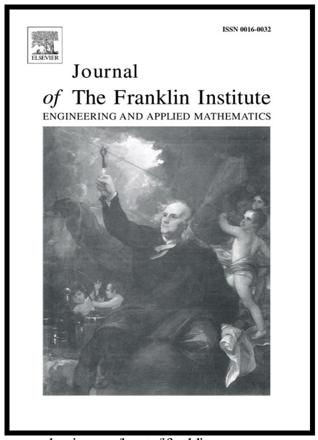

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Group controllability of discrete-time multi-agent systems

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Abstract

Controllability is a fundamental issue concerning control of multi-agent networks and a very important research topic in the modeling, analysis and coordination control of multi-agent systems. Group controllability problem is a further extension of the controllability problem of the general multi-agent systems, which mainly studies the cooperation and control of multi-agent systems with multiple subgroups or multiple intelligence clusters. Comparing with the controllability of the general multi-agent systems, the group controllability is not only to consider the information interaction among the groups, but also to consider the information interaction between different groups, which makes the system reflect the effect of a whole and also the internal structure of the sub-groups. This paper addresses the group controllability problems of discrete-time multi-agent systems with time-delay, in which both switching topology and fixed topology are considered. This paper also proposes the general definition of the group controllability, as well as establishes group controllability criteria from the algebraic and graphical perspectives. Numerical examples and simulations are proposed to illustrate the theoretical results.

Key words: Multi-agent systems; Controllability; Group controllability; Networked dynamic systems.

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1 Introduction

In recent years, the multi-agent system has become a new branch of complex system in control field, which has attracted many researchers' interests and concerns in many areas such as engineering, ecology, biology, sociology, computing science, sensing and communication technologies [1]-[10].

The controllability of the control system is an important basic problem in modern control theory and plays a key role in many research fields. Controllability problem is one of the basic problems for distributed coordinated control of multi-agent systems. In engineering practice, in order to accomplish a specific task and achieve certain performance requirements, by controlling some of the agents, the rest of the group can move or achieve the desired goal of the group via designing the dynamic evolutionary algorithm or control law for each agent. As the evolution behavior of multi-agent system can be affected by many factors, such as the dynamic evolution of the agent itself, the communication topology among agents, the state evolution protocol and the external interference, as well as the structure, parameters, control input (leaders selection), feedback gain and the dimension of the multi-agent system, the controllability of multi agent system becomes a very challenging task.

The controllability problem of multi-agent systems was first proposed by Tanner [11] in 2004, in which one-integrator dynamic model through nearest neighbor rules was presented, where one of the agents was regarded as a leader (external input), and necessary and sufficient conditions of the controllability were derived under a fixed time-invariant nearest-neighbor topology. In what follows, some related results on the controllability of multi-agent systems are investigated, such as continuous-time [12], [13], [14] and discrete-time [15], [16]; first-order [12], [17], [18], second-order [19] and high-order [13]; undirected topology and directed topology [16]; switching topology and time delay in [17] , [20].

At present, the main methods for the controllability of multi-agent systems have concentrated on investigating graph-theoretic characterisation [21], [22] and algebraic criteria [17] of the systems. Much work has focused on the graphical conditions of controllability based on equitable or relaxed equitable partition in [22], external equitable partition in [23] and the references therein. Algebraic conditions characterized the relationship between controllability of topological structures described in [17], [24], [25], [26], [27], [28] , [29] and so on. There is other progress in the controllability of paths [30], multi-chain topologies [31], [32], tree graphs [33], symmetric structures [17], [34], [35], [36], grid graphs [37], Cartesian product networks [38], threshold graphs [39], etc.

For multi-agent systems, the controllability refers to transferring the remaining intelligent agents of such system from any arbitrary initial state to any final state by controlling dynamics of a small amount of intelligent agents under exchanged information between each other. With the improvement of multi-agent systems' complexity, the whole system can be divided into some subgroups. How to investigate the controllability of the multi-agent system containing multiple subgroups, that is, how to make the intelligent agents of each subgroup in the multi-agent system be transferred from any arbitrary initial state to any final state by controlling dynamics of a small amount of intelligent agents? In a recent paper [40], the authors studied the group consensus in multi-agent systems with switching topologies and communication delays.

Motivated by the work in [40], we will investigate the controllability of the general multi-agent systems based on the group consensus. However, comparing with the controllability of the general multi-agent systems, the group controllability is not only to consider the information interaction among the groups, but also to consider the information interaction among different groups, which makes the system reflect the effect of a whole and also the internal structure of the sub-groups, as well as remains to be a challenging task but be highlighted in this paper.

The objective of the paper is to study the group controllability for multi-agent systems with different networked topological structures and communications restrictions in more detail, in which time-delays occur in the coupling information among the sub-groups, propose the general definition of the group controllability, establish group controllability criteria, and provide analysis to a unified theoretical framework for the controllability and the group controllability of multi-agent systems. For a special case, that is, the fixed networked topology, the group controllability of multi-agent systems with time-delays can be tested by its eligible subsystems, which is theoretically proved via PBH rank test.

Compared with the existing works, the main contributions of the paper are summarised as follows.

1. *Different from the controllability problem studied under leader-follower framework [17], where the leaders are acted as inputs, the whole network system without leaders or inputs is divided into some subgroups based on consensus protocol, in which communication delay occurs when information is exchanged between agents in the same subgroup in this paper. The consensus protocol is proposed to figure out how the protocol affect the group controllability of discrete-time multi-agent systems.*

2. *The definitions of the group-controllability of multi-agent systems are first established for switching topology and fixed topology, respectively.*

3. *Sufficient and/or necessary conditions are established for the group controllability of discrete-time multi-agent systems, which means controllability with and without input delay is equivalent to that of an equivalent augmented system without time-delays.*

4. *The group controllability of discrete-time multi-agent system with time delay can be determined only by the information from the subgroup to the other subgroup and it has nothing to do with the inner topology among the subgroups. This important result is valid no matter topology of the network is fixed or switching. This advantage is much desirable in applications as it can provide more freedom to design the configuration of the multi-agent system.*

5. *The PBH rank test is introduced to determine the controllability of discrete-time multi-agent systems, which is very valid to compute the eigenvalues of the system with multiple time-delays and high dimension using the PBH rank method by Matlab.*

The rest of the paper is organized as follows. Section 2 states the problem formulation and some definitions. Section 3 gives the main results. Section 4 presents simulation results. The conclusion is given in Section 5.

2 Problem formulation and Preliminaries

2.1 Graphs preliminaries

Graph theory is an effective tool to study the coupling topology of the communication configuration of the agents. In this section, we briefly review some basic notations and concepts in graph theory that will be used in this paper [17].

A *weighted directed graph* $\mathscr{G}=(\mathscr{V},\mathscr{E},\mathscr{A})$ consists of a *vertex set* $\mathscr{V}=\{v_1,v_2,\cdots,v_N\}$ and an *edge set* $\mathscr{E}=\{(v_i,v_j):v_i,v_j\in\mathscr{V}\}$, where an *edge* is an ordered pair of distinct vertices of $\mathscr{V}$, and the nonsymmetric weighted adjacency matrix $\mathscr{A}=[a_{ij}]$, with $a_{ij}>0$ if and only if $e_{ij}\in\mathscr{E}$ and $a_{ij}=0$ if not. If all the elements of $\mathscr{V}$ are unordered pairs, then the graph is called an *undirected graph*. If $v_i,v_j\in\mathscr{V}$, and $(v_i,v_j)\in\mathscr{E}$, then we say that v_i and v_j are adjacent or v_j is a neighbor of v_i. The neighborhood set of node v_i is denoted by $\mathscr{N}_i=\{v_j\in\mathscr{V}:(v_i,v_j)\in\mathscr{E}\}$. The number of neighbors of each vertex is its *degree*. A graph is called *complete* if every pair of vertices are adjacent. A *path* of length r from v_i to v_j in a graph is a sequence of $r+1$ distinct vertices starting with v_i and ending with v_j such that consecutive vertices are adjacent. If there is a path between any two vertices of $\mathscr{G}$, then $\mathscr{G}$ is *connected*. The *degree matrix* $\triangle(\mathscr{G})$ of $\mathscr{G}$ is a diagonal matrix with rows and columns indexed by $\mathscr{V}$, in which the (v_i,v_i)-entry is the degree of vertex v_i. The symmetric matrix defined as

$$L(\mathscr{G})=\triangle(\mathscr{G})-A(\mathscr{G})$$

is the *Laplacian* of $\mathscr{G}$. The Laplacian is always symmetric and positive semi-definite, and the algebraic multiplicity of its zero eigenvalue is equal to the number of connected components in the graph.

Define a network $(\mathscr{G},x)$ with state $x \in \Re^k$ and topology graph, where the network has N agents and the $i-th$ agent's state is marked as $x_i \in \Re$. Each agent updates its state based on the information available from its neighbors.

Definition 1 *[40] (Subgraph)*
A network with topology $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1, \mathscr{A}_1)$ is said to be a sub-network of a network with topology $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ if (i)$\mathscr{V}_1 \subseteq \mathscr{V}$, (ii) $\mathscr{E}_1 \subseteq \mathscr{E}$ and (iii) the weighted adjacency matrix $\mathscr{A}_1$ inherits $\mathscr{A}$. Correspondingly, we call $\mathscr{G}_1$ a subgraph of $\mathscr{G}$. Furthermore, if the inclusion relations in (i) and (ii) are strict, and $\mathscr{E}_1 = \{(v_i, v_j) : i, j \in \mathscr{V}_1, (v_i, v_j) : i, j \in \mathscr{E}\}$, we say that the first network is a proper sub-network of the second one. Correspondingly, we call $\mathscr{G}_1$ a proper subgraph of $\mathscr{G}$.

Definition 2 *[40] (Balanced couple)*
For any $i \in \ell_1$, the i-th agent v_i in $\mathscr{G}_1$ is said to be in-degree balanced to $\mathscr{G}_2$ at time t if $\sum_{j=n+1}^{n+m} a_{ij} = 0$, and $\mathscr{G}_1$ is said to be in-degree balanced to $\mathscr{G}_2$ at time t if all agents in it are in-degree balanced to $\mathscr{G}_2$ at time t. Similarly, the i-th node v_i in $\mathscr{G}_1$ is said to be out-degree balanced to $\mathscr{G}_2$ at time t if $\sum_{j=n+1}^{n+m} a_{ji} = 0$, and $\mathscr{G}_1$ is said to be out-degree balanced to $\mathscr{G}_2$ at time t if all agents in it are out-degree balanced to $\mathscr{G}_2$ at time t. Furthermore, we say that $\mathscr{G}_1$ and $\mathscr{G}_2$ are an in-degree balanced couple at time t if $\mathscr{G}_1$ is in-degree balanced to $\mathscr{G}_2$ at time t and vice versa. Moreover, we say that $\mathscr{G}_1$ and $\mathscr{G}_2$ are a balanced couple at time t if $\mathscr{G}_1$ is out-degree balanced and in-degree balanced to $\mathscr{G}_2$ at time t and vice versa.

In the following, for the convenience of discussion, we assume that the coupling topology graph is connected throughout this paper.

2.2 Problem formulation

Consider a multi-agent system with N dynamic agents described by

$$x_i(k+1) = u_i(k), \quad i = 1, \cdots, N, \tag{1}$$

where $x_i \in \Re$ is the state of the i-th agent and each agent updates its state based on nearest neighbor-averaging rules, u_i is a state feedback.

In reality, the whole multi-agent system can be divided into some complex subgroups or intelligent clusters, which can make the cooperation and control of such systems have a more practical significance. Without loss of generality, all the agents are in the network $(\mathscr{G},x)$ consisting of $n+m$ ($n,\ m >$

1) agents divided into two different subgroups $(\mathscr{G}, x^1)$ and $(\mathscr{G}, x^2)$ with $x^1 = (x_1, x_2, \ldots, x_n)^T$ and $x^2 = (x_{n+1}, x_{n+2}, \ldots, x_{n+m})^T$, and the agents in each subgroup can establish a subnetwork. An example of two different subgroups connected digraph is displayed in Fig. 1.

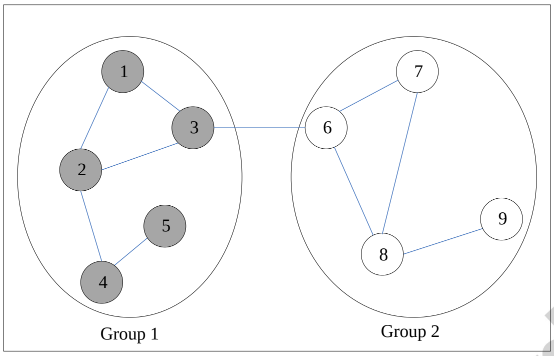

Fig. 1: Topology $\mathscr{G}$.

