Group-bipartite consensus in the networks with cooperative-competitive interactions

Jun Liu, Hengyu Li, Jinchen Ji, and Jun Luo

***Abstract*—This brief addresses the group-bipartite consensus problem of multi-agent systems with cooperative-competitive interactions. By combining the characteristics of group consensus and bipartite consensus, the concept of group-bipartite consensus is introduced to specify multiple bipartite consensus behavior. A distributed control protocol is then proposed for the topology graphs with acyclic partition and sign-balanced couples. The network topology studied in this brief eliminates the constraint that negative links can only exist between different groups, and thus the weights between agents in the same group can be either positive or negative. Some necessary and sufficient conditions for solving group-bipartite consensus problems are established by constructing a new form of the Laplacian matrix associated with the directed communication graphs. A simulation example is given to validate the theoretical results.**

***Index Terms*—Group-bipartite consensus, negative link, acyclic partition, sign-balanced couple topology.**

I. Introduction

Unlike complete consensus or synchronization [1], group (or cluster) consensus [2] and bipartite consensus [3] deal with multiple control objectives for networked multi-agent systems and have received significant interest from research community in recent years. Generally, complete consensus or synchronization is associated with a cooperative network topology whose communication weights are all non-negative. On the contrary, group and bipartite consensus often arises from a network where cooperative and competitive links coexist. A competitive link, which is denoted by a negative weight in a network topology graph, can well represent an antagonistic relationship between two individuals [3]. Many types of networked systems with profound applications, such as in biological systems, communication engineering [4] and social networks [5], are established by introducing negative weights. Although group consensus and bipartite consensus are two important collective behaviors they demonstrate different multi-objective convergence forms. Specifically, group consensus is associated with a group partition topology, which can desynchronize the motion of agents from different groups [6], whereas bipartite consensus requires the agents to achieve a form of "modulus consensus" [3].

In realizing bipartite consensus, structural balance [3] is an inseparable companion to the structure of network topology. Under the structural balance assumption, many studies were devoted to the understanding of the bipartite consensus of networked multi-agent systems from different scenarios (e.g. Refs. [7–13]). On the other hand, for group consensus, the topologies with acyclic partitions and balanced couples are two typical network structures to ensure group consensus in networked multi-agent systems. These two topologies require that the weights between different groups are in-degree balanced [6, 14]. It is noted that the adjacent weights between agents in the same group are nonnegative over acyclic partition and balanced couple topologies. Hence, a network in a single group can be seen as a cooperative network. A question naturally arose: "can networked systems exhibit a compound behavior that combines group consensus and bipartite consensus (hereafter called *group-bipartite consensus*) by introducing competitive relationships between agents in the same group?" The essence to answer this question is an effective combination of bipartite and group consensus topologies. As a matter of fact, many modern production processes consist of several sub-processes and each sub-process can be equivalently categorized as a cooperative-competitive group. Then the agents in each group are required to fulfill a bipartite objective coordinately. Thus, the agents involved in a large-scale production line should be divided into several groups according to multiple bipartite manufacturing tasks to be accomplished. Compared with the traditional bipartite consensus, group-bipartite consensus is more appropriate to represent this type of industrial manufacturing process, and therefore has an advantage over the general set-point control schemes in many practical applications.

Based on the above motivation, this brief mainly studies the group–bipartite consensus problem for multi-agent systems. Specifically, the network topology discussed in the current work eliminates the constraint that negative links can only exist between different groups. This means that the weights between agents can be either positive or negative. A new form of the Laplacian matrix associated with the communication topology graph is introduced by exploiting thoroughly the structure of the group-bipartite network topology. Then, the group–bipartite consensus problem in networked multi-agent systems over acyclic partition and sign-balanced couple topologies is addressed. Additionally, the final states of

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group-bipartite consensus are explicitly given by analyzing the eigenvectors of the Laplacian matrix.

This brief is organized as follows. After providing some preliminaries in Section II, Sections III and IV discuss the group-bipartite consensus problem over acyclic partition and sign-balanced couple topologies, respectively. A simulation example is given to validate the theoretical results in Section V, and conclusion and future work is discussed in Section VI.

Throughout this paper, $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the set of real numbers, the set of n-dimensional Euclidean space, and the set of $m \times n$ real matrices, respectively. $\mathbf{0}_n \in \mathbb{R}^n$ and $\mathbf{1}_n \in \mathbb{R}^n$ are the vectors with all zeros and ones, respectively. $O_{m\times n} \in \mathbb{R}^{m\times n}$ is the zero matrix. For a matrix A, A^{-1} and A^{T} are its transposition and inverse. For a square matrix A, $Eig(A)$ represents the union of the eigenvalue set of A.

