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Leader Group Selection for Energy-Related Controllability of Signed Acyclic Graphs

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***Abstract*—A leader group selection approach that jointly considers network controllability and control energy is investigated in this work. Specifically, a dynamic multi-agent system with signed acyclic topology is considered, where signed edges capture cooperative and competitive interactions among agents. To effectively and efficiently control the multi-agent system, leader group selection in this work focuses on energy-related controllability, which jointly considers two primary objectives: 1) network controllability, i.e., identification of a subset of agents as leaders that can drive the entire network to a desired state even in the presence of antagonistic interactions, and 2) energy efficiency, which takes into account the control cost incurred by the selected leaders in steering the network to the desired state. To achieve these objectives, graph-inspired characterizations of energy-related controllability are developed based on the interaction between the network topology and the agent dynamics. The developed topological characterizations are exploited to derive heuristic leader selection algorithms on signed acyclic graphs. Illustrative examples are provided to demonstrate the effectiveness of the developed leader group selection methods.**

I. INTRODUCTION

Controllability, originally considered for dynamical systems, has recently seen renewed interest in the context of networked systems with applications in robotic networks, brain networks, social networks, and power grids. One popular approach is to cast the network controllability problem into a leader-follower framework, wherein the leaders dictate the overall behavior of the network by influencing the followers via the connectivity characteristics of the network. Identifying the leaders that can render the network controllable has generated substantial research interests [1]–[6]. However, along with controllability, an important aspect, from the practical applicability standpoint, that needs to be considered is the energy needed to control the network. For instance, a network might be controllable by a selected leader group but the control energy required by the leaders might be infeasible to allocate in practice. Therefore, this work is particularly motivated to investigate energy-related controllability in leader-follower networks to identify a group of leaders considering network controllability as well as control energy.

Unsigned graphs that admit exclusively non-negative edge weights are generally used to represent networks with cooperative relationships, such as average consensus [7], formation control [8], and cooperative timing [9]. To investigate control energy in unsigned graphs, controllability Gramian based metrics, such as the worst case control energy, average control energy, average controllability, and volumetric control energy, have been extensively explored in the works of [10]–[13] to provide quantitative measures of energy required in network control. Based on these energy-related metrics, the leader group identification problem was investigated in [14]–[17]. In [14], a projected gradient method was developed to determine the key nodes with minimum control cost in complex networks. The relationship between network structure and minimum energy required to control networked systems was studied in [15] to facilitate the identification of the leader group. In [16], controllability Gramian based metrics were employed to improve system performance via the design of system dynamics and network topology. Despite substantial progress, previous research mainly focused on unsigned cooperative graphs. Many real life networks, on the other hand, exhibit competitive interactions in addition to cooperative behaviors. For instance, social networks [18] are often characterized by amiable and adversary relationships, and multiagent systems [19] can also admit collaborative and competitive interactions. In contrast to unsigned graphs, signed graphs that admit positive and negative edge weights are suited to represent networks with cooperative and competitive interactions [20]. However, the controllability of signed graphs, especially considering control energy expenditure, remains largely unattended in the literature.

The presented work jointly investigates network controllability and control energy for leader group selection in networked systems. In view of the aforementioned practical motivation, a dynamic multi-agent system with signed acyclic topology is considered, where signed edges capture potential cooperative and competitive interactions among agents. The leader group selection is based on two primary considerations: 1) network controllability, i.e., identification of a small subset of agents as leaders such that the selected leaders are able to drive the network to a desired state even in the presence of antagonistic interactions, and 2) control energy required to steer the network to the desired state by the selected leaders. To this end, we consider an augmented

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controllability metric, referred to as energy-related controllability, which considers the classical network controllability along with the notion of average controllability (i.e., controllability Gramian based control energy measures). Graph inspired characterizations of the energy-related controllability are developed based on the interaction between the network topology and the agent dynamics. The developed topological characterizations are then exploited to derive leader selection algorithms on signed acyclic graphs. Illustrative examples are provided to demonstrate the effectiveness of the developed leader group selection methods.

