Laplacian Controllability of Interconnected Graphs

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***Abstract*—We consider the controllability of a graph constructed by interconnecting a finite number of single-input Laplacian controllable (SILC) graphs. We first study the interconnection realized by the composite graph of two SILC graphs, called the structure graph and the cell graph, respectively. Suppose the cell graph is Laplacian controllable by an input connected to some special vertex called the composite vertex. The composite graph is constructed by interconnecting several identical cell graphs through the composite vertices such that connecting these composite vertices alone forms the structure graph. We prove that the structure graph is SILC by an input connected to some vertex of the graph if and only if the composite graph is SILC by the same input connected to the same vertex. The second part of this article generalizes the path structure by viewing it as a serial interconnection of two-vertex antiregular graphs, with or without appending a terminal one-vertex path. We show that its SILC property is preserved if we increase the number of vertices of the antiregular graphs and/or that of the terminal path. Examples are provided to illustrate the novel class of SILC graphs we propose.**

***Index Terms*—Antiregular graph, Laplacian controllability, multiagent system, path graph.**

I. Introduction

MODERN control, communication, and networking technologies admit their integrations to innovative systems composed of the units such as robots or drones [1]. However, the common property that each unit communicates only with other reachable units causes system-wide problems, such as the synchronization, coordination, and formation [2], [3]. Many of the issues have been partially addressed under the formulation of multiagent systems following the leader–follower dynamics. A more fundamental topic closely related to the effective operation of the networked system is on its controllability. This topic has been a main theme of many studies in system and control sciences since Tanner's conference paper was published more than a decade ago [4]. This pioneering paper investigated the interplay between control and communication in a networked system defined on a graph and formulated a nontrivial new problem that inspired many research works in these areas. In the general setting, a linear and time-invariant system observing the consensus policy is proposed to simulate the information propagation across a networked system. Each state variable in the system changes with the difference between its value and that of other variables interacting with it. All state variables become steady as these variables reach a common value, or the consensus. In this scenario, an autonomous system evolved according to the Laplacian dynamics is induced [5]. The controllability issues are raised when input signals are used to maneuver the linear system. Testing the controllability of the system is a classical problem. Several solutions, including Kalman's rank test, are available [6]. Nevertheless, a major challenge to the controllability issue in a networked system is the numerical imprecision caused by the large scale of the system. A standard approach is to adopt the controllability test proposed by Popov, Belevitch, and Hautus. This test sheds some light on the relation between Laplacian eigenspaces and Laplacian controllability of the graph modeling the networked system. Since specific Laplacian eigenspaces can be inferred from the connection topology of the graph, partitioning schemes such as the equitable partition and the almost equitable partition [7] were proposed to identify a certain symmetry property that leads to Laplacian uncontrollability [8], [9]. These schemes helped to find a class of uncontrollable graphs, but failed to recognize a controllable one. This failure is partially compensated by a combined use of the distance partition and almost equitable partition schemes to bound the dimension of controllable subspace of state variables [10]. This expedient is simple and feasible for a large-scale system, but it offers a tight bound only in the extreme case such as a path or called a line graph. In fact, how to use the minimum number of controllers to drive a general networked system to achieve an arbitrary state is a challenging task. The solved cases are restricted to the networks following specific connection structures, including the paths [11], multichain [12], antiregular graphs [13], grids [14], circulant graphs [15], and complete graphs [16]. In these examples, Laplacian spectra and eigenspaces of the graphs are fully available, and thus, their controllability analyses are tractable.

In the previous study, we proposed a sufficient condition for a graph to be controllable by one input [17]. The breakthrough of the result is showing that a partial knowledge of the Laplacian spectrum and eigenspaces might be enough to identify a Laplacian controllable graph. The conclusion applies to a special graph family, in which each graph is constructed by interconnecting a path and an antiregular graph. Recent progress shows

Manuscript received May 14, 2019; revised August 27, 2019; accepted October 3, 2019. Date of publication October 15, 2019; date of current version June 12, 2020. This work was supported in part by the Ministry of Science and Technology in Taiwan under Grant MOST-106-2221-E-005-019. Recommended by Associate Editor A. E. Motter.

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Digital Object Identifier 10.1109/TCNS.2019.2947589

that the conclusion is applicable to the interconnection of two antiregular graphs as well [18]. This article follows the line of our previous study and investigates a richer class of graphs constructed by interconnecting a finite number of single-input Laplacian controllable (SILC) graphs. We first consider two connected simple graphs, called the structure graph and the cell graph, where the cell graph is Laplacian controllable by an input connected to some vertex, called the composite vertex, of the graph. We then interconnect all cell graphs to form a composite graph through the composite vertices such that composite vertices alone form the structure graph. We show that the structure graph is Laplacian controllable by an input connected to some vertex of the graph, if and only if the composite graph is Laplacian controllable by that input connected to that (composite) vertex. Furthermore, we extend our result by showing that using different vertices to interconnect the cell graphs might also lead to the SILC property of the resulting graph.

The contributions of our results are to propose an innovative and simple method to generate a huge class of SILC graphs, whose controllability properties cannot be inferred from existing methods. Moreover, our results provide potential solutions for the design problem that asks to construct a k-vertex Laplacian controllable graph whose edge-related parameters, such as the diameter and maximum degree, are subject to constraints. Specifically, our first result shows that the graph can be generated from the composite graph of a k_1-vertex graph and a k_2-vertex graph, where $k_1 k_2 = k$ as long as each of these two graphs are Laplacian controllable by an input connected to some vertex of the graph. Our second result is based on a novel point of view to explain the connection structure of a path. Observe that two-vertex and three-vertex antiregular graphs are both line graphs. Thus, the paths with $2k$ and $3k$ vertices can be regarded as the graphs constructed by serial connections of k two-vertex and k three-vertex antiregular graphs, respectively. The well-known result that a path is Laplacian controllability by an input connected to its terminal vertex was then generalized to the case in which each antiregular graph has two or more vertices [19]. In this article, we extend this result by providing a more general perspective on the vertices selected for the serial connection and identify a wide class of SILC graphs that are essentially related to a pure path. Additionally, we show that if a graph is Laplacian controllable by an input and its Laplacian eigenvectors satisfy a certain nonzero property, then its controllability can be maintained if it is appropriately connected to a path. These results make it possible to generate a k-vertex SILC graph by interconnecting one k_p-vertex path and c antiregular graphs, each with k_a vertices, such that $ck_a + k_p = k$. (In [17], c is restricted to 1, and in [19], k_p is restricted to 0 or 1. In particular, the interconnection schemes adopted in this article are different from the previous ones.). As k is large, our results become very helpful in constructing a large and feasible class of graphs for the network designers to determine the connection topology that meets the edge or performance requirements.

The rest of this article is organized as follows. In Section II, we recapitulate the essential graph-theoretical notations and concepts as well as related control theories. In Section III, we present our main results on the schemes to interconnect graphs while preserving the Lapalacian controllability. This article is concluded in Section IV, where possible extensions of our results are discussed.

