On the Controllability of Matrix-Weighted Networks

Lulu Pan, Haibin Shao, *Member, IEEE*, Mehran Mesbahi, *Fellow, IEEE*, Yugeng Xi, *Senior Member, IEEE*, and Dewei Li

***Abstract*—This letter examines the controllability of matrix-weighed networks from a graph-theoretic perspective. As distinct from the scalar-weighted networks, the rank of weight matrices introduce additional intricacies into characterizing the dimension of the controllable subspace for such networks. Specifically, we investigate how the definiteness of weight matrices, encoding a generalized characterization of inter-agent connectivity on matrix-weighted networks, influences the lower and upper bounds of the associated controllable subspaces. We show that such a lower bound is determined by the existence of a certain positive path in the distance partition of the network. By introducing the notion of matrix-valued almost equitable partitions, we show that the corresponding upper bound is determined by the product of the dimension of the weight matrices and the cardinality of the associated matrix-valued almost equitable partition. Furthermore, the structure of an uncontrollable input for such networks is examined.**

***Index Terms*—Controllability, matrix-weighed networks, controllable subspace, positive semi-definite matrices.**

I. Introduction

Controllability of a dynamic system is a fundamental notion in control theory [1]. For multi-agent networks, controllability is closely related to the graph-theoretic properties of the underlying network [2], [3], [4]. The controllability of multi-agent networks under nearest-neighbor interactions has initially been examined in [2], where it was shown that network connectivity can have adverse effects on controllability. The influence of network symmetry of leader-following networks on its controllability has been reported in [3]. Graph node partitions were subsequently employed to characterize the upper bounds on the dimension of the controllable subspace of multi-agent networks [5], [6], [7]; analogous lower bounds have also been derived using distance partitions [5], [8]. Due to the difficulty in analyzing controllability of general networks, controllability for special classes of networks has been an active area of research [9], [10], [11], [12], [13], [14]. Recently, controllability of multi-agent system on signed networks (where cooperative and competitive interactions coexist) has also received attention. A graph-theoretic characterization of the upper bound on the dimension of the controllable subspace for signed networks has been proposed using generalized equitable partition in [15]. In [16], sufficient conditions on the controllability of signed path, cycle and tree networks have been derived. The controllability problem on certain classes of signed networks is also studied in [17]; a comprehensive review on network controllability has been provided in [18].

In the meantime, existing works on network controllability are mainly concerned with networks with scalar weighted edges; such network models are restrictive in characterizing interdependence amongst subsets of the underlying node states [19]. Matrix-weighted networks are a natural extension of scalar-valued networks; they have been examined in scenarios such as graph effective resistance (motivated by distributed estimation and control) [20], [21], logical inter-dependency of multiple topics in opinion evolution [22], [23], bearing-based formation control [24], array of coupled LC oscillators [25], as well as multilayer networks [26]. More recently, consensus and synchronization problems on matrix-weighted networks have been examined in [27], [28], [29], [30], [31].

Consensus protocol plays a vital role in cooperative control of multi-agent networks, ensuring asymptotic alignment on the states of the agents required for accomplishing a global task via local interactions [32], [33], [34], [35]. In this letter, we examine the controllability of multi-agent systems governed by consensus-like dynamics on matrix-weighted networks. Although the matrix-weighted setup is a natural extension of scalar-weighted networks, extending network controllability to the former case is non-trivial. An essential distinction in this direction is that the rank of the weight matrix can

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range from zero up to its dimension, introducing additional intricacies into characterizing the dimension of controllable subspace. In this letter, we show how the definiteness of weight matrices, encoding a generalized characterization of inter-agent connectivity on matrix-weighted networks [27], influences the dimension of the controllable subspace for the corresponding network. Moreover, graph theoretic lower and upper bounds on the dimension of the controllable subspace of the influenced consensus are provided–this is achieved by exploiting the distance partition and matrix-valued almost equitable partitions of matrix-weighted networks, extending results for scalar-weighted networks.

The remainder of this note is organized as follows. The preliminary notions used in this letter are introduced in Section II. The problem formulation is discussed in Section III followed by the characterization of lower and upper bounds of the dimension of the controllable subspace in Section IV and V, respectively. The structure of uncontrollable input matrix is further discussed in Section VI followed by concluding remarks in Section VII.

II. Notations

Let $\mathbb{R}$ and $\mathbb{N}$ be the set of real and natural numbers, respectively. Denote $\underline{n} = \{1, 2, \ldots, n\}$ for an $n \in \mathbb{N}$. A symmetric matrix $M \in \mathbb{R}^{n\times n}$ is positive definite, denoted by $M \succ 0$, if $z^T M z > 0$ for all nonzero $z \in \mathbb{R}^n$, and is positive semi-definite, denoted by $M \succeq 0$, if $z^T M z \geq 0$ for all $z \in \mathbb{R}^n$. The image and rank of a matrix M are denoted by $\mathbf{img}(M)$ and $\mathbf{rank}(M)$, respectively. Denote by $\mathbf{dim}(\cdot)$ as the dimension of a vector space (or subspace) and $\mathbf{diag}\{\cdot\}$ as the (block) diagonal matrix comprised from its arguments. For a block matrix Z with $n \in \mathbb{N}$ row partitions and $m \in \mathbb{N}$ column partitions, we denote by $(Z)_{ij}$ as the matrix block on the ith row and jth column in Z, where $i \in \underline{n}$ and $j \in \underline{m}$. Denote by $\mathbf{row}_i(Z)$ as $[(Z)_{i1}, (Z)_{i2}, \ldots, (Z)_{im}]$. Let $\mathbf{gcd}\{k_1, k_2, \ldots, k_m\}$ signify the greatest common divisor of a set of integers $k_1, k_2, \ldots, k_m \in \mathbb{N}$. The $d \times d$ zero matrix and identity matrix are denoted by $\mathbf{0}_{d\times d}$ and $I_{d\times d}$, respectively.

