Influence of Mesh Refinement on Aerodynamic Shape Optimization

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Problem Statement

Inverse Pressure Matching

Inverse Pressure Matching: Given the pressure/C_p distribution, inversely find the shape of the airfoil that produces this distribution.

We cast the problem to be minimizing the pressure mismatch. Specifically we wish to

$$\min_{\mu} \int_{\Gamma} (C_p(u(\mu)) - C_{p,true})^2 dx \tag{1}$$

Overview of solution

- C_p distribution is computed from a reference solution parametrized by Free-Form Deformation (FFD) cages.
- Using adjoint-based method to inversely find the geometry of the airfoil that produces this distribution.

Experimental Setup

Experimental Setup

Governing Equation: Euler's Equation. $M = 0.2$, $AoA = 5^\circ$.

CFD Solver: using aps solver and auxiliary libraries compiled by Aerospace Computational Engineering Lab.

Case: an inverse pressure matching case study prepared by Professor Yano and Ben Gibson using `C++`.

Computation: Using SciNet Niagara cluster.

Experimental Goals

- Explore error convergence rates in both uniformly and adaptively refined meshes.
- More specifically, investigate the relation between degrees of freedom in uniformly and adadptively refined mesh, and errors in C_p mismatch and FFD cage parameter (μ), and compare the difference in their convergence rates.
- Impacts of underlying polynomial degrees are also being investigated.

Terminology

Mesh Fineness: The scale of fineness of a uniform mesh compared to standard mesh generated with `aps` library.

Refinement Iterations: Number of iterations performed in adaptive mesh refinement process. The iteration is terminated when termination tolerance is reached.

Hyperparameter Updates

The experiment hyperparameters are updated for better performance since last meeting.

- Decreased target error tolerance in adaptive mesh refinement process from 1×10^{-3} to 1×10^{-8}.
- Maximum optimization iterations are increased from 50 to 100.

Experiment Procedures

- Set the basis polynomial degree to 2.
- A reference solution is computed using a very fine mesh (fineness 4, iteratively refined 8 times). This mesh has a total of 713,160 DOFs.
- The fineness of solution mesh is increased in each run, while performing no adaptive mesh refinement. The algorithm to increase mesh fineness has been improved, and each increment in fineness results in an increment of DOF by 4 times.
- The inverse pressure matching problem is solved using adjoint-based methods, while errors are being recorded per each iteration of solving.
- Repeat procedure 2 - 4 using basis polynomial with a degree of 1.

