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Extended Structural Balance Theory and Method for Cooperative-Antagonistic Networks

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***Abstract*—For the behavior analysis of cooperative-antagonistic networks (CANs), the structural balance theory has been adopted widely since it can be developed to accommodate the simultaneous existence of competitive interactions. This advantage of structural balance is further explored in the present paper by aiming at the local nodes instead of the global CAN, and a notion of structurally balanced nodes is proposed. It is shown that structurally balanced nodes play a dominant role in determining the dynamic behaviors of CANs. Furthermore, such an extension of structural balance is applied to distinguish the roles of all nodes to create a hierarchical structure decomposition of CANs. Particularly, regarding interval bipartite consensus of quasi-strongly connected CANs, the impact index of root nodes can be calculated directly through counting the number of structurally balanced nodes.**

***Index Terms*—Cooperative-antagonistic network, impact index, interval bipartite consensus, signed digraph, structural balance.**

I. Introduction

Cooperative-antagonistic networks (CANs) represent a class of network systems that involves not only cooperative but also antagonistic interactions among nodes. In the literature, CANs that are alternatively called signed networks [1] or coopetition networks [2] have attracted considerable research interest since they supply good mathematical representations and theoretical guidelines for analysis and problem-solving in many practical fields, especially in the field of social networks. We refer the readers to a recent survey [3] and references therein for more detailed explanations on CANs and their applications.

In CAN theory, structural balance is a fundamental property that yields a feasible way to harness the simultaneous existence of two competitive interactions [4]. It requires making an exact bipartition of the nodes into two disjoint sets such that a CAN corresponds to a signed graph possessing a positive weight for each edge between nodes in the same set and a negative weight for each edge between nodes in two different sets. Considering dynamic behaviors, bipartite consensus (or polarization) arises for the nodes of CANs with structural balance and stability (or neutralization) emerges, otherwise. These behavior results also need to impose a strong connectivity assumption on CANs [4]. In addition, it is worth noting that the structural balance theory is popular in the social network field [5]. For example, it may lead to structural balance owing to the coevolution of appraisal and influence relationships among nodes [6] and may require computing structural balance in some circumstances [7].

However, when the structural balance is broken or the strong connectivity is relaxed to the quasi-strong connectivity (that is, the spanning tree condition), bipartite consensus is difficult to be accomplished for all nodes, instead of which fragmentation or convergence into clusters may emerge in CANs [2], [8], [9]. Because, given the quasi-strong connectivity, the root nodes of CANs can still be driven to reach bipartite consensus, together with the non-root nodes converging into the symmetric interval determined by the polarized values of the root nodes, this class of convergence behaviors is called interval bipartite consensus, e.g., see [3], [8]. Although the structural balance is not satisfied by CANs, interval bipartite consensus admits "partial bipartite consensus" for a portion of nodes, especially including the root nodes. This reveals hints that the structural balance encounters difficulties in characterizing the local properties of CANs even in spite of its notable global property to bridge a close relation between CANs and conventional (unsigned) networks through a gauge transformation or switching equivalence [10]–[13]. In recent results, the structural balance theory has been developed to CANs subject to changing topologies in [14]–[20], all which are devoted to the extension of the global property of structural balance. Other extensions of bipartite consensus or of behavior analysis approaches for CANs generally benefit directly from the classical structural balance theory (see, e.g., [21]–[27]).

Unlike the aforementioned results established with the structural balance theory for CANs, this paper focuses on exploiting a class of extended structural balance properties that is directly devoted to distinguishing the structural balance and unbalance of each node. We thus contribute to investigating a new class of node structural balance properties of CANs through examining whether there exist structurally balanced nodes. The extended structural balance theory is developed to contain the traditional structural balance theory as a trivial case. In particular, it helps to reveal a notable property of interval bipartite consensus that the impact index of the root nodes is exactly the number of the structurally balanced nodes. This greatly improves the relevant result of [8] that does not resolve how to calculate the impact index although it introduces the impact index to evaluate the influences of the root nodes on the non-root nodes.

For approaches to identifying the roles of nodes in a network system, the one that resorts to root and non-root nodes depends heavily on the quasi-strong connectivity of the network system, e.g., see [2], [8], [9]. The most adopted approach able to avoid the hypothesis of connectivity on network systems is to divide the nodes into leaders and followers, which however typically restricts the leaders to "stubborn nodes" without any neighbors (see, e.g., [28]–[30]). By contrast, an extended leader-follower

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framework is proposed in [1] such that all nodes of CANs with any topologies can be admitted to have neighbors. We develop this line of role identification in the current paper and combine it with the idea of structurally balanced nodes to create a novel hierarchical framework to identify the roles of nodes in CANs. There generally exist four mutually exclusive types of roles for the nodes of CANs with any topologies: structurally balanced leaders, structurally unbalanced leaders, structurally balanced followers and structurally unbalanced followers, where the first three types of nodes send information to the last type of nodes, but not vice versa. Furthermore, we reveal that the structurally balanced nodes, especially those structurally balanced leaders, play a major role in analyzing the dynamic behaviors of CANs, regardless of any conditions about the connectivity, balance or sign pattern of them.

Notations: Denote $\mathscr{I}_n=\{1,2,\cdots,n\}$, $1_n=[1,1,\cdots,1]^{\mathrm{T}}\in\mathbb{R}^n$, and $\mathrm{diag}\{\sigma_1,\sigma_2\cdots,\sigma_n\}$ as a diagonal matrix whose diagonal entries are σ_1, σ_2, $\cdots$, σ_n and off-diagonal entries are equal to zero. For any $\gamma\in\mathbb{R}$, $|\gamma|$ and $\mathrm{sgn}(\gamma)$ are the absolute value and sign of γ, respectively. For any $\Gamma=[\gamma_{ij}]\in\mathbb{R}^{p\times q}$, $|\Gamma|=[|\gamma_{ij}|]$, $\Delta_\Gamma=\mathrm{diag}\left\{\sum_{j=1}^q\gamma_{1j},\sum_{j=1}^q\gamma_{2j},\cdots,\sum_{j=1}^q\gamma_{pj}\right\}$, $\det(\Gamma)$ is the determinant of Γ when $p=q$, and Γ denotes a nonnegative matrix (namely, $\Gamma\geq 0$) if $\gamma_{ij}\geq 0$, $\forall i\in\mathscr{I}_p$, $\forall j\in\mathscr{I}_q$. Let us also denote the set of all the n-by-n gauge transformations as $\mathscr{D}_n=\{D=\mathrm{diag}\{d_1,d_2,\cdots,d_n\}:d_i\in\{\pm1\},\forall i\in\mathscr{I}_n\}$.

II. Extended Structural Balance

Our aim is to characterize a new class of extended structural balance properties for signed graphs, with which we can carry out the behavior analysis of CANs with arbitrary topologies. In this section, we first introduce basic notions of signed graphs, and then introduce the structural balance and its extension.

A. Basic Notions of Signed Graphs

A *(weighted) signed digraph*, short for directed graph, is described by a triple $\mathscr{G}=\{\mathscr{V},\mathscr{E},A\}$, where $\mathscr{V}=\{v_1,v_2,\cdots,v_n\}$ is the set of nodes, $\mathscr{E}\subseteq\mathscr{V}\times\mathscr{V}=\{(v_j,v_i):\forall v_i\in\mathscr{V},\forall v_j\in\mathscr{V}\}$ is the set of edges, and $A=[a_{ij}]\in\mathbb{R}^{n\times n}$ is the weighted matrix with signed weights such that $a_{ij}\neq 0\Leftrightarrow(v_j,v_i)\in\mathscr{E}$ and $a_{ij}=0$ otherwise. Assume that $\mathscr{G}$ does not have self-loops (i.e., $a_{ii}=0$ or $(v_i,v_i)\notin\mathscr{E}$, $i\in\mathscr{I}_n$). If there is a (directed) edge $(v_j,v_i)\in\mathscr{E}$, then v_j is a *neighbor* of v_i, by which $\mathscr{N}_{v_i}=\{v_j:(v_j,v_i)\in\mathscr{E}\}$ and $\mathscr{N}_i=\{j:(v_j,v_i)\in\mathscr{E}\}$ are denoted as the set and the label set of all neighbors of v_i, respectively. The Laplacian matrix L_A (or simply L) of $\mathscr{G}$ is an n-by-n matrix defined by

$$L_A=\left[l_{ij}^A\right]\ \text{with}\ l_{ij}^A=\begin{cases}\sum\limits_{h\in\mathscr{N}_i}|a_{ih}|, & i=j\\ -a_{ij}, & i\neq j.\end{cases}$$

An *unsigned digraph* is represented by $\mathscr{G}(|A|)=\{\mathscr{V},\mathscr{E},|A|\}$ that has the same node and edge sets with the signed digraph $\mathscr{G}$ but is associated with a nonnegative weighted matrix $|A|\geq 0$. Clearly, the Laplacian matrix of $\mathscr{G}(|A|)$ is given by $L_{|A|}$ (i.e., it is defined in the same way as L_A and its entries are obtained by only replacing a_{ij} in L_A with $|a_{ij}|$, $\forall i,j\in\mathscr{I}_n$).

