The Graphical Conditions for Controllability of Multiagent Systems Under Equitable Partition

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***Abstract*—In this article, by analyzing the eigenvalues and eigenvectors of Laplacian L, we investigate the controllability of multiagent systems under equitable partitions. Two classes of nontrivial cells are defined according to the different numbers of links between them, which are completely connected nontrivial cells (CCNCs) and incompletely connected nontrivial cells. For the system with CCNCs, a necessary condition for controllability is found to be choosing leaders from each nontrivial cell, the number of which should be one less than the cardinality of the cell. It is shown that the controllability is affected by three factors: 1) the number of the links between nontrivial cells; 2) the rank of the connection matrix; and 3) the odevity of the capacity of the nontrivial cells. In the case of nontrivial cells under the equitable partition, there are automorphisms of interconnection graph $\mathcal{G}$, which induce the eigenvectors of L with zero entries. For the system with automorphisms, by taking advantage of the property of eigenvectors associated with L, we propose several graphical necessary conditions for controllability. In addition, by the PBH rank criterion, the controllable subspaces of the system with different classes of nontrivial cells are compared. Finally, a necessary and sufficient condition for controllability under minimum inputs is given.**

***Index Terms*—Automorphism, controllability, equitable partition, incompletely connected nontrivial cells (ICNCs), multiagent systems.**

I. Introduction

Most studies of multiagent systems (MASs) focus on agents' cooperation to achieve performance, for example, consensus and controllability [1]–[9]. For the work of consensus, most researchers are devoted to discovering the conditions for the stability of systems [10]–[13], [27], [28]. For the controllability of networks, which is another major research topic in the field of MASs, most of these works aimed to find the relationship between the graph property and controllability [18], [30]–[33]. The leader–follower dynamics of MASs were first proposed by Tanner [14], with the states of leaders always being controllable. For this circumstance, Jafari *et al.* [15] showed the equivalence of controllability between the MASs with single integrator and leader–follower interaction. The antagonistic network was both considered in consensus and controllability problems [16]–[18]. The controllability of undirected signed graphs was studied in [17]. It was shown that there is a case of equivalent controllability between signed graphs and unsigned graphs. For directed graphs, an equivalence of controllability between signed graphs and unsigned graphs was also proved [18]. Especially, from the aspect of eigenvalues, Rahmani *et al.* [19] gave a necessary and sufficient condition for controllability while Ji *et al.* [20] developed a necessary and sufficient condition for controllability from the aspect of eigenvectors. Thus, Rahmani and Ji presented the methods in terms of eigenvalues and eigenvectors, where one can take these advantages to investigate the controllability.

Based on the graphical and algebraic characterizations, necessary and sufficient conditions were derived for controllability [21]. By these results, the uncontrollable system can be recognized directly from the identified uncontrollable topologies. Considering a special case of equitable partitions, the minimal set of input nodes was given, which induces a necessary and sufficient condition for controllability [22]. Equitable partition and almost equitable partition are typical ways to discuss the controllability, playing a significant role in graphical characterizations. Cardoso *et al.* [23] put forward a necessary and sufficient condition in terms of almost equitable partition, which explicates the relationship between L and L_π. For equitable partitions, a necessary and sufficient condition was presented in [24].

In this article, we investigate the controllability of MASs by using equitable partition and almost equitable partition. The essence of equitable partitions and almost equitable partitions is to analyze the uncontrollability by nontrivial cells, by which necessary conditions can be derived. Due to the different numbers of communication links between nontrivial cells, there are two classes of nontrivial cells. One is completely connected nontrivial cell (CCNC), and the other is incompletely connected nontrivial cell (ICNC). The two classes of nontrivial cells have a conspicuous discrepancy on controllability. We remark that the ICNCs were ignored in most research. In this article, we first consider the situation of ICNC and fill a primary gap in the corresponding problems. Based on the leader–follower network, Rahmani *et al.* [19] proposed an uncontrollable system with a leader symmetry. The essence of leader symmetry is that there are automorphisms of $\mathcal{G}$ under fixed inputs. If there are automorphisms of $\mathcal{G}$, there is at least

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one nontrivial cell in the graph $\mathcal{G}$, namely, an automorphism is a sufficient condition for nontrivial cells. The automorphisms of the graph induce the isomorphisms of eigenvectors of L, which means there is a node set with labels of these nodes that do not affect the existence of edges. As a consequence, even the positions of the entries in an eigenvector are changed, which is still an eigenvector for the identical eigenvalue.

The multiplicity of the eigenvalue associated with L is related to the number of communication links between nontrivial cells. Accordingly, the formats of eigenvectors are diverse under different numbers of links. Three factors that affect controllability are the number of communication links, the rank of the connection matrix, and the odevity of the capacity of nontrivial cells. The situation for any factor is not unique, which poses challenges to the controllability analysis. For different situations, we present different necessary conditions for controllability. Actually, the PBH criterion can be applied to consider the controllable subspace, and the relationship between PBH criterion and controllable subspace was revealed in this article. In view of the intersections of nonzero entries in eigenvectors associated with L, we can obtain the minimum number of inputs for controllable systems. It is worth noting that the analysis results of matrix eigenvalues and eigenvectors obtained based on equivalent partition are expected to be applied to other problems, such as the leader-following consensus [29].

The remainder of this article is organized as follows. In Section II, the background of graph theory and preliminaries are presented. In Section III, the controllability of the system with CCNCs and ICNCs is studied. We show the effect on controllability caused, respectively, by the number of the links among nontrivial cells, the rank of connection matrix H, and the odevity of the capacity of the nontrivial cells. In Section IV, the controllability of the system under almost equitable partition is studied. In Section V, we reveal the relationship between PBH rank criterion and controllable subspace, and contrast the controllable subspace of the system with different classes of nontrivial cells. A necessary and sufficient condition for controllability is derived under minimum inputs. Finally, the conclusions are given in Section VI.

II. Preliminaries

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$ be an undirected graph, where $\mathcal{V} = \{v_1, \dots, v_n\}$ denotes the set of n nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges, and $A \in \mathbb{R}^n$ is the adjacency matrix of $\mathcal{G}$. If v_i has a communication link with v_j, then there is an edge $e_{ij} \in \mathcal{E}$ between v_i and v_j, with $a_{ij} = a_{ji} = 1$, $i, j \in \{1, \dots, n\}$. The self-loop is meaningless for control, under which $a_{ii} = 0\ \forall i = 1, \dots, n$. So we do not consider this situation.

Let $x_i(t) \in \mathbb{R}$ denote the state of node v_i at time t. Then, based on the feedback law, its dynamics are described by the single integrator

$$\dot{x}_i(t) = -\sum_{j \in \mathcal{N}(i)} \big(x_i(t) - x_j(t)\big) + u_i(t),\ \ i = 1, \dots, n \tag{1}$$

where $\mathcal{N}(i)$ denotes the set of the adjacent nodes of node i, and $u_i(t)$ is the external input for node i. The Laplacian matrix of $\mathcal{G}$ is represented by $L = D - A$, where $D \in \mathbb{R}^{n\times n}$ is the diagonal matrix denoting the connectivity degree. Then, (1) can be written as

$$\dot{x} = -Lx + Bu \tag{2}$$

where B is the input matrix. If one node is injected by input u_i, then the corresponding entry of B is 1.

In a leader–follower network, some nodes take leaders' rule and the others take followers' rule. We use subscript l to denote the affiliation with leaders while f for the followers. A follower subgraph $\mathcal{G}_f$ is the subgraph induced by the follower node set $\mathcal{V}_f$. Similarly, there is a leader subgraph induced by the leader node set $\mathcal{V}_l$. Then, $\mathcal{V}_f \cup \mathcal{V}_l = \mathcal{V}$, and the Laplacian matrix can be accordingly partitioned as

$$L = \begin{bmatrix} L_f & L_{fl} \\ L_{fl}^T & L_l \end{bmatrix}$$

where L_f and L_l correspond to the follower subgraph and leader subgraph, respectively. The nonzero entries of L_{fl} indicate the connection between the follower subgraph and the leader subgraph. In this article, we assume that one external input is only connected with one node. Then

$$\begin{bmatrix} \dot{x}_f \\ \dot{x}_l \end{bmatrix} = -\begin{bmatrix} L_f & L_{fl} \\ L_{fl}^T & L_l \end{bmatrix}\begin{bmatrix} x_f \\ x_l \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u. \tag{3}$$

Let $v = -[L_{fl}^T\ L_l]\ [x_f\ x_l]^T + u$ represent the combinations of leaders and the original inputs. Then, (3) can be written as

$$\begin{bmatrix} \dot{x}_f \\ \dot{x}_l \end{bmatrix} = -\begin{bmatrix} L_f & L_{fl} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_f \\ x_l \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} v \tag{4}$$

which is equivalent to

$$\dot{x}_f = -L_f x_f - L_{fl} x_l \tag{5}$$

where $\dot{x}_l = v$. It follows that the controllability of system (2) and system (5) is equivalent. If input u_i has more than one link with leader nodes, there will arise a situation that the states of some leader nodes are not controllable. For this situation, even system (5) is controllable and the system (2) is still uncontrollable. Therefore, if one input u_i is connected with more than one node, the controllability of system (2) and system (5) is not always equivalent.

Let E denote a matrix with proper dimensions and all entries taking 1. The standard basis vectors in $\mathbb{R}^n$ are denoted by e_1, e_2, ..., e_n. Let J denote a permutation matrix with $J \neq I$, which consists of e_1, e_2, ..., e_n. Every row and column of J is a standard basis vector, which means that J is a unitary matrix with $J^T J = I$.

III. Equitable Partition

Definition 1: $\pi = \{C_1, \dots, C_r\}$ is called an equitable partition if each node in cell C_i has the same number of neighbors in cell C_j, $i, j \in \{1, \dots, r\}$. If each node in C_i has the same number of neighbors in C_j, the partition π is called an almost equitable partition, where $i \neq j, i, j \in \{1, \dots, r\}$. We denote the cardinality of the cell C_r with $|C_r|$.

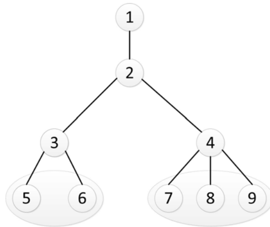

Fig. 1. Equitable partition with nontrivial cells.

