times n} & I_n \\ \alpha I_n & \beta I_n \end{bmatrix} W(s) - \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ k_1 Le^{-\tau s} & k_2 Le^{-\tau s} \end{bmatrix} W(s) \\ &= \begin{bmatrix} 0_{n\times n} & I_n \\ \alpha I_n - k_1 Le^{-\tau s} & \beta I_n - k_2 Le^{-\tau s} \end{bmatrix} W(s). \end{aligned} \tag{39}$$

It can further obtain

$$W(s) = \Xi^{-1}(s) w(0)$$

where

$$\Xi(s) = \begin{bmatrix} sI_n & -I_n \\ -\alpha I_n + k_1 Le^{-\tau s} & (s-\beta)I_n + k_2 Le^{-\tau s} \end{bmatrix}.$$

As a consequence, the stability analysis of system (38) is equal to explore the MIMO transfer function $\Xi^{-1}(s)$. By the same analysis as the structurally balanced case, we can derive

$$\det(\Xi(s)) = \prod_{j=1}^{n} f_j(s,\tau)$$

where $f_j(s,\tau) = s^2 + e^{-\tau s}(k_2 s + k_1)\lambda_j - \beta s - \alpha$.

The following theorem can present the consensus results of second-order SNSs in the presence of a communication delay.

Theorem 3: Consider the system (33) under a signed digraph $\mathcal{G}$ that is strongly connected. For any $\lambda_j \neq 0$ and $j \in \mathcal{F}_n$, let k_1 and k_2 satisfy the condition (8), and $\omega_j > 0$ be the root of the following equation:

$$(\omega_j^2 + \alpha)^2 + \beta^2\omega_j^2 - (k_1^2 + k_2^2\omega_j^2)|\lambda_j|^2 = 0. \tag{40}$$

Define τ_j as

$$\tau_j = \frac{\{b\pi + \arctan \eta_j\}}{\omega_j} \tag{41}$$

where b is the minimum integer such that $\tau_j > 0$ and

$$\eta_j = \frac{(\omega_j^2 + \alpha)\phi_j + \beta\omega_j\psi_j}{\beta\omega_j\phi_j - (\omega_j^2 + \alpha)\psi_j}$$

with $\phi_j = k_2\omega_j \mathrm{Re}(\lambda_j) + k_1 \mathrm{Im}(\lambda_j)$ and $\psi_j = k_2\omega_j \mathrm{Im}(\lambda_j) - k_1 \mathrm{Re}(\lambda_j)$. Then, the following results hold.

1) The system (33) can reach the bipartite consensus if and only if $\mathcal{G}$ is structurally balanced and $\tau \in [0, \tau^*)$ holds, where $\tau^* = \min\{\tau_2, \tau_3, \ldots, \tau_n\}$.
2) The system (33) can reach the state stability if and only if $\mathcal{G}$ is structurally unbalanced and $\tau \in [0, \tau^*)$ holds, where $\tau^* = \min\{\tau_1, \tau_2, \ldots, \tau_n\}$.

Before showing the proof of Theorem 3, we first propose a lemma that can provide the necessary and sufficient condition for the Hurwitz stability of $f_j(s,\tau)$ with respect to nonzero eigenvalue λ_j, $\forall j \in \mathcal{F}_n$.

Lemma 2: For the eigenvalue $\lambda_j \neq 0$, $\forall j \in \mathcal{F}_n$, let the control gains k_1 and k_2 fulfill the condition (8). Then, the polynomial $f_j(s,\tau)$ is Hurwitz stable if and only if the communication delay τ satisfies $\tau \in [0, \tau_j)$, where τ_j is given by (41).

Proof: See Appendix B. ■

With Lemma 2, we can present the proof of Theorem 3.

