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A Design Method for Controllable Topologies of Multi-Agent Networks*

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Abstract: For a multi-agent system with each agent having general linear dynamics, necessary and sufficient algebraic conditions are proposed for controllability. Then the difference in algebraic controllability conditions between single-integrator and generic multi-agent systems is pointed out. Based on these conditions and the identified indecisive graph node, a method is finally developed for the design of controllable topologies. The result also indicates the importance of recognition of uncontrollable topology structures.

Key Words: Controllability, Leader-follower structure, Multi-agent systems, Local interactions

1 Introduction

Designing control strategies directly from network topologies has recently drawn a lot of attention in the literature, which contributes to an efficient manipulation of networks and a better understanding of the nature of the system. The problem requires a study of interplay between network topologies and system dynamics [1, 23]. Recently, a lot of efforts have been made along this line in the multi-agent literature to understand how communication topology structures lead to controllability. This is also the focus of this paper.

The multi-agent controllability was formulated under a leader-follower framework, where the influence over network is achieved by exerting control inputs over leader nodes, see e.g., [2, 3, 4]. Roughly speaking, a multi-agent system is controllable if followers can be steered to proper positions to make up any desirable configuration by regulating the movement of leaders. So far most of results are on the controllability of single-integrator multi-agent systems. The earliest necessary and sufficient algebraic condition on multi-agent controllability was presented in [2], which was expressed in terms of eigenvalues and eigenvectors of submatrices of the Laplacian of the communication graph. Another one is an eigenvector based necessary and sufficient algebraic condition proposed in [5], where a relationship between eigenvectors of the Laplacian and the controllability was presented, which provided a method of determining leaders from the elements of eigenvectors. With these preparations, the virtue that leaders should have in controllability was characterized from both the algebraic and graphical perspectives [6]. There are some other algebraic conditions found in, e.g., [7, 8, 9, 10, 11, 12, 13, 21, 22, 24]. Recently, a protocol design method was proposed for controllability in [3]. It was proved that controllability of single-integrator, high-order and generic linear multi-agent systems all amounted to the topology structure of the communication graph, which could be achieved by proper design of protocols. It is worth noting that in [14], the authors obtained interesting graph-theoretic characterizations of controllability by using the proposed notion of graph controllability classes.

Algebraic conditions lay the foundation for a further investigation of interactions between topology structures and controllability. Although this issue is interesting and meaningful, previous work has suggested that it is quite involved, even for the simplest path topology structure [15]. Special topologies, naturally, were studied first, such as grid graphs [16], switching topologies [17], multi-chain topologies [18] and tree graphs [6]. It is worth noting that controllability can be fully addressed by a complete analysis of eigenvectors of Laplacian, see e.g., [15, 16]. Topologies construction offers another way of attacking the problem, which sometimes relates to the partition of communication graphs. In this regard, topologies are designed by using, for example, the vanishing coordinates based partition [6] and an eigenvector based partition [3, 19]. In particular, the construction of uncontrollable topologies not only facilitates the design of control strategies but also deepens understanding of controllable ones [18, 5].

The above work guides a further study of this issue in the paper. The paper presents a design method for controllable topologies, which implies that the identification of uncontrollable topologies helps to better understand the controllable ones. As far as we know, there are almost no results reported in the literature on how to directly construct controllable topologies. Further contributions of the paper lie in a unified and compact proof to the necessary and sufficient conditions on controllability for generic multi-agent systems, as well as an identification of the difference of algebraic controllability conditions between single-integrator and generic multi-agent systems.

2 Problem formulation

Consider a group of $n + l$ single integrator agents given by

$$\begin{cases} \dot{y}_i = u_i, & i = 1, \dots, n; \\ \dot{z}_j = u_{n+j}, & j = 1, \dots, l, \end{cases} \tag{1}$$

where n and l are the numbers of followers and leaders, respectively; y_i and z_j are the states of the ith and $(n + j)$th

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agent, respectively. Assume without loss of generality that the last l agents $z_1, \cdots, z_l$ act as leaders and are influenced only by external control inputs. The rest of the agents take roles of followers and are governed by the following neighbor based law

$$u_i = \begin{cases} \sum\limits_{j\in\mathcal{N}_i} (y_j - y_i) \text{ if } j \in \{1,\ldots,n\}; \\ \sum\limits_{(n+j)\in\mathcal{N}_i} (z_j - y_i) \text{ if } (n+j) \in \{n+1,\ldots,n+l\}, \end{cases} \tag{2}$$

where $\mathcal{N}_i = \{j \mid v_i \sim v_j; j \neq i\}$ is the neighboring set of v_i; '$\sim$' denotes the neighboring relation.