If a cell contains more than one node, it is said to be a nontrivial cell, otherwise, trivial. The nodes in one nontrivial cell have similar or the same communication links, as well as the dynamics. The similar or the same dynamics are destructive to controllability since leaders cannot identify the difference between these nodes. The weak similarity of dynamics can enhance the dimension of controllable subspace compared with the strong similarity of dynamics. Hence, it is expected to obtain weak similarity for the dynamics of the nodes in one cell.

As Fig. 1 shows, there is a partition $\pi = \{C_1, C_2, C_3, C_4, C_5, C_6\}$, where C_1, C_2, C_3, and C_4 are trivial cells associated with nodes 1, 2, 3, and 4, respectively, and $C_5 = \{5, 6\}$ and $C_6 = \{7, 8, 9\}$ are nontrivial cells. Based on the same communication links of nodes 5 and 6, if we choose node 1 as the leader, then the dynamics of nodes 5 and 6 are equivalent. This situation is the same to nodes 7, 8, and 9. Therefore, it can be imagined that the system is uncontrollable.

A characteristic vector $p_i \in \mathbb{R}^n$ has 1's on the positions associated with the nodes in C_i, and 0's elsewhere. Namely, the entries of p_i are constant on the cells $C_1, \ldots, C_r$. A characteristic matrix $P \in \mathbb{R}^{n\times r}$ of a partition π is the matrix whose columns consist of the characteristic vectors associated with C_i, where

$$p_{ij} = \begin{cases} 1, & i \in C_j; \\ 0, & i \notin C_j \end{cases}$$

and p_{ij} is the entry of P.

For a partition $\pi = \{C_1, \ldots, C_r\}$ of $\mathcal{G}$, let α_{ij} be the number of neighbors in C_j of any node in C_i, and α_{ji} has a similar meaning. The quotient graph of $\mathcal{G}$ over π, denoted by $\mathcal{G}_\pi$, is defined as a weighted digraph with nodes $\mathcal{V}_\pi = \pi = \{C_1, \ldots, C_r\}$, the edge set $\mathcal{E}_\pi = \{(C_i, C_j)|\alpha_{ij} \neq 0\}$, and the weight of $(C_i, C_j) \in \mathcal{E}_\pi$ being α_{ij}.

If there are communication links between two nontrivial cells, we call this kind of cells as connected nontrivial cells. For the connected nontrivial cells, there is $\alpha_{ij}|C_i| = \alpha_{ji}|C_j|$. In view of the different numbers of links, there are two classes of nontrivial cells.

Definition 2: If $\alpha_{ij} = |C_j|(\alpha_{ji} = |C_i|)$, then we call these nontrivial cells CCNCs, otherwise, ICNCs.

Assume that there are two connected nontrivial cells C_1 and C_2. We see that $\alpha_{12}|C_1| = \alpha_{21}|C_2|$ is a necessary condition for equitable partition. Then, based on $\alpha_{12} = \alpha_{21}|C_2|/|C_1|$, only if $|C_1|$ and $|C_2|$ share a common divisor except for 1, there exists ICNCs, otherwise, only CCNCs. In Fig. 2(a), the two nontrivial cells are CCNCs while the nontrivial cells in Fig. 2(b) are ICNCs.

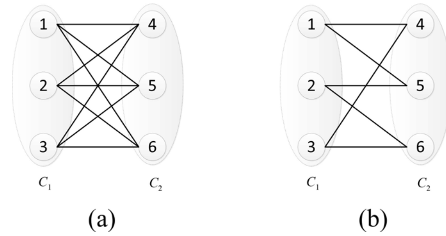

Fig. 2. (a) CCNCs. (b) ICNCs.

Let H denote the connection matrix that represents the connection relationship between the nodes of two nontrivial cells. The connection matrix of the graph in Fig. 2(b) is

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

The rows and columns of H correspond to the nodes in C_1 and C_2, respectively. Accordingly, all entries of the connection matrix between C_1 and C_2 in the graph of Fig. 2(a) is 1.

A. Completely Connected and Almost Completely Connected Nontrivial Cell

It is well known that the controllability is related to eigenvalues and eigenvectors of L. Below, we will present the results in terms of eigenvalues and eigenvectors, as well as the relationship between L and L_π.

Lemma 1 (Rahmani et al. [19]): Given a connected graph, the system (L_f, L_{fl}) is controllable if and only if L and L_f do not share any common eigenvalues.

Lemma 2 (Ji et al. [20]): The MAS is controllable if and only if there is no eigenvector of $\mathcal{G}$ taking 0 on the elements corresponding to the leaders.

Lemma 3 (Cardoso et al. [23]):

$$u \in Ker(L_\pi - \lambda I) \setminus \{0\} \Leftrightarrow Pu \in Ker(L - \lambda I) \setminus \{0\}$$

where L_π is the Laplacian matrix of quotient graph $\mathcal{G}_\pi$ which does not consider the self-loops.

Lemma 4 (Cardoso et al. [23]): Let $\mathcal{G}$ be a graph, L be its Laplacian matrix, $\pi = (V_1, \ldots, V_k)$ be a k-partition of V, and P be the characteristic matrix of π. Then, π is almost equitable k-partition if and only if there is a $k \times k$ matrix L_π such that

$$LP = PL_\pi$$

where L_π is the Laplacian matrix of quotient graph $\mathcal{G}_\pi$ which does not consider the self-loops.

Lemma 5 (Qu et al. [16]): For an equitable weights partition, if there is a nontrivial cell with m nodes, the number of linearly independent *Faria* vectors in eigenvector x corresponding to λ_{ij} is $m - 1$.

Equitable partition is one kind of equitable weights partitions, with all weights taking 1. The nontrivial cells in Lemma 5 are referring to CCNCs so that one nontrivial cell C_i can induce one eigenvalue with multiplicity $|C_i| - 1$, and the corresponding eigenvectors are *Faria* vectors, where *Faria* vector is a vector only containing two opposite nonzero entries 1 and -1.

Lemma 6: For an equitable partition with nontrivial cells, there is $x^T P = 0$, where x is an eigenvector of L with $x \neq Pu$, and u is a right eigenvector of L_π.

Proof: If $\mathcal{G}$ has an equitable partition with nontrivial cells, then the dimension of L_π is less than the dimension of L. There is at least one column of P with more than one nonzero entry such that Pu is an eigenvector of L, and the components $[Pu]_i$ and $[Pu]_j$ are identical, where u is a right eigenvector of L_π, $i, j \in C_r$. Since $LP = PL_\pi$, there is $x^T LP = \lambda x^T P = x^T PL_\pi$ such that $x^T P = 0$ for $x \neq Pu$, which implies that the sum of the entries in x corresponding to the nontrivial cell C_r is 0. ■

Lemma 7: L and L_f share at least one common eigenvalue if and only if $L_{fl}^T x_f = 0$, and the number of the linearly independent vectors x_f is no more than the number of the common eigenvalues, where x_f is an eigenvector of L_f.

Proof: If x_f is an eigenvector of L_f, then $L_f x_f = \lambda x_f$. Let $x = \{x_{lf}, x_{ll}\}^T$ denote an eigenvector of L, where $x_{lf} \in \mathbb{R}^{|\mathcal{V}_f|}$, $x_{ll} \in \mathbb{R}^{|\mathcal{V}_l|}$. Since $Lx = \lambda [x_{lf} \; x_{ll}]^T$, one has

$$\begin{bmatrix} L_f x_{lf} + L_{fl} x_{ll} \\ L_{fl}^T x_{lf} + L_l x_{ll} \end{bmatrix} = \lambda \begin{bmatrix} x_{lf} \\ x_{ll} \end{bmatrix}. \quad (6)$$

In case that $L_{fl}^T x_{lf} = 0$, there is an eigenvector $x = \{x_f, 0\}^T$ of L such that L and L_f share one common eigenvalue. In this case, the leader–follower system is uncontrollable, that is, $\exists \lambda$, $[\lambda I - L_f | L_{fl}]$ is rank deficient. Thus, there is a nonzero solution of $x_f^T [\lambda I - L_f | L_{fl}] = 0$. Let $R = \{I_f, 0\}$, where $R \in \mathbb{R}^{|V_f| \times |V|}$ and $I_f \in \mathbb{R}^{|V_f| \times |V_f|}$ is an identity matrix. Then, there are $L_f = RLR^T$ and $R^T x_f = \{x_f, 0\}$. One has

$$L_f x_f = RLR^T x_f = \lambda x_f \quad (7)$$

and

$$R^T RLR^T x_f = \begin{bmatrix} I_f & 0 \\ 0 & 0 \end{bmatrix} LR^T x_f = \lambda R^T x_f. \quad (8)$$

Since $LR^T x_f = \lambda R^T x_f$ does not always hold for every x_f, it can be derived from (8) that there exists a common eigenvalue between L and L_f, with x not taking the form of $\{x_f, 0\}^T$. Therefore, the number of the linearly independent vector x_f is less than or equal to the number of the common eigenvalues. ■

By Lemma 7, $x_f^T L_{fl} \neq 0$ is a necessary and sufficient condition for controllability. If $|V_l|$ increases and $|V_f|$ decreases, then $\mathrm{rank}(L_{fl}) \to |V_f|$. That is, the probability of $x_f^T L_{fl} \neq 0$ increases with the rank of L_{fl} increasing. Particularly, if the rank of L_{fl} reaches the maximum $|V_f|$, the system is controllable.

Lemma 8: For an equitable partition, if the follower subgraph contains r CCNCs and k trivial cells, then L and L_f share at least $\sum_{i=1}^{r+k} |C_i| - (r + k)$ common eigenvalues λ_{ij} with *Faria* vectors being its eigenvectors.

Proof: If $C_1, \ldots, C_r$ are CCNCs, then there are $|C_m| - 1$ eigenvalues λ_{ij} of the submatrix obtained by selecting the rows and columns corresponding to the nontrivial cell C_d, $m = \{1, \ldots, r\}$. By Lemma 5, L_f has $\sum_{i=1}^{r} |C_i| - r$ eigenvalues λ_{ij}, and all the corresponding eigenvectors x_f are *Faria* vectors. The rows of L_{fl} corresponding to the nodes of C_m are identical such that there is $L_{fl}^T x_f = 0$. Hence, $x = \{x_f, 0\}$ is an eigenvector of L, and the corresponding eigenvalue is λ_{ij}. By Lemma 7, L and L_f share at least $\sum_{i=1}^{r+k} |C_i| - (r + k)$ eigenvalues for the existence of nontrivial cells. ■

Remark 1: The nonzero entries in the *Faria* vectors associated with λ_{ij} correspond to the nodes in one nontrivial cell.

