String theory compactifications

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Abstract

String theory is consistently defined in ten dimensions, six of which should be curled up in some small internal compact manifold. The procedure of linking this manifold to four-dimensional physics is called string compactification, and in these lectures we will review it quite extensively. We will start with a very brief introduction to string theory, in particular we will work out its massless spectrum and show how the condition on the number of dimensions arises. We will then dwell on the different possible internal manifolds, starting from the simplest to the most relevant phenomenologically. We will show that these are most elegantly described by an extension of ordinary Riemannian geometry termed generalized geometry, first introduced by Hitchin. We shall finish by discussing (partially) open problems in string phenomenology, such as the embedding of the Standard Model and obtaining de Sitter solutions.

Contents

Introduction

The search for a unified theory of elementary particles and their interactions has culminated in the last two decades to the spectacular development of string theory. String theory reconciles general relativity with quantum mechanics, and contains the main ingredients of the Standard Model, *i.e.* gauge interactions and parity violation. It furthermore incorporates naturally most of the theoretical ideas for physics beyond the Standard Model – gauge unification, supersymmetry and extra dimensions – while also inspiring new ones such as the possibility of confining our observed world on a brane. Some of these ideas could be tested in the coming years, either in accelerator experiments or by future cosmological and astrophysical observations.

String theory is consistently defined in ten dimensions (see for example [1]- [5]). In order to extract any information about four-dimensional physics, we need to understand the way it is compactified to these four dimensions, or in other words, we need to know the precise way in which the Standard Model and Einstein's gravity are embedded as low-energy limits in string theory. Early attempts involved mostly compactification of the heterotic string on Calabi-Yau manifolds or on exact (2,0) backgrounds, and exceptionally type I theory. With the introduction of D-branes, compactifications of the type-II string theory involving orientifolds and intersecting D-branes became the center of attention.

The current state of the art is that one can find semi-realistic models in both frameworks, but several key issues remain open. Among them, one is the problem of moduli stabilization: in any of these compactifications, the four-dimensional low-energy action has a number of massless fields with no potential. These would lead to long-range scalar forces unobserved in nature. Furthermore, the couplings of other fields (like Yukawa couplings) depend on their vacuum expectation values (VEV's). As a consequence, no predictions can be made in these scenarios since the VEV of the moduli can take any value. Therefore, there should be a mechanism that generates a potential for the moduli, fixing (or "stabilizing") their VEV's. The only known mechanism within perturbative string theory that we know of today is via fluxes: turning on fluxes for some of the field strengths available in the theory (these are generalizations of the electromagnetic field to ten dimensions) generates a non trivial potential for the moduli, which stabilize at their minima. The new "problem" that arises is that fluxes back-react on the geometry, and whatever manifold was allowed in the absence of fluxes, will generically be forbidden in their presence.

Much of what we know about stabilisation of moduli is done in two different contexts: in Calabi-Yau compactifications under a certain combination of three-form fluxes whose back-reaction on the geometry just makes them conformal Calabi-Yau manifolds, and in the context of parallelizable manifolds, otherwise known as "twisted tori". In the former, fluxes stabilize the moduli corresponding to the complex structure of the manifold, as well as the dilaton. To stabilize the other moduli, stringy corrections are invoked. The result is that one can stabilize all moduli in a regime of parameters where the approximations can be somehow trusted, but it is very hard to rigorously prove that the corrections not taken into account do not destabilize the full system. In the latter, one takes advantage of the fact that, similarly to tori, there is a trivial structure group. However, the vectors that are globally defined obey some non-trivial algebra, and therefore compactness is ensured by having twisted identifications. What makes these manifolds amenable to the study of moduli stabilisation is the possibility of analyzing them as if they were a torus subject to twists, or in other words a torus in the presence of "geometric fluxes", which combined to the electromagnetic fluxes can give rise to solutions to the equations of motion. One can then use the bases of cycles of the tori, and see which ones of them get a fixed size (or gets "stabilized"), due to a balance of forces between the gravitational and the electromagnetic ones. It turns out that even in this simple situations, it is impossible

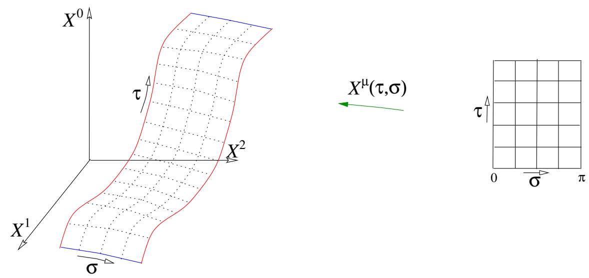

Figure 1: Embedding of an open string in spacetime by the function $X^\mu(\tau, \sigma)$.

to stabilize all moduli in a Minkowski vacuum without the addition of other "non-geometric" fluxes, which arise in some cases as duals to the known fluxes, but whose generic string theory interpretation is still under discussion.

In the last years the framework of generalized complex geometry has turned out to be an excellent tool to study flux backgrounds in more detail, in particular going away from the simplest cases of parallelizable manifolds. In these lectures, we review these tools and discuss the allowed manifolds in the presence of fluxes.

Generalized complex geometry is interesting from a mathematical viewpoint on its own, as it incorporates complex and symplectic geometry into a larger framework [6]- [8], and thereby finds a common language for the two sectors of geometric scalars that typically arise in compactifications. It does so by introducing a new bundle structure that covariantizes symmetries of string compactifications, T-duality among others. In particular, mirror symmetry between the type II string theories appears naturally in this setup.

These lectures shall give a brief introduction into generalized geometry and its appearance in string theory. For more comprehensive reviews and lecture notes, see for instance [11], [12], while introductions to string theory and D-branes can for example be found in [1]- [5].

1 Lecture 1: Introduction to String Theory

In this section we give a basic introduction to string theory, focusing on its low energy effective description. We start with the bosonic string, work out the massless spectrum, and introduce the low energy effective action governing its dynamics. We then continue with the superstring, showing the corresponding massless spectrum and action. More details can be found in [1]- [4].

1.1 Bosonic strings

String theory is a quantum theory of one-dimensional objects (*strings*) moving in a D-dimensional spacetime. Strings sweep a two dimensional surface, the "worldsheet", labeled by the coordinates σ along the string ($0 \leq \sigma \leq \pi$ for an open string and $0 \leq \sigma \leq 2\pi$ for a closed string with 0 and 2π identified), and τ.

The evolution of the worldsheet is given by the "Polyakov" action

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma \mathrm{d}\tau \sqrt{\gamma}\, \gamma^{\alpha\beta} \eta_{MN} \partial_\alpha X^M \partial_\beta X^N \ , \quad M = 0, ..., D-1 \ ; \ \alpha = \tau, \sigma \ , \tag{1.1}$$

which just measures the area of the string worldsheet inside D-dimensional spacetime. In this equation, X^M are the functions defining the embedding of the worldsheet in spacetime (see

Figure 1); $\gamma^{\alpha\beta}$ is the worldsheet metric, η_{MN} is the spacetime Minkowski metric and α' is related to the string tension T by

$$T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_s^2} \ . \tag{1.2}$$

l_s, which has units of length, is called the "string scale". Its inverse gives the "string mass" M_s (we are using units in which $c = \hbar = 1$, to obtain a mass from the inverse of the string scale we should multiply by $\hbar/c$), which defines the typical energy scale of strings.

Varying the action (1.1) with respect to X^M we get the following equations of motion

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^M(\tau,\sigma) = 0 \ . \tag{1.3}$$

By introducing left- and right-moving worldsheet coordinates $\sigma^\pm = \tau \pm \sigma$, the above equation can be rewritten as

$$\frac{\partial}{\partial\sigma^+}\frac{\partial}{\partial\sigma^-} X^M(\tau,\sigma) = 0 \ , \tag{1.4}$$

which means that the embedding vector X^M is the sum of left- and right-moving degrees of freedom, i.e.

$$X^M(\tau,\sigma) \quad = \quad X_R^M(\sigma^-) + X_L^M(\sigma^+) \ . \tag{1.5}$$

We will concentrate on closed strings from now on, and discuss very briefly the open string at the end of this section. Imposing the closed string boundary conditions $X^M(\tau,0) = X^M(\tau,2\pi)$, $X'^M(\tau,0) = X'^M(\tau,2\pi)$ (with prime indicating a derivative along σ) we get the following mode decomposition

$$\begin{aligned} X^M(\tau,\sigma) &= X_R^M(\sigma^-) + X_L^M(\sigma^+) \\ X_R^M(\sigma^-) &= \frac{1}{2}x^M + \alpha' p^M \sigma^- + \mathrm{i}\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0} \frac{1}{n}\alpha_n^M e^{-2\mathrm{i}n\sigma^-} \\ X_L^M(\sigma^+) &= \frac{1}{2}x^M + \alpha' p^M \sigma^+ + \mathrm{i}\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^M e^{-2\mathrm{i}n\sigma^+} \ , \end{aligned} \tag{1.6}$$

where x^M and p^M are the centre of mass position and momentum, respectively. We can see that the mode expansion for the closed string is that of a pair of independent left- and right-moving traveling waves. To ensure a real solution we impose $\alpha_{-n}^M = (\alpha_n^M)^*$ and $\tilde{\alpha}_{-n}^M = (\tilde{\alpha}_n^M)^*$.

Varying (1.1) with respect to the worldsheet metric $\gamma_{\alpha\beta}$ gives the extra constrains that the energy-momentum tensor is vanishing

$$T^{\alpha\beta} \equiv -\frac{2\pi}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma_{\alpha\beta}} = -\frac{1}{\alpha'}\left(\partial^\alpha X^M \partial^\beta X_M - \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\lambda\rho}\partial^\lambda X^M \partial^\rho X_M\right) = 0 \ . \tag{1.7}$$

This enforces the additional conditions

$$\eta_{MN}\partial_{\sigma^+}X_L^M\partial_{\sigma^+}X_L^N = \eta_{MN}\partial_{\sigma^-}X_R^M\partial_{\sigma^-}X_R^N = 0 \ . \tag{1.8}$$

The system governed by the action (1.1) can be quantized in a canonical way in terms of left- and right-moving oscillators, resulting in the following commutators

$$[\alpha_n^P, \alpha_m^Q] = [\tilde{\alpha}_n^P, \tilde{\alpha}_m^Q] = n\delta_{n+m}\eta^{PQ} \ , \quad [x^P, p^Q] = i\eta^{PQ} \ . \tag{1.9}$$

We can therefore interpret $\alpha_n^M, \tilde{\alpha}_n^M$ as creation operators and $\alpha_{-n}^M, \tilde{\alpha}_{-n}^M$ with $n > 0$ as annihilation operators, which create or annihilate a left or right moving excitation at level n.[1] Each mode carries an energy proportional to the level. The mass (energy) of a state is obtained using the operator

$$M^2 = \frac{2}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha_n \cdot \alpha_{-n} + \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n} - 2\right) \tag{1.10}$$

where the -2 comes from normal ordering the operators (corresponding to the zero point energy of all the oscillators). The classical conditions (1.8) in the quantum theory become the vanishing of the so-called Virasoro operators on the physical spectrum. We will not go into more details here (these can be found, for example, in [1]- [3]) but only mention the most important of these contraints, the level-matching condition that says that the operator

$$\hat{L}_0 = \frac{1}{2}\left(\sum_{n=1}^{\infty} \alpha_n \cdot \alpha_{-n} - \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}\right) \tag{1.11}$$

vanishes on the physical states. Thus, we have to impose $N = \tilde{N}$, where N, $\tilde{N}$ are the total sums of oscillator levels excited on the left and on the right, respectively. The massless states have therefore one left-moving and one right-moving excitation, namely

$$|\xi_{MN}\rangle \equiv \xi_M \tilde{\xi}_N \alpha_1^M \tilde{\alpha}_1^N |0\rangle \tag{1.12}$$

and a center of mass momentum k^M, $k \cdot k = 0$. It is not hard to check that the norm of the state is positive only if $\xi, \tilde{\xi}$ are space-like vectors. The classical conditions (1.8) impose $\xi \cdot k = \tilde{\xi} \cdot k = 0$, i.e., the polarization vectors have to be orthogonal to the center of mass momentum. Choosing a frame where $\vec{k} = (k, k, 0, ..., 0)$, we get that $\xi_M, \tilde{\xi}_M$ belong to the $D-2$-dimensional space parametrized by the coordinates $2, ..., D-1$. The states are therefore classified by their SO$(D-2)$ representations. The tensor $\xi_{MN} \equiv \xi_M \tilde{\xi}_N$ decomposes into

$$\xi_{MN} = \xi^s_{MN} + \xi_t\, \eta_{MN} + \xi^a_{MN} \ , \tag{1.13}$$

where we have defined

$$\xi_t \equiv \frac{1}{D}\eta^{MN}\xi_{MN} \ , \quad \xi^s_{MN} = \frac{1}{2}(\xi_{MN} + \xi_{NM} - 2\xi_t\, \eta_{MN}) \ , \ \xi^a_{MN} = \frac{1}{2}(\xi_{MN} - \xi_{NM}) \tag{1.14}$$

The state corresponding to the polarization ξ_{MN} is a massless state of spin 2: the graviton. The state corresponding to the scalar ξ_t is the dilaton, while the one given by an antisymmetric tensor is called the B-field.

The ground state of Hilbert space $|0\rangle$ has a negative mass square (it is $-\frac{4}{\alpha'}$). The appearance of this tachyon means that the bosonic string is unstable and will condensate to the true vacuum of the theory. We will see in the discussion of the superstring below how to remove the tachyon from the spectrum to obtain a well-behaved theory (this cannot be done for the bosonic string).

Strings moving in a curved background can be studied by modifying the action (1.1) by the following "σ-model" action [13]

$$S = -\frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau \sqrt{\gamma}\,\gamma^{\alpha\beta} G_{MN}(X)\partial_\alpha X^M \partial_\beta X^N \ , \quad M = 0, ..., D-1 \ ; \ \alpha = \tau, \sigma \tag{1.15}$$

This action looks like (1.1) but it now has field dependent couplings given by the spacetime metric $G_{MN}(X)$. The curved background can be seen as a coherent state of gravitons in the

[1]Note that the convention used here for creation (positive modes) and annihilation operators (negative modes) is opposite to the one used most often (for example in [2]).

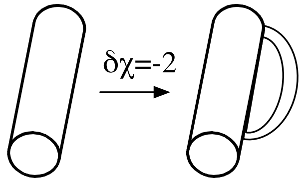

Figure 2: Worldsheet topology change due to the emission and absorption of a closed string. The second diagram has two powers of the string coupling g_s with respect to the first one.

following sense: If we consider a small deviation from flat space, $G_{MN} = \eta_{MN} + h_{MN}(X)$, with h small, and expand the path integral [14]

$$Z = \int \mathcal{D}X\mathcal{D}\gamma e^{-S} = \int \mathcal{D}X\mathcal{D}\gamma e^{-S_0}\left(1 + \frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau\sqrt{\gamma}\,\gamma^{\alpha\beta}h_{MN}\partial_\alpha X^M\partial_\beta X^N + ...\right) , \tag{1.16}$$

we find that the additional terms correspond to inserting a graviton emission vertex operator with $h_{MN} \propto \xi^s_{MN}$. Thus, a background metric is generated by a condensate (or coherent state) of strings with certain excitations. It is natural to try and further generalize the σ-model action (1.15) to include coherent states of the B-field and the dilaton. The natural reparametrization-invariant σ-model action is

$$S = -\frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau\sqrt{\gamma}\left[\left(\gamma^{\alpha\beta}g_{MN}(X) + \mathrm{i}\epsilon^{\alpha\beta}B_{MN}(X)\right)\partial_\alpha X^M\partial_\beta X^N + \alpha'\Phi R\right] . \tag{1.17}$$

The power of α' in the last term should be there in order to get the right dimensions. Note that α' in this action behaves like $\hbar$: the action is large in the limit $\alpha' \to 0$, which makes it a good limit to expand around.

Something remarkable happens with the dilaton Φ. Since it appears multiplied by the Euler density, it couples to the Euler number of the worldsheet

$$\chi = \frac{1}{4\pi}\int d^2\sigma(-\gamma)^{1/2}R . \tag{1.18}$$

In the path integral (1.16) the resulting amplitudes are weighted by a factor $e^{-\Phi\chi}$. On a 2-dimensional surface with h handles, b boundaries and c cross-caps, the Euler number is $\chi = 2 - 2h - b - c$. An emission and reabsorption of a closed string amounts to adding an extra handle on the worldsheet (see Figure 2), and therefore results in $\delta\chi = -2$. Therefore, relative to the tree level closed string diagram, the amplitudes are weighted by $e^{2\Phi}$. At the same time, this diagram has two powers of the closed string coupling constant g_s^2. This means that the closed string coupling is not a free parameter, but it is given by the VEV of one of the background fields: $g_s = e^{<\Phi>}$. Therefore, the only free parameter in the theory is the string tension (1.2), or the string energy scale $(1/\sqrt{\alpha'})$. However, without the introduction of fluxes the VEV of the dilaton is undetermined, as the action does not contain a potential term for it. This is actually the main motivation for considering backgrounds with fluxes, as the latter generate a potential whose minimum determines the dilaton VEV.

Although the σ-model action (1.17) is invariant under Weyl transformations[2] at the classical level, it is not automatically Weyl invariant in the quantum theory, and therefore the theory as

[2] A Weyl transformation is $\gamma_{\alpha\beta} \to e^\omega\gamma_{\alpha\beta}$.

it is is not consistent. Weyl invariance (also called conformal invariance) can be translated as a tracelessness condition on the energy momentum tensor (1.7). The expectation value of the trace of the energy momentum is given by [15]

$$T^{\alpha}_{\ \alpha} = -\frac{1}{2\alpha'}\beta^{G}_{MN} g^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N - \frac{i}{2\alpha'}\beta^{B}_{MN}\epsilon^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N - \frac{1}{2}\beta^{\Phi} R \ . \tag{1.19}$$

where we have defined

$$\begin{aligned}
\beta^{G}_{MN} &= \alpha'\left(R_{MN} + 2\nabla_M\nabla_N\Phi - \frac{1}{4}H_{MPQ}H_N^{\ \ PQ}\right) + O(\alpha'^2),\\
\beta^{B}_{MN} &= \alpha'\left(-\frac{1}{2}\nabla^P H_{PMN} + \nabla^P\Phi H_{PMN}\right) + O(\alpha'^2),\\
\beta^{\Phi} &= \alpha'\left(\frac{D-26}{6\alpha'} - \frac{1}{2}\nabla^2\Phi + \nabla_M\Phi\nabla^M\Phi - \frac{1}{24}H_{MNP}H^{MNP}\right) + O(\alpha'^2) \ ,
\end{aligned} \tag{1.20}$$

and H_{MPQ} is the field strength of B_{PQ}, i.e. $H_{MPQ} = \partial_{[M}B_{PQ]}$. We have written the one-loop contribution, while contributions from higher loops give higher orders in α' and therefore in derivatives. By setting these beta-functions to zero, we obtain *spacetime* equations for the fields. Therefore, Weyl invariance at one loop (in α') gives the spacetime equation of motion for the massless closed fields. The first equation in (1.20) resembles Einstein's equation, and we can see that a gradient of the dilaton as well as H-flux carry energy-momentum. The second equation is the equation of motion for the B-field, and it is the antisymmetric tensor generalization of Maxwell's equation. The third equation is the dilaton equation of motion. The first term in this last equation is striking: if the spacetime dimension D is not 26, then the dilaton or the B-field must have large gradients, of the order $1/\sqrt{\alpha'}$. If we do not allow for this, then the spacetime dimension must be 26.[3] The equations of motion (1.20) can actually be derived from the following spacetime action

$$\mathrm{S} = \frac{1}{2\kappa_0^2}\int \mathrm{d}^D X (-G)^{1/2} e^{-2\Phi}\left[R + 4\nabla_M\Phi\nabla^M\Phi - \frac{1}{12}H_{MNP}H^{MNP} - 2(D-26)3\alpha' + O(\alpha')\right] \ . \tag{1.21}$$

Summarizing, we have found that the massless closed string spectrum contains the graviton, a scalar and an antisymmetric two-form. Demanding Weyl invariance of the the worldsheet action, we have obtained their equations of motion. The equation for the metric resembles Einstein's equation, with the gradient of the other fields acting as sources. The B-field obeys a Maxwell-type equation, while from the equation of motion for the dilaton we have fixed the spacetime dimension to be 26.

One first remark is that the massless closed string spectrum does not contain a regular (one-form) gauge field. There is however a gauge field in the spectrum of massless open strings. The solution to the (worldsheet) equations of motion (1.3) imposing Neumann boundary conditions at $\sigma = 0, \pi$ (i.e. $\partial_\sigma X^M(\tau, 0) = \partial_\sigma X^M(\tau, \pi) = 0$) gives the following mode decomposition

$$X^M(\tau,\sigma) = x^M + 2\alpha' p^M \tau + \mathrm{i}(2\alpha')^{1/2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^M e^{-\mathrm{i}n\tau} cos(n\sigma) \ . \tag{1.22}$$

The mass operator is given by (cf. (1.10)

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_n\cdot\alpha_{-n} - 1\right) \ . \tag{1.23}$$

[3]There are solutions with $D \neq 26$, like the "linear dilaton" [16] (an exact solution with $\Phi = V_M X^M$, $|V|^2 = (D-26)/\alpha'$), but we will not discuss them here.

The massless states are therefore given by

$$|\xi_M\rangle \equiv \xi_M \alpha_1^M |0\rangle \tag{1.24}$$

which is a gauge field A_M. The σ-model action for the gauge field is boundary action

$$\int_{\partial M} d\tau A_M \partial_\tau X^M \tag{1.25}$$

Its equation of motion can be derived from the spacetime action

$$\mathrm{S} = -\frac{1}{4}\int \mathrm{d}^D X\, e^{-\Phi} F_{MN}F^{MN} + O(\alpha') \ , \tag{1.26}$$

where $F = dA$. This is the Yang-Mills action, with a field dependent coupling constant $g_o = e^{<\Phi/2>}$ which is the square root of the closed string coupling constant g_s.

Similarly, one can replace some of the Neumann boundary conditions by Dirichlet boundary conditions, restraining the endpoint of the open string to $p+1$-dimensional surface, a so-called Dp-brane (D for Dirichlet) [17]. The sigma-model action then depends on a $p+1$ dim. vector plus $D-p-1$ scalars on the boundary of the open string, similar to (1.25). It turns out that similar to the closed string, the VEV of these scalars determines the position of the Dp-brane. In other words, the dynamics of the open string corresponds to the dynamics of the Dp-brane, which therefore is a dynamical object.

1.2 Superstrings

So far we have found a graviton, a gauge field, a scalar and a two-form in the massless spectrum of open and closed strings. None of these is a fermion like the electron, or the quarks. In order to have fermions in the spectrum, we should introduce fermions in the worldsheet. We therefore introduce superpartners of X^M (i.e., the field related to X^M by a supersymmetry transformation): Ψ^M and $\tilde{\Psi}^M$. Each of them is a Majorana-Weyl spinor on the worldsheet, which has only one component (times $D-2$ for the index M). The combined action (1.1) for X^M and $\Psi^M, \tilde{\Psi}^M$ is

$$S = \frac{1}{4\pi}\int_M \mathrm{d}^2\sigma\, \eta_{MN}\left[\frac{1}{\alpha'}\partial X^M \bar{\partial} X^N + \Psi^M \bar{\partial}\Psi^N + \tilde{\Psi}^M \bar{\partial}\tilde{\Psi}^N\right] \tag{1.27}$$

where we have used the Euclidean worldsheet by sending $\tau \to \mathrm{i}\tau$, and the derivatives are with respect to the holomorphic coordinate $z = e^{\tau - \mathrm{i}\sigma}$. The equations of motion for Ψ and $\tilde{\Psi}$ admit two possible boundary conditions, denoted "Ramond" (R) and "Neveu-Schwarz" (NS)

$$\begin{aligned} \mathrm{R}: &\qquad \Psi^M(\tau,0) = \Psi^M(\tau,2\pi) \\ \mathrm{NS}: &\qquad \Psi^M(\tau,0) = -\Psi^M(\tau,2\pi) \end{aligned} \tag{1.28}$$

and similarly for $\tilde{\psi}$. When expanding in modes as in (1.6), the first type of boundary condition (R) will give rise to an expansion in integer modes, while the second type (NS) gives an expansion in half-integer modes. The (anti-)commutation relation for the modes are

$$\{\psi_r^M, \psi_s^N\} = \{\tilde{\psi}_r^M, \tilde{\psi}_s^N\} = \eta^{MN}\delta_{r+s} \ . \tag{1.29}$$

In a similar fashion as for the bosonic string, in the superstring conformal anomaly cancellation imposes a fixed dimension for spacetime: $D = 10$.

The quantized states have masses

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_n\cdot\alpha_{-n} + \sum_r r\psi_r\cdot\psi_{-r} - a + \text{same with tilde}\right) \ , \tag{1.30}$$

where the normal ordering constant a is zero for Ramond modes, and $1/2$ for NS modes. Furthermore, the level-matching condition says that physical states vanish under the action of the operator

$$\tilde{L}_0 = \left(\sum_{n=1}^{\infty}\alpha_n\cdot\alpha_{-n} + \sum_r r\psi_r\cdot\psi_{-r} - a - \text{same with tilde}\right) \ . \tag{1.31}$$

The massless states are therefore

$$\begin{aligned} \mathrm{R-R}: &\quad \xi_{MN}\psi_0^M\tilde{\psi}_0^N|0\rangle \ , &\qquad \mathrm{NS-R}: &\xi_{MN}\psi_{1/2}^M\tilde{\psi}_0^N|0\rangle \ , \\ \mathrm{NS-NS}: &\quad \xi_{MN}\psi_{1/2}^M\tilde{\psi}_{1/2}^N|0\rangle \ , &\qquad \mathrm{R-NS}: &\xi_{MN}\psi_0^M\tilde{\psi}_{1/2}^N|0\rangle \ . \end{aligned} \tag{1.32}$$

In the same way as for the bosonic string, the polarizations ξ_M have to be orthogonal to the center of mass momentum $\vec{k}$, and are therefore classified by SO(8) representations.

