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Controllability of multi-agent systems with directed and weighted signed networks[☆]

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ABSTRACT

This paper studies the controllability of multi-agent systems with signed networks, which are represented by directed weighted signed graphs. The adjacency weights of network depict the property of interactions. Positive weight means cooperative interaction while negative weight antagonistic interaction. First, by using switching equivalent transformation, the controllability of three kinds of signed networks is studied: structurally balanced, anti-balanced, and strictly unbalanced. It is shown that the controllability of the structurally balanced network is equivalent to that of the associated underlying network. Then, a graph-theoretic necessary condition for controllability is given by virtue of almost equitable partition of directed weighted signed networks. Furthermore, some necessary and sufficient conditions are given for the controllability of generic linear multi-agent systems. In addition, several examples are presented to illustrate the theoretical results.

1. Introduction

In recent years, cooperative control of multi-agent systems (MASs) has been extensively studied, which finds various applications in areas like mobile vehicle cooperative formation, satellite clusters attitude alignment, multiple robots flocking, and so on. Many basic and important issues have been studied in cooperative control of MASs, including consensus [1,2], formation control [3,4], and stabilizability [5,6], to name a few.

Controllability is a fundamental and important research topic in the cooperative control of MASs, and has drawn much attention of researchers from various scientific communities [7–26]. The MAS controllability was first introduced in [7], where a nearest-neighbor leader–follower framework was proposed and a necessary and sufficient condition was established through the eigenspectrum of the resulting Laplacian submatrix from the algebraic viewpoint. Afterwards, researchers investigated MAS controllability from the graph-theoretic perspective. Specifically, various concepts and properties of graph partitions were employed to study MAS controllability, such as equitable partition [8], almost equitable partition [9–12], distance partition [11–13] (see also surveys [14–16] for more details). Recently, some special graphs were investigated and their controllability issues were partially solved. Examples include the paths and cycles [17], trees [18], grids [19], multi-chain [20]. Additionally, there are some other works addressing the MAS controllability with switching topology, heterogeneous dynamics, protocols design, time-delay (see [21–25] and references therein).

It is noteworthy that the aforementioned results on controllability are derived for MASs with only cooperative interactions among agents. However, in reality, the interactions are not always cooperative, antagonistic (competitive) interactions can also exist among agents especially in the social, biological and information fields [26–28]. In social networks, individual relationships may be friendly or hostile. In international community, the relationship between countries could be cooperative or antagonistic. In biochemical and gene regulatory networks, the interactions among cells are either activations or inhibitions. In the field of information, users can express trust or distrust to other users. In practice, the interactions among agents are generally described by signed graphs, where the positive and negative weights represent the cooperative and antagonistic interactions, respectively. Recently, some researchers have addressed the issue of complex dynamic behaviors of MASs, such as bipartite consensus, interval bipartite consensus, bipartite containment [29–31]. In particular, the controllability problem for multi-agent systems with antagonistic interactions was studied in [26], where the networks were described by an undirected unweighted signed graph. In fact, however, the direction and link weights, which in the process of

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information transmission and exchange among the agents, have an important effect on the behavior of MASs. In general, a directed weighted graph is much more complicated than an undirected unweighted graph, e.g., the adjacency matrix and Laplacian matrix of the directed weighted graph are asymmetric matrices. For the MAS controllability issues, how do the directed weighted antagonistic interactions affect the controllability of MASs? What is the relationship between the controllability with directed weighted antagonistic interactions and the controllability with cooperative interactions? What is the difference between controllability of MASs with single integrator dynamics and controllability of MASs with general linear dynamics under the directed weighted antagonistic interactions? These challenging questions deserve further study.

This study is devoted to addressing the issue of MAS controllability with directed weighted signed networks. Firstly, the switching equivalent transformation is employed to study the controllability of structurally balanced, anti-balanced, and strictly unbalanced networks, respectively. Secondly, by virtue of almost equitable partition of directed weighted signed networks, the controllable subspaces are quantitatively studied. Finally, the controllability of generic linear MASs with directed weighted antagonistic interactions is investigated. The main contributions of this work include: (i) The relationship between controllability with antagonistic interactions and controllability with cooperative interactions is addressed. The results show that the controllability of structurally balanced networks is equivalent to that of traditional networks whose adjacency edge weights are all positive. (ii) The graph-theoretic necessary condition for controllability is proposed by virtue of almost equitable partition of directed weighted signed networks. The result provides a quantitative characterization of MAS controllability. (iii) Several necessary and sufficient conditions for controllability of generic linear MASs with directed weighted antagonistic interactions are established. The results indicate that the controllability of generic linear MASs is congruously determined by the interaction network and the agent dynamics.

Compared with the existing works, the advantages and novelties of this study lie in the following aspects. (1) Taking consideration of the directed weighted signed networks, the theoretical results obtained in this paper generalize the ones in [26], where undirected unweighted signed networks were considered. Moreover, anti-balanced and strictly unbalanced signed networks are further investigated compared to [26]. These results not only strengthen the understanding of the differences between the MAS controllability with antagonistic interactions and cooperative interactions, but also gain new insights into the graph-theoretic characterization of controllability with antagonistic interactions. (2) We generalize the concept of almost equitable partitions to directed weighted signed graphs and derive useful properties of almost equitable partitions. This concept includes as special cases the traditional almost equitable partitions of undirected and/or unsigned graphs [9–33] and provide a new perspective to the characterization of directed weighted signed graphs. We also provide a graph-theoretic necessary condition for controllability. The result gives a quantitative analysis for controllability. In addition, not limited to connected [8,9,11,26] and strongly connected [10] networks, our results can also be applied to arbitrary weighted signed networks. (3) Agents are assumed to take single-integrator dynamics in [8–26], while we consider the scenario of general linear agents, which brings new features for the study of the controllability problem. Additionally, some precise necessary and sufficient conditions on the controllability of general linear MASs with antagonistic interactions are obtained. Compared with the recent results (e.g., Theorem 1 of [11], Theorem 1 of [24]), these conditions here do not involve in the eigenvalue of Laplacian matrix and control protocol design. Therefore, it is more direct and easier to verify.

The rest of this paper is organized as follows. In Section 2, the problem of MAS controllability is formulated, and preliminary signed graph concepts are also presented. Section 3 presents the controllability with switching equivalence. The controllability with signed graph partitions is studied in Section 4. In Section 5, the controllability of generic linear MASs is investigated. Finally, the conclusions are given in Section 6.

Notation: Let I_n be the $n \times n$ identity matrix and $diag\{a_1, a_2, \ldots, a_n\}$ be a diagonal matrix with a_i, $i = 1, 2, \ldots, n$ being the diagonal entries. $\mathbf{1}$ denotes an all-1 vector or matrix and $\mathbf{0}$ represents an all-zero vector or matrix. Let $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ represent the set of real numbers, complex numbers and natural numbers, respectively. For $i \in \mathbb{N}$, e_i denotes the standard basis vectors. Suppose $A \in \mathbb{R}^{n\times n}$, $\lambda \in \mathbb{C}$ and $\beta \in \mathbb{R}^n$, if $Av = \lambda v$, then (λ, v) is said to be an eigenpair of A. $\sigma(A)$ denotes the set of all the eigenvalues of A. For any matrix B, $img(B)$ denotes the column space of B. $\otimes$ denotes the Kronecker product, $|X|$ denotes the cardinality of a set X, $X \setminus Y$ denotes the set $\{x|\ x \in X, x \notin Y\}$, and $sign(x)$ denotes the sign function of a scalar $x \in \mathbb{R}$. Let V be a vector space and let $T : V \to V$ be a linear operator. A vector space $W \subseteq V$ is T-invariant, if $TW \subseteq W$.