In $\mathscr{G}$, a *(directed) path* $\mathscr{P}$ from v_j to v_i is formed by a finite sequence of $m\geq 1$ edges $\{(v_{k_{l-1}},v_{k_l})\in\mathscr{E}:1\leq l\leq m\}$, where $v_{k_0}=v_j$, v_{k_1}, $\cdots$, $v_{k_m}=v_i$ are distinct nodes. If $\mathscr{G}$ admits such a path $\mathscr{P}$, then v_j is called an *origin node* of v_i, and the set of all origin nodes of v_i is denoted by $\mathscr{M}_{v_i}$. For any given pair of distinct nodes, if one of them is an origin node of the other, then $\mathscr{G}$ is *strongly connected*. If there exists some node v_j that is an origin node of all the other nodes v_i $(i\neq j)$, then $\mathscr{G}$ is *quasi-strongly connected* (or has a *spanning tree*), where v_j is called a *root node*.

A signed digraph $\mathscr{G}^s=\{\mathscr{V}^s,\mathscr{E}^s,A^s\}$ is called a *subgraph* of $\mathscr{G}$ if $\mathscr{V}^s\subseteq\mathscr{V}$ and $\mathscr{E}^s\subseteq\mathscr{E}$. In addition, we consider that A^s is induced from A (i.e., the nonzero entry of A^s associated with any edge $(v_j,v_i)\in\mathscr{E}^s$ is equal to a_{ij}). For any given $\mathscr{V}^s\subseteq\mathscr{V}$, $\mathscr{G}^s=\{\mathscr{V}^s,\mathscr{E}^s,A^s\}$ is called a *subgraph of $\mathscr{G}$ over $\mathscr{V}^s$* if $\mathscr{E}^s$ and A^s are obtained from $\mathscr{E}$ and A, respectively, by deleting all edges in $\mathscr{E}$ and all columns and rows in A corresponding to the nodes in $\mathscr{V}\setminus\mathscr{V}^s$. It can be easily seen that the subgraph $\mathscr{G}^s$ of $\mathscr{G}$ over any given $\mathscr{V}^s\subseteq\mathscr{V}$ is uniquely induced from $\mathscr{G}$.

We extend the notion of neighbors to subgraphs such that for any subgraph $\mathscr{G}^s$ of $\mathscr{G}$, $\mathscr{N}_{\mathscr{G}^s}$ denotes its neighbor set involving the neighbors of all nodes of $\mathscr{V}^s$ but excluding those belonging to $\mathscr{V}^s$, namely, $\mathscr{N}_{\mathscr{G}^s}=\left(\cup_{v_i\in\mathscr{V}^s}\mathscr{N}_{v_i}\right)\setminus\mathscr{V}^s$. By following, e.g., [1, Definition 1], we call a node v_i a *leader* of a signed digraph $\mathscr{G}$ if there exists a strongly connected subgraph $\mathscr{G}^s$ of $\mathscr{G}$ such that $v_i\in\mathscr{V}^s$ and $\mathscr{N}_{\mathscr{G}^s}=\emptyset$; and we call this node v_i a *follower* of $\mathscr{G}$, otherwise. The sets of leaders and followers are denoted by $\mathscr{L}$ and $\mathscr{F}$, respectively. Clearly, $\mathscr{L}$ and $\mathscr{F}$ are disjoint subsets of $\mathscr{V}$, i.e., $\mathscr{L}\cup\mathscr{F}=\mathscr{V}$ and $\mathscr{L}\cap\mathscr{F}=\emptyset$, where $\mathscr{L}$ in particular includes all "stubborn nodes," and each of its entries is in some strongly connected component of $\mathscr{G}$ [6].

B. Structural Balance and Its Extension

The property of structural balance plays an important role in investigating the signed digraphs, which is formally introduced in the following definition (see also [4], [10]–[12]).

Definition 1. A signed digraph $\mathscr{G}$ is *structurally balanced* if $\mathscr{V}$ has a bipartition $\{\mathscr{V}_1,\mathscr{V}_2:\mathscr{V}_1\cup\mathscr{V}_2=\mathscr{V},\mathscr{V}_1\cap\mathscr{V}_2=\emptyset\}$ fulfilling $a_{ij}\geq 0$, $\forall v_i$, $v_j\in\mathscr{V}_h$ for any $h\in\{1,2\}$ and $a_{ij}\leq 0$, $\forall v_i\in\mathscr{V}_h$, $\forall v_j\in\mathscr{V}_l$ for any h, $l\in\{1,2\}$ and $h\neq l$; and $\mathscr{G}$ is *structurally unbalanced*, otherwise.

From Definition 1, the structural balance determines a global property of signed digraphs. When given the structural balance of a signed digraph $\mathscr{G}$, it accordingly yields a unique unsigned digraph $\mathscr{G}(|A|)=\{\mathscr{V},\mathscr{E},|A|\}$. In fact, the structural balance of $\mathscr{G}$ implies the existence of some gauge transformation $D\in\mathscr{D}_n$ such that $DAD=|A|$. However, if $\mathscr{G}$ is structurally unbalanced, there may not exist a close relation between $\mathscr{G}$ and $\mathscr{G}(|A|)$. It is obvious that the global requirement of structural balance seems restrictive for its applications in studying signed digraphs (see similar discussions in, e.g., [1], [2], [8], [9], [27]).

We propose a new definition for structural balance, but focus on the structural balance property of nodes in a signed digraph, rather than of the signed digraph itself.

Definition 2. For a signed digraph $\mathscr{G}$, v_i is called a *structurally balanced node* if the subgraph $\mathscr{G}^s$ of $\mathscr{G}$ over $\overline{\mathscr{M}}_{v_i}$ is structurally balanced, where $\overline{\mathscr{M}}_{v_i}=\mathscr{M}_{v_i}\cup\{v_i\}$; and otherwise, v_i is called a *structurally unbalanced node*.

Different from Definition 1, Definition 2 suggests a new idea for the structural balance from the perspective of "every node"

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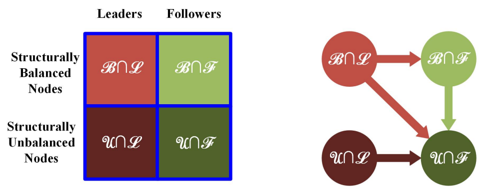

(a) (b)

Fig. 1: (a) A decomposition of $\mathscr{V}$ into four disjoint subsets. (b) Information flow directions of nodes in the four subsets.

of signed digraphs. This brings benefits in distinguishing nodes into two disjoint sets that are closely related with the structural balance and making it possible to exploit the structural balance for local portions of signed digraphs. In the remaining analysis, let us denote $\mathscr{B}$ and $\mathscr{U}$ as the two sets of structurally balanced and unbalanced nodes, respectively, which satisfy $\mathscr{B}\cup\mathscr{U}=\mathscr{V}$ and $\mathscr{B}\cap\mathscr{U}=\emptyset$.