II. Preliminaries

A. Graph theory

Generally, $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is used to denote a weighted directed graph of order n, where $\mathcal{V} = \{1, 2, \cdots, n\}$, $\mathcal{E} \in \mathcal{V}\times\mathcal{V}$ and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ are the node set, the edge set and the weighted adjacency matrix, respectively. Here, node i is employed to denote the i-th agent. Moreover, $(j, i) \in \mathcal{E}$ means that there is a directed path from agent j to agent i, and $(j, i) \in \mathcal{E} \Leftrightarrow a_{ij} \neq 0$. In this brief, it is assumed $a_{ii} = 0$, $i = 1, 2, \cdots, n$. A directed path in $\mathcal{G}$ is a sequence of distinct edges of the form (l_1, l_2), (l_2, l_3), $\cdots$, (l_{k-1}, l_k) if $(l_{j-1}, l_j) \in \mathcal{E}$. The directed graph $\mathcal{G}$ has a directed spanning tree if there exists at least one agent with a directed path to any other agent. A directed graph $\mathcal{G}$ is said to be structurally balanced if $\mathcal{V}$ can be partitioned into two subsets as $\{\mathcal{V}^{(1)}, \mathcal{V}^{(2)}\}$ which satisfy $\mathcal{V}^{(1)} \cup \mathcal{V}^{(2)} = \mathcal{V}$ and $\mathcal{V}^{(1)} \cap \mathcal{V}^{(2)} = \emptyset$, and also satisfy that all $a_{ij} \geq 0$, if i and j are from the same set $\mathcal{V}^{(k)}$, $k = 1, 2$, and all $a_{ij} \leq 0$ if i and j are from different sets $\mathcal{V}^{(k)}$ [5].

B. Problem formulation

Consider a given adjacency matrix $\mathcal{A}$ associated with the network topology under a partition $\{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_k\}$. We use $\mathcal{G}_i$ to denote the underlying topology of the node subsets $\mathcal{V}_i$, $i = 1, 2, \cdots, k$. Without loss of generality, the node set of each group is indexed as $\mathcal{V}_i = \{q_{i-1}+1, q_{i-1}+2, \cdots, q_i\}$, where $q_i = \sum_{j=1}^{i} h_j$, $q_0 = h_0 = 0$, $h_i > 0$, $i \in \{2, 3, \cdots, k\}$; $q_k = \sum_{j=1}^{k} h_j = n$, $i = 1, 2, \cdots, k$. For convenience, let $\bar{i}$ be the index of the group of a partition in which the number i is in, that is, if $i \in \mathcal{V}_j$, then $\bar{i} = j$.

The adjacency matrix $\mathcal{A}$ can then be rewritten in block matrix form as $\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \cdots & \mathcal{A}_{1k} \\ \cdots & \cdots & \cdots \\ \mathcal{A}_{k1} & \cdots & \mathcal{A}_{kk} \end{pmatrix}$, where $\mathcal{A}_{ii} \in \mathbb{R}^{h_i\times h_i}$ indicates the commutation weights between the agents in $\mathcal{V}_i$, and $\mathcal{A}_{ij}$, $i \neq j$, denotes the information commutation weights from the agents in $\mathcal{V}_j$ to the agents in $\mathcal{V}_i$, $i, j \in \{1, 2, \cdots, k\}$.

Assumption 1. *Each of $\mathcal{G}_i$ is structurally balanced.*

Under Assumption 1, we can obtain from Refs. [3, 7] that the node set $\mathcal{V}_i$ in the subgraph $\mathcal{G}_i$ can be partitioned into two subsets as $\{\mathcal{V}_i^{(1)}, \mathcal{V}_i^{(2)}\}$ satisfying $\mathcal{V}_i^{(1)} \cup \mathcal{V}_i^{(2)} = \mathcal{V}_i$ and $\mathcal{V}_i^{(1)} \cap \mathcal{V}_i^{(2)} = \emptyset$. Furthermore, we define a diagonal matrix $\Phi_i = \mathrm{diag}\{\phi_{1+q_{i-1}}, \phi_{2+q_{i-1}}, \cdots, \phi_{q_i}\}$, $\phi_j \in \{1, -1\}$, where $\phi_j = 1$, for $j \in \mathcal{V}_i^{(1)}$, and $\phi_j = -1$, for $j \in \mathcal{V}_i^{(2)}$ such that $\Phi_i\mathcal{A}_{ii}\Phi_i$ has all nonnegative entries.