2. Preliminaries and problem formulation

2.1. Signed graph

A (weighted) signed graph $\mathbb{G} = (\mathcal{G}, \theta)$ consists of an unsigned (weighted) graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ and a signal mapping $\theta : \mathcal{E} \to \{+, -\}$, where $\mathcal{G}$ is said to be the underlying graph of $\mathbb{G}$. The edge set $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$, where $\mathcal{E}_+ = \{(j, i)|\ a_{ij} > 0\}$ and $\mathcal{E}_- = \{(j, i)|\ a_{ij} < 0\}$ denote the sets of positive and negative edges, respectively. Let $\mathcal{N}_i = \mathcal{N}_{i_+} \cup \mathcal{N}_{i_-}$ denote the neighborhood set of vertex i, where $\mathcal{N}_{i_+} = \{j \in \mathcal{V}|\ (j, i) \in \mathcal{E}_+\}$ and $\mathcal{N}_{i_-} = \{j \in \mathcal{V}|\ (j, i) \in \mathcal{E}_-\}$ represent the positive and negative neighborhood set of vertex i, respectively. A path is a sequence of distinct vertices such that any two consecutive vertices are an edge of the graph. A path with identical starting and ending vertices is called as a cycle. A cycle is positive (negative) if $|\mathcal{E}_-|$ is even (odd). Given a signed graph $\mathbb{G}$, for every vertex i, $c_i = \sum_{j\in\mathcal{N}_i} |a_{ij}|$ denotes the degree of i. The Laplacian of $\mathbb{G}$ is defined as $L = C - \mathcal{A}$, where $C = diag\{c_1, c_2, \ldots, c_n\}$ and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ are, respectively, the degree matrix and the adjacency matrix of $\mathbb{G}$. The signature matrix set is denoted by $\mathcal{D} = \{diag\{\delta_1, \delta_2, \ldots, \delta_n\}|\ \delta_i \in \{\pm 1\}\}$. Two matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$ are said to be signature similar if $\exists D \in \mathcal{D}$ such that $M_2 = DM_1D$.

Definition 1 (*[29,32]*)**.** Let $\mathbb{G}$ be a signed graph with the vertex set $\mathcal{V}$.

1. $\mathbb{G}$ is structurally balanced (simply, balanced) if it has a bipartition of the vertices $\mathcal{V}_1, \mathcal{V}_2$, with $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, such that $a_{ij} \geqslant 0$ for $\forall v_i, v_j \in \mathcal{V}_q$ $(q \in \{1, 2\})$ and $a_{ij} \leqslant 0$ for $\forall v_i \in \mathcal{V}_q, v_j \in \mathcal{V}_r, q \neq r$ $(q, r \in \{1, 2\})$.
2. $\mathbb{G}$ is structurally anti-balanced (simply, anti-balanced) if it has a bipartition of the vertices $\mathcal{V}_1, \mathcal{V}_2$, with $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, such that $a_{ij} \leqslant 0$ for $\forall v_i, v_j \in \mathcal{V}_q$ $(q \in \{1, 2\})$ and $a_{ij} \geqslant 0$ for $\forall v_i \in \mathcal{V}_q, v_j \in \mathcal{V}_r, q \neq r$ $(q, r \in \{1, 2\})$.
3. $\mathbb{G}$ is strictly structurally unbalanced (simply, unbalanced) if $\mathbb{G}$ is neither balanced nor anti-balanced.

2.2. Problem formulation

Consider a MAS consisting of n single integrator dynamics agents, which are labeled by set $\mathcal{V} = \{1, 2, \ldots, n\}$. Naturally, we assume that the set of the first m nodes, (i.e., leaders) is denoted by $\mathcal{V}_l$ $(\mathcal{V}_l = \{1, 2, \ldots, m\})$, and the set of the rest agents, (i.e., followers) is denoted by $\mathcal{V}_f = \mathcal{V} \setminus \mathcal{V}_l$. The leaders can be actuated by external inputs, and the followers obey distributed consensus-based protocol. Let $u \in \mathbb{R}^m$ represent the control input. The dynamics of each agent is given by

$$\begin{cases} \dot{x}_i = \sum_{j\in\mathcal{N}_i} a_{ij}[x_j - sign(a_{ij})x_i] + u_i, & i \in \mathcal{V}_l \\ \dot{x}_i = \sum_{j\in\mathcal{N}_i} a_{ij}[x_j - sign(a_{ij})x_i], & i \in \mathcal{V}_f \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}$ is the state of agent i and u_i is the ith entry of the control vector u.

Using a compact notation, the dynamics of the agents can be expressed as

$$\dot{x} = -Lx + Bu \tag{2}$$

where $x = [x_1, x_2, \ldots, x_n]^T$, $u = [u_1, u_2, \ldots, u_n]^T$, $L \in \mathbb{R}^{n\times n}$ is the Laplacian of interaction network $\mathbb{G}$ and $B = [e_1, e_2, \ldots, e_m] \in \mathbb{R}^{n\times m}$. Let $E(L, B) := [B, (-L)B, \ldots, (-L)^{n-1}B]$ be the controllability matrix of system (2). Obviously, system (2) is controllable if and only if $rank(E) = n$. Note that when the leader agents are given, the system (2) has a one-to-one correspondence to an interaction network. To simplify the expression, the interaction network $\mathbb{G}$ is regarded to be controllable/uncontrollable if the system (2) is controllable/uncontrollable.

The following observations follow directly from the classical Popov–Belevitch–Hautus (PBH) rank and eigenvector tests [34].

Proposition 1. *The MAS (2) is controllable if and only if there exists no λ such that any of the following statements is satisfied.*

(i) $rank[\lambda I + L, B] < n$.

(ii) $\bar{v} = [0, \ldots, 0, v_{m+1}, \ldots, v_n]^T$ *is a left eigenvector of L associated with the eigenvalue λ.*

(iii) *the following equations hold.*

$$\sum_{i\in\mathcal{N}_j^{of}} a_{ij}v_i = 0, j \in \mathcal{V}_l, \tag{3}$$

$$c_jv_j - \sum_{i\in\mathcal{N}_j^{of}} a_{ij}v_i = \lambda v_j, j \in \mathcal{V}_f, \tag{4}$$

where $\mathcal{N}_j^{of} = \{k|\, (j, k) \in \mathcal{E}, k \in \mathcal{V}_f\}$.

Remark 1. Proposition 1 provides some necessary and sufficient algebraic conditions for controllability of MASs with single integrator dynamics. However, in Section 5, we will show that these conditions are only necessary for controllability of general linear MASs with antagonistic interactions.

3. Controllability with switching equivalence

In this section, the controllability of the interaction network $\mathbb{G}$ is investigated through switching equivalence. Specifically, the controllability of structurally balanced, anti-balanced, and strictly unbalanced networks is studied. The relationship between the controllability of the interaction network $\mathbb{G}$ and the controllability of the underlying graph $\mathcal{G}$ is addressed.

Definition 2 ([32]). Let $\mathbb{G}_1 = (\mathcal{G}, \theta_1)$ and $\mathbb{G}_2 = (\mathcal{G}, \theta_2)$ be two signed graphs with Laplacian matrices L_1 and L_2, respectively. We call $\mathbb{G}_1$ and $\mathbb{G}_2$ switching equivalent, write $\mathbb{G}_1 \sim \mathbb{G}_2$, if and only if L_1 and L_2 are signature similar, i.e., $L_2 = DL_1D$, $D \in \mathcal{D}$.

