Understanding quantum information and computation

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Lesson 4

Entanglement in action

Alice and Bob

- Alice and Bob are names given to hypothetical entities or agents in systems, protocols, and games that involves the exchange of information.
- They are assumed to be in different locations.
- The specific roles they play must be clarified in different situations.
- Additional characters (e.g., Charlie, Diane, Eve, and Mallory) may be introduced as needed.

Remarks on entanglement

In Lesson 2, we encountered this example of an *entangled state* of two qubits:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

We also encountered this example of a *probabilistic state* of two bits:

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle$$

It is typical in the study of quantum information and computation that we view entanglement as a *resource* that can be used to accomplish different tasks.

When we do this we view the state $|\phi^+\rangle$ as representing one unit of entanglement called an *e-bit.*

Terminology

To say that Alice and Bob *share an e-bit* means that Alice has a qubit A, Bob has a qubit B, and together the pair (A, B) is in the state $|\phi^+\rangle$.

Teleportation set-up

Scenario

Alice has a *qubit* Q that she wishes to transmit to Bob.

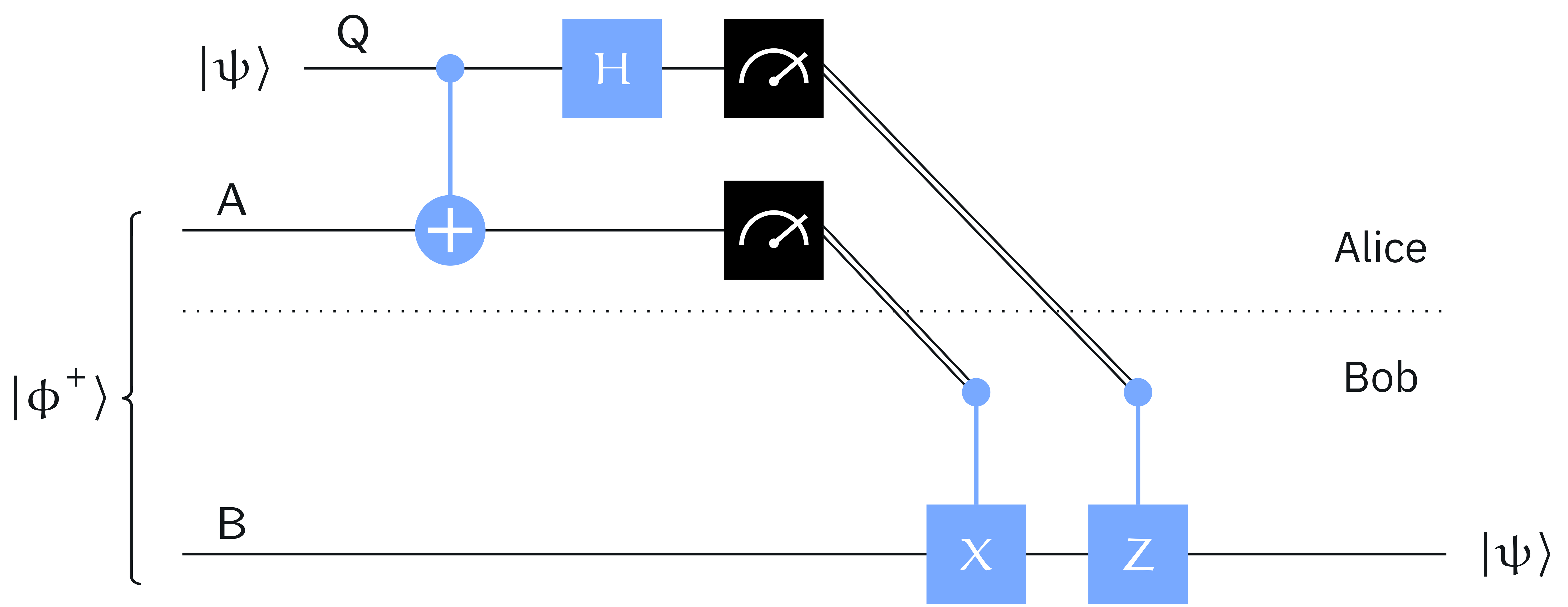
- Alice is unable to physically send Q to Bob — she is only able to send *classical information.*
- Alice and Bob *share an e-bit.*

Remarks

- The state of Q is "unknown" to both Alice and Bob.
- Correlations (including entanglement) between Q and other systems must be preserved by the transmission.
- The *no-cloning theorem* implies that if Bob receives the transmission, Alice must no longer have the qubit in its original state.

Teleportation protocol

Circuit description

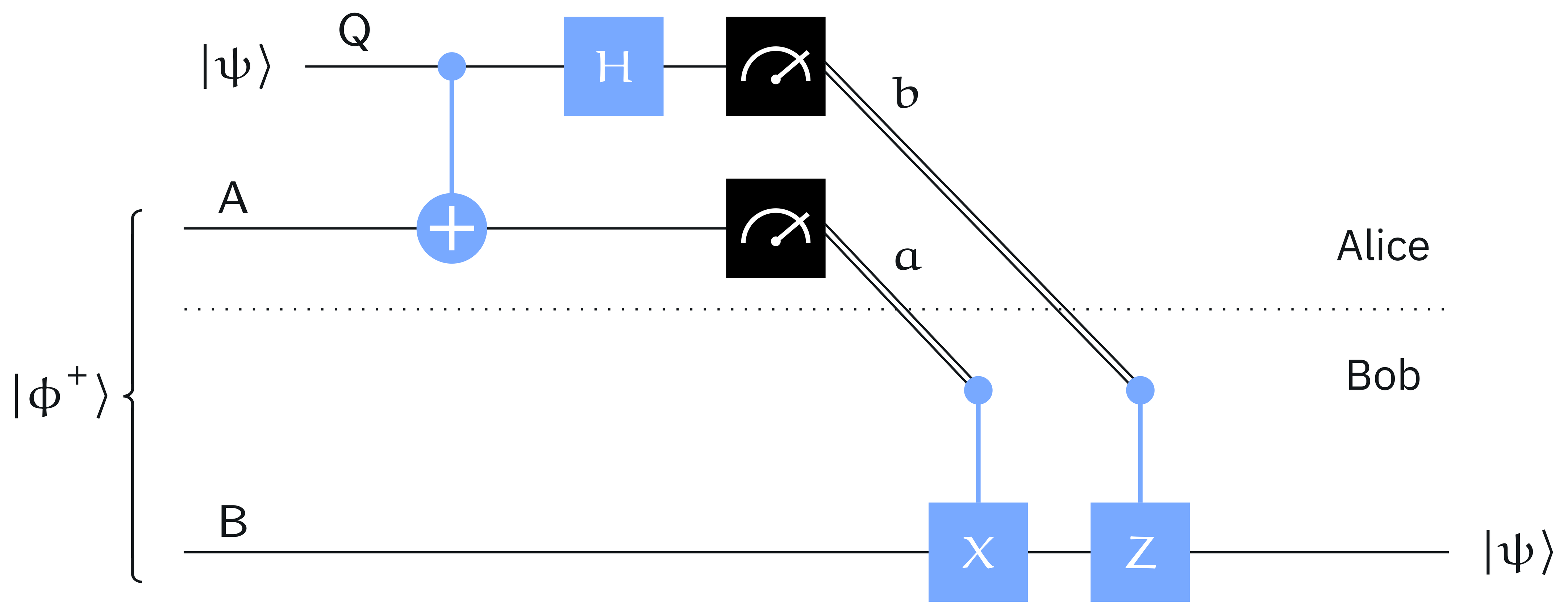


Initial conditions

Alice and Bob share one e-bit: Alice has a qubit A, Bob has a qubit B, and (A,B) is in the state $|\phi^+\rangle$.

Alice also has a qubit Q that she wishes to transmit to Bob.

Teleportation protocol



Operation performed by Bob

$\mathbb{1}$	if $ab = 00$
Z	if $ab = 01$
X	if $ab = 10$
ZX	if $ab = 11$

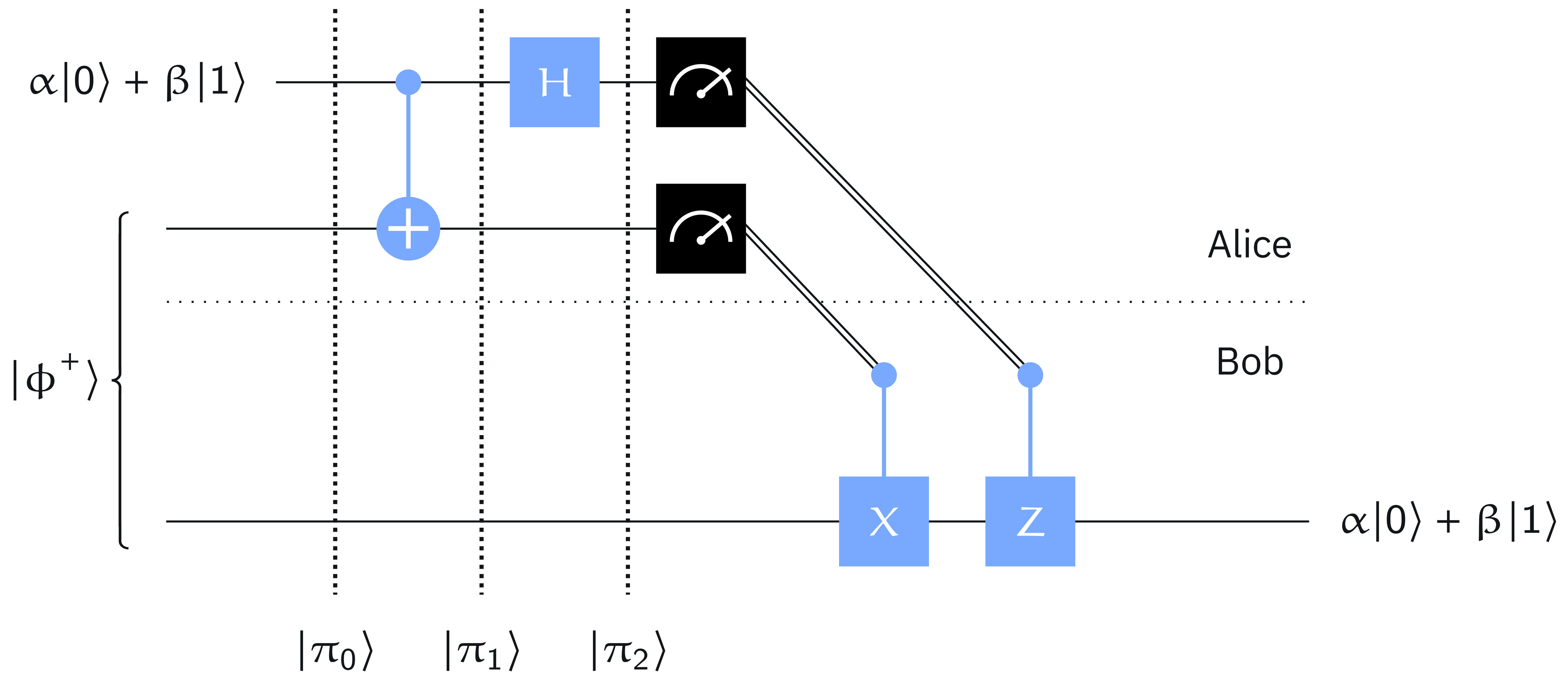
Protocol

1. Alice performs a controlled-NOT operation, where Q is the control and A is the target.
2. Alice performs a Hadamard operation on Q.
3. Alice measures A and Q, obtaining binary outcomes a and b, respectively.
4. Alice sends a and b to Bob.
5. Bob performs these two steps:

   5.1 If $a = 1$, then Bob applies an X operation to the qubit B.

   5.2 If $b = 1$, then Bob applies a Z operation to the qubit B.

Teleportation analysis

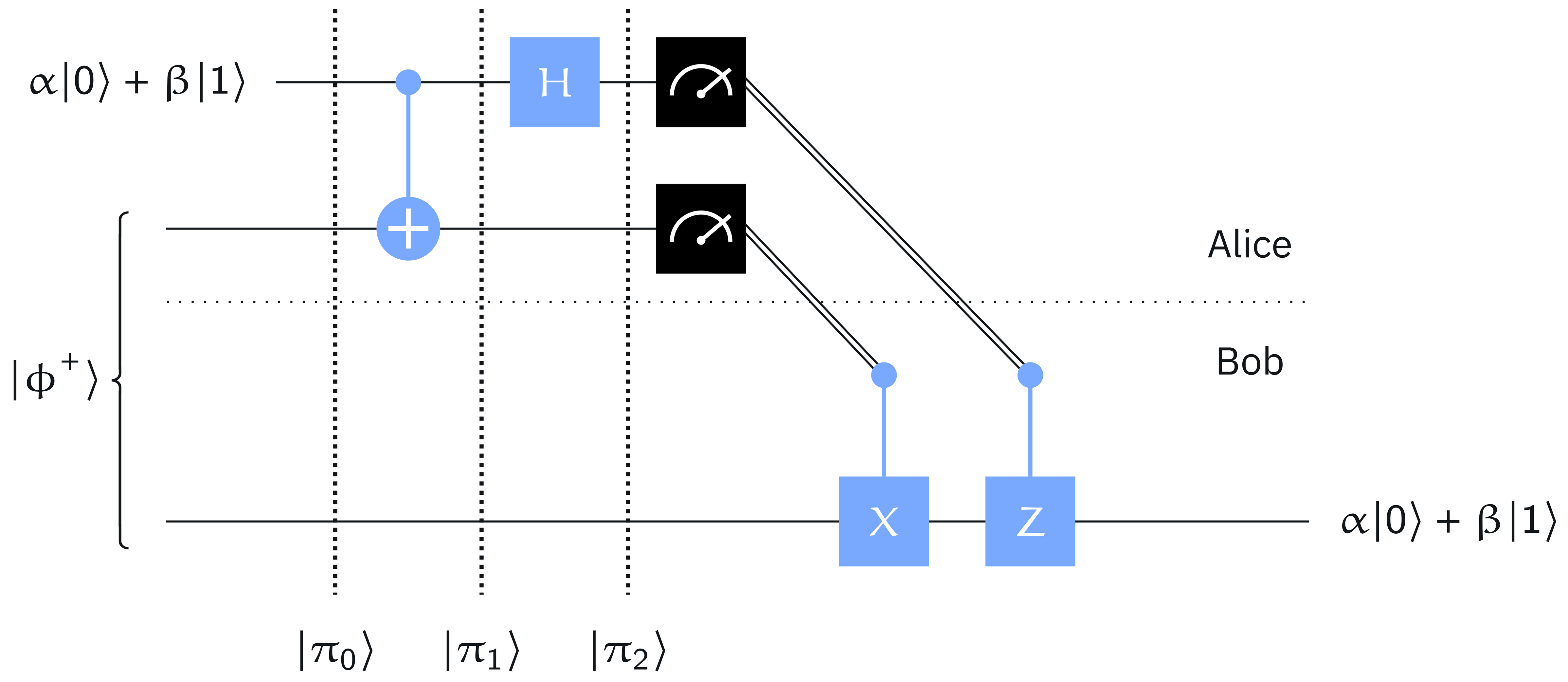


$$|\pi_0\rangle = \frac{\alpha|000\rangle + \alpha|110\rangle + \beta|001\rangle + \beta|111\rangle}{\sqrt{2}}$$

$$|\pi_1\rangle = \frac{\alpha|000\rangle + \alpha|110\rangle + \beta|011\rangle + \beta|101\rangle}{\sqrt{2}}$$

