Controllability of Multiagent Networks With Antagonistic Interactions

Chao Sun, Guoqiang Hu, and Lihua Xie

***Abstract*—This paper addresses the controllability of a class of antagonistic multiagent networks with both positive and negative edges. All the agents of the multiagent network run a consensus algorithm using a signed Laplacian. Based on the generalized equitable partition, we propose a graph-theoretic characterization of an upper bound on the controllable subspace. Then, we provide a necessary condition for the controllability of the system and give an algorithm to compute the partition. Furthermore, we prove that for a structurally balanced network, the controllability is equivalent to that of the corresponding all-positive network, if the leaders are chosen from the same vertex set. Several examples are given to illustrate these results.**

***Index Terms*—Antagonistic networks, controllability, networked control systems, signed graphs.**

I. Introduction

Multiagent systems have attracted much attention [1]–[5], as they have wide range of applications in both civilian and military areas, such as robotics [6]–[8], sensor networks [9] , [10], and power systems [11], [12]. One of the major research topics in this field is the controllability of networks [13]–[27]. Most of these works aimed to find the relationship between the graph property and the controllability of networks modeled by the following dynamics

$$\dot{x} = Ax + Bu \tag{1}$$

where $x \in \mathbb{R}^N$ is the state vector, $u \in \mathbb{R}^m$ is the control input vector, $A \in \mathbb{R}^{N\times N}$ is a matrix associated with the topological structure, and $B \in \mathbb{R}^{N\times m}$ is the control matrix. The controllability of networks has been studied from several distinct perspectives. When A is the Laplacian matrix of a graph, necessary or sufficient conditions for controllability were established in [14]–[21] for a general graph using upper and/or lower bounds on the controllable subspace. For special graphs such as paths, cycles, grids and Cartesian products, necessary and sufficient controllability conditions were presented in [22]–[24]. When A is a structured matrix with elements of either zeros or independent free parameters, researchers sought to find if there exists a group of parameters such that (A, B) is controllable (structural controllability, see [25]) or if (A, B) is controllable for all possible parameters in A (strong structural controllability, see [26], [27]).

Research so far on controllability of multiagent networks focuses on full-cooperative networks, i.e., networks with edges all being nonnegative. Recently, networks with antagonistic interactions have attracted many interests. For social networks, it is common to have both positive and negative edges where a positive edge can be viewed as a connection to a friend whereas a negative edge can be viewed as a connection to an enemy [28], [29]. In studies on consensus problems of multiagent systems, there are also some models having both positive and negative edges. For instance, in [30], Altafini modeled an antagonistic multiagent system based on the signed graph. It was shown that in structurally balanced networks, the states of the agents asymptotically converge to a common nonzero value but with opposite signs, while in structurally unbalanced graphs, all the states converge to zero. In [31], Hu and Zheng extended the results of [30] by considering signed graphs with a directed spanning tree. Similar problems have recently been studied in [32]–[35]. Although networks with possibly negative weights can be studied using the structural controllability scheme, the network model depends on the structure of the graph rather than a particular physical realization [36].

In this paper, we study the controllability problem of multiagent networks defined on an undirected signed graph. We consider a similar model as described in (1) with A replaced by a signed Laplacian matrix. Since the row sum of a signed Laplacian matrix may not be equal to zero, many approaches and concepts for the controllability of networks defined on signless graphs are no longer applicable. The main contributions of this work are threefold. First, we propose a graph-theoretic characterization of an upper bound on the controllable subspace through the so-called generalized almost equitable partition (AEP), and provide a necessary condition for the controllability of the network. Second, we study how to obtain the partition for a given graph and a set of leaders. Third, we explore the controllability relationship between a structurally balanced network and the corresponding all-positive network.

The rest of this paper is organized as follows. In Section II, the background of graph theory and some preliminaries are provided. In Section III, the controllability problem of antagonistic networks is formulated mathematically. In Section IV, the main results are presented. In Section V, several examples are provided to illustrate the main results. Finally, conclusions are given in Section VI.

II. Notations and Preliminaries

In this paper, $\mathbb{R}$ stands for the set of real numbers. 0_m and 1_m are m dimensional column vectors composed of 0 and 1, respectively. I_m denotes an $m \times m$ identity matrix and $0_{m\times n}$ denotes an $m \times n$ zero matrix. A space $\mathcal{X}$ is called an upper bound of a space $\mathcal{Y}$ if $\mathcal{Y} \subseteq \mathcal{X}$. The vector space generated by the columns of the matrix $P$is denoted by im(P). For a square matrix R, $R\times$im(P) denotes a vector space $\{Rx : x \in$im$(P)\}$. Given two sets $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{Y} \setminus \mathcal{X}$ denotes the relative complement of $\mathcal{X}$ in $\mathcal{Y}$.