Lemma 1 ([32]). *Let $\mathbb{G} = (\mathcal{G}, \theta)$ be a signed graph, then the following statements hold:*

(i) *If $\mathbb{G}$ is balanced, then $\mathbb{G} \sim (\mathcal{G}, +)$.*

(ii) *If $\mathbb{G}$ is anti-balanced, then $\mathbb{G} \sim (\mathcal{G}, -)$.*

Lemma 2 ([29]). *Let $\mathbb{G} = (\mathcal{G}, \theta)$ be a signed graph, then it has $2^{|\mathcal{E}|-|\mathcal{V}|+1}$ switching equivalent classes.*

Lemma 3. *Let $\mathbb{G}_1 = (\mathcal{G}, \theta_1)$ and $\mathbb{G}_2 = (\mathcal{G}, \theta_2)$ be two interaction networks of MAS (2). If $\mathbb{G}_1 \backsim \mathbb{G}_2$, then the controllability of $\mathbb{G}_1$ is equivalent to that of $\mathbb{G}_2$.*

Proof. Suppose that $L_1 \in \mathbb{R}^{n\times n}$ and $L_2 \in \mathbb{R}^{n\times n}$ are, respectively, the Laplacians of the interaction networks $\mathbb{G}_1$ and $\mathbb{G}_2$. If $\mathbb{G}_1 \backsim \mathbb{G}_2$, then $\exists D \in \mathcal{D}$ such that $L_2 = DL_1D$. When the leader agents are given, the Laplacian of $\mathbb{G}_1$ can be partitioned into $L_1 = \begin{bmatrix} L_{ll} & L_{lf} \\ L_{fl} & L_{ff} \end{bmatrix}$, where L_{ll} and L_{ff} correspond to the indices of leaders and followers, respectively. Clearly,

$$\begin{aligned} rank[sI_n + L_2, B] &= rank[sI_n + DL_1D, B] \\ &= rank\begin{bmatrix} sI_m + D_1L_{ll}D_1 & D_1L_{lf}D_2 & I_m \\ D_2L_{fl}D_1 & sI_{n-m} + D_2L_{ff}D_2 & \mathbf{0} \end{bmatrix} \\ &= m + rank[D_2L_{fl}D_1, sI_{n-m} + D_2L_{ff}D_2] \\ &= m + rankD_2[L_{fl}, sI_{n-m} + L_{ff}]D \\ &= m + rank[L_{fl}, sI_{n-m} + L_{ff}] \\ &= rank\begin{bmatrix} sI_m + L_{ll} & L_{lf} & I_m \\ L_{fl} & sI_{n-m} + L_{ff} & \mathbf{0} \end{bmatrix} \\ &= rank[sI_n + L_1, B], \end{aligned}$$

where $s \in \mathbb{C}$, $D_1 = diag\{\delta_1, \delta_2, \ldots, \delta_m\}$ and $D_2 = diag\{\delta_{m+1}, \delta_{m+2}, \ldots, \delta_n\}$ such that $D = diag\{D_1, D_2\}$. This completes the proof. □

Theorem 1. *For MAS (2) with the interaction network $\mathbb{G} = (\mathcal{G}, \theta)$, the following statements hold:*

(i) *If $\mathbb{G}$ is balanced, then the controllability of $\mathbb{G}$ is equivalent to that of $\mathcal{G}$.*

(ii) *If $\mathbb{G}$ is anti-balanced, then the controllability of $\mathbb{G}$ is equivalent to that of $(\mathcal{G}, -)$.*

(iii) *If $\mathbb{G}$ is a path or tree, then the controllability of $\mathbb{G}$ is equivalent to that of $\mathcal{G}$.*

(iv) *If $\mathbb{G}$ is a cycle, then the controllability of $\mathbb{G}$ is equivalent to that of $\mathcal{G}$.*

Proof. (i) Suppose that $\mathbb{G}$ is balanced. Then, it follows from Lemma 1(i) that $\mathbb{G} \sim (\mathcal{G}, +)$. Note that $(\mathcal{G}, +)$ is nothing but the underlying graph $\mathcal{G}$. By Lemma 3, the controllability of $\mathbb{G}$ and $\mathcal{G}$ is equivalent.

(ii) Suppose that $\mathbb{G}$ is anti-balanced. Then, it follows from Lemma 1(ii) that $\mathbb{G} \sim (\mathcal{G}, -)$. By Lemma 3, the controllability of $\mathbb{G}$ and $(\mathcal{G}, -)$ is equivalent.

(iii) Suppose that $\mathcal{G}$ is a path (tree). Then, it follows from Lemma 2 that all signed graphs on $\mathcal{G}$ are switching equivalent. Then, for any sign function θ, $L(\mathbb{G})$ and $L(\mathcal{G})$ are signature similar. By Lemma 3, the controllability of $\mathbb{G}$ and $\mathcal{G}$ is equivalent.

(iv) Suppose that $\mathcal{G}$ is a cycle. It follows from Lemma 2 that all signed graphs on $\mathcal{G}$ can be divided into two different switching equivalence classes, one contains the positive cycle and the other contains the negative cycle. By Lemma 3, all the graphs that belong to the same class have the same controllability. On one hand, since the first class contains the graph $(\mathcal{G}, +)$, hence, the controllability of $\mathbb{G}$, which in the first class, is equivalent to that of $\mathcal{G}$. On the other hand, the second class contains the signed cycle graph $\mathbb{G}$ with $\mathcal{E}_- = \{(1, 2)\}$ and $\mathcal{E}_+ = \mathcal{E} \setminus \mathcal{E}_-$. It is easy to prove that the controllability of $\mathbb{G}$ and $\mathcal{G}$ is equivalent. Therefore, it follows from Lemma 3 that the controllability of $\mathbb{G}$, which in the second class, is also equivalent to that of $\mathcal{G}$. The proof is completed. □

Remark 2. As shown in Theorem 1, the controllability of structurally balanced networks is equivalent to that of the traditional networks whose edge weights are all positive. Clearly, this implies that we can use the associated unsigned networks to study the controllability of structurally balanced networks. This merit gives significant convenience for controllability analysis. Further, Theorem 1 indicates that the controllability of anti-balanced networks is equivalent to that of the networks whose edge weights are all negative. In light of this equivalence, one can use the signless Laplacian matrix (i.e., $L = \mathcal{C} + \mathcal{A}$) of anti-balanced graphs to study the controllability of MAS (2). In addition, one can find that the controllability of some special strictly unbalanced graphs, such as path, cycle, start and tree graphs is not affected by edge signal. The results are summarized in Table 1 where some classes of signed networks and unsigned networks are presented.

4. Controllability and graph partitions

In this section, we first introduce the concept of partitions for directed weighted signed graphs. In particular, we generalize the almost equitable partitions for directed weighted signed graphs. Furthermore, by using almost equitable partition, a graph-theoretic necessary condition for controllability is given.

Let $\mathbb{G}$ be a signed graph with the vertex set $\mathcal{V}$. The set $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ is called a *partition* of $\mathbb{G}$ if $\bigcup_{i=1}^{k}\mathcal{C}_i = \mathcal{V}$ and $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for $i \neq j$, where $\mathcal{C}_i \subseteq \mathcal{V}$ is called a *cell*. We use $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ to denote the partition. A partition π is called *nontrivial*, if it contains at least one cell with more than one vertex; otherwise, π is called *trivial*. Let π_1 and π_2 be two partitions of $\mathbb{G}$, if each cell of π_1 is a subset of some cell of π_2, then π_1 is said to be finer than π_2, and is denoted by $\pi_1 \leqslant \pi_2$. For a partition π, the characteristic matrix $P(\pi)$ is given by

$$P_{ij}(\pi) = \begin{cases} 1, & \text{if } i \in \mathcal{C}_j \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4. *Let $\mathbb{G}$ be a signed graph, $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ be a partition of $\mathbb{G}$. Suppose that $P \in \mathbb{R}^{n\times k}$ is the characteristic matrix of π, then the following statements hold:*

(i) $M := P^TP = diag\{|\mathcal{C}_1|, |\mathcal{C}_2|, \ldots, |\mathcal{C}_k|\}$.
(ii) $PP^T = diag\{\mathbf{1}_{\mathcal{C}_1}\mathbf{1}_{\mathcal{C}_1}^T, \mathbf{1}_{\mathcal{C}_2}\mathbf{1}_{\mathcal{C}_2}^T, \ldots, \mathbf{1}_{\mathcal{C}_k}\mathbf{1}_{\mathcal{C}_k}^T\}$.
(iii) $P^+ := M^{-1}P^T$ *is the (left) Moore–Penrose pseudoinverse of* P.
(iv) $PP^+e_i = \frac{1}{|\mathcal{C}_r|}\sum_{j\in\mathcal{C}_r} e_j$, $i \in \mathcal{C}_r$, $r = 1, 2, \ldots, k$.