III. Problem Formulation

Consider a multi-agent network consisting of $n \in \mathbb{N}$ agents where the state of an agent $i \in \underline{n}$ is denoted by the vector $\boldsymbol{x}_i(t) \in \mathbb{R}^d$ with $d \in \mathbb{N}$. The state of the multi-agent network is denoted by $\boldsymbol{x}(t) = [\boldsymbol{x}_1^T(t), \boldsymbol{x}_2^T(t), \ldots, \boldsymbol{x}_n^T(t)]^T \in \mathbb{R}^{dn}$. The interaction topology of the network is characterized by a matrix-weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. The node and edge sets of $\mathcal{G}$ are denoted by $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, respectively. The weight on the edge $(v_i, v_j) \in \mathcal{E}$ in $\mathcal{G}$ is a symmetric matrix $A_{ij} \in \mathbb{R}^{d\times d}$ such that $A_{ij} \succeq 0$ or $A_{ij} \succ 0$ and $A_{ij} = \mathbf{0}_{d\times d}$ if $(v_i, v_j) \notin \mathcal{E}$. Thereby, the matrix-valued adjacency matrix $A = [A_{ij}] \in \mathbb{R}^{dn\times dn}$ is a block matrix such that the matrix block located in the ith row and jth column is A_{ij}. We shall assume that $A_{ij} = A_{ji}$ for all $v_i \neq v_j \in \mathcal{V}$ and $A_{ii} = \mathbf{0}_{d\times d}$ for all $v_i \in \mathcal{V}$. The neighbor set of an agent $v_i \in \mathcal{V}$ is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} \,|\, (v_i, v_j) \in \mathcal{E}\}$. The interaction protocol for each agent in a matrix-weighted network now

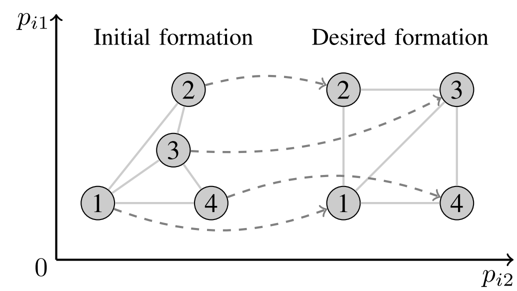

Fig. 1. The initial formation (left) and the destination formation (right) in the 2-dimensional plane. The solid lines illustrate the communication channels amongst agents and the dashed lines depict the trajectories of agents from their initial positions to the desired positions.

assumes the form,

$$\dot{\boldsymbol{x}}_i(t) = -\sum_{j\in\mathcal{N}_i} A_{ij}(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)), \; i \in \mathcal{V}. \tag{1}$$

To see the necessity of using matrix-valued weights between neighboring agents, we provide the following example to demonstrate how matrix-weighted networks arise in bearing-based formation control problems [24].

Example 1: Consider the formation translation (shown in Figure 1) via bearing-based control protocol (2). The unit bearing constraints for the desired formation are $\boldsymbol{g}_{12}^* = -\boldsymbol{g}_{21}^* = [0, 1]^T$, $\boldsymbol{g}_{23}^* = -\boldsymbol{g}_{32}^* = [1, 0]^T$, $\boldsymbol{g}_{34}^* = -\boldsymbol{g}_{43}^* = [0, -1]^T$, $\boldsymbol{g}_{41}^* = -\boldsymbol{g}_{14}^* = [-1, 0]^T$, $\boldsymbol{g}_{13}^* = -\boldsymbol{g}_{31}^* = [\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]^T$, and the control protocol for each agent admits the form,

$$\dot{\boldsymbol{p}}_i(t) = -\sum_{j\in\mathcal{N}_i} P_{\boldsymbol{g}_{ij}^*}(\boldsymbol{p}_i(t) - \boldsymbol{p}_j(t)), \tag{2}$$

where $\boldsymbol{p}_i \in \mathbb{R}^2$ denotes the position of agent i and the projection matrices $P_{\boldsymbol{g}_{ij}^*} = I_{d\times d} - \boldsymbol{g}_{ij}^* \boldsymbol{g}_{ij}^{*T}$ are matrix-valued weights, namely, $P_{\boldsymbol{g}_{12}^*} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_{\boldsymbol{g}_{23}^*} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $P_{\boldsymbol{g}_{34}^*} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_{\boldsymbol{g}_{41}^*} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $P_{\boldsymbol{g}_{13}^*} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Therefore in this example, protocol (2) for bearing-based formation is a special case of the interaction protocol (1). Note that $P_{\boldsymbol{g}_{ij}^*} \succeq 0$ for all $i, j \in \mathcal{V}$; as it turns out, the non-definiteness of these weights can have an adverse effect on the controllability of matrix-weighted networks.

Controllability of a networked system examines whether the state of its nodes can be steered from any initial state to an arbitrary desired state in a finite time by manipulating some of the nodes, referred to as the leader nodes. Let $\boldsymbol{u} = [\boldsymbol{u}_1^\top, \boldsymbol{u}_2^\top, \ldots, \boldsymbol{u}_m^\top]^\top \in \mathbb{R}^{dm}$ be the control input exerted on the leader nodes, where $\boldsymbol{u}_j \in \mathbb{R}^d$ and $j \in \underline{m}$. Denote by $B = [B_{il}] \in \mathbb{R}^{dn\times dm}$ as the matrix-weighted input matrix where $B_{il} \in \{\mathbf{0}_{d\times d}, I_{d\times d}\}$. The set of leaders and "followers" can now be defined as $\mathcal{V}_{\text{leader}} = \{i \in \mathcal{V} \,|\, B_{il} = I_{d\times d}\}$ and $\mathcal{V}_{\text{follower}} = \mathcal{V} \setminus \mathcal{V}_{\text{leader}}$, respectively. Denote by $D = \mathbf{diag}\{D_1, D_2, \ldots, D_n\} \in \mathbb{R}^{dn}$ as the matrix-valued degree matrix of $\mathcal{G}$, where $D_i = \sum_{j\in\mathcal{N}_i} A_{ij} \in \mathbb{R}^{d\times d}$. The matrix-valued Laplacian is defined as $L = D - A$. As such, the leader-following multi-agent system on matrix-weighted networks can be characterized by a linear time-invariant system,