Let $\ell_1 = \{1, 2, \cdots, n\}$, $\ell_2 = \{n+1, n+2, \cdots, n+m\}$, $\mathscr{V}_1 = \{v_1, \ldots, v_n\}$, $\mathscr{V}_2 = \{v_{n+1}, \ldots, v_{n+m}\}$, $\mathscr{V} = \mathscr{V}_1 \cup \mathscr{V}_2$, $\ell = \ell_1 \cup \ell_2$. $\mathscr{N}_i$ represents the neighbor set of agent i in the network, $\mathscr{N}_{1i} = \{v_j \in \mathscr{V}_1 : (v_j, v_i) \in \mathscr{E}\}$, $\mathscr{N}_{2i} = \{v_j \in \mathscr{V}_2 : (v_j, v_i) \in \mathscr{E}\}$, where $\mathscr{N}_i = \mathscr{N}_{1i} \cup \mathscr{N}_{2i}$ and $\mathscr{N}_{1i} \bigcap \mathscr{N}_{2i} = \varnothing$.

Inspired by the group consensus protocol in [40], in this paper, for network $(\mathscr{G}, x)$, the protocol is described as

$$u_i(k) = \begin{cases} x_i(k) + \sum\limits_{j \in \mathscr{N}_{1i}} a_{ij}[x_j(k-h) - x_i(k-h)] + \sum\limits_{j \in \mathscr{N}_{2i}} a_{ij} x_j(k), & i \in \ell_1, \\ x_i(k) + \sum\limits_{j \in \mathscr{N}_{2i}} a_{ij}[x_j(k-h) - x_i(k-h)] + \sum\limits_{j \in \mathscr{N}_{1i}} a_{ij} x_j(k), & i \in \ell_2, \end{cases} \tag{2}$$

where $a_{ij} \geq 0$, $\forall i, j \in \ell_1, \ell_2$, and $a_{ij} \in \mathfrak{R}$, h is the integer time delay.

Remark 1 *Here the weighting factor a_{ij} is permitted to be negative. In fact, it is more difficult to deal with the controllability problem due to the existence of negative factors of the topology between different subgroups, which makes the controllability analysis more complex.*

Remark 2 *Different from the controllability problem studied under leader-follower framework [17], where the leaders are acted as inputs, the whole network system has no leaders or inputs based on consensus protocol.*

Let $x = (x_1, \ldots, x_N)^T$ be the stack vector of all the agent states, then it follows that

$$x(k+1) = -Lx(k) \tag{3}$$

where $L = [l_{ij}] \in \Re^{N\times N}$ with

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum\limits_{k=1,k\neq i} a_{ik}, & i = j, \\ 0, & otherwise. \end{cases}$$

Assume that $x^1 = (x_1, \cdots, x_n)^T$ and $x^2 = (x_{n+1}, \cdots, x_{n+m})^T$ be the state vectors of $\mathscr{G}_1$ and $\mathscr{G}_2$, respectively. Then system (3) becomes

$$\begin{cases} x^1(k+1) = x^1(k) - L_{11}x^1(k-h) - L_{12}x^2(k) \\ x^2(k+1) = x^2(k) - L_{22}x^2(k-h) - L_{21}x^1(k) \end{cases}, \tag{4}$$

where

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}_{(n+m)\times(n+m)},$$

with $L_{11} \in \Re^{n\times n}$, $L_{12} \in \Re^{n\times m}$, $L_{21} \in \Re^{m\times n}$, $L_{22} \in \Re^{m\times m}$.

Remark 3 *Matrix L is not the Laplacian matrix, that is, the the row sums of L are not always all equal to zero. But from (4), matrices $L_{ii}(i = 1,2)$ are the Laplacian matrices.*

Remark 4 *If x is p-dimensional, then system (4) can be rewritten as*

$$\begin{cases} x^1(k+1) = (I_n \otimes I_p)x^1(k) - (L_{11} \otimes I_p)x^1(k-h) - (L_{12} \otimes I_p)x^2(k) \\ x^2(k+1) = (I_m \otimes I_p)x^2(k) - (L_{22} \otimes I_p)x^2(k-h) - (L_{21} \otimes I_p)x^1(k) \end{cases},$$

where $\otimes$ represents Kronecker product, I_p is the p-dimensional identity matrix.

The formation of system (4) is different from the classical formation of the discrete-time model, therefore, it is hard to study the issue of controllability via the classical control theory. For analytic justification, an equivalent augmented system of system (4) are introduced as follows:

$$\begin{cases} x^1(k+1) &= x^1(k) - L_{11}x^1(k-h) - L_{12}x^2(k) \\ x^1(k) &= x^1(k) \\ & \vdots \\ x^1(k-h+1) &= x^1(k-h+1) \end{cases}$$

and

$$\begin{cases} x^2(k+1) &= x^2(k) - L_{11}x^2(k-h) - L_{12}x^1(k) \\ x^2(k) &= x^2(k) \\ &\vdots \\ x^2(k-h+1) &= x^2(k-h+1) \end{cases}.$$

Define new state vectors $X^1(k) = (x^1(k)^T, x^1(k-1)^T, \cdots, x^1(k-h)^T)^T$ and $X^2(k) = (x^2(k)^T, x^2(k-1)^T, \cdots, x^2(k-h)^T)^T$, then we have

$$\begin{cases} X^1(k+1) = \mathscr{L}_{11}X^1(k) + \mathscr{L}_{12}X^2(k) \\ X^2(k+1) = \mathscr{L}_{22}X^2(k) + \mathscr{L}_{21}X^1(k) \end{cases} \tag{5}$$

where

$$\mathscr{L}_{11} = \begin{bmatrix} I & 0 & \cdots & 0 & -L_{11} \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}_{(h+1)n\times(h+1)n}, \quad \mathscr{L}_{12} = \begin{bmatrix} -L_{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(h+1)n\times(h+1)m},$$

$$\mathscr{L}_{22} = \begin{bmatrix} I & 0 & \cdots & 0 & -L_{22} \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}_{(h+1)m\times(h+1)m}, \quad \mathscr{L}_{21} = \begin{bmatrix} -L_{21} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(h+1)m\times(h+1)n},$$

and I is the identity matrix with compatible dimensions.

Remark 5 *System (5) can be regarded as a new system without time-delays, where $X^2(k)$ and $X^1(k)$ are the external inputs, the matrices $\mathscr{L}_{11}$ and $\mathscr{L}_{22}$ are system matrices and matrices $\mathscr{L}_{12}$ and $\mathscr{L}_{21}$ are input matrices. System (4) is controllable if and only if system (5) is controllable. Therefore, the group controllability of system (5) via the classical control theory can be investigated.*

3 Group controllability analysis

The controllability of the multi-agent system refers to transferring the remaining intelligent agents of such system from any arbitrary initial state to any final state by controlling dynamics of a small amount of intelligent agents. Group controllability problem is a further extension of the controllability problem of the general multi-agent systems, that is, the intelligent agents of each subgroup in the multi-agent system can be transferred from any arbitrary initial state to any final state by controlling dynamics of a small amount of intelligent agents. This paper first aims to theoretically investigate the group controllability problems of multi-agent systems with different networked topological structures and communications restrictions to establish group controllability criteria, and design appropriate control inputs to realize the group controllability of multi-agent systems.

3.1 Group controllability of discrete-time multi-agent systems on switching topology

With the transformation from the original dynamics to system (5), the corresponding system via switching topology of system (5) can be described as

$$\begin{cases} X^1(k+1) = \mathscr{L}_{11(\sigma(l))}X^1(k) + \mathscr{L}_{12(\sigma(l))}X^2(k) \\ X^2(k+1) = \mathscr{L}_{22(\sigma(l))}X^2(k) + \mathscr{L}_{21(\sigma(l))}X^1(k) \end{cases}, \tag{6}$$

where the piecewise constant scalar function $\sigma(l) : \Re^+ \to \{1, \cdots, K\}$ is the switching path describing the time-variant coupling of the network, K is the number of possible coupling patterns (i.e. switching topologies). Moreover, $\sigma(l) = i$ means that the system realizations are chosen as $(\mathscr{L}_{11(i)}, \mathscr{L}_{12(i)})$ and $(\mathscr{L}_{22(i)}, \mathscr{L}_{21(i)})$, $i = 1, \cdots, K$.

In the following, we first introduce some necessary concepts and lemmas.

Definition 3 *[17](Switching sequence)*
A switching sequence π is a set with finite scalars

$$\pi \triangleq \{i_0, \ldots, i_{M-1}\}$$

where $0 < M < \infty$ is the length of π, $i_m \in \{1, \ldots, M\}$ the index of the mth realization, for $m \in \underline{M}$, $\underline{M} = 0, 1, \ldots, M-1$.

Given a switching sequence $\pi \triangleq \{i_0, \ldots, i_{M-1}\}$, an associated switching path $\sigma(m)$ can be determined as $\sigma(m) = i_m,\ if\ m \in \underline{M}$.

Definition 4 *(Group Switching Controllability)*
A non-zero state x of system (6) is defined as group switching controllability if the states of agents satisfy:

1. *if there exists a time instant* $0 < M < \infty$ *and a switching path* $\sigma : \underline{M} \to 1, \ldots, K$*, and the control input* X^2 *for* $k \in \underline{M}$*, such that* $X^1(0) = X^1$ *and* $X^1(M) = 0$*; and*

2. *if there exists a time instant* $0 < M < \infty$ *and a switching path* $\sigma : \underline{M} \to 1, \ldots, K$*, and the control input* X^1 *for* $k \in \underline{M}$*, such that* $X^2(0) = X^2$ *and* $X^2(M) = 0$*.*

If any non-zero state x of system (6) attains group switching controllability, then system (6) is said to attain group switching controllability.

Definition 5 *[17](Column space)*
Given a matrix $B_{p\times m} = [b_1, b_2, \cdots, b_m]$*, the column space* $\mathscr{R}(B)$ *is defined as*

$$\mathscr{R}(B) \triangleq span\{b_1, b_2, \cdots, b_m\}.$$

Lemma 1 *[15] Given* $A_i \in \Re^{m\times n_i}$*,* $i \in \underline{r}$*, where* $\underline{r}$ *presents an index set* $(1, 2, \ldots, r)$*, and* $B = [A_1, A_2, \ldots, A_r] \in P^{m\times s}$*,* $s = \sum_{i=1}^{r} n_i$*, then*

$$\mathscr{R}(B) = \sum_{i=1}^{r} \mathscr{R}(A_i).$$

Definition 6 *[17](Cyclic invariant subspace)*
Given a matrix $A \in \Re^{N\times N}$ *and a linear subspace* $\mathscr{W} \subseteq \Re^N$*, the* $\mathscr{K}$*-cyclic invariant subspace* $\langle A|\mathscr{W}\rangle$ *is defined as*

$$\langle A|\mathscr{W}\rangle \triangleq \sum_{i=1}^{K} A^{i-1}\mathscr{W}.$$

For notational simplicity, let $\langle A|B\rangle = \langle A|\mathscr{R}(B)\rangle$, which can be derived by the definition of column space and cyclic invariant subspace. Choose the matrices $\mathscr{L}_{11}, \mathscr{L}_{12}$ and $\mathscr{L}_{22}, \mathscr{L}_{21}$ in system (6), respectively, where $\mathscr{L}_{12(i)} = [l_{12(i_{n+1})}, l_{12(i_{n+2})}, \cdots, l_{12(i_{n+m})}]$, $\mathscr{L}_{21(i)} = [l_{21(i_1)}, l_{21(i_2)}, \cdots, l_{21(i_n)}]$.

