A New Perspective to Graphical Characterization of Multiagent Controllability

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***Abstract*—Recently, graphical characterization of multiagent controllability has been studied extensively. A major effort in the study is to determine controllability directly from topology structures of communication graphs. In this paper, we proposed the concept of controllability destructive nodes, which indicates that the difficulty in graphical characterization turns out to be the identification of topology structures of controllability destructive nodes. It is shown that each kind of double and triple controllability destructive nodes happens to have a uniform topology structure which can be defined similarly. The definition, however, is verified not to be applicable to the topology structures of quadruple controllability destructive (QCD) nodes. Even so, a design method is proposed to uncover topology structures of QCD nodes for graphs with any size, and a complete graphical characterization is presented for the graphs consisting of five vertices. One advantage of the established complete graphical characterization is that the controllability of any graph with any selection of leaders can be determined directly from the identified/defined destructive topology structures. The results generate several necessary and sufficient graphical conditions for controllability. A key step of arriving at these results is the discovery of a relationship between the topology structure of the controllability destructive nodes and a corresponding eigenvector of the Laplacian matrix.**

***Index Terms*—Controllability, destructive nodes, leader–follower structure, local interactions, multiagent systems.**

I. Introduction

THE STUDY of multiagent systems needs to understand the interplay between network topologies and system dynamics. Recently, considerable efforts have been made along this line in multiagent literature to interpret how communication topological structures relate to, e.g., controllability and consensus, which is also the focus here, where destructive nodes are defined to characterize controllability-relevant topologies.

A. Literature Review

Multiagent controllability was formulated under a leader–follower framework in which the influence over network is achieved by exerting control inputs upon leaders [1]. A system is controllable if followers can be steered to proper positions to form any desirable configuration by regulating the movement of leaders. In [1], a necessary and sufficient algebraic condition on controllability was presented, which was expressed in terms of eigenvalues and eigenvectors of submatrices of Laplacian. Another algebraic condition is a relation between controllability and the eigenvectors of the Laplacian, which provided a method of determining leaders from the elements of eigenvectors [2]. Armed with these results, the virtue that leaders should have was characterized from both algebraic and graphical perspectives [3]. Controllability of switching networks was discussed in [4]. Recently, new graph-theoretic characterizations of controllability were developed by means of the proposed notion of graph controllability classes [5], and a unified protocol design method was proposed for controllability in [6]. The presented protocol involves a consensus algorithm. Other algebraic conditions on controllability and consensus exist in [7]–[19].

Algebraic conditions lay the foundation for understanding interactions between topological structures and controllability. Previous work has suggested that the interactions are quite involved, even for the simplest path graph [20]. Special topologies were studied first, such as grid graphs [21], multichain topologies [22], and tree graphs [3]. Controllability can be fully understood by analyzing the eigenvectors of Laplacian as shown in [20] and [21]. It can also be tackled through topological construction which sometimes relates to the partition of graphs. For example, topologies were designed by using the vanishing coordinates-based partition [3] and an eigenvector-based partition [23]. In particular, the construction of uncontrollable topologies not only facilitates the design of control strategies but also deepens understanding of controllable ones [2], [22]. Conditions in terms of graphs have also been formulated for structural controllability and consensus (see [24]–[30]). Recently, it was proved, via a proper design of protocols, that the controllability of single-integrator, high-order, and generic linear multiagent systems is uniquely determined by the topology structure of the communication graph [6]. The above work guides a further study of controllability. Here the focus is on the following problem: identify topology structures formed by, respectively, m ($m = 2, 3, 4$) nodes and prove that the controllability can, and further, only can be destroyed by these identified topology structures. That is, it is to show that the system is uncontrollable only with these topologies. The problem is closely related to the number and locations of leaders. These m nodes are referred

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to as controllability destructive nodes. The information flow inside them, and between them and the others, constitutes the controllability destructive topology structure.

B. Contributions

The contributions of this paper are as follows. First, the efforts made in this paper help us to recognize a fundamental fact—the complete graphical characterization of multiagent controllability boils down to the identification of topology structures of all m controllability destructive nodes, $m = 2, \ldots, \gamma - 1$, if the graph consists of γ nodes. For graphs of given sizes, the word "complete" means that the controllability of any graph with any selection of leaders can be determined directly from the identified/defined destructive topology structures. Second, it can be seen from the derivation of the results that difficulties in complete graphical characterization lie in not only how to find out the destructive topologies, but also how to prove they are the only topologies that destroy controllability. We tackle this problem for $m = 2, 3, 4$ through detailed analysis of graphical implications of a specified eigenvector for each m. Note that the analysis cannot be done by existing techniques, e.g., leader-symmetry or almost equitable partitions. As a consequence, a complete graphical characterization is obtained for the controllability of graphs consisting of five nodes. Even though the number of nodes is 5, there are 1050 different topology structures due to different leader selections. For the same graph, controllability may be different since leaders' selection yields different topology structures for both leaders and followers. We show that 48 topology structures constitute a complete graphical characterization by which controllability of any of 1050 graphs can be directly determined. It can be imagined that the difficulty will rise quickly with the increase of the number of nodes. Third, the concept of controllability destructive nodes is first proposed in this paper by which a complete graphical characterization can be achieved. In the case that the number m of destructive nodes is, respectively, 2 and 3, it is proved that the topology structures of these two kinds of destructive nodes can be defined in the same way. Although the definition cannot be generalized, for the case $m = 4$, a topology construction procedure is still proposed for graphs of any size and all destructive topology structures are identified for graphs of five nodes. We show that each kind of m ($m = 2, 3, 4$) controllability destructive nodes corresponds to an appropriate scope of leaders' selection. For each kind, the specific leaders' selection is not unique and a corresponding necessary and sufficient graphical condition is derived for controllability. The combination of all these conditions results in a complete graphical characterization for graphs of five vertices. This process suggests an approach of complete graphical characterization for graphs with a larger size.

Compared to the existed works, the advantages and innovation points of this paper lie in the following aspects.

1) It is shown for the first time that complete graphical characterizations can be developed, by which the controllability of any graph with any selection of leaders can be determined directly from the identified destructive topology structures. Since topology structures and leaders' selections have to be considered at the same time, the result has also strengthened the understanding of the leaders' role in graphical characterizations.
2) A new concept of controllability destructive nodes is put forward. The topology structures of controllability destructive nodes are verified to play a key role in the graphical characterization of controllability. In addition, the presented results demonstrate how partial/local topology structures formed by the controllability destructive nodes affect the controllability of the overall interconnection graph. This gains new insights into the graph-theoretic characterization of controllability.
3) The results provide a new perspective to the complexity involved in the investigation of graphical characterizations. The derivations show that the complexity comes from the identification of topology structures of controllability destructive nodes. Although more destructive topologies can be designed/identified for a graph of any size by following the similar idea in steps 1–4 proposed in Theorem 3, the most difficult point is how to prove that the identified destructive topologies cover all possible controllability destructive topology structures.
4) To get a controllable graph, the results indicate that any $m(2 \le m \le n-1)$ nodes of an arbitrarily given set of $m+1$ follower nodes should not constitute a controllability destructive topology structure. This is a newly discovered nested relationship required in the graph-theoretic characterization of multiagent controllability.

C. Organization

This paper is organized as follows. In Section II, preliminary concepts and algebraic conditions are given. In Section III, necessary and sufficient graphical conditions are derived by taking advantage of the identified topology structures of double controllability destructive (DCD), triple controllability destructive (TCD), and quadruple controllability destructive (QCD) nodes. In Section IV, it is shown how to combine the previous conditions to yield a complete graphical characterization. Finally, the conclusions are summarized in Section V.

II. Preliminaries

A. Notations and Graph Theory

The cardinality of a finite set $\mathcal{H}$ is denoted by $|\mathcal{H}|$. If $\mathcal{H} \subset \mathcal{K}$, the complement of $\mathcal{H}$ in $\mathcal{K}$ is denoted by $\mathcal{K}\backslash\mathcal{H}$. $\mathrm{diag}\{\beta_1, \ldots, \beta_{n+l}\}$ defines a diagonal matrix with diagonal entries $\beta_1, \ldots, \beta_{n+l}$, and A^{T} denotes the transpose of a matrix A. For a set of $n+l$ agents, the information flow between agents is incorporated in an undirected and unweighted simple graph $\mathcal{G}$, which has no loops and multiple edges, and consists of a node set $\mathcal{V} = \{v_1, \ldots, v_{n+l}\}$ and an edge set $\mathcal{E} = \{(v_i, v_j) \in \mathcal{V} \times \mathcal{V}\}$, with nodes representing agents and edges indicating the interconnections between them. $\mathcal{N}_i = \{j \,|\, v_i \sim v_j; j \neq i\}$ represents the neighboring set of v_i and "$\sim$" denotes the neighboring relation. The adjacency matrix of $\mathcal{G}$ is $A = [a_{ij}]$ satisfying $a_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$, while

$a_{ij} = 0$, otherwise. The degree matrix of $\mathcal{G}$ is defined as $D = \text{diag}\{d_1, \dots, d_{n+l}\}$ with diagonal entries $d_i = |\mathcal{N}_i|$. The Laplacian matrix of $\mathcal{G}$ is given by $\mathcal{L} = D - A$. A path of $\mathcal{G}$ is a sequence of consecutive edges. $\mathcal{G}$ is connected if there is a path between any distinct nodes. A subgraph of $\mathcal{G}$ is a graph whose vertex set is a subset of $\mathcal{V}$ and whose edge set is a subset of $\mathcal{E}$ restricted to this subset. A subgraph is induced from $\mathcal{G}$ if it is obtained by deleting a subset of nodes and all the edges connecting to those nodes. An induced subgraph, which is maximal and connected, is said to be a connected component. Throughout this paper, it is assumed without loss of generality that agents $n+1, \dots, n+l$ play leaders' role. Thus one can define

$$\mathcal{N}_{kf} \triangleq \{i | v_i \sim v_k, v_i \text{ is a node of follower subgraph } \mathcal{G}_f\}$$
$$\mathcal{N}_{kl} \triangleq \{j | v_j \sim v_k, v_j \text{ is a node of leader subgraph } \mathcal{G}_l\}.$$

Then $\mathcal{N}_k = \mathcal{N}_{kf} \cup \mathcal{N}_{kl}$, $\mathcal{N}_{kf} \cap \mathcal{N}_{kl} = \Phi$, where Φ is the empty set. This partition of neighboring set is important and will be used frequently in the derivation. Let $\mathcal{G}_f$ and $\mathcal{G}_l$ represent the subgraphs induced, respectively, from the follower and leader node set. Controllability can be studied under the assumption that $\mathcal{G}$ is connected [2].

B. Problem Formulation and Preliminary Results

Consider $n+l$ single integrator agents given by

$$\begin{cases} \dot{x}_i = u_i, & i = 1, \dots, n \\ \dot{z}_j = u_{n+j}, & j = 1, \dots, l \end{cases} \tag{1}$$

where n and l are the number of followers and leaders, respectively; $x_i \in \mathbb{R}^1$ and $z_j \in \mathbb{R}^1$ are the states of the ith and $(n+j)$th agent, respectively. Let $z_1, \dots, z_l$ act as leaders and be influenced only via external control inputs. The followers are governed by the neighbor-based consensus rule

$$u_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i) + \sum_{(n+j) \in \mathcal{N}_i} (z_j - x_i) \tag{2}$$

where $j \in \{1, \dots, n\}$; $(n+j) \in \{n+1, \dots, n+l\}$. x and z denote the stack vectors of x_i's and z_j's, respectively. Let $\mathcal{F}$ be the matrix obtained from $\mathcal{L}$ after deleting the last l rows and l columns, and $\mathcal{R}$ be the matrix consisting of the first n elements of the deleted columns. Then under (2), the followers' dynamics is

$$\dot{x} = -\mathcal{F}x - \mathcal{R}z. \tag{3}$$

Since (3) captures the followers' dynamics, the controllability of a multiagent system can be realized through (3).

