2019 58th Annual Conference of the Society of Instrument and Control Engineers of Japan (SICE)
September 10-13, 2019, Hiroshima, Japan

Topological Controllability of Undirected Networks of Diffusively-Coupled Agents

Hyo-Sung Ahn[1,2†], Kevin L. Moore[2], Seong-Ho Kwon[1], Quoc Van Tran[1], Byeong-Yeon Kim[3], and Kwang-Kyo Oh[4]

[1]School of Mechanical Engineering, Gwangju Institute of Science and Technology (GIST), Gwangju, Korea. E-mails: `hyosung@gist.ac.kr; seongho@gist.ac.kr; tranvanquoc@gist.ac.kr`
[2]Department of Electrical Engineering, Colorado School of Mines, Golden, CO, USA. E-mails: `kmoore@mines.edu; hahn@mines.edu`
[3] Korea Atomic Energy Research Institute, Daejeon, Korea. E-mail: `tktrktna12@gmail.com`
[4] Department of Electrical Engineering, Sunchon National University, Sunchon, Jeollanam-do, Korea. E-mail: `kwangkyo.oh@gmail.com`

Abstract: This paper presents conditions for establishing topological controllability in undirected networks of diffusively-coupled agents. Specifically, controllability is considered based on the signs of the edges (negative, positive or zero). Our approach differs from well-known structural controllability conditions for linear systems or consensus networks, where controllability conditions are based on edge connectivity (i.e., zero or non-zero edges). Our results first provide a process for merging controllable graphs into a larger controllable graph. Then, based on this process, we provide a graph decomposition process for evaluating the topological controllability of a given network.

Keywords: Diffusive networks, Topological controllability, Structural controllability, Merging process, Decomposition process.

1. INTRODUCTION

This paper studies the controllability of a class of network systems using only knowledge of the sign-connectivity between nodes, without relying upon knowledge of the magnitude of the connections. By *sign-connectivity*, we mean that the knowledge of signs of edges can be known; but the magnitude of the edges are not known. Since the magnitude of the edges weights are not used, it is not an algebraic approach, but rather we call it a topological approach. We call such a topological analysis of the controllability of a network *topological controllability*. Topological controllability has also been studied under the name of *structural controllability* [1], although they do not consider the signs of the edges. In traditional controllability of a network or in linear systems theory [2], both the topology and coupling strengths (i.e., magnitude of the connections) between nodes are taken into account. Thus, in traditional approaches, it is a topological and algebraic solution. However, in certain networks, it may be hard to know the coupling strengths between nodes or there may be uncertainties in the measure or identification of the coupling strengths. Consequently, if the controllability of a network can be evaluated only using the topology or structure, it will be beneficial in some applications, including control of digital and electric circuits [3, 4], power electronics [5], large scale networks [6], complex networks [7, 8], and brain networks [9].

From a review of the literature, we could find many interesting analyses and concepts related to structural controllability or topological controllability. The pioneering analysis was conducted in [1], which introduced the concepts of stem, dilations, bud, origin, cactus, and accessibility. These concepts were used for merging basic controllable graphs into a bigger graph. Since [1], there has been a lot of research on the controllability of network systems. In [10], the concept of maximum matching was introduced to find matched nodes from inputs. The matched nodes are elements of paths. Then, it was argued that unmatched nodes need to be controlled directly by control inputs. In [11], a minimum control structure, i.e., the minimum number of independent control inputs, was further examined on the basis of number of source nodes, and external/internal dilation nodes. Under Laplacian dynamics, input symmetry was characterized for making a network uncontrollable in [12]. We also note that there have been many studies on structural controllability for specific types of dynamic systems, including switching networks [13], high-order dynamic systems [14], random networks [15], and descriptor systems [16]. However, in most of the literature, only the connectivity between nodes are considered. That is, in most of literature, a value of an edge is given as zero or non-zero. But, in many network systems the signs of edge (i.e., positive or negative) are quite critical for evaluating the convergence or stabilization of the overall network. For example, in diffusive coupling networks, the positive edge means a cooperative coupling, while a negative edge means a negative coupling. In some systems (e.g., social networks) there can exist edges with different signs, as some agents are cooperative and others are antagonistic. For such systems it is critical to know the signs of the edge values. Motivated from this observation, in this paper, we assume that edges of a network are classified as

† Hyo-Sung Ahn is the presenter of this paper.

negative, positive, or zero and with only this knowledge we provide conditions for the topological controllability of the network system (for more motivations and advantage of using signs of edges, see *Remark 3*). To deliver our ideas in a simple way, we consider the specific case of diffusively-coupled agents connected as an undirected network, although the techniques developed in this paper can be extended to directed and general networks.