Let us now first discuss the left-moving states. We combine them with the right-moving states afterwards. Let us start with the Ramond states. Since Ψ_0 obeys a Clifford algebra, we can form raising and lowering operators $d^i_\pm = \frac{1}{\sqrt{2}}(\psi_0^{2i} \pm \psi_0^{2i+1})$, with $i = 1,..,4$. The Ramond ground states form a representation of this algebra labeled by s:

$$R: \psi_0^M|0\rangle \to |s\rangle = |\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}\rangle \tag{1.33}$$

where $\pm\frac{1}{2}$ are the chiralities in the four 2-dimensional planes[4]. The state $\psi_0^M|0\rangle$ is therefore a *spacetime* fermion, in the representation $\mathbf{16}$. The $\mathbf{16}$ reduces into $\mathbf{8}_s \oplus \mathbf{8}_c$. In order to get spacetime supersymmetry, we need only 8 physical fermions (to be the superpartners of the 8 X^M). We therefore would like to project out half of the fermions. This is consistently done by an operation called "GSO" projection (named after Gliozzi, Scherk, and Olive [18]), which requires $\sum_{i=1}^4 s_i = 0$ (mod 2). One can view the GSO projection as the projection to the (Majorana-)Weyl spinor subspace of definite chirality.

Let us now turn to the NS states. From the above definition one can see that the GSO projector anti-commutes with single creation and annihilation operators. This means that the GSO projection either projects out the (tachyonic) vacuum state $|0\rangle$ or the massless state

$$NS: \psi_{1/2}^M|0\rangle \ , \tag{1.34}$$

which forms a vector in ten-dimensional spacetime. This means that we can choose the GSO projection such that there is no tachyon in the superstring spectrum while we keep the massless NS states. As we will see, the latter is actually identical to the massless left-moving sector of the bosonic string.

Note that in transverse eight-dimensional spacetime the irreducible spin 1/2 and spin 1 representations have the same number of degrees of freedom: 8. We label the spinors of positive and negative chirality by $\mathbf{8}_s$ and $\mathbf{8}_c$, respectively, while the vector representation is called $\mathbf{8}_v$. Actually, there is a discrete symmetry interchanging the three of them. This is called "triality" (for the trio $\mathbf{8}_s$, $\mathbf{8}_c$, $\mathbf{8}_v$), and plays a very important role in string theory.

[4]The RR massless state $\xi_{MN}\psi_0^M\tilde{\psi}_0^N|0\rangle$ should therefore be undesrtood as $|s\rangle\otimes|\tilde{s}\rangle$.

We are ready to build the whole closed superstring massless spectrum. In the R-R sector, we have the product of two spinorial **8** representations, one for the left and one for the right movers. We can take the same GSO projection for the left movers as for the right movers, but we could as well take opposite projections. The first choice leads to the so-called type IIB superstring (which is chiral, since we chose the same chiralities on the left and on the right), while the other choice leads to the type IIA superstring. Decomposing the products into irreducible representations we get

$$\begin{aligned} \text{type IIB R}-\text{R}: \quad & \mathbf{8}_s \otimes \mathbf{8}_s = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+ = C_0 \oplus C_2 \oplus C_{4+} \\ \text{type IIA R}-\text{R}: \quad & \mathbf{8}_s \otimes \mathbf{8}_c = \mathbf{8}_v \oplus \mathbf{56}_t = C_1 \oplus C_3 \ . \end{aligned} \tag{1.35}$$

In this equation, we have defined C_n to be an n-form, called a R-R potential. In type IIB there are even-degree R-R potentials (with C_4 satisfying a duality condition $dC_4 = *dC_4$). In type IIA, the R-R potentials are odd.

From the NS-R and R-NS sectors we find the fermionic states

$$\begin{aligned} \text{type IIB} \quad & \text{NS}-\text{R}: \ \mathbf{8}_v \otimes \mathbf{8}_s = \mathbf{8}_s \oplus \mathbf{56}_s = \lambda^1 \oplus \Psi^{1\,M} \ , \\ & \text{R}-\text{NS}: \ \mathbf{8}_s \otimes \mathbf{8}_v = \mathbf{8}_s \oplus \mathbf{56}_s = \lambda^2 \oplus \Psi^{2\,M} \ , \\ \text{type IIA} \quad & \text{NS}-\text{R}: \ \mathbf{8}_v \otimes \mathbf{8}_c = \mathbf{8}_c \oplus \mathbf{56}_c = \lambda^1 \oplus \Psi^{1\,M} \ , \\ & \text{R}-\text{NS}: \ \mathbf{8}_s \otimes \mathbf{8}_v = \mathbf{8}_s \oplus \mathbf{56}_s = \lambda^2 \oplus \Psi^{2\,M} \ . \end{aligned} \tag{1.36}$$

The fermions in the **8** representation λ^A (A=1,2) are called the dilatini, and the ones in the **56**, $\Psi^{A,M}$ are the gravitini. The name "type II" refers to the fact that there are two gravitini.

Finally, in the NS-NS sector we get

$$NS-NS: \ \text{type IIA and IIB} \ \ \mathbf{8}_\mathrm{v} \otimes \mathbf{8}_\mathrm{v} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} = \Phi \oplus \mathrm{B}_2 \oplus \mathrm{G}_{\mathrm{MN}} \tag{1.37}$$

This sector is common to both theories and it has the same matter content the as the massless bosonic string: a scalar, the dilaton; a symmetric traceless tensor, the graviton; and a two-form, the B-field.

In the open strings, from the R and NS sectors we get respectively a fermion and a gauge field A_M, whose spacetime action is the supersymmetrized version of (1.26), i.e. the action for an Abelian (U(1)) gauge field and fermionic matter. The discussion of Chan-Paton factors, D-branes etc. carries over from the discussion for the bosonic superstring.

Note the magic that has happened: We started with worldsheet fermions and worldsheet supersymmetry and we ended up with massless spacetime fermions, spacetime bosons and (two) spacetime supersymmetries. This is known as the NSR formulation of superstrings. The spectrum of massive states has masses of order $1/\sqrt{\alpha'}$. At energies much lower than the string scale, these states can be neglected and one can just consider the massless spectrum, which forms the field content of the type II supergravities in ten dimensions.[5] When we discuss compactifications we will work in this limit.

Type IIA and IIB are not two independent theories, but they are related by a symmetry called T-duality (for a review see [19]). This symmetry is inherent to 1-dimensional objects: if we compactify one direction on an S^1 of radius R, the center of mass momentum for the string along that direction will be quantized, in units of $1/R$. On the other hand, there are "winding

[5]Note that the discussion of Weyl anomalies of the superstring is analogous to the one in (1.20) for the bosonic string. The vanishing of the beta functions correspond to the equations of motion of type IIA and IIB supergravity in ten dimensions.

states", namely states with boundary conditions $X^M(\tau, 2\pi) = X^M(\tau, 0) + 2\pi m R$. A string with n units of momentum, m units of winding and N, $\tilde{N}$ total number of oscillators on the left and on the right has a total mass given by (cf. Eq(1.10)

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{1.38}$$

This formula is symmetric under the exchange of n and m (or in other words winding and momentum states) if we also exchange R with α'/R! This means that large and small radius of compactification are dual, on one side the momentum modes are light, and winding modes are heavy, while on the T-dual picture winding modes will be light, and momentum would be the heavy modes. The exchange between winding and momentum amounts to exchanging $X_L^M + X_R^M$ with $X_L^M - X_R^M$, and in the superstring ψ^M with $\bar{\psi}^M$. This exchanges representations $\mathbf{8}_s$ with $\mathbf{8}_c$ and therefore the GSO projections, or in other words type IIA with IIB!

T-duality has also another important consequence for type II string theories: Since T-duality maps type IIA and IIB theory into each other, the composition of two T-dualities on different circles maps the theory non-trivially into itself and therefore provides a symmetry of type II string theory on an n-dimensional torus. These symmetries combine with the translational and rotational symmetries of such a torus into the discrete symmetry group $O(n, n, \mathbb{Z})$, the so-called T-duality group.

At the classical level, the massless superstring spectrum is symmetric even under *continuous* transformations in the group $O(8, 8)$. Apart from Lorentz-transformations in $Gl(8)$ that act equally on left- and right-moving excitations, this group incorporates rotations between left- and right-movers. More precisely, both left- and right-movers admit a Clifford algebra for $O(8)$ on their own. Together, they combine into the Clifford algebra of $O(8, 8)$. Both the R-R and NS-NS sector form irreducible representations of this group: The R-R fields combine into an $O(8, 8)$ spinor, while the metric and the B-field form a symmetric tensor, the $O(8, 8)$ metric. The dilaton remains a singlet under $O(8, 8)$.

The group $O(8, 8)$ is not only a classical symmetry of the massless fields, but is also locally a symmetry of their field equations in ten dimensions.[6] This means that we can make the corresponding type II supergravity and its field equations covariant under this group. The program to understand geometry in this covariant language is called generalized complex geometry and will be discussed in more detail in the next section. Generalized complex geometry turns out to be extremely helpful for understanding more complicated backgrounds of string theory, as we will see in Lecture 4.

Under T-duality, the Neumann boundary conditions along the T-dualized directions for the open string are switched to Dirichlet boundary conditions, or the other way around. This means that under T-duality, a Dp-brane changes its dimension and becomes a D$(p \pm 1)$-brane.

There is an analogous action to (1.21) for Dp-branes, which is called the Born-Infeld action (1.39)

$$S_{DBI} = -T_p \int \mathrm{d}^{p+1}\xi e^{-\Phi} \sqrt{\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} \tag{1.39}$$

(where $T_p = 8\pi^7 \alpha'^{7/2} (4\pi^2\alpha')^{-p/2}$ supplemented by the Chern-Simons term depending on RR potentials

$$S_{CS} = \mathrm{i} T_p \int_{p+1} e^{2\pi\alpha' F} \wedge C \tag{1.40}$$

[6]The symmetry group of the supergravity theory is in fact $O(10, 10)$ and thus larger, but the orthogonality of the oscillators reduces the symmetry on the spectrum to $O(8, 8)$, cf. the paragraph below (1.12).

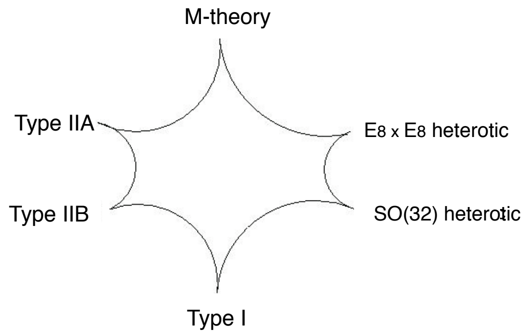

Figure 3: String theories as limits of one theory.

where C is the sum of the RR potentials, and the integral picks only the $p+1$-rank forms. In these expressions, the background fields should be pulled-back to the D-brane world-volume. For example, the pulled-back metric is

$$G_{ab}(\xi) = \frac{\partial X^M}{\partial \xi^a} \frac{\partial X^N}{\partial \xi^b} G_{MN}(X) \tag{1.41}$$

With N D-branes on top of each other, the open strings transform in adjoint representations of U(N). The excitations along the brane represent a U(N) gauge field and gaugino, while excitations orthogonal to the brane are bosons and fermions in the adjoint representation of U(N). The DBI action can actually be generalized to the case of a non-Abelian gauge group. With stacks of D-branes intersecting at angles, or D-branes placed at special singularities, the U(N) symmetry can for instance be broken to the gauge group of the Standard Model of particle physics, namely $SU(3) \times SU(2) \times U(1)$ (see for example [20] for a review).

In summary, we have seen that string theory relies on supersymmetry (the spectrum of the purely bosonic string has a tachyon, which signals an instability). We showed that in the massless spectrum of string theory there is a graviton, and there are also gauge fields and fermions in representations that can be that of the Standard Model of particle physics. As well as type IIA and type IIB, there are other string "theories", consisting, for example, of a mixture of the bosonic string on the left movers and the superstring on the right movers. These other "theories" are connected as well to type IIA and type IIB by dualities. There might therefore be a single string theory and various corners of it, or various low energy versions. One can move from one corner to another by varying the VEVs of scalar fields (in a similar fashion as the string coupling constant, or the radius of a circle). The space of VEVs of massless scalar fields is called moduli space, which is shown in Figure 3.

2 Lecture 2: Compactifications on tori

Compactification is a very old idea in physics. It originates back in 1921 when Theodor Kaluza tried to unify gravity with electromagnetism with the assumption of an extra dimension.[7] This additional dimension should be finite in order not to be observed (Klein 1926). Although the

[7] To be historically fair, the first person to introduce an extra dimension to unify gravity with electromagnetism was Gunnar Nordström in 1914.

original Kaluza-Klein idea did not work out, the concept of compactification became again popular more than half a century later in the framework of string theory. As we saw in the previous chapter, consistent superstring theories can live only in a ten-dimensional spacetime. The compactification scheme then goes as follows: The ten-dimensional spacetime is divided into an external non-compact spacetime $\mathcal{M}_{10-d}$ and an internal compact space $\mathcal{M}_d$, so that it reads

$$\mathcal{M}_{10} = \mathcal{M}_{10-d} \times \mathcal{M}_d \ . \tag{2.1}$$

The physically interesting case is of course $d = 6$. The compactification scale is $M_c = 1/R$ (R the typical length scale associated with the internal space) and is considered much smaller than the string scale $M_s = 1/l_s$. We will work at energies $E \ll M_c \ll M_s$.

2.1 Kaluza-Klein compactifications on S^1 (in field theory)

Example 1

As a warm-up, we consider a real massless scalar field living in five dimensions $x^M = (x^\mu, x^4)$, $\mu = 0, 1, 2, 3$ with a product structure

$$ds^2 = g_{\mu\nu}(x^\mu)dx^\mu dx^\nu + (dx^4)^2 \tag{2.2}$$

and the fifth dimension compactified $x^4 \cong x^4 + 2\pi R$. Then, the five-dimensional field can be expanded in Fourier modes

$$\Phi(x^M) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu)e^{inx^4/R} \tag{2.3}$$

and the 5D equation of motion $\nabla_M\nabla^M\Phi = 0$ splits into

$$(\nabla_\mu\nabla^\mu - n^2/R^2)\phi_n = 0 \tag{2.4}$$

where the field ϕ_0 is real and the fields $\phi_n, n \geq 1$ are complex. We see that the four-dimensional field theory describes particles which have masses integer multiples of the compactification scale $M_c = 1/R$. Hence, it is safe to ignore the massive modes (the so-called Kaluza-Klein tower) if we work at energies $E \ll M_c$.

Example 2

The next example we are going to consider is a five-dimensional Einstein gravity with the fifth dimension compactified as previously. The metric can be parametrized as follows

$$ds^2 = G_{MN}dx^M dx^N = g_{\mu\nu}dx^\mu dx^\nu + g_{44}(dx^4 + A_\mu dx^\mu)^2 \tag{2.5}$$

The above metric is invariant under five-dimensional general coordinate transformations $x^M \to x'^M(x^N)$. Adopting a four-dimensional point of view for our geometry , i.e. restricting ourselves only to transformations $x^\mu \to x'^\mu(x^\nu)$ splits the original metric into

- a scalar field $G_{44} = g_{44} = e^{2\sigma}$
- a vector field $G_{\mu4} = e^{2\sigma}A_\mu$
- a metric $G_{\mu\nu} = g_{\mu\nu} + e^{2\sigma}A_\mu A_\nu$

These fields are naturally invariant under four-dimensional coordinate transformations. Considering $x^4 \to x^4 + \lambda(x^\mu)$, we see that the metric is invariant if the vector field suffers a $U(1)$ gauge transformation

$$A \to A - d\lambda \tag{2.6}$$

implying that electromagnetism can be embedded in a higher-dimensional geometry (Kaluza-Klein idea).

Exercise: Start from $S = \frac{1}{16\pi G_5^N}\int d^5x\sqrt{-G}\left(R_5 + (\partial_M\Phi)^2\right)$ in order to conclude for the four-dimensional action $S = \frac{1}{16\pi G_4^N}\int d^4x\sqrt{-g}\left(R_4 + F_{\mu\nu}^2 + (\partial_\mu\sigma)^2 + (\partial_\mu\Phi_0)^2 + \text{tower}\right)$.[8] Normalize the various terms using an appropriate Weyl transformation.

The field σ is called the *radion* and contributes to the size of the extra dimension through the determinant of the 5D metric. It has no potential (and hence zero mass), so its value cannot be determined only by using the equations of motion. Such massless fields are called *moduli* and they are not observed in nature[9]. The four-dimensional gravitational strength is described by G_4^N and is given by

$$\frac{2\pi R_{\text{phys}}}{G_5^N} = \frac{1}{G_4^N} \tag{2.7}$$

where $R_{\text{phys}} = R\sqrt{G_{44}}$ is the size of the extra dimension containing both the compactification radius R and the radion σ.

2.2 Compactifications of the bosonic string on S^1

After examining the implications of a compactified dimension in field theory, we proceed to the case of string theory. In this section, we will consider a bosonic string moving in $\mathcal{M}_{10} = \mathcal{M}_9 \times S^1$.[10] Without any compactification, the mode expansion would be as in (1.6). From the intuition we obtained from the previous examples, we expect the momentum along the compactified dimension to be quantized $p^9 = n/R$. However, strings are not point particles: they can also wind around S^1. Therefore, instead of the periodicity condition $X^9(\tau, \sigma + 2\pi) = X^9(\tau, \sigma)$, we now require

$$X^9(\tau, \sigma + 2\pi) = X^9(\tau, \sigma) + 2\pi R w \tag{2.8}$$

where w is an integer called the *winding number*. Then, the mode expansion takes the form

$$X_L^9(\sigma^+) = \left(\frac{\alpha' n}{R} + wR\right)\sigma^+ + \text{oscillators} \tag{2.9}$$

$$X_R^9(\sigma^-) = \left(\frac{\alpha' n}{R} - wR\right)\sigma^- + \text{oscillators} \tag{2.10}$$

The states of the theory acquire two additional labels (n, w). We also have to take into account the Virasoro constraints. $L_0 + \tilde{L}_0 = 2$ gives the mass spectrum

$$M^2 = \left(\frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2}\right) + \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right) \tag{2.11}$$

while the $L_0 - \tilde{L}_0 = 0$ yields the *modified* level-matching condition

$$N - \tilde{N} + nw = 0 \tag{2.12}$$

The massless states of the theory are of particular interest here. At a generic R, these can be obtained by setting $n = w = 0$ and $N = \tilde{N} = 1$. Without excitations in the x^9-direction, we

[8] The expressions in the parentheses are schematic. They just denote the field content.

[9] The four-dimensional dilaton is also a modulus.

[10] In order to make contact with the superstring, we assume a ten-dimensional spacetime forgeting for a moment about consistency. The discussion is completely analogous for the NS-NS sector of the superstring.

get a metric $g_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$ and scalar Φ, as would be the case for a string theory in nine flat spacetime directions. In the present case, we also get additional states by acting with the creators $\alpha^9_{-1}, \tilde{\alpha}^9_{-1}$ resulting in two gauge fields $A_{\mu\pm} = g_{\mu 9} \pm B_{\mu 9}$ in the adjoint representation of $U(1) \times U(1)$.

Exercise: Show that at $R = \sqrt{\alpha'}$ new massless states arise and the gauge symmetry is enhanced: $U(1) \times U(1) \to SU(2) \times SU(2)$.

We can easily see from (2.11) that we now have two towers of massive modes, the familiar Kaluza-Klein (KK) states described by n and the winding states described by w. At $R \gg \sqrt{\alpha'}$ the KK states are light and the winding states are heavy while at $R \ll \sqrt{\alpha'}$ the opossite happens. A closer look at (2.11) shows that the mass spectrum is invariant under the change

$$n \leftrightarrow w, \quad R \to \alpha'/R \tag{2.13}$$

This reveals a new stringy symmetry which is called *T-duality*. The effect of T-duality on the string coordinates is

$$X^9_L \to X^9_L \tag{2.14}$$

$$X^9_R \to -X^9_R \tag{2.15}$$

This results for example in the exchange of $g_{\mu 9}$ and $B_{\mu 9}$. We usually demand that $R \gg \sqrt{\alpha'}$ (or $R \ll \sqrt{\alpha'}$ and then work in the T-dual frame). For energies $E \ll 1/R \ll R/\alpha'$, we can forget about both Kaluza-Klein and winding states. If one considers the superstring, the above analysis also holds for the NS sector. In the R sector T-duality has another feature: It changes the chirality of the spinor that is preserved by the GSO-projection. Therefore, T-duality exchanges type IIA and type IIB.

2.3 Toroidal string compactifications

Up to now, the compactifications we have considered were on a single circle. However, in order to have some hope to describe the real world, we should compactify on higher-dimensional manifolds in order to end up with 4 noncompact dimensions. So, as a next step let us try

$$\mathcal{M}_{10} = \mathcal{M}_{10-d} \times T^d \tag{2.16}$$

where T^d is the d-dimensional torus defined by the identification

$$x^m \cong x^m + 2\pi R^m, \quad m = 1, \ldots, d \tag{2.17}$$

Generically, the massive states have at least one of the (w^m, n_m) non-zero. In this case, it is convenient to collect all the KK and winding numbers in a $2d$ vector

$$N^M = (w^m, n_m) \tag{2.18}$$

transforming in the fundamental representation of $SO(d, d, \mathbb{Z})$. The group $SO(d, d, \mathbb{Z})$ is called the T-duality group and is the discrete subgroup with integer-valued matrices in $SO(d, d)$, which leaves the matrix

$$G_{MN} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} , \tag{2.19}$$

invariant. Here the T-duality group leaves invariant the mass spectrum which can be written as

$$M^2 \sim \begin{pmatrix} w & n \end{pmatrix} \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \begin{pmatrix} w \\ n \end{pmatrix} = N^M H_{MN} N^N \tag{2.20}$$

where we introduced

$$H_{MN} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \ . \tag{2.21}$$

For example, T-duality in all directions exchanges ($w^m \leftrightarrow n_m$) and therefore corresponds to

$$T = \begin{pmatrix} 0 & 1_{d\times d} \\ 1_{d\times d} & 0 \end{pmatrix} \in O(d,d) \tag{2.22}$$

Also the level matching condition can be written in a $SO(d,d,\mathbb{Z})$-covariant way as

$$N - \tilde{N} + N^M G_{MN} N^N = 0 \ . \tag{2.23}$$

In general, the T-duality group $SO(d,d,\mathbb{Z})$ is generated by

- discrete (d-dimensional) rotations that map the torus (or the lattice defining the torus) non-trivially onto itself,
- discrete "boosts", that correspond to pairs of T-dualities on two circles on the torus and
- shifts of the B-field by an integer amount of a two-dimensional volume form on the torus (a *large gauge transformation*).

We procceed now to the massless states, arising from two oscillation modes. The following table describes the dimensional reduction of the massless bosonic spectrum in the NSNS sector on T^d. As an illustrative example we take $d = 2$, namely the compactification is performed on a 2-dimensional torus.

Fields in 10-dim. theory	Fields in (10-d)-theory	Example for d=2
metric g_{MN}	$g_{\mu\nu} \to 1$ metric	1 metric
	$g_{\mu m} \to$ d vectors	2 vectors
	$g_{mn} \to$ d(d+1)/2 scalars	3 scalars
Kalb-Ramond B_{MN}	$B_{\mu\nu} \to 1$ Tensor	1 Tensor
	$B_{\mu m} \to$ d vectors	2 vectors
	$B_{mn} \to$ d(d-1)/2 scalars	1 scalar
dilaton Φ	$\Phi \to 1$ scalar	1 scalar

Exercise: consider the full massless spectrum of the superstring and complete the above table with the new states. Show that all these states are organized in multiplets of $\mathcal{N} = 8$ 4-dimensional supergravity.