II. Preliminaries

We start this section by defining and reviewing notations and concepts to be used throughout this article. The presentation is standard, and the materials are available in popular textbooks and in the literature dealing with similar topics. Let $\boldsymbol{e}_i$ be the zero vector except its ith entry being 1. The floor and ceiling functions of x, written as $\lfloor x \rfloor$ and $\lceil x \rceil$, are the largest integer not greater than x and the smallest integer not less than x, respectively. The identity matrix of order k is I_k. Occasionally, we drop the subscript k for simplicity as the context is clear. The difference of sets S_1 and S_2 is $S_1 \backslash S_2$, meaning that $\{s | s \in S_1, s \notin S_2\}$. We say $(\lambda, \boldsymbol{v})$ is an eigenpair of a matrix P if λ is an eigenvalue of P, and $\boldsymbol{v}$ is an eigenvector corresponding to λ. If A_1 and A_2 are square matrices of the same order, A_1 is called a modal matrix of A_2 if the columns of A_1 are independent eigenvectors of A_2. The Kronecker product of matrices P_1 and P_2 is written as $P_1 \otimes P_2$. If $V = \{1, 2, \ldots, k\}$ and E is a subset of $\{ (v_1, v_2) \,|\, v_1, v_2 \in V\}$, then (V, E) describes a k-vertex graph, or a graph on k vertices, where V and E are called the vertex set and edge set, respectively, of the graph. In this article, we consider only the connected simple graphs to avoid trivial cases (e.g., a disconnected graph must be uncontrollable by one input) and to simplify the interconnection pattern. The algebraic aspects of such graphs can be seen, for example, in [20] and [21]. In the context of a simple graph, any element in E can be written as an unordered pair. We say v_1 and v_2 are neighbors if $v_1, v_2 \in V$ and $\{v_1, v_2\} \in E$. The neighbor set of the vertex v is $\mathcal{N}_v := \{u \,|\, \{v, u\} \in E\}$. The degree, or valency, of vertex v is defined as $|\mathcal{N}_v|$, the number of elements in $\mathcal{N}_v$. In a connected simple graph defined by (V, E), at least two vertices share the same degree. The vertex v is called a terminal vertex if $|\mathcal{N}_v| = 1$, it is called a dominating vertex if $|\mathcal{N}_v| = |V| - 1$, and it is called a degree-repeating vertex if its degree is not unique among those of vertices in the graph. The degree sequence $\boldsymbol{d}$ of a k-vertex graph is a nonincreasing sequence $(\, d_1, d_2, \ldots, d_k \,)$, where $d_i = |\mathcal{N}_i|$, the degree of vertex i. The trace $\tau_{\boldsymbol{d}}$ of a degree sequence $\boldsymbol{d}$ is defined as $\tau_{\boldsymbol{d}} := |\, j : d_j \geq j \,|$. The conjugate $\boldsymbol{d}^*$ of $\boldsymbol{d}$ is defined as $(\, d_1^*, d_2^*, \ldots, d_k^* \,)$, where d_i^* is $|\, j : d_j \geq i \,|$, or the number of vertices whose degrees are at least i. For a non-increasing sequence of k nonnegative integers $\boldsymbol{d} = (\, d_1, d_2, \ldots, d_k \,)$, there exists a graph whose degree sequence is exactly $\boldsymbol{d}$ if and only if

$$\sum_{i=1}^{j} (d_i + 1) \leq \sum_{i=1}^{j} d_i^* \quad \forall j \in \{1, 2, \ldots, \tau_{\boldsymbol{d}}\}. \tag{1}$$

If a sequence $\boldsymbol{d}$ satisfies (1), it is said to be graphical [22, p. 72].

Suppose (V, E) determines a connected simple graph, which has the degree sequence $\boldsymbol{d} = (\, d_1, d_2, \ldots, d_k \,)$. The adjacency matrix $\mathcal{A}$ of the graph is the binary matrix, whose (i, j)th element is 1 if $\{i, j\} \in E$ and is 0 otherwise. Let $\mathcal{D}$ be a diagonal matrix, whose ith diagonal term is d_i. The Laplacian matrix $\mathcal{L}$

of the graph is defined as

$$\mathcal{L} := \mathcal{D} - \mathcal{A}.$$

The eigenvalues and eigenvectors of $\mathcal{L}$ are called the Laplacian eigenvalues and Laplacian eigenvectors, respectively, of the graph. It is easy to see that $\mathcal{L}$ is symmetric and has an apparent eigenpair $(0, \mathbf{1})$, where $\mathbf{1}$ is a vector of 1s. Many essential properties concerning the eigenvalues and eigenvectors of $\mathcal{L}$ were presented in [23]. By the Grone–Merris theorem, the spectrum of $\mathcal{L}$ is majorized by the conjugate $\boldsymbol{d}^*$ of the degree sequence of the graph, namely

$$\sum_{i=1}^{t} \ell_{k-i+1} \leq \sum_{i=1}^{t} d_i^* \quad \forall t \in \{1, 2, \dots, k\} \tag{2}$$

where ℓ_i is the ith smallest Laplacian eigenvalue of the graph [24]. In the special case that the equality in (1) holds, the graph defined by $\boldsymbol{d}$ is unique and is called *threshold graph* or *maximal graph*. It can be proved that the equality in (1) is a sufficient condition for the equality in (2) [25]. Thus, we can derive Laplacian eigenvalues of a threshold graph from its degree sequence in a straightforward manner. There exist different definitions for threshold graphs, e.g., the definition based on the Ferrers–Sylvester diagram [22, p. 70], and the one on constructing the graph using join and union operations only [26]. These equivalent definitions shed some light on the simple relation between their Laplacian matrices and eigenpairs. An antiregular graph is a connected simple graph that has exactly one pair of vertices sharing the same degree [27]. An antiregular graph turns out to be a special threshold graph and thus enjoys many excellent Laplacian eigenpair properties. Specifically, let $\mathbb{G}_A^{(k)}$ denote a k-vertex antiregular graph and $\mathcal{L}_A^{(k)}$ be its Laplacian matrix written as

$$\begin{bmatrix} (k-1) & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ -1 & (k-2) & \cdots & -1 & -1 & \cdots & -1 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & ⋰ & & \\ -1 & -1 & \cdots & \lfloor k/2 \rfloor & -\delta_k & & & \\ -1 & -1 & \cdots & -\delta_k & \lfloor k/2 \rfloor & & & \\ \vdots & \vdots & ⋰ & & & \ddots & & \\ -1 & -1 & & & & & 2 & \\ -1 & & & & & & & 1 \end{bmatrix}.$$

Here, δ_k is 1 if k is even and is 0 otherwise. Since an antiregular graph is also a threshold graph, its Laplacian spectrum can be readily derived from its degree conjugate. That is, any element in $\{0, 1, \dots, k\} \setminus \{\lceil k/2 \rceil\}$ is a distinct eigenvalue of $\mathcal{L}_A^{(k)}$. A spanning and orthogonal set of eigenvectors of $\mathcal{L}_A^{(k)}$ can be obtained using the following lemma.

Lemma 2.1 (see [13], [18], and [28]): Let the (i, j)th entry of matrices $T^{(m)}$ be $t_{ij}^{(m)}$ for each $m \in \{1, 2, 3, 4\}$. Suppose $T^{(1)} = \mathcal{L}_A^{(k)}$ and let $T^{(2)}, T^{(3)}$ be generated by

$$t_{ij}^{(2)} = \begin{cases} -1 - t_{ij}^{(1)}, & \text{if } j > i \\ t_{ij}^{(1)}, & \text{otherwise} \end{cases} \tag{3}$$

and

$$t_{ij}^{(3)} = \begin{cases} -\sum_{k, k \neq j} t_{kj}^{(2)}, & \text{if } j = i \\ t_{ij}^{(2)}, & \text{otherwise.} \end{cases} \tag{4}$$

Finally, remove the (unique) zero column of $T^{(3)}$ and append the column of 1s (or -1 s) to the last column to yield $T^{(4)}$. Then, the jth column of $T^{(4)}$ is the eigenvector corresponding to the jth largest eigenvalue of $\mathcal{L}_A^{(k)}$, or the jth entry of the conjugate of the degree sequence of $\mathbb{G}_A^{(k)}$.

Example 2.1: For a six-vertex antiregular graph, the degree sequence is $\boldsymbol{d} = (5, 4, 3, 3, 2, 1)$, and thus, the Laplacian eigenvalues are 6, 5, 4, 2, 1, and 0. By Lemma 2.1, we have

$$T^{(1)} = \mathcal{L}_A^{(6)} = \begin{bmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 & \\ -1 & -1 & 3 & -1 & & \\ -1 & -1 & -1 & 3 & & \\ -1 & -1 & & & 2 & \\ -1 & & & & & 1 \end{bmatrix}$$

$$T^{(2)} = \begin{bmatrix} 5 & & & & & \\ -1 & 4 & & & & -1 \\ -1 & -1 & 3 & & -1 & -1 \\ -1 & -1 & -1 & 3 & -1 & -1 \\ -1 & -1 & & & 2 & -1 \\ -1 & & & & & 1 \end{bmatrix}$$

$$T^{(3)} = \begin{bmatrix} 5 & & & & & \\ -1 & 3 & & & & -1 \\ -1 & -1 & 1 & & -1 & -1 \\ -1 & -1 & -1 & & -1 & -1 \\ -1 & -1 & & & 2 & -1 \\ -1 & & & & & 4 \end{bmatrix}$$

and

$$T^{(4)} = \begin{bmatrix} 5 & & & & & 1 \\ -1 & 3 & & & -1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & & 2 & -1 & 1 \\ -1 & & & & 4 & 1 \end{bmatrix}.$$

The ith column of $T^{(4)}$ is the corresponding Laplacian eigenvector to the ith largest Laplacian eigenvalue of the six-vertex antiregular graph, $i \in \{1, 2, \dots, 6\}$.