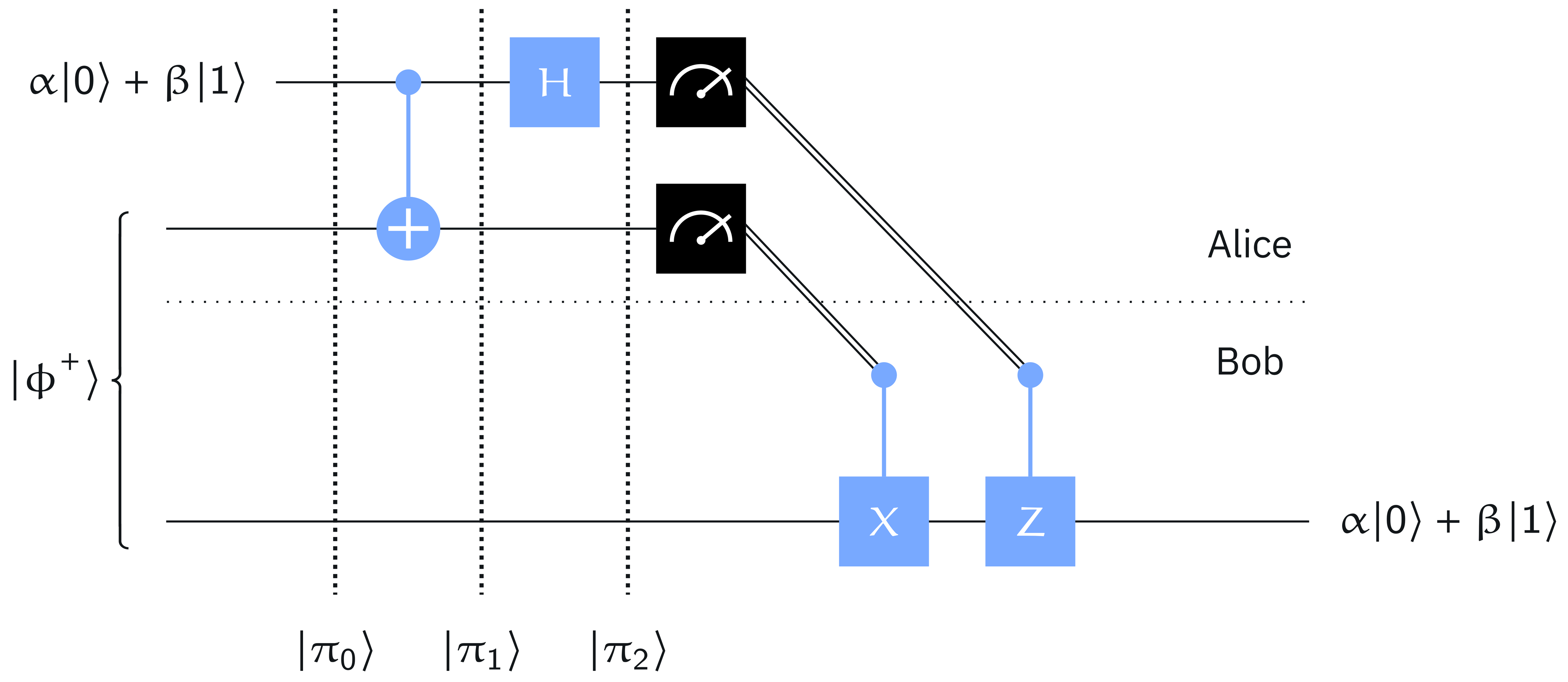
$$|\pi_2\rangle = \frac{\alpha|00\rangle|+\rangle + \alpha|11\rangle|+\rangle + \beta|01\rangle|-\rangle + \beta|10\rangle|-\rangle}{\sqrt{2}}$$

Teleportation analysis



$$
\begin{aligned}
|\pi_2\rangle &= \frac{\alpha|00\rangle|+\rangle + \alpha|11\rangle|+\rangle + \beta|01\rangle|-\rangle + \beta|10\rangle|-\rangle}{\sqrt{2}} \\
&= \frac{\alpha|00\rangle(|0\rangle + |1\rangle) + \alpha|11\rangle(|0\rangle + |1\rangle) + \beta|01\rangle(|0\rangle - |1\rangle) + \beta|10\rangle(|0\rangle - |1\rangle)}{2} \\
&= \frac{\alpha|000\rangle + \alpha|001\rangle + \alpha|110\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|011\rangle + \beta|100\rangle - \beta|101\rangle}{2} \\
&= \frac{1}{2}(\alpha|0\rangle + \beta|1\rangle)|00\rangle + \frac{1}{2}(\alpha|0\rangle - \beta|1\rangle)|01\rangle + \frac{1}{2}(\alpha|1\rangle + \beta|0\rangle)|10\rangle + \frac{1}{2}(\alpha|1\rangle - \beta|0\rangle)|11\rangle
\end{aligned}
$$

Teleportation analysis



$$|\pi_2\rangle = \frac{1}{2}(\alpha|0\rangle + \beta|1\rangle)|00\rangle + \frac{1}{2}(\alpha|0\rangle - \beta|1\rangle)|01\rangle + \frac{1}{2}(\alpha|1\rangle + \beta|0\rangle)|10\rangle + \frac{1}{2}(\alpha|1\rangle - \beta|0\rangle)|11\rangle$$

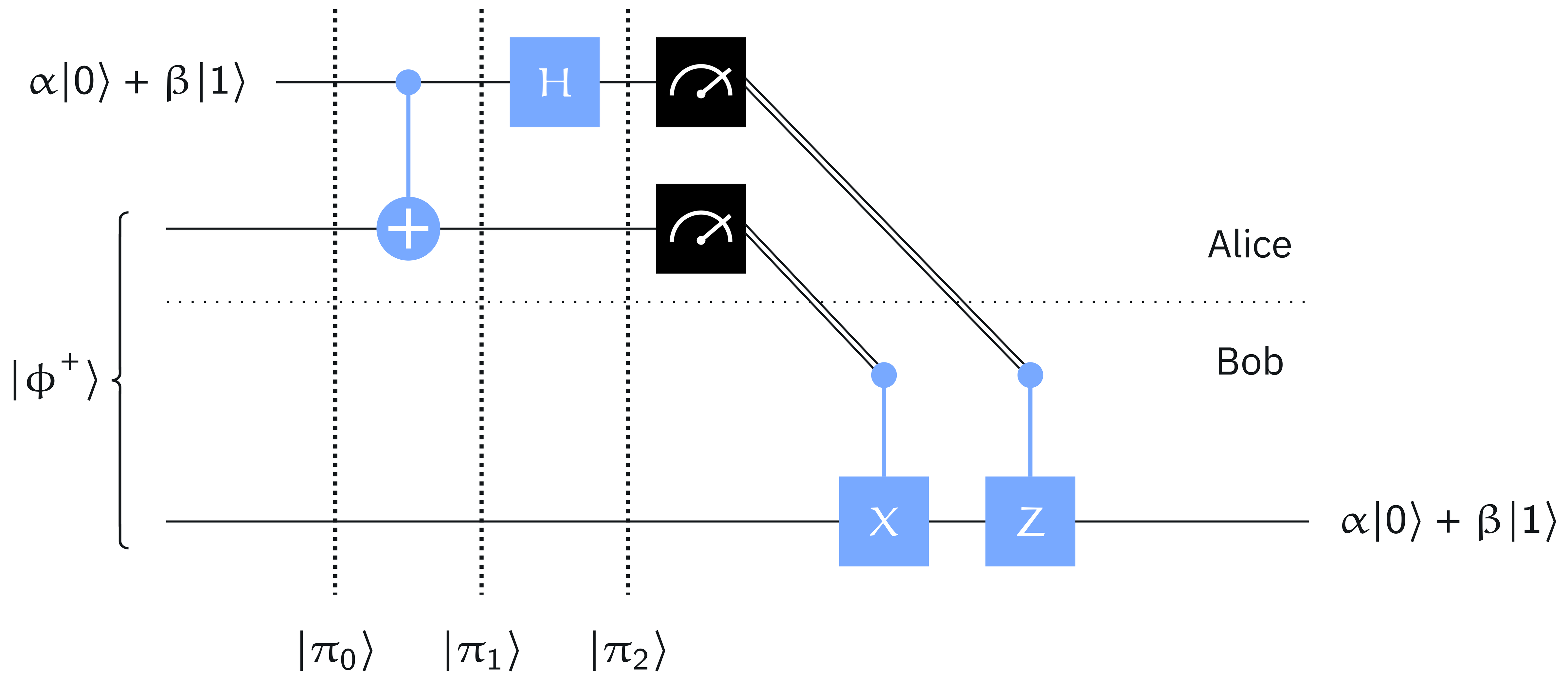
$$\Pr(ab = 00) = \frac{1}{4}\|\alpha|0\rangle + \beta|1\rangle\|^2 = \frac{1}{4}$$

$$\Pr(ab = 01) = \frac{1}{4}\|\alpha|0\rangle - \beta|1\rangle\|^2 = \frac{1}{4}$$

$$\Pr(ab = 10) = \frac{1}{4}\|\alpha|1\rangle + \beta|0\rangle\|^2 = \frac{1}{4}$$

$$\Pr(ab = 11) = \frac{1}{4}\|\alpha|1\rangle - \beta|0\rangle\|^2 = \frac{1}{4}$$

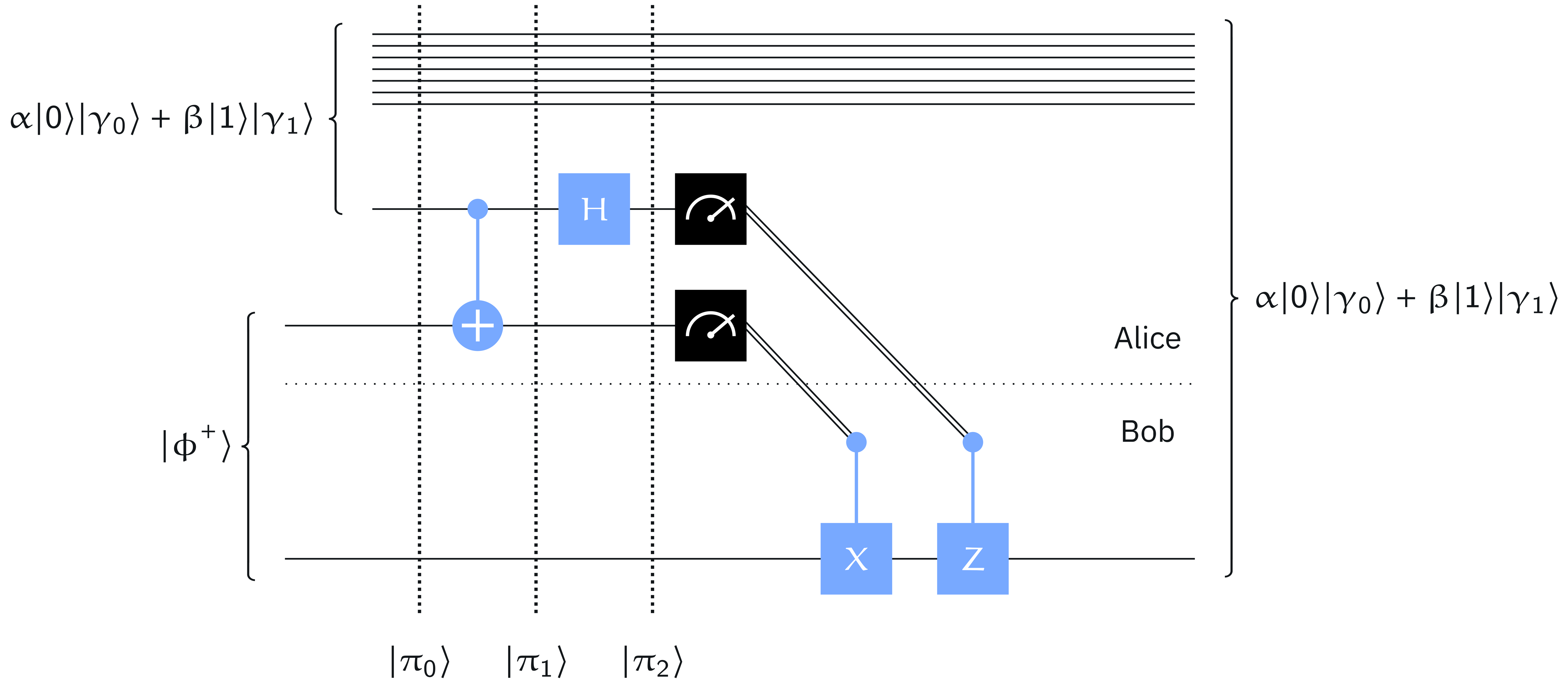
Teleportation analysis



$$|\pi_2\rangle = \frac{1}{2}(\alpha|0\rangle + \beta|1\rangle)\,|00\rangle + \frac{1}{2}(\alpha|0\rangle - \beta|1\rangle)\,|01\rangle + \frac{1}{2}(\alpha|1\rangle + \beta|0\rangle)\,|10\rangle + \frac{1}{2}(\alpha|1\rangle - \beta|0\rangle)\,|11\rangle$$

ab	Probability	Conditional state of (B, A, Q)	Operation on B	Final state of B
00	$\frac{1}{4}$	$(\alpha|0\rangle + \beta|1\rangle)|00\rangle$	$\mathbb{1}$	$\alpha|0\rangle + \beta|1\rangle$
01	$\frac{1}{4}$	$(\alpha|0\rangle - \beta|1\rangle)|01\rangle$	Z	$\alpha|0\rangle + \beta|1\rangle$
10	$\frac{1}{4}$	$(\alpha|1\rangle + \beta|0\rangle)|10\rangle$	X	$\alpha|0\rangle + \beta|1\rangle$
11	$\frac{1}{4}$	$(\alpha|1\rangle - \beta|0\rangle)|11\rangle$	ZX	$\alpha|0\rangle + \beta|1\rangle$