Signed graph [30], [37]: Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \sigma\}$ represent a signed graph, where $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ indicate the set of vertices and edges, respectively, and σ: $\mathcal{E} \to \{+, -\}$ is the mapping of the edges to the signs $\{+, -\}$. An edge is an ordered pair $(i, j) \in \mathcal{E}$ if agent j can be directly supplied with information from agent i. $\mathcal{N}_i =$

Manuscript received March 26, 2016; revised December 31, 2016; accepted April 18, 2017. Date of publication April 24, 2017; date of current version September 25, 2017. This work was supported by Singapore MOE AcRF Tier 1 Grant RG60/12(2012-T1-001-158). Recommended by Associate Editor N. Chopra. *(Corresponding author: Guoqiang Hu.)*

The authors are with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798 (e-mail: csun002@e.ntu.edu.sg; gqhu@ntu.edu.sg; elhxie@ntu.edu.sg).

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Digital Object Identifier 10.1109/TAC.2017.2697202

$\{j \in \mathcal{V} \mid (j,i) \in \mathcal{E}\}$ represents the neighborhood set of vertex i. An $N \times N$ matrix $A = [a_{ij}]$ represents the adjacency matrix of $\mathcal{G}$, where $a_{ij} \in \{0, 1,-1\}$. If $a_{ij} = 1$, agent j is called a positive neighbor of agent i; if $a_{ij} = -1$, agent j is called a negative neighbor of agent i; else $a_{ij} = 0$, agent j is not a neighbor of agent i. We further assume that there is no self-loop in the graph, i.e., $a_{ii} = 0$. If $a_{ij} \geq 0$ for all i and j, the graph is called an all-positive graph. Let $L \triangleq D - A \in \mathbb{R}^{N\times N}$ be the signed Laplacian matrix of $\mathcal{G}$, where $D = [d_i] \in \mathbb{R}^{N\times N}$ is a diagonal matrix with $d_i = \sum_{j=1}^{N} |a_{ij}|$ being the degree of i. Thus, the entries of the matrix L can be written as

$$[L]_{ij} = \begin{cases} \sum_{s=1}^{N} |a_{i,s}| & i = j \\ -a_{i,j} & i \neq j \end{cases}. \tag{2}$$

Graph partition: For an undirected graph $\mathcal{G}$ with the vertex set $\mathcal{V}$, a subset V of $\mathcal{V}$ is called a class (also called a cell [19], [21]). If a class contains only one node, it is called a singleton class. If a class contains more than one node, it is called a nonsingleton class. A class V_1 is called a subclass of class V_2 if for any node $i \in V_1$, $i \in V_2$. A collection of classes $\pi = \{V_1, V_2, \ldots, V_k\}$ is called a partition if $V_i \cap V_j = \varnothing$ for all $i \neq j$ and $\cup_i V_i = \mathcal{V}$. The $N \times k$ matrix $P(\pi) = [P_{ij}]$ is called the characteristic matrix of π, where

$$P_{ij} \triangleq \begin{cases} 1 & i \in V_j \\ 0 & i \notin V_j \end{cases}.$$

Example 1: For a partition $\pi = \{\{1, 2\}, \{3, 4, 5\}\}$, its characteristic matrix $P(\pi) \in \mathbb{R}^{5\times 2}$ can be written as

$$P(\pi) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

III. System Description

Consider a multiagent network with node set $\mathcal{V} = \{1, \ldots, N\}$. We suppose that m ($m \leq N$) agents in the network are selected as leaders and each leader is assigned with a control input. The other nodes are followers. Without loss of generality, assume that the first m agents $1, \ldots, m$ correspond to the leaders. Let $\mathcal{V}_m \triangleq \{1, \ldots, m\}$ and $\mathcal{V}_F \triangleq \mathcal{V} \backslash \mathcal{V}_m$ be the sets of leaders and followers, respectively.

Each follower $i \in \mathcal{V}_F$ is governed by the updating law

$$\dot{x}_i = -d_i x_i + \sum_{j=1}^{N} a_{ij} x_j \tag{3}$$

where $x_i \in \mathbb{R}$ is the state of each agent, $d_i = \sum_{j=1}^{N} |a_{ij}|$ represents the degree of agent i, and a_{ij} is defined in Section II.