Choose $i = 1$, and let $\mathscr{L}_{12(1)} = \mathscr{L}_{12}, \mathscr{L}_{21(1)} = \mathscr{L}_{21}$ for simplicity, where

$$\mathscr{L}_{12} = [l_{12(n+1)}, l_{12(n+2)}, \cdots, l_{12(n+m)}],\ \ \mathscr{L}_{21} = [l_{21(1)}, l_{21(2)}, \cdots, l_{21(n)}].$$

Therefore,

$$\begin{aligned}\langle \mathscr{L}_{11}|\mathscr{L}_{12}\rangle &= \sum_{i=1}^{n} \mathscr{L}_{11}^{i-1}[l_{12(n+1)}, l_{12(n+2)}, \cdots, l_{12(n+m)}] \\ &= [\sum_{i=1}^{n} \mathscr{L}_{11}^{i-1} l_{12(n+1)}, \sum_{i=1}^{n} \mathscr{L}_{11}^{i-1} l_{12(n+2)}, \cdots, \sum_{i=1}^{n} \mathscr{L}_{11}^{i-1} l_{12(n+m)}] \\ &= [\langle \mathscr{L}_{11}|l_{12(n+1)}\rangle, \langle \mathscr{L}_{11}|l_{12(n+2)}\rangle, \cdots, \langle \mathscr{L}_{11}|l_{12(n+m)}\rangle] \\ &= \sum_{j=n+1}^{n+m} \langle \mathscr{L}_{11}|l_{12(j)}\rangle \\ &= \langle \mathscr{L}_{11}|\mathscr{R}(\mathscr{L}_{12})\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle \mathscr{L}_{22}|\mathscr{L}_{21}\rangle &= \sum_{i=1}^{m} \mathscr{L}_{22}^{i-1}[l_{21(1)}, l_{21(2)}, \cdots, l_{21(n)}] \\ &= [\sum_{i=1}^{m} \mathscr{L}_{22}^{i-1} l_{21(1)}, \sum_{i=1}^{m} \mathscr{L}_{22}^{i-1} l_{21(2)}, \cdots, \sum_{i=1}^{m} \mathscr{L}_{22}^{i-1} l_{21(n)}] \\ &= [\langle \mathscr{L}_{22}|l_{21(1)}\rangle, \langle \mathscr{L}_{22}|l_{21(2)}\rangle, \cdots, \langle \mathscr{L}_{22}|l_{21(n)}\rangle] \\ &= \sum_{j=1}^{n} \langle \mathscr{L}_{22}|l_{21(j)}\rangle \\ &= \langle \mathscr{L}_{22}|\mathscr{R}(\mathscr{L}_{21})\rangle.\end{aligned}$$

For system (6), subspace sequences are defined as

$$\mathscr{W}_{11} = \sum_{i=1}^{K} \langle \mathscr{L}_{11(i)}|\mathscr{L}_{12(i)}\rangle, \mathscr{W}_{12} = \sum_{i=1}^{K} \langle \mathscr{L}_{11(i)}|\mathscr{W}_{11}\rangle, \cdots, \mathscr{W}_{1n} = \sum_{i=1}^{K} \langle \mathscr{L}_{11(i)}|\mathscr{W}_{1(n-1)}\rangle,$$

$$\mathscr{W}_{21} = \sum_{i=1}^{K} \langle \mathscr{L}_{22(i)}|\mathscr{L}_{21(i)}\rangle, \mathscr{W}_{22} = \sum_{i=1}^{K} \langle \mathscr{L}_{22(i)}|\mathscr{W}_{21}\rangle, \cdots, \mathscr{W}_{2m} = \sum_{i=1}^{K} \langle \mathscr{L}_{22(i)}|\mathscr{W}_{2(m-1)}\rangle.$$

Lemma 2 *System (6) attains group switching controllability if and only if $\mathscr{W}_{1n} = \Re^n$ and $\mathscr{W}_{2m} = \Re^m$.*

Proof. According to [41], for every subsystem $(\mathscr{L}_{11(m)}, \mathscr{L}_{12(m)})(m = 1, \cdots, K)$, there exist a constant $h_m > 0$, such that $\langle \mathscr{L}_{11(m)}|\mathscr{W}\rangle = \langle exp(\mathscr{L}_{11(m)}h_m)|\mathscr{W}\rangle$. Then

$$\mathscr{W}_{11} = \sum_{i=1}^{K} \langle \mathscr{L}_{11(i)}|\mathscr{L}_{12(i)}\rangle, \;\; \mathscr{W}_{1p} = \sum_{i=1}^{K} \langle exp(\mathscr{L}_{11(i)}h_i)|\mathscr{W}_{1(p-1)}\rangle (p = 2, \cdots, n),$$

and

$$\mathscr{W}_{1n} = \sum_{j=1}^{K} \sum_{i_1,\cdots,i_{n-1}\in\{1,\cdots,K\}}^{l_1,\cdots,l_{n-1}\in\{1,\cdots,n\}} \left[exp(\mathscr{L}_{11(i_{n-1})}h_{i_{n-1}})\right]^{l_{n-1}} \cdots \left[exp(\mathscr{L}_{11(i_1)}h_{i_1})\right]^{l_1} \langle \mathscr{L}_{11_{(j)}} | \mathscr{L}_{12_{(j)}} \rangle.$$

Assume that $dim(\mathscr{W}_{1n}) = d$, there exist subspaces $\mathscr{V}_1, \cdots, \mathscr{V}_d$, such that

$$\mathscr{W}_{1n} = \sum_{l=1}^{d} \mathscr{V}_l,$$

where $\mathscr{V}_l = \Pi_{m=1}^{M} exp(\mathscr{L}_{11(i_m)}h_{i_m})\langle \mathscr{L}_{11(j)} | \mathscr{L}_{12(j)} \rangle$ for $i_1, \cdots, i_M, j \in \{1, \cdots, K\}, 0 \leq M \leq n^{n-1}$. We select switching sequences as

$$\pi_\alpha = \{j, \cdots, j\}_{1\times n} \ \ and \ \ \pi_\beta = \{i_1, \cdots, i_M\},$$

such that

$$\Pi_{m=1}^{M} exp(\mathscr{L}_{11(i_m)}h_{i_m})\langle \mathscr{L}_{11(j)} | \mathscr{L}_{12(j)} \rangle \subseteq \mathscr{R}(\pi_\alpha \wedge \pi_\beta),$$

where $\mathscr{R}(\pi_\alpha \wedge \pi_\beta)$ means the space with a switching sequence $\pi_\alpha \wedge \pi_\beta \triangleq \{j, \cdots, j, i_1, \cdots, i_M\}$. Thus we can also choose switching sequences $\pi_1, \cdots, \pi_d$, such that $\mathscr{V}_l \subseteq \mathscr{R}(\pi_l)(l = 1, \cdots, d)$, and then

$$\mathscr{W}_{1n} \subseteq \sum_{l=1}^{d} \mathscr{R}(\pi_l).$$

Because it is easy to know $\sum_{l=1}^{d} \mathscr{R}(\pi_l) \subseteq \mathscr{W}_{1n}$, then $\mathscr{W}_{1n} = \sum_{l=1}^{d} \mathscr{R}(\pi_l)$. Now we can construct switching sequence π_b as follows. If $\mathscr{R}(\pi_1^{\wedge n}) = \mathscr{W}_{1n}$, we take $\pi_b = \pi_1^{\wedge n}$. If not, there exist $k \in [2, \cdots, d]$, without loss of generality, let k=2, we have

$$\mathscr{R}(\pi_2) \nsubseteq \mathscr{R}(\pi_1^{\wedge n}).$$

Then

$$\begin{aligned}\mathscr{R}(\pi_2 \wedge \pi_1^{\wedge n}) &= \mathscr{R}(\pi_1^{\wedge n}) + exp(\pi_1^{\wedge n})\mathscr{R}(\pi_2) \\ &= exp(\pi_1^{\wedge n})[\mathscr{R}(\pi_1^{\wedge n}) + \mathscr{R}(\pi_2)],\end{aligned}$$

and then

$$dim(\mathscr{R}(\pi_2 \wedge \pi_1^{\wedge n})) = dim(\mathscr{R}(\pi_1^{\wedge n}) + \mathscr{R}(\pi_2)) \geq dim(\mathscr{R}(\pi_1^{\wedge n})) + 1 = 2.$$

Let $\bar{\pi}_1 = \pi_1, \ \ \bar{\pi}_2 = \pi_2 \wedge \pi_1^{\wedge n}, \ \ \cdots, \ \ \bar{\pi}_d = \pi_d \wedge \pi_1^{\wedge n}$. Taking $\pi_b = \bar{\pi}_d$, we have

$$dim\mathscr{R}(\pi_b) \geq d,$$

therefore $\mathscr{R}(\pi_b) = \mathscr{W}_{1n}$. Hence we can know that there must exist a switching sequence π, such that

$$\mathscr{R}(\pi) = \Re^n = \mathscr{W}_{1n}.$$

Similarly, we can also have $\mathscr{W}_{2m} = \Re^m$. Thus system (6) can attain group switching controllability. This completes the proof.

Based on Lemma 2, we can obtain the following main results for system (6) with switching topologies.

Theorem 1 *System (6) attains group switching controllability if*

$$\sum_{i=1}^{K} \mathscr{R}(\mathscr{L}_{12(i)}) = \Re^n \text{ and } \sum_{i=1}^{K} \mathscr{R}(\mathscr{L}_{21(i)}) = \Re^m.$$

Proof. It is obvious that $\mathscr{W}_n \subseteq \Re^n$. On the other hand, it is easy to prove

$$\mathscr{R}(\mathscr{L}_{12(i)}) \subseteq \mathscr{R}(\mathscr{L}_{12(i)}) + \mathscr{R}(\mathscr{L}_{11(i)}\mathscr{L}_{12(i)}) + \cdots + \mathscr{R}(\mathscr{L}_{11(i)}^{n-1}\mathscr{L}_{12(i)}) = \langle \mathscr{L}_{11(i)} | \mathscr{L}_{12(i)} \rangle$$

for all $i = 1, 2, \cdots, K$. Then

$$\begin{aligned}
\Re^n &= \mathscr{R}(\mathscr{L}_{12(1)}) + \mathscr{R}(\mathscr{L}_{12(2)}) + \cdots + \mathscr{R}(\mathscr{L}_{12(K)}) \\
&\subseteq \langle \mathscr{L}_{11(1)} | \mathscr{L}_{12(1)} \rangle + \langle \mathscr{L}_{11(2)} | \mathscr{L}_{12(2)} \rangle + \cdots + \langle \mathscr{L}_{11(K)} | \mathscr{L}_{12(K)} \rangle \\
&= \mathscr{W}_{11} \\
&\subseteq \mathscr{W}_{12} \\
&\subseteq \cdots \\
&\subseteq \mathscr{W}_{1n}.
\end{aligned}$$

Therefore we can have $\mathscr{W}_{1n} = \Re^n$. For the subspace $\mathscr{W}_{2m}$, we can also have the similar result, that is, $\mathscr{W}_{2m} = \Re^m$. From Lemma 2, system (6) can achieve group switching controllability.

Theorem 2 *System (6) attains group switching controllability if*

$$\sum_{i=1}^{K} \sum_{p=n+1}^{n+m} \mathscr{R}(l_{1i_p}) = \Re^n \text{ and } \sum_{i=1}^{K} \sum_{p=1}^{n} \mathscr{R}(l_{2i_p}) = \Re^m.$$

Proof. It is obvious to see that $\mathscr{W}_{1n} \subseteq \Re^n$. On the other hand, it is easy to prove

$$\mathscr{R}(l_{1i_p}) \subseteq \mathscr{R}(l_{1i_p}) + \mathscr{R}(\mathscr{L}_{11(i)} l_{1i_p}) + \cdots + \mathscr{R}(\mathscr{L}_{11(i)}^{n-1} l_{1i_p}) = \langle \mathscr{L}_{11(i)} | l_{1i_p} \rangle$$

for all $i = 1, 2, \cdots, K$. Then

$$\begin{aligned}
\Re^n &= \mathscr{R}(l_{11_{n+1}}) + \cdots + \mathscr{R}(l_{11_{n+m}}) + \mathscr{R}(l_{12_{n+1}}) + \cdots + \mathscr{R}(l_{12_{n+m}}) + \cdots + \mathscr{R}(l_{1K_{n+1}}) + \cdots + \mathscr{R}(l_{1K_{n+m}}) \\
&\subseteq \langle \mathscr{L}_{11(1)} | l_{11_{n+1}} \rangle + \cdots + \langle \mathscr{L}_{11(1)} | l_{11_{n+m}} \rangle + \cdots + \langle \mathscr{L}_{11(K)} | l_{1K_{n+1}} \rangle + \cdots + \langle \mathscr{L}_{11(K)} | l_{1K_{n+m}} \rangle \\
&= \sum_{p=n+1}^{n+m} \langle \mathscr{L}_{11(1)} | l_{11_p} \rangle + \sum_{p=n+1}^{n+m} \langle \mathscr{L}_{11(2)} | l_{12_p} \rangle + \cdots + \sum_{p=n+1}^{n+m} \langle \mathscr{L}_{11(K)} | l_{1K_p} \rangle \\
&= \langle \mathscr{L}_{11(1)} | \mathscr{L}_{12(1)} \rangle + \langle \mathscr{L}_{11(2)} | \mathscr{L}_{12(2)} \rangle + \cdots + \langle \mathscr{L}_{11(K)} | \mathscr{L}_{12(K)} \rangle \\
&= \mathscr{W}_{11} \\
&\subseteq \mathscr{W}_{12} \\
&\subseteq \cdots \\
&\subseteq \mathscr{W}_{1n}.
\end{aligned}$$

Then we can have $\mathscr{W}_{1n} = \Re^n$. For the subspace $\mathscr{W}_{2m}$, we can also have $\mathscr{W}_{2m} = \Re^m$. From Lemma 2, system (6) attains group switching controllability.

