Neurocomputing 409 (2020) 316–328

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Controllability of discrete-time multi-agent systems based on absolute protocol with time-delays

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ARTICLE INFO

Article history:
Received 12 February 2020
Revised 23 March 2020
Accepted 6 May 2020
Available online 26 May 2020
Communicated by Lei Zou

Keywords:
MASs
Controllability
Absolute protocol
Time-delays

ABSTRACT

This paper investigates complete controllability and structural controllability of discrete-time multi-agent systems (MASs) under a leader-follower framework with time-delays based on absolute protocol, respectively, where both a single time-delay and multiple time-delays are investigated. By using the equivalent augmented systems without time-delays, some new graph-theoretic and algebraic characterizations are respectively built for complete controllability and structural controllability depending on communication flows of MASs. In addition, the simulation results are given.

1. Introduction

Controllability problem is a very challenging task in modelling, analysis and coordination control of MASs [1–9]. Distinguishing the usual control systems, the evolutionary behavior of MASs is influenced by various factors, such as the dynamic characteristics of the agent itself, the information communication topology between agents, and the protocols followed by the evolution of agents' states, the raise and loss of the agents' numbers and the connecting edges, and the interference of the external environment, etc., all of which make it more difficult to study in controllability problems.

Most of the existing studies on coordinated control of MASs mainly focus on agent-based discrete-time/continuous-time dynamics from single-integrator/double-integrator/high-order-integrator to general linear dynamics, from particle dynamics to rigid-body dynamics, from homogeneous dynamics to heterogeneous dynamics, and so on [10–17]. In addition, due to the actual factors existing in various physical systems, such as input saturation, communication information delay, quantization and loss, it is harder to investigate controllability of MASs.

In 2004, Tanner [18] first put forward the controllability of a classical leader-follower first-order MASs. In reality, the controllability of MASs reflects the leaders' control ability, that is, whether the leaders can control the followers to any given positions in a finite time. The influence of the number and positions of leaders on controllability of MASs based on consensus protocols was discussed in [19]. Ji et al. [20] studied the influence of interconnection topologies on the controllability for MASs. In [21], the authors gave the technique of selecting leaders and investigated the influence of leaders on undirected tree graph by means of algebra and graph theory. Guan and Wang provided qualitative and quantitative description of the leaders' number needed to attain controllability/target controllability of MASs in [22,23]. The authors [24–27] mainly studied the first-order controllability of continuous-time/discrete-time MASs and established a series of sufficient and necessary criteria from algebraic and graphic points of view, respectively, and further pointed out that the complete graph with the same weights among agents can not be controlled on the contrary. In [28], the second-order controllability of discrete-time MASs with time-delays was discussed and some criteria of graph theory were obtained. In particular, [19] addressed the controllability of high-order-integrator and general linear MASs based on linear agreement protocol, respectively, and showed that the controllability of high-order and general linear MASs were both equivalent to that of first-order MASs under the same communication information

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https://doi.org/10.1016/j.neucom.2020.05.046
0925-2312/

topology and the same leader–follower partition. In [29], the authors discussed the formation control of time-invariant linear high-order MASs. Liu et al. [30–33] studied the group controllability of continuous-time/ discrete-time MASs for single-time-scale and two-time-scale features, respectively. In [34], the controllability of MASs with node dynamics and edge dynamics was studied and the algebraic conditions were given. In [26], the authors studied the controllability of continuous/ discrete heterogeneous MASs composed of single-integrator and double-integrator, respectively.

The essence of the controllability of the control system is the structural quality, which is closely related to the system's structure, parameters, control inputs as well as transferring gains, etc. Lin [35] first put forward the concept of structural controllability of control systems. [36] first investigated the structural controllability of MASs, which characterizes the graph-theoretic controllable conditions for MASs. In [37], the structural controllability of high-order dynamic MASs based on fixed topology was studied, while similar problem was considered for switching topology in [38]. The structural controllability of complex networks was established in [39] via the maximum matching and the minimum input principles, and the minimum number of control agents required to steer the whole network was studied. [40,41] discussed the structural controllability of MASs under directed weighted topologies. At present, most studies on the structural controllability of MASs mainly focus on some simple topological graphs, such as path graph [42], ring graph [42], tree graph [43], complete graph [43], grid graph [44], chain graph [45], symmetric graph, cartesian product network, threshold graph, etc.

This paper mainly aims at the complete controllability and structural controllability of discrete-time MASs with communications restrictions and establishes controllable criteria. The main contributions of this paper include:

1. *Different from the controllability problem studied based on the relative protocol, which is represented by the Laplacian matrix in Ref. [21], the current work has considered the complete controllability and the structural controllability of MASs with the absolute protocol, which is represented by the adjacency matrix. It is obvious that different protocols can lead to completely different properties of system matrices.*
2. *The models of discrete-time MASs with a single time-delay and multiple time-delays are put forward based on absolute protocol, respectively.*
3. *An equivalent lower-dimensional system without time-delay is deduced to investigate the controllability of discrete-time MASs. In addition, the equivalent low-dimensional system is proved to be both completely controllable and structurally controllable.*
4. *Moreover, the complete controllability and the structural controllability of MASs are compared further. To the best of our knowledge, the comparison the complete controllability and the structural controllability of MASs has not been studied yet.*

The remainder of this work is organized as follows. Some preliminaries and models are given in Section 2. Section 3 and Section 4 state main results of MASs for a single time-delay and multiple time-delays, respectively. Some examples and simulations are given in Section 5. Finally, the conclusion is summarised in Section 6.

2. Preliminaries and models

2.1. Graph theory

$\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ is called as a directed weighted graph, in which $\mathscr{V} = \{v_1, \cdots, v_N\}$ represents the vertex set and $\mathscr{E} = \{(v_i, v_j) : v_i, v_j \in \mathscr{V}\}$ represents the edge set, respectively; $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix, where $a_{ij} > 0$ if $(v_i, v_j) \in \mathscr{E}$ and $a_{ij} = 0$, otherwise. The set of neighbors of agent v_i is represented as $\mathscr{N}_i = \{v_j \in \mathscr{V} | (v_j, v_i) \in \mathscr{E}\}$. The subgraph is denoted as $\mathscr{G}_s = (\mathscr{V}_s, \mathscr{E}_s)$ if $\mathscr{V}_s \subseteq \mathscr{V}$ and $\mathscr{E}_s \subseteq \mathscr{E}$. If there exists a directed path or link between any two distinct vertices of graph $\mathscr{G}$, the directed graph $\mathscr{G}$ is defined as a strongly connected graph, where the maximal strongly connected subgraph of graph $\mathscr{G}$ is called as a strongly connected component (SCC). If $\widetilde{\mathscr{G}}_s$ is a SCC, the induced subgraph $\widetilde{\mathscr{G}}_s = \left(\widetilde{\mathscr{V}}_s, \widetilde{\mathscr{E}}_s, \widetilde{\mathscr{A}}_s\right)$ is called as an independent strongly connected component ($iSCC$) of graph $\mathscr{G}$, where $(v_i, v_j) \notin \mathscr{E}$ for any $v_j \in \mathscr{V}/\widetilde{\mathscr{V}}_s$ and $v_i \in \mathscr{V}$.

2.2. Structured system

A structured matrix means that its entries are either independent free parameters or fixed zeros. A structured discrete-time system: $x(k+1) = Ax(k) + By(k)$ is described as a directed graph $\mathscr{G}^c(A, B) = (\mathscr{V}^c, \mathscr{E}^c)$, where $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{m\times p}$ are structured matrices. If a structured discrete-time system (A, B) can choose a set of free parameters or fix to make such system be controllable, the system (A, B) is structurally controllable.

Definition 1. [46] For the structured matrix $[A, B] \in \mathbb{R}^{m\times(m+q)}$, supposed that there is a permutation matrix Q such that

$$QAQ^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \; QB = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix},$$

$[A, B]$ is said to be reducible, where $A_{11} \in \mathbb{R}^{l\times l}, A_{21} \in \mathbb{R}^{(m-l)\times l}, A_{22} \in \mathbb{R}^{(m-l)\times(m-l)}$ and $B_{22} \in \mathbb{R}^{(m-l)\times q}$ are structured matrices.