Teleportation analysis

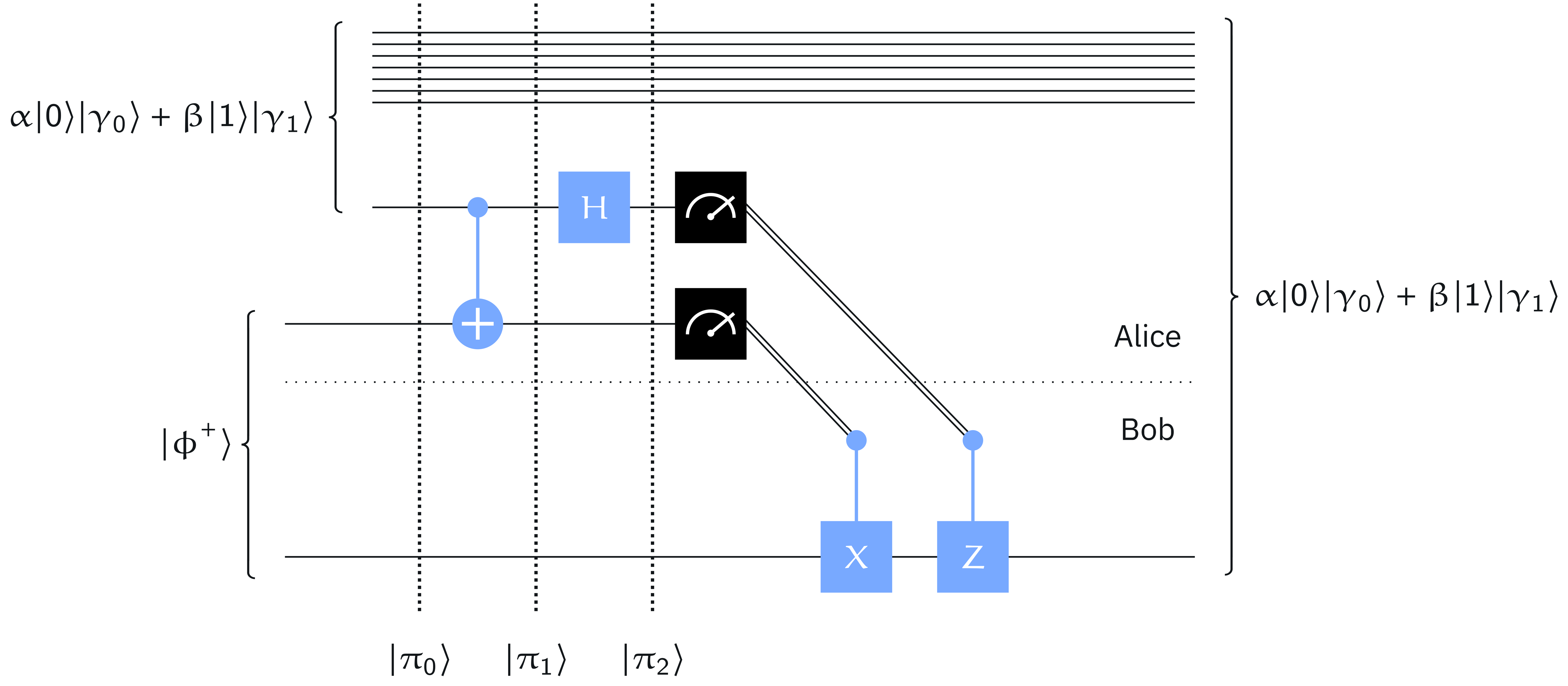


$$|\pi_0\rangle = \frac{1}{\sqrt{2}}\Big(\alpha|00\rangle|0\rangle|\gamma_0\rangle + \alpha|11\rangle|0\rangle|\gamma_0\rangle + \beta|00\rangle|1\rangle|\gamma_1\rangle + \beta|11\rangle|1\rangle|\gamma_1\rangle\Big)$$

$$|\pi_1\rangle = \frac{1}{\sqrt{2}}\Big(\alpha|00\rangle|0\rangle|\gamma_0\rangle + \alpha|11\rangle|0\rangle|\gamma_0\rangle + \beta|01\rangle|1\rangle|\gamma_1\rangle + \beta|10\rangle|1\rangle|\gamma_1\rangle\Big)$$

$$\begin{aligned}
|\pi_2\rangle &= \frac{1}{\sqrt{2}}\Big(\alpha|00\rangle|+\rangle|\gamma_0\rangle + \alpha|11\rangle|+\rangle|\gamma_0\rangle + \beta|01\rangle|-\rangle|\gamma_1\rangle + \beta|10\rangle|-\rangle|\gamma_1\rangle\Big)\\
&= \frac{1}{2}\Big(\alpha|0\rangle|00\rangle|\gamma_0\rangle + \alpha|0\rangle|01\rangle|\gamma_0\rangle + \alpha|1\rangle|10\rangle|\gamma_0\rangle + \alpha|1\rangle|11\rangle|\gamma_0\rangle\\
&\qquad + \beta|1\rangle|00\rangle|\gamma_1\rangle - \beta|1\rangle|01\rangle|\gamma_1\rangle + \beta|0\rangle|10\rangle|\gamma_1\rangle - \beta|0\rangle|11\rangle|\gamma_1\rangle\Big)
\end{aligned}$$

Teleportation analysis



$$\begin{aligned}|\pi_2\rangle = \frac{1}{2}\Big(&\alpha|0\rangle|00\rangle|\gamma_0\rangle + \alpha|0\rangle|01\rangle|\gamma_0\rangle + \alpha|1\rangle|10\rangle|\gamma_0\rangle + \alpha|1\rangle|11\rangle|\gamma_0\rangle\\ &+\beta|1\rangle|00\rangle|\gamma_1\rangle - \beta|1\rangle|01\rangle|\gamma_1\rangle + \beta|0\rangle|10\rangle|\gamma_1\rangle - \beta|0\rangle|11\rangle|\gamma_1\rangle\Big)\end{aligned}$$

ab	Probability	Conditional state of $(\mathsf{B},\mathsf{R},\mathsf{A},\mathsf{Q})$	Operation on B	Final state of (B,R)
00	$\frac{1}{4}$	$\big(\alpha|0\rangle|\gamma_0\rangle + \beta|1\rangle|\gamma_1\rangle\big)|00\rangle$	$\mathbb{1}$	$\alpha|0\rangle|\gamma_0\rangle + \beta|1\rangle|\gamma_1\rangle$
01	$\frac{1}{4}$	$\big(\alpha|0\rangle|\gamma_0\rangle - \beta|1\rangle|\gamma_1\rangle\big)|01\rangle$	Z	$\alpha|0\rangle|\gamma_0\rangle + \beta|1\rangle|\gamma_1\rangle$
10	$\frac{1}{4}$	$\big(\alpha|1\rangle|\gamma_0\rangle + \beta|0\rangle|\gamma_1\rangle\big)|10\rangle$	X	$\alpha|0\rangle|\gamma_0\rangle + \beta|1\rangle|\gamma_1\rangle$
11	$\frac{1}{4}$	$\big(\alpha|1\rangle|\gamma_0\rangle - \beta|0\rangle|\gamma_1\rangle\big)|11\rangle$	ZX	$\alpha|0\rangle|\gamma_0\rangle + \beta|1\rangle|\gamma_1\rangle$

Remarks on teleportation

- Teleportation is not an application of quantum information — it's a way to perform *quantum communication.*
- Teleportation motivates *entanglement distillation* as a means to reliable quantum communication.
- Beyond its potential for communication, teleportation also has fundamental importance in the study of quantum information and computation.

Superdense coding set-up

Scenario

Alice has *two classical bits* that she wishes to transmit to Bob.

- Alice is able to send a *single qubit* to Bob.
- Alice and Bob *share an e-bit.*

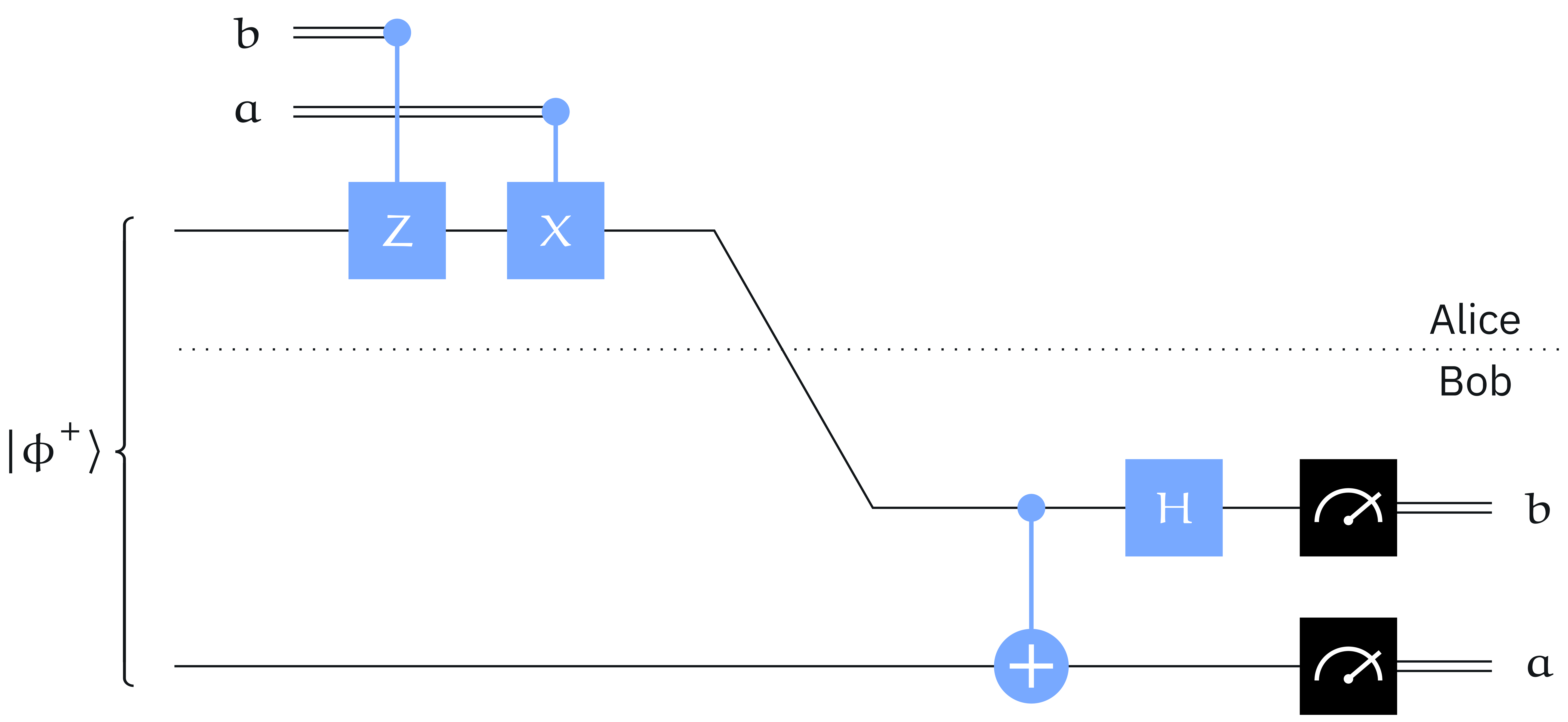
Remark

Without the e-bit, Alice and Bob's task would be impossible...

