Observability Through a Matrix-Weighted Graph

S. Emre Tuna

***Abstract*—Observability of an array of identical linear time-invariant systems with incommensurable output matrices is studied, where an array is called observable when identically zero relative outputs imply synchronized solutions for the individual systems. It is shown that the observability of an array is equivalent to the connectivity of its interconnection graph, whose edges are assigned matrix weights. Moreover, to better understand the relative behavior of distant units, pairwise observability that concerns with the synchronization of a certain pair of individual systems in the array is studied. This milder version of observability is shown to be closely related to certain connectivity properties of the interconnection graph as well. Pairwise observability is also analyzed using the circuit theoretic tool effective conductance. The observability of a certain pair of units is proved to be equivalent to the nonsingularity of the (matrix-valued) effective conductance between the associated pair of nodes of a resistive network (with matrix-valued parameters) whose node admittance matrix is the Laplacian of the array's interconnection graph.**

***Index Terms*—Connectivity, effective conductance, matrix-weighted graph, observability, synchronization.**

I. Introduction

OBSERVABILITY is one of the central concepts in systems theory which, for linear time-invariant (LTI) systems, can be expressed in many seemingly different yet mathematically equivalent forms [8]. One alternative is the following. *A pair* $[C,\,A]$ *is observable if* $C(x_1(t)-x_2(t))\equiv 0$ *implies* $x_1(t)\equiv x_2(t)$, where $x_i(t)$ denote the solutions of two identical systems $\dot{x}_i=Ax_i$, $i=1,\,2$. Admittedly, this appears to be an uneconomical definition, for the implication therein employs two systems where one would have sufficed. The overuse, however, has a relative advantage: It points in an interesting direction of generalization. Namely, for a pair $[(C_{ij})_{i,j=1}^q,\,A]$, the following condition suggests itself as a natural extension:

$$C_{ij}(x_i(t)-x_j(t))\equiv 0\;\forall (i,\,j)\implies x_i(t)\equiv x_j(t)\;\forall (i,\,j)\quad (1)$$

where $x_i(t)$ are the solutions of q identical systems $\dot{x}_i=Ax_i$, $i=1,\,2,\,\ldots,\,q$. Aside from its theoretical allure, this particular choice of generalization is not without practical motivation; the condition (1) happens to be both necessary and sufficient for synchronization of certain arrays of oscillators. We give two examples in the sequel.

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The author is with the Department of Electrical and Electronics Engineering, Middle East Technical University, Ankara 06800, Turkey (e-mail: etuna@metu.edu.tr).

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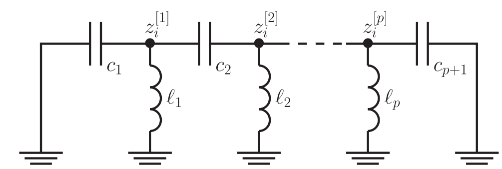

Fig. 1. Electrical oscillator.

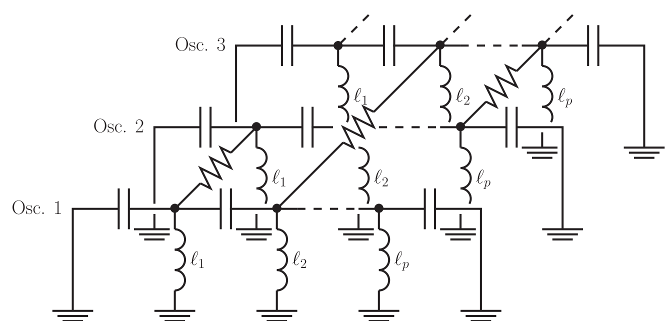

Fig. 2. Array of coupled electrical oscillators.

Coupled Electrical Oscillators: Consider a collection of q identical oscillators, where each oscillator comprises LTI (passive) inductors and LTI (passive) capacitors connected as shown in Fig. 1. The node voltages of the ith individual oscillator are denoted by $z_i^{[\ell]}\in\mathbb{R}$. The state of the ith oscillator can then be represented by $x_i=[z_i^T\;\dot{z}_i^T]^T\in\mathbb{R}^{2p}$, where $z_i=[z_i^{[1]}\;z_i^{[2]}\;\cdots\;z_i^{[p]}]^T$ is the node voltage vector. Note that we can find a matrix $A\in\mathbb{R}^{2p\times 2p}$ satisfying $\dot{x}_i=Ax_i$ when the oscillators are not coupled; see [19]. Let now an array be constructed by coupling certain pairs of oscillators via unit $(1\,\Omega)$ resistors as shown in Fig. 2. The voltages across these resistors are clearly of the form $z_i^{[\ell]}-z_j^{[\ell]}$. Hence, stacking the resistor voltages associated to a particular pair $(i,\,j)$ of indices yields the (relative) output signal $y_{ij}=C_{ij}(x_i-x_j)$ where the rows of $C_{ij}\in\mathbb{R}^{m_{ij}\times 2p}$ are of the form $[e_\ell^T\;\;0_{1\times p}]$. ($e_\ell\in\mathbb{R}^p$ denotes the unit vector whose ℓth entry is one.) Now suppose that the condition (1) holds for our array. Then, (and only then) the oscillators synchronize, i.e., $\|x_i(t)-x_j(t)\|\to 0$ for all $(i,\,j)$ and all initial conditions $x_1(0),\,x_2(0),\,\ldots,\,x_q(0)$. The reason is the following. All the components in our array are passive. Therefore, the total energy stored in the capacitors and inductors throughout the network is nonincreasing. Since the energy is never negative, in the steady state, it must have converged to some fixed value. In this constant energy state, the resistors throughout the network can no longer dissipate energy for otherwise the energy would further decrease, which it cannot because it is constant. A resistor with zero power output means that the

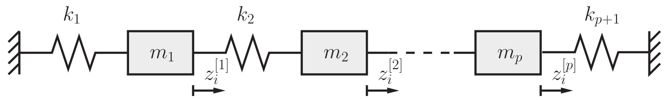

Fig. 3. Mechanical oscillator.

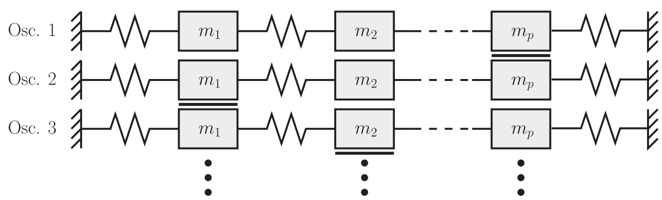

Fig. 4. Array of coupled mechanical oscillators.

current passing through it is zero, i.e., it imitates an open circuit. Therefore, in the steady state, the oscillator dynamics become decoupled $\dot{x}_i = Ax_i$. Moreover, a resistor with zero power output also means that the voltage across its terminals is zero, i.e., it imitates a short circuit. Therefore, (in the steady state) we have to have $C_{ij}(x_i(t) - x_j(t)) \equiv 0$ for all pairs (i, j). These explain why the condition (1) means synchronization for our coupled electrical oscillators. A similar case can be encountered also in the mechanical domain.

Coupled Mechanical Oscillators: Consider a collection of q identical oscillators, where each oscillator is a chain of masses and linear springs as shown in Fig. 3. Such chains are used to model the interaction of atoms in a crystal [6]. Let $z_i^{[\ell]} \in \mathbb{R}$ be the displacement of the mass m_ℓ of the ith individual oscillator from the equilibrium. Then, the state of the ith oscillator is $x_i = [z_i^T \ \dot{z}_i^T]^T \in \mathbb{R}^{2p}$, where $z_i = [z_i^{[1]} \ z_i^{[2]} \ \cdots \ z_i^{[p]}]^T$. As before, we let $A \in \mathbb{R}^{2p\times 2p}$ represent the decoupled dynamics $\dot{x}_i = Ax_i$. Consider now the array under the arrangement shown in Fig. 4, where certain pairs of oscillators are coupled through viscous friction between certain masses. (The coupling is represented by a dark line between blocks.) It is not difficult to see that the distribution of the friction associated to a certain pair (i, j) gives rise to the output matrix $C_{ij} \in \mathbb{R}^{m_{ij}\times 2p}$ with rows of the form $[e_\ell^T \ \ 0_{1\times p}]$. The passivity argument previously adopted for the synchronization analysis of the coupled electrical oscillators is valid here, too. The outcome is the same. Namely, under the condition (1) the mechanical oscillators synchronize. Having motivated the condition (1) in the context of synchronization, we will next try to explain its relation to certain assumptions that appear in the literature.

Synchronization of linear systems is a broad area of research, where one of the main goals of the researcher is to unearth conditions under which the solutions of coupled units converge to a common trajectory. Different sets of assumptions have led to a rich collection of results, bringing our understanding on the subject closer to complete; see, for instance, [7], [11], [12], [16], [22], and [23]. Despite their differences in degree and direction of generality, all these works share two assumptions in common: 1) the graph describing the interconnection contains a spanning tree and 2) the individual system is observable (detectable). We intend to emphasize in this paper that these two separate assumptions, the former on *connectivity* and the latter on *observability*, dissolve inseparably in the condition (1). In particular, for an array represented by the pair $[(C_{ij})_{i,j=1}^q, A]$, it is in general not meaningful to search for a spanning tree because the interconnection graph will be matrix weighted, whereas a tree is well defined for a scalar-weighted graph only. As for the second assumption, requiring the individual systems to be observable also falls prey to ambiguity since there is not a single output matrix for each system; instead every system is coupled to each of its neighbors through a different matrix C_{ij}. It is true that separation is possible in the special case $C_{ij} = w_{ij}C$ with $C \in \mathbb{C}^{m\times n}$ and $w_{ij} \in \mathbb{R}$. In this much-studied scenario, where the output matrices are commensurable, the scalar weights w_{ij} are used to construct the interconnection graph, which can be checked to contain a spanning tree; and the pair $[C, A]$ can separately be checked for observability. However, in general, the condition (1) in its entirety is what we have to deal with, which requires that we work with the matrix-weighted graphs. We will explain how these matrix-valued weights emerge soon. But first, let us review the scarce literature on observability over networks.

Observability over networks, motivated in general by synchronization (consensus) of coupled systems, and in particular, by distributed estimation [1], [17], [20], is largely an unexplored area of research. Among the few works is [9], where the observability of sensor networks is studied by means of equitable partitions of graphs. This tool is employed also in [14]. The observability of path and cycle graphs is studied in [15] and of grid graphs in [13]. Recently, the networks whose individual systems' dynamics are allowed to be nonidentical is covered in [24]. Each of the aforementioned investigations deals with a different case, yet they all consider interconnections that can be described by graphs with scalar-weighted edges. At this point, our work is located relatively far from the reported results. In particular, to the best of our knowledge, observability over matrix-weighted graphs has not yet been studied in detail.

In the first half of this paper, we report conditions on the array $[(C_{ij})_{i,j=1}^q, A]$ that imply observability in the sense of (1). To this end, we construct a graph Γ (with q vertices), where to each pair of vertices (v_i, v_j), we assign a weight that is a Hermitian positive semidefinite matrix, whose null space is the unobservable subspace corresponding to the individual pair $[C_{ij}, A]$. We reveal that the array $[(C_{ij})_{i,j=1}^q, A]$ is observable if and only if the interconnection graph Γ is connected. Also, we notice that for each distinct eigenvalue of A there exists a graph (we call it an *eigengraph*) and the observability of $[(C_{ij})_{i,j=1}^q, A]$ is ensured if no eigengraph is disconnected. For our analysis, we define the connectivity of a graph through a certain spectral property of its Laplacian. We note that the connectivity of a matrix-weighted graph cannot in general be characterized by the standard tools of the graph theory such as path and tree. The reason is that the meaning or function of an edge (out of which paths and trees are constructed) becomes equivocal when one has to allow semidefinite weights.