Lemma 1. [46] *The given system (A, B) is structurally controllable iff matrix pair $[A, B]$ is irreducible and the generic rank g-rank$([A, B]) = m$, where g-rank$([A, B]) = m$ refers to a maximum rank obtained by a function of the free parameters in $[A, B]$.*

2.3. Models

For multi-agent systems, there are mainly two kinds of reasonable control protocols designed: one is to use the relative state information between the agent itself and its neighbors, which is called the relative protocol; the other is to use the absolute state information between the agent itself and its neighbors, which is called the absolute protocol. The models of the discrete-time MASs based on absolute protocol with different time-delays will be established in the following.

2.3.1. A single time-delay

A single time-delayed MAS is governed by

$$x_i(k+1) = x_i(k) + \sum_{i\in\mathscr{N}_{ij}} a_{ij}x_j(k-h) + \sum_{p\in\mathscr{N}_{ip}} b_{ip}x_p(k), \;\; k \in \mathscr{I}_k, \tag{1}$$

where $x_i \in \mathbb{R}$ and $x_p \in \mathbb{R}$ are the states of follower i and leader p for $i \in \underline{m} \triangleq \{1, 2, \cdots, m\}$, $p \in \underline{m+q} - \underline{m}$; $h > 0$ is an integer; $\mathscr{N}_i$ is the i-th agent's neighbor set consisting of followers and leaders, where $\mathscr{N}_{i_j} \bigcup \mathscr{N}_{i_p} = \mathscr{N}_i$ and $\mathscr{N}_{i_j} \bigcap \mathscr{N}_{i_p} = \emptyset$; $\mathscr{I}_k$ is an index set of discrete time; $A = [a_{ij}] \in \mathbb{R}^{m\times m}$ is a coupling information weight matrix with $a_{ij} \geqslant 0$ and $a_{ii} = 0$; $B = [b_{ip}] \in \mathbb{R}^{m\times q}$ is a matrix with $b_{ip} \geqslant 0$.

Denote $x(k) \triangleq (x_1(k), x_2(k), \cdots, x_m(k))^T$ and $y(k) \triangleq (x_{m+1}(k), x_{m+2}(k), \cdots, x_{m+q}(k))^T$. Then system (1) follows that

$$x(k+1)=x(k)+Ax(k-h)+By(k),\ k\in\mathscr{I}_k. \tag{2}$$

In order to better study the controllability problem of system (2), we established its equivalent augmented system using the classical control theory as follows:

$$\begin{cases} x(k+1) &= x(k)+Ax(k-h)+By(k)\\ x(k) &= x(k)\\ &\cdots\\ x(k-h+1) &= x(k-h+1)\end{cases}.$$

Furthermore, let $z(k)\triangleq\left(x(k)^T,x(k-1)^T,\cdots,x(k-h)^T\right)^T$, then system (2) can be rewritten as

$$z(k+1)=\mathscr{A}z(k)+\mathscr{B}y(k),\ k\in\mathscr{I}_k, \tag{3}$$

where

$$\mathscr{A}=\begin{bmatrix}E&0&\dots&0&A\\E&0&\dots&0&0\\0&E&\dots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\dots&E&0\end{bmatrix}_{(h+1)m\times(h+1)m},\quad \mathscr{B}=\begin{bmatrix}B\\0\\\vdots\\0\end{bmatrix}_{(h+1)m\times q},$$

and E is the identity matrix with appropriate dimensions.

2.3.2. Multiple time-delays

In fact, there may be a variety of time delays in real life, but they are not always equal. Therefore, an multi-delayed MAS is governed by

$$x_i(k+1)=x_i(k)+\sum_{i\in\mathscr{N}_{ij}}a_{ij}x_j(k-h_{ij})+\sum_{p\in\mathscr{N}_{ip}}b_{ip}x_p(k),\ k\in\mathscr{I}_k, \tag{4}$$

where the time-delays $h_{ij}\geqslant 0$ and $h_{ii}=0$. Let $h_{max}=\max_{\forall i\neq j,i,j\in\underline{m}}\{h_{ij}\}$, then a positive integer $h_{ij}\in\underline{h}=\{0,1,\dots,h_{max}\}$ is a time-delay. Similarly, system (4) can be redescribed as

$$\begin{aligned}x(k+1)=x(k)+A_1x(k-1)+A_2x(k-2)+\cdots\\+A_{h_{max}}x(k-h_{max})+By(k),\ k\in\mathscr{I}_k,\end{aligned} \tag{5}$$

where $A=A_1+A_2+\cdots+A_{h_{max}}$.

Similarly, let

$$\begin{cases}x(k+1)=x(k)+A_1x(k-1)+A_2x(k-2)+\cdots\\+A_{h_{max}}x(k-h_{max})+By(k)x(k)=x(k)\\\cdots\\x(k-h_{max}+1)=x(k-h_{max}+1)\end{cases},$$

then system (5) can be rewritten as

$$z(k+1)=\tilde{\mathscr{A}}z(k)+\mathscr{B}y(k),\ k\in\mathscr{I}_k, \tag{6}$$

where

$$\tilde{\mathscr{A}}=\begin{bmatrix}E&A_1&\dots&A_{h_{max}-1}&A_{h_{max}}\\E&0&\dots&0&0\\0&E&\dots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\dots&E&0\end{bmatrix}_{(h_{max}+1)m\times(h_{max}+1)m},$$

$z(k)$ and $\mathscr{B}$ are defined as above.

Remark 1. Through a particular linear transformation, the original time-delayed system (2) (or (5)) is transformed into a new delay-free system (3) (or (6)), whose controllability is equivalent. Moreover, it is obvious to see that $\left(\tilde{\mathscr{A}},\mathscr{B}\right)$ of MAS (6) with multiple time-delays is more complex than $(\mathscr{A},\mathscr{B})$ of MAS (3) a single time-delay under the condition $A=A_1+A_2+\cdots+A_{h_{max}}$. As a result, the analysis of controllability of MAS (6) is more complex and difficult than that of MAS (3) rather than a simple direct derivation.

3. Controllability analysis of discrete-time MASs with a single time-delay

Complete controllability and structural controllability of discrete-time MASs based on absolute protocol with a single time-delay, respectively, will be analyzed in the following.

3.1. Complete controllability

According to known results in linear time-invariant system theory, the following propositions can be obtained at once.

Proposition 1. *System (3) is completely controllable iff controllability matrix $\mathscr{Q}$ has full row rank, where*

$$\mathscr{Q}=\left[\mathscr{B}\ \mathscr{A}\mathscr{B}\ \mathscr{A}^2\mathscr{B}\cdots\mathscr{A}^{(h+1)m-1}\mathscr{B}\right].$$

Proposition 2. *System (3) is completely controllable iff system (3) satisfies*

(1) *$rank(sE-\mathscr{A},\mathscr{B})=(h+1)m$ for $\forall s\in\mathbb{C}$; or*
(2) *$rank(\lambda_iE-\mathscr{A},\mathscr{B})=(h+1)m$, where λ_i is an eigenvalue of $\mathscr{A}$ for $\forall i=1,\cdots,(h+1)m$.*

From Propositions 1,2, it is easy to see that the amount of calculation is relatively large since the system has time-delay and higher dimension. Moreover, the influence of matrix $[A,B]$ on controllability can not be found directly. Therefore, we will investigate the direct influence of system matrix $[A,B]$ on controllability in the following.