Remark 6 *Notice from the aforementioned analysis that we can find the group controllability of discrete-time multi-agent system with time delay can be determined only by the information from the subgroup to the other subgroup and it has nothing to do with the inner topology among the subgroups. This important result is valid no matter topology of the network is fixed or switching. Moreover, for the conditions given in Lemma 2, Theorem 1 and Theorem 2 do not require the controllability of the subgroups. This merit is much desirable in applications as it can provide more freedom to design the configuration of the multi-agent system. This feature also gives important convenience for the design of a switching path to guarantee the group switching controllability of the discrete-time multi-agent system.*

3.2 Group controllability of discrete-time multi-agent systems on fixed topology

For the special case $\sigma(l) = 1$, system (6) describes a multi-agent network on fixed topology. The group controllability is not only to consider the information interaction among the groups, but also to consider the information interaction among different groups, which makes the system reflect the effect of a whole and also the internal structure of the sub-groups, therefore, it is necessary to discuss the group controllability of system (6) with fixed topology, which is equivalent to system (5).

In the following discussion, the group controllability matrix and rank test are fundamental notations, which are introduced as follows.

Definition 7 *(Group Controllability)*

A non-zero state x of system (5) is defined as group controllability if the states of agents satisfy:

1. *If there exist a finite time $T \in J$, and a piecewise input x^2, such that $x^1(0) = x^1$ and $x^1(T) = 0$; and*

2. *if there exist a finite time $T \in J$, and a piecewise input x^1, such that $x^2(0) = x^2$ and $x^2(T) = 0$,*

where J is the discrete time region.

Theorem 3 *System (5) attains group controllability if and only if*

$$rank(Q_1) = (h+1)n, \;\; rank(Q_2) = (h+1)m,$$

where the controllability matrices of system (5) are defined as $Q_1 = [\mathscr{L}_{12}, \mathscr{L}_{11}\mathscr{L}_{12}, \cdots, \mathscr{L}_{11}^{n-1}\mathscr{L}_{12}]$, and $Q_2 = [\mathscr{L}_{21}, \mathscr{L}_{22}\mathscr{L}_{21}, \cdots, \mathscr{L}_{22}^{m-1}\mathscr{L}_{21}]$.

This proof is obvious, here omitted.

Since the controllable matrices of system (5) are so complex, so it is difficult to study the controllability of multi-agent system (5) by rank test. In the following, we will introduce a more concise and effective method–the PBH rank test.

Theorem 4 *(PBH rank test for multi-agent system) System (5) attains group controllability if and only if system (5) satisfies one of the following conditions:*
(1) $rank(sI - \mathscr{L}_{11}, \mathscr{L}_{12}) = (h+1)n$ and $rank(tI - \mathscr{L}_{22}, \mathscr{L}_{21}) = (h+1)m$, for all s, $t \in \mathbb{C}$ (where $\mathbb{C}$ is a complex number);
(2) $rank(\lambda_i I - \mathscr{L}_{11}, \mathscr{L}_{12}) = (h+1)n$ and $rank(\mu_i I - \mathscr{L}_{22}, \mathscr{L}_{21}) = (h+1)m$, where $\lambda_i (\forall i = 1, 2, \cdots, n)$ and $\mu_i (\forall i = 1, 2, \cdots, m)$ are the eigenvalues of matrices $\mathscr{L}_{11}$ and $\mathscr{L}_{22}$, respectively.

Proof. If condition (1) is true, condition (2) is absolutely true. It is therefore to prove the condition (1). Necessity: By contradiction, if $\exists \lambda_1 \in \mathbb{C}$ such that

$$rank(\lambda_1 I - \mathscr{L}_{11}, \mathscr{L}_{12}) < (h+1)n.$$

So the rows of $[\lambda_1 I - \mathscr{L}_{11}, \mathscr{L}_{12}]$ are linear dependent. Therefore, there is a vector $\alpha \neq 0$ such that $\alpha'[\lambda_1 I - \mathscr{L}_{11}, \mathscr{L}_{12}] = 0$, then we can obtain

$$\lambda_1 \alpha' = \alpha' \mathscr{L}_{11}, \;\; \alpha' \mathscr{L}_{12} = 0.$$

Furthermore,

$$\alpha'\mathscr{L}_{12}=0,\ \ \alpha'\mathscr{L}_{11}\mathscr{L}_{12}=0,\ \cdots,\ \alpha'\mathscr{L}_{11}^{n-1}\mathscr{L}_{12}=0.$$

According to the PBH rank test for controllability, we have

$$\alpha'[\mathscr{L}_{12},\mathscr{L}_{11}\mathscr{L}_{12},\cdots,\mathscr{L}_{11}^{n-1}\mathscr{L}_{12}]=0.$$

Since $\alpha\neq 0$, then there must be

$$rank([\mathscr{L}_{12},\mathscr{L}_{11}\mathscr{L}_{12},\cdots,\mathscr{L}_{11}^{n-1}\mathscr{L}_{12}])<(h+1)n.$$

Then, system (5) is uncontrollable, which contradicts to the fact that system (5) can attain group controllability. Therefore, this completes the proof of the necessity of condition (1).
Sufficiency: By contradiction, if the system (5) is uncontrollable. By Theorem 3, we can know that

$$rank([\mathscr{L}_{12},\ \mathscr{L}_{11}\mathscr{L}_{12},\ \cdots,\ \mathscr{L}_{11}^{n-1}\mathscr{L}_{12}])<(h+1)n.$$

There must be an uncontrollable eigenvalue $\lambda_2\in\mathbb{C}$ belongs to $\mathscr{L}_{11}$, whose corresponding eigenvector is denoted as $\beta\in\mathbb{C}$. Then

$$\beta'[\mathscr{L}_{12},\ \ \mathscr{L}_{11}\mathscr{L}_{12},\ \ \cdots,\ \ \mathscr{L}_{11}^{n-1}\mathscr{L}_{12}]=0.$$

Furthermore,

$$\beta'\mathscr{L}_{12}=0,\ \ \beta'\mathscr{L}_{11}\mathscr{L}_{12}=0=\lambda_2\beta'\mathscr{L}_{12},\ \ \cdots,\ \ \beta'\mathscr{L}_{11}^{n-1}\mathscr{L}_{12}=0,$$

which imply that there exists a vector $\beta\neq 0$, such that $\beta'[\lambda_2 I-\mathscr{L}_{11},\mathscr{L}_{12}]=0$, and then there is a $\lambda_2\in\mathbb{C}$ such that $rank(\lambda_2 I-\mathscr{L}_{11},\mathscr{L}_{12})<(h+1)n$. This contradicts to $rank(sI-\mathscr{L}_{11},\mathscr{L}_{12})=(h+1)n$ for all $s\in\mathbb{C}$. Thus, this completes the proof of the sufficiency of condition (1).

Remark 7 *Note that the group controllability depends on not only the information interaction among the groups, but also the information interaction between different groups.*

Remark 8 *Most of the existing literatures investigating the controllability of multi-agent system use the rank criterion of the controllable matrix, such as Theorem 3. In this paper, we introduce the PBH rank test to determine the group controllability of discrete-time multi-agent systems. The PBH rank rest is very useful since it only depends on the eigenvalues of the network. It is well known that, for the multi-agent system with multiple time-delays and high dimension, the controllable matrix of such system is too complex to calculate. However, it is so easy to compute the eigenvalues of the system using the PBH rank method by Matlab.*

The study of the group controllability of multi-agent systems keeps to be a challenging task due to the fact that system is high dimensional and time-delayed. In the following, we will show some main results on the group controllability of discrete-time multi-agent systems, which are theoretically proved via PBH rank test.

Theorem 5 *System (5) attains group controllability if and only if the matrices* $Y_i = \begin{bmatrix} L_{ii} - (\lambda^{h+1} - \lambda^h)I, & -L_{ij} \end{bmatrix}$ *have full row rank at every root of* $det(L_{ii} - (\lambda^{h+1} - \lambda^h)I) = 0$ *(*$i, j = 1, 2$ *and* $i \neq j$*).*

Proof. From Theorem 4, system (5) attains group controllability if and only if the matrix

$$\begin{bmatrix} \lambda I - \mathscr{L}_{11}, & \mathscr{L}_{12} \end{bmatrix} = \begin{bmatrix} \lambda I - I & 0 & 0 & \cdots & 0 & L_{11} & -L_{12} & \cdots & 0 \\ -I & \lambda I & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -I & \lambda I & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ddots & \lambda I & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -I & \lambda I & 0 & \cdots & 0 \end{bmatrix}_{(h+1)n \times (h+1)(m+n)} \tag{7}$$

has full row rank for every $\lambda \in \mathbb{C}$. By elementary row transformation, then

$$\begin{bmatrix} -I & \lambda I & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -I & \lambda I & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -I & \ddots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -I & \lambda I & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & L_{11} - (\lambda^{h+1} - \lambda^h)I & -L_{12} & \cdots & 0 \end{bmatrix}, \tag{8}$$

which has the same rank with matrix (7). Based on the basic matrix theory, it is easy to know that the first hn rows of matrix (8) are linearly independent. If the last n rows are linearly independent, matrix (8) has full row rank, which implies that

$$Y_1 = \begin{bmatrix} L_{11} - (\lambda^{h+1} - \lambda^h)I, & -L_{12} \end{bmatrix}$$

has full row rank, i.e. $rank(Y_1) = n$ for every $\lambda \in \mathbb{C}$. For matrix Y_1, if $det(L_{11} - (\lambda^{h+1} - \lambda^h)I) \neq 0$ or matrix L_{12} is full row rank, matrix Y_1 must be full row rank. So, we only need verify that the matrix Y_1 is full row rank at the roots of $det(L_{11} - (\lambda^{h+1} - \lambda^h)I) = 0$ instead of every $\lambda \in \mathbb{C}$. Similarly, we can prove the condition $i = 2$. This completes the proof.

Remark 9 *In this context, the group controllability only depends on the coupling communications from the subgroup to the other subgroup.*

Specially, note that it is more easy for us to compute the roots of $det(L_{11}-(\lambda^{h+1}-\lambda^h)I)=0$ than all the eigenvalues of the matrix $\mathscr{L}_{11}$ or every $\lambda\in\mathbb{C}$. We can immediately get the following corollaries from Theorem 5.

Corollary 1 *System (5) attains group controllability if* $det(L_{ii}-(\lambda^{h+1}-\lambda^h)I)\neq 0\ \ (i=1,2)$.

Corollary 2 *System (5) attains group controllability if matrices* $L_{ij}\ \ (i,j=1,2,i\neq j)$ *have full row rank.*

Corollary 3 *The roots of* $det([L_{ii}-(\lambda^{h+1}-\lambda^h)I])=0$ *are some of the eigenvalues of the matrix* $\mathscr{L}_{ii}$ *(*$i=1,2$*).*

Proof. The eigenvalues of $\mathscr{L}_{ii}$ are the roots of

$$det\left(\begin{bmatrix} \lambda I-I & 0 & 0 & \cdots & 0 & L_{ii} \\ -I & \lambda I & 0 & \cdots & 0 & 0 \\ 0 & -I & \lambda I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda I & 0 \\ 0 & 0 & 0 & \cdots & -I & \lambda I \end{bmatrix}\right)=0,$$

which is equivalent to the following form

$$(-1)^h det\left(\begin{bmatrix} -I & \lambda I & 0 & \cdots & 0 & 0 \\ 0 & -I & \lambda I & \cdots & 0 & 0 \\ 0 & 0 & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -I & \lambda I \\ 0 & 0 & 0 & \cdots & 0 & L_{ii}-(\lambda^{h+1}-\lambda^h)I \end{bmatrix}\right)=0,$$

and also equivalent to $det([L_{ii}-(\lambda^{h+1}-\lambda^h)I])=0$. Then we can obtain the roots of $det([L_{ii}-(\lambda^{h+1}-\lambda^h)I])=0$, which are some of the eigenvalues of the matrix $\mathscr{L}_{ii}$.

Corollary 4 λ *is called a controllable eigenvalue if* λ *satisfies* $det([L_{ii}-(\lambda^{h+1}-\lambda^h)I])=0$ *such that system (5) attains group controllability; otherwise an uncontrollable eigenvalue.*

Notice from corollaries 1–4 that the high dimension of multi-agent systems is so high to discuss the basic problems in control theory. In the following, we can be transformed it into an eligible subsystem with less dimension. Based on this subsystem, we can easily deduce the group controllability of system (5).

Corollary 5 *System (5) attains group controllability if and only if the following system*

$$\begin{cases} X^1(k+1) = [-L_{11} + (\lambda^{h+1} - \lambda^h)I + \lambda I]X^1(k) - L_{12}X^2(k) \\ X^2(k+1) = [-L_{22} + (\mu^{h+1} - \mu^h)I + \mu I]X^2(k) - L_{21}X^1(k) \end{cases} \tag{9}$$

attains group controllability.