In the case of T^2, parametrized by the coordinates x^8 and x^9, from the 4 massless scalars g_{88}, g_{89}, g_{99} and B_{89}, we can form 2 complex scalars ρ and τ. The first is defined by

$$\rho = B_{89} + \mathrm{vol}(T^2) \tag{2.24}$$

and is called the *Kähler modulus*. The *complex structure modulus* τ is then defined through its appearance in the T^2-metric as

$$ds_2^2 = \frac{\mathrm{Im}\rho}{\mathrm{Im}\tau}|dx^8 + \tau dx^9|^2 \equiv \frac{\mathrm{Im}\rho}{\mathrm{Im}\tau}\, dz d\bar{z} \tag{2.25}$$

or in matrix form

$$g_2 = \frac{\mathrm{Im}\rho}{\mathrm{Im}\tau} \begin{pmatrix} 1 & \mathrm{Re}\tau \\ \mathrm{Re}\tau & |\tau|^2 \end{pmatrix} \ . \tag{2.26}$$

Exercise: take $\mathrm{Im}\rho = 1$ and prove that we can write $g_2 = \partial_\tau \partial_{\bar{\tau}} K$ where $K = -\log\left(i \int \Omega \wedge \bar{\Omega}\right)$ and $\Omega = dx^8 + \tau dx^9$.

$$\frac{SL(2,\mathbb{R})\times SL(2,\mathbb{R})}{U(1)\times U(1)}\ .$$

However, there are discrete symmetries (like T-duality) that give equivalent backgrounds. A coordinate transformation $X\to LX$, where $X=(x^8,x^9)$ and $L\in SL(2,\mathbb{Z})$ gives the same physics, and transforms τ by fractional linear transformations. Therefore the inequivalent (in the sense of corresponding to different backgrounds) values of τ live in the coset

$$\frac{SL(2,\mathbb{R})}{U(1)\times SL(2,\mathbb{Z})}\ .$$

Similarly, the Kähler modulus enjoys the same symmetries, as on one hand shifting the B-field by an integer number (i.e. $\rho\to\rho+N$) does not change the physics, and on the other it is not hard to check that T-duality on T^2 acts by $\rho\to -1/\rho$. These two generate an $SL(2,\mathbb{Z})$ on ρ. Therefore, the moduli space describing inequivalent compactifications is

$$\frac{SL(2,\mathbb{R})}{SL(2,\mathbb{Z})\times U(1)}\times\frac{SL(2,\mathbb{R})}{SL(2,\mathbb{Z})\times U(1)}\cong\frac{SO(2,2,\mathbb{R})}{SO(2,\mathbb{R})\times SO(2,\mathbb{R})\times SO(2,2,\mathbb{Z})}\tag{2.27}$$

As an aside note, in the case of T^2 compactifications, if we add the RR sector as well, the symmetry group enlarges to $SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$ and is called *U-duality.*

The expression on the right hand side of (2.27) generalizes directly to d dimensions, namely the moduli space of compactifications of the metric and B-field on a T^d is

$$\frac{SO(d,d,\mathbb{R})}{SO(d,\mathbb{R})\times SO(d,\mathbb{R})\times SO(d,d,\mathbb{Z})}\ .\tag{2.28}$$

This group will come back repeatedly on Lecture 4.

3 Lecture 3: Calabi-Yau Compactifications

Compactifying on tori leaves all supersymmetries unbroken, which means that there are still 32 supercharges in four dimensions, or in other words $\mathcal{N}=8$ supersymmetry. This theory is very interesting for its simplicity and uniqueness in admitting such an amount of supersymmetry, but not so much for phenomenology, as it is unable to mimic most of the physics observed in nature. Requiring the compactification scheme to preserve less supersymmetry than $\mathcal{N}=8$ in four dimensions calls for more sophisticated manifolds than tori. Here we review these constructions.

In general one would be most interested in non-supersymmetric backgrounds of string theory, as supersymmetry is not observed at low energies. However, supersymmetry provides us with a very strong tool to understand string backgrounds and their corrections. Therefore, in general those models are preferred where supersymmetry is broken at scales comparably low to the compactification scale.

Obviously, not just any manifold can be picked as internal space. For starters, for the ten-dimensional spacetime to be a product space $M_{10}=M_4\times M_6$, the first equation of motion in (1.20) (or rather its version in the superstring, whose NSNS piece looks like that in (1.20), but also has contributions from the RR fields) tells us that in the absence of flux the manifolds must be Ricci-flat, i.e.

$$R_{mn}=0\ .\tag{3.1}$$

In the following we want to find a four-dimensional vacuum, in other words a solution where M_4 is four-dimensional Minkowski spacetime and all fields have an expectation value invariant under the Poincar'e group.[11] This means that the vacuum expectation value of any field except scalar fields is zero, while scalar expectation values must be constant.

Supersymmetry imposes an additional constraint: a supersymmetric vacuum where only bosonic fields have non-vanishing vacuum expectation values should obey $< Q_\epsilon \chi > = < \delta_\epsilon \chi > = 0$, where Q is the supersymmetry generator, ϵ is the supersymmetry parameter and χ is any fermionic field[12]. In type II theories, the fermionic fields are two gravitini ψ_M^A, $A = 1, 2$ and two dilatini λ^A. In the supergravity approximation, the bosonic part of the supersymmetry transformation of the gravitini is

$$\delta\psi_M = \nabla_M \epsilon + \frac{1}{4} \not{H}_M \mathcal{P} \epsilon + \frac{1}{16} e^{\phi} \sum_n \not{F}_n^{(10)} \, \Gamma_M \mathcal{P}_n \, \epsilon \,, \tag{3.2}$$

where $M = 0, ..., 9$, ψ_M stands for the column vector

$$\psi_M = \begin{pmatrix} \psi_M^1 \\ \psi_M^2 \end{pmatrix} \,, \tag{3.3}$$

containing the two Majorana-Weyl spinors of the same chirality in type IIB, and of opposite chirality in IIA, and similarly for ϵ, the 2×2 matrices $\mathcal{P}$ and $\mathcal{P}_n$ are different in IIA and IIB: for IIA $\mathcal{P} = \Gamma_{11}$ and $\mathcal{P}_n = \Gamma_{11}^{(n/2)} \sigma^1$, while for IIB $\mathcal{P} = -\sigma^3$, $\mathcal{P}_n = \sigma^1$ for $\frac{n+1}{2}$ even and $\mathcal{P}_n = i\sigma^2$ for $\frac{n+1}{2}$ odd. A slash means a contraction with gamma matrices in the form $\not{F}_n = \frac{1}{n!} F_{P_1 \ldots P_N} \Gamma^{P_1 \ldots P_N}$, and $H_M \equiv \frac{1}{2} H_{MNP} \Gamma^{NP}$. When no fluxes are present, demanding zero VEV for the gravitino variation (3.2) requires the existence of a covariantly constant spinor on the ten-dimensional manifold, i.e. $\nabla_M \epsilon = 0$. Spliting the supersymmetry spinors into four-dimensional and six-dimensional spinors in the form

$$\begin{aligned} \epsilon_1 &= \xi_{1+}^i \otimes \eta_+^i + h.c., \\ \epsilon_2 &= \xi_{2+}^i \otimes \eta_\mp^i + h.c. \;, \end{aligned} \tag{3.4}$$

where $i = 1, .., N$ ($N \leq 4$) will indicate the number of preserved supersymmetries in four-dimensions, the subindices denote chirality in four and six dimensions, and upper (lower) sign is for type IIA (type IIB), each four and six-dimensional pieces must be covariantly constant. In particular if the four-dimensional space is Minkowski, then the four-dimensional spinor is just constant. The six-dimensional space is then required to have a covariantly constant spinor, i.e. there should exist at least one spinor on the compactification manifold η such that

$$\nabla_m \eta = 0 \;. \tag{3.5}$$

This puts a strong requirement on the internal geometries. Such manifolds are called "Calabi-Yau" (CY).[13] It is not hard to show that (3.5) implies that the manifold is Ricci-flat, but the converse is not necessarily true. For each covariantly constant spinor on the internal manifold, we recover two four-dimensional supersymmetry parameters. In this case we find two four-dimensional supersymmetry parameters ξ_1 and ξ_2, and therefore the four-dimensional supersymmetry preserved is $\mathcal{N} = 2$. We will now dwell into the properties of Calabi-Yau manifolds. Excellent, comprehensible lecture notes on the subject of Calabi-Yau manifolds are [21].

[11] More generally also de Sitter and anti de Sitter and their corresponding symmetry groups can be chosen as vacuum state.

[12] The same should be true for bosonic fields, but their supersymmetry transformation is proportional to fermionic fields whose vacuum expectation value has to vanish in order to preserve Lorentz symmetry in four dimensions

[13] In the strict sense, Calabi-Yau's are manifolds that have only one covariantly constant spinor.

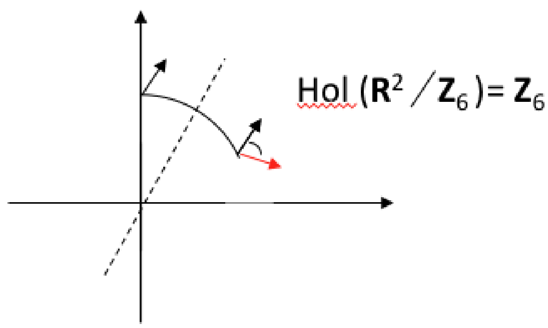

Figure 4: A vector parallel transported around the closed loop in $\mathbb{R}^2/Z_6$ comes back to itself up to a Z_6 rotation.

3.1 The geometry of Calabi-Yau manifolds

As we said, the internal manifold in the compactification scheme is required to have a covariantly constant spinor. This puts both an algebraic (topological) and a differential condition on the structure of the manifold.

The algebraic condition is that there should exist an everywhere non-vanishing Weyl spinor. More precisely, this means that at each point contained in the overlap of two patches, the spinor should be identical in both patches. Therefore, the transition functions (which form a representation of $Spin(6) \cong SU(4)$) should respect this property. In other word they can rotate the other three spinors among themselves, but should leave this spinor invariant. This results in a reduction of the group of transition functions, the *structure group*, to $SU(3)$.

The differential condition is the integrability of the spinor, which is expressed as $\nabla_m \eta = 0$. This imposes stronger constraints on the manifold, namely that it should have $SU(3)$ holonomy. The *holonomy group* is defined as the group of transformations suffered by vectors or spinors when they are parallel transported with the Levi-Civita connection around a closed loop. In order to understand what holonomy is, we will examine the following illustrative example.

It is obvious that two-dimensional flat space $\mathbb{R}^2$ has trivial holonomy (only the identity element), as it has trivial connection. However, if we consider $\mathbb{R}^2/\mathbb{Z}_6$ (where points are identified if the angle they form with the origin is 60 degrees), the holonomy group reduces to $\mathbb{Z}_6$, i.e. vectors parallel transported around closed loops (in the $\mathbb{Z}_6$ sense) can be rotated in multiples of $2\pi/6$. This is illustrated in Figure 4.

An alternative (and equivalent) definition of CY manifolds is that these are Kähler (i.e. complex and symplectic) manifolds with $c_1 = 0$. Let us define these in turn, and then see how this definition relates to the one of $SU(3)$ holonomy.

3.1.1 Complex manifolds

Almost complex manifolds of even dimension $d = 2n$ are those where one can define a real map I from the tangent space to itself such that

$$I^2 = -\mathbf{1} \ .$$

This map is called the almost complex structure, as having n $(+i)$ eigenvalues and n $(-i)$ eigenvalues, allows to define holomorphic and antiholomorphic vectors. Let us see how this works with a very simple example in $d = 2$, with the almost complex structure given by

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

The eigenvectors are then given by

$$\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} \equiv \frac{\partial}{\partial z} , \qquad \frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2} \equiv \frac{\partial}{\partial \bar{z}} .$$

Given an almost complex structure for the tangent space, one can define the projectors

$$P_\pm = \frac{1}{2}(\mathbf{1} \mp iI)$$

which project onto the holomorphic and anti-holomorphic bundles.

There are a few equivalent ways of defining integrability of an almost complex structure. Here we use the one that is easily generalized to generalized almost complex structures that will appear in the next lecture. We say that I is integrable if and only if

$$P_-[P_+v, P_+w] = 0 , \tag{3.6}$$

i.e. the Lie bracket[14] of two holomorphic vectors has to be holomorphic as well. The adjective "integrable" is used because if (and only if) I is integrable, there are complex coordinates z^i ($i = 1, .., n$) such that $(dz)^i = d(z^i)$ give a set of holomorphic one-forms. In other words, the holomorphic one-forms dz can be *integrated* globally to find the coordinates z. An integrable almost complex structure is called a complex structure (with the corresponding manifold being called complex manifold).

On an almost complex manifold, any n-form A

$$A = \frac{1}{n!}A_{m_1\ldots m_n} dx^{m_1} \wedge \ldots \wedge dx^{m_n} ,$$

can be decomposed into its holomorphic and anti-holomorphic components, as:

$$A_{p,q} = \frac{1}{p!q!}A_{i_1\ldots i_p \bar{\jmath}_i \ldots \bar{\jmath}_q} dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{\jmath}_1} \wedge \ldots \wedge d\bar{z}^{\bar{\jmath}_q} , \tag{3.7}$$

where $p + q = n$. If the manifold is even complex, the exterior derivative

$$dA = \frac{1}{n!}\partial_{[m_1} A_{m_2\ldots m_{n+1}]} dx^{m_1} \wedge \ldots \wedge dx^{m_{n+1}} , \tag{3.8}$$

can be split into a holomorphic and an antiholomorphic pieces, namely $d = \partial + \bar{\partial}$ where

$$\partial : (p, q) \to (p + 1, q) , \qquad \bar{\partial} : (p, q) \to (p, q + 1) .$$

Complex manifolds look locally like $\mathbb{C}^n$, and have holonomy $GL(n, \mathbb{C})$ (or a subgroup thereof), since parallel transport preserves the complex structure (or in other words it preserves holomorphicity).

[14] The definition of Lie bracket is as follows:

$$[v, w]^n = (v^m\partial_m w^n - w^m\partial_m v^n)\partial_n .$$

3.1.2 Symplectic manifolds

A manifold is symplectic if there is a globally-defined nowhere vanishing two-form J such that

$$dJ = 0 \ . \tag{3.9}$$

This defines a symplectic product

$$< v, w >_J \equiv J(v, w) = v^m J_{mn} w^n \tag{3.10}$$

Similarly to complex coordinates z^i for complex manifolds, in symplectic manifolds one can define Darboux coordinates (x^i, y^i) such that

$$J = \sum_{i=1}^{n} dx^i \wedge dy^i \tag{3.11}$$

A symplectic manifold has holonomy $Sp(2n)$ (or a subgroup thereof), as parallel transport preserves J.

3.1.3 Kähler manifolds

Käher manifolds are complex and symplectic manifolds for which the complex and symplectic structures are compatible. This means that J is $(1,1)$ in terms of I, i.e.

$$J = J_{i\bar{\jmath}} dz^i \wedge d\bar{z}^{\bar{\jmath}} \ . \tag{3.12}$$

This also implies that the complex and symplectic groups intersect in $U(3)$, i.e. $GL(n, \mathbb{C}) \cap Sp(2n) = U(3)$, and the holonomy of the manifold must be contained in this $U(3)$.

The complex structure I and the symplectic two-form J together define the metric g via

$$g_{mn} = J_{mp} I^p{}_n \ .$$

This metric can be derived from a real scalar function K called the Kähler potential, namely[15]

$$J_{i\bar{\jmath}} = i g_{i\jmath} = i\partial_i \bar{\partial}_{\bar{\jmath}} K$$

We are almost there on our way to Calabi-Yau manifolds. Only the last requirement, $c_1 = 0$, remains to be explained. To understand what it means, we need to introduce the notion of cohomology classes.

3.1.4 Cohomology classes

The exterior derivative given in (3.8) is a nilpotent operator (of order two), i.e. $d^2 = 0$, and as such it allows to define cohomology classes, as will be explained now. Let us make the following definitions:

- Closed p-forms A_p are p-forms such that $dA_p = 0$. Let us call $C^p(\mathcal{M})$ the space of closed p-forms.

[15] Note that the Kähler potential is not uniquely defined, but a so-called Kähler transformation $K \to K + f(z) + \bar{f}(\bar{z})$ leaves the metric invariant. Moreover, the Kähler potential might only be locally defined.

- Exact p-forms B_p are such that $B_p = dC_{p-1}$ for some globally defined $(p-1)$-form C_{p-1}.[16] Let us call $Z^p(\mathcal{M})$ the space of exact forms.

Then the p-th de Rham cohomology H^p is defined by

$$H^p(\mathcal{M}) = C^p(\mathcal{M})/Z^p(\mathcal{M}) \ .$$

The elements of H^p are equivalence classes of closed forms and are called cohomology classes. Two forms A_p and $\tilde{A}_p$ are in the same cohomology class ($A \simeq \tilde{A}$) if $A_p = \tilde{A}_p + dC_{p-1}$ for some $(p-1)$-form C_{p-1}. Note that though both the closed and exact forms form vector spaces of infinite dimension, the dimension of the H^p are finite and define the Betti numbers b_p, i.e.

$$\mathrm{dim} H^p(\mathcal{M}) = b_p \ . \tag{3.13}$$

These are topological invariants, i.e. independent of the metric that we define on the manifold. If the manifold is orientable (and compact), Betti numbers have the property[17]

$$b_p = b_{d-p} \ .$$

On complex manifolds one can define cohomology classes for (p,q) forms. Furthermore, one can define another cohomology called the Dolbeault cohomology, which is analogous to the de Rham cohomology but with the exterior derivative d replaced by the holomorphic derivative ∂. On Kähler manifolds, the de Rham and the Dolbeault cohomology theories coincide, namely

$$H_d^{p,q} = H_\partial^{p,q} = H_{\bar{\partial}}^{p,q} \ . \tag{3.14}$$

The dimensions of these cohomology classes are denoted by the Hodge numbers $h^{p,q}$ and satisfy the following properties

$$\sum_{k=0}^{p} h^{k,\,p-k} = b_p \ , \qquad h^{p,q} = h^{q,p} = h^{n-p,n-q} \ . \tag{3.15}$$

It is customary to arrange these numbers in the following diamond (we give the example for six-dimensional manifolds)

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array} \tag{3.16}$$

Using (3.15) one can see that the diamond is symmetric under reflections through the horizontal and vertical axes. Furthermore, $h^{0,0}$ simply counts the connected components so that for connected manifolds one finds $h^{0,0} = h^{3,3} = 1$.

[16] Note that (as stated by the Poincar'e Lemma) all closed forms are *locally* exact, i.e. exact in a patch. Here by exact we mean *globally* exact, i.e. exact on the whole manifold.

[17] This fact follows from the properties of the Hodge star operator that we will define in (3.28). This property is not true for non-orientable manifolds: since the manifold is non-orientable, the volume form is not well-defined and we have $b_d = 0$, while of course b_0 is larger than zero and counts the number of connected components.

A particularly important cohomology class is the one in which the Ricci two-form lives. The latter is defined as

$$\mathfrak{R} \equiv R_{mnpq} J^{pq} dx^m \wedge dx^n \ .$$

This form is closed on Kähler manifolds, $d\mathfrak{R} = 0$, and therefore allows to define a cohomology class, the first Chern class[18]

$$c_1 = \frac{1}{2\pi}[\mathfrak{R}] \ . \tag{3.17}$$

3.1.5 Calabi-Yau manifolds

On Calabi-Yau manifold the holonomy is $SU(3)$. Equivalently, CY manifolds are Kähler manifolds on which the first Chern class is trivial ($c_1 = 0$), or in other words $\mathfrak{R} = dA$ for some globally defined one-form A. Calabi conjectured, and Yau proved that these manifolds always admit a Ricci-flat metric (i.e. there is always a metric for which $A = 0$).

The Hodge diamond (3.16) is quite simple for CY manifolds. First there is $h^{1,0} = 0$ and $h^{2,0} = 0$, with the former equation saying that all closed one-forms are exact (or "trivial in cohomology"). Furthermore, one finds $h^{3,0} = 1$. A representative of this cohomology class is called the holomorphic three-form Ω,

$$\Omega = \frac{1}{6}\Omega_{ijk} dz^i \wedge dz^j \wedge dz^k \ . \tag{3.18}$$

Equivalently to the complex structure I, this form tells us what the complex coodinates are.[19] If the complex structure is integrable then $d\Omega = \xi \wedge \Omega$ for some one-form ξ. For a Calabi-Yau there is $\xi = 0$ so that $d\Omega = 0$.

Using the properties (3.15), the Hodge diamond of a CY manifold is then simply of the form

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{3.19}$$

and is therefore completely determined by two independent topological numbers $h^{1,1}$ and $h^{2,1}$.

It is customary to denote the elements in $H^{1,1}$ as ω_a, $a = 1, ..., h^{1,1}$, and those of $H^{2,2}$ as $\tilde{\omega}^a$. They can be chosen such that

$$\int \omega_a \wedge \tilde{\omega}^b = \delta_a^b \ , \tag{3.20}$$

while for the real 3-forms we use α_K, $\beta^K, K = 0, .., h^{2,1}$ (note that since these forms are real, they belong to $H^{2,1} \oplus H^{1,2}$, or $H^{3,0} \oplus H^{0,3}$). They can also be chosen to satisfy

$$\int \alpha_K \wedge \beta^L = \delta_K^L \ . \tag{3.21}$$

One can also choose a basis of complex forms, which are purely in $H^{2,1}$. These forms are called χ_k. Table 3.1 summarizes all this.

[18] There are also higher Chern classes constructed out of the curvature tensor, but we will not use or introduce them.

[19] To be more precise, Ω defines a complex structure, but the opposite is not necessarily true. Only when the $U(1)$ piece of the $U(3)$ holonomy of Kähler manifolds is trivial (i.e. when the holonomy is reduced to $SU(3)$), one has such a closed $(3,0)$-form Ω globally defined. This statement is equivalent to the requirement $c_1 = 0$.

Cohomology group	basis	
$H^{(1,1)}$	w_a	$a = 1, .., h^{(1,1)}$
$H^{(0,0)} \oplus H^{(1,1)}$	$w_A = (1, w_a)$	$A = 0, .., h^{(1,1)}$
$H^{3,3} \oplus H^{(2,2)}$	$\tilde{w}^A = (\text{vol}, \tilde{w}^a)$	$A = 0, .., h^{(1,1)}$
$H^{(2,1)}$	χ_k	$k = 1, .., h^{(2,1)}$
$H^{(3)}$	(α_K, β^K)	$K = 0, .., h^{(2,1)}$

Table 3.1: Basis of harmonic forms in a Calabi–Yau manifold.

Going back to the covariantly constant spinor, we find that J and Ω can be written as bilinears of the spinor, namely

$$J_{mn} = \mp 2i\, \eta_\pm^\dagger \gamma_{mn} \eta_\pm \qquad \Omega_{mnp} = -2i\, \eta_-^\dagger \gamma_{mnp} \eta_+ \ . \tag{3.22}$$

Taking η_+ to be the Clifford vacuum, i.e. such that it annihilated by gamma matrices with holomorphic indices ($\gamma^i \eta_+ = 0$), it is not hard to see that J_{mn} is a $(1,1)$-form with respect to this complex structure, while Ω is a $(3,0)$-form.

3.1.6 Examples of Calabi-Yau manifolds

It is time to give someexamples of Calabi-Yau manifolds. In two dimensions, the only CY manifold is the torus. In four dimensions, there is T^4, which in the strict definition of Calabi-Yau n-folds to exactly have $SU(n)$ (and not smaller) holonomy is not a Calabi-Yau. The only 'exactly' $SU(2)$ holonomy manifold in four dimensions is called $K3$. To be more precise, $K3$ is not a single manifold, but a class of manifolds, which are all diffeomorphic to one another and all have the same Betti numbers $b_1 = 0$ and $b_2 = 22$. This class is a 58-parameter family of $K3$, where these 58 parameters each deform the metric and correspond to moduli in the lower-dimensional theory. One can describe $K3$ as the resolution of the singular space $T^4/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by inverting all four directions on the torus. This involution has 16 singularities which must be blown up into two-spheres to make the manifold smooth. These two-spheres each define non-trivial homology classes, so that together with the six two-cycles on $T^4/\mathbb{Z}_2$ we find 22 non-trivial homology classes, giving $b_2 = 22$. On the other hand, all one-cycles are projected out by the $\mathbb{Z}_2$ so that $K3$ has $b_1 = 0$, similar to Calabi-Yau threefolds.

In six dimensions, there are many Calabi-Yau manifolds known and their number could possibly be infinite. There are various ways of constructing Calabi-Yau manifolds. In the following we will give an example by constructing a family of Calabi-Yau three-folds by taking the zeros of certain polynomials in the complex projective space $\mathbb{C}P^4$. The Complex projective space $\mathbb{C}P^{n+1}$ is the space that results from taking $\mathbb{C}^{n+2} \setminus \{0\}$ and identifying points if they are related by a complex scaling, i.e.

$$(z^1, ..., z^{n+2}) \sim \lambda(z^1, ..., z^{n+2}) \ , \quad \text{for } \lambda \in \mathbb{C}^* \ . \tag{3.23}$$

To visualize this, we can think of the real projective space $\mathbb{R}P^1$, in which two points in $\mathbb{R}^2\{0\}$ are identified if they are related by a real rescaling, i.e. if they lie on a line. We can take a representative of this line as a point that is at unit distance from the origin. The set of inequivalent points therefore lie on the circle of unit radius, hence $\mathbb{P}^1 \cong S^1$.[20]

[20] The inequivalent points actually lie on half of the circle, as one half is identified with the other, so technically $\mathbb{P}^1 \cong S^1/\mathbb{Z}_2$, but this is just isomorphic to S^1, since $\mathbb{Z}_2$ is just a rotation by 180 degrees.

The complex projective spaces $\mathbb{C}P^{n+1}$ are Kähler manifolds, but not Calabi-Yau. But the n-dimensional subspace defined by

$$G(z^1, ..., z^{n+2}) = 0 \ , \tag{3.24}$$

where G is a holomorphic homogeneous polynomial of degree $n+2$ (i.e. $G(\lambda z^1, ..., \lambda z^{n+2}) = \lambda^{n+2} G(z^1, ..., z^{n+2})$), is a Calabi-Yau n-fold.

For example, for $n = 3$, the "quintic CY" is the family of subspaces in $\mathbb{C}P^4$ defined as the zero set of holomorphic homogeneous polynomial G of degree five, for instance by

$$G = \sum_{a=1}^{4} (z^a)^5 \ . \tag{3.25}$$

It has $b_2 = 1$ and $b_3 = 204$, where the former is the two-cycle inherited by $\mathbb{C}P^4$ and the latter correspond to the number of changes in the coefficients in the polynomial G.