Theorem 1. *System (3) attains complete controllability iff matrix $Y=\left[A+\left(\lambda^h-\lambda^{h+1}\right)E,B\right]$ has full row rank at each root of $det\left(\left[A+\left(\lambda^h-\lambda^{h+1}\right)E\right]\right)=0$.*

Proof. From Proposition 2, system (3) is completely controllable iff matrix

$$(\mathscr{A}-\lambda E,\mathscr{B})=\begin{bmatrix}E-\lambda E&0&\cdots&\cdots&A&B\\E&-\lambda E&\cdots&\cdots&0&0\\0&E&\ddots&\ddots&\vdots&\vdots\\\vdots&\ddots&\ddots&\ddots&\vdots&\vdots\\0&\cdots&0&E&-\lambda E&0\end{bmatrix}$$
$$\rightarrow\begin{bmatrix}E&0&\cdots&0&0\\0&E&\cdots&0&0\\0&0&\cdots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&A+\left(\lambda^h-\lambda^{h+1}\right)E&B\end{bmatrix}$$

has full row rank. Through matrix elementary transformation, if matrix $Y=\left[A+\left(\lambda^h-\lambda^{h+1}\right)E,B\right]$ has full row rank, matrix

$(\mathscr{A} - \lambda E, \mathscr{B})$ has full row rank. The following similar proof can be seen from Theorem 4 in [21], omitted. □

For system (3), the complete controllability is determined by the system matrix pair $(\mathscr{A}, \mathscr{B})$. Obviously, the dimension of the complete controllability matrix pair $(\mathscr{A}, \mathscr{B})$ in Theorem 1 is higher than that of (A, B), which increases the difficulties of numerical calculation and theoretical analysis. Nevertheless, through the analytical process of Theorem 1, we find that the controllability of $(\mathscr{A}, \mathscr{B})$ is depended on matrix $Y = \left[A + \left(\lambda^h - \lambda^{h+1}\right)E, B\right]$. Corollaries 1–5 respectively discussed several special cases to satisfy the condition that matrix $Y = \left[A + \left(\lambda^h - \lambda^{h+1}\right)E, B\right]$ has full row rank in the following, which make it be more concise and straightforward in judging and testing the controllability of systems.

Corollary 1. *System (3) attains complete controllability if* $det\left(\left[A + \lambda^h(1-\lambda)E\right]\right) \neq 0, \forall \lambda \in \mathbb{C}$.

Corollary 2. *System (3) attains complete controllability if matrix B has full row rank.*

Corollary 3. *System (3) attains complete controllability if matrix* $[A, B]$ *has full row rank.*

Corollary 4. *The roots of* $det\left(\left[A + \left(\lambda^h - \lambda^{h+1}\right)E\right]\right) = 0$ *are some eigenvalues of* $\mathscr{A}$.

Proof. Since

$$det(\mathscr{A} - \lambda E) = det\left(\begin{bmatrix} E-\lambda E & 0 & \cdots & \cdots & A \\ E & -\lambda E & \cdots & \cdots & 0 \\ 0 & E & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & E & -\lambda E \end{bmatrix}\right) = 0,$$

whose equivalent form is

$$det\left(\begin{bmatrix} E & 0 & \cdots & 0 \\ 0 & E & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A + \left(\lambda^h - \lambda^{h+1}\right)E \end{bmatrix}\right) = 0,$$

which follows that $\det\left(\left[A + \left(\lambda^h - \lambda^{h+1}\right)E\right]\right) = 0$, the roots of $\det\left(\left[A + \left(\lambda^h - \lambda^{h+1}\right)E\right]\right) = 0$ are some eigenvalues of $\mathscr{A}$. The proof is completed. □

Corollary 5. *System (3) attains complete controllability iff system*

$$x(k+1) = \left(-A - \lambda^h E + \lambda^{h+1}E + \lambda E\right)x(k) + By(k)$$

attains complete controllability.

Proof. From Proposition 2, system (3) attains complete controllability iff $\left[\lambda E - \left(-A - \lambda^h E + \lambda^{h+1}E + \lambda E\right), B\right] = \left[A + \left(\lambda^h - \lambda^{h+1}\right)E, B\right]$ has full row rank $(\forall \lambda \in \mathbb{C})$. Based on Theorem 1, the conclusion is true. □

Furthermore, the complete controllability of system (3) can also be judged by the eigenvectors of the system matrices.

Theorem 2. *System (3) attains complete controllability if system (3) satisfies*

(1) *The eigenvalues of* $\mathscr{A}$ *are all distinct; and*
(2) *All the row vectors of* U^{-1} *are not orthogonal to at least one column in* $\mathscr{B}$ *at the same time, where U is made up of the eigenvectors of* $\mathscr{A}$.

Proof. Assumed that condition (1) is satisfied, then $\mathscr{A} = UDU^{-1}$, where $D = diag(\lambda_1, \cdots, \lambda_{(h+1)m})$ with $\lambda_1, \cdots, \lambda_{(h+1)m}$ being the eigenvalues of $\mathscr{A}$, and nonsingular U is made up of the eigenvectors of $\mathscr{A}$. Therefore, the controllability matrix $\mathscr{Q}$ can be represented as

$$\begin{aligned} \mathscr{Q} &= \left[\mathscr{B} \vdots \left(UDU^{-1}\right)\mathscr{B} \vdots \cdots \vdots \left(UDU^{-1}\right)^{(h+1)m-1}\mathscr{B}\right] \\ &= \left[\mathscr{B} \vdots UDU^{-1}\mathscr{B} \vdots \cdots \vdots UD^{(h+1)m-1}U^{-1}\mathscr{B}\right] \\ &= U\left[U^{-1}\mathscr{B} \vdots DU^{-1}\mathscr{B} \vdots \cdots \vdots D^{(h+1)m-1}U^{-1}\mathscr{B}\right] \\ &= U\widehat{\mathscr{Q}}, \end{aligned}$$

where $\widehat{\mathscr{Q}} = \left[U^{-1}\mathscr{B} \vdots DU^{-1}\mathscr{B} \vdots \cdots \vdots D^{(h+1)m-1}U^{-1}\mathscr{B}\right]$.

Since U is nonsingular, then $rank(\mathscr{Q}) = rank\left(\widehat{\mathscr{Q}}\right)$, which yields to

$$w(k+1) = Dw(k) + U^{-1}\mathscr{B}y(k), \ k \in \mathscr{I}_k. \tag{7}$$

Next, complete controllability of system (3) will be discussed. Denote $U^{-1} \triangleq \left[\eta_1 \eta_2 \cdots \eta_{(h+1)m}\right]^T \in \mathbb{R}^{(h+1)m \times (h+1)m}$, $\mathscr{B} \triangleq \left[B_{(h+1)m+1} B_{(h+1)m+2} \cdots B_{(h+1)m+q}\right] \in \mathbb{R}^{(h+1)m \times q}$, then

$$\begin{aligned} \widehat{\mathscr{Q}} &= \left[U^{-1}\mathscr{B} \vdots DU^{-1}\mathscr{B} \vdots \cdots \vdots D^{(h+1)m-1}U^{-1}\mathscr{B}\right] \\ &= \left[\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{(h+1)m} \end{bmatrix} \left[B_{(h+1)m+1} B_{(h+1)m+2} \cdots B_{(h+1)m+q}\right] \vdots \right. \\ & \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{(h+1)m} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{(h+1)m} \end{bmatrix} \\ & \left[B_{(h+1)m+1} B_{(h+1)m+2} \cdots B_{(h+1)m+q}\right] \vdots \cdots \vdots \\ & \begin{bmatrix} \lambda_1^{(h+1)m-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{(h+1)m-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{(h+1)m}^{(h+1)m-1} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{(h+1)m} \end{bmatrix} \\ & \left.\left[B_{(h+1)m+1} B_{(h+1)m+2} \cdots B_{(h+1)m+q}\right]\right]. \end{aligned}$$

Through elementary transformation for $\widehat{\mathscr{Q}}$, we can obtain

$$\hat{\mathcal{Q}} \to \left[\left[\begin{bmatrix} (\eta_1, B_{(h+1)m+1}) & & & \\ & (\eta_2, B_{(h+1)m+1}) & & \\ & & \ddots & \\ & & & (\eta_{(h+1)m}, B_{(h+1)m+1}) \end{bmatrix} M \vdots \cdots \vdots \begin{bmatrix} (\eta_1, B_{(h+1)m+q}) & & & \\ & (\eta_2, B_{(h+1)m+q}) & & \\ & & \ddots & \\ & & & (\eta_{(h+1)m}, B_{(h+1)m+q}) \end{bmatrix} M\right]\right],$$

where $(\eta_i, B_{(h+1)m+j})$ is vector inner product $(i = 1, \cdots, (h+1)m, j = 1, \cdots, q)$, M is transpose of Vandermonde matrix

$$M = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{(h+1)m-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{(h+1)m-1} \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & \lambda_{(h+1)m} & \lambda_{(h+1)m}^2 & \cdots & \lambda_{(h+1)m}^{(h+1)m-1} \end{bmatrix}.$$

Since $\lambda_1, \lambda_2, \cdots, \lambda_{(h+1)m}$ are all distinct, then M is nonsingular. So $\hat{\mathcal{Q}}$ has full row rank if $\eta_i (i = 1, \cdots, (h+1)m)$ are not orthogonal to $B_{(h+1)m+j} (j = 1, \cdots, q)$. Therefore, system (3) is completely controllable. The proof is completed. □

Remark 2. Note that Theorem 1 provides a sufficient and necessary condition, that is, the complete controllability of system (3) is depended on whether matrix $Y = \left[A + \left(\lambda^h - \lambda^{h+1}\right)E, B\right]$ has full row rank, which is not specifically required for the eigenvalues of matrix $\mathscr{A}$. But Theorem 2 only gives a sufficient condition needing that the eigenvalues of matrix $\mathscr{A}$ are all distinct, which is a relatively intuitive and easy condition to judge.