In the sequel there are two main results. First, we interpret the results of [17] in the sense of a graph. We then present some conditions for merging subgraphs under the condition of topological controllability. Then, by the merging rules, we can gradually enlarge the network while keeping the topological controllability. However, this merging process does not provide a direct method for evaluating a topological controllability of a given network. Thus, as the second goal of this paper, we present some ideas to decompose a graph into subgraphs, which are path graphs. Then, starting again from the decomposed subgraphs, we gradually again add the edges to merge the subgraphs under the topological controllability condition. By this way, we can find a largest subgraph, which can be called a subgraph induced by the controllability. This allows us to develop an algorithm for testing the topological controllability of a given network.

2. PRELIMINARIES AND PROBLEM FORMULATIONS

Let an undirected network of diffusively-coupled agents x_i with direct nodes inputs u_i be given by:

$$\dot{x}_i = -\sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j) + b_i u_i \tag{1}$$

where $\mathcal{N}_i$ is the set of neighboring nodes of i, and a_{ij} are diffusive couplings and b_i are input couplings. We define the network $T = [L, B]$ concisely as the Laplacian dynamics:

$$\dot{x} = Lx + Bu \tag{2}$$

where $x = (x_1, \ldots, x_n)^T$, $u = (u_1, \ldots, u_m)^T$, $L \in \mathbb{R}^{n\times n}$ is a Laplacian matrix with possible negative edges, and $B \in \mathbb{R}^{n\times m}$ is the input coupling matrix. The Laplacian matrix L is a matrix defined by the interactions of n state nodes and the matrix B defines input couplings from m input nodes to state nodes. So, there are $n+m$ nodes in the network. The interactions among state nodes are undirected (thus L is a symmetric row- and column-stochastic matrix) while the interactions from the input nodes to state nodes are directed. It is also assumed that each input node is connected to only one state node by one-to-one mapping (injective).

Definition 1: Controllability: An undirected network $T = [L, B]$ of diffusively-coupled agents with directed input nodes given by 2 is said to be controllable if there exists an input vector $u(t)$ such that $x(t) \to x^*$ for any desired vector x^*.

Definition 2: Topological controllability: A controllable undirected network $T = [L, B]$ of diffusively-coupled agents with directed input nodes given by 2 is said to be topologically controllable if all other undirected networks $\bar{T} = [\bar{L}, \bar{B}]$ whose edges have the same signs (positive, negative, or zero) as $T = [L, B]$ are also controllable.

To characterize the topological controllability of an undirected network of diffusively-coupled agents, we borrow the analysis given in [17]. Thus, this paper is a kind of an interpretation of the analaysis of [17]. Let the network can be re-defined as a *graph*, denoted

$$\mathcal{G}(T) = (\mathcal{V}, \mathcal{E}) \tag{3}$$

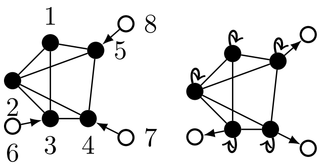
where $T = [L, B]$, the set of vertices $\mathcal{V}$ is the set of indices of nodes as $\mathcal{V} = \{\underbrace{1, \ldots, n}_{\text{state nodes}=\mathcal{V}^{\mathcal{S}}}, \underbrace{n+1, \ldots, n+m}_{\text{input nodes}=\mathcal{V}^{\mathcal{I}}}\}$, and the set of edges $\mathcal{E}$ is determined from the interaction characteristics between nodes. Fig. 1 depicts a network and a graph. It is necessary to distinguish the concepts of *network* and *graph*. The network is a relationship of physical interactions among nodes, while the graph is a representation of the network as a set of vertices and edges. To illustrate, consider a network depicted in Fig. 1(a). With some edge weightings, for example, let the Laplacian matrix corresponding to the network in Fig. 1(a) be given as:

$$L = \begin{bmatrix} -2 & 2 & 1 & 0 & -1 \\ 2 & -3 & 1 & 1 & -1 \\ 1 & 1 & -3 & 1 & 0 \\ 0 & 1 & 1 & -5 & 3 \\ -1 & -1 & 0 & 3 & -1 \end{bmatrix} \tag{4}$$

and the input coupling matrix B be given as:

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Then, the interaction characteristics of a graph, which is a representation of a network, are decided by the matrices L and B. That is, given $T = [t_{ij}] = [L, B]$, if $t_{ij} \neq 0$, then there exists an edge (i, j), which is the directed edge from i to j. For undirected edges (i.e., when $i, j \in \{1, \ldots, n\}$), if there exists (i, j) in $\mathcal{E}$, then there also exists (j, i). If $t_{ii} \neq 0$, $i \in \{1, \ldots, n\}$, then there exists a self-loop at node i. We assume there is no edge between the input nodes. In the graph, there are edges from $\mathcal{V}^{\mathcal{S}}$ to $\mathcal{V}^{\mathcal{I}}$ as $(i, j) \in \mathcal{E}$ where $i \in \mathcal{V}^{\mathcal{S}}$ and $j \in \mathcal{V}^{I}$. The edge (i, j) from a state node to an input node in the graph implies that the node i is influenced by j. For a node i, if there exists an edge (i, j), then j is a neighboring node (the set of neighboring nodes of node i is denoted as $\mathcal{N}_i$) in the graph $\mathcal{G}$, i.e., $j \in \mathcal{N}_i$. Fig. 1(b) depicts a graph, which is a representation of the network in Fig. 1(a). The edge directions in the graph and the network are reversed. It is shown that $\mathcal{V}^{S} = \{1, 2, 3, 4, 5\}$



