Structural Controllability of Undirected Diffusive Networks With Vector-Weighted Edges

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***Abstract*—In this letter, controllability of undirected networked systems with diffusively coupled subsystems is considered, where each subsystem is of identically *fixed* general high-order single-input-multi-output dynamics. The underlying graph of the network topology is *vector-weighted*, rather than scalar-weighted. The aim is to find conditions under which the networked system is structurally controllable, i.e., for almost all vector values for interaction links of the network topology, the corresponding system is controllable. It is proven that, the networked system is structurally controllable, if and only if each subsystem is controllable and observable, and the network topology is globally input-reachable. These conditions are further extended to the cases with multi-input-multi-output subsystem dynamics and matrix-weighted edges, or where both directed and undirected interaction links exist.**

***Index Terms*—Undirected diffusive network, structural controllability, network analysis and control, vector-weighted Laplacian.**

I. Introduction

ANALYSIS and synthesize of networked systems with diffusively coupled subsystems, also known as *diffusive networks*, have received much attention in the fields of synchronization, consensus, stability, as well as controllability and observability [1]–[3]. This is because the diffusive coupling mechanism frequently arises naturally in thermal systems, power systems, car-following traffic systems, etc. [2], [4]. As is known to all, controllability is a fundamental system property. Particularly, controllability of a leader-follower multi-agent system (MAS) running the consensus protocol guarantees that the system can reach agreement subspace arbitrarily fast [5].

Many works have focused on controllability of leader-follower MASs [3], [5]–[7]. For example, [6] gave conditions for controllability of such networked systems in terms of eigenvectors of the Laplacian matrices. The works [3], [5], [7] studied the same problem by means of graph-theoretic tools. However, except [3], all the above works assume that each subsystem is a single-integrator. On the other hand, controllability of networks with high-order subsystems has also attracted much research interests in [8]–[13]. To be specific, [8], [11], [13] focused on networked systems with identical subsystems (called homogeneous networks), whiles [9], [10], [12] on networked systems with general heterogeneous subsystems (called heterogeneous networks). Particularly, controllability as a generic property for a networked system is studied in [12], [13].

However, except [9], [10], [12] which focus on heterogeneous networks, almost all results on controllability of homogeneous networks are built on the condition that all weights of edges in the network topology belong to {0, 1} or some given scalars [3], [5]–[8], [11], [13]. Notice that, when each subsystem is not of single-input-single-output (SISO), there is a typical situation that different interaction channels between two subsystems are weighted differently, either because of differences in the nature of physical variables they convey, or the variants of the channels themselves. For example, in some networks consisting of both physical couplings and cyber couplings, the physical channels and the cyber ones between two subsystems can have different weights. See more examples in [14]–[16]. If we use a graph to denote the subsystem interaction topology, then each edge of the graph may have a *vector-valued* or *matrix-valued* weight as introduced in [14], [15]. In such case, some existing approaches for controllability analysis for networks with scalar-weighted edges may not be applicable (such as the spectrum-based approaches in [3], [11]).

In this letter, we study structural controllability of an undirected diffusive networked system with high-order subsystems, where the underlying graph of the network topology has symmetric vector-weighted edges. Our purpose is to find conditions under which the networked system is structurally controllable, i.e., for almost all vector values for edges of the network topology, the whole system is controllable. The main contributions of this letter are three-fold. First, we prove that, an undirected diffusive networked system with identical *single-input-multi-output (SIMO)* subsystems (leading to vector-weighted edges) is structurally controllable, if and only if each subsystem is controllable and observable, and the network topology is globally input-reachable. Second, we show that our conditions are still valid even when both directed edges and undirected ones exist. Third, we extend our results to the case with arbitrarily dimensional matrix-weighted edges.

It is remarkable that some relations between connectivity and observability have been revealed in [15] for a networked system where subsystems are decoupled whereas their outputs are relatively measured by sensor networks, and each interconnection edge defined therein has a *semi-definite*

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weight, which is obviously different from the systems studied herein. It is also mentionable that (strong) structural controllability of networks of single-integrators with symmetric weights (or more complicated parameter dependencies) has recently received much attention in [17]–[20]. Although heterogeneous networks in [9], [10], [12] may cover the system model studied in this letter, their results are essentially rank conditions [9], [10] whose verifications usually require algebraic calculations in the global system level, or some combinatorial tools like matroid [12], rather than simple topological conditions herein. Finally, if different interacted variables among subsystems are conveyed over networks with different structures, then the associated system could not be modeled by a single network with vector-weighted edges, which is beyond the scope of this letter.

Notations: For a set, $|\cdot|$ denotes its cardinality. For a matrix M, M_{ij} or $[M]_{ij}$ denotes the entry in the ith row and jth column of M. $\sigma(M)$ denotes the set of eigenvalues of the square matrix M, and $\mathbf{diag}\{X_i|_{i=1}^n\}$ the block diagonal matrix whose ith diagonal block is X_i, $\mathbf{col}\{X_i|_{i=1}^n\}$ the matrix stacked by $X_i|_{i=1}^n$. By $e_i^{[N]}$ we denote the ith column of the N dimensional identify matrix I_N, and $\mathbf{1}_{m\times n}$ the $m\times n$ matrix with entries all being one.