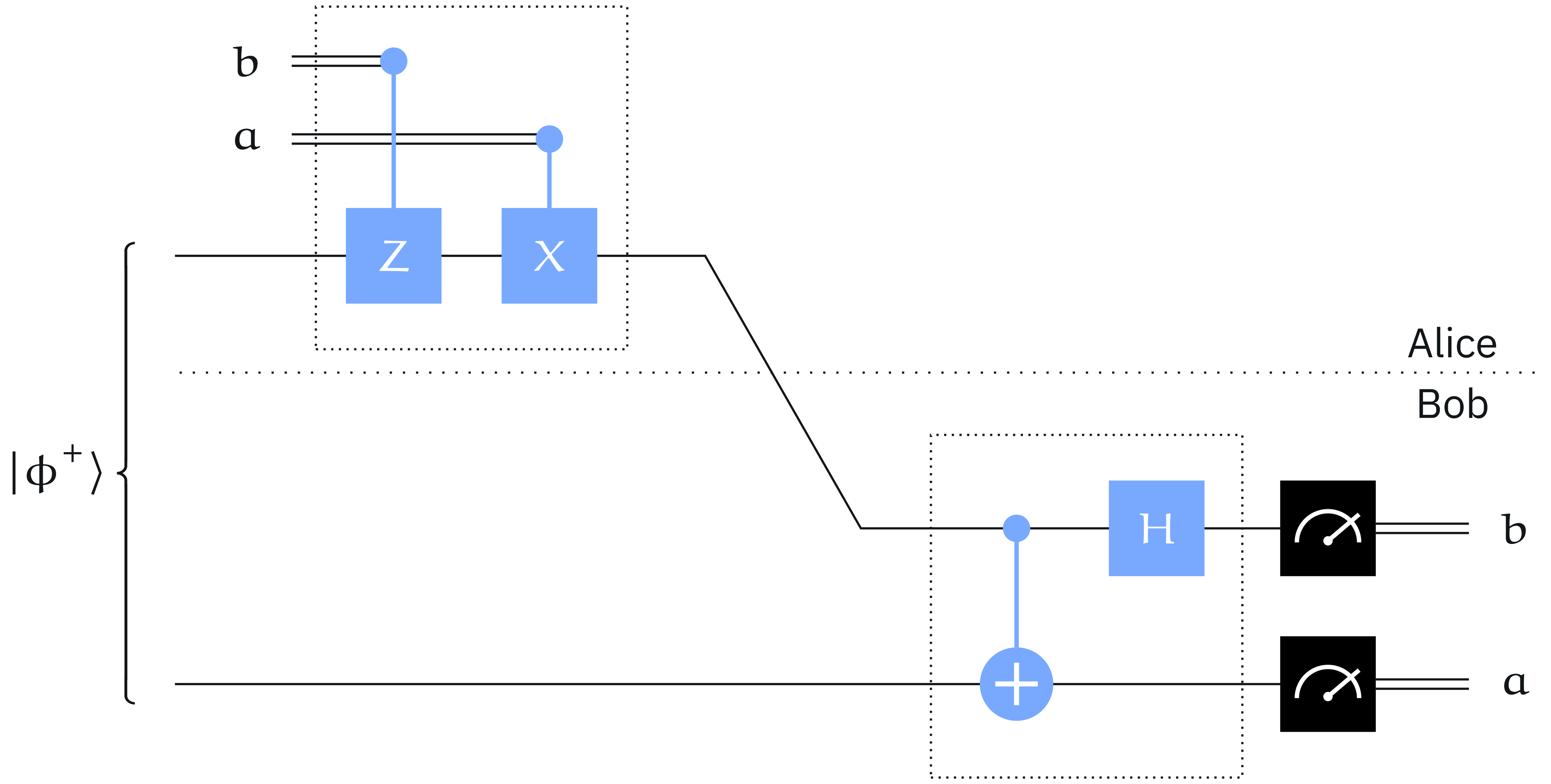
Holevo's theorem implies that two classical bits of communication cannot be reliably transmitted by a single qubit alone.

Superdense coding protocol

Circuit description



Superdense coding analysis



$$|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

ab	Alice's action	Bob's action
00	$\lvert\phi^+\rangle \mapsto \lvert\phi^+\rangle$	$\lvert\phi^+\rangle \mapsto \lvert 00\rangle$
01	$\lvert\phi^+\rangle \mapsto \lvert\phi^-\rangle$	$\lvert\phi^-\rangle \mapsto \lvert 01\rangle$
10	$\lvert\phi^+\rangle \mapsto \lvert\psi^+\rangle$	$\lvert\psi^+\rangle \mapsto \lvert 10\rangle$
11	$\lvert\phi^+\rangle \mapsto \lvert\psi^-\rangle$	$\lvert\psi^-\rangle \mapsto -\lvert 11\rangle$

Remarks on superdense coding

- Superdense coding seems unlikely to be useful in a practical sense.
- The underlying idea is fundamentally important, and illustrates an interesting aspect of entanglement.
- Together with teleportation, superdense coding establishes an equivalence:

$$\text{1 qubit of quantum communication} \overset{\text{1 ebit}}{\Longleftrightarrow} \text{2 bits of classical communication}$$

Nonlocal games

Mathematical abstractions of *games* are both important and useful.

The CHSH game is an example of a *nonlocal game.*

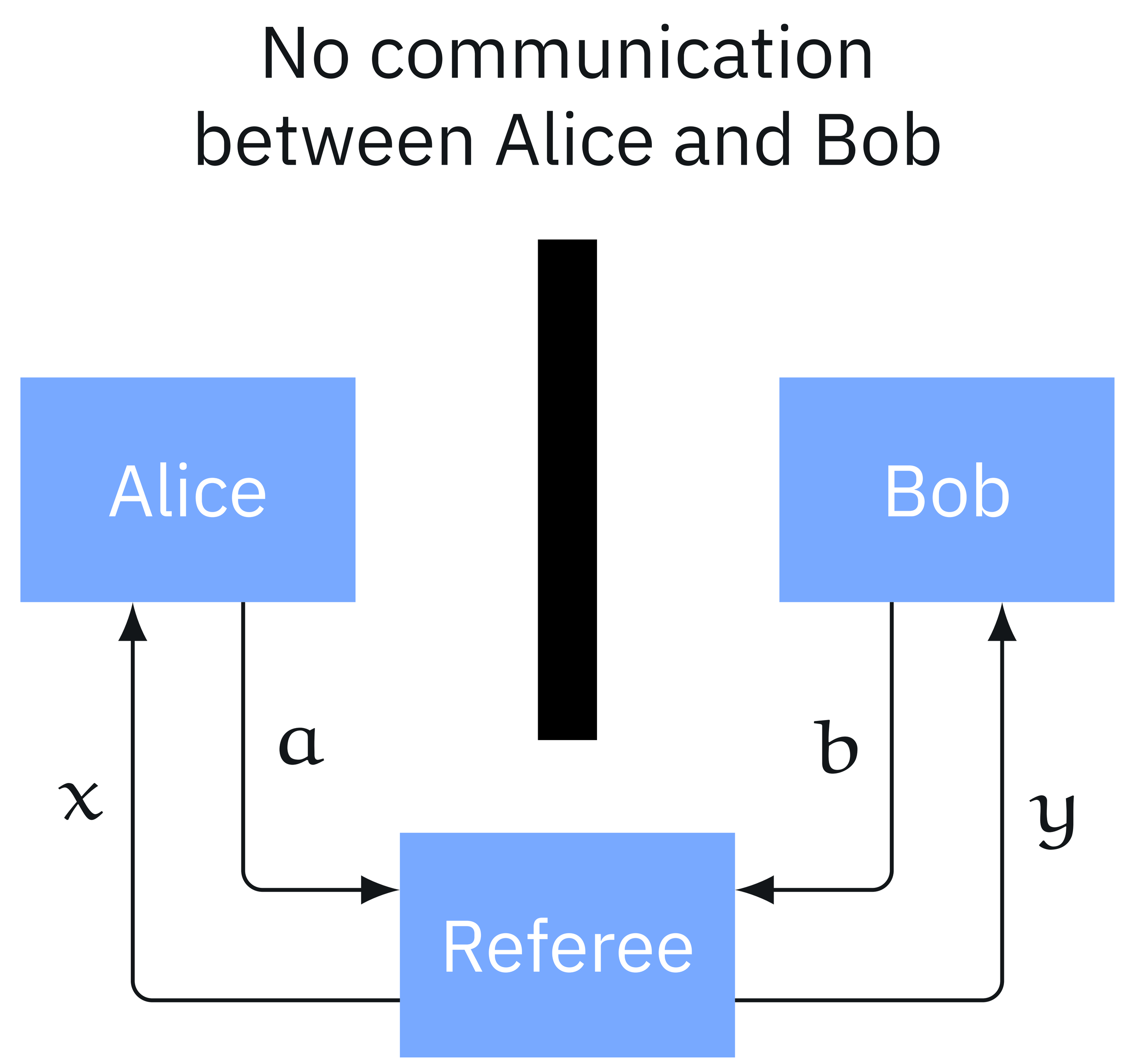
Set-up

The players are Alice and Bob, who *cooperate as a team.*

The game is run by a *referee.*

Alice and Bob can prepare for the game however they choose...

...but once the game starts they are *forbidden from communicating.*



Nonlocal games

Mathematical abstractions of *games* are both important and useful.

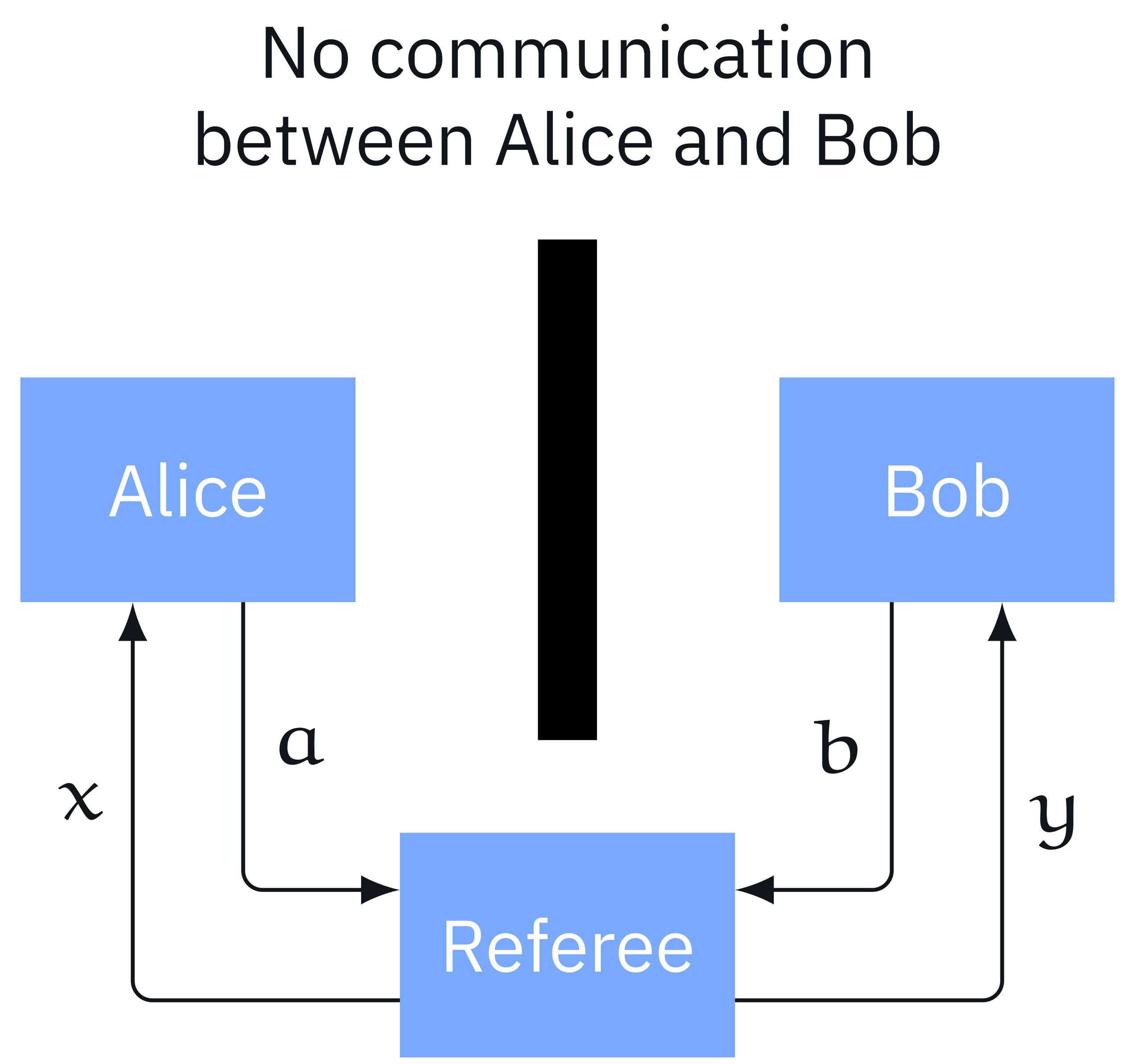
The CHSH game is an example of a *nonlocal game.*

The referee

The referee uses *randomness* to select the questions x and y.

The referee determines whether a pair of answers (a, b) *wins or loses* for the questions pair (x, y) according to some fixed rule.

(A precise description of the referee defines an instance of a nonlocal game.)