Different protocols lead to different $\mathcal{F}$ and $\mathcal{R}$ in (3). In this way, protocols have an impact on controllability. Under the given protocol (3), the interplay between communication topology and controllability is to be studied below. The study may also be applicable to other protocols.

Proposition 1: The multiagent system with dynamics (1) is controllable if and only if there does not exist some β such that any of the following statements is satisfied.

1) β is an eigenvalue of $\mathcal{F}$ associated with eigenvector $y = [y_1, \dots, y_n]^T$ and y is orthogonal to all columns of $\mathcal{R}$.
2) $\bar{y} = [y_1, \dots, y_n, 0, \dots, 0]^T$ is an eigenvector of the Laplacian $\mathcal{L}$ associated with the eigenvalue at β.
3) $\mathcal{F}$ and $\mathcal{L}$ share a common eigenvalue at β.
4) The following equations hold:

$$d_k y_k - \sum_{i \in \mathcal{N}_{kf}} y_i = \beta y_k, \ k = 1, \dots, n \tag{4}$$
$$\sum_{i \in \mathcal{N}_{kf}} y_i = 0, \ k = n+1, \dots, n+l. \tag{5}$$

Proof: 2) and 3) were proved, respectively, in [2] and [31]. The remaining is to show that the four statements are equivalent. 1)⇔2) and 2)⇔3) follow from $\mathcal{L}\bar{y} = \beta\bar{y}$ and [32, Th. 9.5.1]. Next we show 2)⇔4). $\mathcal{L}\bar{y} = \beta\bar{y}$ yields $\mathcal{F}y = \beta y$, $\mathcal{R}^T y = 0$, which, respectively, leads to (4) and (5). On the contrary, since $y_i = 0$ for $i = n+1, \dots, n+l$; $\sum_{i \in \mathcal{N}_{kl}} y_i = 0$. Then, by (4), for $k = 1, \dots, n$, $d_k y_k - \sum_{i \in \mathcal{N}_k} y_i = d_k y_k - \sum_{i \in \mathcal{N}_{kf}} y_i - \sum_{i \in \mathcal{N}_{kl}} y_i = \beta y_k$. For $k = n+1$ to $n+l$, since $y_k = 0$ and $\sum_{i \in \mathcal{N}_{kl}} y_i = 0$, by (5), $d_k y_k - \sum_{i \in \mathcal{N}_k} y_i = \beta y_k$ also holds. Thus the eigen-condition is met for each k, i.e., $\mathcal{L}\bar{y} = \beta\bar{y}$. ∎

III. Controllability Destructive Nodes

This section includes the identification/definition of destructive topologies and a proof that only these topologies destroy the controllability if leaders are properly selected. This is achieved via a detailed analysis of graphical implications for a specified eigenvector $\bar{y}$.

A. Double and Triple Destructive Nodes

Definition 1: (v_p, v_q) is called a DCD node tuple if for any v_k with $k \neq p, q$, the corresponding $\mathcal{N}_{kf}$ either contains both p and q or neither of them; and (v_p, v_q, v_r) a TCD node tuple if for any v_k with $k \neq p, q, r$; $\mathcal{N}_{kf}$ either contains all the p, q, r or none of them.

Remark 1: The proof of subsequent Lemma 2 indicates that DCD and TCD nodes happen to have the above defined topology structure. Unfortunately, the structure cannot be generalized to QCD nodes. DCD and TCD nodes can only be generated from follower nodes. Lemma 1 can be proved similarly as Lemma 2, while the proof of the latter is much more complicated (see a preliminary version of this paper [33]). Note that Lemma 1 coincides with [34, Th. 3.31].

Lemma 1: $\bar{y} = [0, \dots, 0, \ y_p, 0, \dots, 0, y_q, 0, \dots, 0]^T$ with $y_p, y_q \neq 0$ is an eigenvector of $\mathcal{L}$ if and only if v_p, v_q are DCD nodes. Moreover, if $p \in \mathcal{N}_{qf}$, then $y_p = -y_q$ and $d_p = d_q$ with the corresponding $\lambda = d_p + 1$; otherwise, $\lambda = d_q$.

Theorem 1: There exists a nonempty set of leaders selected from $\Gamma_{p,q} \triangleq \{1, \dots, n+l\} \backslash \{p, q\}$ such that the multiagent system with single-integrator dynamics (1) is controllable if and only if there does not exist a follower DCD node tuple (v_p, v_q) with $p \neq q$.

Proof: The proof is done by contradiction.

1) *Necessity:* Suppose by contradiction that the follower subgraph $\mathcal{G}_f$ contains DCD nodes v_p, v_q. Lemma 1 shows that $\mathcal{L}$ has an eigenvector $\bar{y} = [0, \dots, 0, y_p, 0, \dots, 0, y_q, 0, \dots, 0]^T$ with $y_p = -y_q \neq 0$. By Proposition 1, system (1) is uncontrollable with

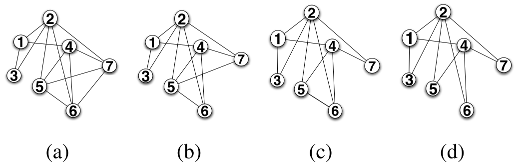

Fig. 1. (a)–(d) Subsequent topologies I–IV with v_5, v_6, and v_7 being the TCD nodes.

any leaders selected from $\Gamma_{p,q}$. This contradicts the assumption.

2) *Sufficiency:* Suppose by contradiction that the system is uncontrollable with any leaders selected from $\Gamma_{p,q}$. Then the system is uncontrollable with all the elements of $\Gamma_{p,q}$ playing leaders' role. By Proposition 1, $\mathcal{L}$ has an eigenvector $\bar{y} = [0, \ldots, 0, y_p, 0, \ldots, 0, y_q, 0, \ldots, 0]^{\mathrm{T}}$. Next we show $y_p, y_q \neq 0$. Suppose by contradiction $y_p = 0$, then $y_q \neq 0$ because $\bar{y}$ is an eigenvector. Since the graph is connected, $\lambda = 0$ is a simple eigenvalue associated with the all-one eigenvector 1. Thus the eigenvalue β associated with $\bar{y}$ is not zero. In addition, there exist at least one $k \neq q$ with $k \in \mathcal{N}_q$; otherwise, v_q will be isolated from all the other nodes. The special form of $\bar{y}$ then results in $\sum_{i\in\mathcal{N}_{kl}} y_i = 0, \sum_{i\in\mathcal{N}_{kf}} y_i = y_q$. Since $y_k = 0$, $d_k y_k - \sum_{i\in\mathcal{N}_k} y_i = -y_q$. The eigen-condition in (22) is not met for v_k since $y_k = 0$ and $y_q \neq 0$. This contradicts with the fact that $\bar{y}$ is an eigenvector. Therefore $y_p \neq 0$. Similar arguments yield $y_q \neq 0$. Finally, it follows from Lemma 1 that v_p and v_q are DCD nodes since $\bar{y}$ with $y_p, y_q \neq 0$ is an eigenvector of $\mathcal{L}$. This is in contradiction with the assumption. The proof is completed. ■

Remark 2: By Theorem 1, if the system is controllable, any two followers are not DCD nodes. Conversely, if v_p, v_q are not DCD notes, there is at least one set of leaders, say $\mathcal{V}\backslash\{v_p, v_q\}$ such that the system is controllable.

Remark 3: Next, many situations have to be analyzed to show that among all topology structures formed by three nodes, only those of TCD nodes destroy controllability. This is the difficulty of graphical characterization.

Topologies of Fig. 1(a)–(d) correspond to, respectively, the following four cases of $\mathcal{N}_{kf}$ for v_p, v_q, and v_r: 1) for any $k \in \{p, q, r\}$, $\mathcal{N}_{kf}$ contains the other two of p, q, and r; 2) there is a $k \in \{p, q, r\}$(say $k = p$) with $\mathcal{N}_{pf}$ containing q, r and each of $\mathcal{N}_{qf}, \mathcal{N}_{rf}$ contains only p in $\{p, q, r\}$; 3) there is a $k \in \{p, q, r\}$(say $k = p$) with $\mathcal{N}_{kf}$ containing one and only one of the other two of p, q, r; and its single neighbor node of p, q, r(say q) also has k as its single neighbor node in $\{p, q, r\}$; and 4) for any $k \in \{p, q, r\}$, $\mathcal{N}_{kf}$ contains none of p, q, and r.

Lemma 2: $\bar{y} = [0, \ldots, y_p, 0, \ldots, y_q, 0, \ldots, y_r, 0, \ldots, 0]^{\mathrm{T}}$ with $y_p, y_q, y_r \neq 0$ and all the other elements being zero is an eigenvector of $\mathcal{L}$ if and only if v_p, v_q, v_r are TCD nodes. Moreover, $y_p + y_q + y_r = 0, y_k \neq 0, k = p, q, r$, and for topology I, $d_p = d_q = d_r$ with the corresponding eigenvalue $\lambda = d_p + 1$; for topology II, $y_q = y_r$, $d_p = d_q + 1 = d_r + 1$ with $\lambda = d_p + 1$; for topology III, $y_p = y_q$, $d_p = d_q = d_r + 1$ with $\lambda = d_r$; and for topology IV, $d_p = d_q = d_r$ with $\lambda = d_r$.

Remark 4: By the proof (see the Appendix), v_p, v_q, v_r can only have the topological structure of TCD nodes when $\bar{y}$ is an eigenvector. To show this, all other topologies have to be excluded by checking the eigen-condition for each node. This graphical implication of $\bar{y}$ is the most valuable and difficult finding here, which constitutes a proof of necessity and cannot be analyzed by existing techniques, such as leader-symmetry or almost equitable partitions.

Theorem 2: There exist a group of leaders selected from $\Gamma_{p,q,r}$ such that the multiagent system with single-integrator dynamics (1) is controllable if and only if the following two conditions are met simultaneously.

1) The follower node set does not contain TCD nodes v_p, v_q, and v_r, where $p, q, r \in \{1, \ldots, n + l\}$, $\Gamma_{p,q,r} \triangleq \{1, \ldots, n + l\}\backslash\{p, q, r\}$.
2) Any two of v_p, v_q, and v_r are not DCD nodes.

Proof: The proof is based on Theorem 1 and Lemma 2.