Proof of Theorem 3: 1) *Sufficiency:* When $\mathcal{G}$ is structurally balanced, based on the abovementioned analyses, we can know that the bipartite consensus of the system (33) is equivalent to the asymptotical stability of the subsystem (36). Furthermore, three equivalent statements are proposed as follows.

a) The subsystem (36) is asymptotically stable.
b) All poles of $\Gamma^{-1}(s)$ are on the open left half-plane (LHP).
c) $f_j(s, \tau)$ is Hurwitz stable, $\forall j \in \{2, 3, \ldots, n\}$

where the equivalence among the statements a)–c) can be developed in a similar way as in [2, proof of Theorem 10]. Thus, it indicates that the system (33) can reach the bipartite consensus if and only if $f_j(s, \tau)$ is Hurwitz stable, $\forall j \in \{2, 3, \ldots, n\}$. With the condition $\tau \in [0, \tau^*)$, it follows from Lemma 2 that the system (33) can reach the bipartite consensus.

2) *Sufficiency:* When $\mathcal{G}$ is structurally unbalanced, we can directly derive the following three equivalent conditions.

a) The system (33) can achieve the state stability.
b) All poles of $\Xi^{-1}(s)$ are on the open LHP.
c) $f_j(s, \tau)$ is Hurwitz stable, $\forall j \in \mathcal{F}_n$.

The rest proof is similar to the structurally balanced case, and we omit it for simplicity.

1) and 2) *Necessity:* On the one hand, we suppose $\tau \notin [0, \tau^*]$ and there exists at least one $f_i(s, \tau)$ that is Hurwitz unstable, which causes a contradiction that the system (33) can achieve the bipartite consensus (or state stability). On the other hand, there exists the mutually exclusive relation between the structural balance and unbalance of the signed digraph $\mathcal{G}$. Hence, the necessity can be derived. ■

Remark 6: Through Theorem 3, we can solve the consensus problems of second-order SNSs in the presence of a communication delay, with which a time margin is developed. When given topological structure conditions, the second-order SNS can achieve the bipartite consensus (or state stability) if and only if its communication delay is less than the time margin. It can further extend the consensus results of Theorem 1 to the case with a communication delay.

Remark 7: In this article, we mainly pay attention to exploring the distributed control problems of second-order SNSs with a single communication delay. When considering second-order SNSs in face of multiple communication delays, the control protocol (32) can be generalized as

$$\begin{aligned} u_i(t) = &-k_1 \sum_{j \in N(i)} |a_{ij}|[x_i(t-\tau_{ij}) - \operatorname{sgn}(a_{ij})x_j(t-\tau_{ij})] \\ &- k_2 \sum_{j \in N(i)} |a_{ij}|[v_i(t-\tau_{ij}) - \operatorname{sgn}(a_{ij})v_j(t-\tau_{ij})] \\ &\qquad \forall i \in \mathcal{F}_n \qquad (42) \end{aligned}$$

where $\tau_{ij} = \tau_{ji}$ represents the communication delay between the node υ_i and the node υ_j. If the control protocol (42) is applied to the second-order system (1), then we can derive the conditions to guarantee the bipartite consensus (or state stability) objective by taking advantage of the techniques used in [14], which is our future work.

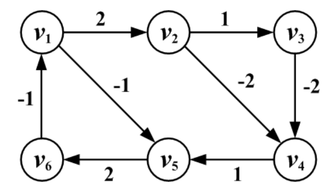

Fig. 1. Signed digraph $\mathcal{G}_1$ that is structurally balanced.

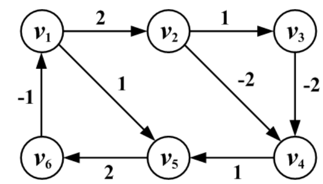

Fig. 2. Signed digraph $\mathcal{G}_2$ that is structurally unbalanced.