Each leader $i \in \mathcal{V}_m$ is governed by the updating law

$$\dot{x}_i = -d_i x_i + \sum_{j=1}^{N} a_{ij} x_j + u_i \tag{4}$$

where $u_i \in \mathbb{R}$ is the control input.

Remark 1: Networks with the form (3) were studied in the bipartite consensus problems in [30]–[35], where a node sends the opposite of its true state to its antagonistic neighbors. A similar model was utilized in [38] to model the process of opinion forming. State x_i represents the opinion of the individual i. Positive edges represent cooperative interactions (individuals transmit their "true" opinions), whereas negative edges represent antagonistic interactions (individuals transmit their "false" opinions). Each individual updates its opinion based on the interactions with neighbors and finally forms its own opinion. In addition, some social networks are well known for containing both positive and negative links, such as the Slashdot Zoo [1] [39].

Let $x = [x_1, \ldots, x_N]^T \in \mathbb{R}^N$ be the aggregated state vector and $u = [u_1, \ldots, u_m]^T \in \mathbb{R}^m$ denote the control input vector. The dynamics of the antagonistic network can be rewritten into a concatenated form as

$$\dot{x} = -Lx + Mu \tag{5}$$

where L is the signed Laplacian matrix defined in (2), and M is an $N \times m$ matrix satisfying $M = [\begin{smallmatrix} I_m \\ 0_{(N-m)\times m} \end{smallmatrix}]$.

IV. Controllability of Antagonistic Networks

The objective of this section is to investigate how to characterize the controllability of an antagonistic network. According to Kalman's controllability condition, (5) is controllable from the controlled nodes if and only if the matrix $[M \; LM \; L^2 M \cdots L^{N-1} M]$ has full rank. However, besides the algebraic methods, graphic characterization methods are highly demanded because they can avoid complicated matrix computations.

A. Upper Bound on the Controllable Subspace

In this section, we give a graph-theoretic characterization of an upper bound on the controllable subspace for system (5). It is well known that the AEP can be used to determine upper bounds on the controllable subspace when the network is all-positive [16], [21]. A partition $\pi = \{V_1, \ldots, V_k\}$ is called an AEP if for every pair of distinct i,$j \in \{1, \ldots, k\}$, there exists a nonnegative number d_{ij} such that any vertex in V_i has d_{ij} neighbors in V_j. However, this partition does not work for a signed graph due to the existence of negative weights, which motivates us to give a generalized definition.

Definition 1: 1) $\pi = \{V_1, \ldots, V_k\}$ is called a generalized AEP (GAEP) if for every pair of distinct i, $j \in \{1, \ldots, k\}$, there exists a nonnegative number d_{ij+} such that any vertex in V_i has d_{ij+} positive neighbors in V_j and for every pair of $i, j \in \{1, \ldots, k\}$ (not necessarily distinct), there exists a nonnegative number d_{ij-} such that any vertex in V_i has d_{ij-} negative neighbors in V_j.

2) A partition π_L is said to be a leader-isolated GAEP if π_L is a GAEP and each leader is in a singleton class.

Definition 2: A partition π_L^* is called the coarsest leader-isolated GAEP if for any leader-isolated GAEP π_L, each class in π_L is a subclass of some class in π_L^*.

An example of the coarsest leader-isolated GAEP is given in Fig. 1. In the following, we prove the existence and uniqueness of it.

Lemma 1: For each signed graph $\mathcal{G}$, there exists a unique coarsest leader-isolated GAEP.

Proof: If the graph has only one leader-isolated GAEP, then it has to be the coarsest leader-isolated GAEP. Suppose that the graph has more than one leader-isolated GAEP and the coarsest leader-isolated GAEP does not exist. Then, there exist at least two leader-isolated GAEPs π_{L1}^* and π_{L2}^* such that 1) each class in each leader-isolated GAEP π_L is a subclass of some class in π_{L1}^* or π_{L2}^*; 2) not the all classes in π_{L1}^* (π_{L2}^*) are subclasses of the classes in π_{L2}^* (π_{L1}^*).

[1] http://slashdot.org/

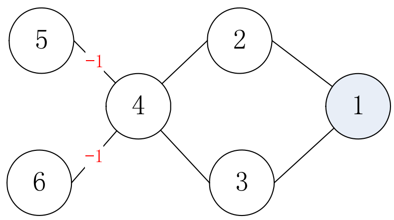

Fig. 1. Signed graph with a single leader namely 1. The coarsest leader-isolated GAEP is $\pi_L^* = \{\{1\}, \{2, 3\}, \{4\}, \{5, 6\}\}$.