1) *Necessity:* Suppose by contradiction that two of v_p, v_q, and v_r are DCD nodes, then necessity can be proved in the same vein as that of Theorem 1. In case v_p, v_q, and v_r are TCD nodes, the proof can be carried out in the same way by using Lemma 2.
2) *Sufficiency:* Suppose by contradiction that the system is uncontrollable with any leaders selected from $\Gamma_{p,q,r}$. Then the same arguments as the sufficiency proof of Theorem 1 show that $\bar{y} = [0, \ldots, y_p, 0, \ldots, y_q, 0, \ldots, y_r, 0, \ldots, 0]^{\mathrm{T}}$ is an eigenvector of $\mathcal{L}$. Next, it is to verify $y_p, y_q, y_r \neq 0$. First, we show that two of y_p, y_q, and y_r cannot be zero. Suppose by contradiction that two of y_p, y_q, and y_r take zero, say $y_p = y_q = 0$. Then $y_r \neq 0$, or else $\bar{y}$ is a zero vector. Since $\mathcal{G}$ is connected, $\lambda = 0$ is a simple eigenvalue associated with the all one eigenvector 1. Thus the eigenvalue β associated with $\bar{y}$ is not zero. Since $\mathcal{G}$ is connected, there is a $k \neq r$ with $k \in \mathcal{N}_r$, i.e., the corresponding v_k is incident to v_r. Otherwise, v_r turns to be an isolated node. The special form of $\bar{y}$ then leads to $\sum_{i\in\mathcal{N}_{kl}} y_i = 0, \sum_{i\in\mathcal{N}_{kf}} y_i = y_r$. From $y_k = 0$, one has $d_k y_k - \sum_{i\in\mathcal{N}_k} y_i = -y_r$. Since $y_k = 0, y_r \neq 0$, this equation means that the eigen-condition (22) of v_k is not met. This contradicts with the condition that $\bar{y}$ is an eigenvector. So any two of y_p, y_q, and y_r cannot take the value of zero. Second, suppose there is one and only one of y_p, y_q, and y_r taking zero, say $y_p = 0$ and $y_q \neq 0, y_r \neq 0$. By Lemma 1, the corresponding v_q and v_r constitute a pair of DCD nodes. This contradicts with the condition that any two of v_p, v_q, and v_r are not DCD nodes. Since $y_p, y_q, y_r \neq 0$, Lemma 2 shows that v_p, v_q, and v_r constitute a triple of TCD nodes. This also contradicts with the condition. ■

Remark 5: By Theorem 2, any topology not formed by controllability destructive nodes cannot destroy controllability. This leads to a complete graphical characterization for controllability, as verified in Section IV. Thus the difficulty in complete graphical characterization lies in uncovering the topologies of destructive nodes.

B. Quadruple Destructive Nodes

The identification of topology structures for QCD nodes becomes very complicated. Since Definition 1 is not applicable to QCD nodes; we will first identify topology structures via the following construction procedure. Then all topologies of QCD nodes will be found out in the next section for graphs of five vertices, which also suggests an approach for graphs with a larger size.

Below s_1, s_2, t_1, and t_2 are used to represent the indices of the desired QCD nodes. Let η be a vector with entries $\eta_p = \eta_q = 0$ and

$$\eta_{s_1} = \eta_{s_2} = -\eta_{t_1} = -\eta_{t_2} \neq 0 \tag{6}$$

where p, q, s_1, s_2, t_1, and t_2 are distinct and all the other entries of η are zero. The node set of $\mathcal{G}$ can be broken down into four parts: $\{v_p, v_q\}, \{v_{s_1}, v_{s_2}\}$, $\{v_{t_1}, v_{t_2}\}$, and the others. In subsequent topology design procedure, v_p, v_q are fixed in advance to assist in designing neighbor relationship of $\{v_{s_1}, v_{s_2}\}$ and $\{v_{t_1}, v_{t_2}\}$. The neighbor topology structure of $\{v_{s_1}, v_{s_2}\}$ to $\{v_p, v_q\}$ and $\{v_{t_1}, v_{t_2}\}$ is constructed below, where v_{s_2} follows the same rule as v_{s_1}. So the rule is stated only for v_{s_1}. A topology design procedure for QCD nodes is as follows.

Case 1: v_{s_1} has no neighbor relationship with v_{s_2}, and so has v_{t_1} with v_{t_2}. The design is divided into four steps.

Step 1: The construction of neighbor nodes of v_{s_1} conforms to one of the following cases.

1) v_{s_1} is a neighbor of both v_p and v_q. In this case, v_{s_1} is required to have neighbor relationship with only one of v_{t_1} and v_{t_2}.
2) v_{s_1} has neighbor relationship to neither v_p nor v_q. In this case, v_{s_1} is required to have neighbor relationship with both v_{t_1} and v_{t_2}.

Step 2: The design of the neighbor topology structure of $\{v_{t_1}, v_{t_2}\}$ to $\{v_p, v_q\}$ and $\{v_{s_1}, v_{s_2}\}$ is in the same vein as that of $\{v_{s_1}, v_{s_2}\}$ to $\{v_p, v_q\}$ and $\{v_{t_1}, v_{t_2}\}$.

Step 3: For $k = p, q$, $\mathcal{N}_{kf}$ contains exactly one of s_1 and s_2 and one of t_1 and t_2.

Step 4: For $k \in \Omega \overset{\Delta}{=} \{1, \ldots, n+l\} \backslash \{p, q, s_1, s_2, t_1, t_2\}$, the design of neighbors of v_k conforms to the following cases.

1) v_k is a neighbor of both v_p and v_q.
2) v_k is a neighbor of all of $v_{s_1}, v_{s_2}, v_{t_1}$, and v_{t_2}.
3) v_k does not have neighbor relationship to any of $v_p, v_q, v_{s_1}, v_{s_2}, v_{t_1}$, and v_{t_2}.
4) v_k has arbitrary neighbor relationship with any other nodes except $v_p, v_q, v_{s_1}, v_{s_2}, v_{t_1}$, and v_{t_2}.

Any number of conditions 1)–4) can be satisfied simultaneously.

Case 2: At least one of the following two cases occur: v_{s_1} is a neighbor of v_{s_2}; or v_{t_1} is a neighbor of v_{t_2}. The remaining construction is the same as case 1.

Remark 6: The neighbor topology structure of $\{v_{s_1}, v_{s_2}\}$ to $\{v_p, v_q\}$ is designed to be the same as that of $\{v_{t_1}, v_{t_2}\}$ to $\{v_p, v_q\}$. This kind of equivalence of neighbor topology between $\{v_{s_1}, v_{s_2}\}$ and $\{v_{t_1}, v_{t_2}\}$ discourages leaders of tearing open them and therefore destroys controllability.

Theorem 3: If system (1) is controllable, then the follower node set does not contain $v_{s_1}, v_{s_2}, v_{t_1}$, and v_{t_2} with the topology structure of $v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}$ agreeing with any of those designed via steps 1–4, where $s_1, s_2, t_1, t_2 \in \{1, \ldots, n+l\}$ are distinct indices.

Proof: The η in (6) is shown to be an eigenvector of $\mathcal{L}$. The result will then follow from Proposition 1.

For $k = s_1, s_2$, if the neighbor nodes of v_k to $\{v_p, v_q\}$ and $\{v_{t_1}, v_{t_2}\}$ are designed according to 1) of step 1, there are three neighbors of v_k in $\{v_p, v_q, v_{t_1}, v_{t_2}\}$. In addition, denote by σ the number of neighbor nodes of v_k in $\mathcal{V} \backslash \{v_p, v_q, v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}\}$. Then the node degree of v_k is $d_k = \sigma + 3$. Note that 2) of step 4 means that the value of σ remains unchanged for each $v_k, k = s_1, s_2$. Since all the elements of η are zero except $\eta_{s_1}, \eta_{s_2}, \eta_{t_1}$, and η_{t_2}; $\sum_{i\in\mathcal{N}_k} \eta_i = \eta_t$, where $t = t_1$ or t_2 depending on the specific situation of item 1). Then $\eta_k = -\eta_t$ yields that

$$d_k\eta_k - \sum_{i\in\mathcal{N}_k} \eta_i = (d_k + 1)\eta_k = (\sigma + 4)\eta_k, \quad k = s_1, s_2. \tag{7}$$

If the neighbors of v_{s_k} are designed via 2) of step 1, $d_k = \sigma + 2$. In this case, $\sum_{i\in\mathcal{N}_k} \eta_i = \eta_{t_1} + \eta_{t_2}$. By (6), $d_k\eta_k - \sum_{i\in\mathcal{N}_k} \eta_i = d_k\eta_k + 2\eta_k = (\sigma + 4)\eta_k$, $k = s_1, s_2$. For $k = t_1, t_2$, the neighbor nodes of $\{v_{t_1}, v_{t_2}\}$ to $\{v_p, v_q\}$ and $\{v_{s_1}, v_{s_2}\}$ is designed in the same way as that of $\{v_{s_1}, v_{s_2}\}$ to $\{v_p, v_q\}$ and $\{v_{t_1}, v_{t_2}\}$. In addition, step 4 means that the aforementioned σ is also the number of neighbors of v_k in $\mathcal{V} \backslash \{v_p, v_q, v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}\}$. Then the proof can be carried out in the same manner as the case of $k = s_1, s_2$. Accordingly

$$d_k\eta_k - \sum_{i\in\mathcal{N}_k} \eta_i = (\sigma + 4)\eta_k, \quad k = t_1, t_2. \tag{8}$$

For $k = p, q$, it follows from step 3 that:

$$\sum_{i\in\mathcal{N}_k} \eta_i = \sum_{i\in\mathcal{N}_{kf}} \eta_i = \eta_s + \eta_t, \quad k = p, q \tag{9}$$

where $s = s_1$ or s_2; $t = t_1$ or t_2 depending on the specific situation of step 3. By (6), $\eta_s = -\eta_t$. Then (9) yields $\sum_{i\in\mathcal{N}_k} \eta_i = 0$. By $\eta_k = 0$, (8) also holds for $k = p, q$.

For $k \in \Omega$, step 4 means $\sum_{i\in\mathcal{N}_k} \eta_i = \eta_{s_1} + \eta_{s_2} + \eta_{t_1} + \eta_{t_2} = 0$ if 2) is involved; and $\sum_{i\in\mathcal{N}_k} \eta_i = 0$ if 2) is not involved. This together with $\eta_k = 0$ also leads to (8) for $k \in \Omega$. The above arguments show that η is an eigenvector of $\mathcal{L}$.

For case 2, the above proof for case 1 needs a bit of alteration. Below the discussion focuses on the situation that v_{s_1} is a neighbor of v_{s_2}. The result can be shown in the same way when v_{t_1} is a neighbor of v_{t_2}. For $k = s_1, s_2$, the node degree of v_k is changed to be $\sigma + 4$ and $\sum_{i\in\mathcal{N}_k} \eta_i = 0$ since there is an additional edge between v_{s_1} and v_{s_2}. Thus (8) holds for $k = s_1, s_2$. If the neighbors of v_{s_k} are designed according to 2) of step 1, $d_k = \sigma + 3$. In this case, $\sum_{i\in\mathcal{N}_k} \eta_i = \eta_t$, where $t = t_1$ or t_2 depending on the specific construction. By (6), (8) still holds. For $k = t_1, t_2$, the proof is in the same manner as $k = s_1, s_2$. The remaining proof is the same as case 1. This completes the proof. ■

Example 1: The example is to illustrate Theorem 3.

