

CALABI-YAU MANIFOLDS

A Bestiary for Physicists

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World Scientific

Singapore • New Jersey • London • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.

P O Box 128, Farrer Road, Singapore 9128

USA office: Suite 1B, 1060 Main Street, River Edge, NJ 07661

UK office: 73 Lynton Mead, Totteridge, London N20 8DH

Library of Congress Cataloging-in-Publication Data

Hübsch, Tristan.

Calabi-Yau manifolds. A bestiary for physicists / Tristan Hübsch.

p. cm.

Includes bibliographical references.

ISBN 9810206623 ISBN 981021927X (pbk)

1. Calabi-Yau manifolds. 2. Mathematical physics. I. Title.

QC20.7.M24H83 1991

530.1--dc20

91-35759

CIP

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First published: 1992

Paperback edition: 1994

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Preface

Bestiary (bestiāry) [ad. L. *bestiarius* ‘a fighter with beasts in public spectacles,’ and med. L. *bestiārium* a menagerie, also ‘liber de bestiis compositus,’ etc., . . .]. . .
1871 *Sacristy* I. 7/1 The Bestiaries..are natural histories of animals treated so that the peculiarities of animals shall convey a wholesome moral.

—The Oxford English Dictionary

THE RAPIDLY INCREASING number of articles about Calabi-Yau spaces makes it very difficult, if not impossible, to fruitfully use and learn from the existing results without being forced to spend too much time merely in getting oriented in the literature. Yet, the physics application in compactifying (super)strings has attracted the attention of many a physicist, more often than not quite unprepared for the rather specialized techniques (and even jargon) with the use of which results can be obtained most readily. It is my sincerest hope that the present effort may provide the necessary ingredients for a bridge over the troubled waters that divide the existing results on Calabi-Yau spaces and the continually developing machinery of algebraic geometry from their vigorous physics application.

From the point of view of building superstring models with possibly realistic physics in 3+1-dimensional spacetime, Calabi-Yau spaces of complex 3-dimensions are the most interesting. In complex one and two dimensions, such spaces are unique up to diffeomorphisms : they are the torus (T^2) and the Kummer surface (K_3), respectively. These have been encountered previously in various branches of physics and are relatively well known. On the other hand, the number of *known* topological types of Calabi-Yau 3-folds is well in the thousands and more and more new constructions are being uncovered daily.

Yau's proof [54] of Calabi's conjecture [83] guarantees the existence of a unique Ricci-flat Kähler metric (per choice of the Kähler class) on any compact Kähler space, precisely if the first Chern class vanishes. Regardless of dimension, all these are then called Calabi-Yau spaces. Motivated primarily by immediate physics application, however, we shall be interested in compact Calabi-Yau spaces almost exclusively of three complex dimensions. Accordingly, unless explicitly stated, by a Calabi-Yau space we shall understand the three dimensional case.

Most of this volume is dedicated to methods of constructing and analyzing various Calabi-Yau 3-folds. To begin with, however, we first review some basics of the physics application in superstring models to note the properties of the Calabi-Yau 3-folds that have so far been recognized as physically interesting. The reader should be aware that string theory with an 'internal' Calabi-Yau space is rapidly developing but rather incomplete and we necessarily rely on approximations. It is gratifying to learn, through particular examples and also general results, that the geometric analysis is quite robust and here we shall adhere to it. The last chapter is intended as a brief directory of some more "stringy" aspects of our subject.



*Algebraic curves were created by God, algebraic surfaces by the Devil**; it is then only human to create algebraic 3-folds. The world of Calabi-Yau 3-folds, even if only a very restricted set of algebraic 3-folds, is in its present stage of infancy most similar to a Universe still in its "first three minutes". New manifolds are being created every moment, joining the menagerie, some of them readily finding their place in this jig-saw puzzle while others seed entire new families, seemingly disconnected from the rest.

Our task therefore necessarily becomes taxonomy in part, guiding the interested reader through the web of known Calabi-Yau manifolds and preparing the more adventurous reader for a voyage into the jungle with many more beasts to be discovered. It should be obvious that this could not possibly have been intended as a definitive account on Calabi-Yau 3-folds; not even a compilation of all known facts and results. Rather, the focus is primarily on techniques and methods which will hopefully have a long-lasting application.