Proof. From Theorem 5, system (5) attains group controllability if and only if matrices

$$[\lambda I - (-L_{11} + (\lambda^{h+1} - \lambda^h)I + \lambda I), \quad -L_{12}], \quad [\mu I - (-L_{22} + (\mu^{h+1} - \lambda^h)I + \mu I), \quad -L_{12}]$$

have full row rank for any $\lambda, \mu \in \mathbb{C}$, i.e.

$$[L_{11} - (\lambda^{h+1} - \lambda^h)I_{nn}, \quad -L_{12}], \quad [L_{22} - (\mu^{h+1} - \mu^h)I_{nn}, \quad -L_{21}]$$

have full row rank for any $\lambda, \mu \in \mathbb{C}$. This concludes the proof.

Sometimes, some special structures should be of more concern, because there may be more specific and practical results. In the following, we will investigate the group controllability for some special structures.

Case I: $L_{12}^T = L_{21}$

If the the information interaction between different groups are the same, that is $L_{12}^T = L_{21}$, and then $\mathscr{L}_{12}^T = \mathscr{L}_{21}$. Then we can obtain the following special result.

Theorem 6 *system (5) attains group controllability if and only if $\mathscr{L}_{ii}(i = 1,2)$ and $\mathscr{L}$ have no common eigenvalues.*

Proof. Sufficiency: By contradiction, suppose system (5) is uncontrollable, by the PBH rank test for multi-agent system, there exists a vector $\alpha \in \mathbb{C}$ and $\alpha \neq 0$, such that

$$\alpha'(\lambda_1 I - \mathscr{L}_{11}, \mathscr{L}_{12}) = 0,$$

so,

$$\alpha'\mathscr{L}_{11} = \lambda_1 \alpha', \alpha'\mathscr{L}_{12} = 0.$$

Then

$$\begin{bmatrix}\alpha' & 0\end{bmatrix}\begin{bmatrix}\mathscr{L}_{11} & \mathscr{L}_{12}\\ \mathscr{L}_{21} & \mathscr{L}_{22}\end{bmatrix} = \begin{bmatrix}\alpha'\mathscr{L}_{11} & \alpha'\mathscr{L}_{12}\end{bmatrix} = \begin{bmatrix}\lambda_1\alpha' & 0\end{bmatrix} = \lambda_1\begin{bmatrix}\alpha' & 0\end{bmatrix}$$

So, matrix $\mathscr{L}_{11}$ has the same eigenvalue with matrix $\mathscr{L}$. It is contradictory to the fact that matrix $\mathscr{L}_{11}$ and matrix $\mathscr{L}$ have no common eigenvalues. This completes the proof of the sufficiency.

Necessity: By contradiction, suppose $\mathscr{L}_{11}$ and $\mathscr{L}$ at least have a common eigenvalue λ, and there is q_n, which is the eigenvector corresponding to eigenvalue λ of $\mathscr{L}_{11}$. And there exists $p^T = [I_{(h+1)n} \quad 0] \in R^{(h+1)n\times(h+1)(m+n)}$ such that $\mathscr{L}_1 1 = p^T\mathscr{L}p$. Moreover,

$$q = pq_n = \begin{bmatrix} q_n \\ 0 \end{bmatrix} = \begin{bmatrix} I_{(h+1)n}q_n \\ 0 \end{bmatrix}_{(h+1)(n+m)\times(h+1)n},$$

where q is the eigenvector corresponding to eigenvalue λ of $\mathscr{L}$. Then

$$\mathscr{L}q = \begin{bmatrix}\mathscr{L}_{11} & \mathscr{L}_{12}\\ \mathscr{L}_{21} & \mathscr{L}_{22}\end{bmatrix}\begin{bmatrix} q_n \\ 0 \end{bmatrix} = \begin{bmatrix}\mathscr{L}_{11}q_n \\ \mathscr{L}_{21}q_n\end{bmatrix} = \lambda\begin{bmatrix} q_n \\ 0 \end{bmatrix} = \lambda q,$$

then

$$\mathscr{L}_{21}q_n = 0,$$

that is

$$\mathscr{L}_{12}^T q_n = 0,$$

which means $q_n^T\mathscr{L}_{12} = 0$ when $\mathscr{L}_{12}^T = \mathscr{L}_{21}$. So,

$$q_n^T\begin{bmatrix}\lambda I - \mathscr{L}_{11}, & \mathscr{L}_{12}\end{bmatrix} = 0.$$

Then,

$$rank\begin{bmatrix}\lambda I - \mathscr{L}_{11}, & \mathscr{L}_{12}\end{bmatrix} < (h+1)n.$$

By PBH rank test, group $(\mathscr{G}, x^1)$ is uncontrollable, which contradicts to the fact that group $(\mathscr{G}, X^1)$ is controllable. So $\mathscr{L}_{11}$ has no common eigenvalues with $\mathscr{L}$. Similarly, $\mathscr{L}_{22}$ has no common eigenvalues with $\mathscr{L}$. This completes the proof of the necessity.

Remark 10 *Note that Theorem 6 provides a simpler and more easily checkable method since it only involves the couplings of the whole group and subgroups.*

Case II: $L_{ii}^T = L_{ii}(i = 1,2)$

Under the symmetry condition of group $(\mathscr{G}_i, x)(i = 1,2)$, one can obtain the following result.

Theorem 7 *If group $(\mathscr{G}_1, x)$ and group $(\mathscr{G}_2, x)$ are both symmetric, that is, $L_{ii}^T = L_{ii}(i = 1,2)$, then system (5) attains group controllability if and only if the following conditions:*
(1) the eigenvalues of $L_{ii}(i = 1,2)$ are all distinct, and
(2) the eigenvector of $L_{ii}(i = 1,2)$ are not orthogonal to at least one column of $L_{ij}(i, j = 1,2, i \neq j)$
both hold.

Proof. From Theorem 5, system (5) attains group controllability if and only if matrices

$$Y_i = \begin{bmatrix} L_{ii} - (\lambda^{h+1} - \lambda^h)I, & -L_{ij} \end{bmatrix}$$

have full row rank at every roots of $det([L_{ii} - \lambda^h(\lambda - 1)I]) = 0$ where every $\lambda^h(\lambda - 1)$ is the eigenvalue of L_{ii}. Since L_{ii} is symmetric, L_{ii} can be decomposed by $L_{ii} = U_i \Lambda_i U_i^T$, $\Lambda_i = diag\{\lambda_{i1}, \cdots, \lambda_{ik}\}$ where $\lambda_{if}(f = 1, \ldots, k)$ are the eigenvalues of L_{ii} and the columns of U_i is orthogonal eigenvectors of L_{ii}. Then, for $i, j = 1,2, i \neq j$,

$$\begin{aligned} Y_i &= [U_i \Lambda_i U_i^T - \lambda_{if} U_i U_i^T, \quad -U_i U_i^T L_{ij}] \\ &= [U_i(\Lambda - \lambda_{if} I)U_i^T, \quad -U_i U_i^T L_{ij}] \\ &= U_i [(\Lambda - \lambda_{if} I)U_i^T, \quad -U_i^T L_{ij}] \ . \end{aligned}$$

Since U_i is reversible, so

$$\widetilde{Y}_i = [(\Lambda_i - \lambda_{if} I)U_i^T, \quad -U_i^T L_{ij}] = \begin{bmatrix} \begin{bmatrix} \lambda_{i1} - \lambda_{if} & & & \\ & \lambda_{i2} - \lambda_{if} & & \\ & & \ddots & \\ & & & \lambda_{ik} - \lambda_{if} \end{bmatrix} U_i^T, & -U_i^T L_{ij} \end{bmatrix}.$$

Let $-U_i^T L_{ij} \triangleq [b_1, b_2, \cdots, b_k]^T$, and since the eigenvalues of $L_{ii}(i = 1,2)$ are all distinct, so

$$\widetilde{Y}_i = \begin{bmatrix} & & & & b_1 \\ & & * & & \vdots \\ 0 & 0 & \cdots & 0 & b_f \\ & & * & & \vdots \\ & & & & b_n \end{bmatrix}.$$

$b_f \neq 0 (f = 1, 2, \cdots, k)$, iff the eigenvectors of L_{ii} are not orthogonal to at least one column of L_{ij}, therefore we know that $\widetilde{Y}_i$ have the row full rank. This completes the proof.

In particular, we can have some interesting results that show that a system divided into some subgroups will be uncontrollable under certain conditions.

Case III: Balanced couple

Theorem 8 *If group $\mathscr{G}_1$ and group $\mathscr{G}_2$ are out-degree (in-degree) balanced couple, system (5) is uncontrollable.*

Proof. Denote

$$L_{12} \triangleq \begin{bmatrix} a_{1(n+1)} & a_{1(n+2)} & \cdots & a_{1(n+m)} \\ a_{2(n+1)} & a_{2(n+2)} & \cdots & a_{2(n+m)} \\ \vdots & \vdots & & \vdots \\ a_{n(n+1)} & a_{n(n+2)} & \cdots & a_{n(n+m)} \end{bmatrix},$$

and the controllability matrix of group $\mathscr{G}_1$ can be rewritten as

$$Q_1 = [\mathscr{L}_{12},\ \mathscr{L}_{11}\mathscr{L}_{12}, \cdots,\ \mathscr{L}_{11}^{n-1}\mathscr{L}_{12}] = \begin{bmatrix} -L_{12} & \cdots & 0 & -L_{12} & \cdots & 0 & \cdots & -L_{12} & \cdots & 0 \\ 0 & \cdots & 0 & -L_{12} & \cdots & 0 & \cdots & -L_{12} & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & -L_{12} & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & -L_{12} & \cdots & 0 \end{bmatrix}.$$

Since group $(\mathscr{G}_1, x)$ is out-degree balance, then $\sum_{j=1}^{n} a_{ji} = 0 \ \ (i = n+1, \cdots, n+m)$. The k-th column sum of L_{12} is

$$\sum_{i=1}^{n} a_{i(n+k)} = 0,$$

which implies that there is a row consisting of zero elements. Then the controllability matrix Q_1 has no full row rank as similar as Q_2. Therefore, system (5) is uncontrollable.

Case IV: Star graph

Corollary 6 *If $(\mathscr{G}, x^1)$ and $(\mathscr{G}, x^2)$ are both the star graphs (described as Fig. 2, where $n, m \geq 4$), and all $a_{ij} (\forall i, j \in \mathscr{C}_1, \mathscr{C}_2)$ are the same, no matter how to connect $(\mathscr{G}, x^1)$ and $(\mathscr{G}, x^2)$, system (5) is uncontrollable.*

Proof. Without loss of generality, suppose $a_{ij} = 1 (\forall i, j \in \mathscr{C}_1, \mathscr{C}_2)$, and $(\mathscr{G}, x^1)$ and $(\mathscr{G}, x^2)$ are both the star

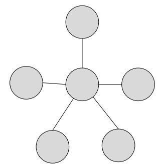

Fig. 2: Star topology.

graphs, then

$$L_{11}=\begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{bmatrix},\ L_{22}=\begin{bmatrix} m-1 & -1 & \cdots & -1 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{bmatrix}.$$

It is easy to find that some of the eigenvalues of $L_{ii}(i=1,2)$ are the same (i.e. $\lambda(L_{ii})=1$). From condition (1) of Theorem 7, no matter how to connect $(\mathscr{G},x^1)$ and $(\mathscr{G},x^2)$, system (5) is always uncontrollable.

Case V: Isolated agent

Definition 8 *An agent is called as an isolated agent if it has no any information with other agents.*

Theorem 9 *System (5) is uncontrollable if there exists one isolated agent in group $(\mathscr{G}_i,x)$ $(i=1,2)$ or there is no information between two subgroups.*

Proof. Case 1. If there exists one isolated agent in group $(\mathscr{G}_1,x)$ $(i=1,2)$, without loss of generality, we assume that agent 1 in group $(\mathscr{G}_1,x)$, then

$$\mathscr{L}_{11}=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ & & * & \end{bmatrix},\ \mathscr{L}_{12}=\begin{bmatrix} 0 & \cdots & 0 \\ & * & \end{bmatrix}.$$

The controllability matrix $Q_1=[\mathscr{L}_{12},\mathscr{L}_{11}\mathscr{L}_{12},\cdots,\mathscr{L}_{11}^{n-1}\mathscr{L}_{12}]$, whose elements in the first row are all zeros. Therefore, system (5) is uncontrollable.

Case 2. If there has no information between two sub-groups, then

$$L_{12}=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Obviously, system (5) is uncontrollable.

3.3 Special case: $h = 0$

Notice that system (4) has no time delay when $h = 0$, then system (4) can be rewritten as

$$\begin{cases} x^1(k+1) = x^1(k) - L_{11}x^1(k) - L_{12}x^2(k) = (I - L_{11})x^1(k) - L_{12}x^2(k), \\ x^2(k+1) = x^2(k) - L_{22}x^2(k) - L_{21}x^1(k) = (I - L_{22})x^2(k) - L_{21}x^1(k), \end{cases} \tag{10}$$

where I is compatible dimension identity matrix.