Denote $\pi_{L1}^* = \{V_1, V_2, \ldots, V_{l_1}, \bar{\pi}_L\}$ and $\pi_{L2}^* = \{V_1', V_2', \ldots, V_{l_2}', \bar{\pi}_L\}$, where $\bar{\pi}_L$ is the common part of π_{L1}^* and π_{L2}^* and l_1 and l_2 are two positive integers. Since each leader is in a singleton class, $\bar{\pi}_L$ is nonempty. According to Definition 1, each node in $V_1, V_2, \ldots, V_{l_1}$, as well as in $V_1', V_2', \ldots, V_{l_2}'$, has the same number of positive and negative neighbors in $\bar{\pi}_L$. Then, there exist at least two classes in π_{L1}^*, as well as in π_{L2}^*, that can be grouped into one class such that the newly generated partition has a class which is the union of some classes in π_{L1}^* (or π_{L2}^*). This contradicts to condition 1).

By Definition 2, if the coarsest leader-isolated GAEP exists, then it must be unique. ∎

For the system (5), the controllable subspace K can be written as

$$K = \mathrm{im}(M) + L \times \mathrm{im}(M) + \cdots + L^{N-1} \times \mathrm{im}(M)$$

which is the smallest L-invariant subspace that contains $\mathrm{im}(M)$ [40], [41]. Here, the operator "+" represents the union of two spaces.

Let $\tilde{D}_+$ and $\tilde{D}_-$ be diagonal matrices with the nonnegative diagonal elements $\tilde{d}_{i+}$ and $\tilde{d}_{i-}$ representing the number of positive and negative neighbors of agent i, respectively. Define $n_{i,j} = d_{ij+} - d_{ij-}$ if $i \neq j$, $i, j \in \{1, \ldots, k\}$, and $n_{i,i} = 0$, where d_{ij+} and d_{ij-} are defined in Definition 1. Next, we prove that $\mathrm{im}(P_\pi)$ is L -invariant.

Lemma 2: Let $\pi = \{V_1, \ldots, V_k\}$ be a GAEP of the graph $\mathcal{G}$ and P_π be the characteristic matrix of π. Then, there exists a matrix L_π such that the signed Laplacian L satisfies $(L - 2\tilde{D}_-)P_\pi = P_\pi L_\pi$. Moreover, $\mathrm{im}(P_\pi)$ is L-invariant.

Proof: Define a matrix $L_\pi \in \mathbb{R}^{k\times k}$ as

$$[L_\pi]_{ij} = \begin{cases} \sum_{s=1}^{k} n_{i,s} & i = j \\ -n_{i,j} & i \neq j \end{cases}.$$

Without loss of generality, we assume that the nodes belonging to V_1 are indexed by $\{n_0 + 1, n_0 + 2, \ldots, n_0 + |V_1|\}$, where n_0 is some nonnegative integer. Thus, the matrix $P_\pi \in \mathbb{R}^{N\times k}$ can be written as

$$P_\pi = \begin{bmatrix} 0_{n_0} & * \\ 1_{|V_1|} & 0_{|V_1|\times(k-1)} \\ 0_{N-n_0-|V_1|} & * \end{bmatrix} \tag{6}$$

where $|V_1|$ denotes the cardinality of V_1.

Assuming that node p belongs to V_1, the pth row of $P_\pi L_\pi$ can be written as

$$\begin{aligned} \mathrm{row}_p(P_\pi L_\pi) &= \mathrm{row}_1(L_\pi) \\ &= \left[\sum_{s=1}^{k} n_{1,s}, \ -n_{1,2}, \ldots, \ -n_{1,k}\right]. \end{aligned}$$

The pth row of $(L - 2\tilde{D}_-)$ is

$$\begin{aligned} &\mathrm{row}_p(L - 2\tilde{D}_-) \\ &= [-a_{p1}, \ldots, -a_{p(p-1)}, \ (\tilde{d}_{p+} - \tilde{d}_{p-}), \\ &\quad -a_{p(p+1)}, \ldots, -a_{pN}]. \end{aligned}$$

According to (6) and the definition of a_{ij} in Section II, the pth row of $(L - 2\tilde{D}_-)P_\pi$ can be written as

$$\begin{aligned} &\mathrm{row}_p((L - 2\tilde{D}_-)P_\pi) \\ &= [(\tilde{d}_{p+} - \tilde{d}_{p-} - (\tilde{d}_{p1+} - \tilde{d}_{p1-})), \\ &\quad -(d_{12+} - d_{12-}), \ldots, -(d_{1k+} - d_{1k-})] \end{aligned}$$

where $\tilde{d}_{p1+}$ denotes the number of positive neighbors of node p in V_1, whereas $\tilde{d}_{p1-}$ denotes its number of negative neighbors.