The interested reader undoubtedly has at least nodding acquaintance with the basics of differential and algebraic geometry which are by now fairly standard in theoretical physics. As a prerequisite we have in mind texts like the Physics

*This saying is attributed to Max Noether; see p. 272 of Ref. [37].

Report by Eguchi, Gilkey and Hanson [19] and the last couple of chapters of *String Theory II* by Green, Schwarz and Witten [22], to name a few. With a little group theory, linear algebra and an open mind in one's arsenal, no problem should appear in mastering this Bestiary.



The recent literature on Calabi-Yau spaces is simply too abundant for the Bibliography included at the end to be complete in any sense; competing with the databases @BABBAGE.SISSA.IT, @CERNVM, @PUBLICATIONS.MATH.DUKE.EDU, @SLACVM, @XXX.LANL.GOV and others was not the intention. It has been compiled as a suggestion for possible further reading and references for details which could not be covered in sufficient detail. It necessarily reflects my personal preference and convenience and should by no means be taken as a value judgment. As regards the structure of the book and the selection and presentation of topics, these have grown out of my own learning and in this respect the innumerable discussions foremost with Paul Green but also with Philip Candelas, Michael Eastwood, Sylvester James Gates, Jr., Joseph Harris, Shin-Tung Yau and many others have taught me and helped a lot. Clearly however, they could not correct all faults of my design and misconceptions. In addition, I'm afraid, all the errors (typographical and otherwise) are entirely mine[†]. Comments and corrections are most welcome; I hope an opportunity will emerge to incorporate them. Special thanks to Ms. H.M. Ho of the World Scientific Pub. Co., for correcting more than a lion's share of typos and stylistic errors.

Also, I am indebted to all of my research collaborators and colleagues for sharing the joys of research, for countless explanations, discussions, arguments and disagreements—I know I have benefited from all of it and perhaps especially so from the repeated questions which have often helped me to understand my own thoughts better. A simple "thank you" is ... well, perhaps too simple to convey my gratitude, but in printed form, that will have to do. Special thanks to Donna D'Finis, for the cover photograph. Thank you all, very much indeed. The support by the DOE grant DE-FG02-88-ER25065 is gratefully acknowledged.

Tristan Hübsch
Cambridge and Austin, 1991
updated: Silver Spring, 1994

[†]ERRARE DIVINE EST, ALITER NOS NON SIMVS.

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Chapter 0

Spiritus Movens

To begin, we briefly review supersymmetric compactification on Calabi-Yau spaces, having superstring models in mind. To avoid delving too deeply into the tangle of string theory and 2-dimensional superconformal field theories, we propose a deal : we begin with a pointillist sketch, shaded however by world-sheet instanton retouch. This will suffice to determine which characteristics of Calabi-Yau spaces are relevant for their immediate physics application and will provide a subject for most part of this volume. Towards the end, we return to more “stringy” aspects of Calabi-Yau models and include some more recent advances in this field.

0.1 A Pointillist Sketch

From the usual point of view, one begins with string theory in Minkowski spacetime of 9+1 dimensions (denoted M_{9+1}) and then attempts to replace M_{9+1} with $M_{3+1} \times \mathcal{M}$. The phenomenologically motivated N=1 local supersymmetry and various consistency requirements then imply that \mathcal{M} is a Calabi-Yau space. This line of analysis was originally carried out by Candelas, Horowitz, Strominger and Witten [94] in the ‘point-field limit’, i.e., restricting in the outset to the massless modes of the 9+1-dimensional string theory; they also assumed \mathcal{M} to be smooth.

Instead of a system of (point-like) massless particles, for a string theory, one should consider a system of closed strings which propagate in some general 9+1-dimensional spacetime, X . As time elapses, we obtain a one-parameter family of strings, the world-sheet, immersed in X . In this approach, it is natural to consider the so-called based loop space, $\Omega_* X$, and construct the Hilbert space from functions on $\Omega_* X$ [127]. $\Omega_* X$ is the space of all closed paths which begin

and end at some point $*$ $\in X$. Notably, this space appears as the fibre of two very different *bundles*^(L) over X :

$$\begin{array}{ccc} \Omega_*X & \rightarrow & \mathcal{L}X \\ \downarrow \pi & \text{and} & \downarrow \pi \\ X & & X \end{array} \quad (0.1.1)$$

In the first one, we simply sweep the base point, $*$, through X and obtain the so-called free loop space $\mathcal{L}X$; after all, there appears to exist no preferred point in our Universe X . $\mathcal{L}X$ consists of all closed paths in X and is most naturally the (canonical) configuration space for a theory of closed strings propagating in X . In the second fibration, \mathcal{P}_*X is the space of all paths $*\tilde{p}$ (open if $p \neq *$, and closed if $p = *$) in which Ω_*X is embedded as the closed paths. The projection π is given by the endpoints p ranging over all of X (see p.197–199 of Ref. [9]).