Let y and z denote the stack vectors of all y_i's and z_j's, respectively. The information flow between agents is incorporated in an interconnection graph $\mathcal{G}$, which consists of a set of nodes $\mathcal{V} = \{v_1,\ldots,v_{n+l}\}$ and a set of edges $\mathcal{E} = \{(v_i,v_j) \in \mathcal{V}\times\mathcal{V} | v_i \sim v_j\}$, with nodes representing dynamic agents, edges indicating the interconnections between them. The graph $\mathcal{G}$ is said to be fixed if each node of $\mathcal{G}$ has a fixed neighbor set. Let $\mathcal{L} = D - A$ be the Laplacian matrix, where A is the adjacency matrix of $\mathcal{G}$, D is the diagonal matrix whose diagonal entries are $d_i = |\mathcal{N}_i|$ and $|\mathcal{N}_i|$ is the cardinality of $\mathcal{N}_i$. Then, under protocol (2), the dynamics of followers read

$$\dot{y} = -\mathcal{F}y - \mathcal{R}z,$$

where $\mathcal{F}$ is the matrix obtained from $\mathcal{L}$ after deleting the last l rows and l columns; $\mathcal{R}$ is the $n \times l$ submatrix consisting of the first n elements of the deleted columns.

In a graph $\mathcal{G}$, a path is a sequence of consecutive edges, and $\mathcal{G}$ is connected if there is a path between any pair of distinct nodes. A subgraph of $\mathcal{G}$ is a graph whose vertex set is a subset of that of $\mathcal{G}$, and whose adjacency relation is a subset of that of $\mathcal{G}$ restricted to this subset. A subgraph $\mathcal{G}'$ is said to be induced from $\mathcal{G}$ if it is obtained by deleting a subset of nodes and all the edges connecting to those nodes. An induced subgraph of an undirected graph, which is maximal and connected, is said to be a connected component of the graph. It was shown that controllability can be studied under the assumption that $\mathcal{G}$ is connected [5].

Definition 1. *A multi-agent system is said to be controllable if for any initial state $x_f(t_0)$ of followers and any final state $x_f(t_1)$, $x_f(t_0)$ can be transferred to $x_f(t_1)$ in a finite time $t_1 > t_0$ by regulating the moving of leaders.*

3 Algebraic conditions

The algebraic conditions will be derived first for single-integrator multi-agent systems and then for generic systems with linear dynamics. Let agents $n+1,\ldots,n+l$ play roles of leaders, and for node v_k define the two node sets $\mathcal{N}_{kf} \overset{\Delta}{=} \{i|v_i \sim v_k, v_i$ is a node of the follower subgraph $\mathcal{G}_f\}$, $\mathcal{N}_{kl} \overset{\Delta}{=} \{j|v_j \sim v_k, v_j$ is a node of the leader subgraph$\mathcal{G}_l\}$. It can be seen that the neighbor set $\mathcal{N}_k$ of v_k can be broken down into two sets $\mathcal{N}_{kf}$ and $\mathcal{N}_{kl}$ with $\mathcal{N}_k = \mathcal{N}_{kf} \cup \mathcal{N}_{kl}, \mathcal{N}_{kf} \cap \mathcal{N}_{kl} = \Phi$, where Φ represents the empty set.

Lemma 1. *Given β as a complex number, the following statements are equivalent:*

i) β is an eigenvalue of $\mathcal{F}$ associated with eigenvector $y = [y_1,\ldots,y_n]^T$ and y is orthogonal to all the columns of $\mathcal{R}$;

ii) $\overline{y} = [y_1,\ldots,y_n,\underbrace{0,\ldots,0}_{l}]^T$ is an eigenvector of Laplacian $\mathcal{L}$ associated with the eigenvalue at β;

iii) $d_k y_k - \sum_{i\in\mathcal{N}_{kf}} y_i = \beta y_k$ for $k = 1,\ldots,n$ and $\sum_{i\in\mathcal{N}_{kf}} y_i = 0$ for $k = n+1,\ldots,n+l$, where d_k is the valency of node v_k.