From Theorem 2, a corollary can be got immediately.

Corollary 6. *System (3) attains complete controllability if system (3) satisfies*

(1) *The eigenvalues of $\mathscr{A}$ are all distinct; and*
(2) *Each row of $U^{-1}\mathscr{B}$ has at least one nonzero element, where U is made up of the eigenvectors of $\mathscr{A}$.*

Proof. Inspired by Proposition 2, for system (7), we have

$$\left[\lambda_i E_{(h+1)m} - D, U^{-1}\mathscr{B}\right] = \begin{bmatrix} \lambda_i - \lambda_1 & 0 & \cdots & 0 & (\eta_1, P_{(h+1)m+1}) & (\eta_1, P_{(h+1)m+2}) & \cdots & (\eta_1, P_{(h+1)m+q}) \\ 0 & \lambda_i - \lambda_2 & \cdots & 0 & (\eta_2, P_{(h+1)m+1}) & (\eta_2, P_{(h+1)m+2}) & \cdots & (\eta_2, P_{(h+1)m+q}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i - \lambda_{(h+1)m} & \left(\eta_{(h+1)m}, P_{(h+1)m+1}\right) & \left(\eta_{(h+1)m}, P_{(h+1)m+2}\right) & \cdots & \left(\eta_{(h+1)m}, P_{(h+1)m+q}\right) \end{bmatrix}.$$

Because the eigenvalues of $\mathscr{A}$ are all distinct, $\left[\lambda_i E_{(h+1)m} - D, U^{-1}\mathscr{B}\right]$ has row full rank if each row of $U^{-1}\mathscr{B}$ has at least one nonzero element. Thus system (7) is completely controllable, and system (3) is completely controllable. □

By observing the relationship between system $(\mathscr{A}, \mathscr{B})$ and system (A, B), an important result is obtained as follows.

Theorem 3. *For system (3), complete controllability of system $(\mathscr{A}, \mathscr{B})$ is equivalent to that of system (A, B).*

Proof. Let λ be the eigenvalue of $\mathscr{A}$, which corresponds to the left eigenvector denoted as $\beta^T = \left[\beta_1^T \cdots \beta_{h+1}^T\right]^T$ with $\beta_i^T \in \mathbb{R}^m$ for $i \in \{1, \cdots, h+1\}$, then

$$\left[\beta_1^T \beta_2^T \cdots \beta_{h+1}^T\right]\mathscr{A} = \lambda\left[\beta_1^T \beta_2^T \cdots \beta_{h+1}^T\right],$$

and then

$$\begin{cases} \beta_1^T + \beta_2^T & = \lambda\beta_1^T \\ \beta_3^T & = \lambda\beta_2^T \\ & \vdots \\ \beta_{h+1}^T & = \lambda\beta_h^T \\ \beta_1^T A & = \lambda\beta_{h+1}^T \end{cases},$$

which can be deduced the following formula

$$\lambda\beta_1^T A + \lambda^{h+1}\beta_1^T = \lambda^{h+2}\beta_1^T,$$

that is

$$\beta_1^T A = \left(\lambda^{h+1} - \lambda^h\right)\beta_1^T \triangleq \nu\beta_1^T,$$

which implies that $\nu = \lambda^{h+1} - \lambda^h$ is the eigenvalue of A, which corresponds to the left eigenvector denoted as β_1^T. From the above formula, it is obvious to find that system $(\mathscr{A}, \mathscr{B})$ attains complete controllability if system (A, B) attains complete controllability. More details of similar proof can be seen in [46], here omitted. □

Remark 3. From Theorem 3, it is easy to find that the complete controllability of such MAS only depends on the information flows between followers and information flows from leaders to followers, but has nothing to do with the system's dimension and time-delay. In other words, the information topology structure of such system can characterize the complete controllability of time-delayed MASs. Therefore, it is simpler and more convenient to judge the controllability and also plays a key role in modeling and protocol design for MASs.

Furthermore, the complete controllability of discrete-time MASs is mainly studied from the perspective of algebra. A parallel research method is from graphical perspectives, which depicts the graph-theoretic features of making MASs controllable for structural controllability. Next, the controllability of such MAS from the perspective of graph theory–structural controllability will be investigated.

3.2. Structural controllability

Definition 2. [47] System (1) is structurally controllable if system (3) is completely controllable by selecting a set of a_{ij}, b_{ip}.

It is noted from Theorem 3 that for system (3), system $(\mathscr{A},\mathscr{B})$ attains complete controllability iff system (A,B) attains complete controllability, so we only need to analyze the structural controllability of (A,B) for system (3).

Lemma 2. [48] *The following three presentations are equivalent:*
(1) *Matrix pair* $[A,B]$ *is structurally controllable;*
(2) *Matrix pair* $[A,B]$ *is irreducible and g-rank*$([A,B]) = m$;
(3) *There exists a cacti which spans* $\mathscr{G}(A,B)$.

Theorem 4. *System (3) attains structural controllability iff the information topology graph* $\mathscr{G}$ *is connected.*

Proof. Necessity: By contradiction, suppose that the information topology graph $\mathscr{G}$ is disconnected, there are two possible cases: one case is at least one leader is isolated, that is, $b_{ip} = 0$, then there is at least one row vector of B be, and then system (3) is uncontrollable regardless of connectivity among the other agents. Another case is that at least one follower is isolated, then $[A,B]$ is reducible, which means system (3) is uncontrollable. The necessity is completed.

Sufficiency: Suppose that $[A,B]$ is reducible, from Definition 1, there must be a permutation matrix Q such that

$$QAQ^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad QB = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}.$$

For A_{11}, it is easy to see that there is no any information from leaders to followers, which implies that graph $\mathscr{G}$ is disconnected. The sufficiency is completed.

From Lemma 1 and Theorem 4, the following results can be got immediately.

Corollary 7. *System (3) attains structural controllability if the information topology graph* $\mathscr{G}$ *is strongly connected.*

Corollary 8. *System (3) attains structural controllability if the information topology graph* $\mathscr{G}$ *can be spanned by a cacti.*

Corollary 9. *System (3) is uncontrollable if the information topology graph* $\mathscr{G}$ *contains isolated agents.*

From Theorem 2 in [24], we can find the fact that even if the information communication topology graph $\mathscr{G}$ is connected, the MAS is not completely controllable if the leader sends the same communication to each follower (that is, $b_{ip} = b$). Notice that the structural controllability only depends on the connectivity of $\mathscr{G}$ regardless of the size of the weights. However, the complete controllability not only depends on the connectivity of $\mathscr{G}$, but also the size of the weights. Complete controllability aims to a fixed system, whose corresponding topology graphs of the system are all kinds, so the controllable conditions are very strong, while structural controllability aims to a kind of systems, so the requirement the structurally controllable conditions is weaker. Therefore, the MAS must be structural controllable if it is complete controllable. In addition, the complete controllability of the MAS is studied from the angle of algebra, but the structural controllability is described from the topological graph structure. Therefore, it is hard to establish the sufficient and necessary conditions between algebra criterion and graph structure.

4. Controllability analysis of discrete-time MASs with multiple time-delays

In real life, different time-delays are general in networked systems. Although the situation is more complex, we can have similar results corresponding to the case of a single time delay.

4.1. Complete controllability

Some main results for controllability of MASs with multiple time-delays are given as follows.