II. Problem Formulation

Consider a networked system consisting of N subsystems. Let $\mathcal{G}_{\rm sys}=(\mathcal{V}_{\rm sys},\mathcal{E}_{\rm sys})$ be an *undirected graph* without self-loops describing the subsystem interconnection topology (i.e., underlying graph of the network topology), with $\mathcal{V}_{\rm sys}=\{1,\ldots,N\}$, and an undirected edge $(i,j)\in\mathcal{E}_{\rm sys}$ if the jth subsystem and the ith one are directly influenced by each other. Dynamics of the ith subsystem is described by

$$\dot{x}_i(t)=Ax_i(t)+bv_i(t), \tag{1}$$

where $A\in\mathbb{R}^{n\times n}$, $b\in\mathbb{R}^n$, $x_i(t)\in\mathbb{R}^n$ is the state vector, $v_i(t)\in\mathbb{R}$ is the input injected to the ith subsystem. The input $v_i(t)$ may contain both subsystem interactions and the external control inputs, expressed as

$$v_i(t)=\delta_iu_i(t)+\sum_{j=1,j\neq i}^{N}W_{ij}C(x_j(t)-x_i(t)), \tag{2}$$

where $u_i(t)$ is the external control input, $C\doteq[c_1^{\intercal},\ldots,c_r^{\intercal}]^{\intercal}\in\mathbb{R}^{r\times n}$ with $c_k\in\mathbb{R}^{1\times n}$, and $W_{ij}\in\mathbb{R}^{1\times r}$ is the vector-valued weight of edge from the jth subsystem to the ith one. Denote the kth element of W_{ij} by $w_{ij}^{[k]}$, $k=1,\ldots,r$, i.e., $w_{ij}^{[k]}\in\mathbb{R}$ is the scalar weight imposed on the relative information $c_k(x_j(t)-x_i(t))$. Moreover, $\delta_i\in\{0,1\}$, $\delta_i=1$ means that the ith subsystem is directly controlled by the external input $u_i(t)$, and $\delta_i=0$ means the contrary. In addition, $W_{ij}=W_{ji}$ for $i,j\in\{1,\ldots,N\}$, and $W_{ij}\neq 0$ only if $(j,i)\in\mathcal{E}_{\rm sys}$ ($i\neq j$). Let $\Delta=\mathbf{diag}\{\delta_i|_{i=1}^N\}$, $u(t)=[u_1(t),\ldots,u_N(t)]^{\intercal}$, $x(t)=[x_1^{\intercal}(t),\ldots,x_N^{\intercal}(t)]^{\intercal}$. Define (scalar-weighted) Laplacian matrices $L_k\in\mathbb{R}^{N\times N}$ associated with $\mathcal{G}_{\rm sys}$ as $[L_k]_{ij}=-w_{ij}^{[k]}$ if $i\neq j$, and $[L_k]_{ij}=\sum_{p=1,p\neq i}^N w_{ip}^{[k]}$ if $i=j$, for $k=1,\ldots,r$. The lumped state-space representation of (1)-(2) then is

$$\dot{x}(t)=A_{\rm sys}x(t)+B_{\rm sys}u(t), \tag{3}$$

with

$$A_{\rm sys}=I_N\otimes A-\sum_{k=1}^{r}L_k\otimes(bc_k)=I_N\otimes A-(I_N\otimes b)L_g(I_N\otimes C),\quad B_{\rm sys}=\Delta\otimes b, \tag{4}$$

where $\otimes$ denotes the Kronecker product, and

$$L_g=\begin{bmatrix}\sum_{j=2}^N W_{1j} & \cdots & -W_{1N}\\ \vdots & \ddots & \vdots\\ -W_{N1} & \cdots & \sum_{j=1}^{N-1}W_{Nj}\end{bmatrix}\in\mathbb{R}^{N\times Nr}$$

is a *vector-weighted Laplacian* [14]. Throughout this letter, without losing of generality, assume that $c_k\neq 0$, $\forall k=1,\ldots,r$.

The (1)-(2) models a diffusive networked system with identical subsystems, which arises in modeling interacted liquid tanks [4], synchronizing networks of linear oscillators [1], [12], electrical networks [15], consensus-based MASs [2], etc. Specially, when $r=1$, (1)-(2) becomes a networked system with SISO subsystems. Readers are referred to [14]–[16] for more examples for networked systems with vector-weighted edges.

Definition 1: Given A,b,C,Δ and an undirected $\mathcal{G}_{\rm sys}$, the networked system (1)-(2) is said to be structurally controllable, if there exists a set of values for $\{W_{ij}\}_{(j,i)\in\mathcal{E}_{\rm sys}}$ with $W_{ij}=W_{ji}$, such that the associated $(A_{\rm sys},B_{\rm sys})$ is controllable.

In line with [18], it can be shown that controllability of the networked system (1)-(2) is a *generic property* in the sense that, if this system is structurally controllable, then for almost all values for $\{W_{ij}\}_{(j,i)\in\mathcal{E}_{\rm sys}}$ with $W_{ij}=W_{ji}$, the corresponding system is controllable. In practise, due to parameter uncertainties or geographical distance between subsystems, the exact weights W_{ij} might be hard to know. Under such circumstance, structural controllability may be a good alternative for controllability evaluation. The main problem considered in this letter is as follows.

Problem 1: Given A,b,C,Δ and an undirected subsystem interaction topology $\mathcal{G}_{\rm sys}$, find necessary and sufficient conditions under which the system (1)-(2) is structurally controllable.

III. Main Results

In this section, we first give necessary and sufficient conditions for Problem 1. We then extend them to the case with semi-symmetric topologies. All proofs are given in Section IV.

Let $\mathcal{I}_u=\{i:\delta_i\neq 0\}$ be the set of indices of subsystems that are directly influenced by external inputs, and $\mathcal{U}=\{u_i:i\in\mathcal{I}_u\}$. Let $\bar{\mathcal{G}}_{\rm sys}=(\mathcal{V}_{\rm sys}\cup\mathcal{U},\mathcal{E}_{\rm sys}\cup\mathcal{E}_{ux})$, where $\mathcal{E}_{ux}=\{(u_i,i),i\in\mathcal{I}_u\}$. Then, $\bar{\mathcal{G}}_{\rm sys}$ reflects the information flows of the networked system. A path from vertex i_1 to vertex i_p is a sequence of edges $(i_1,i_2),(i_2,i_3),\ldots,(i_{p-1},i_p)$, where each edge is either directed or undirected.