The contributions of this work are multi-fold. First, in contrast to most existing approaches that focus on unsigned graphs, this work characterizes network controllability and control energy on signed networks, where classical methods are no longer applicable due to the presence of cooperative and competitive interactions. Second, graph-inspired understandings of the relationship between leader roles and network topology are developed to facilitate the development of leader selection rules that jointly consider network controllability and control energy. To the best of our knowledge, the existing results do not consider energy-related controllability for leader selection on signed graphs. In addition, constructive examples are provided to illustrate how the developed leader group selection algorithms can be applied to general signed acyclic graphs.

II. Problem Formulation

Consider a multi-agent system represented by an undirected signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where the node set $\mathcal{V} = \{v_1, \ldots, v_n\}$ and the edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ represent the agents and their interactions, respectively. The network-wide interactions are captured by the adjacency matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$, where $a_{ij} \neq 0$ if $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. No self-loop is considered, i.e., $a_{ii} = 0$ $\forall i = 1, \ldots, n$. Without loss of generality, it is assumed that $a_{ij} : \mathcal{E} \to \{\pm 1\}$, where $a_{ij} = 1$ if v_i and v_j are positive neighbors with cooperative interactions and $a_{ij} = -1$ if v_i and v_j are negative neighbors with antagonistic interactions. A path of length $k-1$ in $\mathcal{G}$ is a concatenation of distinct edges $\{(v_1, v_2), (v_2, v_3), \cdots, (v_{k-1}, v_k)\} \subset \mathcal{E}$. A cycle is a path with identical starting and end node, i.e, $v_1 = v_k$. In this work, we focus on a particular class of graphs, namely acyclic graphs, which does not have cycles. Tree or path graphs are typical examples of acyclic graphs.

Graph $\mathcal{G}$ is connected if there exists a path between any pair of nodes in $\mathcal{V}$. The neighbor set of v_i is defined as $\mathcal{N}_i = \{v_j | (v_i, v_j) \in \mathcal{E}\}$, and the degree of v_i, denoted as $d_i \in \mathbb{Z}^+$, is defined as the number of its neighbors, i.e., $d_i = |\mathcal{N}_i|$, where $|\mathcal{N}_i|$ denotes the cardinality of $\mathcal{N}_i$. The signed graph Laplacian of $\mathcal{G}$ is defined as $\mathcal{L}(\mathcal{G}) \triangleq \mathcal{D} - \mathcal{A}$, where $\mathcal{D} \triangleq \operatorname{diag}\{d_1, \ldots, d_n\}$ is a diagonal matrix.

Let $\boldsymbol{x}(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ denote the stacked system states, where each entry $x_i(t) \in \mathbb{R}$ represents the state of node v_i. Suppose the system states evolve according to the following Laplacian dynamics:

$$\dot{\boldsymbol{x}}(t) = -\mathcal{L}(\mathcal{G})\,\boldsymbol{x}(t), \tag{1}$$

where the graph Laplacian $\mathcal{L}(\mathcal{G})$ indicates that each node v_i updates its state x_i taking into account the states of its neighboring nodes, i.e., $x_j \in \mathcal{N}_i$. It is assumed that a set $\mathcal{K} = \{k_1, \ldots, k_m\} \subseteq \mathcal{V}$ of nodes, referred as leaders in the multi-agent system, can be endowed with external controls. With external inputs, the system dynamics in (1) can be rewritten as

$$\dot{\boldsymbol{x}}(t) = -\mathcal{L}(\mathcal{G})\,\boldsymbol{x}(t) + B\boldsymbol{u}(t), \tag{2}$$

where $B = \begin{bmatrix} e_{k_1} & \cdots & e_{k_m} \end{bmatrix} \in \mathbb{R}^{n\times m}$ is the input matrix with basis vectors e_i, $i = k_1, \ldots, k_m$, indicating that the ith node is endowed with external controls $u(t) \in \mathbb{R}^m$. The dynamics of (2) indicate that the network behavior is not only driven by the graph Laplacian $\mathcal{L}$, but also depends on the input matrix B via the leader set $\mathcal{K}$. Different leader sets can result in different B, leading to drastic differences in the capability of controlling a network, which is elucidated by introducing the definition of leader-follower controllability.

Definition 1 (Leader-Follower Controllability)**.** Provided that the leaders are endowed with an exogenous input $\boldsymbol{u}(t)$, a leader-follower network with dynamics in (2) is controllable, if the system state $\boldsymbol{x}(t)$ can be driven to any target state by a proper design of $\boldsymbol{u}(t)$.