In general, a fibration $A \rightarrow B \rightarrow X$ allows one to determine the homotopy and (co)homology groups of any one of A, B, X in terms of the other two. Notice however that each path $*\tilde{p}$ may be contracted to $* \in X$. That is, \mathcal{P}_*X is contractible and so has trivial homotopy and (co)homology. As a result, using the right-hand-side fibration, the homotopy and (co)homology of Ω_*X is determined in terms of $\pi_*(X)$, $H_*(X)$ and $H^*(X)$, respectively. Using then the left-hand-side fibration, the analogous data is also determined for $\mathcal{L}X$. For essentially flat spaces, this relation is straightforward and simple; depending on X , however, it may well be beyond our present means of calculation. Nevertheless, from the general framework of geometric quantization, we expect the Hilbert space of the complete string theory on X to be largely determined in a similar fashion. In other words, even though the fully fledged string theory is much more involved than the ‘point-field’ limit model, the spectrum of massless states and many physical observables can reliably be determined ignoring the stringiness.



In the other, perhaps presently more popular approach to (super)strings, one considers 1+1-dimensional, (super)conformally invariant quantum field theories over the world-sheet [201]. A sector of each of these theories is given by a (flat) σ -model with the 3+1-dimensional Minkowski spacetime as the target space. The ‘rest’ of such a quantum field theory is then just about anything, as long as the complete quantum theory is consistent. It is the particular choice of the ‘rest’ of the theory which determines the physical observables in the effective, low-energy, 3+1-dimensional model. Consequently, two such models with iden-

0.1. A Pointillist Sketch

tical 3+1-dimensional physics are considered equivalent even though they might appear as rather different 1+1-dimensional quantum field theories.

For our purposes, the ‘rest’ of the theory will always contain a ‘Calabi-Yau sector’ and a ‘Yang-Mills sector’. The former is simply a non-linear σ -model, the target space of which is a Calabi-Yau space. The latter is more involved and gives rise to the Yang-Mills interactive fields in the effective model. In this approach, it is natural to consider $\Sigma\mathcal{M}$, the space of mappings $Z^\mu(\sigma)$ of the world-sheet Σ into the Calabi-Yau space \mathcal{M} , equipped with the (Euclidean) action functional

$$S_{\mathcal{M}} = \frac{1}{\pi\alpha'} \int_{\Sigma} \left\{ (B_{\mu\bar{\nu}} + iG_{\mu\bar{\nu}})(\bar{\partial}Z^\mu \partial\bar{Z}^{\bar{\nu}}) - (B_{\mu\bar{\nu}} - iG_{\mu\bar{\nu}})(\partial Z^\mu \bar{\partial}\bar{Z}^{\bar{\nu}}) \right\} + \text{susy completion} \dots \quad (0.1.2)$$

$S_{\mathcal{M}}$ is the integrated total ‘energy’ of Σ immersed in \mathcal{M} ; here \int_{Σ} is the reparametrization-invariant integration over the world-sheet Σ and $G_{\mu\bar{\nu}}(Z, \bar{Z})$ is a Hermitian metric on \mathcal{M} . In practice, the σ -model is chosen to have full (2,2)-supersymmetry on the world-sheet, the background (vacuum) metric $G_{\mu\bar{\nu}}$ is Kähler and the background value of $B_{\mu\bar{\nu}}$ vanishes :

$$\begin{aligned} S_{\mathcal{M}} &= \int_{\Sigma} d^2\sigma d^2\zeta d^2\bar{\zeta} K(Z, \bar{Z}), \\ G_{\mu\bar{\nu}}(Z, \bar{Z}) &= \partial_\mu \partial_{\bar{\nu}} K(Z, \bar{Z}) \Big|_{\zeta=\bar{\zeta}=0}, \end{aligned} \quad (0.1.3)$$

where $(\sigma^0, \sigma^1; \zeta^\pm, \bar{\zeta}^\pm)$ are coordinates on the (2,2)-super Riemann surface, Z^μ are the coordinate (chiral) superfields and $\partial_\mu = \partial/\partial Z^\mu$. Furthermore, the required conformal invariance of the σ -model implies that $G_{\mu\bar{\nu}}$ also must be Ricci-flat, to lowest order.