Proposition 3. *System (6) is completely controllable iff controllability matrix* $\tilde{\mathscr{Q}}$ *has full row rank, where*

$$\tilde{\mathscr{Q}} = \left[\mathscr{B}\ \tilde{\mathscr{A}}\mathscr{B}\ \tilde{\mathscr{A}}^2\mathscr{B}\cdots\tilde{\mathscr{A}}^{(h_{max}+1)m-1}\mathscr{B}\right].$$

Proposition 4. *System (6) is completely controllable iff system (6) satisfies*

(1) $rank\left(sE - \tilde{\mathscr{A}}, \mathscr{B}\right) = (h_{max}+1)m, \forall s \in \mathbb{C}$; *or*
(2) $rank\left(\lambda_i E - \tilde{\mathscr{A}}, \mathscr{B}\right) = (h_{max}+1)m$, *where* λ_i *is the eigenvalue of matrix* $\tilde{\mathscr{A}}, \forall i = 1, \cdots, (h_{max}+1)m$.

In order to make the calculation simpler and more convenient, the main results of system (6) are obtained by Propositions 3,4.

Theorem 5. *System (6) attains complete controllability iff matrix* $\tilde{Y} = \left[\sum_{i=1}^{h_{max}} \lambda^{h_{max}-i}A_i + \left(\lambda^{h_{max}} - \lambda^{h_{max}+1}\right)E, B\right]$ *has full row rank at every root of* $det\left(\left[\sum_{i=1}^{h_{max}} \lambda^{h_{max}-i}A_i + \left(\lambda^{h_{max}} - \lambda^{h_{max}+1}\right)E\right]\right) = 0$.

Proof. From Proposition 4, system (6) attains complete controllability iff matrix

$$\left(\tilde{\mathscr{A}} - \lambda E, \mathscr{B}\right) = \begin{bmatrix} E-\lambda E & A_1 & \cdots & A_{h_{max}-1} & A_{h_{max}} & B \\ E & -\lambda E & \cdots & \cdots & 0 & 0 \\ 0 & E & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & E & -\lambda E & 0 \end{bmatrix},$$

$$\rightarrow \begin{bmatrix} E & 0 & \cdots & 0 & 0 \\ 0 & E & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sum_{i=1}^{h_{max}} \lambda^{h_{max}-i}A_i + \left(\lambda^{h_{max}} - \lambda^{h_{max}+1}\right)E & B \end{bmatrix}$$

has full row rank. Through elementary transformation of matrix, if $\tilde{Y} = \left[\sum_{i=1}^{h_{max}} \lambda^{h_{max}-i}A_i + \left(\lambda^{h_{max}} - \lambda^{h_{max}+1}\right)E, B\right]$ has full row rank, then $\left[\tilde{\mathscr{A}} - \lambda E, \mathscr{B}\right]$ has full row rank. The following similar proof can be found from Theorem 4 in [21], here omitted. □

We can have similar parallel conclusions for multiple time-delays.

Theorem 6. *System (6) attains complete controllability if*

(1) *The eigenvalues of $\widetilde{\mathscr{A}}$ are all distinct; and*

(2) *All the row vectors of $\widetilde{U}^{-1}$ are not orthogonal to at least one column of $\mathscr{B}$ at the same time, where $\widetilde{U}^{-1}$ is made up of the eigenvectors of $\widetilde{\mathscr{A}}$.*

Corollary 10. *System (6) attains complete controllability if*

(1) *The eigenvalues of $\widetilde{\mathscr{A}}$ are all distinct; and*

(2) *Each row of $\widetilde{U}^{-1}\mathscr{B}$ has at least one nonzero element, where $\widetilde{U}^{-1}$ is made up of the eigenvectors of $\widetilde{\mathscr{A}}$.*

Specially, we find that the relationship of controllability between matrix pair $\left[\widetilde{\mathscr{A}},\mathscr{B}\right]$ and matrix pairs $[A_i, B]$ for $i\in\{1,2,\cdots,h_{max}+1\}$.

Theorem 7. *For system (6), complete controllability of system $(\mathscr{A},\mathscr{B})$ is equivalent to that of system $\left(\sum_{i=1}^{h_{max}}\tilde{\lambda}^{h_{max}-i}A_i, B\right)$, where $\tilde{\lambda}$ is an eigenvalue of $\widetilde{\mathscr{A}}$.*

Proof. Similar to the proof of Theorem 3, suppose that $\tilde{\lambda}$ is an eigenvalue of $\widetilde{\mathscr{A}}$, whose corresponding left eigenvector is denoted as $\beta^T=\left[\beta_1^T\beta_2^T\cdots\beta_{h_{max}+1}^T\right]$ with $\beta_i^T\in\mathbb{R}^m$ for $i\in\{1,2,\cdots,h_{max}+1\}$, then

$$\left[\beta_1^T\beta_2^T\cdots\beta_{h_{max}+1}^T\right]\widetilde{\mathscr{A}}=\tilde{\lambda}\left[\beta_1^T\beta_2^T\cdots\beta_{h_{max}+1}^T\right],$$

and then

$$\begin{cases}\beta_1^T+\beta_2^T & =\tilde{\lambda}\,\beta_1^T\\ \beta_1^TA_1+\beta_3^T & =\tilde{\lambda}\,\beta_2^T\\ \quad\vdots & \\ \beta_1^TA_{h_{max}-1}+\beta_{h_{max}+1}^T & =\tilde{\lambda}\,\beta^T{}_{h_{max}}\\ \beta_1^TA_{h_{max}} & =\tilde{\lambda}\,\beta_{h_{max}+1}^T\end{cases},$$

which can be deduced the following formula

$$\beta_1^TA_{h_{max}}=\tilde{\lambda}^{h_{max}+1}\beta_1^T-\tilde{\lambda}^{h_{max}}\beta_1^T-\tilde{\lambda}^{h_{max}-1}\beta_1^TA_1-\cdots-\tilde{\lambda}\beta_1^TA_{h_{max}-1},$$

i.e.,

$$\beta_1^T\left(\sum_{i=1}^{h_{max}}\tilde{\lambda}^{h_{max}-i}A_i\right)=\left(\tilde{\lambda}^{h_{max}+1}-\tilde{\lambda}^{h_{max}}\right)\beta_1^T,$$

which implies that $\left(\tilde{\lambda}^{h_{max}+1}-\tilde{\lambda}^{h_{max}}\right)$ is an eigenvalue of $\sum_{i=1}^{h_{max}}\tilde{\lambda}^{h_{max}-i}A_i$, whose corresponding left eigenvector is β_1^T. From the above formula, we can know that system $\left(\widetilde{\mathscr{A}},\mathscr{B}\right)$ is completely controllable iff system $\left(\sum_{i=1}^{h_{max}}\tilde{\lambda}^{h_{max}-i}A_i, B\right)$ is completely controllable. More details of similar proof can be seen in [46], here omitted. □

4.2. Structural controllability

Inspired by the case of a single time delay, we can have the following results for multiple time-delays.

Lemma 3. *If $A_1, A_2, \ldots, A_m$ are all the same order structure matrices and irreducible, then $\sum_{i=1}^m c_iA_i$ is irreducible, where c_i is a constant.*

Proof. For given matrix $A\in\mathbb{R}^{m\times m}$, A is irreducible iff $\mathscr{G}=(\mathscr{V},\mathscr{E})$ corresponding to A, denoted as $\mathscr{G}(A)$, is a SCC. If $\mathscr{G}(A_i)$ is iSCC, then $\mathscr{G}(A_1+A_2+\cdots+A_m)=(\mathscr{V},\mathscr{E})$ is a SCC, where $\mathscr{G}(A_i)=(\mathscr{V}_i,\mathscr{E}_i)$ with $\mathscr{V}=\mathscr{V}_1=\cdots=\mathscr{V}_m$ and $\mathscr{E}=\mathscr{E}_1\cup\cdots\cup\mathscr{E}_m$ for $i=1,\ldots,m$. Therefore, $\sum_{i=1}^m c_iA_i$ is irreducible. □

Combining Lemma 2, Definition 3 and Theorem 7, we can have the following results immediately.