The leader-follower controllability in Def. 1 indicates that a network can be controllable in theory with appropriate selection of leader nodes. However, it does not tell how difficult it is to control the network in practice, i.e., how much energy is needed to drive the network to the target state. To provide energy-related quantification of network control, the total control energy over the time interval $[0, t]$ is given by $E(t) = \int_0^t \|\boldsymbol{u}(\tau)\|^2 d\tau$. Assuming the initial state $\boldsymbol{x}(0) = \mathbf{0}$[1] and the optimal control $\boldsymbol{u}(t)$ in [21], the minimum control energy required to drive the system of (2) from $\boldsymbol{x}(0)$ to a desired target state $\boldsymbol{x}_t$ is

$$E(t) = \boldsymbol{x}_t^T W^{-1}(t)\,\boldsymbol{x}_t, \tag{3}$$

where

$$W(t) = \int_0^t e^{-\mathcal{L}\tau} BB^T e^{-\mathcal{L}^T\tau} d\tau, \tag{4}$$

is the controllability Gramian at time t, which is positive definite if and only if the system of (2) is leader-follower controllable. In this work, we will focus on the infinite-horizon Gramian, i.e., the case when $t \to \infty$ in (3), due to the consideration of asymptotic or exponential convergence/stability of dynamics systems.

Since the controllability Gramian W provides an energy-related measure of network control in (3), various energy

[1]For $\boldsymbol{x}(0) \neq \mathbf{0}$, the minimum control energy required to drive the system of (2) from $\boldsymbol{x}(0)$ to a desired target state $\boldsymbol{x}_t$ is $E(t) = \left(\boldsymbol{x}_t - e^{-\mathcal{L}t}\boldsymbol{x}(0)\right)^T W^{-1}(t)\left(\boldsymbol{x}_t - e^{-\mathcal{L}t}\boldsymbol{x}(0)\right)$.

metrics were developed based on W [10]–[13]. In this work, a particular energy metric, namely average controllability [11], is considered, which is defined as the trace of the controllability Gramian $\operatorname{tr}(W)$. If the multi-agent network (2) is uncontrollable in certain direction in its state space, then the eigenvalue of W corresponding to that direction will be zero, resulting in infinite control energy in that direction due to the inverse W^{-1} in (3). Likewise, if the system (2) is controllable in a certain direction but with a small eigenvalue, then more energy will be expanded for network control in that direction due to (3). Consequently, as a sum of the eigenvalues of W, $\operatorname{tr}(W)$ characterizes the average difficulty of network control in all directions and is inversely related to the energy required in network control by (3). Although various methods have been developed to maximize $\operatorname{tr}(W)$ to improve energy efficiency, the issue with these approaches is that the leader-follower controllability is generally not guaranteed, since maximizing $\operatorname{tr}(W)$ can not rule out the possibility of having zero eigenvalues of W. Therefore, the goal of this work is to develop leader group selection approaches such that the resulting leader-follower system has guaranteed network controllability and improved energy efficiency in terms of the average controllability $\operatorname{tr}(W)$.

III. Topological Characterizations of Energy-Related Controllability

Topological characterizations of the energy-related controllability of signed acyclic graphs are investigated in this section. Based on the topological structures, signed graphs can be classified as either structurally balanced or structurally unbalanced.

Definition 2 (Structural Balance)**.** A signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is structurally balanced if the node set $\mathcal{V}$ can be partitioned into $\mathcal{V}_1$ and $\mathcal{V}_2$ with $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, where $a_{ij} > 0$ if $v_i, v_j \in \mathcal{V}_q$, $q \in \{1, 2\}$, and $a_{ij} < 0$ if $v_i \in \mathcal{V}_q$ and $v_j \in \mathcal{V}_r$, $q \neq r$, and $q, r \in \{1, 2\}$.

Def. 2 indicates that v_i and v_j are positive neighbors if they are from the same subset, i.e., either $\mathcal{V}_1$ or $\mathcal{V}_2$, and negative neighbors if v_i and v_j are from different subset. To characterize structural balance, necessary and sufficient conditions are provided.