Results

Reference Mesh

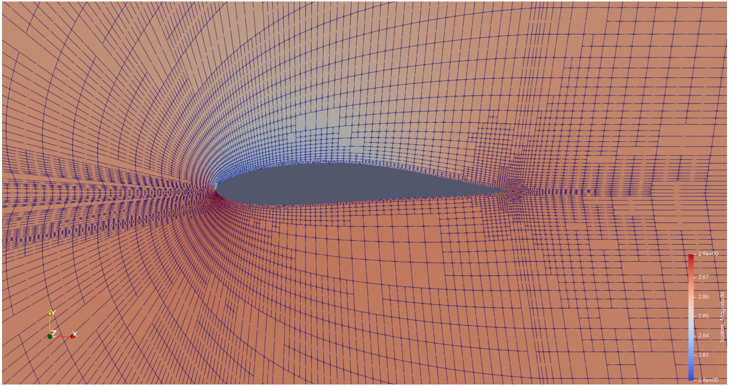

Figure: Reference Mesh, dof = 713160

Solution Meshes

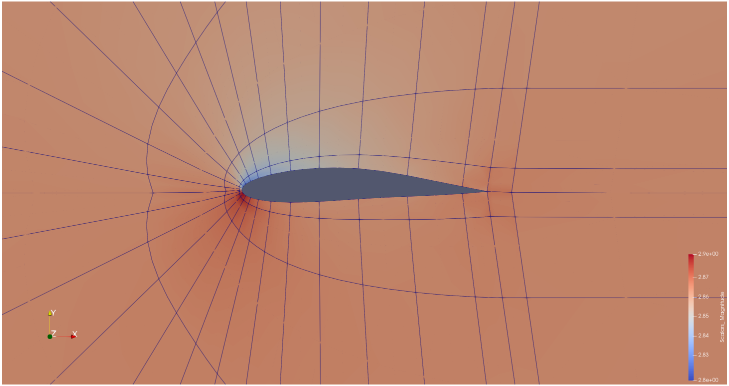

Figure: Mesh with refinement 1, dof = 4608

Solution Meshes

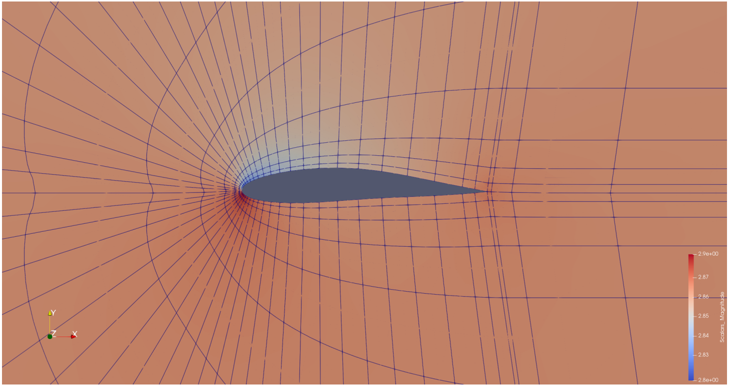

Figure: Mesh with refinement 2, dof = 18432

Solution Meshes

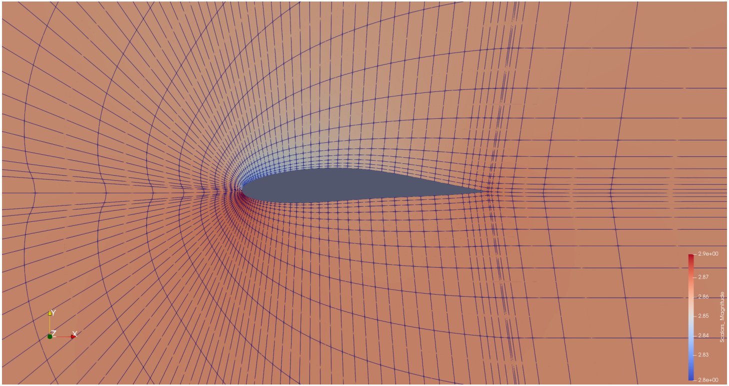

Figure: Mesh with refinement 3, dof = 73728

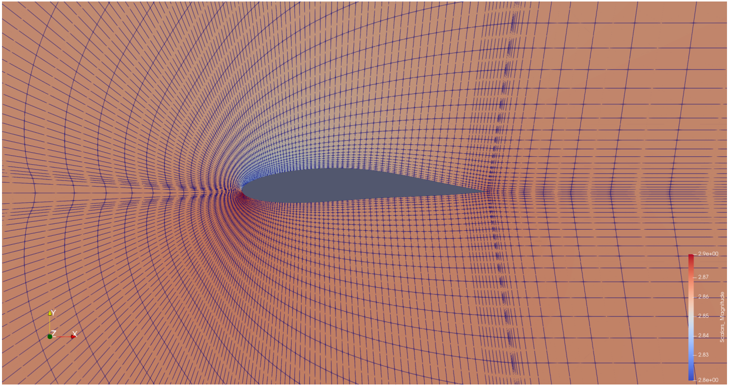

Figure: Mesh with refinement 4, dof = 294912

Experimental Outcome: p=2

h/h_0	DOF	C_p error	μ error
1	4608	7.91939×10^{-4}	$6.57186498 \times 10^{-3}$
1/2	18432	2.00889×10^{-4}	$6.48498527 \times 10^{-4}$
1/4	73728	7.65156×10^{-5}	$3.23398942 \times 10^{-4}$
1/8	294912	3.46435×10^{-5}	$1.40202012 \times 10^{-4}$

Table: Error in uniformly refined mesh using basis polynomials with p=2.

Initially, each time h value decrease by 2 times, the error drops by approximately 4 times (quadratic in h, linear in N). Later, each time the DOF decrease by 2 times, the error drops by factor of 2 (linear in h, square root in N).

Experimental Outcome: p=2

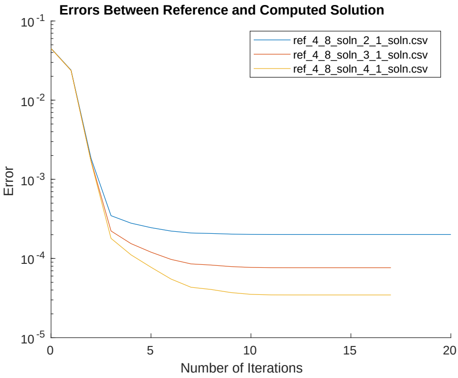

Figure: Pressure mismatch in uniform mesh during solving using basis polynomials with p=2.