In the second half of this paper, we focus on the so-called (k, ℓ)-observability of $[(C_{ij})_{i,j=1}^q, A]$. Namely, for a given pair of indices $(k,\,\ell)$, we search for conditions under which $x_k(t) \equiv x_\ell(t)$ provided that $C_{ij}(x_i(t) - x_j(t)) \equiv 0$ for all $(i,\,j)$. To this end, we define $(k,\,\ell)$-connectivity of a matrix-weighted graph through its Laplacian. We show the equivalence between the $(k,\,\ell)$-observability of the array $[(C_{ij})_{i,j=1}^q,\,A]$ and the $(k,\,\ell)$-connectivity of the interconnection graph Γ as well as the *nonequivalence* between the $(k,\,\ell)$-observability of the array and the $(k,\,\ell)$-connectivity of its eigengraphs. Moreover, we present the interesting interchangeability between the $(k,\,\ell)$-observability of an array and the nonsingularity of the matrix $\Gamma_{k\ell}$, where $\Gamma_{k\ell}$ is the (matrix-valued) effective conductance between the nodes k and ℓ of a resistive network (with matrix-valued parameters) whose node admittance matrix is the Laplacian of the array's interconnection graph Γ. From a graph-theoretic point of view, the nonsingularity of the effective conductance Γ_{kl} may be interpreted to indicate that the pair of vertices $(v_k,\,v_\ell)$ of the matrix-weighted graph Γ are connected. This, therefore, allows one to study connectivity of vertices without employing paths; which is potentially useful since defining a path, as mentioned previously, is problematic for matrix-weighted graphs. One may ask why our formulation is in terms of effective conductance instead of the commoner effective resistance, e.g., [21]. The reason is that the conductances we work with are matrix valued and not necessarily invertible. That is, since resistance is the inverse of conductance, we would have run into certain difficulties had we chosen to employ effective resistance instead. Potential applications of generalized electrical circuits with matrix-valued parameters seem to have so far gone unnoticed by the control theorists. Notable exceptions are the works [2]–[4] on the problem of estimation over networks.

II. Preliminaries and Notation

In this section, we provide the formal definitions for the observability of an array and the connectivity of an n-graph through its Laplacian matrix. (The reader should be warned that the term *n-graph* has appeared in the literature in different meanings. In this paper, it means a weighted graph, where each pair of vertices is assigned an n-by-n matrix.)

A pair $[(C_{ij})_{i,j=1}^q,\,A]$ is meant to represent the array of $q \geq 2$ identical systems

$$\dot{x}_i = Ax_i \tag{2a}$$

$$y_{ij} = C_{ij}(x_i - x_j), \qquad i,\,j = 1,\,2,\,\ldots,\,q \tag{2b}$$

where $x_i \in \mathbb{C}^n$ is the state of the ith system with $A \in \mathbb{C}^{n\times n}$ and $y_{ij} \in \mathbb{C}^{m_{ij}}$ is the ijth relative output with $C_{ij} \in \mathbb{C}^{m_{ij}\times n}$. We let $C_{ii} = 0$. In our paper, we will solely be studying the case $y_{ij}(t) \equiv 0$ for all $(i,\,j)$. Hence, we suppose $C_{ij} = C_{ji}$ without loss of generality. The generality is not lost because if $C_{ij} \neq C_{ji}$, then we can always redefine $C_{ij}^{\rm new} = C_{ji}^{\rm new} := [C_{ij}^T\; C_{ji}^T]^T$; and then, $y_{ij}^{\rm new}(t) \equiv 0$ for all $(i,\,j)$ if and only if $y_{ij}(t) \equiv 0$ for all $(i,\,j)$. The ordered collection $(C_{ij})_{i,j=1}^q$ will sometimes be compactly written as (C_{ij}) when there is no risk of ambiguity. We now define observability and detectability of the array (2) as follows.

Definition 1: An array $[(C_{ij}),\,A]$ is said to be *observable* if

$$y_{ij}(t) \equiv 0 \text{ for all } (i,\,j) \Longrightarrow x_i(t) \equiv x_j(t) \text{ for all } (i,\,j)$$

and *detectable* if

$$y_{ij}(t) \equiv 0 \text{ for all } (i,\,j) \Longrightarrow \|x_i(t) - x_j(t)\| \to 0 \text{ for all } (i,\,j)$$

for all initial conditions $x_1(0),\,x_2(0),\,\ldots,\,x_q(0)$.

For each $(i,\,j)$ we denote by W_{ij} the observability matrix of the individual pair $[C_{ij},\,A]$. Namely

$$W_{ij} = \begin{bmatrix} C_{ij} \\ C_{ij}A \\ \vdots \\ C_{ij}A^{n-1} \end{bmatrix}.$$

By $\mu_1,\,\mu_2,\,\ldots,\,\mu_m$ $(1 \leq m \leq n)$, we denote the distinct eigenvalues of A. By $V_\sigma \in \mathbb{C}^{n\times n_\sigma}$, $\sigma = 1,\,2,\,\ldots,\,m$, we denote a full column rank matrix satisfying $\mathrm{range}\,V_\sigma = \mathrm{null}\,[A - \mu_\sigma I_n]$, where I_n is the n-by-n identity matrix. Note that the columns of V_σ are the linearly independent eigenvectors of A corresponding to the eigenvalue μ_σ. In particular, we have $AV_\sigma = \mu_\sigma V_\sigma$. This yields

$$\begin{aligned} V_\sigma^* W_{ij}^* W_{ij} V_\sigma &= V_\sigma^* \left(\sum_{k=0}^{n-1} A^{k*} C_{ij}^* C_{ij} A^k\right) V_\sigma \\ &= V_\sigma^* \left(\sum_{k=0}^{n-1} \mu_\sigma^{k*} C_{ij}^* C_{ij} \mu_\sigma^k\right) V_\sigma \\ &= \left(\sum_{k=0}^{n-1} |\mu_\sigma|^{2k}\right) V_\sigma^* C_{ij}^* C_{ij} V_\sigma \end{aligned} \tag{3}$$

where V_σ^* is the conjugate transpose of V_σ.

Sometimes due to notational convenience, we will represent the array (2) as a single big system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{4a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{4b}$$

where $\mathbf{x} = [x_1^T\;\; x_2^T\;\; \ldots\;\; x_q^T]^T$ is the state and the output is $\mathbf{y} = [y_{12}^T\; y_{13}^T\; \ldots\; y_{1q}^T \mid y_{23}^T\; y_{24}^T\; \ldots\; y_{2q}^T \mid \ldots \mid y_{(q-1)q}^T]^T$. (Note that only the outputs y_{ij} with indices $i < j$ are included in our big output $\mathbf{y}$ since y_{ji} contains no information that is not already contained in y_{ij}. Also, the order of the outputs y_{ij} as they appear in $\mathbf{y}$ is immaterial.) Clearly, we have

$$\mathbf{A} = [I_q \otimes A]$$

and $\mathbf{C}$ enjoys the following structure:

$$\mathbf{C} = \left[\begin{array}{ccccccc} C_{12} & -C_{12} & 0 & 0 & \cdots & 0 & 0 \\ C_{13} & 0 & -C_{13} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{1q} & 0 & 0 & 0 & \cdots & 0 & -C_{1q} \\ \hline 0 & C_{23} & -C_{23} & 0 & \cdots & 0 & 0 \\ 0 & C_{24} & 0 & -C_{24} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & C_{2q} & 0 & 0 & \cdots & 0 & -C_{2q} \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & \cdots & C_{(q-1)q} & -C_{(q-1)q} \end{array}\right].$$

At this point, we define the synchronization subspace as $\mathcal{S}_n = \{\mathbf{x} \in (\mathbb{C}^n)^q : x_i = x_j \text{ for all } (i,\, j)\}$. The distance of $\mathbf{x}$ to $\mathcal{S}_n$ we denote by $\|\mathbf{x}\|_{\mathcal{S}_n}$. Note that by construction, we have $\operatorname{null} \mathbf{C} \supset \mathcal{S}_n$. The observability matrix $\mathbf{W}$ associated to the pair $[\mathbf{C},\, \mathbf{A}]$ reads $[\mathbf{C}^T\; \mathbf{A}^T \mathbf{C}^T\; \cdots\; \mathbf{A}^{(qn-1)T} \mathbf{C}^T]^T$. However, being equal to $[I_q \otimes A]$, the matrix $\mathbf{A}$ satisfies the characteristic equation of A (which is of order n), and therefore, the observability index for the pair $[\mathbf{C},\, \mathbf{A}]$ is at most n. Hence, for all practical purposes, we can (and we will) take

$$\mathbf{W} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}.$$

In particular, $\operatorname{null} \mathbf{W}$ is the unobservable subspace for the pair $[\mathbf{C},\, \mathbf{A}]$. Let $\mathbf{1}_q \in \mathbb{C}^q$ denote the vector of all ones. Define $K = \mathbf{1}_q / \sqrt{q}$ (note that $K^* K = 1$) and let $G \in \mathbb{C}^{q \times (q-1)}$ be any matrix whose columns make an orthonormal basis for $\operatorname{null} K^*$. Then, the columns of the matrix $[G\; K]$ make an orthonormal basis for $\mathbb{C}^n$. The following identities are easy to show and will find frequent use in the sequel.

1) $GG^* + KK^* = I_q$.
2) $\operatorname{range} [G \otimes I_n] = \mathcal{S}_n^{\perp}$.
3) $\operatorname{null} [G^* \otimes I_n] = \mathcal{S}_n$.
4) $\mathbf{C}[K \otimes I_n] = 0$.

Finally, define the *reduced* parameters as

$$\mathbf{A}_{\mathrm{r}} = [I_{q-1} \otimes A] \text{ and } \mathbf{C}_{\mathrm{r}} = \mathbf{C}[G \otimes I_n]\,.$$

The next two lemmas give a useful interpretation of Definition 1.

Lemma 1: The following are equivalent.

1) The array $[(C_{ij}),\, A]$ is observable.
2) $\operatorname{null} \mathbf{W} \subset \mathcal{S}_n$.

Proof: Consider the system (4). Recall that $\operatorname{null} \mathbf{W}$ equals the unobservable subspace for the pair $[\mathbf{C},\, \mathbf{A}]$. Hence, for any initial condition $\mathbf{x}(0)$, we can write

$$\begin{aligned} \mathbf{x}(0) \in \operatorname{null} \mathbf{W} &\iff \mathbf{y}(t) \equiv 0 \\ &\iff y_{ij}(t) \equiv 0 \text{ for all } (i,\, j)\,. \end{aligned}$$

Also, since $\mathcal{S}_n$ is invariant with respect to $[I_q \otimes A] = \mathbf{A}$, we have

$$\begin{aligned} \mathbf{x}(0) \in \mathcal{S}_n &\iff \mathbf{x}(t) \in \mathcal{S}_n \text{ for all } t \\ &\iff x_i(t) \equiv x_j(t) \text{ for all } (i,\, j)\,. \end{aligned}$$

Therefore, if $[(C_{ij}),\, A]$ is observable, then $\operatorname{null} \mathbf{W} \subset \mathcal{S}_n$ because

$$\begin{aligned} \mathbf{x}(0) \in \operatorname{null} \mathbf{W} &\implies y_{ij}(t) \equiv 0 \text{ for all } (i,\, j) \\ &\implies x_i(t) \equiv x_j(t) \text{ for all } (i,\, j) \\ &\implies \mathbf{x}(0) \in \mathcal{S}_n\,. \end{aligned}$$

Likewise, if $\operatorname{null} \mathbf{W} \subset \mathcal{S}_n$, then the array $[(C_{ij}),\, A]$ has to be observable because

$$\begin{aligned} y_{ij}(t) \equiv 0 \text{ for all } (i,\, j) &\implies \mathbf{x}(0) \in \operatorname{null} \mathbf{W} \\ &\implies \mathbf{x}(0) \in \mathcal{S}_n \\ &\implies x_i(t) \equiv x_j(t) \text{ for all } (i,\, j)\,. \end{aligned}$$

Hence, the result. ∎

Lemma 2: The following are equivalent.

1) The array $[(C_{ij}),\, A]$ is detectable.
2) The pair $[\mathbf{C}_{\mathrm{r}},\, \mathbf{A}_{\mathrm{r}}]$ is detectable.

