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# Observability of leader-based discrete-time multi-agent systems over signed networks

Bo Liu, Housheng Su, Licheng Wu, Xixi Shen

***Abstract*—This paper models a class of leader-based discrete-time multi-agent systems (MASs) over signed networks with both positive edges and negative edges, and addresses the observability of such multi-agent network based on an agreement protocol. Some new algebraic and graphical characterisations of the observability are given using a signed Laplacian. In addition, for a structurally balanced network, the observability of a leader-follower discrete-time signed network with structurally balanced condition is equivalent to that of the corresponding unsigned network, if the leaders are selected from the same partitioned subset. Some numerical examples and simulations are given to demonstrate the analysis.**

***Index Terms*—Observability, multi-agent systems, signed networks, structural balance.**

## I. Introduction

Controllability and observability are the core concepts in modern control theory and play the fundamental roles in the synthesis and decomposition of linear control systems. For MASs, the controllability problem is primarily by steering a small number of leaders to bring followers from any initial state to any specified terminal state (See more examples in [1]–[9]). However, in the practical control process, due to the limitation of measuring equipments, the system state is difficult to measure directly, yet the state can be determined by observing the system output. The observability of MASs is to observe the state of the entire multiagent network as far as possible from a subset of the agents.

The controllability problem of the leader-follower MASs was first taken into account in [10], in which an algebraic feature on controllability was established by eigenvalues and eigenvectors of Laplacian matrix. Since then, the controllability and observability of MASs have been widely concerned by scholars. Liu et al [11] discussed the observability of a complex system by using a graphical method obtained from the dynamical laws and reconstructing the entire internal state of a complex network. In [12], the observability analysis and estimation were applied for distributed sensor networks. [13] studied the observability of switched linear systems, obtained the observability criteria and provided the unobservable subspace algorithms. The authors [14] proposed a decentralized interaction strategy to guarantee preservation of the controllability and observability for networked systems by choosing the edge-weighted values. Most of the existing works mainly focused on the controllability and observability of MASs in the case of fixed topology. However, in real life, network communication or information transmission is not always fixed and will change according to the actual situation, so it is of great significance to study the controllability and observability of MASs under the condition of switching topology. This paper gives the concept of observability of MASs with switching topology and establishes a switching subspace to make the system be observable by designing the valid switching sequence, which is a challenging work. The observability of MASs with switching topology was considered even though each subsystem is unobservable in [15], while the observability of switched MASs was investigated in [16]. The observability of heterogeneous MASs was respectively studied in [17] and [18]. [19] established a unified framework for the observability of leader-based MASs with first-order/ second-order/ high-order dynamics. [20] gave a decentralized condition for the observability by the estimated eigenvalues via local interaction rule.

Many researchers have studied the controllability and observability of discrete-time MASs from the perspective of graph theory (e.g., [21]–[25] and the references therein). The authors studied the reachability and observability of path and cycle graphs based on average consensus protocol and provided a closed form expression for the unobservable subsystem in [21]. [22] presented a composition approach for observability of networks by Cartesian products. The observability of a class of linear dynamical systems was investigated according to the Laplacian matrix of a grid graph in [23]. [24] characterized the observability of MASs with agreement dynamics using the external equitable partitions. [25] focused on the observability of mulitiagent networks with strongly regular graphs or distance regular graphs and obtained an observable algebraic criterion for distance regular graphs.

It is worth noting that for a long time, the majority of the existing research results in the coordination control of MASs, especially controllability and observability, have mainly focused on cooperative interaction networks with Laplacian matrix, that is, the interactions between all agents are cooperative and non-antagonistic. In other words, the interaction coupling weights between agents in the network are characterized by

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B. Liu and L. Wu are with the School of Information Engineering, Minzu University of China, Beijing 100081, China. Email: boliu2020@muc.edu.cn, wulicheng@tsinghua.edu.cn.

H. Su is with the School of Artificial Intelligence and Automation, Huazhong University of Science and Technology, and also with the Key Laboratory of Image Processing and Intelligent Control of Education Ministry of China, Wuhan 430074, China. Email: houshengsu@gmail.com.

X. Shen is with the College of Science, North China University of Technology, Beijing 100144, China. Email: 2100782828@qq.com

non-negative real numbers. Typically, positive weights mean that the two associated agents are cooperative, friendly, and trusted to each other. Hence, the nonnegative matrix theory plays a key role in analyzing the coordination control problem of MASs. In such system, agents are often in a collaborative manner to achieve a common goal or collective behavior, such as swarming, consensus, synchronization, etc. In fact, cooperation and antagonism (competition) are important factors driving the dynamic evolution of natural society or ecosystem. The networked MASs with competitive interactions can be represented by signed graphs [26], and the edge-weighted values of the adjacency matrix have both positive and negative numbers to show the cooperative and competitive relationship between agents, such as friends and enemies, cooperation and competition, and attraction and repulsion. In the real world, these phenomena are widespread, such as the antagonism and competition between the two competition teams in sports competitions, the betrayal between individuals in social networks, the suppression of cells in neural networks and the competition between major countries. Therefore, the agents with antagonistic interactions have stronger application in the real world. The coordination control of MASs over signed networks has recently become a research hotspot, such as bipartite consensus [27]–[32], decentralized stabilizability and formation control [33], [34], controllability [35]–[39].

In addition, the observability is an important concept in modern control theory, which describes the ability whether all the system dynamics from the output can be reconstructed, that is, whether a dynamic system can correctly know the internal state of the system by observing the external variables of the system. The observability has important applications not only in system engineering and control theory, but also in chemical process control, power systems, aerospace and quantum systems, nanotechnology, etc. Specially, as the basic and important problems of MASs, the observability can provide a more reasonable explanation for the formation mechanism of these natural phenomena and engineering problems. Therefore, it will be of great theoretical value and practical significance to study the observability of MASs with antagonistic interactions. However, the limited cognition of signed graphs makes it difficult to study the observability of signed network, involving how to describe the antagonistic relationship between agents; what evolutionary behaviors do agents with antagonistic interactions produce; and what are the synergistic effects of node dynamics evolution, network topology, number and location of negative edge weights in the network on system observability? These are the big open challenges of the observability problem of MASs.

To the best of our knowledge, up to now, the observability of MASs over signed works is still in its infancy, so that very few results of observability with antagonistic interactions can have been available. One of the first attempts to study the observability to leader-based discrete-time MASs over signed networks can be found in this paper. The key of studying signed networks lies in the comprehensive utilization of positive- and negative-edged information. For this reason, this paper aims at finding the edge-weighted conditions of signed graphs to make the overall system be observable. The main contributions of this work are summed up as: (i) A new discrete-time MAS model with pinning control is established to address the observability and discuss effects of topologies of the signed graph, the agent's dynamics as well as the preset leaders on the observability. (ii) Leader selection to guarantee the observability of topologies signed networks is investigated in the presence of antagonistic interactions. In addition, the existing results on the observability and controllability of MASs with switching topologies concerned on interactions from followers to followers (e.g., [16], [17]), while this work will investigate the observability involving the interactions not only from followers to followers, but also from leaders to followers, which makes the problem more difficult. (iii) A simultaneously structurally balanced signed MAS on switching topology can be observable even though each subsystem is unobservable. Moreover, a switching unobservable subspace is obtained by designing a switching sequence, which is an interesting and challenging work.

## II. Preliminaries and Problem Statement

### A. Preliminaries

A weighted signed graph is denoted as $\mathbb{G} = (\mathcal{G}, \eta)$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is $\mathbb{G}$'s underlying graph, $\eta : \mathcal{E} \to \{+, -\}$ represents a signal mapping, $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$ and $\mathcal{E} = \{(v_i, v_j) : v_i, v_j \in \mathcal{V}\}$ are the vertex set and the edge set, respectively. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the adjacency matrix, where $a_{ij} \neq 0$ if and only if $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. Let $\mathcal{E} = \mathcal{E}_+ \bigcup \mathcal{E}_-$ with $\mathcal{E}_+ = \{(v_j, v_i)|a_{ij} > 0\}$ and $\mathcal{E}_- = \{(v_j, v_i)|a_{ij} < 0\}$. Denote $\mathcal{N}_i = \mathcal{N}_{i+} \bigcup \mathcal{N}_{i-}$ with $\mathcal{N}_{i+} = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}_+\}$ and $\mathcal{N}_{i-} = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}_-\}$. The Laplacian matrix of graph $\mathbb{G}$ is $L_s \triangleq \mathcal{C} - \mathcal{A}$, where $\mathcal{C}$ is the diagonal degree matrix and $L_s = [l^s_{ij}] \in \mathbb{R}^{n\times n}$ with

$$l^s_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum\limits_{j=1}^{n} |a_{ij}|, & i = j. \end{cases}$$

From the definition of $L_s$, it is easy to see that $L_s$ is a main diagonally dominant matrix and then $L_s$ is semi-positive definite (positive definite).

*Definition 1:* For signed graph $\mathbb{G}$, suppose that there is a bipartition $\{\mathcal{V}_1, \mathcal{V}_2\}$ of $\mathcal{V}$ such that $a_{ij} > 0$ when $\forall v_i, v_j \in \mathcal{V}_p,\ (p \in \{1, 2\})$, $a_{ij} < 0$ when $\forall v_i \in \mathcal{V}_p,\ v_j \in \mathcal{V}_q,\ (p, q \in \{1, 2\},\ p \neq q)$, where $\mathcal{V} = \mathcal{V}_1 \bigcup \mathcal{V}_2$ and $\mathcal{V}_1 \bigcap \mathcal{V}_2 = \emptyset$, it is structurally balanced, otherwise, structurally unbalanced (See Fig. 1).

*Definition 2:* A Gauge Transformation is expressed by diagonal matrix $D \in \mathcal{D}$, where $\mathcal{D} = \{\mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}|\sigma_i \in \{\pm 1\}\} \in \mathbb{R}^{n\times n}$. Suppose that there is $D \in \mathcal{D}$ such that $M_1 = DM_2D$, $M_1 \in \mathbb{R}^{n\times n}$ is signature similar to $M_2 \in \mathbb{R}^{n\times n}$, denoted as $M_1 \sim M_2$.

*Definition 3:* For $\mathbb{G}_1 = (\mathcal{G}, \eta_1)$ and $\mathbb{G}_2 = (\mathcal{G}, \eta_2)$ with the corresponding Laplacian matrices $L_{s1}$ and $L_{s2}$, suppose that $L_{s1} \sim L_{s2}$ ($\exists D \in \mathcal{D}$ such that $L_{s2} = DL_{s1}D$), $\mathbb{G}_1$ is switching equivalent to $\mathbb{G}_2$, denoted as $\mathbb{G}_1 \sim \mathbb{G}_2$.