Theorem 8. *Matrix pair $\left[\sum_{i=1}^{h_{max}}\tilde{\lambda}^{h_{max}-i}A_i, B\right]$ attains structural controllability iff the information topology graph $\mathscr{G}$ is connected.*

Theorem 9. *System (6) attains structural controllability iff the information topology graph $\mathscr{G}$ is connected.*

At present, to the best of our knowledge, the comparison of the complete controllability and the structural controllability of MASs has not been studied yet. This is first time to study the problem. First, the complete controllability of discrete-time MASs is mainly studied from the perspective of algebra, while the research method for the structural controllability is mainly from graphical perspectives. Second, for system (3), the structural controllability can be realized from the complete controllability by selecting a set of a_{ij}, b_{ip}. Third, for system (3), the complete controllability and the structural controllability are respectively depended on system matrix pairs $(\mathscr{A},\mathscr{B})$ and (A, B) (or $\left(\widetilde{\mathscr{A}},\mathscr{B}\right)$ and (A, B) for multiple time-delays). The main contribution of this work lies on: the equivalent conditions from algebraic and graphical perspectives to study the controllability are found, that is, a method to build a bridge for the controllability between algebra and graph theory is found. At the same time, the relationship between system matrix pairs $(\mathscr{A},\mathscr{B})$ and (A, B) is found by a series of equivalent algebraic transformations. However, the studies in the controllability of MASs is still in its infancy and presents new features and theoretical difficulties lacking more effective tools for theoretical analysis.

5. Examples and simulations

Example 1. This numerical example demonstrates the theoretical results of controllability for a connected network with a single time-delay. Consider an MAS (1) with followers 1–4 and leaders 5–6 described by Fig. 1. For the sake of simplicity, let time-delay $h=1$ and $a_{ij}=b_{ip}=1$ if there exist information flows between agents i.

The corresponding structural matrices of Fig. 1 can be given by

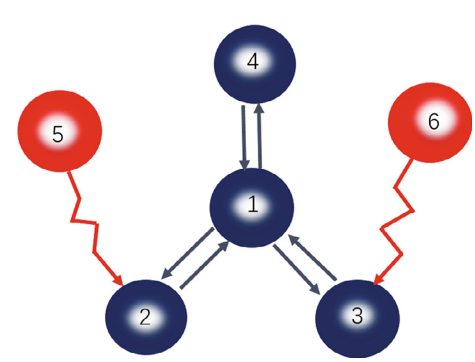


Fig. 1. Topology with multiple leaders.

$$A=\begin{bmatrix}0&1&1&1\\1&0&0&0\\1&0&0&0\\1&0&0&0\end{bmatrix},\ B=\begin{bmatrix}0&0\\1&0\\0&1\\0&0\end{bmatrix},$$

and then

$$\mathscr{A}=\begin{bmatrix}E&A\\E&0\end{bmatrix},\ \mathscr{B}=\begin{bmatrix}B\\0\end{bmatrix}.$$

Through a series of elementary transformation

$$(\mathscr{A}-\lambda E,\mathscr{B})=\begin{bmatrix}E-\lambda E&A&B\\E&-\lambda E&0\end{bmatrix}\rightarrow\begin{bmatrix}E&-\lambda E&0\\0&A+\lambda E-\lambda^2E&B\end{bmatrix},$$

we can know

$$A+\lambda(1-\lambda)E=\begin{bmatrix}\lambda(1-\lambda)&1&1&1\\1&\lambda(1-\lambda)&0&0\\1&0&\lambda(1-\lambda)&0\\1&0&0&\lambda(1-\lambda)\end{bmatrix}.$$

Computing $det([A+\lambda(1-\lambda)E])=0$, all roots are $\lambda_1=\lambda_2=0$, $\lambda_3=\lambda_4=1, \lambda_5=-0.9079, \lambda_6=1.9079, \lambda_7=0.500+1.2174i, \lambda_8=0.500-1.2174i$.

For $\lambda_1=\lambda_2=0$, then

$$Y_1=Y_2=[A,B]=\begin{bmatrix}0&1&1&1&0&0\\1&0&0&0&1&0\\1&0&0&0&0&1\\1&0&0&0&0&0\end{bmatrix},$$

and then $rank(Y_1)$=$rank(Y_2)$=4.

For $\lambda_3=\lambda_4=1$, then

$$Y_3=Y_4=[A,B]=\begin{bmatrix}0&1&1&1&0&0\\1&0&0&0&1&0\\1&0&0&0&0&1\\1&0&0&0&0&0\end{bmatrix},$$

and then $rank(Y_3)$=$rank(Y_4)$=4.

For $\lambda_5=-0.9079$, then

$$Y_5=\left[A+(-0.9079)E-(-0.9079)^2E,B\right]$$
$$=\begin{bmatrix}-1.7322&1&1&1&0&0\\1&-1.7322&0&0&1&0\\1&0&-1.7322&0&0&1\\1&0&0&-1.7322&0&0\end{bmatrix},$$

and then $rank(Y_5)=4$.

For $\lambda_6=1.9079$, then

$$Y_6=\left[A+(1.9079)E-(1.9079)^2E,B\right]$$
$$=\begin{bmatrix}-1.7322&1&1&1&0&0\\1&-1.7322&0&0&1&0\\1&0&-1.7322&0&0&1\\1&0&0&-1.7322&0&0\end{bmatrix},$$

and then $rank(Y_6)=4$.

For $\lambda_7=0.500+1.2174i$, then

$$Y_7=\left[A+(0.500+1.2174i)E-(0.500+1.2174i)^2E,B\right]$$
$$=\begin{bmatrix}1.7321&1&1&1&0&0\\1&1.7321&0&0&1&0\\1&0&1.7321&0&0&1\\1&0&0&1.7321&0&0\end{bmatrix},$$

and then $rank(Y_7)=4$.

For $\lambda_8=0.500-1.2174i$, then

$$Y_8=\left[A+(0.500-1.2174i)E-(0.500-1.2174i)^2E,B\right]$$
$$=\begin{bmatrix}1.7321&1&1&1&0&0\\1&1.7321&0&0&1&0\\1&0&1.7321&0&0&1\\1&0&0&1.7321&0&0\end{bmatrix},$$

and then $rank(Y_8)=4$.

It is obvious to obtain the conclusion that $Y=[A+\lambda E-\lambda^2E,B]$ has full row rank at every root of $det\left(\left[A+\lambda^h(1-\lambda)E\right]\right)=0$, then system (1) is completely controllable from Theorem 1.

Besides,

$$rank(A,B)=rank\begin{bmatrix}0&1&1&1&0&0\\1&0&0&0&1&0\\1&0&0&0&0&1\\1&0&0&0&0&0\end{bmatrix}=4,$$

then system (1) is also completely controllable from Theorem 3. From this example, we can see that Theorem 3 is simpler and more practical.

On the other hand, we can also know that system (1) is structurally controllable from Theorem 4. Therefore, such system is not only completely controllable, but also structurally controllable. Moreover, Remark 3 tells us that the complete controllability of such MAS has nothing to do with time-delay and only depends on the system structure, so it is more interesting to see the fact the same system can still be complete controllable with larger time-delay $h=10$ from the following simulations.

Figs. 2,3 are the followers trajectories in two-dimensional and three-dimensional spaces, respectively, where the black stars are random initial configuration and the circular dots are desired configuration. It is easy to see that all the followers can be controlled to the desired positions by adjusting the leaders.

Example 2. This example demonstrates the theoretical conclusions of controllability for a connected network with multiple time-delays. Consider an MAS (4) with followers 1–4 and leaders 5–6 described by Fig. 1. Let time-delays $h_{12}=h_{21}=1, h_{13}=h_{31}=2, h_{14}=h_{41}=3$, and $a_{12}=a_{21}=0.9, a_{13}=a_{31}=3.3, a_{14}=a_{41}=4.6$.