Nonlocal games

CHSH game referee

1. The questions and answers are all bits:

$$x, y, a, b \in \{0, 1\}$$

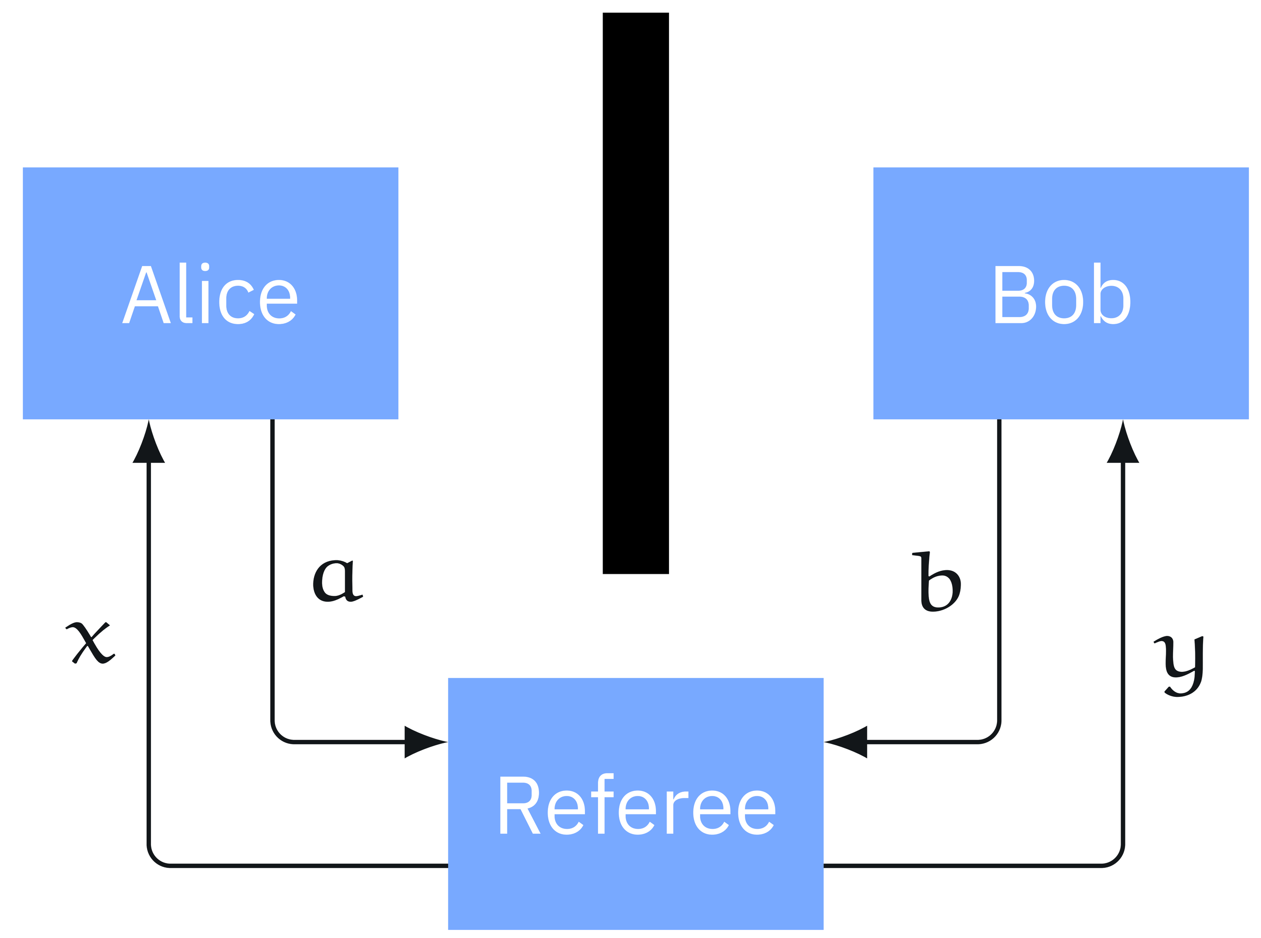
2. The questions x and y are chosen *uniformly at random.*

3. A pair of answers (a, b) *wins* for (x, y) if

$$a \oplus b = x \wedge y$$

and *loses otherwise.*

(x, y)	winning condition
$(0, 0)$	$a = b$
$(0, 1)$	$a = b$
$(1, 0)$	$a = b$
$(1, 1)$	$a \neq b$



Deterministic strategies

No deterministic strategy can win every time.

$$\begin{aligned} a(0) \oplus b(0) &= 0 \\ a(0) \oplus b(1) &= 0 \\ a(1) \oplus b(0) &= 0 \\ a(1) \oplus b(1) &= 1 \end{aligned}$$

It follows that no deterministic strategy can with with probability greater than $3/4$.

Nonlocal games

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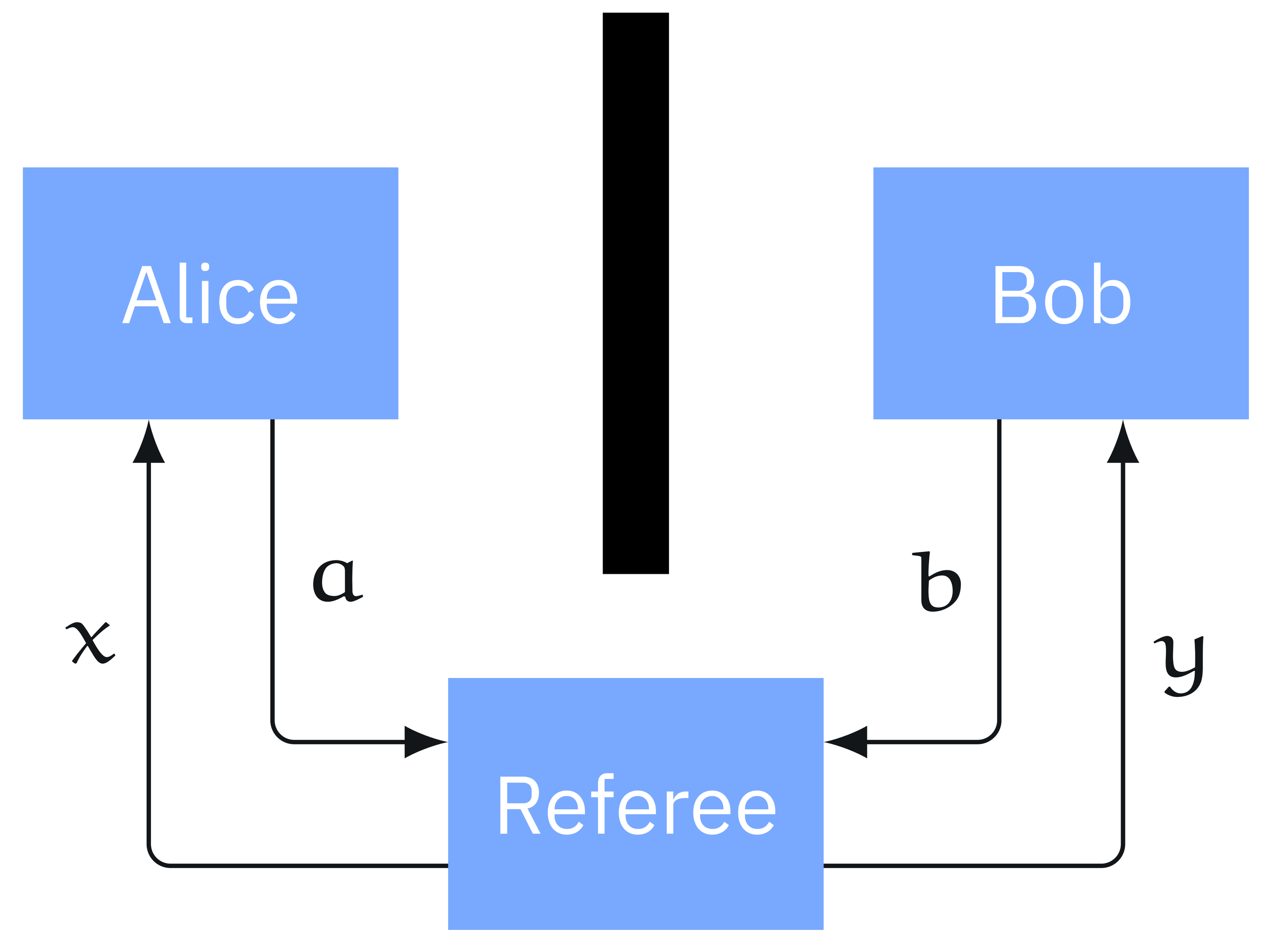

Probabilistic strategies

Every probabilistic strategy can be viewed as a *random choice* of a *deterministic* strategy.

It follows that no probabilistic strategy can win with probability greater than $3/4$.

Nonlocal games

CHSH game referee

1. The questions and answers are all bits:

$$x, y, a, b \in \{0, 1\}$$

2. The questions x and y are chosen *uniformly at random.*

3. A pair of answers (a, b) *wins* for (x, y) if

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and *loses otherwise.*

(x, y)	winning condition
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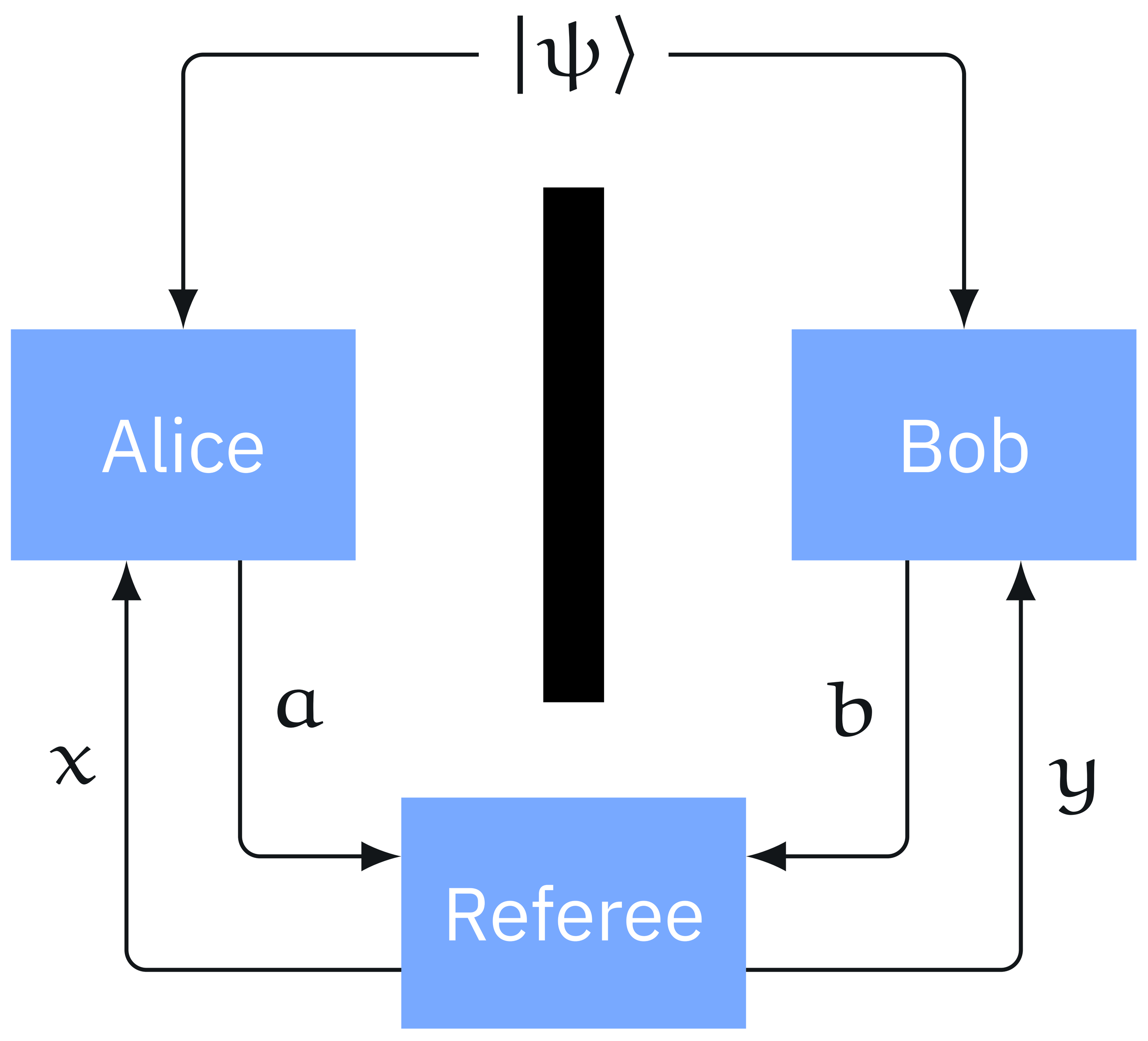
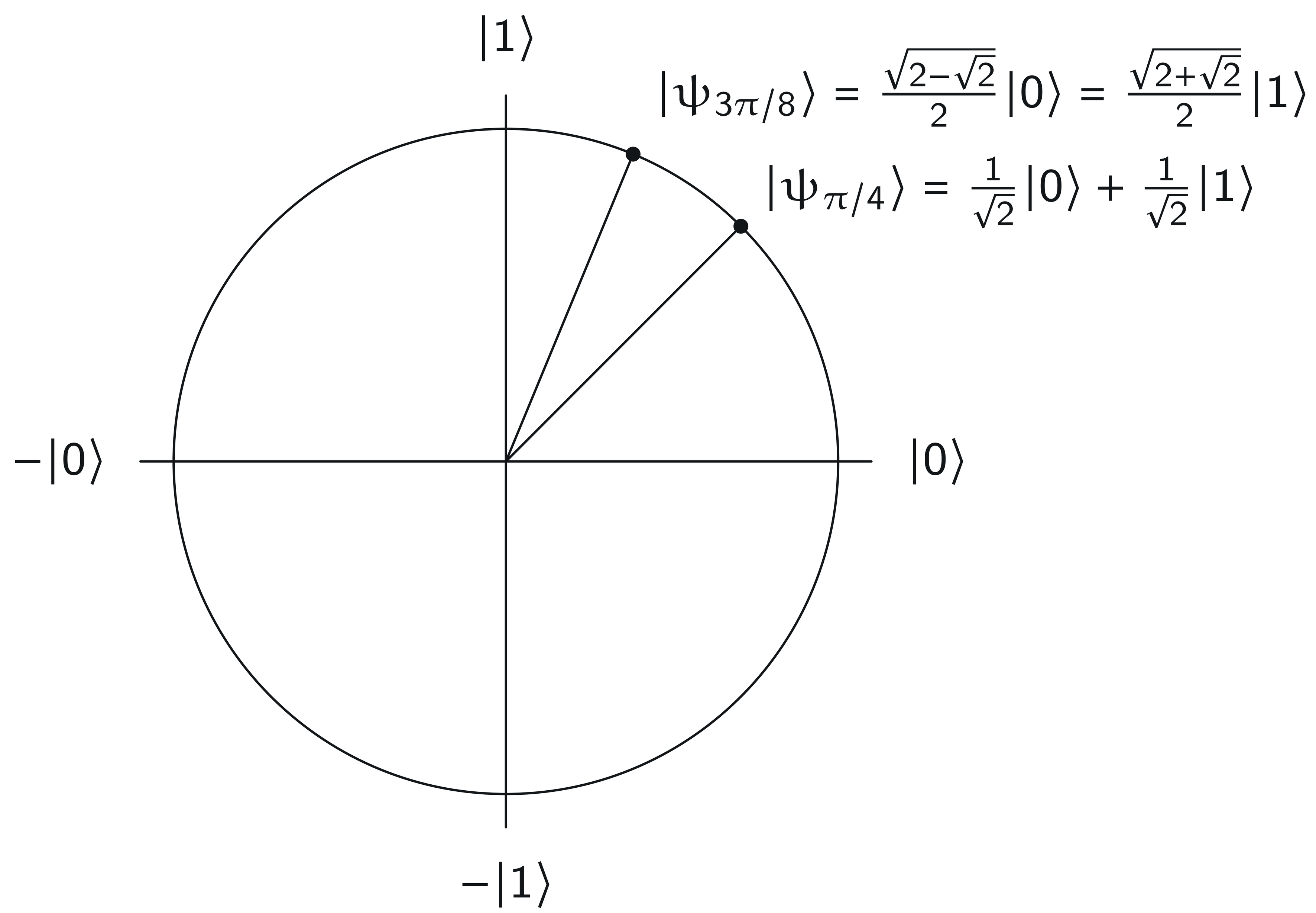


Can a *quantum strategy* do better?