Consider a network of n agents, and the dynamics of i-th agent are governed by

$$\dot{x}_i(t) = u_i(t), \tag{1}$$

where $x_i(t) \in \mathbb{R}$ is the state of agent i, and $u_i(t) \in \mathbb{R}$ is the designed control input based on the states of node i and its neighbors. Here we consider one dimensional integrator model for brevity. The corresponding results can be easily extended to higher-dimensional case by using Kronecker Product.

The Laplacian matrix $\mathcal{L}$ associated with $\mathcal{G}$ in this brief is defined as $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$, where

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{j\notin\mathcal{V}_{\bar{i}}} \phi_j a_{ij} + \sum_{j\in\bar{i}} |a_{ij}|, & i = j. \end{cases} \tag{2}$$

$\mathcal{L}$ has a block form according to the adjacency matrix $\mathcal{A}$ as

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \cdots & \mathcal{L}_{1k} \\ \cdots & \cdots & \cdots \\ \mathcal{L}_{k1} & \cdots & \mathcal{L}_{kk} \end{pmatrix}, \tag{3}$$

where $\mathcal{L}_{ii}$ corresponds to the graphs $\mathcal{G}_i$, and $\mathcal{L}_{ij}$ describes the information exchange from group $\mathcal{G}_j$ to $\mathcal{G}_i$, $i, j \in \{1, 2, \cdots, k\}$.

Remark 1. *Although the Laplacian matrix $\mathcal{L}$ may have negative eigenvalues, the diagonal dominance of the Laplacian matrix associated with subgraph $\mathcal{G}_i$ can be guaranteed. Furthermore, two typical network topologies, i.e., acyclic partition and balanced couple topologies, which are frequently used to investigate group consensus [6], can ensure that no negative eigenvalues corresponding to the Laplacian matrix $\mathcal{L}$ appear under some reasonable assumptions. This point is useful for convergence analysis in the cases of acyclic partition and balanced couple topologies.*

Consider a graph shown in Fig. 1 with partition $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2\}$, where $\mathcal{V}_1 = \{1, 2\}$, $\mathcal{V}_2 = \{3, 4, 5\}$. Obviously, $\phi_1 = \phi_3 = \phi_4 = 1$, and $\phi_2 = \phi_5 = -1$. The adjacency matrix is $\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix}$, where $\mathcal{A}_{11} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, $\mathcal{A}_{12} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathcal{A}_{21} = \begin{pmatrix} -1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathcal{A}_{22} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$. The corresponding Laplacian matrix is $\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}$, where $\mathcal{L}_{11} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $\mathcal{L}_{12} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathcal{L}_{21} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, and $\mathcal{L}_{22} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. From Fig. 1, it is easy to see that each of the subgroups $\mathcal{G}_1$ and $\mathcal{G}_2$ is structurally balanced, indicating that bipartite consensus can be realized in each individual subgroup. Therefore, the communication formations between $\mathcal{G}_1$ and $\mathcal{G}_2$ in Fig. 1 might cancel each other out, meaning that the states of system (1) present a behavior involving not only group consensus but also bipartite

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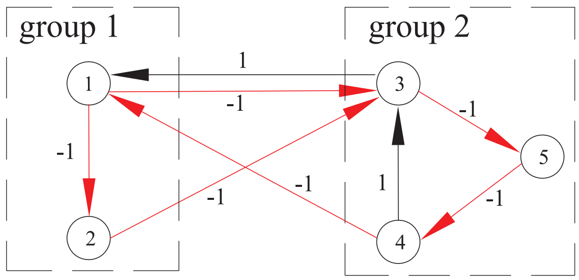

Fig. 1. Topology graph satisfying Assumption 1.

consensus features. This compound behavior has a multi-symmetric convergence feature, which will be formally defined below.