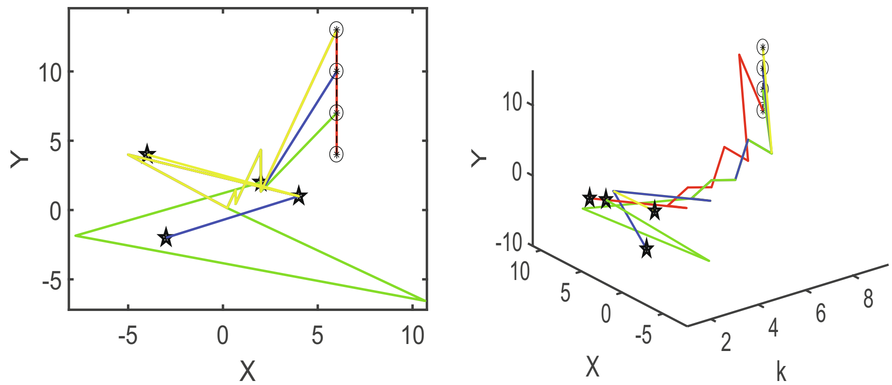
The corresponding structural matrices of Fig. 1 can be given by

$$A_1=\begin{bmatrix}0&0.9&0&0\\0.9&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\ A_2=\begin{bmatrix}0&0&3.3&0\\0&0&0&0\\3.3&0&0&0\\0&0&0&0\end{bmatrix},$$
$$A_3=\begin{bmatrix}0&0&0&4.6\\0&0&0&0\\0&0&0&0\\4.6&0&0&0\end{bmatrix},\ B=\begin{bmatrix}0&0\\1&0\\0&1\\0&0\end{bmatrix},$$

and then

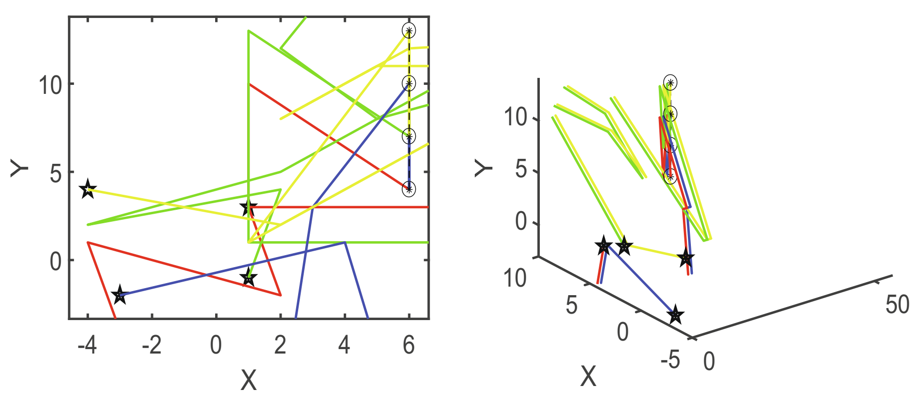
$$\tilde{\mathscr{A}}=\begin{bmatrix}E&A_1&A_2&A_3\\E&0&0&0\\0&E&0&0\\0&0&E&0\end{bmatrix},\ \mathscr{B}=\begin{bmatrix}B\\0\\0\\0\end{bmatrix}.$$

Through a series of elementary transformation



(a) 2-dimensional configuration. (b) 3-dimensional configuration.

Fig. 2. Configurations with a single time-delay $h = 1$.



(a) 2-dimensional configuration. (b) 3-dimensional configuration.

Fig. 3. Configurations with a single time-delay $h = 10$.

$$\left(\tilde{\mathscr{A}} - \lambda E, \mathscr{B}\right) = \begin{bmatrix} E-\lambda E & A_1 & A_2 & A_3 & B \\ E & -\lambda E & 0 & 0 & 0 \\ 0 & E & -\lambda E & 0 & 0 \\ 0 & 0 & E & -\lambda E & 0 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} E & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & \lambda^3 E + \lambda^2 A_1 + \lambda A_2 + A_3 - \lambda^4 E & B \end{bmatrix},$$

we can know

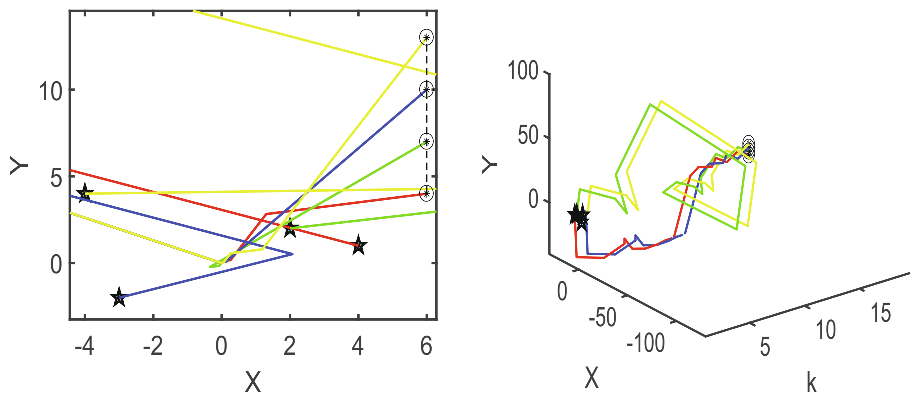
$$\lambda^3 E + \lambda^2 A_1 + \lambda A_2 + A_3 - \lambda^4 E$$
$$= \begin{bmatrix} (1-\lambda)\lambda^3 & 0.9\lambda^2 & 3.3\lambda & 4.6 \\ 0.9\lambda^2 & (1-\lambda)\lambda^3 & 0 & 0 \\ 3.3\lambda & 0 & (1-\lambda)\lambda^3 & 0 \\ 4.6 & 0 & 0 & (1-\lambda)\lambda^3 \end{bmatrix}.$$

Computing $det([\lambda^3 E + \lambda^2 A_1 + \lambda A_2 + A_3 - \lambda^4 E]) = 0$, we can obtain that all roots are $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0, \lambda_7 = \lambda_8 = 1, \lambda_9 = -1.4119, \lambda_{10} = 2.0481, \lambda_{11} = -0.6939 + 1.0915i, \lambda_{12} = -0.6939 - 1.0915i, \lambda_{13} = 0.1567 + 1.2193i, \lambda_{14} = 0.1567 - 1.2193i, \lambda_{15} = 1.2191 + 1.1868i, \lambda_{16} = 1.2191 - 1.1868i$. In the following, we bring each eigenvalue λ_i into the $Y_i = [\lambda_i^3 E + \lambda_i^2 A_1 + \lambda_i A_2 + A_3 - \lambda_i^4 E, B]$, and always obtain each matrix Y_i has full row rank; and then system (4) attains complete controllability based on Theorem 5. Similarly, according to Theorem 9, we can know that the connected network (4) is structurally controllable. Moreover, for multiple time-delays, the complete controllability of such MAS also has nothing to do with time-delay and only depends on the system structure, so it is also interesting to see the fact the same system can still be complete controllable with larger time-delay $h_{12} = h_{21} = 8, h_{13} = h_{31} = 9$ and $h_{14} = h_{41} = 10$ from the following simulations.

Figs. 4,5 are the followers trajectories in two-dimensional and three-dimensional spaces, respectively, where the black stars are random initial configuration and the circular dots are desired configuration. It is easy to see that all the followers can be controlled to the desired positions by adjusting the leaders. Obviously, the trajectories of all followers in Fig. 4 (or 5) with multiple time-delays are more complex than these of all followers in Fig. 2 (or 3) with a single time-delay, although they can all be controllable.

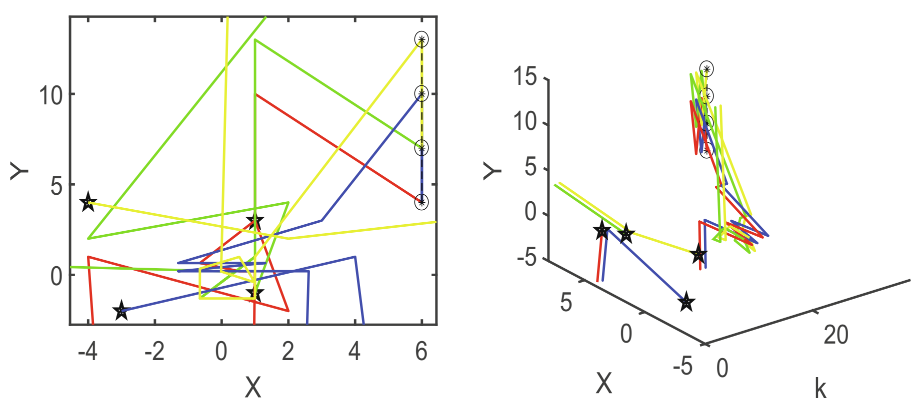
Example 3. This numerical example demonstrates the theoretical results of uncontrollability for a connected network with a single time-delay. Consider an MAS (1) with followers 1–4 and leader 5 described by Fig. 6. For the sake of simplicity, let time-delay $h = 1$ and $a_{ij} = b_{ip} = 1$.

The corresponding structural matrices of Fig. 6 can be given by



(a) 2-dimensional configuration. (b) 3-dimensional configuration.