Next, we give a lemma to better reveal the relation between the structural balance of signed digraphs and of their nodes.

Lemma 1. For any signed digraph $\mathscr{G}$, it is:

1) structurally balanced if and only if $\mathscr{B}=\mathscr{V}$ ($\mathscr{U}=\emptyset$);
2) structurally unbalanced if and only if $\mathscr{B}\subset\mathscr{V}$ ($\mathscr{U}\neq\emptyset$).

Proof: A consequence of Definitions 1 and 2. ■

From Lemma 1, it is worth emphasizing that for a signed digraph $\mathscr{G}$, the structural balance is a global property concerning all nodes of $\mathscr{G}$. This may inevitably cause inconvenience to the application of structural balance theory, especially when only a portion of nodes are of interest. Clearly, such a drawback can be overcome through the introduction of structurally balanced nodes in Definition 2. Further, Lemma 1 ensures that the global benefits of structural balance can be retained when considering the notion of structurally balanced nodes.

Another benefit of Definition 2 is that for any signed digraph $\mathscr{G}$, the classification of all nodes into structurally balanced and unbalanced nodes, as well as into leaders and followers, results in a specific decomposition of the node set $\mathscr{V}$ into four disjoint subsets: $\mathscr{B}\cap\mathscr{L}$, $\mathscr{B}\cap\mathscr{F}$, $\mathscr{U}\cap\mathscr{L}$ and $\mathscr{U}\cap\mathscr{F}$ (see Fig. 1 (a)). We can see that this decomposition helps to clearly distinguish the roles of all nodes into four different subsets. For the sake of clarity, the nodes of $\mathscr{B}\cap\mathscr{L}$, $\mathscr{B}\cap\mathscr{F}$, $\mathscr{U}\cap\mathscr{L}$ and $\mathscr{U}\cap\mathscr{F}$ will be called *structurally balanced leaders*, *structurally balanced followers*, *structurally unbalanced leaders*, and *structurally unbalanced followers*, respectively.

When we focus on specific portions of nodes, we can obtain local structural balance properties for signed digraphs, together with the equivalence characterizations. Toward this end, let the four subgraphs of any signed digraph $\mathscr{G}$ over $\mathscr{U}\cap\mathscr{L}$, $\mathscr{B}\cap\mathscr{L}$, $\mathscr{B}\cap\mathscr{F}$, and $\mathscr{U}\cap\mathscr{F}$ be, respectively, denoted by

$$\mathscr{G}_{ul}^s=\{\mathscr{U}\cap\mathscr{L},\mathscr{E}_{ul}^s,A_{ul}^s\},\qquad \mathscr{G}_{bl}^s=\{\mathscr{B}\cap\mathscr{L},\mathscr{E}_{bl}^s,A_{bl}^s\}$$
$$\mathscr{G}_{bf}^s=\left\{\mathscr{B}\cap\mathscr{F},\mathscr{E}_{bf}^s,A_{bf}^s\right\},\quad \mathscr{G}_{uf}^s=\left\{\mathscr{U}\cap\mathscr{F},\mathscr{E}_{uf}^s,A_{uf}^s\right\}$$

where $\eta_1=|\mathscr{U}\cap\mathscr{L}|$, $\eta_2=|\mathscr{B}\cap\mathscr{L}|$, $\eta_3=|\mathscr{B}\cap\mathscr{F}|$, and $\eta_4=|\mathscr{U}\cap\mathscr{F}|$ are the numbers of nodes in them, respectively. If the four subgraphs exist, then $\mathscr{G}_{ul}^s$ is structurally unbalanced, $\mathscr{G}_{bl}^s$ and $\mathscr{G}_{bf}^s$ are structurally balanced, and $\mathscr{G}_{uf}^s$ can be either structurally balanced or unbalanced. Moreover, $\mathscr{G}_{bl}^s$ and $\mathscr{G}_{ul}^s$ formed by leaders have equivalence characterizations with the signs of cycles, gauge transformations, and eigenvalues of Laplacian matrices, whereas $\mathscr{G}_{bf}^s$ and $\mathscr{G}_{uf}^s$ formed by followers generally may not have. These properties can be validated with Lemma 1 and the structural balance results of, e.g., [4], [27].

In addition, we propose the following lemma corresponding to these four subgraphs of signed digraphs.

Lemma 2. For any signed digraph $\mathscr{G}$, if the nodes are arranged in order of their label indexes such that

$$\mathscr{U}\cap\mathscr{L}=\{v_i:1\leq i\leq\eta_1\}$$
$$\mathscr{B}\cap\mathscr{L}=\{v_i:\eta_1+1\leq i\leq\eta_1+\eta_2\}$$
$$\mathscr{B}\cap\mathscr{F}=\{v_i:\eta_1+\eta_2+1\leq i\leq\eta_1+\eta_2+\eta_3\}$$
$$\mathscr{U}\cap\mathscr{F}=\{v_i:\eta_1+\eta_2+\eta_3+1\leq i\leq\eta_1+\eta_2+\eta_3+\eta_4\}$$

then the weighted matrix A is correspondingly structured into

$$A=\begin{bmatrix}A_{ul}^s & 0 & 0 & 0\\ 0 & A_{bl}^s & 0 & 0\\ 0 & A_{32} & A_{bf}^s & 0\\ A_{41} & A_{42} & A_{43} & A_{uf}^s\end{bmatrix}$$

where $A_{32}\in\mathbb{R}^{\eta_3\times\eta_2}$ and $A_{4i}\in\mathbb{R}^{\eta_4\times\eta_i}$, $i=1,\,2,\,3$ satisfy:

1) $A_{32}\neq 0$, and otherwise $A_{32}=0\Leftrightarrow\eta_3=0$;
2) $[A_{41}\ A_{42}\ A_{43}]\neq 0$, and otherwise $[A_{41}\ A_{42}\ A_{43}]=0\Leftrightarrow\eta_4=0$.

In particular, for the subgraph $\mathscr{G}_b^s=\left\{\mathscr{B},\mathscr{E}_b^s,A_b^s\right\}$ of $\mathscr{G}$ over $\mathscr{B}$, it follows that $\mathscr{G}_b^s$ is structurally balanced, together with

$$\mathscr{B}=\{v_i:\eta_1+1\leq i\leq\eta_1+\eta_2+\eta_3\}\,,\ A_b^s=\begin{bmatrix}A_{bl}^s & 0\\ A_{32} & A_{bf}^s\end{bmatrix}.$$

Proof: This lemma can be established by considering the definitions of the leaders and structurally balanced nodes. ■

Remark 1. The result of Lemma 2 actually hints a hierarchical structure of a signed digraph, in addition to the decomposition of nodes in Fig. 1(a). Generally, any signed digraph admits two layers: leaders and followers. The layer of leaders contains two disjoint sets consisting of structurally balanced and unbalanced leaders, respectively. By contrast, the layer of followers further separates into two sublayers composed of structurally balanced and unbalanced followers, respectively. In addition, it is easy to see that structurally unbalanced followers may get information from all other nodes, whereas structurally balanced followers can acquire information only from structurally balanced leaders (see Fig. 1(b) for an illustration).

For the decomposition of $\mathscr{V}$ in Fig. 1, we propose two useful properties in the following lemma.

Lemma 3. For any signed digraph $\mathscr{G}$, its nodes satisfy:

1) $\mathscr{L}\neq\emptyset$;
2) $\mathscr{B}\cap\mathscr{L}=\emptyset\Leftrightarrow\mathscr{B}=\emptyset$.

Proof: 1): Notice a property of any signed digraph $\mathscr{G}$ (see also [1, Condition 1]) that for each follower $v_j\in\mathscr{F}$ (if exists), there exists at least one leader $v_i\in\mathscr{L}$ to admit at least one path from v_i to v_j. It is clear that $\mathscr{L}\neq\emptyset$ follows straightforwardly as a consequence of this property.