It can be shown that the manifolds just described have $SU(n)$ holonomy and therefore admit a Ricci-flat metric. However, the explicit metric is not known, not even for the only (complex) two-dimensional Calabi-Yau manifold $K3$. However, we can go a far way along without knowing the metric explicitly, as we will see in the following.

3.2 Effective theory for compactifications of type II theories on $\mathbf{CY}_3$

In the following we want to discuss how Calabi-Yau compactifications of type II string theory can be effectively described at low energies. By decoupling all modes with a mass at least of the order of the compactification scale, one can in general find an effective action. In the particular case when the internal manifold is Calabi–Yau and no fluxes are turned on, the four-dimensional effective action is well known: It corresponds to an $\mathcal{N} = 2$ ungauged supergravity whose matter content is completely determined by the Hodge numbers $h^{(1,1)}$ and $h^{(2,1)}$, depending on which type II theory we look at.

This effective action arises from expanding all ten-dimensional fields in a basis of "massless" forms, i.e. in the higher dimensional equivalent to the zero mode $n = 0$ of the S^1 compactification. One can show that the operator whose eigenvalues correspond to four-dimensional masses is[21]

$$\triangle = d\, d^\dagger + d^\dagger\, d \ , \tag{3.26}$$

where d is the exterior derivative and $d^\dagger$ is its "adjoint", given by

$$d^\dagger \equiv (-1)^{dp+p+1} * d* \tag{3.27}$$

where p is the degree of the form that is acting on, d the dimension of the space, and $*$ is the Hodge star operator, which on a p form acts as

$$(*A)_{m_{p+1}\ldots m_{d-p}} = \frac{1}{p!} \epsilon_{m_{p+1},\ldots,m_{d-p}}{}^{m_1,\ldots,m_p} A_{m_1,\ldots,m_p} \tag{3.28}$$

and $\epsilon_{m_1\ldots m_d}$ is the volume density, with values $\pm\sqrt{g}$ (and whose indices are raised with the inverse metric). Forms which satisfy $d^\dagger A = 0$ are called coclosed.

It is easy to see that

[21] This can be shown for example by splitting the term $\int \sqrt{-G} H^2$ in the Lagrangian into external and internal components, and integrating by parts the contribution from the internal piece.

- $d: p \to p+1$
- $d^\dagger: p \to p-1$
- $\triangle: p \to p$

The operator $\triangle$ replaces then the Laplacian that appears in S^1 compactifications.[22] Harmonic forms are those that satisfy $\triangle A = 0$. It is not hard to show that a form is harmonic exactly if it is both closed and coclosed

$$\triangle A = 0 \Leftrightarrow (dA = 0 \text{ and } d^\dagger A = 0) \; . \qquad (3.29)$$

The so-called Hodge theorem states that any p-form D_p can be decomposed uniquely into an harmonic, a (globally) exact, and a coexact piece, i.e.

$$D_p = A_p + dB_{p-1} + d^\dagger C_{p+1} \text{ where } \triangle A_p = 0 \; . \qquad (3.30)$$

This is called the Hodge decomposition. If D_p is closed, then $C_{p+1} = 0$, and D_p is in the same cohomology class as A_p. This means that each cohomology class has exactly one harmonic form, which we can take to be a representative of it. Therefore the space of harmonic p-forms is isomorphic to the p-th cohomology. The forms in Table 3.1 can therefore be chosen to be harmonic forms, and will in the following be used to expand all the ten-dimensional fields in, in order to obtain the four-dimensional low-energy action. We will now show how this works.

We start with the NSNS sector. The dilaton is "expanded" in the only scalar harmonic form w_0, which we can identify with the constant 1. As there are no harmonic one-forms on a CY in which one could expand the mixed components of the B-field, it can have only purely external or internal components. The former is expanded in the only internal scalar 1, while the latter is expanded in the basis w_a (this makes sense, as B is a gauge field, i.e. $B \simeq B + d\lambda$, which is precisely the definition of a cohomology class). As for the metric, the four-dimensional massless fields correspond to deformations that respect the Calabi-Yau condition. It is not hard to see that the deformations $\delta g_{i\bar{j}}$ correspond to deformations of the fundamental form J_2, which is closed and coclosed ($*J = J^2$, so $d^\dagger J = 0$), and therefore expanded in the basis of $h^{(1,1)}$ harmonic forms. δg_{ij} correspond on the contrary to deformations of the complex structure, which are in one-to-one correspondence with the harmonic $(2,1)$-forms. We have therefore the following expansions for the deformations of the fields in the NS sector:

$$\begin{aligned} \phi(x,y) &= \phi(x) \, , \\ g_{i\bar{j}}(x,y) &= i v^a(x) (\omega_a)_{i\bar{j}}(y) \; , \qquad \delta g_{ij}(x,y) = i \bar{z}^k(x) \left(\frac{(\bar{\chi}_k)_{i\bar{k}\bar{l}} \, \Omega^{\bar{k}\bar{l}}{}_j}{|\Omega|^2} \right)(y) \; , \\ B_2(x,y) &= B_2(x) + b^a(x) \omega_a(y) \, , \end{aligned} \qquad (3.31)$$

where x parameterizes the external coordinates, while y parameterize the internal ones. Here, all x-dependent fields are massless scalars in the four-dimensional theory, so-called "moduli" [23]. In the NSNS sector we get a total of $2\,(h^{(1,1)} + h^{(2,1)} + 1)$ moduli.[24]

[22] For scalar fields, ∇^2 and $\triangle$ coincide, while for higher order p-forms they differ by terms involving the curvature.

[23] The terminology "modulus" is sometimes used also to denote the (unfixed) VEV of the scalar field.

[24] Note that the $z^k(x)$ are complex scalar fields and therefore have two degrees of freedom. Moreover, the field $B_2(x)$ is a four-dimensional tensor field, which however can be dualized into a four-dimensional scalar field.

gravity multiplet	1	$(g_{\mu\nu}, C_1^0)$
vector multiplets	$h^{(1,1)}$	(C_1^a, v^a, b^a)
hypermultiplets	$h^{(2,1)}$	$(z^k, \xi^k, \tilde{\xi}_k)$
tensor multiplet	1	$(B_2, \phi, \xi^0, \tilde{\xi}_0)$

Table 3.2: Type IIA massless fields arranged in $\mathcal{N}=2$ multiplets.

gravity multiplet	1	$(g_{\mu\nu}, V_1^0)$
vector multiplets	$h^{(2,1)}$	(V_1^k, z^k)
hypermultiplets	$h^{(1,1)}$	(v^a, b^a, c^a, ρ_a)
tensor multiplet	1	(B_2, C_2, ϕ, C_0)

Table 3.3: Type IIB massless fields arranged in $\mathcal{N}=2$ multiplets.

In the RR sector, we perform the following expansions

$$\begin{aligned} C_1(x,y) &= C_1^0(x) \ , \\ C_3(x,y) &= C_1^a(x)\,\omega_a(y) + \xi^K(x)\alpha_K(y) - \tilde{\xi}_K(x)\beta^K(y) \end{aligned} \tag{3.32}$$

for type IIA, and

$$\begin{aligned} C_0(x,y) &= C_0(x) \ , \\ C_2(x,y) &= C_2(x) + c^a(x)\omega_a(y) \, , \\ C_4(x,y) &= V_1^K(x)\,\alpha_K(y) + \rho_a(x)\tilde{\omega}^a(y) \end{aligned} \tag{3.33}$$

for type IIB. In the expansion of C_4 we have used the self duality of $F_5^{(10)}$, which connects the terms expanded in α_K to the ones that would be expanded in in the forms β_K, and similarly for ρ_a, which are dual to $h^{(1,1)}$ tensors D_2 coming from expanding in the basis ω_a.

These moduli arrange into the $\mathcal{N}=2$ multiplets shown in Tables 3.2 and 3.3.

Inserting the expansions (3.31), (3.32) in the ten-dimensional type IIA action, and (3.31), (3.33) in the IIB one and integrating over the Calabi-Yau, one obtains a standard four-dimensional $\mathcal{N}=2$ ungauged supergravity action whose forms is

$$\begin{aligned} S^{(4)}_{\rm IIA} &= \int_{M_4} -\tfrac{1}{2}R*\mathbf{1} + \tfrac{1}{2}\mathrm{Re}\mathcal{N}_{AB}\,F^A\wedge F^B + \tfrac{1}{2}\mathrm{Im}\mathcal{N}_{AB}\,F^A\wedge *F^B \\ &\quad -G_{ab}\,dt^a\wedge *d\bar{t}^b - h_{uv}\,dq^u\wedge *dq^v \ , \end{aligned} \tag{3.34}$$

for type IIA, and

$$\begin{aligned} S^{(4)}_{\rm IIB} &= \int_{M_4} -\tfrac{1}{2}R*\mathbf{1} + \tfrac{1}{2}\mathrm{Re}\mathcal{M}_{KL}\,F^K\wedge F^L + \tfrac{1}{2}\mathrm{Im}\mathcal{M}_{KL}\,F^K\wedge *F^L \\ &\quad -G_{kl}\,dz^k\wedge *d\bar{z}^l - h_{pq}\,d\tilde{q}^p\wedge *d\tilde{q}^q \ . \end{aligned} \tag{3.35}$$

Let us explain these expressions. In the gauge kinetic part, the field strengths are $F^A = dC_1^A = (dC_1^0, dC_1^a)$ in the IIA action (3.34), and $F^K = dV_1^A = (dV_1^0, dV_1^k)$ in the IIB action (3.35). The gauge kinetic coupling matrices $\mathcal{N}$ and $\mathcal{M}$ depend only on the scalars in the respective vector multiplets. In IIA, these are the complex combination of Kähler and B-field deformations t^a, called complexified Kähler deformations, and defined

$$B + iJ = (b^a + iv^a)\,\omega_a \equiv t^a\,\omega_a \ . \tag{3.36}$$

In IIB, the scalars in the vector multiplet moduli space are the complex structure deformations z^k, or the periods, defined as

$$Z^K = \int \Omega \wedge \beta^K \ . \tag{3.37}$$

The scalars in the vector multiplets span a special Kähler manifold of complex dimension $h^{1,1}$ and $h^{2,1}$ in IIA and IIB respectively, whose metric G_{ab} and G_{kl} will be given shortly. The scalars in the hypermultiplets span a quaternionic manifold (a manifold that looks locally like copies of the quaternions) whose coordinates are q^u, $u = 0, ..., h^{(2,1)}$ and $u = 0, ..., h^{(1,1)}$ for IIA and IIB respectively.

The Kähler potential in the vector multiplet moduli space in IIA, which is spanned by the complexified Kähler deformations t^a, is given by

$$K = -\ln\left[\frac{4}{3}\int J \wedge J \wedge J\right] , \tag{3.38}$$

In type IIB, the scalars in the vector multiplet moduli space are the complex structure deformations z^k. They span again a Kähler manifold, with Kähler potential given by

$$K = -\ln\left[i\int \Omega \wedge \bar{\Omega}\right] . \tag{3.39}$$

These spaces are "special Kähler", which means that the Kähler potentials can be derived from a holomorphic function called the prepotential. The matrices $\mathcal{M}$ and $\mathcal{N}$ in the four-dimensional effective actions (3.34) and (3.34) are obtained from derivatives of these prepotentials. For the interested reader, more details are given in [11], and references therein.

4 Lecture 4: Fluxes and Generalized Geometry

4.1 Charges and fluxes

4.1.1 Homology

Similar to the concept of cohomology classes for differential forms, which we introduced in Section 3.1.4, there is a similar characterization for submanifolds. The role of the exterior derivative on forms is played in this case by the boundary operator δ, that maps a p-dimensional submanifold to its $(p-1)$-dimensional boundary. As a submanifold that is itself a boundary cannot have any boundary, the operator δ is nilpotent and obeys

$$\delta^2 = \{\} \ . \tag{4.1}$$

Closed submanifolds (called *cycles*) are defined as those manifolds that have no boundary, while an exact submanifold is a boundary of another submanifold of one dimension higher. Moreover, we can define addition and subtraction on submanifolds: The sum $\mathcal{C}^p + \tilde{\mathcal{C}}^p$ of two p-dimensional submanifolds $\mathcal{C}^p$ and $\tilde{\mathcal{C}}^p$ is the union of the two, respecting the orientation of both, while the difference $\mathcal{C}^p - \tilde{\mathcal{C}}^p$ is the union of $\mathcal{C}^p$ and $\tilde{\mathcal{C}}^p$, but with the orientation of $\tilde{\mathcal{C}}^p$ inverted. Homology

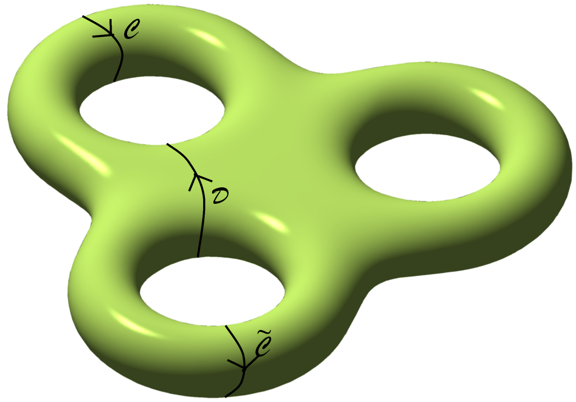

Figure 5: The cycles $\mathcal{C}^p$ and $\tilde{\mathcal{C}}^p$ together are homologous to $\mathcal{D}^p$.

classes are now defined as equivalence classes of closed p-dimensional submanifolds, where two submanifolds $\mathcal{C}^p$ and $\tilde{\mathcal{C}}^p$ are equivalent (or "homologous") if the difference $\mathcal{C}^p - \tilde{\mathcal{C}}^p = \delta\mathcal{D}^{(p+1)}$ is a boundary. Therefore, homology is defined as

$$H_p = \frac{\{\mathcal{C}^p | \delta\mathcal{C}^p = \{\}\}}{\{\delta\mathcal{D}^{(p+1)}\}} \ . \tag{4.2}$$

An illustration of equivalent cycles is given in Figure 5, where the sum of the cycles $\mathcal{C}^p$ and $\tilde{\mathcal{C}}^p$ is homologous to $\mathcal{D}^p$.

There is a canonical pairing between a p-form α_p and p-dimensional submanifold $\mathcal{C}^p$, given by integrating α_p over $\mathcal{C}^p$, i.e.

$$(\mathcal{C}^p, \alpha_p) = \int_{\mathcal{C}^p} \alpha_p \ . \tag{4.3}$$

This scalar product also relates the exterior derivative d and the boundary operator δ as dual operators, by Stokes theorem

$$\int_{\mathcal{C}^p} d\alpha_{p-1} = \int_{\delta\mathcal{C}^p} \alpha_{p-1} \ . \tag{4.4}$$

Using Stokes theorem, it is easy to see that the scalar product is well-defined on $H_p(M) \times H^p(M)$, as $([\mathcal{C}], [\alpha])$ is independent of the respective representatives. Furthermore, it gives an invertible $b_p \times b_p$ matrix. This means that homology and cohomology are in fact dual spaces to each other, and they are therefore isomorphic to one another, i.e.

$$H^p(M) \cong H_{d-p}(M) \ , \tag{4.5}$$

by the identification

$$\int_{\mathcal{C}^p} \alpha_p = \int_M C_{d-p} \wedge \alpha_p \ . \tag{4.6}$$

4.1.2 D-branes as Dirac monopoles

In the following we want to discuss the field configuration sourced by a D-brane. This field configuration resembles just a higher-dimensional realization of the magnetic monopole configuration in electrodynamics that was first discussed by Dirac.

D-branes are charged objects. From (1.40) we see that a Dp-brane electrically sources the field C_{p+1} or, equivalently, magnetically sources C_{10-p-3}. If we consider now a Dp-brane in flat ten-dimensional spacetime, it will fill out $p+1$ dimensions and create in the $9-p$ other directions a non-trivial field configuration for C_{10-p-3} so that for any $(8-p)$-sphere S^{8-p} surrounding the Dp-brane we find

$$\int_{S^{8-p}} F_{8-p} = T_p \ , \tag{4.7}$$

where F_{8-p} is the field strength of C_{7-p}. If C_{7-p} would be globally defined, then from Stokes theorem (4.4) we would find that

$$\int_{S^{8-p}} F_{8-p} = \int_{S^{8-p}} dC_{7-p} = \int_{\delta S^{8-p}} C_{7-p} = 0 \ , \tag{4.8}$$

as the sphere has no boundary, therefore being in conflict with (4.7). Therefore, the gauge field C_{7-p} is only locally defined, i.e. in the northern and southern hemisphere $D^{8-p}_{(N)}$ and $D^{8-p}_{(S)}$ of S^{8-p}. These two gauge fields must then be related by a gauge transformation, thus

$$d(C^{(N)}_{7-p} - C^{(S)}_{7-p}) = 0 \ . \tag{4.9}$$

Locally, we can again write

$$C^{(N)}_{7-p} - C^{(S)}_{7-p} = d\Lambda_{6-p} \ . \tag{4.10}$$

This means that we can reduce

$$\begin{aligned} \int_{S^{8-p}} F_{8-p} &= \int_{D^{8-p}_{(N)}} dC^{(N)}_{7-p} + \int_{D^{8-p}_{(S)}} dC^{(S)}_{7-p} = \int_{S^{7-p}} (C^{(N)}_{7-p} - C^{(S)}_{7-p}) \\ &= \int_{D^{7-p}_{(N)}} d\Lambda^{(N)}_{6-p} + \int_{D^{7-p}_{(S)}} d\Lambda^{(S)}_{6-p} \ , \end{aligned} \tag{4.11}$$

where S^{7-p} is the equator of S^{8-p} and $D^{7-p}_{(N)}$ and $D^{7-p}_{(S)}$ its corresponding hemispheres. This reduction can be continued in an inductive way so that we end up with

$$\int_{S^{8-p}} F_{8-p} = \int_{S^1} d\lambda = T_p \ , \tag{4.12}$$

where we have $\lambda = T_p \frac{\phi}{2\pi}$. Effectively, the field strength F_{8-p} defines a non-trivial cohomology class in $H^{8-p}(S^{8-p})$.[25] If we now forget about the Dp-brane and the ambient space and focus on the field strength configuration on the sphere, we see that it can be non-trivial. This field strength configuration is called a *flux*.

This solution tells us a number of things. For cycles that belong to non-trivial homology classes, the field strength can have a non-vanishing flux on it. This flux through non-trivial cycles is measured by the cohomology class of the form field strength on M. In general all form fields of string theory, being harmonic according to their EOM, can admit such a flux. Fluxes have an important consequence: as the cohomology class of the field strength is non-trivial, the field strength can never become zero everywhere on the compactification manifold. Therefore, the supersymmetric solutions to (3.2) will not be Calabi-Yau any more. Next we will briefly discuss the geometry of these solutions before introducing new tools to deal with them.

4.1.3 Compactifications with flux

As we have seen in Lecture 3, compactifications on Calabi-Yau manifolds have great mathematical sophistication but suffer from various phenomenological problems. First of all, the

[25] Since S^{8-p} is orientable, we know that $b_{8-p} = 1$ and therefore $H^{8-p}(S^{8-p}) \cong \mathbb{R}$.

supersymmetry preserved in four dimensions is $\mathcal{N}=2$, which does not allow for chiral interactions like that mediated by the electro-weak force. Supersymmetric extensions of the standard model can therefore have at most $\mathcal{N}=1$ supersymmetry. Besides, compactifications on Calabi-Yau manifolds have a number of moduli (massless fields with no potential), like the complex structure or the Kähler deformations of the metric [26]. The magnitude of four dimensional masses and interactions depends on the vacuum expectation value of these moduli, which can take any value. Furthermore, massless fields lead to a long range fifth force unobserved so far in nature. Therefore, there should be some mechanism creating a potential for these scalar fields, such that their vacuum expectation value is fixed or "stabilized". The only mechanism within perturbative string theory creating a potential for the fields that we know of today is to turn on fluxes for the field strengths F_n, H_3. String theory in non-trivial backgrounds with fluxes, known as flux compactifications, has a number of novel features that make them particularly interesting. First of all, fluxes break supersymmetry partially or completely in a stable way, leading to $\mathcal{N}=1$ (and thus phenomenologically viable) backgrounds. Second, some or all of the moduli of traditional compactifications get fixed, thus limiting the arbitrariness of the vacuum. Besides, fluxes allow for a non-trivial warp factor $A(y)$ in the ten-dimensional metric

$$ds^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + g_{mn}(y) dy^m dy^n \ , \tag{4.13}$$

which could be responsible for the observed hierarchy between the Planck and the electroweak scale.[26]

For general fluxes, the equations of motion and even the supersymmetry conditions (3.2) are much more difficult to solve. As we saw in Lecture 3, in the absence of fluxes, supersymmetry requires a covariantly constant spinor on the internal manifold. This condition actually splits into two parts, first the existence of such a spinor (i.e., the existence of a non-vanishing globally well defined section on the spinor bundle over T), and second the condition that it is covariantly constant. A generic spinor such as the supercurrent can be decomposed in the same way as the supersymmetry parameters, Eq. (3.4). The first condition implies then the existence of two four-dimensional supersymmetry parameters and thus an effective $\mathcal{N}=2$ four-dimensional action, while the second implies that this action has an $\mathcal{N}=2$ Minkowski vacuum. As far as the internal manifold is concerned, the first condition is a topological requirement on the manifold, while the second one is a differential condition on the metric, or rather, on its connection. In general however, Eq. (3.2) will only have a solution with a spinor that is not covariantly constant, in other words $\nabla\eta \neq 0$. Therefore, we can only impose the first condition for compactifications with fluxes.

A globally well defined non-vanishing spinor exists only on manifolds that have a reduced structure group [27]. Here, the structure group of a manifold is the group of linear transformations $G \subset GL(d)$ that is needed to glue together objects in different patches. In particular, the vielbein (i.e. the set of d independent sections in the frame bundle) transforms under the structure group when going from one patch to another. Therefore the manifold has G-structure if the frame bundle admits a subbundle with fiber group G.

A Riemannian manifold of dimension d has automatically structure group $SO(d)$. All vector, tensor and spinor representations can therefore be decomposed in representations of $SO(d)$. If the manifold has reduced structure group G, then every representation can be further decomposed in representations of G. For example, Weyl spinors have four complex components in six dimensions, and the structure group acts via its double cover $SU(4) \cong SO(6)$. If we know that one spinor η is globally defined, it does not need to transform under the structure group. Therefore, the structure group must actually be contained in $SU(3) \subset SU(4)$.

[26] Non-trivial fluxes also appear in most string backgrounds with field theory duals, and they therefore play an important role in the present understanding of holography.

As in the Calabi-Yau case, we can construct a real two-form J and a complex three-form Ω from η. For a Calabi-Yau, η was covariantly constant and therefore J and Ω where closed forms. Therefore Calabi-Yau manifolds are both symplectic and complex (and has additionally $c_1 = 0$). For general flux compactifications, this is not true any more, and both J and Ω can have a non-trivial exterior derivative.

Moreover, there is another generalization that did not play a role in the absence of fluxes. In (3.4) we assumed that we could expand both ten-dimensional supersymmetry parameters ϵ^i in terms of the same internal spinor η. As long as η must be covariantly constant, this is indeed the most general ansatz. In the presence of R-R fluxes however the symmetry between left- and right-movers can be broken and the most general ansatz for a compactification with minimal supersymmetry is

$$\begin{aligned} \epsilon_1 &= \xi_{1+} \otimes \eta^1_+ + h.c., \\ \epsilon_2 &= \xi_{2+} \otimes \eta^2_\mp + h.c. \ , \end{aligned} \tag{4.14}$$

where in general η^1 and η^2 might coincide at some points on the internal manifold, but differ at generic points. In order to describe such backgrounds in a more natural way, we need to introduce new tools. Next we will introduce the mathematical formalism of generalized geometry and show that flux compactifications are naturally described in this language. We will see that general flux backgrounds can be understood as complex or symplectic manifolds in a generalized sense.

4.2 Generalized geometry

As we have just discussed, for general flux compactifications the supersymmetry condition (3.2) is difficult to solve. In order to do so, we will use a formalism that geometrically covariantizes the symmetries of string theory and that is called generalized complex geometry. Generalized complex geometry was first proposed by Hitchin and his students in [6, 7] in order to describe complex and symplectic geometry in a unifying formalism, before being utilized to describe flux compactifications of string theory [9, 10]. In this section we introduce the mathematical formalism and discuss its relevance in string theory in the subsequent section.

First let us review what are the additional symmetries that we have in string theory, compared to a conventional point particle theory. In Lecture 2 we discussed compactifications of string theory on tori. One of the major results was that the full string spectrum was invariant under the group $SO(d, d, \mathbb{Z})$. Momentum and winding formed a $SO(d, d, \mathbb{Z})$ vector N^M, see (2.18), and the mass spectrum was given in terms of a metric H_{MN} for this vector, cf. (2.21). Moreover, the massless spectrum on the torus was even symmetric under the continuous group $SO(d, d)$, as the massless scalar fields arrange in the metric H_{MN} and span the moduli space (2.28). Here, the two copies of $SO(d)$ in the denominator are actually two copies of the internal Lorentz group. These two Lorentz groups are acting on left- and right-movers individually. String theory on a torus is therefore highly symmetric.

The torus is however a very special background and we cannot expect any of the symmetries discussed to carry over to a general compactification manifold. However, locally (i.e. in a patch) any manifold looks like a torus, up to second derivatives like the curvature. Therefore, we expect for any manifold to locally (i.e. over a point) find the symmetry group $SO(d, d)$. Generalized geometry is the formalism that covariantizes $SO(d, d)$ by defining a geometry that transforms covariantly under it.