Proof. i)$\Leftrightarrow$ii). First, i) $\Rightarrow$ii) follows from the following equation:

$$\mathcal{L}\overline{y} = \begin{bmatrix} \mathcal{F} & \mathcal{R} \\ \mathcal{R}^T & \mathcal{L}_l \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{F}y \\ \mathcal{R}^T y \end{bmatrix} = \begin{bmatrix} \beta y \\ 0 \end{bmatrix} = \beta\overline{y}$$

where the second to last equalities due to the assumption that β is an eigenvalue of $\mathcal{F}$ associated with eigenvector $y = [y_1,\ldots,y_n]^T$ and y is orthogonal to all the columns of $\mathcal{R}$.

On the contrary, if $\overline{y} = [y_1,\ldots,y_n,0,\ldots,0]^T_{(n+l)\times 1}$ is an eigenvector of $\mathcal{L}$ associated with the eigenvalue at β, it follows from $\mathcal{L}\overline{y} = \beta\overline{y}$ that

$$\begin{bmatrix} \mathcal{F} & \mathcal{R} \\ \mathcal{R}^T & \mathcal{L}_l \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} = \beta \begin{bmatrix} y \\ 0 \end{bmatrix} \tag{3}$$

which means

$$\mathcal{F}y = \beta y \text{ and } \mathcal{R}^T y = 0. \tag{4}$$

ii)$\Leftrightarrow$iii). If $\overline{y} = [y_1,\ldots,y_n,0,\ldots,0]^T_{(n+l)\times 1}$ is an eigenvector of $\mathcal{L}$ associated with the eigenvalue at β, it follows from $\mathcal{L}\overline{y} = \beta\overline{y}$ that

$$\mathcal{F}y = \beta y \text{ and } \mathcal{R}^T y = 0.$$

From $\mathcal{F}y = \beta y$, one has

$$d_k y_k - \sum_{i\in\mathcal{N}_{kf}} y_i = \beta y_k \quad \text{for} \quad k = 1,\ldots,n. \tag{5}$$

From $\mathcal{R}^T y = 0$, we see that y is orthogonal to all the columns of $\mathcal{R}$. Denote by $\mathcal{R} = [r_1,\ldots,r_l]$. It can be seen from $y^T r_1 = 0,\ldots,y^T r_l = 0$ that

$$\sum_{i\in\mathcal{N}_{kf}} y_i = 0 \quad \text{for} \quad k = n+1,\ldots,n+l. \tag{6}$$

On the contrary, we show that

$$\overline{y} = [y_1,\ldots,y_n,0,\ldots,0]^T_{(n+l)\times 1}$$

is an eigenvector of $\mathcal{L}$. Consider the following two situations:

1) $k = 1,\ldots,n$. In this case, since $y_i = 0$ for $i = n+1,\ldots,n+l$ and $\mathcal{N}_{kl}$ only contains nodes from $\{v_{n+1},\ldots,v_{n+l}\}$, one has

$$\sum_{i\in\mathcal{N}_{kl}} y_i = 0$$

Then for $k = 1,\ldots,n$,

$$\begin{aligned} d_k y_k - \sum_{i\in\mathcal{N}_k} y_i =& d_k y_k - \sum_{i\in\mathcal{N}_{kf}} y_i - \sum_{i\in\mathcal{N}_{kl}} y_i \\ =& d_k y_k - \sum_{i\in\mathcal{N}_{kf}} y_i \\ =& \beta y_k \end{aligned}$$

where the last equality follows from (5).

2) $k = n+1, \ldots, n+l$. In this case, $y_k = 0$ and $\sum_{i\in\mathcal{N}_{kl}} y_i = 0$ also holds. By (6)

$$\begin{aligned} d_k y_k - \sum_{i\in\mathcal{N}_k} y_i =& d_k \cdot 0 - \sum_{i\in\mathcal{N}_{kf}} y_i - \sum_{i\in\mathcal{N}_{kl}} y_i \\ =& \sum_{i\in\mathcal{N}_{kf}} y_i \\ =& 0 \\ =& \beta y_k \end{aligned}$$

The above arguments show that for any $k = 1, \ldots, n+l$, the following eigen-condition is satisfied.

$$d_k y_k - \sum_{i\in\mathcal{N}_k} y_i = \beta y_k.$$

Accordingly $\mathcal{L}\overline{y} = \beta\overline{y}$. □

Proposition 1. *The multi-agent system with single-integrator dynamics (1) is controllable if and only if there does not exist some β such that any of statements i), ii) and iii) of Lemma 1 is satisfied.*