Fig. 4. Configurations with multiple time-delays $h_{12} = h_{21} = 1, h_{13} = h_{31} = 2$ and $h_{14} = h_{41} = 3$.



(a) 2-dimensional configuration. (b) 3-dimensional configuration.

Fig. 5. Configurations with multiple time-delays $h_{12} = h_{21} = 8, h_{13} = h_{31} = 9$ and $h_{14} = h_{41} = 10$.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Obviously, $rank(A, B) = rank \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = 3 < 4$, then MAS (1) is completely uncontrollable from Theorem 3 (Since $rank(\mathscr{A}, \mathscr{B}) = 12 < 16$).

Fig. 7 denotes the followers trajectories, where the black dot stars are random initial configuration and the black asterisk stars are desired configuration, such as a straight-line configuration and a quadrilateral configuration, respectively. The circular dots are final actual configuration, which is obviously different from the desired configuration. It is easy to see that all the followers cannot be controlled to the desired configuration.

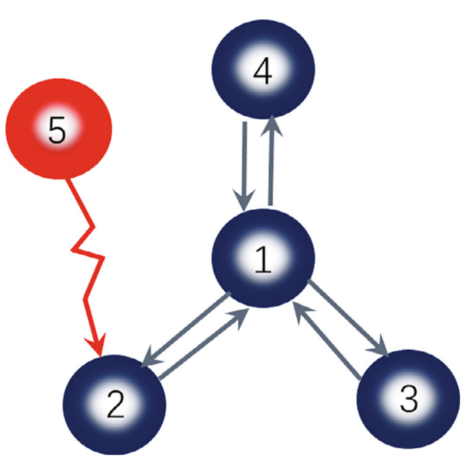
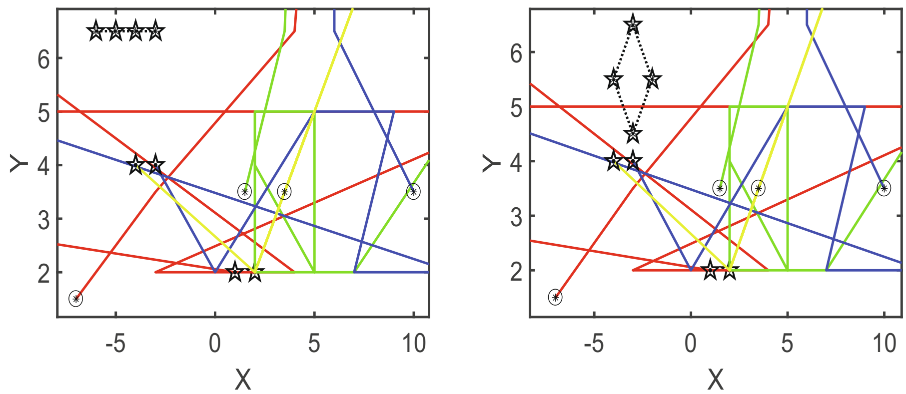


Fig. 6. Topology with a single leader.

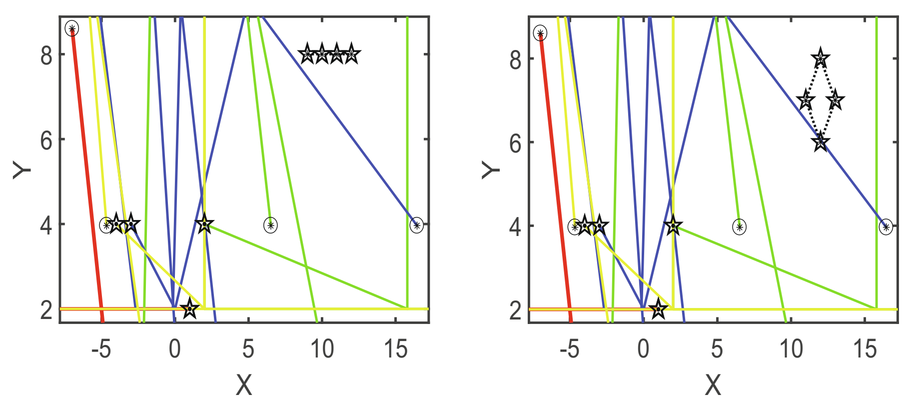
Example 4. This numerical example demonstrates the theoretical results of uncontrollability for a connected network with multiple time-delays. Consider an MAS (4) with followers 1–4 and leader 5 described by Fig. 6. For the sake of simplicity, let time-delay $h_{12} = h_{21} = 1, h_{13} = h_{31} = 2, h_{14} = h_{41} = 3$ and $a_{12} = a_{21} = 0.9$, $a_{13} = a_{31} = 3.3, a_{14} = a_{41} = 4.6$.



(a) A straight-line configuration.

(b) A quadrilateral configuration.

Fig. 7. Uncontrollable configurations with a single time-delay $h = 1$.



(a) A straight-line configuration.

(b) A quadrilateral configuration.

Fig. 8. Uncontrollable configurations with multiple time-delays $h_{12} = h_{21} = 1, h_{13} = h_{31} = 2, h_{14} = h_{41} = 3$.

The corresponding structural matrices of Fig. 6 can be given by

$$A_1 = \begin{bmatrix} 0 & 0.9 & 0 & 0 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 3.3 & 0 \\ 0 & 0 & 0 & 0 \\ 3.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 4.6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4.6 & 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

and then

$$\tilde{\mathscr{A}} = \begin{bmatrix} E & A_1 & A_2 & A_3 \\ E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & E & 0 \end{bmatrix}, \; \mathscr{B} = \begin{bmatrix} B \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Through a series of elementary transformation

$$\left(\tilde{\mathscr{A}} - \lambda E, \mathscr{B}\right) = \begin{bmatrix} E-\lambda E & A_1 & A_2 & A_3 & B \\ E & -\lambda E & 0 & 0 & 0 \\ 0 & E & -\lambda E & 0 & 0 \\ 0 & 0 & E & -\lambda E & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} E & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & \lambda^3 E + \lambda^2 A_1 + \lambda A_2 + A_3 - \lambda^4 E & B \end{bmatrix},$$

we can compute $det\left(\left[\lambda^3 E + \lambda^2 A_1 + \lambda A_2 + A_3 - \lambda^4\right]\right) = 0$, then all roots are $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0, \lambda_7 = \lambda_8 = 1, \lambda_9 = -0.6939 + 1.0915i, \lambda_{10} = -0.6939 - 1.0915i, \lambda_{11} = 0.1567 + 1.2193i, \lambda_{12} = 0.1567 - 1.2193i, \lambda_{13} = 1.2191 + 1.1868i, \lambda_{14} = 1.2191 - 1.1868i, \lambda_{15} = -1.4119, \lambda_{16} = 2.0481$. We put each eigenvalue λ_i into $Y_i = \left[\lambda^3 E + \lambda^2 A_1 + \lambda A_2 + A_3 - \lambda^4 E, B\right]$, we can find that $rank(Y_i)$ has no full row rank. Then MAS (4) with multiple time-delays is completely uncontrollable from Theorem 5.

Fig. 8 denotes the followers trajectories, where the black dot stars are random initial configuration and the black asterisk stars are desired configuration, such as a straight-line configuration and a quadrilateral configuration, respectively. The circular dots are final actual configuration, which is obviously different from the desired configuration. It is easy to see that all the followers cannot be controlled to the desired configuration.

6. Conclusion

In this paper, we have studied the complete controllability and structural controllability of discrete-time MASs under leader–follower framework with absolute protocol for a single time-delay and multiple time-delays, respectively. For a connected network, some conditions for complete controllability and structural controllability have been established. Moreover, the comparisons between complete controllability and structural controllability have been also given. Future research will consider the more complex cases with taking into account the communication scheduling protocols.

CRediT authorship contribution statement

Bo Liu: Conceptualization, Methodology, Investigation, Writing - review & editing. **Yaoyao Ping:** Software, Investigation. **Licheng Wu:** Investigation. **Housheng Su:** Investigation, Writing - original draft.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

This work was supported by the National Natural Science Foundation of China under Grant Nos. 61773023, 61991412, 61773416, and 61873318, the Program for HUST Academic Frontier Youth Team under Grant No. 2018QYTD07, and the Frontier Research Funds of Applied Foundation of Wuhan under Grant No. 2019010701011421.

The authors declare that there is no conflict of interest regarding the publication of this paper.

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