Definition 2: We say a vertex i is input-reachable, if there exists a path beginning from any $u\in\mathcal{U}$ and ending at i in $\bar{\mathcal{G}}_{\rm sys}$. If every vertex of $i\in\mathcal{V}_{\rm sys}$ is input-reachable, we just say that $\bar{\mathcal{G}}_{\rm sys}$ (or the network topology) is globally input-reachable (i.e., global input-reachability means that $\bar{\mathcal{G}}_{\rm sys}$ can be decomposed into a collection of disjoint spanning trees rooted at $\mathcal{U}$).

Theorem 1: Assume that $|\mathcal{I}_u|<N$. Then the networked system (1)-(2) is structurally controllable, if and only if

1) (A, b) is controllable, and $(A, [c_1^\intercal, \dots, c_r^\intercal]^\intercal)$ is observable;

2) $\bar{\mathcal{G}}_{\rm sys}$ is globally input-reachable.

In the above theorem, we have ruled out the trivial case where $|\mathcal{I}_u| = N$, under which the networked system is always structurally controllable whenever (A, b) is controllable (which is always necessary for the networked system to be controllable [9]). The above theorem implies that, if each subsystem is controllable and observable, and the networked topology is globally input-reachable, then for almost all vector-valued weights, the corresponding networked system is controllable. We refer readers to [21] for an application example of Theorem 1 on a practical system. Two direct corollaries are as follows.

Corollary 1: Let L be the weighted Laplacian matrix of a connected undirected graph $\mathcal{G}$ with N vertices. Then, for almost all weights of edges of $\mathcal{G}$, $(-L, e_i^{[N]})$ is controllable, $\forall i \in \{1, \dots, N\}$.

Corollary 2: Suppose in the networked system (1)-(2), each subsystem is SISO (i.e., $r = 1$), and $|\mathcal{I}_u| < N$. Then the system is structurally controllable, if and only if (A, b) is controllable, (A, C) is observable, and $\bar{\mathcal{G}}_{\rm sys}$ is globally input-reachable.

Remark 1: Structural controllability of undirected networks of single-integrators running the consensus protocol has been discussed in [22]. Corollary 1 differs from [22], as the result of [22] is under the condition that the total sum of each row of the lumped state transition matrix $A_{\rm sys}$ and input matrix $B_{\rm sys}$ is zero, rather than that the sum of each row of $A_{\rm sys}$ is zero.

Remark 2: If the underlying graph of the network topology is scalar-weighted, i.e., $L_1 = \cdots = L_k = \bar{L}$, structural controllability of this kind of networked systems falls into the SISO case with subsystem output matrix being $c_1 + \cdots + c_k$ and subsystem input matrix being b, noting that in this case $A_{\rm sys} = I \otimes A - \sum_{k=1}^r L_k \otimes (bc_k) = I \otimes A - \bar{L} \otimes (b(c_1 + \cdots c_k))$. It is easy to see that observability of $(A, c_1 + \cdots + c_r)$ implies that $(A, [c_1^\intercal, \dots, c_r^\intercal]^\intercal)$ is observable, while the converse is not necessarily true. This verifies the intuition that allowing vector-valued weights makes the conditions for structural controllability less restrictive than that of scalar-valued ones.

We are now extending Theorem 1 to the case where $\mathcal{G}_{\rm sys}$ contains both directed and undirected edges. That is to say, not all off-diagonal entries of the Laplacian matrices $L_i|_{i=1}^r$ need to be equal to their symmetric ones, and a nonzero entry of $L_i|_{i=1}^r$ may even have a fixed zero symmetric entry. A pair of symmetrically equal entries of L_i correspond to an undirected edge, whiles an entry not equaling its symmetrical one corresponds to a directed edge. We call such topology as the *semi-symmetric topology* with a little abuse of terminology. Semi-symmetric topologies may emerge, such as, in a networked system where both bidirectional and unidirectional interactions exist, which cover both the directed topologies and undirected ones, and are more general than them.

Given a semi-symmetric topology $\mathcal{G}_{\rm sys}$, let $\bar{\mathcal{G}}_{\rm sys}$ be defined in the same way as before in this section. We say $\bar{\mathcal{G}}_{\rm sys}$ is globally input-reachable, if for each vertex $i \in \mathcal{V}_{\rm sys}$, there is a path consisting of either directed, undirected, or both directed and undirected edges beginning from a $u \in \mathcal{U}$ ending at i.

Theorem 2: Consider the networked system (1)-(2) with semi-symmetric topology $\mathcal{G}_{\rm sys}$. Suppose $|\mathcal{I}_u| < N$. The system is structurally controllable, if and only if (A, b) is controllable, (A, C) is observable, and $\bar{\mathcal{G}}_{\rm sys}$ is globally input-reachable.

IV. Analysis

This section gives the proofs of Theorem 1 and Theorem 2.

Proof of Necessary Part of Theorem 1: The proof for necessity seems quite standard. We refer readers to [21]. ■

Our proof for sufficient part of Theorem 1 is based on the linear parameterization [23]. Consider a linear-parameterized pair (A, B) modeled as

$$A = A_0 + \sum_{i=1}^k g_i s_i h_{1i}^\intercal, B = B_0 + \sum_{i=1}^k g_i s_i h_{2i}^\intercal. \tag{5}$$

where $A_0 \in \mathbb{R}^{n\times n}$, $B_0 \in \mathbb{R}^{n\times m}$, $g_i, h_{1i} \in \mathbb{R}^n$, $h_{2i} \in \mathbb{R}^m$, and $s_1, \dots, s_k$ are real free parameters. The pair (A, B) in (5) is said to be structurally controllable, if there exists one set of values for $s_1, \dots, s_k$, such that the associated system is controllable.