An available lemma is given in the following .

*Lemma 1:* [35] A connected signed graph $\mathbb{G}(\mathcal{A})$ is structurally balanced if and only if any of the following equivalent

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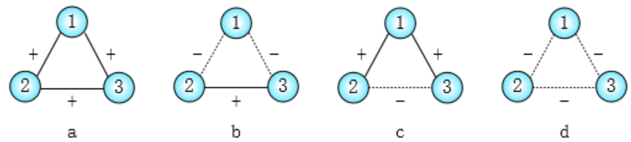


**Fig. 1:** Diagrams of structurally balanced $(a, b)$ and structurally unbalanced $(c, d)$ triangles.

conditions holds:
(1) All cycles of $\mathbb{G}(\mathcal{A})$ are positive;
(2) Let $D = \text{diag}(\sigma) \in \mathcal{D}$ with $\sigma$ satisfying $\sigma_i = 1$ if $v_i \in \mathcal{V}_1$ and $\sigma_i = -1$ if $v_i \in \mathcal{V}_2$; then $D\mathcal{A}D$ (and then $DL_sD$) has all off-diagonal entries nonnegative;
(3) 0 is an eigenvalue of $L_s$.

### B. Problem statement

A signed MAS consists of $n + n_l$ agents, denoted as $\mathcal{V} = \{v_1, v_2, \ldots, v_{n+n_l}\} \triangleq \mathcal{V}_f \bigcup \mathcal{V}_l$, where the follower's set is $\mathcal{V}_f$ labelled from 1 to $n$ and the leader's set is $\mathcal{V}_l$ labelled from $n+1$ to $n + n_l$. The followers are governed by

$$\begin{aligned} x_i(k+1) = x_i(k) - &\sum_{j\in\mathcal{N}_{i_j}} a_{ij}[(\text{sign}(a_{ij})x_i(k) - x_j(k)] \\ - &\sum_{l\in\mathcal{N}_{i_l}} b_{il}[(\text{sign}(b_{il})x_i(k) - x_l(k)], \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}$ is the state of agent $i$ ($i \in \underline{n}$) and $x_l \in \mathbb{R}$ is the state of leader $l$ ($l \in \underline{n+n_l} - \underline{n}$), where $\underline{n} \triangleq \{1, 2, \ldots, n\}$ and $\underline{n+n_l} \triangleq \{1, 2, \ldots, n_l\}$. The neighbor set of agent $i$ is $\mathcal{N}_i = \mathcal{N}_{i_j} \bigcup \mathcal{N}_{i_l}$ with $\mathcal{N}_{i_j} \bigcap \mathcal{N}_{i_l} = \emptyset$. $k \in J_k$ (a discrete-time index set). $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the adjacency matrix with $a_{ij} \neq 0$ and $a_{ii} = 0$ and $B = [b_{il}] \in \mathbb{R}^{n\times n_l}$ with $b_{il} \in \{\pm 1\}$ is the coupling matrix from leaders to followers. It is noted that $a_{ij}$ and $b_{il}$ can be allowed to be negative. sign$(\cdot)$ is defined by

$$\text{sign}(\xi) = \begin{cases} 1, & \xi > 0, \\ 0, & \xi = 0, \\ -1, & \xi < 0. \end{cases}$$

Let $x(k) \triangleq [x_1(k), x_2(k), \ldots, x_n(k)]^T$ and $u(k) \triangleq [x_{n+1}(k), x_{n+2}(k), \ldots, x_{n+n_l}(k)]^T$, then MAS (1) can be rewritten as

$$x(k+1) = Fx(k) + Bu(k), \quad k \in J_k, \tag{2}$$

where $F = I - L_s - BB^T$, $I \in \mathbb{R}^{n\times n}$ is the identity matrix and $L_s \in \mathbb{R}^{n\times n}$ is the Laplacian matrix of $\mathbb{G}$. In this work, we assume that each follower can have information link with at most one leader, then $BB^T = \text{diag}\{\delta_1, \ldots, \delta_n\}$ with $\delta_i \in \{0, 1\}$, where $\delta_i = 1$ as follower $i$ receives information from leader $l$, otherwise 0.

*Remark 1:* Note that Laplacian matrix $L_s$ of $\mathbb{G}$ is isomorphic to $L$ of $\mathbb{G}$'s underlying graph $\mathcal{G}$, therefore, $L_s$ and $L$ have the same eigenvalues. Because $I - BB^T$ is a diagonal matrix, there exists $D \in \mathcal{D}$, such that $D(I - L_s - BB^T)D = I - L - BB^T$ using Gauge Transformation, which is the analytical foundation for observability of signed networks under structurally balanced condition.

## III. Observability analysis

Observability is a basic problem for MASs over signed networks, which follows with interests in whether such MAS can be reconstructed by observed agent(s) after a period of observation. In other words, whether the non-zero initial states can be uniquely determined by the output. An MAS is observable if there exist no unobservable states. This section will respectively investigate the observability of MASs over signed networks with fixed topology and switching topology.

### A. Fixed topology

Inspired by [19], we take $y(k) = B^Tx(k)$ as the output vector measured by the leaders, then the dynamics of the signed MAS (2) can be regarded as a classical LTI system

$$\begin{cases} x(k+1) = Fx(k) + Bu(k), \\ \quad y(k) = B^Tx(k), \end{cases} \tag{3}$$

where $y \in \mathbb{R}^{n_l}$ is the output corresponding to output matrix $B^T$, which consists of all the state vectors measured by the leaders.

Different from the classical LTI systems, MAS (3) faces many factors, such as nodal dynamics, network topology, communication limitation and system dimension, etc., therefore, the observability presents new features and theoretical difficulties. This paper is devoted to carrying out a systematic and comprehensive study of the intricate interplay between the node-system dynamics, network topology and the observability of networked MASs, establishes the observability criteria for fixed topology, and then gives a switching sequence and a switching subspace to study the observability for switching topology. On the other hand, from the classical control theory, it is well known that a LTI system is controllable if and only if its controllability matrix has full rank. Duality principle can be used to prove that a networked LTI system is controllable if and only if it is observable. However, for MAS (3), the observability and controllability of MAS (3) are not equivalent, that is, MAS (3) is observable, not necessarily controllable, and vice versa, unless the system matrix $F$ is symmetric (i. e., $F^T = F$, see Theorems 1-2). More specific explanations can be found in Example 1.

*Definition 4:* [19] Any initial state $x(0)$ ($k \in \{0, 1, \ldots, n-1\}$) is said to be observable if $x(0)$ can be uniquely determined by the sequence $\{y(0), y(1), \ldots, y(n-1)\}$ within $n$ sampling periods.

The following proposition for MAS (3) based on the LTI system theory can be obtained immediately.

*Proposition 1:* For MAS (3), the following assertions are equivalent:
(1) MAS (3) is observable;
(2) Observability matrix $Q_o = \begin{bmatrix} B^T \\ B^TF \\ \vdots \\ B^TF^{n-1} \end{bmatrix}$ has full rank;
(3) Matrix $\begin{bmatrix} \lambda I - F \\ B^T \end{bmatrix}$ has full rank for $\forall\lambda \in \mathbb{R}$;
(4) Matrix $F$ has no non-zero right eigenvectors orthogonal to

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all rows of $B^T$. Namely, for all eigenvalues ($\lambda_i,\ i=1,\ldots,n$) of matrix $F$, if

$$F\alpha=\lambda\alpha,\quad B^T\alpha=0,$$

both hold, then $\alpha=0$.

Together with Lemma 1 and Proposition 1, the following results for $\mathbb{G}$ can be obtained.

*Theorem 1:* MAS (3) under $\mathfrak{L}^T=\mathfrak{L}$ is observable if and only if $\mathfrak{L}$ and $F$ do not share common eigenvalues, where $\mathfrak{L}\triangleq\begin{bmatrix} F & B\\ B^T & L_l\end{bmatrix}$ with $L_l$ corresponding to the index of leaders to show the interactions between the leaders, i.e., the leader-to-leader weighted factors.

*Proof:* Necessity: By contradiction, if MAS (3) is unobservable, based on Proposition 1 (4), there must be a vector $v_f\in\mathbb{R}^n$ such that,

$$\begin{cases} Fv_f=\lambda v_f,\\ B^Tv_f=0,\end{cases}$$

for $\lambda\in\mathbb{R}$. And since

$$\mathfrak{L}\begin{bmatrix} v_f\\ 0\end{bmatrix}=\begin{bmatrix} F & B\\ B^T & L_l\end{bmatrix}\begin{bmatrix} v_f\\ 0\end{bmatrix}=\begin{bmatrix} Fv_f\\ B^Tv_f\end{bmatrix}=\begin{bmatrix} \lambda v_f\\ 0\end{bmatrix}=\lambda\begin{bmatrix} v_f\\ 0\end{bmatrix},$$

therefore, $\lambda$ is also an eigenvalue of $\mathfrak{L}$ corresponding to eigenvector $\begin{bmatrix} v_f\\ 0\end{bmatrix}$, which contradicts to the fact that $\mathfrak{L}$ and $F$ do not share common eigenvalues.

Sufficiency: By contradiction, let $P=[I_n\ 0]^T\in\mathbb{R}^{(n+n_l)\times n}$, then

$$F=P^T\mathfrak{L}P.$$

Assume that $\lambda$ is a common eigenvalue of $\mathfrak{L}$ and $F$, then the corresponding eigenvector satisfies $v=Pv_f=\begin{bmatrix} v_f\\ 0\end{bmatrix}$, where $v$ is the eigenvector of $\mathfrak{L}$ and $v_f$ is the eigenvector of $F$ corresponding to the common eigenvalue $\lambda$, which implies $B^Tv_f=0$. Therefore, MAS (3) is unobservable, which contradicts to the fact that MAS (3) is observable. □

The following result can be obtained immediately from Theorem 1.

*Theorem 2:* MAS (3) under $\mathfrak{L}^T=\mathfrak{L}$ is observable if and only if there exists no eigenvector of $\mathfrak{L}$ taking zero on the element corresponding to the leader, where $\mathfrak{L}$ is defined as Theorem 1.

*Proof:* More detailed proof can be seen in Theorem 2 in [19]. □

From Proposition 1, we can get the following result.

*Theorem 3:* MAS (3) is observable if
(*i*) The eigenvalues of $F$ are all distinct, and
(*ii*) All the corresponding eigenvectors of $F$ are un-orthogonal to at least one row in $B^T$ simultaneously.