Consider a linear time-invariant system, in which each state variable interacts with some other state variables, and in the long run, all variables reach the zero-input consensus. A simple model for the system evolution is the Laplacian dynamics [5, p. 1613]

$$\dot{x}_i = -\sum_{j\in\mathcal{N}_i}(x_i - x_j) \tag{5}$$

where $\mathcal{N}_i$ is the index set of state variables with which x_i interacts. In a matrix form, we have

$$\dot{\boldsymbol{x}} = -\mathcal{L}\boldsymbol{x} \tag{6}$$

where $\mathcal{L}$ is the Laplacian matrix of the graph describing the interactions of state variables of the system. This model fits into the operation scenario of a dynamic networked system, in which each vertex state interacts with its neighboring vertex states and expects to reach a common state as time goes by. The autonomous system in (6) can be controlled by input signals to achieve some particular goal. In the single-input case, the controlled dynamics can be written as

$$\dot{\boldsymbol{x}} = -\mathcal{L}\boldsymbol{x} + \boldsymbol{b}u(t) \tag{7}$$

where $\boldsymbol{b}$ is called the control coefficient vector. The ith element of $\boldsymbol{b}$ is 1 if vertex i is connected to the input directly; otherwise, it is 0. For simplicity, the notation $(\mathcal{L}, \boldsymbol{b})$ is used to represent the controlled graph model of the dynamic system in (7). A graph is called SILC if its corresponding $(\mathcal{L}, \boldsymbol{b})$ is controllable. The focus of this article is on the controllability of $(\mathcal{L}, \boldsymbol{b})$, where $\mathcal{L}$ is the Laplacian matrix of a graph constructed by interconnecting a finite sequence of SILC graphs. Apparently, how the sequence of graphs is interconnected and how the input is applied affect the Laplacian controllability of the resulting graph. We first consider a scheme to form a composite graph of two graphs and present a necessary and sufficient condition for the composite graph to be SILC. Following this result, we show for a special case that this scheme, which relies on specific vertices for interconnection, can be modified to allow more flexibility while preserving the SILC property. Before presenting our main results, we mention a version of the classical Popov–Belevitch–Hautus test on which our studies are based.

Theorem 2.2 (see [6, p. 145]): A graph is Laplacian controllable by an input represented by the control coefficient vector $\boldsymbol{b}$ if and only if $\boldsymbol{b}$ is not orthogonal to any Laplacian eigenvector of the graph.

III. Main Results

A. Composite Graph

Suppose $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ are the Laplacian matrices of connected simple graphs $\mathbb{G}_1$ and $\mathbb{G}_2$, with k_1 and k_2 vertices, respectively. If we interconnect k_1 copies of $\mathbb{G}_2$ via the sth vertex of each $\mathbb{G}_2$ such that these interconnecting vertices alone form $\mathbb{G}_1$, the resulting graph, written as $\mathcal{G}_1(\mathbb{G}_2, s)$, is called the composite graph of $\mathbb{G}_1$ by $\mathbb{G}_2$ via vertex s. Graphs $\mathbb{G}_1$ and $\mathbb{G}_2$ are called the structure graph and the cell graph, respectively, of $\mathcal{G}_1(\mathbb{G}_2, s)$, and the vertex s in $\mathbb{G}_2$ the composite vertex. Let the Laplacian matrix of $\mathcal{G}_1(\mathbb{G}_2, s)$ be $\mathcal{L}_s^{(1,2)}$. We, thus, have

$$\mathcal{L}_s^{(1,2)} := I_{k_1} \otimes \mathcal{L}^{(2)} + \mathcal{L}^{(1)} \otimes \boldsymbol{e}_s\boldsymbol{e}_s^T. \tag{8}$$

Example 3.1: Consider the example that $\mathbb{G}_1$ and $\mathbb{G}_2$ are the five-vertex and four-vertex antiregular graphs, respectively. The corresponding Laplacian matrices can be written as

$$\mathcal{L}^{(1)} = \mathcal{L}_A^{(5)} = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & \\ -1 & -1 & 2 & & \\ -1 & -1 & & 2 & \\ -1 & & & & 1 \end{bmatrix}$$

$$\mathcal{L}^{(2)} = \mathcal{L}_A^{(4)} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & \\ -1 & -1 & 2 & \\ -1 & & & 1 \end{bmatrix}.$$

Suppose we choose one of the two vertices with the same degree 2 in the cell graph $\mathbb{G}_2$ as the composite vertex. Thus, the composite graph $\mathcal{G}_1(\mathbb{G}_2, s)$ has the Laplacian matrix

$$\begin{aligned}\mathcal{L}_2^{(1,2)} :&= I_5 \otimes \mathcal{L}^{(2)} + \mathcal{L}^{(1)} \otimes \boldsymbol{e}_2\boldsymbol{e}_2^T \\ &= \begin{bmatrix} \mathcal{L}^{(2)} & & & & \\ & \mathcal{L}^{(2)} & & & \\ & & \mathcal{L}^{(2)} & & \\ & & & \mathcal{L}^{(2)} & \\ & & & & \mathcal{L}^{(2)} \end{bmatrix} \\ &+ \begin{bmatrix} 4\mathscr{I} & -\mathscr{I} & -\mathscr{I} & -\mathscr{I} & -\mathscr{I} \\ -\mathscr{I} & 3\mathscr{I} & -\mathscr{I} & -\mathscr{I} & \\ -\mathscr{I} & -\mathscr{I} & 2\mathscr{I} & & \\ -\mathscr{I} & -\mathscr{I} & & 2\mathscr{I} & \\ -\mathscr{I} & & & & \mathscr{I} \end{bmatrix}\end{aligned}$$

where

$$\mathscr{I} := \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}.$$

In the following, we present several matrix properties that are closely related to the Laplacian eigenspace of the composite graph.

Lemma 3.1: If A is a diagonalizable matrix of order k and the ith entry of every eigenvector of A is nonzero for some $i \in \{1, 2, \ldots, k\}$, then all but the ith column of $A - \alpha I$ are independent for any α.

Proof: If for some $i \in \{1, 2, \ldots, k\}$, the ith entry of every eigenvector of A is nonzero, then every eigenvalue of A is not repeated (cf. [8, p. 169]). In case α is not an eigenvalue of A, $A - \alpha I$ is nonsingular, and thus, all columns of $A - \alpha I$ are independent. If α is an eigenvalue of A, the nonzero entry in the eigenvector implies that the ith column of $A - \alpha I$ is a linear combination of the remaining columns. If all but the ith column of $A - \alpha I$ are dependent, then the rank of $A - \alpha I$ is at most $k - 2$, and α is an eigenvalue with its algebraic multiplicity at least 2, a contradiction. ∎

Lemma 3.2: Let λ be an eigenvalue of $\mathcal{L}_{s_2}^{(1,2)}$. If the s_ith entry of every eigenvector of $\mathcal{L}^{(i)}$ is nonzero for every i in $\{1,2\}$, the matrix obtained by removing the s_2th row and s_2th column of $\mathcal{L}^{(2)} - \lambda I$ is nonsingular.