For system (10) on fixed topology, we can have more special results.

Proposition 1 *System (10) attains group controllability if and only if the matrixes Q_1 and Q_2 have full row rank, where*

$$Q_1 = -[L_{12}, (I - L_{11})L_{12}, (I - L_{11})^2 L_{12}, \cdots, (I - L_{11})^{n-1} L_{12}],$$

and

$$Q_2 = -[L_{21}, (I - L_{22})L_{21}, (I - L_{22})^2 L_{21}, \cdots, (I - L_{22})^{m-1} L_{21}].$$

Corollary 7 *System (10) attains group controllability if and only if system (10) satisfies one of the following conditions:*
(1) $rank[(s-1)I + L_{11}, -L_{12}] = n$ and $rank[(t-1)I + L_{22}, -L_{21}] = m$, $\forall s,t \in \mathbb{C}$;
(2) $rank[(\lambda_i - 1)I + L_{11}, -L_{12}] = n$ and $rank[(\mu_i - 1)I + L_{22}, -L_{21}] = m$, where $\lambda_i(\forall i = 1,2,\cdots,n), \mu_i(\forall i = 1,2,\cdots,m)$ are the eigenvalues of matrices L_{11} and L_{22}, respectively.

Corollary 8 *System (10) attains group controllability if and only if $I - L_{ii}(i = 1,2)$ and L have no common eigenvalues.*

Theorem 10 *If group $(\mathscr{G}_1, x)$ and group $(\mathscr{G}_2, x)$ are both symmetric, and group $(\mathscr{G}_1, x)$ and group $(\mathscr{G}_2, x)$ are connected by any one agent in $(\mathscr{G}_1, x)$ and any one agent in $(\mathscr{G}_2, x)$, then system (10) attains group controllability if and only if the eigenvalues of $I - L_{ii}(i = 1,2)$ are all distinct.*

Proof. The controllable matrix

$$Q_1 = -[L_{12}, (I_{11} - L_{11})L_{12}, \cdots, (I_{11} - L_{11})^{n-1} L_{12}].$$

If $L_{11}^T = L_{11}$, then L_{11} can be decomposed by $L_{11} = U\Lambda U^T$, where $\Lambda = diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, $\lambda_i(i = 1,2,\cdots,n)$ are eigenvalues of $I - L_{11}$ and U is reversible. Then

$$\begin{aligned} Q_1 &= -[UU^T L_{12}, U\Lambda U^T L_{12}, \cdots, U\Lambda^{n-1} U^T L_{12}] \\ &= -U[U^T L_{12}, \Lambda U^T L_{12}, \cdots, \Lambda^{n-1} U^T L_{12}]. \end{aligned}$$

Because U is reversible, we only consider

$$\tilde{Q}_1 = [U^T L_{12}, \Lambda U^T L_{12}, \cdots, \Lambda^{n-1} U^T L_{12}].$$

Suppose $U^T L_{12} = [r_1, r_2, \cdots, r_m]$, where $r_i = [r_{1i}, r_{2i}, \cdots, , r_{ni}] (i = 1, 2, \cdots, m)$. Then

$$\tilde{Q}_1 = \{[[r_1, r_2, \cdots, r_m], \Lambda[r_1, r_2, \cdots, r_m], \cdots, \Lambda^{n-1}[r_1, r_2, \cdots, r_m]\}.$$

According to matrix elementary column transformation, we know that

$$\begin{aligned} \tilde{Q}_1 &\to \{[r_1, \Lambda r_1, \cdots, \Lambda^{n-1} r_1], \cdots, [r_m, \Lambda r_m, \cdots, \Lambda^{n-1} r_m]\} \\ &\to \{diag(r_{11}, \cdots, r_{n1})C, \cdots, diag(r_{1m}, \cdots, r_{nm})C\}, \end{aligned}$$

where $C = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix}$.

Since the eigenvalues of $I_{11} - L_{11}$ are all different, C has full rank. Group $(\mathscr{G}_1, x)$ and group $(\mathscr{G}_2, x)$ are connected by one agent, then there exists at least $r_{i1} \neq 0$. Then the rank of $diag(r_{11}, r_{21}, \cdots, , r_{n1})$ is full, therefore the rank of Q_1 is full. According to Q_1 , we have the same discussion to the controllability matrix Q_2.

Corollary 9 *System (10) attains controllability if and only if*
(1) the eigenvalues of $I - L_{ii} (i = 1, 2)$ are all distinct,and
(2) the eigenvectors of $I - L_{ii} (i = 1, 2)$ are not orthogonal to at least one column of $L_{ij} (i, j = 1, 2, \ i \neq j)$.

3.4 Some further discussions

In the real world, the agents emerged diversities such dividing into some subgroups. In this subsection, We choose the N agents are divided into s subgroups for example, see Fig. 3. We omit the time-delay among the agents, just pay attention to the grouping situation. The network protocol of s subgroups is designed as

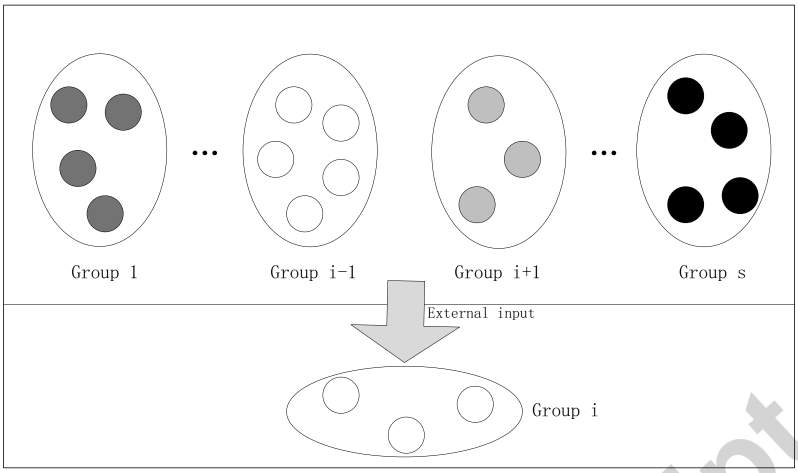

Fig. 3: Other groups effect on group i.

$$x_i(k+1)=\begin{cases} x_i(k)+\sum\limits_{v_j\in\mathcal{N}_{1i}}a_{ij}[x_j(k)-x_i(k)]+\sum\limits_{v_j\in\mathcal{N}_{2i}}a_{ij}x_j(k)+\cdots+\sum\limits_{v_j\in\mathcal{N}_{si}}a_{ij}x_j(k), i\in l_1\\ \vdots\\ x_i(k)+\sum\limits_{v_j\in\mathcal{N}_{1i}}a_{ij}x_j(k)+\cdots+\sum\limits_{v_j\in\mathcal{N}_{(m-1)i}}a_{ij}x_j(k)+\sum\limits_{v_j\in\mathcal{N}_{mi}}a_{ij}[x_j(k)-x_i(k)]\\ +\sum\limits_{v_j\in\mathcal{N}_{(m+1)i}}a_{ij}x_j(k)+\cdots+\sum\limits_{v_j\in\mathcal{N}_{si}}a_{ij}x_j(k), i\in l_m\\ \vdots\\ x_i(k)+\sum\limits_{v_j\in\mathcal{N}_{1i}}a_{ij}x_j(k)+\cdots+\sum\limits_{v_j\in\mathcal{N}_{(s-1)i}}a_{ij}x_j(k)+\sum\limits_{v_j\in\mathcal{N}_{si}}a_{ij}[x_j(k)-x_i(k)], i\in l_s \end{cases}. \quad (11)$$

Let $x^1=[x_{n_0+1},x_{n_0+2},\cdots,x_{n_1}]^T$, $x^2=[x_{n_1+1},x_{n_1+2},\cdots,x_{n_2}]^T$, $\cdots$, $x^s=[x_{n_{(s-1)}+1},x_{n_{(s-1)}+2},\cdots,x_{n_s}]^T$, then system (11) can become

$$\begin{cases} x^1(k+1)=(I_{11}-L_{11})x^1(k)-L_{12}x^2(k)-\cdots-L_{1s}x^s(k)\\ \vdots\\ x^m(k+1)=(I_{mm}-L_{mm})x^m(k)-L_{m1}x^1(k)-\cdots-L_{m(m-1)}x^{m-1}(k)\\ -L_{m(m+1)}x^{m+1}(k)-\cdots-L_{ms}x^s(k)\\ \vdots\\ x^s(k+1)=(I_{ss}-L_{ss})x^s(k)-L_{s1}x^1(k)-\cdots-L_{s(s-1)}x^{s-1}(k) \end{cases}. \quad (12)$$

Similarly, for subgroup i, the other subgroup can be regarded as the inputs, then system (12) can be rewritten as

$$
\begin{cases}
x^1(k+1) = (I_{11} - L_{11})x^1(k) - [L_{12}, L_{13}, \cdots, L_{1s}] \begin{bmatrix} x^2(k) \\ x^3(k) \\ \vdots \\ x^s(k) \end{bmatrix} \\
\qquad \vdots \\
x^m(k+1) = (I_{mm} - L_{mm})x^m(k) - [L_{m1}, \cdots, L_{m(m-1)}, L_{m(m+1)}, \cdots, L_{ms}] \begin{bmatrix} x^1(k) \\ \vdots \\ x^{m-1}(k) \\ x^{m+1}(k) \\ \vdots \\ x^s(k) \end{bmatrix} \\
\qquad \vdots \\
x^s(k+1) = (I_{ss} - L_{ss})x^s(k) - [L_{s1}, L_{s2}, \cdots, L_{s(s-1)}] \begin{bmatrix} x^1(k) \\ x^2(k) \\ \vdots \\ x^{s-1}(k) \end{bmatrix}
\end{cases}
. \tag{13}
$$

For the more subgroups in the network, we can have the following result.

Theorem 11 *System (13) attains group controllability if and only if rank$(Q_i) = n_i$ $(i = 1, 2, \cdots, s)$, where*

$$Q_i = [-(L_{i1}, \cdots, L_{ik}, \cdots, L_{is}), -(I_{ii} - L_{ii})(L_{i1}, \cdots, L_{ik}, \cdots, L_{is}), \cdots, -(I_{ii} - L_{ii})^{n_i - 1}(L_{i1}, \cdots, L_{ik}, \cdots, L_{is})],$$

$i \neq k$ and Q_i is the controllability matrices of $(\mathscr{G}_i, x_i)$.

Remark 11 *In fact, it is of physical interest and of theoretical interest to investigate the group controllability for networks of multiple subgroups. In addition, Theorem 5 and Theorem 6 imply that the connectivity of the graph among different subgroups has great influence on the controllability, as well as indicate that the positions of agents have important effect on the controllability. Considering the physical and theoretical significance of multi-agent systems with multiple subgroups, there exit some essential differences between 2 subgroups and s subgroups. It is complicated and difficult to make system (13) be appropriately changed into further discussion in the form of a matrix. In the future work, we will consider the group controllability of multi-agent system consisting of s subgroups.*

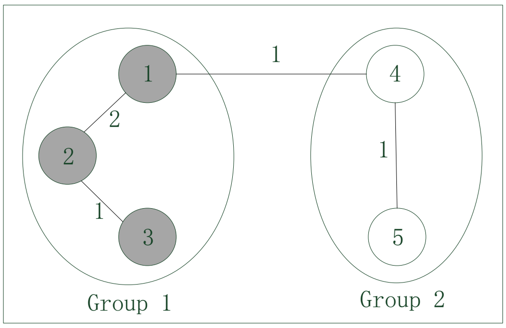

Fig. 4: Fixed topology of the network.

4 Examples and Simulations

In this section, we give some examples to illustrate the effective of the proposed theoretical results.

4.1 Example 1

Consider a five-agent network divided into two sub-groups with a fixed topology and time-delay $h = 1$ described by Fig. 4. From Fig. 4, the fixed system (5) is defined by

$$L_{11} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix},\ L_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ L_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ L_{22} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

According to Theorem 5, then

$$det(L_{11} - (\lambda^2 - \lambda)I) = -\lambda^6 + 3\lambda^5 + 3\lambda^4 - 11\lambda^3 + 6\lambda = 0$$

and

$$det(L_{22} - (\mu^2 - \mu)I) = \mu^4 - 2\mu^3 - \mu^2 + 2\mu = 0.$$

By computing, we can obtain the root are

$$\{0, 2.7321, -1.7321, 1.7321, 1.0000, -0.7321\}$$

and

$$\{0, -1.0000, 2.0000, 1.0000\}.$$

For $\lambda_1 = 0$,

$$Y_{1(\lambda_1)} = [L_{11} - (\lambda_1^2 - \lambda_1)I, -L_{12}] = \begin{bmatrix} 2 & -2 & 0 & -1 & 0 \\ -2 & 3 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix},$$

by computing, $rank(Y_{1(\lambda_1)}) = 3$.