*Proof:* Since $F$'s eigenvalues are all distinct, $F$ can be represented as $F=U\Theta U^{-1}$, where $\Theta\triangleq\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, $\lambda_i$ is $F$'s eigenvalue for $i=1,\ldots,n$ and $U$ is composed of $F$'s eigenvectors. The observability matrix $Q_o$ can be expressed as

$$\begin{aligned} Q_o &= \begin{bmatrix} B^T\\ B^TF\\ \vdots\\ B^TF^{n-1}\end{bmatrix}=\begin{bmatrix} B^TUU^{-1}\\ B^TU\Theta U^{-1}\\ \vdots\\ B^T(U\Theta U^{-1})^{n-1}\end{bmatrix}\\ &= \begin{bmatrix} B^TUU^{-1}\\ B^TU\Theta U^{-1}\\ \vdots\\ B^TU\Theta^{n-1}U^{-1}\end{bmatrix}=\begin{bmatrix} B^TU\\ B^TU\Theta\\ \vdots\\ B^TU\Theta^{n-1}\end{bmatrix}U^{-1}\\ &\triangleq \breve{Q}_oU^{-1},\end{aligned}$$

where $\breve{Q}_o\triangleq\begin{bmatrix} B^TU\\ B^TU\Theta\\ \vdots\\ B^TU\Theta^{n-1}\end{bmatrix}$. Because $|U^{-1}|\neq 0$, $\mathrm{rank}(Q_o)=\mathrm{rank}(\breve{Q}_o)$. Let $U=[t_1,t_2,\ldots,t_n]$ and $B^T=[b_1^T,b_2^T,\ldots,b_{n_l}^T]^T$, then

$$\breve{Q}_o=\begin{bmatrix} B^TU\\ B^TU\Theta\\ \vdots\\ B^TU\Theta^{n-1}\end{bmatrix}=\begin{bmatrix} \begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_{n_l}\end{bmatrix}[t_1,t_2,\ldots,t_n]\\ \begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_{n_l}\end{bmatrix}[t_1,t_2,\ldots,t_n]\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)\\ \begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_{n_l}\end{bmatrix}[t_1,t_2,\ldots,t_n]\mathrm{diag}(\lambda_1^{n-1},\lambda_2^{n-1},\ldots,\lambda_n^{n-1})\end{bmatrix}.$$

By the elementary row transformation, we can get

$$\breve{Q}_o=\begin{bmatrix} D\cdot\mathrm{diag}(b_1t_1,b_1t_2,\ldots,b_1t_n)\\ D\cdot\mathrm{diag}(b_2t_1,b_2t_2,\ldots,b_2t_n)\\ \vdots\\ D\cdot\mathrm{diag}(b_{n_l}t_1,b_{n_l}t_2,\ldots,b_{n_l}t_n)\end{bmatrix},$$

where $D$ is a Vandermonde matrix with

$$D=\begin{bmatrix} 1 & 1 & \ldots & 1\\ \lambda_1 & \lambda_2 & \ldots & \lambda_n\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1}\end{bmatrix}.$$

Obviously, if the eigenvalues of $F$ are all distinct and all the corresponding eigenvectors of $F$ are un-orthogonal to at least one row in $B^T$ simultaneously, observability matrix $Q_o$ has full rank. Thus, MAS (3) is observable. □

*Remark 2:* Obviously, the observability of MAS (3) depends on the leader-to-follower and follower-to-follower interactions from Theorems 1–3, which can be seen from Examples

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2 and 3. It is easy to see that Theorems 1–2 both offer a sufficient and necessary condition satisfying symmetrical weights, while Theorem 3 only provides a sufficient condition, which does not require symmetrical weights so that it is a relatively intuitive and easy condition to be satisfied. In addition, we can have $F = P^T \mathfrak{L} P$, which does not involve the interactions between the leaders.

Together with Theorem 3 and Proposition 1, the following result can be obtained.

*Theorem 4:* MAS (3) is observable if
($i$) The eigenvalues of $F$ are all distinct, and
($ii$) Each column of $B^T U$ has at least one nonzero element, where $U$ is made up of the eigenvectors of $F$.

*Proof:* From Proposition 1 (3) and Theorem 3, we can have

$$\text{rank}\begin{bmatrix} \lambda_i I - \Theta \\ B^T U \end{bmatrix} = \text{rank}\begin{bmatrix} \lambda_i - \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_i - \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_i - \lambda_n \\ t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n_l 1} & t_{n_l 2} & \dots & t_{n_l n} \end{bmatrix}.$$

Since $\lambda_i \neq \lambda_j$ for $i \neq j$, and each column of $B^T U$ has at least one nonzero element, then

$$\text{rank}\begin{bmatrix} \lambda_i I - \Theta \\ B^T U \end{bmatrix} = n.$$

This conclusion holds. □

*Remark 3:* Theorems 3–4 only provide sufficient conditions to make MAS (3) be observable, therefore, MAS (3) may be observable or unobservable if $F$ has some identical eigenvalues, which can not be used to judge whether the system is observable or not. For example, we consider a five-agent signed MAS on fixed topology with the system matrices being

$$F = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix},$$

obviously, $\text{rank}(Q_o) = \text{rank}\left(\begin{bmatrix} B^T \\ B^T F \\ B^T F^2 \end{bmatrix}\right) = 3$, then the given system can be observable even if $F$ has some identical eigenvalues (since the eigenvalues of $F$ are $\{-2, 0, 0\}$). On the other hand, we also consider a five-agent signed MAS on fixed topology with the system matrices being

$$F = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix},$$

obviously, $\text{rank}(Q_o) = \text{rank}\left(\begin{bmatrix} B^T \\ B^T F \\ B^T F^2 \end{bmatrix}\right) = 2 < 3$, then the given system is unobservable even if $F$ has some identical eigenvalues (since the eigenvalues of $F$ are $\{-1, 0, 0\}$).

*Remark 4:* Note that the observability of MAS (3) only depends on the interactions between agents, i.e., $(B^T, F)$. Based on Theorems 1–4, we can find that the interaction between leaders can also affect the observability of MAS (3). However, how to select leader(s) to make MAS (3) be more quickly observable? Here we will discuss the problem.

*Theorem 5:* For structurally balanced MAS (3), the observability of $(B^T, F)$ is equivalent to that of $(B^T, \bar{F})$ if $\mathcal{V}_l \subseteq \mathcal{V}_1$ (or $\mathcal{V}_l \subseteq \mathcal{V}_2$) satisfying $b_{il} = 1$ (or $b_{il} = -1$), where $\bar{F} = I - L - BB^T$.

*Proof:* From the definition of $L_s$, $\exists D \in \mathcal{D}$, such that $L_s = DLD$ and $D(I - BB^T)D = I - BB^T$ (since $I - BB^T$ is a diagonal matrix), then $F = D\bar{F}D$. Thus

$$\begin{aligned}
&\text{rank}\left\{\begin{bmatrix} B^T \\ B^T F \\ \vdots \\ B^T F^{n-1} \end{bmatrix}\right\} \\
&= rank\left\{\begin{bmatrix} B^T \\ B^T(I - L_s - BB^T) \\ \vdots \\ B^T(I - L_s - BB^T)^{n-1} \end{bmatrix}\right\} \\
&= \text{rank}\left\{\begin{bmatrix} B^T DD \\ B^T D(I - L - BB^T)D \\ \vdots \\ B^T D(I - L - BB^T)^{n-1} D \end{bmatrix}\right\} \\
&= \text{rank}\left\{\begin{bmatrix} B^T D \\ B^T D(I - L - BB^T) \\ \vdots \\ B^T D(I - L - BB^T)^{n-1} \end{bmatrix} D\right\} \\
&= \text{rank}\left\{\begin{bmatrix} B^T \\ B^T(I - L - BB^T) \\ \vdots \\ B^T(I - L - BB^T)^{n-1} \end{bmatrix} (\pm D)\right\} \\
&= \text{rank}\left\{\begin{bmatrix} B^T \\ B^T \bar{F} \\ \vdots \\ B^T \bar{F}^{n-1} \end{bmatrix} (\pm D)\right\},
\end{aligned}$$

if $\mathcal{V}_l \subseteq \mathcal{V}_1$, $B^T D = B^T$ ($\mathcal{V}_l \subseteq \mathcal{V}_2$, $B^T D = -B^T$) with $b_{il} = 1$; or $\mathcal{V}_l \subseteq \mathcal{V}_1$, $B^T D = -B^T$ ($\mathcal{V}_l \subseteq \mathcal{V}_2$, $B^T D = B^T$) with $b_{il} = -1$. □

*Remark 5:* Theorem 5 implies that structurally balanced MAS (3) can be observable by selecting the leaders from the same partitioned subsets.

*Theorem 6:* For MAS (3), the observability of $(B^T, F)$ is equivalent to that of $(B^T, -L_s)$ if each different follower can only receive information from each different leader as $n_l = n$.

*Proof:* If each different follower can only receive information from each different leader as $n_l = n$, then $I - BB^T = 0$, and then $I - L_s - BB^T = -L_s$. Therefore $\text{rank}(Q_o(B^T, F)) = \text{rank}(Q_o(B^T, -L_s))$. □

*Theorem 7:* Let $\mathbb{G}_1 = (\mathcal{G}, \eta_1)$ and $\mathbb{G}_2 = (\mathcal{G}, \eta_2)$ be two different coupling networks of MAS (3). The observability of $\mathbb{G}_1$ is equivalent to that of $\mathbb{G}_2$ if $\mathbb{G}_1 \sim \mathbb{G}_2$.

*Proof:* Suppose that $L_{s1} \in \mathbb{R}^{n \times n}$ and $L_{s2} \in \mathbb{R}^{n \times n}$ are the Laplacian matrices of $\mathbb{G}_1$ and $\mathbb{G}_2$, then $L_{s1} \sim L_{s2}$ if

$\mathbb{G}_1 \sim \mathbb{G}_2$. Therefore, $\exists D \in \mathcal{D}$ such that $L_{s2} = DL_{s1}D$, and then $F_2 \triangleq I - L_{s2} - BB^T = D(I - L_{s1} - BB^T)D \triangleq DF_1D$, i.e., $F_1 \sim F_2$ when the leaders are given. From Theorem 5 and Lemma 1, the observability of $\mathbb{G}_1$ is equivalent to that of $\mathbb{G}_2$. □

*Corollary 1:* Let $\mathbb{G} = (\mathcal{G}, \eta)$ be the coupling network of MAS (3), then the following assertions hold:
$(i)$ The observability of structurally balanced graph $\mathbb{G}$ is equivalent to that of graph $\mathcal{G}$.
$(ii)$ The observability of a path graph $\mathbb{G}$ is equivalent to that of graph $\mathcal{G}$.
$(iii)$ The observability of a tree graph $\mathbb{G}$ is equivalent to that of graph $\mathcal{G}$.
$(iv)$ The observability of a cycle graph $\mathbb{G}$ is equivalent to that of graph $\mathcal{G}$.