Proof. The assertion ii) was proved in [5]. Then Lemma 1 means that the result also holds for conditions i) and iii). □

Remark 1. *Lemma 1 and Proposition 1 provide a unified and compact proof to the existing and newly proposed algebraic conditions.*

Remark 2. *One may wonder whether Proposition 1 still holds for generic multi-agent systems. Below it will be shown that for multi-agent systems with general linear dynamics, the conditions are only necessary. To make them further sufficient, an additional condition is required.*

Consider generic multi-agent systems with dynamics of each agent described by

$$\dot{x}_i = Ax_i + Bu_i, \quad i \in \mathcal{I}_{n+l}, \tag{7}$$

where $\mathcal{I}_{n+l} \triangleq \{1, \ldots, n+l\}$; $A \in R^{m\times m}, B \in R^{m\times p}$. The interactions between agents are realized through the following protocol

$$u_i = \sum_{j\in\mathcal{N}_i} K a_{ij}(x_j - x_i),$$

where $K \in R^{p\times m}$ is the feedback gain. Then, by following the same arguments as the single integrator case, the follower dynamics is

$$\dot{y} = [(I_n \otimes A) - \mathcal{F} \otimes (BK)]y - [\mathcal{R} \otimes (BK)]z.$$

In [20], the following result was established.

Lemma 2. *The multi-agent system with linear dynamics (7) is controllable if and only if the following two conditions are met simultaneously:*

- *(A, BK) is controllable;*
- *there is no eigenvector of $\mathcal{F}$ orthogonal to all the columns of $\mathcal{R}$.*

Combing Lemma 2 with Lemma 1 gives rise to the following result.

Theorem 1. *The multi-agent system with generic dynamics (7) is controllable if and only if (A, BK) is controllable and there does not exist some β such that any of statements i), ii) and iii) of Lemma 1 is satisfied.*

Remark 3. *Proposition 1 and Theorem 1 present the difference in algebraic conditions between single-integrator and generic multi-agent systems.*

Remark 4. *Another necessary and sufficient algebraic condition was derived in [9] for the controllability of multi-agent systems with general linear dynamics. In this condition, each eigenvalue λ of Laplacian is involved in the verification of controllability for a pair of matrices. This pair of matrices plays a similar role as (A, BK) in the above Theorem 1, while (A, BK) does not involve any eigenvalue λ of Laplacian.*

Example 1. *The example is used to show that system (7) is not controllable even if there is no eigenvector of $\mathcal{F}$ orthogonal to all the columns of $\mathcal{R}$. This means that the uncontrollability of system (7) can also be caused by the uncontrollability of (A, BK). Thus the difference in algebraic controllability conditions between single-integrator and generic multi-agent systems is verified by this example. Consider the*

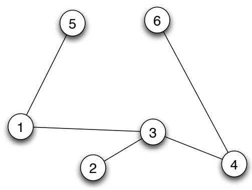


Fig. 1: An information flow graph, with nodes 5 and 6 playing leaders' role.

topology depicted by Fig.1. In this case,

$$\mathcal{F} = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \mathcal{R} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

It can be verified that there is no eigenvector of $\mathcal{F}$ orthogonal to all the columns of $\mathcal{R}$. Set

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, K = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

It can be verified that (A, BK) is not controllable. Calculations show that

$$A_c = I_4 \otimes A - \mathcal{F} \otimes (BK)$$

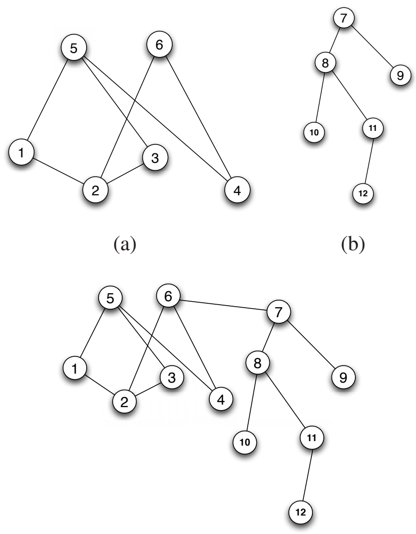
$$
=\begin{bmatrix}
1 & -2 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -1 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & -3 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\begin{aligned}
B_c =& -\mathcal{R}\otimes(BK)\\
=&\begin{bmatrix}
0 & 1 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 1\\
0 & 0 & 0 & 0
\end{bmatrix}
\end{aligned}
$$

and the rank of the controllability matrix of (A_c, B_c) is 2. Therefore the generic system is not controllable even if there is no eigenvector of $\mathcal{F}$ orthogonal to all the columns of $\mathcal{R}$, which means that the uncontrollability of the system is due to the uncontrollability of (A, BK).