(a)Network (b)Graph

Fig. 1 (a) A network with five state nodes and three input nodes. (b) Graph representation of the network.

and $\mathcal{V}^I = \{6, 7, 8\}$, and the edges from $\mathcal{V}^\mathcal{S}$ to $\mathcal{V}^\mathcal{I}$ are $(3, 6), (4, 7), (5, 8)$. For a set α, which is a subset of $\mathcal{V}$ (i.e., $\alpha \subseteq \mathcal{V}$), the set of neighboring nodes of the set α is defined as $\mathcal{N}(\alpha) = \cup \mathcal{N}_i,\ \forall i \in \alpha$.

The graph $\mathcal{G}$ can be decomposed as $\mathcal{G} = \mathcal{G}^S \cup \mathcal{G}^I$, where $\mathcal{G}^S$ is the induced subgraph by $\mathcal{V}^S$, and $\mathcal{G}^I$ is the interaction graph between the set of vertices $\mathcal{V}^\mathcal{S}$ and set of vertices $\mathcal{V}^\mathcal{I}$. Thus, $\mathcal{G}^\mathcal{S} = (\mathcal{V}^\mathcal{S}, \mathcal{E}^\mathcal{S})$ and $\mathcal{G}^\mathcal{I} = (\mathcal{V}, \overrightarrow{\mathcal{E}}^\mathcal{I})$, where $\overrightarrow{\mathcal{E}}^I$ is the set of directed edges. Note that $\mathcal{E} = \mathcal{E}^\mathcal{S} \cup \overrightarrow{\mathcal{E}}^\mathcal{I}$. For Fig. 1, the matrix T is a $5 \times (5+3)$ matrix, i.e., $T \in \mathbb{R}^{5\times(5+3)}$.

Next, we say that any matrix with the same sign as T is contained in the set of sign pattern matrices $Q(T)$. So, any matrix $T' \in Q(T)$ has the same sign as T in an elementwise fashion. We also say that if the row vectors of $T', \forall T' \in Q(T)$ are linearly independent, then the matrix T is called an L-matrix. From the perspective of control system design, since the matrix B can be designed, we assume that the input coupling matrix B is fixed, while the Laplacian matrix is a sign pattern matrix. Thus, $Q(T)$ is defined as

$$Q(T) := [Q(L), B] \tag{6}$$

The matrix $T = [L, B]$ is called nominal graph matrix and $Q(T)$ is called a family of sign pattern matrices. It is certain that $\text{rank}(T) = n$ if and only if the row vectors are linearly independent. The following assumptions are necessary for simplicity.

Assumption 1: The values of off-diagonal elements of L may change; but their signs do not change (i.e., sign fixed). The diagonal elements, $l_{ii} = -\sum_{j\in\mathcal{N}_i} a_{ij}$ where a_{ij} are edge weights of the network, of L are non-zero and also sign fixed.
This assumption means that the sign of the summation of incident edge weights does not vary, even though each edge weight does vary under the same sign.

Assumption 2: Given a nominal graph matrix $T = [L, B]$, the Laplacian dynamics (2) is controllable.

Assumption 3: For any $T' \in Q(T)$, the row vectors of T' are linearly independent.
It is clear that these assumptions are necessary conditions for ensuring controllability for all $T' \in Q(T)$. In [17], *Assumption 2* is required to ensure *accessibility*[1] of the graph $\mathcal{G}$. If there is no path connecting an input node to a state node, the state is not controllable. The *Assumption 3* means that the matrix $T = [L, B]$ is an L-matrix. *Assumption 2* and *Assumption 3* are basic requirements for ensuring the topological controllability of a graph.

Remark 1: To guarantee the L-matrixness of T, one idea is to design the matrix B. For example, from the relationship:

$$\begin{aligned}\text{rank}[T] &= \text{rank}[L, B] \\ &= \text{rank}(L) + \text{rank}(B) \ \text{ if } \ \mathbf{R}(L) \cap \mathbf{R}(B) = \emptyset\end{aligned} \tag{7}$$

where $\mathbf{R}(\cdot)$ is the range of the matrix $\cdot$, if $\text{rank}(L) = n - d$, it is required to design B such that $\text{rank}(B) = d$ with the property $\mathbf{R}(L) \cap \mathbf{R}(B) = \emptyset$.