Since $\tilde{d}_{p+} - \tilde{d}_{p-} - (\tilde{d}_{p1+} - \tilde{d}_{p1-}) = \sum_{s=1}^{k} n_{1,s}$, it gives $\mathrm{row}_p((L - 2\tilde{D}_-)P_\pi) = \mathrm{row}_p(P_\pi L_\pi)$. Using similar analysis to all the other nodes yields $(L - 2\tilde{D}_-)P_\pi = P_\pi L_\pi$.

According to the definition of the GAEP, the nodes in class V_i, $i \in \{1, \ldots, k\}$ have the same number of negative neighbors, denoted by κ_i. Thus, $\tilde{d}_{p1-} = \kappa_1$ and the first column of $\tilde{D}_- P_\pi$ can be written as

$$\mathrm{column}_1(\tilde{D}_- P_\pi) = \begin{bmatrix} 0_{n_0} \\ \kappa_1 * 1_{|V_1|} \\ 0_{N-n_0-|V_1|} \end{bmatrix}.$$

Let $D_\pi \in \mathbb{R}^{k\times k}$ be a diagonal matrix defined as

$$[D_\pi]_{ij} = \begin{cases} 0 & i \neq j \\ \kappa_i & i = j \end{cases}. \tag{7}$$

Thus, $\mathrm{column}_1(\tilde{D}_- P_\pi) = \mathrm{column}_1(P_\pi D_\pi)$. Furthermore, $\tilde{D}_- P_\pi = P_\pi D_\pi$ and $LP_\pi = P_\pi(2D_\pi + L_\pi)$. For a matrix E, the vector space generated by columns of E is L-invariant if and only if there exists a matrix B such that $LE = EB$ [42]. Therefore, $\mathrm{im}(P_\pi)$ is L -invariant. ∎

Based on Lemma 2, the following result can be obtained.

Lemma 3: For any leader-isolated GAEP π_L, the controllable subspace K satisfies $K \subseteq \mathrm{im}(P_{\pi_L})$.

Proof: From (5), it can be easily obtained that each column of the matrix M is a column of P_{π_L}, which indicates that $\mathrm{im}(M) \subseteq \mathrm{im}(P_{\pi_L})$. Since $\mathrm{im}(P_{\pi_L})$ is L -invariant, we have

$$\begin{aligned} K &= \mathrm{im}(M) + L \times \mathrm{im}(M) + \cdots + L^{N-1} \times \mathrm{im}(M) \\ &\subseteq \mathrm{im}(P_{\pi_L}) + L \times \mathrm{im}(P_{\pi_L}) + \cdots + L^{N-1} \times \mathrm{im}(P_{\pi_L}) \\ &= \mathrm{im}(P_{\pi_L}). \end{aligned} \tag{8}$$

∎

Lemma 3 provides an upper bound on the controllable subspace based on the leader-isolated GAEP. In the following, we utilize the coarsest leader-isolated GAEP to get a tighter upper bound.

Theorem 1: The controllable subspace K of the system (5) satisfies $K \subseteq \mathrm{im}(P_{\pi_L^*})$, where π_L^* is the coarsest leader-isolated GAEP.

Proof: According to Lemma 3, $K \subseteq \mathrm{im}(P_{\pi_L^*})$ and $K \subseteq \mathrm{im}(P_{\pi_L})$. Since each class in π_L is a subclass of some class in π_L^*, it can be obtained that $\mathrm{im}(P_{\pi_L^*}) \subseteq \mathrm{im}(P_{\pi_L})$. Thus, $K \subseteq \mathrm{im}(P_{\pi_L^*}) \subseteq \mathrm{im}(P_{\pi_L})$. ∎

Remark 2: Theorem 1 applies when the network has both positive and negative edges. It extends the results in [21] which dealt with an all-positive network.

Based on Theorem 1, we further get the following necessary condition for the controllability of the system (5).

Proposition 1: If the system (5) is controllable, then each class in the coarsest leader-isolated GAEP π_L^* is a singleton class.