The procedure of making $SO(d, d)$ manifestly covariant can be viewed analogously to general relativity. There the Lorentz group (in Euclidean signature) $SO(d)$ is a local symmetry over each point, though globally the space admits no symmetries. The symmetry is realized as a symmetry of the tangent space T_pM over each point p. The tangent space admits a $GL(d)$ action, with $SO(d)$ being the subgroup that leaves the metric invariant. The metric itself therefore forms the coset $GL(d)/SO(d)$ over each point of the space.

In generalized geometry we follow this procedure in exactly the same way, but now for the group $SO(d,d)$ and for two copies of the Lorentz group, i.e. $SO(d) \times SO(d)$. The construction is based on the generalized tangent bundle.

4.2.1 The generalized tangent bundle

Usual complex geometry deals with the tangent bundle of a manifold T, whose sections are vector fields X, and separately, with the cotangent bundle T^*, whose sections are 1-forms ζ. In generalized complex geometry the tangent and cotangent bundle are joined as a single bundle, $T \oplus T^*$. Its sections are the sum of a vector field plus a one-form $X + \zeta$. The bundle $T \oplus T^*$ transforms under a more general group of transformations. While diffeomorphisms act on the tangent and cotangent bundle independently, there are more general transformations that mix both components. However, all these tranformations should preserve the canonical pairing of tangent and cotangent space, represented by the split-signature metric (2.19). The transformations that preserve this metric are $O(n,n)$, where n is the dimension of the manifold. In the following we will concentrate on manifolds of dimension $n = 6$.

Of particular interest is the nilpotent subgroup of $O(6,6)$ defined by the generator

$$\mathcal{B} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} , \tag{4.15}$$

with B an antisymmetric 6×6 matrix, or equivalently a two-form. This transformation is used to patch the generalized tangent bundle. To be more precise the generalized tangent bundle E is a particular extension of T by T^*

$$0 \longrightarrow T^*M \longrightarrow E \xrightarrow{\pi} TM \longrightarrow 0 \, . \tag{4.16}$$

In going from one coordinate patch U_α to another U_β, we do not only patch vectors and one-forms by a diffeomorphism-induced $Gl(6)$-matrix $a_{(\alpha\beta)}$, but also use such two-form shifts $b_{(\alpha\beta)}$ to fiber T^* over T. In total, the patching reads

$$x_{(\alpha)} + \xi_{(\alpha)} = a_{(\alpha\beta)} \cdot x_{(\beta)} + \left[a^{-T}_{(\alpha\beta)} \xi_{(\beta)} - \iota_{a_{(\alpha\beta)} x_{(\beta)}} b_{(\alpha\beta)} \right] , \tag{4.17}$$

where $a^{-T} = (a^{-1})^T$. In fact, for consistency the two-form shift must satisfy $b_{(\alpha\beta)} = -\mathrm{d}\Lambda_{(\alpha\beta)}$, where $\Lambda_{(\alpha\beta)}$ are required to satisfy

$$\Lambda_{(\alpha\beta)} + \Lambda_{(\beta\gamma)} + \Lambda_{(\gamma\alpha)} = g_{(\alpha\beta\gamma)} \mathrm{d} g_{(\alpha\beta\gamma)} \tag{4.18}$$

on any triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ and $g_{\alpha\beta\gamma} := \mathrm{e}^{\mathrm{i}\alpha}$ is a $U(1)$ element. This is analogous to the patching of a $U(1)$ bundle, except that the transition "functions" are one-forms, $\Lambda_{(\alpha\beta)}$. Formally it is called the "connective structure" of a gerbe. This is exactly the structure that we encountered already in Section 4.1.2 when we described the field configuration for flux through the sphere. Therefore, this gerbe structure reflects exactly the presence of flux for H_3.

4.2.2 Generalized complex geometry

The standard machinery of complex geometry can be generalized to this even-dimensional bundle. One can construct a generalized almost complex structure $\mathcal{J}$, which is a map from E

to itself that squares to $-\mathbb{I}_{2d}$ (d is real the dimension of the manifold). This is analogous to an almost complex structure $I_m{}^n$, which is a bundle map from T to itself that squares to $-\mathbb{I}_d$. As for an almost complex structure, $\mathcal{J}$ must also satisfy the hermiticity condition $\mathcal{J}^t G \mathcal{J} = G$, with respect to the natural metric G on $T \oplus T^*$ defined in (2.19).

Usual complex structures I are naturally embedded into generalized ones $\mathcal{J}$: take $\mathcal{J}$ to be

$$\mathcal{J}_1 \equiv \begin{pmatrix} I & 0 \\ 0 & -I^t \end{pmatrix} , \tag{4.19}$$

with $I_m{}^n$ a regular almost complex structure (i.e. $I^2 = -\mathbb{I}_d$). This $\mathcal{J}$ satisfies the desired properties, namely $\mathcal{J}^2 = -\mathbb{I}_{2d}$, $\mathcal{J}^t G \mathcal{J} = G$. Another example of generalized almost complex structure can be built using a non degenerate two–form J_{mn},

$$\mathcal{J}_2 \equiv \begin{pmatrix} 0 & -J^{-1} \\ J & 0 \end{pmatrix} . \tag{4.20}$$

Given an almost complex structure $I_m{}^n$, one can build holomorphic and antiholomorphic projectors $\pi_\pm = \frac{1}{2}(\mathbb{I}_d \pm \mathrm{i} I)$. Correspondingly, projectors can be built out of a generalized almost complex structure, $\Pi_\pm = \frac{1}{2}(\mathbb{I}_{2d} \pm \mathrm{i}\mathcal{J})$. There is an integrability condition for generalized almost complex structures, analogous to the integrability condition for usual almost complex structures. For the usual complex structures, integrability, namely the vanishing of the Nijenhuis tensor, can be written as the condition $\pi_\mp[\pi_\pm X, \pi_\pm Y] = 0$, i.e. the Lie bracket of two holomorphic vectors should again be holomorphic. For generalized almost complex structures, integrability condition reads exactly the same, with π and X replaced respectively by Π and $X + \zeta$, and the Lie bracket replaced by the Courant bracket on $TM \oplus T^*M$, which is defined as follows

$$[X + \zeta, Y + \eta]_C = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \zeta - \frac{1}{2}\mathrm{d}(\iota_X \eta - \iota_Y \zeta) . \tag{4.21}$$

The Courant bracket does not satisfy Jacobi identity in general, but it does on the i–eigenspaces of $\mathcal{J}$. In case these conditions are fulfilled, we can drop the "almost" and speak of generalized complex structures.

For the two examples of generalized almost complex structure given above, $\mathcal{J}_1$ and $\mathcal{J}_2$, integrability condition turns into a condition on their building blocks, $I_m{}^n$ and J_{mn}. Integrability of $\mathcal{J}_1$ enforces I to be an integrable almost complex structure on T, and hence I is a complex structure, or equivalently the manifold is complex. For $\mathcal{J}_2$, which was built from a two-form J_{mn}, integrability imposes $dJ = 0$, thus making J into a symplectic form, and the manifold a symplectic one.

These two examples are not exhaustive, and the most general generalized complex structure is partially complex, partially symplectic. Explicitly, a generalized complex manifold is locally equivalent to the product $\mathbb{C}^k \times (\mathbb{R}^{d-2k}, J)$, where $J = \mathrm{d}x^{2k+1} \wedge \mathrm{d}x^{2k+2} + ... + \mathrm{d}x^{d-1} \wedge \mathrm{d}x^d$ is the standard symplectic structure and $k \leq d/2$ is called rank, which can be constant or even vary over the manifold (jump by two at certain special points or planes).

On this generalized tangent bundle E we can also define a positive definite metric $\mathcal{H}$. On $T \oplus T^*$ a standard choice would be to combine the ordinary metric g and its inverse in a block-diagonal metric. However, this metric can be rotated by some two-form shift B as given in (4.15). Indeed it turns out that locally a generalized metric on the gerbe E can always be written in the form (2.21). If we have now a pair of *commuting* generalized almost complex structures $\mathcal{J}_i$, $i = 1, 2$, we can define from this a generalized metric via

$$\mathcal{H} = G \mathcal{J}_1 \mathcal{J}_2 . \tag{4.22}$$

Commutation and Hermiticity of the $\mathcal{J}_i$ ensures that $\mathcal{H}$ is indeed a generalized metric.

There is an algebraic one-to-one correspondence between generalized almost complex $\mathcal{J}$ structures and (lines of) pure spinors of Clifford(6,6) Φ.[27] It maps the $+i$ eigenspace of $\mathcal{J}$ to the annihilator space of the spinor Φ. In string theory, the picture of generalized almost complex structures emerges naturally from the worldsheet point of view [23], while that of pure spinors arises on the spacetime side, as we discuss in Section 4.1.3.

One can build Clifford$(6,6)$Weyl pure spinors by tensoring Clifford(6) Weyl spinors (which are automatically pure), namely

$$\Phi_\pm = \eta^1_+ \otimes \eta^{2\dagger}_\pm \; . \tag{4.23}$$

Using Fierz identities, this tensor product can be written in terms of bilinears of the spinors by

$$\eta^1_+ \otimes \eta^{2\dagger}_\pm = \frac{1}{4}\sum_{k=0}^{6} \frac{1}{k!}\eta^{2\dagger}_\pm \gamma_{i_1\ldots i_k}\eta^1_+ \gamma^{i_k\ldots i_1} \tag{4.24}$$

A Clifford$(6,6)$ spinor can also be mapped to a formal sum of forms via the Clifford map

$$\sum_k \frac{1}{k!} C^{(k)}_{i_1\ldots i_k}\gamma^{i_i\ldots i_k}_{\alpha\beta} \qquad \longleftrightarrow \qquad \sum_k \frac{1}{k!} C^{(k)}_{i_1\ldots i_k}\mathrm{d}x^{i_i}\wedge\ldots\wedge \mathrm{d}x^{i_k} \; . \tag{4.25}$$

The tensor products in (4.23) are then identified with sums of regular forms. From now on, we will use $\Phi_\pm$ to denote just the forms. The subindices plus and minus in $\Phi_\pm$ denote the Spin(6,6) chirality: positive corresponds to an even form, and negative to an odd form. Irreducible Spin(6,6) representations are actually "Majorana-Weyl", namely they are of definite parity (”Weyl”) and real (”Majorana”). The usual spinor bilinear form $\psi^t\cdot\chi$ is then related to the Mukai pairing $\langle\cdot,\cdot\rangle$ on forms by

$$(\psi^t\cdot\chi)\,\epsilon = \langle\psi,\chi\rangle = \sum_p (-)^{[p/2]}\psi_p\wedge\chi_{6-p} \; , \tag{4.26}$$

where the subscripts denote the degree of the component forms and $[p/2]$ takes the integer part of $p/2$.

The B-transform (4.15) on spinors amounts to the exponential action

$$\Phi_\pm \to e^{-B}\Phi_\pm \equiv \Phi^D_\pm \tag{4.27}$$

where on the polyform associated to the spinor, the action is $e^{-B}\Phi = (1-B\wedge+\frac{1}{2}B\wedge B\wedge+...)\Phi$. We will refer to Φ as naked pure spinor, while Φ^D will be called dressed pure spinor. The former lives in the spinor bundle over $T\oplus T^*$, the latter lives in the spinor bundle over E.

For the particular case $\eta^1=\eta^2\equiv\eta$, the pure spinors in (4.23) reduce to

$$\Phi_+ = e^{-\mathrm{i}J} \; , \qquad \Phi_- = -\mathrm{i}\Omega \; , \tag{4.28}$$

i.e. the symplectic and the complex structure on the manifold (the relation between J and Ω and the spinor η is given in (3.22)).

Under the one-to-one correspondence between spinors and generalized almost complex structures we have

$$\begin{aligned} \Phi_- &= -\frac{\mathrm{i}}{8}\,\Omega \quad \leftrightarrow \quad \mathcal{J}_1 \\ \Phi_+ &= \frac{1}{8}\,e^{-\mathrm{i}J} \quad \leftrightarrow \quad \mathcal{J}_2 \end{aligned} \tag{4.29}$$

[27] A spinor is said to be pure if its anihilator space is maximal (i.e. 6-dimensional in this case)

where $\mathcal{J}_1$ and $\mathcal{J}_2$ are defined in (4.19, 4.20). These spinors are of special type: they correspond to a purely complex or purely symplectic generalized almost complex structure. In a more general case, such as the one obatined by doing tensor products of two different Clifford(6) spinors, one obtains pure spinors which are a mixture of a complex and a symplectic structure. For example, if the two Clifford(6) spinors are orthogonal, they define a complex vector (i.e., one can write $\eta^2 = (v + \mathrm{i}v')_m \gamma^m \eta^1$, and the corresponding pure spinors would read

$$\begin{aligned}\Phi_+ &= \eta^1_+ \otimes \eta^{2\,\dagger}_+ = -\frac{\mathrm{i}}{8}\omega \wedge e^{-\mathrm{i}v\wedge v'} \ , \\ \Phi_- &= \eta^1_+ \otimes \eta^{2\,\dagger}_- = -\frac{1}{8} e^{-\mathrm{i}\,j} \wedge (v + \mathrm{i}v') \ . \end{aligned} \tag{4.30}$$

These are given in terms of the local SU(2) structure defined by (η^1, η^2): j and ω are the (1,1) and (2,0)-forms on the local four dimensional space orthogonal to v and v'. Φ_+ describes therefore a generalized almost complex structure "of rank 2" (i.e., of complex type in two dimensions), while the rank of Φ_- is one. In the most generic case, the rank of the pure spinors need not be constant over the manifold, but can be point-dependent, and jump across the manifold.

Integrability condition for the generalized complex structure corresponds on the pure spinor side to the condition

$$\mathcal{J} \text{ is integrable} \Leftrightarrow \exists \text{ vector } v \text{ and 1-form } \zeta \text{ such that } \mathrm{d}\Phi = (v\llcorner + \zeta\wedge)\Phi$$

A generalized Calabi-Yau [6] is a manifold on which a closed pure spinor exists:

$$\text{Generalized Calabi-Yau} \Leftrightarrow \exists\ \Phi \text{ pure such that } \mathrm{d}\Phi = 0$$

From the previous property, a generalized Calabi-Yau has obviously an integrable generalized complex structure. Examples of Generalized Calabi-Yau manifolds are symplectic manifolds and complex manifolds with trivial torsion class W_5 (i.e. if $\exists\ f$ such that $\Phi = e^{-f}\Omega$ is closed). More generally, if the integrable pure spinor has rank k, then the manifold looks locally like $\mathbb{C}^k \times (\mathbb{R}^{d-2k}, J)$, as we mentioned before.

As an alternative to the use of the generalized tangent bundle E, one can just use $T \oplus T^*$ and twist the differential d by a closed three–form H such that the differential becomes $d - H\wedge$. Similar, the Courant bracket is then modified to

$$[X + \zeta, Y + \eta]_H = [X + \zeta, Y + \eta]_C + \iota_X \iota_Y H \ , \tag{4.31}$$

and with it the integrability condition. In terms of "integrability" of the pure spinors Φ, adding H amounts to twisting the differential condititions for integrability and for generalized Calabi-Yau. More precisely,

$$\text{"twisted" generalized Calabi-Yau} \Leftrightarrow \exists\ \Phi \text{ pure, and } H \text{ closed s.t. } (\mathrm{d} - H\wedge)\Phi = 0$$

Alternatively, a twisted generalized Calabi-Yau can be defined by the existence of a pure spinor Φ^D on the generalized tangent bundle E such that $\mathrm{d}\Phi^D = 0$.

4.3 Flux compactifications and generalized complex geometry

As we reviewed in the previous sections, as a result of demanding $\delta\Psi_m = \delta\lambda = 0$, supersymmetry imposes differential conditions on the internal spinor η. These differential conditions turn into differential conditions for the pure Clifford$(6,6)$ spinors $\Phi_\pm$, defined in (4.23). We quote the results of Ref. [10], skipping the technical details of the derivation. $\mathcal{N}=1$ supersymmetry on warped Minkowski 4D vacua requires

$$\begin{aligned} e^{-2A+\phi}(\mathrm{d}+H\wedge)(e^{2A-\phi}\Phi_+) &= 0\,, \\ e^{-2A+\phi}(\mathrm{d}+H\wedge)(e^{2A-\phi}\Phi_-) &= \mathrm{d}A\wedge\bar{\Phi}_- + \frac{\mathrm{i}}{16}e^{\phi+A} * F_{\mathrm{IIA}\,+} \end{aligned} \tag{4.32}$$

for type IIA, and

$$\begin{aligned} e^{-2A+\phi}(\mathrm{d}-H\wedge)(e^{2A-\phi}\Phi_+) &= \mathrm{d}A\wedge\bar{\Phi}_+ - \frac{\mathrm{i}}{16}e^{\phi+A} * F_{\mathrm{IIB}\,-} \\ e^{-2A+\phi}(\mathrm{d}-H\wedge)(e^{2A-\phi}\Phi_-) &= 0\,, \end{aligned} \tag{4.33}$$

for type IIB, and the algebraic constrain

$$|\Phi_+| = |\Phi_-| = e^A \tag{4.34}$$

In these equations

$$F_{\mathrm{IIA}\,\pm} = F_0 \pm F_2 + F_4 \pm F_6 \ , \qquad F_{\mathrm{IIB}\,\pm} = F_1 \pm F_3 + F_5 \ . \tag{4.35}$$

These F are purely internal forms, that is related to the total ten–dimensional RR field strength by

$$F^{(10)} = F + \mathrm{vol}_4 \wedge \lambda(*F) \ , \tag{4.36}$$

where $*$ the six–dimensional Hodge dual, we have used the self-duality property of the ten-dimensional RR fields

$$F^{(10)}_n = (-1)^{\mathrm{Int}[n/2]} \star F^{(10)}_{10-n} \ , \tag{4.37}$$

and λ is

$$\lambda(A_n) = (-1)^{\mathrm{Int}[n/2]} A_n \ . \tag{4.38}$$

According to the definitions given in the previous section, Eqs. (4.32) and (4.33) tell us that all $\mathcal{N}=1$ vacua on manifolds with SU(3)×SU(3) structure on $T\oplus T^*$ are *twisted Generalized Calabi-Yau's*, which we discussed in Section 4.2. We can also see from (4.32), (4.33) that RR fluxes act as an obstruction for the integrability of the second pure spinor.

Specializing to the pure SU(3) structure case, i.e. for $\Phi_\pm$ given by Eq.(4.28), and looking at (4.29), we see that the Generalized Calabi-Yau manifold is complex[28] in IIB and (twisted) symplectic in IIA. For the general SU(3)×SU(3) case, $\mathcal{N}=1$ vacua can be realized in hybrid complex–symplectic manifolds, i.e. manifolds with k complex dimensions and $6-2k$ (real) symplectic ones. In particular, given the chiralities of the preserved Clifford$(6,6)$ spinors, the rank k must be even in IIA and odd in IIB (equal respectively to 0 and 3 in the pure SU(3) case).

A final comment is that $\mathcal{N}=2$ vacua with NS fluxes only, where shown to satisfy Eqs. (4.32, 4.33) for $\tilde{\Phi}_\pm = \Phi_\pm$, $F_{\mathrm{IIA}} = F_{\mathrm{IIB}} = A = 0$ [25].

[28] H in Eq.(4.33) does not "twist" the (usual) complex structure, as $(d-H\wedge)\Omega = 0$ implies in particular $d\Omega = 0$.

5 Lecture 5: 4D Effective actions for compactifications on manifolds of reduced structure

In Lectures 2 and 3 we reviewed compactifications on tori and on Calabi-Yau manifolds, and their corresponding four-dimensional low energy actions. In this lecture we will see how this changes when the corresponding compactification manifolds do not have reduced holonomy (like tori, which have trivial holonomy, or Calabi-Yau manifolds, which have SU(3) holonomy), but reduced structure. We start by reviewing the case of trivial structure, which corresponds to the so-called "twisted tori", that give rise to four-dimensional $N = 8$ gauged supergravity, and then move on to compactifications on manifolds of $SU(3)$ structure, or more generally, as we explained in the previous lecture, on manifolds of $SU(3)\times SU(3)$ structure. These give effective actions in four-dimensions which have the form of $N = 2$ gauged supergravity.

5.1 Compactifications on twisted tori and $N = 8$ gauged supergravity

5.1.1 Twisted tori

Let us start with a Lie group G of dimension d viewed as a manifold (also called a group manifold).[29] From the Maurer–Cartan equations, G has a set of d globally defined one forms e^a that satisfy

$$de^a = \frac{1}{2} f^a{}_{bc}\, e^b \wedge e^c \ , \tag{5.1}$$

with the $f^a{}_{bc}$ being constant. Such a basis is obviously very useful: it can reduce many differential problems to algebraic ones. One can also define a basis of vectors $\hat{e}_a$ dual to the e^a (i.e. $\langle \hat{e}_a, e^b\rangle = \delta_a{}^b$). This basis obeys

$$[\hat{e}_b, \hat{e}_c] = f^a{}_{bc}\, \hat{e}_a \ . \tag{5.2}$$

This shows that the constants $f^a{}_{bc}$ are indeed the structure constants of the group G.

Conversely, if we are looking for a manifold M with a basis e^a of globally defined one–forms, we are providing a global section to the frame bundle, hence trivializing it. The cotangent and tangent bundle will therefore be topologically trivial. Such manifolds are called *parallelizable*. One can of course always expand de^a in the basis of two–forms $e^b\wedge e^c$, which would give us (5.1), but in general the $f^a{}_{bc}$ will not be constant. If they are constant, the manifold is homogeneous. Imposing $d^2e^a = 0$ results in

$$f^a{}_{[bc} f^e{}_{d]a} = 0 \ , \tag{5.3}$$

i.e. the $f^a{}_{bc}$ satisfy Jacobi identities, and are therefore structure constants of a real Lie algebra $\mathcal{G}$. The vectors $\hat{e}_a$, when exponentiated, give then an action of G over M. One can see that this action is transitive (it sends any point into any other) and hence $M = G/\Gamma$, where Γ is a discrete subgroup.

Let us show the simplest example of this type of manifolds, which will explain the name "twisted torus", the so–called Heisenberg algebra. The structure constants in the form language of (5.1) are:

$$de^1 = 0 \ ; \qquad de^2 = 0 \ ; \qquad de^3 = N e^1 \wedge e^2 \ . \tag{5.4}$$

[29] A Lie group is a continuous group that can also be understood as a manifold, i.e. there are charts and a tangent bundle. The tangent space at the neutral element of the group and the commutation relations of its tangent vectors define the Lie algebra.

One can see this is a torus fibration over a circle ($Ne^1 \wedge e^2$ being the curvature of the fibration). To see this more clearly, let us choose a gauge where

$$e^1 = dx^1 \ ; \qquad e^2 = dx^2 \ ; \qquad e^3 = dx^3 + Nx^1 e^2 \ . \tag{5.5}$$

We can compactify G by making the identifications $(x^1, x^2, x^3) \simeq (x^1, x^2 + a, x^3) \simeq (x^1, x^2, x^3 + b)$, with a, b integer, but cannot do the same for x^1, because the one form e^3 would not be single–valued. For that, we need to "twist" the identification by $(x^1, x^2, x^3) \simeq (x^1 + c, x^2, x^3 - N\, c\, x^2)$. This means that when moving once around in the x^1 plane, the x^2-x^3 torus is tilted N times in such a way that it gets identified with itself again (i.e. the tilt maps the lattice that defines the torus non-trivially onto itself). In this way, the resulting G/Γ turns out to be a T^2 fibration over S^1 whose $c_1 = N$, and hence topologically distinct from T^3. Such a quotient of a nilpotent group by a discrete subgroup is called a *nilmanifold* or sometimes more loosely a "twisted torus", as such a fibration twists some torus directions over others. The structure constants are often referred to in the literature of flux compactifications as "metric fluxes". Some twisted tori are T–dual to regular tori with NS 3-form fluxes, as we will see. For instance, the manifold given in this example T–dual to a T^3 with N units of H flux.

Another example of a parallelizable group manifold is S^3, which is identical with the group $SU(2)$. The structure constants in this case are $f^a{}_{bc} = \epsilon_{abc}$. The globally-defined one-forms e^a are obtained by demanding that they behave in a natural way under the pull-back of the group action. If the e^a are constant under the group action from the left, they are called left-invariant forms. In this case the group $SU(2)$ can act from the left or from the right onto $\sigma_a e^a$, where σ_a are the Pauli matrices. If the forms are left-invariant, they obey

$$g^{-1} dg = \sigma_a e^a \ . \tag{5.6}$$

Note that, despite semi-simple groups, which are always compact, it is in general a non-trivial problem which non-compact Lie groups can be made compact by dividing by a lattice Γ. In the following, we will always assume that we have such a Γ at our disposal, in other words that the manifold is compact.

5.1.2 Reductions on twisted tori

Reductions on twisted tori work very similar to torus compactifications, doing the replacement $dx^m \to e^m$. For instance, the metric for a twisted torus is of the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \mathcal{M}_{mn}(x) \left(e^m + A^m(x)\right) \left(e^n + A^n(x)\right) \ , \tag{5.7}$$

where

- $g_{\mu\nu}$ is the four-dimensional metric,
- $A^m = A^m_\mu dx^\mu$ are the four-dimensional Kaluza-Klein (KK) vector fields and
- $\mathcal{M}_{mn}$ is an arbitrary symmetric matrix whose coefficients are four-dimensional scalar fields.