With the above assumptions, the following theorem for the topological controllability of a graph is given in [17] as a sufficient condition.

Theorem 1: [17] Let us suppose that *Assumption 1*, *Assumption 2*, and *Assumption 3* are satisfied. Then, for all $\alpha \subseteq \mathcal{V}^\mathcal{S}$ satisfying $\alpha \subset \mathcal{N}(\alpha)$ in $\mathcal{G}$, if there exists at least one $j \in \mathcal{N}(\alpha) \setminus \alpha$ and there exists exactly one $i \in \alpha$ such that $(i, j) \in \mathcal{E}$ exists, then the graph $\mathcal{G}(T)$ determined from $T = [L, B]$ is topologically controllable.

3. TOPOLOGICALLY CONTROLLABLE GRAPHS

This section is dedicated to an elaboration of the condition of *Theorem 1*. The condition of *Theorem 1* can be modified from an algorithm perspective as:

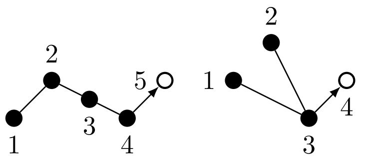
Corollary 1: Under the same conditions as *Theorem 1*, $\forall \alpha \subseteq \mathcal{V}^\mathcal{S}$ satisfying $\alpha \subset \mathcal{N}(\alpha)$, if there exists $i \in \alpha$ such that $\mathcal{N}_i \cap (\mathcal{N}(\alpha) \setminus \alpha) \neq \emptyset$ and $\{\mathcal{N}_i \cap (\mathcal{N}(\alpha) \setminus \alpha)\} \setminus \{\mathcal{N}_j \cap (\mathcal{N}(\alpha) \setminus \alpha), \forall j \in \alpha \setminus \{i\}\} \neq \emptyset$, then the graph $\mathcal{G}(\mathcal{T})$ determined from $T = [L, B]$ is topologically controllable.
It is relatively easy to check the statement of *Corollary 1*, since we examine $i \in \alpha$ rather than $j \in \mathcal{N}(\alpha) \setminus \alpha$. It means that if there exists $i \in \alpha$, which is connected to $j \in \mathcal{N}(\alpha) \setminus \alpha$ and j is not connected to other nodes in α, the statement of *Corollary 1* is satisfied. Such a node, i.e., node j, is called a *dedicated node* to i. Consequently, for any $i \in \alpha$ (at least one i, i.e, $\exists i \in \alpha$), if there exists a dedicated node $j \in \mathcal{N}(\alpha) \setminus \alpha$, then the grouping α is considered to satisfy the statement. We call a graph topologically controllable if the condition of *Corollary 1* is satisfied.

Remark 2: A sufficient condition for satisfying the condition of *Corollary 1* is that there exists $i \in \alpha$ such that $\mathcal{N}_i \cap (\mathcal{N}(\alpha) \setminus \alpha) \neq \emptyset$ and $\{\mathcal{N}_i \cap (\mathcal{N}(\alpha) \setminus \alpha)\} \cap \{\mathcal{N}_j \cap (\mathcal{N}(\alpha) \setminus \alpha), \forall j \in \alpha \setminus \{i\}\} = \emptyset$.

Lemma 1: Let us suppose that any diagonal element of L is not identically zero. Then, under the undirected interactions in $\mathcal{V}^\mathcal{S}$ and directed interactions between $\mathcal{V}^\mathcal{S}$ and $\mathcal{V}^\mathcal{I}$, for any choice α, it is true that $\alpha \subset \mathcal{N}(\alpha)$.

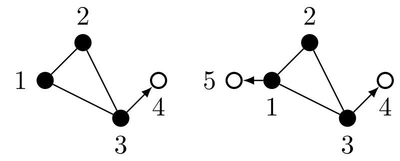
Proof: For any L', $L' \in Q(L)$, since the diagonal elements are non-zero, all the state nodes have self-loops. Thus, each state node has at least two neighbor-

[1] Accessibility means that for any $i \in \mathcal{V}^\mathcal{S}$, there is a path from i to $j \in \mathcal{V}^\mathcal{I}$ in the graph $\mathcal{G}$.



(a)Path graph (b)Tree graph

Fig. 2 Graphs without cycle (for a simplicity, the self-loops are omitted in the figure).



(a)Cycle with one input (b)Cycle with two inputs

Fig. 3 Graphs with three state nodes (cycle) and one input node (Left), or two input nodes (Right).

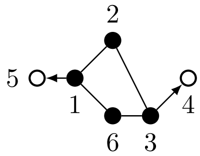


Fig. 4 A graph with four state nodes (cycle) and two input node.