Experimental Outcome: p=2

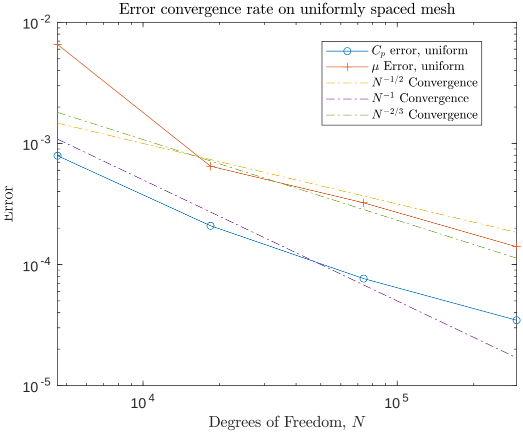

Figure: Final error in uniformly refined mesh using basis polynomials with p=2.

Experimental Outcome: p=1

h/h_0	DOF	C_p error	μ error
1	2304	1.78×10^{-2}	8.14×10^{-2}
1/2	9216	1.046×10^{-3}	1.05×10^{-2}
1/4	36864	1.86×10^{-4}	3.211×10^{-3}
1/8	147456	9.81×10^{-5}*	4.84×10^{-3}*

Table: Error in uniformly refined mesh using basis polynomials with p=1.

Error in parameter is observed to increase even though DOF increase by 4. Likely due to approximation error associated using p = 1 polynomials in both reference and inverse solutions.

Experimental Outcome: p=1

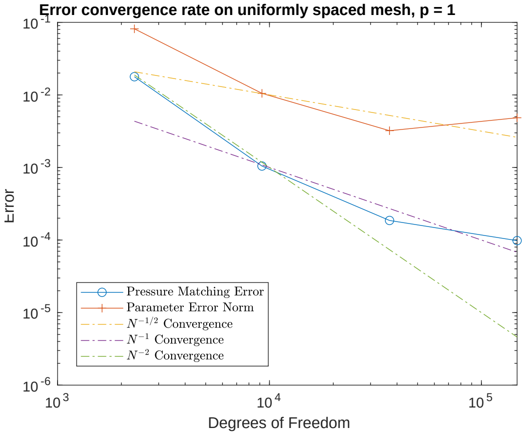

Figure: Final error in uniformly refined mesh using basis polynomials with p=1.

Experimental Outcome: Compared with AMR

Solutions are computed with AMR to show convergence rate difference between adaptively and uniformly refined meshes.

- Polynomials with order 2 are used as basis function. Regions with errors in the top 15% are refined each iteration.
- The initial mesh is the standard mesh generated from aps libraries, with mesh fineness of 1.
- Solve the inverse pressure matching problem with adaptive refinement with different refinement iterations. Error in both C_p and μ at the final step is acquired.
- Repeat last step using an initial mesh with fineness of 2.

Experimental Outcome: AMR with fineness 1

Case	DOF	C_p error	μ error
1	6696	$2.949\,77 \times 10^{-4}$	$2.326\,283\,14 \times 10^{-3}$
2	9792	$1.012\,02 \times 10^{-4}$	$1.601\,682\,55 \times 10^{-3}$
3	30312	1.0968×10^{-5}	$2.525\,043\,21 \times 10^{-4}$
4	63216	$5.792\,24 \times 10^{-6}$	$7.964\,994\,31 \times 10^{-5}$
5	277320	$5.249\,42 \times 10^{-6}$*	$3.146\,868\,83 \times 10^{-5}$

Table: Error in adaptively refined mesh, initial mesh refinement 1.

The error converges linearly in N, but ultimately stalls. The error possibly stems from geometry difference between the reference and the inverse solution.

Experimental Outcome: AMR with fineness 1

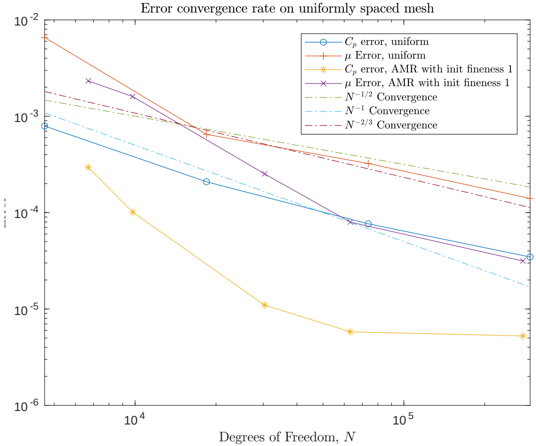

Figure: Error comparison with uniform and adaptively refined meshes with initial mesh refinement of 1

Experimental Outcome: AMR with fineness 2

Case	DOF	C_p error	μ error
1	27072	$9.314\,01 \times 10^{-5}$	$7.869\,154\,59 \times 10^{-4}$
2	39240	$3.884\,39 \times 10^{-5}$	$5.346\,351\,46 \times 10^{-4}$
3	57024	$1.877\,61 \times 10^{-5}$	$4.117\,972\,27 \times 10^{-4}$
4	120720	$5.165\,44 \times 10^{-6}$	$2.697\,024\,11 \times 10^{-5}$
5	243384	$2.947\,32 \times 10^{-6}$	$1.065\,679\,37 \times 10^{-5}$

Table: Error in adaptively refined mesh, initial mesh refinement 2.