As for the torus compactification, the internal metric spans the space $\frac{Gl(6)}{SO(6)}$. However, there will also be some differences. As the one-forms e^m are not closed any more, the curvature tensor contains additional terms that are linear and quadratic in f^m_{np}. As the ten-dimensional Ricci scalar appears in the action, these additional terms will appear as additional couplings in four dimensions. It turns out that these couplings can be associated with gaugings and potential terms. The reduction however is rather involved and we therefore will not go into details. The situation can be seen more clearly from the expansion of the form fields, as the following exercise shows.

Exercise: Expand B_2 and C in e^m, $e^m \wedge e^n$ etc. and show that H_3 and F contain *covariant* derivatives for the 4d fields. For instance, the NS-NS two-form is expanded as

$$B \sim B_{\mu\nu} dx^\mu \wedge dx^\nu + B_{\mu m} dx^\mu \wedge e^m + b_{mn} e^m \wedge e^n$$

then the 3-form flux H contains $\mathcal{D}_\mu b_{mn}\, dx^\mu \wedge e^m \wedge e^n$, with the covariant derivative given by

$$\mathcal{D}_\mu b_{mn} = \partial_\mu b_{mn} + B_{\mu p} f^p_{mn} \ . \tag{5.8}$$

Therefore the four-dimensional scalars will now be gauged under the four-dimensional vector fields. It turns out that all modifications from the torus reduction can be summarized by gauging certain scalars under the vector fields, parameterized by the f^p_{mn}.

Note that we can also incorporate fluxes in the reduction, by writing the form field strengths H_3 and F as harmonic piece plus the exterior derivative of the potential, in the fashion of (3.30). The flux numbers $h_{mnp} = H_3(\hat{e}_m, \hat{e}_n, \hat{e}_p)$ (and their analogues for F) also appear as gaugings in the four-dimensional supergravity.

Apart from multiplet numbers and moduli space data, gaugings are the only free parameters of supergravity (for $N \geq 2$).[30] These gaugings lead to a non-trivial potential for the scalars that is helpful for moduli stabilization and spontaneous supersymmetry breaking (also to $N = 1$). We will give more details on these topics in Lecture 6.

In the special case that type II theories are reduced on twisted tori, the resulting action is an $N = 8$ gauged supergravity. In Section 5.2 we will discuss more general reductions to $N = 2$ gauged supergravity, which allows for more freedom and therefore for more interesting theories from the point of view of phenomenology.

In general one has to be careful with reductions on twisted tori. There is usually no mass gap between the masses induced by the gaugings and the masses of the Kaluza-Klein modes. Consequently, the derived action does not contain the light modes of the theory, compared to heavy Kaluza-Klein modes. In particular, the gaugings give not only extra masses to scalars in the action, also the masses of modes in the massive towers can change due to these gaugings and some modes can even become massless. Therefore the theory is not an effective action in the Wilsonian sense. This makes the action only partly useful for understanding low-energy effects. Furthermore, the vacuum of the supergravity action might appear to be non-supersymmetric though it is supersymmetric in the full ten-dimensional theory. The four-dimensional action is usually rather interesting as a consistent truncation: Solutions to its equations of motion are always solutions to the (rather complicated) ten-dimensional equations of motion. Therefore, the four-dimensional action is useful to generate classical solutions for ten-dimensional string theory.

5.1.3 T-duality, non-geometric fluxes and T-folds

One of the new phenomena of string theory for a torus background was T-duality, relating torus compactifications with large and small radii. An interesting question is what happens to

[30] For $N = 1$ supergravity, the only additional parameter is the holomorphic superpotential that also contributes to the scalar potential.

fluxes and the structure constants under T-duality. Let us discuss this for a simple example: A three-dimensional torus with H_3 flux on it. In other words, h_{123} is nonzero. If we now perform T-duality, we can read off from (2.21) the transformation behavior of H_3 and g_{mn}. It turns out that the T-dual geometry has no flux in H_3. However, some of the torus directions are fibred over the others. A closer look reveals that the geometry T-dual (on x^3) to a torus with H_3-flux is the twisted torus with $f^3{}_{12}$ being equal to the flux number h_{123} of the original torus. This is the example we already discussed in Section 5.1.1.

A natural question to ask is now which geometry we find if we perform another T-duality, but now along x^2. From (2.21) we find again an answer, though it is more puzzling than before: The resulting background has no well-defined metric any more. In other words, when one encircles once the x^1 direction, not only transforms the x^2-x^3 torus non-trivially onto itself, but (among other transformations) it also transforms its volume up to two T-dualities. Though this is not a valid background for point-particles, it is a well-defined background for propagating strings, as their spectrum remains invariant under these changes. This is reflected by the fact that the metric (and the B-field) are individually not well-defined any more, but the matrix H_{MN} in (2.21) is. It remains a difficult task to find a valid global description of such backgrounds that grasps all its features. String worldsheet computations suggest that such backgrounds render the string coordinates non-commutative.

In principle one could imagine even performing a third T-duality, this time in the x^1 direction. Not much is known about these backgrounds, but there exist arguments that such backgrounds would even locally not be well-defined for a point particle, and worldsheet reasonings suggest that string coordinates would become non-associative.

In total, we find the following chain of dualities

$$h_{123} \leftrightarrow f^3{}_{12} \leftrightarrow \underbrace{Q^{23}{}_1 \leftrightarrow R^{123}}_{\text{non-geometric flux}} \quad . \tag{5.9}$$

Both numbers Q_1^{23} and R^{123} are called "non-geometric fluxes". It still remains an open problem to fully describe backgrounds with these features. A first step into this direction is the idea of T-folds.

Similarly to generalized geometry, T-folds covariantize more string symmetries, in this case $SO(n, n, \mathbb{Z})$. More particularly, one takes two tori of opposite signature

$$\underbrace{T^6}_{momentum} \times \underbrace{\tilde{T}^6}_{winding} \quad , \tag{5.10}$$

that should reflect the dual coordinates to momentum and winding modes of the string. This covariantizes T-duality: Performing a T-duality corresponds in this language to exchanges of circles between T^6 and $\tilde{T}^6$.

However, similar to twisted tori, we can also twist these T-folds. In other words, we define one-forms

$$E^A = \begin{pmatrix} e^a \\ \tilde{e}_a \end{pmatrix} \quad , \tag{5.11}$$

such that

$$dE^A = F^A{}_{BC} E^B \wedge E^C \quad . \tag{5.12}$$

If we decompose the $2n$-dimensional structure constants $F^A{}_{BC}$, we can write them as

$$\begin{aligned} de^a &= f^a_{bc} e^b \wedge e^c + Q^{ab}_c \tilde{e}_b \wedge e^c + R^{abc} \tilde{e}_b \wedge \tilde{e}_c 0 \, , \\ d\tilde{e}_a &= Q^{bc}_a \tilde{e}_b \wedge \tilde{e}_c + f^c_{ab} e^b \wedge \tilde{e}_c + H_{abc} e^b \wedge e^c \end{aligned} \tag{5.13}$$

One can show that these F^A_{BC} just become the structure constants of the supergravity gauge group which is a subgroup of $SO(6,6)$.

On a first glimpse, this seems to be able to describe non-geometric fluxes, as it puts all fluxes on an equal footing. However, this is only formal. Strings should see only six of the twelve coordinates, but in order to reproduce all components of the $2n$-dimensional structure constants, the exterior derivative must necessarily depend on all twelve coordinates. Furthermore, this formalism can formally describe non-geometric generalizations of twisted tori, but it is difficult to generalize this idea to non-parallelizable manifolds (with less than $N = 8$ supersymmetry). Some attempts towards this direction are collectively termed "doubled geometry". Also there basic problems arise that make it unclear how to generalize the formalism beyond twisted tori. Effectively, one can conclude that T-folds are defined "bottom-up". As the gauged supergravity is well-understood for general F^A_{BC}, one postulates a ten-dimensional origin by formally defining a T-fold. It remains however unclear how to understand string dynamics in such backgrounds, including string corrections to the classical action.

5.2 Compactifications on Generalized Complex Geometries and $N = 2$ Gauged Supergravity

Let us now get to reductions within generalized geometry, in particular reductions that lead to $N = 2$ rather than $N = 8$ gauged supergravity. This follows very much the procedure of reductions for Calabi-Yau manifolds: we expand all fields in some set of forms, with the coefficients being four-dimensional fields. In this case we expand the pure spinors Φ_+ and Φ_- in a set of even and odd forms, respectively. These are given by

$$\Sigma_+ = (\omega_A, \tilde{\omega}^A) , \qquad \Sigma_- = (\alpha_I, \beta^I) , \tag{5.14}$$

which we require to form symplectic vectors under the Mukai pairing (4.26). In other words, the Mukai pairing pulls back to non-degenerate symplectic forms $\mathcal{S}_+$ and $\mathcal{S}_-$ on Σ_+ and Σ_-, respectively, where the pullback is given by

$$\mathcal{S}_\pm = \langle \Sigma_\pm, \Sigma_\pm^T \rangle . \tag{5.15}$$

In general we can choose the vectors $\Sigma_\pm$ such that $\mathcal{S}_\pm$ is of the standard form, i.e. so that

$$\mathcal{S}_\pm = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} . \tag{5.16}$$

In general now the modes in $\Sigma_\pm$ are not closed. We have seen in the case of twisted tori that the non-closure of forms led to gaugings in the four-dimensional theory. This will also happen in this case. Before we do so however, we first discuss how ten-dimensional fields assemble in four-dimensional $N = 2$ supergravity multiplets in general, and we will come back in section 5.2.2 to the conditions required on these forms.

5.2.1 Group theory and multiplets

In this section we discuss the group-theoretical properties of massless type II supergravity fields in a generalized geometry background. In particular we show how the fields assemble in $N = 2$-like multiplets.

Let us start by decomposing even and odd forms in terms of $SU(3) \times SU(3)$ representations. Here, even forms are given by tensoring spinors of opposite chirality while odd forms are formed by tensoring spinors of the same chirality. In other words, $\Lambda^{\text{even}}T^*M$ decomposes under $SO(6,6) \to SO(6) \times SO(6) \to SU(3) \times SU(3)$ as

$$\mathbf{32}^+ \to (\mathbf{4}, \bar{\mathbf{4}}) \oplus (\bar{\mathbf{4}}, \mathbf{4}) \to 2(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \bar{\mathbf{3}}) \oplus (\bar{\mathbf{3}}, \mathbf{3}) \oplus (\mathbf{3}, \bar{\mathbf{3}}) \ , \tag{5.17}$$

while $\Lambda^{\text{odd}}T^*M$ is given by

$$\mathbf{32}^- \to (\mathbf{4}, \mathbf{4}) \oplus (\bar{\mathbf{4}}, \bar{\mathbf{4}}) \to 2(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \bar{\mathbf{3}}) \oplus (\mathbf{3}, \mathbf{3}) \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}}) \ . \tag{5.18}$$

We can collect all these representations in a 'generalized diamond' in Table 5.1, which resembles very much the Hodge diamond of Calabi-Yau compactifications. Note however that each entry in the diamond contains poly-forms, ie. sums of forms of different degree. For example Φ^+, which is a singlet under both $SU(3)$ structures, belong to $U_{1,\bar{1}}$.

$$\begin{array}{ccccccc}
 & & & U_{\mathbf{1},\bar{\mathbf{1}}} & & & \\
 & & U_{\bar{\mathbf{3}},\bar{\mathbf{1}}} & & U_{\mathbf{1},\mathbf{3}} & & \\
 & U_{\mathbf{3},\bar{\mathbf{1}}} & & U_{\bar{\mathbf{3}},\mathbf{3}} & & U_{\mathbf{1},\bar{\mathbf{3}}} & \\
U_{\bar{\mathbf{1}},\bar{\mathbf{1}}} & & U_{\mathbf{3},\mathbf{3}} & & U_{\bar{\mathbf{3}},\bar{\mathbf{3}}} & & U_{\mathbf{1},\mathbf{1}} \\
 & U_{\bar{\mathbf{1}},\mathbf{3}} & & U_{\mathbf{3},\bar{\mathbf{3}}} & & U_{\bar{\mathbf{3}},\mathbf{1}} & \\
 & & U_{\bar{\mathbf{1}},\bar{\mathbf{3}}} & & U_{\mathbf{3},\mathbf{1}} & & \\
 & & & U_{\bar{\mathbf{1}},\mathbf{1}} & & &
\end{array}$$

Table 5.1: *Generalized SU(3)×SU(3) diamond.*

Now let us see how these reprentations link to four-dimensional supergravity multiplets.

In Lecture 4 we have seen that a pair of compatible pure spinors define an $SU(3) \times SU(3)$. Here, the two $SU(3)$ structures correspond to the two internal spinors η_1 and η_2. The interpretation in terms of string theory is that left-movers actually see the supersymmetry defined by η_1 while right-movers the one of η_2. Following this point of view, we can decompose the ten-dimensional string modes in terms of four-dimensional fields times some $SU(3) \times SU(3)$ representations. Therefore, we consider the decomposition of ten-dimensional string modes under $SO(8) \to SO(2) \times SU(3) \times SU(3)$, where $SO(8)$ is the Lorentz group acting on transversal excitations. In order to do so let us first recall the decomposition of the two 8-dimensional inequivalent Majorana-Weyl representations $\mathbf{8_S}$ and $\mathbf{8_C}$ and the vector representation $\mathbf{8_V}$ of $SO(8)$ under $SO(8) \to SO(2) \times SO(6) \to SO(2) \times SU(3)$. One has

$$\begin{aligned}
\mathbf{8_S} \ &\to \ \mathbf{4}_{\frac{1}{2}} \oplus \bar{\mathbf{4}}_{-\frac{1}{2}} \ \to \ \mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}} \oplus \mathbf{3}_{\frac{1}{2}} \oplus \bar{\mathbf{3}}_{-\frac{1}{2}} \ , \\
\mathbf{8_C} \ &\to \ \mathbf{4}_{-\frac{1}{2}} \oplus \bar{\mathbf{4}}_{\frac{1}{2}} \ \to \ \mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}} \oplus \mathbf{3}_{-\frac{1}{2}} \oplus \bar{\mathbf{3}}_{\frac{1}{2}} \ , \\
\mathbf{8_V} \ &\to \ \mathbf{1_1} \oplus \mathbf{1_{-1}} \oplus \mathbf{6_0} \ \to \ \mathbf{1_1} \oplus \mathbf{1_{-1}} \oplus \mathbf{3_0} \oplus \bar{\mathbf{3}}_\mathbf{0} \ .
\end{aligned} \tag{5.19}$$

where the subscript denotes the helicity of $SO(2)$.

Let us start with the decomposition of the fermions which arise in the (NS,R) and (R,NS) sector. More precisely, in type IIA the two gravitinos together with the two dilatinos are in the $(\mathbf{8_S}, \mathbf{8_V})$ and $(\mathbf{8_V}, \mathbf{8_C})$ of $SO(8)_L \times SO(8)_R$ while in type IIB they come in the $(\mathbf{8_S}, \mathbf{8_V})$ and $(\mathbf{8_V}, \mathbf{8_S})$ representations. The decomposition of these representations under $SO(8)_L \times$

$SO(8)_R \to SO(2) \times SU(3)_L \times SU(3)_R$ yields[31]

$$\begin{aligned}
(\mathbf{8_S}, \mathbf{8_V}) \ &\to (\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\frac{3}{2},-\frac{1}{2}} \oplus (\bar{\mathbf{3}},\mathbf{1})_{-\frac{3}{2},\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\bar{\mathbf{3}})_{\pm\frac{1}{2}} \\
&\qquad \oplus (\mathbf{3},\mathbf{3})_{\frac{1}{2}} \oplus (\bar{\mathbf{3}},\mathbf{3})_{-\frac{1}{2}} \oplus (\mathbf{3},\bar{\mathbf{3}})_{\frac{1}{2}} \oplus (\bar{\mathbf{3}},\bar{\mathbf{3}})_{-\frac{1}{2}} \ , \\
(\mathbf{8_V}, \mathbf{8_S}) \ &\to (\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\bar{\mathbf{3}},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{\frac{3}{2},-\frac{1}{2}} \oplus (\mathbf{1},\bar{\mathbf{3}})_{-\frac{3}{2},\frac{1}{2}} \\
&\qquad \oplus (\mathbf{3},\mathbf{3})_{\frac{1}{2}} \oplus (\bar{\mathbf{3}},\mathbf{3})_{\frac{1}{2}} \oplus (\mathbf{3},\bar{\mathbf{3}})_{-\frac{1}{2}} \oplus (\bar{\mathbf{3}},\bar{\mathbf{3}})_{-\frac{1}{2}} \ , \\
(\mathbf{8_V}, \mathbf{8_C}) \ &\to (\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\bar{\mathbf{3}},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{-\frac{3}{2},\frac{1}{2}} \oplus (\mathbf{1},\bar{\mathbf{3}})_{\frac{3}{2},-\frac{1}{2}} \\
&\qquad \oplus (\mathbf{3},\mathbf{3})_{-\frac{1}{2}} \oplus (\bar{\mathbf{3}},\mathbf{3})_{-\frac{1}{2}} \oplus (\mathbf{3},\bar{\mathbf{3}})_{\frac{1}{2}} \oplus (\bar{\mathbf{3}},\bar{\mathbf{3}})_{\frac{1}{2}} \ .
\end{aligned} \tag{5.20}$$

We see that the gravitini (the $\pm\frac{3}{2}$ representations) assemble both in type IIA and in type IIB into two singlet and two triplet representations. As will be confirmed below, the singlet representation includes the $N = 2$ gravitational multiplet, while the triplet representations consist of $N = 2$ gravitino multiplets. As the $N = 2$ gravitino multiplets will be massive for generic $SU(3) \times SU(3)$-structure compactifications, we see that the triplet representations will consist of only massive multiplets, and will be ignored in the mode expansion. In other words, we project out all $SU(3)\times SU(3)$ triplets. This will ensure to lead to a 'standard' gauged $N = 2$ supergravity.

After this projection both type II theories have two gravitinos and two Weyl fermions (dilatinos) in the $(\mathbf{1},\mathbf{1})$ representations. They reside in the gravitational multiplet and the 'universal' tensor multiplet respectively. Furthermore, eq. (5.20) shows that there is a pair of Weyl fermions in the representations $(\mathbf{3},\mathbf{3})\oplus(\bar{\mathbf{3}},\bar{\mathbf{3}})$ and a pair in the $(\bar{\mathbf{3}},\mathbf{3})\oplus(\mathbf{3},\bar{\mathbf{3}})$. These fermions are members of vector- or hypermultiplets depending on which type II theory is being considered.

The bosonic fields in the NS sector can be similarly decomposed under SU(3)×SU(3). It is convenient to use the combination $E_{MN} = g_{MN} + B_{MN}$ of the metric and the B-field since from a string theoretical point it is a tensor product of a left and a right NS-mode excitation. As a consequence it decomposes under SU(3)×SU(3) as

$$\begin{aligned}
E_{\mu\nu} &: (\mathbf{1},\mathbf{1})_{\pm\mathbf{2}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{T}} \ , \\
E_{\mu m} &: (\mathbf{1},\mathbf{3})_{\pm\mathbf{1}} \oplus (\mathbf{1},\bar{\mathbf{3}})_{\pm\mathbf{1}} \ , \\
E_{m\mu} &: (\mathbf{3},\mathbf{1})_{\pm\mathbf{1}} \oplus (\bar{\mathbf{3}},\mathbf{1})_{\pm\mathbf{1}} \ , \\
E_{mn} &: (\mathbf{3},\mathbf{3})_{\mathbf{0}} \oplus (\bar{\mathbf{3}},\bar{\mathbf{3}})_{\mathbf{0}} \oplus (\bar{\mathbf{3}},\mathbf{3})_{\mathbf{0}} \oplus (\mathbf{3},\bar{\mathbf{3}})_{\mathbf{0}} \ ,
\end{aligned} \tag{5.21}$$

where $\mathbf{T}$ denotes the antisymmetric tensor. Projecting out the triplet representations leaves only $E_{\mu\nu}$ and E_{mn} in the spectrum. From a four-dimensional point of view $E_{\mu\nu}$ corresponds to the graviton and an antisymmetric tensor while E_{mn} represent scalar fields. The latter can be viewed as parameterizing the deformations of the SU(3)×SU(3) structure or equivalently as deformations of the pure spinors $\Phi^{\pm}$. More precisely, keeping the normalization of the pure spinors fixed, $\delta\Phi^+$ transforms in the $(\bar{\mathbf{3}},\mathbf{3})$, while $\delta\Phi^-$ transforms in the $(\bar{\mathbf{3}},\bar{\mathbf{3}})$ (and $\delta\bar{\Phi}^+$, $\delta\bar{\Phi}^-$ transform in the complex conjugate representations, $(\mathbf{3},\bar{\mathbf{3}})$ and $(\mathbf{3},\mathbf{3})$ respectively).

Finally we decompose the fields in the RR-sector. Here the bosonic fields arise from the decomposition of $(\mathbf{8}_S,\mathbf{8}_C)$ for type IIA and $(\mathbf{8}_S,\mathbf{8}_S)$ for type IIB. One finds (after projecting out the triplets)

$$\begin{aligned}
\text{IIA}: &\quad (\mathbf{8}_S,\mathbf{8}_C) \to (\mathbf{1},\mathbf{1})_{\pm\mathbf{1},\mathbf{0}} \oplus (\mathbf{3},\mathbf{3})_{\mathbf{0}} \oplus (\bar{\mathbf{3}},\bar{\mathbf{3}})_{\mathbf{0}} \oplus (\mathbf{3},\bar{\mathbf{3}})_{\mathbf{1}} \oplus (\bar{\mathbf{3}},\mathbf{3})_{-\mathbf{1}} \ , \\
\text{IIB}: &\quad (\mathbf{8}_S,\mathbf{8}_S) \to (\mathbf{1},\mathbf{1})_{\pm\mathbf{1},\mathbf{0}} \oplus (\mathbf{3},\mathbf{3})_{\mathbf{1}} \oplus (\bar{\mathbf{3}},\bar{\mathbf{3}})_{-\mathbf{1}} \oplus (\mathbf{3},\bar{\mathbf{3}})_{\mathbf{0}} \oplus (\bar{\mathbf{3}},\mathbf{3})_{\mathbf{0}} \ .
\end{aligned} \tag{5.22}$$

[31]The $SO(2)$ factors for left- and right-movers in four-dimensional spacetime are identified.

In type IIA the RR sector contains gauge potentials of odd degree. The decomposition (5.22) naturally groups these into vectors (helicity ± 1) and scalars (helicity 0) from a four-dimensional point of view. This leads us to define

$$\begin{aligned} \mathcal{C}_0^- &= A_{(0,1)} + A_{(0,3)} + A_{(0,5)} \simeq (\mathbf{1},\mathbf{1})_\mathbf{0} \oplus (\mathbf{3},\mathbf{3})_\mathbf{0} \oplus (\bar{\mathbf{3}},\bar{\mathbf{3}})_\mathbf{0} \ , \\ \mathcal{C}_1^+ &= A_{(1,0)} + A_{(1,2)} + A_{(1,4)} + A_{(1,6)} \simeq (\mathbf{1},\mathbf{1})_{\pm\mathbf{1}} \oplus (\mathbf{3},\bar{\mathbf{3}})_\mathbf{1} \ \oplus (\bar{\mathbf{3}},\mathbf{3})_{-\mathbf{1}}, \end{aligned} \tag{5.23}$$

where $A_{(p,q)}$ is a 'four-dimensional' p-form and a 'six-dimensional' q-form.[32] $\mathcal{C}_0^-$ contains 'four-dimensional' scalar degrees of freedom and is a sum of odd 'six-dimensional' forms while $\mathcal{C}_1^+$ contains 'four-dimensional' vectors and is a sum even 'six-dimensional' forms.

In type IIB the situation is exactly reversed. Here we define

$$\begin{aligned} \mathcal{C}_0^+ &= A_{(0,0)} + A_{(0,2)} + A_{(0,4)} + A_{(0,6)} \simeq (\mathbf{1},\mathbf{1})_\mathbf{0} \oplus (\mathbf{3},\bar{\mathbf{3}})_\mathbf{0} \oplus (\bar{\mathbf{3}},\mathbf{3})_\mathbf{0} \ , \\ \mathcal{C}_1^- &= A_{(1,1)} + A_{(1,3)} + A_{(1,5)} \simeq (\mathbf{1},\mathbf{1})_\mathbf{1} \oplus (\mathbf{3},\mathbf{3})_\mathbf{1} \oplus (\bar{\mathbf{3}},\bar{\mathbf{3}})_{-\mathbf{1}} \ . \end{aligned} \tag{5.24}$$

As expected all these fields combine into $N = 2$ multiplets, as shown in Tables 5.2 and 5.3. We see that the fields arrange nicely and symmetrically into multiplets of a given Spin(6,6)

multiplet	SU(3)×SU(3)rep.	bosonic field content
gravity multiplet	$(\mathbf{1},\mathbf{1})$	$g_{\mu\nu}, \mathcal{C}_1^+$
tensor multiplet	$(\mathbf{1},\mathbf{1})$	$B_{\mu\nu}, \phi, \mathcal{C}_0^-$
vector multiplets	$(\mathbf{3},\bar{\mathbf{3}})$	$\mathcal{C}_1^+, \delta\Phi^+$
hypermultiplets	$(\mathbf{3},\mathbf{3})$	$\delta\Phi^-, \mathcal{C}_0^-$

Table 5.2: *N=2 multiplets in type IIA*

multiplet	SU(3)×SU(3)rep.	bosonic field content
gravity multiplet	$(\mathbf{1},\mathbf{1})$	$g_{\mu\nu}, \mathcal{C}_1^-$
tensor multiplet	$(\mathbf{1},\mathbf{1})$	$B_{\mu\nu}, \phi, \mathcal{C}_0^+$
vector multiplets	$(\mathbf{3},\mathbf{3})$	$\mathcal{C}_1^-, \delta\Phi^-$
hypermultiplets	$(\mathbf{3},\bar{\mathbf{3}})$	$\delta\Phi^+, \mathcal{C}_0^+$

Table 5.3: *N=2 multiplets in type IIB*

chirality. From these tables it should be clear that SU(3)×SU(3) structure is the relevant one for $N = 2$ effective actions coming from type II theories.

