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Dynamics over Signed Networks*

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Abstract. A signed network is a network in which each link is associated with a positive or negative sign. Models for nodes interacting over such signed networks arise from various biological, social, political, and economic systems. As modifications to the conventional DeGroot dynamics for positive links, two basic types of negative interactions along negative links, namely, the opposing rule and the repelling rule, have been proposed and studied in the literature. This paper reviews a few fundamental convergence results for such dynamics over deterministic or random signed networks under a unified algebraic-graphical method. We show that a systematic tool for studying node state evolution over signed networks can be obtained utilizing generalized Perron–Frobenius theory, graph theory, and elementary algebraic recursions.

Key words. networks, consensus, clustering

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Contents

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1. Introduction. In recent decades, the study of network dynamics has attracted tremendous research attention from a variety of scientific disciplines [14]. In particular, with roots traceable back to topics such as the 1960s products of stochastic matrices [54], the 1970s DeGroot social interaction models [16], and the 1980s distributed optimization problem [52], consensus algorithms serve as a primary model for social network dynamics as well as a foundation for some prominent engineering applications of large-scale complex networks [26, 41, 37, 27, 21].

It is now commonly understood how cooperative node dynamics lead to the emergence of collective network behaviors. On the other hand, in various biological, social, political, and economic systems, there are often two different types of node interactions: activatory or inhibitory, trustful or mistrustful, cooperative or antagonistic [19, 30, 1]. Using a positive or negative sign to represent the type of a link, the structure of these systems can be modeled as signed graphs. The dynamics of such signed networks follow from the dynamical relationships specified along the positive and negative links. For instance, different consensus algorithms with positive and negative links have been recently proposed and investigated [2, 42, 45, 46, 28, 32, 33, 44, 55, 24]. There exist two basic types of interactions along the negative links: the "opposing negative dynamics" [2] where nodes are attracted by the opposite values of the neighbors, and the "repelling negative dynamics" [42] where nodes tend to be repulsive of the relative position of the states with respect to the neighbors.

1.1. Signed Graphs. Consider a network with n nodes indexed in the set $\mathrm{V} = \{1, \dots, n\}$. The structure of the network is represented as an undirected graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$, where an edge (link) $\{i, j\} \in \mathrm{E}$ is an unordered pair of two distinct nodes in the set V. Each edge in E is associated with a sign, positive or negative, defining G as a signed graph. The positive and negative edges are collected in the sets E^+ and E^-, respectively. Then $\mathrm{G}^+ = (\mathrm{V}, \mathrm{E}^+)$ and $\mathrm{G}^- = (\mathrm{V}, \mathrm{E}^-)$ are, respectively, termed positive and negative subgraphs. Throughout the paper and without further mentioning it we assume that G is connected and G^- contains at least one edge.

For a node $i \in \mathrm{V}$, its positive neighbors are the nodes that share a positive link with i, forming the set $\mathrm{N}_i^+ := \{j : \{i,j\} \in \mathrm{E}^+\}$. Similarly, the negative neighbor set of node i is denoted as $\mathrm{N}_i^- := \{j : \{i,j\} \in \mathrm{E}^-\}$. The set $\mathrm{N}_i = \mathrm{N}_i^+ \bigcup \mathrm{N}_i^-$ then contains all nodes that interact with node i over the graph G. We use $\deg_i = |\mathrm{N}_i|$ to denote the degrees of node i, i.e., the number of neighbors of node i. Similarly, $\deg_i^+ = |\mathrm{N}_i^+|$ and $\deg_i^- = |\mathrm{N}_i^-|$ represent the positive and negative degrees of node i, respectively.

1.2. Signed Laplacian. Let $D_{\mathrm{G}^+} = \mathrm{diag}(\deg_1^+, \dots, \deg_n^+)$ and $D_{\mathrm{G}^-} = \mathrm{diag}(\deg_1^-, \dots, \deg_n^-)$ be the degree matrices of the positive subgraph and negative subgraph, respectively. Let A_{G^+} be the adjacency matrix of the graph G^+ with $[A_{\mathrm{G}^+}]_{ij} = 1$ if $\{i,j\} \in \mathrm{E}^+$ and $[A_{\mathrm{G}^+}]_{ij} = 0$ otherwise. The adjacency matrix A_{G^-} of the negative subgraph G^- is defined by $[A_{\mathrm{G}^-}]_{ij} = -1$ for $\{i,j\} \in \mathrm{E}^-$ and $[A_{\mathrm{G}^-}]_{ij} = 0$ for $\{i,j\} \notin \mathrm{E}^-$.

The Laplacian plays a central role in the algebraic representation of structural properties of graphs [18]. In the presence of negative edges, more than one definition of Laplacian is possible; see, e.g., [2, 3, 11]. The Laplacian of the positive subgraph G^+ is $L_{\mathrm{G}^+} := D_{\mathrm{G}^+} - A_{\mathrm{G}^+}$, while for the negative subgraph G^- the following two variants can be used: $L^{\mathrm{o}}_{\mathrm{G}^-} := D_{\mathrm{G}^-} - A_{\mathrm{G}^-}$ and $L^{\mathrm{r}}_{\mathrm{G}^-} := -D_{\mathrm{G}^-} - A_{\mathrm{G}^-}$. Consequently, we have the following definitions.

DEFINITION 1. *Given the signed graph* G*, its opposing Laplacian is defined as*

$$L^{\mathrm{o}}_{\mathrm{G}} := L_{\mathrm{G}^+} + L^{\mathrm{o}}_{\mathrm{G}^-} = D_{\mathrm{G}^+} + D_{\mathrm{G}^-} - A_{\mathrm{G}^+} - A_{\mathrm{G}^-}, \tag{1}$$

and its repelling Laplacian is defined as

$$L^{\mathrm{r}}_{\mathrm{G}} = L_{\mathrm{G}^+} + L^{\mathrm{r}}_{\mathrm{G}^-} := D_{\mathrm{G}^+} - D_{\mathrm{G}^-} - A_{\mathrm{G}^+} - A_{\mathrm{G}^-}. \tag{2}$$

The two superindexes "o" and "r" stand for "opposing" and "repelling" rules, terminology which will be introduced in section 1.4 and used throughout the paper.[1] The two Laplacians $L^{\mathrm{o}}_{\mathrm{G}}$ and $L^{\mathrm{r}}_{\mathrm{G}}$ have different properties. For instance, $L^{\mathrm{o}}_{\mathrm{G}}$ is always diagonally dominant, while $L^{\mathrm{r}}_{\mathrm{G}}$ may or may not be; $L^{\mathrm{r}}_{\mathrm{G}}$ always has zero as an eigenvalue, while $L^{\mathrm{o}}_{\mathrm{G}}$ may or may not do so. Denote $\mathbf{x} = (x_1 \ \dots \ x_n)^\top$. Then we have the

[1] We prefer to avoid ambiguous terms like "signed Laplacian," which has been used in the literature to indicate both $L^{\mathrm{o}}_{\mathrm{G}}$ and $L^{\mathrm{r}}_{\mathrm{G}}$.

two induced quadratic forms

$$\mathbf{x}^\top L_{\mathrm{G}}^{\circ}\mathbf{x} = \sum_{\{i,j\}\in \mathrm{E}^+} (x_i - x_j)^2 + \sum_{\{i,j\}\in \mathrm{E}^-} (x_i + x_j)^2, \tag{3}$$

$$\mathbf{x}^\top L_{\mathrm{G}}^{\mathrm{r}}\mathbf{x} = \sum_{\{i,j\}\in \mathrm{E}^+} (x_i - x_j)^2 - \sum_{\{i,j\}\in \mathrm{E}^-} (x_i - x_j)^2. \tag{4}$$

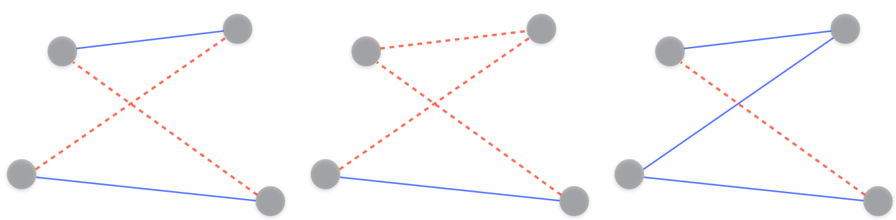
The two definitions (1) and (2) can be straightforwardly generalized to the weighted sign graph case in which each link is associated with a positive or negative real number as its weight.

1.3. Structural Balance Theory. Introduced in the 1940s [23] and primarily motivated by social-interpersonal and economic networks, a fundamental notion in the study of signed graphs is the so-called structural balance. We recall the following definition (see [14] for a detailed introduction).

DEFINITION 2. *A signed graph* G *is structurally balanced if there is a partition of the node set into* $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2$ *with* V_1 *and* V_2 *being nonempty and mutually disjoint, where any edge between the two node subsets* V_1 *and* V_2 *is negative, and any edge within each* V_i *is positive.*

Known as Harary's balance theorem, a signed graph G is structurally balanced if and only if there is no cycle with an odd number of negative edges in G [12]. If G is a complete graph, it turns out that we can verify its structural balance property by simply checking all triangles: G is structurally balanced if and only if among every set of three nodes there are either one or three positive edges [14]. The notion of structural balance can be weakened in the following definition [15].

DEFINITION 3. *A signed graph* G *is weakly structurally balanced if there is a partition of* V *into* $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2 \cdots \bigcup \mathrm{V}_m, m \geq 2$ *with* $\mathrm{V}_1, \ldots, \mathrm{V}_m$ *being nonempty and mutually disjoint, where any edge between different* V_i*'s is negative, and any edge within each* V_i *is positive.*



(a) Strong balance. (b) Weak balance. (c) Unbalanced.

Figure 1 *Examples of strongly balanced (left), weakly balanced (middle), and unbalanced signed graphs (right). Here solid lines represent positive edges and dashed lines represent negative edges.*

It is known that G is weakly structurally balanced if and only if no cycle has exactly one negative edge in G [15]. When G is a complete graph, this condition is equivalent to the fact that there is no set of three nodes among which there is exactly one negative edge [14]. In Figure 1, three basic examples are presented illustrating graph balance.

1.4. Positive/Negative Interactions. Time is slotted at $t = 0, 1, \ldots$. Each node i holds a state $x_i(t) \in \Re$ at time t and interacts with its neighbors at each time to revise its state. The interaction rule is specified by the sign of the links. Let $\alpha, \beta \geq 0$. We first focus on a particular link $\{i, j\} \in \mathrm{E}$ and specify for the moment the dynamics along this link isolating all other interactions.

- If the sign of $\{i, j\}$ is positive, each node $s \in \{i, j\}$ updates its value by
  - The DeGroot Rule:

$$x_s(t+1) = x_s(t) + \alpha\big(x_{-s}(t) - x_s(t)\big) = (1-\alpha)x_s(t) + \alpha x_{-s}(t), \tag{5}$$

    where $-s \in \{i, j\} \setminus \{s\}$ with $\alpha \in (0, 1)$.
- If the sign of $\{i, j\}$ is negative, each node $s \in \{i, j\}$ updates its value by either
  - The Opposing Rule:

$$x_s(t+1) = x_s(t) + \beta\big(-x_{-s}(t) - x_s(t)\big) = (1-\beta)x_s(t) - \beta x_{-s}(t); \tag{6}$$

    or
  - The Repelling Rule:

$$x_s(t+1) = x_s(t) - \beta\big(x_{-s}(t) - x_s(t)\big) = (1+\beta)x_s(t) - \beta x_{-s}(t). \tag{7}$$

The positive interaction is consistent with DeGroot's rule of social interactions, which indicates that the opinions of trustful social members are attractive to each other [16]. Along a negative link, the opposing rule (introduced in [2] in the form of continuous-time dynamics) indicates that the interaction will drive a node state to be attracted by the opposite of its neighbor's state; the repelling rule [42] indicates that the two node states will repel each other instead of being attractive. The two parameters α and β mark the strength of positive and negative links, respectively. There can indeed be various types of negative interactions. As the DeGroot rule is the (discrete-time) gradient flow of the Laplacian quadratic form for networks with only positive links [18], the opposing rule and the repelling rule define network gradient flows from the quadratic forms by the opposing and repelling Laplacians of signed graphs in (3) and (4), respectively. Therefore, these opposing/repelling rules can quite naturally be considered as the primary signed dynamic models, especially from the perspective of social opinion dynamics [2, 44].