### *B. Switching topology*

The signed network $\mathbb{G}$ of MAS (3) may be time-variant in reality, which can be depicted by switching topologies, i.e., $a_{ij}(k)$, $\mathcal{N}_i(k)$ and $L_s(k)$ are all time-variant. MAS (3) can be regarded as the following form of switching network

$$\begin{cases} x(k+1) = F_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \\ \quad y(k) = B^T_{\sigma(k)}x(k), \end{cases} \tag{4}$$

where $F_{\sigma(k)} \in \mathbb{R}^{n\times n}$, $B_{\sigma(k)} \in \mathbb{R}^{n\times m}$, $\sigma(k) : \mathbb{R}^+ \to \mathcal{K} \triangleq \{1, \ldots, K\}$ is a piecewise constant switching path depicting the time-variable coupling to be designed, $K$ is the index set of possible switching topologies of the signed network. Furthermore, $\sigma(k) = i$ means that the subsystem $(B_i^T, F_i)$ can be realized at time $k$. As in [1], here we always suppose that each $F_i$ is reversible.

*Definition 5:* [17] A state $x_0$ of MAS (4) is unobservable if for given $x(k_0) = x_0$, then $y(k) = 0$ for $k \in [k_0, k_M]$ and any switching sequence $\pi = \{(i_m, h_m)\}_{m=1}^M$ with $h_m = k_m - k_{m-1}$. If any nonzero state is observable, MAS (4) is observable.

*Definition 6:* [1] For matrix $B = [b_{n+1}, b_{n+2}, \ldots, b_{n+m}] \in \mathbb{R}^{n\times m}$, its column space is defined as $\mathcal{R}(B) = \text{span}\{b_{n+1}, b_{n+2}, \ldots, b_{n+m}\}$.

*Definition 7:* $\langle F \mid \mathcal{O}\rangle \triangleq \sum_{i=1}^K F^{i-1}\mathcal{O}$ is said to be the $\mathcal{K}$–*cyclic invariant subspace* for $F \in \mathbb{R}^{n\times n}$ and a linear subspace $\mathcal{O} \subseteq \mathbb{R}^n$. For simplicity, let $\langle F \mid \mathcal{R}(B)\rangle = \langle F \mid B\rangle$.

A subspace sequence for MAS (4) is defined as

$$\mathcal{O}_1 = \sum_{i=1}^K \langle F_i^T | B_i\rangle, \mathcal{O}_2 = \sum_{i=1}^K \langle F_i^T | \mathcal{O}_1\rangle, \cdots, \mathcal{O}_n = \sum_{i=1}^K \langle F_i^T | \mathcal{O}_{n-1}\rangle.$$

*Proposition 2:* [16] MAS (4) is observable if and only if $\mathcal{O}_n = \mathbb{R}^n$.

From Proposition 2, the following result can be got immediately.

*Theorem 8:* MAS (4) is observable if

$$\mathcal{R}(B_1) + \mathcal{R}(B_2) + \cdots + \mathcal{R}(B_K) = \mathbb{R}^n.$$

*Proof:* For all $i$, since

$$\mathcal{R}(B_i) \subseteq \mathcal{R}(B_i) + \mathcal{R}(F_i^T B_i) + \cdots + \mathcal{R}((F_i^T)^{n-1}B_i) = \langle F_i^T | B_i\rangle,$$

we can have

$$\begin{aligned} \mathbb{R}^n &= \mathcal{R}(B_1) + \mathcal{R}(B_2) + \cdots + \mathcal{R}(B_K) \\ &\subseteq \langle F_1^T | B_1\rangle + \cdots + \langle F_K^T | B_K\rangle \\ &= \mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq \cdots \subseteq \mathcal{O}_n. \end{aligned}$$

On the other side, obviously $\mathcal{O}_n \subseteq \mathbb{R}^n$. Therefore $\mathcal{O}_n = \mathbb{R}^n$. The statement holds. □

*Remark 6:* Theorem 8 only focuses on the leader-to-follower coupling links regardless of the follower-to-follower coupling links. It can offer more freedom and convenience to design switching paths.

*Remark 7:* Indeed, the observability of MASs with a fixed topology needs $B^T \neq 0$. But for a switching topology, even though $B^T_{\sigma(k)} = 0$ is possible to appear for some $\sigma(k) = i$ during the switching process, the whole system can still be observable. This assertion can be confirmed in Example 8 of the simulation part.

*Proposition 3:* MAS (4) is observable if and only if $\text{rank}(\tilde{Q}_o) = n$, where observability matrix

$$\tilde{Q}_o = \begin{bmatrix} Q_1 \\ Q_2 F_{i_1}^{k_1-1} \\ \vdots \\ Q_M F_{i_{M-1}}^{k_{M-1}-k_{M-2}} \cdots F_{i_2}^{k_2-k_1} F_{i_1}^{k_1-1} \end{bmatrix}$$

with the observability matrices of the subsystems being expressed by

$$Q_1 = \begin{bmatrix} B_{i_1}^T \\ B_{i_1}^T F_{i_1} \\ \vdots \\ B_{i_1}^T F_{i_1}^{k_1-1} \end{bmatrix}, \; Q_2 = \begin{bmatrix} B_{i_2}^T F_{i_2} \\ B_{i_2}^T F_{i_2}^2 \\ \vdots \\ B_{i_2}^T F_{i_2}^{k_2-k_1} \end{bmatrix}, \ldots,$$

$$Q_M = \begin{bmatrix} B_{i_M}^T F_{i_M} \\ B_{i_M}^T F_{i_M}^2 \\ \vdots \\ B_{i_M}^T F_{i_M}^{k_M-k_{M-1}} \end{bmatrix}.$$

*Proof:* Necessity: It is well known that the system is observable if there exist no unobservable states and only necessary to consider the zero input state space model to study the state estimation or observation of the system, therefore,

$$\begin{aligned} y(0) &= B_{i_1}^T x(0), \\ y(1) &= B_{i_1}^T x(1) = B_{i_1}^T F_{i_1} x(0), \\ y(2) &= B_{i_1}^T F_{i_1}^2 x(0), \\ &\vdots \\ y(k_1 - 1) &= B_{i_1}^T F_{i_1}^{k_1-1} x(0), \\ y(k_1) &= B_{i_2}^T F_{i_2} F_{i_1}^{k_1-1} x(0), \\ &\vdots \\ y(k_2 - 1) &= B_{i_2}^T F_{i_2}^{k_2-k_1} F_{i_1}^{k_1-1} x(0), \\ &\vdots \\ y(k_M - 1) &= B_{i_M}^T F_{i_M}^{k_M-k_{M-1}} F_{i_{M-1}}^{k_{M-1}-k_{M-2}} \cdots F_{i_1}^{k_1-1} x(0), \end{aligned}$$

that is,

$$y \triangleq \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k_M-1) \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 F_{i_1}^{k_1-1} \\ \vdots \\ Q_M F_{i_{M-1}}^{k_{M-1}-k_{M-2}} \dots F_{i_2}^{k_2-k_1} F_{i_1}^{k_1-1} \end{bmatrix} x(0) \triangleq \tilde{Q}_o x(0),$$

where $\tilde{Q}_o \triangleq \begin{bmatrix} Q_1 \\ Q_2 F_{i_1}^{k_1-1} \\ \vdots \\ Q_M F_{i_{M-1}}^{k_{M-1}-k_{M-2}} \dots F_{i_2}^{k_2-k_1} F_{i_1}^{k_1-1} \end{bmatrix}$, with

$$Q_1 = \begin{bmatrix} B_{i_1}^T \\ B_{i_1}^T F_{i_1} \\ \vdots \\ B_{i_1}^T F_{i_1}^{k_1-1} \end{bmatrix}, Q_2 = \begin{bmatrix} B_{i_2}^T F_{i_2} \\ B_{i_2}^T F_{i_2}^2 \\ \vdots \\ B_{i_2}^T F_{i_2}^{k_2-k_1} \end{bmatrix}, \dots,$$

$$Q_M = \begin{bmatrix} B_{i_M}^T F_{i_M} \\ B_{i_M}^T F_{i_M}^2 \\ \vdots \\ B_{i_M}^T F_{i_M}^{k_M-k_{M-1}} \end{bmatrix}.$$

We know that $y(k)$ can not uniquely determine the initial state $x(0)$ if rank$(\tilde{Q}_o) < n$, which contradicts with the fact that MAS (4) is observable.

Sufficiency: Let the two sides of equation $y = \tilde{Q}_o x(0)$ multiply $\tilde{Q}_o^T$, thus

$$\tilde{Q}_o^T y = \tilde{Q}_o^T \tilde{Q}_o x(0).$$

If rank$(\tilde{Q}_o) = n$, then $|\tilde{Q}_o^T \tilde{Q}_o| \neq 0$, and then

$$x(0) = (\tilde{Q}_o^T \tilde{Q}_o)^{-1} \tilde{Q}_o^T y,$$

which means that $y(k)$ uniquely determines the initial state $x(0)$, therefore, MAS (4) is observable. □

For MAS (4), given a switching sequence $\pi = \{(i_m, h_m)\}_{m=1}^M$, the output $y(k)$ is defined as

$$y(k) = B_{i_m}^T F_{i_m}^{(k-k_{m-1})} F_{i_{m-1}}^{h_{m-1}} \dots F_{i_1}^{h_1} x(k_0),$$

for $k \in [k_{m-1}, k_m),\ m = 1, ..., M$.

Let $Ker(B_i^T)$ be the zero space of $B_i^T$ and $\langle Ker(B_i^T)|F_i\rangle = \bigcap_{j=0}^{n-1} F_i^{-j} Ker(B_i^T)$ be the largest $F_i$-invariant subspace containing $Ker(B_i^T)$, where $Ker(B_i^T) = \{x \in \mathbb{R}^n | B_i^T x = \mathbf{0}\}$ and $F_i^{-j} Ker(B_i^T) = \{x \in \mathbb{R}^n | F_i^j x \in Ker(B_i^T)\}$.