4 Controllable topologies with a heuristic example

Based on the algebraic conditions developed in the previous section, this section provides a design method for controllable topologies, which in turn lines the value of identifying the uncontrollable topology structures. Let us start from the following example in Fig. 2.



(a) (b) (c)

Fig. 2: Graph (c) is obtained by connecting v_6 and v_7 between graphs (a) and (b). In graphs (a) and (c), v_1, v_2 play leaders' role and in graph (b), v_7 plays the single leader's role.

With v_1, v_2 playing leaders' role, graph (a) depicted in Fig. 2 is controllable. By Proposition 1, the Laplacian $\mathcal{L}_a$ of graph (a) has no eigenvector $\eta = [\eta_1, \ldots, \eta_6]^T$ with all the elements corresponding to the leaders being zero, i.e., $\eta_1 = \eta_2 = 0$. That is to say, with $\eta_1 = \eta_2 = 0$, there always exist indices $j_1, \cdots, j_s$, and for each $j = j_1, \cdots, j_s$, the corresponding eigen-condition

$$
d_j\eta_j - \sum_{i\in\mathcal{N}_j}\eta_i = \beta\eta_j \tag{8}
$$

does not hold, where β is an arbitrary eigenvalue of $\mathcal{L}_a$.

More specifically, for graph (a), $\eta_1 = \eta_2 = 0$ and the eigen-conditions $d_1\eta_1 - \sum_{i\in\mathcal{N}_1}\eta_i = \beta\eta_1$ and $d_2\eta_2 - \sum_{i\in\mathcal{N}_2}\eta_i = \beta\eta_2$ yield that

$$
\sum_{i\in\mathcal{N}_1}\eta_i = 0, \quad \sum_{i\in\mathcal{N}_2}\eta_i = 0. \tag{9}
$$

Since $\mathcal{N}_1 = \{2,5\}$, $\mathcal{N}_2 = \{1,3,6\}$, it follows from (9) that $\eta_2 + \eta_5 = 0$, $\eta_1 + \eta_3 + \eta_6 = 0$. Then $\eta_1 = \eta_2 = 0$ implies $\eta_5 = 0$ and $\eta_3 + \eta_6 = 0$. Accordingly, η can be written as $\eta = [0, 0, \eta_3, \eta_4, 0, -\eta_3]^T$. As to $\eta_5 = 0$, the eigen-condition yields that $\sum_{i\in\mathcal{N}_5}\eta_i = 0$. Since $\mathcal{N}_5 = \{1,3,4\}$ and $\eta_1 = 0$, one has $\eta_3 + \eta_4 = 0$. Hence, η can be further written as

$$
\eta = [0, 0, \eta_3, -\eta_3, 0, -\eta_3]^T. \tag{10}
$$

This implies that $\eta_6 = -\eta_3 \neq 0$. Otherwise, η will be a zero vector, which contradicts with the assumption that η is an eigenvector.

Now let us consider the eigen-condition for node v_3. Since $\mathcal{N}_3 = \{2,5\}$ and $\eta_2 = \eta_5 = 0$,

$$
\begin{aligned}
d_3\eta_3 - \sum_{i\in\mathcal{N}_3}\eta_i =& 2\eta_3 - (\eta_2 + \eta_5)\\
=& 2\eta_3
\end{aligned}
$$

Hence, the eigen-condition

$$
d_3\eta_3 - \sum_{i\in\mathcal{N}_3}\eta_i = \beta\eta_3 \tag{11}
$$

holds only for $\beta = 2$. For node v_4, $\mathcal{N}_4 = \{5,6\}$. Then it follows from (10) that

$$
\begin{aligned}
d_4\eta_4 - \sum_{i\in\mathcal{N}_4}\eta_i =& 2\eta_4 - (\eta_5 + \eta_6)\\
=& -2\eta_3 - (0 - \eta_3)\\
=& -\eta_3\\
=& \eta_4
\end{aligned}
$$

Hence, the associated eigen-condition

$$
d_4\eta_4 - \sum_{i\in\mathcal{N}_4}\eta_i = \beta\eta_4 \tag{12}
$$

holds only for $\beta = 1$.

