1(a) $p = l_1(x) f(x_1) + l_2(x) f(x_2) + l_3(x) f(x_3)$

where $l_1(x) = \dfrac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$

$l_2(x) = \dfrac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$

$l_3(x) = \dfrac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$

$$w_1 = \int_0^1 l_1(x)\,dx = \int_0^1 \frac{x^2 - (x_2+x_3)x + x_2x_3}{K_1}\,dx$$

$$= \frac{\frac{x^3}{3} - \frac{(x_2+x_3)}{2}x^2 + x_2x_3 \cdot x}{K}\Bigg|_0^1$$

$$= \frac{\frac{1}{3} - \frac{1}{2}\cdot\left(\frac{1}{2}+\frac{3}{4}\right) + \frac{1}{2}\cdot\frac{3}{4}}{\left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right)}$$

$$= \left(\frac{1}{3} - \frac{1}{4} - \frac{3}{8} + \frac{3}{8}\right)\cdot 8$$

$$= \frac{8}{3} - 2$$

$$\boxed{w_1 = \frac{2}{3}}$$

$$w_2 = \int_0^1 l_2(x)\,dx = \int_0^1 \frac{(x-x_1)(x-x_3)}{K_2}\,dx$$

$$= \frac{1}{K_2}\int_0^1 x^2 - (x_1+x_3)x + x_1x_3\,dx$$

$$= \frac{1}{K_2}\cdot\left(\frac{x^3}{3} - \frac{(x_1+x_3)}{2}x + x_1x_3x\right)\Bigg|_0^1$$

$$= -16\left(\frac{1}{3} - \frac{1}{2} + \frac{3}{16}\right)$$

16

$$-16\left(\frac{}{3} - \frac{}{2} + \frac{}{16}\right)$$

$$= -\frac{16}{3} + 8 - 3$$

$$= 5 - \frac{16}{3} = \boxed{-\frac{1}{3}}$$

From symmetry we know $w_3 = w_1 = \frac{2}{3}$

$$\boxed{\therefore [w_1 \quad w_2 \quad w_3] = \left[\frac{2}{3} \quad -\frac{1}{3} \quad \frac{2}{3}\right]}$$

b) Single Seg trapezoidal rule:

$$|I - Q_{trap}| = \left|\int_a^b f(x)\,dx - \frac{b-a}{2}(f(a) + f(b))\right| = \frac{1}{12} \max_{s \in [a,1]} |f''(s)|$$

$$\Rightarrow \left|\int_0^1 f(x)\,dx - \frac{1}{2}(f(0) + f(1))\right| = \frac{1}{12} \max_{s \in [0,1]} |f''(s)|$$

We need to find the point that maximizes the 2^{nd} derivative

assume $f''(s) = K < 0$, take derivative on both side

$$f(1) - f(0) - \frac{1}{2}(f'(0) + f'(1)) = 0 \qquad (1)$$

Since we know f is a 2^{nd} order function

Some characteristics of parabolas are: they are symmetric

if we have the 2 roots at 1 and 0

$f(1) = f(0) = 0$, $f'(0) = -f'(1)$

$\therefore f(1) - f(0) = 0$, $f'(0) + f'(1) = 0$, satisfies condition (1).

we can have

$$\boxed{f(x) = ax(1-x)}$$

To verify:

To verify: $a\int_0^1 (x - x^2)\,dx - \frac{1}{2}(0+0)$

$$= \frac{a}{2}x^2 - \frac{ax^3}{3}\Big|_0^1$$

$$= \frac{a}{6}$$

$$f''(x) = -2a$$

$$\frac{1}{12}\max_{x\in(0,1)} |f''(x)| = \frac{2a}{12} = \frac{a}{6} = \text{LHS}$$

2. a. $\tilde{x}_i = a + \frac{b-a}{2}(x_i+1)$

$$\tilde{w}_i = \frac{b-a}{2}w_i$$

```
function q = composite_gauss2(f, a, b, n)
    x1 = -1/sqrt(3);
    x2 = 1/sqrt(3);
    w = (b-a)/(2*n);
    x = linspace(a, b, n+1)';
    sums = zeros(n,1);
    for i = 1:n
        c = x(i);
        d = x(i+1);
        x1_new = c + (d-c)./2 .*(x1 + 1);
        x2_new = c + (d-c)./2 .*(x2 + 1);
        sums(i) = w .* f(x1_new) + f(x2_new) .* w;
    end
    q = sum(sums);
end

Not enough input arguments.

Error in composite_gauss2 (line 4)
    w = (b-a)/(2*n);
```