Analogously, for $\lambda_2 = 2.7321$,

$$Y_{1(\lambda_2)} = [L_{11} - (\lambda_2^2 - \lambda_2)I, -L_{12}] = \begin{bmatrix} -2.7323 & -2.0000 & 0 & -1 & 0 \\ -2.0000 & -1.7323 & -1.0000 & 0 & 0 \\ 0 & -1.0000 & -3.7323 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{1(\lambda_2)}) = 3$;

for $\lambda_3 = -1.7321$,

$$Y_{1(\lambda_3)} = [L_{11} - (\lambda_3^2 - \lambda_3)I, -L_{12}] = \begin{bmatrix} -2.7323 & -2.0000 & 0 & -1 & 0 \\ -2.0000 & -1.7323 & -1.0000 & 0 & 0 \\ 0 & -1.0000 & -3.7323 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{1(\lambda_3)}) = 3$;

for $\lambda_4 = 1.7321$,

$$Y_{1(\lambda_4)} = [L_{11} - (\lambda_4^2 - \lambda_4)I, -L_{12}] = \begin{bmatrix} 0.7319 & -2.0000 & 0 & -1 & 0 \\ -2.0000 & 1.7319 & -1.0000 & 0 & 0 \\ 0 & -1.0000 & -0.2681 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{1(\lambda_4)}) = 3$;

for $\lambda_5 = 1$,

$$Y_{1(\lambda_5)} = [L_{11} - (\lambda_5^2 - \lambda_5)I, -L_{12}] = \begin{bmatrix} 2 & -2 & 0 & -1 & 0 \\ -2 & 3 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{1(\lambda_5)}) = 3$;

for $\lambda_6 = -0.7321$,

$$Y_{1(\lambda_6)} = [L_{11} - (\lambda_6^2 - \lambda_6)I, -L_{12}] = \begin{bmatrix} 0.7319 & -2.0000 & 0 & -1.0000 & 0 \\ -2.0000 & 1.7319 & -1.0000 & 0 & 0 \\ 0 & -1.0000 & -0.2681 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{1(\lambda_6)}) = 3$;

for $\mu_1 = 0$,

$$Y_{2(\mu_1)} = [L_{22} - (\mu_1^2 - \mu_1)I, -L_{21}] = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{2(\mu_1)}) = 2$;

for $\mu_2 = -1$,

$$Y_{2(\mu_2)} = [L_{22} - (\mu_2^2 - \mu_2)I, -L_{21}] = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{2(\mu_2)}) = 2$;

for $\mu_3 = 2$,

$$Y_{2(\mu_3)} = [L_{22} - (\mu_3^2 - \mu_3)I, -L_{21}] = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{2(\mu_3)}) = 2$;

for $\mu_4 = 1$,

$$Y_{2(\mu_4)} = [L_{22} - (\mu_4^2 - \mu_4)I, -L_{21}] = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{2(\mu_4)}) = 2$.

It can be easily found that matrices $Y_i = \begin{bmatrix} L_{ii} - (\lambda^2 - \lambda)I, & -L_{ij} \end{bmatrix}$ $(i, j = 1, 2$ and $i \neq j)$ have full row rank at every root of $det(L_{ii} - (\lambda^2 - \lambda)I) = 0$ $(i, j = 1, 2$ and $i \neq j)$, then system (5) attains group controllability based on Theorem 5.

Figs. 5–8 show the simulation results when $h = 1$. The vertices (the agents of group 1, the black star dots; the agents of group 2, the black circle dots) begin from random initial positions. Interconnections are depicted as dotted (red, green, blue, pink and wathet, respectively) lines connecting the corresponding vertices. Beginning from this initial configuration, the vertices are ultimately being controlled to a straight-line configuration and a regular-triangle configuration, respectively. Fig. 6 is a magnification of a portion of Fig. 5, and Fig. 8 is a magnification of a portion of Fig. 7, respectively.

4.2 Example 2

Consider a five-agent network described as example 1 when $h = 2$.

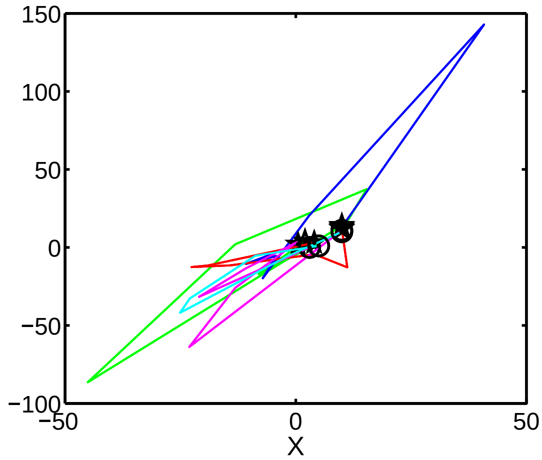

Fig. 5: Alignment in a straight line on a fixed network with $h = 1$.

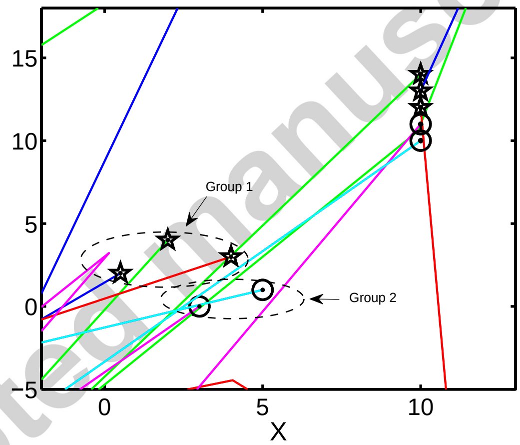

Fig. 6: A magnification of Fig. 5.

According to Theorem 5, we need to compute the rank of matrix $Y_i = \left[L_{ii} - (\lambda^3 - \lambda^2)I, \quad -L_{ij}\right]$ $(i, j = 1, 2$ and $i \neq j)$ at every root of $det(L_{ii} - (\lambda^2 - \lambda)I) = 0$ $(i, j = 1, 2$ and $i \neq j)$, then

$$det(L_{11} - (\lambda^3 - \lambda^2)I) = -\lambda^9 + 3\lambda^8 - 3\lambda^7 + 7\lambda^6 - 12\lambda^5 + 6\lambda^4 - 6\lambda^3 + 6\lambda^2 = 0$$

and

$$det(L_{22} - (\mu^3 - \mu^2)I) = \mu^6 - 2\mu^5 + \mu^4 - 2\mu^3 + 2\mu^2 = 0.$$

By computing, we can obtain the roots are

$$\{0, 0, 2.0867, -0.5434 + 1.4044i, -0.5434 - 1.4044i, 1.5368, 1.0000, -0.2684 + 0.8677i, -0.2684 - 0.8677i\}$$

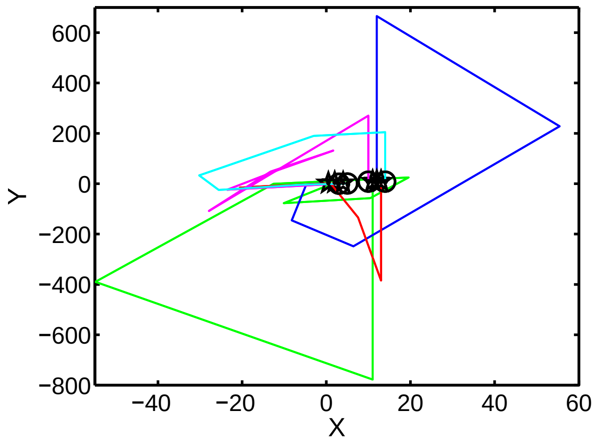

Fig. 7: Forming a regular triangle on a fixed network with $h = 1$.

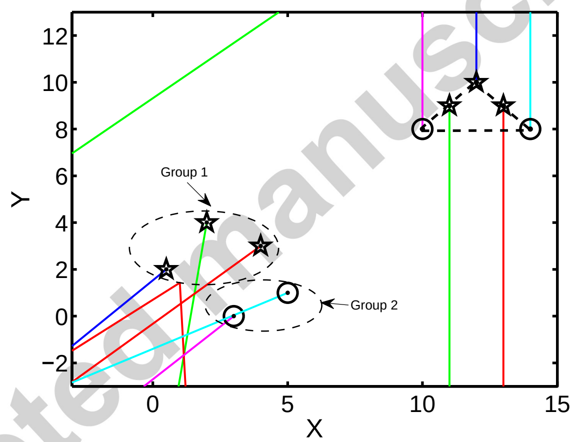

Fig. 8: A magnification of Fig. 7.

and

$$\{0, 0, -0.3478 + 1.0289i, -0.3478 - 1.0289i, 1.6956, 1.0000\},$$

respectively.

For $\lambda_1 = \lambda_2 = 0$,

$$Y_{1(\lambda_1)} = Y_{1(\lambda_2)} = [L_{11} - (\lambda_1^3 - \lambda_1^2)I, -L_{12}] = \begin{bmatrix} 2 & -2 & 0 & -1 & 0 \\ -2 & 3 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix},$$

by computing, $rank(Y_{1(\lambda_1)}) = rank(Y_{1(\lambda_2)}) = 3$.

Analogously, for $\lambda_3 = 2.0867$,

$$Y_{1(\lambda_3)} = [L_{11} - (\lambda_3^3 - \lambda_3^2)I, -L_{12}] = \begin{bmatrix} -2.7318 & -2.0000 & 0 & -1 & 0 \\ -2.0000 & -1.7318 & -1 & 0 & 0 \\ 0 & -1 & -3.7318 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{1(\lambda_3)}) = 3$;

for $\lambda_4 = -0.5434 + 1.4044i$,

$$Y_{1(\lambda_4)} = [L_{11} - (\lambda_4^3 - \lambda_4^2)I, -L_{12}] = \begin{bmatrix} -2.7319 & -0.0004i - 2 & 0 & -1 & 0 \\ -2 & -1.7319 - 0.0004i & -1 & 0 & 0 \\ 0 & -1 & -3.7319 - 0.0004i & 0 & 0 \end{bmatrix},$$

and $rank(Y_{1(\lambda_4)}) = 3$;

for $\lambda_5 = -0.5434 - 1.4044i$,

$$Y_{1(\lambda_5)} = [L_{11} - (\lambda_5^3 - \lambda_5^2)I, -L_{12}] = \begin{bmatrix} -2.7319 + 0.0004i & -2 & 0 & -1 & 0 \\ -2 & -1.7319 + 0.0004i & -1 & 0 & 0 \\ 0 & -1 & -3.7319 + 0.0004i & 0 & 0 \end{bmatrix},$$

and $rank(Y_{1(\lambda_5)}) = 3$;

for $\lambda_6 = 1.5368$,

$$Y_{1(\lambda_6)} = [L_{11} - (\lambda_6^3 - \lambda_6^2)I, -L_{12}] = \begin{bmatrix} 0.7322 & -2 & 0 & -1 & 0 \\ -2 & 1.7322 & -1 & 0 & 0 \\ 0 & -1 & -0.2678 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{1(\lambda_6)}) = 3$;

for $\lambda_7 = 1$,

$$Y_{1(\lambda_7)} = [L_{11} - (\lambda_7^3 - \lambda_7^2)I, -L_{12}] = \begin{bmatrix} 2 & -2 & 0 & -1 & 0 \\ -2 & 3 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{1(\lambda_7)}) = 3$;

for $\lambda_8 = -0.2684 + 0.8677i$,

$$Y_{1(\lambda_8)} = [L_{11} - (\lambda_8^3 - \lambda_8^2)I, -L_{12}] = \begin{bmatrix} 0.7322 - 0.00001i & -2 & 0 & -1 & 0 \\ -2 & 1.7322 - 0.00001i & -1 & 0 & 0 \\ 0 & -1 & -0.2678 - 0.00001i & 0 & 0 \end{bmatrix},$$

and $rank(Y_{1(\lambda_8)}) = 3$;

for $\lambda_9 = -0.2684 - 0.8677i$,

$$Y_{1(\lambda_9)} = [L_{11} - (\lambda_9^3 - \lambda_9^2)I, -L_{12}] = \begin{bmatrix} 0.7322+0.00001i & -2 & 0 & -1 & 0 \\ -2 & 1.7322+0.00001i & -1 & 0 & 0 \\ 0 & -1 & -0.2678+0.00001i & 0 & 0 \end{bmatrix},$$

and $rank(Y_{1(\lambda_9)}) = 3$;

for $\mu_1 = \mu_2 = 0$,

$$Y_{2(\mu_1)} = [L_{22} - (\mu_1^3 - \mu_1^2)I, -L_{21}] = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{2(\mu_1)}) = 2$;

for $\mu_3 = -0.3478 + 1.0289i$,

$$Y_{2(\mu_3)} = [L_{22} - (\mu_3^3 - \mu_3^2)I, -L_{21}] = \begin{bmatrix} -1.0002+0.0001i & -1 & -1 & 0 & 0 \\ -1 & -1.0002+0.0001i & 0 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{2(\mu_3)}) = 2$;

for $\mu_4 = -0.3478 - 1.0289i$,

$$Y_{2(\mu_4)} = [L_{22} - (\mu_4^3 - \mu_4^2)I, -L_{21}] = \begin{bmatrix} -1.0002-0.0001i & -1 & -1 & 0 & 0 \\ -1 & -1.0002-0.0001i & 0 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{2(\mu_4)}) = 2$;

for $\mu_5 = 1.6956$,

$$Y_{2(\mu_5)} = [L_{22} - (\mu_5^3 - \mu_5^2)I, -L_{21}] = \begin{bmatrix} -0.9999 & -1 & -1 & 0 & 0 \\ -1 & -0.9999 & 0 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{2(\mu_5)}) = 2$;

for $\mu_6 = 1.6956$,

$$Y_{2(\mu_6)} = [L_{22} - (\mu_6^3 - \mu_6^2)I, -L_{21}] = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and $rank(Y_{2(\mu_6)}) = 2$.