Combining (11) with (12) yields that there does not exist an identical β so that these two equations hold simultaneously provided that $\eta_3 \neq 0$. Hence the vector η with $\eta_1 = \eta_2 = 0$ cannot be an eigenvector of the Laplacian $\mathcal{L}$ of graph (a) depicted by Fig. 2. As a consequence, the system is controllable with agents 1 and 2 taking leaders' role.

The above arguments show that with $\eta_1 = \eta_2 = 0$, there exist indices $j_1 = 3, j_2 = 4$ and the corresponding eigen-conditions (11) and (12) cannot be met simultaneously. So we call j_1, j_2 the $\eta_1 = \eta_2 = 0$ induced inverse eigen-condition indices. The key point is that the neighbor sets $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5$ together with $\eta_1 = \eta_2 = 0$ determine the indices j_1 and j_2, and the neighbor set $\mathcal{N}_6$ of the remaining node v_6 is not involved in the determination. At the same time, it can be seen from (10) that $\eta_6 = -\eta_3 = 0$ leads to $\eta = 0$. In what follows, we call v_6 an indecisive node.

In general, for a controllable graph with leaders, say $v_1, \ldots, v_l$, and any candidated vector η with the elements corresponding to all leaders being zero, i.e., $\eta_1 = \cdots = \eta_l = 0$, Proposition 1 implies that there always exist node indices $j_1, \ldots, j_s$ whose eigen-conditions cannot be met simultaneously. Below, $j_1, \ldots, j_s$ are said to be zero leader elements induced inverse eigen-condition indices. The above discussion inspires the following two definitions.

Definition 2. *For a controllable communication graph and any candidated vector whose entries corresponding to all leaders are taken as zero, $j_1, \ldots, j_s$ are said to be zero leader elements induced inverse eigen-condition indices if the eigen-conditions associated with $j_1, \ldots, j_s$ cannot be met simultaneously.*

Remark 5. *Proposition 1 affirms the existence of inverse eigen-condition node indices. To determine them, a group of neighbor node sets, say $\mathcal{N}_{l_1}, \ldots, \mathcal{N}_{l_t}$, need to be examined. For example, in the above example, $l_1 = 1, \ldots, l_t = 5$. In what follows, we focus on the nodes in $\mathcal{V}_f \setminus \{v_{l_1}, \ldots, v_{l_t}\}$, where $\mathcal{V}_f$ is the node set of follower subgraph $\mathcal{G}_f$.*

Definition 3. *For a controllable communication graph and any candidated vector η with elements corresponding to all leaders being zero, a node v is said to be an indecisive node if v does not affect the determination of all the inverse eigen-condition node indices $j_1, \ldots, j_s$.*

Graph (b) depicted in Fig. 2 is controllable with the first agent v_7 taking the single leader's role. Now a new graph in (c) depicted in Fig. 2 can be designed by connecting the indecisive node v_6 of graph (a) and the leader node v_7 of graph (b). With v_1 and v_2 still playing leaders' role, it can be verified that the designed graph in (c) is controllable. In this way, a new controllable graph is constructed by connecting two controllable smaller graphs. This is proved below to be a general property.

Theorem 2. *For a controllable graph $\mathcal{G}_a$ with multiple leaders and a controllable graph $\mathcal{G}_b$ with a single leader, a newly designed graph $\mathcal{G}$ is controllable with the same leaders as $\mathcal{G}_a$ if $\mathcal{G}$ is obtained by adding an information exchange edge between an indecisive node of $\mathcal{G}_a$ and the single leader node of $\mathcal{G}_b$.*

Proof. Suppose n_1 and n_2 are respectively the number of nodes of $\mathcal{G}_a$ and $\mathcal{G}_b$. The design means that $n_1 + n_2$ is the number of nodes of the constructed graph $\mathcal{G}$. Assume without loss of generality that the last agent node v_{n_1} of $\mathcal{G}_a$ is the indecisive node of $\mathcal{G}_a$ and the single leader, labeled as v_{n_1+1}, is the first agent node of $\mathcal{G}_b$. Denote by $v_1, \ldots, v_l$ the leader nodes of $\mathcal{G}_a$. In what follows, it will be shown that any vector η with the first l zero elements corresponding to all leaders, i.e.,

$$\eta = [\underbrace{0, \ldots, 0}_{l}, \eta_{l+1}, \ldots, \eta_{n_1}, \eta_{n_1+1}, \ldots, \eta_{n_1+n_2}]^T \quad (13)$$

is not an eigenvector of $\mathcal{G}$. To this end, let us distinguish between two cases $\eta_{n_1} \neq 0$ and $\eta_{n_1} = 0$.