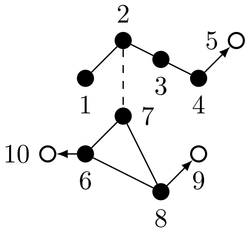


Fig. 5 A graph merged by two topologically controllable graphs.

ing nodes including itself, if the underlying graph is connected. Also, when $\alpha = \mathcal{V}^{\mathcal{S}}$, the neighboring set $\mathcal{N}(\alpha)$ includes all the nodes in $\mathcal{V}^{\mathcal{S}}$ and at least one node in $\mathcal{V}^{\mathcal{I}}$. Thus, $\alpha \subset \mathcal{N}(\alpha)$. ■

The above lemma shows that we need to check whether each α, for all $\alpha \subseteq \mathcal{V}^{\mathcal{S}}$, would satisfy the condition of *Corollary 1*. For example, let $\mathcal{G}^{\mathcal{S}}$ be a path graph, or a tree graph, with an input at terminal node. Fig. 2(a) shows a path graph. In this case, whatever taking α, it is clear that $\{\mathcal{N}_i \cap (\mathcal{N}(\alpha) \setminus \alpha)\} \cap \{\mathcal{N}_j \cap (\mathcal{N}(\alpha) \setminus \alpha)\} = \emptyset$ for any $i, j \in \alpha$. That is, $\exists i \in \alpha$ and $j, \forall j \in \alpha \setminus \{i\}$ such that $\mathcal{N}_i \cap (\mathcal{N}(\alpha) \setminus \alpha) \neq \emptyset$. Consequently, a path graph is topologically controllable, which is coincident with the result in [10]. Fig. 2(b) shows a tree graph. The node 3 is devided into two paths, i.e., $3 \leftrightarrow 1$ and $3 \leftrightarrow 2$, where the symbol $\leftrightarrow$ is used to denote the connection in the undirected path. In this tree, if we take $\alpha = \{1, 2\}$, then the nodes 1 and 2 share a common neighboring node 3, and they do not have any dedicated node. Thus, in general, a tree graph with a single input node is not topologically controllable.

Let $\mathcal{G}$ be a undirected cycle graph. Then, the condition is also not satisfied, without properly located input nodes. The graphs depicted in Fig. 3 include an undirected cycle. For Fig. 3(a), when choosing $\alpha = \{1, 2\}$, the nodes 1 and 2 share 3 as the common node in $\mathcal{N}(\alpha) \setminus \alpha$. So, it does not satisfy the condition. For Fig. 3(b), we have two input nodes. When choosing $\alpha = \{1, 2\}$, the nodes 1 and 2 share 3 as the common node in $\mathcal{N}(\alpha) \setminus \alpha$; but the node 1 has a dedicated node 5. In more detail, when choosing $\alpha = \{1, 2\}$, we obtain $\mathcal{N}(\alpha) \setminus \alpha = \{\ni, \bigtriangledown\}$. For $i = 1$ and $j = 2$, we obtain $\mathcal{N}_i = \{2, 3, 5\}$ and $\mathcal{N}_j = \{1, 3\}$. Then, it follows that $\mathcal{N}_i \cap (\mathcal{N}(\alpha) \setminus \alpha) \neq \emptyset$ and $\{\mathcal{N}_i \cap (\mathcal{N}(\alpha) \setminus \alpha)\} \setminus \{\mathcal{N}_j \cap (\mathcal{N}(\alpha) \setminus \alpha), \forall j \in \alpha \setminus \{i\}\} = \{5\} \neq \emptyset$. Likewise, we can see that, for $\alpha = \{1\}$, $\alpha = \{2\}$, $\alpha = \{3\}$, $\alpha = \{2, 3\}$, $\alpha = \{1, 3\}$, and $\alpha = \{1, 2, 3\}$, there is at least one dedicated node. Thus, the graph in Fig. 3(b) satisfies the condition. However, when a node is added between the nodes 1 and 3, as shown in Fig. 4, the graph does not satisfy the condition, i.e., if we choose $\alpha = \{2, 6\}$, then the nodes 2 and 6 share 1 as a commone node and 3 also as a common node, i.e., there is no dedicated node for 2 or for 6. It is remarkable that a directed cycle, with the same directions, satisfies the controllability condition since whatever choosing α, there is a dedicated node for at least one $i \in \alpha$ (such a directed cycle is called bud in [1]).

As analyzed in the above examples, it is hard to generate a general rule for the topological controllability. It is observed that the graph in Fig. 3(b) is a merged graph of two paths $5 \leftarrow 1 \leftrightarrow 2$ and $4 \leftarrow 3$ where the symbol $\leftarrow$ is used to denote a connection in directed connection in a path. Also, the graph in Fig. 4 is a merged graph of two paths $5 \leftarrow 1 \leftrightarrow 2$ and $4 \leftarrow 3 \leftrightarrow 6$. The graph in Fig. 3(b) is topologically controllable, while the graph in Fig. 4 is not topologically controllable. If we can generate a graph by merging simple graphs, we may obtain some general rules.