Definition 1. *The system (1) is said to reach group-bipartite consensus under the partition* $\{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_k\}$, *if* $\lim_{t\to\infty} |x_i(t) - \alpha_{\bar{i}}| = 0$, *for* $i \in \mathcal{V}_{\bar{i}}^{(1)}$, *and* $\lim_{t\to\infty} |x_i(t) + \alpha_{\bar{i}}| = 0$, *for* $i \in \mathcal{V}_{\bar{i}}^{(2)}$, *where* $i = 1, 2, \cdots, n$, α_j, $j = 1, 2, \cdots, k$, *are some constants.*

To achieve group-bipartite consensus in the networked multi-agent systems, the control protocol for the i-th agent is designed as:

$$u_i(t) = \sum_{j\in\bar{i}} a_{ij}\left[x_j(t) - \mathrm{sgn}(a_{ij})x_i(t)\right] + \sum_{j\notin\bar{i}} a_{ij}\left[x_j(t) - \phi_j x_i(t)\right]. \tag{4}$$

Let $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^{\mathrm{T}}$, then we have

$$\dot{x}(t) = -\mathcal{L}x(t). \tag{5}$$

Remark 2. *The concept of group-bipartite consensus given in Definition 1 demonstrates multiple origin-symmetric states, which distinguishes from bipartite consensus [3, 8–13] and group (or cluster) consensus [2, 4, 6, 14]. Moreover, by introducing the parameters* $\mathrm{sgn}(a_{ij})$ *and* ϕ_j, *the control protocol proposed in Eq. (4) can be regarded as an effective combination and generalization of bipartite consensus and group (or cluster) consensus control protocols under which the networked multi-agent systems can realize the group-bipartite consensus in the sense of Definition 1.*

III. Group-bipartite consensus over acyclic partition

Suppose that the partition $\{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_k\}$ is acyclic. This indicates that the Laplacian matrix $\mathcal{L}$ has a lower block-triangular form as [6]: $\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \cdots & O_{h_1\times h_k} \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{k1} & \cdots & \mathcal{L}_{kk} \end{pmatrix}$. Here, $\mathcal{L}_{st}$, $s < t$, $s, t \in \{1, 2, \cdots, k\}$, are zero submatrices, ensuring that the agents in the lower ranked groups cannot transmit their information to the groups ahead of them. The following two assumptions are made for acyclic partition topology [6].

Assumption 2. *Sum of each row of* $\Phi_i\mathcal{L}_{ij}\Phi_j$ *is zero,* $(i \neq j)$.

Assumption 3. *Every* $\mathcal{G}_i$ *has a spanning tree.*

Remark 3. *Assumption 2 is associated with the topological structure of networks and is a natural generalization of the assumptions for group (or cluster) consensus, which was adopted in many existing studies (e.g., [6]). In fact, Assumption 2 shows that the effects on one agent from the agents in different groups cancel each other out in the sense of group consensus.*

Lemma 1. *Suppose that* $\{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_k\}$ *is an acyclic partition and that Assumption 2 holds, then Assumption 3 holds if and only if the Laplacian* $\mathcal{L}$ *has a zero eigenvalue whose algebraic and geometric multiplicity are both* k *and all the other eigenvalues have positive real parts. Furthermore,* $p_1 = \left(\mathbf{1}_{h_1}^{\mathrm{T}}\Phi_1, \mathbf{0}_{n-h_1}^{\mathrm{T}}\right)^{\mathrm{T}}$, $p_2 = \left(\mathbf{0}_{h_1}^{\mathrm{T}}, \mathbf{1}_{h_2}^{\mathrm{T}}\Phi_2, \mathbf{0}_{n-h_1-h_2}^{\mathrm{T}}\right)^{\mathrm{T}}$, $\cdots$, $p_k = \left(\mathbf{0}_{n-h_k}^{\mathrm{T}}, \mathbf{1}_{h_k}^{\mathrm{T}}\Phi_k\right)^{\mathrm{T}}$ *are the* k *linearly independent right eigenvectors of* $\mathcal{L}$ *associated with the zero eigenvalue. The* k *linearly independent left eigenvectors associated with the zero eigenvalue can be expressed as* $\pi_1 = \left(\alpha_1^{\mathrm{T}}, \mathbf{0}_{n-h_1}^{\mathrm{T}}\right)^{\mathrm{T}}$, $\pi_2 = \left((\beta_1^{[2]})^{\mathrm{T}}, \alpha_2^{\mathrm{T}}, \mathbf{0}_{n-h_1-h_2}^{\mathrm{T}}\right)^{\mathrm{T}}$, $\cdots$, $\pi_k = \left((\beta_1^{[k]})^{\mathrm{T}}, (\beta_2^{[k]})^{\mathrm{T}}, \cdots, (\beta_{k-1}^{[k]})^{\mathrm{T}}, \alpha_k^{\mathrm{T}}\right)^{\mathrm{T}}$, *where* $\Phi_i = \mathrm{diag}\{\phi_{n_{i-1}}, \phi_{n_{i-1}+1}, \cdots, \phi_{n_i}\} \in \mathbb{R}^{h_i}$, $i = 1, 2, \cdots, k$, α_i, $\beta_i^{[j]} \in \mathbb{R}^{h_i}$ *satisfying that* $\Phi_i\alpha_i$ *are nonnegative vectors,* $\alpha_i^{\mathrm{T}}\Phi_i\mathbf{1}_{h_i} = 1$ *and* $(\beta_i^{[j]})^{\mathrm{T}}\Phi_i\mathbf{1}_{h_i} = 0$, $i, j \in \{1, 2, \cdots, k\}$.