V. Simulations

In this section, we will introduce four examples for second-order SNSs with six nodes to illustrate the developed results. The communication topologies of second-order SNSs are described by the signed digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Figs. 1 and 2, respectively. We select the initial states of nodes as

$$x(0) = [1, -2, 3, -4, 5, -6]^T$$
$$v(0) = [1, 2, 3, -2, -4, -6]^T.$$

Example 1: Consider the second-order system (19) under a signed digraph $\mathcal{G}_1$ in Fig. 1. We can clearly see that the signed digraph $\mathcal{G}_1$ is strongly connected and structurally balanced. Let L_1 be the Laplacian matrix of $\mathcal{G}_1$. The eigenvalues of L_1 are $\lambda_1 = 0$, $\lambda_2 = 1.6522 + 1.0289\imath$, $\lambda_3 = 1.6522 - 1.0289\imath$, $\lambda_4 = 2$, $\lambda_5 = 3$, and $\lambda_6 = 3.6956$. The protocol (2) with $k_1 = 1$ is employed. It follows from Theorem 2 that the protocol (2) can guarantee the system (19) to achieve bipartite consensus if and only if $k_2 > 0.41$ holds.

We plot the state evolutions of all nodes in Fig. 3. Obviously, it can be found in Fig. 3 that the system (19) achieves the bipartite consensus if $k_2 > 0.41$ and the states of all nodes are divergent, otherwise. The simulation results are consistent with Theorem 2.

Example 2: Consider a signed digraph $\mathcal{G}_2$ in Fig. 2 that is strongly connected and structurally unbalanced. We adopt the signed digraph $\mathcal{G}_2$ to denote the communication topology of the system (19). Let L_2 denote the Laplacian matrix of $\mathcal{G}_2$ and the eigenvalues of L_2 are given by $\lambda_1 = 3.6797 + 0.3440\imath$, $\lambda_2 = 3.6797 - 0.3440\imath$, $\lambda_3 = 1.1276 + 1.3294\imath$, $\lambda_4 = 1.1276 - 1.3294\imath$, $\lambda_5 = 0.3855$, and $\lambda_6 = 2$. We apply the control protocol (2) with $k_1 = 1$ to the system (19). Based on Theorem 2, we realize that the system (19) can accomplish the state stability if and only if the gain $k_2 > 0.72$ holds.

The state evolutions of all nodes can be plotted in Fig. 4. It is obvious from Fig. 4 that all nodes can converge to zero when $k_2 = 0.8$ holds and are divergent when k_2 satisfies $k_2 = 0.65$. These behaviors coincide with the results of Theorem 2.

Example 3: Let the signed digraph $\mathcal{G}_1$ represent the communication topology of the system (19). Since $\mathcal{G}_1$ is structurally balanced, there exists a gauge transformation $D_6 = \operatorname{diag}\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\} = \operatorname{diag}\{1, 1, 1, -1, -1, -1\}$

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

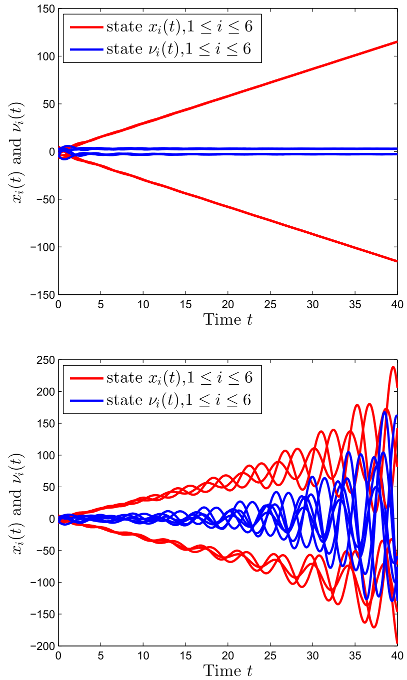

Fig. 3. Example 1—state evolution of system (19) under the protocol (2). Top: $k_1 = 1$ and $k_2 = 0.5$. Bottom: $k_1 = 1$ and $k_2 = 0.3$.