Definition 3: Given an $n \times n$ matrix H and an $n \times m$ matrix P, the auxiliary digraph associated with (H, P) is denoted by $\mathcal{G}_{\rm aux}(H, P)$, which is defined as the digraph $(\mathcal{V}_H \cup \mathcal{V}_P, \mathcal{E}_{HH} \cup \mathcal{E}_{HP})$, where $\mathcal{V}_H = \{v_1, \dots, v_n\}$, $\mathcal{V}_P = \{u_1, \dots, u_m\}$, and $\mathcal{E}_{HH} = \{(v_i, v_j){:}H_{ji} \neq 0\}$, $\mathcal{E}_{HP} = \{(u_i, v_j){:}P_{ji} \neq 0\}$.[1] A vertex $v \in \mathcal{V}_H$ is input-reachable in $\mathcal{G}_{\rm aux}(H, P)$, if there is a path from one vertex in $\mathcal{V}_P$ ending at v. A cycle of $\mathcal{G}_{\rm aux}(H, P)$ is said to be input-reachable, if there is at least one vertex in this cycle that is input-reachable.

Define two transfer function matrices (TFMs) as $G_{zv}(\lambda) = [h_{11}, \dots, h_{1k}]^\intercal(\lambda I - A_0)^{-1}[g_1, \dots, g_k]$, $G_{zu}(\lambda) = [h_{11}, \dots, h_{1k}]^\intercal(\lambda I - A_0)^{-1}B_0 + [h_{21}, \dots, h_{2k}]^\intercal$. The following lemma gives necessary and sufficient conditions for (A, B) in (5) to be structurally controllable.

Lemma 1 [12], [23]: The pair (A, B) in (5) is structurally controllable, if and only if

1) Every cycle is input-reachable in $\mathcal{G}_{\rm aux}(G_{zv}(\lambda), G_{zu}(\lambda))$;

2) For each $\lambda_0 \in \sigma(A_0)$, $\text{grank}[\lambda_0 I - A_0 - \sum_{i=1}^k g_i s_i h_{1i}^\intercal, B_0 + \sum_{i=1}^k g_i s_i h_{2i}^\intercal] = n$, where grank($\bullet$) is the maximum rank a matrix can achieve as the function of its free parameters.

Due to dependencies among nonzero entries of L_i, before utilizing Lemma 1, we need to diagonalize L_i, $i = 1, \dots, r$. To this end, first arbitrarily assign an orientation to each undirected edge of $\mathcal{G}_{\rm sys}$, and let $\mathcal{E}_{\rm sys} = \{e_1, \dots, e_{|\mathcal{E}_{\rm sys}|}\}$. Then construct the $|\mathcal{E}_{\rm sys}| \times |\mathcal{V}_{\rm sys}|$ incidence matrix K_I as follows: $[K_I]_{ij} = 1$ ($[K_I]_{ij} = -1$) if vertex j is the starting vertex (ending vertex) of e_i, and the remaining entries are zero. Then, define a $|\mathcal{V}_{\rm sys}| \times |\mathcal{E}_{\rm sys}|$ matrix K as $K = -K_I^\intercal$. It can be validated that $L_i = -K\Lambda_i K_I$, where Λ_i is a diagonal matrix whose jth diagonal equals the scalar weight of e_j associated with L_i, $j = 1, \dots, |\mathcal{E}_{\rm sys}|$. We then have the following linear parameterization of $(A_{\rm sys}, B_{\rm sys})$

$$\begin{aligned}[A_{\rm sys}, B_{\rm sys}] &= [I_N \otimes A, \Delta \otimes b] + [K \otimes b, \dots, K \otimes b] \\ &\quad \times \mathbf{diag}\{\Lambda_1, \dots, \Lambda_r\}[[K_I^\intercal \otimes c_1^\intercal, \dots, K_I^\intercal \otimes c_r^\intercal]^\intercal, 0]. \end{aligned} \tag{6}$$

Regarding the linear parameterization (6), direct algebraic manipulations show that the associated TFMs are respectively

$$G_{zv}(\lambda) = \begin{bmatrix} K_I \otimes c_1 \\ \vdots \\ K_I \otimes c_r \end{bmatrix} (\lambda I - I_N \otimes A)^{-1}[K \otimes b, \dots, K \otimes b]$$

[1] Here, $H_{ji} \neq 0$ means that H_{ji} is not identically zero, so is with $P_{ji} \neq 0$.

$$= \mathbf{1}_{1\times r} \otimes \begin{bmatrix} (K_I K) \otimes [c_1(\lambda I - A)^{-1} b)] \\ \vdots \\ (K_I K) \otimes [c_r(\lambda I - A)^{-1} b)] \end{bmatrix},$$

$$G_{zu}(\lambda) = \begin{bmatrix} (K_I \Delta) \otimes (c_1(\lambda I - A)^{-1} b) \\ \vdots \\ (K_I \Delta) \otimes (c_r(\lambda I - A)^{-1} b) \end{bmatrix}. \tag{7}$$

To proceed with our proof, we need the following lemma.

Lemma 2 [16, Lemma 8]: Given four matrices H, P, G and Λ, suppose the following conditions hold: 1) H, P and G are of the dimensions $k \times n$, $k \times m$ and $n \times k$ respectively; 2) Whenever there exists one $l \in \{1, \ldots, k\}$ such that $G_{il} \neq 0$ and $H_{li} \neq 0$ (resp. $G_{il} \neq 0$ and $P_{li} \neq 0$), it implies that $[GH]_{ij} \neq 0$ (resp. $[GP]_{ij} \neq 0$); 3) Λ is an $n \times n$ diagonal matrix whose diagonal entries are free parameters. Then, every cycle is input-reachable in $\mathcal{G}_{\text{aux}}(GH, GP)$, if and only if such property holds in $\mathcal{G}_{\text{aux}}(H\Lambda G, P)$.

Proposition 1: If (A, b) is controllable and the network topology is globally input-reachable, then every cycle of $\mathcal{G}_{\text{aux}}(G_{zv}(\lambda), G_{zu}(\lambda))$ is input-reachable.