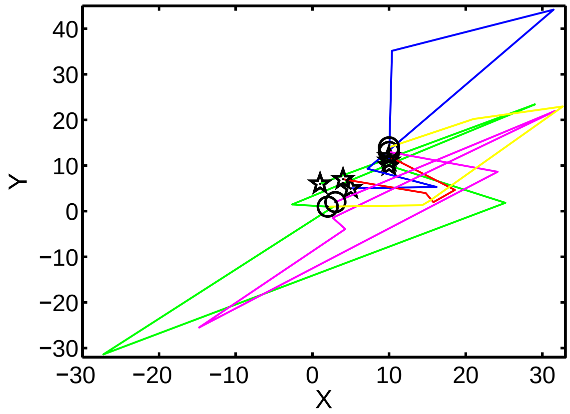

Fig. 9: Alignment in a straight line on a fixed network with $h = 2$.

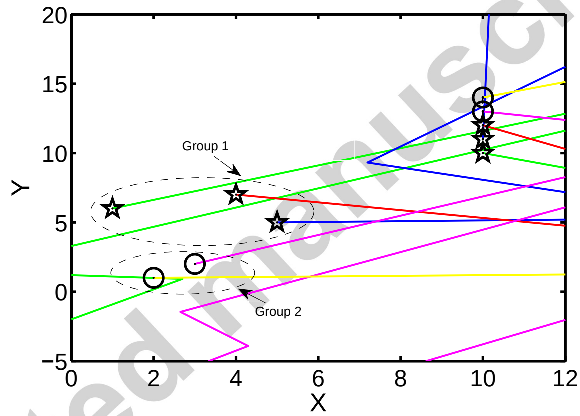

Fig. 10: A magnification of Fig. 9.

By computing, we can find that matrix $Y_i = \begin{bmatrix} L_{ii} - (\lambda^3 - \lambda^2)I, & -L_{ij} \end{bmatrix}$ $(i, j = 1, 2$ and $i \neq j)$ have full row rank at every root of $det(L_{ii} - (\lambda^3 - \lambda^2)I) = 0$ $(i, j = 1, 2$ and $i \neq j)$. According to Theorem 5, the system (5) shown in Fig. 4 also attains group controllability when $h = 2$.

Figs. 9–12 show the simulation results when $h = 2$. The vertices (the agents of group 1, the black star dots; the agents of group 2, the black circle dots) begin from random initial positions. Interconnections are depicted as dotted (red, green, blue, pink and wathet, respectively) lines connecting the corresponding vertices. Beginning from this initial configuration, the vertices are ultimately being controlled to a straight-line configuration and a regular-triangle configuration, respectively. Fig. 10 is a magnification of a portion of Fig. 9, and Fig. 12 is a magnification of a portion of Fig. 11, respectively.

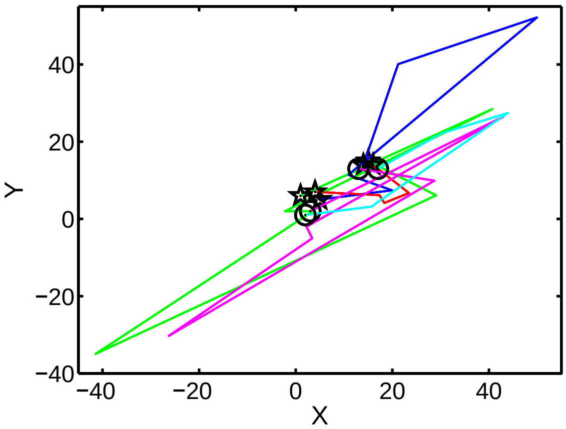

Fig. 11: Forming a regular triangle on a fixed network with $h = 2$.

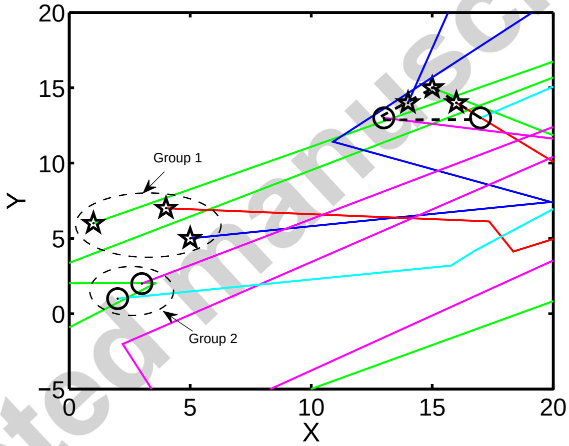

Fig. 12: A magnification of Fig. 11.

4.3 Example 3

Consider a network of five agents divided into two sub-groups with agents 1, 2, and 3 as group 1 (black dots), agents 4 and 5 (white dots) as group 2 and with a switching topology as shown in Fig. 13.

For simplicity, the switched system (6) is defined by

$$L_{11_{(1)}} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix},\ L_{12_{(1)}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ L_{22_{(1)}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ L_{21_{(1)}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

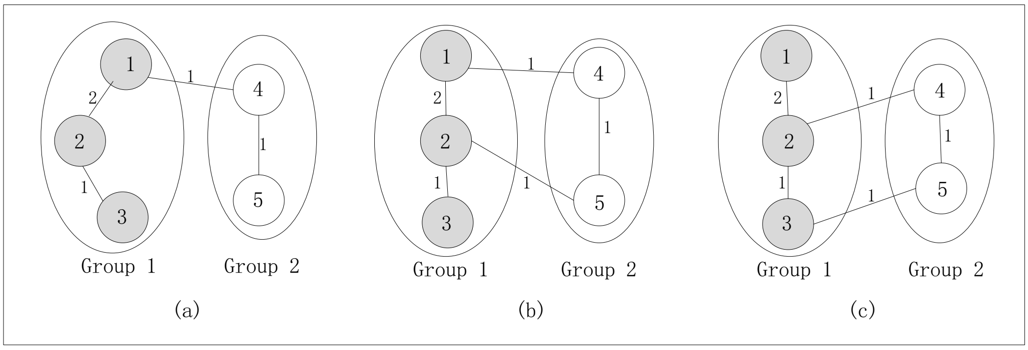

Fig. 13: Switching topology of the network.

$$L_{11_{(2)}} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix},\ L_{12_{(2)}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ L_{22_{(2)}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ L_{21_{(2)}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix};$$

$$L_{11_{(3)}} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix},\ L_{12_{(3)}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ L_{22_{(3)}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ L_{21_{(3)}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By computing,

$$\mathscr{W}_{11} = span\{L_{12_{(1)}}, L_{12_{(2)}}, L_{12_{(3)}}\} = span\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \mathscr{W}_{12} = \mathscr{W}_{13} = \Re^3,$$

$$\mathscr{W}_{21} = span\{L_{21_{(1)}}, L_{21_{(2)}}, L_{21_{(3)}}\} = span\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} = \mathscr{W}_{22} = \mathscr{W}_{23} = \Re^2.$$

According to Theorem 1 , the system (6) described by Fig. 13 attains group switching controllability.

Figs. 14–17 are the simulation results of the system with a switching topology. The agents (the agents of group 1, the black star dots; the agents of group 2, the black circle dots) begin from random initial positions. Interconnections are depicted as lines (red, green, blue, pink and wathet, respectively) which connect the corresponding vertices. They show that the member agents can be controlled to a straight-line and a regular-triangle configuration, respectively.

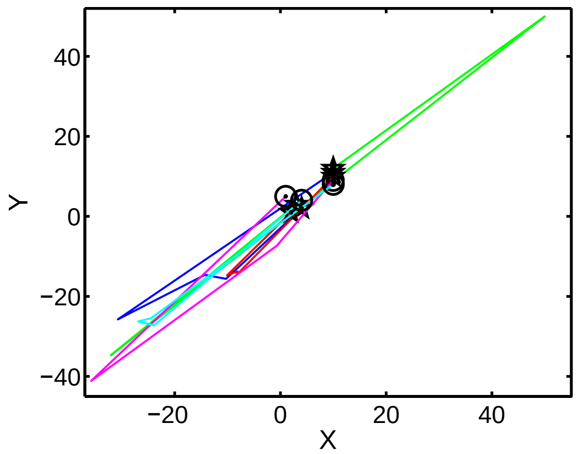

Fig. 14: Alignment in a straight line on a switching network.

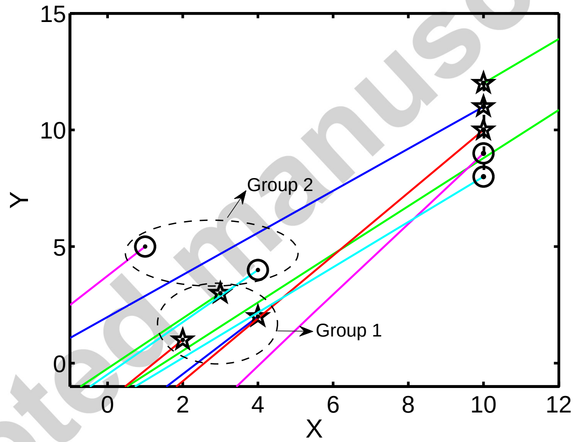

Fig. 15: A magnification of Fig. 14.

5 Conclusion

In this paper, we have investigated the group controllability of multi-agent discrete-time systems with time-delays. Both fixed topology and switching topology have been studied for the multi-agent discrete-time systems, respectively. Through the theoretical analysis, we have derived some effective conditions for the group switching controllability of the multi-agent discrete-time systems with time-delays. Moreover, we have used an equivalent augmented system to study the group controllability of multi-agent discrete-time systems on fixed network via the PBH test. Our results give further insights into the effect of connec-

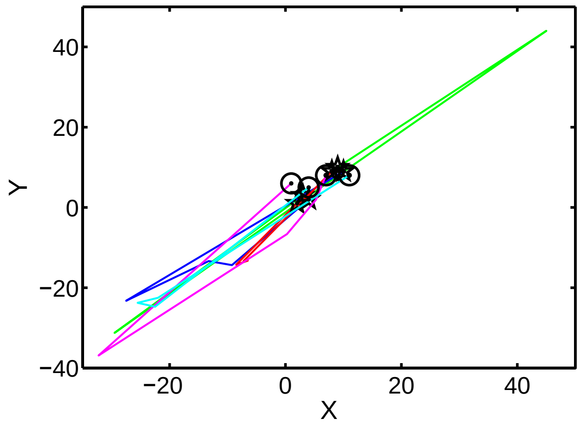

Fig. 16: Forming a regular triangle on a switching network.

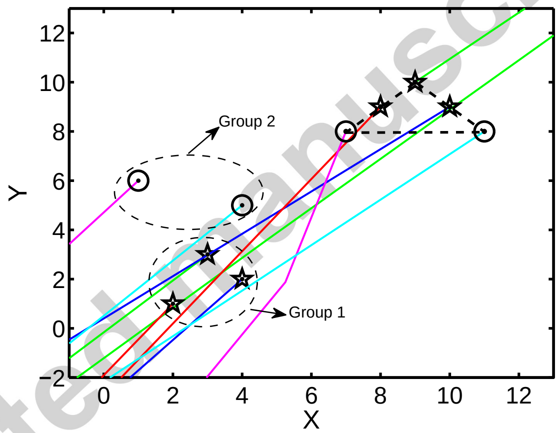

Fig. 17: A magnification of Fig. 16.

tivity on the controllability of the network.

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