CHSH game strategy

For each angle θ (measured in radians), define a unit vector

$$|\psi_\theta\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$$

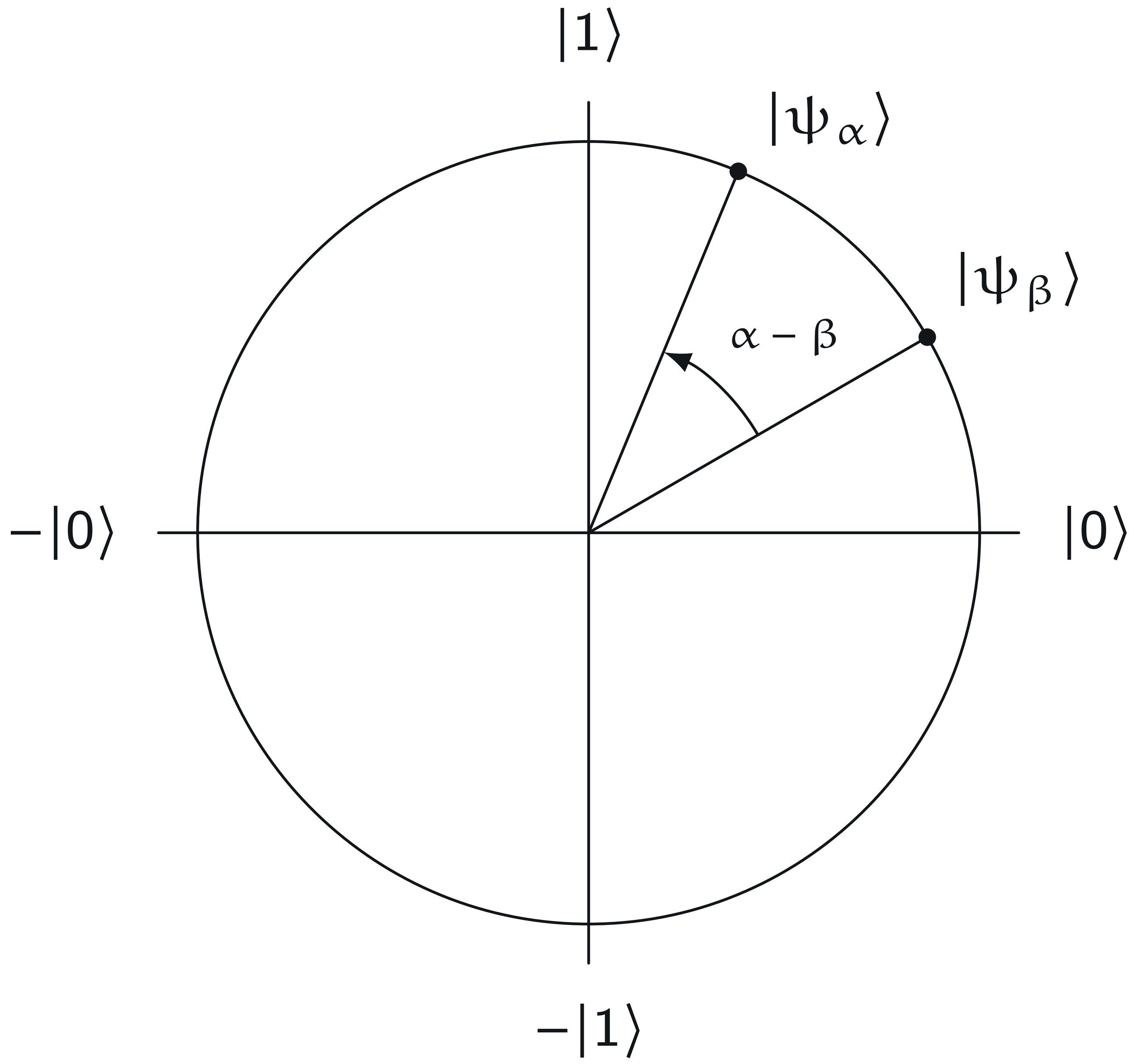


θ	$\cos(\theta)$	$\sin(\theta)$
0	1	0
$\frac{\pi}{8}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{3\pi}{8}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$
$\frac{\pi}{2}$	0	1

CHSH game strategy

For each angle θ (measured in radians), define a unit vector

$$|\psi_\theta\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$$



θ	$\cos(\theta)$	$\sin(\theta)$
0	1	0
$\frac{\pi}{8}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{3\pi}{8}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$
$\frac{\pi}{2}$	0	1

By one of the *angle addition formulas* we have

$$\langle\psi_\alpha|\psi_\beta\rangle = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta)$$

$$\langle\psi_\alpha \otimes \psi_\beta|\phi^+\rangle = \frac{\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)}{\sqrt{2}} = \frac{\cos(\alpha - \beta)}{\sqrt{2}}$$

CHSH game strategy

For each angle θ (measured in radians), define a unit vector

$$|\psi_\theta\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$$

Also define a unitary matrix

$$U_\theta = |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}|$$

Alice and Bob's strategy

Alice and Bob share an e-bit (A, B).

Alice's actions

Alice applies an operation to A as follows:

$$\begin{cases} U_0 & \text{if } x = 0 \\ U_{\pi/4} & \text{if } x = 1 \end{cases}$$

She then measures A and sends the result to the referee.

Bob's actions

Bob applies an operation to B as follows:

$$\begin{cases} U_{\pi/8} & \text{if } y = 0 \\ U_{-\pi/8} & \text{if } y = 1 \end{cases}$$

He then measures B and sends the result to the referee.

CHSH game strategy

Alice and Bob's strategy

Alice and Bob share an e-bit (A, B).

Alice's actions

Alice applies an operation to A:

$$\begin{cases} U_0 & \text{if } x = 0 \\ U_{\pi/4} & \text{if } x = 1 \end{cases}$$

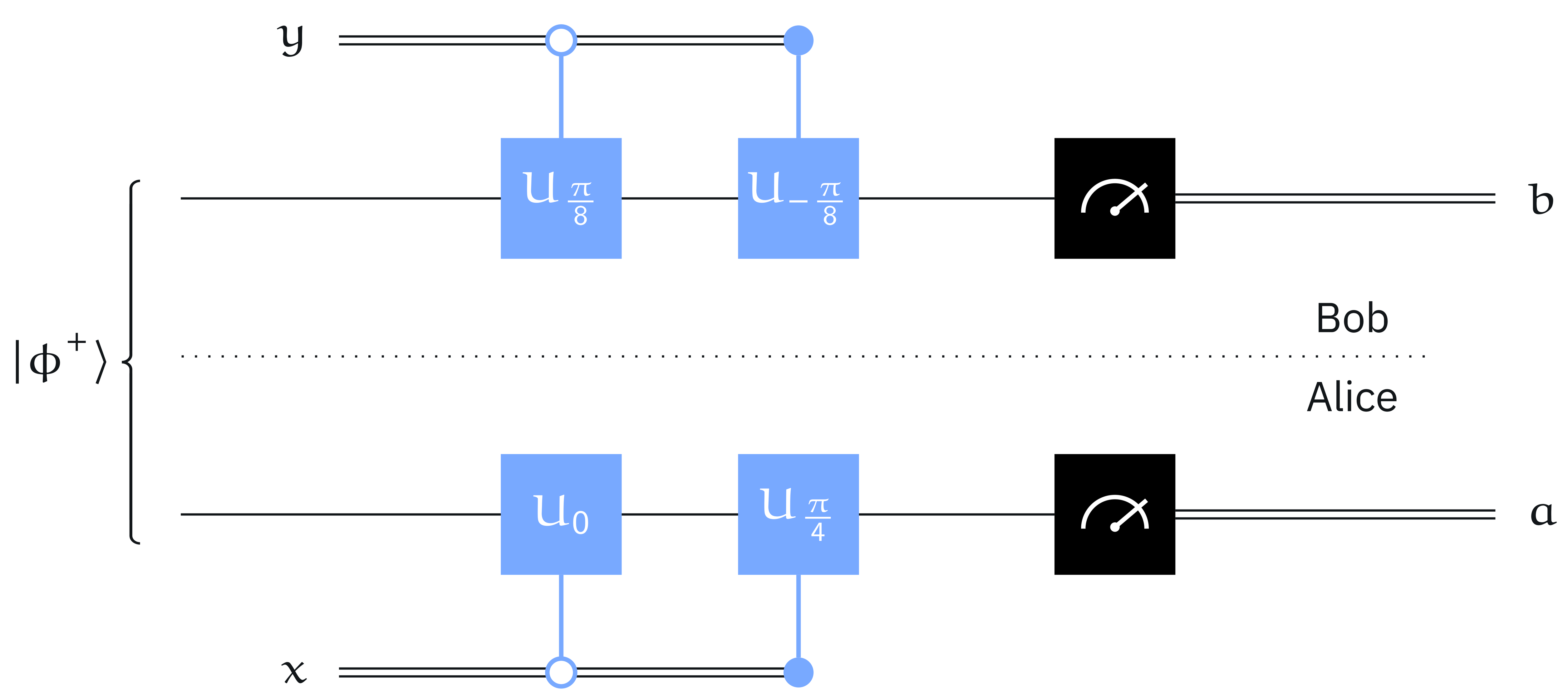
She then measures A and sends the result to the referee.

Bob's actions

Bob applies an operation to B:

$$\begin{cases} U_{\pi/8} & \text{if } y = 0 \\ U_{-\pi/8} & \text{if } y = 1 \end{cases}$$

He then measures B and sends the result to the referee.



Analysis of the strategy

$$U_\theta = |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}| \qquad \langle\psi_\alpha \otimes \psi_\beta|\phi^+\rangle = \frac{\cos(\alpha-\beta)}{\sqrt{2}}$$

Case 1: $(x, y) = (0, 0)$

Alice performs U_0 and Bob performs $U_{\frac{\pi}{8}}$.

$$\begin{aligned}(U_0 \otimes U_{\frac{\pi}{8}})|\phi^+\rangle = {} & |00\rangle\langle\psi_0 \otimes \psi_{\frac{\pi}{8}}|\phi^+\rangle + |01\rangle\langle\psi_0 \otimes \psi_{\frac{5\pi}{8}}|\phi^+\rangle \\ & + |10\rangle\langle\psi_{\frac{\pi}{2}} \otimes \psi_{\frac{\pi}{8}}|\phi^+\rangle + |11\rangle\langle\psi_{\frac{\pi}{2}} \otimes \psi_{\frac{5\pi}{8}}|\phi^+\rangle \\ = {} & \frac{\cos(-\frac{\pi}{8})|00\rangle + \cos(-\frac{5\pi}{8})|01\rangle + \cos(\frac{3\pi}{8})|10\rangle + \cos(-\frac{\pi}{8})|11\rangle}{\sqrt{2}}\end{aligned}$$

(a, b)	Probability	Simplified
$(0, 0)$	$\frac{1}{2}\cos^2(-\frac{\pi}{8})$	$\frac{2+\sqrt{2}}{8}$
$(0, 1)$	$\frac{1}{2}\cos^2(-\frac{5\pi}{8})$	$\frac{2-\sqrt{2}}{8}$
$(1, 0)$	$\frac{1}{2}\cos^2(\frac{3\pi}{8})$	$\frac{2-\sqrt{2}}{8}$
$(1, 1)$	$\frac{1}{2}\cos^2(-\frac{\pi}{8})$	$\frac{2+\sqrt{2}}{8}$

$$\Pr(a = b) = \frac{2+\sqrt{2}}{4} \approx 0.85$$

$$\Pr(a \neq b) = \frac{2-\sqrt{2}}{4} \approx 0.15$$

They win with probability $\frac{2+\sqrt{2}}{4} \approx 0.85$.

Analysis of the strategy

$$U_\theta = |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}| \qquad \langle\psi_\alpha \otimes \psi_\beta|\phi^+\rangle = \frac{\cos(\alpha-\beta)}{\sqrt{2}}$$

Case 2: $(x, y) = (0, 1)$

Alice performs U_0 and Bob performs $U_{-\frac{\pi}{8}}$.

$$\begin{aligned}(U_0 \otimes U_{-\frac{\pi}{8}})|\phi^+\rangle &= |00\rangle\langle\psi_0 \otimes \psi_{-\frac{\pi}{8}}|\phi^+\rangle + |01\rangle\langle\psi_0 \otimes \psi_{\frac{3\pi}{8}}|\phi^+\rangle \\ &\quad + |10\rangle\langle\psi_{\frac{\pi}{2}} \otimes \psi_{-\frac{\pi}{8}}|\phi^+\rangle + |11\rangle\langle\psi_{\frac{\pi}{2}} \otimes \psi_{\frac{3\pi}{8}}|\phi^+\rangle \\ &= \frac{\cos(\frac{\pi}{8})|00\rangle + \cos(-\frac{3\pi}{8})|01\rangle + \cos(\frac{5\pi}{8})|10\rangle + \cos(\frac{\pi}{8})|11\rangle}{\sqrt{2}}\end{aligned}$$

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$(1,0)$	$\frac{1}{2}\cos^2(\frac{5\pi}{8})$	$\frac{2-\sqrt{2}}{8}$
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$$\Pr(a = b) = \frac{2+\sqrt{2}}{4} \approx 0.85$$

$$\Pr(a \neq b) = \frac{2-\sqrt{2}}{4} \approx 0.15$$

They win with probability $\frac{2+\sqrt{2}}{4} \approx 0.85$.