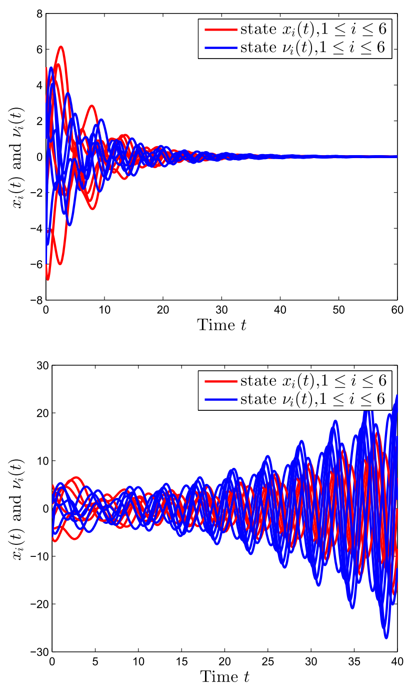

Fig. 4. Example 2—state evolution of system (19) under the protocol (2). Top: $k_1 = 1$ and $k_2 = 0.8$. Bottom: $k_1 = 1$ and $k_2 = 0.65$.

such that $\overline{L}_1 = D_6 L_1 D_6$. From Example 1, we realize that the protocol (32) with $k_1 = k_2 = 1$ can guarantee the bipartite consensus of the system (19) when the communication delay τ is zero. With (41), we can calculate $\tau_2 = 0.7872$, $\tau_3 = 0.2691$, $\tau_4 = 0.3442$, $\tau_5 = 0.4013$, and $\tau_6 = 0.5205$. It can derive the delay margin $\tau^* = \min\{\tau_2, \tau_3, \tau_4, \tau_5, \tau_6\} = 0.2691$.

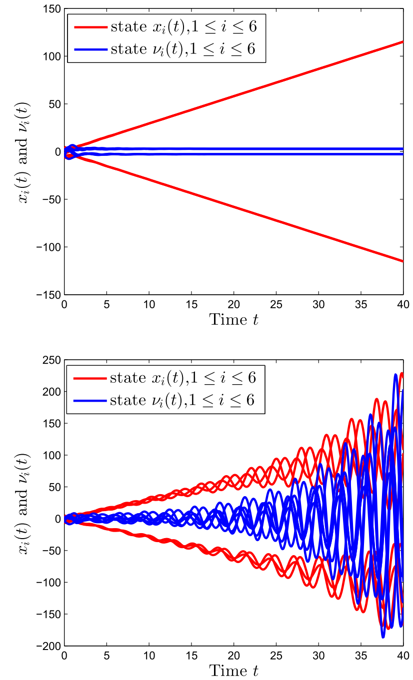

Fig. 5. Example 3—state evolution of system (19) under the protocol (32). Top: $k_1 = 1$, $k_2 = 1$ and $\tau = 0.2$. Bottom: $k_1 = 1$, $k_2 = 1$ and $\tau = 0.3$.

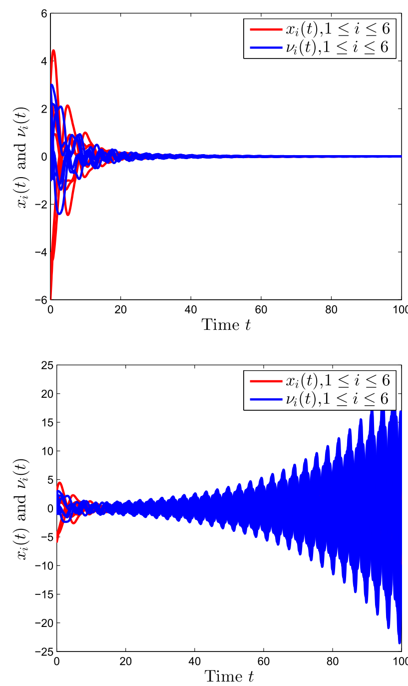

Fig. 6. Example 4—state evolution of system (19) under the protocol (32). Top: $k_1 = 1$, $k_2 = 1$, and $\tau = 0.1$. Bottom: $k_1 = 1$, $k_2 = 1$, and $\tau = 0.13$.