Proof: 2⟹1. Suppose the pair $[\mathbf{C}_{\mathrm{r}},\, \mathbf{A}_{\mathrm{r}}]$ is detectable. Consider the array (2) and choose arbitrary initial conditions $x_1(0),\, x_2(0),\, \ldots,\, x_q(0)$ yielding $y_{ij}(t) \equiv 0$ for all $(i,\, j)$. Define the signals $\mathbf{x}_{\mathrm{r}}(t) = [G^* \otimes I_n]\mathbf{x}(t)$ and $\mathbf{y}_{\mathrm{r}}(t) = \mathbf{C}_{\mathrm{r}} \mathbf{x}_{\mathrm{r}}(t)$. We notice two things. First, $\mathbf{x}_{\mathrm{r}}(t)$ satisfies $\dot{\mathbf{x}}_{\mathrm{r}} = \mathbf{A}_{\mathrm{r}} \mathbf{x}_{\mathrm{r}}$ as can be seen through

$$\begin{aligned} \dot{\mathbf{x}}_{\mathrm{r}} &= [G^* \otimes I_n]\dot{\mathbf{x}} \\ &= [G^* \otimes I_n][I_q \otimes A]\mathbf{x} \\ &= [I_{q-1} \otimes A][G^* \otimes I_n]\mathbf{x} \\ &= \mathbf{A}_{\mathrm{r}} \mathbf{x}_{\mathrm{r}}\,. \end{aligned}$$

Therefore, $\mathbf{y}_{\mathrm{r}}(t) \equiv 0$ implies $\mathbf{x}_{\mathrm{r}}(t) \to 0$ under the detectability of the pair $[\mathbf{C}_{\mathrm{r}},\, \mathbf{A}_{\mathrm{r}}]$. Second, $\mathbf{y}_{\mathrm{r}}(t) = \mathbf{y}(t)$ because

$$\begin{aligned} \mathbf{y}_{\mathrm{r}} &= \mathbf{C}_{\mathrm{r}} \mathbf{x}_{\mathrm{r}} \\ &= \mathbf{C}[G \otimes I_n][G^* \otimes I_n]\mathbf{x} \\ &= \mathbf{C}[GG^* \otimes I_n]\mathbf{x} + \underbrace{\mathbf{C}[K \otimes I_n]}_{0}[K^* \otimes I_n]\mathbf{x} \\ &= \mathbf{C}[(\underbrace{GG^* + KK^*}_{I_q}) \otimes I_n]\mathbf{x} \\ &= \mathbf{C}\mathbf{x} \\ &= \mathbf{y}\,. \end{aligned}$$

Now, we can write the following, since $\mathrm{null}\,[G^* \otimes I_n] = \mathcal{S}_n$:

$$\begin{aligned} y_{ij}(t) \equiv 0\ \forall (i,\, j) &\implies \mathbf{y}(t) \equiv 0 \\ &\implies \mathbf{y}_{\mathrm{r}}(t) \equiv 0 \\ &\implies \mathbf{x}_{\mathrm{r}}(t) \to 0 \\ &\implies [G^* \otimes I_n]\mathbf{x}(t) \to 0 \\ &\implies \|\mathbf{x}(t)\|_{\mathcal{S}_n} \to 0 \\ &\implies \|x_i(t) - x_j(t)\| \to 0\ \forall (i,\, j)\,. \end{aligned}$$

1⟹2. Suppose now the pair $[\mathbf{C}_{\mathrm{r}},\, \mathbf{A}_{\mathrm{r}}]$ is not detectable. Then, by Popov–Belevitch–Hautus (PBH) test, there exists an eigenvector $\eta \in (\mathbb{C}^n)^{q-1}$ of $\mathbf{A}_{\mathrm{r}}$ satisfying $\mathbf{C}_{\mathrm{r}}\eta = 0$ and $\mathbf{A}_{\mathrm{r}}\eta = \lambda\eta$ with $\mathrm{Re}\,\lambda \geq 0$. Define $\xi = [G \otimes I_n]\eta$, which clearly belongs to $\mathcal{S}_n^\perp = \mathrm{range}\,[G \otimes I_n]$. Since $\eta \neq 0$ and $[G \otimes I_n]$ is full column rank, we have $\xi \neq 0$. Then, $\xi \in \mathcal{S}_n^\perp$ implies $\xi \notin \mathcal{S}_n$. Also, ξ is an eigenvector of $\mathbf{A}$ with eigenvalue λ because

$$\begin{aligned} \mathbf{A}\xi &= [I_q \otimes A][G \otimes I_n]\eta \\ &= [G \otimes I_n][I_{q-1} \otimes A]\eta \\ &= [G \otimes I_n]\mathbf{A}_{\mathrm{r}}\eta \\ &= \lambda[G \otimes I_n]\eta \\ &= \lambda\xi\,. \end{aligned}$$

Let us consider now the system (4). Set the initial condition to the eigenvector ξ, i.e., $\mathbf{x}(0) = \xi$. This yields the solution $\mathbf{x}(t) = e^{\lambda t}\xi$. Since $\xi \notin \mathcal{S}_n$, we have $\|\xi\|_{\mathcal{S}_n} > 0$. Then, we can write $\|\mathbf{x}(t)\|_{\mathcal{S}_n} = |e^{\lambda t}| \cdot \|\xi\|_{\mathcal{S}_n} \not\to 0$ because $\mathrm{Re}\,\lambda \geq 0$. This means

$$\|x_k(t) - x_\ell(t)\| \not\to 0 \text{ for some } (k,\, \ell)\,. \tag{5}$$

Finally, we notice $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) = e^{\lambda t}\mathbf{C}\xi = e^{\lambda t}\mathbf{C}[G \otimes I_n]\eta = e^{\lambda t}\mathbf{C}_{\mathrm{r}}\eta = 0$. This means

$$y_{ij}(t) \equiv 0 \text{ for all } (i,\, j)\,. \tag{6}$$

Combining (5) and (6) shows us that the array $[(C_{ij}),\, A]$ cannot be detectable. ■

An *n-graph* $\Gamma = (\mathcal{V},\, w)$ has a finite set of vertices $\mathcal{V} = \{v_1,\, v_2,\, \ldots,\, v_q\}$ and a weight function $w : \mathcal{V} \times \mathcal{V} \to \mathbb{C}^{n\times n}$ with the following properties:

1) $w(v,\, v) = 0$;
2) $w(u,\, v) = w(v,\, u)$;
3) $w(u,\, v) = w(u,\, v)^* \geq 0$.

Let $G_{ij} = w(v_i,\, v_j)$. The nq-by-nq matrix

$$\mathrm{lap}\,\Gamma = \begin{bmatrix} \sum_j G_{1j} & -G_{12} & \cdots & -G_{1q} \\ -G_{21} & \sum_j G_{2j} & \cdots & -G_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ -G_{q1} & -G_{q2} & \cdots & \sum_j G_{qj} \end{bmatrix}$$

is called the *Laplacian* of Γ. Let $L = \mathrm{lap}\,\Gamma$. By construction the Laplacian is Hermitian, i.e., $L^* = L$, and enjoys some other desirable properties. We see that $\mathrm{null}\,L \supset \mathcal{S}_n$. Letting $\xi = [z_1^T\ z_2^T\ \cdots\ z_q^T]^T$ with $z_i \in \mathbb{C}^n$ we can write $\xi^* L\xi = \sum_{i>j}(z_i - z_j)^* G_{ij}(z_i - z_j) \geq 0$, i.e., the Laplacian is positive semidefinite. Therefore, all its eigenvalues are real and nonnegative, thanks to which the ordering $\lambda_1(L) \leq \lambda_2(L) \leq \ldots \leq \lambda_{qn}(L)$ is not meaningless. In the sequel, $\lambda_k(L)$ denotes the kth smallest eigenvalue of L.

The *interconnection graph* of the array (2) is denoted by $\Gamma(W_{ij})_{i,j=1}^q$ (or by $\Gamma(W_{ij})$ when there is no risk of confusion). The n-graph $\Gamma(W_{ij})$ has the vertex set $\mathcal{V} = \{v_1,\, v_2,\, \ldots,\, v_q\}$ and its weight function is such that $w(v_i,\, v_j) = W_{ij}^* W_{ij}$. The *eigengraph* corresponding to the eigenvalue μ_σ is denoted by $\Gamma(C_{ij}V_\sigma)_{i,j=1}^q$ (or by $\Gamma(C_{ij}V_\sigma)$ sometimes) and defined similarly. Namely, the n_σ-graph $\Gamma(C_{ij}V_\sigma)$ has the vertex set $\{v_1,\, v_2,\, \ldots,\, v_q\}$ and its weight function is the mapping $(v_i,\, v_j) \mapsto V_\sigma^* C_{ij}^* C_{ij} V_\sigma$. The following identities can be established by carrying out the multiplication explicitly.

$$\begin{aligned} \mathrm{lap}\,\Gamma(W_{ij}) &= \mathbf{W}^*\mathbf{W} \\ \mathrm{lap}\,\Gamma(C_{ij}V_\sigma) &= [I_q \otimes V_\sigma^*]\mathbf{C}^*\mathbf{C}[I_q \otimes V_\sigma]\,. \end{aligned}$$

Furthermore, using (3), we can write

$$\begin{aligned} &[I_q \otimes V_\sigma^*][\mathrm{lap}\,\Gamma(W_{ij})][I_q \otimes V_\sigma] \\ &\quad= \mathrm{lap}\,\Gamma(W_{ij}V_\sigma) \\ &\quad= \left(\sum_{k=0}^{n-1} |\mu_\sigma|^{2k}\right) \mathrm{lap}\,\Gamma(C_{ij}V_\sigma)\,. \end{aligned} \tag{7}$$

In the graph theory [5], connectivity (in the classical sense) is characterized by means of adjacency. A connected graph is said to have a path between each pair of its vertices, where a path is a sequence of adjacent vertices. For 1-graphs, the definition of adjacency is unequivocal: a pair of vertices $(u,\, v)$ are adjacent if $w(u,\, v) > 0$ and nonadjacent if $w(u,\, v) = 0$. (Adjacent vertices are said to have an edge between them.) For n-graphs ($n \geq 2$), however, since we have the in-between semidefinite case $w(u,\, v) \geq 0$, how to define adjacency and, in turn, connectivity becomes a matter of choice. For our purposes, in this paper, we (inevitably) abandon the concept of adjacency altogether and define connectivity of a graph through its Laplacian. Recall that a 1-graph Γ is connected if and only if $\lambda_2(\mathrm{lap}\,\Gamma) > 0$. Since this is an equivalence result it can replace the definition of connectivity for 1-graphs. This substitute turns out to be much easier to generalize than the standard definition that uses paths.

Definition 2: An n-graph Γ is said to be *connected* if $\lambda_{n+1}(\mathrm{lap}\,\Gamma) > 0$.

We could have given an equivalent definition in terms of the null space of the Laplacian:

Lemma 3: An n-graph Γ is connected iff $\mathrm{null}\,\mathrm{lap}\,\Gamma = \mathcal{S}_n$.

Proof: Let $L = \mathrm{lap}\,\Gamma$ denote the Laplacian. Suppose $\mathrm{null}\,L = \mathcal{S}_n$. By definition $\dim \mathcal{S}_n = n$. Therefore, L has n linearly independent eigenvectors whose eigenvalues are zero. Since $L^* = L$, this means that L has exactly n eigenvalues at the origin. That all the eigenvalues of L are nonnegative, then yields $\lambda_{n+1}(L) > 0$. To show the other direction this time we begin by letting $\lambda_{n+1}(L) > 0$. Then, since it is Hermitian positive semidefinite, L has at most n eigenvalues at the origin. In other words, $\dim \mathrm{null}\,L \leq n$. This implies, in the light of the facts $\mathrm{null}\,L \supset \mathcal{S}_n$ and $\dim \mathcal{S}_n = n$, that $\mathrm{null}\,L = \mathcal{S}_n$. ■

III. Observability and Connectivity

In this section, we establish the equivalence between observability and connectivity. Then, we present a corollary on an interesting special case followed by a relevant numerical example. We start with a theorem on detectability.

Theorem 1: The following are equivalent.

1) The array $[(C_{ij}),\, A]$ is detectable.
2) All the eigengraphs $\Gamma(C_{ij}V_\sigma)$ with $\mathrm{Re}\,\mu_\sigma \geq 0$ are connected.