[32] There is an ambiguity in the representation of the scalar degrees of freedom arising in the RR-sector. They can be equally well written as a four-dimensional two-form. On the other hand, A_1^+ includes both the vector and dual vector degrees of freedom.

5.2.2 Dimensional reduction and consistent truncations

Now let us come to the reduction ansatz. As already anticipated, we expand the ten-dimensional fields in the modes Σ_+ and Σ_-, given in (5.14). In order to find a consistent theory in four dimensions, we have to make a few assumptions, beyond $\Sigma_\pm$ forming symplectic vectors. In particular, we want to ensure that the four-dimensional equations of motion imply the ten-dimensional ones. This is called a consistent truncation. We already mentioned that there should be no triplets of $SU(3) \times SU(3)$ inside $\Sigma_\pm$. Moreover, the forms in $\Sigma_\pm$ should form a closed set under the exterior derivative d and under the Hodge star. The former implies

$$\begin{aligned} d\alpha_I &= p_I^A \omega_A + e_{IA}\tilde{\omega}^A \ , \qquad & d\beta^I &= q^{IA}\omega_A + m_A^I \tilde{\omega}^A, \\ d\omega_A &= m_A^I \alpha_I - e_{IA}\beta^I \ , \qquad & d\tilde{\omega}^A &= -q^{IA}\alpha_I + p_I^A \beta^I \ , \end{aligned} \tag{5.25}$$

where $p_I^A, e_{IA}, q^{IA}, m_A^I$ are constant coefficients. In terms of Σ_+ and Σ_- Eq. (5.25) turns into

$$d_H \Sigma_- = \mathcal{Q}\,\Sigma_+ \ , \qquad d_H \Sigma_+ = \mathcal{S}_+ \mathcal{Q}^T (\mathcal{S}_-)^{-1}\,\Sigma_- \tag{5.26}$$

where

$$\mathcal{Q} = \begin{pmatrix} p_I{}^A & e_{IB} \\ q^{JA} & m^J{}_B \end{pmatrix} \ . \tag{5.27}$$

Note that in both equations of (5.26) the same matrix $\mathcal{Q}$ has to appear in order to ensure consistency of partial integration, i.e. $\int_{M^6} \langle \Sigma_+, d\Sigma_- \rangle = \int_{M^6} \langle d\Sigma_+, \Sigma_- \rangle$. Furthermore $d^2 = 0$ implies two additional quadratic constraints

$$\mathcal{Q}\mathcal{S}_+\mathcal{Q}^T \ = \ 0 \ = \ \mathcal{Q}^T(\mathcal{S}_-)^{-1}\mathcal{Q} \ . \tag{5.28}$$

The matrix $\mathcal{Q}$ determines the four-dimensional gaugings. The quadratic constraints will turn into the quadratic constraints of gauged $N = 2$ supergravity.

Exercise: Expand the R-R fields C in Σ_+ and Σ_- and show that the field strength $F = dC$ contains covariant derivatives for the scalars $\mathcal{C}_0$ that couple them to the vectors $\mathcal{C}_1$ via the charge matrix $\mathcal{Q}$.

Now let us expand Φ_+ and Φ_- in Σ_+ and Σ_-, respectively. This gives

$$\Phi_+ = Z^A \omega_A + \mathcal{G}_A \tilde{\omega}^A \ , \qquad \Phi_- = X^I \alpha_I + \mathcal{F}_I \beta^I \ . \tag{5.29}$$

We therefore see that both $\Phi_\pm$ form holomorphic symplectic vectors. This is exactly the structure of special-Kähler geometry. Special-Kähler manifolds are Kähler manifolds that admit a symplectic holomorphic vector Z^A such that the Kähler potential reads

$$K_+ = -\log(i(\bar{Z}^A \mathcal{G}_A - Z^A \bar{\mathcal{G}}_A)) \ . \tag{5.30}$$

This means that

$$K_\pm = -\log(i\langle \Phi_\pm, \bar{\Phi}_\pm \rangle) \ . \tag{5.31}$$

In most cases, one can locally choose the Z^A to be homogeneous coordinates and find a holomorphic function $\mathcal{G}$ such that

$$\mathcal{G}_A = \frac{\partial \mathcal{G}}{\partial Z^A} \ . \tag{5.32}$$

One can furthermore show that $\mathcal{G}$ is homogeneous of degree two, i.e. for $Z^A \to \lambda Z^A$ we find $\mathcal{G} \to \lambda^2 \mathcal{G}$ with $\lambda \in \mathbb{C}^*$.

For a single $SU(3)$ structure, i.e. for $\Phi^+ = \frac{1}{8}\mathrm{e}^{-(B+\mathrm{i}J)}$, $\Phi^- = -\frac{\mathrm{i}}{8}\mathrm{e}^{-B}\Omega$, the Kähler potentials are given respectively by the familiar expressions

$$\mathrm{e}^{-K^+} = \tfrac{1}{48} J\wedge J\wedge J, \qquad\qquad \mathrm{e}^{-K^-} = \tfrac{\mathrm{i}}{64}\Omega\wedge\bar{\Omega}. \tag{5.33}$$

Note that B drops from these expressions (which is easy to see since $\left\langle \mathrm{e}^{-B}\psi, \mathrm{e}^{-B}\chi\right\rangle = \left\langle \psi, \mathrm{e}^{B}\mathrm{e}^{-B}\chi\right\rangle = \langle\chi,\psi\rangle$).

It turns out that the Kähler potential coincides with the Hitchin functional of a real pure spinor. Let us review this connection. Working at a fixed point in the manifold, one starts with a real stable $Spin(6,6)$ spinor, or its associated form $\chi^\pm$. Such form is stable if it lies in an open orbit of $Spin(6,6)$. One can construct a $Spin(6,6)$ invariant six-form, known as the Hitchin function $H(\chi^\pm)$, which is homogeneous of degree two as a function of $\chi^\pm$. One can get a second real form by derivation of the Hitchin function: $\hat{\chi}^\pm(\chi) := -\partial H(\chi^\pm)/\partial\chi^\pm$. This form $\hat{\chi}^\pm$ has the same parity as $\chi^\pm$, and can be used to define the complex spinors $\Phi^\pm = \chi^\pm + \mathrm{i}\hat{\chi}^\pm$. Hitchin showed that the complex spinors built in this form are pure. Since H is homogeneous of degree two in $\chi^\pm$, we have

$$H(\Phi^\pm) = \frac{1}{2}\left\langle \chi^\pm, \hat{\chi}^\pm\right\rangle = \mathrm{i}\left\langle \Phi^\pm, \bar{\Phi}^\pm\right\rangle\ . \tag{5.34}$$

There is a symplectic structure on the space of stable spinors given by the Mukai pairing and a complex structure corresponding to the complex spinor $\Phi^\pm$. Both complex and symplectic structures are integrable, and therefore the space of stable forms (or pure spinors) is Kähler, or rather it is rigid special-Kähler. Quotenting this space by the $\mathbb{C}^*$ action $\Phi^\pm \to \lambda\Phi^\pm$ for $\lambda \in \mathbb{C}^*$ (i.e., modding out by rescalings of the pure spinor), gives a space with a the Kähler potential K related to the Hitchin function by

$$\mathrm{e}^{-K^\pm} = H(\Phi^\pm) = \mathrm{i}\left\langle \Phi^\pm, \bar{\Phi}^\pm\right\rangle\ , \tag{5.35}$$

which defines a local special-Kähler metric.

Now let us turn to the R-R fields. As already discussed, from the expansion of C we find a symplectic vector of vector fields $\mathcal{C}_1^\pm$ and a symplectic vector of scalar fields $\mathcal{C}_0^\mp$ (in type IIA/IIB). The former combines with the scalars from $\Phi^\pm$ into vector multiplets, while the latter combine with $\Phi^\mp$, the dilaton ϕ and the four-dimensional tensor $B_{\mu\nu}$ into hypermultiplets. The space of hypermultiplet scalars therefore forms a fibration over $\Phi^\mp$, which is usually denoted as the "c-map". This c-map maps the special-Kähler space of $\Phi^\mp$ to a quaternion-Kähler space, whose geometry is completely determined by the Kähler potential $K_\mp$.

The resulting $N = 2$ supergravity is gauged, with gauging parameters $\mathcal{Q}$. These gaugings introduce non-trivial couplings between vector and hypermultiplets.[33] Furthermore, it leads to a potential that can break (part of) supersymmetry.

Let us close this section by remarking that there is a symmetry between type IIA and type IIB that just corresponds to the exchange of $SO(6,6)$ spinor chiralities. In other words,

$$\begin{aligned}\Phi_+ &\leftrightarrow \Phi_- \\ F_+ &\leftrightarrow F_-\ .\end{aligned} \tag{5.36}$$

This symmetry between type IIA and type IIB is believed to hold even at the quantum level and is denoted by mirror symmetry. We see that in the framework of generalized geometry, this symmetry is manifest.

[33] Remember that ungauged $N = 2$ supergravity has no couplings between vector and hypermultiplets.

5.3 Exceptional generalized geometry

Generalized geometry is manifest under symmetries that are common to both type II and heterotic (and even bosonic) string theories, this means those symmetries that transform NS-NS sector and R-R sector (or the gauge sector) individually without mixing them. This means that each individual string symmetry should actually have further symmetries, depending on its extension to a supersymmetric (and anomaly-free) theory. As we could already understand the T-duality group $SO(d,d,\mathbb{Z})$ and its continuous partner $SO(d,d)$ from its appearance in torus compactifications, also the full symmetry group of type II reductions can be anticipated in this case: If we compactify type II string theory on a d-dimensional torus, we find the *U-duality group* $E_{d+1(d+1)}(\mathbb{Z})$, and the moduli space is given by the coset space $E_{d+1(d+1)}/K$, where K is the maximal compact subgroup of $E_{d+1(d+1)}$. Let us first introduce these groups before we define a geometric theory in which these symmetries are manifestly covariant.

5.3.1 Exceptional Lie groups and string dualities

The group $E_{n(n)}$ is a non-compact version of the Lie group E_n that belongs to the exceptional cases in the classification of simple Lie algebras and Lie groups. Here, simple Lie groups are all compact non-Abelian Lie groups that cannot be decomposed into a product of two groups. Among the simple Lie groups are the infinite series $SO(n)$, $Sp(n)$ and $SU(n)$, but there are also exceptional cases that are denoted by G_2, F_4 and E_n, with $n = 6, 7, 8$.[34] As complex groups, this list is even exhaustive, but there are various real Lie groups related to each of them. As $SO(n,n)$ is a non-compact version of $SO(2n)$ (which are both defined by certain real forms of $SO(2n,\mathbb{C})$), the group $E_{n(n)}$ is a non-compact version of E_n. The maximal compact subgroups of $E_{n(n)}$ are $USp(8)$, $SU(8)$ and $SO(16)$ for $n = 6, 7, 8$, respectively.[35]

Similar to the T-duality group $SO(d,d,\mathbb{Z})$, the discrete subgroup $E_{d+1(d+1)}(\mathbb{Z})$ of $E_{d+1(d+1)}$ can be understood as the subgroup that maps integer quantum number of string states to integer quantum numbers.[36] Let us discuss the simplest case, which is $E_{2(2)} \cong SL(2,\mathbb{Z})$, which is usually denoted as the *S-duality* group, and exists in type IIB even in the ten dimensional theory. It acts on the axiodilaton

$$S = C_0 + \mathrm{i}e^{-\phi} \ , \tag{5.37}$$

with C_0 being the axionic scalar from the R-R sector and ϕ being the dilaton, by

$$S \to \frac{aS+b}{cS+d} \ , \tag{5.38}$$

where a, b, c and d are integers obeying $ad - bc = 1$ and thereby defining a matrix in $SL(2,\mathbb{Z})$. This group is spanned by large gauge transformations $C_0 \to C_0 + 1$ and a strong-weak duality, that for vanishing axion reads $e^{\phi} \to e^{-\phi}$, which maps strongly and weakly coupled theory to each other.

An important feature of $E_{d+1(d+1)}$ is for $d \leq 5$ that it admits a maximal subgroup $\mathbb{R}_+ \times SO(d,d)$. The second factor can be easily identified with the continuous T-duality group of generalized geometry, while the first one acts on the four-dimensional dilaton ϕ. The group

[34] One can define the groups E_n also for $n \leq 5$, but in these cases they always coincide with other, regular Lie groups, for instance $E_3 \cong SU(2) \times SU(3)$, $E_4 \cong SU(5)$ and $E_5 \cong SO(10)$. Generalizations for $n > 8$ are known (and interesting for string theory), but fall into the more general class of Kac-Moody algebras, which can be understood as infinite-dimensional generalizations of Lie groups.

[35] Here, $USp(8)$ is a compact symplectic group that is also often denoted by $Sp(4)$.

[36] Depending on d, the interpretation of $E_{d+1(d+1)}(\mathbb{Z})$ is however less obvious, as for instance $E_{8(8)}$ (and also E_8) does not admit a representation in terms of matrices.

$E_{7(7)}$ that is relevant for compactifications to four dimensions even has the maximal subgroup $SL(2,\mathbb{R}) \times SO(6,6)$, where the first factor is the continuous S-duality group acting on the four-dimensional axiodilaton[37] $b + \mathrm{i}e^{-\phi}$, where b is the axion that is dual to B_2 in four dimensions. Therefore the group $SL(2,\mathbb{R}) \times SO(6,6)$ is the one that acts on NS-NS and R-R sector individually. On top of these transformations, $E_{d+1(d+1)}$ consists of large gauge transformations for the R-R form fields and of string dualities that mix the NS-NS and R-R sectors.

5.3.2 Embedding generalized geometry into E7

Similar to generalized geometry, we can define a formalism that makes the $E_{7(7)}$ symmetry manifest for general type II compactifications to four dimensions by identifying the correct extension of the generalized tangent bundle. By using the representation theory for the subgroup $SL(2,\mathbb{R}) \times SO(6,6) \subset E_{7(7)}$, one can identify this bundle locally as[38]

$$F \sim TM \oplus T^*M \oplus \Lambda^5 TM \oplus \Lambda^5 T^*M \oplus \Lambda^{\text{even}} T^*M \ . \tag{5.39}$$

As in generalized geometry, globally F should be viewed as a fibration of these factors over each other, defined in terms of a gerbe structure that corresponds to the fluxes for F and H. Similar to generalized geometry and T-folds, where TM and T^*M corresponded to momentum and winding modes for the string, one can associate certain charged string objects to each of the components above. So corresponds the last term $\Lambda^{\text{even}} T^*M$ for instance to D-branes wrapping even cycles in the internal geometry. Furthermore, F has dimension 56, which is exactly the dimension of the fundamental representation of $E_{7(7)}$.

The discussion of generalized geometry carries over from E to F. Most importantly, the pure spinor Φ^+ and the vectors $\mathcal{C}_\mu$ from the R-R sector can both be embedded into the last component $\Lambda^{\text{even}} T^*M$ in (5.39), giving the vector multiplets of $N = 2$ supergravity. In particular, the vector multiplet scalars can be assembled in a vector L in the complex fundamental representation. The pure spinor Φ^- and the R-R scalars $\mathcal{C}$ however do not fit into F. They are embedded into three K_a, $a = 1, 2, 3$, in the adjoint representation of $E_{7(7)}$ which has dimension 133. Together they form an $SU(2)$ subalgebra in the Lie algebra of $E_{7(7)}$ and define a quaternion-Kähler manifold. The incorporation of the R-R fields to form out of $SL(2,\mathbb{R}) \times SO(6,6)$ the group $E_{7(7)}$ exactly corresponds to the c-map construction in supergravity, cf. 5.2.2. It turns out that the supersymmetry conditions for vacua can be stated in a very simple way in terms of "integrability conditions", schematically as

$$D_L K_a = 0 \ , \qquad D_K L = 0 \ . \tag{5.40}$$

6 Lecture 6: Open problems in phenomenology

In this lecture we will explain the basic idea and address the current status of moduli stabilization and de Sitter vacua in string theory.

We will first consider moduli stabilization in the context of Calabi-Yau compactifications with NS-NS and R-R flux. We already pointed out that the introduction of fluxes leads to gaugings in the four-dimensional action and that these give rise to a potential. We will focus here on the form of this potential, in particular on its explicit dependence on moduli.

[37] The dilaton in $10 - d$ dimensions consists of the ten-dimensional dilaton and the internal volume of the compactification manifold.

[38] The following expressions are for type IIA. For type IIB, just exchange all chiralities as discussed at the end of Section 5.2.2.

As we explained in Section 4.1.3, NS-NS flux is expanded in a basis for the odd cohomology, namely

$$H = m^K \alpha_K - e_K \,\beta^K \ . \tag{6.1}$$

Upon reduction on Calabi-Yau manifolds, the kinetic term for H in the ten-dimensional action gives rise to a potential

$$\int H \wedge *H \longrightarrow V \sim (e_K + \mathcal{M}_{KM}\, m^M)(\mathrm{Im}\mathcal{M})^{KL}(e_L + \bar{\mathcal{M}}_{LN}\, m^N) \tag{6.2}$$

where $\mathcal{M}$ was introduced in (3.35) and depends on the complex structure moduli. This can be understood since the Hodge star in the ten-dimensional action introduces a dependence on the metric, and furthermore since it is acting on a 3-form, it only depends on the complex structure part of it. The NS-NS fluxes e, m create therefore a non-trivial potential for the complex structure moduli, which can acquaint therefore a preferred vacuum expectation value, or in other words these moduli are "stabilized". Similarly, R-R fluxes in type IIB, being odd as well, give rise to a potential for complex structure moduli, while in IIA they create a potential for Kähler moduli.

In general compactifications on manifolds of $SU(3)$ structure, the NS-NS H-flux units combine with those coming from the non-closure of the form basis (a.k.a. "metric fluxes") to build matrices m_A^I, e_{AI}, where m_0^I is the H-flux. Upon reduction of (6.2), we get extra terms in the potential, since

$$H = dB + H_{\text{flux}} \ , \tag{6.3}$$

where H_{flux} is the piece we have just discussed, and[39]

$$B = b^a \omega_a \Rightarrow dB = b^a(m_a^I \alpha_I + e_{Ia}\beta^I) \ , \tag{6.4}$$

where we have used (5.25). Inserting this into (6.2) we get a potential for the axions b^a that depends on the complex structure moduli.

All the different pieces of the potential should assemble such that the latter comes from the Killing prepotentials (or moment maps), as dictated by $N = 2$ supersymmetry (remember that the deformations in $N = 2$ cannot be arbitrary, but are completely determined by the gaugings. The latter create a potential which is uniquely determined by the moment maps corresponding to the gaugings). However, for phenomenological applications we are not so much interested in compactifications with $N = 2$ supersymmetry, but rather on those with $N = 1$. The latter can be obtained as a projection on the former by an "orientifold action".

6.1 Calabi-Yau orientifold reductions

Orientifolding amounts to keeping states that are invariant under the action of an involutive symmetry σ (i.e. such that $\sigma^2 = 1$), that leaves the metric invariant, combined with the world-sheet parity action Ω_p that exchanges left- and right-movers. For consistency, certain orientifold actions have furthermore to be combined with $(-1)^{F_L}$, the fermionic index on the left-moving sector. The latter is $+1$ for NS-NS states, and -1 for R-R ones. Since Ω_p exchanges left- and right-movers, it acts as $+1$ on the metric and as -1 on the B-field (remember the former has a symmetric combination of a left- and a right-mover, while for the latter they are antisymmetric). The action on the R-R states alternates sign, for type IIB for instance it is $+1$ for C_2 and -1 for C_0, C_4.

[39] Here and in the following, we neglect any dependence on four-dimensional spacetime coordinates and focus on the potential terms.

	O3	
gravity multiplet	1	$g_{\mu\nu}$
vector multiplets	$\cancel{h_+^{(2,1)}}$	$\cancel{V_1^\alpha}$
chiral multiplets	$h_-^{(2,1)}$	z^k
	$h_+^{(1,1)}$	(v^a, ρ_a)
	$\cancel{h_-^{(1,1)}}$	$(\cancel{b^\alpha}, \cancel{c^\alpha})$
	1	(ϕ, C_0)
	$\cancel{1}$	$(\cancel{B_{\mu\nu}}, \cancel{C_{\mu\nu}})$

Table 6.1: Type IIB moduli arranged in $\mathcal{N} = 1$ multiplets for O3 orientifolds. The fielkds crossed out are projected out from the spectrum.

On Calabi-Yau compactifications in type IIB, such a orientifold projection is consistent if the involution σ is holomorphic (i.e. $\sigma^*\Omega = \pm\Omega$ and $\sigma^* J = J$), while in type IIA it has to be anti-holomorphic ($\sigma {I_n}^m = -{I_n}^m$), which acts on the three-form Ω as $\sigma^*\Omega = e^{2i\theta}\bar{\Omega}$, where θ is some phase, while $\sigma^* J = -J$. The different signs lead to fixed points (or rather fixed planes) of different dimensionality. These planes are called "orientifold planes". We will only discuss projections with fixed planes that span the four-dimensional space-time but are point-like inside the Calabi-Yau, i.e. where the involution σ consists of inverting all the internal coordinates close to the fixed plane. These fixed planes are called O3-planes (the notations is as in D_p-branes, denoting only the spatial directions wrapped by the plane) and exist only in type IIB.

Besides projecting out half of the supersymmetries (which one can understand as the left- and right-moving supersymmetries get related by Ω_p), orientifold planes have negative D-brane charge and tension. They are therefore a necessary ingredient in a compactification, where tadpole cancelation requires all charges and masses of spacetime-filling objects to add up to zero (in other words, if there is a source of charge or mass on a compact manifold, there should be a sink for it somewhere else).

The massless states that survive the orientifold projection are those that are even under the combined action of $(-1)^{F_L}\Omega_p\,\sigma$ for an O3 projection. For example, the B-field is odd under $(-1)^{F_L}\Omega_p$ (as it is odd under Ω_p and it has $F_L = 0$). This implies that in the presence of any O-plane, the only components of B that survive are those that are odd under σ.

The space of harmonic p-forms, H^p, splits into two eigenspaces under the action of σ with eigenvalues plus and minus one. From the massless modes of the B-field given in Table 3.2, b^a and B_2, only a subset of b^a, namely those multiplying a two-form in $H_-^{(1,1)}$, survive. The same is true for v^a, the Kähler deformations of the metric. Opposite to this, the vector C_1^a, which is in the same $\mathcal{N} = 2$ multiplet as the complexified Kähler deformations and comes from C_3, should be expanded in harmonic forms in $H_+^{(1,1)}$ The case of O3-planes is particularly simple, as even forms are even under σ, while odd forms are odd. Therefore, certain fields like for instance b^a are projected out.

In general, an $N = 2$ gravity multiplet splits into an $N = 1$ gravity and an $N = 1$ vector multiplet. An $N = 2$ vector multiplet splits into a vector and a chiral multiplet in $N = 1$. A hypermultiplet splits into two $N = 1$ chiral multiplets. For orientifolding, for any $N = 2$ multiplet these two $N = 1$ components differ in parity under σ. For the $O3$ projection only one of them will survive the projection. Table 6.1 shows the surviving IIB multiplets after the O3 projection.

The moduli spaces spanned by the scalars are Kähler manifolds, and are appropriate subspaces of the special-Kähler and quaternionic spaces of the $\mathcal{N}=2$ moduli spaces of Calabi-Yau compactifications. Before moving on, let us comment that the orientifold projection can be analogously defined in spaces of $SU(3)$ or even $SU(3)\times SU(3)$ structures, by the action of the involution on the pure spinors. We will however focus on Calabi-Yau manifolds from now on and so will not discuss this further.

The Kähler potential determining the geometry of moduli space is for the O3 projection

$$K_{\mathrm{O3}} = -\ln\Big[-i\int \Omega(z)\wedge\bar{\Omega}(\bar{z})\Big] - \ln\big[-i(S-\bar{S})\big] - 2\ln\tfrac{1}{6}\int J\wedge J\wedge J \qquad (6.5)$$

where the axion-dilaton field S was defined in (5.37), and J depends on the Kähler moduli.[40]

In $N=1$ the potential is uniquely determined by a holomorphic function of the chiral fields called the superpotential W, and a real function called the D-term D. The latter is zero for Calabi-Yau compatifications with O3 planes, so we will not discuss it. The superpotential (together with the Kähler potential defining the moduli space metric) determines the potential by

$$V = e^{K}\left(K^{I\bar{J}}D_I W D_{\bar{J}}\bar{W} - 3|W|^2\right)\,, \qquad (6.6)$$

where K is the Kähler potential, $K^{I\bar{J}}$ is the inverse of the metric in moduli space, and we have defined the Kähler covariant derivatives as

$$D_I W = \partial_I W + W\partial_I K\,. \qquad (6.7)$$

A solution to the equations of motion satisfies $\partial_I V=0$, while a supersymmetric solution should additionally satisfy $D_I W=0$. On the contrary, a supersymmetric vacuum with $D_I W=0$ for all I automatically satisfies $\partial_I V=0$.