Proof: Without loss of generality, we let $s_2 = 1$ and write

$$\mathcal{L}^{(2)} = \begin{bmatrix} \ell_{11} & \boldsymbol{l}_{12} \\ \boldsymbol{l}_{21} & L_{22} \end{bmatrix} \tag{9}$$

where L_{22} is a principal submatrix of $\mathcal{L}^{(2)}$ of order $k_2 - 1$ and $\boldsymbol{l}_{12}^T = \boldsymbol{l}_{21}$. Our goal is to show that $L_{22} - \lambda I$ is nonsingular. Note that the nonzeroness of the first entry of every eigenvector of $\mathcal{L}^{(2)}$ implies the nonorthogonality of $\boldsymbol{l}_{21}$ to the eigenvectors of L_{22} [5], meaning that $\mathcal{L}^{(2)}$ and L_{22} have different eigenvalues [29]. In case the eigenvalue λ of $\mathcal{L}_1^{(1,2)}$ is also an eigenvalue of $\mathcal{L}^{(2)}$, then λ is not an eigenvalue of L_{22}, and thus, $L_{22} - \lambda I$ is nonsingular. If λ is not an eigenvalue of $\mathcal{L}^{(2)}$, $\mathcal{L}^{(2)} - \lambda I$ is nonsingular, meaning that all rows of $\mathcal{L}^{(2)} - \lambda I$ are independent. If we replace its first row with $\boldsymbol{e}_1^T = [1\,0\,0 \cdots 0]$ to yield

$$\hat{\mathcal{L}}^{(2)}(\lambda) := \begin{bmatrix} 1 & \\ \boldsymbol{l}_{21} & L_{22} - \lambda I \end{bmatrix} \tag{10}$$

then $\hat{\mathcal{L}}^{(2)}(\lambda)$ is nonsingular. This can be verified with the following argument: if $\hat{\mathcal{L}}^{(2)}(\lambda)$ is singular, its first row must be a linear combination of other rows. Note that

$$\mathcal{L}_1^{(1,2)} - \lambda I = I_{k_1} \otimes (\mathcal{L}^{(2)} - \lambda I) + \mathcal{L}^{(1)} \otimes \boldsymbol{e}_1 \boldsymbol{e}_1^T.$$

We can thus perform Gaussian elimination on $\mathcal{L}_1^{(1,2)} - \lambda I$ to eliminate $\mathcal{L}^{(1)} \otimes \boldsymbol{e}_1 \boldsymbol{e}_1^T$ and obtain that the null space of $\mathcal{L}_1^{(1,2)} - \lambda I$ is that of $I_{k_1} \otimes (\mathcal{L}^{(2)} - \lambda I)$, or equivalently, $\mathcal{L}_1^{(1,2)} - \lambda I$ and $I_{k_1} \otimes (\mathcal{L}^{(2)} - \lambda I)$ have the same rank, which is $k_1 \times k_2$, a contradiction, since λ is an eigenvalue of $\mathcal{L}_{s_2}^{(1,2)}$. Noting that $|\hat{\mathcal{L}}^{(2)}(\lambda)| = |L_{22} - \lambda I|$, we conclude that $L_{22} - \lambda I$ is nonsingular. ■

Theorem 3.3: Suppose for every eigenvector of $\mathcal{L}^{(1)}$, there exists a fixed position in which the entry of the eigenvector is always nonzero. In addition, the s_2th entry of every eigenvector of $\mathcal{L}^{(2)}$ is nonzero. Then, all eigenvalues of $\mathcal{L}_{s_2}^{(1,2)}$ are distinct. In particular, if a vertex in the structure graph corresponds to a fixed position, in which the entry of every eigenvector of $\mathcal{L}^{(1)}$ is nonzero, the vertex corresponds to a fixed position, in which the entry of every eigenvector of $\mathcal{L}_{s_2}^{(1,2)}$ is nonzero.

Proof: Let λ be the eigenvalue of $\mathcal{L}_{s_2}^{(1,2)}$ and assume $s_2 = 1$ to simplify the presentation. Instead of considering $\boldsymbol{v} := [v_1 \ \ldots, v_{k_1 k_2}]$ in the null space of $\mathcal{L}_{s_2}^{(1,2)} - \lambda I$, we permutate the entries of $\mathcal{L}_{s_2}^{(1,2)}$ and $\boldsymbol{v}$ to study $\tilde{\mathcal{L}}_{\tilde{s}_2}^{(1,2)}$ and $\tilde{\boldsymbol{v}} := [\,\bar{\boldsymbol{v}}\,\hat{\boldsymbol{v}}\,]$, where

$$\begin{aligned} \bar{\boldsymbol{v}} &:= \left[\, v_1\, v_{k_2+1} \ \cdots \ v_{(k_1-1)k_2+1} \,\right] \\ \hat{\boldsymbol{v}} &:= \left[\, v_2\, v_3 \ \cdots \ v_{k_2}\, v_{k_2+2}\, v_{k_2+3} \ \cdots \ v_{2k_2}\, v_{2k_2+2} \cdots v_{k_1 k_2} \,\right]. \end{aligned} \tag{11}$$

The nonsingularity of $L_{22} - \lambda I$ by Lemma 3.2 allows Gaussian elimination on $\tilde{\mathcal{L}}_{\tilde{s}_2}^{(1,2)} - \lambda I$ such that $\boldsymbol{v}$ is in the null space of $\mathcal{L}_{s_2}^{(1,2)} - \lambda I$ if and only if $\tilde{\boldsymbol{v}}$ is in the null space of the matrix

$$\tilde{\mathscr{L}}(\lambda) := \begin{bmatrix} \mathscr{L}_{11}(\lambda) & \\ \mathscr{L}_{21} & \mathscr{L}_{22}(\lambda) \end{bmatrix} \tag{12}$$

where

$$\begin{aligned} \mathscr{L}_{22}(\lambda) :&= I_{k_1} \otimes (L_{22} - \lambda I) \\ \mathscr{L}_{21} :&= I_{k_1} \otimes \boldsymbol{l}_{21} \\ \mathscr{L}_{11}(\lambda) :&= \mathcal{L}^{(1)} - \left(\lambda + \boldsymbol{l}_{21}^T (L_{22} - \lambda I)^{-1} \boldsymbol{l}_{21} - \tilde{d}\right) I \end{aligned} \tag{13}$$

and $\tilde{d}$ is the degree of the composite vertex in $\mathbb{G}_2$. This implies that $\bar{\boldsymbol{v}}$ is in the null space of $\mathscr{L}_{11}(\lambda)$. By Lemma 3.2, $L_{22} - \lambda I$ is nonsingular; thus, $\mathscr{L}_{22}(\lambda)$ has its rank $k_1(k_2 - 1)$. Note that $\tilde{\mathscr{L}}(\lambda)$ is blockwise lower triangular. If λ is repeated, the rank of $\tilde{\mathscr{L}}(\lambda)$ is at most $k_1 k_2 - 2$, and thus, the rank of $\mathscr{L}_{11}(\lambda)$ is at most $k_1 k_2 - 2 - k_1(k_2 - 1) = k_1 - 2$. This is contradictory to Lemma 3.1 that the rank of $\mathscr{L}_{11}(\lambda)$ is at least $k_1 - 1$. We conclude that $\mathcal{L}_{s_2}^{(1,2)}$ does not have a repeated eigenvalue. Finally, observe that all columns of $\mathscr{L}_{22}(\lambda)$ are independent. Suppose the ith entry of every eigenvector of $\mathcal{L}^{(1)}$ is nonzero. If the ith entry of the eigenvector of $\mathscr{L}_{11}(\lambda)$ corresponding to some λ is zero, then by Lemma 3.1, the null space of $\tilde{\mathscr{L}}(\lambda)$ is $\mathbf{0}$, a contradiction. Thus, for all eigenvectors of $\mathcal{L}_{s_2}^{(1,2)}$, the p entries corresponding to their composite vertices of $\mathcal{G}_1(\mathbb{G}_2, s_2)$ cannot be zero. ■

Theorem 3.4: Under the assumptions made in Theorem 3.3 for $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$, the structure graph $\mathbb{G}_1$ is Laplacian controllable by an input connected to one of its vertices if and only if the composite graph $\mathcal{G}_1(\mathbb{G}_2, s_2)$ is Laplacian controllable by the input connected to that vertex.