Lemma 1 (Gauge Transformation)**.** *[20] A connected signed graph $\mathcal{G}$ is structurally balanced if and only if there exists a diagonal matrix $\Phi = \operatorname{diag}\{\phi_1, \ldots, \phi_n\}$ with $\phi_i \in \{\pm 1\}$ such that $\Phi \mathcal{A} \Phi$ has non-negative entries.*

Lemma 1 indicates that, if $\mathcal{G}$ is structurally balanced, there always exists a gauge transformation Φ and a corresponding graph $\bar{\mathcal{G}} = \left(\mathcal{V}, \mathcal{E}, \bar{\mathcal{A}}\right)$ with $\bar{\mathcal{A}} = [\bar{a}_{ij}] \in \mathbb{R}^{n \times n} = \Phi \mathcal{A} \Phi$. Clearly, $\bar{\mathcal{G}}$ is an unsigned correspondence of $\mathcal{G}$, since they share the same node and edge sets except that the edge weights in $\bar{\mathcal{A}}$ are all non-negative, i.e., $\bar{a}_{ij} = \operatorname{abs}(a_{ij})$, where $\operatorname{abs}(a_{ij})$ denotes the absolute value of a_{ij}.

Theorem 1. *Consider a signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ and its corresponding gauge transformed unsigned graph $\bar{\mathcal{G}} = \left(\mathcal{V}, \mathcal{E}, \bar{\mathcal{A}}\right)$ with the controllability Gramian W and $\bar{W}$, respectively. Let the nodes in $\mathcal{G}$ and $\bar{\mathcal{G}}$ evolve according to the dynamics (2) but with different adjacency matrices $\mathcal{A}$ and $\bar{\mathcal{A}}$, respectively. If $\mathcal{G}$ is structurally balanced, W has the same matrix spectrum as $\bar{W}$.*

Proof: For the unsigned graph $\bar{\mathcal{G}}$, its nodes evolve according to the following dynamics:

$$\dot{\boldsymbol{x}}(t) = -\mathcal{L}_u(\mathcal{G})\, \boldsymbol{x}(t) + B \boldsymbol{u}(t), \tag{5}$$

where $\mathcal{L}_u = \mathcal{D} - \bar{\mathcal{A}}$ is the graph Laplacian of $\bar{\mathcal{G}}$. The controllability Gramian of (5) is defined in (4) as $\bar{W} = \int_0^\infty e^{-\mathcal{L}_u \tau} B B^T e^{-\mathcal{L}_u \tau} d\tau$. From Lemma 1, if $\mathcal{G}$ is structurally balanced then $\mathcal{L}_u = \mathcal{D} - \Phi(\mathcal{A})\Phi = \Phi \mathcal{L} \Phi$, which indicates that

$$\begin{aligned} \bar{W} &= \int_0^\infty e^{-\Phi \mathcal{L} \Phi \tau} B B^T e^{-\Phi \mathcal{L} \Phi \tau} d\tau \\ &= \int_0^\infty \Phi e^{-\mathcal{L} \tau} \Phi B B^T \Phi e^{-\mathcal{L} \tau} \Phi d\tau \\ &= \Phi W \Phi, \end{aligned} \tag{6}$$

where $W = \int_0^\infty e^{-\mathcal{L}\tau} B B^T e^{-\mathcal{L}\tau} d\tau$ is the controllability Gramian associated with $\mathcal{G}$. Since gauge transformation preserves the matrix spectrum [20], i.e., $\Phi W \Phi$ has the same set of eigenvalues with W, it is clear from (6) that W has the same matrix spectrum as $\bar{W}$. ■

Since W and $\bar{W}$ have the same set of eigenvalues, Theorem 1 indicates that $\mathcal{G}$ and its corresponding unsigned graph $\bar{\mathcal{G}}$ are equivalent in terms of average controllability, i.e., $\operatorname{tr}(W) = \operatorname{tr}\left(\bar{W}\right)$. In addition, for other energy-related metrics [11] that are based on the controllability Gramian, such as the smallest eigenvalue $\lambda_{\min}(W)$, the trace of the inverse of the controllability Gramian $\operatorname{tr}\left(W^{-1}\right)$, the determinant $\det(W)$, and the rank $\operatorname{rank}(W)$, Theorem 1 implies that $\mathcal{G}$ and $\bar{\mathcal{G}}$ are equivalent in these metrics if $\mathcal{G}$ is structurally balanced. Therefore, Theorem 1 provides a means to investigate the energy-related controllability of signed networks by examining its corresponding unsigned graph $\bar{\mathcal{G}}$, where many existing analysis and design methods developed for unsigned graphs become available.