The ‘point-field limit’ results are obtained by restricting to the constant maps z^μ . The effects of non-constant maps are then treated as (σ -model) corrections. In particular, holomorphic maps which are not continuous deformations of the constant z^μ give rise to the so-called ‘world-sheet instanton’ corrections. In addition, choosing $\text{genus}(\Sigma) = 0$ corresponds to the string tree level, while the choice of a genus- n Riemann surface for Σ leads to the n^{th} order string-corrections. Nevertheless, many of the results obtained in the lowest approximation and corrected for the world-sheet instanton effects are actually quite robust [240,111,113] in view of the required world-sheet and spacetime supersymmetry. In particular, this level of accuracy will perfectly suffice for our present purpose : to determine which geometrical characteristics of Calabi-Yau manifolds are physically relevant.

Associated to the action (0.1.2), the partition function is defined

$$\mathcal{Z}_{CY} = \int D[\Sigma] D[\mathbf{Z}] e^{-S_{\mathcal{M}}}, \quad (0.1.4)$$

where the integration over $D[\Sigma]$ is taken to include a sum over genera and spin structures, integration over moduli and metrics of Riemann surfaces Σ ; the action $S_{\mathcal{M}}$ is taken in the Euclidean formulation. Path-integration over $D[\mathbf{Z}]$ ranges over the maps $Z^\mu: \Sigma \hookrightarrow \mathcal{M}$ and the superpartners $\zeta^\mu, \zeta^{\bar{\nu}}$. Including the sector pertaining to spacetime, M_{3+1} , and also the Yang-Mills sector,

$$\mathcal{Z}_{tot.} = \mathcal{Z}_{3+1\text{-spacetime}} \cdot \mathcal{Z}_{CY} \cdot \mathcal{Z}_{YM}; \quad (0.1.5)$$

all of these sectors interact with each other. For example, the metric $G_{\mu\bar{\nu}}$ on the internal Calabi-Yau manifold depends on a number of parameters, which are spacetime dependent in general. Also, the Yang-Mills gauge fields are functions of both the 3+1-dimensional spacetime and the ‘internal’ Calabi-Yau space. The complete target space of the total σ -model is then a combined space in which possibly different Calabi-Yau spaces \mathcal{M} are fibered over M_{3+1} and the Yang-Mills sector is then fibered over both.

Of course, this hierarchical structure of the target space of the total σ -model is a biased choice. From the point of view of the 1+1-dimensional field theory, each of these sectors—the spacetime M_{3+1} , the compactification space \mathcal{M} and also the space of Yang-Mills degrees of freedom—is ‘internal’ and arises as spanned by the vibrations of the string. Generalizing, one should consider the space of all possible consistent world-sheet quantum field theories. Some of them will span the Minkowski spacetime, M_{3+1} , moreover with $N=1$ local supersymmetry and also possibly realistic gauge-interacting matter and so may come close to a theoretical model of the real world.



Many such models have been constructed in a wide variety of quite dissimilar ways, in some of which the symmetries alone determine all physically relevant data and moreover at the fully fledged string level. In this respect, such constructions clearly outflank the geometrical σ -model description. Increasing evidence is however being accumulated indicating that these exactly soluble models correspond to certain very special (highly symmetric and possibly singular) Calabi-Yau spaces. Motivated by this, it has been conjectured [139,30] that in fact all models with $N=1$ locally supersymmetric spacetime M_{3+1} also contain an ‘internal’

0.2. Why Calabi-Yau Spaces?

Calabi-Yau space. The converse, namely that every Calabi-Yau space offers a possibly realistic compactification, is beyond every physicist’s doubt, although this has not been proven to every mathematician’s satisfaction. Since there is a true *embarras des richesses* of Calabi-Yau spaces, we shall focus here on these but keep in mind that many other, possibly equivalent, constructions exist.