Proof. These assertions follow directly from the definition of π, P and Moore–Penrose pseudoinverse. □

Definition 3. Let $\mathbb{G}$ be a signed graph, $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ be a partition of $\mathbb{G}$. If for any $r, s \in \mathcal{C}_i$, $i \neq j$, $i, j = 1, 2, \ldots, k$,

$$\sum_{t_1\in\mathcal{C}_j, t_1\in\mathcal{N}_{r+}} a_{rt_1} = \sum_{t_2\in\mathcal{C}_j, t_2\in\mathcal{N}_{s+}} a_{st_2} \tag{5a}$$

and for any $r, s \in \mathcal{C}_i$, $i, j = 1, 2, \ldots, k$,

$$\sum_{t_1\in\mathcal{C}_j, t_1\in\mathcal{N}_{r-}} a_{rt_1} = \sum_{t_2\in\mathcal{C}_j, t_2\in\mathcal{N}_{s-}} a_{st_2} \tag{5b}$$

where $(t_1, r), (t_2, s) \in \mathcal{E}$, (5a)–(5b) are satisfied, the partition π is said to be an almost equitable partition (AEP) of $\mathbb{G}$.

Remark 3. Note that if the graph $\mathbb{G}$ is unsigned, namely, all the edge weights are positive, then (5b) is naturally satisfied. In this case, Definition 3 collapses into the concept of AEP for unsigned graph (see e.g. [9–33] for more details). Obviously, Definition 3 presents a more general AEP concept, including AEP for undirected and/or unsigned graph as a special case.

Given an AEP $\pi_{AEP} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ of $\mathbb{G}$, we denote by $\alpha_{ij} := \sum_{t\in\mathcal{C}_j} a_{rt}$, for any $r \in \mathcal{C}_i$, and define $\alpha_{ii} = 0$, for $\forall i = 1, 2, \ldots, k$.

Definition 4. The quotient graph of $\mathbb{G}$ relative to the AEP π_{AEP}, denoted by $\mathbb{G}_{\pi_{AEP}}$, is defined as a graph with vertices $\mathcal{V}(\mathbb{G}_{\pi_{AEP}}) = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$, edge set $\mathcal{E}(\mathbb{G}_{\pi_{AEP}}) = \{(\mathcal{C}_j, \mathcal{C}_i) | \alpha_{ij} \neq 0\}$, and the weight of $(\mathcal{C}_j, \mathcal{C}_i) \in \mathcal{E}(\mathbb{G}_{\pi_{AEP}})$ is α_{ij}.

Note that, $\mathbb{G}_{\pi_{AEP}}$ is also a directed weighted signed graph and $\alpha_{ij} \neq \alpha_{ji}$ in general.

Example 1. Some examples are used to illustrate Definitions 3 and 4. Let $\mathbb{G}$ be a directed signed graph shown in Fig. 1(a), one partition $\pi(\mathbb{G})$ of $\mathbb{G}$ is given by the four cells, namely $\pi(\mathbb{G}) = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$, where $\mathcal{C}_1 = \{1\}, \mathcal{C}_2 = \{2, 3\}, \mathcal{C}_3 = \{4\}, \mathcal{C}_4 = \{5\}$. By Definition 3, it is straightforward to verify that $\pi(\mathbb{G})$ is an AEP of $\mathbb{G}$. The quotient graph of $\mathbb{G}$ relative to $\pi(\mathbb{G})$ is shown in Fig. 1(b). The underlying graph $\mathcal{G}$ of $\mathbb{G}$ is displayed in Fig. 1(c). It can be verified that $\pi(\mathcal{G}) = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}, \mathcal{C}_1 = \{1\}, \mathcal{C}_2 = \{2, 3, 4\}, \mathcal{C}_3 = \{5\}$ is an AEP of $\mathcal{G}$. The corresponding quotient graph is displayed in Fig. 1(d). Clearly, the partition $\pi(\mathbb{G})$ is fine that $\pi(\mathcal{G})$, i.e., $\pi(\mathbb{G}) \leqslant \pi(\mathcal{G})$.

Lemma 5. *Given a signed graph $\mathbb{G}$, let L be the corresponding Laplacian, $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ be an AEP of $\mathbb{G}$, P be the characteristic matrix of π. Then, there exists a matrix $Q \in \mathbb{R}^{k\times k}$ such that satisfying*

$$LP = PQ. \tag{6}$$

Furthermore, $Q = P^+LP$ and $img(P)$ is L-invariant.

Proof. The proof is similar to the proof of Proposition 1 of [33], and thus is omitted. □

Remark 4. In fact, Lemma 5 describes one way in which L-invariant subspaces arise with respect to an AEP of directed weighted signed graph. It can be seen as a natural generalization of the undirected and/or unsigned graph. However, there are some differences between them, in particular, when the graph $\mathbb{G}$ is an unsigned graph, the matrix Q is nothing but the Laplacian matrix of quotient graph of $\mathbb{G}$ relative to π, i.e., $Q = P^+LP = L(\mathbb{G}_\pi)$, which is consistent with Theorem 2 of [10] and Proposition 1 of [33]. However, when the graph $\mathbb{G}$ is a signed graph, the matrix Q is not equal to the Laplacian matrix of quotient graph, i.e., $Q = P^+LP \neq L(\mathbb{G}_\pi)$. This fact shows that the AEP of signed graph has some special properties compared with the AEP of unsigned graph, which to our knowledge has not been studied yet.

Example 2. Consider the directed weighted signed graph $\mathbb{G}$ shown in Fig. 2(a). The underlying graph $\mathcal{G}$ of $\mathbb{G}$ is displayed in Fig. 2(c). It is easy to verify that the partitions $\pi_1(\mathbb{G}) = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}, \mathcal{C}_1 = \{1\}, \mathcal{C}_2 = \{2, 3\}, \mathcal{C}_3 = \{4, 5\}, \mathcal{C}_4 = \{6\}$ and $\pi_2(\mathcal{G}) = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$, $\mathcal{C}_1 = \{1\}, \mathcal{C}_2 = \{2, 3, 4, 5\}, \mathcal{C}_3 = \{6\}$ are the AEP of $\mathcal{G}$ and $\mathbb{G}$, respectively. The quotient graphs of $\mathbb{G}$ and $\mathcal{G}$ are shown in Fig. 2(b) and (d), respectively. By calculating, we get that two matrices $Q(\mathbb{G}_{\pi_1})$ and $Q(\mathcal{G}_{\pi_2})$ such that Eq. (6) are satisfied for $\pi_1(\mathbb{G})$ and $\pi_2(\mathcal{G})$, respectively. In this case, $Q(\mathbb{G}_{\pi_1}) = P^+(\pi_1(\mathbb{G}))L(\mathbb{G})P(\pi_1(\mathbb{G})) \neq L(\mathbb{G}_{\pi_1})$ and $Q(\mathcal{G}_2) = P^+(\pi_2(\mathcal{G}))L(\mathcal{G})\ P(\pi_2(\mathcal{G})) = L(\mathcal{G}_{\pi_2})$, which is consistent with Lemma 5.