Proof: The set of eigenvalues of $\mathcal{L}_{s_2}^{(1,2)}$ can be derived from the set of eigenvalues of $\mathcal{L}^{(2)} + \lambda_i \boldsymbol{e}_{s_2} \boldsymbol{e}_{s_2}^T$, where λ_i is the ith smallest eigenvalue of $\mathcal{L}^{(1)}$, $i \in \{1, 2, \ldots, k_1\}$. Suppose $(\lambda_i, \boldsymbol{v}^{(i)})$ is an eigenpair of $\mathcal{L}^{(1)}$, and $U^{(i)}$ is an orthogonal modal matrix that diagonalizes $\mathcal{L}^{(2)} + \lambda_i \boldsymbol{e}_{s_2} \boldsymbol{e}_{s_2}^T$. Then, an orthogonal set of eigenvectors of $\mathcal{L}_{s_2}^{(1,2)}$ is

$$\left\{ \boldsymbol{v}^{(1)} \otimes U^{(1)}, \boldsymbol{v}^{(2)} \otimes U^{(2)}, \ldots, \boldsymbol{v}^{(k_1)} \otimes U^{(k_1)} \right\}. \tag{14}$$

In light of Theorem 3.3, it remains to show the case that $\mathbb{G}_1$ is not Laplacian controllable by the input connected to that particular vertex. In this case, the (composite) vertex of $\mathbb{G}_1$ induces a zero entry in some $\boldsymbol{v}^{(i)}$, where $i \in \{1, 2, \ldots, k_1\}$. Thus, the same vertex induces a zero entry in some eigenvector in (14), which leads to the Laplacian uncontrollability of the composite graph by the input. ■

Example 3.2: In Fig. 1, we establish the composite graph in (c) using the five-vertex antiregular graph shown in (a) for the cell graph, and the seven-vertex antiregular graph in (b) for the structure graph. By Theorem 3.4, the composite graph is Laplacian controllable by the specified input u.

Example 3.3: In Fig. 2, we illustrate the first two steps of a recursive application of our scheme to generate a complicated graph from a simple one, while preserving the single-input Laplacian controllability. Specifically, we start from the four-vertex antiregular graph in (a). Using this graph as the cell graph and the structure graph, we obtain the 16-vertex composite graph in (b). Using the four-vertex graph again as the structure graph but the 16-vertex graph as the cell graph, we obtain the 64-vertex composite graph. By Theorem 3.4, we can easily identify each graph's vertex to connect the input signal that renders the graph SILC.

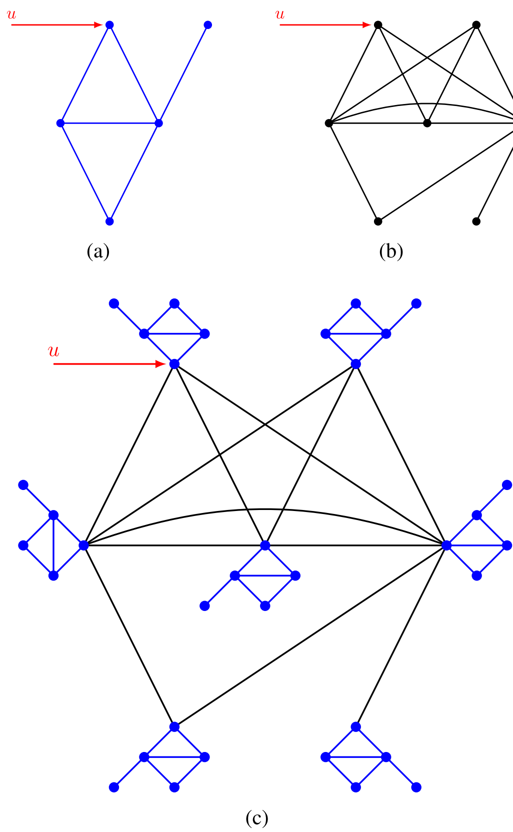

Fig. 1. Example of a composite graph. (a) Five-vertex antiregular graph. (b) Seven-vertex antiregular graph. Both graphs are SILC by the input u connected to the specified vertices. (c) Composite graph whose cell graph and structure graph are (a) and (b), respectively. By Theorem 3.4, the composite graph is also Laplacian controllable by the specified input u. See Example 3.2 as well.

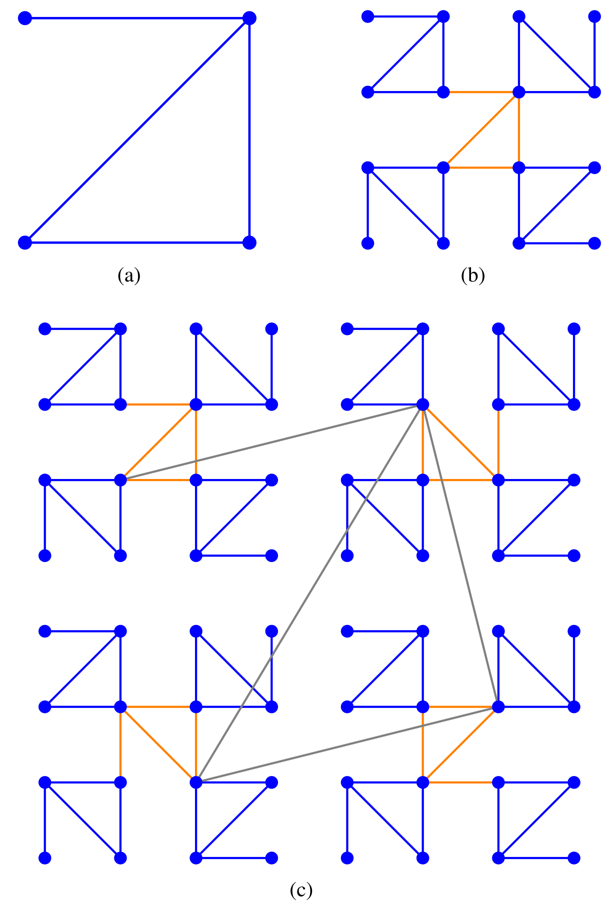

Fig. 2. Constructing a composite graph using another composite graph. (a) Four-vertex antiregular graph. (b) Composite graph whose cell graph and structure graph are (a). (c) Composite graph whose cell graph is (b) and structure graph is (a). See Example 3.3 also.

Let j be a positive integer and the set $\mathcal{C}_j$ be defined as

$$\mathcal{C}_j := \{j, j + (2j+1), j + 2(2j+1), j + 3(2j+1), \dots\}. \tag{15}$$

A special case of Theorem 3.3 follows.

Corollary 3.5: Suppose $\mathbb{G}_1$ in Theorem 3.3 is a path with k_1 vertices, and an input is connected to one of these vertices. The input divides the vertices of the path into two parts: one with k_{11} vertices and the other with k_{12}, where $k_{11} + k_{12} = k_1 - 1$. If $\mathbb{G}_2$ and s_2 satisfy the conditions in Theorem 3.3, then $\mathcal{G}_1(\mathbb{G}_2, s_2)$ is SILC if and only if k_{11} and k_{12} do not appear in the same $\mathcal{C}_j$ in (15), $j \in \{1, 2, \dots\}$.

Proof: It can be proved that $\mathbb{G}_1$ has a Laplacian eigenvector, whose entry corresponding to the vertex connected to the input is zero if and only if k_{11} and k_{12} are both in $\mathcal{C}_j$ where j is some positive integer [12]. This can also be seen from [30, Th. 5] that for a set of orthogonal Laplacian eigenvectors $\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}, \dots, \boldsymbol{v}^{(k_1)}$ of $\mathbb{G}_1$, $v_j^{(i)}$, the jth entry of $\boldsymbol{v}^{(i)}$, can be written as

$$v_j^{(i)} = \cos\frac{(i-1)(2j-1)\pi}{2k_1} \tag{16}$$

where $i, j \in \{1, 2, \dots, k_1\}$. If k_1 and k_2 are in $\mathcal{C}_{j^*}$, we can write $k_{11} = j^* + n_1(2j^*+1)$, $k_{12} = j^* + n_2(2j^*+1)$, where n_1 and n_2 are some non-negative integers, and thus, $k_1 = (n_1 + n_2 + 1)(2j^* + 1)$. The entry corresponding to the vertex connected to the input can be written as

$$v_{k_{11}+1}^{(i)} = \cos\left\{(i-1)\frac{2n_1+1}{n_1+n_2+1}\frac{\pi}{2}\right\} \tag{17}$$

for $i \in \{1, 2, \dots, (n_1+n_2+1)(2j^*+1)\}$. Thus, we have

$$v_{k_{11}+1}^{(n_1+n_2+2)} = \cos\left\{(2n_1+1)\frac{\pi}{2}\right\} = 0. \tag{18}$$

For the same corresponding vertex, Theorem 3.3 shows that the nonzero property of some entries in Laplacian eigenvectors of $\mathbb{G}_1$ is preserved in those of $\mathcal{G}_1(\mathbb{G}_2, s_2)$. It can also be seen that if some vertex induces a zero entry in some Laplacian eigenvector of $\mathbb{G}_1$, it will also induce a zero entry in some Laplacian eigenvector of $\mathcal{G}_1(\mathbb{G}_2, s_2)$. Thus, the proof is complete. ∎

In this subsection, we have proposed a condition for a class of SILC graph to maintain its single-input controllability after interconnecting a finite number of its identical copies. This condition requires the interconnection via the composite vertices. In the following, we show that this requirement can be relaxed.