The error roughly converges linearly in N, even though a singularity exists. We can observe the geometry mismatch error is less dominant due to our use of a finer mesh.

Experimental Outcome: AMR with fineness 2

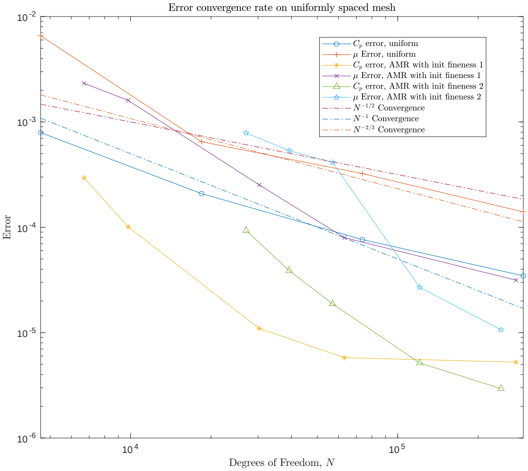

Figure: Error comparison with uniform and adaptively refined meshes with initial mesh refinement of 1 and 2

Conclusion

Conclusions

- Error in uniform meshes decreases slowly when a singularity exists. The mesh will be refined as resolution increase, yet no targeted refinements to regions of interest is applied, causing the error to still be large in those areas.
- Mesh adaptation decreases error rapidly compared to uniform meshes. When resolution increased in areas of interest, there could be sudden error drops. Desired method to use when solving CFD.
- Degree of polynomials influences solution accuracy noticeably. Error in solution can compound globally, resulting an increment in error even though DOF is increased.
- A true representation of the airfoil geometry that is independent of the mesh should be present.

Revisiting This Summer

Recapitulation

Variational Methods

find a function u such that it satisfies

$$a(u, v) = l(v) \quad \forall v \in \mathcal{V}$$

where $a(u, v)$ is the bilinear form, $l(v)$ is the linear form of the variational problem. $\mathcal{V}$ is the function space.

Finite Element Method

After discretizing our domain, we wish to find a piecewise continuous function $u_h \in \mathcal{V}_h$ that satisfies

$$a(u_h, v) = l(v) \quad \forall v \in \mathcal{V}_h$$

Where $\mathcal{V}_h$ is a space of piecewise linear functions defined over our triangulation of the domain.

Recapitulation

Adjoint-based methods: For an ODE/PDE constrained optimization problem, we wish to:

$$\text{Minimize } J(u) = q(u(\mu); \mu)$$
$$\text{Subject to } R(u(\mu); \mu) = 0$$

We can have the Jacobian to the functional to be:

$$\begin{aligned}\frac{\partial J}{\partial \mu} &= -\frac{\partial q}{\partial u}\left(\frac{\partial R}{\partial u}\right)^{-1}\frac{\partial R}{\partial \mu} + \frac{\partial q}{\partial \mu} \\ &= -\psi^{\mathsf{T}}\frac{\partial R}{\partial \mu} + \frac{\partial q}{\partial \mu}\end{aligned}$$

where ψ is the adjoint that satisfies

$$\left(\frac{\partial R}{\partial u}\right)^{\mathsf{T}}\psi = \left(\frac{\partial q}{\partial u}\right)^{\mathsf{T}}$$

Recapitulation

Solved 3 problems to understand the idea of finite element method, and adjoint-based optimization:

- 1-D linear reaction-diffusion system
- Inverse diffusivity matching on 1-D linear reaction-diffusion system
- 1-D non-linear reaction-diffusion system and its inverse diffusivity matching.

Recapitulation

Applied these methods to a more complex problem: inverse pressure matching to a 2-D airfoil. Investigations are made in the following fields:

- How the problem is formalized and solved.
- Impacts of the difference between reference and solution meshes to the result.
- Impacts of mesh fineness on solution accuracy and runtime.
- Impacts of adaptive mesh refinement processes.
- Error convergence rate in both uniformly and adaptively refined meshes.

Acknowledgement

Acknowledgement

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