Proof: The structure of $\mathcal{L}$ means that $Eig(\mathcal{L}) = \cup_{i=1}^{k} Eig(\mathcal{L}_{ii})$. Then, from the definition of $\mathcal{L}$ and Assumption 1, we can say that $\Phi_i\mathcal{L}_{ii}\Phi_i$ is a traditional Laplacian matrix with nonnegative weights. Note that $\Phi_i\mathcal{L}_{ii}\Phi_i$ is similar to $\mathcal{L}_{ii}$, then it is obvious that $\mathcal{L}$ has k zero eigenvalues, and all the other eigenvalues have positive real parts if and only if Assumption 3 holds.

Since $l_{ii} = -\sum_{j\notin\mathcal{V}_{\bar{i}}}\phi_j a_{ij} + \sum_{j\in\bar{i}}|a_{ij}|$ and Assumption 2 hold, then $p_1, p_2, \cdots, p_k$ are the k right eigenvectors of $\mathcal{L}$ corresponding to the zero eigenvalues. For the matrix $\mathcal{L}_{ii}$, obviously, under Assumptions 3, we can obtain that $\Phi_i\mathcal{L}_{ii}\Phi_i$ has a nonnegative left eigenvector $\gamma_i \in \mathbb{R}^{h_i}$ associated with the zero eigenvalue, satisfying $\gamma_i^{\mathrm{T}}\mathbf{1}_{h_i} = 1$. Define $\alpha_1 = \Phi_1\gamma_1$, and from the lower triangular block structure of $\mathcal{L}$, π_1 is a left eigenvector of $\mathcal{L}$ associated with the zero eigenvalue. For block $B_2 = \begin{pmatrix} \mathcal{L}_{11} & O_{h_1\times h_2} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}$, $\tilde{B} = \mathrm{diag}\{\Phi_1, \Phi_2\}B_2\mathrm{diag}\{\Phi_1, \Phi_2\} = \begin{pmatrix} \Phi_1\mathcal{L}_{11}\Phi_1 & O_{h_1\times h_2} \\ \Phi_2\mathcal{L}_{21}\Phi_1 & \Phi_2\mathcal{L}_{22}\Phi_2 \end{pmatrix}$. Moreover, from Assumption 2, the sum of each row of $\mathcal{L}_{21}\Phi_1$ is zero, as is the matrix $\Phi_2\mathcal{L}_{21}\Phi_1$. This, combined with the self-similarity of $\mathrm{diag}\{\Phi_1, \Phi_2\}$, gives that $(\mathbf{1}_{h_1}^{\mathrm{T}}, \mathbf{0}_{h_2}^{\mathrm{T}})^{\mathrm{T}}$ and $(\mathbf{0}_{h_1}^{\mathrm{T}}, \mathbf{1}_{h_2}^{\mathrm{T}})^{\mathrm{T}}$ are two linearly independent right eigenvectors of $\tilde{B}$ associated with the zero eigenvalue, and $\tilde{B}$ has a left eigenvector $(c^{\mathrm{T}}, \gamma_2^{\mathrm{T}})^{\mathrm{T}}$ corresponding to the zero eigenvalue, satisfying $c^{\mathrm{T}}\mathbf{1}_{h_1} = 0$. Therefore, by defining $\beta_1^{[2]} = \Phi_1 c$ and $\alpha_2 = \Phi_2\gamma_2$, we can conclude that π_2 is a left eigenvector of $\mathcal{L}$ associated with the zero eigenvalues. Then, the results of this lemma can be easily obtained by repeating the above procedures. ■

We are now ready to solve the group-bipartite consensus problem under an acyclic partition topology.