Proof: Since (A, b) is controllable, (I, A, b) is output-controllable (see [4, Sec. 9.6]). According to [4], this requires that the rows of $(\lambda I - A)^{-1} b$ are linearly independent in the field of complex values. That is, there cannot exist $x \in \mathbb{C}^n \backslash \{0\}$ making $x^\mathsf{T}(\lambda I - A)^{-1} b \equiv 0$. As $c_i \neq 0$, $c_i(\lambda I - A)^{-1} b \neq 0$. Hence, $[G_{zv}(\lambda), G_{zu}(\lambda)]$ has the same sparsity pattern as

$$[\mathbf{1}_{r\times r} \otimes (K_I K), \mathbf{1}_{r\times 1} \otimes (K_I \Delta)]. \tag{8}$$

Let $\widetilde{\mathcal{G}}_{\text{aux}}$ be the auxiliary digraph associated with (8). Define matrices $G \doteq \mathbf{diag}\{K_I|_{i=1}^r\}$, $H \doteq \mathbf{1}_{r\times r} \otimes K$, $P \doteq \mathbf{1}_{r\times 1} \otimes \Delta$. Then, (8) can be expressed as $[GH, GP]$. Notice that every row of K_I, as well as every column of K, contains only two nonzero entries. Moreover, no nonzero addends can cancel each other out in obtaining $[K_I K]_{ij}$, $1 \leq i, j \leq |\mathcal{E}_{\text{sys}}|$. Hence, Condition 2) of Lemma 2 holds with respect to (G, H, P).

Let L be a Laplacian matrix associated with $\mathcal{G}_{\text{sys}}$. Using Lemma 2 on (G, H, P), one will obtain the following matrix

$$[(\mathbf{1}_{r\times r} \otimes K)\mathbf{diag}\{\Lambda_i K_I|_{i=1}^r\}, \mathbf{1}_{r\times 1} \otimes \Delta]$$

which has the same sparsity pattern as

$$[-(\mathbf{1}_{r\times r} \otimes L), \mathbf{1}_{r\times 1} \otimes \Delta], \tag{9}$$

where Λ_i is defined just before (6). Denote by $\hat{\mathcal{G}}_{\text{aux}}$ the auxiliary digraph associated with (9). According to Lemma 2, every cycle in $\widetilde{\mathcal{G}}_{\text{aux}}$ is input-reachable, if and only if such property holds in $\hat{\mathcal{G}}_{\text{aux}}$. Notice that $\mathcal{G}_{\text{aux}}(-I_r \otimes L, \mathbf{1}_{r\times 1} \otimes \Delta)$ is a subgraph of $\hat{\mathcal{G}}_{\text{aux}}$. As $\bar{\mathcal{G}}_{\text{sys}}$ is the auxiliary digraph associated with $[-L, \Delta]$ by eliminating all self-loops, global reachability of $\bar{\mathcal{G}}_{\text{sys}}$ indicates every vertex is input-reachable in $\mathcal{G}_{\text{aux}}(-I_r \otimes L, \mathbf{1}_{r\times 1} \otimes \Delta)$, so is with $\hat{\mathcal{G}}_{\text{aux}}$. The proposition follows then from Lemma 2. ■

Proposition 2: Consider the networked system (1)-(2). If $\bar{\mathcal{G}}_{\text{sys}}$ is globally input-reachable, whiles (A, b) is controllable and $(A, [c_1^\mathsf{T}, \ldots, c_r^\mathsf{T}]^\mathsf{T})$ is observable. Then, for each $\lambda_i \in \sigma(A)$, $\text{grank}[\lambda_i I - A_{\text{sys}}, B_{\text{sys}}] = Nn$.

To prove Proposition 2, we need the following lemma.

Lemma 3: Given matrices $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $C \in \mathbb{R}^{r\times n}$, suppose that (A, b) is controllable and (A, C) is observable. Let $\Lambda = [s_1, \ldots, s_r]$ be a $1 \times r$ matrix whose entries are all free parameters $s_i|_{i=1}^r$. Then, for arbitrary $Q_0 \in \mathbb{C}^{r\times 1}$,

$$\text{grank}\begin{bmatrix} \lambda_i I - A & b\Lambda \\ -C & I_r - Q_0 \Lambda \end{bmatrix} = n + r$$

holds for each $\lambda_i \in \sigma(A)$.

Proof: Define $S \doteq \mathbf{diag}\{s_i|_{i=1}^r\}$ and $M_0 = -\mathbf{1}_{1\times r} \otimes Q_0$. Let $\bar{s}_i \doteq s_i^{-1}$, $i = 1, \ldots, r$. We then have

$$\begin{bmatrix} \lambda_i I - A & b\Lambda \\ -C & I_r - Q_0\Lambda \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_i I - A & \mathbf{1}_{1\times r} \otimes b \\ -C & S^{-1} + M_0 \end{bmatrix}}_{\doteq F(\lambda_i)} \begin{bmatrix} I_n & \\ & S \end{bmatrix}.$$

It follows that, if $F(\lambda_i)$ is full column generic rank for each $\lambda_i \in \sigma(A)$, then the proposed statement is proved. Define

$$\Gamma = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & \cdots & 1 \\ 0 & & \cdots & -1 \end{bmatrix} \in \mathbb{R}^{r\times(r-1)}. \tag{10}$$

As (A, b) is controllable, it can be shown that, for any $\lambda_i \in \sigma(A)$, a basis matrix spanning the null space of $[\lambda_i I - A, \mathbf{1}_{1\times r} \otimes b]$ can be $\mathbf{diag}\{x_i, \Gamma\}$, where x_i is the eigenvector of A associated with λ_i. Notice that x_i is unique if scaled by certain scalars. Then, by the property of null space [24, Ch. 0.2], $F(\lambda_i)$ is of full generic rank, if and only if

$$[-C, S^{-1} + M_0]\mathbf{diag}\{x_i, \Gamma\} = [-Cx_i, S^{-1}\Gamma + M_0\Gamma]$$

is of full generic rank. As (A, C) is observable, there exists one $j \in \{1, \ldots, r\}$, such that $[Cx_i]_j \neq 0$. According to the structure specificity of Γ in (10), there exists one and only one nominal $\prod_{k=1,k\neq j}^r \bar{s}_k$ in $\det[-Cx_i, S^{-1}\Gamma + M_0\Gamma]$, whose coefficient is $[Cx_i]_j$ (the sign is ignored). This can be validated by the definition of determinant [24, Ch. 0.3]. Notice that $\prod_{k=1,k\neq j}^{k=r} \bar{s}_k$ has the maximum degree $r-1$ such that arbitrary constant M_0 cannot violate that fact. Hence, $[-Cx_i, S^{-1}\Gamma + M_0\Gamma]$ is of full generic rank, proving Lemma 3. ■