1.5. Paper Organization. This paper reviews the existing results on fundamental convergence properties of signed dynamical networks [1, 2, 42, 45, 46, 28, 32, 33, 44, 55, 24, 4]. In the past few years, a variety of signed network models have appeared in the literature that fall into the categories of the above opposing or repelling rules. Various treatments ranging from Lyapunov direct methods [2] to graph lifting [24] and even analysis based on complete observability theory [4] have been used to answer questions concerning node state consensus or clustering in the asymptotic limit. We form a general signed network model by collecting the node interactions at the individual links of an underlying graph. Then an algebraic-graphical method is provided serving as a system-theoretic tool for studying consensus dynamics over signed networks. Combining generalized Perron–Frobenius theory, graph theory, and elementary algebraic recursions, we show that this approach provides simple yet unified proofs to a series of basic convergence results for networks with deterministic or random node interactions.

The remainder of the paper is organized as follows. Section 2 presents a series of basic results for dynamics over deterministic networks. Section 3 extends the discussion to random networks with convergence results established using similar algebraic-graphical analysis and a few additional probabilistic ingredients. Finally, section 4 concludes the paper with a few concluding remarks in addition to some discussion on open problems and future directions.

1.6. Notation. Real numbers are in general denoted by lowercase letters $x, y, a, b, c, \dots$ and lowercase Greek letters $\alpha, \beta, \gamma, \dots$. All vectors are column vectors denoted by bold lowercase letters $\mathbf{x}, \mathbf{y}, \dots$. Matrices are denoted with upper case letters such as $A, B, C, \dots$. All matrices are real. Given a matrix A, $A^\top$ denotes its transpose and A^k denotes the kth power of A when it is a square matrix. Likewise, the transpose of a vector $\mathbf{x}$ is denoted by $\mathbf{x}^\top$. The ijth entry of a matrix A is denoted by $[A]_{ij}$; the spectrum and spectral radius of a matrix A are denoted by $\sigma(A)$ and $\rho(A)$, respectively; the largest eigenvalue of a symmetric matrix A is denoted by $\lambda_{\max}(A)$. The n-dimensional all-one vector is denoted by $\mathbf{1}$, and the n-dimensional unit vector with the ith entry being one is $\mathbf{e}_i$. The node set is always $\mathrm{V} = \{1, \dots, n\}$, over which a deterministic graph is denoted as G and a random graph is denoted as $\mathcal{G}$. Depending on the argument, $|\cdot|$ stands for the absolute value of a real number or the cardinality of a set. The Euclidean norm of a vector is $\|\cdot\|$.

2. Deterministic Networks.

In this section, we investigate the evolution of the node states with deterministic interactions. The pairwise interactions among the signed links are collected over a deterministic network. We are interested in characterizing the asymptotic limits of the node states and providing some basic convergence theorems. Relevant results in the literature can be seen, for instance, in [2, 28, 33, 55, 24].

2.1. Fundamental Convergence Results.

2.1.1. Opposing Negative Dynamics. With the opposing rule (6) along the negative links, the update of $x_i(t)$ reads as

$$
\begin{aligned}
x_i(t+1) &= x_i(t) + \alpha \sum_{j \in \mathrm{N}_i^+} \Big(x_j(t) - x_i(t)\Big) - \beta \sum_{j \in \mathrm{N}_i^-} \Big(x_j(t) + x_i(t)\Big) \\
&= \Big(1 - \alpha \deg_i^+ - \beta \deg_i^-\Big) x_i(t) + \alpha \sum_{j \in \mathrm{N}_i^+} x_j(t) - \beta \sum_{j \in \mathrm{N}_i^-} x_j(t). \qquad (8)
\end{aligned}
$$

Denote $\mathbf{x}(t) = (x_1(t) \dots x_n(t))^\top$. We can now rewrite (8) in the compact form

$$
\mathbf{x}(t+1) = W_{\mathrm{G}} \mathbf{x}(t) = \big(I - \alpha L_{\mathrm{G}^+} - \beta L^{\circ}_{\mathrm{G}^-}\big)\mathbf{x}(t), \qquad (9)
$$

where L_{G^+} and $L^{\circ}_{\mathrm{G}^-}$ are the opposing Laplacians of G^+ and G^-, respectively. Also note that

$$
W_{\mathrm{G}} = I - \alpha L_{\mathrm{G}^+} - \beta L^{\circ}_{\mathrm{G}^-} = I - L^{\mathrm{ow}}_{\mathrm{G}},
$$

with $L^{\mathrm{ow}}_{\mathrm{G}} = \alpha L_{\mathrm{G}^+} + \beta L^{\circ}_{\mathrm{G}^-}$ being the opposing weighted Laplacian of G.

Recall that a real matrix (or vector) is called positive (nonnegative) if all its entries are positive (nonnegative); a stochastic matrix is a nonnegative matrix with row sum equal to one [25]. A key property of the matrix W_{G} lies in the fact that for small α and β (e.g., $0 < \alpha + \beta < 1/\max_{i \in \mathrm{V}} \deg_i$),

$$
\sum_{j=1}^{n} \big|[W_{\mathrm{G}}]_{ij}\big| = 1, \ i \in \mathrm{V}, \qquad (10)
$$

which indicates that W_{G} will become a stochastic matrix if all its entries are put into their absolute values. The following result holds relating the structural balance of G with the notion of bipartite consensus, i.e., node states are asymptotically clustered into two values with opposite signs. This type of result was first presented in [2] for continuous-time node dynamics based on Lyapunov analysis. Here we provide a proof by incorporating graphical analysis into plain algebraic inequalities.

THEOREM 1. *Assume that* $0 < \alpha + \beta < 1/\max_{i\in \mathrm{V}} \deg_i$. *Then along* (8) *the following statements hold for any initial value* $\mathbf{x}(0)$.

(i) *If* G *is structurally balanced subject to the partition* $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2$, *then* $\lim_{t\to\infty} x_i(t) = \big(\sum_{j\in \mathrm{V}_1} x_j(0) - \sum_{j\in \mathrm{V}_2} x_j(0)\big)/n,\ i \in \mathrm{V}_1$, *and* $\lim_{t\to\infty} x_i(t) = -\big(\sum_{j\in \mathrm{V}_1} x_j(0) - \sum_{j\in \mathrm{V}_2} x_j(0)\big)/n,\ i \in \mathrm{V}_2$.

(ii) *If* G *is not structurally balanced, then* $\lim_{t\to\infty} x_i(t) = 0,\ i \in \mathrm{V}$.

Proof. (i) Let G be structurally balanced with partition $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2$. Consider a gauge transformation given by

$$z_i(t) = x_i(t),\ i \in \mathrm{V}_1; \quad z_i(t) = -x_i(t),\ i \in \mathrm{V}_2.$$

The evolution of the $z_i(t)$ becomes a standard consensus algorithm, whose convergence follows from, for instance, Theorem 2 in [37]. The convergence of $x_i(t)$ can then be inferred.

(ii) Let $0 < \alpha + \beta < 1/\deg_i$ for all i. Applying Geršhgorin's Circle Theorem (see, e.g., Theorem 6.1.1 in [25]), it is easy to see that $-1 < \lambda_i(W_{\mathrm{G}}) \le 1$ for all $\lambda_i \in \sigma(W_{\mathrm{G}})$. This immediately implies that for any initial value $\mathbf{x}(0)$, there exists $\mathbf{y}(\mathbf{x}(0)) = (y_1(\mathbf{x}(0)) \ \dots \ y_n(\mathbf{x}(0)))^\top$ satisfying $W_{\mathrm{G}}\mathbf{y} = \mathbf{y}$ such that $\lim_{t\to\infty} x_i(t) = y_i$.

Claim. $|y_1| = \cdots = |y_n|$ for any $\mathbf{x}(0)$.

Suppose there are two distinct nodes i and j with $|y_i| \neq |y_j|$. The fact that $W_{\mathrm{G}}\mathbf{y} = \mathbf{y}$ gives

$$|y_i| \le \sum_{j=1}^{n} \big|[W_{\mathrm{G}}]_{ij}\big| \cdot |y_j|,\ i \in \mathrm{V}. \tag{11}$$

This is impossible for a connected graph G, noting (10), which proves the above claim.

Now let $y_* = |y_1| = \cdots = |y_n| \neq 0$ for some $\mathbf{x}(0)$. There must be a set V_* (which, of course, may be an empty set at this point) with

$$y_i = y_*,\ i \in \mathrm{V}_*; \qquad y_i = -y_*,\ i \in \mathrm{V} \setminus \mathrm{V}_*.$$

It is straightforward to verify that in order for $W_{\mathrm{G}}\mathbf{y} = \mathbf{y}$ to hold, all links (if any) in either V_* or $\mathrm{V}\setminus\mathrm{V}_*$ must be positive, and the links (if any) between V_* and $\mathrm{V}\setminus\mathrm{V}_*$ must be negative. That is to say, G must be structurally balanced since by our standing assumption G^- is nonempty. We have now completed the proof. ∎

We remark that the condition $0 < \alpha + \beta < 1/\max_{i\in\mathrm{V}} \deg_i$ in Theorem 1 can certainly be relaxed; e.g., a straightforward relaxation would be $0 < \alpha\deg_i^+ + \beta\deg_i^- < 1$ for all i. Further relaxations can be obtained making use of the structure of L_{G^+} and $L^{\circ}_{\mathrm{G}^-}$ and the fact that the spectrum of W_{G} will be restricted within the unit cycle for sufficiently small α and β. The essential message of Theorem 1 is that structural balance of G determines whether one is within the spectrum of W_{G}. In fact, there holds

$$\big\|\mathbf{x}(t+1)\big\|^2 \le \lambda_{\max}\big(W_{\mathrm{G}}^2\big)\big\|\mathbf{x}(t)\big\|^2 \le \big\|\mathbf{x}(t)\big\|^2 \tag{12}$$

with sufficiently small α and β guaranteeing $\lambda_{\max}(W_{\mathrm{G}}^2) \le 1$. Therefore, the algorithm (9) defines an overall contraction mapping, consistent with the standard consensus algorithms without negative links.

2.1.2. Repelling Negative Dynamics. Now consider the repelling rule (7) for negative links. The update of $x_i(t)$ reads as

$$\begin{aligned} x_i(t+1) &= x_i(t) + \alpha \sum_{j\in \mathrm{N}_i^+} \Big(x_j(t) - x_i(t)\Big) - \beta \sum_{j\in \mathrm{N}_i^-} \Big(x_j(t) - x_i(t)\Big) \\ &= \Big(1 - \alpha \mathrm{deg}_i^+ + \beta \mathrm{deg}_i^-\Big) x_i(t) + \alpha \sum_{j\in \mathrm{N}_i^+} x_j(t) - \beta \sum_{j\in \mathrm{N}_i^-} x_j(t). \end{aligned} \tag{13}$$

The algorithm (13) can be written as

$$\mathbf{x}(t+1) = M_{\mathrm{G}}\mathbf{x}(t) = \big(I - \alpha L_{\mathrm{G}+} - \beta L^{\mathrm{r}}_{\mathrm{G}-}\big)\mathbf{x}(t). \tag{14}$$

Here,

$$M_{\mathrm{G}} = I - \alpha L_{\mathrm{G}+} - \beta L^{\mathrm{r}}_{\mathrm{G}-} = I - L^{\mathrm{rw}}_{\mathrm{G}},$$

with $L^{\mathrm{rw}}_{\mathrm{G}} = \alpha L_{\mathrm{G}+} + \beta L^{\mathrm{r}}_{\mathrm{G}-}$ being the repelling weighted Laplacian of G. From (14), $M_{\mathrm{G}}\mathbf{1} = \mathbf{1}$ always holds. We present the following result, which by itself is merely a straightforward look into the spectrum of the repelling Laplacian $L^{\mathrm{rw}}_{\mathrm{G}}$.

THEOREM 2. *Suppose* G^+ *is connected. Then along* (13) *for any* $0 < \alpha < 1/\max_{i\in \mathrm{V}} \mathrm{deg}_i^+$*, there exists a critical value* $\beta_* > 0$ *for* β *such that*

(i) *if* $\beta < \beta_*$*, then average consensus is reached in the sense that* $\lim_{t\to\infty} x_i(t) = \sum_{j=1}^n x_i(0)/n$ *for all initial values* $\mathbf{x}(0)$*;*

(ii) *if* $\beta > \beta_*$*, then* $\lim_{t\to\infty} \|\mathbf{x}(t)\| = \infty$ *for almost all initial values w.r.t. Lebesgue measure.*

Proof. Define $J = \mathbf{1}\mathbf{1}^\top/n$. Fix $\alpha \in (0, 1/\max_{i\in\mathrm{V}} \mathrm{deg}_i^+)$ and consider

$$f(\beta) := \lambda_{\max}\Big(I - \alpha L_{\mathrm{G}+} - \beta L^{\mathrm{r}}_{\mathrm{G}-} - J\Big),\quad g(\beta) := \lambda_{\min}\Big(I - \alpha L_{\mathrm{G}+} - \beta L^{\mathrm{r}}_{\mathrm{G}-} - J\Big).$$

The Courant–Fischer Theorem (see Theorem 4.2.11 in [25]) implies that both $f(\cdot)$ and $g(\cdot)$ are continuous and nondecreasing functions over $[0,\infty)$. The matrix J always commutes with $I - \alpha L_{\mathrm{G}+} - \beta L^{\mathrm{r}}_{\mathrm{G}-}$, and 1 is the only nonzero eigenvalue of J. Moreover, the eigenvalue 1 of J shares a common eigenvector $\mathbf{1}$ with the eigenvalue 1 of $I - \alpha L_{\mathrm{G}+} - \beta L^{\mathrm{r}}_{\mathrm{G}-}$.