Corollary 1. *Consider a signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ and its unsigned correspondence $\bar{\mathcal{G}} = \left(\mathcal{V}, \mathcal{E}, \bar{\mathcal{A}}\right)$. If $\mathcal{G}$ is structurally balanced, the leader-follower controllability of $\mathcal{G}$ is equivalent to that of $\bar{\mathcal{G}}$ under the the same leader set.*

It is well known that a system is controllable if and only if its controllability Gramian is positive definite. Thus, Corollary 1 follows immediately from Theorem 1 by the fact that W and $\bar{W}$ share the same set of eigenvalues when $\mathcal{G}$ is structurally balanced.

Remark 1. Leader group selection on structurally balanced signed graphs has been partially studied in the work of [22], which requires the leaders to be selected from the same partitioned set (i.e., $\mathcal{V}_1$ or $\mathcal{V}_2$) to ensure network controllability. Corollary 1 relaxes such a constraint allowing leaders to be selected from different partitioned sets, as long as the corresponding unsigned graph is controllable under the selected leader group. In addition, the discovered equivalence of controllability between $\mathcal{G}$ and $\bar{\mathcal{G}}$ enables the use of existing leader group selection methods developed for unsigned graphs [1]–[4].

Corollary 2. *If $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is a signed acyclic graph, there always exists an unsigned correspondence $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \bar{\mathcal{A}})$, such that the controllability Gramian W of $\mathcal{G}$ has the same matrix spectrum as the $\bar{W}$ of $\bar{\mathcal{G}}$.*

Since acyclic graphs, e.g., tree or path graphs, are inherently structurally balanced [20], Corollary 2 is another immediate result of Theorem 1. Corollary 2 indicates that, instead of investigating the signed acyclic graph $\mathcal{G}$, its unsigned correspondence $\bar{\mathcal{G}}$ can be explored to enable leader group selection for improved control energy in terms of $\operatorname{tr}(W)$.

Specifically, let $d_{ij} \in \mathbb{R}^+$ denote the the distance between v_i and v_j in $\mathcal{G}$, i.e., the number of edges of the shortest path connecting v_i and v_j. The total distance of node i to all other nodes, often referred as the closeness of node i, is defined as $\sigma_i = \sum_{j \neq i} d_{ij}$. An important result from [17] is that, for unsigned graphs, $\operatorname{tr}(W)$ is proportional to the node closeness on acyclic graphs. In other words, selecting nodes with higher closeness values can improve $\operatorname{tr}(W)$, which implies reduced control energy, since $\operatorname{tr}(W)$ is inversely related to the total control energy by (3). However, as discussed earlier, simply selecting leaders that maximize $\operatorname{tr}(W)$ may not guarantee the leader-follower controllability. Therefore, based on our recent result [23], the subsequent section focuses on the development of leader group selection algorithms that jointly consider the leader-follower controllability and the average controllability $\operatorname{tr}(W)$.

IV. Leaders Selection on Signed Acyclic Graphs

The leader group selection method developed in this section consists of two steps. The first step is to identify a set of leader group candidates that can render the network controllable. Among these candidates, the energy metric $\operatorname{tr}(W)$ will be exploited in the second step to refine the initial selection by identifying leaders that require less control energy. Specifically, based on the result of [23] that a network is controllable if each sub-graph is controllable and the sub-graphs are connected via the leaders only, the first step is to partition the graph into a set of sub-graphs, where the sub-graphs are connected via leaders. Further, acyclic graphs generally take the form of a tree structure, which has path graphs as its basic building block. Therefore, acyclic signed graphs such as tree graphs can always be partitioned into a set of path graphs using the available partition methods [24]. The following lemma from [23] summarizes the leader group selection rules on signed path graphs.