For a given partition π and nodes $1, 2, \ldots, m \in \mathcal{V}$, if the partition π is an AEP and the nodes $1, 2, \ldots, m$ are its cells, i.e., $\{1\}, \{2\}, \ldots, \{m\} \in \pi$, then the partition π is called an AEP relative to nodes $1, 2, \ldots, m$ and denoted by $\pi_{AEP}(1, 2, \ldots, m)$. Let $\pi^*_{AEP}(1, 2, \ldots, m)$ represent the maximal partitions with respect to the partial order "$\leqslant$" for AEPs relative to nodes $1, 2, \ldots, m$. In fact, for a given signed graph $\mathbb{G}$, $\pi^*_{AEP}(1, 2 \ldots, m)$ is always existent uniquely (see e.g. [11,35] for more details). Define $\Pi := \{\pi \,|\, \pi \text{ is a partition of } \mathbb{G}\}$. Let $\mathbb{R}^{n\times\bullet}$ denote all matrices with n rows and $\psi : \mathbb{R}^{n\times\bullet} \to \Pi$ be the mapping such that $\forall X \in \mathbb{R}^{n\times\bullet}$, $i, j \in \pi_0$, $\pi_0 \in \psi(X)$ if and only if the ith and jth rows of X are the same.

Table 1
The relationship between the controllability of signed networks and the controllability of unsigned networks.

Signed networks	Unsigned networks	Leaders	Controllability relationship
Path graph $(\mathbb{P}_n, \theta)$	Path graph $\mathbb{P}_n$	$\mathcal{V}_l(\mathbb{P}_n, \theta) = \mathcal{V}_l(\mathbb{P}_n)$	$E(\mathbb{P}_n, \theta) \sim E(\mathbb{P}_n)$ (for any θ)
Cycle graph $(\mathbb{C}_n, \theta)$	Cycle graph $\mathbb{C}_n$	$\mathcal{V}_l(\mathbb{C}_n, \theta) = \mathcal{V}_l(\mathbb{C}_n)$	$E(\mathbb{C}_n, \theta) \sim E(\mathbb{C}_n)$ (for any θ)
Start graph $(\mathbb{S}_n, \theta)$	Start graph $\mathbb{S}_n$	$\mathcal{V}_l(\mathbb{S}_n, \theta) = \mathcal{V}_l(\mathbb{S}_n)$	$E(\mathbb{S}_n, \theta) \sim E(\mathbb{S}_n)$ (for any θ)
Tree graph $(\mathbb{T}_n, \theta)$	Tree graph $\mathbb{T}_n$	$\mathcal{V}_l(\mathbb{T}_n, \theta) = \mathcal{V}_l(\mathbb{T}_n)$	$E(\mathbb{T}_n, \theta) \sim E(\mathbb{T}_n)$ (for any θ)
Balanced graph $\mathbb{G} = (\mathcal{G}, \theta)$	The underlying graph $\mathcal{G}$	$\mathcal{V}_l(\mathbb{G}) = \mathcal{V}_l(\mathcal{G})$	$E(\mathbb{G}) \sim E(\mathcal{G})$ $(\theta(\mathcal{E}) = +)$
Anti-balanced graph $\mathbb{G} = (\mathcal{G}, \theta)$	The underlying graph $\mathcal{G}$	$\mathcal{V}_l(\mathbb{G}) = \mathcal{V}_l(\mathcal{G})$	$E(\mathbb{G}) \sim E(\mathcal{G}, -)$ $(\theta(\mathcal{E}) = -)$

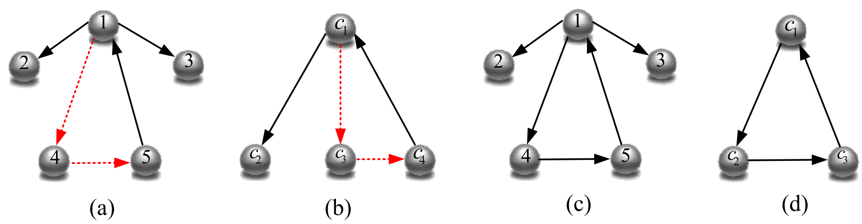

Fig. 1. Examples illustrating Definitions 3 and 4. The black solid and red dotted lines represent the positive and negative edges, respectively.

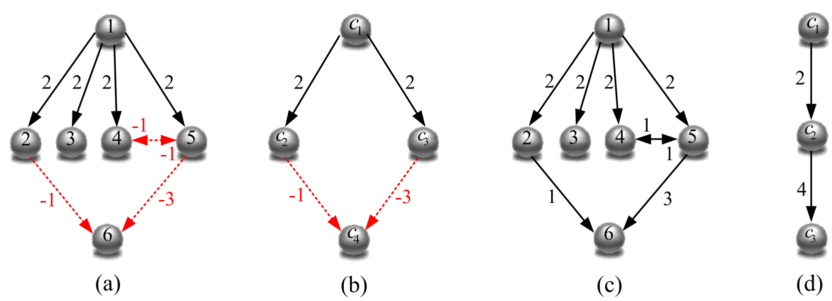

Fig. 2. Examples illustrating Lemma 5. The black solid and red dotted lines represent the positive and negative edges, respectively.

Next, inspired by [11], an algorithm for computing the maximal AEP for a given directed weighted signed graph is proposed.

Algorithm for computing the maximal AEP relative to $\mathcal{V}_l$:

Input: The signed graph $\mathbb{G}$, $\mathcal{V}_l = \{1, 2, \ldots, m\}$, and the Laplacian L
Output: The maximal AEP π^*_{AEP}, and the characteristic matrix $P(\pi^*_{AEP})$
Initialization: $\pi_0 \leftarrow \{\{1\}, \{2\}, \ldots, \{m\}, \mathcal{V} \setminus \mathcal{V}_l\}$, $j \leftarrow 0$
while $(\pi_{k+1} \neq \pi_k)$
  $\pi_j \leftarrow \psi[P(\pi_j), LP(\pi_j)]$, $j \leftarrow j + 1$
end while
return π^*_{AEP} and $P(\pi^*_{AEP})$

Example 3. The example is employed to illustrate the algorithm for computing the maximal AEP relative to $\mathcal{V}_l$. Consider a weighted signed graph is displayed in Fig. 3(a), where all weights are ones except the two labeled 2 and −1. Assume that $\mathcal{V}_l = \{1, 2\}$, thus $\pi_0(1, 2) = \{\{1\}, \{2\}, \{3, 4, 5, 6, 7\}\}$. The partitions obtained by carrying out the algorithm are shown in Fig. 3(b–e), where agents in the same cell are denoted by the some color. The maximal AEP relative to $\mathcal{V}_l$ is $\pi^*_{AEP}(1, 2) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}$.

Next, we will present some results relating controllability (controllable subspaces) of system (2) with AEP. Suppose that $\pi^*_{AEP}(1, 2, \ldots, m)$ is a maximal AEP of $\mathbb{G}$ relative to $\mathcal{V}_l$ and let $P \in \mathbb{R}^{n\times k}$ be the characteristic matrix of $\pi^*_{AEP}(1, 2, \ldots, m)$, where $1 \leq k < n$.

Theorem 2. *For MAS (2) with the interaction network $\mathbb{G}$, we have*
(i) *the interaction network $\mathbb{G}$ is uncontrollable.*
(ii) $img(E) \subseteq img(P)$.
(iii) $rank(E) \leqslant |\pi^*_{AEP}(1, 2, \ldots, m)|$.

Proof. Because $\pi^*_{AEP}(1, 2, \ldots, m)$ is an AEP of $\mathbb{G}$ and P is the corresponding characteristic matrix, according to Lemma 5, $img(P)$ is L-invariant. Since $B = [e_1, e_2, \ldots, e_m]$, it follows that $img(B) \subseteq img(P)$. Note that $img(E)$ is the smallest L-invariant subspace containing $img(B)$, therefore $img(E) \subseteq img(P)$ and $rank(E) \leqslant rank(P) < n$. This implies that $rank(E) \leqslant |\pi^*_{AEP}(1, 2, \ldots, m)|$ and the interaction network $\mathbb{G}$ is uncontrollable. □

Remark 5. Controllability problems have been studied for MASs based on graph-theoretic partition schemes in the literature (see e.g. [8–26]). However, the existing controllability results for MASs on one hand apply only to unsigned networks with all positive edge weights, and on the other hand require connectivity of undirected networks [8,9,11,26] and strong connectivity of directed networks [10]. In contrast, Theorem 2 makes a slight improvement and develops a general controllable subspace result on MASs with directed weighted signed networks, regardless of whether the interactions are connected or strong connected. In particular, if all the edge weights of the interaction network $\mathbb{G}$ are positive, then $\mathbb{G}$ becomes an unsigned network. In this case, the main results on the controllable subspaces of MASs with undirected and/or unsigned network (e.g., Theorem 4 of [11], Theorem 1 of [12]) can be viewed as a special case of Theorem 2.