Since G^+ is connected, the second smallest eigenvalue of $L_{\mathrm{G}+}$ is positive. Since $0 < \alpha < 1/\max_{i\in\mathrm{V}} \mathrm{deg}_i^+$, there holds $\lambda_{\min}(I - \alpha L_{\mathrm{G}+}) > -1$, again due to the Geršhgorin's Circle Theorem. Therefore, $f(0) < 1$, $g(0) > -1$. Noticing $f(\infty) = \infty > 1$, there exists $\beta_* > 0$ satisfying $f(\beta_*) = 1$. We can then verify the following facts:

- There hold $f(\beta) < 1$ and $g(\beta) > -1$ if $\beta < \beta_*$. In this case, along (14) $\lim_{t\to\infty}(I - J)\mathbf{x}(t) = 0$, which in turn implies that $\mathbf{x}(t)$ converges to the eigenspace corresponding to the eigenvalue 1 of M_{G}. This leads to the average consensus statement in (i).
- There holds $f(\beta) > 1$ if $\beta > \beta_*$. In this case, along (14) $\mathbf{x}(t)$ diverges as long as the initial value $\mathbf{x}(0)$ has a nonzero projection onto the eigenspace corresponding to $\lambda_{\max}(M_{\mathrm{G}})$ of M_{G}. This leads to the almost everywhere divergence statement in (ii).

The proof is now complete. ∎

The condition that G^+ is a connected graph is crucial for Theorem 2. Once G^+ becomes disconnected, it is easy to see that one single negative link and an arbitrarily small $\beta > 0$ will drive the network state to diverge for almost all initial values. Necessary and sufficient conditions are established in [13, 56] on when the repelling Laplacian $L_{\rm G}^{\rm rw}$ is positive semidefinite from linear matrix inequalities, which can be utilized to establish deeper results compared to Theorem 2. See also [11] for a much more detailed analysis of the spectrum of repelling Laplacians.

2.2. Mathematical Reasoning: Eventually Positive Matrices. Theorems 1 and 2 provide some basic yet informative characterizations of how negative links influence the network dynamics in the two models:

- With the opposing rule, both the positive and negative links contribute to state convergence of the nodes. The overall dynamics has a contraction nature for small α and β. As long as the overall graph G is connected, the absolute values of node states asymptotically agree; structural balance of the graph further determines the existence of nontrivial absolute value agreement in the sense that a bipartite consensus is achieved.
- With repelling negative dynamics, the negative links produce repulsive interactions with a divergence nature. These negative links are therefore essentially perturbations: the positive links must generate convergence with sufficient speed that the negative links can be overcome. This requires that the positive graph G^+ must be connected by itself and leads to a critical value of β below which convergence to consensus still holds.

It is well known that convergence of standard consensus algorithms is closely related to the Perron–Frobenius Theory [37]. Consider a graph G (unsigned) with Laplacian $L_{\rm G}$. A standard consensus algorithm over the graph G, is defined as

$$x_i(t+1) = x_i(t) + \alpha \sum_{j\in {\rm N}_i} \big(x_j(t) - x_i(t)\big),\ i \in {\rm V}, \tag{15}$$

or in vector form,

$$\mathbf{x}(t+1) = S_{\rm G}\mathbf{x}(t), \tag{16}$$

where $S_{\rm G} = I - \alpha L_{\rm G}$. Obviously, $S_{\rm G}$ is a nonnegative matrix for $\alpha < 1/\max_{i\in{\rm V}} \deg_i$. Perron–Frobenius Theory is the fundamental reasoning behind the convergence of the algorithm (15) [37]: if and only if G is connected, there holds

$$\lim_{t\to\infty} S_{\rm G}^t = \mathbf{1}\mathbf{1}^\top/n.$$

In fact, $\mathbf{1}^\top$ and $\mathbf{1}$ are the left and right eigenvector corresponding to eigenvalue 1 of $S_{\rm G}$, known as its Perron–Frobenius eigenvalue.

A matrix A is called eventually positive if there exists an integer $k_0 \in \mathbb{N}^+$ such that A^k is positive for all $k \geq k_0$. If G is structurally balanced subject to the node set partition ${\rm V}_1$ and ${\rm V}_2$, it is easy to see that $KW_{\rm G}K^{-1}$ defines a nonnegative stochastic matrix, which is eventually positive if $0 < \alpha + \beta < 1/\max_{i\in{\rm V}} \deg_i$, where $K = {\rm diag}(k_1, \dots, k_n)$ with $k_i = 1, i \in {\rm V}_1$, and $k_i = -1, i \in {\rm V}_2$. On the other hand, the matrix $M_{\rm G}$ for the repelling rule would contain negative values. Letting β_* be the critical value established in Theorem 2, the following conclusion shows that $M_{\rm G}$ is also eventually positive when convergence is achieved. We refer to [3] for a deeper investigation on the eventual positiveness of signed network dynamics.

PROPOSITION 1. *Let* G^{+} *be connected. Then* $M_{\mathrm{G}} = I - \alpha L_{\mathrm{G}^+} - \beta L^{\mathrm{r}}_{\mathrm{G}^-}$ *is eventually positive if* $0 < \alpha < 1/\max_{i\in\mathrm{V}} \deg_i$ *and* $\beta < \beta_*$.

Proof. Note that (see Theorem 2.2 in [36]) a matrix $A \in \mathbb{R}^{n\times n}$ is eventually positive if both A and $A^\top$ have the strong Perron–Frobenius property: (i) $\rho(A)$ is a simple positive eigenvalue of A; (ii) the right eigenvector related to $\rho(A)$ is positive. The statement is immediate upon verifying that M_{G} has the Perron–Frobenius property under the given conditions, respectively, from the proof of Theorem 2. □

2.3. Directed Graphs. Directional links in a network can also be associated with signs [53]. We now present generalizations of the previous model and results to signed directed networks. For ease of presentation, we keep the previous notation and simply adapt it to the directed graph case. Its usage is of course restricted to the current subsection.

Now let the graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a directed graph (digraph), where a link $(i, j) \in \mathrm{E}$ is directed starting from i and pointing to j. A diagraph is termed a signed digraph if each of its links has a positive or negative sign. By revising the definition of positive and negative neighbor sets of node i to

$$\mathrm{N}_i^+ := \big\{j : (j,i) \in \mathrm{E}^+\big\}; \quad \mathrm{N}_i^- := \big\{j : (j,i) \in \mathrm{E}^-\big\},$$

the network dynamics (8) and (13) are then readily defined for the digraph G. The set $\mathrm{N}_i = \mathrm{N}_i^+ \bigcup \mathrm{N}_i^-$ continues to represent the overall neighbor set of node i. In this directed graph case we continue to define $\deg_i^+ = |\mathrm{N}_i^+|$, $\deg_i^- = |\mathrm{N}_i^-|$, and $\deg_i = |\mathrm{N}_i|$ as the positive, negative, and overall degrees of node i. We can also define the degree matrices D_{G^+} and D_{G^-} based on these positive or negative degrees.

The concept of structural balance can be generalized to digraphs by replacing the undirected edges with directional links.

DEFINITION 4. *A signed digraph* G *is structurally balanced if there is a partition of the node set into* $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2$ *with* V_1 *and* V_2 *being nonempty and disjoint, such that any directional link between* V_1 *and* V_2 *is negative, and any link with two end nodes belonging to the same* V_i *is positive.*

For a digraph G, the adjacency matrix A_{G^+} of G^+ is given by $[A_{\mathrm{G}^+}]_{ij} = 1$ if $(j,i) \in \mathrm{E}^+$ and $[A_{\mathrm{G}^+}]_{ij} = 0$ otherwise; the adjacency matrix A_{G^-} of G^- is given by $[A_{\mathrm{G}^-}]_{ij} = -1$ if $(j,i) \in \mathrm{E}^-$ and $[A_{\mathrm{G}^+}]_{ij} = 0$ otherwise. Then $L_{\mathrm{G}^+} := D_{\mathrm{G}^+} - A_{\mathrm{G}^+}$ is the Laplacian of the directed positive subgraph, and

$$L^{\circ}_{\mathrm{G}^-} := D_{\mathrm{G}^-} - A_{\mathrm{G}^-}$$

is the opposing Laplacian of the directed negative subgraph. The dynamics (8) can still be written into the form of (9) with $W_{\mathrm{G}} = I - \alpha L_{\mathrm{G}^+} - \beta L^{\circ}_{\mathrm{G}^-}$. The following theorem is a generalization of Theorem 1 for signed digraphs.

THEOREM 3. *Consider network dynamics* (8) *over a digraph* G. *Assume that* $0 < \alpha + \beta < 1/\max_{i\in\mathrm{V}} \deg_i$. *Suppose* G *is strongly connected. The following statements hold for any initial value* $\mathbf{x}(0)$.

(i) *If* G *is structurally balanced subject to partition* $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2$, *then there are* n *positive numbers* $w_1, \dots, w_n$ *with* $\sum_{i=1}^n w_i = 1$ *such that* $\lim_{t\to\infty} x_i(t) = \big(\sum_{j\in\mathrm{V}_1} w_j x_j(0) - \sum_{j\in\mathrm{V}_2} w_j x_j(0)\big)/n$, $i \in \mathrm{V}_1$, *and* $\lim_{t\to\infty} x_i(t) = -\big(\sum_{j\in\mathrm{V}_1} w_j x_j(0) - \sum_{j\in\mathrm{V}_2} w_j x_j(0)\big)/n$, $i \in \mathrm{V}_2$.

(ii) *If* G *is not structurally balanced, then* $\lim_{t\to\infty} x_i(t) = 0,\ i \in \mathrm{V}$.

The $(w_1 \dots w_n)$ in Theorem 3 is the left eigenvector relative to the eigenvalue 1 of the matrix $KW_{\mathrm{G}}K^{-1}$, which of course depends on α and β. Again, Geršhgorin's Circle Theorem leads to $\rho\big(W_{\mathrm{G}}\big) \le 1$. However, the matrix W_{G} of a directed graph G is no longer necessarily symmetric. We cannot immediately conclude from $\rho\big(W_{\mathrm{G}}\big) \le 1$ the state convergence of the nodes as in the proof of Theorem 1 for undirected graphs. We can, however, bypass this obstacle by imposing a contradiction argument, again from an algebraic-graphical recursion.

For a diagraph G^-,

$$L^{\mathrm{r}}_{\mathrm{G}^-} = -D_{\mathrm{G}^-} - A_{\mathrm{G}^-}$$

is its repelling Laplacian. The network dynamics (13) can be again represented by (14) with

$$M_{\mathrm{G}} = I - \alpha L_{\mathrm{G}^+} - \beta L^{\mathrm{r}}_{\mathrm{G}^-}.$$

With G being directed, M_{G} is not necessarily symmetric, but $M_{\mathrm{G}}\mathbf{1} = \mathbf{1}$ continues to hold. The following theorem corresponds to Theorem 2 for signed digraphs.

THEOREM 4. *Consider network dynamics* (13) *over a digraph* G. *Suppose* G^+ *is strongly connected and fix* $0 < \alpha < 1/\max_{i\in\mathrm{V}} \deg_i^+$. *There exists* $\beta_* > 0$ *such that for any* $\beta < \beta_*$, *there are* $q_1(\beta), \dots, q_n(\beta) \in \mathbb{R}^+$ *with* $\sum_{i=1}^n q_i(\beta) = 1$ *for which a consensus is reached at*

$$\lim_{t\to\infty} x_i(t) = \sum_{j=1}^n q_i(\beta) x_i(0),\ \ i \in \mathrm{V},$$

for all initial values $\mathbf{x}(0)$.