Case i. $\eta_{n_1} \neq 0$. In this case, since $\mathcal{G}_a$ is controllable, $\eta_1 = \cdots = \eta_l = 0$ implies that there are inverse eigen-condition indices $j_1, \ldots, j_s$ of $\mathcal{G}_a$. By the construction of $\mathcal{G}$ and the definition of indecisive nodes, the neighbor sets $\mathcal{N}_1, \ldots, \mathcal{N}_{n_1-1}$ remain unchanged from $\mathcal{G}_a$ to $\mathcal{G}$. As a consequence, the eigen-conditions in (8) associated with $j_1, \ldots, j_s$ still cannot be met simultaneously in $\mathcal{G}$, and accordingly η is not an eigenvector of $\mathcal{G}$. More specifically, denote by $\mathcal{L}_a, \mathcal{L}_b$ the Laplacian of the graph $\mathcal{G}_a$ and $\mathcal{G}_b$, respectively. It can be seen that

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_a' & \begin{bmatrix} 0_{\tilde{n}_1\times 1} & \cdots & 0_{\tilde{n}_1\times 1} \\ -1 & \cdots & 0 \end{bmatrix} \\ \begin{bmatrix} 0_{1\times \tilde{n}_1} & -1 \\ \vdots & \vdots \\ 0_{1\times \tilde{n}_1} & 0 \end{bmatrix} & \mathcal{L}_b' \end{bmatrix}$$

where $\tilde{n}_1 \triangleq n_1 - 1$, $\mathcal{L}$ is the Laplacian of $\mathcal{G}$,

$$\mathcal{L}_a' = \mathcal{L}_a + \begin{bmatrix} 0_{\tilde{n}_1\times 1} & \cdots & 0_{\tilde{n}_1\times 1} \\ 0 & \cdots & -1 \end{bmatrix}$$

$$\mathcal{L}_b' = \mathcal{L}_b + \begin{bmatrix} -1 & 0_{1\times \tilde{n}_2} \\ \vdots & \vdots \\ 0 & 0_{1\times \tilde{n}_2} \end{bmatrix}$$

and $\tilde{n}_2 \triangleq n_2 - 1$. Denote $\eta^{(n_1)} \triangleq [0, \ldots, 0, \eta_{l+1}, \ldots, \eta_{n_1}]^T$, $\eta^{(n_2)} \triangleq [\eta_{n_1+1}, \ldots, \eta_{n_1+n_2}]^T$. Calculations show that

$$\mathcal{L}\eta = \begin{bmatrix} \mathcal{L}_a\eta^{(n_1)} + \begin{bmatrix} 0_{(n_1-1)\times 1} \\ \eta_{n_1} - \eta_{n_1+1} \end{bmatrix} \\ \mathcal{L}_b\eta^{(n_2)} + \begin{bmatrix} \eta_{n_1+1} - \eta_{n_1} \\ 0_{(n_2-1)\times 1} \end{bmatrix} \end{bmatrix}$$

Hence the elements of $\mathcal{L}_a\eta^{(n_1)}$ and $\mathcal{L}_b\eta^{(n_2)}$ remain unchanged in $\mathcal{L}\eta$ except the last element of $\mathcal{L}_a\eta^{(n_1)}$ and the first element of $\mathcal{L}_b\eta^{(n_2)}$. As a consequence, the eigen-conditions in (8) associated with the inverse eigen-condition indices $j_1, \ldots, j_s$ of $\mathcal{G}_a$ still cannot be met simultaneously in $\mathcal{G}$, and accordingly η is not an eigenvector of $\mathcal{G}$.