Lemma 2. *Consider a signed path graph $\mathcal{G}_p = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ with the node set $\mathcal{V} = \{1, \ldots, n\}$, the edge set $\mathcal{E} = \{(i, i+1) | \, i \in \{1, \ldots, n-1\}\}$, and the weight matrix $\mathcal{W} \in \mathbb{R}^{n \times n}$ indicating associated positive or negative weights in $\mathcal{E}$. Let the nodes in $\mathcal{G}_p$ evolve according to the dynamics in (2). The signed path graph $\mathcal{G}_p$ is controllable if the leader group is selected according to either of the following rules: 1) selecting an end node as the leader, i.e. v_1 or v_n; or 2) selecting any two connected nodes as the leaders, i.e., $v_i, v_j \in \mathcal{V}_l$ with $(v_i, v_j) \in \mathcal{E}$.*

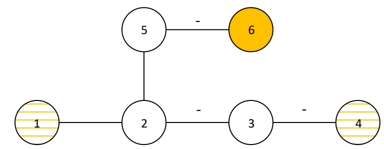

Figure 1. Node closeness on the signed acyclic graph.

Once the initial selection following Lemma 2 is complete, the next step is to refine the initial selection based on $\operatorname{tr}(W)$ for improved energy efficiency. Note that other leader group selection approaches originally developed for unsigned path graphs [5] are also applicable due to Corollary 2.

Lemma 3. *For a signed path graph $\mathcal{G}_p$, the closer the node i to the end node of $\mathcal{G}_p$, the higher its closeness σ_i.*

Lemma 3 is an immediate result of the fact that, if a node is closer to the end of the path graph, it has greater total distance to the other nodes, i.e., a higher closeness value. It was proven in [17] that selecting nodes with higher closeness as leaders can improve the average controllability. Although Lemma 3 is applicable to path graphs, it is alone insufficient to characterize the energy-related controllability of general acyclic graphs consisting of multiple path graphs.

Example 1. Consider a signed acyclic graph $\mathcal{G}_T$ composed of 6 nodes as shown in Fig 1, where negative edges are marked with "-". Suppose v_6 is a leader and it can be verified that $\mathcal{G}_T$ is controllable if either v_1 or v_4 is selected as an additional leader. Due to the relationship between node closeness and average controllability, it is desirable to select nodes with larger closeness values. For the sub-path graph $\{v_1, v_2, v_3, v_4\}$, clearly v_1 and v_4 has the same closeness from Lemma 3. However, due to the additional sub-path graph $\{v_5, v_6\}$, it can be verified that the closeness of v_1 and v_4 in $\mathcal{G}_T$ are $\sigma_1 = 11$ and $\sigma_4 = 13$, respectively. Thus v_4 should be selected as the additional leader to reduce the control cost.

Motivated by the discussion in Example 1, the subsequent lemma quantifies the influence of sub-path graphs on node closeness.

Lemma 4. *Consider a sub-path graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1)$ and let $\bar{\sigma}_{ij}$ be the closeness difference of v_i and v_j on $\mathcal{G}_1$. Suppose $\mathcal{G}_1$ is augmented by connecting another sub-path graph $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{A}_2)$ with cardinality $h = |\mathcal{V}_2|$ via an edge (v_p, v_q), where $v_p \in \mathcal{V}_1$* and *$v_q \in \mathcal{V}_2$. In the augmented* *graph, the closeness difference of v_i and v_j, denoted as σ_{ij}, is obtained as*

$$\sigma_{ij} = \bar{\sigma}_{ij} + h\left(d_{ip} - d_{jp}\right), \tag{7}$$

where d_{ip} and d_{jp} represent the distance from v_i to v_p and v_j to v_p, respectively.

Proof: From the definition of closeness, $\sigma_i = \sum_{k\in\mathcal{V}_1} d_{ik} + \sum_{l\in\mathcal{V}_2} d_{il}$ and $\sigma_j = \sum_{k\in\mathcal{V}_1} d_{jk} + \sum_{l\in\mathcal{V}_2} d_{jl}$. Therefore, σ_{ij} can be obtained as

$$\sigma_{ij} = \sigma_i - \sigma_j = \bar{\sigma}_{ij} + \left(\sum_{l\in\mathcal{V}_2} d_{il} - \sum_{l\in\mathcal{V}_2} d_{jl}\right). \tag{8}$$

Note that, for any node $v_o \in \mathcal{V}_2$ in $\mathcal{G}_2$, one has $d_{io} - d_{jo} = (d_{ip} + d_{po}) - (d_{jp} + d_{po}) = (d_{ip} - d_{jp}) + (d_{po} - d_{po}) = d_{ip} - d_{jp}$. Therefore, (8) can be simplified as $\sigma_{ij} = \bar{\sigma}_{ij} + h\left(d_{ip} - d_{jp}\right)$, since $\mathcal{G}_2$ consists of h nodes. ■