Next, the control protocol (32) with $k_1 = k_2 = 1$ is applied and the simulation results are shown in Fig. 5. We can easily see from Fig. 5 that all nodes can reach the bipartite consensus if and only if the communication delay τ is less than the delay margin τ^*, which coincides with Theorem 3.

Example 4: Consider the communication topology for the system (19) described by $\mathcal{G}_2$. It follows from Example 2 that when $\tau = 0$ holds, the protocol (32) with $k_1 = 1$ and $k_2 = 1$ can make sure the state stability of the system (19). By (41), we can develop $\tau_1 = 0.3686$, $\tau_2 = 0.3198$, $\tau_3 = 1.004$, $\tau_4 = 0.1181$, $\tau_5 = 0.8772$, and $\tau_6 = 0.5205$. With Theorem 3, we have $\tau^* = \min\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6\} = 0.1181$.

Under the protocol (32) with $k_1 = 1$ and $k_2 = 1$, the state evolution of the system (19) is shown in Fig. 6. It is clearly seen from Fig. 6 that if τ is less than τ^*, then the system (19) can achieve the state stability. Otherwise, the system (19) is divergent. The simulation results of Example 4 are consistent with the developed results of Theorem 3.

VI. Conclusion

In this article, we have investigated the distributed control issues of second-order SNSs under strongly connected signed digraphs. We have provided a distributed control protocol by using the nearest neighbor rule and further given the necessary and sufficient conditions for consensus results of second-order SNSs, in which new dynamic behaviors of SNSs are disclosed by picking out different system parameters. Besides, we have introduced a distributed control protocol with a communication delay, for which a time margin of the delay can be developed. Befitting from this time margin, we have derived the necessary and sufficient conditions for consensus results of SNSs with a communication delay. In addition, we have provided four simulation examples to illustrate the correctness of our derived theoretical results.

In our future work, we will target at addressing distributed control problems of second-order SNSs with multiple communication delays and external disturbances, in addition to those with switching topologies and time-varying topologies.

Appendix

A. Proof of Lemma 1

To exhibit the proof of Lemma 1, we need a useful lemma that is provided as follows.

Lemma 3 [15]: For a polynomial $f(s)$, let $f(\omega\imath) = f_1(\omega) + f_2(\omega)\imath$, where $\omega \in \mathbb{R}$ and $\imath^2 = -1$. The polynomial $f(s)$ is Hurwitz stable if and only if the following conditions hold.

1) $f_1(0)((\partial f_2(\omega)/\partial\omega)|_{\omega=0}) - ((\partial f_1(\omega)/\partial\omega)|_{\omega=0}) f_2(0) > 0$.
2) $f_1(\omega)$ and $f_2(\omega)$ cross the x axis alternately as ω traverses from $-\infty$ to $+\infty$.

With the help of Lemma 3, we can give the proof of Lemma 1 as follows.

Proof of Lemma 1: For $\lambda_i \neq 0$, $\forall i \in \mathcal{F}_n$, we can deduce

$$\begin{aligned} f_i(\omega\imath) &= -\omega^2 - \beta\omega\imath - \alpha + (k_2\omega\imath + k_1)\lambda_i \\ &= -\omega^2 - k_2\mathrm{Im}(\lambda_i)\omega + k_1\mathrm{Re}(\lambda_i) - \alpha \\ &\quad + [k_2\mathrm{Re}(\lambda_i)\omega - \beta\omega + k_1\mathrm{Im}(\lambda_i)]\imath. \end{aligned}$$

Let $\mathrm{Re}(f_i(\omega\imath)) = -\omega^2 - k_2\mathrm{Im}(\lambda_i)\omega + k_1\mathrm{Re}(\lambda_i) - \alpha$ and $\mathrm{Im}(f_i(\omega\imath)) = k_2\mathrm{Re}(\lambda_i)\omega - \beta\omega + k_1\mathrm{Im}(\lambda_i)$. $\mathrm{Re}(f_i(\omega\imath)) = 0$ is a quadratic equation and its discriminant is

$$\Delta_i = k_2^2\mathrm{Im}^2(\lambda_i) + 4k_1\mathrm{Re}(\lambda_i) - 4\alpha.$$