Specially, a subspace sequence for MAS (4) is denoted as

$$\mathcal{L}_{0,i} = \langle Ker(B_i^T)|F_i\rangle,\ \mathcal{L}_{s,i} = \bigcap_{j=1}^{K} \langle \mathcal{L}_{s-1,i}|F_j\rangle,\ \mathcal{L}_s = \bigcap_{l=0}^{s} \bigcap_{j=1}^{K} \mathcal{L}_{l,j},$$

where $i = 1, ..., K$, and $s = 0, ..., n$. Obviously, $\mathcal{L}_{s+1} \subseteq \mathcal{L}_s \subseteq \mathcal{L}_{l,j}$ for $s = 0, ..., n,\ l = 0, ..., s,\ j = 1, ..., K$.

*Proposition 4:* MAS (4) is observable if and only if $\mathfrak{D} = \{\mathbf{0}\}$, where $\mathfrak{D}$ is the set of all the unobservable states of MAS (4).

*Theorem 9:* MAS (4) is observable if and only if $\mathcal{L}_n = \mathfrak{D} = \{\mathbf{0}\}$.

*Proof:* First, we will prove that $\mathfrak{D} \subseteq \mathcal{L}_n$. For $\forall x \in \mathfrak{D}$, let $F_s x$ be the initial state $x(k_0), s = 1, ..., K$. For $(k \in [k_{m-1}, k_m), m = 1, ..., M)$, $y(k) = B_{i_m}^T F_{i_m}^{(k-k_{m-1})} F_{i_{m-1}}^{h_{m-1}} \dots F_{i_1}^{h_1} F_s x$ is the corresponding output with arbitrary switching sequence $\pi = \{(i_m, h_m)\}_{m=1}^M$, which can also be regarded as the output of MAS (4) with the initial state $x(k_0 - 1) = x$ under $\pi = \{(s, 1), (i_1, h_1)..., (i_M, h_M)\}$. Since $x \in \mathfrak{D}$, $y(k) = 0$ for all $k \in [k_0 - 1, k_M]$, which yields that $F_s x \in \mathfrak{D}$. This implies that $\mathfrak{D}$ is a $F_s$-invariant subspace for $s = 1, ..., K$, then $F_s^l x \in \mathfrak{D} \subseteq \bigcap_{j=1}^K Ker(B_j^T)$ for $\forall x \in \mathfrak{D}$ and $l = 0, ..., n-1$. And then $F_s^l x \in Ker(B_s^T)$, for $s = 1, 2, \cdots, K$, which yields that $x \in \bigcap_{l=0}^{n-1} F_s^{-l} Ker(B_s^T) = \mathcal{L}_{0,s}$, that is, $F_i^l x \in \mathfrak{D} \subseteq \mathcal{L}_{0,s}$, which implies that $x \in F_i^{-l} \mathcal{L}_{0,s}$ for $i, s \in \{1, ..., K\},\ l \in \{0, ..., n-1\}$. Thus, $x \in \bigcap_{i=1}^K \bigcap_{l=0}^{n-1} F_i^{-l} \mathcal{L}_{0,s} = \bigcap_{i=1}^K \langle \mathcal{L}_{0,s}|F_i\rangle = \mathcal{L}_{1,s}$ for $s = 1, ...K$. Similarly, we can prove that $x \in \mathcal{L}_{i,s}, i = 2, ..., n, s = 1, ..., K$. Therefore, for $\forall x$, we can have $x \in \mathcal{L}_n$, and then $\mathfrak{D} \subseteq \mathcal{L}_n$. On the other hand, for $\forall x \in \mathcal{L}_n$ and arbitrary switching sequence $\pi = \{(i_m, h_m)\}_{m=1}^M$, let $x(k_0) = x$. Since $x \in \mathcal{L}_{0,i_1} = \bigcap_{l=0}^{n-1} F_{i_1}^{-l} Ker(B_{i_1}^T)$, we can have $B_{i_1}^T F_{i_1}^l x = 0,\ l = 0, ..., n-1$, which implies that $y(k) = B_{i_1}^T F_{i_1}^{(k-k_0)} x = 0,\ k \in [k_0, k_1),\ k - k_0 = 0, 1, \cdots$. Moreover, $x \in \mathcal{L}_{1,i_2} = \bigcap_{j=1}^K \langle \mathcal{L}_{0,i_2}|F_j\rangle \subseteq \langle \mathcal{L}_{0,i_2}|F_{i_1}\rangle = \langle \langle Ker(B_{i_2}^T)|F_{i_2}\rangle | F_{i_1}\rangle$, which implies that $F_{i_1}^l x \in \langle Ker(B_{i_2}^T)|F_{i_2}\rangle, l = 0, ..., n-1$, then we can get $F_{i_1}^l x \in \langle Ker(B_{i_2}^T)|F_{i_2}\rangle,\ l = 0, 1, \cdots$, and then, $F_{i_1}^{h_1} x \in \langle Ker(B_{i_2}^T)|F_{i_2}\rangle$. Based on $\langle Ker(B_{i_2}^T)|F_{i_2}\rangle = \bigcap_{l=0}^{n-1} p_{i_2}^{-l}\ Ker(B_{i_2}^T)$, we can also have $B_{i_2}^T F_{i_2}^l F_{i_1}^{h_1} x = 0,\ l = 0, ..., n-1$. Therefore, $y(k) = B_{i_2}^T F_{i_2}^{(k-k_1)} F_{i_1}^{h_1} x = 0,\ k \in [k_1, k_2),\ k - k_1 = 0, 1, \cdots$. Similarly, $y(k) = B_{i_m}^T F_{i_m}^{(k-k_{m-1})} F_{i_{m-1}}^{h_{m-1}} \dots F_{i_1}^{h_1} x = 0,\ k \in [k_{m-1},\ k_m),\ m = 3, ..., M$. Thus, $x \in \mathfrak{D}$ from the arbitrariness of $x$. Then we can obtain $\mathcal{L}_n \subseteq \mathfrak{D}$. So $\mathcal{L}_n = \mathfrak{D}$. Therefore, MAS (4) is observable if and only if $\mathcal{L}_n = \{\mathbf{0}\}$ based on Proposition 4. The assertion holds. □

*Remark 8:* Theorem 9 mainly studies the unobservable states of MAS (4). A subspace sequence is constructed from Definition 5 and the concept of kernel space, and it is proved that $\mathcal{L}_n$ is an unobservable subspace of MAS (4). Moreover, based on Proposition 2, it is easy to obtain that $\mathcal{L}_n$ and $\mathcal{O}_n$ are orthogonal complementary spaces, that is, $\mathcal{L}_n = \mathcal{O}_n^{\perp}$.

*Lemma 2:* MAS (4) is switching observable if and only if rank$(\tilde{\tilde{Q}}_o(B_i, F_i^T)) = n$, where

$$\tilde{\tilde{Q}}_o(B_i, F_i^T) \triangleq [B_1, B_2, \dots, B_K, F_1^T B_1, F_2^T B_2, \dots, F_K^T B_K, (F_1^T)^2 B_1, \dots, F_K^T F_1^T B_1, \dots, (F_K^T)^2 B_K, \dots, (F_1^T)^{n-1} B_1, \dots, (F_K^T)^{n-1} B_K].$$

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*Proof:* From Proposition 2, we can know

$$\begin{aligned}\mathcal{O}_n &= \sum_{i_1=1}^{K}\langle F_{i_1}^T|\mathcal{O}_{n-1}\rangle\\ &=\sum_{l_1=0}^{n-1}\sum_{i_1=1}^{K}(F_{i_1}^T)^{l_1}\mathcal{O}_{n-1}\\ &=\sum_{\substack{i_1,i_2\in\{1,\ldots,K\}}}^{l_1,l_2\in\{0,\ldots,n-1\}}(F_{i_1}^T)^{l_1}(F_{i_2}^T)^{l_2}\mathcal{O}_{n-2}\\ &\ldots\\ &=\sum_{i_1,\ldots,i_{n-1}\in\{1,\ldots,K\}}^{l_1,\ldots,l_{n-1}\in\{0,\ldots,n-1\}}(F_{i_1}^T)^{l_1}\ldots(F_{i_{n-1}}^T)^{l_{n-1}}\mathcal{O}_1\\ &=\sum_{j=1}^{K}\sum_{i_1,\ldots,i_{n-1}\in\{1,\ldots,K\}}^{l_1,\ldots,l_{n-1}\in\{0,\ldots,n-1\}}\\ &\quad\times\mathcal{R}((F_{i_1}^T)^{l_1}\ldots(F_{i_{n-1}}^T)^{l_{n-1}}[B_j,F_j^TB_j,\ldots,(F_j^T)^{n-1}B_j])\\ &=\mathcal{R}(\tilde{\tilde{Q}}_o(B_i,F_i^T)),\end{aligned}$$

therefore, MAS (4) is switching observable if and only if $\text{rank}(\tilde{\tilde{Q}}_o(B_i,F_i^T))=n$, i.e. $\mathcal{O}_n=\mathbb{R}^n$. □

Furthermore, let $\tilde{u}(t)\triangleq[u_1(t),\ldots,u_M(t)]^T\in\mathbb{R}^M$, $\tilde{B}\triangleq[B_1,\ldots,B_K]\in\mathbb{R}^{n\times M}$, where $M\triangleq\sum_{i=1}^K m_i$.

*Theorem 10:* MAS (4) is switching observable if and only if system

$$\begin{cases}x(k+1)=F_{\sigma(k)}x(k)+\tilde{B}_{\sigma(k)}\tilde{u}(k),\\ \quad y(k)=\tilde{B}_{\sigma(k)}^Tx(k),\end{cases}\tag{5}$$

is observable, where

$$\begin{aligned}\tilde{\tilde{Q}}_o(\tilde{B},F_i^T)\ \triangleq\ &[\tilde{B},F_1^T\tilde{B},F_2^T\tilde{B},\ldots,F_K^T\tilde{B},\\ &(F_1^T)^2\tilde{B},\ldots,F_K^TF_1^T\tilde{B},\ldots,(F_K^T)^2\tilde{B},\ldots,\\ &(F_1^T)^{n-1}\tilde{B},\ldots,(F_K^T)^{n-1}\tilde{B}].\end{aligned}$$

*Proof:* It is easy to show that $rank(\tilde{\tilde{Q}}_o(\tilde{B},F_i^T))=rank(\tilde{\tilde{Q}}_o(B_i,F_i^T))$. □

*Theorem 11:* Suppose that MAS (4) is simultaneously structurally balanced and $\mathcal{V}_l\subseteq\mathcal{V}_1$ (or $\mathcal{V}_l\subseteq\mathcal{V}_2$) satisfying $b_{il}=1$ ($b_{il}=-1$), then the observability of $(B_{\sigma(k)},\ F_{\sigma(k)}^T)$ is equivalent to that of $(B_{\sigma(k)},\ \bar{F}_{\sigma(k)}^T)$.