For graphs in Fig. 2, $p = 1, q = 3$; $s_1 = 2, s_2 = 4, t_1 = 5$, and $t_2 = 6$. In (a), $v_{s_1} = v_2$ is a neighbor of both $v_p = v_1$ and $v_q = v_3$; and it is incident to v_6, i.e., only one of $v_{t_1} = v_5$ and $v_{t_2} = v_6$. This corresponds to case 1) of step 1. Similarly, v_{s_2} corresponds to 2) of step 1. These arguments

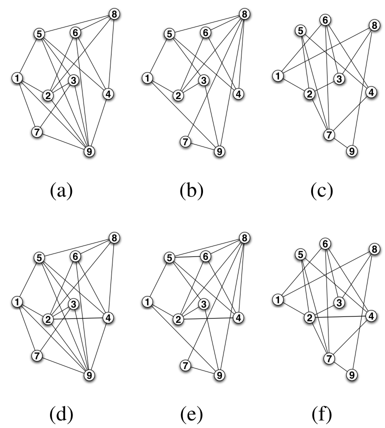

Fig. 2. (a)–(f) Designed according to cases 1 and 2, respectively, with QCD nodes v_2, v_4, v_5, and v_6.

exhibit the neighbor topology structure of $\{v_{s_1}, v_{s_2}\}$ to $\{v_p, v_q\}$ and $\{v_{t_1}, v_{t_2}\}$. That of $\{v_{t_1}, v_{t_2}\}$ to $\{v_p, v_q\}$ and $\{v_{s_1}, v_{s_2}\}$ can be illustrated in the same manner. For graph (a), $\sigma = 2$ since the number of neighbors of each $v_{s_k} (k = 1, 2)$ in $\mathcal{V}\backslash\{v_p, v_q, v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}\}$ is 2. The neighbor topology structures of v_7, v_8, and v_9 are designed in accordance with step 4. For $k = p, q$, exactly one of $v_{s_1} = v_2, v_{s_2} = v_4$ (v_2 here) and one of $v_{t_1} = v_5, v_{t_2} = v_6$ (v_5 here) are included in the neighbor set of v_k. This is consistent with step 3. It can be verified that $\eta = [0, -0.5, 0, -0.5, 0.5, 0.5, 0, 0, 0]^{\mathrm{T}}$ is an eigenvector of $\mathcal{L}$ of graph (a) associated with eigenvalue $\sigma + 4 = 6$. For graph (b), $\sigma = 1$ and $\eta = [0, 0.5, 0, 0.5, -0.5, -0.5, 0, 0, 0]^{\mathrm{T}}$ is an eigenvector of $\mathcal{L}$ of graph (b) associated with eigenvalue $\sigma + 4 = 5$. For graph (c), $\sigma = 1$ as well, and $\eta = [0, -0.5, 0, -0.5, 0.5, 0.5, 0, 0, 0]^{\mathrm{T}}$ is an eigenvector of $\mathcal{L}$ associated with eigenvalue 5. Hence for graphs (a)–(c), the system is not controllable whenever leaders are selected from $\mathcal{V}\backslash\{v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}\}$. For graphs (d)–(f), there is a similar explanation.

IV. Complete Graphical Characterization

In this section, we show that the complete graphical characterization consists of 48 controllability destructive topology structures which are identified from a total of 1050 different topologies of five nodes.

A. QCD Nodes

Definition 2: For a graph consisting of five vertices v_1, v_2, v_3, v_4, and v_5, any four of them, say v_2, v_3, v_4, and v_5 are said to be QCD nodes if they conform to any of the topologies depicted in Figs. 3 and 4.

Lemma 3: For a graph $\mathcal{G}$ consisting of five vertices, $\bar{y} = [0, y_2, y_3, y_4, y_5]$ with $y_2, y_3, y_4, y_5 \neq 0$ is an eigenvector of $\mathcal{L}$ if and only if v_2, v_3, v_4, and v_5 are QCD nodes of $\mathcal{G}$.

The lemma is used to develop Theorem 4. To prove Lemma 3, we need to establish subsequent Proposition 2 and Lemma 4 first.

Theorem 4: For a communication graph consisting of five vertices, there is a single leader, denoted by v_1, such that the multiagent system with single-integrator dynamics (1) is

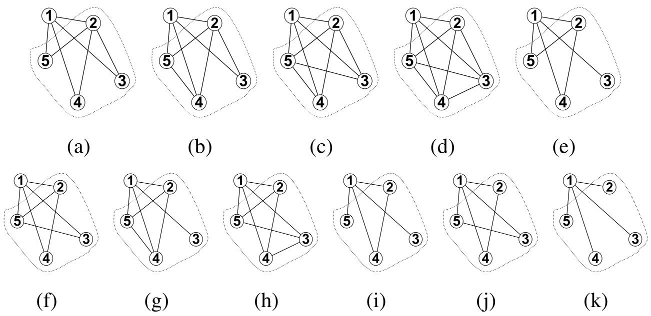

Fig. 3. Topologies of QCD notes of graphs consisting of five nodes.

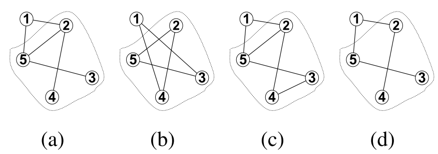

Fig. 4. Topologies of QCD notes of graphs consisting of five nodes.

controllable if and only if the following three conditions are met simultaneously.

1) $\mathcal{V}\backslash\{v_1\} = \{v_2, v_3, v_4, v_5\}$ do not constitute a group of QCD nodes.
2) Any three of v_2, v_3, v_4, and v_5 are not TCD nodes.
3) Any two of v_2, v_3, v_4, and v_5 are not DCD nodes.

Proof: Based on Lemma 3, the result can be proved in the same vein as Theorem 2. ■

Remark 7: The difficulty lies in the identification of all topology structures of QCD nodes depicted in Figs. 3 and 4. We tackle this problem via exploring the graphical implications of eigenvector $\bar{y}$, which also constitutes the necessity proof of Lemma 3. To this end, we need to build the following Proposition 2 and Lemma 4 first.

Consider an eigenvector $\bar{y}$ of $\mathcal{L}$ with $\bar{y} = [0, \ldots, y_{s_1}, \ldots, y_{s_2}, \ldots, y_{t_1}, \ldots, y_{t_2}, \ldots, 0]^{\mathrm{T}}$, $y_{s_1}, y_{s_2}, y_{t_1}, y_{t_2} \neq 0$ and all the other elements being zero. $\bar{y}$ does not necessarily meet (6) and each entry of it ought to satisfy the eigen-condition. For each $k \neq s_1, s_2, t_1, t_2$; $\mathcal{N}_{kf}$ has five cases.

1) $s_1, s_2, t_1, t_2 \in \mathcal{N}_{kf}$.
2) Any three and only three of s_1, s_2, t_1, t_2 belong to $\mathcal{N}_{kf}$.
3) Any two and only two of s_1, s_2, t_1, t_2 belong to $\mathcal{N}_{kf}$.
4) Any one and only one of s_1, s_2, t_1, t_2 belongs to $\mathcal{N}_{kf}$.
5) None of s_1, s_2, t_1, t_2 belongs to $\mathcal{N}_{kf}$.

The cases 1)–5) are involved in the derivation of the following proposition.

Proposition 2: Suppose leaders are selected from $\mathcal{V}\backslash\{v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}\}$ and $\bar{y}$ is an eigenvector of $\mathcal{L}$, then:

1) for any given $k \neq s_1, s_2, t_1, t_2$; $\mathcal{N}_{kf}$ falls into one and only one of the following two situations.
   a) At least one of cases 1), 3), and 5) occurs.
   b) At least one of cases 2), 3), and 5) occurs.

   Moreover, if 2) arises, there are at most three different $k \neq s_1, s_2, t_1, t_2$ with each $\mathcal{N}_{kf}$ containing a different set of three indices of $\{s_1, s_2, t_1, t_2\}$; and so is to 3) with each set containing two indices of $\{s_1, s_2, t_1, t_2\}$;

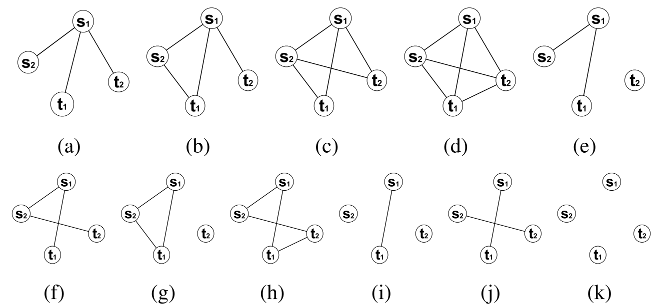

Fig. 5. All topology structures consisting of s_1, s_2, t_1, and t_2.

2) for $k = s_1, s_2, t_1, t_2$; all possible topologies consisting of $v_{s_1}, v_{s_2}, v_{t_1}$, and v_{t_2} are depicted in Fig. 5.

Proof: Consider $k \neq s_1, s_2, t_1, t_2$ and $k = s_1, s_2, t_1, t_2$. In case $k \neq s_1, s_2, t_1, t_2$, $\sum_{i\in\mathcal{N}_{kf}} y_i = 0$ which can be shown in the same way as (24). If circumstance 1) arises, the same arguments as (25) yield

$$y_{s_1} + y_{s_2} + y_{t_1} + y_{t_2} = 0. \tag{10}$$

If circumstance 2) arises and $s_1, s_2, t_1 \in \mathcal{N}_{kf}$, it follows from $\sum_{i\in\mathcal{N}_{kf}} y_i = 0$ that:

$$y_{s_1} + y_{s_2} + y_{t_1} = 0. \tag{11}$$

Situations (10) and (11) cannot occur simultaneously, or else, $y_{t_2} = 0$. Similarly, if another $\mathcal{N}_{kf}(k \neq s_1, s_2, t_1, t_2)$ contains, say s_2, t_1, and t_2, one has

$$y_{s_2} + y_{t_1} + y_{t_2} = 0. \tag{12}$$

Equations (11) and (12) lead to $y_{s_2} + y_{t_1} = -y_{s_1} = -y_{t_2}$. If there is the third $k \neq s_1, s_2, t_1, t_2$ with its $\mathcal{N}_{kf}$ containing, say s_1, s_2, t_2, one has $y_{s_1} + y_{s_2} + y_{t_2} = 0$. Combining this equation with (11) yields $y_{s_1} + y_{s_2} = -y_{t_1} = -y_{t_2}$. If there is the fourth $k \neq s_1, s_2, t_1, t_2$ with $s_1, t_1, t_2 \in \mathcal{N}_{kf}$, then $y_{s_1} + y_{t_1} + y_{t_2} = 0$. This together with (12) yields $y_{s_1} = y_{s_2}$. Thus, if the above four situations arise at the same time, then $y_{s_1} = y_{s_2} = y_{t_1} = y_{t_2} = 0$, which contradicts to the assumption. Therefore, at most three of the above four situations occur.

If circumstance 3) arises, there are totally $C_4^2 = 6$ situations, i.e., $s_1, s_2 \in \mathcal{N}_{kf}; s_1, t_1 \in \mathcal{N}_{kf}; s_1, t_2 \in \mathcal{N}_{kf}$; $s_2, t_1 \in \mathcal{N}_{kf}; s_2, t_2 \in \mathcal{N}_{kf}; t_1, t_2 \in \mathcal{N}_{kf}$. The same discussion as circumstance 2) shows that the eigen-condition allows at most three of the above situations occur. The circumstance 4) cannot occur. This follows from the same discussion as 3) of the case 1 of TCD nodes. For circumstance 5), the special form of $\bar{y}$ means that the condition $\sum_{i\in\mathcal{N}_{kf}} y_i = 0$ is always satisfied. Thus for any given $k \neq s_1, s_2, t_1, t_2$, $\mathcal{N}_{kf}$ conforms to one and only one of the above two cases 1) and 2).