Thus, the necessary and sufficient condition that the quadratic equation $\mathrm{Re}(f_i(\omega\imath)) = 0$ contains two different real roots is $\Delta_i > 0$. Two real roots of $\mathrm{Re}(f_i(\omega\imath)) = 0$ are provided by

$$r_{i1} = \frac{-k_2\mathrm{Im}(\lambda_i) - \sqrt{\Delta_i}}{2} \text{ and } r_{i2} = \frac{-k_2\mathrm{Im}(\lambda_i) + \sqrt{\Delta_i}}{2}.$$

$\mathrm{Im}(f_i(\omega\imath)) = 0$ is a linear equation and its solution is

$$r_{i3} = \frac{k_1\mathrm{Im}(\lambda_i)}{\beta - k_2\mathrm{Re}(\lambda_i)}.$$

This, together with Lemma 3, guarantees the Hurwitz stability of $f_i(s)$ if and only if the following conditions hold:

$$\begin{aligned} &\Delta_i > 0, \quad \beta - k_2\mathrm{Re}(\lambda_i) \neq 0, \ r_{i1} < r_{i3} < r_{i2} \\ &k_1k_2|\lambda_i|^2 + \alpha\beta - (k_1\beta + k_2\alpha)\mathrm{Re}(\lambda_i) > 0. \end{aligned} \tag{43}$$

Hence, the necessary and sufficient condition (8) for Hurwitz stability of $f_i(s)$ can be induced from (43). ■

B. Proof of Lemma 2

Proof: For $\lambda_j \neq 0$, $\forall j \in \mathcal{F}_n$, we explore how to develop the delay margin τ_j such that $f_j(s, \tau)$ is Hurwitz stable if and only if $\tau < \tau_j$. For $\tau = 0$, the control gains k_1 and k_2 satisfying (8) can guarantee that two roots of $f_j(s, \tau) = 0$ are on the LHP. When $\tau > 0$ holds, it follows from [9, Lemma 4] that all the roots of $f_j(s, \tau) = 0$ still lie in the LHP for $\forall\tau \in [0, \tau_j)$, and at least one root of $f_j(s, \tau) = 0$ has positive real part for $\forall\tau > \tau_j$ if and only if the equation $f_j(s, \tau_j) = 0$ has a root on the imaginary axis. In the following, we only need to find τ_j such that $f_j(s, \tau_j) = 0$ contains a root on the imaginary axis. We assume that $s_j = \omega_j\imath$, $\omega_j \in \mathbb{R}$, is a purely imaginary root of $f_j(s, \tau_j) = 0$ and thus derive

$$\omega_j^2 + \beta\omega_j\imath + \alpha - (k_2\omega_j\imath + k_1)e^{-\tau_j\omega_j\imath}\lambda_j = 0. \tag{44}$$

Based on $\lambda_j = \mathrm{Re}(\lambda_j) + \mathrm{Im}(\lambda_j)\imath$ and $e^{-\tau_j\omega_j\imath} = \cos(\tau_j\omega_j) - \sin(\tau_j\omega_j)\imath$, (44) can be written as

$$\begin{aligned} &\omega_j^2 + \alpha - \sin(\tau_j\omega_j)\phi_j + \cos(\tau_j\omega_j)\psi_j \\ &\quad + \imath[\beta\omega_j - \cos(\tau_j\omega_j)\phi_j - \sin(\tau_j\omega_j)\psi_j] = 0 \end{aligned} \tag{45}$$

where $\phi_j = k_2\omega_j\mathrm{Re}(\lambda_j) + k_1\mathrm{Im}(\lambda_j)$ and $\psi_j = k_2\omega_j\mathrm{Im}(\lambda_j) - k_1\mathrm{Re}(\lambda_j)$. From (45), we can deduce