*Proof:* Since $\mathbb{G}_i$ $(i=1,\ldots,K)$ are simultaneously structurally balanced, from the definition of $L_s$ and the fact that $I-BB^T$ be a diagonal matrix, it is well known that there exists a common signature matrix $D\in\mathcal{D}$, such that $DL_{si}D=L_i$ and $D(I-B_iB_i^T)D=I-B_iB_i^T$, then $DF_iD=\bar{F}_i$, $i=1,\ldots,K$, and then

$$\begin{aligned}&\text{rank}(\tilde{\tilde{Q}}_o(B_i,F_i^T))\\ &=\text{rank}(\tilde{\tilde{Q}}_o(\tilde{B},F_i^T))\\ &=\text{rank}\{[\tilde{B},(D\bar{F}_1D)^T\tilde{B},\ldots,(D\bar{F}_KD)^T\tilde{B},\\ &\qquad((D\bar{F}_1D)^T)^2\tilde{B},\ldots,(D\bar{F}_KD)^T(D\bar{F}_1D)^T\tilde{B},\\ &\qquad\ldots,((D\bar{F}_KD)^T)^2\tilde{B},\ldots,((D\bar{F}_1D)^T)^{n-1}\tilde{B},\\ &\qquad\ldots,((D\bar{F}_KD)^T)^{n-1}\tilde{B}]\}\\ &=\text{rank}\{[\tilde{B},D\bar{F}_1^TD\tilde{B},\ldots,D\bar{F}_K^TD\tilde{B},D(\bar{F}_1^T)^2D\tilde{B},\\ &\qquad\ldots,D\bar{F}_K^T\bar{F}_1^TD\tilde{B},\ldots,D(\bar{F}_K^T)^2D\tilde{B},\ldots,\\ &\qquad D(\bar{F}_1^T)^{n-1}D\tilde{B},\ldots,D(\bar{F}_K^T)^{n-1}D\tilde{B}]\}\\ &=\text{rank}\{D[D\tilde{B},\bar{F}_1^TD\tilde{B},\ldots,\bar{F}_K^TD\tilde{B},(\bar{F}_1^T)^2D\tilde{B},\\ &\qquad\ldots,\bar{F}_K^T\bar{F}_1^TD\tilde{B},\ldots,(\bar{F}_K^T)^2D\tilde{B},\ldots,\\ &\qquad(\bar{F}_1^T)^{n-1}D\tilde{B},\ldots,(\bar{F}_K^T)^{n-1}D\tilde{B}]\}\\ &=\text{rank}\{\pm D[\tilde{B},\bar{F}_1^T\tilde{B},\ldots,\bar{F}_K^T\tilde{B},(\bar{F}_1^T)^2\tilde{B},\\ &\qquad\ldots,\bar{F}_K^T\bar{F}_1^T\tilde{B},\ldots,(\bar{F}_K^T)^2\tilde{B},\ldots,\\ &\qquad(\bar{F}_1^T)^{n-1}\tilde{B},\ldots,(\bar{F}_K^T)^{n-1}\tilde{B}]\}\\ &=\text{rank}(\tilde{\tilde{Q}}_o(\tilde{B},\bar{F}_i^T))\\ &=\text{rank}(\tilde{\tilde{Q}}_o(B_i,\bar{F}_i^T)),\end{aligned}$$

if $\mathcal{V}_l\subseteq\mathcal{V}_1$, $D\tilde{B}=\tilde{B}$ (or $\mathcal{V}_l\subseteq\mathcal{V}_2$, $D\tilde{B}=-\tilde{B}$) with $b_{il}=1$; or $\mathcal{V}_l\subseteq\mathcal{V}_1$, $D\tilde{B}=-\tilde{B}$ (or $\mathcal{V}_l\subseteq\mathcal{V}_2$, $D\tilde{B}=\tilde{B}$) with $b_{il}=-1$, which implies that the observability of $(B_{\sigma(k)},\ F_{\sigma(k)}^T)$ is equivalent to that of $(B_{\sigma(k)},\ \bar{F}_{\sigma(k)}^T)$. □

## IV. Examples and Simulations

The controllability of MAS (3) refers to that the system orbit can be driven by an input from any initial state to any other state in finite time, while the observability of MAS (3) refers to that input-output pair $(u(k),y(k))$ on $[0,T]$ uniquely determines the initial state $x(0)$. From equation (3) with fixed topology, one has

$$\begin{cases}x(k+1)=Fx(k),\\ \quad y(k)=B^Tx(k),\end{cases}$$

then the output response of the system can be obtained as

$$y(k)=B^TF^kx(0).$$

On the other hand, for switching topology, one has

$$y(k)=B_{i_m}^TF_{i_m}^{(k-k_{m-1})}F_{i_{m-1}}^{h_{m-1}}\ldots F_{i_1}^{h_1}x(k_0),$$

for $k\in[k_{m-1},k_m)$, $m=1,...,M$. Since matrices $F$ and $B$ are known, we can estimate the unknown state $x(0)$ whether the topology is fixed or switching. Notice that $x(0)$ completely relies on matrix pair $(B^T,F)$, therefore, any initial state $x(0)$ can be uniquely determined by $y(k)$, if the given MAS with given topologies is observable. Then for given any initial state $x(0)$, we can get the followers' trajectories (the system orbit) by iterating on $x(k)$. However, to the best of our knowledge, almost all of the existing studies on observability of MASs only have given theoretical results, but not do some numerical simulations since it is very difficult to show the outputs and

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recover the (initial) states from the outputs. In order to express intuitively the validity of the obtained results on observability from the numerical simulations, we need to solve the following three problems.
(1) How to show the outputs?
(2) How to recover the (initial) states from the outputs?
(3) How to select the leaders?
First, since $B = [b_{il}] \in \mathbb{R}^{n\times n_l}$ with $b_{il} \in \{\pm 1\}$ is the coupling matrix from leaders to followers, then

$$y(k) = B^T x(k) = \pm x_i(k),$$

if the leader acts on follower $i$, that is, the output $y(k)$ is equal to the state of follower $i$. From this, we can know that the output $y(k)$ is equal to the state of the followers acted by the leaders, therefore, it is a configuration control problem. Besides, from Proposition 3, we can know that

$$y \triangleq \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k_M-1) \end{bmatrix} = \tilde{Q}_o x(0),$$

which implies that $x(0)$ can be uniquely determined by $y(k)$ if and only if $\tilde{Q}_o$ has full rank, so the (initial) states can be recovered from the outputs. Finally, notably, in our research models, the followers are controlled by the leaders as external control inputs. So, it is necessary to study how leaders work on followers so that the whole system can be observable, that is, how to design the internal structure of matrix $B$, which is provided with a good idea in Theorems 5, 8 and 11. For equation (3), if the states of the followers are described by illustrations, as a result, the followers' trajectories can be depicted but the leaders as the external control inputs can not be reflected in the simulations. Of course, the leaders can also be reflected in the simulations if the leaders can receive the external input signals. In particular, for a structurally balanced or simultaneously structurally balanced work, the condition for selecting leaders from the same subset is satisfied (See Examples 4 and 7), which is described in Theorem 5, Theorem 11 and Remark 5.

Based on the control theory, network science theory, linear algebra and graph theory, this paper deeply studies the observability of large-scale networked MASs from the perspective of the combination of the intrinsic dynamic complexity of agents and the complexity of the whole network topology, aiming to reveal the qualitative and quantitative relationship between the intrinsic dynamics of agents, network topology and system observability. By analysing, the observability of the system depends on the system matrix pair $(B^T, F)$, which is a fundamental problem for MASs involving the dynamics and evolution protocols of agents, the complexity of the topology structure, the high-dimension of the system, the leaders' selections, time-delays and many other factors, but the observability of an MAS is essentially independent of the number of agents. Theoretically, as long as the system is observable, we can simulate the system orbit for large-scale MASs. Nevertheless, in the simulation process, the internal structure of the system becomes more and more complex if the number of agents is more and more. In order to maintain the observability, the internal topology design of the system becomes more and more difficult, including re-designing the connection between agents, positive and negative weights, size of weights and leader selections, the switching sequence for MASs with switching topology, which will change the evolutionary behavior of agents with antagonistic interactions and increase the difficulty of simulations. In fact, we only use numerical simulation to verify the validity of the theoretical results. Therefore, we will give some examples with simple topologies and a small number of agents to illustrate the validity of theoretical results in the following examples and simulations. Of course, with the increasing cognition of MASs with signed networks, we will simulate large-scale MAS with complex topologies and a large number of agents in future numerical simulation work. Therefore, we will give some examples (Examples 1-8) with simple topologies to illustrate the validity of theoretical results in the following examples and simulations, where the topology is fixed in Examples 1-5 and the topology is switching in Examples 6-8. Specially, the following simulation results (Examples 2-8) show the straight-line configurations from an arbitrary given initial state if the system is observable. Example 1 shows the fact that the observability and controllability of MAS (3) are not equivalent. Theorems 1 (or 2), 3 (or 4), 5, 7, 8 and 11 are analyzed and simulated in Examples 2, 3, 4, 5, 6 and 7, respectively. Example 8 considers a special case, that is, $B^T_{\sigma(k)} = 0$ for some $\sigma(k) = i$.

*Example 1:* A five-agent signed MAS with fixed topology is considered, where the followers and the leaders are agents 1-3 and 4-5, respectively. The network topology is described by Fig. 2. The given system matrices are

$$L_s = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is easy to calculate

$$F = I - L_s - BB^T = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and rank $\left(\begin{bmatrix} B^T \\ B^T F \\ B^T F^2 \end{bmatrix}\right) = 3$, therefore, MAS (3) is observable. However, the controllability matrix of MAS (3) has no full rank, that is, $\mathrm{rank}([B, FB, F^2B]) = 2 < 3$, which implies that MAS (3) is observable, but it is uncontrollable. In other words, the controllability and observability of MAS (3) are not equivalent.

*Example 2:* A five-agent signed MAS with fixed topology is considered, where the followers and the leaders are agents 1-3 and 4-5, respectively. The network topology is described by Fig. 3$(a)$. The given system matrices are

$$L_s = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

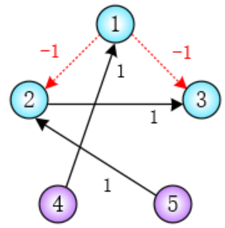



**Fig. 2:** Topology of a signed MAS.

It is easy to calculate

$$F = I - L_s - BB^T = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix},$$

$$\mathfrak{L} = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ -1 & -2 & 1 & 1 & 0 \\ -1 & 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

By computing, we can obtain that the eigenvalues of $F$ and $\mathfrak{L}$ are respectively $\lambda(F) = \{-3, -2.4142, 0.4142\}$ and $\lambda(\mathfrak{L}) = \{-3.3028, -2.6180, -0.3820, 0.3028, 1\}$, which implies that MAS (3) is observable from Theorem 1.