$$\dot{\mathbf{x}}(t) = -L\mathbf{x}(t) + B\mathbf{u}(t). \tag{3}$$

Hence, the network (3) is controllable from the leader set $\mathcal{V}_{\text{leader}}$ if and only if the associated controllability matrix,

$$\mathcal{K}(L, B) = \left[B \;\; -LB \;\; L^2B \;\; \cdots \;\; (-L)^{dn-1}B\right], \tag{4}$$

has a full row rank, i.e., $\mathbf{rank}(\mathcal{K}(L, B)) = dn$.

Definition 1: The controllable subspace of the system (3) is defined as the range space of $\mathcal{K}(L, B)$, namely,

$$\langle L|B\rangle = \mathbf{img}(B) + L\mathbf{img}(B) + \cdots + L^{dn-1}\mathbf{img}(B), \tag{5}$$

where the summation is with respect to subspace addition.

In our subsequent discussion, we provide graph-theoretic lower and upper bounds on the dimension of the controllable subspace $\langle L|B\rangle$和符号网络一样的概念，注意矩阵的权重的处理。实际用途？

IV. Lower Bound on the Dimension of the Controllability Subspace

In this section, we examine the lower bound on the dimension of $\langle L|B\rangle$; first, let us introduce the necessary graph-theoretic concepts.

Definition 2: For a matrix-weighted network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, a node partition $\pi = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_s\}$ is a collection of subsets $\mathcal{V}_i \subset \mathcal{V}$ such that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s$ and $\mathcal{V}_1 \cap \mathcal{V}_2 \cap \cdots \cap \mathcal{V}_s = \emptyset$, where $i \in \underline{s}$ and $s \in \mathbb{N}$. The matrix-weighted characteristic matrix $P(\pi) = [P_{ij}(\pi)] \in \mathbb{R}^{dn\times ds}$ of a node partition $\pi = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_s\}$ is now defined as,

$$P_{ij}(\pi) = \begin{cases} I_{d\times d}, & v_i \in \mathcal{V}_j \\ \mathbf{0}_{d\times d}, & v_i \notin \mathcal{V}_j \end{cases}.$$

For any $\mathcal{Q} \subseteq \mathcal{V}$, denote by $\delta_{|\mathcal{V}|,\mathcal{Q}}$ as a block matrix with $|\mathcal{V}|$ row partitions and one column partition such that the qth $d\times d$ block in $\delta_{|\mathcal{V}|,\mathcal{Q}}$ is $I_{d\times d}$, and all the remaining blocks are $\mathbf{0}_{d\times d}$, where $v_q \in \mathcal{Q}$.

Example 2: Consider a 5-node matrix-weighted network with a node partition $\pi = \{\{1\}, \{2, 3\}, \{4, 5\}\}$ and the dimension of weight matrices on edges is $d = 2$. Then $P(\pi) = [\delta_{5,\{1\}},\ \delta_{5,\{2,3\}},\ \delta_{5,\{4,5\}}]$.

A path in a matrix-weighted network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is a sequence of edges of the form $\mathcal{P}_{v_{i_1},v_{i_p}} = (v_{i_1}, v_{i_2}), \ldots, (v_{i_{p-1}}, v_{i_p})$, where nodes $v_{i_1}, v_{i_2}, \ldots, v_{i_p} \in \mathcal{V}$ are all distinct and it is said that v_{i_p} is reachable from v_{i_1}; a path $\mathcal{P}_{v_{i_1},v_{i_p}}$ turns to a cycle if $v_{i_1} = v_{i_p}$. The network $\mathcal{G}$ is connected if any two distinct nodes in $\mathcal{G}$ are reachable from each other. A tree is a connected graph with n nodes and $n-1$ edges where $n \in \mathbb{N}$. All networks discussed in this letter are assumed to be connected. The shortest path between two nodes $v_i, v_j \in \mathcal{V}$ is a path that contains the least number of the edges; the number of the edges on this shortest path is referred to as the distance between nodes v_i and v_j, denoted by $\mathbf{dist}(v_i, v_j)$. The diameter of $\mathcal{G}$ is then defined as $\mathbf{diam}(\mathcal{G}) = \max_{v_i,v_j\in\mathcal{V}}\mathbf{dist}(v_i, v_j)$. An edge $(v_i, v_j) \in \mathcal{E}$ is positive definite or positive semi-definite if its weight matrix A_{ij} is positive definite or positive semi-definite.

Definition 3 (Positive Definite Path): A positive definite path in a matrix-weighted network $\mathcal{G}$ is a path for which every edge has a positive definite weight.

In the subsequent discussion, we will characterize a lower bound on the dimension of the controllable subspace of (3) for acyclic networks, followed by cycle and complete networks. In particular, we examine the influence of the positive definiteness of weight matrices on $\mathbf{dim}(\langle L|B\rangle)$.

Definition 4 (Distance Partition): Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-weighted network. The distance partition relative to an agent $v_i \in \mathcal{V}$ consists of the subsets,

$$\mathcal{C}_r = \left\{v_j \in \mathcal{V} \,|\, \mathbf{dist}(v_i, v_j) = r\right\},$$

where $0 \le r \le \mathbf{diam}(\mathcal{G})$.

Theorem 1: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-weighted tree network whose dimension of the weight matrix is $d \in \mathbb{N}$. Let $v_l \in \mathcal{V}$ be the leader agent and denote the distance partition relative to v_l as $\pi_D(v_l) = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_r\}$, where $0 \le r \le \mathbf{diam}(\mathcal{G})$. If there exists an agent v_i in $\mathcal{C}_r$ such that the path $\mathcal{P}_{v_l,v_i}$ is positive definite, then $\mathbf{dim}(\langle L|B\rangle) \ge d\,|\pi_D(v_l)|$.