Proof: 1⟹2. Suppose for some $\mathrm{Re}\,\mu_\sigma \geq 0$ the eigengraph $\Gamma(C_{ij}V_\sigma)$ is not connected. Since $\mathrm{null}\, L_\sigma \supset \mathcal{S}_{n_\sigma}$, by Lemma 3, there exists $h \in (\mathbb{C}^{n_\sigma})^q$ that satisfies $h \notin \mathcal{S}_{n_\sigma}$ and $L_\sigma h = 0$, where $L_\sigma = \mathrm{lap}\,\Gamma(C_{ij}V_\sigma)$. We have $\mathbf{C}[I_q \otimes V_\sigma]h = 0$ because, employing the identity $L_\sigma = [I_q \otimes V_\sigma]^*\mathbf{C}^*\mathbf{C}[I_q \otimes V_\sigma]$, we can write $\|\mathbf{C}[I_q \otimes V_\sigma]h\|^2 = h^* L_\sigma h = 0$. Note that since $h \notin \mathcal{S}_{n_\sigma}$ and $\mathrm{null}\,[G^* \otimes I_{n_\sigma}] = \mathcal{S}_{n_\sigma}$, we have $[G^* \otimes I_{n_\sigma}]h \neq 0$. Define $\eta = [G^* \otimes V_\sigma]h$. Then, we have $\eta \neq 0$ because $[I_{q-1} \otimes V_\sigma]$ is full column rank and we can write $\eta = [I_{q-1} \otimes V_\sigma][G^* \otimes I_{n_\sigma}]h$. Now, we proceed as

$$\begin{aligned}
\mathbf{C}_{\rm r}\eta &= \mathbf{C}[G \otimes I_n][G^* \otimes V_\sigma]h \\
&= \mathbf{C}[GG^* \otimes V_\sigma]h + \underbrace{\mathbf{C}[K \otimes I_n]}_{0}[K^* \otimes V_\sigma]h \\
&= \mathbf{C}[GG^* \otimes V_\sigma]h + \mathbf{C}[KK^* \otimes V_\sigma]h \\
&= \mathbf{C}[(\underbrace{GG^* + KK^*}_{I_q}) \otimes V_\sigma]h \\
&= \mathbf{C}[I_q \otimes V_\sigma]h \\
&= 0\,.
\end{aligned} \tag{8}$$

Also

$$\begin{aligned}
\mathbf{A}_{\rm r}\eta &= [I_{q-1} \otimes A][G^* \otimes V_\sigma]h \\
&= [G^* \otimes AV_\sigma]h \\
&= [G^* \otimes \mu_\sigma V_\sigma]h \\
&= \mu_\sigma[G^* \otimes V_\sigma]h \\
&= \mu_\sigma \eta\,.
\end{aligned} \tag{9}$$

The PBH test allows us to assert by (8) and (9) that the pair $[\mathbf{C}_{\rm r},\, \mathbf{A}_{\rm r}]$ is not detectable because $\mathrm{Re}\,\mu_\sigma \geq 0$. Then, by Lemma 2, we see that the array $[(C_{ij}),\, A]$ cannot be detectable.

2⟹1. Suppose the array $[(C_{ij}),\, A]$ is not detectable. Then, by Lemma 2, the pair $[\mathbf{C}_{\rm r},\, \mathbf{A}_{\rm r}]$ is not detectable. Hence, by the PBH test, there should exist an eigenvector $\eta \in (\mathbb{C}^n)^{q-1}$ and $\mathrm{Re}\,\lambda \geq 0$ such that $\mathbf{A}_{\rm r}\eta = \lambda\eta$ and $\mathbf{C}_{\rm r}\eta = 0$. Employing the partition $\eta = [z_1^T\ z_2^T\ \ldots\ z_{q-1}^T]^T$ with $z_i \in \mathbb{C}^n$, we can write

$$\begin{bmatrix} (A - \lambda I_n)z_1 \\ (A - \lambda I_n)z_2 \\ \vdots \\ (A - \lambda I_n)z_{q-1} \end{bmatrix} = [\mathbf{A}_{\rm r} - \lambda I_{(q-1)n}]\eta = 0$$

that allows us to see that λ is an eigenvalue of A. In particular, we have $\lambda = \mu_\sigma$ for some $\sigma \in \{1, 2, \ldots, m\}$, and for each z_i, we can find $\rho_i \in \mathbb{C}^{n_\sigma}$ such that $z_i = V_\sigma \rho_i$. Let us stack these ρ_i into $g = [\rho_1^T\ \rho_2^T\ \ldots\ \rho_{q-1}^T]^T$. This lets us write $\eta = [I_{q-1} \otimes V_\sigma]g$. Note that η is nonzero because it is an eigenvector. Therefore, $g \neq 0$. Now define $h = [G \otimes I_{n_\sigma}]g$. We have $h \neq 0$ because g is nonzero and $[G \otimes I_{n_\sigma}]$ is full column rank. Then, that $h \in \mathrm{range}\,[G \otimes I_{n_\sigma}] = \mathcal{S}_{n_\sigma}^\perp$ allows us to write $h \notin \mathcal{S}_{n_\sigma}$. Now, we can proceed as

$$\begin{aligned}
\mathbf{C}[I_q \otimes V_\sigma]h &= \mathbf{C}[I_q \otimes V_\sigma][G \otimes I_{n_\sigma}]g \\
&= \mathbf{C}[G \otimes I_n][I_{q-1} \otimes V_\sigma]g \\
&= \mathbf{C}_{\rm r}\eta \\
&= 0\,.
\end{aligned}$$

This yields $L_\sigma h = [I_q \otimes V_\sigma^*]\mathbf{C}^*\mathbf{C}[I_q \otimes V_\sigma]h = 0$. Then, $h \notin \mathcal{S}_{n_\sigma}$ gives us $\mathrm{null}\, L_\sigma \neq \mathcal{S}_{n_\sigma}$. By Lemma 3, therefore, the eigengraph $\Gamma(C_{ij}V_\sigma)$, for which we have $\mathrm{Re}\,\mu_\sigma \geq 0$, is not connected. ∎

Theorem 2: The following are equivalent.

1) The array $[(C_{ij}),\, A]$ is observable.
2) All the eigengraphs $\Gamma(C_{ij}V_1), \Gamma(C_{ij}V_2), \ldots, \Gamma(C_{ij}V_m)$ are connected.
3) The interconnection graph $\Gamma(W_{ij})$ is connected.

Proof: 1⟺2. For $\lambda \in \mathbb{R}$ define the matrix $\tilde{A} = A + \lambda I_n$. The following facts are easy to show. First, $[(C_{ij}),\, A]$ is observable if and only if $[(C_{ij}),\, \tilde{A}]$ is observable. Second, $\tilde{\mu}_\sigma = \mu_\sigma + \lambda$ and $\tilde{V}_\sigma = V_\sigma$ for all $\sigma = 1, 2, \ldots, m$. Now choose λ large enough so that $\mathrm{Re}\,\tilde{\mu}_\sigma \geq 0$ for all σ. This choice allows us to claim that the array $[(C_{ij}),\, \tilde{A}]$ is observable if and only if it is detectable. Then, by Theorem 1, we can write

$$\begin{aligned}
[(C_{ij}),\, A]\ \text{observable} &\iff [(C_{ij}),\, \tilde{A}]\ \text{observable} \\
&\iff [(C_{ij}),\, \tilde{A}]\ \text{detectable} \\
&\iff \Gamma(C_{ij}\tilde{V}_\sigma)\ \text{connected for all}\ \sigma \\
&\iff \Gamma(C_{ij}V_\sigma)\ \text{connected for all}\ \sigma\,.
\end{aligned}$$

1⟺3. Recall $\mathrm{null}\,\mathbf{C} \supset \mathcal{S}_n$. Also, $\mathcal{S}_n$ is clearly invariant with respect to $[I_q \otimes A] = \mathbf{A}$. Hence, the unobservable subspace for the pair $[\mathbf{C},\, \mathbf{A}]$ must contain the synchronization subspace, i.e., $\mathrm{null}\,\mathbf{W} \supset \mathcal{S}_n$. This implies

$$\mathrm{null}\,\mathbf{W} \subset \mathcal{S}_n \iff \mathrm{null}\,\mathbf{W} = \mathcal{S}_n\,. \tag{10}$$

Using the identity $\mathrm{lap}\,\Gamma(W_{ij}) = \mathbf{W}^*\mathbf{W}$ and the fact that $\mathbf{W}$ and $\mathbf{W}^*\mathbf{W}$ share the same null space we at once have

$$\mathrm{null}\,\mathrm{lap}\,\Gamma(W_{ij}) = \mathrm{null}\,\mathbf{W}\,. \tag{11}$$

Now, Lemmas 1 and 3, (10), and (11) yield

$$\begin{aligned}
[(C_{ij}),\, A]\ \text{observable} &\iff \mathrm{null}\,\mathbf{W} \subset \mathcal{S}_n \\
&\iff \mathrm{null}\,\mathbf{W} = \mathcal{S}_n \\
&\iff \mathrm{null}\,\mathrm{lap}\,\Gamma(W_{ij}) = \mathcal{S}_n \\
&\iff \Gamma(W_{ij})\ \text{connected}\,.
\end{aligned}$$

Hence, the result. ∎

Theorem 2 has an interesting implication concerning 1-graphs. Let $d_A(s)$ and $m_A(s)$, respectively, denote the characteristic polynomial and the minimal polynomial of the matrix A. Note that if each V_σ consists of a single column, i.e., each eigenvalue μ_σ has a unique (up to a scaling) eigenvector, then all the eigengraphs $\Gamma(C_{ij}V_1),\,\Gamma(C_{ij}V_2),\,\ldots,\,\Gamma(C_{ij}V_m)$ become 1-graphs. A sufficient condition for this is that the eigenvalues $\mu_1,\,\mu_2,\,\ldots,\,\mu_m$ are all simple, i.e., $m=n$. More generally, the following holds.

Corollary 1: Suppose $m_A(s)=d_A(s)$. Then, the array $[(C_{ij}),\,A]$ is observable iff all the 1-graphs $\Gamma(C_{ij}V_1)$, $\Gamma(C_{ij}V_2),\ldots,\Gamma(C_{ij}V_m)$ are connected.

An example. Consider the array $[(C_{ij})_{i,j=1}^4,\,A]$ with

$$A=\begin{bmatrix}0&1&-7&-14&21&31\\1&1&1&3&-7&-11\\3&6&-28&-43&7&5\\-2&-4&18&28&-7&-7\\-2&-4&-2&1&-32&-49\\1&2&3&2&20&31\end{bmatrix}$$

and

$$\begin{aligned}C_{12}&=\begin{bmatrix}2&3&8&12&6&10\end{bmatrix}\\C_{23}&=\begin{bmatrix}2&3&4&6&6&9\end{bmatrix}\\C_{34}&=\begin{bmatrix}4&6&6&10&6&9\end{bmatrix}\\C_{41}&=\begin{bmatrix}1&2&6&9&4&7\end{bmatrix}.\end{aligned}$$

The matrices C_{13} and C_{24} are zero. (Recall $C_{ij}=C_{ji}$ and $C_{ii}=0$.) The characteristic polynomial of A reads $d_A(s)=s^6-s^2=s^2(s-1)(s+1)(s-j)(s+j)$. That is, A has $m=5$ distinct eigenvalues: $\mu_1=0$, $\mu_2=1$, $\mu_3=-1$, and $\mu_{4,5}=\pm j$. The eigenvalue at the origin is repeated, yet $\dim\operatorname{null}A=1$. Therefore, there is a single eigenvector corresponding to μ_1. Consequently, we have $m_A(s)=d_A(s)$. The matrices (or, in this case, vectors) V_σ corresponding to the eigenvalues μ_σ are given as follows:

$$V_1=\begin{bmatrix}-5\\2\\-4\\3\\5\\-3\end{bmatrix},\quad V_2=\begin{bmatrix}1\\-1\\-5\\3\\-4\\3\end{bmatrix},\quad V_3=\begin{bmatrix}-2\\1\\2\\-1\\3\\-2\end{bmatrix}$$

$$V_{4,5}=\begin{bmatrix}-17\\4\\-19\\14\\22\\-13\end{bmatrix}\pm j\begin{bmatrix}0\\1\\8\\-5\\-3\\1\end{bmatrix}.$$

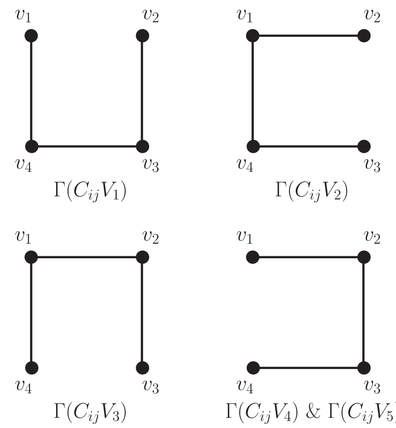

Fig. 5. Connected eigengraphs.

Now let us determine whether the array $[(C_{ij}),\,A]$ is observable or not. Thanks to Corollary 1, we can do this by checking the connectivities of the 1-graphs $\Gamma(C_{ij}V_1),\,\Gamma(C_{ij}V_2),\,\ldots,\,\Gamma(C_{ij}V_5)$. A pleasant thing about a 1-graph is that its connectivity can be read from its visual representation. In this universal picture, every vertex is represented by a dot and a line (called an edge) is drawn connecting a pair of vertices $(v_i,\,v_j)$ if the value of the weight function (which is scalar for a 1-graph) is positive, i.e., $w(v_i,\,v_j)>0$. Then, the graph is connected if we can reach from any dot to any other dot by tracing the lines. Our 1-graphs are shown in Fig. 5. Clearly, all of them are connected. Hence, we conclude that the array $[(C_{ij}),\,A]$ is observable. Note that for some nonzero C_{ij} certain edges $(v_i,\,v_j)$ are missing. For instance, for the graph $\Gamma(C_{ij}V_1)$, the edge $(v_1,\,v_2)$ is absent despite $C_{12}\neq 0$. The reason is that $\mu_1=0$ is an unobservable eigenvalue for the pair $[C_{12},\,A]$. In particular, we have $C_{12}V_1=0$, i.e., the eigenvector V_1 belongs to the null space of C_{12}. Interestingly, the intersection of the graphs in Fig. 5 yields an empty set of edges. This is because there is not a single pair $[C_{ij},\,A]$ that is observable. Still, that does not prevent the array $[(C_{ij}),\,A]$ from being observable.