Moreover, the eigenvalues of $\mathfrak{L}$ are

$$\lambda(\mathfrak{L}) = \{-3.3028, -2.6180, -0.3820, 0.3028, 1\},$$

and their corresponding eigenvectors are the columns of matrix

$$\begin{bmatrix} 0 & -0.6325 & 0.6325 & 0 & 0.4472 \\ -0.6768 & -0.5117 & -0.1954 & -0.2049 & -0.4472 \\ 0.6768 & -0.5117 & -0.1954 & 0.2049 & -0.4472 \\ 0.2049 & 0.1954 & 0.5117 & -0.6768 & -0.4472 \\ -0.2049 & 0.1954 & 0.5117 & 0.6768 & -0.4472 \end{bmatrix},$$

which implies that MAS (3) is observable from Theorem 2.

Fig. 3$(b)$ depicts the random initial positions (the black s-tars), trajectories (pink, green, blue lines) and desired positions (the black circles) of the followers, and it can be seen that the black stars finally reach to the black circles.

*Example 3:* A five-agent signed MAS with fixed topology is considered, where the followers and the leaders are agents 1-3 and 4-5, respectively. The network topology is described by Fig. 4$(a)$. The given system matrices are

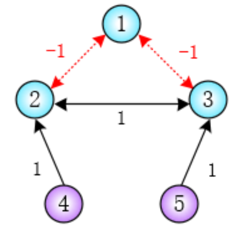
$$L_s = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to calculate

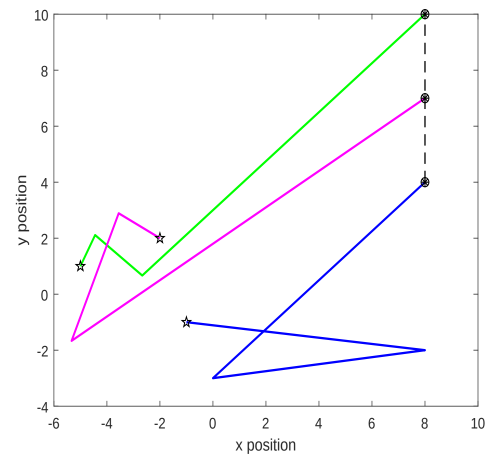
$$F = I - L_s - BB^T = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

By computing, we can obtain that

$$\lambda(F) = \{-0.1226 + 0.7449i, -0.1226 - 0.7449i, -1.7549\},$$




(a) Topology of a signed MAS.




(b) A straight-line configuration.

**Fig. 3:** A straight-line configuration $(b)$ corresponding to topology $(a)$.

and

$$B^TU = \begin{bmatrix} -0.6559 & -0.6559 & -0.4140 \\ -0.0804+0.4885i & -0.0804-0.4885i & -0.7265 \\ -0.4344+0.3688i & -0.4344-0.3688i & 0.5484 \end{bmatrix},$$

which implies that MAS (3) is observable from Theorem 3 or Theorem 4.

Fig. 4$(b)$ depicts the random initial positions (the black s-tars), trajectories (pink, green, blue lines) and desired positions (the black circles) of the followers, and it can be seen that the black stars finally reach to the black circles.

*Example 4:* A five-agent structurally balanced signed MAS with fixed topology and its corresponding underlying graph are considered, where the followers and the leaders are agents 1-3 and 4-5, respectively. The structurally balanced signed network topology (i.e., $(L_s, B, ))$ and its corresponding underlying graph (i.e., $(L, B, ))$ are described by Fig. 5$(a)$ and Fig. 6$(a)$, respectively. The given system matrices are

$$L_s = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \ L = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is easy to calculate

$$F = I - L_s - BB^T = \begin{bmatrix} -2 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix},$$

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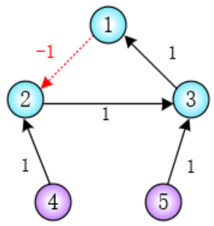
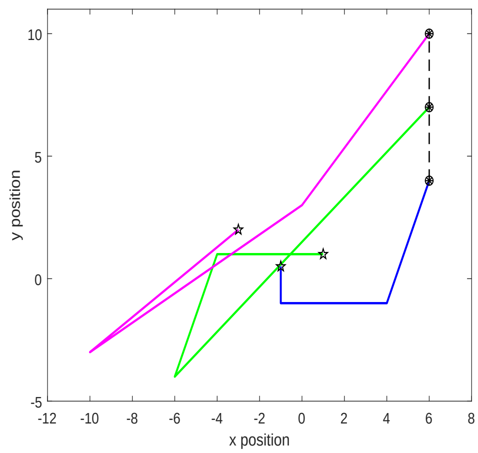



(a) Topology of a signed MAS.




(b) A straight-line configuration.

**Fig. 4:** A straight-line configuration $(b)$ corresponding to topology $(a)$.

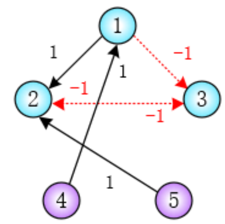
$$\bar{F} = I - L - BB^T = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and then $\operatorname{rank}\left(\begin{bmatrix} B^T \\ B^T F \\ B^T F^2 \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} B^T \\ B^T \bar{F} \\ B^T \bar{F}^2 \end{bmatrix}\right) = 3$, therefore, the observability of $(B^T, F)$ is equivalent to that of $(B^T, \bar{F})$ based on Theorem 5.

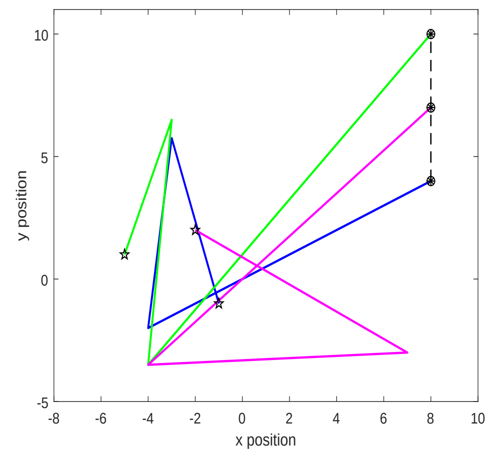
Figs. $5(b)$ and $6(b)$ depict the random initial positions (the black stars), trajectories (pink, green, blue lines) and desired positions (the black circles) of the followers in a structurally balanced signed MAS and its corresponding underlying graph, respectively, and it can be seen that the black stars finally reach to the black circles. Specially, we can find from Fig. $5(a)$ that the leaders are chosen from the same partitioned subset, but the leaders can be permitted to be chosen from different partitioned subsets for signed MASs for Fig. 2, thus the observability of MAS (3) can be influenced by the topology structure between the followers and the selections of the leaders.

*Example 5:* Two five-agent signed MASs with fixed topology are considered, where the followers and the leader are agents 1-4 and 0, respectively. The network topologies are described by Fig. $7(a)$ and Fig. $8(a)$, respectively. For Fig. $7(a)$, the given system matrices are

$$L_{s1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$




(a) Topology of a structurally balanced signed MAS with the leaders from the same partitioned subset.




(b) A straight-line configuration.

**Fig. 5:** A straight-line configuration $(b)$ corresponding to topology $(a)$.

It is easy to calculate

$$F_1 = I - L_{s1} - B_1 B_1^T = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

and $\operatorname{rank}\left(\begin{bmatrix} B_1^T \\ B_1^T F_1 \\ B_1^T F_1^2 \\ B_1^T F_1^3 \end{bmatrix}\right) = 4$, therefore, MAS (3) is observable.

For Fig. $8(a)$, the given system matrices are

$$L_{s2} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For matrix $L_{s2}$, there is matrix $D = \operatorname{diag}\{1, 1, 1, -1\}$ such that $L_{s2} = DL_{s1}D$, therefore, $L_{s2} \sim L_{s1}$, the observability of Fig. $8(a)$ is equivalent to that of Fig. $7(a)$ according to Theorem 7. In fact,

$$F_2 = I - L_{s2} - B_2 B_2^T = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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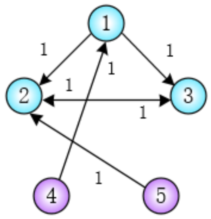
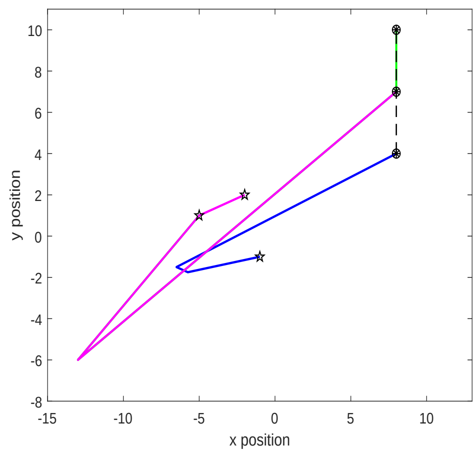


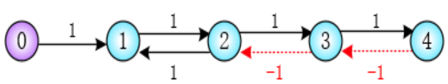

(a) The underlying graph corresponding to Fig. 5(a).



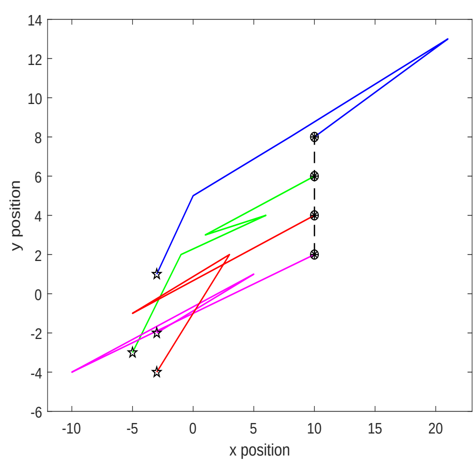

(b) A straight-line configuration.

**Fig. 6:** A straight-line configuration $(b)$ corresponding to topology $(a)$.

and rank $\left(\begin{bmatrix} B_2^T \\ B_2^T F_2 \\ B_2^T F_2^2 \\ B_2^T F_2^3 \end{bmatrix}\right) = 4$, therefore, MAS (3) is also observable.

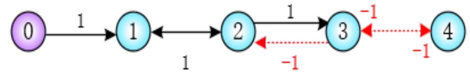



(a) A path with signed networks.