In case $k = s_1, s_2, t_1, t_2$, all possible topologies consisting of s_1, s_2, t_1, and t_2 are generated by following the same discussion as case II in the proof of Lemma 2, which are depicted in Fig. 5. ■

Remark 8: Proposition 2 greatly reduces the number of graphs required in the identification of QCD nodes.

Definition 3: A graph is said to be designed from Fig. 5(a) if the topology structure of $v_{s_1}, v_{s_2}, v_{t_1}$, and v_{t_2} accords with Fig. 5(a) and the graph is obtained by adding edges between $\{v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}\}$ and $\mathcal{V}\backslash\{v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}\}$. The definition applies to other topologies of Fig. 5.

Lemma 4: Suppose $\bar{y}$ is an eigenvector of a graph designed from Fig. 5(a). The following assertions hold.

1) If the situation 1) of Proposition 2 arises, then

$$\frac{1}{d_{t_2} - d_{s_1} - 1} + \frac{1}{d_{t_1} - d_{s_1} - 1} + \frac{1}{d_{s_2} - d_{s_1} - 1} = -1. \tag{13}$$

2) If situation 2) arises with a $v_k \in \mathcal{V}\backslash\{v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}\}$ incident to only three of $v_{s_1}, v_{s_2}, v_{t_1}$, and v_{t_2}, say v_{s_1}, v_{s_2}, and v_{t_1}, then one of the following four equations must occur:

$$\lambda_1 = \tilde{\lambda}_1, \lambda_1 = \tilde{\lambda}_2, \lambda_2 = \tilde{\lambda}_1, \lambda_2 = \tilde{\lambda}_2 \tag{14}$$

where

$$\lambda_{1,2} = \frac{d_{t_1} + d_{s_2} + 2 \pm \sqrt{\left(d_{s_2} - d_{t_1}\right)^2 + 4}}{2} \tag{15}$$

$$\tilde{\lambda}_{1,2} = \frac{d_{s_1} + d_{t_2} + 1 \pm \sqrt{\left[\left(d_{s_1} - d_{t_2}\right) + 1\right]^2 + 4}}{2}. \tag{16}$$

3) If 3) arises with a $v_k \in \mathcal{V}\backslash\{v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}\}$ incident to only two of $v_{s_1}, v_{s_2}, v_{t_1}$, and v_{t_2}, say, v_{s_1} and v_{s_2}, then

$$d_{s_1} - d_{s_2} = \frac{1}{d_{t_1} - d_{s_2} - 1} + \frac{1}{d_{t_2} - d_{s_2} - 1}. \tag{17}$$

Proof: Suppose any of situations 1)–3) of Proposition 2 arises and the graph is designed from topology of Fig. 5(a). The eigen-condition is to be computed for $v_{s_1}, v_{s_2}, v_{t_1}$, and v_{t_2}, respectively. First, for node v_{t_2}, since $y_k = 0$ for any $k \neq s_1, s_2, t_1, t_2$, it follows that $\sum_{i\in\mathcal{N}_{t_2 l}} y_i = 0, \sum_{i\in\mathcal{N}_{t_2 f}} y_i = y_{s_1}$. Accordingly $d_{t_2} y_{t_2} - \sum_{i\in\mathcal{N}_{t_2}} y_i = d_{t_2} y_{t_2} - y_{s_1}$. So the eigen-condition requires

$$\left(d_{t_2} - \lambda\right) y_{t_2} = y_{s_1}. \tag{18}$$

Similarly, the eigen-conditions of v_{t_1} and v_{s_2} require that

$$\left(d_{t_1} - \lambda\right) y_{t_1} = y_{s_1} \text{ and } \left(d_{s_2} - \lambda\right) y_{s_2} = y_{s_1}. \tag{19}$$

For v_{s_1}, since $\sum_{i\in\mathcal{N}_{s_1 l}} y_i = 0, \sum_{i\in\mathcal{N}_{s_1 f}} y_i = y_{s_2} + y_{t_1} + y_{t_2}$, one has $d_{s_1} y_{s_1} - \sum_{i\in\mathcal{N}_{s_1}} y_i = d_{s_1} y_{s_1} - (y_{s_2} + y_{t_1} + y_{t_2})$. Then the eigen-condition associated with v_{s_1} requires

$$\left(d_{s_1} - \lambda\right) y_{s_1} = y_{s_2} + y_{t_1} + y_{t_2}. \tag{20}$$

Since $y_{s_1} \neq 0$ and $\bar{y}$ is an eigenvector, it can be assumed that $y_{s_1} = 1$. Consider the following circumstances.

1) Situation 1) of Proposition 2 arises with a $v_k \in \mathcal{V}\backslash\{v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}\}$ incident to all $v_{s_1}, v_{s_2}, v_{t_1}$, and v_{t_2}. In this situation, (10) holds. By (20), $(d_{s_1} - \lambda + 1) y_{s_1} = 0$. Since $y_{s_1} \neq 0$, $\lambda = d_{s_1} + 1$. Substituting λ, (18), and (19) into (10) yields (13). Thus, if $\bar{y}$ is an eigenvector, condition (13) ought to be satisfied.
2) Situation 2) arises with a $v_k \in \mathcal{V}\backslash\{v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}\}$ incident to only three of $v_{s_1}, v_{s_2}, v_{t_1}$, and v_{t_2}, say v_{s_1}, v_{s_2}, and v_{t_1}. In this situation, (11) holds. Substituting (11) into (19) yields $(d_{t_1} - \lambda + 1) y_{t_1} = -y_{s_2}$

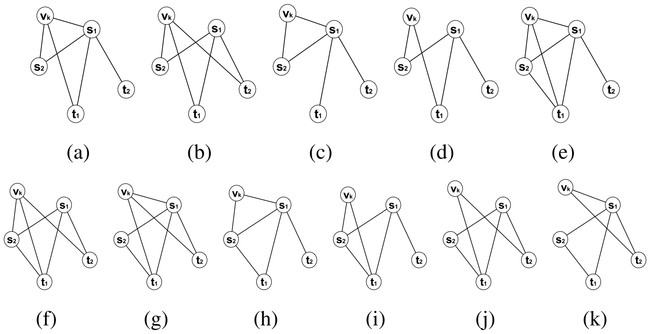

Fig. 6. Graphs abiding by situations 2) or 3) of Proposition 2, where (a)–(d) and (e)–(k) are designed, respectively, from the topology structures of Fig. 5(a) and (b).

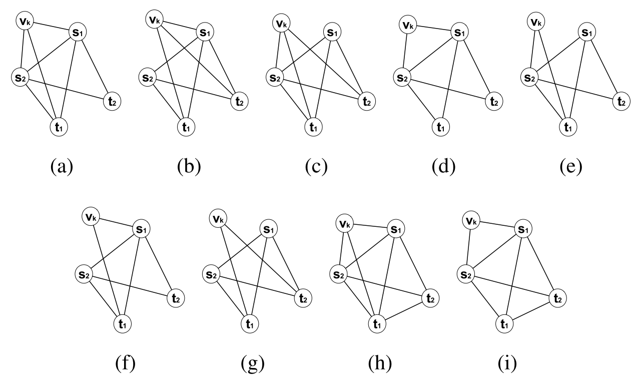

Fig. 7. Graphs abiding by situations 2) or 3) of Proposition 2, where (a)–(g), and (h) and (i) are designed, respectively, from the topology structures Fig. 5(c) and (d).

and $(d_{s_2} - \lambda + 1)y_{s_2} = -y_{t_1}$. Thus $(d_{s_2}-\lambda+1)(d_{t_1}-\lambda+1)y_{t_1} = y_{t_1}$. Since $y_{t_1} \neq 0$, $(d_{s_2}-\lambda+1)(d_{t_1}-\lambda+1) = 1$ whose roots are (15). On the other hand, combining (20) with (11) yields $y_{t_2} = d_{s_1} - \lambda + 1$. By (18), $y_{t_2} = (1/(d_{t_2} - \lambda))$. Thus $d_{s_1} - \lambda + 1 = (1/(d_{t_2} - \lambda))$, that is

$$\lambda^2 - \left(d_{s_1} + d_{t_2} + 1\right)\lambda + d_{t_2}d_{s_1} + d_{t_2} - 1 = 0. \quad (21)$$

The two roots of (21) are (16). Because the eigen-condition of each node holds for the same eigenvalue λ, it follows from (15) and (16) that one of the four cases of (14) must occur.

3) Situation c) arises with a $v_k \in \mathcal{V}\backslash\{v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}\}$ incident to only two of $v_{s_1}, v_{s_2}, v_{t_1}, v_{t_2}$, say v_{s_1}, v_{s_2}. Similar arguments as (11) yields $y_{s_1} + y_{s_2} = 0$. Substituting this with $y_{s_1} = 1$ into (18)–(20) results in $\lambda = d_{s_2} + 1$ and accordingly (17) should be met. ■

Remark 9: Lemma 4 serves to check whether $\bar{y}$ is an eigenvector of a graph designed from Fig. 5(a) and accordingly contributes to the discrimination of topologies of QCD nodes. Graphs designed from other topologies of Fig. 5 can be analyzed in the same manner.

By Proposition 2, the following candidate graphs are designed to discriminate topologies of QCD nodes. First, graphs abiding by situation 1) of Proposition 2 are depicted in Fig. 3, where Fig. 3(a)–(k) are designed, respectively, from the topology structures of Fig. 5(a)–(k) with $v_1 = v_k, v_2 = v_{s_1}$,

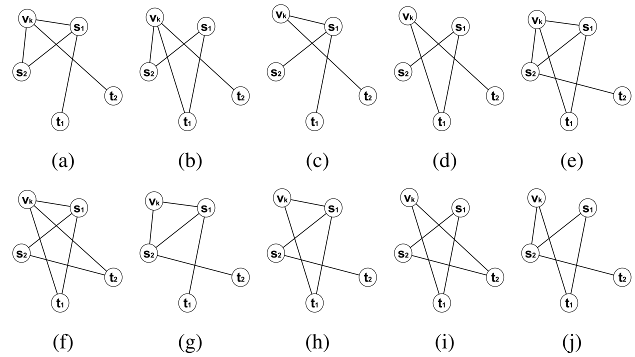

Fig. 8. Graphs abiding by situations 2) or 3) of Proposition 2, where (a)–(d) and (e)–(j) are designed, respectively, from the topology structures of Fig. 5(e) and (f).

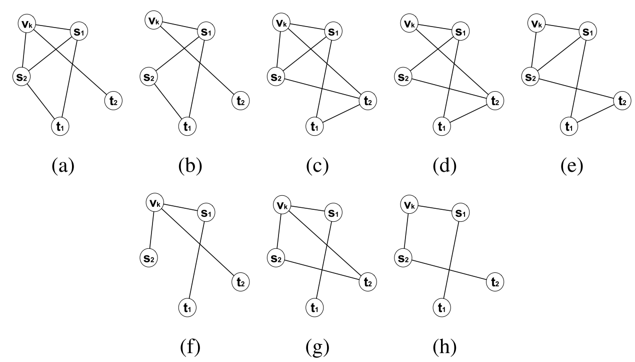

Fig. 9. Graphs abiding by situations 2) or 3) of Proposition 2, where (a) and (b), (c) and (d), (e), and (f) and (g) are designed, respectively, from the topology structures of Fig. 5(g)–(j).