Proof: The adopted line of reasoning is similar to that presented in [36] for the scalar-weights. Without loss of generality, let v_1 be the leader agent. Denote the distance partition relative to v_1 as $\pi_D(v_1) = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_r\}$. Specifically,

$$\begin{aligned} \mathcal{C}_0 &= v_1, \\ \mathcal{C}_q &= \{v_i \,|\, v_i \in \mathcal{V}, \mathbf{dist}(v_i, v_1) = q\} \\ &= \left\{v_1^{(q)}, v_2^{(q)}, \ldots, v_{|\mathcal{C}_q|}^{(q)}\right\}, q \in \underline{r}. \end{aligned}$$

According to Definition 4, there does not exist agents in $\mathcal{C}_i$ with a neighbor in $\mathcal{C}_j$ if $|i-j| > 1$, where $i, j \in \underline{r}$. Then the matrix-weighted Laplacian of $\mathcal{G}$ admits the form,

$$L = \begin{bmatrix} L_{00} & L_{01} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ L_{10} & L_{11} & L_{12} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & L_{21} & L_{22} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & L_{r-1,r-1} & L_{r-1,r} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & L_{r,r-1} & L_{r,r} \end{bmatrix},$$

where $L_{kl} \in \mathbb{R}^{d\,|\mathcal{C}_k|\times d\,|\mathcal{C}_l|}$ for all $0 \le k, l \le r$ and $\mathbf{0}$'s are zero matrices with proper dimensions.

Let $E = [\,B \;\; LB \;\; \cdots \;\; L^rB\,]$ be a block matrix with $r+1$ row partitions and $r+1$ column partitions. Note that as agent v_1 is the leader, $B = [\,I_{d\times d} \;\; 0_{d\times d} \;\; \cdots \;\; 0_{d\times d}\,]^\top$ and,

$$E = \begin{bmatrix} E_{00} & E_{01} & \cdots & E_{0,r} \\ E_{10} & E_{11} & \cdots & E_{1,r} \\ \vdots & \vdots & \ddots & \vdots \\ E_{r,0} & E_{r,1} & \cdots & E_{r,r} \end{bmatrix},$$

where $E_{00} = I_{d\times d}$, E_{qq} is a block matrix with $|\mathcal{C}_q|$ row partitions and 1 column partition where $q \in \underline{r}$ and E_{pq} with $p > q$ are matrices with proper size and all elements equal to 0 where $p \in \underline{r}$. In particular, we are interested in those blocks located in qth row and qth column in E since they are crucial in determining $\mathbf{rank}(E)$.

Denote the block in sth row block in E_{qq} as $E_{qq}^{(s)}$, where $s \in \{1, 2, \ldots, |\mathcal{C}_q|\}$ and $q \in \underline{r}$; then

$$E_{qq}^{(s)} = \prod_{(i,j)\in\mathcal{P}_{v_1,v_s^{(q)}}} A_{ij}.$$

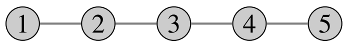

Fig. 2. A matrix-weighted path network with 5 nodes.

By our standing assumption, there exists one node in $\mathcal{C}_r$ such that the path between this node and v_l are positive definite. As the product of positive definite matrices has full rank, one has $\mathbf{rank}(E_{qq}) = d$ for all $q \in \underline{r}$. Hence,

$$\mathbf{rank}(E) = d\,|\pi_D(v_1)|,$$

completing the proof. ■

Corollary 1 (Path Network): Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a path network in the form of $\mathcal{P}_{v_{i_1}, v_{i_p}} = (v_{i_1}, v_{i_2}), \ldots, (v_{i_{p-1}}, v_{i_p})$, where nodes $v_{i_1}, v_{i_2}, \ldots, v_{i_p} \in \mathcal{V}$. Then $\mathcal{G}$ is controllable from v_{i_1} (or v_{i_p}) if and only if the path $\mathcal{P}_{v_{i_1}, v_{i_p}}$ is positive definite.

Example 3: Consider the matrix-weighted path network in Figure 2. Choose agent 1 as leader, the weight matrices on edges are set as $A_{12} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, A_{23} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, and $A_{34} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, A_{45} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. As we can see that the weight matrices are all positive definite.

The matrix-weighted input matrix can be written as $B = [\delta_{5,\{1\}}]$. The dimension of the controllable subspace $\langle L|B\rangle$ in this case is 10 and therefore (L, B) is controllable. We proceed to replace the weight matrix between agent 2 and agent 3 by a positive semi-definite matrix $A_{23} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. In this case, $\mathbf{dim}(\langle L|B\rangle) = 9$, implying (L, B) is uncontrollable.

Corollary 2 (Cycle Network): Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-weighted cycle with dimension of the weight matrix as $d \in \mathbb{N}$. Let $v_l \in \mathcal{V}$ be a leader agent and denote the distance partition relative to v_l as $\pi_D(v_l) = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_r\}$, where,

$$r = \begin{cases} \frac{|\mathcal{V}|}{2}, & |\mathcal{V}| \text{ is even;} \\ \frac{|\mathcal{V}|-1}{2}, & |\mathcal{V}| \text{ is odd.} \end{cases}$$

If there exists an agent v_i in $\mathcal{C}_r$ such that the shortest path between v_i and v_l are positive definite, then

$$\mathbf{dim}(\langle L|B\rangle) \geq \begin{cases} d\,\frac{|\mathcal{V}|}{2} + 1, & |\mathcal{V}| \text{ is even;} \\ d\,\frac{|\mathcal{V}|+1}{2}, & |\mathcal{V}| \text{ is odd.} \end{cases}$$

Corollary 3 (Complete Network): Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-weighted complete network with the dimension of the weight matrix as $d \in \mathbb{N}$. Let $v_l \in \mathcal{V}$ be a leader agent and denote the distance partition relative to v_l as $\pi_D(v_l) = \{\mathcal{C}_0, \mathcal{C}_1\}$. If there exists an agent v_i in $\mathcal{C}_1$ such that the path $\mathcal{P}_{v_l, v_i}$ is positive definite, then $\mathbf{dim}(\langle L|B\rangle) \geq d$.