Lemma 4 shows that the closeness difference of any two nodes in a general acyclic graph depends on h, $\bar{\sigma}_{ij}$, and $d_{ip} - d_{jp}$, each of which can easily be obtained from its sub-path graph. For instance, in Example 1, the closeness difference of v_1 and v_4 in $\mathcal{G}_T$ can be calculated from the sub-path graph with nodes $\{v_1, v_2, v_3, v_4\}$. According to (7), $\sigma_{14} = \bar{\sigma}_{14} + h\left(d_{12} - d_{42}\right)$, where $\bar{\sigma}_{14} = \sum_{k\in\{1,2,3,4\}} d_{1k} - \sum_{k\in\{1,2,3,4\}} d_{4k} = 6 - 6 = 0$, $d_{12} = 1$, $d_{42} = 2$, and $h = 2$ is the cardinality of graph $\{v_5, v_6\}$. Thus, $\sigma_{14} = -2$, which indicates v_4 is a better leader candidate than v_1 in terms of average controllability. Note that following similar analysis, Lemma 4 can be extended to the case of multi sub-path graphs.

Motivated by the discussion above, Algorithm 1 summarizes the heuristic leader selection rules. It starts by identifying nodes in a given acyclic signed graph whose node degree is greater than two. Since the joint nodes in the acyclic graph have degree higher than two, the reason to identify nodes degree more than two is to facilitate the partition of the acyclic graph into paths. Once the graph is partitioned into a set of path graphs, the leader selection rules developed in Lemma 2 and Lemma 4 can be applied to identify leader groups with ensured network controllability and improved average controllability. The following example is provided to elucidate the approach.

Example 2. Consider a signed acyclic graph $\mathcal{G}$ in Fig. 2 (a) with the objective of selecting a set of leaders with ensured network controllability and improved energy efficiency. Based on Algorithm 1 the initial leader set is identified as nodes with degrees more than two, i.e., the nodes $\{1, 2, 4, 10, 16\}$ in Fig.2 (b). Since a graph can be safely partitioned without affecting the network controllability using leader to leader connections [23], $\mathcal{G}$ is partitioned into two sub-path graphs $\{1, 3, 7, 12, 17\}$ and $\{4, 8, 9, 13\}$, respectively, and a sub-tree with the rest nodes as shown in Fig. 2 (c). The next step is to update the leader set ensuring each partitioned graph is controllable. Specifically, the sub-path graph $\{1, 3, 7, 12, 17\}$ is controllable with leader v_1

Algorithm 1 Leader Selection for Signed Acyclic Graph

procedure INPUT:(Graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$);
  Output: The set of leaders $\mathcal{V}_l$
  Calculate the node degree d_i for each node $v_i \in \mathcal{V}$;
  Select nodes v_i with $d_i > 2$ to form $\mathcal{V}_l$;
  Use graph partition techniques (e.g., [24]) to partition $\mathcal{G}$ into path graphs;
  for Each sub-path graph **do**
    if The path is controllable **then**
      Keep the selected leaders in $\mathcal{V}_l$;
    else
      Update $\mathcal{V}_l$ by including nodes that can render the sub-path graphs controllable (i.e., Lemma 2);
      Further update $\mathcal{V}_l$ for improved average controllability (i.e., Lemma 4);
    end if
  end for
  Output $\mathcal{V}_l$;
end procedure