2): If $\mathscr{B}=\emptyset$, then $\mathscr{B}\cap\mathscr{L}=\emptyset$. Conversely, when $\mathscr{B}\cap\mathscr{L}=\emptyset$, we assume $\mathscr{B}\neq\emptyset$ by contradiction. We consequently have

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$\eta_3 = |\mathscr{B} \cap \mathscr{F}| \neq 0$, i.e., $A_{32} \neq 0$, which together with Lemma 2 leads to $\eta_2 = |\mathscr{B} \cap \mathscr{L}| \neq 0$, i.e., $\mathscr{B} \cap \mathscr{L} \neq \emptyset$. A contradiction to $\mathscr{B} \cap \mathscr{L} = \emptyset$ occurs, by which we can conclude $\mathscr{B} = \emptyset$. ■

In addition to Lemma 3, the following lemma presents an algebraic criterion to examine whether there exist structurally balanced nodes or not in signed digraphs.

Lemma 4. For any signed digraph $\mathscr{G}$, it follows:

1) $\mathscr{B} = \emptyset \Leftrightarrow \det(L) > 0$;
2) $\mathscr{B} \neq \emptyset \Leftrightarrow \det(L) = 0$.

Proof: With Lemmas 2 and 3, this lemma can be proved by noting [1, Lemma 1] and [27, Theorems 4.1 and 4.2]. ■

As the counterpart results of 1) and 2) in Lemma 3, $\mathscr{F} \subset \mathscr{V}$ and $\mathscr{B} \cap \mathscr{L} \neq \emptyset \Leftrightarrow \mathscr{B} \neq \emptyset$ hold, respectively. By $\mathscr{F} \subset \mathscr{V}$, there may or may not exist followers in $\mathscr{G}$, where the nodes can not be all followers. By the equivalence $\mathscr{B} \cap \mathscr{L} \neq \emptyset \Leftrightarrow \mathscr{B} \neq \emptyset$, it means that $\mathscr{G}$ has structurally balanced leaders if and only if $\mathscr{G}$ has structurally balanced nodes. Further, we can use Lemma 4 to verify either $\det(L) = 0$ or $\det(L) \neq 0$ for any signed digraph $\mathscr{G}$ to algebraically determine whether there exist structurally balanced nodes or not in $\mathscr{G}$. These equivalent facts can help to predict what class of dynamic behaviors will emerge in CANs, which will be exploited theoretically in the following section.

III. Convergence Behaviors of CANs

In this section, we explore properties derived for structurally balanced nodes to provide a new perspective for the behavior analysis of CANs. Consider a CAN under any signed digraph $\mathscr{G}$, and let $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$ be its state that evolves dynamically with the type of Laplacian flow. Namely, the CAN is described by (see also [1], [4], [8])

$$\dot{x}(t) = -Lx(t) \tag{1}$$

which is written in components as

$$\dot{x}_i(t) = \sum_{j \in \mathscr{N}_i} a_{ij} \left[x_j(t) - \operatorname{sgn}(a_{ij}) x_i(t) \right], \quad i \in \mathscr{I}_n.$$

For behaviors of the CAN (1), the converged state exists for any $x(0) = x_0$ (see [3, Proposition 28]). Let $\theta_i \triangleq \lim_{t \to \infty} x_i(t)$, $\forall i \in \mathscr{I}_n$, and denote $\theta = [\theta_1, \theta_2, \cdots, \theta_n]^{\mathrm{T}}$. Our aim is to explore deterministic relations among the converged states of all nodes in CANs. Of special interest is to reach four popular behaviors. For any $x_0 \in \mathbb{R}^n$, if $\theta_i = 0$, $\forall i \in \mathscr{I}_n$ holds, then the CAN (1) is *stable*; and if $\theta_i \in \cup_{v_j \in \mathscr{L}} \left[-|\theta_j|, |\theta_j| \right]$, $\forall v_i \in \mathscr{F}$ holds, then it achieves *bipartite containment tracking*. In particular, if there exists a positive scalar $\bar{\theta} \triangleq \bar{\theta}(x_0) > 0$ for any $x_0 \in \mathbb{R}^n$ such that $\theta_i \in \{-\bar{\theta}, \bar{\theta}\}$, $\forall v_i \in \mathscr{L}$ and $\theta_i \in [-\bar{\theta}, \bar{\theta}]$, $\forall v_i \in \mathscr{F}$, then the CAN (1) achieves *interval bipartite consensus*, and further the number of entries in θ whose moduli are equal to $\bar{\theta}$ is called the *impact index* of the leaders (see also [8, Definition 1]). For the case when $\theta_i \in \{-\bar{\theta}, \bar{\theta}\}$, $\forall i \in \mathscr{I}_n$, the CAN (1) is said to accomplish *bipartite consensus* (see, e.g., [4, Definition 1]).

A. General Behavior Analysis by Extended Structural Balance

With the notion of structurally balanced nodes, we develop a general theorem for the dynamic behaviors of CANs described by (1) in the sense of whether there exist structurally balanced leaders or structurally balanced nodes.

Theorem 1. For any signed digraph $\mathscr{G}$ with arbitrary topology, the state components of the CAN (1) all converge together with their converged values θ_i, $i \in \mathscr{I}_n$ satisfying

$$\theta_i \begin{cases} \in \bigcup\limits_{v_j \in \mathscr{B} \cap \mathscr{L}} \left[-|\theta_j|, |\theta_j| \right], & \mathscr{B} \cap \mathscr{L} \neq \emptyset \\ = 0, & \mathscr{B} \cap \mathscr{L} = \emptyset \end{cases}, \quad i \in \mathscr{I}_n \tag{2}$$

or equivalently,

$$\theta_i \begin{cases} \in \bigcup\limits_{v_j \in \mathscr{B}} \left[-|\theta_j|, |\theta_j| \right], & \mathscr{B} \neq \emptyset \\ = 0, & \mathscr{B} = \emptyset \end{cases}, \quad i \in \mathscr{I}_n. \tag{3}$$

Proof: From [1, Theorem 8], we can validate that for the CAN (1) under any $\mathscr{G}$, θ_i exists for all $i \in \mathscr{I}_n$ and satisfies

$$\theta_i \in \bigcup_{v_j \in \mathscr{L}} \left[-|\theta_j|, |\theta_j| \right], \quad i \in \mathscr{I}_n. \tag{4}$$

We consider the result (4) by resorting to the set decomposition $\mathscr{L} = (\mathscr{B} \cap \mathscr{L}) \cup (\mathscr{U} \cap \mathscr{L})$ (see Fig. 1). Since each $v_j \in \mathscr{U} \cap \mathscr{L}$ is a leader node, it is included in a strongly connected subgraph of $\mathscr{G}$ without neighbors (i.e., if this subgraph is denoted as $\mathscr{G}_j^s$, then $\mathscr{G}_j^s$ is strongly connected and $\mathscr{N}_{\mathscr{G}_j^s} = \emptyset$ holds). In addition, $v_j \in \mathscr{U} \cap \mathscr{L}$ is a structurally unbalanced node. It hence follows from Lemma 1 that $\mathscr{G}_j^s$ is structurally unbalanced. With these properties of $\mathscr{G}_j^s$, (1) contains a subsystem associated with $\mathscr{G}_j^s$, which depends only on the Laplacian matrix L_j^s of $\mathscr{G}_j^s$. Namely, if we denote the state of this subsystem of (1) as $x_j^s(t)$, then

$$\dot{x}_j^s(t) = -L_j^s x_j^s(t)$$

which is stable according to, e.g., [27, Theorem 4.3]. Further, the stability result ensures that for any $v_j \in \mathscr{U} \cap \mathscr{L}$, the related state is stable, and consequently

$$\bigcup_{v_j \in \mathscr{U} \cap \mathscr{L}} \left[-|\theta_j|, |\theta_j| \right] = \{0\}. \tag{5}$$

Case 1): $\mathscr{B} \cap \mathscr{L} = \emptyset$. It is equivalent to $\mathscr{B} = \emptyset$ from Lemma 3. In this case, we can see clearly from Fig. 1 that $\mathscr{U} = \mathscr{V}$, and then $\mathscr{U} \cap \mathscr{L} = \mathscr{L}$, hold. Thus, it immediately follows from (5) that all leaders are stable, i.e.,

$$\bigcup_{v_j \in \mathscr{L}} \left[-|\theta_j|, |\theta_j| \right] = \{0\}.$$

As a consequence of the stability result of all leaders, we know from (4) that $\theta_i = 0$, $i \in \mathscr{I}_n$.