Theorem 1. *Suppose that* $\mathcal{G}$ *has an acyclic partition* $\{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_k\}$ *and that Assumptions 1 and 2 hold. Then, under the control protocol (4), system (1) admits a group-bipartite consensus solution in the sense of Definition 1 if and only if Assumption 3 holds. Furthermore, under Assumptions 1, 2 and 3,* $\lim_{t\to\infty} x_i(t) = \phi_i\pi_{\bar{i}}^{\mathrm{T}}x(0)$, $i = 1, 2, \cdots, n$.

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Proof: (Sufficient) Define $\mathcal{L}_\Phi = \Phi\mathcal{L}\Phi$, where $\Phi = \text{diag}\{\Phi_1, \Phi_2, \cdots, \Phi_k\}$. It is easy to see that the eigenvalues of $\mathcal{L}_\Phi$ are $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$, $\lambda_{k+1}, \lambda_{k+2}, \cdots, \lambda_n$, which satisfy that λ_j, $j = k+1, \cdots, n$, have positive real parts. From the structure of the vectors p_i and π_i, $i = 1, 2, \cdots, k$, $\mathcal{L}_\Phi$ has a Jordan canonical form as $\mathcal{L}_\Phi = C\begin{pmatrix} O_{k\times k} & O_{k\times(n-k)} \\ O_{(n-k)\times k} & J \end{pmatrix}C^{-1}$, where $C = \left(\frac{1}{h_1}\Phi p_1, \frac{1}{h_2}\Phi p_2, \cdots, \frac{1}{h_k}\Phi p_k, \omega_1, \cdots, \omega_{n-k}\right)$, $C^{-1} = [\Phi\pi_1, \Phi\pi_2, \cdots, \Phi\pi_k, \nu_1, \cdots, \nu_{n-k}]^{\mathrm{T}}$, p_i and π_i, $i = 1, 2, \cdots, k$, are defined in Lemma 1, $\omega_i \in \mathbb{R}^n$ can be chosen to be the right eigenvectors or the generalized eigenvectors, $\nu_i \in \mathbb{R}^n$, $i = 1, 2, \cdots, n-k$, can be the left eigenvectors or the generalized eigenvectors of $\mathcal{L}_\Phi$, and J is the Jordan upper-diagonal block matrix associated with the eigenvalues $\lambda_{k+1}, \cdots, \lambda_n$. Note that all $\lambda_{k+1}, \cdots, \lambda_n$ have positive real parts; then, by using $\exp(-\mathcal{L}_\Phi t) = C\exp(-J_\Phi t)C^{-1}$, we have

$$\lim_{t\to\infty} \exp(-\mathcal{L}_\Phi t) = \left(\Phi\pi_1\mathbf{1}_{h_1}^{\mathrm{T}}, \Phi\pi_2\mathbf{1}_{h_2}^{\mathrm{T}}, \cdots, \Phi\pi_k\mathbf{1}_{h_k}^{\mathrm{T}}\right)^{\mathrm{T}}. \quad (6)$$

Then, the solution of system (1) has the limit $\lim_{t\to\infty} x(t) = \Phi\left(\pi_1\mathbf{1}_{h_1}^{\mathrm{T}}, \pi_2\mathbf{1}_{h_2}^{\mathrm{T}}, \cdots, \pi_k\mathbf{1}_{h_k}^{\mathrm{T}}\right)^{\mathrm{T}} x(0)$.

(Necessity) Otherwise, there exists at least one agent which cannot receive any information from other agents in the same group. Then, group-bipartite consensus cannot be reached in this case ■

Remark 4. *If groups $\mathcal{G}_i$, $i = 1, 2, \cdots, k$, are all cooperation-type networks, then the traditional group consensus can be reached if each of $\mathcal{G}_i$, $i = 1, 2, \cdots, k$, has a spanning tree, which means that group consensus is a special case of group-bipartite consensus. Furthermore, the bipartite property of each group plays an important role in reaching group-bipartite consensus. Without the assumption, the agents in one group may display complete consensus or fragmental behaviour.*

IV. Group-bipartite consensus over sign-balanced couple

A balanced couple graph with a partition $\{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_k\}$ was introduced in Ref. [14] to describe the weight balanced between different groups and the agents in the same group. While, the bi-partition of the group $\mathcal{G}_i$ implies that the final bipartite consensus may occur. Therefore, the definition of the balanced couple graph should be amended according to the group-bipartite graph. We now define the sign-balanced couple graph with group-bipartite interaction under Assumption 1.