$v_3 = v_{t_2}, v_4 = v_{t_1}$, and $v_5 = v_{s_2}$. The others are shown in subsequent Figs. 6–9.

Relabel $v_k = v_1, v_{s_1} = v_2, v_{t_2} = v_3, v_{t_1} = v_4$, and $v_{s_2} = v_5$.

Proof of Lemma 3: Now we can prove Lemma 3.

1) *Necessity:* Let $\bar{y}$ be an eigenvector of $\mathcal{L}$. Since $\mathcal{V}\backslash\{v_2, v_3, v_4, v_5\}$ contains only one element v_1 for a graph of five vertices, situation 5) of Proposition 2 cannot occur (or else, v_1 will be isolated), and any two of 1)–3) do not arise simultaneously. Thus all connected graphs complying with 1) or 2) of Proposition 2 can be generated by just following one and only one of 1)–3), and accordingly, by Proposition 2, constitute all the possible graphs of five nodes with $\bar{y}$ being an eigenvector. All these graphs are shown in Figs. 3 and 6–9. First, consider graphs designed from Fig. 5(a). Calculations show that the necessary condition (13) of Lemma 4 is met by graph of Fig. 3(a), and condition (14) is not met by Fig. 6(a) and (b), nor is condition (17) met by Fig. 6(c) and (d). Thus Fig. 6(a)–(d) are excluded from the graphs with $\bar{y}$ being an eigenvector. For graphs designed from the other topologies of Fig. 5, similar arguments yield that only Figs. 8(g) and (i) and 9(e) and (h) satisfy the associated necessary conditions of $\bar{y}$ being an eigenvector. Thus if $\bar{y}$ is an eigenvector, v_2, v_3, v_4, and v_5 are QCD nodes.

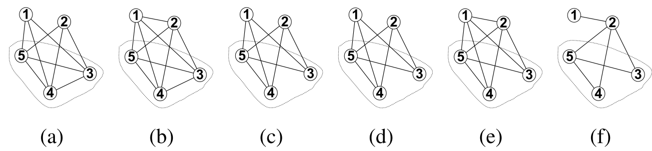

Fig. 10. Topology structures of TCD nodes v_3, v_4, and v_5 with v_1 and v_2 playing leaders' role.

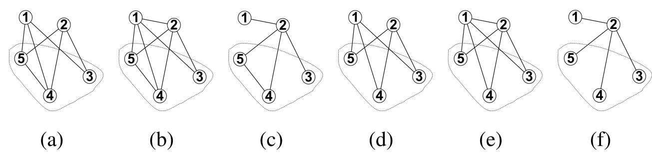

Fig. 11. Topology structures of TCD nodes v_3, v_4, and v_5 with v_1 and v_2 playing leaders' role.

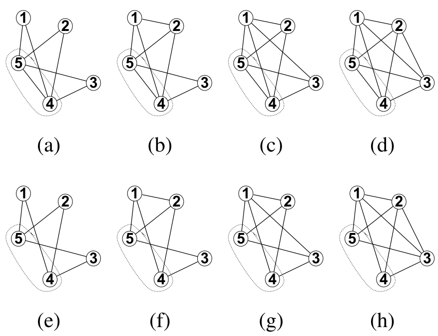

Fig. 12. Topology structures of DCD nodes v_4 and v_5 with v_1, v_2, and v_3 playing leaders' role.

2) *Sufficiency:* For graph Fig. 3(a) with QCD nodes v_2, v_3, v_4, and v_5; $d_1 = d_2 = 4$, $d_3 = d_4 = d_5 = 2$. For v_1, the eigencondition requires $4y_1 - (y_2 + y_3 + y_4 + y_5) = \lambda y_1$. Set $y_1 = 0$, then $y_2 + y_3 + y_4 + y_5 = 0$. For v_5 and v_4, the eigencondition, respectively, yields $2y_5 - y_2 = \lambda y_5$ and $2y_4 - y_2 = \lambda y_4$. Thus $(2 - \lambda)(y_4 - y_5) = 0$. Similarly, for v_3, $(2 - \lambda)(y_3 - y_4) = 0$ and for v_2, $4y_2 - (y_3 + y_4 + y_5) = \lambda y_2$. Take $y_3 = y_4 = y_5$, the above arguments show that $y_2 = -3y_3$. Hence $\bar{y} = [0, -3, 1, 1, 1]$ is an eigenvector of graph of Fig. 5(a) with the corresponding eigenvalue $\lambda = 5$. It can be verified in the same way for the other graphs with QCD nodes that $\mathcal{L}$ has an eigenvector $\bar{y}$. ■

Remark 10: For a graph consisting of five vertices, Theorems 1, 2, and 4 conspire to answer the following question: with all different selections of leaders, what is the graph topology-based necessary and sufficient condition for controllability? Theorems 4, 2, and 1 answer this question with respect to, respectively, the case of single, double, and triple leaders. Thus, the three theorems together provide a complete graphical characterization for controllability, as shown in subsequent analysis.

B. TCD and DCD Nodes

By definition of TCD nodes, for graphs consisting of five nodes, all topology structures of TCD nodes v_3, v_4, and v_5 are listed below in Figs. 10 and 11. They are uncontrollable when any of v_1 and v_2 are selected as leaders.

All topology structures of DCD nodes v_4 and v_5 are listed below in Figs. 12–14. They are uncontrollable when any of v_1, v_2, and v_3 are leaders.

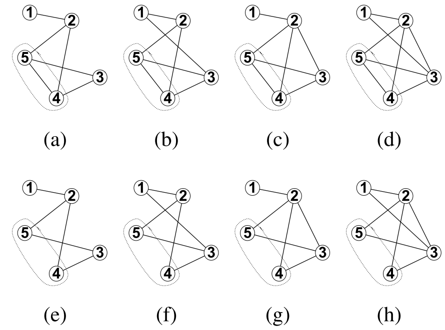

Fig. 13. Topology structures of DCD nodes v_4 and v_5 with v_1, v_2, and v_3 playing leaders' role.

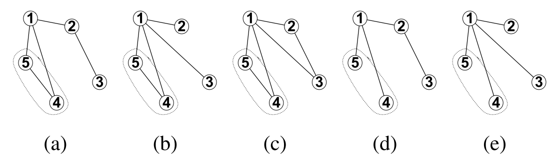

Fig. 14. Topology structures of DCD nodes v_4 and v_5 with v_1, v_2, and v_3 playing leaders' role.

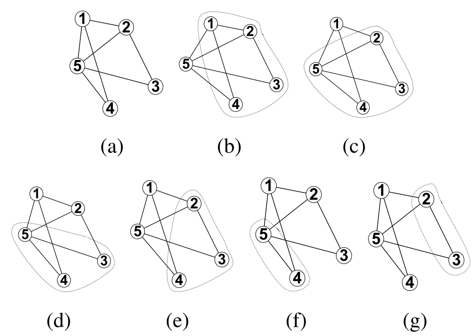

Fig. 15. v_5 and v_1 are, respectively, the single leader of the (b) uncontrollable graph and (c) controllable graph. v_1 and v_2, and v_1 and v_5 are, respectively, the leaders of (d) and (e) controllable graphs. v_1, v_2, and v_3, and v_1, v_4 and v_5 are, respectively, the leaders of (f) and (g) controllable graphs.

C. Concrete Analysis

Armed with the above topology structures of DCD, TCD, and QCD nodes, controllability with any selection of leaders can be determined directly from them. Take an arbitrarily given graph, say, the one depicted in Fig. 15(a), as an example, let us consider its controllability under any selection of leaders. In case v_5 is selected as the single leader [see Fig. 15(b)], the topology structure of v_1, v_2, v_3, and v_4 coincides with that of v_2, v_3, v_4, and v_5 in Fig. 3(f). Hence, v_1, v_2, v_3, and v_4 constitute a set of QCD nodes in Fig. 15(a). Accordingly, Fig. 15(a) is uncontrollable with v_5 playing the single leader. Instead, Fig. 15(c) is controllable since v_2, v_3, v_4, and v_5 do not constitute any set of QCD nodes. Similar analysis can be applied to graphs depicted in Fig. 15(d)–(g). It turns out that all of them are controllable with the corresponding selection of leaders.

Remark 11: Careful and patient calculation shows that there are 35 different graphs consisting of five nodes, which are not listed here because of space limitation. For each graph, there are $C_5^1 + C_5^2 + C_5^3 + C_5^4 = 30$ possible ways of leaders' selection. Therefore, the controllability of a total of $35 * 30 = 1050$ different topology structures of five nodes is completely determined by the 48 topology structures of DCD, TCD, and QCD nodes.

V. Conclusion

The increasingly widespread use of networks calls for reasonable design and organization of network topologies. For controllability of multiagent networks, the problem was tackled in this paper by identifying topology structures formed by the proposed controllability destructive nodes. The identified structures revealed the only topologies that destroy controllability, which results in several necessary and sufficient graphical conditions on controllability for any graph, as well as a complete graph-based controllability characterization for graphs of five nodes. In addition, a topology construction procedure was proposed for QCD nodes in graphs of any size. Although the complete graphical characterization on controllability was presented for graphs of only five nodes, it outlined a novel idea applicable to larger graphs. Moreover, the results indicate that the underlying difficulty in graphical characterization of multiagent controllability lies in the identification of topology structures of controllability destructive nodes.

Future works will deal with more flexible and general approaches to determine the sets of destructive nodes. For any given number of nodes, constructive procedure can be developed to find out the controllability destructive topology structures for a graph of any size. This is verified by the steps 1–4 proposed in Theorem 3 and more topologies may be constructed by the following similar logic rules. The most difficult point concerning this issue is how to design more topologies to cover all possible controllability destructive topology structures. Moreover, the specific circumstances associated with this problem may be different for different graphs even for the destructive topology consisting of the same number of nodes. These may be the main difficulties encountered in developing the general approach. Other research directions include extending the results to directed and weighted graphs, as well as investigating the stabilizability and consensus problems by utilizing the concept of destructive nodes.

Appendix

Proof of Lemma 2: Since v_p, v_q, and v_r are follower nodes, the special form of $\bar{y}$ leads to $\sum_{i\in\mathcal{N}_{kl}} y_i = 0$ for any k.

Necessity: $\mathcal{L}\bar{y} = \lambda\bar{y}$ means

$$d_k y_k - \sum_{i\in\mathcal{N}_k} y_i = \lambda y_k, \quad k = 1, \ldots, n+l. \tag{22}$$

Case 1 ($k \neq p, q, r$): In this case, $y_k = 0$. Then

$$d_k y_k - \sum_{i\in\mathcal{N}_k} y_i = -\sum_{i\in\mathcal{N}_{kf}} y_i. \tag{23}$$

Combining (22) with (23) yields

$$\sum_{i\in\mathcal{N}_{kf}} y_i = 0. \tag{24}$$

Each $\mathcal{N}_{kf} (k \neq p, q, r)$ falls into one of the four cases.