Proof of Proposition 2: We will use mathematical induction. Since $\bar{\mathcal{G}}_{\text{sys}}$ is globally input-reachable, first assume that $\bar{\mathcal{G}}_{\text{sys}}$ has one spanning tree rooted at u_1. Denote this spanning tree by $\mathcal{T}$, and vertices $u_1, 1, \ldots, N$ are in lexicographic order in the sense that the parent of vertex i is among vertices $i-1, \ldots, 1, u_1$ in $\mathcal{T}$, $i = 1, \ldots, N$. Moreover, let $\mathcal{T}_i$ be the subgraph of $\mathcal{T}$ induced by vertices $1, \ldots, i$, and K_{Ii} be the incidence matrix associated with $\mathcal{T}_i$, which is defined similarly to K_I of $\mathcal{G}_{\text{sys}}$. In this sense, K_{Ii} can be recursively constructed as

$$K_{I,i+1} = \begin{bmatrix} K_{Ii} & 0 \\ (e_{(i+1)^*}^{[i]})^\mathsf{T} & -1 \end{bmatrix}, \tag{11}$$

where $(i+1)^*$ is the parent of vertex $i+1$ in $\mathcal{T}$, $i = 0, \ldots, N-1$, and K_{I0} is empty. Let $K_i = -K_{Ii}^\mathsf{T}$. Let weights of edges not in $\mathcal{T}$ be zero. Remember that W_{ii^*} is the vector-valued weight of the edge connecting vertex i and its parent i^*. Let $S_i = \mathbf{diag}\{W_{jj^*}|_{j=2}^{i+1}\}$, $i = 1, \ldots, N-1$, and $S_0 = \emptyset$, i.e., S_i stores all weight vectors of edges in $\mathcal{T}_{i+1}$. Then, similar to (6)

$$A_{\text{sys}} = I_N \otimes A - (K_{IN}^\mathsf{T} \otimes b)S_{N-1}(K_{IN} \otimes C) \tag{12}$$

We will prove by induction that, for each $\lambda_0 \in \sigma(A)$

$$[I_i \otimes A - (K_{Ii}^\mathsf{T} \otimes b)S_{i-1}(K_{Ii} \otimes C) - \lambda_0 I, e_1^{[i]} \otimes b] \tag{13}$$

is of full row generic rank for $i = 1, \dots, N$. Since (A, b) is controllable, the base case where $i = 1$ is obviously true. Now suppose that (13) is of full row generic rank for some i between 1 and $N - 1$. Rewrite (13) as

$$[I_i \otimes A - \lambda_0 I, e_1^{[i]} \otimes b] + (K_i \otimes b)S_{i-1}[K_{Ii} \otimes C, 0]. \quad (14)$$

Using Schur complement [24, Ch. 0.8] on the above formula, we have that (13) is of full row generic rank, if and only if

$$\Psi_i \doteq \left[\begin{array}{cc:c} I_i \otimes A - \lambda_0 I & e_1^{[i]} \otimes b & (K_{Ii}^{\mathsf{T}} \otimes b)S_{i-1} \\ \hdashline K_{Ii} \otimes C & 0 & I \end{array}\right]$$

is so. Substituting (11) into Ψ_{i+1} and after some elementary permutations, Ψ_{i+1} is of full row generic rank, if and only if

$$\left[\begin{array}{ccc:cc} I_i \otimes A - \lambda_0 I & e_1^{[i]} \otimes b & (K_{Ii}^{\mathsf{T}} \otimes b)S_{i-1} & 0 & e_{(i+1)^*}^{[i]} \otimes b\Lambda_{i+1} \\ K_{Ii} \otimes C & 0 & I & 0 & 0 \\ \hdashline 0 & 0 & 0 & A - \lambda_0 I & -b\Lambda_{i+1} \\ (e_{(i+1)^*}^{[i]})^{\mathsf{T}} \otimes C & 0 & 0 & -C & I_r \end{array}\right] \quad (15)$$

is of full row generic rank, where $\Lambda_{i+1} \doteq W_{(i+1)(i+1)^*}$ for notation simplicity. Let S_{i-1} take some value such that Ψ_i is of full row rank. Then for arbitrary $\Lambda_{i+1} \in \mathbb{R}^{1\times r}$, $\Psi_i \bar{\Psi}_i = \begin{bmatrix} e_{(i+1)^*}^{[i]} \otimes b\Lambda_{i+1} \\ 0 \end{bmatrix}$, where $\bar{\Psi}_i \doteq \Psi_i^{\dagger} \begin{bmatrix} e_{(i+1)^*}^{[i]} \otimes b\Lambda_{i+1} \\ 0 \end{bmatrix}$, and $(\bullet)^{\dagger}$ denotes the Moore-Penrose inverse. Hence, post-multiplying $\begin{bmatrix} I & 0 & -\bar{\Psi}_i \\ 0 & I_n & 0 \\ 0 & 0 & I_r \end{bmatrix}$ to (15), one will obtain