Definition 3: Assume that there are connected nontrivial cells with the same capacity m. If each node of C_i has $m - 1$ neighbors in its connected nontrivial cells, then we call these nontrivial cells almost completely connected nontrivial cells (ACNCs).

Theorem 1: For a given graph under equitable partition, the following statements hold.

1) If all nontrivial cells are CCNCs, then a necessary condition for controllability is choosing $|C_i| - 1$ leaders from each nontrivial cell C_i.
2) If all nontrivial cells are ACNCs and $H = E - I$, then a necessary condition for controllability is choosing $|C_i| - 1$ leaders from any nontrivial cell.

Proof:

1) If all nontrivial cells are CCNCs, then λ_{ij} is an eigenvalue of the submatrix L_r obtained by selecting the rows and columns corresponding to C_r, $i, j \in C_r$. The *Faria* vector x_r is an eigenvector of L_r, and the number of linearly independent x_r is $|C_r| - 1$. By Lemma 8, if there are CCNCs in a partition, then L has a multiple eigenvalue λ_{ij}, and the number of multiple eigenvalues is that of nontrivial cells. The eigenvector of λ_{ij} associated with L is $x = \{0, \ldots, 0, x_r, 0 \ldots, 0\}$, and the nonzero entry pair of x corresponds to C_r. Then, if the system is controllable, there are $|C_i| - 1$ leaders in each nontrivial cell.
2) Let $d(C_i)$ denote the degree of any node in nontrivial cell C_i, with $d(C_i) < d(C_{i+1})$. Assume that all the nontrivial cells are ACNCs, and the corresponding connection matrices can be transformed into $H = E - I$. Let L_i denote a submatrix obtained by selecting the rows and columns corresponding to C_i. Since $H = E - I$, then $L_i x_i = c_1 x_i$, $H x_i = c_2 x_i$, $c_1, c_2 \in \mathbb{R}$, and x_i is a *Faria* vector, $i = \{1, \ldots, r\}$. There is an eigenvalue λ_{C_i} of L with eigenvector $x = \{0_t, x_1, \ldots, x_r\}^T$, where 0_t is a zero vector with dimension equaling to the number of trivial cells. Once the positions of nonzero entries in x_i are fixed, the positions of nonzero entries in x_j are also fixed, where $i \neq j$. Thus, there are $|C_i| - 1$ linearly independent eigenvectors associated with λ_{C_i}. Accordingly, the multiplicity of eigenvalues $\lambda_{C_i}(L)$ is $|C_i| - 1$. For $\lambda_{C_1}(L) < \cdots < \lambda_{C_r}(L)$, the signs of diagonal entries of $[\lambda_{C_i} I - L]$ and $[\lambda_{C_{i+1}} I - L]$ corresponding to C_i and C_{i+1} are different, with the signs of the other diagonal entries being the same. Thus, the signs associated with the entries of the eigenvectors of $\lambda_{C_i}(L)$ and $\lambda_{C_{i+1}}(L)$ corresponding to C_i are different, with the signs of the other entries being the same. So there are r eigenvalues with multiplicity $|C_i| - 1$, and accordingly $|C_i| - 1$ leaders in one nontrivial cell for controllability. ■

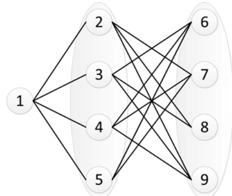

Fig. 3. Nontrivial cells with $|C_i| - 1$ links.

For a system without any ICNC, if the number of the leaders satisfies $|\mathcal{V}_l| < \sum_{i=1}^{r}(|C_i| - 1)$, then there is at least one nontrivial cell C_l in the follower subgraph, where C_l is a subcell of C_r. By Lemma 8, if there is a CCNC, then L and L_f share common eigenvalue λ_{ij}, and accordingly the system is uncontrollable. $|\mathcal{V}_l| = \sum_{i=1}^{r}(|C_i| - 1)$ ensures that there is not any nontrivial cell in the follower subgraph.

Example 1: There are three cells in the graph of Fig. 3, with $C_1 = \{1\}$, $C_2 = \{2, 3, 4, 5\}$, $C_3 = \{6, 7, 8, 9\}$, and $\alpha_{23} = \alpha_{32} = 3$. Hence, there are two eigenvalues 2.382 and 4.618 with multiplicity 3, and its eigenvectors consist of the linear combination of two orthogonal *Faria* vectors. The eigenvectors of 2.382 are $\{0, -2.2475, 2.2475, 0, 0, 3.367, -3.367, 0, 0\}^T$, $\{0, -2.2475, 0, 2.2475, 0, 3.367, 0, -3.367, 0\}^T$, and $\{0, -2.2475, 0, 0, 2.2475, 3.367, 0, 0, -3.367\}^T$.

B. Incompletely Connected Nontrivial Cell and Automorphism

The rank of the connection matrix will be discussed as follows.

1) Connection Matrix:

Proposition 1: Let $H \in \mathbb{R}^{b\times b}$ be a connection matrix. Then $\operatorname{rank}(H) = \operatorname{rank}(E - H)$, where $E \in \mathbb{R}^{b\times b}$. In addition, H is rank deficient if any of the following conditions hold.

1) $[(b + 1)/(c + 1)] < \alpha_{12} < [(b - 1)/c]$ for $[b/(c + 1)] < \alpha_{12} < (b/c)$.
2) $\alpha_{12} = (b/c)$

where $\alpha_{12} = \alpha_{21}$, $c \in \mathbb{N}$ is more than 2, and $\operatorname{rank}(H) = b - \min\{\alpha_{12}, b - \alpha_{12}\} - 1$.

Proof: The number of nonzero entries in every row of $H \in \mathbb{R}^{b\times b}$ is the same, so is to the matrix $(E - H) \in \mathbb{R}^{b\times b}$. Thus, $Hx = \mathbf{1}$ and $(E - H)x = \mathbf{1}$ have nonzero solutions, where $\mathbf{1}$ is a vector with all entries taking 1. Since $\operatorname{rank}(H) = \operatorname{rank}(H, E)$ and $\operatorname{rank}(E - H) = \operatorname{rank}(E - H, E)$, one has $\operatorname{rank}(E - H, E) = \operatorname{rank}(-H, E) = \operatorname{rank}(H, E)$. Hence, $\operatorname{rank}(H) = \operatorname{rank}(E - H)$.

The permutation of the nodes does not affect the controllability of a system. So let

$$H = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

with α_{12} nonzero entries in every row. Considering the solutions of $Hx = 0$, it follows that $x_i = x_{i+\alpha_{12}}$, where x_i and $x_{i+\alpha_{12}}$ are the components of x. Since $x_i = x_{i+\alpha_{12}}$ and $\alpha_{12} > b/2$, one has $x_i = x_{i+\alpha_{12}} = x_{i+(b-\alpha_{12})}$. If $Hx = 0$ has nonzero solutions, then $\alpha_{12} > 1$, $|2\alpha_{12} - b| > 1$. In view of $\operatorname{rank}(H) = \operatorname{rank}(E - H)$, we only consider the case of $\alpha_{12} \le b/2$. If $\alpha_{12} = b/c$, $c \in \mathbb{N}$, then $x_i = x_{i+c}$. If $\alpha_{12} > b/3$ and H is rank deficient, then $(b + 1)/3 < \alpha_{12} < (b - 1)/2$. Therefore, if $b/(c + 1) < \alpha_{12} < b/c$ and H is rank deficient, it follows $(b + 1)/(c + 1) < \alpha_{12} < (b - 1)/c$.

Assume $m = \min\{\alpha_{12}, b - \alpha_{12}\}$. Since $x_i = x_{i+\alpha_{12}}$, let $\bar{x}_1 = \{\underbrace{1, -1, 0, \ldots, 0}_{m}, 1, -1, 0, \ldots, 1, -1, 0, \ldots, 0\}$, $\bar{x}_2 = \{\underbrace{1, 0, -1, 0, \ldots, 0}_{m}, 1, -1, 0, \ldots, 0, 1, -1, \ldots, 0\}$, $\cdots$, $\bar{x}_{m-1} = \{\underbrace{1, 0, \ldots, 0, -1}_{m}, \underbrace{1, 0, \ldots, 0, -1, 1}_{m+1}, 0, \ldots, -1, 1, 0, \ldots, 0, -1\}$, and $\bar{x}_{\alpha_{12}} = \{\underbrace{1, 0, \ldots, 1}_{m}, 0, \ldots, 1, 0, \ldots, 0\}$. Thus, there are only $m - 1$ linearly independent vectors $\bar{x}_i$, which are the solutions of $Hx = 0$ for $\operatorname{rank}(H) < b$. Therefore, the number of the solutions of $Hx = 0$ is $m - 1$, and $b - \operatorname{rank}(H) = m - 1$. ■

If the nontrivial cells are completely connected, then *Faria* vectors are the solutions of $Hx = 0$ and the number of solutions is $|C_r| - 1$. Since *Faria* vectors are the eigenvectors of L, the corresponding eigenvalues are λ_{ij}, $i, j \in C_r$.

Proposition 2: Let H be a connection matrix between two ICNCs C_1 and C_2 with $|C_1| \ne |C_2|$. The following statements hold for *rank*(H).

1) If $c|C_1| = |C_2|$ and $c \in \mathbb{N} \setminus \{0, 1\}$, then $\operatorname{rank}(H) = |C_1|$.
2) If $c|C_1| = |C_2|$ and $c \notin \mathbb{N}$, with $\alpha_{21} = 2$, then $\operatorname{rank}(H) = |C_1| - 1$.