Case 2): $\mathscr{B} \cap \mathscr{L} \neq \emptyset$. Since $\{0\} \subseteq \left[-|\theta_j|, |\theta_j| \right]$, $\forall v_j \in \mathscr{V}$ holds, we can benefit from (5) to get

$$\begin{aligned} &\bigcup_{v_j \in \mathscr{L}} \left[-|\theta_j|, |\theta_j| \right] \\ &= \left(\bigcup_{v_j \in \mathscr{B} \cap \mathscr{L}} \left[-|\theta_j|, |\theta_j| \right] \right) \bigcup \left(\bigcup_{v_j \in \mathscr{U} \cap \mathscr{L}} \left[-|\theta_j|, |\theta_j| \right] \right) \\ &= \bigcup_{v_j \in \mathscr{B} \cap \mathscr{L}} \left[-|\theta_j|, |\theta_j| \right]. \end{aligned} \tag{6}$$

Based on (6), a straightforward consequence of (4) is

$$\theta_i \in \bigcup_{v_j \in \mathscr{B} \cap \mathscr{L}} \left[-|\theta_j|, |\theta_j| \right], \quad i \in \mathscr{I}_n.$$

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With the discussions of *Cases 1)* and *2)*, we can immediately conclude (2). Next, we exploit the equivalence between (2) and (3) to complete this proof.

Due to $\mathscr{B}=\emptyset \Leftrightarrow \mathscr{B}\cap\mathscr{L}=\emptyset$ based on Lemma 3, we can develop $\theta_i=0$, $\forall i\in\mathscr{I}_n$ for $\mathscr{B}=\emptyset$ straightforwardly from (2). For $\mathscr{B}\neq\emptyset$, we revisit (4) and can insert (6) to derive

$$\begin{aligned}\bigcup_{v_j\in\mathscr{B}\cap\mathscr{F}}\left[-\left|\theta_j\right|,\left|\theta_j\right|\right] &\subseteq \bigcup_{v_j\in\mathscr{F}}\left[-\left|\theta_j\right|,\left|\theta_j\right|\right]\\ &\subseteq \bigcup_{v_j\in\mathscr{L}}\left[-\left|\theta_j\right|,\left|\theta_j\right|\right] = \bigcup_{v_j\in\mathscr{B}\cap\mathscr{L}}\left[-\left|\theta_j\right|,\left|\theta_j\right|\right].\end{aligned} \tag{7}$$

If we consider the set decomposition $\mathscr{B}=(\mathscr{B}\cap\mathscr{L})\cup(\mathscr{B}\cap\mathscr{F})$ (see Fig. 1), then the fact of (7) leads to

$$\begin{aligned}&\bigcup_{v_j\in\mathscr{B}}\left[-\left|\theta_j\right|,\left|\theta_j\right|\right]\\ &=\left(\bigcup_{v_j\in\mathscr{B}\cap\mathscr{L}}\left[-\left|\theta_j\right|,\left|\theta_j\right|\right]\right)\bigcup\left(\bigcup_{v_j\in\mathscr{B}\cap\mathscr{F}}\left[-\left|\theta_j\right|,\left|\theta_j\right|\right]\right)\\ &=\bigcup_{v_j\in\mathscr{B}\cap\mathscr{L}}\left[-\left|\theta_j\right|,\left|\theta_j\right|\right].\end{aligned} \tag{8}$$

A consequence of (2) and (8) is that for $\mathscr{B}\neq\emptyset$, we can deduce

$$\theta_i\in\bigcup_{v_j\in\mathscr{B}}\left[-\left|\theta_j\right|,\left|\theta_j\right|\right],\quad i\in\mathscr{I}_n.$$

Thus, (3) is obtained from (2). ■

Remark 2. It is worth highlighting that Theorem 1 effectively works, regardless of any conditions of CANs on connectivity, balance or sign pattern, and shows new comprehensive convergence behaviors for CANs by integrating the ideas of extended leader-follower description and extended structural balance of signed digraphs. If we combine Lemmas 3 and 4 with Theorem 1, then we can gain an alternative description of (2) or (3) as

$$\theta_i\begin{cases}\in\bigcup_{v_j\in\mathscr{B}\cap\mathscr{L}}\left[-\left|\theta_j\right|,\left|\theta_j\right|\right], & \det(L)=0\\ =0, & \det(L)>0\end{cases},\quad i\in\mathscr{I}_n. \tag{9}$$

With the help of (9), we can particularly examine whether the CAN (1) is stable or not by calculating $\det(L)$.

Remark 3. By (2) in Theorem 1, we emphasize a convergence behavior of the CAN (1) that the converged states of all nodes lie within the minimal convex hull spanned by those symmetric intervals dependent only on the converged states of structurally balanced leaders. This, in particular, is a more precise behavior than the bipartite containment tracking of CANs (see, e.g., [1]) and also provides it with a new perspective from the viewpoint of the extended structural balance for signed digraphs.

B. Specific CAN Behaviors with Extended Structural Balance

Given the strong connectivity, the following theorem reveals that the bipartite consensus of CANs depends on whether there exist structurally balanced nodes.

Theorem 2. Consider any strongly connected signed digraph $\mathscr{G}$. Then the CAN (1) achieves

1) bipartite consensus if and only if $\mathscr{B}=\mathscr{V}$;
2) stability if and only if $\mathscr{B}=\emptyset$.

Proof: Since $\mathscr{G}$ is strongly connected, we can validate the following equivalent relations:

$$\begin{aligned}\mathscr{B}=\mathscr{V} &\Leftrightarrow \mathscr{G}\text{ is structurally balanced}\\ &\Leftrightarrow \exists\bar{\theta}>0\text{ such that }\theta_i\in\left\{-\bar{\theta},\bar{\theta}\right\},\forall i\in\mathscr{I}_n.\end{aligned} \tag{10}$$

Based on (10), we consider Theorem 1 to gain the equivalence between bipartite consensus of the CAN (1) and $\mathscr{B}=\mathscr{V}$. Note that $\mathscr{B}=\mathscr{V}$ and $\mathscr{B}=\emptyset$ are mutually exclusive for any strongly connected $\mathscr{G}$. It is immediate to derive the equivalence between stability of the CAN (1) and $\mathscr{B}=\emptyset$ (see also Theorem 1). ■

Actually, Theorem 2 presents the bipartite consensus results of CANs but with the extended structural balance that concerns local nodes. If we consider Lemma 1, then we can easily verify that Theorem 2 is consistent with classical bipartite consensus results in, e.g., [4], [27] obtained by benefiting from the global structural balance theory for CANs.

When given the quasi-strong connectivity, the CANs possess more complex behaviors than bipartite consensus, as presented in the following theorem.

Theorem 3. For any quasi-strongly connected signed digraph $\mathscr{G}$, the CAN (1) achieves

1) bipartite consensus if and only if $\mathscr{B}=\mathscr{V}$;
2) interval bipartite consensus if and only if $\emptyset\neq\mathscr{B}\subset\mathscr{V}$ (or $\mathscr{B}\neq\emptyset$ and $\mathscr{U}\neq\emptyset$);
3) stability if and only if $\mathscr{B}=\emptyset$.

Proof: 1): By Lemma 1, the equivalence between $\mathscr{B}=\mathscr{V}$ and the structural balance of $\mathscr{G}$ can be obtained. This, together with the quasi-strong connectivity of $\mathscr{G}$, implies that bipartite consensus of the CAN (1) holds if and only if $\mathscr{B}=\mathscr{V}$ (see, e.g., [27, Corollary 5.2]).