1) $p, q, r \in \mathcal{N}_{kf}$. Since $\sum_{i\in\mathcal{N}_{kf}} y_i = y_p + y_q + y_r$, by (24)

$$y_p + y_q + y_r = 0. \tag{25}$$

2) Any two and only two of p, q, and r belong to $\mathcal{N}_{kf}$. Suppose $p, q \in \mathcal{N}_{kf}$, then $\sum_{i\in\mathcal{N}_{kf}} y_i = y_p + y_q$. By (24)

$$y_p + y_q = 0. \tag{26}$$

Equations (25) and (26) cannot be met simultaneously, or else, $y_r = 0$. This contradicts with $y_r \neq 0$. If there is another $k \neq p, q, r$ with $\mathcal{N}_{kf}$ containing p, r, by (24)

$$y_p + y_r = 0. \tag{27}$$

From (26) and (27), $y_p = -y_q = -y_r$. If (26) and (27) are met simultaneously, there does not exist the third $k \neq p, q, r$ with $\mathcal{N}_{kf}$ containing q, r. Otherwise

$$y_q + y_r = 0. \tag{28}$$

This, however, is impossible because $y_q + y_r = 0$ and $y_p = -y_q = -y_r$ lead to $y_q = y_r = 0$, which is incompatible with $y_k \neq 0, k = p, q, r$. Hence, at most two of (26)–(28) take place.

3) Any one and only one of p, q, and r belongs to $\mathcal{N}_{kf}$, say $p \in \mathcal{N}_{kf}$. In this case, $\sum_{i\in\mathcal{N}_{kf}} y_i = y_p$. To satisfy (24), it requires $y_p = 0$. This is impossible because $y_p \neq 0$.

4) None of p, q, and r belongs to $\mathcal{N}_{kf}$. In this case, the special form of $\bar{y}$ implies $\sum_{i\in\mathcal{N}_{kf}} y_i = 0$, i.e., (24) is met.

Since (25) and (26) cannot be met simultaneously, 1) and 2) cannot occur at once. That is, there do not exist different v_{k_1} and v_{k_2} in $\mathcal{G}$ with v_{k_1} and v_{k_2} consistent with cases 1) and 2), respectively. Thus, with given $k \neq p, q, r$; $\mathcal{N}_{kf}$ conforms to one and only one of the following cases: 1) at least one of 1) and 4) occurs and 2) at least one of 2) and 4) occurs.

Case 2 ($k = p, q, r$): Since $\sum_{i\in\mathcal{N}_{kl}} y_i = 0$, by (22)

$$(d_k - \lambda)y_k = \sum_{i\in\mathcal{N}_{kf}} y_i. \tag{29}$$

1) There is at least one $k \in \{p, q, r\}$ with $\mathcal{N}_{kf}$ containing the other two indices of p, q, and r.
   1a) Only one $k \in \{p, q, r\}$ is of this kind.
   1b) There are two $k's \in \{p, q, r\}$ of this kind. Fig. 16(a) and (b) corresponds to 1a) and 1b), respectively.
   1c) Each $k \in \{p, q, r\}$ is of this kind. Note that 1b) and 1c) are equivalent.
2) There is at least one $k \in \{p, q, r\}$ with $\mathcal{N}_{kf}$ containing one and only one of the other two indices of $\{p, q, r\}$.
   2a) Only one $k \in \{p, q, r\}$(say $k = p$) is of this kind and its single neighbor node in $\{p, q, r\}$, say q, also has k as its single neighbor node in $\{p, q, r\}$.
   2b) There are two $k's \in \{p, q, r\}$ of this kind. 1a) coincides with 2b). That each $k \in \{p, q, r\}$ is of this kind does not occur.

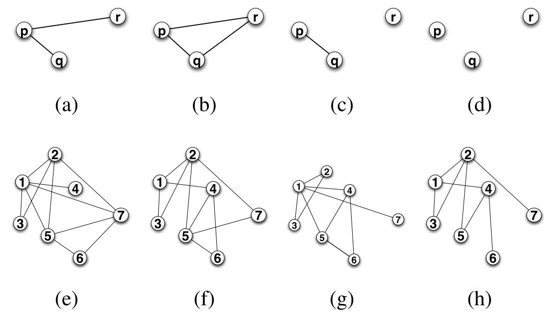

Fig. 16. (a)–(d) Correspond to 1a) with $k = p$; 1b) [or 1c)]; 2a) with $k = p$ (or q) and 3b). (e)–(h) Topologies V–VIII with v_5, v_6, and v_7 the destructive nodes.

3) For each $k = p, q, r$; $\mathcal{N}_{kf}$ contains none of the other two indices of p, q, r.
   3a) Only one $k \in \{p, q, r\}$ is of this kind, which coincides with 2a).
   3b) There are two $k's \in \{p, q, r\}$ of this kind [see Fig. 16(d)].

Item 1) of case 1, together with 1b), 1a), 2a), and 3b) of case 2, respectively, results in topologies I–IV (see Fig. 1). If the "item 1) of case 1" is replaced by "item 2) of case 1," then topologies V–VIII are generated [see Fig. 16(e)–(h)]. So, if $\bar{y}$ is an eigenvector of $\mathcal{L}$, then v_p, v_q, and v_r have maximum of eight possible topologies. Moreover, it will be shown that topologies V–VIII are redundant.

Fact 1: If $\bar{y}$ is an eigenvector of $\mathcal{L}$, then v_p, v_q, and v_r cannot have topology structures V–VIII.

Case 1 ($k \neq p, q, r$): It is to be proved by contradiction first for V. In this case, (23) holds. Since the graph is connected, one of v_p, v_q, and v_r, say v_q in subsequent arguments, must have a neighbor in $\mathcal{V}\backslash\{v_p, v_q, v_r\}$. By the topology structure of V, there is a node of v_p, v_q, and v_r, say v_p with v_p and v_q sharing at least one common neighbor node in $\mathcal{V}\backslash\{v_p, v_q, v_r\}$. Suppose this node is v_k, then $p, q \in \mathcal{N}_{kf}$. Since 1) of case 1 does not arise, $\sum_{i\in\mathcal{N}_{kf}} y_i = y_p + y_q$. Then by (22) and (23), (26) holds. Now there are two situations for v_p and v_r. One is that there is another $k \neq p, q, r$ with v_k incident to both v_p and v_r; the other is that none of $v_k (k \neq p, q, r)$ is incident to both v_p and v_r. For the first situation, similar arguments to (26) yield that the eigen-condition requires (27) to be true. $\{v_p, v_q\}$ and $\{v_p, v_r\}$ cannot be incident to the same $v_k (k \neq p, q, r)$ because a) of case 1 does not arise in topology V. For $k \neq p, q, r$, with $\mathcal{N}_{kf}$ containing none of p, q, and r; $\sum_{i\in\mathcal{N}_{kf}} y_i = 0$. It follows from $y_k = 0 (k \neq p, q, r)$ and (23) that for these $k's$ the eigen-condition (22) is met.

Case 2 ($k = p, q, r$): Let us first consider the first situation of v_p, v_r. Since $\sum_{i\in\mathcal{N}_{kl}} y_i = 0$, one has

$$d_k y_k - \sum_{i\in\mathcal{N}_k} y_i = d_k y_k - \sum_{i\in\mathcal{N}_{kf}} y_i. \tag{30}$$

In topology V, each $\mathcal{N}_{kf} (k = p, q, r)$ contains two indices of p, q, and r, which are different from k. Thus, for a $k \in \{p, q, r\}$, say $k = p$, $\sum_{i\in\mathcal{N}_{kf}} y_i = y_q + y_r$. By (26) and (27), $y_p = -y_q = -y_r$. So $\sum_{i\in\mathcal{N}_{kf}} y_i = -2y_p$. By (30)

$$d_p y_p - \sum_{i\in\mathcal{N}_p} y_i = (d_p + 2) y_p. \tag{31}$$

Thus, for $k = p$, the eigen-condition is met for $\lambda = d_p + 2$. For $k = q$, $\sum_{i\in\mathcal{N}_{qf}} y_i = y_p + y_r = 0$. From (30)

$$d_q y_q - \sum_{i\in\mathcal{N}_q} y_i = d_q y_q. \tag{32}$$

Similarly, for $k = r$, $\sum_{i\in\mathcal{N}_{rf}} y_i = y_p + y_q = 0$. Thus

$$d_r y_r - \sum_{i\in\mathcal{N}_r} y_i = d_r y_r. \tag{33}$$

To satisfy (31)–(33) simultaneously, it requires $d_p + 2 = d_q = d_r$. Below shows that this is not possible. If there is a node v_h in $\mathcal{V}\backslash\{v_p, v_q, v_r\}$ which is incident to both v_q and v_r, then (28) should also be met. However, the arguments of 2) of case 1 show that (26)–(28) cannot be satisfied simultaneously. Hence this cannot be happening. In this situation, to satisfy $d_q = d_r$, the number of v_k in $\mathcal{V}\backslash\{v_p, v_q, v_r\}$ which is incident to both v_p and v_q is required to be equal to the number of v_h in $\mathcal{V}\backslash\{v_p, v_q, v_r\}$ which is incident to both v_p and v_r, where $k \neq h$. As a consequence, $d_p \geq d_q$. Accordingly $d_p + 2 > d_q$. Hence (31)–(33) cannot be met at the same time, and accordingly $\bar{y}$ is not an eigenvector of Laplacian. This contradicts with the assumption.

Next, for the second situation of $\{v_p, v_r\}$, i.e., none of $v_k (k \neq p, q, r)$ is incident to both v_p and v_r, (26) still holds. In this situation, we further distinguish between two circumstances: one is that there is a $v_k \in \mathcal{V}\backslash\{v_p, v_q, v_r\}$ which is incident to both v_q and v_r, the other is the reversal. For the first circumstance, relabelling v_p as v_q and vice-versa, the proof is the same as that of the aforementioned first situation of $\{v_p, v_r\}$. For the second circumstance, it can be seen that $d_p = d_q$. By (26) and (30), $d_r y_r - \sum_{i\in\mathcal{N}_r} y_i = d_r y_r - (y_p + y_q) = d_r y_r$. Hence, to satisfy the eigen-condition, it requires $\lambda = d_r$. Consider the eigen-condition of v_p. By (30), $d_p y_p - \sum_{i\in\mathcal{N}_p} y_i = d_p y_p - (y_q + y_r) = (d_p + 1) y_p - y_r$. To satisfy the eigen-condition, it requires

$$(d_p + 1) y_p - y_r = \lambda y_p. \tag{34}$$

With $\lambda = d_r$, the above equation means $y_r = (d_p + 1 - d_r) y_p$. Thus, for node v_q, $\sum_{i\in\mathcal{N}_{qf}} y_i = y_p + y_r = (d_p + 2 - d_r) y_p$. By (26) and (30), $d_q y_q - \sum_{i\in\mathcal{N}_q} y_i = d_q y_q + (d_p + 2 - d_r) y_q = (2d_q + 2 - d_r) y_q$. Hence, to satisfy the eigen-condition, it requires $2d_q + 2 - d_r = \lambda = d_r$, i.e., $d_q + 1 = d_r$. However, it will be shown $d_q > d_r$. Since none of $v_k (k \neq p, q, r)$ is incident to both v_p and v_r and 1) and 2) of case 1 cannot arise simultaneously, then a node v_h in $\mathcal{V}\backslash\{v_p, v_q, v_r\}$ which is incident to v_r is also incident to v_q. In addition, there is already at least one v_k in $\mathcal{V}\backslash\{v_p, v_q, v_r\}$ which is incident to v_q and v_p. Hence $d_q > d_r$ and accordingly $\bar{y}$ cannot be an eigenvector of $\mathcal{L}$. This contradicts with the assumption.