In this section, we studied array observability (detectability) from the graph connectivity point of view. For instance, in Theorem 2, we presented the equivalence between the observability of the array $[(C_{ij}),\,A]$ and the connectivity of the graph $\Gamma(W_{ij})$. Now, the connectivity of $\Gamma(W_{ij})$ is determined by the positivity of the $n+1$st eigenvalue of its Laplacian and this eigenvalue is no other than the square of the $n+1$st (listed in increasing order) singular value of the observability matrix $\mathbf{W}$. This brings up the following question. *Why did we not study observability directly through* $\mathbf{W}$, *without ever resorting to tools like graph, connectivity, and Laplacian?* More simply put: *When is graph approach useful?* As an answer to this question consider the simple scenario where we have (say) three (sub)arrays, each of which is known to be observable. Suppose we are to design (determine) relative outputs between these three subarrays (all having the same $A\in\mathbb{R}^{n\times n}$ matrix) so that their union considered as a single big array is observable. This is an easy problem to deal with by the graph approach. Since each subarray is observable, the corresponding components of the graph representing the big array are connected. Therefore, we can treat each of those three components as a single node and study the connectivity of the big graph through the connectivity of the much simpler three-node n-graph, whose Laplacian is of size $3n \times 3n$. If, however, we choose to work with the matrix $\mathbf{W}$ representing the big array, we have to deal with a matrix of arbitrarily large size. Still worse, we may not know exactly the subarray parameters, which are required to construct $\mathbf{W}$. Note that with graph approach, we need not know anything except that the subarrays are connected.

IV. Pairwise Observability and Effective Conductance

Hitherto, regarding the array (2), we have focused solely on the total synchronization, i.e., $x_i(t) \equiv x_j(t)$ for all pairs (i, j). Henceforth, we consider partial synchronization. In particular, we study the synchronization of a certain pair of systems under the condition that all the relative outputs throughout the array are identically zero. The results we establish in this section will put the previous analysis on a more complete footing. But not only that, studying partial synchronization may be interesting on its own right since it is not difficult to imagine situations where total synchronization is not desired for reasons of security. For instance, in a secure communication scenario, there may be an array with many identical systems among which there is a pair of distant units that are desired to synchronize without synchronizing with their immediate neighbors.

The following definition is our starting point.

Definition 3: An array $[(C_{ij}), A]$ is said to be (k, ℓ)-*observable* if

$$y_{ij}(t) \equiv 0 \text{ for all } (i, j) \Longrightarrow x_k(t) \equiv x_\ell(t)$$

and (k, ℓ)-*detectable* if

$$y_{ij}(t) \equiv 0 \text{ for all } (i, j) \Longrightarrow \|x_k(t) - x_\ell(t)\| \to 0$$

for all initial conditions $x_1(0), x_2(0), \ldots, x_q(0)$.

As before, where we studied the observability of an array by means of the connectivity of interconnection graph, we will again approach the problem from graphic angle. Recall that a 1-graph is connected when the null space of its Laplacian is spanned by the vector of all ones, which led us to the generalization stated in Lemma 3. Likewise, a pair of vertices (v_k, v_ℓ) of a 1-graph is connected if any vector that belongs to the null space of the Laplacian is with identical kth and ℓth entries. The reason is the following. Let Γ be a 1-graph with vertex set $\{v_1, v_2, \ldots, v_q\}$, weight function w, and Laplacian $L = \operatorname{lap}\Gamma$, a q-by-q matrix. (Note that, w takes real scalar values since Γ is a 1-graph.) For the graph Γ, the connectedness of the pair (v_k, v_ℓ) means that there exists a path between the kth and ℓth vertices. That is, we can find a sequence of distinct indices $(i_1, i_2, \ldots, i_p)$ with $i_1 = k$ and $i_p = \ell$ such that $w_{i_j i_{j+1}} = w(v_{i_j}, v_{i_{j+1}}) > 0$ for all $j \in \{1, 2, \ldots, p-1\}$. Choose now an arbitrary $\xi \in \mathbb{C}^q$ that belongs to $\operatorname{null} L$. Let $\xi = [z_1\ z_2\ \ldots\ z_q]^T$. We can write

$$\begin{aligned} 0 = \ & \xi^* L \xi = \sum_{i>j} w_{ij} |z_i - z_j|^2 \\ \geq & \sum_{j \in \{1, 2, \ldots, p-1\}} w_{i_j i_{j+1}} |z_{i_j} - z_{i_{j+1}}|^2 . \end{aligned}$$

Since $w_{i_j i_{j+1}} > 0$, this implies $z_{i_j} = z_{i_{j+1}}$, for $j \in \{1, 2, \ldots, p-1\}$ yielding $z_k = z_\ell$, which was to be shown. Let $e_i \in \mathbb{C}^q$ be the unit vector with ith entry one and the remaining entries zero. Observe that $z_k = z_\ell$ implies $(e_k - e_\ell)^* \xi = 0$. Since $\xi \in \operatorname{null} L$ was arbitrary, we can write

$$\operatorname{null} \operatorname{lap} \Gamma \subset \operatorname{null} (e_k - e_\ell)^* \tag{12}$$

which is just another way of saying that the pair of vertices (v_k, v_ℓ) is connected (for a 1-graph). Let us introduce the notation

$$\mathcal{S}_n^{(k, \ell)} = \operatorname{null} [(e_k - e_\ell)^* \otimes I_n] .$$

The condition (12) suggests the following natural generalization of pairwise connectivity for n-graphs:

Definition 4: An n-graph Γ is said to be (k, ℓ)-*connected* if $\operatorname{null} \operatorname{lap} \Gamma \subset \mathcal{S}_n^{(k, \ell)}$.

It may seem reasonable to expect that Theorems 1 and 2 of the previous section can be effortlessly converted into "pairwise" statements by simply replacing the words *observable, detectable, connected* therein with (k, ℓ)-*observable,* (k, ℓ)-*detectable,* (k, ℓ)-*connected*, respectively. This, however, is not the case; certain associations disappear in the pairwise domain. In particular, an array that is not (k, ℓ)-observable may still have all its eigengraphs (k, ℓ)-connected. We now proceed by establishing the remaining links. Then, we provide evidence (counterexample) for the missing implications.

Theorem 3: The following are equivalent.

1) The array $[(C_{ij}), A]$ is (k, ℓ)-observable.

2) The n-graph $\Gamma(W_{ij})$ is (k, ℓ)-connected.

Proof: 1⟹2. Suppose the array $[(C_{ij}), A]$ is (k, ℓ)-observable. Consider the system (4). Recall that $\operatorname{null} \mathbf{W}$ equals the unobservable subspace for the pair $[\mathbf{C}, \mathbf{A}]$. Let $\mathbf{x}(0) \in \operatorname{null} \mathbf{W}$ be arbitrary. We can write

$$\begin{aligned} \mathbf{x}(0) \in \operatorname{null} \mathbf{W} \Longrightarrow\ & \mathbf{y}(t) \equiv 0 \\ \Longrightarrow\ & y_{ij}(t) \equiv 0 \text{ for all } (i, j) \\ \Longrightarrow\ & x_k(t) \equiv x_\ell(t) \\ \Longrightarrow\ & \mathbf{x}(t) \in \mathcal{S}_n^{(k, \ell)} \text{ for all } t \\ \Longrightarrow\ & \mathbf{x}(0) \in \mathcal{S}_n^{(k, \ell)} . \end{aligned}$$

Therefore, $\operatorname{null} \mathbf{W} \subset \mathcal{S}_n^{(k, \ell)}$. Let $L = \operatorname{lap} \Gamma(W_{ij})$. Using $L = \mathbf{W}^* \mathbf{W}$, we can write $\operatorname{null} L = \operatorname{null} \mathbf{W}^* \mathbf{W} = \operatorname{null} \mathbf{W} \subset \mathcal{S}_n^{(k, \ell)}$.

2⟹1. Suppose the n-graph $\Gamma(W_{ij})$ is (k, ℓ) connected. Then, we have $\operatorname{null} \mathbf{W} = \operatorname{null} \mathbf{W}^* \mathbf{W} = \operatorname{null} L \subset \mathcal{S}_n^{(k, \ell)}$. Choose now arbitrary initial conditions $x_1(0), x_2(0), \ldots, x_q(0)$ yielding $y_{ij}(t) \equiv 0$ for all (i, j).

Since $\mathrm{null}\,\mathbf{W} \subset \mathcal{S}_n^{(k,\,\ell)}$, we can write

$$\begin{aligned} y_{ij}(t) \equiv 0 \text{ for all } (i,\,j) &\implies \mathbf{y}(t) \equiv 0 \\ &\implies \mathbf{x}(t) \in \mathcal{S}_n^{(k,\,\ell)} \text{ for all } t \\ &\implies x_k(t) \equiv x_\ell(t) \end{aligned}$$

which was to be shown. ■

Theorem 4: If the array $[(C_{ij}),\, A]$ is $(k,\,\ell)$-detectable, then all the eigengraphs $\Gamma(C_{ij}V_\sigma)$ with $\mathrm{Re}\,\mu_\sigma \geq 0$ are $(k,\,\ell)$-connected.

Proof: Suppose that for some $\sigma \in \{1,\,2,\,\ldots,\,m\}$, the graph $\Gamma(C_{ij}V_\sigma)$ is not $(k,\,\ell)$-connected and $\mathrm{Re}\,\mu_\sigma \geq 0$. Then, there exists $h \in (\mathbb{C}^{n_\sigma})^q$ that satisfies $[(e_k - e_\ell)^* \otimes I_{n_\sigma}]h \neq 0$ and $L_\sigma h = 0$, where $L_\sigma = \mathrm{lap}\,\Gamma(C_{ij}V_\sigma)$. Let $\xi \in (\mathbb{C}^n)^q$ be defined as $\xi = [I_q \otimes V_\sigma]h$. We can write

$$\begin{aligned} [(e_k - e_\ell)^* \otimes I_n]\xi &= [(e_k - e_\ell)^* \otimes I_n][I_q \otimes V_\sigma]h \\ &= V_\sigma[(e_k - e_\ell)^* \otimes I_{n_\sigma}]h \\ &\neq 0 \end{aligned}$$

because $[(e_k - e_\ell)^* \otimes I_{n_\sigma}]h \neq 0$ and the matrix V_σ is full column rank. Consider the system (4). Observe that ξ is an eigenvector since

$$\begin{aligned} \mathbf{A}\xi &= [I_q \otimes A][I_q \otimes V_\sigma]h \\ &= [I_q \otimes AV_\sigma]h \\ &= [I_q \otimes \mu_\sigma V_\sigma]h \\ &= \mu_\sigma[I_q \otimes V_\sigma]h \\ &= \mu_\sigma \xi\,. \end{aligned}$$

Moreover, ξ belongs to the unobservable subspace $\mathrm{null}\,\mathbf{W}$, because we can write using (7)

$$\begin{aligned} \|\mathbf{W}\xi\|^2 &= \xi^*\mathbf{W}^*\mathbf{W}\xi \\ &= h^*[I_q \otimes V_\sigma^*]L[I_q \otimes V_\sigma]h \\ &= \left(\sum_{k=0}^{n-1}|\mu_\sigma|^{2k}\right) h^* L_\sigma h \\ &= 0 \end{aligned}$$

where $L = \mathrm{lap}\,\Gamma(W_{ij})$. Let us now consider the solution $\mathbf{x}(t)$ for the initial condition $\mathbf{x}(0) = \xi$. Since $\mathbf{x}(0) \in \mathrm{null}\,\mathbf{W}$, we have $\mathbf{y}(t) \equiv 0$, i.e.,

$$y_{ij}(t) \equiv 0 \text{ for all } (i,\,j)\,. \tag{13}$$

Also, due to $\mathbf{A}\xi = \mu_\sigma\xi$, we have $\mathbf{x}(t) = e^{\mu_\sigma t}\xi$. Now, we can proceed as

$$\begin{aligned} \|x_k(t) - x_\ell(t)\| &= \|[(e_k - e_\ell)^* \otimes I_n]\mathbf{x}(t)\| \\ &= |e^{\mu_\sigma t}| \cdot \|[(e_k - e_\ell)^* \otimes I_n]\xi\| \\ &\not\to 0 \end{aligned} \tag{14}$$

because $[(e_k - e_\ell)^* \otimes I_n]\xi \neq 0$ and $\mathrm{Re}\,\mu_\sigma \geq 0$. Combining (13) and (14), we see that the array $[(C_{ij}),\, A]$ cannot be $(k,\,\ell)$-detectable. ■

Theorem 5: If the array $[(C_{ij}),\, A]$ is $(k,\,\ell)$-observable, then all the eigengraphs $\Gamma(C_{ij}V_\sigma)$ are $(k,\,\ell)$-connected.