Theorem 2 provides a graph-theoretic necessary condition for controllability involving in AEP. In particular, if the interaction network $\mathbb{G}$ is controllable, then $\pi^*_{AEP}(1, 2, \ldots, m)$ is trivial. In addition, a tight bound for controllable subspaces of system (2) is quantitatively given. Next, we give an example to illustrate the results.

Example 4. Consider the MAS (2) with the interaction network shown in Fig. 1(a). If agent 1 is chosen as the leader, then system

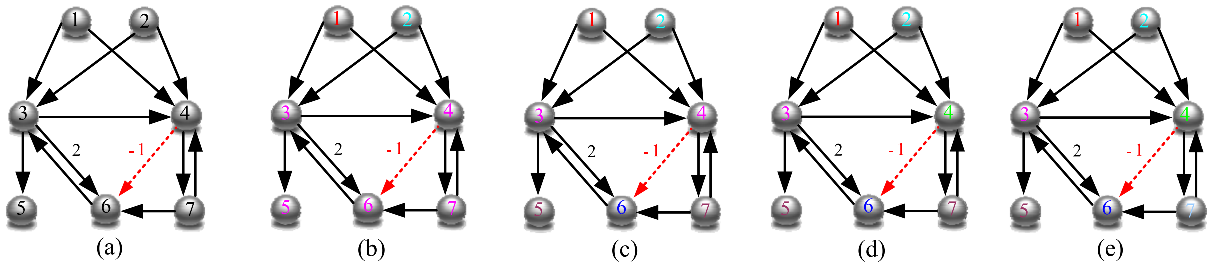

Fig. 3. Example for the algorithm. (a) A weighted signed graph with 7 nodes. (b) $\pi_0(1, 2) = \{\{1\}, \{2\}, \{3, 4, 5, 6, 7\}\}$. (c) $\pi_1(1, 2) = \{\{1\}, \{2\}, \{3, 4\}, \{5, 7\}, \{6\}\}$. (d) $\pi_2(1, 2) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5, 7\}, \{6\}\}$. (e) $\pi^*_{AEP}(1, 2) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}$. The black solid and red dotted lines represent the positive and negative edges, respectively.

(2) is uncontrollable and $rank(E)$ is strictly less than the upper bound of controllable subspaces of system (2). In fact, $rank(E) = 3 < 4 = |\pi^*_{AEP}(1)|$, where $\pi^*_{AEP}(1) = \{\{1\}, \{2, 3\}, \{4\}, \{5\}\}$ is nontrivial. For the MAS (2), assume that the interaction network is depicted in Fig. 1(c). Similarly, if we take agent 1 as leader, then system (2) is also uncontrollable, while the upper bound for controllable subspaces of system (2) is achieved, i.e., $rank(E) = |\pi^*_{AEP}(1)| = 3$, where $\pi^*_{AEP}(1) = \{\{1\}, \{2, 3, 4\}, \{5\}\}$ is nontrivial. For the MAS (2), assume that the interaction network is depicted in Fig. 3(a). If agents 1 and 2 are chosen as the leaders, then $\pi^*_{AEP}(1, 2, \ldots, m)$ is trivial and the upper bound is achieved and tight, i.e., $rank(E) = |\pi^*_{AEP}(1, 2)| = 7$, where $\pi^*_{AEP}(1, 2) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}$ is trivial.

5. Controllability of generic linear mass

In this section, controllability of generic linear MASs will be studied. In particular, the relationships among the controllability of generic linear MASs, the interaction network, and the agent dynamics will be discussed.

Consider an MAS consisting n agents with general linear dynamics. The dynamics of each agent is described by equations of the form

$$\dot{x}_i = Ax_i + Hz_i, i \in \mathcal{V}, \tag{7}$$

where $x_i \in \mathbb{R}^N$ denotes the state of agent i, and $z_i \in \mathbb{R}^N$ denotes the control input. $A \in \mathbb{R}^{N\times N}$ and $H \in \mathbb{R}^{N\times M}$ are the system matrix and the input matrix, respectively. Without loss of generality, we assume the first m nodes are leaders and the remaining $n-m$ nodes are followers.

Consider the following control protocol:

$$y_i = \begin{cases} \sum\limits_{j\in\mathcal{N}_i} a_{ij}[x_j - sign(a_{ij})x_i] + u_i, & i \in \mathcal{V}_l \\ \sum\limits_{j\in\mathcal{N}_i} a_{ij}[x_j - sign(a_{ij})x_i], & i \in \mathcal{V}_f \end{cases} \tag{8}$$

$$z_i = Ky_i \tag{9}$$

where $K \in \mathbb{R}^{M\times N}$ is a feedback gain matrix to be designed and $u_i \in \mathbb{R}^M$ denotes the ith external control input.

Let $X = [x_1^T, x_2^T, \ldots, x_n^T]^T$ and $U = [u_1^T, u_2^T, \ldots, u_m^T]^T$. MAS (7) together with protocol (8) can be rewritten as

$$\dot{X} = \tilde{A}X + \tilde{B}U, \tag{10}$$

where

$$\tilde{A} := I_n \otimes A - L \otimes HK, \quad \tilde{B} := B \otimes HK. \tag{11}$$

First, we give a sufficient and necessary condition for the controllability of system (10).

Theorem 3. *The system (10) is controllable if and only if the pairs (L, B) and (A, HK) are simultaneously controllable.*

Proof. There is a nonsingular matrix S such that $L = S^{-1}JS$, where $J = diag\{J_{n_1}, J_{n_2}, \ldots, J_{n_t}\}$ is a Jordan matrix with $n_1+n_2+\cdots+n_t = n$ and J_{n_i} is the Jordan block, i.e.,

$$J_{n_i} = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}_{n_i\times n_i}, \quad \lambda_i \in \sigma(L), i = 1, 2, \ldots, t.$$

Let $S = [\xi_1, \xi_2, \ldots, \xi_n]^T, \xi_i \in \mathbb{R}^n, i = 1, 2, \ldots, n$. Denote $k_0 = 0, k_1 = n_1, k_2 = k_1 + n_2, \ldots, k_t = k_{t-1} + n_t$. Obviously, $(\lambda_1, \xi_{k_1}), (\lambda_2, \xi_{k_2}), \ldots, (\lambda_t, \xi_{k_t})$ are the left-eigenpairs of L and $\xi_{k_1}, \xi_{k_2}, \ldots, \xi_{k_t}$ are linearly independent. Let $SB = [\bar{B}_1, \bar{B}_2, \ldots, \bar{B}_t]^T$ (where $\bar{B}_i^T \in \mathbb{R}^{n_i\times m}$ is composed of the $(k_{i-1}+1)$th, $(k_{i-1}+2)$th, $\ldots, k_i$-th row vectors of SB, i.e., $\bar{B}_i = [B^T\xi_{k_{i-1}+1}, B^T\xi_{k_{i-1}+2}, \ldots, B^T\xi_{k_i}]^T$). By the PBH rank test, system (10) is controllable if and only if for $\forall s \in \mathbb{C}$, $rank[sI_{nN} - \tilde{A}, \tilde{B}] = nN$. Clearly (see the equation in Box I).