Lemma 2: Let there be two disconnected topologically controllable graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. If a state node i in $\mathcal{G}_1$ and a state node j in $\mathcal{G}_2$ are connected by an undirected edge, then the merged graph $G = \mathcal{G}_1 \cup \mathcal{G}_2$ is topologically controllable.

Proof: When $\alpha = \{i, j\}$, $i \in G_1$ and $j \in G_2$, it is true that $\{\mathcal{N}_i \cap (\mathcal{N}(\alpha) \setminus \alpha)\} \cap \{\mathcal{N}_j \cap (\mathcal{N}(\alpha) \setminus \alpha)\} = \emptyset$ since they do not share a common neighbor. Also all the nodes other than i in $\mathcal{G}_1$ and all the nodes other than j in $\mathcal{G}_2$ do not have a common neighbor node (for example, as shown in Fig. 5, the nodes 2 and 7 do not have a common neighbor node).

Let us choose arbitrary $\alpha \subseteq \mathcal{G}$, where $\alpha = \alpha_1 \cup \alpha_2$, and $\alpha_1 \subseteq \mathcal{G}_1$ and $\alpha_2 \subseteq \mathcal{G}_2$. When we choose $\alpha = \alpha_1$ or $\alpha = \alpha_2$, for any $i \in \alpha$, there is at least one dedicated node $j \in \mathcal{N}(\alpha) \setminus \alpha$ since $\mathcal{G}_1$ and $\mathcal{G}_2$ are topologically controllable. In the case there exist i and j such that $i, j \in \alpha$, and $i \in \alpha_1$ and $j \in \alpha_2$, there is still no chance of having $\{\mathcal{N}_i \cap (\mathcal{N}(\alpha) \setminus \alpha)\} \setminus \{\mathcal{N}_j \cap (\mathcal{N}(\alpha) \setminus \alpha), \forall i, j\} = \emptyset$. Moreover, for all $\alpha_1 \subset \alpha$, and for all $\alpha_2 \subset \alpha$, it

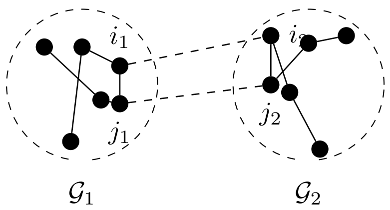


Fig. 6 A topologically controllable graph merged by two topologically controllable graphs with two edges.

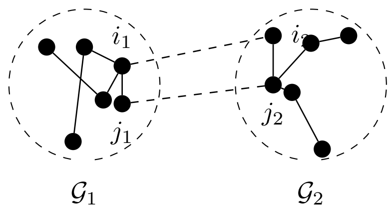


Fig. 7 Not topologically controllable graph when merged by two topologically controllable graphs with two edges.

is certain that either in α_1 or in α_2, there is a node that has a dedicated node in $\mathcal{N}(\alpha_1) \setminus \alpha_1$ or in $\mathcal{N}(\alpha_2) \setminus \alpha_2$, respectively. Thus, the merged graph $\mathcal{G}$ is topologically controllable. ■

With the above lemma, the following corollary is directly obtained.

Corollary 2: Let there be two disconnected path graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. If a state node i in $\mathcal{G}_1$ and a state node j in $\mathcal{G}_2$ are connected by an undirected edge, then the merged graph $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ is topologically controllable.

Remark 3: In structural controllability, a path with an input node (called stem) and directed cycle with the same direction with an input node (called bud) are basic controllable elements [1]. In maximum mathcing process [10], the key issue is to find paths that are controllable. Similarly to the structural controllability, in topological controllability, the paths are key elements for enlarging the network. However, in our approach, i.e., topological controllability, we are not limited to the paths. The path graph is a special case for controllable graphs. That is, although the path graphs are important for enlarging a graph, as far as the condition of *Corollary 1* is satisfied, any graph can be used as a basic element for controllable graph or for enlarging the network. This superiority, in fact, can be used for merging two controllable graphs in a much general way than the cases in structural controllability, as stated in *Corollary 3*.

The *Lemma 2* may provide an intuition for a more general case for merging two graphs. Next, let us consider a case of merging by connecting two edges.

Lemma 3: Let us consider two disconnected topologically controllable graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. Let the state nodes i_1, j_1 in $\mathcal{G}_1$ and state nodes i_2, j_2 in $\mathcal{G}_2$ be connected by undirected edges as (i_1, i_2) and (j_1, j_2). Then the merged graph $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ is topologically controllable, if α, $\forall \alpha \subseteq \{i_1, i_2, j_1, j_2\}$, has at least one dedicated node in $\mathcal{N}(\alpha) \setminus \alpha$.