2): Based on Lemma 3, we can verify $\emptyset\neq\mathscr{B}\subset\mathscr{V}\Leftrightarrow\emptyset\neq\mathscr{B}\cap\mathscr{L}\subset\mathscr{V}$. Because $\mathscr{G}$ is quasi-strongly connected, it follows that when $\emptyset\neq\mathscr{B}\cap\mathscr{L}\subset\mathscr{V}$ holds, $\mathscr{B}\cap\mathscr{L}=\mathscr{L}$ consists of all root nodes of $\mathscr{G}$. This actually guarantees $\left|\theta_i\right|=\left|\theta_j\right|$ for all $v_i,v_j\in\mathscr{B}\cap\mathscr{L}$, and consequently,

$$\bigcup_{v_j\in\mathscr{B}\cap\mathscr{L}}\left[-\left|\theta_j\right|,\left|\theta_j\right|\right]=\left[-\bar{\theta},\bar{\theta}\right] \tag{11}$$

holds for some $\bar{\theta}>0$ depending upon the initial state x_0. With (11), we apply Theorem 1 and conclude that interval bipartite consensus of the CAN (1) holds if and only if $\emptyset\neq\mathscr{B}\subset\mathscr{V}$.

3): A consequence of (3) and the above results 1) and 2) in this theorem. ■

Another benefit of the extended structural balance is that for interval bipartite consensus, a specific characterization can be established for the impact index of root nodes. Furthermore, it is worth emphasizing that this issue has not been addressed in [8] but can be conveniently handled by considering the idea of structurally balanced nodes, as given in the following theorem.

Theorem 4. For any quasi-strongly connected signed digraph $\mathscr{G}$, the impact index μ of the root nodes (namely, leaders) in the CAN (1) satisfies

$$\mu=\eta_2+\eta_3 \tag{12}$$

where $\eta_2=|\mathscr{B}\cap\mathscr{L}|$ and $\eta_3=|\mathscr{B}\cap\mathscr{F}|$ denote the number of structurally balanced leaders and that of structurally balanced

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followers, respectively. Namely, $\mu = |\mathscr{B}|$ is exactly the number of structurally balanced nodes in $\mathscr{G}$.

Remark 4. It is worth highlighting that the impact index helps to estimate the disagreement number of CANs and, thus, plays an important role in disagreement behavior analysis of opinion dynamics in social networks [8]. Though the impact index and its importance have been discussed in [8], how to calculate the impact index is an unsolved problem. This can be solved under Theorem 4 that provides a feasible way to calculate the impact index by counting the number of structurally balanced nodes.

To prove Theorem 4, some preliminary discussions are made on the CAN (1). The quasi-strong connectivity of $\mathscr{G}$ guarantees that the root nodes are leaders, where either $\mathscr{L} \subseteq \mathscr{U}$ or $\mathscr{L} \subseteq \mathscr{B}$ holds. Since the impact index reflects influences of root nodes on the other nodes when interval bipartite consensus emerges, $\mathscr{B} \neq \emptyset$ follows straightforwardly from the result 2) of Theorem 3. Consequently, $\mathscr{U} \cap \mathscr{L} = \emptyset$ (namely, $\eta_1 = 0$) and $\mathscr{B} \cap \mathscr{L} \neq \emptyset$ (namely, $\eta_2 \neq 0$) hold in Lemma 2. This thus leads to that the weighted matrix $A = [a_{ij}]$ can generally be written as

$$A = \begin{bmatrix} A_{bl}^s & 0 & 0 \\ A_{21} & A_{bf}^s & 0 \\ A_{31} & A_{32} & A_{uf}^s \end{bmatrix} \triangleq \begin{bmatrix} A_b^s & 0 \\ A_{21}^* & A_{uf}^s \end{bmatrix} \tag{13}$$
$$\text{with } A_b^s = \begin{bmatrix} A_{bl}^s & 0 \\ A_{21} & A_{bf}^s \end{bmatrix}, A_{21}^* = \begin{bmatrix} A_{31} & A_{32} \end{bmatrix}$$

where $A_{21} \in \mathbb{R}^{\eta_3 \times \eta_2}$, $A_{31} \in \mathbb{R}^{\eta_4 \times \eta_2}$ and $A_{32} \in \mathbb{R}^{\eta_4 \times \eta_3}$ fulfill

1) $A_{21} \neq 0$, and otherwise, $\eta_3 = 0$;
2) $A_{21}^* \neq 0$, and otherwise, $\eta_4 = 0$.

Correspondingly to (13), the Laplacian matrix L satisfies

$$L = \begin{bmatrix} L_{A_{bl}^s} & 0 & 0 \\ -A_{21} & L_{A_{bf}^s} + \Delta_{|A_{21}|} & 0 \\ -A_{31} & -A_{32} & L_{A_{uf}^s} + \Delta_{|A_{31}|} + \Delta_{|A_{32}|} \end{bmatrix} \tag{14}$$
$$= \begin{bmatrix} L_{A_b^s} & 0 \\ -A_{21}^* & L_{A_{uf}^s} + \Delta_{|A_{21}^*|} \end{bmatrix}.$$

From [4], [27], $L_{A_{bl}^s}$ has zero eigenvalues. It actually follows that L has exactly one zero eigenvalue and its other eigenvalues have positive real parts due to the quasi-strong connectivity of $\mathscr{G}$. This property of eigenvalue distribution also holds for $L_{A_{bl}^s}$ and $L_{A_b^s}$. Furthermore, we consider (14) and can hence obtain that $L_{A_{bf}^s} + \Delta_{|A_{21}|}$ and $L_{A_{uf}^s} + \Delta_{|A_{21}^*|}$ are invertible. In addition, we can verify

$$\lim_{t \to \infty} e^{-Lt} = \frac{w_\mathrm{r} w_\mathrm{l}^\mathrm{T}}{w_\mathrm{r}^\mathrm{T} w_\mathrm{l}} \tag{15}$$

where $w_\mathrm{r} \in \mathbb{R}^n$ and $w_\mathrm{l} \in \mathbb{R}^n$ represent the right eigenvector and the left eigenvector of L associated with the zero eigenvalue, respectively. For the formulation (15), we propose a candidate of how to choose w_r and w_l in the following lemma.

Lemma 5. Let any quasi-strongly connected signed digraph $\mathscr{G}$ be considered. If $\mathscr{B} \neq \emptyset$ holds, then for the Laplacian matrix L, a candidate of the left eigenvector w_l of the zero eigenvalue is given by

$$w_\mathrm{l} = \begin{bmatrix} D_1 \beta \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

for some $D_1 \in \mathscr{D}_{\eta_2}$ and some $\beta \in \mathbb{R}^{\eta_2}$ satisfying

$$\beta^\mathrm{T} L_{|A_{bl}^s|} = 0, \quad \beta^\mathrm{T} 1_{\eta_2} = 1. \tag{17}$$

Moreover, the right eigenvector w_r of the zero eigenvalue such that $w_\mathrm{r}^\mathrm{T} w_\mathrm{l} = 1$ is given by

$$w_\mathrm{r} = \begin{bmatrix} D_1 1_{\eta_2} \\ D_2 1_{\eta_3} \\ \xi \end{bmatrix} \text{ with } \xi = \left(L_{uf}^s + \Delta_{|A_{21}^*|} \right)^{-1} A_{21}^* D_3 1_{\eta_2 + \eta_3} \tag{18}$$

where $D_2 \in \mathscr{D}_{\eta_3}$, together with $D_1 \in \mathscr{D}_{\eta_2}$, renders $D_3 A_b^s D_3 = \left|A_b^s\right|$ for $D_3 = \mathrm{diag}\{D_1, D_2\}$ and $\xi \in \mathbb{R}^{\eta_4}$ fulfills

$$|\xi| < 1_{\eta_4}. \tag{19}$$

Proof: See Appendix A. ■

With Lemma 5, we can prove Theorem 4 as follows.