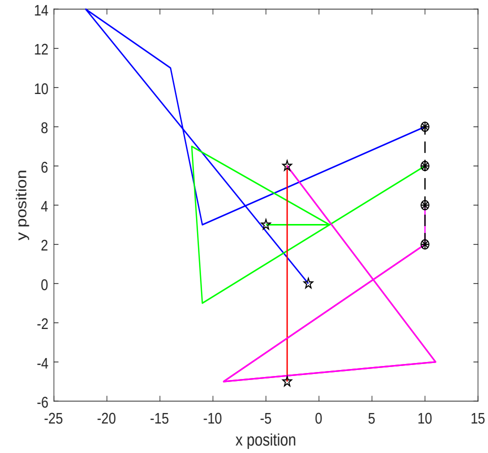



(b) A straight-line configuration.

**Fig. 7:** A straight-line configuration $(b)$ corresponding to topology $(a)$.




(a) A path with signed networks.




(b) A straight-line configuration.

**Fig. 8:** A straight-line configuration $(b)$ corresponding to topology $(a)$

Figs. $7(b)$ and Fig. $8(b)$ depict the random initial positions (the black stars), trajectories (red, green, pink, blue lines) and desired positions (the black circles) of the followers corresponding to Figs. $7(a)$ and Fig. $8(a)$, respectively, and it can be seen that the black stars finally reach to the black circles.

*Example 6:* A four-agent signed MAS with switching topology is considered, where the followers and the leader are agents 1-3 and 0, respectively. The network topology is described by Fig. $9(a)$. The switched linear subsystems are

$$F_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$
$$F_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};$$
$$F_3 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \; B_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

By computing,

$$\begin{aligned} \mathcal{O}_1 &= \operatorname{span}\{B_1, B_2, B_3\} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \\ &= \mathcal{O}_2 = \mathcal{O}_3 = \mathbb{R}^3, \end{aligned}$$

therefore, MAS (4) is observable. We can also obtain that $\mathcal{R}(B_1) + \mathcal{R}(B_2) + \mathcal{R}(B_3) = \mathbb{R}^3$, which implies that MAS (4) is observable according to Theorem 8.

Fig. $9(b)$ depicts the random initial positions (the black s-tars), trajectories (green, pink, blue lines) and desired positions (the black circles) of the followers, and it can be seen that

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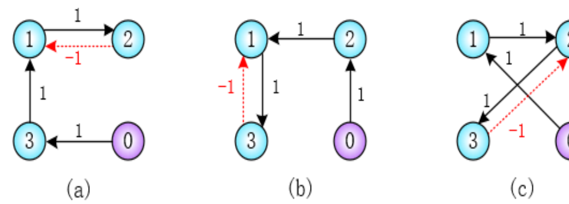
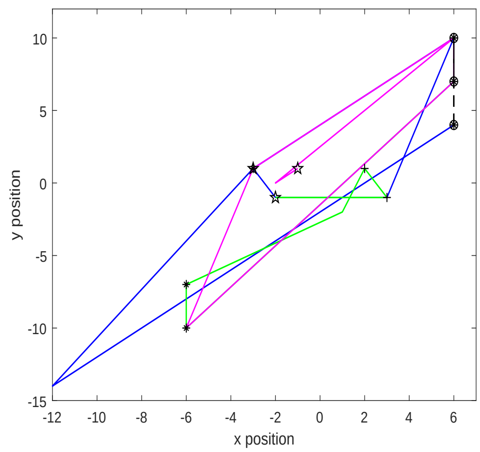


(a) The switching topology of the network.




(b) A straight-line configuration.

**Fig. 9:** A straight-line configuration $(b)$ corresponding to topology $(a)$

the black stars finally reach to the black circles with a given switching path.

*Example 7:* An eight-agent structurally balanced signed MAS with switching topology is considered, where the followers and the leaders are agents 1-6 and 7-8, respectively. The network topology is described by Fig. $10(a)$. The switched linear subsystems are

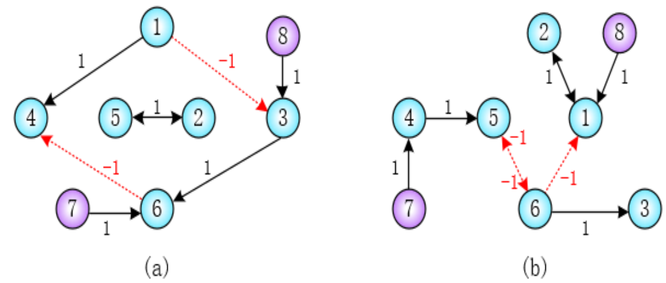
$$F_1 = \begin{bmatrix} -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix};$$

$$F_2 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & -1 & -2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

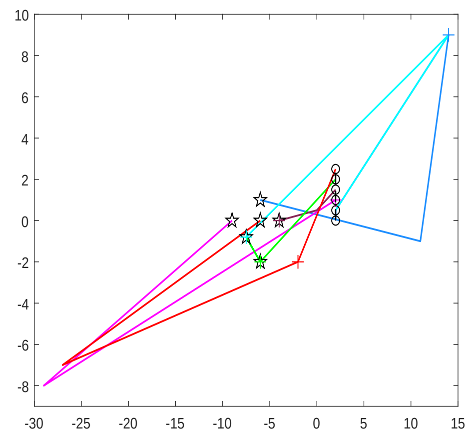
By computing, $\text{rank}(\tilde{\tilde{Q}}_o(B_i, F_i^T)) = 6$, therefore, MAS (4) is observable from Theorem 11.

Fig. $10(b)$ depicts the random initial positions (the black stars), trajectories (red, green, pink, blue, purple, turquoise lines) and desired positions (the black circles) of the followers, and it can be seen that the black stars finally reach to the black circles. Specially, we can find from Fig. $10(a)$ that the leaders are chosen from the same partitioned subset.

*Example 8:* A four-agent signed MAS with switching topology is considered, where the followers and the leader



(a) The switching topology of the network.



(b) A straight-line configuration.

**Fig. 10:** A straight-line configuration $(b)$ corresponding to topology $(a)$.

are agents 1-3 and 0, respectively. The network topology is described by Fig. $11(a)$. The switched linear subsystems are

$$F_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

$$F_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

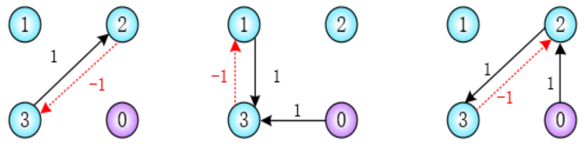
$$F_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Obviously, realizations $(B_i^T, F_i)$ $(i = 1, 2, 3)$ all are unobservable and $B_1^T = 0$. However,

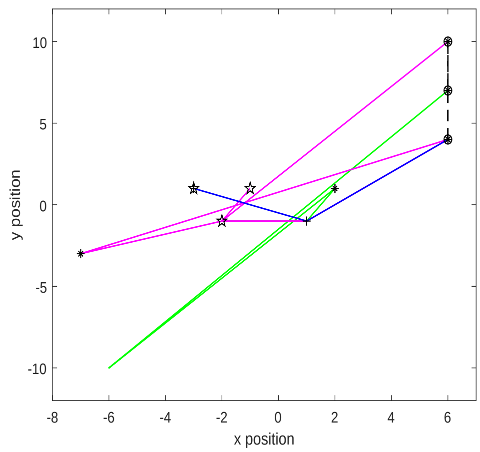
$$\begin{aligned} \mathcal{O}_1 &= \text{span}\{B_2, (F_2^T)^2 B_2, B_3\} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \\ &= \mathcal{O}_2 = \mathcal{O}_3 = \mathbb{R}^3, \end{aligned}$$

therefore, MAS (4) is observable based on Theorem 8.

Fig. $11(b)$ depicts the random initial positions (the black stars), trajectories (green, pink, blue lines) and desired positions (the black circles) of the followers, and it can be seen that the black stars finally reach to the black circles. Specially, we can find that the whole system can still be observable even though $B_{\sigma(k)}^T = 0$ is possible to appear for some $\sigma(k) = i$ during the switching process.



(a) The switching topology of the network.



(b) A straight-line configuration.

**Fig. 11:** A straight-line configuration $(b)$ corresponding to topology $(a)$.

## V. Conclusion

This paper has considered the observability of signed MASs with fixed topology and switching topology, respectively. It has proved that the observability of signed MASs only lies on structural pair $(B^T, F)$. We have obtained some effective conditions for judging the switching observability of MASs over signed networks even though each subsystem is unobservable. Our main results show some advantages of a unified programme for the observability of MASs and increase the theoretical studying on the observability of MASs.

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**Bo Liu** was born in 1977. She received the Ph.D. degree in Dynamics and Control from Peking University in 2007. She is currently a Professor in Minzu University of China. Her research interests include swarm dynamics, networked systems, collective behavior and coordinate control of multi-agent systems.

**Housheng Su** received his B.S. degree in automatic control and his M.S. degree in control theory and control engineering from Wuhan University of Technology, Wuhan, China, in 2002 and 2005, respectively, and his Ph.D. degree in control theory and control engineering from Shanghai Jiao Tong University, Shanghai, China, in 2008. From December 2008 to January 2010, he was a Postdoctoral researcher with the Department of Electronic Engineering, City University of Hong Kong, Hong Kong. Since November 2014, he has been a full professor with the School of Automation, Huazhong University of Science and Technology, Wuhan, China. He is an Associate Editor of IET Control Theory and Applications. His research interests lie in the areas of multi-agent coordination control theory and its applications to autonomous robotics and mobile sensor networks.

**Licheng Wu** received the B. S. Degree from Beijing University of Aeronautics and Astronautics (BUAA), China in 1995, and the Ph. D. degree in robotics from the Institute of Robotics of BUAA in 2000. He is currently a full professor and dean of School of Information Engineering, Minzu University of China, China. Previously, in 2001 and 2002, he was a postdoc at State Key Laboratory of Intelligent Technology and Systems, Department of Computer Science and Technology, Tsinghua University, China. He was promoted as an associate professor of Tsinghua University in 2008. He moved to School of Information Engineering, Minzu University of China on November 2009. At Minzu University of China, he was promoted as full professor from January 2013. From November 2006 to November 2007, he was a visiting scholar at Polytechnic di Milano and Cassino University, Italy. His research interests include artificial intelligence and robotics, especially computer games, and intelligent robot.

**Xixi Shen** was born in 1996. She received the B.S. degree in mathematics and applied mathematics from Dongchang College of Liaocheng University, Shandong, China, in 2018, and will receive the M. S. degree in applied mathematics at North China University of Technology, Beijing, China, in 2021. Her research interests are in the fields of networked systems and collective intelligence.