Note that from Theorem 1, the rank of weight matrices influences the lower bound on the dimension of the controllable subspace of (3); this is distinct from the scalar-weighted case. As such, the semi-definiteness of weight matrices plays an important role in the controllability of matrix-weighted networks.

V. Upper Bound on the Dimension of the Controllability Subspace

We now proceed to examine graph-theoretic characterizations of the upper bound of the controllable subspace of system (3) in terms of the matrix-valued almost equitable partition. For a given subset $\mathcal{Q} \in \mathcal{V}$ in a matrix-weighted network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ and an agent $v_i \in \mathcal{V}$, denote the matrix-valued degree of v_i relative to $\mathcal{Q}$ as $D(v_i, \mathcal{Q}) = \sum_{v_j \in \mathcal{Q}} A_{ij}$.

Definition 5: An $s-$partition $\pi = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_s\}$ of a matrix-weighted network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is a matrix-valued almost equitable partition (matrix-valued AEP) if for $\forall i \neq j \in \underline{s}$ and $\forall v, w \in \mathcal{V}_i$ one has $D(v, \mathcal{V}_j) = D(w, \mathcal{V}_j)$.

According to Definition 5, if an $s-$partition, $\pi = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_s\}$ is a matrix-valued AEP, then one can denote $D(\mathcal{V}_i, \mathcal{V}_j) = D(v, \mathcal{V}_j)$ for $\forall v \in \mathcal{V}_i$. Next, we proceed to define the quotient graph of a matrix-weighted network based on the matrix-valued AEP.

Definition 6: For a given matrix-valued AEP $\pi = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_s\}$ of a matrix-weighted network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, the quotient graph of $\mathcal{G}$ over π is a matrix-weighted network denoted by $\mathcal{G}/\pi$ with the node set $\mathcal{V}(\mathcal{G}/\pi) = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_s\}$, whose edge set is $\mathcal{E}(\mathcal{G}/\pi) = \{(\mathcal{V}_i, \mathcal{V}_j) \,|\, D(\mathcal{V}_i, \mathcal{V}_j) \neq \mathbf{0}_{d\times d}\}$, and the weight on edge $(\mathcal{V}_i, \mathcal{V}_j)$ is $D(\mathcal{V}_i, \mathcal{V}_j)$ for $i \neq j \in \underline{s}$.

Note that the condition $D(\mathcal{V}_i, \mathcal{V}_j) = D(\mathcal{V}_j, \mathcal{V}_i)$ does not necessary hold; as such, the quotient graph $\mathcal{G}/\pi$ can be directed. The following result provides the relationship between the $L-$invariant subspace and the matrix-valued AEP of matrix-weighted networks.

Lemma 1: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-weighted network with the dimension of edge weight $d \in \mathbb{N}$, L be the matrix-valued Laplacian of $\mathcal{G}$, $\pi = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_s\}$ be a $s-$partition of $\mathcal{V}(\mathcal{G})$ and $P(\pi)$ be the characteristic matrix of π. Then π is a matrix-valued AEP of $\mathcal{G}$ if and only if $\mathbf{img}(P(\pi))$ is $L-$invariant, i.e., there exists a matrix $L^{\pi} \in \mathbb{R}^{ds\times ds}$ such that $LP(\pi) = P(\pi)L^{\pi}$.

Proof (Necessity): Define the matrix $L^{\pi} \in \mathbb{R}^{ds\times ds}$ as

$$(L^{\pi})_{ij} = \begin{cases} \sum_{\mathcal{V}_j \in \mathcal{V}(\mathcal{G}/\pi)} D(\mathcal{V}_i, \mathcal{V}_j), & i = j; \\ -D(\mathcal{V}_i, \mathcal{V}_j), & i \neq j. \end{cases}$$

Suppose that $\pi = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_s\}$ is a matrix-valued AEP of the matrix-weighted network $\mathcal{G}$ and $v_p \in \mathcal{V}_k$, where $p \in \underline{n}$ and $k \in \underline{s}$. On one hand, the p-th block row of $LP(\pi)$ can be characterized by,

$$\begin{aligned}(LP(\pi))_p = [&-\sum_{j\in\mathcal{V}_1\cap\mathcal{N}_{\mathcal{G}}(v_p)} A_{pj}, -\sum_{j\in\mathcal{V}_2\cap\mathcal{N}_{\mathcal{G}}(v_p)} A_{pj}, \ldots, \\ &D_p - \sum_{j\in\mathcal{V}_k\cap\mathcal{N}_{\mathcal{G}}(v_p)} A_{pj}, -\sum_{j\in\mathcal{V}_{k+1}\cap\mathcal{N}_{\mathcal{G}}(v_p)} A_{pj}, \\ &\ldots, -\sum_{j\in\mathcal{V}_s\cap\mathcal{N}_{\mathcal{G}}(v_p)} A_{pj}].\end{aligned}$$

On the other hand, the entries in the p-th block row of $P(\pi)L^{\pi}$ are,

$$\begin{aligned}[-D(\mathcal{V}_k, \mathcal{V}_1), \ldots, -D(\mathcal{V}_k, \mathcal{V}_{k-1}), \sum_{r\neq k} D(\mathcal{V}_k, \mathcal{V}_r), \\ -D(\mathcal{V}_k, \mathcal{V}_{k+1}), \ldots, -D(\mathcal{V}_k, \mathcal{V}_s)].\end{aligned}$$

According to Definition 5, we have

$$\sum_{j\in\mathcal{V}_r\cap\mathcal{N}_{\mathcal{G}}(p)} A_{pj} = D(\mathcal{V}_k, \mathcal{V}_r),$$

and

$$D_p - \sum_{j\in\mathcal{V}_k\cap\mathcal{N}_{\mathcal{G}}(p)} A_{pj} = \sum_{r\neq k} D(\mathcal{V}_k, \mathcal{V}_r).$$

Then $\mathbf{row}_p(LP(\pi)) = \mathbf{row}_p(P(\pi)L^{\pi})$, which implies that $LP(\pi) = P(\pi)L^{\pi}$.