The Kähler potential for the chiral multiplets in type IIB O3 compactifications coming from $\mathcal{N}=2$ hypermultiplets, i.e. the last term in Eq. (6.5), satisfies a very important property, namely

$$\partial_{t_a}K\partial_{\bar{t}_b}K\,K^{t_a\bar{t}_b} = 3\ . \qquad (6.8)$$

This is a no-scale type condition. This condition (6.8) implies that the positive contribution to the potential (6.6) offsets the negative one $-3|W|^2$, and we therefore get $V\geq 0$. This equality can be easily checked in the simple case of one Kähler modulus (see footnote 40). Note that Kähler potentials are independent of the axions due to their shift symmetry.

The type IIB superpotential for compactifications on Calabi-Yau O3 generated by the fluxes is

$$W_{O3/O7} = \int G_3\wedge\Omega\ , \qquad \text{where } G_3 = F_3 - \mathrm{i}SH_3 \qquad (6.9)$$

This superpotential depends on the complex structure moduli through Ω, and on the axiodilaton, by the definition of G_3. On the contrary, the Kähler moduli t do not appear in the superpotential. Therefore, $D_{t^a}W = W\partial_{t^a}K$, and the no-scale property of the Kähler potential means that the potential is positive semi-definite. In particular, it is zero (i.e. Minkowski) for supersymmetric vacua, for which $D_{t^a}W=0$ implies $W=0$. The potential has therefore the form illustrated in Figure 6.

In the following section we will show how moduli stabilisation works on tori, as this is far more intuitive than general Calabi-Yau manifolds, but we will work on an $N=1$ subset of the torus moduli.

[40]In O3 compactifications the Kähler moduli are not just $\rho_a + \mathrm{i}v^a$ (which can be guessed from the index structure), but $t_a = \rho_a + \mathrm{i}\kappa_{abc}v^b v^c$, where κ_{abc} are the intersection numbers. In other words, the imaginary part of the Kähler moduli are not the sizes of two-cycles, but the sizes of four-cycles. Expressing J^3 in terms of the Kḧler moduli cannot be done explicitly in general. For the case in which there is only one v (and therefore one $t_a\equiv t$), i.e when $h^{(1,1)}=1$, we get $K_t = -3\ln\left[-i(t-\bar{t})\right]$.

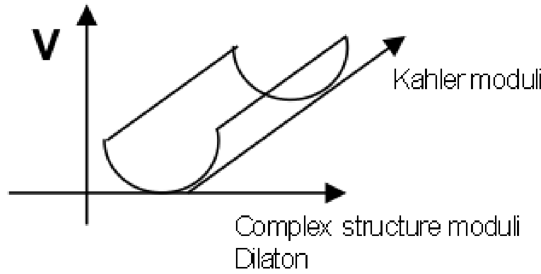

Figure 6: Potential for Calabi-Yau O3 compactifications with fluxes.

6.2 Moduli stabilization in type IIB

In this section we discuss moduli stabilization in compactifications of type IIB, where we start from the example of tori, since they are far more intuitive, and one can get the hands dirty and work on them explicitly. We will present one of the first examples discussed in the literature, that of Kachru-Schulz-Trivedi (KST) [28], consisting of $N = 1$ compactifications on a $T^6/\mathbb{Z}_2$ orientifold with NS-NS flux H_3 and R-R flux F_3.

The moduli for O3 compactifications in Calabi-Yau manifolds are given in Table 6.1. In the case of a torus, we have to take into account that the structure group is trivial. This implies that besides $h^{(2,1)} = 9$, $h^{(1,1)} = 9$, we have to consider also the cohomologies $h^{(1,0)} = h^{(2,0)} = h^{(3,1)} = h^{(3,2)} = 3$ and their conjugates ($h^{(0,1)} = 3$, etc.).

The explicit solution is constructed as follows. First, let x^i, y^i, $i = 1, 2, 3$ be six real coordinates on the torus, with periodicities $x^i \equiv x^i + 1$, $y^i \equiv y^i + 1$, and take the holomorphic 1-forms to be

$$dz^i = dx^i + \tau^{ij} dy^j \ . \tag{6.10}$$

The matrix τ^{ij} specifies the complex structure. The holomorphic 3-form is

$$\Omega = dz^1 \wedge dz^2 \wedge dz^3 \,. \tag{6.11}$$

The basis (α_K, β^L) from Eq. (3.21), where $K = 0, .., 9$ is taken to be

$$\begin{aligned} \alpha_0 &= dx^1 \wedge dx^2 \wedge dx^3 \ , \\ \alpha_{ij} &= \frac{1}{2}\epsilon_{ilm} dx^l \wedge dx^m \wedge dy^j \ , \\ \beta^{ij} &= -\frac{1}{2}\epsilon_{jlm} dy^l \wedge dy^m \wedge dx^j \\ \beta^0 &= dy^1 \wedge dy^2 \wedge dy^3 \ . \end{aligned} \tag{6.12}$$

The holomorphic 3-form Ω in (6.11) is given in this basis by

$$\Omega = \alpha_0 + \alpha_{ij}\tau^{ij} - \beta^{ij}(\mathrm{cof}\tau)_{ij} + \beta^0(\det \tau) \ , \tag{6.13}$$

where

$$(\mathrm{cof}\tau)_{ij} \equiv (\det \tau)\tau^{-1,\mathrm{T}} = \frac{1}{2}\,\epsilon_{ikm}\epsilon_{jpq}\tau^{kp}\tau^{mq} \ . \tag{6.14}$$

The NS-NS three-form fluxes along these three-cycles are denoted e_0, e_{ij}, m^0 and m^{ij} (see Eq. 6.1), and similarly for the R-R fluxes, adding a subindex "RR".

For the given setup, the superpotential (6.9) is

$$W = (m^0_{\rm RR} - Sm^0)\det\tau - (m^{ij}_{\rm RR} - Sm^{ij})({\rm cof}\tau)_{ij} - (e_{ij\,{\rm RR}} - Se_{ij})\tau^{ij} - (e_{0\,{\rm RR}} - Se_0)\ . \quad (6.15)$$

The supersymmetry conditions reduce to eleven (complex) equations, namely

$$D_{t^a}W = 0 \ \Leftrightarrow W = 0\ , \qquad \partial_S W = 0\ , \qquad \partial_{\tau^{ij}} W = 0 \qquad (6.16)$$

The equations resulting from here are eleven complex coupled non-linear equations for ten complex variables, namely the axiodilaton and the nine complex structure moduli τ^{ij}. They depend on twenty flux parameters. Generically, these cannot be solved and supersymmetry is broken, or even more drastic, for a given set of fluxes there might be no solution at all to the equations of motion.[41]

Exercise: Show that for $(e_{ij}, m^{ij}, e_{ij\,{\rm RR}}, m^{ij}_{\rm RR}) = (e, m, e_{\rm RR}, m_{\rm RR})\,\delta_{ij}$, the complex structure matrix that solves (6.16) should be proportional to the identity $\tau^{ij} = \tau\,\delta^{ij}$. This means that the torus factorizes as $T^6 = T^2 \times T^2 \times T^2$ with respect to the complex structure. Show that τ is the root of a third degree polynomial equation where the coefficients are given by the fluxes. Taking for example the set of fluxes $(e_0, e_{ij}, m^{ij}, m^0, e_{0\,{\rm RR}}, e_{ij\,{\rm RR}}, m^{ij}_{\rm RR}, m^0_{\rm RR}) = (2, -2\delta_{ij}, -2\delta^{ij}, -4, 2, 0, 0, 2)$, show that the complex structure and the dilaton are fixed at $S = \frac{1}{2}\tau = e^{\frac{2\pi i}{3}}$.

Note that the coupling constant g_s in this example is fixed at a value where perturbative corrections are important, namely $g_s = 1/\sqrt{3}$. It is possible however to chose fluxes such that g_s is fixed in the perturbative regime.

Exercise Show that for the solution just found, the complex 3-form flux is imaginary-self dual, i.e.

$$*G_3 = {\rm i}G_3\ . \qquad (6.17)$$

The result of these exercises should be surprising: we just found a solution with fluxes on a torus, though we initially said that fluxes back-react on the geometry, and tori (or more generally Calabi-Yau manifolds) are no longer solutions to the equations of motion. The solution found lies actually in a very particular class of solutions where the back-reaction on the geometry is minimal. The internal manifolds are conformal Calabi-Yau, where the conformal factor is the inverse of the warp factor introduced in (4.13), i.e. the ten-dimensional metric is of the form

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A}\tilde{g}_{mn}(y)dy^m dy^n\ , \qquad (6.18)$$

where $\tilde{g}$ is the metric on a Calabi-Yau manifold, and A depends on the internal coordinates. In the effective action, such warp factor is taken to be constant (we are expanding in the harmonic forms of the Calabi-Yau manifold, which for a scalar function corresponds to a constant), but uplifting the solution back to ten-dimensions, one can see that it has to be a non-trivial function of the coordinates. The solution has also RR five-form flux associated to the variation of this warp factor, namely

$$\tfrac{1}{4}F_5^{(10)} = dA \wedge {\rm vol}_4 + *_6 dA\ , \qquad (6.19)$$

and from the equation of motion for F_5, A (or rather e^{-4A}) satisfies a Laplace equation with sources, where the sources are D3-branes (if any), the O3 planes and the three-form fluxes.

Exercise Show that this solution satisfies the $N = 1$ equations (4.33).

[41] Since the structure group is trivial and not $SU(3)$, one should additionally impose the three-form flux G_3 to be primitive, i.e. $J_2 \wedge G_3 = 0$. These are six real equations for the nine Kähler moduli v^a, which means that generically, and differently from the Calabi-Yau case, only three of them remain unfixed.

Going back to the problem of moduli stabilization, this class of solutions are no-scale, i.e. dilaton and complex structure moduli are stabilized, but Kähler moduli are not. There are further (or different) steps to stabilize the rest of the moduli, e.g.

- Use fluxes in type IIA on T^6, where one can have supersymmetric AdS vacua where all moduli except a few axions are stabilized. This cannot be done however on a Calabi-Yau threefold.
- Add "geometric fluxes" (i.e. go to $SU(3)$ structure manifolds) and different orientifold projections. The problem there is that one does not know the basis of forms which one should keep in the spectrum, except in the case of twisted tori. Furthermore, solutions have typically still some moduli unfixed.
- Add "non-geometric" fluxes. The solution in four-dimensional gauged supergravity indicates that all moduli can be stabilized in this context, but then it is not clear whether this four-dimensional gauged supergravity is still a valid description.
- Stay within the class described in this section, but take into account the α' and g_s corrections to the action.

We will not review explicitly the works done around the first three possibilities, but we just give some of the tools needed, in particular the superpotential, which is nicely encoded in the language of generalized complex geometry. Taking the IIB/O3 superpotential (6.9) and following the mirror symmetry rules (5.36), we get exactly the RR contribution to the IIA superpotential on Calabi-Yau, namely

$$W_{\mathrm{IIA,RR}} = \langle F_+, e^{iJ} \rangle \ . \tag{6.20}$$

This can be easily generalized to manifolds of $SU(3) \times SU(3)$ structure by replacing e^{iJ} by Φ_+. The NS-NS contribution to (6.9) does not transform as naively expected from the exchange of pure spinors, as the NS-NS flux contribution in type IIA is still of the form $H \wedge \Omega$. One may wonder then what is the mirror counterpart of this term, that should depend on the Kähler moduli. The answer should not come as a total surprise: the mirror of NS-NS flux is purely in the geometry, i.e. is "geometric flux". The best way to see it it to write

$$H \wedge \Omega = \langle d\mathrm{Re}(\Phi_+^D), \Phi_-^D \rangle \tag{6.21}$$

where $\Phi_\pm^D$ are the B-field dressed pure spinors, defined in (4.27). We can now follow the mirror symmetry rules, and we get that the mirror of $H \wedge \Omega$ is $d\Omega \wedge J$. The full IIA and IIB superpotentials, where $SO(6,6)$ symmetry and mirror symmetry are manifest, are

$$\begin{aligned} W_{\mathrm{IIA}} &= \int \langle \Phi_+, d\,\mathrm{Re}(e^{i\theta} e^{-\phi} \Phi_-) \rangle + i\, \langle \Phi_+, F_+ \rangle \,, \\ W_{\mathrm{IIB}} &= \int \langle \Phi_-, d\,\mathrm{Re}(e^{i\theta} e^{-\phi} \Phi_+) \rangle + i\, \langle \Phi_-, F_- \rangle \ , \end{aligned} \tag{6.22}$$

where θ is a phase that depends on the type of orientifold projection ($\theta = 0$ for O3 projection).

We will now discuss the last item in the above list of possible ways to achieve moduli stabilisation, namely taking into account the corrections to the low-energy action.

6.3 Moduli stabilization including non perturbative effects and De Sitter vacua

We saw in the previous sections that fluxes are usually not enough to stabilize all moduli. In particular, we reviewed in the previous section that in type IIB compactifications on Calabi-Yau orientifolds fluxes stabilize the complex structure moduli and the dilaton, but leave the Kähler moduli unfixed. There are nevertheless perturbative and non-perturbative corrections to the tree-level Kähler potential and superpotential considered in previous sections that can help in stabilizing the remaining moduli. In this section we discuss these corrections, concentrating on their effect on moduli stabilization, and whether they lead to de Sitter vacua. We will mainly focus on type IIB compactifications on Calabi-Yau orientifolds. Nothing is known so far about corrections on more general manifolds of $SU(3)$ structure.

Corrections to the low-energy effective supergravity action are governed by the Planck scale, which in string theory is given by $M_P^8 = \frac{1}{g_s^2(\alpha')^4}$. In the low-energy limit, the dimensionless parameter $\frac{l_P}{R}$, where R is a characteristic length of the solution, controls the corrections. The corrections can then be understood as a double series expansion in g_s and α'. There are perturbative and non-perturbative corrections to the supergravity action. The non-perturbative arise ones from world-sheet or brane instantons. A world-sheet or a p-brane wrapping a topologically non-trivial space-like 2-cycle or p-cycle Σ on the internal manifold gives instanton corrections which are suppressed by $e^{-\frac{Vol(\Sigma)}{2\pi\alpha'}}$. This will be the main effect stabilizing the Kähler moduli.

Let us discuss the perturbative corrections in the case of $\mathcal{N}=1$ compactifications, concentrating on compactifications of IIB on Calabi-Yau orientifolds. The ten-dimensional supergravity action is corrected by a series of α' terms, coming from higher-derivative terms in the action:

$$S = S_{(0)} + \alpha'^3 S_{(3)} + ... + \alpha'^n S_{(n)} + ... \, . \tag{6.23}$$

In addition to higher-derivative corrections contributing to (6.23), there are string loop corrections to the action, suppressed by powers of g_s. String loop corrections to the bulk effective action appear at order α'^3, so their effects are subsumed in the expansion (6.23), as a further g_s expansion of each term.

The term $S_{(3)}$ contains $\mathcal{R}^4$ corrections to the action (where $\mathcal{R}$ is the Ricci scalar), as well as $\mathcal{R} - F_p$ terms mixing flux and curvature, whose form is not yet known.

These higher-order terms lead to corrections to the four-dimensional Kähler potentials and superpotentials. The $\mathcal{N}=1$ Kähler potential receives corrections at every order in perturbation theory, while the superpotential receives non-perturbative corrections only. Considering leading α' corrections, the Kähler potential and superpotential can be written

$$\begin{aligned} K &= K_0 + K_{\rm p} + K_{\rm np} \\ W &= W_0 + W_{\rm np} \, . \end{aligned} \tag{6.24}$$

where $K_{\rm p}$ comes from the corrections discussed above, $K_{\rm np}$ comes mainly from fundamental string worldsheet instantons, and $W_{\rm np}$ comes from non-perturbative effects of string theory such as D-brane instantons (or similarly, from gaugino condensation on D-branes).

In type IIB on Calabi-Yau O3 orientifolds, the $\mathcal{R}^4$ corrections to the ten-dimensional action correct the last piece of the Kähler potential (6.5) in the form

$$K = -2 \ln\left[\frac{1}{6}\int J \wedge J \wedge J - i\,\alpha'^3 \frac{k\,\chi}{(S-\bar{S})^{3/2}}\right] \, , \tag{6.25}$$

where k is some real positive number and χ is the Euler characteristic of the Calabi-Yau. We notice immediately that there is no longer a separation between the moduli space for the dilaton and that of the Kähler moduli. Furthermore, the perturbative correction breaks the no-scale structure of the potential. In other words Eq. (6.8) is no longer true, and therefore the Kähler moduli appear explicitly in the potential and not just as an overall multiplicative volume factor e^K any more. Fluxes on an α'-corrected moduli space have therefore the capability of stabilizing Kähler moduli.

As for corrections to the superpotential, these have two origins: gaugino condensation and D-brane instantons. We will briefly discuss the latter, coming from Euclidean D3 branes wrapping four-cycles in the Calabi-Yau. Their contribution to the superpotential is

$$W_{\rm np} = A_n e^{-2\pi n^a t_a} \;, \tag{6.26}$$

where n^a is the wrapping number on the cycle whose volume is measured by the imaginary part of the Kähler modulus t_a and A_n are one-loop determinants that depend on the expectation values of the complex structure moduli.[42] The non-perturbative superpotential depends therefore on Kähler moduli, which were absent in the flux induced GVW superpotential (6.9). Taking into account this non-perturbative correction to the superpotential can lead to Calabi-Yau O3 compactifications with all moduli stabilized. Using these corrections, there are two types of moduli stabilization scenarios widely used these days, the so-called "KKLT-type" and the "large volume" scenarios.

Considering the non-perturbative corrections to the superpotential, all moduli can be stabilized in type IIB compactifications on Calabi-Yau orientifolds. The idea was first put up by Kachru, Kallosh, Linde and Trivedi (KKLT) [29], where it was argued that besides having all moduli stabilized in this type of compactifications, it is possible to obtain de Sitter vacua by adding a small number of anti-D3-branes. If in these setups $W_0 \ll 1$ holds in suitable units, then the tree level superpotential can have similar magnitude as the non-perturbative superpotential, leading to

$$W_0 \sim W_{\rm np} \;\Rightarrow W_{\rm np}^2 \sim V_{W_{\rm np}} \gg V_{K_{\rm p}} \sim W_{\rm np}^2 K_{\rm p} \;, \tag{6.27}$$

and K_p can safely be ignored.

KKLT study IIB flux compactifications on Calabi-Yau O3 orientifolds with $h^{(2,1)}$ arbitrary, and $h^{(1,1)} = 1$, i.e. with any number of complex structure moduli but with only a single Kähler modulus t. A single instanton leads to a total superpotential of the form (6.24), namely

$$W = W_0 + Be^{-2\pi t} \tag{6.28}$$

where W_0 is the contribution coming from the fluxes. Setting the axion $\rho = 0$, a supersymmetric minimum satisfying $D_t W = 0$ is attained at

$$\mathrm{Im}t \equiv \sigma_{cr} = \tfrac{1}{2} V_{cr}^{2/3} \;, \qquad W_0 = -Be^{-2\pi\sigma_{cr}} \left(1 + \frac{4\pi}{3}\sigma_{cr}\right) \;, \tag{6.29}$$

where V is the overall volume. The volume, stabilized at $V_{cr} = (2\sigma_{cr})^{3/2}$, can take reasonably large values for sufficiently small $|W_0|$. Inserting this in the potential, we get that the minimum leads to an AdS vacuum

$$V_{\rm min} = -3(e^K|W|^2)_{\rm min} = -\frac{2\pi^2 B^2 e^{-4\pi\sigma_{cr}}}{3\sigma_{cr}} \;. \tag{6.30}$$

[42]The explicit form of the coefficients A_n is often difficult to determine.

A few comments are in order. First, note that we need $W_0 \neq 0$ for the complete stabilization to work. In the previous section, when we discussed the supersymmetry conditions, we set $D_I W_0 = 0$, as well as $W_0 = 0$. Looking at the Eqs. $D_I W = 0$, we see however that if we do not consider derivatives along the Kähler moduli, the supersymmetry conditions are just $\int \bar{G}_3 \wedge \Omega = \int G_3 \wedge \chi_k = 0$, or in other words, the complex three-form flux G_3 must be imaginary self-dual (see Eq. (6.17)), which means that G_3 is has a (2,1) component that is primitive and an additional (0,3) component. If we consider a (0,3) piece in G_3, the superpotential does not vanish. If we had just the flux superpotential, this will break supersymmetry, as $D_{T_\alpha} W_0 \neq 0$. However, KKLT have shown that taking into account the non-perturbative corrections to the superpotential, (0,3) pieces of the three-form flux can lead to supersymmetric (AdS) vacua. This piece has to be fine-tuned, though, to give $e^K |W_0|^2 \ll 1$, otherwise there is no large radius minimum of the potential.

KKLT further argued that the negative AdS potential can be uplifted to a positive value by adding a small number of anti-D3-branes at the bottom of one of the warped throats, where the warp factor e^{2A} is very small and which are typically supported by the presence of flux. The potential energy of such $\overline{\text{D3}}$-brane should be proportional to e^{4A} at the location of the brane and inversely proportional to the square of the volume. Adding a small number n of $\overline{\text{D3}}$-branes, there is an extra contribution to the potential of the previous section, given by

$$V_{\overline{D}3} = \frac{D}{\sigma^3} = \frac{D}{(\mathrm{Im}T)^3} \ , \tag{6.31}$$

where the coefficient D is proportional to n and e^{4A} at the position of the branes. Adding this to the previous potential, we get

$$V = \frac{A\pi e^{-2\pi\sigma}}{\sigma^2} \left(A e^{-2\pi\sigma} \left(1 + \frac{2\pi}{3}\sigma \right) + W_0 \right) + \frac{D}{\sigma^3} \ . \tag{6.32}$$

There are two extrema of the potential, a local minimum at positive energy and a maximum separating the de Sitter minimum from the vanishing potential at infinity. By fine-tuning D, it is easy to get very small positive energy, at large values of σ, i.e. at large volume.

Note that the de Sitter vacuum just obtained is metastable, as there is a runaway behavior to infinite volume. This is expected for many reasons. On one hand, it has become clear on entropy grounds that de Sitter space cannot be a stable state in any theory of quantum gravity. On the other hand, the runaway behavior is a standard feature of all string theories, i.e. a positive vacuum energy in a space with extra dimensions implies an unstable universe towards decompactification. KKLT showed nevertheless that the lifetime of the dS vacuum is large in Planck times (which can be much greater than the age of our universe).

There are several critiques to the KKLT procedure, to be discussed shortly. These critiques do not affect the main results, but they do affect the detailed physics and therefore tell us that KKLT should be taken only as a toy model of complete moduli stabilization in compactifications of IIB Calabi-Yau orientifolds. The first critique is that the procedure of obtaining an effective potential for light moduli via non-perturbative corrections after integrating out moduli that are assumed to be heavy at the classical level is in general not correct. In some cases, this two-step procedure can fail, giving rise to tachyonic directions. One should instead minimize the full potential, which has additional terms (mixing the light and heavy modes). This is a highly involved procedure, which has not been carried out in the explicit examples, where a full stabilization had been worked out in the framework of KKLT in the literature. Another critique is that the corrections to the Kähler potential, both perturbative and non-perturbative in nature, have not been taken into account. As reviewed, the α' corrections to the Kähler potential are subleading whenever $W_0 \sim W_{np}$. Otherwise, the perturbative corrections to the

Kähler potential dominate, and one should include them in order to analyze the details of the potential. At large volumes, the vacuum expectation value of the GVW superpotential, cf. Eq. (6.9), is indeed larger than the non-perturbative one, and perturbative α' corrections start to take over. Taking into account the known α' correction to the Kähler potential (6.25), it was shown that there is a large-volume minimum which for sufficiently small values of W_0 coexists with the KKLT minimum. Finally, some arguments show that in order to stabilize the Kähler moduli at strictly positive radii, one needs a sufficient number of distinct four cycles, which excludes the case of internal manifolds with $h^{(1,1)} = 1$, as in the toy model of KKLT.

Let us note again that due to the critiques discussed in the previous section, KKLT is a toy model for getting de Sitter vacua in IIB compactifications. Differently from the case of AdS vacua, no explicit models with dS vacua were constructed so far. On top of the difficulties already discussed in stabilizing all moduli in a controlled way, there is an extra fine tuning needed in order to make the constant D sufficiently small and to get at the same time a long lived vacuum. More severe arguments resulting from studying the back-reaction of anti-D3-branes on the geometry suggest the possibility that such a metastable vacuum does not even exist.

In the second scenario usually termed large volume stabilization, explicit examples of type IIB orientifold compactifications were constructed with all moduli stabilized on some particular Calabi-Yau manifolds. In these setups, one uses both Kähler and superpotential corrections. At least two Kähler moduli are needed, and they are stabilized such that one takes large values (t_l) while the other one takes small values (t_s). The zeroth order Kähler potential written in terms of these moduli is of the form (see footnote 40)

$$K = -2 \ln \left[\tfrac{1}{6} \int J \wedge J \wedge J \right] \sim -2 \ln \left(t_l^{3/2} - t_s^{3/2} \right) \ . \tag{6.33}$$

We can therefore picture the small cycles as holes, since they subtract in the formula for the volume. Manifolds with these property are referred to as "swiss cheese" Calabi-Yau. In these scenarios, which need additionally $h^{(2,1)} > h^{(1,1)} > 1$ to have negative Euler characteristic (the α' correction to the Kähler potential is proportional to it), all moduli can be stabilised in a non-supersymmetric AdS vacuum at exponentially large volume. It has recently been shown that with a certain amount of fine-tuning, one can get also de Sitter vacua this way. In these scenarios, moduli stabilization also works as a two-step procedure (first stabilizing dilaton and complex structure moduli, and in a second step stabilizing Kähler moduli), so the same critiques as in the KKLT scenario regarding this apply (though here we do not need $|W_0| \ll 1$).

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