Analysis of the strategy

$$U_\theta = |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}| \qquad \langle\psi_\alpha \otimes \psi_\beta|\phi^+\rangle = \frac{\cos(\alpha-\beta)}{\sqrt{2}}$$

Case 3: $(x, y) = (1, 0)$

Alice performs $U_{\frac{\pi}{4}}$ and Bob performs $U_{\frac{\pi}{8}}$.

$$\begin{aligned}(U_{\frac{\pi}{4}} \otimes U_{\frac{\pi}{8}})|\phi^+\rangle &= |00\rangle\langle\psi_{\frac{\pi}{4}} \otimes \psi_{\frac{\pi}{8}}|\phi^+\rangle + |01\rangle\langle\psi_{\frac{\pi}{4}} \otimes \psi_{\frac{5\pi}{8}}|\phi^+\rangle \\ &\quad + |10\rangle\langle\psi_{\frac{3\pi}{4}} \otimes \psi_{\frac{\pi}{8}}|\phi^+\rangle + |11\rangle\langle\psi_{\frac{3\pi}{4}} \otimes \psi_{\frac{5\pi}{8}}|\phi^+\rangle \\ &= \frac{\cos(\frac{\pi}{8})|00\rangle + \cos(-\frac{3\pi}{8})|01\rangle + \cos(\frac{5\pi}{8})|10\rangle + \cos(\frac{\pi}{8})|11\rangle}{\sqrt{2}}\end{aligned}$$

(a, b)	Probability	Simplified
$(0, 0)$	$\frac{1}{2}\cos^2(\frac{\pi}{8})$	$\frac{2+\sqrt{2}}{8}$
$(0, 1)$	$\frac{1}{2}\cos^2(-\frac{3\pi}{8})$	$\frac{2-\sqrt{2}}{8}$
$(1, 0)$	$\frac{1}{2}\cos^2(\frac{5\pi}{8})$	$\frac{2-\sqrt{2}}{8}$
$(1, 1)$	$\frac{1}{2}\cos^2(\frac{\pi}{8})$	$\frac{2+\sqrt{2}}{8}$

$$\Pr(a = b) = \frac{2+\sqrt{2}}{4} \approx 0.85$$

$$\Pr(a \neq b) = \frac{2-\sqrt{2}}{4} \approx 0.15$$

They win with probability $\frac{2+\sqrt{2}}{4} \approx 0.85$.

Analysis of the strategy

$$U_\theta = |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}| \qquad \langle\psi_\alpha \otimes \psi_\beta|\phi^+\rangle = \frac{\cos(\alpha-\beta)}{\sqrt{2}}$$

Case 4: $(x, y) = (1, 1)$

Alice performs $U_{\frac{\pi}{4}}$ and Bob performs $U_{-\frac{\pi}{8}}$.

$$\begin{aligned}
(U_{\frac{\pi}{4}} \otimes U_{-\frac{\pi}{8}})|\phi^+\rangle &= |00\rangle\langle\psi_{\frac{\pi}{4}} \otimes \psi_{-\frac{\pi}{8}}|\phi^+\rangle + |01\rangle\langle\psi_{\frac{\pi}{4}} \otimes \psi_{\frac{3\pi}{8}}|\phi^+\rangle \\
&\quad + |10\rangle\langle\psi_{\frac{3\pi}{4}} \otimes \psi_{-\frac{\pi}{8}}|\phi^+\rangle + |11\rangle\langle\psi_{\frac{3\pi}{4}} \otimes \psi_{\frac{3\pi}{8}}|\phi^+\rangle \\
&= \frac{\cos(\frac{3\pi}{8})|00\rangle + \cos(-\frac{\pi}{8})|01\rangle + \cos(\frac{7\pi}{8})|10\rangle + \cos(\frac{3\pi}{8})|11\rangle}{\sqrt{2}}
\end{aligned}$$

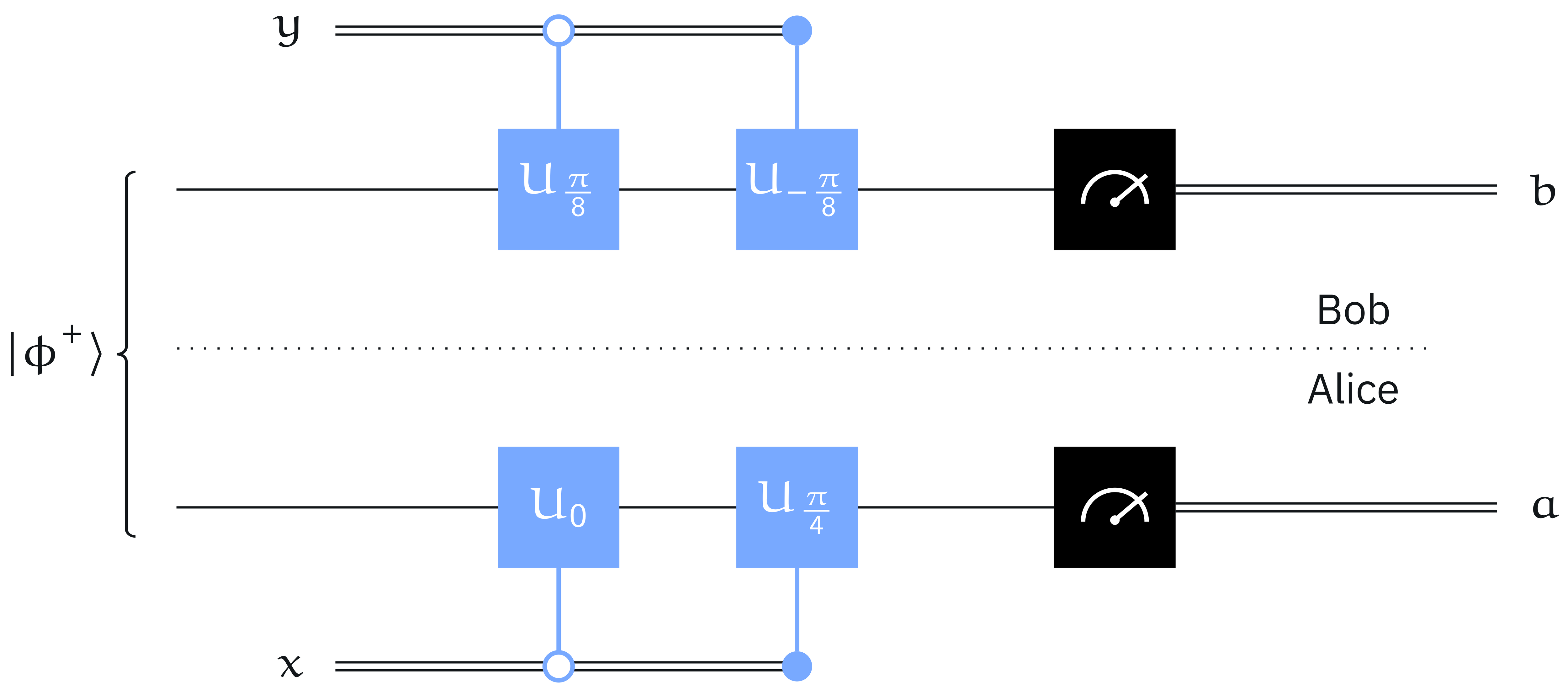
(a, b)	Probability	Simplified
$(0, 0)$	$\frac{1}{2}\cos^2(\frac{3\pi}{8})$	$\frac{2-\sqrt{2}}{8}$
$(0, 1)$	$\frac{1}{2}\cos^2(-\frac{\pi}{8})$	$\frac{2+\sqrt{2}}{8}$
$(1, 0)$	$\frac{1}{2}\cos^2(\frac{7\pi}{8})$	$\frac{2+\sqrt{2}}{8}$
$(1, 1)$	$\frac{1}{2}\cos^2(\frac{3\pi}{8})$	$\frac{2-\sqrt{2}}{8}$

$$\Pr(a = b) = \frac{2-\sqrt{2}}{4} \approx 0.15$$

$$\Pr(a \neq b) = \frac{2+\sqrt{2}}{4} \approx 0.85$$

They win with probability $\frac{2+\sqrt{2}}{4} \approx 0.85$.

Analysis of the strategy



The strategy wins with probability $\frac{2+\sqrt{2}}{4} \approx 0.85$ (in all four cases, and therefore overall).

Analysis of the strategy

We can also think about the strategy geometrically.

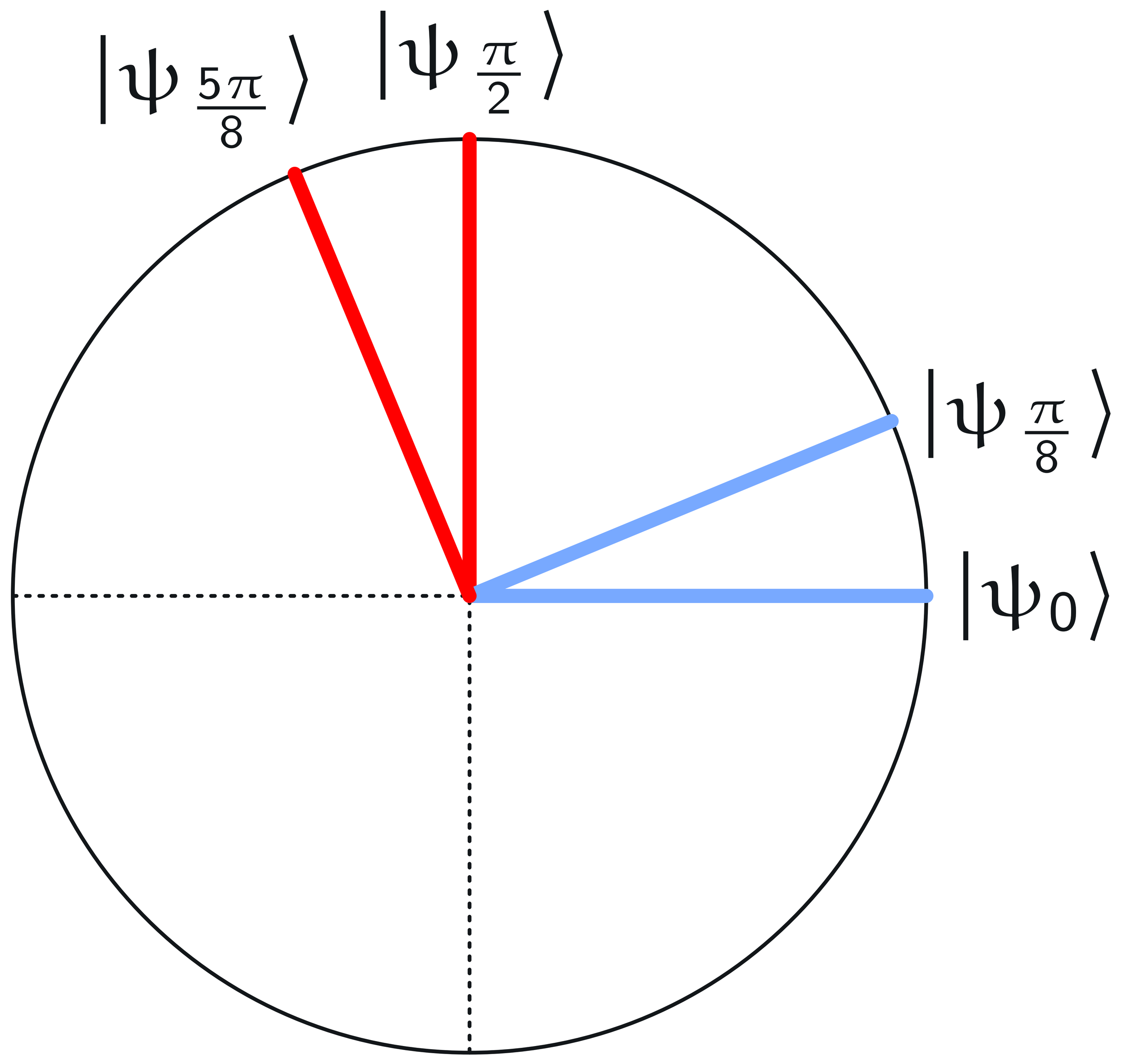
Using the formula

$$\langle \psi_\alpha \otimes \psi_\beta | \phi^+ \rangle = \frac{1}{\sqrt{2}} \langle \psi_\alpha | \psi_\beta \rangle$$

the probabilities of different measurement outcomes can be expressed as follows.

$(x, y) = (0, 0)$

(a, b)	Probability
$(0, 0)$	$\frac{1}{2} \lvert \langle \psi_0 \vert \psi_{\frac{\pi}{8}} \rangle \rvert^2$
$(0, 1)$	$\frac{1}{2} \lvert \langle \psi_0 \vert \psi_{\frac{5\pi}{8}} \rangle \rvert^2$
$(1, 0)$	$\frac{1}{2} \lvert \langle \psi_{\frac{\pi}{2}} \vert \psi_{\frac{\pi}{8}} \rangle \rvert^2$
$(1, 1)$	$\frac{1}{2} \lvert \langle \psi_{\frac{\pi}{2}} \vert \psi_{\frac{5\pi}{8}} \rangle \rvert^2$



Analysis of the strategy

We can also think about the strategy geometrically.