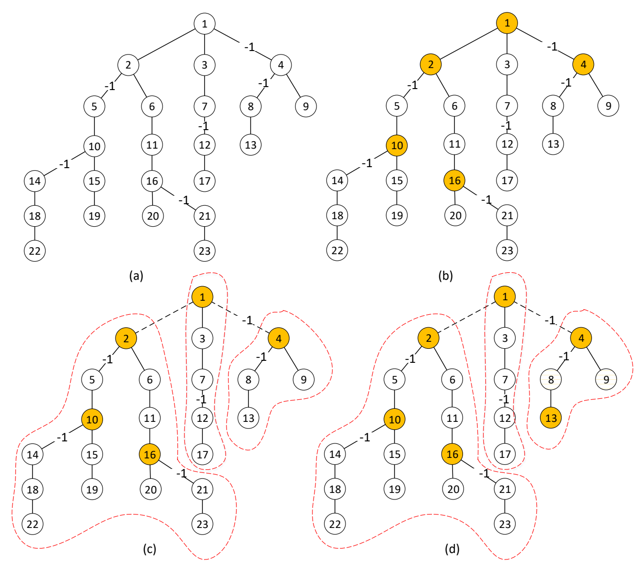

Figure 2. (a) A signed tree with 23 nodes. (b) The initial selection of leader nodes (i.e., nodes with degree higher than two) are marked. (c) The signed tree is partitioned into a set of path graphs. (d) Based on Lemma 2 and 4, the leader set is updated for ensured leader-follower controllability and improved average controllability.

from Lemma 2. For the other sub-path graph $\{4, 8, 9, 13\}$, selecting any one node from the set $\{v_8, v_9, v_{13}\}$ along with v_4 can ensure its controllability. According to Lemma 4, v_{13} is the best candidate, since it has a higher closeness value than v_8 and v_9, and thus provides better average controllability. Therefore, v_{13} is selected as an additional leader, as shown in Fig.2 (d).

Now consider sub-tree of $\mathcal{G}$ in Fig. 3 (a), which can be further partitioned into sub-path graphs according to Algorithm 1. Suppose that the sub-tree is partitioned into $\{15, 19\}$, $\{20\}$, and a long sub-path graph shown within the dashed region, as shown in Fig. 3 (b). According to Lemma 2, v_{15} and v_{20} have to be selected as leaders so that the two sub-graphs $\{15, 19\}$ and $\{20\}$ are controllable and the sub-graphs are connected by leaders only. For the sub-path graph within the dashed region, it is observed from Lemma 2 that any node except v_{18} from the path can render this sub-path

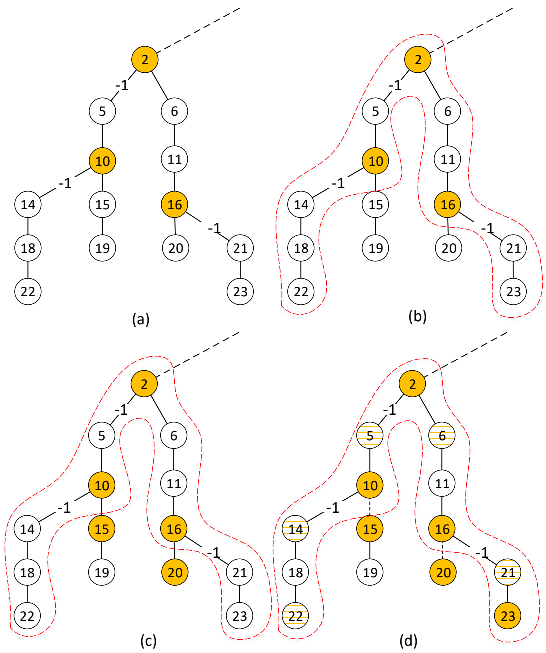

Figure 3. (a) The sub-tree of $\mathcal{G}$. (b) Identification of a sub-path graph. In (c) and (d), the leader group is updated based on Algorithm 1.

graph controllable. To improve average controllability, the node with the highest closeness value should be selected as the leader. It can be verified from Lemma 4 that v_{23} has the highest closeness value, and thus selected as the leader, as shown in Fig. 3 (d). Therefore, the marked nodes in Fig. 2 and 3 indicate the final selection of leaders that can ensure network controllability with improved control cost.

Remark 2. Note that Algorithm 1 is based on partitioning an acyclic graph into a set of sub-path graphs. There may exist different ways to partition an acyclic graph, resulting in different leader sets. In addition, the leader selection rules developed in Algorithm 1 are only sufficient conditions for network controllability. The selected leader group are by no means the optimal set, in terms of the minimum number of leaders or the minimum control cost. Nevertheless, this work is one of the first attempts to address leader group selection on signed graphs that jointly considers network controllability and control energy. Future research will continue along this direction to further refine the developed leader selection methods.

V. Conclusion

In this paper, leader group selection on multi-agent systems with signed acyclic topology is investigated. Graph-inspired approaches are developed to facilitate the identification of leaders that can ensure network controllability and improve energy expenditure required in network control. Since the current work is developed based on acyclic graphs, additional research will focus on extending the developed leader selection approach to general signed graphs.

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