Definition 2. *The directed graph $\mathcal{G}$ is said to be sign balanced if $\sum_{j=1}^n |a_{ij}| = \sum_{j=1}^n |a_{ji}|$.*

Definition 3. *Suppose that the node set $\mathcal{V}$ in graph $\mathcal{G}$ has a partition $\{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_k\}$. If the weights a_{ij}, $i, j \in \{1, 2, \cdots, n\}$, satisfy that $\sum_{l\in\mathcal{V}_j} \phi_l a_{lk} = 0$ for $\forall k \in \mathcal{V}_i$, $i \neq j$, then $\mathcal{G}_i$ is sign out–degree balanced with $\mathcal{G}_j$. $\mathcal{G}_i$ and $\mathcal{G}_j$ are a sign-balanced couple if $\mathcal{G}_i$ and $\mathcal{G}_j$ are sign out-degree balanced with each other. Furthermore, $\mathcal{G}$ is called a sign-balanced couple graph if each of the $\mathcal{G}_i$, $i = 1, 2, \cdots, n$, is sign balanced and for $\forall i, j \in \{1, 2, \cdots, k\}$, $i \neq j$, $\mathcal{G}_i$ and $\mathcal{G}_j$ are a sign-balanced couple.*

Remark 5. *Definitions 2 and 3 can be seen as an extension of the traditional balanced and balanced couple graphs with all nonnegative weights. Obviously, if all groups $\mathcal{G}_i$, $i = 1, 2, \cdots, k$, are of the cooperation type, then $\phi_j = 1$, $j = 1, 2, \cdots, n$. Accordingly, Definitions 2 and 3 will be reduced to the traditional balanced graph and balanced couple graphs as defined in Ref. [14]. Therefore, the theoretical results obtained in this brief will also be applicable for the traditional balanced and balanced couple graphs with nonnegative weights between agents in the same group.*

The following assumption is introduced in order to study group-bipartite consensus in this section.

Assumption 4. *$\mathcal{G}$ with partition $\{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_k\}$ is a sign-balanced couple graph.*

Following the same analysis method of Lemma 1, we have that $p_1 = \left(\mathbf{1}_{h_1}^{\mathrm{T}}\Phi_1, \mathbf{0}_{n-h_1}^{\mathrm{T}}\right)^{\mathrm{T}}$, $p_2 = \left(\mathbf{0}_{h_1}^{\mathrm{T}}, \mathbf{1}_{h_2}^{\mathrm{T}}\Phi_2, \mathbf{0}_{n-h_1-h_2}^{\mathrm{T}}\right)^{\mathrm{T}}$, $\cdots$, $p_k = \left(\mathbf{0}_{n-h_k}^{\mathrm{T}}, \mathbf{1}_{h_k}^{\mathrm{T}}\Phi_k\right)^{\mathrm{T}}$ are k linearly independent right and left eigenvectors of $\mathcal{L}$ associated with the zero eigenvalue if Assumptions 1, 2, 3 and 4 hold. This gives the conditions required to solve the group-bipartite consensus problem over sign-balanced couple topology.

Theorem 2. *Under Assumptions 1, 2, 3 and 4, system (1) can reach group-bipartite consensus if and only if the matrix $\mathcal{L}$ has exactly k zero eigenvalues and all the remaining eigenvalues of $\mathcal{L}$ have positive real parts. In addition, the group-bipartite solution of (1) is given by $\lim_{t\to\infty} x(t) = \Phi\left(\frac{1}{h_1}p_1\mathbf{1}_{h_1}^{\mathrm{T}}, \frac{1}{h_2}p_2\mathbf{1}_{h_2}^{\mathrm{T}}, \cdots, \frac{1}{h_k}p_k\mathbf{1}_{h_k}^{\mathrm{T}}\right)^{\mathrm{T}} x(0)$.*

Proof: The proof is a straightforwad process similar to the proof of Theorem 1. ■

Remark 6. *Unlike the acyclic partition case, Assumption 3 cannot guarantee that $\mathcal{L}$ has exactly k zero eigenvalues and that all the remaining eigenvalues of $\mathcal{L}$ have positive real parts. Therefore, Theorem 2 only gives an algebraic condition to ensure group-bipartite consensus for a sign-balanced couple network.*