$$\Pi \doteq \begin{bmatrix} \Psi_i & 0 \\ \Pi_{21} & \Pi_{22} \end{bmatrix},$$

where $\Pi_{21} \doteq \begin{bmatrix} 0 & 0 & 0 \\ (e_{(i+1)^*}^{[i]})^{\mathsf{T}} \otimes C & 0 & 0 \end{bmatrix}$, $\Pi_{22} \doteq \begin{bmatrix} A - \lambda_0 I & -b\Lambda_{i+1} \\ -C & I_r - Q_0\Lambda_{i+1} \end{bmatrix}$, with constant matrix $Q_0 \in \mathbb{C}^{r\times 1}$ satisfying $[(e_{(i+1)^*}^{[i]})^{\mathsf{T}} \otimes C, 0, 0]\bar{\Psi}_i = Q_0\Lambda_{i+1}$. By Lemma 3, Π_{22} is of full generic rank. Hence, Π is of full row generic rank, which means that (15) is so, too. Thus, replacing i with $i + 1$, (13) is of full row generic rank. Inducing from $i = 1$ to $i = N$, the proposed statement is proved. The case that $\bar{\mathcal{G}}_{\text{sys}}$ can be decomposed into more than one disjoint spanning trees rooted at $\mathcal{U}$ follows immediately from the former case. ■

Proof of Sufficient Part of Theorem 1: By Lemma 1, Propositions 1 and 2 assure the sufficiency of conditions in Theorem 1 for structural controllability. ■

Proof of Theorem 2: To handle with semi-symmetric topologies, we shall modify the diagonalization of $L_i|_{i=1}^r$. To this end, first for each undirected edge of $\mathcal{G}_{\text{sys}}$, arbitrarily assign an orientation, whereas each directed edge remains unchanged. Then the incidence matrix K_I is defined as a $|\mathcal{E}_{\text{sys}}| \times |\mathcal{V}_{\text{sys}}|$ matrix (each undirected edge counts one for $|\mathcal{E}_{\text{sys}}|$), where $[K_I]_{ij} = 1$ (resp. $[K_I]_{ij} = -1$) if vertex j is the starting vertex (resp. ending vertex) of the ith edge. Moreover, K is a $|\mathcal{V}_{\text{sys}}| \times |\mathcal{E}_{\text{sys}}|$ matrix defined as follows

$$K_{ij} = \begin{cases} 1, & \text{if } [K_I]_{ji} = -1 \\ -1, & \text{if the jth edge is undirected and } [K_I]_{ji} = 1 \\ 0, & \text{otherwise} \end{cases}$$

Let Λ_i be the diagonal matrix whose jth diagonal entry is the weight of the jth edge of $\mathcal{G}_{\text{sys}}$ associated with L_i, $j = 1, \dots, |\mathcal{E}_{\text{sys}}|$. Then $L_i = -K\Lambda_i K_I$. Based on such diagonalization, the rest of the proof follows similar arguments to that of Theorem 1. Details are omitted due to their similarities. ■

V. Extension With Matrix-Weighted Edges

This section extends Theorem 1 to the case with arbitrarily dimensional matrix-weighted edges. We will give a sufficient condition for structural controllability. Let us modify subsystem dynamics in (1)-(2) into multi-input-multi-output (MIMO)

$$x_i(t) = Ax_i(t) + Bv_i(t), \quad (16)$$

$$v_i(t) = \sum_{j=1}^{N} W_{ij}C(x_j(t) - x_i(t)) + \delta_i u_i(t), \quad (17)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{r\times n}$, $\delta_i \in \{0, 1\}$, and $u_i(t), v_i(t) \in \mathbb{R}^p$. Matrix $W_{ij} \in \mathbb{R}^{p\times r}$ is a *matrix-valued weight* of edge $(j, i) \in \mathcal{E}_{\text{sys}}$ with symmetric constraint $W_{ij} = W_{ji}$. Moreover, $W_{ij} = 0$ if $(j, i) \notin \mathcal{E}_{\text{sys}}$. The lumped representation of the system becomes

$$A_{\text{sys}} = I_N \otimes A - (I_N \otimes B)L_m(I_N \otimes C), B_{\text{sys}} = \Delta \otimes (B), \quad (18)$$

where Δ is defined in the same way as Section II, and L_m is

$$L_m = \begin{bmatrix} \sum_{j=2}^{N} W_{1j} & \cdots & -W_{1N} \\ \vdots & \ddots & \vdots \\ -W_{N1} & \cdots & \sum_{j=1}^{N-1} W_{Nj} \end{bmatrix} \in \mathbb{R}^{Np\times Nr},$$

which could be called a matrix-weighted Laplacian for $\mathcal{G}_{\text{sys}}$. Examples of the above networked system include coupled mass-spring systems, coupled electrical oscillators, and multi-agent systems coupled by linear feedback; see [14].

To give a linear parameterization of (18), introduce two matrices $T \doteq I_p \otimes \mathbf{1}_{1\times r}$ and $Q \doteq \mathbf{1}_{p\times 1} \otimes I_r$. For each $(i, j) \in \mathcal{E}_{\text{sys}}$, let Λ_{ij} be a $pr \times pr$ dimensional diagonal matrix whose diagonal entries consist of all entries of W_{ij}, i.e., $\Lambda_{ij} = \mathbf{diag}\{[W_{ij}]_{11}, [W_{ij}]_{12}, \dots, [W_{ij}]_{1r}, \dots, [W_{ij}]_{pr}\}$. Then, we have $W_{ij} = T\Lambda_{ij}Q$. Moreover, let K_I be the incidence matrix of $\mathcal{G}_{\text{sys}}$, whose definition is given in Section IV, and $K = -K_I^{\mathsf{T}}$. Then, it can be validated that, (18) has the following linear parameterization,

$$\begin{aligned} A_{\text{sys}} = I_N \otimes A - (I_N \otimes B)(K \otimes T)\mathbf{diag}\{\Lambda_{ij}|_{(i,j)\in\mathcal{E}_{\text{sys}}}\} \\ (K_I \otimes Q)(I_N \otimes C), \quad B_{\text{sys}} = \Delta \otimes B, \end{aligned} \quad (19)$$

where the diagonal entries of $\mathbf{diag}\{\Lambda_{ij}|_{(i,j)\in\mathcal{E}_{\text{sys}}}\}$ are placed in the order consistent with the incidence matrix K_I.