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b. The error converges very rapidly for every function below order $p = 2\times2-1 = 3$, the function is fit perfectly as expected (other than a blip in $p=2$ & 3 but I would assume it's due to machine precision)

For $p = 4$ & 5, the error converged rapidly. Initially the convergence rate is at $\boxed{O(h^{-4})}$.

$$Q_{G(2)} = \sum w_i f(x_i) = (b-a)\sum \tilde{w}_i f(\tilde{x}_i)$$

Recall in composite rule, error goes down by factor

Recall in composite rule, error goes down by factor of $p+1$, 1 more than the highest polynomial it can perfectly represent. The idea is similar here, except the upper bound is given by $2m-1+1=2m$. For $m=2$, $2m=4$.

	1	x	x^2	x^3	x^4	x^5
1	0	0	0	2.7756e-17	5.5556e-3	1.3889e-2
2	0	0	0	0	3.4722e-4	8.6806e-4
4	0	0	-5.5511e-17	0	2.1701e-5	5.4253e-5
8	0	0	0	0	1.3563e-6	3.3908e-6

c. functions with order $p = 2\times1-1 = 1$

it gets anything linear perfectly.

The error should converge by $\boxed{O(h^2)}$

3. See MATLAB Code

```
function [q,e,cnt] = adaptive_gauss_kronrod(f,a,b,tol)
    cnt = 0;
    param = importdata('gauss_kronrod.dat').data;
    xi = a + (b-a)/2 * (param(:, 1) + 1);
    wi = (b-a)/2 * param(:, 2:3);
    q_gauss = arrayfun(f, xi') * wi(:, 1);
    q = arrayfun(f, xi') * wi(:, 2);
    e = (200 * abs(q - q_gauss))^(3/2);
    if e > tol
        c = a + (b-a)/2;
        tol_new = tol/2;
        [q1, e1, cnt1] = adaptive_gauss_kronrod(f,a,c,tol_new);
        [q2, e2, cnt2] = adaptive_gauss_kronrod(f,c,b,tol_new);
        [q, e, cnt] = deal(q1+q2, e1+e2, cnt1+cnt2);
    end
    cnt = cnt+15;
end
```

AER336 Problem Set 2

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```
clear
```

Question 2b

```
points = [1 2 4 8]';

Q = zeros(4,6);
for i = 1:6
    f2 = @(p) composite_gauss2(@(x) x^(i-1), 0, 1, p);
    Q(:, i) = 1/i - arrayfun(f2, points);
end
disp(Q)
```

```
     0          0          0   2.7756e-17   5.5556e-03   1.3889e-02
     0          0          0            0   3.4722e-04   8.6806e-04
     0          0 -5.5511e-17            0   2.1701e-05   5.4253e-05
     0          0          0            0   1.3563e-06   3.3908e-06
```

Question 3b

```
format shorte
points = [1e-2 1e-5 1e-8]';
f = @(p)adaptive_gauss_kronrod(@(x)2 * sqrt(1 -x^2), -1, 1, p);
[q, e, cnt] = arrayfun(f, points);
ans_real = pi;
err = abs(q - ans_real);
disp('value (Q) Error (|I - Q|) Estimated Error(E) Function Interation');
```

```
value (Q) Error (|I - Q|) Estimated Error(E) Function Interation
```

```
disp([q err e cnt])
```

```
   3.1416e+00   3.3649e-06   8.8956e-04   2.2500e+02
   3.1416e+00   8.2146e-10   3.3910e-09   7.0500e+02
   3.1416e+00   1.9096e-13   1.3547e-14   1.1850e+03
```

Question 3c

choose the error bound to be 10^{-5}, since we need 4 significant digits.

```
f = @(x) ktgeom(x);
[v, err, count] = adaptive_gauss_kronrod(f, 0, 1, 1e-5);
v = v*2
```

```
v =
   7.6955e-02
```

```
err
```

```
err =
   1.7759e-07
```

```
count
```

```
count =
   195
```