Proof:

1) The number of nonzero entries in each row and column of H is identical, so is to the zero entries. There exists a permutation of nodes such that the nonzero entries of H can be adjacent. The number of nonzero entries in each row of $H \in \mathbb{R}^{|C_1|\times|C_2|}$ is at least 2 for $|C_1| \ne |C_2|$. Therefore, if $c|C_1| = |C_2|$, and $c \in \mathbb{N}\setminus\{0, 1\}$, then the columns of H can be divided into $|C_2| - z$ parts averagely, where z is the number of nonzero entries in each row of H. The columns of every part partitioning from H are identical, and every part is independent from each other. Hence, there is at least one row with all entries taking 0 and the other rows taking 1 in the parts. The existence of each part induces a *Faria* vector which is a solution of $Hx = 0$. Thus, for $Hx = 0$, *Faria* vectors are the solutions with the simplest format of nonzero entries. The number of linearly independent solutions of $Hx = 0$ is $(|C_2|/\theta) \cdot (\theta - 1)$ such that $\operatorname{rank}(H) = |C_2| - |C_2|(\theta - 1)/\theta$, $\theta = \min\{\bar{z}, z\}$, $H \in \mathbb{R}^{|C_1|\times|C_2|}$, where $\bar{z}$ is the number of zero entries in each row of H. Since $\alpha_{12}|C_1| = \alpha_{21}|C_2|$ and $\operatorname{rank}(H) = \operatorname{rank}(E - H)$, we only consider the case of $\theta = z$. In this case, for $z = \alpha_{12}$ and $\alpha_{21} = 1$, it can be derived that $|C_2|/\alpha_{12}(\alpha_{12} - 1) = |C_2| - |C_1|$ and $\operatorname{rank}(H) = |C_1|$.
2) Assume $c|C_1| = |C_2|$, and $c \notin \mathbb{N}$, with $\alpha_{21} = 2$. Then, the parts partitioned from H can be divided into two classes of H_1 and H_2, with $H_1 \in \mathbb{R}^{|C_1|\times\alpha_1}$, $H_2 \in \mathbb{R}^{|C_1|\times\alpha_2}$, $\alpha_1 + \alpha_2 = \alpha_{12}$, and $\alpha_2 - \alpha_1 = 1$. Each column of H_1 is identical, so is the same to the columns of H_2. The solutions of $Hx = 0$ can be spanned by the solutions of $H_1x_1 = 0$ and $H_2x_2 =$

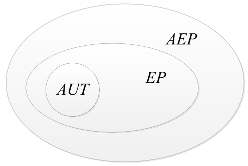

Fig. 4. Relationship among AEP, EP, and AUT.

0 such that $x = \{0, \dots, 0, x_1, 0, \dots, 0\}^T$ and $x = \{0, \dots, 0, x_2, 0, \dots, 0\}^T$. If $\alpha_1, \alpha_2 > 0$, then there are $|C_2|/b[(\alpha_1 - 1) + (\alpha_2 - 1)]$ *Faria* vectors which are the solutions of $Hx = 0$, and there is an extra solution constructed by $[H_1, H_2]x_{12} = 0$. Thus, there are $|C_2|/ \cdot (\alpha_1 - 1) + (\alpha_2 - 1)] + 1$ linearly independent solutions of $Hx = 0$, and $|C_2|/\alpha_{12}(\alpha_{12} - 2) + 1 = |C_2| - \text{rank}(H)$. Then, $\text{rank}(H) = |C_1| - 1$. ■

In part 2) of Proposition 2, α_1 and α_2 are divided from $|C_2|$ such that the numbers of H_1 and H_2 are the same. Therefore, there is $\alpha_{12} \cdot d = |C_2|$, $d \in \mathbb{N}$, which means that $|C_2|$ can be divisible by the number of nonzero entries in any row of H. It can be seen from $\alpha_{21} = 2$ that whatever $|C_2|$ is, $|C_1|$ is always even.

The solutions of $Hx = 0$ are spanned by the solutions of $H_i x_i = 0$ with $H = \{H_1, H_2, \dots, H_{d-1}, H_d\}$. Then, $x = \{0_1, \dots, 0_{i-1}, x_i, 0_{i+1}, \dots, 0_d\}^T$. Assume $H \in \mathbb{R}^{|C_1| \times |C_2|}$ and $|C_1| < |C_2|$. If $|C_1|/|C_2| \in \mathbb{N}$, then $H_1, H_2, \dots, H_d \in \mathbb{R}^{|C_1| \times \min\{z, |C_2| - z\}}$. In particular, if $|C_1|/|C_2| \notin \mathbb{N}$, with $\alpha_{21} = 2$, there is an extra solution spanned by $[\, H_1 \; H_2 \,]x_{12} = 0$, and accordingly $x = \{x_{12}, \dots, x_{12}\}^T$ for $Hx = 0$.

In the circumstance that some nontrivial cells are connected with more than one nontrivial cell, let $\hat{H}_i$ denote the connection matrix between C_i and all its connected nontrivial cells. Then, $\hat{H}_i = \{H_{i1}, \dots, H_{ig}\} = \{\hat{H}_{i1}, \dots, \hat{H}_{id}\}$, where H_{ij} is the connection matrix between C_i and C_j, $i \neq j$, $j \in \{1, \dots, g\}$. $\hat{H}_{iq}$ is one part of $\hat{H}_i$ which induces the solution of $\hat{H}_i x = 0$, $q \in \{1, \dots, d\}$, where d is the maximum number of the parts divided from $\hat{H}_i$.

Remark 2: For the case of $(\alpha_{21}/\alpha_{12}) \notin \mathbb{N}$, *rank*$(H)$ is related to the magnitude of α_{21}.

2) Automorphism:

Definition 4: The topology of the system has automorphisms, if there is a permutation matrix J such that $J^T L J = L$.

It is worth to mention that $J^T L J = L$ is a necessary and sufficient condition for topologies having automorphism. Definition 4 is defined only from the perspective of algebra. Note that there is another definition from graphics. If there is a permutation σ of the node set $\mathcal{V}$ such that the pair of nodes (u, v) form an edge if and only if the pair $(\sigma(u), \sigma(v))$ form an edge for $v, \sigma(v) \in S_1$, $u, \sigma(u) \in S_2$, then we say that there are automorphisms of $\mathcal{G}$. In this circumstance, the labels of $\sigma(v)$ and $\sigma(u)$ are different from v and u. The nodes in the sets S_1 and S_2 are called automorphism nodes.

An automorphism of graph $\mathcal{G}$ is an isomorphism from $\mathcal{G}$ to itself. The labels of automorphism nodes do not affect the existence of the edges, which means that the automorphism nodes have similar or the same communication links with the other nodes. Hence, one set of automorphism nodes induces one nontrivial cell under the equitable partition, and we can consider controllability by an automorphism. The relationship among almost equitable partition, equitable partition, and automorphism is shown as Fig. 4, where *AEP*, *EP*, and *AUT* denote almost equitable partition, equitable partition, and automorphism, respectively. Equitable partition is one kind of almost equitable partitions, and automorphism is one kind of equitable partitions.

Let J denote a permutation matrix. For the nodes in one nontrivial cell, its connection links are similar. As a consequence, if the labels of the nodes in the nontrivial cell are changed, the corresponding L remains unchanged. If there is one nontrivial cell, then there is a unitary matrix J with $JL = LJ$. Thus, $JLx = \lambda Jx = LJx$, where x is an eigenvector of L. It is easy to see that Jx is also an eigenvector of L. The entry of J is represented by

$$[J]_{ij} = \begin{cases} 1 & \sigma(i) = j; \\ 0 & \text{otherwise.} \end{cases}$$

Let J_1, J_2, and J_3 represent *recursion*, *symmetry*, and *strong symmetry*, respectively, where

$$J_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, J_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$J_3 = \begin{bmatrix} 0 & I_c \\ I_c & 0 \end{bmatrix}.$$

Proposition 3: The following assertions hold.

1) Automorphism nodes have *recursion*, *symmetry*, and *strong symmetry*.
2) The entries of eigenvectors x corresponding to automorphism nodes have *recursion*, *symmetry*, and *strong symmetry*.

Proof:

1) In view of the structure associated with nontrivial cells, the recurrent change of the positions associated with nodes in one nontrivial cell does not affect the structure of nontrivial cell. The *recursion* for automorphism nodes is identified. For CCNCs, the reversion of the positions of automorphism nodes does not affect the structure of edges, which follows *symmetry*. Once the nontrivial cells contain even number of nodes, then the exchanging between positions of v_i and $v_{i+|C_1|/2}$ does not affect the structure of edges, which indicates *strong symmetry*, where $i = 1, \dots, |C_1|/2$, $v_1, \dots, v_{|C_1|} \in C_1$.
2) If $J = \text{diagblocks}[I_1, J_1, I_2]$, then there are $JL = LJ$ and $JLx = \lambda Jx = LJx$, which implies *recursion* for eigenvectors, where I_1 and I_2 are identity matrices with proper dimensions. For the case of ICNCs, one has that the number of J_1 in J is equal to the number of nontrivial cells, and J_1s' positions correspond to nontrivial cells. Let $J = \text{diagblocks}[I_1, J_2, I_2]$. $JLx = \lambda Jx = LJx$ induces the *symmetry* for CCNCs. If the number of the nodes in nontrivial cells is even, there is $J = \text{diagblocks}[I_1, J_3, I_2]$ such that $JL = LJ$, where $J_3 = \begin{bmatrix} 0 \; I_c \\ I_c \; 0 \end{bmatrix}$ corresponds

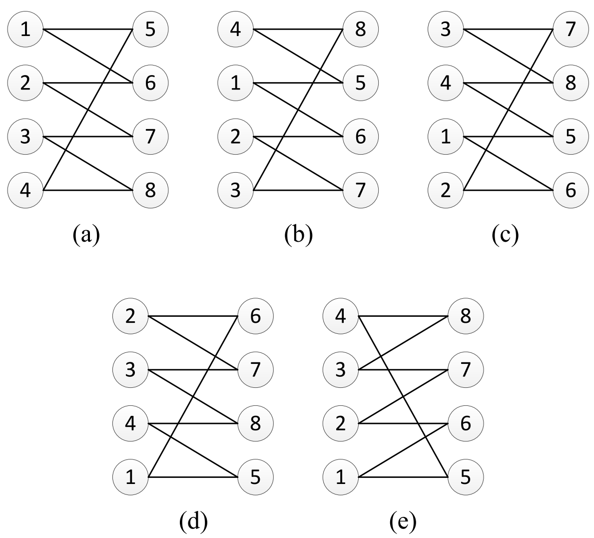

Fig. 5. Automorphisms. (a) Original graph with Laplacian L. (b) Automorphism of $\bar{J}_1^T L \bar{J}_1$. (c) Automorphism of $(\bar{J}_1^T)^2 L \bar{J}_1^2$, $\bar{J}_3^T L \bar{J}_3$. (d) Automorphism of $(\bar{J}_1^T)^3 L \bar{J}_1^3$. (e) Symmetric graph of $\mathcal{G}$.

to nontrivial cell C_1 and I_c is an identity matrix with dimension $|C_1|/2$. Assume that all nontrivial cells are incompletely connected. If there is an eigenvector x of L with $x^T P = 0$, then the eigenvector consists of orthogonal *Faria* vectors. By $J^T L J = L$, there is $J_2 L_i J_2 = L_i$, where L_i is a submatrix obtained by selecting the rows and columns corresponding to C_i. $J_2 H_i J_2$ is the elementary transformation of H_i, with only the permutations of the columns or rows of H_i changing. Due to $H_i x_i = c x_i$, it follows that $(\beta I - L_i) x_i - H_i x_i = 0$, where x_i is a *Faria* vector. Assume that there is a vector $\hat{x}_i = c_1 x_1 + \cdots + c_r x_r$, where x_i's are *Faria* vectors and orthogonal to each other. Then, one has $(\beta L_i - H_i)\hat{x}_i = 0$, $\beta = \lambda$. Thus, if $(\beta I - L_i - H_i)\hat{x}_i = 0$, then $(\beta L_i - H_i) J_2 \hat{x}_i = 0$. Accordingly, for $J = \text{diagblocks}[I_1, J_2, I_2]$ or $J = \text{diagblocks}[I_1, J_3, I_2]$, even $JL \neq LJ$, $JLx = \lambda Jx = LJx$ still holds. The above argument indicates the *symmetry* and *strong symmetry* of the eigenvectors of L. ■

Remark 3: The law of the position changing associated with automorphism nodes is related with the common divisor of nodes between nontrivial cells.