Proof: If a nonsingleton class in the coarsest leader-isolated GAEP exists, then the dimension of $\mathrm{im}(P_{\pi_L^*})$ is less than N. Then, based on Theorem 1, the dimension of the controllable subspace K is less than N. This contradicts to the supposition that the system is controllable. ■

B. Algorithm to Compute π_L^*

The computation for the upper bound in Theorem 1 requires the coarsest leader-isolated GAEP π_L^*. In the following, we propose an algorithm to obtain π_L^*, which is motivated by the algorithms presented in [20] and [43] to find the coarsest leader-isolated AEP. The algorithm is described as follows.

Step 1: Let $\pi_{L0} = \{\{1\}, \{2\}, \ldots, \{m\}, \mathcal{V}_F\}$ be the initial partition.

Step 2: Relabel the classes in the current partition by $V_1, \ldots, V_k$, $k > 1$ and select an arbitrary non-singleton class, e.g., V_i, $1 \leq i \leq k$. Then, for each node in V_i, calculate the number of its positive neighbors in $V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_k$ and the number of its negative neighbors in $V_1, \ldots, V_k$. Nodes with the same neighbor sequence are grouped into one class. Replace the old class with the newly created classes.

Step 3: Repeat Step 2 until no class can be split.

Theorem 2: For a signed graph $\mathcal{G}$, the partition obtained via Steps 1–3 is the coarsest leader-isolated GAEP.

Proof: Based on Definition 1, it can be seen that a leader-isolated GAEP can be obtained by Steps 1–3. In the following, we will show that the obtained leader-isolated GAEP π_L^* is the coarsest leader-isolated GAEP of the graph. Suppose that another partition $\pi_{Lr}^* \neq \pi_L^*$ is the coarsest leader-isolated GAEP. Then, each class in π_L^* is a subclass of some class in π_{Lr}^*. Suppose that two nodes p, q, located in two distinct classes V_1, V_2 in π_L^*, belong to the same class V_{1r} in π_{Lr}^*, and they are separated from a class V_0 in a certain iteration. According to Step 2, there exist three cases such that p, q in V_0 are separated: 1) p, q have different numbers of positive neighbors in V_0; 2) p, q have different numbers of positive and/or negative neighbors in another class, e.g., V_3; 3) both 1) and 2) hold. For case 1), V_0 is neither a class in π_{Lr}^* nor a union of classes in π_{Lr}^*. Otherwise, p, q have the same number of positive neighbors in V_0. Then, in this iteration, there must exist another two nodes located in two distinct classes V_0, V_4, which are in the same class in π_{Lr}^*. For case 2), V_3 is neither a class in π_{Lr}^* nor a union of classes in π_{Lr}^*. Otherwise, p, q cannot be separated. Thus, there must exist another two nodes located in two distinct classes V_3, V_5, which are in the same class in π_{Lr}^*. The analysis of case 3) is similar to case 1). Overall, if p, q can be separated, there must exist another two nodes p', q' that belong to two distinct classes in π_L^* and the same class in π_{Lr}^*. Applying this logic iteratively and noticing that there are no such two nodes in the initial partition, we can get the conclusion. ■

Example 2: In order to illustrate the algorithm, we consider a signed network shown in Fig. 2 (a) with agent 1 chosen as the leader. The initial partition is chosen as $\pi_{L0} = \{\{1\}, \{2, 3, 4, 5, 6, 7\}\}$. Fig. 2(b)–(d) shows the process of finding the coarsest leader-isolated GAEP, where nodes in the same class are denoted by the same color. The coarsest leader-isolated GAEP is $\pi_L^* = \{\{1\}, \{2, 3\}, \{4\}, \{5, 6, 7\}\}$.

C. Controllability of Structurally Balanced Signed Networks

Due to the difference between the signed Laplacian and the signless Laplacian, the controllability of an antagonistic network is not necessarily equivalent to the controllability of its corresponding all-positive network (see the counterexample in Example 3). In this section, we use the structural balance theory to explore the relationship between them.

Structural balance is a basic concept in social network analysis. In a structurally balanced graph, the vertex set $\mathcal{V}$ can be partitioned into two disjoint subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that every edge between $\mathcal{V}_1$ and $\mathcal{V}_2$ is negative and every edge within $\mathcal{V}_1$ or $\mathcal{V}_2$ is positive [37]. A connected and structurally balanced graph has the following property.