In the statement of Theorem 4, for any $\beta < \beta_*$, $\big(q_1(\beta) \dots q_n(\beta)\big)$ is a left eigenvector relative to the eigenvalue 1 of M_{G}. It is worth emphasizing that the β_* in Theorem 4 is merely an upper bound for β under which the network can still reach a consensus in the presence of negative links, and it is unclear whether such a β_* would remain a critical value as in the undirected case. From the proof, the actual value of β_* can be expressed by

$$\sup_{\eta} \Big\{ \eta : \max_{\lambda \in \sigma(M_{\mathrm{G}})\setminus\{1\}} \big|\lambda\big| < 1 \text{ for all } \beta < \eta \Big\}.$$

Theorem 3 is a special case of various results in the literature [33, 55, 24], for which the same algebraic-graphical analysis can be adopted. Theorem 4 follows from a straightforward matrix perturbation analysis. The proofs of Theorems 3 and 4 are given in the appendix.

2.4. Rates of Convergence. The convergence statements throughout Theorems 1–4 are of course exponential since the network dynamics are linear time-invariant. In either the undirected or the directed case, the rate of convergence of the network dynamics (whenever convergence has been assured) is specified by

- $\rho(W_{\mathrm{G}})$ under the opposing rule without structural balance;
- $\rho(KW_{\mathrm{G}}K^{-1} - \mathbf{1}\mathbf{1}^\top/n)$ under the opposing rule with structural balance, where K is the corresponding Gauge transform;
- $\rho(M_{\mathrm{G}} - \mathbf{1}\mathbf{1}^\top/n)$ under the repelling rule.

From the structure of W_{G} and M_{G}, one can infer that for small α, β and with undirected node interactions, adding one link (positive or negative) for the opposing negative dynamics with structural balance will accelerate the convergence if structural balance is preserved; adding one negative link for the repelling rule will always slow down convergence. The interplay between the weights α and β and the positioning of the positive and negative links is, however, rather complex, and relies on how much the spectrum analysis of the repelling Laplacian as in [13, 56, 11] can be pushed forward.

2.5. Weighted Signs, Continuous-Time Dynamics, Switching Structures. More sophisticated signed networks can certainly be studied using similar tools and analysis. This subsection covers some related results in the literature.

2.5.1. Weighted Signs. The strength of positive and negative links, represented by α and β, can also be link dependent. This means that for the positive and negative dynamics (5), (6), and (7) along the edge $\{i,j\}$, α and β will be replaced by α_{ij} and β_{ij}, respectively. The results of Theorems 1–4 can be extended to networks with weighted signs straightforwardly [2].

2.5.2. Continuous-Time Dynamics. The signed network dynamics considered above clearly have their continuous-time counterpart. For the opposing negative dynamics (9), the corresponding node state evolution in continuous time reads as

$$\frac{d}{dt}\mathbf{x}(t) = -\Big(\alpha L_{\mathrm{G}^+} + \beta L^{\circ}_{\mathrm{G}^-}\Big)\mathbf{x}(t) = -L^{\mathrm{ow}}_{\mathrm{G}}\mathbf{x}(t). \tag{17}$$

On the other hand, the continuous-time counterpart of the repelling dynamics (14) is

$$\frac{d}{dt}\mathbf{x}(t) = -\Big(\alpha L_{\mathrm{G}^+} + \beta L^{\mathrm{r}}_{\mathrm{G}^-}\Big)\mathbf{x}(t) = -L^{\mathrm{rw}}_{\mathrm{G}}\mathbf{x}(t). \tag{18}$$

Evidently, the asymptotic behavior of (17) and (18) is fully determined by the spectrum of the opposing Laplacian $L^{\mathrm{ow}}_{\mathrm{G}}$ and of the repelling Laplacian $L^{\mathrm{rw}}_{\mathrm{G}}$. They are in fact shifts of the spectrum of W_{G} and M_{G}, respectively. With continuous-time dynamics, we no longer need to worry that certain eigenvalues are outside the unit disk for large α and β. Consequently, Theorems 1 and 2 can be immediately translated to the following statements.

PROPOSITION 2. (i) *Along the continuous-time evolution* (17)*, the following hold for any initial value* $\mathbf{x}(0)$*:*

- *If* G *is structurally balanced subject to partition* $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2$*, then* $\lim_{t\to\infty} x_i(t) = (\sum_{j\in \mathrm{V}_1} x_j(0) - \sum_{j\in\mathrm{V}_2} x_j(0))/n,\ i\in \mathrm{V}_1$*, and* $\lim_{t\to\infty} x_i(t) = -(\sum_{j\in\mathrm{V}_1} x_j(0) - \sum_{j\in\mathrm{V}_2} x_j(0))/n,\ i \in \mathrm{V}_2$.
- *If* G *is not structurally balanced, then* $\lim_{t\to\infty} x_i(t) = 0,\ i\in\mathrm{V}$.

(ii) *Consider* (18) *and suppose* G^+ *is connected. Then for any* $\alpha > 0$*, there exists a critical value* $\beta_* > 0$ *for* β *such that*

- *if* $\beta < \beta_*$*, then an average consensus is reached, i.e., for all initial values* $\mathbf{x}(0)$, $\lim_{t\to\infty} x_i(t) = \sum_{j=1}^n x_i(0)/n$*;*
- *if* $\beta > \beta_*$*, then* $\lim_{t\to\infty} \big\|\mathbf{x}(t)\big\| = \infty$ *for almost all initial values w.r.t. Lebesgue measure.*

The results for opposing negative dynamics can even be extended to nonlinear node interactions [1, 32]. As illustrated in (12), under the opposing negative dynamics, both positive and negative links lead to nonexpansive network state evolution.[2] The mathematical reasoning behind those nonlinear generalizations lies in the fact that the nonexpansive property can be preserved for suitable nonlinear interaction rules.

[2]With directed graphs, (12) in general no longer holds under the opposing negative dynamics. However, it still holds that $\max_{i\in\mathrm{V}} \big|x_i(t+1)\big| \le \max_{i\in\mathrm{V}} \big|x_i(t)\big|$, as shown in the proof of Theorem 3. Therefore, the network state evolution continues to be nonexpansive.

2.5.3. Switching Network Structures. In the study of standard consensus algorithms, particular interest lies in establishing convergence conditions under time-varying network structures [26, 8, 41, 34], with earlier work dating back to the 1960s [54]. Such analysis can be challenging due to the absence of a common convergence metric that works for all possible choices of interaction graphs. Nevertheless, possibilities for generalizing the analysis of time-varying network structures have been shown in the literature [2, 39, 32, 55, 28, 4].

Let $\mathrm{G}_t = (\mathrm{V}, \mathrm{E}_t)$, $t = 0, 1, \dots$, be a sequence of graphs with each G_t being a (directed or undirected) signed graph. Then the positive and negative neighbor sets of node i are determined by connections in G_t and therefore become time-dependent, denoted $\mathrm{N}_i^+(t)$ and $\mathrm{N}_i^-(t)$, respectively. The network dynamics under the opposing rule (6) are then represented by

$$x_i(t+1) = x_i(t) + \alpha \sum_{j\in \mathrm{N}_i^+(t)} \Big(x_j(t) - x_i(t)\Big) - \beta \sum_{j\in \mathrm{N}_i^-(t)} \Big(x_j(t) + x_i(t)\Big). \tag{19}$$

We cite the following result from Theorems 2.1 and 2.2 in [33].

PROPOSITION 3. *Suppose there exists a constant* $0 < \delta < 1$ *such that* $\alpha\big|\mathrm{N}_i^+(t)\big| + \beta\big|\mathrm{N}_i^-(t)\big| \le 1 - \delta$ *for all* $i \in \mathrm{V}$ *and all* $t \ge 0$*.*

(i) *Let there exist* $T \ge 0$ *such that the graph* $\mathrm{G}_{[s,s+T]} := \big(\mathrm{V}, \bigcup_{t=s}^{s+T} \mathrm{E}_t\big)$ *is strongly connected for all* $s \ge 0$*. Then along* (19)*, for any initial value* $\mathbf{x}(0)$*, there exists* $y_*(\mathbf{x}(0)) \ge 0$ *such that* $\lim_{t\to\infty} \big|x_i(t)\big| = y_*(\mathbf{x}(0))$ *for all* $i \in \mathrm{V}$*.*

(ii) *Suppose* G_t *is undirected for all* $t \ge 0$*. Let the graph* $\mathrm{G}_{[s,\infty]} := \big(\mathrm{V}, \bigcup_{t=s}^{\infty} \mathrm{E}_t\big)$ *be connected for all* $s \ge 0$*. Then along* (19)*, for any initial value* $\mathbf{x}(0)$*, there exists* $y_*(\mathbf{x}(0)) \ge 0$ *such that* $\lim_{t\to\infty} \big|x_i(t)\big| = y_*(\mathbf{x}(0))$ *for all* $i \in \mathrm{V}$*.*

The structural balance condition can be generalized to the sequence of graphs $\mathrm{G}_t = (\mathrm{V}, \mathrm{E}_t)$, under which a bipartite consensus result can be similarly established for opposing negative dynamics [39, 55, 28]. On the other hand, for repelling negative dynamics, analysis for switching network structures can be extremely challenging since the network state is no longer nonexpansive in the presence of one single negative link. It turns out that in order to preserve convergence to consensus, it is important that at each time step, the influence of the negative links can be overcome by the positive links. We refer to [4, 6] for such treatment of continuous-time node dynamics.

3. Random Networks. Node interactions happen randomly in many real-world networks, and how consensus can be reached over a random node interaction process has been extensively studied [22, 10, 20, 50, 51, 27, 43]. We now discuss network dynamics over signed random graph processes, for which relevant results have appeared in [42, 45, 46, 28, 44].

We use the following gossiping model [10] to describe the random node interactions. The undirected, signed graph, $\mathrm{G} = (\mathrm{V}, \mathrm{E})$, continues to define the world of the network where interactions take place. Each node initiates interactions at the instants of a rate-one Poisson process, and at each of these instants, picks a node at random to interact with. Under this model, at a given time, at most one node initiates an interaction. This allows us to order interaction events in time and to focus on modeling the node pair selection at the interaction times. The node pair selection is then characterized as follows.

DEFINITION 5. *Independently at each interaction event* $t \ge 0$*,* (i) *a node* $i \in \mathrm{V}$ *is drawn uniformly at random, i.e., with probability* $1/n$*;* (ii) *node* i *picks a neighbor* j

uniformly with probability $1/\mathrm{deg}_i$ *for* $j \in \mathrm{N}_i$*. In this case, we say that the unordered node pair* $\{i, j\}$ *is selected.*

Let $(\mathrm{E}, \mathscr{S}, \mu)$ be the probability space, where $\mathscr{S}$ is the discrete σ-algebra on E and μ is the probability measure defined by $\mu(\{i,j\}) = (1/\mathrm{deg}_i + 1/\mathrm{deg}_j)/n$ for all $\{i,j\} \in \mathrm{E}$. The node selection process can then be seen as a random event in the product probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\Omega = \mathrm{E}^{\mathbb{N}} = \{\omega = (\omega_0, \omega_1, \dots) : \forall t, \omega_t \in \mathrm{E}\}$, $\mathscr{F} = \mathscr{S}^{\mathbb{N}}$, and $\mathbb{P}$ is the product probability measure (uniquely) defined as follows: for any finite subset $K \subset \mathbb{N}$, $\mathbb{P}((\omega_t)_{t\in K}) = \prod_{t\in K} \mu(\omega_t)$ for any $(\omega_t)_{t\in K} \in \mathrm{E}^{|K|}$. For any $t \in \mathbb{N}$, we define the coordinate mapping $\mathcal{G}_t : \Omega \to \mathrm{E}$ by $\mathcal{G}_t(\omega) = \omega_t$ for all $\omega \in \Omega$. Then, formally, $\mathcal{G}_t,\ t = 0, 1, \dots,$ describes the node pair selection process. We denote $\mathscr{F}_t = \sigma(\mathcal{G}_0, \dots, \mathcal{G}_t)$ as the σ-algebra capturing the $t+1$ first interactions of the selection process.

After the pair of nodes $\{i, j\}$ has been selected at time t, the nodes update their states $x_i(t)$ and $x_j(t)$ according to the sign of the link that they share: if the link is positive, they update their states by (5); if the link is negative, they update their states by either (6) or (7). The nodes that are not selected at time t will keep their states unchanged. In this way, $\mathbf{x}(t)$, $t = 0, 1, \dots,$ specifies a random process over the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and we are interested in the mean, mean-square, and almost sure convergence of $\mathbf{x}(t)$. We note that this signed random gossiping model has been adopted by [44] and is a special case of the work presented in [45, 46], in which switching environments and sign-dependent interaction probabilities are taken into consideration. The current presentation aims for a direct exposure of the same algebraic-graphic analysis for random models utilizing the ease that arises in a simplified model.

3.1. State Convergence. For opposing and repelling negative dynamics models, we present the following results, respectively, for the mean-square and almost sure convergence of $\mathbf{x}(t)$.