B. Interconnecting Antiregular Graphs

A $2k$-vertex path can be viewed as the resulting graph after interconnecting k antiregular graphs, each with two vertices. Suppose the two vertices of each antiregualr graph are interpreted as the terminal vertex and the degree-repeating vertex, respectively, and the interconnection starts from one antiregular graph. To add in the sth antiregular graph, the terminal vertex of the $(s-1)$th antiregular graph is connected to the degree-repeating vertex of the sth antiregular graph, $s \in \{2, 3, \ldots, k\}$. It was shown that if the number of vertices for each antiregular graphs is increased, the controllability property remains [19]. In the following, we show that for the two-vertex antiregular graph, the vertex interpreted as the terminal vertex can also be interpreted as the dominating vertex. Thus, the existing result can be further extended.

Let S be a real and symmetric matrix of order k_2 and

$$\mathscr{S} := I_{k_1} \otimes S + \sum_{i=1}^{k_1-1} \boldsymbol{z}_i \boldsymbol{z}_i^T \tag{19}$$

where $\boldsymbol{z}_i := c_1 \boldsymbol{e}_{(i-1)k_2+p} + c_2 \boldsymbol{e}_{ik_2+1}$, $p \in \{1, 2, \ldots, k_2\}$, and c_1, c_2 nonzero.

Lemma 3.6: If in (19), the first entry of every eigenvector of S is nonzero, then every eigenvalue λ of S is a distinct eigenvalue of $\mathscr{S}$, and the first entry of the eigenvector $\boldsymbol{v}$ corresponding to λ is nonzero. If, in addition, the pth entry of every eigenvector of S is also nonzero, then so are the $((i-1)k_2+1)$th entries, for every $i \in \{2, 3, \ldots, k_1\}$, of the corresponding eigenvectors of $\mathscr{S}$.

Proof: Write

$$S := \begin{bmatrix} s_{11} & \boldsymbol{s}_{12} \\ \boldsymbol{s}_{21} & S_{22} \end{bmatrix}$$

where S_{22} is a principal submatrix of S of order $k_2 - 1$ and $\boldsymbol{s}_{12}^T = \boldsymbol{s}_{21}$. Since the first entry of every eigenvector of S is nonzero, the eigenvalues of S are distinct [8]. Let $(\lambda_i, \boldsymbol{v}_i)$ be an eigenpair of S for every $i \in \{1, 2, \ldots, k_2\}$, where $\boldsymbol{v}_i = [\, v_{i1}\, v_{i2} \,\cdots\, v_{ik_2} \,]^T$ and $v_{i1} \neq 0$. An apparent eigenpair of $\mathscr{S}$ is $(\lambda_i, \boldsymbol{u}_i)$, where $\boldsymbol{u}_i := [\, 1\, t\, t^2 \cdots t^{k_1-1}]^T \otimes \boldsymbol{u}_i^T$ with $t = -(c_1 v_{ip})/(c_2 v_{i1})$ due to the orthogonality of $\boldsymbol{z}_i$ to $\boldsymbol{u}_j$ for every $i \in \{1, 2, \ldots, k_1 - 1\}$ and $j \in \{1, 2, \ldots, k_2\}$. Clearly, the first entry of $\boldsymbol{u}_i$ is nonzero. If $v_{ip} \neq 0$, then the $((j-1)k_2+1)$th entry, for every $j \in \{2, 3, \ldots, k_1\}$ of $\boldsymbol{u}_i$, is also nonzero. Now, we show that λ is a unique eigenvalue of $\mathscr{S}$ if it is an eigenvalue of S. We first consider the case that $p = 1$. The result in this case follows from Theorem 3.3. However, we provide a different proof to make the argument applicable to the cases not covered by Theorem 3.3. Similar to the reason in proving Lemma 3.2, the nonzeroness of the first entry of every eigenvector of S suggests that S and S_{22} do not share a common eigenvalue.

After some row operations, we can write the null space of $\mathscr{S} - \lambda I$ as that of $\tilde{\mathscr{S}}(\lambda)$ defined in (20) shown at bottom of this page, whose first row is a zero row. If $\boldsymbol{u}^T \tilde{\mathscr{S}}(\lambda) = \boldsymbol{0}^T$, where $\boldsymbol{u} = [\, u_1\, u_2 \,\cdots\, u_{k_1 k_2} \,]$, the row independence of $S_{22} - \lambda I$ implies that $u_{(k_1-1)k_2+i} = 0$ for every $i \in \{2, 3, \ldots, k_2\}$, and thus, $u_{(k_1-1)k_2+1} = 0$. In fact, from the structure of $\tilde{\mathscr{S}}(\lambda)$, we obtain that $u_j = 0$ for $j \in \{2, 3, \ldots, k_1 k_2\}$. That is, the rank of $\tilde{\mathscr{S}}(\lambda)$ is $k_1 k_2 - 1$ as λ is an eigenvalue of S. If $p \neq 1$, the only differences in $\tilde{\mathscr{S}}(\lambda)$ are the positions of $c_1 c_2$. Similar arguments can be applied to prove that the rank of $\tilde{\mathscr{S}}(\lambda)$ is $k_1 k_2 - 1$, and thus, λ is not repeated. ■

Now, we consider a different interconnection scheme from that used to generate a composite graph. To interconnect a finite number of identical antiregular graphs, we start with one k_2-vertex antiregular graph and repeatedly add in a new k_2-vertex antiregular graph. When the mth antiregular graph is added, $m = 2, 3, \ldots$, one of its two degree-repeating vertices is connected to the terminal or dominating vertex of the $(m-1)$th antiregular graph that was added. Let $\mathbb{G}_A^{(1,2)}$ be the resulting graph of interconnecting k_1 antiregular graphs, each with k_2 vertices. The corresponding Laplacian matrices $\mathcal{L}_A^{(1,2)}$ can be written as

$$\mathcal{L}_A^{(1,2)} := I_{k_1} \otimes \mathcal{L}_A^{(k_2)} + \sum_{i=1}^{k_1-1} \boldsymbol{z}_i \boldsymbol{z}_i^T \tag{21}$$

where $\boldsymbol{z}_i \in \{\boldsymbol{e}_{(i-1)k_2+1} - \boldsymbol{e}_{ik_2+\bar{\kappa}_2},\ \boldsymbol{e}_{ik_2} - \boldsymbol{e}_{ik_2+\bar{\kappa}_2}\}$ and $\bar{\kappa}_i = \lceil k_i/2 \rceil$, $i \in \{1, 2\}$. For integers a, b with $a < b$, let

$$\mathcal{R}_b^a := R_{b,b-1}^{(-1)} \cdots R_{a+2,a+1}^{(-1)} R_{a+1,a}^{(-1)}$$

where $R_{\alpha,\beta}^{(\gamma)}$ is the elementary row operation matrix, namely, the identity matrix except its (β, α)th entry being γ. As a result, we

$$\tilde{\mathscr{S}}(\lambda) := \begin{bmatrix}
0 & \cdots & 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots \\
\boldsymbol{s}_{21} & S_{22} - \lambda I & & & & & & & \\
c_1 c_2 & & c_2^2 & & & & & & \\
 & & \boldsymbol{s}_{21} & S_{22} - \lambda I & & & & & \\
 & & c_1 c_2 & & c_2^2 & & & & \\
 & & & \ddots & & \ddots & & & \\
 & & & & & \boldsymbol{s}_{21} & S_{22} - \lambda I & & \\
 & & & & & c_1 c_2 & & c_2^2 & \\
 & & & & & & & \boldsymbol{s}_{21} & S_{22} - \lambda I
\end{bmatrix} \tag{20}$$