(Sufficiency) Suppose that π is an $s-$partition of the matrix-weighted network $\mathcal{G}$ satisfying $LP(\pi) = P(\pi)L^{\pi}$. Then each column in $LP(\pi)$ is the linear combination of the columns in $P(\pi)$. For each block column of $LP(\pi)$, the matrix blocks corresponding to the agents belonging to the same subset in π are identical. Therefore one has,

$$(LP(\pi))_{ij} = -\sum_{r\in\mathcal{V}_j\cap\mathcal{N}_{\mathcal{G}}(i)} A_{ir}, \ \forall i\neq j,$$

and for any k in the same subset as i,

$$(LP(\pi))_{kj} = -\sum_{r\in\mathcal{V}_j\cap\mathcal{N}_{\mathcal{G}}(k)} A_{kr}.$$

Note that $(LP(\pi))_{ij} = (LP(\pi))_{kj}$ implies that,

$$\sum_{r\in\mathcal{V}_j\cap\mathcal{N}_{\mathcal{G}}(i)} A_{ir} = \sum_{r\in\mathcal{V}_j\cap\mathcal{N}_{\mathcal{G}}(k)} A_{kr},$$

for any k in the same subset as i. Therefore, π is a matrix-valued AEP. ■

Lemma 1 has the following immediate consequence.

Theorem 2: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-weighted network with the dimension of edge weight $d\in\mathbb{N}$. Suppose that $\pi = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_s\}$ is a matrix-valued AEP of $\mathcal{G}$ with the characteristic matrix $P(\pi)$ where $1\leq s < n$. Denote $B = [b_1, b_2, \ldots, b_m] \in \mathbb{R}^{dn\times dm}$ as the input matrix where $b_i \in \{\mathbf{0}_{d\times d}, I_{d\times d}\}^n$, the matrix blocks in b_i corresponding to the agents belonging to the same subset in π are the same and $i\in \underline{m}$. Then, (1) $\langle L|B\rangle \subseteq \mathbf{img}(P(\pi))$, (2) $\mathbf{dim}(\langle L|B\rangle) \leq ds$, and (3) the pair (L, B) is uncontrollable.

Proof: Since the matrix blocks in b_i corresponding to the agents belonging to the same subset in π are the same where $i\in\underline{m}$, then $\mathbf{img}(B)\subseteq \mathbf{img}(P(\pi))$. In the meantime, $\mathbf{img}(P(\pi))$ is $L-$invariant according to Lemma 1; thus we have,

$$\begin{aligned}\langle L|B\rangle &= \mathbf{img}(B) + L\mathbf{img}(B) + \cdots + L^{dn-1}\mathbf{img}(B)\\ &\subseteq \mathbf{img}(P(\pi)) + L\mathbf{img}(P(\pi)) + \cdots + L^{dn-1}\mathbf{img}(P(\pi))\\ &= \mathbf{img}(P(\pi)),\end{aligned}$$

implying $\mathbf{dim}(\langle L|B\rangle)\leq ds$. Since $1\leq ds < dn$, the pair (L, B) is uncontrollable. ■

Remark 1: According to the above analysis, the upper bound of $\mathbf{dim}(\langle L|B\rangle)$ on the scalar-weighted networks is a special case of Theorem 2 when $d = 1$.

VI. On Uncontrollable Input Matrix

Note from that Theorem 2 provides an upper bound on the controllable subspace using the range space of the characteristic matrix of the matrix-valued AEP. It is shown that $\mathbf{img}(B)\subseteq \mathbf{img}(P(\pi))$ can directly lead to the uncontrollability of the network when the matrix-valued AEP $\pi = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_s\}$ is non-trivial. However, is there any other leader selections that induces the uncontrollability of (L, B)? In the following discussions, we proceed to provide the structure of the uncontrollable matrix B.

Theorem 3: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-weighted network with the dimension of edge weight $d\in\mathbb{N}$. Suppose that $\pi = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_s\}$ is a matrix-valued AEP of $\mathcal{G}$ with the characteristic matrix $P(\pi) = [P_1, P_2, \ldots, P_s]$, where $P_1, P_2, \ldots, P_s \in \mathbb{R}^{dn\times d}$ and $1\leq s < n$. Let π be reducible and $q_j = \frac{|\mathcal{V}_j|}{\mathbf{gcd}(\pi)}$ where $j\in\underline{s}$. Let $B\in\{0_{d\times d}, I_{d\times d}\}^n$ be such that

$$\boldsymbol{p}_{j1}^{\top}\boldsymbol{b}_1 = \boldsymbol{p}_{j2}^{\top}\boldsymbol{b}_2 = \cdots = \boldsymbol{p}_{jd}^{\top}\boldsymbol{b}_d = cq_j, \tag{6}$$

where c is an integer such that $1\leq c\leq \mathbf{gcd}(\pi) - 1$, $\boldsymbol{p}_{j1}, \boldsymbol{p}_{j2}, \ldots, \boldsymbol{p}_{jd} \in \mathbb{R}^{dn\times 1}$ and $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_d \in \mathbb{R}^{dn\times 1}$ are columns of matrices P_j and B, respectively. Then (L, B) is uncontrollable.