Regarding (19), by some algebraic manipulations, the associated TFMs in Lemma 1 are respectively

$$\begin{aligned} G_{zv}^m(\lambda) &= (K_I \otimes QI_N \otimes C)(\lambda I - I_N \otimes A)^{-1}(I_N \otimes BK \otimes T) \\ &= (K_I K) \otimes (QC(\lambda I - A)^{-1}BP) \\ G_{zu}^m(\lambda) &= (K_I \otimes QI_N \otimes C)(\lambda I - I_N \otimes A)^{-1}(\Delta \otimes B) \\ &= (K_I \Delta) \otimes (QC(\lambda I - A)^{-1}B). \end{aligned}$$

Definition 4 (Fixed Mode, [25]): Given a triple $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times p} \times \mathbb{R}^{r\times n}$, (A, B, C) is said to have no fixed mode, if $\bigcap_{F\in\mathbb{R}^{p\times r}} \sigma(A + BFC) = \emptyset$.

Proposition 3: Suppose that for the networked system (16)-(17), (A, B, C) has no fixed mode. If the network topology is globally input-reachable, then every cycle of $\mathcal{G}_{\rm aux}(G_{zv}^m(\lambda), G_{zu}^m(\lambda))$ is input-reachable.

Proof: For space consideration, the proof is given in [21]. ■

Proposition 4: Suppose that for the networked system (16)-(17), (A, B, C) has no fixed mode. If the network topology is globally input-reachable, then ${\rm grank}[\lambda_i I - A_{\rm sys}, B_{\rm sys}] = Nn$ for each $\lambda_i \in \sigma(A)$.

Proof: Observe that $A_{\rm sys}$ in (18) can be rewritten as

$$A_{\rm sys} = I_N \otimes A - (K_I^\intercal \otimes B)\mathbf{diag}\{W_{ij}|_{(i,j)\in\mathcal{E}_{\rm sys}}\}(K_I \otimes C), \quad (20)$$

where the diagonal blocks of $\mathbf{diag}\{W_{ij}|_{(i,j)\in\mathcal{E}_{\rm sys}}\}$ are in the order consistent with K_I. Notice that (20) has the same form as (12). This means that, if we replace the vector $b \in \mathbb{R}^n$ in Lemma 3 with a matrix $B \in \mathbb{R}^{n\times p}$ and show that the associated implications in that lemma still hold under the proposed condition in Proposition 4, then we could prove Proposition 4 in the same line as that of Proposition 2. For this purpose, we will prove that, for any $\lambda_0 \in \sigma(A)$, if $\bigcap_{F\in\mathbb{R}^{p\times r}} \sigma(A + BFC) = \emptyset$, then for arbitrary $Q_0 \in \mathbb{R}^{r\times p}$, there exists a $F_0 \in \mathbb{R}^{p\times r}$, such that matrix

$$M(F_0) = \begin{bmatrix} A - \lambda_0 I & -BF_0 \\ -C & I - Q_0F_0 \end{bmatrix}$$

has full row rank. In fact, if $\bigcap_{F\in\mathbb{R}^{p\times r}} \sigma(A + BFC) = \emptyset$, there exists $W \in \mathbb{R}^{p\times r}$, such that $A - \lambda_0 I - BWC$ and $I + WQ_0$ are simultaneously invertible. This can be justified by the following analysis. Suppose that a matrix W_0 exists such that $A - \lambda_0 I - BW_0C$ is invertible. Then, it is an easy manner to see that the set $\Delta_0 \doteq \{\Delta W \in \mathbb{R}^{p\times r} : A - \lambda_0 I - B(W_0 + \Delta W)C \text{ is singluar}\}$ has zero Lebesgue measure in $\mathbb{R}^{p\times r}$. On the other hand, the set $\Delta_1 \doteq \{\Delta W \in \mathbb{R}^{p\times r} : I + (W_0 + \Delta W)Q_0 \text{ is singluar}\}$ also has zero Lebesgue measure in $\mathbb{R}^{p\times r}$, noting that when $\Delta W = -W_0$, $I+(W_0+\Delta W)Q_0 = I$ is invertible. Hence, each element $\Delta W \in \mathbb{R}^{p\times r}\backslash(\Delta_0 \bigcup \Delta_1)$ making $A - \lambda_0 I - BWC$ and $I + WQ_0$ simultaneously invertible, with $W = W_0 + \Delta W$. Let $F_0 = (I + WQ_0)^{-1}W$. Then, direct manipulations show

$$A - \lambda_0 I - BF_0(I - Q_0F_0)^{-1}C = A - \lambda_0 I - BWC.$$

That means, $A - \lambda_0 I - BF_0(I - Q_0F_0)^{-1}C$ is invertible, which according to the property of Schur complement, indicates that $M(F_0)$ is invertible. Afterwards, the proposed statement follows similar arguments to the proof of Proposition 2. ■

By Lemma 1, the following theorem follows immediately from Propositions 3-4.

Theorem 3: Consider the networked system (16)-(17) with undirected $\mathcal{G}_{\rm sys}$. Suppose that (A, B, C) has no fixed mode. Then, this system is structurally controllable, if and only if the network topology $\bar{\mathcal{G}}_{\rm sys}$ is globally input-reachable.

By characterizations of fixed mode [25], it can be validated that Theorem 1 is a special case of Theorem 3. However, unlike Theorem 1, the nonexistence of fixed mode is not necessarily necessary in the case with matrix-weighted edges.

VI. Conclusion

In this letter, structural controllability of undirected diffusive networks with vector-weighted edges is investigated. It is proven that, an undirected networked system with diffusively coupled identical high-order SIMO subsystems is structurally controllable, if and only if each subsystem is controllable and observable, and the network topology is globally input-reachable. It is also demonstrated that, such conditions are still valid when both directed and undirected edges exist. An extension has been further given when each subsystem is MIMO and the interaction links are matrix-weighted.

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