Example 2: An original graph is shown in Fig. 5, with its automorphisms being depicted as Fig. 5(b)–(e). $\bar{J}_1$, $\bar{J}_2$, and $\bar{J}_3$ are

$$\begin{aligned}\bar{J}_1 &= \begin{bmatrix} e_2 & e_3 & e_4 & e_1 & e_6 & e_7 & e_8 & e_5 \end{bmatrix}\\ \bar{J}_2 &= \begin{bmatrix} e_4 & e_3 & e_2 & e_1 & e_8 & e_7 & e_6 & e_5 \end{bmatrix}\\ \bar{J}_3 &= \begin{bmatrix} e_3 & e_4 & e_1 & e_2 & e_7 & e_8 & e_5 & e_6 \end{bmatrix}.\end{aligned}$$

For Fig. 5, a partition is $\pi = \{C_1, C_2\}$, where $C_1 = \{1, 2, 3, 4\}$ and $C_2 = \{5, 6, 7, 8\}$. If the positions of nodes in C_1 and C_2 are changed circularly, the structure of edges remains unchanged, as well as the connection of the nodes.

Based on the properties of automorphism, we claim that there are multiple eigenvalues of L. The odevity of the capacity of nontrivial cells determines whether there is a *strong symmetry* as well as the eigenvalues with multiplicity 3. The nontrivial cells with an odd number of nodes can only generate eigenvalues with multiplicity 2 for the lack of *strong symmetry*. The details will be represented in the next scenario.

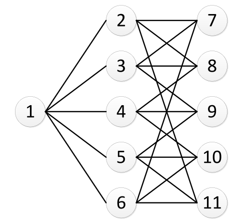

Fig. 6. Nontrivial cells with odd nodes.

For leader–follower systems, Rahmani *et al.* [19] proposed the concept of leader symmetry. The essence is that leader symmetry is one kind of automorphisms with fixed input nodes. Let $J = \text{diagblocks}[J_f, I_l]$, where $J_f \in \mathbb{R}^{|\mathcal{V}_f| \times |\mathcal{V}_f|}$, $I_l \in \mathbb{R}^{|\mathcal{V}_l| \times |\mathcal{V}_l|}$, and J and J_f are both permutation matrices. The system is leader symmetric if and only if

$$\begin{bmatrix} J_f^T & 0 \\ 0 & I_l \end{bmatrix} \begin{bmatrix} L_f & L_{fl} \\ L_{fl}^T & L_l \end{bmatrix} \begin{bmatrix} J_f & 0 \\ 0 & I_l \end{bmatrix} = \begin{bmatrix} L_f & L_{fl} \\ L_{fl}^T & L_l \end{bmatrix}$$

that is, if and only if there is a permutation matrix J_f with $J_f^T L_f J_f = L_f$, the system is leader symmetric.

Once a system is leader symmetric, then the system is uncontrollable. The intention of leader symmetry is to identify the uncontrollable system from the aspect of topologies. Hence, we propose an improved concept of leader symmetry, which is more feasible for the present work.

Definition 5: If the follower subgraph contains nontrivial cells, the system is leader symmetric.

Remark 4: The cases in Definition 5 contain ICNCs.

3) Incompletely Connected Nontrivial Cells: In this section, we analyze controllability by taking the advantage of automorphism. Only if all the connection matrices can be transformed into a symmetric form, there are automorphisms of $\mathcal{G}$. Therefore, we only consider the case of symmetric H_{ij} in subsequent arguments.

Lemma 9: If all nontrivial cells are ICCNs, not ACCNs, with the same capacity and $\text{rank}(H_{ij}) = |C_i|$, $i, j = \{1, \ldots, r\}$, the following statements hold.

1) For the odd capacity, there are $\sum_{i=1}^{r} (|C_i| - 1)/2$ eigenvalues with multiplicity 2, and the corresponding eigenvectors consist of $\sum_{i=1}^{r} (|C_i| - 1)/2$ orthogonal *Faria* vectors.
2) For the even capacity, there are eigenvalues with multiplicity 2 and 3. The corresponding eigenvectors consist of, respectively, $\sum_{i=1}^{r} (|C_i| - 2)/2$ and $\sum_{i=1}^{r} (|C_i| - 4)/2$ orthogonal *Faria* vectors.

Proof: The proof is presented in the Appendix. ■

Example 3: A graph with nontrivial cells containing odd nodes is shown in Fig. 6. There is a partition $\pi = \{C_1, C_2, C_3\}$, with $C_1 = \{1\}$, $C_2 = \{2, 3, 4, 5, 6\}$, and $C_3 = \{7, 8, 9, 10, 11\}$, where C_2 and C_3 are ICNCs, and $\alpha_{23} = \alpha_{32} = 3$.

For the case mentioned in part 2) of Lemma 9 , the nontrivial cells contain at least eight nodes, leading to the case of $\hat{H}_{ij} = |C_i|$. Assume that there is a partition $\pi = \{C_1, C_2, C_3\}$ with ICNCs, where $C_1 = \{1\}$, $C_2 = \{2, 3, 4, 5, 6, 7, 8, 9\}$,

$C_3 = \{10, 11, 12, 13, 14, 15, 16, 17\}$, and $\text{rank}(H) = 8$. Then, the corresponding L has eigenvalues with multiplicity 2 and 3. The eigenvectors take, respectively, the following form, $x_{m2} = \{0, 0, a, b, a, 0, -a, -b, -a, 0, c, d, c, 0, -c, -d, -c\}$ and $x_{m3} = \{0, l, 0, 0, m, -l, 0, 0, -m, p, 0, 0, q, -p, 0, 0, -q\}$.

Definition 6: For a graph with trivial cells being removed, the connected components associated with subgraphs are called connected components of nontrivial cells.

Let $\mathcal{C}_o^p$ and $\mathcal{C}_e^q$ denote, respectively, the set of a connected component of ICNCs with odd and even number of nodes, with $p = \{1, \ldots, c_o\}$ and $q = \{1, \ldots, c_e\}$, where c_o and c_e are the numbers of the connected components associated with $\mathcal{C}_o^p$ and $\mathcal{C}_e^q$, respectively.

Due to the existence of ICNCs, the columns of L_{fl} are not equal to each other anymore. Accordingly, the ranks of the connection matrices of ICNCs are not 1 anymore. For different cases of ICNCs, the ranks of the connection matrices are also different, which can take dissimilar effect on controllability.

Theorem 2: If all nontrivial cells are ICCNs, not ACCNs, with the same capacity and $\text{rank}(H_{ij}) = |C_i|$, then a necessary condition for controllability is choosing two leaders from one cell of $\mathcal{C}_o^p$ and three leaders from one cell of $\mathcal{C}_e^q$, $p = \{1, \ldots, c_o\}$, $q = \{1, \ldots, c_e\}$.

Proof: Assume that the nontrivial cells with the same capacity are incompletely connected, not almost completely connected. By Lemma 9, the nontrivial cells of $\mathcal{C}_o^p$ induce the eigenvectors consisting of orthogonal *Faria* vectors, and only one zero entry corresponding to one nontrivial cell of $\mathcal{C}_o^p$. The cells of $\mathcal{C}_e^q$ induce the eigenvectors with only four 0 entries corresponding to one nontrivial cell and only two zero entries in adjacent $|C_i|/2$ entries corresponding to C_i of $\mathcal{C}_e^q$, which means that every adjacent $|C_i|/2 - 1$ entries corresponding to the cell C_i of $\mathcal{C}_e^q$ have only one zero entry. The connected components of $\mathcal{C}_o^p$ yield the eigenvectors consisting of *Faira* vectors independently, so is to $\mathcal{C}_e^q$. Thus, a necessary condition for controllability is choosing two leaders from one cell of $\mathcal{C}_o^p$ and three leaders from one cell of $\mathcal{C}_e^q$, where $p = \{1, \ldots, c_o\}$ and $q = \{1, \ldots, c_e\}$. ■

If the connection matrices of some ICNCs are rank deficient, the controllability can be analyzed in the same way for the CCNCs. However, these ICNCs can destruct the controllability by one of them. Hence, we claim that the rank deficient $\hat{H}_i$ takes more similarity of links of the nodes in C_i, as well as the dynamics.

Theorem 3: If $\hat{H}_i$ is rank deficient, and the number of leaders in C_i is less than $|C_i| - \text{rank}(\hat{H}_i)$, then the system is uncontrollable, $i = \{1, \ldots, r\}$.