THEOREM 5. *Let* $0 < \alpha, \beta < 1$ *and consider opposing rule* (6) *for dynamics over negative links.*

(i) *If* G *is structurally balanced subject to partition* $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2$*, then in both the mean-square and the almost sure sense there hold*

$$x_i(t) \to \left(\sum_{j\in \mathrm{V}_1} x_j(0) - \sum_{j \in \mathrm{V}_2} x_j(0) \right) /n,\ \ i \in \mathrm{V}_1, \tag{20}$$

and

$$x_i(t) \to - \left(\sum_{j\in \mathrm{V}_1} x_j(0) - \sum_{j \in \mathrm{V}_2} x_j(0) \right) /n,\ \ i \in \mathrm{V}_2. \tag{21}$$

(ii) *If* G *is not structurally balanced, then* $x_i(t) \to 0$ *in both the mean-square and the almost sure sense for all* $i \in \mathrm{V}$*.*

THEOREM 6. *Suppose* G^+ *is connected and consider the repelling rule* (7)*. For any* $0 < \alpha < 1$*, there exists* $\beta^*(\alpha) > 0$ *such that* $x_i(t) \to \sum_{j=1}^n x_i(0)/n$ *in both the mean-square sense and almost surely for all initial value* $\mathbf{x}(0)$ *if* $\beta < \beta^*$*.*

The almost sure convergence statement of Theorem 5 was reported in [45], while the almost sure convergence statement of Theorem 6 was reported in [44]. As the current model gives a stationary graph process, we enjoy the convenience of establishing their proofs using the same mean-square error analysis.

3.2. Almost Sure Divergence. The following results characterize possible almost sure divergence of $\mathbf{x}(t)$ caused by large β related to the negative links, respectively, for opposing and repelling models.

THEOREM 7. *Fix $0 < \alpha < 1$ with $\alpha \neq 1/2$.*

(i) *Suppose both G^+ and G^- are connected. Then under the opposing negative dynamics* (6)*, there exists $\beta_\flat$ such that whenever $\beta > \beta_\flat$, there holds*

$$\mathbb{P}\Big(\limsup_{t\to\infty} \max_{i\in \mathrm{V}} \big|x_i(t)\big| = \infty \Big) = 1 \tag{22}$$

for almost all initial values w.r.t. Lebesgue measure.

(ii) *Suppose G^+ is connected. Under the repelling negative dynamics* (7)*, there exists $\beta_\dagger$ such that whenever $\beta > \beta_\dagger$, there holds*

$$\mathbb{P}\Big(\limsup_{t\to\infty} \max_{i,j\in \mathrm{V}} \big|x_i(t) - x_j(t)\big| = \infty \Big) = 1 \tag{23}$$

for almost all initial values w.r.t. Lebesgue measure.

The same type of almost sure divergence results can be seen in [42, 45, 44, 46] under different random network models. Here, $\alpha \neq 1/2$ is a technical assumption to exclude the case where the positive graph admits finite-time convergence so that the influence of all negative edges is nullified [44]. In fact, for both of the two negative dynamics (6) and (7), the node states under random node interactions follow a so-called *No-Survivor Property* [44], which indicates that every node state (or relative state) will diverge almost surely if the maximum node state (or relative state) diverges almost surely across the entire network. This property is summarized in the following result.

THEOREM 8. *The following statements hold:*

(i) *Under the opposing negative dynamics* (6)*, it holds for any $k \in \mathrm{V}$ that*

$$\mathbb{P}\Big(\limsup_{t\to\infty} \big|x_k(t)\big| = \infty \Big| \limsup_{t\to\infty} \max_{i\in \mathrm{V}} \big|x_i(t)\big| = \infty \Big) = 1. \tag{24}$$

(ii) *Suppose G^+ is connected. Under the repelling negative dynamics* (7)*, it holds for any $k \neq m \in \mathrm{V}$ that*

$$\mathbb{P}\Big(\limsup_{t\to\infty} \big|x_k(t) - x_m(t)\big| = \infty \Big| \limsup_{t\to\infty} \max_{i,j\in \mathrm{V}} \big|x_i(t) - x_j(t)\big| = \infty \Big) = 1. \tag{25}$$

Theorem 8(i) is a special case of Theorem 3 in [45], where general random graph processes are investigated. Theorem 8(ii) is quoted directly from Theorem 1 in [44]. The two statements are established using a sample-path analysis in light of the Borel–Cantelli Lemma (see, e.g., Theorem 2.3.6 in [17]). The "lim sup" in the above two theorems can be replaced by "lim inf" and the results continue to hold.

3.3. Bounded States for Repelling Dynamics. Let $A > 0$ be a constant and define $\mathscr{P}_A(\cdot)$ by $\mathscr{P}_A(z) = -A, z < -A$, $\mathscr{P}_A(z) = z, z \in [-A, A]$, and $\mathscr{P}_A(z) = A, z > A$. Define the function $\theta : \mathrm{E} \to \mathbb{R}$ so that $\theta(\{i,j\}) = \alpha$ if $\{i,j\} \in \mathrm{E}^+$ and $\theta(\{i,j\}) = -\beta$ if $\{i,j\} \in \mathrm{E}^-$. Assume that node i interacts with node j at time t. We now consider the following node interaction under the repelling rule:

$$x_s(t+1) = \mathscr{P}_A\big((1-\theta)x_s(t) + \theta x_{-s}(t)\big),\ s \in \{i,j\}. \tag{26}$$

Now the node dynamics in (26) become nonlinear due to the state constraint. The following result shows that with structural balance of G, state clustering is reached almost surely at the two state boundaries.

THEOREM 9. *Consider node dynamics* (26) *and let* $\alpha \in (0, 1/2)$. *Assume that* G *is a structurally balanced complete graph under the partition* $\mathrm{V} = \mathrm{V}_1 \cup \mathrm{V}_2$. *When* β *is sufficiently large, for almost all initial values* $\mathbf{x}(0)$ *w.r.t. Lebesgue measure, there exists a binary random variable* $l(\mathbf{x}(0))$ *taking values in* $\{-A, A\}$ *such that*

$$\mathbb{P}\Big(\lim_{t\to\infty} x_i(t) = l(\mathbf{x}(0)), i \in \mathrm{V}_1;\ \lim_{t\to\infty} x_i(t) = -l(\mathbf{x}(0)), i \in \mathrm{V}_2\Big) = 1. \tag{27}$$

It is interesting to note that the node state clustering results in Theorems 1 and 9, for the opposing and repelling rules, respectively, both rely on structural balance of G. It turns out that when G is a complete graph, weak structural balance also leads to clustering of node states.

THEOREM 10. *Consider node dynamics* (26) *and let* $\alpha \in (0, 1/2)$. *Assume that* G *is a weakly structurally balanced complete graph under the partition* $\mathrm{V} = \mathrm{V}_1 \cup \mathrm{V}_2 \cdots \cup \mathrm{V}_m$ *with* $m \geq 2$. *When* β *is sufficiently large, almost sure boundary clustering is achieved in the sense that for almost all initial values* $\mathbf{x}(0)$ *w.r.t. Lebesgue measure, there are* m *random variables,* $l_1(\mathbf{x}(0)), \dots, l_m(\mathbf{x}(0))$, *each taking values in* $\{-A, A\}$, *such that*

$$\mathbb{P}\Big(\lim_{t\to\infty} x_i(t) = l_j(\mathbf{x}(0)),\ i \in \mathrm{V}_j,\ j = 1, \dots, m\Big) = 1. \tag{28}$$

When the positive graph G^+ is connected—and so there is no structural balance—any node state will touch the two boundaries $-A$ and A an infinite number of times. Recall that the vertex connectivity $\kappa(\mathrm{G})$ of a graph G is the minimum number of nodes whose removal disconnects G. The result is summarized below.

THEOREM 11. *Consider node dynamics* (26) *and let* $\alpha \in (1/2, 1)$. *Assume that* G *is a complete graph and the positive graph* G^+ *is connected with* $\kappa(\mathrm{G}^+) \geq 2$. *When* β *is sufficiently large, for almost all initial values* $\mathbf{x}(0)$ *w.r.t. Lebesgue measure, it holds for all* $i \in \mathrm{V}$ *that*

$$\mathbb{P}\Big(\liminf_{t\to\infty} x_i(t) = -A,\ \limsup_{t\to\infty} x_i(t) = A\Big) = 1. \tag{29}$$

Results of a similar type to Theorems 9, 10, and 11 were established in [44] for a model in which asymmetric node updates were also taken into consideration. The current simplified model allows for more direct analysis along the same line of mathematical machinery. The assumptions that G is a complete graph and α takes a specific range of values are technical assumptions to simplify the analysis, which can be further relaxed. The proofs of Theorems 9, 10, and 11 are based on stopping time analysis for the process $\mathcal{G}_t, t = 0, 1, \dots$, in light of the second Borel–Cantelli Lemma, and they are given in the appendix.

4. Conclusions. We have surveyed a few fundamental results on the convergence properties of dynamics over signed networks. A unified approach has been provided in view of generalized Perron–Frobenius theory, graph theory, and elementary algebraic recursions. The results illustrate that dynamical properties of a network depend crucially on the sign structure of the network links, for both deterministic and random node interactions. Many interesting future research directions emerge naturally after the connection between such basic convergence conditions have been clarified. First of all, inverse problems such as estimating characteristics of the annotations of links and nodes from observations of various network characteristics at a subset of nodes are of primary interest. Typical questions would include the reconstruction of node initial values, identification of edge signs, and the test of structural balance through a perhaps finite sequence of measurements of the node states [5, 31, 35, 7, 4]. Another interesting research direction would be the investigation of controllability issues related to signed networks along the line of research on network controllability [40, 29, 49, 38]. How sign structure of a network system relates to the network controllability or structural controllability is still an open problem. Finally, it would be of interest to look into the scenario where the evolving node states generate feedback to the signs of the network edges. The closed-loop network dynamics will lead to Krause's type of multiagent systems where state-dependent interaction structure will inevitably cause high nonlinearity [9] in the state update at the nodes.

Appendix A. Proof of Theorem 3. The statement (i) again follows directly from Theorem 2 in [37] after applying a gauge transformation

$$z_i(t) = x_i(t),\ i \in \mathrm{V}_1; \quad z_i(t) = -x_i(t),\ i \in \mathrm{V}_2.$$

We now prove the statement (ii) through a contradiction argument. We proceed in three steps.

Step 1. Define $h(t) := \max_{i\in \mathrm{V}} |x_i(t)|$. Observing that (10) continues to hold with a digraph G, we have $h(t+1) \le h(t)$ for all $t \ge 0$. Consequently, there is a constant $h_*(\mathbf{x}(0)) > 0$ such that $\lim_{t\to\infty} h(t) = h_*$ for any initial value $\mathbf{x}(0)$. We only need to consider the case with $h_* > 0$, and by the definition of h_*, for any $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that

$$|x_i(t)| \le h_* + \epsilon,\ t \ge T. \tag{30}$$

Step 2. Define $g_i := \liminf_{t\to\infty} |x_i(t)|$. In this step, we show that $g_i = h_*$ for all $i \in \mathrm{V}$. Suppose $g_{i_0} < h_*$ for some $i_0 \in \mathrm{V}$. By the definition of g_i, for any $\epsilon > 0$, there always exists $t_1 \ge T$ such that

$$|x_{i_0}(t_1)| \le g_{i_0} + \epsilon. \tag{31}$$

The graph G is strongly connected. Therefore, the set $\mathrm{V}_1^* := \{j : i_0 \in \mathrm{N}_j\}$ is nonempty. Based on (30), (31), and the fact that $i_0 \in \mathrm{N}_{i_1}$, we then have

$$\begin{aligned}
\big|x_{i_1}(t_1+1)\big| &= \left|\Big(1-\alpha\big|\mathrm{N}_{i_1}^+\big| - \beta\big|\mathrm{N}_{i_1}^-\big|\Big)x_{i_1}(t) + \alpha\sum_{j\in\mathrm{N}_{i_1}^+} x_j(t) - \beta\sum_{j\in\mathrm{N}_{i_1}^-} x_j(t)\right|\\
&\le \Big|1-\alpha\big|\mathrm{N}_{i_1}^+\big| - \beta\big|\mathrm{N}_{i_1}^-\big|\Big|\cdot|x_{i_1}(t)| + \alpha\sum_{j\in\mathrm{N}_{i_1}^+}\big|x_j(t)\big| + \beta\sum_{j\in\mathrm{N}_{i_1}^-}\big|x_j(t)\big|\\
&\le \gamma\big(g_{i_0}+\epsilon\big) + (1-\gamma)\big(h_*+\epsilon\big)\\
&= \gamma g_{i_0} + (1-\gamma)h_* + \epsilon
\end{aligned} \tag{32}$$

for any $i_1 \in \mathrm{V}_1^*$, where $\gamma = \min\{\alpha, \beta\}$.