have

$$\begin{aligned}\mathscr{L}_A^{(1,2)}(\lambda) :&= \mathcal{R}_{k_1 k_2}^{(k_1-1)k_2+1} \cdots \mathcal{R}_{2k_2}^{k_2+1} \mathcal{R}_{k_2}^{1} \left(\mathcal{L}_A^{(1,2)} - \lambda I\right) \\ &= I_{k_1} \otimes \tilde{\mathcal{L}}_A^{(k_2)}(\lambda) + \prod_{i=0}^{k_1-1} \mathcal{R}_{(k_1-i)k_2}^{(k_1-i-1)k_2+1} \sum_{i=1}^{k_1-1} \boldsymbol{z}_i \boldsymbol{z}_i^T \\ &= I_{k_1} \otimes \tilde{\mathcal{L}}_A^{(k_2)}(\lambda) + \sum_{i=1}^{k_1-1} \tilde{\boldsymbol{z}}_i \boldsymbol{z}_i^T\end{aligned}$$

where $\tilde{\mathcal{L}}_A^{(k_2)}$ is shown in (22) at the bottom of this page, and

$$\tilde{\boldsymbol{z}}_i = \begin{cases} \boldsymbol{z}_i + \boldsymbol{e}_{ik_2+\bar{\kappa}_2-1}, & \text{if } \boldsymbol{z}_i = \boldsymbol{e}_{(i-1)k_2+1} - \boldsymbol{e}_{ik_2+\bar{\kappa}_2} \\ \boldsymbol{z}_i - \boldsymbol{e}_{ik_2-1} + \boldsymbol{e}_{ik_2+\bar{\kappa}_2-1}, & \text{if } \boldsymbol{z}_i = \boldsymbol{e}_{ik_2} - \boldsymbol{e}_{ik_2+\bar{\kappa}_2}. \end{cases}$$

Observe that $\tilde{\mathcal{L}}_A^{(k_2)}(\lambda)$ has a special structure that only two rows have two entries and other rows three entries. More importantly, the positions of nonzero entries in these rows facilitate the identification of nonzero entries of eigenvectors of $\mathcal{L}_A^{(1,2)}$, as shown in the following lemma.

Lemma 3.7: The $\bar{\kappa}_2$th and $(\bar{\kappa}_2+1)$th entries of eigenvectors of $\mathcal{L}_A^{(1,2)}$ are nonzero.

Proof: By Lemma 3.6, every eigenvalue of $\mathcal{L}_A^{(k_2)}$ is a distinct eigenvalue of $\mathcal{L}_A^{(1,2)}$, and the eigenvectors of $\mathcal{L}_A^{(1,2)}$ corresponding to these eigenvalues have their $\bar{\kappa}_2$th and $(\bar{\kappa}_2+1)$th entries nonzero. Now, we consider the eigenvectors corresponding to the λ s that are not the eigenvalues of $\mathcal{L}_A^{(k_2)}$. The $\bar{\kappa}_2$th row of $\tilde{\mathcal{L}}_A^{(k_2)}(\lambda)$ in (22) implies that the $\bar{\kappa}_2$th and $(\bar{\kappa}_2+1)$th entries of the eigenvectors have the same value, say c. The positions of nonzero entries in the rows of $\tilde{\mathcal{L}}_A^{(k_2)}(\lambda)$ imply that the first k_2 entries of the eigenvectors can be written as $[\,(1-\lambda)c\; c\; c \cdots c\,]$. If c is zero, the first row of $\mathscr{L}_A^{(1,2)}$ suggests that the $(k_2+\bar{\kappa}_2)$th entry of the eigenvector is also zero. The special structure of $\tilde{\mathcal{L}}_A^{(k_2)}(\lambda)$ implies that the first $2k_2$ entries of the eigenvectors are zero. Continuing this argument yields that the eigenvectors are zero vectors, a contradiction. We thus conclude that c is not zero, and the proof is complete. ■

The nonzero property in Lemma 3.7 not only implies the distinctness of Laplacian eigenvalues of $\mathbb{G}_A^{(1,2)}$, but also gives a hint on the vertex selection that ensures the SILC property of the graph. It was shown in [31, Th. 2.1] that a necessary and sufficient condition for a control vector to render the graph Laplacian controllable can be derived by analyzing eigenvectors of the bordered matrix [32, p. 26] composed of the Laplacian matrix of the graph and the control vector. However, analyzing this bordered matrix is challenging, since some nice properties of Laplacian matrices no longer exist. In our case, we relate the selection of control vectors for $\mathbb{G}_A^{(1,2)}$ to that of $\mathbb{G}_A^{(k_2)}$. We first differentiate the following two cases of $\mathcal{L}_A^{(1,2)}$ in (21):

$$\mathcal{L}_A^{(1,2)} = \begin{cases} \bar{\mathcal{L}}_A^{(1,2)}, & \text{if } \boldsymbol{z}_1 = \boldsymbol{e}_1 - \boldsymbol{e}_{k_2+\bar{\kappa}_2} \\ \underline{\mathcal{L}}_A^{(1,2)}, & \text{if } \boldsymbol{z}_1 = \boldsymbol{e}_{k_2} - \boldsymbol{e}_{k_2+\bar{\kappa}_2}. \end{cases} \tag{23}$$

Theorem 3.8: Suppose $\tilde{\boldsymbol{b}}$ is a binary vector of size k_2 and $\bar{\boldsymbol{b}} := [\,\tilde{\boldsymbol{b}}^T\; \boldsymbol{0}^T\,]^T$ is a zero-padding expansion of $\tilde{\boldsymbol{b}}$, with size $k_1 k_2$. The following statements are equivalent:

1) The sum of the $\bar{\kappa}_2$th and $(\bar{\kappa}_2+1)$th entries of $\boldsymbol{b}_1$ is 1;
2) $(\mathcal{L}_A^{(k_2)}, \tilde{\boldsymbol{b}})$ is Laplacian controllable; and
3) $(\bar{\mathcal{L}}_A^{(1,2)}, \bar{\boldsymbol{b}})$ is Laplacian controllable.

Similarly, let $\underset{\sim}{\boldsymbol{b}}$ be a binary vector of size k_2-1 and $\underline{\boldsymbol{b}} := [\,\underset{\sim}{\boldsymbol{b}}^T\; \boldsymbol{0}^T\,]^T$ is a zero-padding expansion of $\underset{\sim}{\boldsymbol{b}}$, with size $k_1 k_2$. The following statements are equivalent;

1) The sum of the $\bar{\kappa}_2$th and $(\bar{\kappa}_2+1)$th entries of $\underset{\sim}{\boldsymbol{b}}$ is 1;
2) $(\mathcal{L}_A^{(k_2)}, [\,\underset{\sim}{\boldsymbol{b}}^T\; 0\,]^T)$ is Laplacian controllable; and
3) $(\underline{\mathcal{L}}_A^{(1,2)}, \underline{\boldsymbol{b}},)$ is Laplacian controllable.

Proof: Consider the case of $\bar{\mathcal{L}}_A^{(1,2)}$. The equivalence of the first two conditions is a well-known result that follows directly from Theorem 2.2 and Lemma 2.1. In proving Theorem 3.7, we have shown that the first k_2 entries of eigenvector of $\bar{\mathcal{L}}_A^{(1,2)}$ is either an eigenvector of $\mathcal{L}_A^{(k_2)}$, or in the form of $[\,(1-\lambda)c\; c\; c \cdots c\,]$, where c is nonzero and λ is not an eigenvalue of $\mathcal{L}_A^{(k_2)}$. This established the equivalence of the three conditions. The difference between the cases of and $\bar{\mathcal{L}}_A^{(1,2)}$ and $\underline{\mathcal{L}}_A^{(1,2)}$ is that if the first k_2 entry of eigenvector of $\underline{\mathcal{L}}_A^{(1,2)}$ is not an eigenvector of $\mathcal{L}_A^{(k_2)}$, it has the form of $\boldsymbol{u} := [\,(1-\lambda)c\; c\; c \cdots c\; f(\lambda)c\,]^T$, where c is nonzero,

$$\begin{bmatrix} k_2-\lambda & 1+\lambda-k_2 & & & & & & & -1 \\ & k_2-1-\lambda & 2+\lambda-k_2 & & & & & -1 & \\ & & k_2-2-\lambda & \ddots & & & -1 & & \\ & & & \ddots & \ddots & \iddots & & & \\ & & & & \ddots & \ddots & & & \\ & & & \iddots & & \ddots & \lambda-3 & & \\ & & -1 & & & & 3-\lambda & \lambda-2 & \\ & -1 & & & & & & 2-\lambda & \lambda-1 \\ -1 & & & & & & & & 1-\lambda \end{bmatrix} \tag{22}$$