Proof: Let us choose arbitrary $\alpha \subseteq \mathcal{G}$, where $\alpha = \alpha_1 \cup \alpha_2$, and $\alpha_1 \subseteq \mathcal{G}_1$ and $\alpha_2 \subseteq \mathcal{G}_2$. When we choose $\alpha = \alpha_1$ or $\alpha = \alpha_2$, for any $i \in \alpha$, there is at least one dedicated node $j \in \mathcal{N}(\alpha) \setminus \alpha$ in $\mathcal{G}_1$ or in $\mathcal{G}_2$.

In the case there exist i and j such that $i, j \in \alpha \subseteq \mathcal{G} \setminus \{i_1, i_2, j_1, j_2\}$, and $i \in \alpha_1 \subset \alpha$ and $j \in \alpha_2 \subset \alpha$, there is still no chance of having $\{\mathcal{N}_i \cap (\mathcal{N}(\alpha) \setminus \alpha)\} \cap \{\mathcal{N}_j \cap (\mathcal{N}(\alpha) \setminus \alpha), \forall i, j\} \neq \emptyset$. Moreover, for all $\alpha_1 \subset \alpha$, and for all $\alpha_2 \subset \alpha$, it is certain that either in α_1 or in α_2, there is a node that has a dedicated node in $\mathcal{N}(\alpha_\infty) \setminus \alpha_\infty$ or in $\mathcal{N}(\alpha_\in) \setminus \alpha_\in$, respectively. Next, let $\alpha = \alpha' \cup \alpha''$, where $\alpha' \subseteq \mathcal{G} \setminus \{i_1, i_2, j_1, j_2\}$ and $\alpha'' \subseteq \{i_1, i_2, j_1, j_2\}$, and $\alpha'' \neq \emptyset$. If $\alpha' \neq \emptyset$, it is clear that α has at least one dedicated node. Otherwise, if $\alpha' = \emptyset$, then it is required that whatever we choose α'', where $\alpha'' \subseteq \{i_1, i_2, j_1, j_2\}$, it needs to have at least one dedicated node, which completes the proof. ■

Fig. 6 depicts a topologically controllable graph produced by merging two topologically controllable graphs with two edges. Whatever $\alpha \subseteq \{i_1, i_2, j_1, j_2\}$, it has at least one dedicated node. However, in the case of Fig. 7, when we choose $\alpha = \{j_1, i_2\}$, these nodes have i_1, j_2 as the common neighbor nodes. Thus, they do not have any dedicated node. Now, with the above lemmas, by induction, we can make the following theorem.

Theorem 2: Let two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ be topologically controllable, respectively. Let q nodes from $\mathcal{G}_1$ (i.e., let them be denoted as $i_1, i_2, \ldots, i_q$) and another q nodes from $\mathcal{G}_2$ (i.e., let them be denoted as $j_1, j_2, \ldots, j_q$) be connected one by one. Then, the merged graph $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ is topologically controllable[2] if and only if α, $\forall \alpha \subseteq \{i_1, \ldots, i_q, j_1, \ldots, j_q\}$, has at least one dedicated node in $\mathcal{N}(\alpha) \setminus \alpha$.

Proof: The *if* condition can be proved by an induction of the proof of *Lemma 3*. For the *only if* condition, let there exist α, $\alpha \subseteq \{i_1, \ldots, i_q, j_1, \ldots, j_q\}$, that does not have a dedicated node. Then, there exists at least one $\alpha \subset \mathcal{G}$, which does not satisfy the condition of *Corollary 1*. ■

The above theorem can be further generalized as:

Corollary 3: Let two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ be topologically controllable, respectively. Let q nodes from $\mathcal{G}_1$ (i.e., let them be denoted as $i_1, i_2, \ldots, i_q$) and p nodes from $\mathcal{G}_2$ (i.e., let them be denoted as $j_1, j_2, \ldots, j_p$), where $p \neq q$, be connected. Then, the merged graph $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ is topologically controllable if and only if α, $\forall \alpha \subseteq \{i_1, \ldots, i_q, j_1, \ldots, j_p\}$, has at least one dedicated node in $\mathcal{N}(\alpha) \setminus \alpha$.

4. TOPOLOGICALLY CONTROLLABILITY OF A GRAPH

In the previous section, we have developed conditions for the topologically controllability when merging two graphs. So, starting from a nominaly controllable graph (ex, a path graph), we can enlarge the graph gradually to make a bigger controllable graph. However, the conditions given in the previous section are not applicable for

[2] Here, achieving the topological controllability is equivalent to satisfying the condition of *Corollary 1*.

checking the topological controllability of a given network. This section provides algorithms for examining the topological controllability of a graph. That is, given a big size graph $\mathcal{G}$, we would like to examine the topological controllability of the graph. It is not computationally feasible to check all $\alpha \subseteq \mathcal{G}$ whether each α would satisfy the condition of *Corollary 1*. We propose an algorithm to solve this issue. Due to the page limitation, the details are omitted. It is recommended to refer to [18].