Proof: Since $\mathbf{img}(P(\pi))$ is $L-$invariant, there exists an eigenvector $\boldsymbol{w}\in \mathbf{img}(P(\pi))$ of L satisfying $\boldsymbol{w}\notin \mathbf{span}\{\mathbf{1}_n\otimes I_d\}$ and $\boldsymbol{w}^{\top}\mathbf{1}_{dn} = 0$. Note that $\{\boldsymbol{p}_{11}, \ldots, \boldsymbol{p}_{1d}, \boldsymbol{p}_{21}, \ldots, \boldsymbol{p}_{2d}, \ldots, \boldsymbol{p}_{s1}, \ldots, \boldsymbol{p}_{sd}\}$ forms a basis of $\mathbf{img}(P(\pi))$; as such $\boldsymbol{w} = \sum_{j=1}^{s}\sum_{t=1}^{d}\alpha_{jt}\boldsymbol{p}_{jt}$ for some α_{jt}'s. Due to the fact that

$$\begin{aligned}\boldsymbol{w}^{\top}(\mathbf{1}_n\otimes I_d) &= \left(\sum_{j=1}^{s}\alpha_{j1}|\mathcal{V}_j|, \ldots, \sum_{j=1}^{s}\alpha_{jd}|\mathcal{V}_j|\right)\\ &= [0, \ldots, 0]\in\mathbb{R}^{1\times dn},\end{aligned}$$

if we choose $B = [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d]$ satisfying (6), then we have

$$\begin{aligned}\boldsymbol{w}^{\top}\boldsymbol{b}_k &= \left(\sum_{j=1}^{s}\sum_{t=1}^{d}\alpha_{jt}\boldsymbol{p}_{jt}^{\top}\right)\boldsymbol{b}_k = \left(\sum_{j=1}^{s}\alpha_{jk}\boldsymbol{p}_{jk}^{\top}\right)\boldsymbol{b}_k\\ &= \sum_{j=1}^{s}\alpha_{jk}cq_j = \sum_{j=1}^{s}\alpha_{jk}c\frac{|\mathcal{V}_j|}{\mathbf{gcd}(\pi)}\\ &= \frac{c}{\mathbf{gcd}(\pi)}\sum_{j=1}^{s}\alpha_{jk}|\mathcal{V}_j| = 0,\end{aligned}$$

for any $k\in\underline{d}$. Therefore, (L, B) is uncontrollable. ■

We proceed to provide an example to illustrate the results presented in the Theorem 3.

Example 4: Consider the matrix-weighted network in Figure 3. The weight matrices on edges in the network are $A_{16} = A_{25} = A_{14} = A_{23} = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$ and $A_{12} = A_{34} = A_{45} = A_{56} = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$.

Note that the network in Figure 3 has a matrix-valued AEP $\pi = \{\mathcal{V}_1, \mathcal{V}_2\}$ where $\mathcal{V}_1 = \{1, 2\}$ and $\mathcal{V}_2 = \{3, 4, 5, 6\}$. The characteristic matrix $P(\pi) = [P_1, P_2]$ where $P_1 = [\delta_{6,\{1,2\}}]$ and $P_2 = [\delta_{6,\{3,4,5,6\}}]$. Since $\mathbf{gcd}(\pi) = 2$, then $q_1 = 1$, $q_2 = 2$. Choose input matrix $B = [\delta_{6,\{1,3,6\}}]$, which satisfies

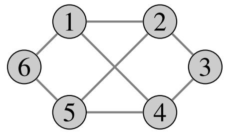

Fig. 3. A 5-node matrix-weighted network that has a matrix-valued AEP.

$\boldsymbol{p}_{11}^{\top}\boldsymbol{b}_1 = \boldsymbol{p}_{12}^{\top}\boldsymbol{b}_2 = 1$ and $\boldsymbol{p}_{21}^{\top}\boldsymbol{b}_1 = \boldsymbol{p}_{22}^{\top}\boldsymbol{b}_2 = 2$. Then $\textbf{rank}(\mathcal{K}(L, B)) = 9$, implying that the (L, B) in this example is uncontrollable, which coincides with Theorem 3.

VII. Conclusion

This letter examines the controllability problem of multi-agent system on matrix-weighed networks. Both lower and upper bounds on the dimension of the controllable subspace–associated with controlled consensus-like dynamics on matrix-weighted networks–is provided from a graph-theoretic perspective. The structure of an uncontrollable input matrix is further investigated. Examples are provided to demonstrate the theoretical results. In our further work, we will examine the graph-theoretic characterizations of lower/upper bound of controllable subspace of matrix-weighted networks allowing both positive (semi-)definite and negative (semi-)definite weight matrices.

References

[1] R. E. Kalman, "Mathematical description of linear dynamical systems," *J. Soc. Ind. Appl. Math. Series A Control*, vol. 1, no. 2, pp. 152–192, 1963.

[2] H. G. Tanner, "On the controllability of nearest neighbor interconnections," in *Proc. 43rd IEEE Conf. Decis. Control*, vol. 3. Nassau, Bahamas, 2004, pp. 2467–2472.

[3] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, "Controllability of multi-agent systems from a graph-theoretic perspective," *SIAM J. Control Optim.*, vol. 48, no. 1, pp. 162–186, 2009.

[4] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, "Controllability of complex networks," *Nature*, vol. 473, no. 7346, p. 167, 2011.

[5] S. Zhang, M. Cao, and M. K. Camlibel, "Upper and lower bounds for controllable subspaces of networks of diffusively coupled agents," *IEEE Trans. Autom. Control*, vol. 59, no. 3, pp. 745–750, Mar. 2014.

[6] C. O. Aguilar and B. Gharesifard, "Almost equitable partitions and new necessary conditions for network controllability," *Automatica*, vol. 80, pp. 25–31, Jun. 2017.

[7] M. Egerstedt, S. Martini, M. Cao, K. Camlibel, and A. Bicchi, "Interacting with networks: How does structure relate to controllability in single-leader, consensus networks?" *IEEE Control Syst. Mag.*, vol. 32, no. 4, pp. 66–73, Aug. 2012.