Continuing, we define $\mathrm{V}_2^* := \{j : \exists i_1 \in \mathrm{V}_1^*,\ i_1 \in \mathrm{N}_j\}$ as the nodes that have a neighbor in the set V_1^*. Again, the set V_2^* is nonempty because the graph G is strongly connected. Repeating the above analysis we have

$$|x_{i_2}(t_1+2)| \le \gamma^2 g_{i_0} + (1-\gamma^2)h_* + \epsilon \tag{33}$$

for any $i_2 \in \mathrm{V}_1^* \bigcup \mathrm{V}_2^*$. This process can be repeated recursively, and eventually it must hold that

$$|x_i(t_1+n-1)| \le \gamma^{n-1} g_{i_0} + (1-\gamma^{n-1})h_* + \epsilon,\ i \in \mathrm{V}. \tag{34}$$

Therefore,

$$h_* \le \gamma^{n-1} g_{i_0} + (1-\gamma^{n-1})h_* + \epsilon, \tag{35}$$

or equivalently,

$$\gamma^{n-1}(h_* - g_{i_0}) \le \epsilon. \tag{36}$$

This leads to a contradiction if $h_* > g_{i_0}$, because ϵ in (36) can be arbitrary.

Step 3. The fact that $g_i = h_*$ for all $i \in \mathrm{V}$ immediately leads to $\lim_{t\to\infty} |x_i(t)| = h_*$ for all $i \in \mathrm{V}$, since $\limsup_{t\to\infty} |x_i(t)| \le h_*$ by the definition of h_*. It is easy to exclude the case where $\liminf_{t\to\infty} x_i(t) = -h_*$ and $\limsup_{t\to\infty} x_i(t) = h_*$ for some i directly from the dynamics (9). In other words, all node states asymptotically converge. From this point, we can define

$$\mathrm{V}_1 := \{i \in \mathrm{V} : \liminf_{t\to\infty} x_i(t) = h_*\},\ \mathrm{V}_2 := \{i \in \mathrm{V} : \liminf_{t\to\infty} x_i(t) = -h_*\}.$$

It is then clear that the links between V_1 and V_2 can only be negative, and the links inside each subset can only be positive. This proves that the graph G is structurally balanced.

We have now concluded the proof. ∎

Appendix B. Proof of Theorem 4. With G being directed, it still holds that $M_{\mathrm{G}}\mathbf{1} = \mathbf{1}$, since $M_{\mathrm{G}} = I - \alpha L_{\mathrm{G}^+} - \beta L^{\mathrm{r}}_{\mathrm{G}^-}$, where $L_{\mathrm{G}^+}\mathbf{1} = 0$ and $L^{\mathrm{r}}_{\mathrm{G}^-}\mathbf{1} = 0$ for digraphs G^+ and G^-. Therefore, 1 is always an eigenvalue of M_{G}.

Fix α with $0 < \alpha < 1/\max_{i\in\mathrm{V}} \deg_i^+$. We can define the following two functions:

$$r(\beta) := \max\Big\{|\lambda_i(M_{\mathrm{G}})| :\ \lambda_i(M_{\mathrm{G}}) \in \sigma(M_{\mathrm{G}}) \setminus \{1\}\Big\} \tag{37}$$

as the largest magnitude of the eigenvalues of M_{G} which are not equal to one, and

$$\mathbf{q}(\beta) := (q_1(\beta) \dots q_n(\beta)) \tag{38}$$

with $\mathbf{q}(\beta)M_{\mathrm{G}} = \mathbf{q}(\beta)$ and $\sum_{j=1}^n q_j(\beta) = 1$.

The following facts hold: (i) $r(0) < 1$, and 1 is a simple eigenvalue of $I - \alpha L_{\mathrm{G}^+}$ if G^+ is strongly connected;[3] (ii) $q(0)$ is a positive row vector. Noticing that both $r(\cdot)$ and $q(\cdot)$ are continuous functions, there exists a sufficiently small β_* such that both the two facts hold for $\beta < \beta_*$, i.e., 1 is a simple eigenvalue of M_{G} with $r(\beta) < 1$, and

[3]In fact, 1 is a simple eigenvalue of $I - \alpha L_{\mathrm{G}^+}$ if G^+ has a directed spanning tree (see, e.g., Proposition 3.8 in [18]).

$\mathbf{q}(\beta)$ is positive. Therefore, through the Jordan decomposition of M_{G}, it is easy to see that

$$\lim_{t\to\infty} M_{\mathrm{G}}^t = \mathbf{1}\mathbf{q}(\beta),$$

and this concludes the proof. (See the same treatment applied to continuous-time dynamics in Theorem 3.12 of [18].) ∎

Appendix C. Proof of Theorem 5. Let $\mathbf{e}_m = (0\dots1\dots0)^\top$ be the n-dimensional unit vector whose mth entry is 1. Under the pair selection process and the opposing rule for negative links, the evolution of the node states can be written as

$$\mathbf{x}(t+1) = \mathcal{W}_t\mathbf{x}(t), \tag{39}$$

where $\mathcal{W}_t$, $t = 0, 1, \dots$, is an i.i.d. random matrix process. The distribution of $\mathcal{W}_t$ is given by

$$\mathbb{P}\Big(\mathcal{W}_t = I - \alpha(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^\top\Big) = p_{ij},\ \{i,j\} \in \mathrm{E}^+, \tag{40}$$

and

$$\mathbb{P}\Big(\mathcal{W}_t = I - \beta(\mathbf{e}_i + \mathbf{e}_j)(\mathbf{e}_i + \mathbf{e}_j)^\top\Big) = p_{ij},\ \{i,j\} \in \mathrm{E}^-. \tag{41}$$

(i) Let G be structurally balanced subject to partition $\mathrm{V} = \mathrm{V}_1 \bigcup \mathrm{V}_2$. Introduce $J = \mathbf{1}\mathbf{1}^\top/n$, $K = \mathrm{diag}(k_1, \dots, k_n)$ with $k_i = 1$ for $i \in \mathrm{V}_1$ and $k_i = -1$ for $i \in \mathrm{V}_2$, and $\mathbf{k} = (k_1, \dots, k_n)^\top$. Note that, for any realization of $\mathcal{W}_t$, it holds that $JK\mathcal{W}_t = JK$. Thus, $K\mathcal{W}_tKJ = JK\mathcal{W}_tK = J$, which implies

$$(I - J)\big(K\mathcal{W}_tK\big) = \big(K\mathcal{W}_tK\big)(I - J). \tag{42}$$

Consider $V(t) = \big\|(I - J)K\mathbf{x}(t)\big\|^2$. Then

$$\begin{aligned}
\mathbb{E}\Big\{V(t+1)\Big|\mathbf{x}(t)\Big\} &= \mathbb{E}\Big\{\mathbf{x}^\top(t)\mathcal{W}_tK(I-J)K\mathcal{W}_t\mathbf{x}(t)\Big\}\\
&\overset{\text{(a)}}{=} \mathbb{E}\Big\{\big(\mathbf{x}^\top(t)K\big)\big(K\mathcal{W}_tK\big)(I-J)\big(K\mathcal{W}_tK\big)\big(K\mathbf{x}(t)\big)\Big\}\\
&\overset{\text{(b)}}{=} \mathbb{E}\Big\{\big(\mathbf{x}^\top(t)K(I-J)\big)\big(K\mathcal{W}_tK\big)(I-J)\big(K\mathcal{W}_tK\big)\big((I-J)K\mathbf{x}(t)\big)\Big\}\\
&\overset{\text{(c)}}{=} \mathbb{E}\Big\{\big(\mathbf{x}^\top(t)K(I-J)\big)\big(K\mathcal{W}_t^2K - J\big)\big((I-J)K\mathbf{x}(t)\big)\Big\}\\
&= \big(\mathbf{x}^\top(t)K(I-J)\big)\big(\mathbb{E}\big\{K\mathcal{W}_t^2K\big\} - J\big)\big((I-J)K\mathbf{x}(t)\big),
\end{aligned} \tag{43}$$

where (a) holds because $K^2 = I$, (b) is due to the equalities $(I - J)^2 = I - J$ and (42), and (c) is obtained by applying $JK\mathcal{W}_t = JK$.

Based on (40) and (41), we have

$$\begin{aligned}
P_{\mathrm{G}}^* :&= \mathbb{E}\big\{K\mathcal{W}_t^2K\big\}\\
&= \sum_{\{i,j\}\in\mathrm{E}^+} p_{ij}\big(I - 2\alpha(1-\alpha)(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^\top\big)\\
&\quad + \sum_{\{i,j\}\in\mathrm{E}^-} p_{ij}\big(I - 2\beta(1-\beta)(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^\top\big)\\
&= I - 2\alpha(1-\alpha)L_{\mathrm{G}^+}^{\mathrm{p}} + 2\beta(1-\beta)L_{\mathrm{G}^-}^{\mathrm{pr}},
\end{aligned} \tag{44}$$

where $L^{\rm p}_{\rm G+}$ is the probabilistically weighted Laplacian of $\rm G^+$ with $[L^{\rm p}_{\rm G+}]_{ij}=-p_{ij}$ for $\{i,j\}\in \rm E^+$, $[L^{\rm p}_{\rm G+}]_{ij}=0$ for $\{i,j\}\notin \rm E^+$ with $i\neq j$, and $[L^{\rm p}_{\rm G+}]_{ii}=\sum_{j\neq i\in {\rm N}_i^+}p_{ij}$; $L^{\rm pr}_{\rm G-}$ is the probabilistically weighted repelling Laplacian of $\rm G^-$ with $[L^{\rm pr}_{\rm G-}]_{ij}=p_{ij}$ for $\{i,j\}\in \rm E^-$, $[L^{\rm pr}_{\rm G-}]_{ij}=0$ for $\{i,j\}\notin \rm E^-$ with $i\neq j$, and $[L^{\rm pr}_{\rm G-}]_{ii}=-\sum_{j\neq i\in {\rm N}_i^-}p_{ij}$. We note the fact that both $L^{\rm p}_{\rm G+}$ and $-L^{\rm pr}_{\rm G-}$ are standard weighted Laplacians, and the implication of the properties of their spectrum [18] including bounds on their eigenvalues from $\sum_j p_{ij}=1$. Also noticing that $\alpha,\beta\in(0,1)$ implies $0<2\alpha(1-\alpha)\leq 1/2$ and $0<2\beta(1-\beta)\leq 1/2$, the following facts hold.

F1. $0\leq \lambda_i\big(P^*_{\rm G}\big)\leq 1$ for all $\lambda_i\big(P^*_{\rm G}\big)\in\sigma\big(P^*_{\rm G}\big)$; $1\in\sigma\big(P^*_{\rm G}\big)$ is a simple eigenvalue with $\mathbf{1}$ being a corresponding eigenvector.

F2. 1 is an eigenvalue of $\mathbf{1}\mathbf{1}^\top/n$ with multiplicity one and $\mathbf{1}$ is an associated eigenvector; $\mathbf{1}\mathbf{1}^\top/n$ also has zero as its eigenvalue with multiplicity $n-1$.

F3. $P^*_{\rm G}$ and $\mathbf{1}\mathbf{1}^\top$ commute, i.e., $P^*_{\rm G}\mathbf{1}\mathbf{1}^\top=\mathbf{1}\mathbf{1}^\top P^*_{\rm G}$.

Consequently, all eigenvalues of $P^*_{\rm G}-\mathbf{1}\mathbf{1}^\top/n$ are strictly less than one. We can therefore further conclude that

$$\mathbb{E}\Big\{V(t+1)\big|\mathbf{x}(t)\Big\}\leq \lambda_{\max}\big(P^*_{\rm G}-\mathbf{1}\mathbf{1}^\top/n\big)V(t). \tag{45}$$

This immediately yields that $\mathbb{E}\big\{V(t)\big\}$ converges to zero or, equivalently, (20) and (21) hold in the mean-square sense.

Moreover, (45) means that $V(t)$ is a supermartingale, which converges to a limit almost surely by the martingale convergence theorem (see Theorem 5.2.9 of [17]). Such a limit must be zero since $0<\lambda_{\max}\big(P^*_{\rm G}-\mathbf{1}\mathbf{1}^\top/n\big)<1$ (which implies $\mathbb{E}\big\{V(t)\big\}$ converges to zero exponentially), and this means that (20) and (21) hold in the almost sure sense.