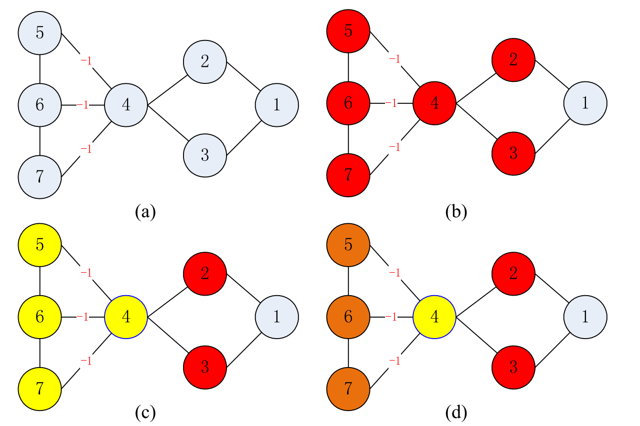

Fig. 2. (a) Network with seven nodes, (b) $\pi_{L0} = \{\{1\}, \{2, 3, 4, 5, 6, 7\}\}$, (c) $\pi_{L1} = \{\{1\}, \{2, 3\}, \{4, 5, 6, 7\}\}$, (d) $\pi_L^* = \{\{1\}, \{2, 3\}, \{4\}, \{5, 6, 7\}\}$.

Lemma 4 (see [30]): A connected signed graph $\mathcal{G}(A)$ is structurally balanced if and only if any of the following equivalent conditions holds:

1) all cycles of $\mathcal{G}(A)$ are positive;
2) let $E = \mathrm{diag}(\sigma)$ with σ satisfying $\sigma_i = 1$ if $v_i \in \mathcal{V}_1$ and $\sigma_i = -1$ if $v_i \in \mathcal{V}_2$; then EAE (and thus ELE) has all off-diagonal entries nonnegative;
3) 0 is an eigenvalue of L.

Denote $L_E = ELE$. The entries of L_E satisfy

$$[L_E]_{ij} = \begin{cases} \sum_{s=1}^{N} |a_{i,s}| & i = j \\ -|a_{i,j}| & i \neq j \end{cases}.$$

Thus, L_E equals to the Laplacian of the network formed by replacing $a_{i,j}$ with $|a_{i,j}|$.

Theorem 3: Suppose that the interconnected antagonistic network in (5) is structurally balanced. If the leaders are chosen from the same subset, i.e., $\mathcal{V}_L \subseteq \mathcal{V}_1$ or $\mathcal{V}_L \subseteq \mathcal{V}_2$, then the controllability of (L, M) is equivalent to that of (L_E, M).

Proof: According to the definition of L_E and the fact that $E = E^{-1}$, we have

$$\begin{aligned} &\mathrm{rank}\{[M \; L_E M \; L_E^2 M \; \cdots \; L_E^{N-1} M]\} \\ &= \mathrm{rank}\{[M \; ELEM \; EL^2EM \; \cdots \; EL^{N-1}EM]\}. \end{aligned} \tag{9}$$

If $\mathcal{V}_L \subseteq \mathcal{V}_1$, $EM = M$. If $\mathcal{V}_L \subseteq \mathcal{V}_2$, $EM = -M$. It follows from (9) that

$$\begin{aligned} &\mathrm{rank}\{[M \; ELEM \; EL^2EM \; \cdots \; EL^{N-1}EM]\} \\ &= \mathrm{rank}\{E[EM \; LEM \; L^2EM \; \cdots \; L^{N-1}EM]\} \\ &= \mathrm{rank}\{\pm E[M \; LM \; L^2M \; \cdots \; L^{N-1}M]\} \\ &= \mathrm{rank}\{[M \; LM \; L^2M \; \cdots \; L^{N-1}M]\}. \end{aligned}$$

The proof is, thus, completed. ■

Remark 3: Suppose that the interconnected antagonistic network in (5) is structurally balanced and only one agent is chosen as the leader, then the system (L, M) is controllable if and only if the corresponding (L_E, M) is controllable. For tree networks with only one leader, the conclusion always holds since any acyclic signed graph is structurally balanced [30].

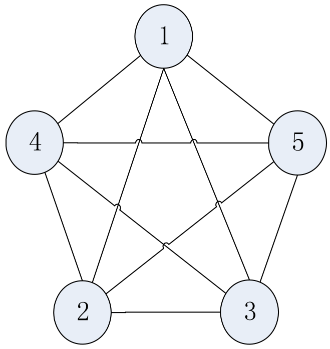

Fig. 3. All-positive network is uncontrollable when agents 3, 4, 5 are chosen as leaders.