Case ii. $\eta_{n_1} = 0$. In this case, there are two kinds of situations. One is that $\eta_{l+1}, \cdots, \eta_{n_1-1}$ are not all zero. In such a case, the proof is the same as Case i. The other is the opposite side, i.e., $\eta_{l+1} = \cdots = \eta_{n_1-1} = 0$. In this situation, we first recall that $\eta_1 = \cdots = \eta_l = 0$. Denote by $\mathcal{L}$ the Laplacian of $\mathcal{G}$ and $\mathcal{N}_{n_1}^{(a)}, \mathcal{N}_{n_1}$ the neighbor set of node v_{n_1}, respectively, in $\mathcal{G}_a$ and $\mathcal{G}$, it follows from the design process of $\mathcal{G}$ that $\mathcal{N}_{n_1} = \mathcal{N}_{n_1}^{(a)} \cup \{n_1 + 1\}$. Since $\eta_1 = \cdots = \eta_{n_1} = 0$, $\sum_{i\in\mathcal{N}_{n_1}^{(a)}} \eta_i = 0$. Thus

$$\begin{aligned} d_{n_1}\eta_{n_1} - \sum_{i\in\mathcal{N}_{n_1}} \eta_i &= d_{n_1}\eta_{n_1} - \sum_{i\in\mathcal{N}_{n_1}^{(a)}} \eta_i - \eta_{n_1+1} \\ &= -\eta_{n_1+1}. \end{aligned}$$

Since graph $\mathcal{G}$ is connected, $\beta = 0$ is a simple eigenvalue of $\mathcal{L}$ associated with the all one eigenvector $[1, \ldots, 1]^T$. Hence, $\beta \neq 0$ if the eigenvector η takes form of (13). Thus, to make v_{n_1} satisfy the eigen-condition $d_{n_1}\eta_{n_1} - \sum_{i\in\mathcal{N}_{n_1}} \eta_i = \beta\eta_{n_1}$, one must set $\eta_{n_1+1} = 0$. This, however, is impossible. The reasons are as follows.

Since $\mathcal{G}_b$ is controllable with v_{n_1+1} playing the leaders' role, it follows from Proposition 1 that the Laplacian $\mathcal{L}_b$ of $\mathcal{G}_b$ has no eigenvector $\eta_b = [\eta_{n_1+1}, \eta_{n_1+2}, \ldots, \eta_{n_1+n_2}]^T$ with the element corresponding to the leader taking zero, i.e., $\eta_{n_1+1} = 0$. That is to say, in case that $\eta_{n_1+1} = 0$, there does not exist an eigenvalue β such that the following eigen-conditions hold simultaneously for each k

$$d_k\eta_k - \sum_{i\in\mathcal{N}_k^{(b)}} \eta_i = \beta\eta_k, \quad k = n_1 + 2, \ldots, n_1 + n_2. \quad (14)$$

In view of the design process of $\mathcal{G}$, $\mathcal{N}_k^{(b)} = \mathcal{N}_k$ for each $k = n_1 + 2, \ldots, n_1 + n_2$. Accordingly, for the eigen-condition of $\mathcal{L}$ of $\mathcal{G}$, the left hand side of (14), i.e.,

$$d_k\eta_k - \sum_{i\in\mathcal{N}_k} \eta_i = d_k\eta_k - \sum_{i\in\mathcal{N}_k^{(b)}} \eta_i$$

remains unchanged for $n_1 + 2, \ldots, n_1 + n_2$. As a consequence, with $\eta_{n_1+1} = 0$, the vector η of (13) does not satisfy the eigen-condition. This means that η could not be an eigenvector of $\mathcal{L}$.

To sum up, with $\eta_1 = \cdots = \eta_l = 0$, the vector η cannot be an eigenvector of $\mathcal{L}$ of $\mathcal{G}$. Hence, Proposition 1 means that the designed graph $\mathcal{G}$ is controllable with $v_1, \ldots, v_l$ taking the leaders' role. □

Remark 6. *In Theorem 2, if $\mathcal{G}_b$ is not a controllable graph with a single leader, the graph $\mathcal{G}$ designed in the same way can be shown to be uncontrollable. This underlines the importance of identification of uncontrollable topologies, which not only provides necessary conditions for controllability but also contributes to the construction of controllable topologies by avoiding the uncontrollable situations.*

5 Conclusions

In response to the increasingly widespread use of networks, effective control of networks, especially direct from their topology structures, are important. This calls for a rational design and organization of network topology. For the controllability of a multi-agent network, a design method is proposed in the paper to link two controllable graphs into a larger controllable one. The result is based on the identified indecisive graph node and the derived necessary and sufficient algebraic controllability conditions for multi-agent systems with generic linear dynamics.

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