For topology VI, only the proof different from that of topology V is given. As topology V, it can be assumed without loss of generality that v_p, v_q share at least one common node in $\mathcal{V}\backslash\{v_p, v_q, v_r\}$. Consider the first situation of $\{v_p, v_r\}$, i.e., there is a $v_k(k \neq p, q, r)$ incident to both v_p and v_r. In this situation, (26) and (27) still hold for $k = p, q, r$. Then $y_p = -y_q = -y_r$. For $k = p$, (31) still holds. For $k = q$, $\sum_{i \in \mathcal{N}_{qf}} y_i = y_p = -y_q$. Thus

$$d_q y_q - \sum_{i \in \mathcal{N}_q} y_i = (d_q + 1) y_q. \tag{35}$$

Similarly, for $k = r$

$$d_r y_r - \sum_{i \in \mathcal{N}_r} y_i = (d_r + 1) y_r. \tag{36}$$

The remaining discussion is the same as topology V. Next consider the second situation of $\{v_p, v_r\}$. In this case, (26) still holds. It can be seen that for v_r, $d_r y_r - \sum_{i \in \mathcal{N}_r} y_i = d_r y_r - y_p$. The eigen-condition requires $d_r y_r - y_p = \lambda y_r$, i.e., $y_p = (d_r - \lambda) y_r$. For v_p, it still requires (34). So $y_r = (d_p + 1 - \lambda) y_p = (d_p + 1 - \lambda)(d_r - \lambda) y_r$. Since $y_r \neq 0$

$$(d_p + 1 - \lambda)(d_r - \lambda) = 1. \tag{37}$$

For v_q, since (26) still holds, $d_q y_q - \sum_{i \in \mathcal{N}_q} y_i = d_q y_q - y_p = (d_q + 1) y_q$. Thus, to satisfy the eigen-condition, it requires $\lambda = d_q + 1$. By (37), $(d_p - d_q)(d_r - d_q - 1) = 1$, which cannot be satisfied because $d_q > d_r$ (as topology V) and d_p and d_q are all integers. Accordingly, $\bar{y}$ cannot be an eigenvector of $\mathcal{L}$. This contradicts with the assumption.

For topology VII and the first situation of $\{v_p, v_r\}$, there does not exist node v_h in $\mathcal{V} \backslash \{v_p, v_q, v_r\}$ which is incident to both v_q and v_r because (26)–(28) cannot be satisfied simultaneously. Hence $d_p > d_r$ and $d_p > d_q$. Note that $\sum_{i \in \mathcal{N}_{pf}} y_i = y_q = -y_p$. By (30), $d_p y_p - \sum_{i \in \mathcal{N}_p} y_i = (d_p + 1) y_p$. Similarly, for $k = q$, $d_q y_q - \sum_{i \in \mathcal{N}_q} y_i = (d_q + 1) y_q$. Since $d_p + 1 > d_q + 1$, the eigen-condition of v_p, v_q cannot be met for the same eigenvalue. For the second situation of $\{v_p, v_r\}$, $d_q > d_r$. Since $d_q y_q - \sum_{i \in \mathcal{N}_q} y_i = (d_q + 1) y_q$; $d_r y_r - \sum_{i \in \mathcal{N}_r} y_i = d_r y_r$ and $d_q + 1 > d_r$, the eigen-condition of v_q, v_r cannot be met for the same eigenvalue as well. This contradicts the assumption that $\bar{y}$ is an eigenvector.

For topology VIII, $\sum_{i \in \mathcal{N}_{kl}} y_i = \sum_{i \in \mathcal{N}_{kf}} y_i = 0 (k = p, q, r)$. By (30)

$$d_k y_k - \sum_{i \in \mathcal{N}_k} y_i = d_k y_k. \tag{38}$$

Since each $v_k(k = p, q, r)$ has no neighbor nodes in $\{v_p, v_q, v_r\}$ and $\mathcal{G}$ is connected, it has at least one neighbor node in $\mathcal{V} \backslash \{v_p, v_q, v_r\}$; or else, v_k will be an isolated node. With v_p and v_q sharing a common neighbor node in $\mathcal{V} \backslash \{v_p, v_q, v_r\}$, the previous arguments show that v_q, v_r do not share a common neighbor node in $\mathcal{V} \backslash \{v_p, v_q, v_r\}$ if the first situation of v_p, v_r arises. In this circumstance, $d_p > d_q$ and $d_p > d_r$. By (38), the eigen-condition requires $d_p = d_q = d_r$, which cannot be met since $d_p > d_q$. If the second situation of v_p, v_r arises, the connectedness of $\mathcal{G}$ means there exist at least one v_k in $\mathcal{V} \backslash \{v_p, v_q, v_r\}$ which is incident to both v_q and v_r. Since this v_k cannot be incident to v_p and v_q simultaneously, $d_q > d_p$ and $d_q > d_r$. By (38), the eigen-condition cannot be met simultaneously for v_p, v_q, and v_r. This contradicts the assumption that $\bar{y}$ is an eigenvector. Above all, if $\bar{y}$ is an eigenvector of $\mathcal{L}$, then the topology of v_p, v_q, v_r accords with one of I–IV, i.e., they constitute a set of TCD nodes.

Sufficiency: First, suppose v_p, v_q, and v_r are TCD nodes with topology I. The corresponding topology structure means $d_p = d_q = d_r$. For $k \neq p, q, r$; the special form of $\bar{y}$ yields $\sum_{i \in \mathcal{N}_{kl}} y_i = 0$ and $y_k = 0$. Then (23) holds. Since the topology structure of v_p, v_q, and v_r accords with type I, for any $k \neq p, q, r$, either $p, q, r \in \mathcal{N}_{kf}$ or $p, q, r \notin \mathcal{N}_{kf}$. If $p, q, r \in \mathcal{N}_{kf}$, then $\sum_{i \in \mathcal{N}_{kf}} y_i = y_p + y_q + y_r$. Since $y_p + y_q + y_r = 0$, by (23)

$$d_k y_k - \sum_{i \in \mathcal{N}_k} y_i = 0. \tag{39}$$

If $p, q, r \notin \mathcal{N}_{kf}$, (39) still holds. Since $y_k = 0 (k \neq p, q, r)$, $\lambda y_k = 0$. Then, for any $k \neq p, q, r$ and any number λ, the eigen-condition (22) holds. For $k = p, q, r$, it follows from $\sum_{i \in \mathcal{N}_{kl}} y_i = 0$ that:

$$d_k y_k - \sum_{i \in \mathcal{N}_k} y_i = d_k y_k - \sum_{i \in \mathcal{N}_{kf}} y_i. \tag{40}$$

Since $\mathcal{N}_{kf}$ contains the other two indices of p, q, and r, for any given $k \in \{p, q, r\}$, say $k = p$, it follows $\sum_{i \in \mathcal{N}_{kf}} y_i = y_q + y_r$. By $y_p + y_q + y_r = 0$ and (40), $d_k y_k - \sum_{i \in \mathcal{N}_k} y_i = (d_k + 1) y_k$. Thus, for any k, the eigen-condition (22) is met for $\lambda = d_p + 1$. So the result holds for topology I.

Second, if v_p, v_q, and v_r are TCD nodes with topology II, the associated topology structure implies $\mathcal{N}_{pl} = \mathcal{N}_{ql} = \mathcal{N}_{rl}$ and $\mathcal{N}_{pf} \backslash \{p, q, r\} = \mathcal{N}_{qf} \backslash \{p, q, r\} = \mathcal{N}_{rf} \backslash \{p, q, r\}$. Moreover, since $q, r \in \mathcal{N}_{pf}$, $p \in \mathcal{N}_{qf}$, $p \in \mathcal{N}_{rf}$, and $\mathcal{N}_k = \mathcal{N}_{kl} + \mathcal{N}_{kf}$, it follows that $d_p = d_q + 1 = d_r + 1$. For $k \neq p, q, r$, the same arguments as topology I yield that the eigen-condition is met for any number λ. For $k = p$, since $\sum_{i \in \mathcal{N}_{kl}} y_i = 0$, $q, r \in \mathcal{N}_{pf}$ and $y_p + y_q + y_r = 0$, $\sum_{i \in \mathcal{N}_{pf}} y_i = y_q + y_r = -y_p$. By (40), $d_p y_p - \sum_{i \in \mathcal{N}_p} y_i = (d_p + 1) y_p$. For $k = q$, since $p \in \mathcal{N}_{qf}$, $d_q y_q - \sum_{i \in \mathcal{N}_q} y_i = d_q y_q - y_p$. From $y_p + y_q + y_r = 0$ and $y_q = y_r$, one has $d_q y_q - \sum_{i \in \mathcal{N}_q} y_i = (d_q + 2) y_q$. For $k = r$, the same arguments as $k = q$ gives $d_r y_r - \sum_{i \in \mathcal{N}_r} y_i = (d_r + 2) y_r$. The previous arguments show that $\bar{y}$ is an eigenvector of $\mathcal{L}$ with $d_p + 1$ the corresponding eigenvalue.

Third, if v_p, v_q, and v_r are TCD nodes with topology III, $d_p = d_q = d_r + 1$, which can be verified in the same way as the beginning part of proof of topology II. For $k \neq p, q, r$, the same proof as that of topology I yields that the eigen-condition holds for any number λ if $y_p + y_q + y_r = 0$. For $k = p, q, r$, (30) holds. For $k = p$, $\sum_{i \in \mathcal{N}_{pf}} y_i = y_q$ and for $k = q$, $\sum_{i \in \mathcal{N}_{qf}} y_i = y_p$. By (30) and $y_p = y_q$, it follows $d_p y_p - \sum_{i \in \mathcal{N}_p} y_i = (d_p - 1) y_p$. Similarly, for $k = q$, $d_q y_q - \sum_{i \in \mathcal{N}_q} y_i = (d_q - 1) y_q$. For $k = r$, since $\sum_{i \in \mathcal{N}_{rl}} y_i = \sum_{i \in \mathcal{N}_{rf}} y_i = 0$, $\sum_{i \in \mathcal{N}_r} y_i = 0$, it can be seen that $d_r y_r - \sum_{i \in \mathcal{N}_r} y_i = d_r y_r$. Since $d_p = d_q = d_r + 1$, the above arguments show that the eigen-condition holds for each k and the corresponding eigenvalue is $\lambda = d_r$.

Finally, if v_p, v_q, and v_r are with topology IV, $d_p = d_q = d_r$. In addition, for $k \neq p, q, r$, the eigen-condition still holds for any number λ if $y_p + y_q + y_r = 0$; and for $k = p, q, r$, $\sum_{i \in \mathcal{N}_{kl}} y_i = \sum_{i \in \mathcal{N}_{kf}} y_i = 0$. Thus $\sum_{i \in \mathcal{N}_k} y_i = 0 (k = p, q, r)$, and accordingly $d_k y_k - \sum_{i \in \mathcal{N}_k} y_i = d_k y_k$. Thus the eigen-condition is met for each k if the eigenvalue $\lambda = d_p$. Therefore,

$\bar{y}$ is an eigenvector of $\mathcal{L}$ if v_p, v_q, v_r are TCD nodes with one of topologies I–IV.

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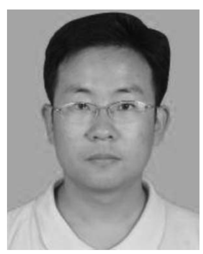

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