It follows from (12) that $rank[sI_{nN} - \tilde{A}, \tilde{B}] = nN$ if and only if for each $i = 1, 2, \ldots, t$ and $\forall s \in \mathbb{C}$ $rank[sI_{n_iN} - (I_{n_i} \otimes A - J_{n_i} \otimes HK), \bar{B}_i^T \otimes HK] = n_iN$.

(Sufficiency). Suppose that the pairs (L, B) and (A, HK) are simultaneously controllable. Consider for $\forall i \in \{1, 2, \ldots, t\}$ (see the equation in Box II).

The controllability of (L, B) implies that $\xi_{k_i}^T B \neq \mathbf{0}$. Without loss of generality, assume that $\xi_{k_i}^T B = [\bar{\xi}_{i1}, \bar{\xi}_{i2}, \ldots, \bar{\xi}_{im}]$ and $\bar{\xi}_{i1} \neq 0$. Then

$$\begin{aligned} & rank[sI_N - (A - \lambda_i HK), \xi_{k_i}^T B \otimes HK] \\ &= rank[sI_N - (A - \lambda_i HK), \bar{\xi}_{i1}HK, \bar{\xi}_{i2}HK, \ldots, \bar{\xi}_{im}HK] \\ &= rank[sI_N - (A - \lambda_i HK), \bar{\xi}_{i1}HK, \bar{\xi}_{i2}HK, \ldots, \bar{\xi}_{im}HK] \\ &\quad \cdot \begin{bmatrix} I_N & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \frac{-\lambda_i}{\bar{\xi}_{i1}} I_N & \frac{1}{\bar{\xi}_{i1}} I_N & \frac{-\bar{\xi}_{i2}}{\bar{\xi}_{i1}} I_N & \cdots & \frac{-\bar{\xi}_{im}}{\bar{\xi}_{i1}} I_N \\ \mathbf{0} & \mathbf{0} & I_N & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & I_N \end{bmatrix} \\ &= rank[sI_N - A, HK], \end{aligned} \tag{14}$$

which means that controllability of $(A-\lambda_i HK,\ \xi_{k_i}^T B \otimes HK)$ is equivalent to that of (A, HK). Thus, $rank[sI_N - (A - \lambda_i HK), \xi_{k_i}^T B \otimes HK] = N$. On one hand, if $\xi_{k_{i-1}+j}^T B \neq \mathbf{0}$, for $j = 1, 2, \ldots, n_i - 1$. Similarly, we can prove that $rank[sI_N - (A - \lambda_i HK), \xi_{k_{i-1}+j}^T B \otimes HK] = N$ for $j = 1, 2, \ldots, n_i - 1$ and $\forall s \in \mathbb{C}$. Therefore, system (10) is

$$\begin{aligned}
&rank[sI_{nN} - \tilde{A}, \tilde{B}] \\
&= rank[sI_{nN} - (I_n \otimes A - L \otimes HK), B \otimes HK] \\
&= rank(S^{-1} \otimes I_N)[sI_{nN} - (I_n \otimes A - J \otimes HK), SB \otimes HK] \begin{bmatrix} (S \otimes I_N) & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \\
&= rank[sI_{nN} - (I_n \otimes A - J \otimes HK), SB \otimes HK] \\
&= rank \left[\begin{bmatrix} sI_{n_1N} - (I_{n_1} \otimes A - J_{n_1} \otimes HK) & & \\ & \ddots & \\ & & sI_{n_sN} - (I_{n_t} \otimes A - J_{n_t} \otimes HK) \end{bmatrix}, \begin{bmatrix} \bar{B}_1^T \otimes HK \\ \vdots \\ \bar{B}_t^T \otimes HK \end{bmatrix} \right].
\end{aligned} \tag{12}$$

Box I.

$$\begin{aligned}
&rank[sI_{n_iN} - (I_{n_i} \otimes A - J_{n_i} \otimes HK), \bar{B}_i^T \otimes HK] \\
&= rank \left[\begin{bmatrix} sI_N - (A - \lambda_i HK) & -HK & & \\ & sI_N - (A - \lambda_i HK) & \ddots & \\ & & \ddots & -HK \\ & & & sI_N - (A - \lambda_i HK) \end{bmatrix}, \begin{bmatrix} \xi_{k_{i-1}+1}^T B \otimes HK \\ \xi_{k_{i-1}+2}^T B \otimes HK \\ \vdots \\ \xi_{k_i}^T B \otimes HK \end{bmatrix} \right].
\end{aligned} \tag{13}$$

Box II.

controllable. On the other hand, suppose without loss of generality that, there exist a $j \in \{1, 2, \dots, n_i - 1\}$, such that $\xi_{k_{i-1}+j}^T B = \mathbf{0}$. The general case can be proved in the same way. Then

$$\begin{aligned}
&rank[sI_N - (A - \lambda_i HK), -HK] \\
&= rank[sI_N - (A - \lambda_i HK), -HK] \begin{bmatrix} I_N & \mathbf{0} \\ -\lambda_i I_N & I_N \end{bmatrix} \\
&= rank[sI_N - A, HK],
\end{aligned}$$

which means controllability of $(A - \lambda_i HK, \ -HK)$ is equivalent to that of (A, HK). It follows from the controllability of (A, H) that $rank[sI_N - (A - \lambda_i HK), -HK] = N$. Therefore, system (10) is controllable.

(Necessity). Suppose that system (10) is controllable. Recall that the fact for each $i = 1, 2, \dots, t$ and $\forall s \in \mathbb{C}$, $rank[sI_{n_iN} - (I_{n_i} \otimes A - J_{n_i} \otimes HK), \bar{B}_i^T \otimes HK] = n_i N$, we can conclude that $\xi_{k_i}^T B \neq \mathbf{0}$ (In fact, if there exists $k \in \{1, 2, \dots, t\}$ such that $\xi_{k_k}^T B = \mathbf{0}$, it follows from (13) and the PBH rank test that $rank[sI_{n_iN} - (I_{n_i} \otimes A - J_{n_i} \otimes HK), \bar{B}_i^T \otimes HK] < n_i N$, if s is taken as eigenvalues of $A - \lambda_i HK$. Thus system (10) is not controllable). In this case, we get that (L, B) is controllable. On the other hand, if $\xi_{k_i}^T B \neq \mathbf{0}$, it follows from (13) that $rank[sI_N - (A - \lambda_i HK), \xi_{k_i}^T B \otimes HK] = N$. By (14), we get that (A, HK) is controllable. □

Combining Theorem 3 with Proposition 1 gives rise to the following results.

Corollary 1. *The system (10) is controllable if and only if the pair (A, HK) is controllable and there does not exist some λ such that any of the statements* (i), (ii) *and* (iii) *of Proposition 1 is satisfied.*

Next, we consider the case that the matrix pair (A, HK) is uncontrollable. In this case, there exists an invertible matrix T such that

$$\bar{A} = T^{-1}AT = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ \mathbf{0} & \bar{A}_{\bar{c}} \end{bmatrix}, \quad \bar{H} = T^{-1}HK = \begin{bmatrix} \bar{H}_c \\ \mathbf{0} \end{bmatrix},$$

where the pair $(\bar{A}_c, \bar{H}_c)$ is controllable, and $\bar{A}_c \in \mathbb{R}^{N_c \times N_c}$, $\bar{H}_c \in \mathbb{R}^{N_c \times N}$. Take the transformation $\bar{x}_i = T^{-1}x_i$, under which, system (10) is transformed into

$$\dot{\hat{X}} = \hat{A}\hat{X} + \hat{B}U, \tag{15}$$

with

$$\begin{aligned}
&\hat{A} := I_n \otimes \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ \mathbf{0} & \bar{A}_{\bar{c}} \end{bmatrix} - L \otimes \begin{bmatrix} \bar{H}_c T \\ \mathbf{0} \end{bmatrix}, \hat{B} := B \otimes \begin{bmatrix} \bar{H}_c \\ \mathbf{0} \end{bmatrix}, \\
&T := [\bar{T}_1 \ \bar{T}_2],
\end{aligned} \tag{16}$$

where $\hat{X} = (I_n \otimes T^{-1})X$.