5. CONCLUSION

This paper has presented conditions to establish the controllability of an undirected networks of diffusively-coupled agents using only the knowledge of the signs of edges, motivated by and based on results in [17]. Because the resulting conditions are computationally-hard, we developed a merging process for creating an enlarged network starting from a basic controllable graph. The merging process was then used to develop a decomposition process for evaluating the topological controllability of a given network. Through numerical simulations, we could verify the effectiveness of the proposed algorithms. However, there are still many open problems. For example, if we could find basic path graphs in the decomposition process in an optimal way (i.e., minimizing the number of nodes that are not included in the final paths), then we may be able to find a more bigger subgraph induced by the controllability[3]. In this paper, we have focused on undirected diffusive-coupled networks, but we believe we can easily extend the results to the directed case. These extensions will be studied in our future research.

ACKNOWLEDGMENT

The work of this paper has been supported by GIST Research Institute (GRI) grant funded by the GIST in 2018 and the National Research Foundation (NRF) of Korea under the grant NRF-2017R1A2B3007034.

REFERENCES

[1] C.-T. Lin, "Structural controllability," *IEEE Trans. Automatic Control*, vol. 19, no. 3, pp. 201–208, 1974.

[2] P. J. Antsaklis and A. N. Michel, *A Linear Systems Primer*. Birkhäuser Basel, 2007.

[3] L. Goldstein, "Controllability/observability analysis of digital circuits," *IEEE Transactions on Circuits and Systems*, vol. 26, no. 9, pp. 685–693, 1979.

[4] X.-Y. Feng and K.-S. Lu, "Structural controllability and reducibility of RLC networks with bipolar transistor," in *Proc. of the 2005 International Conference on Machine Learning and Cybernetics*, 2005, pp. 1015–1020.

[5] T. H. Summers, F. L. Cortesi, and J. Lygeros, "On submodularity and controllability in complex dynamical networks," *IEEE Transactions on Control of Network Systems*, vol. 3, no. 1, pp. 91–101, 2015.

[6] L. Wang, G. Chen, X. Wang, and W. K. S. Tang, "Controllability of networked mimo systems," *Automatica*, vol. 69, no. 7, pp. 405–409, 2016.

[7] L.-Z. Wang, Y.-Z. Chen, W.-X. Wang, and Y.-C. Lai, "Physical controllability of complex networks," *Scientific Reports*, vol. 7, pp. –, Jan. 2017.

[8] C. Shen, Z. Ji, and H. Yu, "The structural controllability of edge dynamics in complex networks," in *Proc. of the 2018 Chinese Control and Decision Conference (CCDC)*, 2018, pp. 5356–5360.

[9] S. Gu, F. Pasqualetti, M. Cieslak, Q. K. Telesford, A. B. Yu, A. E. Kahn, J. D. Medaglia, J. M. Vettel, M. B. Miller, S. T. Grafton, and D. S. Bassett, "Controllability of structural brain networks," *Nature Communications*, vol. 6, pp. –, Oct. 2015.

[10] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, "Controllability of complex networks," *Nature*, vol. 473, pp. 167–173, May, 2011.

[11] J. Ruths and D. Ruths, "Control profiles of complex networks," *Science*, vol. 343, pp. 1373–1375, March, 2014.

[12] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[13] B. Hou, X. Li, and G. Chen, "Structural controllability of temporally switching networks," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 63, no. 10, pp. 1771–1781, 2016.

[14] A. Partovi, H. Lin, and Z. Ji, "Structural controllability of high order dynamic multi-agent systems," in *Proc. of the 2010 IEEE Conference on Robotics, Automation and Mechatronics*, 2010, pp. 327–332.

[15] M. K. S. Faradonbeh, A. Tewari, and G. Michailidis, "Optimality of fast-matching algorithms for random networks with applications to structural controllability," *IEEE Transactions on Control of Network Systems*, vol. 4, no. 4, pp. 770–780, 2017.

[16] J. xiong Yang, D. yun Lin, and M. qing Li, "Analysis on controllability of descriptor systems under structural decomposition," in *Proc. of the IEEE International Conference on Control and Automation*, 2007, pp. 3169–3172.

[17] M. Tsatsomeros, "Sign controllability: Sign patterns that require complete controllability," *SIAM Journal on Matrix Analysis and Applications*, vol. 19, pp. 355–364, 1998.

[18] H.-S. Ahn, K. L. Moore, S.-H. Kwon, Q. V. Tran, B.-Y. Kim, and K.-K. Oh, "Topological controllability of undirected networks of diffusively-coupled agents," 2019, arXiv:1903.11246 [cs.SY].

[3]It appears that the process for finding the paths in an optimal way looks similar to the maximum matching process proposed in [10]. However, it seems that the merging and decomposition algorithms proposed here are more efficient and general. Also, we do not claim that the path graphs are only basic controllable subgraphs, although our algorithms were developed from path graphs. Thus, in our future efforts, we would like to develop new decomposition and merging algorithms from more general basic controllable subgraphs.