Proof: By Proposition 1, if $\hat{H}_r$ is rank deficient, L has an eigenvector x with entries $x_j = x_{j+|C_r|-\text{rank}(\hat{H}_r)}$. For $\lambda_{ij}I - L_r$, there is $\text{rank}(\lambda_{ij}I - L_r) = 1$, where L_r is a submatrix obtained by selecting the rows and columns corresponding to C_r, and $i, j \in C_r$. Thus, if $\hat{H}_r$ is rank deficient, L has an eigenvalue $\lambda_{ij} = l_{ii} - l_{ij}$ with multiplicity $|C_r| - \text{rank}(\hat{H}_r)$. For each $|C_r| - \text{rank}(\hat{H}_r)$ adjacent entries of eigenvectors of λ_{ij}, there are $|C_r| - \text{rank}(\hat{H}_r) - 1$ zero entries, where the $|C_r| - \text{rank}(\hat{H}_r)$ adjacent entries correspond to C_r, and the entries corresponding to the other cells are 0. Thus, if the number of leaders in C_i is less than $|C_i| - \text{rank}(\hat{H}_i)$, then the system is uncontrollable, $i = \{1, \ldots, r\}$. ■

The connection matrix of the nontrivial cells with different capacities can also be rank deficient. Therefore, once $\hat{H}_i$ is rank deficient, we need to choose leaders from C_i for controllability, and the number of leaders is dependent on the number of $|C_i| - \text{rank}(\hat{H}_i)$. The number of leaders in C_i is determined by the law of automorphism which induces the special eigenvector with 0 entries.

The essential discrepancy between CCNCs and ICNCs is on the rank of $\hat{H}_i$. The rank of $\hat{H}_i$ of CCNC is always 1. For different numbers of communication links between nontrivial cells, the situation for $\text{rank}(\hat{H}_i)$ can be different.

Proposition 4: Choosing leaders from nontrivial cells is not a sufficient condition for controllability.

Proof: If the right eigenvectors of L_π corresponding to the nontrivial cells have zero entries, then it gives rise to a circumstance that all leaders correspond to the zero entries of one eigenvector. It follows that choosing leaders from nontrivial cells is only a necessary condition to ensure systems being controllable. Only for the case that there is any zero entry corresponding to nontrivial cells in the right eigenvectors of L_π, the condition of choosing leaders from nontrivial cells is sufficient. ■

Assume that all nontrivial cells are ICNCs. Once we choose a leader from one nontrivial cell, then the structure of the nontrivial cell is destructed. Thus, to ensure controllability, we need to choose more than one leader from a nontrivial cell. Hence, it can be seen that symmetry is not a necessary condition for uncontrollability, namely, even all cells are trivial, there is a case that the system is uncontrollable. The disadvantage of equitable partition and almost equitable partition is that we cannot identify the principle of the right eigenvectors of L_π, which is the key obstacle for establishing a sufficient and necessary condition of graphics for controllability.

IV. Almost Equitable Partition

If a system is controllable, the corresponding graph must be asymmetric. For an asymmetric graph, if there is a nontrivial cell under the almost equitable partition, the corresponding system is uncontrollable. Asymmetry is only a necessary condition for controllability.

Lemma 10 (Qu et al. [16]): Assume that there is only one nontrivial cell associated with $L_r \in \mathbb{R}^{m\times m}$. The Laplacian matrices L and L_r share $m - 1$ common eigenvalues λ_{ij}. Moreover, for matrix L, its x's associated with λ_{ij} take x_r as a subvector, where x_r is an eigenvector of λ_{ij} associated with L_r, that is, $x^T = \{x_r^T, 0, \ldots, 0\}$.

The existence of nontrivial cells in L_r makes that part of eigenvalues and eigenvectors of L rely on L_r. It can be seen that Lemma 8 holds for almost equitable partition with ICNCs while the corresponding eigenvectors are not *Faria* vectors anymore.

Lemma 11 (Rahmani et al. [19]): Let P be the characteristic matrix of an equitable partition with nontrivial cells. $\mathcal{R}(P)$ is K-invariant, where $K = \text{diag}\{k_i I_{n_i}\}_{i=1}^r$, $k_i \in \mathbb{R}$, $n_i = |C_i|$ is the cardinality of the cell, and $r = |\pi|$ is the cardinality of the partition.

Theorem 4: For a system under an almost equitable partition, if the follower subgraph contains one CCNC or all ICNCs, the system is uncontrollable.

Proof: By Lemma 10, if the follower subgraph contains one CCNC C_r, then L and L_f share at least $|C_r| - 1$ common eigenvalues. Accordingly, the system is uncontrollable.

Since $u \in Ker(L_\pi - \lambda I)\backslash\{0\} \Leftrightarrow Pu \in Ker(L - \lambda I)\backslash\{0\}$, if the nonzero entry in one column of P is not unique, then there is an eigenvector $x = Pu$ with $x_i = x_j$, where x_i and x_j are the components of x, $i \neq j$, $i, j \in C_r$. Let P_f denote the characteristic matrix of the follower subgraph. By Lemmas 4 and 11, there is $L_f P_f = P_f L_{f\pi}$, $L_{f\pi} = L_{f\pi} + D_r$, and $D_r = \text{diag}\{N(1) \cap \mathcal{V}_l, \ldots, N(r) \cap \mathcal{V}_l\}$, where $N(i) \cap \mathcal{V}_l$ denotes the neighbors in $\mathcal{V}_l$ of the node j, $j \in C_i$, and $L_{f\pi}$ is the Laplacian of quotient graph associated with $\mathcal{G}_f$. If $\mathcal{G}_f$ contains one nontrivial cell, then the dimension of $L_{f\pi}$ is less than that of L_f. There is at least one eigenvalue belonging to L_f, not $L_{f\pi}$. Let x_f denote an eigenvector of λ associated with L_f, where λ is not an eigenvalue of $L_{f\pi}$. Then, by Lemma 6, there is $x_f^T P_f = 0$. Therefore, $x = \{x_f, 0_l\}^T$ with $x_f^T 1_f = 0$ is an eigenvector of L. L_f shares $r+k$ common eigenvalues with $L_{f\pi}$ for $r+k$ cells, and there is a case that L and L_f share one common eigenvalue which is one of the $r+k$ eigenvalues. Therefore, L and L_f share at least $\sum_{i=1}^{r} |C_i| - (r+k)$ common eigenvalues. Thus, if the follower subgraph contains all ICNCs, the system is uncontrollable. ■

By nontrivial cells, the equitable partition and almost equitable partition reveal uncontrollable topologies. No matter what type of nontrivial cells, the nodes in nontrivial cells are destructive to the controllability. The aim of the above necessary condition for controllability is to destruct the uncontrollable topologies such that the dynamics of the nodes in one nontrivial cell are not linearly dependent anymore.

V. Controllable Subspace and Minimum Inputs

In the following, we will discuss the controllable subspace of MASs by the PBH criterion.

Lemma 12 (Dullerud and Paganini [25]): The pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is controllable if and only if, for each $\lambda \in \mathbb{C}$, the rank condition

$$\text{rank}[\lambda I - A|B] = n \text{holds}.$$

Lemma 13 (Polderman and Willems [26]): Let $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ and assume that there exists a nonzero vector $v_0 \in \mathbb{C}^n$ such that $v_0^T A^k B = 0$, $k = 0, \ldots, n-1$. Define the linear subspace $\mathcal{V}$ of $\mathbb{C}^n$ as

$$\mathcal{V} := \underset{k\geq 0}{\text{span}}\left\{\left(A^T\right)^k v_0\right\}.$$

Then, there exists a nonzero vector $v_1 \in \mathcal{V}$ and $\lambda_1 \in \mathbb{C}$ such that

$$v_1^T A = \lambda_1 v_1^T \text{ and } v_1^T B = 0.$$

Proposition 5: The dimension w of controllable subspace satisfies

$$w = \sum_{i=1}^{m} \text{rank}[L - \lambda_i I|B] - n(m-1)$$

where $L \in \mathbb{R}^{n\times n}$, λ_i's are the distinct eigenvalues of L, $i = 1, \ldots, m$, $m \in \{1, \ldots, n\}$.

Proof: Suppose the system is uncontrollable, then there is an eigenvalue λ_i of $L \in \mathbb{R}^{n\times n}$ such that $\text{rank}[L - \lambda_i I|B] < n$. Hence, there exists a nonzero vector x_i such that $x_i^T[L - \lambda_i I|B] = 0$. The number of linearly independent solutions x_i^T is determined by $n - \text{rank}[L - \lambda_i I|B]$. Since $x_i^T B = 0$ and $x_i^T[L - \lambda_i I] = 0$, x_i is an eigenvector associated with λ_i and is orthogonal to each column of B. $x_i^T L^k = \lambda_i^k x_i^T$

$$x_i^T\begin{bmatrix} B & LB & \cdots & L^{n-1}B \end{bmatrix} = 0$$

and the number of linearly independent solutions x_i^T is determined by $n - w$, where w is the dimension of controllable subspace. By Lemma 13, one solution of $x_i^T[B\ LB \cdots L^{n-1}B] = 0$ corresponds to one solution of $x_i^T[L - \lambda_i I|B] = 0$ such that $\text{rank}[L - \lambda_i I|B] = n$ only determines the controllability of one-dimensional subspace. Therefore, the dimension of controllable subspace w is determined by the number of x_i with $x_i^T B = 0$. Consequently, $w = n - \sum_{i=1}^{m} [n - \text{rank}(L - \lambda_i I|B)]$, where λ_i's are distinct eigenvalues, $i = 1, \ldots, m$, $m \in \{1, \ldots, n\}$. ■

Let w_c, w_{ac}, and w_{in} denote the dimension of controllable subspace of the system with only CCNCs, ACNCs, and ICNCs under equitable partition, respectively. Based on the different numbers of nodes in ICNCs, w_{in} can be divided into w_{ino} and w_{ine}, which correspond to, respectively, the nontrivial cells with odd number of nodes and even number of nodes. For ICNCs, we only consider the case that all connection matrices are full rank.

Theorem 5: Under fixed partition, if all nontrivial cells share a common capacity, then $w_c \leq w_{ac} \leq w_{\text{in}}$.

Proof: Assume that there is an equitable partition $\pi = \{C_1, C_2^*, C_3^*\}$, where $|C_1| = 1$, $* \in \{c, ac, in\}$. Let C_i^c, C_i^{ac}, and C_i^{in} denote, respectively, completely connected, almost completely connected, and ICNC, $i = 2, 3$. The existence of C_i^c induces the eigenvectors $x_{c1} = \{0, x_F, 0_2\}$ and $x_{c2} = \{0, 0_1, x_F\}$, where 0 corresponds to C_1, 0_i is a zero vector corresponding to C_i^c, and x_F is a *Faria* vector. There is an eigenvector $x_{ac} = \{0, \alpha_1 x_F, \alpha_2 x_F\}$ of L for C_i^{ac}, where $\alpha_i \in \mathbb{R}$, and x_F corresponds to nontrivial cells. For C_i^{in} with odd number of nodes and $\text{rank}(H) = |C_i^{\text{in}}|$, there is an eigenvector $x_{ino} = \{0, x_{o1}, x_{o2}\}$, where x_{oi} corresponds to C_i^{in} with only one zero entry. For C_i^{in} with even number of nodes and $\text{rank}(H) = |C_i^{\text{in}}|$, there is an eigenvector $x_{ine} = \{0, x_{e1}, x_{e2}\}$, where x_{ei} corresponds to C_i^{in}, and only two zero entries in each $|C_i^{\text{in}}|/2$ adjacent entries of x_{ei}.