Proof: The result follows from Theorem 4 by arguments similar to those we employed in the first part of the proof of Theorem 2. ■

As mentioned earlier, an array that is not $(k,\,\ell)$-observable may still have all its eigengraphs $(k,\,\ell)$-connected. This can be seen through the following counterexample.

A counterexample. Consider the pair $[(C_{ij})_{i,j=1}^3,\, A]$ with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned} C_{12} &= \begin{bmatrix} 0\ 0\ 1\ 0 \end{bmatrix} \\ C_{23} &= \begin{bmatrix} 1\ 0\ 0\ 0 \end{bmatrix} \\ C_{31} &= \begin{bmatrix} 0\ 1\ 1\ 0 \end{bmatrix}. \end{aligned}$$

(Recall $C_{ij} = C_{ji}$ and $C_{ii} = 0$.) Clearly, A has a single ($m = 1$) distinct eigenvalue $\mu_1 = 0$. The corresponding matrix V_1 satisfying $\mathrm{range}\,V_1 = \mathrm{null}\,[A - \mu_1 I_4]$ reads

$$V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Now consider the solutions

$$x_1(t) = \begin{bmatrix} t \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad x_2(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad x_3(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

It is not difficult to check that these $x_i(t)$ satisfy $\dot{x}_i = Ax_i$ as well as $C_{ij}(x_i(t) - x_j(t)) \equiv 0$ for all $(i,\,j)$. Noting $x_2(t) \not\equiv x_3(t)$, we conclude, therefore, that the array is not $(2,\,3)$-observable. Now let us see what the eigengraphs say on the matter. In fact, the 2-graph $\Gamma(C_{ij}V_1)$ is the only eigengraph of the array. The associated Laplacian can be computed to equal

$$\mathrm{lap}\,\Gamma(C_{ij}V_1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

whose null space is spanned by the columns of the matrix N given as follows:

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Observe $[(e_2 - e_3)^* \otimes I_2]N = 0$. Hence, $\operatorname{null} \operatorname{lap} \Gamma(C_{ij}V_1) \subset \mathcal{S}_2^{(2,3)}$. That is, the eigengraph $\Gamma(C_{ij}V_1)$ *is* $(2, 3)$-connected despite that the array $[(C_{ij}), A]$ is *not* $(2, 3)$-observable.

To better understand (k, ℓ)-connectivity, we now direct our attention to the circuit theory, which has been a fruitful source of ideas for the graph theory. A significant example has been reported in [10] where the effective resistance between two nodes of a resistive network is shown to be a meaningful tool to measure distance between two vertices of a graph. That work inspires us to employ effective conductance over an n-graph, which we will eventually show to be closely related to pairwise observability. Let us, however, first remember the definition of effective conductance for a 1-graph, which will be our starting point for generalization.

Let Γ be a 1-graph with vertex set $\{v_1, v_2, \ldots, v_q\}$ and weight function w. Then, $\operatorname{lap} \Gamma$ equals the node admittance matrix of a resistive network $\mathcal{N}$ with q nodes, where the resistor connecting the nodes i and j has the conductance value $w(v_i, v_j) =: g_{ij}$ (in mhos). For the network $\mathcal{N}$, the effective conductance $\gamma_{k\ell}$ between the nodes k and ℓ is equal to the value of the current (in amps) leaving a 1-volt voltage source connected to the node k, while the node ℓ is grounded, see Fig. 6. Regarding the case depicted in Fig. 6, observe that the effective conductance γ_{14} satisfies the following equation:

$$[\operatorname{lap} \Gamma] \begin{bmatrix} 1 \\ z_2 \\ z_3 \\ 0 \\ z_5 \\ z_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \gamma_{14}$$

where $z_i \in \mathbb{R}$ denote the appropriate node voltages. Imitating this equation yields the following.

Definition 5: Given an n-graph Γ with set of vertices $\mathcal{V} = \{v_1, v_2, \ldots, v_q\}$, the *effective conductance* $\Gamma_{k\ell} \in \mathbb{C}^{n\times n}$ associated to the pair of distinct vertices (v_k, v_ℓ) satisfies

$$[\operatorname{lap} \Gamma] \begin{bmatrix} Z_1 \\ \vdots \\ Z_q \end{bmatrix} = (e_k - e_\ell) \otimes \Gamma_{k\ell}$$
$$\text{subject to } Z_k = I_n \text{ and } Z_\ell = 0 \qquad (15)$$

for some $Z_i \in \mathbb{C}^{n\times n}$.

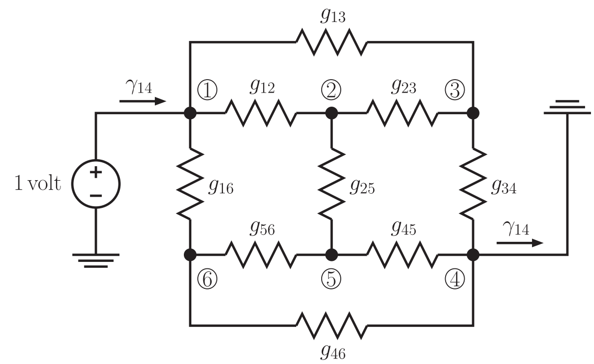

Fig. 6. Effective conductance between the first and fourth nodes equals the current γ_{14}.

It is not evident that this definition is unambiguous. Hence, we have to make sure that effective conductance always exists, preferably uniquely. To be able to do this, we need to introduce some notation. Let $e_{\bar{k}\bar{\ell}} \in \mathbb{C}^{q\times(q-2)}$ denote the matrix obtained from the identity matrix I_q by removing the kth and ℓth columns, i.e., $e_{\bar{k}\bar{\ell}} = [e_1 \ \ldots \ e_{k-1} \ e_{k+1} \ \ldots \ e_{\ell-1} \ e_{\ell+1} \ \ldots \ e_q]$. Furthermore, for $L = \operatorname{lap} \Gamma$, where Γ is an n-graph with q vertices, we adopt the following shortcuts:

$$\begin{aligned} L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}} &= [e^*_{\bar{k}\bar{\ell}} \otimes I_n]L[e_{\bar{k}\bar{\ell}} \otimes I_n] \\ L_{\bar{k}\bar{\ell},k} &= [e^*_{\bar{k}\bar{\ell}} \otimes I_n]L[e_k \otimes I_n] \\ L_{k,\bar{k}\bar{\ell}} &= [e^*_k \otimes I_n]L[e_{\bar{k}\bar{\ell}} \otimes I_n] \\ L_{k,\ell} &= [e^*_k \otimes I_n]L[e_\ell \otimes I_n]. \end{aligned}$$

Finally, $(L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}})^+$ indicates the pseudoinverse of $L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}}$.

Theorem 6: Let Γ be an n-graph with set of vertices $\mathcal{V} = \{v_1, v_2, \ldots, v_q\}$ and $L = \operatorname{lap} \Gamma$. For each pair of (distinct) vertices (v_k, v_ℓ), the effective conductance $\Gamma_{k\ell}$ uniquely exists and satisfies $\Gamma_{\ell k} = \Gamma_{k\ell} = \Gamma^*_{k\ell} \geq 0$. In particular

$$\Gamma_{k\ell} = L_{k,k} - L_{k,\bar{k}\bar{\ell}} (L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}})^+ L_{\bar{k}\bar{\ell},k}.$$

Proof: Let (v_k, v_ℓ) be given. For $q = 2$, the result follows trivially, since $\Gamma_{12} = w(v_1, v_2)$, where w is the weight function associated to Γ. In the sequel, we will consider the case $q \geq 3$.

Existence. We first show that (15) can always be solved. For $E \in \mathbb{C}^{n(q-2)\times n}$, consider the equation

$$L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}} E + L_{\bar{k}\bar{\ell},k} = 0. \qquad (16)$$

Note that a solution E exists for (16) if $\operatorname{range} L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}} \supset \operatorname{range} L_{\bar{k}\bar{\ell},k}$, which is equivalent to the condition $\operatorname{null} L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}} \subset \operatorname{null} L_{k,\bar{k}\bar{\ell}}$ since $(L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}})^* = L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}}$ and $(L_{\bar{k}\bar{\ell},k})^* = L_{k,\bar{k}\bar{\ell}}$. Let us now take an arbitrary $\eta \in \mathbb{C}^{n(q-2)}$ satisfying $L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}}\eta = 0$ and define $\zeta = [e_{\bar{k}\bar{\ell}} \otimes I_n]\eta$. We can write $\zeta^* L\zeta = \eta^*[e^*_{\bar{k}\bar{\ell}} \otimes I_n]L[e_{\bar{k}\bar{\ell}} \otimes I_n]\eta = \eta^* L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}}\eta = 0$, which implies $L\zeta = 0$ since L is Hermitian positive semidefinite. Observe $L_{k,\bar{k}\bar{\ell}}\eta = [e^*_k \otimes I_n]L[e_{\bar{k}\bar{\ell}} \otimes I_n]\eta = [e^*_k \otimes I_n]L\zeta = 0$. Therefore, $\operatorname{null} L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}} \subset \operatorname{null} L_{k,\bar{k}\bar{\ell}}$ and we can find E satisfying (16).

Choose now some E satisfying (16) and define

$$\Gamma_{kl} = L_{k,\bar{k}\bar{\ell}} E + L_{k,k}. \qquad (17)$$

Observe that we have $[\mathbf{1}_q^* \otimes I_n]L = 0$ since $\mathrm{null}\, L \supset \mathcal{S}_n$ and $L^* = L$. Also, $e_k + e_\ell = \mathbf{1}_q - e_{\bar{k}\bar{\ell}}\mathbf{1}_{q-2}$. We can now write using (16) and (17)

$$\begin{aligned}
&L_{\ell,\bar{k}\bar{\ell}}\, E + L_{\ell,k}\\
&\quad= [e_\ell^* \otimes I_n]\,(L[e_{\bar{k}\bar{\ell}} \otimes I_n]E + L[e_k \otimes I_n])\\
&\quad= [(\mathbf{1}_q - e_{\bar{k}\bar{\ell}}\mathbf{1}_{q-2} - e_k)^* \otimes I_n]\\
&\qquad\times (L[e_{\bar{k}\bar{\ell}} \otimes I_n]E + L[e_k \otimes I_n])\\
&\quad= \underbrace{[\mathbf{1}_q^* \otimes I_n]L}_{0}\,([e_{\bar{k}\bar{\ell}} \otimes I_n]E + [e_k \otimes I_n])\\
&\qquad-[\mathbf{1}_{q-2}^* \otimes I_n]\,\big([e_{\bar{k}\bar{\ell}}^* \otimes I_n]L[e_{\bar{k}\bar{\ell}} \otimes I_n]E\\
&\qquad\quad+[e_{\bar{k}\bar{\ell}}^* \otimes I_n]L[e_k \otimes I_n]\big)\\
&\qquad-([e_k^* \otimes I_n]L[e_{\bar{k}\bar{\ell}} \otimes I_n]E\\
&\qquad\quad+[e_k^* \otimes I_n]L[e_k \otimes I_n])\\
&\quad= -[\mathbf{1}_{q-2}^* \otimes I_n](\underbrace{L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}}\, E + L_{\bar{k}\bar{\ell},k}}_{0})\\
&\qquad-(\underbrace{L_{k,\bar{k}\bar{\ell}}\, E + L_{k,k}}_{\Gamma_{kl}})\\
&\quad= -\Gamma_{kl}\,.
\end{aligned}\tag{18}$$