Lemma 6. *The controllability of system (10) is equivalent to that of system (15).*

Proof. Clearly, the controllability of system (10) is invariant under equivalent transformation $\bar{x}_i = T^{-1}x_i$. Hence, the controllability of system (10) and system (15) is equivalent. □

Theorem 4. *The controllability of system (10) is equivalent to that of the pair $(\tilde{A}_{11}, \tilde{B}_c)$, where $\tilde{A}_{11} := I_n \otimes \bar{A}_c - L \otimes \bar{H}_c\bar{T}_1$, $\tilde{B}_c := B \otimes \bar{H}_c$. Furthermore, $rankE(\tilde{A}, \tilde{B}) = N_c \cdot rankE(L, B)$.*

Proof. By Lemma 3 of [24], one has

$$\begin{aligned}
\hat{A} &= W^T \begin{bmatrix} I_n \otimes \bar{A}_c - L \otimes \bar{H}_c\bar{T}_1 & I_n \otimes \bar{A}_{12} - L \otimes \bar{H}_c\bar{T}_2 \\ \mathbf{0} & I_n \otimes \bar{A}_{\bar{c}} \end{bmatrix} W \\
&= W^T \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \mathbf{0} & \tilde{A}_{22} \end{bmatrix} W, \\
\hat{B} &= V^T \begin{bmatrix} B \otimes \bar{H}_c \\ \mathbf{0} \end{bmatrix} V = V^T \begin{bmatrix} \tilde{B}_c \\ \mathbf{0} \end{bmatrix} V
\end{aligned}$$

where W and V are permutation matrices. Let $\tilde{A}_{11} := I_n \otimes \bar{A}_c - L \otimes \bar{H}_c\bar{T}_1$, $\tilde{A}_{12} := I_n \otimes \bar{A}_{12} - L \otimes \bar{H}_c\bar{T}_2$, $\tilde{A}_{22} := I_n \otimes \bar{A}_{\bar{c}}$, and $\tilde{B}_c := B \otimes \bar{H}_c$.

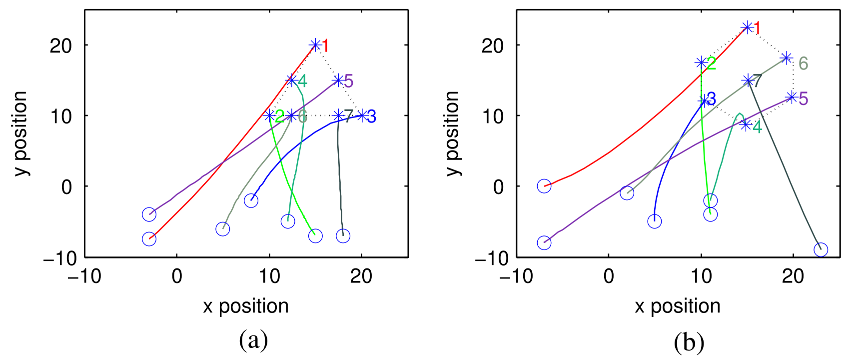

Fig. 4. The motion trajectories of all agents in plane. (a) A triangle configuration. (b) A hexagon configuration. The initial state and the final desired configuration are represented using the circle and asterisk, respectively.

By Lemma 6, one has

$$\begin{aligned} rankE(\tilde{A}, \tilde{B}) &= rankE(\hat{A}, \hat{B}) = rank[\hat{B}, \hat{A}\hat{B}, \ldots, \hat{A}^{nN-1}\hat{B}] \\ &= rank[\tilde{B}_c, \tilde{A}_{11}\tilde{B}_c, \ldots, \tilde{A}_{11}^{nN-1}\tilde{B}_c] \\ &= rank[\tilde{B}_c, \tilde{A}_{11}\tilde{B}_c, \ldots, \tilde{A}_{11}^{nN_c-1}\tilde{B}_c] = rankE(\tilde{A}_{11}, \tilde{B}_c), \end{aligned} \tag{17}$$

which implies that system (10) and the pair $(\tilde{A}_{11}, \tilde{B}_c)$ have the same controllability. On the other hand, since the pair $(\bar{A}_c, \bar{H}_c)$ is controllable, the rest assertion is a direct consequence of Theorem 3. □

Remark 6. Controllability of generic linear MASs have been studied in the literature (see e.g. Theorem 1 of [11], Theorem 1 of [24]). Compared with their results, Theorem 3 has two different points. One is that the controllability of generic linear MASs is studied under directed weighted signed networks instead of undirected and/or unsigned graphs, which the Laplacian matrix is asymmetrical. Subsequent arguments show that weighted signed networks bring new features for the study of this problem. The other is that the results here are only related to controllability of interaction network and the agent dynamics, they do not involve in the eigenvalue of Laplacian matrix and control protocol design. Therefore, it is more direct and easier to verify.

Theorems 3 and 4 indicate that the controllability of system (10) is congruously determined by the interaction network and the agent dynamics. Hence, the controllability of system (10) can be decoupled into the controllability of two low-dimensional systems, i.e., the pairs (L, B) and (A, HK). This merit can reduce the computational cost for checking controllability of system (10) and give significant convenience for the design of generic linear MAS to guarantee the controllability. Next, we give an example to demonstrate Theorem 3.

Example 5. Case 1. Assume that the interaction network of system (7) is depicted in Fig. 3(a), where agents 1 and 2 take the leader's role. The system matrices of each agent are given as follows

$$A_1 = \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}, \; H_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let $K_1 = [1\ 1]$. Obviously the pairs (A_1, H_1K_1) and (L, B) are controllable. By calculating, we obtain $rankE(\tilde{A}, \tilde{B}) = 14$, this means that the system (10) is controllable, which is consistent with Theorem 3. The trajectories of all agents are depicted in Fig. 4.

Case 2. For system (7) with the interaction network shown in Fig. 3(a), assume that agent 1 takes the leader's role. Let A, H and K be the same as in case 1. The control matrix of the interaction network Fig. 3(a) is given by $B = [1\,0\,0\,0\,0\,0\,0]^T$. By calculating, one has $rankE(L, B) = 6 < 7$. Thus, the pair (L, B) is uncontrollable. It follows from Theorem 3 that system (10) is uncontrollable. Next, by calculating, we get that $rank\ C(\tilde{A}, \tilde{B}) = 12 < 14$. This implies that system (10) is uncontrollable, which is consistent with Theorem 3.

Case 3. We still consider MAS (7) with the interaction network shown in Fig. 3(a), where agents 1 and 2 are chosen as leaders. Let the system matrices of each agent be as follows

$$A_2 = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \; H_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; K_2 = [2\ 1].$$

Obviously, the pair (A_2, H_2K_2) is uncontrollable. Thus, by Theorem 3, system (10) is uncontrollable. Next, we check the rank of matrix $E(\tilde{A}, \tilde{B})$. By calculating, we find that $rankE(\tilde{A}, \tilde{B}) = 7 < 14$, which is consistent with Theorem 3.

6. Conclusion

In this paper, the MAS controllability with directed weighted signed networks has been investigated. We have studied the controllability of structural balance, anti-balance, and special strictly unbalanced signed networks and discussed the relationship between the controllability with antagonistic interactions and the controllability with cooperative interactions. The results show that the controllability of the structurally balanced networks is equivalent to that of traditional networks whose edge weights are all positive. Then, we have generalized the almost equitable partitions to directed weighted signed graphs and proposed a graph-theoretic necessary condition for controllability. The result provides a quantitative analysis for controllability. In addition, we have established several necessary and sufficient conditions for the controllability of generic linear MASs with antagonistic interactions. The results indicate that the controllability of generic linear MASs is congruously determined by the interaction network and the agent dynamics. In our further work, we will focus on the controllability of signed networks with communication delays and switching topology, which will bring new challenges.

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