$$\begin{cases} \omega_j^2 + \alpha - \sin(\tau_j\omega_j)\phi_j + \cos(\tau_j\omega_j)\psi_j = 0 \\ \beta\omega_j - \cos(\tau_j\omega_j)\phi_j - \sin(\tau_j\omega_j)\psi_j = 0 \end{cases}$$

which leads to

$$\begin{cases} \sin(\tau_j\omega_j) = \dfrac{(\omega_j^2 + \alpha)\phi_j + \beta\omega_j\psi_j}{(k_1^2 + k_2^2\omega_j^2)|\lambda_j|^2} \\ \cos(\tau_j\omega_j) = \dfrac{\beta\omega_j\phi_j - (\omega_j^2 + \alpha)\psi_j}{(k_1^2 + k_2^2\omega_j^2)|\lambda_j|^2}. \end{cases}$$

Because of $\sin^2(\tau_j\omega_j) + \cos^2(\tau_j\omega_j) = 1$, we can validate that ω_j satisfies the following equation:

$$(\omega_j^2 + \alpha)^2 + \beta^2\omega_j^2 - (k_1^2 + k_2^2\omega_j^2)|\lambda_j|^2 = 0. \tag{46}$$

With $\sin(\tau_j\omega_j)$ and $\cos(\tau_j\omega_j)$, we have

$$\tan(\tau_j\omega_j) = \frac{(\omega_j^2+\alpha)\phi_j + \beta\omega_j\psi_j}{\beta\omega_j\phi_j - (\omega_j^2+\alpha)\psi_j} \triangleq \eta_j. \tag{47}$$

It is immediate from (47) to obtain

$$\tau_j = \frac{b\pi + \arctan(\eta_j)}{\omega_j}$$

where b is the smallest integer such that $\tau_j > 0$. Next, we aim to explain that calculating τ_j only needs to consider $\omega_j > 0$. There exist the following two cases via the real number and complex number of the eigenvalue λ_j of L.

Case 1: λ_j is a real number, i.e., $\mathrm{Im}(\lambda_j) = 0$. Then, (46) and (47) can become

$$(\omega_j^2+\alpha)^2 + \beta^2\omega_j^2 - (k_1^2 + k_2\omega_j^2)\lambda_j^2 = 0 \tag{48}$$

and

$$\tan(\tau_j\omega_j) = \frac{(\omega_j^2+\alpha)k_2\omega_j - \beta\omega_j k_1}{\beta\omega_j^2 k_2 + (\omega_j^2+\alpha)k_1} \triangleq \eta_j. \tag{49}$$

Due to $\omega_j \in \mathbb{R}$, (48) has two real number roots $\omega_{j1} > 0$ and $\omega_{j2} = -\omega_{j1}$. With (49), we can deduce $\eta_{j1} = -\eta_{j2}$, which causes $\omega_{j1}^{-1}\arctan(\eta_{j1}) = \omega_{j2}^{-1}\arctan(\eta_{j2})$. The time delays τ_{j1} and τ_{j2} are

$$\tau_{j1} = \frac{b_1\pi}{\omega_{j1}} + \frac{\arctan(\eta_{j1})}{\omega_{j1}} \text{ and } \tau_{j2} = \frac{b_2\pi}{\omega_{j2}} + \frac{\arctan(\eta_{j2})}{\omega_{j2}}$$

where b_1 and b_2 are the minimum integers such that $\tau_{j1} > 0$ and $\tau_{j2} > 0$, respectively. We continue to explain $\tau_{j1} = \tau_{j2}$.

1) If $\omega_{j1}^{-1}\arctan(\eta_{j1}) > 0$, then $b_1 = b_2 = 0$ holds. We directly obtain $\tau_{j1} = \tau_{j2}$.
2) If $\omega_{j1}^{-1}\arctan(\eta_{j1}) < 0$, then $b_1 = \mathrm{sgn}(\omega_{j1})$ and $b_2 = \mathrm{sgn}(\omega_{j2})$ hold. Thus, we have $\tau_{j1} = \tau_{j2}$.