[8] A. Y. Yazicioglu, W. Abbas, and M. Egerstedt, "A tight lower bound on the controllability of networks with multiple leaders," in *Proc. IEEE 51st IEEE Conf. Decis. Control*, Maui, HI, USA, 2012, pp. 1978–1983.

[9] C. O. Aguilar and B. Gharesifard, "Graph controllability classes for the Laplacian leader-follower dynamics," *IEEE Trans. Autom. Control*, vol. 60, no. 6, pp. 1611–1623, Jun. 2015.

[10] G. Parlangeli and G. Notarstefano, "On the reachability and observability of path and cycle graphs," *IEEE Trans. Autom. Control*, vol. 57, no. 3, pp. 743–748, Mar. 2012.

[11] Z. Ji, H. Lin, and H. Yu, "Leaders in multi-agent controllability under consensus algorithm and tree topology," *Syst. Control Lett.*, vol. 61, no. 9, pp. 918–925, 2012.

[12] A. Chapman, M. Nabi-Abdolyousefi, and M. Mesbahi, "Controllability and observability of network-of-networks via cartesian products," *IEEE Trans. Autom. Control*, vol. 59, no. 10, pp. 2668–2679, Oct. 2014.

[13] G. Notarstefano and G. Parlangeli, "Controllability and observability of grid graphs via reduction and symmetries," *IEEE Trans. Autom. Control*, vol. 58, no. 7, pp. 1719–1731, Jul. 2013.

[14] Y. Hao, Q. Wang, Z. Duan, and G. Chen, "Controllability of Kronecker product networks," *Automatica*, vol. 110, Dec. 2019, Art. no. 108597.

[15] C. Sun, G. Hu, and L. Xie, "Controllability of multiagent networks with antagonistic interactions," *IEEE Trans. Autom. Control*, vol. 62, no. 10, pp. 5457–5462, Oct. 2017.

[16] B. She, S. Mehta, C. Ton, and Z. Kan, "Topological characterizations of leader-follower controllability on signed path and cycle networks," in *Proc. IEEE Conf. Decis. Control*, Miami Beach, FL, USA, 2018, pp. 6157–6162.

[17] Y. Guan and L. Wang, "Controllability of multi-agent systems with directed and weighted signed networks," *Syst. Control Lett.*, vol. 116, pp. 47–55, Jun. 2018.

[18] L. Xiang, F. Chen, W. Ren, and G. Chen, "Advances in network controllability," *IEEE Circuits Syst. Mag.*, vol. 19, no. 2, pp. 8–32, 2nd Quart., 2019.

[19] Z. Sun and C. B. Yu, "Dimensional-invariance principles in coupled dynamical systems: A unified analysis and applications," *IEEE Trans. Autom. Control*, vol. 64, no. 8, pp. 3514–3520, Aug. 2019.

[20] P. Barooah and J. P. Hespanha, "Graph effective resistance and distributed control: Spectral properties and applications," in *Proc. 45th IEEE Conf. Decis. Control*, San Diego, CA, USA, 2006, pp. 3479–3485.

[21] S. E. Tuna, "Observability through matrix-weighted graph," 2016. [Online]. Available: arXiv:1603.07637.

[22] N. E. Friedkin, A. V. Proskurnikov, R. Tempo, and S. E. Parsegov, "Network science on belief system dynamics under logic constraints," *Science*, vol. 354, no. 6310, pp. 321–326, 2016.

[23] M. Ye, M. H. Trinh, Y.-H. Lim, B. Anderson, and H.-S. Ahn, "Continuous-time opinion dynamics on multiple interdependent topics," 2018. [Online]. Available: arXiv:1805.02836.

[24] S. Zhao and D. Zelazo, "Bearing rigidity and almost global bearing-only formation stabilization," *IEEE Trans. Autom. Control*, vol. 61, no. 5, pp. 1255–1268, May 2016.

[25] S. E. Tuna, "Synchronization of small oscillations," *Automatica*, vol. 107, pp. 154–161, Sep. 2019.

[26] M. Kivelä, A. Arenas, M. Barthelemy, J. P. Gleeson, Y. Moreno, and M. A. Porter, "Multilayer networks," *J. Complex Netw.*, vol. 2, no. 3, pp. 203–271, 2014.

[27] M. H. Trinh, C. Van Nguyen, Y.-H. Lim, and H.-S. Ahn, "Matrix-weighted consensus and its applications," *Automatica*, vol. 89, pp. 415–419, Mar. 2018.

[28] S. E. Tuna, "Synchronization under matrix-weighted Laplacian," *Automatica*, vol. 73, pp. 76–81, Nov. 2016.

[29] L. Pan, H. Shao, M. Mesbahi, Y. Xi, and D. Li, "Bipartite consensus on matrix-valued weighted networks," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 66, no. 8, pp. 1441–1445, Aug. 2019.

[30] L. Pan, H. Shao, M. Mesbahi, Y. Xi, and D. Li, "Consensus on matrix-weighted time-varying networks," 2020. [Online]. Available: arXiv:2001.11179.

[31] H. Su, J. Chen, Y. Yang, and Z. Rong, "The bipartite consensus for multi-agent systems with matrix-weight-based signed network," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, early access, doi: 10.1109/TCSII.2019.2943245.

[32] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton, NJ, USA: Princeton Univ. Press, 2010.

[33] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[34] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[35] S.-J. Chung, A. A. Paranjape, P. Dames, S. Shen, and V. Kumar, "A survey on aerial swarm robotics," *IEEE Trans. Robot.*, vol. 34, no. 4, pp. 837–855, Aug. 2018.

[36] S. Zhang, M. K. Camlibel, and M. Cao, "Controllability of diffusively-coupled multi-agent systems with general and distance regular coupling topologies," in *Proc. 50th IEEE Conf. Decis. Control Eur. Control Conf.*, Orlando, FL, USA, 2011, pp. 759–764.