(ii) Now we move on to the case where G is not structurally balanced. Consider instead $V_*(t)=\big\|\mathbf{x}(t)\big\|^2$. We have

$$\mathbb{E}\Big\{V_*(t+1)\big|\mathbf{x}(t)\Big\}=\mathbf{x}^\top(t)\mathbb{E}\big\{\mathcal{W}_t^2\big\}\mathbf{x}(t). \tag{46}$$

Based on (40) and (41), we have

$$\begin{aligned}P_{\rm G}:&=\mathbb{E}\big\{\mathcal{W}_t^2\big\}\\&=\sum_{\{i,j\}\in{\rm E}^+}p_{ij}\big(I-2\alpha(1-\alpha)(\mathbf{e}_i-\mathbf{e}_j)(\mathbf{e}_i-\mathbf{e}_j)^\top\big)\\&\quad+\sum_{\{i,j\}\in{\rm E}^-}p_{ij}\big(I-2\beta(1-\beta)(\mathbf{e}_i+\mathbf{e}_j)(\mathbf{e}_i+\mathbf{e}_j)^\top\big)\\&=I-2\alpha(1-\alpha)L^{\rm p}_{\rm G+}-2\beta(1-\beta)L^{\rm po}_{\rm G-},\end{aligned} \tag{47}$$

where $L^{\rm po}_{\rm G-}$ is the probabilistically weighted (opposing) Laplacian of $\rm G^-$ as a signed graph with $[L^{\rm po}_{\rm G-}]_{ij}=p_{ij}$ for $\{i,j\}\in\rm E^-$, $[L^{\rm po}_{\rm G-}]_{ij}=0$ for $\{i,j\}\notin\rm E^-$ with $i\neq j$, and $[L^{\rm po}_{\rm G-}]_{ii}=\sum_{j\neq i\in{\rm N}_i^-}p_{ij}$. The main difference between $W_{\rm G}$ and $P_{\rm G}$ lies in the weighted edges in $P_{\rm G}$. Noticing that $\alpha,\beta\in(0,1)$ implies $0<2\alpha(1-\alpha)\leq 1/2$ and $0<2\beta(1-\beta)\leq 1/2$, it holds that

$$\sum_{j=1}^n\big|[P_{\rm G}]_{ij}\big|=1. \tag{48}$$

As discussed previously, the absence of structural balance of G implies that

$$\lambda_{\max}\big(P_{\mathrm{G}}\big) < 1$$

as long as G is a connected graph. Consequently, we have

$$\mathbb{E}\big\{V_*(t+1)\big|\mathbf{x}(t)\big\} \le \lambda_{\max}\big(P_{\mathrm{G}}\big)V_*(t), \tag{49}$$

which in turn implies that $\mathbb{E}\{V_*(t)\}$ tends to zero and that $V_*(t)$ goes to zero almost surely from the same analysis applied for $V(t)$. Equivalently, we have proved that $\mathbf{x}(t)$ converges to zero in the mean-square and almost surely sense.

We have now completed the proof of Theorem 5. ∎

Appendix D. Proof of Theorem 6. Let $x_{\mathrm{ave}} = \sum_{i\in\mathsf{V}} x_i(0)/n$ be the average of the initial beliefs. We introduce $V_\flat(t) = \sum_{i=1}^n |x_i(t) - x_{\mathrm{ave}}|^2 = \big\|(I-J)\mathbf{x}(t)\big\|^2$. Similar to (43), we have

$$\mathbb{E}\Big\{V_\flat(t+1)\Big|\mathbf{x}(t)\Big\} \le \lambda_{\max}\big(\mathbb{E}\{\mathcal{W}^2(t)\} - J\big)V_\flat(t). \tag{50}$$

Under the repelling rule for negative dynamics, the distribution of $\mathcal{W}_t$ is given by

$$\mathbb{P}\Big(\mathcal{W}_t = I - \alpha(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^\top\Big) = p_{ij} \tag{51}$$

if $\mathrm{Sgn}(\{i,j\}) = +$, and

$$\mathbb{P}\Big(\mathcal{W}_t = I + \beta(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^\top\Big) = p_{ij} \tag{52}$$

if $\mathrm{Sgn}(\{i,j\}) = -$. As a result, we have

$$\mathbb{E}\{\mathcal{W}^2(t)\} = I - 2\alpha(1-\alpha)L^{\mathrm{p}}_{\mathrm{G}^+} - 2\beta(1+\beta)L^{\mathrm{pr}}_{\mathrm{G}^-}, \tag{53}$$

where $L^{\mathrm{p}}_{\mathrm{G}^+}$ and $L^{\mathrm{pr}}_{\mathrm{G}^-}$ are defined in (47).

Since G^+ is connected, $\lambda_{\max}\big(I - 2\alpha(1-\alpha)L^{\mathrm{p}}_{\mathrm{G}^+}\big) < 1$, noticing $0 < \alpha < 1$. Consequently, $\lambda_{\max}\big(\mathbb{E}\{\mathcal{W}^2(t)\} - J\big) < 1$ for all β satisfying

$$\beta(1+\beta) < \frac{\lambda_2(L^{\mathrm{p}}_{\mathrm{G}^+})}{\lambda_{\max}(-L^{\mathrm{pr}}_{\mathrm{G}^-})}\alpha(1-\alpha), \tag{54}$$

where $\lambda_2(L^{\mathrm{p}}_{\mathrm{G}^+})$ is the second smallest eigenvalue of $L^{\mathrm{p}}_{\mathrm{G}^+}$. Since $g(\beta) = \beta(1+\beta)$ is nondecreasing, $\lambda_{\max}\big(\mathbb{E}\{\mathcal{W}^2(t)\} - J\big) < 1$ for all $0 \le \beta < \beta^*$ with

$$\beta^* := \sup_\beta \left\{\beta(1+\beta) < \frac{\lambda_2(L^{\mathrm{p}}_{\mathrm{G}^+})}{\lambda_{\max}(-L^{\mathrm{pr}}_{\mathrm{G}^-})}\alpha(1-\alpha)\right\}. \tag{55}$$

Applying the same analysis that is used for $V(t)$ and $V_*(t)$, for any $0 \le \beta < \beta^*$ and from (50), it holds that $\mathbb{E}\big\{V_\flat(t)\big\}$ converges to zero and that $V_\flat(t)$ tends to zero almost surely. This completes the proof of Theorem 6. ∎

Appendix E. Proof of Theorem 7. (i) Define $h(t) := \max_{i\in\mathrm{V}} \big|x_i(t)\big|$. The proof is based on the following lemma.

LEMMA 1. *Let* $\alpha \neq 1/2 \in (0,1)$ *and* $\beta \geq 3$. *Then* $\{h(t+1) \geq \min\{|2\alpha - 1|, 1/2\}h(t)\}$ *is a sure event.*

Proof. We discuss two cases.

C1. Suppose a pair of nodes $\{i,j\}$ sharing a positive link is selected at time t. If both $|x_i(t)| < h(t)$ and $|x_j(t)| < h(t)$ hold, then $h(t+1) \geq h(t)$. Therefore, we assume without loss of generality that $|x_i(t)| = h(t)$. This leads to two scenarios.

(a) Let $0 < \alpha < 1/2$. Then

$$\begin{aligned} |x_i(t+1)| &= |(1-\alpha)x_i(t) + \alpha x_j(t)| \\ &\geq (1-\alpha)|x_i(t)| - \alpha|x_j(t)| \geq (1-2\alpha)h(t). \end{aligned} \tag{56}$$

(b) Let $1/2 < \alpha < 1$. Then

$$\begin{aligned} |x_j(t+1)| &= |(1-\alpha)x_j(t) + \alpha x_i(t)| \\ &\geq \alpha|x_i(t)| - (1-\alpha)|x_j(t)| \geq (2\alpha-1)h(t). \end{aligned} \tag{57}$$

We see that (56) and (57) lead to $h(t+1) \geq |2\alpha - 1|h(t)$.

C2. Suppose a pair of nodes $\{i,j\}$ sharing a negative link is selected at time t. Again we assume without loss of generality that $|x_i(t)| = h(t)$. We define $y_i(t) = x_i(t)$ and $y_j(t) = -x_j(t)$. Then the update of $y_i(t)$ and $y_j(t)$ is described by

$$\begin{aligned} y_i(t+1) &= y_i(t) + \beta\big(y_j(t) - y_i(t)\big), \\ y_j(t+1) &= y_j(t) + \beta\big(y_i(t) - y_j(t)\big). \end{aligned} \tag{58}$$

(a) If $|y_j(t)| \geq h(t)/2$, we see clearly from (58) that

$$h(t+1) \geq |y_j(t+1)| \geq h(t)/2 \tag{59}$$

if $y_i(t)$ and $y_j(t)$ have the same sign. Otherwise, without loss of generality, let $y_i(t) > 0$ and $y_j(t) < 0$. Then from (58)

$$\begin{aligned} |y_i(t+1)| &= |y_i(t) + \beta\big(y_j(t) - y_i(t)\big)| \\ &\geq \beta|y_j(t) - y_i(t)| - |y_i(t)| \\ &\geq \frac{3}{2}\beta h(t) - h(t) \\ &\geq h(t)/2 \end{aligned} \tag{60}$$

for $\beta \geq 1$.

(b) If $|y_j(t)| < h(t)/2$, then it holds for $\beta \geq 3$ that

$$\begin{aligned} |y_i(t+1)| &= |(1-\beta)y_i(t) + \beta y_j(t)| \\ &\geq (\beta - 1)|y_i(t)| - \beta|y_j(t)| \\ &\geq \left(\frac{1}{2}\beta - 1\right)h(t) \\ &\geq h(t)/2. \end{aligned} \tag{61}$$

We see that (59), (60), and (61) lead to $h(t+1) \geq h(t)/2$ if $\beta \geq 3$.

We have now proved the desired lemma. ∎

With Lemma 1 serving the same role as Lemma 5 in [45], the desired conclusion follows in view of the strong law of large numbers from the same argument as the proof of Proposition 1 of [45]. We therefore omit the remaining details.

(ii) The result comes from Theorem 3 in [44]. We therefore refer to the proof therein, which is also based on the strong law of large numbers. ∎

Appendix F. Proof of Theorem 9. We quote the following lemma, which is Lemma 7 in [44]. Note that the proof of Lemma 7 in [44] does not rely on the asymmetric node updates, and therefore the lemma continues to hold for (26).

Lemma 2. *Fix $\alpha \in (0,1)$ with $\alpha \neq 1/2$. For the dynamics* (26) *with the random pair selection process, there exists $\beta^{\diamond}(\alpha) > 0$ such that*

$$\mathbb{P}\Big(\limsup_{t\to\infty} \max_{i,j\in \mathrm{V}} \big|x_i(t) - x_j(t)\big| = 2A\Big) = 1$$

for almost all initial beliefs if $\beta > \beta^{\diamond}$.

We establish another technical lemma.

Lemma 3. *Fix $\alpha \in (0,1/2)$ and $\beta \geq 1/\alpha$. Consider the dynamics* (26) *with the random pair selection process. Assume that* G *is a structurally balanced complete graph under the partition $\mathrm{V} = \mathrm{V}_1 \cup \mathrm{V}_2$. If there are $i_1 \in \mathrm{V}_1$, $j_1 \in \mathrm{V}_2$, and $t \geq 0$ with $x_{i_1}(t) = -A$ and $x_{j_1}(t) = A$, then for $Z = 3(n-2)$, there exists a sequence of node pair realizations, $\mathcal{G}_{t+s}(\omega)$ for $s = 0, 1, \ldots, Z-1$, under which it holds that*

$$x_i(t+Z)(\omega) = -A,\ i \in \mathrm{V}_1; \qquad x_i(t+Z)(\omega) = A,\ i \in \mathrm{V}_2. \tag{62}$$

Proof. We recursively construct such a sequence of node pair realizations $\mathcal{G}_{t+s}(\omega)$ for $s = 0, 1, \ldots, Z-1$. Without loss of generality we let V_1 contain at least two nodes.

Take $i_2 \neq i_1 \in \mathrm{V}_1$ and let

$$\mathcal{G}_t(\omega) = \{i_1, i_2\}, \mathcal{G}_{t+1}(\omega) = \{j_1, i_1\}, \mathcal{G}_{t+2}(\omega) = \{j_1, i_2\}. \tag{63}$$

Now we investigate the outcome of the above pair selection process. Since $i_1, i_2 \in \mathrm{V}_1$, they share a positive link whose interaction is defined by (5). Consequently, we conclude from $x_{i_1}(t) = -A$ and $\alpha \in (0,1/2)$ that

$$x_{i_1}(t+1)(\omega) \leq 0,\ x_{i_2}(t+1)(\omega) \leq (1-2\alpha)A. \tag{64}$$

Further, since $\beta \geq 1/\alpha \geq 2$ and $x_{j_1}(t) = A$, with the chosen $\mathcal{G}_{t+1}(\omega)$ we have

$$x_{i_2}(t+2)(\omega) \leq (1-2\alpha)A,\ x_{i_1}(t+2)(\omega) = -A,\ x_{j_1}(t+2)(\omega) = A. \tag{65}$$

Finally, noticing the fact that $\beta \geq 1/\alpha$, it holds that

$$x_{i_1}(t+3)(\omega) = -A,\ x_{i_2}(t+3)(\omega) = -A,\ x_{j_1}(t+3)(\omega) = A. \tag{66}$$

Next, we recursively apply the above pair selections for other nodes in V_1 and obtain $x_{j_1}(t+3n_1)(\omega) = A$ and

$$x_i(t+3n_1)(\omega) = -A,\ i \in \mathrm{V}_1, \tag{67}$$

with $n_1 = |\mathrm{V}_1| - 1$.