Motivated by the abovementioned analyses, we can calculate τ_j by only considering $\omega_j > 0$ when λ_j is a real number.

Case 2: $\lambda_j = \mathrm{Re}(\lambda_j) + \imath\mathrm{Im}(\lambda_j)$ is a complex number, i.e., $\mathrm{Re}(\lambda_j) > 0$ and $\mathrm{Im}(\lambda_j) \neq 0$. There must exists the other eigenvalue $\overline{\lambda}_j = \mathrm{Re}(\lambda_j) - \imath\mathrm{Im}(\lambda_j)$ since the conjugate roots are in pair. For $\overline{\lambda}_j$, (46) turns into

$$(\overline{\omega}_j^2+\alpha)^2 + \beta^2\overline{\omega}_j^2 - (k_1^2 + k_2^2\overline{\omega}_j^2)|\overline{\lambda}_j|^2 = 0. \tag{50}$$

Due to $\omega_j \in \mathbb{R}$, $\overline{\omega}_j \in \mathbb{R}$ and $|\lambda_j|^2 = |\overline{\lambda}_j|^2$, the solutions of (46) and (50) corresponding to λ_j and $\overline{\lambda}_j$, respectively, satisfy

$$\omega_{j1} = \overline{\omega}_{j1}, \quad \omega_{j2} = \overline{\omega}_{j2} \text{ and } \omega_{j1} = -\omega_{j2}. \tag{51}$$

Besides, we can get

$$\begin{aligned}
\phi_{j1} &= k_2\omega_{j1}\mathrm{Re}(\lambda_j) + k_1\mathrm{Im}(\lambda_j)\\
\phi_{j2} &= -k_2\omega_{j1}\mathrm{Re}(\lambda_j) + k_1\mathrm{Im}(\lambda_j)\\
\overline{\phi}_{j1} &= k_2\omega_{j1}\mathrm{Re}(\lambda_j) - k_1\mathrm{Im}(\lambda_j)\\
\overline{\phi}_{j2} &= -k_2\omega_{j1}\mathrm{Re}(\lambda_j) - k_1\mathrm{Im}(\lambda_j)\\
\psi_{j1} &= k_2\omega_{j1}\mathrm{Im}(\lambda_j) - k_1\mathrm{Re}(\lambda_j)\\
\psi_{j2} &= -k_2\omega_{j1}\mathrm{Im}(\lambda_j) - k_1\mathrm{Re}(\lambda_j)\\
\overline{\psi}_{j1} &= -k_2\omega_{j1}\mathrm{Im}(\lambda_j) - k_1\mathrm{Re}(\lambda_j)\\
\overline{\psi}_{j2} &= k_2\omega_{j1}\mathrm{Im}(\lambda_j) - k_1\mathrm{Re}(\lambda_j)
\end{aligned}$$

which indicates $\phi_{j1} = -\overline{\phi}_{j2}$, $\phi_{j2} = -\overline{\phi}_{j1}$, $\psi_{j1} = \overline{\psi}_{j2}$, and $\psi_{j2} = \overline{\psi}_{j1}$. This, together with (47) and (51), guarantees $\eta_{j1} = -\overline{\eta}_{j2}$ and $\overline{\eta}_{j1} = -\eta_{j2}$. It thus can develop $\omega_{j1}^{-1}\arctan(\eta_{j1}) = \overline{\omega}_{j2}^{-1}\arctan(\overline{\eta}_{j2})$ and $\overline{\omega}_{j1}^{-1}\arctan(\overline{\eta}_{j1}) = \omega_{j2}^{-1}\arctan(\eta_{j2})$. Similar to Case 1, we can derive $\tau_{j1} = \overline{\tau}_{j2}$ and $\overline{\tau}_{j1} = \tau_{j2}$. Hence, we can find τ_j and $\overline{\tau}_j$ by only considering $\omega_j > 0$ and $\overline{\omega}_j > 0$, respectively. This proof is complete. ■

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