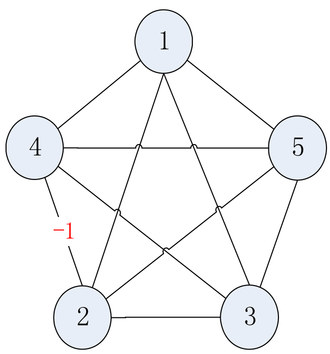

Fig. 4. If $a_{24} = a_{42} = -1$, the network becomes controllable.

Remark 4: In Theorem 3, it is proven that the controllability of a structurally balanced network is equivalent to the controllability of its corresponding all-positive network, if the leaders are chosen from the same subset. In this case, the controllability can be checked using graphic methods for all-positive networks [14]–[20] by treating negative edges as positive ones.

V. Simulation

Example 3: Consider the all-positive multiagent network shown in Fig. 3 .

If the agents 1, 2, and 3 are chosen as leaders, the Laplacian matrix L and the control matrix M can be written as follows

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix} \quad M = \begin{bmatrix} 1\,0 & & \\ 0\,1 & 0 & 0 \\ 0\,0 & 1 & 0 \\ 0\,0 & 0 & 0 \\ 0\,0 & 0 & 0 \end{bmatrix} . \tag{10}$$

The rank of the controllability matrix of system (10) is 4. Thus, the system is uncontrollable.

Replacing the weight between agent 2 and agent 4 by -1, we obtain a new network shown in Fig. 4, where

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & 1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & 1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{11}$$

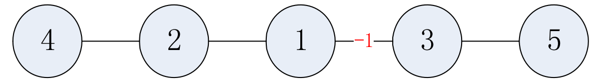

Fig. 5. Path graph network with five agents.

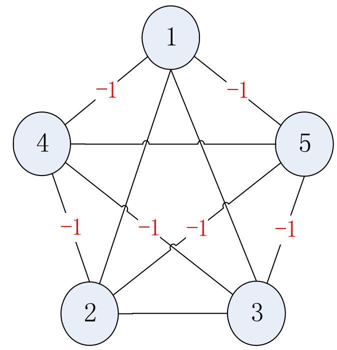

Fig. 6. Structurally balanced network with five agents.

Through a direct calculation of the rank of the controllability matrix, we can conclude that system (11) is controllable. This example shows that the controllability of an antagonistic network is not always equivalent to the controllability of the corresponding all-positive network.

Example 4: Consider the multiagent network shown in Fig. 5 .

If agent 1 is chosen as the leader, then the Laplacian matrix L and the control matrix M are

$$L = \begin{bmatrix} 2 & -1 & 1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \quad M = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

The rank of the controllability matrix of the system is 3 and, thus, the system is uncontrollable.

If the agents 1 and 2 are chosen as leaders, then the Laplacian matrix L and the control matrix M are

$$L = \begin{bmatrix} 2 & -1 & 1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

The dimension of the controllable subspace is 5 and, thus, the system is controllable.

Using the algorithm presented in Section IV-B, we get that the partition $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$ is the coarsest leader-isolated GAEP and each class in this partition is a singleton class. Thus, we can conclude that the condition in Proposition 1 is not sufficient for the controllability of system (5).

Example 5: Consider the multiagent network shown in Fig. 6 .

The network is structurally balanced since it can be partitioned into two subsets $\{1, 2, 3\}$ and $\{4, 5\}$. The network shown in Fig. 3 is the corresponding all-positive network. If we choose the agents 1, 2 and 3 as leaders, the Laplacian matrix and the control matrix are

$$L = \begin{bmatrix} 4 & -1 & -1 & 1 & 1 \\ -1 & 4 & -1 & 1 & 1 \\ -1 & -1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 4 & -1 \\ 1 & 1 & 1 & -1 & 4 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (12)$$

The rank of the controllability matrix is 4, which is the same as that of system (10). This validates Theorem 3.

VI. Conclusion

In this paper, we study the controllability problem for multiagent networks described by signed graphs with both positive and negative weights. We provide a graph-theoretic characterization method for an upper bound on the controllable subspace. Based on this upper bound, we present a necessary condition for the controllability of the network. Furthermore, we study the relationship between the controllability of a structurally balanced signed graph network and the controllability of the corresponding all-positive network. Studying the effects of adding negative weights to some special graphs will be our future work.

Acknowledgment

The authors would like to thank the reviewers and the Associate Editor for their valuable comments.

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