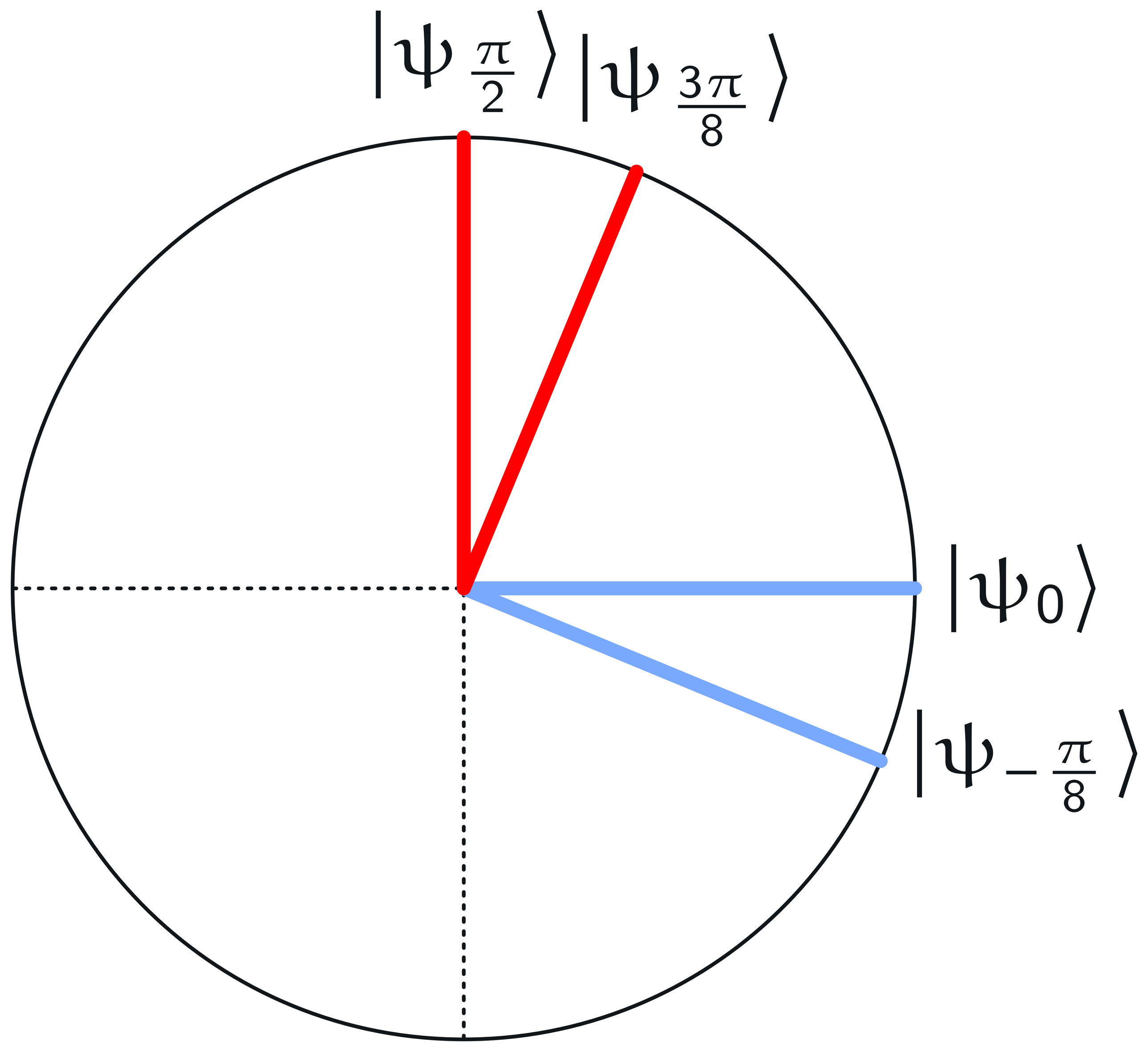
Using the formula

$$\langle \psi_\alpha \otimes \psi_\beta | \phi^+ \rangle = \frac{1}{\sqrt{2}} \langle \psi_\alpha | \psi_\beta \rangle$$

the probabilities of different measurement outcomes can be expressed as follows.

$(x, y) = (0, 1)$

(a, b)	Probability
$(0, 0)$	$\frac{1}{2} \lvert \langle \psi_0 \lvert \psi_{-\frac{\pi}{8}} \rangle \rvert^2$
$(0, 1)$	$\frac{1}{2} \lvert \langle \psi_0 \lvert \psi_{\frac{3\pi}{8}} \rangle \rvert^2$
$(1, 0)$	$\frac{1}{2} \lvert \langle \psi_{\frac{\pi}{2}} \lvert \psi_{-\frac{\pi}{8}} \rangle \rvert^2$
$(1, 1)$	$\frac{1}{2} \lvert \langle \psi_{\frac{\pi}{2}} \lvert \psi_{\frac{3\pi}{8}} \rangle \rvert^2$



Analysis of the strategy

We can also think about the strategy geometrically.

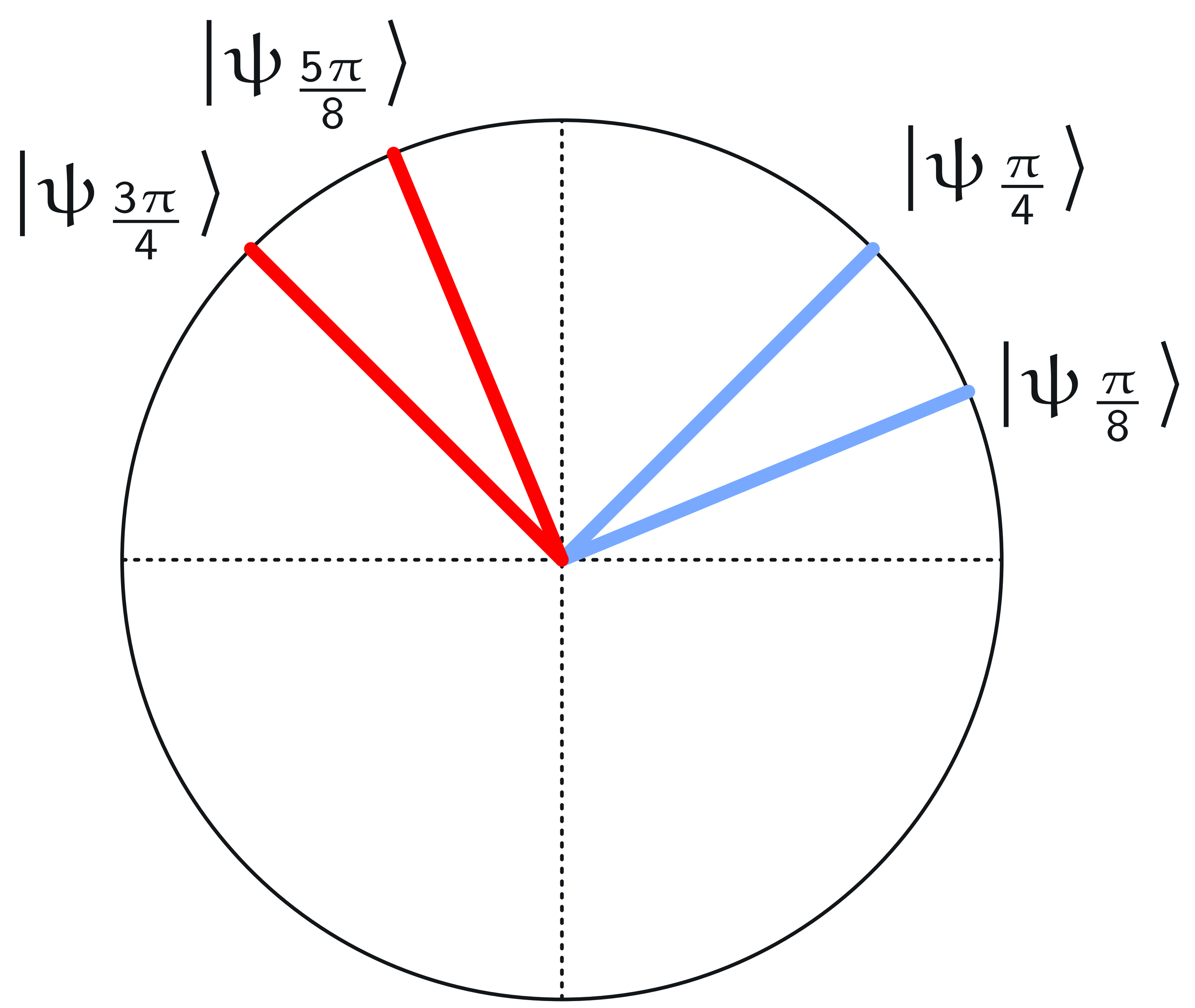
Using the formula

$$\langle \psi_\alpha \otimes \psi_\beta | \phi^+ \rangle = \frac{1}{\sqrt{2}} \langle \psi_\alpha | \psi_\beta \rangle$$

the probabilities of different measurement outcomes can be expressed as follows.

$(x, y) = (1, 0)$

(a, b)	Probability
$(0, 0)$	$\frac{1}{2} \lvert \langle \psi_{\frac{\pi}{4}} \vert \psi_{\frac{\pi}{8}} \rangle \rvert^2$
$(0, 1)$	$\frac{1}{2} \lvert \langle \psi_{\frac{\pi}{4}} \vert \psi_{\frac{5\pi}{8}} \rangle \rvert^2$
$(1, 0)$	$\frac{1}{2} \lvert \langle \psi_{\frac{3\pi}{4}} \vert \psi_{\frac{\pi}{8}} \rangle \rvert^2$
$(1, 1)$	$\frac{1}{2} \lvert \langle \psi_{\frac{3\pi}{4}} \vert \psi_{\frac{5\pi}{8}} \rangle \rvert^2$



Analysis of the strategy

We can also think about the strategy geometrically.

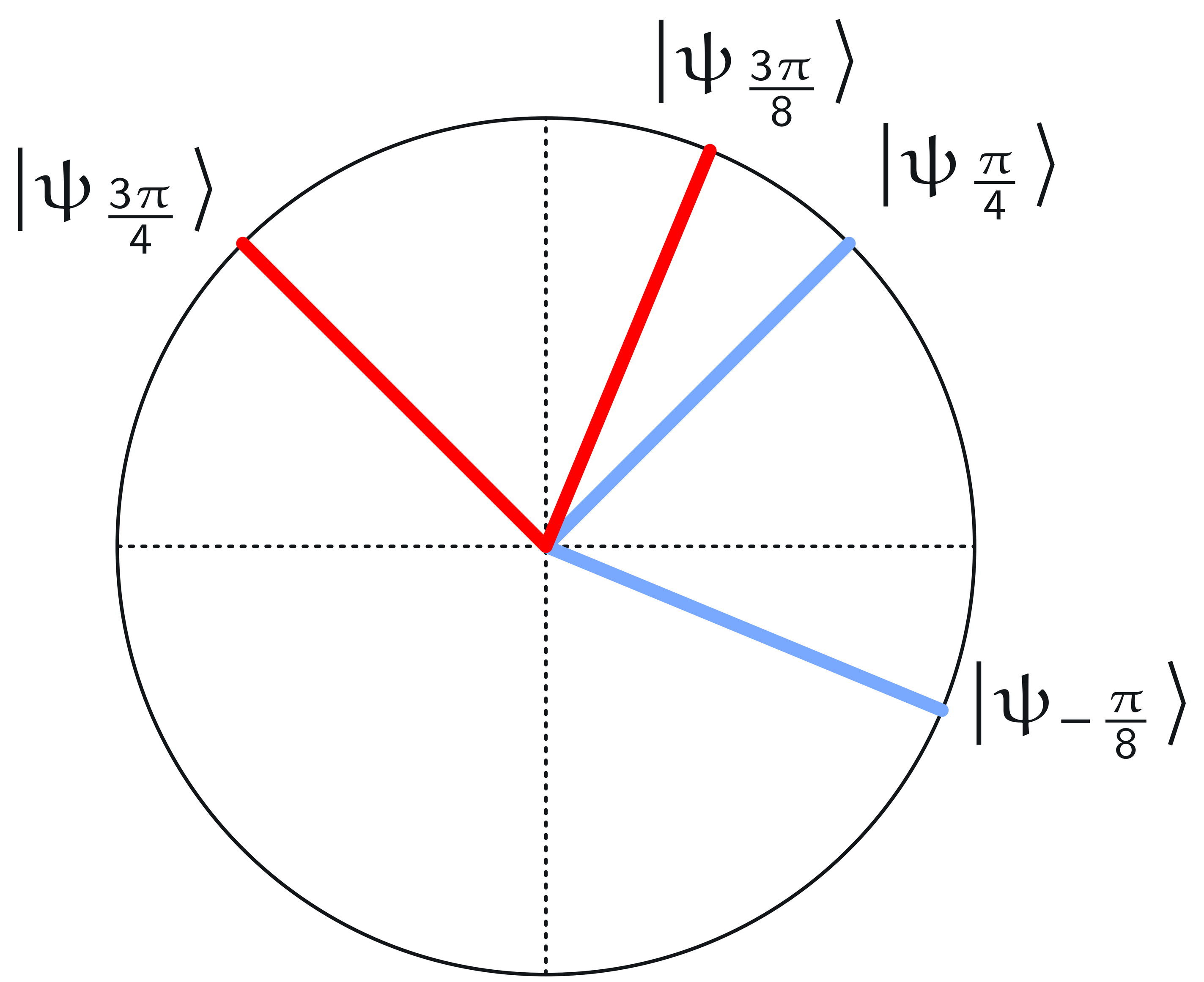
Using the formula

$$\langle \psi_\alpha \otimes \psi_\beta | \phi^+ \rangle = \frac{1}{\sqrt{2}} \langle \psi_\alpha | \psi_\beta \rangle$$

the probabilities of different measurement outcomes can be expressed as follows.

$(x, y) = (1, 1)$

(a, b)	Probability
$(0, 0)$	$\frac{1}{2} \lvert \langle \psi_{\frac{\pi}{4}} \vert \psi_{-\frac{\pi}{8}} \rangle \rvert^2$
$(0, 1)$	$\frac{1}{2} \lvert \langle \psi_{\frac{\pi}{4}} \vert \psi_{\frac{3\pi}{8}} \rangle \rvert^2$
$(1, 0)$	$\frac{1}{2} \lvert \langle \psi_{\frac{3\pi}{4}} \vert \psi_{\frac{-\pi}{8}} \rangle \rvert^2$
$(1, 1)$	$\frac{1}{2} \lvert \langle \psi_{\frac{3\pi}{4}} \vert \psi_{\frac{3\pi}{8}} \rangle \rvert^2$



Remarks on the CHSH game

- The CHSH game is not always described as a game — it’s often described as an experiment, or an example of a *Bell test.*
- The CHSH game offers a way to *experimentally test* the theory of quantum information.

  (The 2022 Nobel Prize in Physics was awarded to Alain Aspect, John Clauser, and Anton Zeilinger for experiments that do this with entangled photons.)
- The study of nonlocal games more generally is a fascinating and active area of research that still holds many mysteries.