Proof of Theorem 4: From (15), we can validate for the CAN (1) that

$$\theta = \lim_{t \to \infty} e^{-Lt} x(0) = \left(\frac{w_\mathrm{l}^\mathrm{T} x(0)}{w_\mathrm{r}^\mathrm{T} w_\mathrm{l}} \right) w_\mathrm{r}. \tag{20}$$

Let us consider the candidates of w_l and w_r in (16) and (18), respectively. Then with Lemma 5, we use (20) to further derive

$$\theta = \theta_{ss} \begin{bmatrix} D_1 1_{\eta_2} \\ D_2 1_{\eta_3} \\ \xi \end{bmatrix} \tag{21}$$

where $\theta_{ss} = \beta^\mathrm{T} D_1 x_1(0) \in \mathbb{R}$ holds for $x_1(0) = \left[x_1(0), x_2(0), \cdots, x_{\eta_2}(0)\right]^\mathrm{T}$. By inserting (19) into (21), we can verify

$$|\theta_i| \begin{cases} = |\theta_{ss}|, & 1 \le i \le \eta_2 + \eta_3 \\ < |\theta_{ss}|, & \eta_2 + \eta_3 + 1 \le i \le n \end{cases}, \quad i \in \mathscr{I}_n$$

which leads to (12) based on [8, Definition 1]. Owing to $|\mathscr{B}| = \eta_2 + \eta_3$, (12) means also that the impact index is actually the number of structurally balanced nodes. ■

Remark 5. For the formulation (15), an additional hypothesis that w_r and w_l satisfy $w_\mathrm{r}^\mathrm{T} w_\mathrm{l} = 1$ is usually adopted without loss of generality. With this hypothesis, let $w_\mathrm{r} = [w_{\mathrm{r}1}, w_{\mathrm{r}2}, \cdots, w_{\mathrm{r}n}]^\mathrm{T}$, and then a general representation of Lemma 5 is that each entry of w_r fulfills

$$|w_{\mathrm{r}i}| \begin{cases} = 1, & v_i \in \mathscr{B} \\ < 1, & v_i \in \mathscr{U} \end{cases}, \quad i \in \mathscr{I}_n. \tag{22}$$

Clearly, (22) together with (15) suggests a direct interpretation of (12) on the impact index in interval bipartite consensus. The meaning behind (22) reflects a common phenomenon of CANs that the structurally balanced nodes possess complete belief in leaders, whereas the structurally unbalanced nodes do not and their beliefs in leaders are weakened owing to the antagonistic interactions among nodes.

From Theorems 3 and 4, we find that the extended structural balance properties are helpful to more specifically characterize the interval bipartite consensus behaviors for CANs than those exploited in, e.g., [8], [9] with typical structural balance theory. Furthermore, the presented idea of structurally balanced nodes not only enhances the leader-decided principle for the behavior analysis of CANs but also helps to determine more general and specific behaviors of CANs.

IV. Conclusions

In this paper, how to extend the structural balance theory for behavior analysis of CANs has been discussed. By introducing structurally balanced nodes, the structural balance property has been developed with a focus on local nodes. This may provide new alternative ways to study signed digraphs and then induce new behavior analysis methods for CANs. It has been revealed that the extended structural balance can be applied to describe the convergence behaviors for CANs. Moreover, the extended structural balance has been exploited for the role identification of nodes in CANs, which helps to distinguish different roles of nodes in the behaviors of CANs. Particularly, the structurally balanced nodes play a crucial role in investigating the dynamic behaviors of CANs, and especially are effective in calculating the impact index.

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Appendix A
Proof of Lemma 5

Proof: Denote

$$w_{\mathrm{l}}=\begin{bmatrix}w_{\mathrm{l}}^1\in\mathbb{R}^{\eta_2}\\ w_{\mathrm{l}}^2\in\mathbb{R}^{\eta_3}\\ w_{\mathrm{l}}^3\in\mathbb{R}^{\eta_4}\end{bmatrix},\quad w_{\mathrm{r}}=\begin{bmatrix}w_{\mathrm{r}}^1\in\mathbb{R}^{\eta_2}\\ w_{\mathrm{r}}^2\in\mathbb{R}^{\eta_3}\\ w_{\mathrm{r}}^3\in\mathbb{R}^{\eta_4}\end{bmatrix}.$$

By inserting the invertibility of $L_{A_{bf}^s}+\Delta_{|A_{21}|}$ and $L_{A_{uf}^s}+\Delta_{|A_{21}^*|}$, we consider (14) and can deduce from $w_{\mathrm{l}}^{\mathrm{T}}L=0$ that

$$\left(w_{\mathrm{l}}^1\right)^{\mathrm{T}}L_{A_{bl}^s}=0,\quad \left(w_{\mathrm{l}}^2\right)^{\mathrm{T}}\left(L_{A_{bf}^s}+\Delta_{|A_{21}|}\right)=0$$
$$\left(w_{\mathrm{l}}^3\right)^{\mathrm{T}}\left(L_{A_{uf}^s}+\Delta_{|A_{21}^*|}\right)=0.$$

We clearly have $w_{\mathrm{l}}^2=0$ and $w_{\mathrm{l}}^3=0$. Because $\mathscr{G}_{bl}^s$ is structurally balanced and $L_{A_{bl}^s}$ has only one zero eigenvalue, we can verify

$$\begin{aligned}&\exists w_{\mathrm{l}}^1\in\mathbb{R}^{\eta_2}\ \text{such that}\ \left(w_{\mathrm{l}}^1\right)^{\mathrm{T}}L_{A_{bl}^s}=0\\ \Leftrightarrow\ &\exists\beta\in\mathbb{R}^{\eta_2}\ \text{such that}\ \beta^{\mathrm{T}}\left(D_1L_{A_{bl}^s}D_1\right)=0\\ &\quad\text{for some}\ D_1\in\mathscr{D}_{\eta_2}\ \text{fulfilling}\ D_1A_{bl}^sD_1=|A_{bl}^s|\\ \Leftrightarrow\ &\exists\beta\in\mathbb{R}^{\eta_2}\ \text{such that}\ \beta^{\mathrm{T}}L_{|A_{bl}^s|}=0.\end{aligned}$$

With these facts, w_{l} satisfying (16) and (17) can be a candidate for the left eigenvector of the zero eigenvalue of L.

Note that $\mathscr{G}_b^s$ is structurally balanced (i.e., $D_3A_b^sD_3=\left|A_b^s\right|$ for some $D_3\in\mathscr{D}_{\eta_2+\eta_3}$), and $L_{A_b^s}$ has only one zero eigenvalue. From (14), $Lw_{\mathrm{r}}=0$ leads to

$$L_{A_{bl}^s}w_{\mathrm{r}}^1=0,\quad L_{A_b^s}\begin{bmatrix}w_{\mathrm{r}}^1\\ w_{\mathrm{r}}^2\end{bmatrix}=0$$

which, together with $L_{A_{bl}^s}D_11_{\eta_2}=0$ and $L_{A_b^s}D_31_{\eta_2+\eta_3}=0$ for $D_1\in\mathscr{D}_{\eta_2}$ and $D_3\in\mathscr{D}_{\eta_2+\eta_3}$, yields

$$w_{\mathrm{r}}^1=\alpha_1D_11_{\eta_2},\quad \begin{bmatrix}w_{\mathrm{r}}^1\\ w_{\mathrm{r}}^2\end{bmatrix}=\alpha_2D_31_{\eta_2+\eta_3}$$

for some $\alpha_1\in\mathbb{R}$ and $\alpha_2\in\mathbb{R}$. If we insert $w_{\mathrm{r}}^{\mathrm{T}}w_{\mathrm{l}}=1$, then based on (16) and (17), we can obtain $\alpha_1=\alpha_2=1$ and consequently,

$$w_{\mathrm{r}}^1=D_11_{\eta_2},\quad w_{\mathrm{r}}^2=D_21_{\eta_3}$$

where $D_2\in\mathscr{D}_{\eta_3}$ is such that $D_3=\operatorname{diag}\{D_1,D_2\}$.