If the system is controllable, then whatever case of C_i^* is in, the dimension of controllable subspace is always n. If the system is uncontrollable under fixed B, the number of solutions of $x^T[L - \lambda I|B] = 0$ varies along with different cases of C_i^*. If the format of x is determined, the more zero entries in x, the more solutions of $x^T B = 0$ arise. Let $x_c \in \{x_{c1}, x_{c2}\}$. Then for different cases, the numbers of the eigenvectors with respect to three formats of x_c, x_{ac}, and $x_{ino}(x_{ine})$ are the same. The number of zero entries in x_c is larger than that in x_{ac}, and x_{ac} contains more zero entries than $x_{\rm in}^o(x_{\rm in}^e)$. Then, the number of solutions of $x_c^T B = 0$ and $x_{ac}^T B = 0$ is no less than that of $x_{ac}^T B = 0$ and $x_{ino}^T B = 0 (x_{ine}^T B = 0)$, respectively. The circumstance of more trivial cells and nontrivial cells can be proved in the same way. Hence, $w_c \le w_{ac} \le w_{\rm in}$. ■

Let $\mathcal{X}_i$ denote a node set, with the labels of these nodes corresponding to the nonzero entries in an eigenvector x_i. For example, let $x_1 = \{0, 1, -1, 0, 0, -1, 1\}$, then $\mathcal{X}_1 = \{v_2, v_3, v_6, v_7\}$. However, there are some sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$ sharing common nodes. So let $\hat{\mathcal{X}}_j$ denote the intersection of $\mathcal{X}_i$'s, namely, $\hat{\mathcal{X}}_j = \mathcal{X}_p \cap \cdots \cap \mathcal{X}_{p+q}$; $\mathcal{X}_p, \ldots, \mathcal{X}_{p+q}$ are the sets sharing common nodes, where $p, (p + q) \in \{1, \ldots, n\}$. Especially, for the sets $\hat{\mathcal{X}}_1, \ldots, \hat{\mathcal{X}}_m$ induced by $\mathcal{X}_i$, there is not any common nodes among $\hat{\mathcal{X}}_j$s, $j = \{1, \ldots, m\}$, $m \in \{1, \ldots, n-1\}$. Note that the number of $\hat{\mathcal{X}}_j$'s is unique, while the form of $\hat{\mathcal{X}}_j$'s is not unique because a set $\hat{\mathcal{X}}_j$ can be constructed by different $\mathcal{X}_i$'s.

Theorem 6: The MAS is controllable under the minimum inputs if and only if one leader is chosen from each set $\hat{\mathcal{X}}_i$, $i = \{1, \ldots, m\}, m \in \{1, \ldots, n-1\}$.

Proof (Sufficiency): If each column of B has one nonzero entry corresponding to the nonzero entry of all eigenvectors of L. Then, $\text{rank}[\lambda_j I - L|B] = n$, $j = \{1, \ldots, n\}$. Let one input only be exerted on one node of $\hat{\mathcal{X}}_i$, that is, choosing one leader from each set $\hat{\mathcal{X}}_i$ such that the system is controllable, $i = \{1, \ldots, m\}, m \in \{1, \ldots, n-1\}$. If there are only $m - 1$ inputs in the system, then there is at least one matrix $[\lambda_j I - L|B]$ being rank deficient. Therefore, there is at least one leader in each $\hat{\mathcal{X}}_i$ for controllability, and the minimum number of inputs to make system controllable is m.

Necessity: Assume that the system is controllable, and the leaders are arbitrarily selected from $\mathcal{X}_i$, then there can be more than one leader selected from some $\hat{\mathcal{X}}_i$'s. Thus, under m inputs, the system is controllable only if one leader is chosen from each set $\hat{\mathcal{X}}_i$, $i = \{1, \ldots, m\}, m \in \{1, \ldots, n-1\}$. ■

Here, one leader is only injected by one input. Thus, once the minimum of leaders is determined for controllability, the minimum of inputs is also determined.

Proposition 6: For a connected undirected graph with n nodes, if there are $n - 2$ leaders and the two followers do not constitute a nontrivial cell, then the system is controllable.

Proof: Let $R^T = [I_{n-1}, 0_{n-1}]$, $I_{n-1} \in \mathbb{R}^{n-1 \times n-1}$ and $0_{n-1} \in \mathbb{R}^{n-1}$. For the connected graph with n nodes, if there are $n-1$ nodes taking leaders' rule, one has the dynamic equation $\dot{x} = -Lx + Bu$, with $B = R$. By the PBH rank condition, if the system is uncontrollable, then there is one eigenvector x of L, with $x^T B = 0$, which means that the first $n-1$ entries of x are 0, and only the nth entry is nonzero. For any eigenvector x of L except for 1_n, there is $x^T 1_n = 0$. So there is no eigenvector x with only one nonzero entry. If there are $n - 1$ nodes to be taken as leaders, the system is controllable.

Let $B = [I_{n-2}\ 0]^T$ for $n - 2$ leaders. Assume that there is an eigenvector x of L with $x^T B = 0$. It can be seen from $x^T 1_n = 0$ and $x^T B = 0$ that x is a *Faria* vector such that the two followers can be shaped into one nontrivial cell. Thus, for $n - 2$ leaders, if the two followers cannot be constructed into a nontrivial cell, the system is controllable. ■

By Proposition 6 and Theorem 6, we can derive the upper and lower bounds of the number of inputs for a controllable system, that is, $m \le N_I \le n - 1$, where N_I is the number of inputs. If the followers cannot be partitioned into a nontrivial cell, then the bound of inputs is $m \le N_I \le n - 2$.

Remark 5: The significant contribution of this article is to propose an eigenvalue and eigenvector analysis method of controllability based on equitable partitions. The superiority of the results is that they provide the judgment conditions of controllability in multiple directions, including the selection number of leaders, the dimension of controllable subspace, the eigenvector-based partition of topology, as well as their relation with automorphism. The sensitivity of the obtained results is mainly due to the change of eigenvectors caused by different topological structures.

VI. Conclusion

In this article, we mainly studied the controllability by using equitable partition with CCNCs and ICNCs. For the system with CCNCs and ICCNs, necessary conditions for controllability are derived. To ensure the controllability of the systems with only CCNC, there are $|C_i| - 1$ leaders in each nontrivial cell. Especially, for the case of ICNCs: 1) the controllability was shown to be affected by three factors the number of links between nontrivial cells; 2) the odevity of the capacity of nontrivial cells; and 3) the rank of connection matrix $\hat{H}_i$. Different from the case of CCNCs, a necessary condition for controllability of the system with only ACNCs is choosing $|C_i| - 1$ leaders from any nontrivial cell. Under $\text{rank}(\hat{H}_i) = |C_i|$, for the case of ICNCs with an odd number of nodes and even number of nodes, a necessary condition for controllability is choosing any two and any three leaders, respectively, from any nontrivial cell. In addition, we revealed the relationship between the PBH rank criterion and the controllable subspace such that we can analyze the controllable subspace by the PBH rank criterion. Finally, a necessary and sufficient condition for controllability under minimum inputs is given.

In future work, we will investigate the controllability of directed and weighted graphs, and analyze the differences between undirected graphs and directed graphs on controllability, as well as the effect of weights to controllability.

Appendix A
Proof of Lemma 9

1) Assume H_{ij} is symmetric, and there are three cells $C_1 = \{1\}$, $C_2 = \{2, \ldots, k\}$, and $C_3 = \{k+1, \ldots, n\}$, where $n = 2k - 1$, and $k - 1$ is an odd number. Then, $x = \{x_1, x_2, \ldots, x_{[(k+2)/2]}, \ldots, x_k, x_{k+1}, \ldots, x_{[(n+k+1)/2]}, \ldots, x_n\}^T$ is an eigenvector of L. It follows from the *symmetry* of eigenvectors and $J = \text{diagbolcks}\{1, J_2, J_2\}$, $J_2 \in \mathbb{R}^{(k-1)\times(k-1)}$ that $Jx = \{x_1, x_k, \ldots, x_{[(k+2)/2]}, \ldots, x_2, x_n, \ldots, x_{[(n+k+1)/2]}, \ldots, x_{k+1}\}^T$ is still an eigenvector of L, and so is $Jx - x = \{0, x_k - x_2, \ldots, x_{[(k+4)/2]} - x_{(k/2)}, 0, x_{(k/2)} - x_{[(k+4)/2]},$

$\ldots, x_2 - x_k, x_n - x_{k+1}, \ldots, x_{[(n+k+3)/2]} - x_{[(n+k+1)/2]}, 0, x_{[(n+k+1)/2]} - x_{[(n+k+3)/2]}, \ldots, x_{k+1} - x_n\}^T$. Hence, if $x \neq Pu$, then the eigenvectors consist of $\sum_{i=1}^{3}(|C_i| - 1)/2$ orthogonal *Faria* vectors. Let $\tilde{x}_1 = x$, $\tilde{x}_2 = Jx$, and $\tilde{x}_3$ be an eigenvector of the same eigenvalue with zero entries taking different positions from $\tilde{x}_1$ and $\tilde{x}_2$; Once there are adjacent nonzero entries of *Faria* pair in x, $\tilde{x}_3$ is a linear combination of $\tilde{x}_1$ and $\tilde{x}_2$, otherwise, $\tilde{x}_3$ is a vector with all entries taking 0. Hence, there are $\sum_{i=1}^{r}(|C_i| - 1)/2$ eigenvalues with multiplicity 2. The case of more than one trivial cell and two nontrivial cells can be proved in the same way.