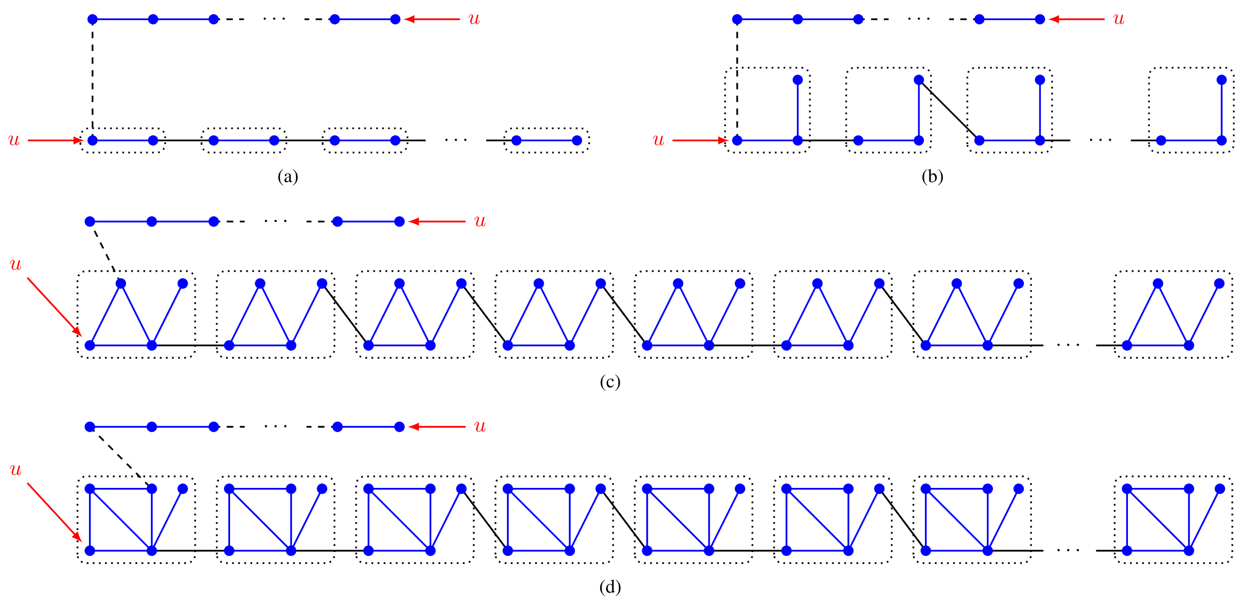

Fig. 3. Illustration of extending the single-input Laplacian controllability of a path. In (a), if the black edges connecting dotted frames are removed, each frame has a two-vertex antiregular graph inside. These frames are connected to form a path, which is Laplacian controllable by the left input u connected to the terminal vertex of the path. The path is also controllable by the input u specified in the upper part of the subfigure. This input is connected to the path via another path. (b)–(d) are generalized versions of (a) in the sense that, after removing the black edges connecting the dotted frames, each frame in each subfigure has an n-vertex antiregular graph, where $n = 3, 4,$ and 5 for (b)–(d), respectively. Note that in connecting two antiregular graphs inside two neighboring frames, the vertex in the right graph should be the degree-repeating vertex, and the vertex in the left graph can be the terminal vertex or the dominating vertex. Lemma 3.7 and Theorems 3.8 and 3.9 suggest that (b)–(d) are all SILC by the u specified in the left or in the upper part similar to the case in (a). See Example 3.4 also.

$f(\lambda) = \lambda^2 - k_2\lambda + 1$, and λ is not an eigenvalue of $\mathcal{L}_A^{(k_2)}$. The existence of $f(\lambda)$ makes it difficult to determine if $\underline{\mathcal{L}}_A^{(1,2)}$ has an eigenvalue leading to a zero sum of some entries in $\boldsymbol{u}$. We thus assign a zero to the k_2th entry of $\underline{\boldsymbol{b}}$ to restrict the control vector and avoid the difficulty. The equivalence of the three conditions then follows again from Theorem 2.2 and Lemma 2.1. ■

C. Appending a Path

The controllability result derived in the previous subsection is a generalized version of that for a $2k$-vertex path, where k is any positive integer. Now, we consider the generalization from a path with $2k + 1$, where the additional vertex is interpreted as a one-vertex path. We will show that an SILC graph could preserve its single-input controllability after interconnecting a path.

Theorem 3.9: Let $A_1, A_2,$ and Z be square matrices of order $n_1, n_2,$ and $n_1 + n_2$, respectively, and a_{ij} and z_{ij} be the (i, j)th entries of A_1 and Z, respectively. Suppose, for each $i \in \{1, 2, \dots, n_1 - 2\}$, that

$$a_{ij} : \begin{cases} \neq 0, & \text{if } j = i+1 \\ = 0, & \text{if } j \geq i+2 \, . \end{cases}$$

In addition, z_{ij} is nonzero if $(i, j) = (n_1, n_1 + 1)$ and is zero if $(i, j) \notin \{(n_1, n_1), (n_1 + 1, n_1), (n_1 + 1, n_1 + 1)\}$. If the first entry of every eigenvector of A_2 is nonzero, then so is that of the matrix

$$\mathcal{A} = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} + Z. \tag{24}$$

Proof: If the first entry of some eigenvector of $\mathcal{A}$ is zero, then all but the first column of $\mathcal{A} - \lambda I$ are dependent for some λ. The special structures of A_1 and Z then imply that all but the first column of $A_2 - \lambda I$ are dependent for some λ. However, this is a contradiction to Lemma 3.1. ■

Example 3.4: In Fig. 3, we illustrate how the single-input Laplacian controllability of a path can be generalized. Specifically, we show how to generate a k-vertex SILC graph by interconnecting one k_p-vertex path and c antiregular graphs, each with k_a vertices, where k_p and c are non-negative integers, $k_a \in \{2, 3, \dots\}$, and $ck_a + k_p = k$. It can be observed from the dotted frames that $k_a = 2, 3, 4,$ and 5 for Fig. 3(a)–(d), respectively. We use black lines to show how to connect two neighboring antiregular graphs inside two different frames: Any black line should connect the degree-repeating vertex in the right graph and the terminal vertex *or* the dominating vertex in the left graph. In each subfigure, if $k_p = 0$, the graph is Laplacian controllable by the input u that is specified in the left; if $k_p \neq 0$, the graph is Laplacian controllable by the input u, specified in the upper part, that is connected via a k_p-vertex path.

IV. Conclusion

We have studied the Laplacian controllability of a novel class of graphs constructed by interconnecting several SILC graphs. The concepts of the composite graph, composite vertex, cell graph, and structure graph have been introduced and used to generate the first type of graphs. Suppose the cell graph is Laplacian controllable by an input connected to the composite vertex. It has been shown that the structure graph is Laplacian controllable by an input connected to some vertex of the graph, if and only if the composite graph is Laplacian controllable by the input connected to that (composite) vertex. Constructing the second type of graphs is motivated by the observation that a path is actually the resulting graph after interconnecting a finite number of two-vertex antiregular graphs and possibly with one more vertex, which can be viewed as a one-vertex path. Combining the results in Sections III-B and III-C, we have shown that with a similar connection scheme, the SILC property of a path is preserved in its generalized version as long as the number of vertices of the antiregular graphs and that of the path are any two non-negative integers but not zero simultaneously. Our results expand the class of SILC graphs significantly and have potential applications to the design of a Laplacian controllable graph under edge constraints. For instance, in designing a network with hundreds of nodes that interact with their neighbors to achieve consensus, our result provides a simple method to construct a feasible composite SILC graph that can meet the constraints on the vertex degree or graph diameter by selecting the proper pair of structure and cell graphs. Furthermore, our proposed composite graph can be expanded by recursively interconnecting itself, in a way resembling the growth of some biological organizations. This way of graph construction might see its application in some area of the systems biology, where researchers design, fabricate, or control certain biological components via the so-called synthetic biological circuits to improve or achieve certain biological functionality. Note that a graph can be SILC even if its Laplacian eigenvectors do not satisfy the nonzero condition required in our work. Exploring the method of interconnecting such graphs to preserve the SILC property is interesting. Multi-input versions of our results on Laplacian controllability of graphs are also interesting to pursue.

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