Let us define $\Xi = [Z_1^T\ Z_2^T\ \ldots\ Z_q^T]^T$ with $Z_i \in \mathbb{C}^{n\times n}$ as $\Xi = [e_{\bar{k}\bar{\ell}} \otimes I_n]E + [e_k \otimes I_n]$. We claim that this choice Ξ and Γ_{kl} defined in (17) together satisfy (15), i.e.,

$$L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl} = 0 \quad \text{subj. to} \quad Z_k = I_n \text{ and } Z_\ell = 0\,.$$

Note that $Z_k = [e_k^* \otimes I_n]\Xi = [e_k^* \otimes I_n]([e_{\bar{k}\bar{\ell}} \otimes I_n]E + [e_k \otimes I_n]) = I_n$ since $e_k^* e_{\bar{k}\bar{\ell}} = 0$ and $e_k^* e_k = 1$. Likewise, $Z_\ell = [e_\ell^* \otimes I_n]\Xi = 0$ since $e_\ell^* e_{\bar{k}\bar{\ell}} = 0$ and $e_\ell^* e_k = 0$. To establish $L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl} = 0$, it suffices that we show $[e_i^* \otimes I_n](L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl}) = 0$ for all $i \in \{1, 2, \ldots, q\}$. Consider $i \neq k, \ell$ as the first case. Then, we can write $[e_i^* \otimes I_n] = [e_i^* \otimes I_n][e_{\bar{k}\bar{\ell}} \otimes I_n][e_{\bar{k}\bar{\ell}}^* \otimes I_n]$. Thence, using (16) and $e_{\bar{k}\bar{\ell}}^*(e_\ell - e_k) = 0$, we obtain

$$\begin{aligned}
&[e_i^* \otimes I_n](L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl})\\
&= [e_i^* \otimes I_n][e_{\bar{k}\bar{\ell}} \otimes I_n][e_{\bar{k}\bar{\ell}}^* \otimes I_n](L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl})\\
&= [e_i^* \otimes I_n][e_{\bar{k}\bar{\ell}} \otimes I_n](L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}}\, E + L_{\bar{k}\bar{\ell},k})\\
&= 0\,.
\end{aligned}$$

Consider $i = k$, the second case. Using (17) and $e_k^*(e_\ell - e_k) = -1$, we can write

$$\begin{aligned}
[e_k^* \otimes I_n](L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl}) &= L_{k,\bar{k}\bar{\ell}}\, E + L_{k,k} - \Gamma_{kl}\\
&= 0\,.
\end{aligned}$$

Finally, let $i = \ell$, the third case. Using (18) and $e_\ell^*(e_\ell - e_k) = 1$, we reach

$$\begin{aligned}
[e_\ell^* \otimes I_n](L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl}) &= L_{\ell,\bar{k}\bar{\ell}}\, E + L_{\ell,k} + \Gamma_{kl}\\
&= 0\,.
\end{aligned}$$

Hermitian pos. semidef. Let $[Z_1^T\ Z_2^T\ \ldots\ Z_q^T]^T = \Xi$ with $Z_i \in \mathbb{C}^{n\times n}$ and $\Gamma_{kl} \in \mathbb{C}^{n\times n}$ satisfy (15). Since $Z_k = I_n$ and $Z_\ell = 0$, we have $\Xi^*[(e_k - e_\ell) \otimes \Gamma_{kl}] = \Gamma_{kl}$. We can write

$$\begin{aligned}
\Gamma_{kl} &= \Xi^*[(e_k - e_\ell) \otimes \Gamma_{kl}]\\
&= \Xi^* L \Xi\,.
\end{aligned}$$

Since L is Hermitian positive semidefinite, so is $\Xi^* L \Xi$.

Uniqueness. Let $[Z_1^T\ Z_2^T\ \ldots\ Z_q^T]^T = \Xi$ with $Z_i \in \mathbb{C}^{n\times n}$ and $\Gamma_{kl} \in \mathbb{C}^{n\times n}$ satisfy (15). Suppose that Γ_{kl} is not unique. Then, we can find $[\tilde{Z}_1^T\ \tilde{Z}_2^T\ \ldots\ \tilde{Z}_q^T]^T = \tilde{\Xi}$ and $\tilde{\Gamma}_{kl} \neq \Gamma_{kl}$ that also satisfy (15). Note that $Z_k = \tilde{Z}_k = I_n$ and $Z_\ell = \tilde{Z}_\ell = 0$. Using $L^* = L$ and $\Gamma_{kl}^* = \Gamma_{kl}$, one can generate the following contradiction:

$$\begin{aligned}
\tilde{\Gamma}_{kl} &= \Xi^*[(e_k - e_\ell) \otimes \tilde{\Gamma}_{kl}] = \Xi^* L \tilde{\Xi} = (L\Xi)^* \tilde{\Xi}\\
&= [(e_k - e_\ell) \otimes \Gamma_{kl}]^* \tilde{\Xi} = (\tilde{\Xi}^*[(e_k - e_\ell) \otimes \Gamma_{kl}])^* = \Gamma_{kl}^*\\
&= \Gamma_{kl}\,.
\end{aligned}$$

Thanks to uniqueness, we can combine (16) and (17) to write *the* expression for effective conductance

$$\Gamma_{k\ell} = L_{k,k} - L_{k,\bar{k}\bar{\ell}}\,(L_{\bar{k}\bar{\ell},\bar{k}\bar{\ell}})^+ L_{\bar{k}\bar{\ell},k}\,.$$

Reciprocity. Let $[Z_1^T\ Z_2^T\ \ldots\ Z_q^T]^T = \Xi$ with $Z_i \in \mathbb{C}^{n\times n}$ and $\Gamma_{kl} \in \mathbb{C}^{n\times n}$ satisfy (15). Define $\tilde{Z}_i = I_n - Z_i$, for $i = 1, 2, \ldots, q$. Note that $\tilde{Z}_\ell = I_n$ and $\tilde{Z}_k = 0$, because $Z_\ell = 0$ and $Z_k = I_n$. Let $\tilde{\Xi} = [\tilde{Z}_1^T\ \tilde{Z}_2^T\ \ldots\ \tilde{Z}_q^T]^T$. Recall that, we have $L[\mathbf{1}_q \otimes I_n] = 0$. We can write

$$L\tilde{\Xi} = L([\mathbf{1}_q \otimes I_n] - \Xi) = -L\Xi = (e_\ell - e_k) \otimes \Gamma_{kl}$$

which implies $\Gamma_{\ell k} = \Gamma_{k\ell}$. ■

Effective conductance turns out to be a definite indicator of pairwise connectivity. In particular, it allows us to improve Theorem 3.

Theorem 7: The following are equivalent ($k \neq \ell$).

1) The array $[(C_{ij}),\, A]$ is $(k,\, \ell)$-observable.
2) The interconnection graph $\Gamma(W_{ij})$ is $(k,\, \ell)$-connected.
3) The effective conductance $\Gamma_{k\ell}(W_{ij})$ is full rank.

Proof: 1 $\Longleftrightarrow$ 2. By Theorem 3.

2$\Longrightarrow$3. Let us employ the shortcuts $\Gamma_{k\ell} = \Gamma_{k\ell}(W_{ij})$ and $L = \mathrm{lap}\,\Gamma(W_{ij})$. Suppose that $\Gamma(W_{ij})$ is $(k,\, \ell)$-connected. Then, $\mathrm{null}\, L \subset \mathrm{null}\,[(e_k - e_\ell)^* \otimes I_n]$ that (since $L^* = L$) implies $\mathrm{range}\, L \supset \mathrm{range}\,[(e_k - e_\ell) \otimes I_n]$. Therefore, we can find matrices $\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_q \in \mathbb{C}^{n\times n}$ such that

$$L \begin{bmatrix} \tilde{Z}_1 \\ \vdots \\ \tilde{Z}_q \end{bmatrix} = (e_k - e_\ell) \otimes I_n\,.$$

Define $\hat{Z}_i = \tilde{Z}_i - \tilde{Z}_\ell$, for $i = 1, 2, \ldots, q$. Note that $\hat{Z}_\ell = 0$ and we can write

$$\begin{aligned}
L \begin{bmatrix} \hat{Z}_1 \\ \vdots \\ \hat{Z}_q \end{bmatrix} &= L \left(\begin{bmatrix} \tilde{Z}_1 \\ \vdots \\ \tilde{Z}_q \end{bmatrix} - [\mathbf{1}_q \otimes I_n]\, \tilde{Z}_\ell \right)\\
&= (e_k - e_\ell) \otimes I_n
\end{aligned}\tag{19}$$

since $L[\mathbf{1}_q \otimes I_n] = 0$. Recalling that L is Hermitian positive semidefinite, we proceed as follows:

$$\begin{aligned} n &= \operatorname{rank}[(e_k - e_\ell) \otimes I_n] \\ &= \operatorname{rank}\left(L \begin{bmatrix} \hat{Z}_1 \\ \vdots \\ \hat{Z}_q \end{bmatrix} \right) \\ &= \operatorname{rank}\left(\begin{bmatrix} \hat{Z}_1 \\ \vdots \\ \hat{Z}_q \end{bmatrix}^* L \begin{bmatrix} \hat{Z}_1 \\ \vdots \\ \hat{Z}_q \end{bmatrix} \right). \end{aligned} \tag{20}$$

Then, we write (recalling $\hat{Z}_\ell = 0$)

$$\begin{aligned} \begin{bmatrix} \hat{Z}_1 \\ \vdots \\ \hat{Z}_q \end{bmatrix}^* L \begin{bmatrix} \hat{Z}_1 \\ \vdots \\ \hat{Z}_q \end{bmatrix} &= \begin{bmatrix} \hat{Z}_1 \\ \vdots \\ \hat{Z}_q \end{bmatrix}^* [(e_k - e_\ell) \otimes I_n] \\ &= \hat{Z}_k^* . \end{aligned} \tag{21}$$

Combining (20) and (21), we deduce that $\hat{Z}_k$ is nonsingular. This allows us to define $Z_i = \hat{Z}_i \hat{Z}_k^{-1}$ for $i = 1, 2, \ldots, q$. Note that $Z_k = I_n$ and $Z_\ell = 0$. Revisiting (19), we can write

$$\begin{aligned} L \begin{bmatrix} Z_1 \\ \vdots \\ Z_q \end{bmatrix} &= L \begin{bmatrix} \hat{Z}_1 \\ \vdots \\ \hat{Z}_q \end{bmatrix} \hat{Z}_k^{-1} \\ &= [(e_k - e_\ell) \otimes I_n] \hat{Z}_k^{-1} \\ &= (e_k - e_\ell) \otimes \hat{Z}_k^{-1} \end{aligned}$$

which implies (since $Z_k = I_n$ and $Z_\ell = 0$) that $\Gamma_{kl} = \hat{Z}_k^{-1}$. Clearly, Γ_{kl} is full rank.

3⟹2. Suppose $\operatorname{rank} \Gamma_{k\ell} = n$. Then, $\Gamma_{k\ell}^{-1}$ exists. Recall that Γ_{kl} satisfies (15) for some $Z_1, Z_1, \ldots, Z_q \in \mathbb{C}^{n \times n}$. We can write

$$\begin{aligned} L \begin{bmatrix} Z_1 \Gamma_{k\ell}^{-1} \\ \vdots \\ Z_q \Gamma_{k\ell}^{-1} \end{bmatrix} &= L \begin{bmatrix} Z_1 \\ \vdots \\ Z_q \end{bmatrix} \Gamma_{k\ell}^{-1} \\ &= [(e_k - e_\ell) \otimes \Gamma_{k\ell}] \, \Gamma_{k\ell}^{-1} \\ &= (e_k - e_\ell) \otimes I_n . \end{aligned}$$

Hence, $\operatorname{range} L \supset \operatorname{range}[(e_k - e_\ell) \otimes I_n]$ yielding $\operatorname{null} L \subset \operatorname{null}[(e_k - e_\ell)^* \otimes I_n]$. That is, $\Gamma(W_{ij})$ is (k, ℓ)-connected. ■

Finally, we present a result on detectability, which can be considered as an extension of the well-known PBH test.

Theorem 8: The following are equivalent ($k \neq \ell$).

1) The array $[(C_{ij}), A]$ is (k, ℓ)-detectable.

2) $\operatorname{rank} \begin{bmatrix} A - \lambda I_n \\ \Gamma_{k\ell}(W_{ij}) \end{bmatrix} = n$ for all $\operatorname{Re} \lambda \geq 0$.