V. Simulations

As an application of Theorem 1, we consider a network with an acyclic partition having three subgraphs $\mathcal{V}_1 = \{1, 2, 3\}$, $\mathcal{V}_2 = \{4, 5, 6, 7\}$ and $\mathcal{V}_3 = \{8, 9, 10, 11\}$. The interaction topology is shown in Fig. 2. Obviously, Assumptions 1, 2 and 3 hold. Moreover, by straightforward calculation, we have $\phi_i = \begin{cases} 1, & i = 1, 3, 4, 5, 8, 9, \\ -1, & i = 2, 6, 7, 10, 11. \end{cases}$ The initial value $x(0)$ is taken as $x(0) = (1, -2, 3, 1.5, 4, 6, -3.5, -4, 5, -1, -3)^{\mathrm{T}}$. According to Theorem 1, the final group-bipartite state of $x(t)$ is $\Phi\left(\pi_1\mathbf{1}_3^{\mathrm{T}}, \pi_2\mathbf{1}_4^{\mathrm{T}}, \pi_3\mathbf{1}_4^{\mathrm{T}}\right)^{\mathrm{T}} x(0)$ with $\Phi = \text{diag}\{\phi_1, \cdots, \phi_{11}\}$, $\pi_1 = (\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0)^{\mathrm{T}}$, $\pi_2 = (\frac{1}{6}, \frac{1}{12}, -\frac{1}{12}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0, 0, 0)^{\mathrm{T}}$, and $\pi_3 = (\frac{1}{8}, \frac{1}{16}, -\frac{1}{16}, -\frac{1}{16}, -\frac{1}{16}, \frac{1}{16}, -\frac{3}{16}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})^{\mathrm{T}}$. Specifically, $\lim_{t\to\infty} x_1 = \lim_{t\to\infty} x_3 = -\lim_{t\to\infty} x_2 = 2$, $\lim_{t\to\infty} x_4 = \lim_{t\to\infty} x_5 = -\lim_{t\to\infty} x_6 = -\lim_{t\to\infty} x_7 = 0.5$, and $\lim_{t\to\infty} x_8 = \lim_{t\to\infty} x_9 = -\lim_{t\to\infty} x_{10} = -\lim_{t\to\infty} x_{11} = 1.75$. As shown in Fig. 3, the trjectories of all the agents gradually approach to a triple-bipartite final convergent state, indicating that group-bipartite has been achieved.

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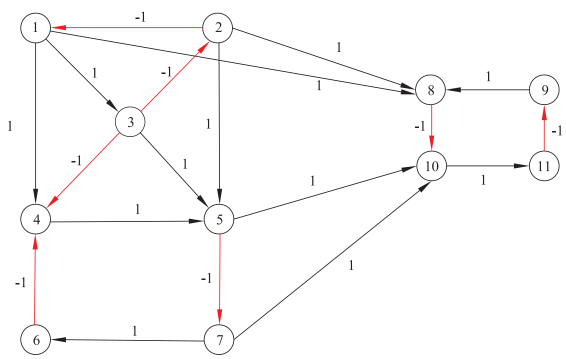

Fig. 2. The network topology with an acyclic partition

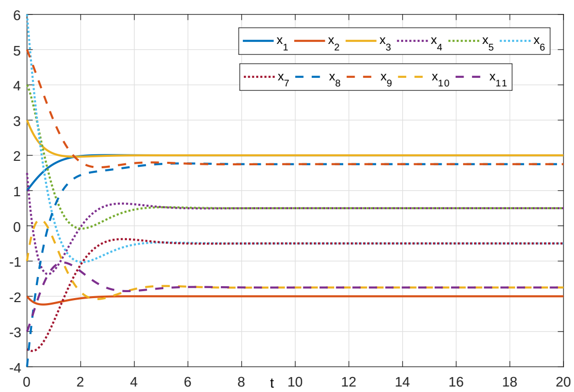

Fig. 3. State evolution of the agents.

VI. Conclusion

This brief introduced the definition of the group-bipartite consensus of networked multi-agent systems. A neighbor-based control protocol was then proposed to realize group-bipartite consensus. By modifying the diagonal elements of the traditional Laplacian matrix, a new form of Laplacian matrix was constructed for group-bipartite convergence analysis. The final group-bipartite solution was explicitly given for the networked systems under acyclic partitioned and sign-balanced couple network topologies. It should be noted that the present research is only the first step in the group-bipartite control of networked multi-agent systems in this direction, and the topologies discussed in this brief are special to some extent. Moreover, when the topology is a sign-balanced couple structure, the algebraic condition given in Theorem 2 requires global information, such as the details of the Laplacian matrix, and therefore for a large sized network of multi-agent systems, the condition presented in Theorem 2 would be difficult to verify. Our future work will be focused on considering more general network topologies and developing fully distributed group-bipartite consensus protocols.

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