Finally, we repeat the same pair selection process for nodes in V_2. This will yield

$$x_i(t+3(n-2))(\omega) = -A,\ i \in \mathrm{V}_1; \quad x_i(t+3(n-2))(\omega) = A,\ i \in \mathrm{V}_2. \tag{68}$$

This proves the desired lemma. ∎

We now have the necessary tools in hand for the proof of Theorem 9. By Lemma 2, there are two nodes i_* and j_* such that with probability one,

$$\limsup_{t\to\infty} \big|x_{i_*}(t) - x_{j_*}(t)\big| = 2A. \tag{69}$$

We define

$$\mathcal{T}_1 : \inf_{t\geq 0} \big|x_{i_*}(t) - x_{j_*}(t)\big| \geq A$$

and then recursively define

$$\mathcal{T}_{m+1} : \inf_{t\geq \mathcal{T}_m+1} \big|x_{i_*}(t) - x_{j_*}(t)\big| \geq A$$

for $m = 2, 3, \dots$. Evidently, they form a sequence of stopping times [17] in the random node pair process $\mathcal{G}_t, t = 0, 1, \dots$. From the fact that (69) holds with probability one, $\mathcal{T}_m$ is almost surely finite for any $m = 1, 2, \dots$.

There will be two cases.

C1. Let i_* and j_* belong to different subgroups, say, $i_* \in \mathrm{V}_1$ and $j_* \in \mathrm{V}_2$. Then by selecting $\{i_*, j_*\}$ at time $\mathcal{T}_m$, we have

$$x_{i_1}(\mathcal{T}_m + 1) = -A,\ x_{j_1}(\mathcal{T}_m + 1) = A, \tag{70}$$

where i_1 and j_1 are from the set $\{i_*, j_*\}$ sharing a negative link. Let $i_1 \in \mathrm{V}^{i_1}$ and $i_2 \in \mathrm{V}^{i_2}$, where each V^{i_1} and V^{i_2} is either V_1 or V_2. Then Lemma 3 suggests from (70) that

$$\begin{aligned} &\mathbb{P}\Big(x_i\big(\mathcal{T}_m + Z + 1\big) = -A,\ i \in \mathrm{V}^{i_1};\ x_i\big(\mathcal{T}_m + Z + 1\big) = A,\ i \in \mathrm{V}^{i_2}\Big) \\ &\geq \big(\min_{\{i,j\}\in \mathrm{E}} p_{ij}\big)^{Z+1}. \end{aligned} \tag{71}$$

Note that, since the $\mathcal{T}_m$ are stopping times of $\mathcal{G}_t, t = 0, 1, \dots$, by the strong Markov property we can invoke the second Borel–Cantelli Lemma (e.g., Theorem 2.3.6 in [17]) to conclude from (71) that almost surely there is $m_0 \in \mathbb{Z}^+$ such that

$$x_i\big(\mathcal{T}_{m_0} + Z + 1\big) = -A,\ i \in \mathrm{V}^{i_1};\ x_i\big(\mathcal{T}_{m_0} + Z + 1\big) = A,\ i \in \mathrm{V}^{i_2},$$

and therefore

$$x_i(t) = -A,\ i \in \mathrm{V}^{i_1};\ x_i(t) = A,\ i \in \mathrm{V}^{i_2}$$

for all $t \geq \mathcal{T}_{m_0} + Z + 1$ from the structure of the dynamics.

C2. Let i_* and j_* belong to the same subgroup, say, V_1. There must be another node $k_* \in \mathrm{V}_2$ such that either $\big|x_{i_*}(\mathcal{T}_m) - x_{k_*}(\mathcal{T}_m)\big| \geq A/2$ or $\big|x_{j_*}(\mathcal{T}_m) - x_{k_*}(\mathcal{T}_m)\big| \geq A/2$. No matter which case holds, by selecting the corresponding pair $\{i_*, k_*\}$ or $\{j_*, k_*\}$ for time $\mathcal{T}_m$, we obtain two nodes i_1 ($= i_*$ or j_*) and j_1 ($= k_*$) so that

$$x_{i_1}(\mathcal{T}_m + 1) = -A,\ x_{j_1}(\mathcal{T}_m + 1) = A. \tag{72}$$

Consequently, this case also ends up with condition (70) and therefore the rest of the treatment remains the same.

We have now completed the proof of Theorem 9. ∎

Appendix G. Proof of Theorem 10. Following Lemma 3, another lemma can be established.

LEMMA 4. *Fix $\alpha \in (0, 1/2)$ and $\beta \geq 1/\alpha$. Consider the dynamics (26) with the random pair selection process. Let G be a weakly structurally balanced complete graph under the partition $\mathrm{V} = \mathrm{V}_1 \cup \mathrm{V}_2 \cdots \cup \mathrm{V}_m$ for $m \geq 2$. If there are $i_1 \in \mathrm{V}_1$, $j_1 \in \mathrm{V}_2$, and $t \geq 0$ with $x_{i_1}(t) = -A$ and $x_{j_1}(t) = A$, then for $Z = 3n - 2m - 2$, there exists a sequence of node pair realizations, $\mathcal{G}_{t+s}(\omega)$ for $s = 0, 1, \ldots, Z-1$, under which it holds that*

$$\begin{aligned}
&x_i(t+Z)(\omega) = -A, \quad i \in \mathrm{V}_1;\\
&x_i(t+Z)(\omega) = A, \quad i \in \mathrm{V}_2;\\
&x_i(t+Z)(\omega) = \mathcal{I}_0 A, \quad i \in \mathrm{V}_m, m \geq 3,
\end{aligned}$$

where $\mathcal{I}_0$ takes its value from $\{-1, 1\}$ relying on $\mathbf{x}(t)$.

Proof. First of all we apply the node pair selection process in the proof of Lemma 3 and find with $Z_1 = 3(|\mathrm{V}_1| + |\mathrm{V}_2| - 2)$ that

$$x_i(t+Z_1)(\omega) = -A,\ i \in \mathrm{V}_1; \qquad x_i(t+Z_1)(\omega) = A,\ i \in \mathrm{V}_2. \tag{73}$$

Now take $k_1 \in \mathrm{V}_3$. Either $x_{k_1}(t) = x_{k_1}(t+Z_1) < 0$ or $x_{k_1}(t) = x_{k_1}(t+Z_1) \geq 0$ must hold. If $x_{k_1}(t+Z_1) < 0$, then letting $\mathcal{G}_{t+Z_1} = \{k_1, j_1\}$ we have $x_{k_1}(t+Z_1+1) = -A,\ x_{j_1}(t+Z_1+1) = A$. Applying the proof of Lemma 3 to V_3, there is a sequence of node pairs leading to

$$x_i(t + Z_1 + 3|\mathrm{V}_3| - 2) = -A,\ i \in \mathrm{V}_3.$$

Similarly, the other case with $x_{k_1}(t) = x_{k_1}(t+Z_1) \geq 0$ leads to

$$x_i(t + Z_1 + 3|\mathrm{V}_3| - 2) = A,\ i \in \mathrm{V}_3.$$

The process can be carried out recursively to the rest of the nodes. The whole process counts $3(n-m) + m - 2 = 3n - 2m - 2$ node pairs. The desired conclusion holds. ∎

The same argument based on stopping times of $\mathcal{G}_t$ and the second Borel–Cantelli Lemma in the proof of Theorem 9 can now be applied to the weakly structurally balanced case with the help of Lemma 4, and then Theorem 10 holds. ∎

Appendix H. Proof of Theorem 11. The proof is based on the following lemma.

LEMMA 5. *Fix $\alpha \in (1/2, 1)$ and $\beta \geq 2/(2\alpha - 1)$. Consider the dynamics (26) with the random pair selection process. Let G be the complete graph with $\kappa(\mathrm{G}^+) \geq 2$. Suppose for time t there are $i_1, j_1 \in \mathrm{V}$ with $x_{i_1}(t) = -A$ and $x_{j_1}(t) = A$. Then for any $\epsilon \in [0, (2\alpha - 1)A/2\alpha]$ and any $i_\star \in \mathrm{V}$, the following statements hold.*

(i) *There exist an integer $Z(\epsilon)$ and a sequence of node pair realizations, $\mathcal{G}_{t+s}(\omega)$ for $s = 0, 1, \ldots, Z-1$, under which $x_{i_\star}(t+Z)(\omega) \leq -A + \epsilon$.*

(ii) *There exist an integer $Z(\epsilon)$ and a sequence of node pair realizations, $\mathcal{G}_{t+s}(\omega)$ for $s = 0, 1, \ldots, Z-1$, under which $x_{i_\star}(t+Z)(\omega) \geq A - \epsilon$.*

Proof. From our standing assumption, the negative graph G^- contains at least one edge. Let $k_*, m_* \in \mathrm{V}$ share a negative link. We assume the two nodes $i_1, j_1 \in \mathrm{V}$

defined in the lemma are different from k_*, m_*, for ease of presentation. We can then analyze all possible sign patterns among the four nodes i_1, j_1, k_*, m_*. We present here just the analysis for the case with

$$\{i_1, k_*\} \in \mathrm{E}^+, \quad \{i_1, m_*\} \in \mathrm{E}^+, \quad \{j_1, k_*\} \in \mathrm{E}^+, \quad \{j_1, m_*\} \in \mathrm{E}^+.$$

The other cases are indeed simpler and can be studied via similar techniques.

Without loss of generality we let $x_{m_*}(t) \geq x_{k_*}(t)$. First of all we select $\mathcal{G}_t = \{i_1, k_*\}$ and $\mathcal{G}_{t+1} = \{j_1, m_*\}$. It is then straightforward to verify that

$$x_{m_*}(t+2) \geq x_{k_*}(t+2) + 2\alpha A.$$

By selecting $\mathcal{G}_{t+2} = \{m_*, k_*\}$ we know from $\beta \geq 2/(2\alpha - 1) \geq 1/\alpha$ that

$$x_{k_*}(t+3) = -A, \quad x_{m_*}(t+3) = A.$$

There will be two cases.

(a) Let $i_\star \notin \{m_*, k_*\}$. Noting that $\kappa(\mathrm{G}^+) \geq 2$, there will be a path connecting to k_* from $i_\star$ without passing through m_* in G^+. It is then obvious that we can select a finite number Z_1 of links which alternate between $\{m_*, k_*\}$ and the edges over that path so that $x_{i_\star}(t+3+Z_1) \geq A - \epsilon$. Here Z_1 depends only on α and n. Similarly, there is also a path connecting to m_* from $i_\star$ without passing through k_* in G^+, and based on this we can select realizations of node pairs guaranteeing $x_{i_\star}(t+3+Z_1) \leq -A + \epsilon$.

(b) Let $i_\star \in \{m_*, k_*\}$. We only need to show that we can select pair realizations so that x_{m_*} can get close to $-A$, and x_{k_*} gets close to A after $t+3$. Since G^+ is connected, either m_* or k_* has at least one positive neighbor. For the moment assume m' is a positive neighbor of m_* and k' is a positive neighbor of k_* with $m' \neq k'$. Then from part (a) we can select Z_2 node pairs so that

$$x_{m'}(t+3+Z_2) \leq -A + \epsilon, \; x_{k'}(t+3+Z_2) \geq A - \epsilon.$$

Thus, selecting $\{m', m_*\}$ and $\{k', k_*\}$ for the next two time instances leads to

$$\begin{aligned} x_{m_*}(t+5+Z_2) &\leq (1-2\alpha)A + \alpha\epsilon \leq (1-2\alpha)A/2, \\ x_{k_*}(t+5+Z_2) &\geq (2\alpha-1)A - \alpha\epsilon \geq (2\alpha-1)A/2. \end{aligned}$$

Selecting the negative edge $\{m_*, k_*\}$ for $t+5+Z_2$ implies $x_{m_*}(t+6+Z_2) = -A, \; x_{k_*}(t+6+Z_2) = A$ for $\beta \geq 2/(2\alpha-1)$. The case with $m' = k'$ can be dealt with by a similar treatment, leading to the same conclusion.

This concludes the proof of the lemma. ∎

In view of Lemmas 2 and 5, the desired theorem is, again, a consequence of the second Borel–Cantelli Lemma. ∎

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