Proof: 1⟹2. Suppose the second condition fails. This implies that we can find an eigenvalue μ_σ on the closed right half-plane ($\operatorname{Re} \mu_\sigma \geq 0$) and an eigenvector $f \in \mathbb{C}^n$ satisfying $\Gamma_{kl} f = 0$ and $[A - \mu_\sigma I_n] f = 0$, where we adopt the shorthand notation $\Gamma_{kl} = \Gamma_{kl}(W_{ij})$. Let $Z_1, Z_2, \ldots, Z_q \in \mathbb{C}^{n \times n}$ satisfy (15). Define $\xi = [Z_1^T \; Z_2^T \; \cdots \; Z_q^T]^T f$. Note that $\xi \in \operatorname{null} L$ because by (15), we can write

$$\begin{aligned} L\xi &= L \begin{bmatrix} Z_1 \\ \vdots \\ Z_q \end{bmatrix} f \\ &= [(e_k - e_\ell) \otimes \Gamma_{kl}] f \\ &= (e_k - e_\ell) \otimes (\Gamma_{kl} f) \\ &= 0 \end{aligned}$$

where $L = \operatorname{lap} \Gamma(W_{ij})$. Then, the identity $L = \mathbf{W}^* \mathbf{W}$ yields $\xi \in \operatorname{null} \mathbf{W}$. Consider now the system (4). Set the initial condition $\mathbf{x}(0) = [x_1(0)^T \; x_2(0)^T \; \cdots \; x_q(0)^T]^T = \xi$. Since $\mathbf{x}(0)$ belongs to the unobservable subspace $\operatorname{null} \mathbf{W}$, it produces $\mathbf{y}(t) \equiv 0$. That is

$$y_{ij}(t) \equiv 0 \text{ for all } (i, j) . \tag{22}$$

Moreover, we have $x_k(0) = Z_k f = f$ (because $Z_k = I_n$) and $x_\ell(0) = Z_\ell f = 0$ (because $Z_\ell = 0$). Hence, $x_k(t) = e^{\mu_\sigma t} f$ (because $[A - \mu_\sigma I_n] f = 0$) and $x_\ell(t) = 0$. As a result

$$\|x_k(t) - x_\ell(t)\| = |e^{\mu_\sigma t}| \cdot \|f\| \not\to 0 \tag{23}$$

due to $\operatorname{Re} \mu_\sigma \geq 0$. Combining (22) and (23), we see that the array $[(C_{ij}), A]$ is not (k, ℓ)-detectable.

2⟹1. Suppose the array $[(C_{ij}), A]$ is not (k, ℓ)-detectable. Then, for the system (4), we can find an initial condition belonging to the unobservable subspace $\mathbf{x}(0) \in \operatorname{null} \mathbf{W}$ yielding

$$[(e_k - e_\ell)^* \otimes I_n] \mathbf{x}(t) \not\to 0 .$$

Note that $\mathbf{x}(t)$ identically stays in the unobservable subspace $\operatorname{null} \mathbf{W}$. In particular, we can write $L\mathbf{x}(t) \equiv 0$ thanks to $L = \mathbf{W}^* \mathbf{W}$. Since $\mathbf{A} = [I_q \otimes A]$ and A share the same eigenvalues, the solution $\mathbf{x}(t)$ enjoys the structure

$$\mathbf{x}(t) = p_1(t) e^{\mu_1 t} + p_2(t) e^{\mu_2 t} + \cdots + p_m(t) e^{\mu_m t}$$

for some polynomials $p_1(t), p_2(t), \ldots, p_m(t)$ whose coefficients are vectors in $(\mathbb{C}^n)^q$. Let us here make a few observations. Since $\mu_1, \mu_2, \ldots, \mu_m$ are distinct, the collection of mappings $\{t \mapsto p_\sigma(t) e^{\mu_\sigma t} : p_\sigma(t) \not\equiv 0, \sigma = 1, 2, \ldots, m\}$ are linearly independent. Therefore, $L\mathbf{x}(t) \equiv 0$ implies

$$L p_\sigma(t) e^{\mu_\sigma t} \equiv 0 \tag{24}$$

for all σ. Moreover, $[(e_k - e_\ell)^* \otimes I_n] \mathbf{x}(t) \not\to 0$ implies

$$[(e_k - e_\ell)^* \otimes I_n] p_\sigma(t) e^{\mu_\sigma t} \not\to 0 \tag{25}$$

for some σ. Finally, $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ implies

$$\frac{d}{dt} \left\{ p_\sigma(t) e^{\mu_\sigma t} \right\} = \mathbf{A} p_\sigma(t) e^{\mu_\sigma t} \tag{26}$$

for all σ. Let us now fix an index $\sigma \in \{1, 2, \ldots, m\}$ that satisfies (25). Clearly, our choice comes with $\operatorname{Re} \mu_\sigma \geq 0$. Let $p_\sigma(t) = \zeta_r t^r + \cdots + \zeta_1 t + \zeta_0$ with $\zeta_0, \zeta_1, \ldots, \zeta_r \in$

$(\mathbb{C}^n)^q$ and $\zeta_r \neq 0$. By (26), we can write $p_\sigma(t)e^{\mu_\sigma t} = e^{\mathbf{A}t}\zeta_0 = [I_q \otimes e^{At}]\zeta_0$. This implies

$$[(e_k - e_\ell)^* \otimes I_n]\zeta_0 \neq 0$$

for otherwise ($[(e_k - e_\ell)^* \otimes I_n]\zeta_0 = 0$), we would have had

$$\begin{aligned}
&[(e_k - e_\ell)^* \otimes I_n]p_\sigma(t)e^{\mu_\sigma t}\\
&\quad = [(e_k - e_\ell)^* \otimes I_n][I_q \otimes e^{At}]\zeta_0\\
&\quad = [(e_k - e_\ell)^* \otimes e^{At}]\zeta_0\\
&\quad = e^{At}[(e_k - e_\ell)^* \otimes I_n]\zeta_0\\
&\quad = 0
\end{aligned}$$

which contradicts (25). Note that the maps $t \mapsto \zeta_0 e^{\mu_\sigma t}$, $t \mapsto \zeta_1 t e^{\mu_\sigma t}, \ldots, t \mapsto \zeta_r t^r e^{\mu_\sigma t}$ are linearly independent. Therefore, (24) yields $L\zeta_\nu t^\nu e^{\mu_\sigma t} \equiv 0$ for all $\nu \in \{0, 1, \ldots, r\}$. Consequently

$$L\zeta_\nu = 0$$

for all $\nu \in \{0, 1, \ldots, r\}$. Since $t \mapsto (\zeta_r t^r + \cdots + \zeta_1 t + \zeta_0)e^{\mu_\sigma t}$ solves (26), we have the following chain

$$\begin{aligned}
[\mathbf{A} - \mu_\sigma I_{nq}]\,\zeta_0 &= \zeta_1\\
[\mathbf{A} - \mu_\sigma I_{nq}]\,\zeta_1 &= 2\zeta_2\\
&\vdots\\
[\mathbf{A} - \mu_\sigma I_{nq}]\,\zeta_{r-1} &= r\zeta_r\\
[\mathbf{A} - \mu_\sigma I_{nq}]\,\zeta_r &= 0\,.
\end{aligned}$$

Let us now fix an index $\nu \in \{0, 1, \ldots, r\}$ that satisfies $[(e_k - e_\ell)^* \otimes I_n]\zeta_\nu \neq 0$ and $[(e_k - e_\ell)^* \otimes I_n][\mathbf{A} - \mu_\sigma I_{nq}]\zeta_\nu = 0$. Such ν should exist because $[(e_k - e_\ell)^* \otimes I_n]\zeta_0 \neq 0$ and $[\mathbf{A} - \mu_\sigma I_{nq}]\zeta_r = 0 \in \operatorname{null}[(e_k - e_\ell)^* \otimes I_n]$. Define the nonzero vector $f = [(e_k - e_\ell)^* \otimes I_n]\zeta_\nu$. It turns out that f is an eigenvector of A because

$$\begin{aligned}
[A - \mu_\sigma I_n]f &= [A - \mu_\sigma I_n][(e_k - e_\ell)^* \otimes I_n]\zeta_\nu\\
&= [(e_k - e_\ell)^* \otimes I_n]([I_q \otimes A] - \mu_\sigma I_{nq})\zeta_\nu\\
&= [(e_k - e_\ell)^* \otimes I_n][\mathbf{A} - \mu_\sigma I_{nq}]\zeta_\nu\\
&= 0\,.
\end{aligned} \tag{27}$$

Let now $\Gamma_{kl} = \Gamma_{kl}(W_{ij})$ be the effective conductance satisfying (15) for some $Z_1, Z_2, \ldots, Z_q \in \mathbb{C}^{n\times n}$. Using $L\zeta_\nu = 0$ and recalling $\Gamma_{kl} = \Gamma_{kl}^*$ and $L = L^*$, we can write

$$\begin{aligned}
\Gamma_{kl}f &= \Gamma_{kl}^*[(e_k - e_\ell)^* \otimes I_n]\zeta_\nu\\
&= [(e_k - e_\ell) \otimes \Gamma_{kl}]^*\zeta_\nu\\
&= \left(L\begin{bmatrix} Z_1\\ \vdots\\ Z_q\end{bmatrix}\right)^*\zeta_\nu = \begin{bmatrix} Z_1\\ \vdots\\ Z_q\end{bmatrix}^* L\zeta_\nu\\
&= 0\,.
\end{aligned} \tag{28}$$

Combining (27) and (28), yields

$$\operatorname{rank}\begin{bmatrix} A - \mu_\sigma I_n\\ \Gamma_{k\ell}\end{bmatrix} \neq n\,.$$

Hence, the result. ■

V. Summary

In this paper, we studied the observability of an array of LTI systems with identical individual dynamics, where an array was called observable when identically zero relative outputs implied identical solutions for the individual systems. In our setup, the relative output for each pair of units admitted a (possibly) different matrix. This incommensurability of the output matrices made it necessary to study the observability of the array via the connectivity of a matrix-weighted interconnection graph instead of the usual scalar-weighted topologies. In the first part of this paper, we established the equivalence between the observability of an array and the connectivity of its interconnection graph. In addition, we showed that the observability of an array could be studied also through the connectivity of the so-called eigengraphs, each of which corresponded to a particular eigenvalue of the system matrix of the individual dynamics.

In the second part, we investigated the pairwise observability of an array, where an array was called (k, ℓ)-observable when identically zero relative outputs implied identical solutions for the kth and ℓth individual systems. There too we addressed the problem from the graph connectivity point of view. Our findings were partially parallel to those in the first part. In particular, we obtained the equivalence between the (k, ℓ)-observability of an array and the (k, ℓ)-connectivity of its interconnection graph. However, in contrast with the first part, the (k, ℓ)-observability of an array was not in general guaranteed by the (k, ℓ)-connectivity of its eigengraphs. Moreover, we showed that pairwise observability could be studied via (matrix-valued) effective conductance, which was obtained from the interconnection graph by treating its Laplacian as the node admittance matrix of some resistive network, where nodes were connected by resistors with matrix-valued conductances. We found that an array was (k, ℓ)-observable if and only if the associated effective conductance was full rank.

VI. Notes

Although we consider continuous-time systems in this paper, how observability (detectability) is defined would also be valid for discrete-time systems. Then, the results on the relation between observability and graph connectivity would still hold true in discrete time. As for the detectability theorems, they would remain correct once the condition $\operatorname{Re}\lambda \geq 0$ on the eigenvalues is replaced by its discrete-time counterpart $|\lambda| \geq 1$.

The duality between observability and controllability for linear systems make the following question worthwhile to ask. What is the dual of the array (2)? The natural candidate seems to be the array

$$\dot{x}_i = A^* x_i + \sum_{j=1}^{q} C_{ij}^* u_{ij}\,, \qquad i = 1, 2, \ldots, q \tag{29}$$

under the constraint $u_{ij} = -u_{ji}$. (Recall $C_{ij} = C_{ji}$ and $C_{ii} = 0$.) If the controllability for this array is defined through the existence of control inputs $t \mapsto u_{ij}(t)$ that steer the states $x_1(t),\, x_2(t),\, \ldots,\, x_q(t)$ to a common trajectory in finite time, then it is not difficult to see that the array (29) is controllable if and only if the array (2) is observable. Designing distributed control laws for the synchronization of arrays having the structure (29) is an interesting problem. A preliminary attempt toward the solution can be found in [18].

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S. Emre Tuna was born in 1979, in Iskenderun, Turkey. He received the B.S. degree in electrical and electronics engineering from Middle East Technical University, Ankara, Turkey, in 2000, and the Ph.D. degree in electrical and computer engineering from the University of California, Santa Barbara, CA, USA, in 2005. He currently teaches at Middle East Technical University.