Next, we are left to prove (19). We know from [8, Theorem 5] that $|\xi|\le 1_{\eta_4}$. Clearly, $\xi=w_{\mathrm{r}}^3$ holds, which is denoted as

$$\xi=\left[\xi_{\eta_2+\eta_3+1},\xi_{\eta_2+\eta_3+2},\cdots,\xi_n\right]^{\mathrm{T}}$$

and hence, it is left to only show $|\xi_i|<1$, $\forall\eta_2+\eta_3+1\le i\le n$. We adopt the *proof-by-contradiction*, and assume that $\left|\xi_{i_0}\right|=1$ holds for some i_0, $\eta_2+\eta_3+1\le i_0\le n$. With (18), we consider $Lw_{\mathrm{r}}=0$ and can deduce

$$\left(\sum_{j\in\mathscr{N}_{i_0}}^n\left|a_{i_0j}\right|\right)\xi_{i_0}-\sum_{j=\eta_2+\eta_3+1,j\ne i_0}^n a_{i_0j}\xi_j-\sum_{j=1}^{\eta_2+\eta_3}a_{i_0j}d_j=0 \tag{23}$$

where $d_j\in\{\pm1\}$, $\forall 1\le j\le\eta_2+\eta_3$ is such that

$$D_3=\left\{d_1,d_2,\cdots,d_{\eta_2+\eta_3}\right\}.$$

Due to $\left|\xi_{i_0}\right|=1$ and $\sum_{j\in\mathscr{N}_{i_0}}^n\left|a_{i_0j}\right|=\sum_{j=1,j\ne i_0}^n\left|a_{i_0j}\right|$, we employ (23) to obtain

$$\begin{aligned}\sum_{j=1,j\ne i_0}^n\left|a_{i_0j}\right|&=\left|\left(\sum_{j\in\mathscr{N}_{i_0}}^n\left|a_{i_0j}\right|\right)\xi_{i_0}\right|\\ &=\left|\sum_{j=\eta_2+\eta_3+1,j\ne i_0}^n a_{i_0j}\xi_j+\sum_{j=1}^{\eta_2+\eta_3}a_{i_0j}d_j\right|\\ &\le\sum_{j=\eta_2+\eta_3+1,j\ne i_0}^n\left|a_{i_0j}\right|\left|\xi_j\right|+\sum_{j=1}^{\eta_2+\eta_3}\left|a_{i_0j}\right|\end{aligned}$$

which leads to

$$\sum_{j=\eta_2+\eta_3+1,j\ne i_0}^n\left|a_{i_0j}\right|\le\sum_{j=\eta_2+\eta_3+1,j\ne i_0}^n\left|a_{i_0j}\right|\left|\xi_j\right|. \tag{24}$$

Owing to $|\xi|\le 1_{\eta_4}$, we can validate from (24) that

$$\left|\xi_j\right|=1,\quad\forall v_j\in\mathscr{N}_{v_{i_0}}\cap\mathscr{U}\cap\mathscr{F}. \tag{25}$$

In the same way as the derivation of (25), we can further derive

$$\left|\xi_j\right|=1,\quad\forall v_j\in\mathscr{M}_{v_{i_0}}\cap\mathscr{U}\cap\mathscr{F}$$

which together with $\left|\xi_{i_0}\right|=1$ yields

$$\left|\xi_j\right|=1,\quad\forall v_j\in\overline{\mathscr{M}}_{v_{i_0}}\cap\mathscr{U}\cap\mathscr{F}. \tag{26}$$

Note that for any $v_i\in\overline{\mathscr{M}}_{v_{i_0}}$, $\mathscr{N}_{v_i}\subseteq\overline{\mathscr{M}}_{v_{i_0}}$ holds. Consequently, we have $\mathscr{N}_{v_i}\cap\mathscr{U}\cap\mathscr{F}\subseteq\overline{\mathscr{M}}_{v_{i_0}}\cap\mathscr{U}\cap\mathscr{F}$. This actually implies that $(\mathscr{U}\cap\mathscr{F})\setminus\left(\overline{\mathscr{M}}_{v_{i_0}}\cap\mathscr{U}\cap\mathscr{F}\right)$ does not have nodes admitting paths to any nodes in $\overline{\mathscr{M}}_{v_{i_0}}\cap\mathscr{U}\cap\mathscr{F}$. Namely, it means

$$\begin{aligned}a_{ij}=0,\quad&\forall v_i\in\overline{\mathscr{M}}_{v_{i_0}}\cap\mathscr{U}\cap\mathscr{F}\\ &\forall v_j\in(\mathscr{U}\cap\mathscr{F})\setminus\left(\overline{\mathscr{M}}_{v_{i_0}}\cap\mathscr{U}\cap\mathscr{F}\right).\end{aligned} \tag{27}$$

Let $m=\left|\overline{\mathscr{M}}_{v_{i_0}}\cap\mathscr{U}\cap\mathscr{F}\right|$ $(1\le m\le\eta_4)$, and without any loss of generality we denote

$$\overline{\mathscr{M}}_{v_{i_0}}\cap\mathscr{U}\cap\mathscr{F}=\{v_{\eta_2+\eta_3+1},v_{\eta_2+\eta_3+2},\cdots,v_{\eta_2+\eta_3+m}\}.$$

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By noting this fact and inserting (27) into (13), we can obtain

$$A^s_{uf} = \begin{bmatrix} A^s_{uf,11} & 0 \\ A^s_{uf,21} & A^s_{uf,22} \end{bmatrix}$$

which together with (14) leads to that L can be expressed as

$$L = \begin{bmatrix} L_{A^s_b} & 0 & 0 \\ -A^*_{21,1} & L_{A^s_{uf,11}} + \Delta_{\left|A^*_{21,1}\right|} & 0 \\ -A^*_{21,2} & -A^s_{uf,21} & L_{A^s_{uf,22}} + \Delta_{\left|A^*_{21,2}\right|} + \Delta_{\left|A^s_{uf,21}\right|} \end{bmatrix} \quad (28)$$

for $A^s_{uf,11} \in \mathbb{R}^{m\times m}$, $A^s_{uf,21} \in \mathbb{R}^{(\eta_4-m)\times m}$, $A^s_{uf,22} \in \mathbb{R}^{(\eta_4-m)\times(\eta_4-m)}$, $A^*_{21,1} \in \mathbb{R}^{m\times(\eta_2+\eta_3)}$ and $A^*_{21,2} \in \mathbb{R}^{(\eta_4-m)\times(\eta_2+\eta_3)}$. If we combine (26) with (18), then with (28), $Lw_{\mathrm{r}} = 0$ ensures that there exists a gauge transformation

$$D = \left\{d_1, \cdots, d_{\eta_2+\eta_3}, \xi_{\eta_2+\eta_3+1}, \cdots, \xi_{\eta_2+\eta_3+m}\right\} \in \mathscr{D}_{\eta_2+\eta_3+m}$$

such that

$$\begin{bmatrix} L_{A^s_b} & 0 \\ -A^*_{21,1} & L_{A^s_{uf,11}} + \Delta_{\left|A^*_{21,1}\right|} \end{bmatrix} D 1_{\eta_2+\eta_3+m} = 0. \quad (29)$$

From (29), we can equivalently deduce

$$D \begin{bmatrix} A^s_b & 0 \\ A^*_{21,1} & A^s_{uf,11} \end{bmatrix} D = \left|\begin{bmatrix} A^s_b & 0 \\ A^*_{21,1} & A^s_{uf,11} \end{bmatrix}\right|. \quad (30)$$

By (30), the subgraph $\widehat{\mathscr{G}}^s_{i_0}$ of $\mathscr{G}$ over $\left(\overline{\mathscr{M}}_{v_{i_0}} \cap \mathscr{U} \cap \mathscr{F}\right) \cup \mathscr{B}$ is structurally balanced. For the subgraph $\mathscr{G}^s_{i_0}$ of $\mathscr{G}$ over $\overline{\mathscr{M}}_{v_{i_0}}$, it is also a subgraph of $\widehat{\mathscr{G}}^s_{i_0}$, and therefore is structurally balanced, which guarantees that v_{i_0} is a structurally balanced node based on Definition 2. This obviously contradicts with $v_{i_0} \in \mathscr{U} \cap \mathscr{F}$. By contradiction, we can conclude that (19) holds. ■

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