2) Assume that there are three cells, $V_1 = \{v_1\}$, $V_2 = \{v_2, \ldots, v_k\}$, and $V_3 = \{v_{k+1}, \ldots, v_n\}$, $V_1 \in C_1$, $V_2 \in C_2$, and $V_3 \in C_3$, where $k - 1 = n - k$ is an even number. If H is symmetric, and there is an eigenvector $x = \{x_1, x_2, \ldots, x_{[(k+1)/2]}, x_{[(k+3)/2]}, \ldots, x_k, x_{k+1}, \ldots, x_{[(n+k)/2]}, x_{[(n+k+2)/2]}, \ldots, x_n\}^T$ of L, then $Jx = \{x_1, x_{[(k+3)/2]}, \ldots, x_k, x_2, \ldots, x_{[(k+1)/2]}, x_{[(n+k+2)/2]}, \ldots, x_n, x_{k+1}, \ldots, x_{[(n+k)/2]}\}^T$ is also an eigenvector of L because of the *symmetry* of eigenvectors. $Jx - x = \{0, x_{[(k+1)/2]} - x_2, \ldots, x_k - x_{[(k+3)/2]}, x_2 - x_{[(k+1)/2]}, \ldots, x_{[(k+3)/2]} - x_k, x_{[(n+k+2)/2]} - x_{k+1}, \ldots, x_n - x_{[(n+k)/2]}, x_{k+1} - x_{[(n+k+2)/2]}, \ldots, x_{[(n+k)/2]} - x_n\}^T$ and $Jx + x = \{0, x_{[(k+1)/2]} + x_2, \ldots, x_k + x_{[(k+3)/2]}, x_2 + x_{[(k+1)/2]}, \ldots, x_{[(k+3)/2]} + x_k, x_{[(n+k+2)/2]} + x_{k+1}, \ldots, x_n + x_{[(n+k)/2]}, x_{k+1} + x_{[(n+k+2)/2]}, \ldots, x_{[(n+k)/2]} + x_n\}^T$. $Jx - x$ and $Jx + x$ induce, respectively, the eigenvectors with $x_i^j = x_{i+(|C_j|/2)}^j$ and $x_i^j = -x_{i+(|C_j|/2)}^j$, where $j \in C_j$, $x_i^j = \{x_i | v_i \in V_{(C_j/2)}\}$, $V_{(C_1/2)} = \{v_2, \ldots, v_{[(k-1)/2]}\}$, and $V_{(C_2/2)} = \{v_{k+1}, \ldots, v_{[(n+k/2]}\}^T$. Due to the *symmetry* and *recursion* of eigenvectors, $x_s = \{x_1, x_k, \ldots, x_{[(k+1/2)}, x_{[(k-1)/2]}, \ldots, x_2, x_n, \ldots, x_{[(n+k+2)/2]}, x_{[(n+k)/2]}, \ldots, x_{k+1}\}^T$ and $x_r = \{x_1, x_3, \ldots, x_{[(k+1)/2]}, x_{[(k+3)/2]}, \ldots, x_2, x_{k+2}, \ldots, x_{[(n+k)/2]}, x_{[(n+k+2)/2]}, \ldots, x_{k+1}\}^T$ are the eigenvectors associated with x. $x_s - x_r = \{0, \bar{x}_2, \ldots, \bar{x}_{k-1}, 0, -\bar{x}_2, \ldots, -\bar{x}_{k-1}, 0\}^T$ yields an eigenvector which is a linear combination of $\sum_{i=1}^{r}(|C_i| - 2)/2$ orthogonal *Faria* vectors. Let $\bar{x}_{sr1} = x_s - x_r$, and $\bar{x}_{sr2} = \{0, 0, \bar{x}_2, \ldots, \bar{x}_{k-1}, 0, -\bar{x}_2, \ldots, -\bar{x}_{k-1}\}^T$. If the zero entries of $\bar{x}_{sr3}$ take different positions from $\bar{x}_{sr1}$ and $\bar{x}_{sr2}$, $\bar{x}_{sr3}$ can be constructed by the linear combination of $\bar{x}_{sr1}$ and $\bar{x}_{sr2}$. Then, the multiplicity of the corresponding eigenvalue is 2.

For the eigenvector $\tilde{x} = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_{[(k-1)/2]}, \tilde{x}_{[(k+1)/2]}, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_{[(k-1)/2]}, \tilde{x}_{[(k+1)/2]}, \tilde{x}_{k+1}, \tilde{x}_{k+2}, \ldots, \tilde{x}_{[(n+k)/2]-1}, \tilde{x}_{[(n+k)/2]}, \tilde{x}_{k+1}, \tilde{x}_{k+2}, \ldots, \tilde{x}_{[(n+k)/2]-1}, \tilde{x}_{[(n+k)/2]}\}^T$ of L, based on the *symmetry* of eigenvectors, there is an eigenvector $\tilde{x}_s$ of L, with $\tilde{x}_s = \{\tilde{x}_1, \tilde{x}_{[(k+1)/2]}, \tilde{x}_{[(k-1)/2]}, \ldots, \tilde{x}_3, \tilde{x}_2, \tilde{x}_{[(k+1)/2]}, \tilde{x}_{[(k-1)/2]}, \ldots, \tilde{x}_3, \tilde{x}_2, \tilde{x}_{[(n+k)/2]}, \tilde{x}_{[(n+k)/2]-1}, \ldots, \tilde{x}_{k+2}, \tilde{x}_{k+1}, \tilde{x}_{[(n+k)/2]}, \tilde{x}_{[(n+k)/2]-1}, \ldots, \tilde{x}_{k+2}, \tilde{x}_{k+1}\}^T$. $\tilde{x}_{sr} = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_{[(k+1)/2]}, \ldots, \tilde{x}_4, \tilde{x}_3, \tilde{x}_2, \tilde{x}_{[(k+1)/2]}, \ldots, \tilde{x}_4, \tilde{x}_3, \tilde{x}_{k+1}, \tilde{x}_{[(n+k)/2]}, \ldots, \tilde{x}_{k+3}, \tilde{x}_{k+2}, \tilde{x}_{k+1}, \tilde{x}_{[(n+k)/2]}, \ldots, \tilde{x}_{k+3}, \tilde{x}_{k+2}\}^T$ is an eigenvector of L for the *recursion* of eigenvector. Thus, both $\tilde{x}_s$ and $\tilde{x}_{sr}$ are the eigenvectors of L for the same eigenvalue associated with $\tilde{x}$. $\tilde{x} - \tilde{x}_{sr} = \{0, 0, \tilde{x}_3 - \tilde{x}_{[(k+1)/2]}, \ldots, \tilde{x}_{[(k-1)/2]} - \tilde{x}_4, \tilde{x}_{[(k+1)/2]} - \tilde{x}_3, 0, \tilde{x}_3 - \tilde{x}_{[(k+1)/2]}, \ldots, \tilde{x}_{[(k-1)/2]} - \tilde{x}_4, \tilde{x}_{[(k+1)/2]} - \tilde{x}_3, 0, \tilde{x}_{k+2} - \tilde{x}_{[(n+k)/2]}, \ldots, \tilde{x}_{[(n+k)/2]-1} - \tilde{x}_{k+3}, \tilde{x}_{[(n+k)/2]} - \tilde{x}_{k+2}, 0, \tilde{x}_{k+2} - \tilde{x}_{[(n+k)/2]}, \ldots, \tilde{x}_{[(n+k)/2]-1} - \tilde{x}_{k+3}, \tilde{x}_{[(n+k)/2]} - \tilde{x}_{k+2}\}^T$ is an eigenvector of L. By the above manner, there is an eigenvector $\hat{x} = (\tilde{x} - \tilde{x}_{sr}) - (\tilde{x} - \tilde{x}_{sr})_{sr} = \{0, 0, \hat{x}_3, \ldots, \hat{x}_{g-1}, 0, \hat{x}_{g+1}, \ldots, \hat{x}_{[(k+1)/2]}, 0, -\hat{x}_3, \ldots, -\hat{x}_{g-1}, 0, -\hat{x}_{g+1}, \ldots, -\hat{x}_{[(k+1)/2]}, 0, \hat{x}_{k+2}, \ldots, \hat{x}_{h-1}, 0, \hat{x}_{h+1}, \ldots, \hat{x}_{[(n+k)/2]}, 0, -\hat{x}_{k+2}, \ldots, -\hat{x}_{h-1}, 0, -\hat{x}_{h+1}, \ldots, -\hat{x}_{[(n+k)/2]}\}^T$ of L, where $|C_1|/4 + 1 \leq g \leq (|C_1| + 2)/4 + 1$ and $|C_2|/4 + k \leq h \leq (|C_2| + 2)/4 + k$. After transformation, the 0 entries of eigenvector $\hat{x}$ can be adjacent such that there is an eigenvector $\hat{x}_1 = \{0, 0, 0, \hat{x}_4, \ldots, \hat{x}_{[(k+1)/2]}, 0, 0, -\hat{x}_4, \ldots, -\hat{x}_{[(k+1)/2]}, 0, 0, \hat{x}_{k+3}, \ldots, \hat{x}_{[(n+k)/2]}, 0, 0, -\hat{x}_{k+3}, \ldots, -\hat{x}_{[(n+k)/2]}\}^T$. $\hat{x}_2 = \{0, 0, \hat{x}_4, \ldots, \hat{x}_{[(k+1)/2]}, 0, 0, -\hat{x}_4, \ldots, -\hat{x}_{[(k+1)/2]}, 0, 0, \hat{x}_{k+3}, \ldots, \hat{x}_{[(n+k)/2]}, 0, 0, -\hat{x}_{k+3}, \ldots, -\hat{x}_{[(n+k)/2]}, 0\}^T$ and $\hat{x}_3 = \{0, \hat{x}_4, \ldots, \hat{x}_{[(k+1)/2]}, 0, 0, -\hat{x}_4, \ldots, -\hat{x}_{[(k+1)/2]}, 0, 0, \hat{x}_{k+3}, \ldots, \hat{x}_{[(n+k)/2]}, 0, 0, -\hat{x}_{k+3}, \ldots, -\hat{x}_{[(n+k)/2]}, 0, 0\}^T$ are the eigenvectors of the same eigenvalue associated with $\hat{x}_1$. If $\hat{x}_4$ is an eigenvector of the identical eigenvalue with 0 entries taking different positions from $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$, then $\hat{x}_4$ can be constructed by a linear combination of $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$. Thus, the multiplicity of the eigenvalue corresponding to $\hat{x}$ is 3. The case of more than one trivial cell and two nontrivial cells can be proved in the same manner.

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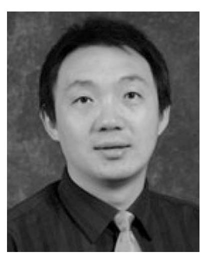

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