0.2 Why Calabi-Yau Spaces?

For ease of notation, we follow here the first attempt [94,137] to make the heterotic string model [160] physically realistic. We restrict to the massless modes of the $E_8 \times E_8$ heterotic string and replace the 9+1-dimensional Minkowski spacetime, M_{9+1} , with a product of a 3+1-dimensional maximally symmetric spacetime and some ‘internal’ compact manifold \mathcal{M} . For the resulting 3+1-dimensional effective model to possess precisely $N=1$ local supersymmetry, consistency will require the 3+1-dimensional spacetime to be Minkowskian, M_{3+1} , and \mathcal{M} to be Ricci-flat and Kähler if Riemannian.

0.2.1 $N=1$ supersymmetry

Recall that the supersymmetry parameter in the original heterotic string model is a spinor transforming as the 8_S representation of the *SO(8) helicity subgroup*^(L) of the *Lorentz group*^(L) $SO(1, 9)$. Upon passing to the product $M_{3+1} \times \mathcal{M}$, the helicity subgroup is broken to $SO(2) \times SO(6)$ and the 8-component spinor gives rise to *four* 2-component spinors of $SO(2)$, corresponding to four independent supersymmetries in the 3+1-dimensional sense. One—and only one—linear combination of these is needed to parametrize the desired $N=1$ (super)symmetry of the effective ‘low energy’ physics.

To this end, the background values of the fermionic fields are set to vanish (no preferred spinor in M_{3+1}). For consistency then, the supersymmetry transformations are required to maintain this Ansatz precisely ‘along’ one of the four component supersymmetries. Now, supersymmetry transformations act by

$$\delta_\varepsilon \langle \mathbf{B} \rangle = \varepsilon \cdot \langle \mathbf{F} \rangle, \quad \delta_\varepsilon \langle \mathbf{F} \rangle = \varepsilon \cdot \langle \mathbf{B}' \rangle, \quad (0.2.1)$$

where the quantities $\langle \mathbf{B} \rangle$ ($\langle \mathbf{F} \rangle$) depend on the *background* fields and transform as tensorial (spinorial) representations of the Lorentz group. Since $\langle \mathbf{F} \rangle = 0$, we also need $\langle \mathbf{B}' \rangle = 0$, which then restricts the various background fields relating to the internal manifold \mathcal{M} and thereby determines its geometry, i.e., the vacuum

configuration. To stabilize this Ansatz, it turns out, one needs to modify the original action by adding certain curvature-squared terms; this in turn modifies the supersymmetry transformation rules ... Fortunately, these feedback corrections can be treated perturbatively [137].

Assuming maximal symmetry for the 3+1-dimensional spacetime,

$$R_{\mu\nu\rho\sigma} = \kappa(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \quad (0.2.2)$$

explicit calculation yields that $\kappa = 0$ and the 3+1-dimensional space-time is Minkowskian order by order in the string tension, α' . Also, suffice it here just to state that the connection on \mathcal{M} can consistently be chosen torsion-free so that we are left with Riemannian candidates for \mathcal{M} . Such manifolds are partially classified by their holonomy groups [8]. Since spinors in odd-dimensional spaces are (pseudo)real, if the internal space contains an odd-dimensional factor the resulting 3+1-dimensional model will necessarily have particles of both helicities. In particular, the left-handed neutrino would always be accompanied by its right-handed mirror image. It would be very hard, if not impossible, to resolve this violent contradiction with experiment in the present context; the possibilities at hand are then listed in Table 0.1.

Dim.	Holonomy algebra	Tangent spinor
6	$\mathfrak{so}(6)$	$\mathbf{4} \oplus$ c.c.
	$\mathfrak{u}(3)$	$\mathbf{3}_{-2} \oplus \mathbf{1}_6 \oplus$ c.c.
	$\mathfrak{su}(3)$	$\mathbf{3} \oplus \mathbf{1} \oplus$ c.c.
4+2	$\mathfrak{so}(4) \times \mathfrak{so}(2)$	$(\mathbf{2}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{2})_{-1} \oplus$ c.c.
	$\mathfrak{su}(2) \times \mathfrak{so}(2)$	$\mathbf{2}_2 \oplus \mathbf{1}_{-1} \oplus \mathbf{1}_{-1} \oplus$ c.c.
	$[\mathfrak{su}(2) \times \mathfrak{u}(1)] \times \mathfrak{so}(2)$	$(\mathbf{2}, \mathbf{0})_1 \oplus (\mathbf{1}, \mathbf{3})_{-1} \oplus (\mathbf{1}, \mathbf{-3})_{-1} \oplus$ c.c.
2+2+2	$\mathfrak{so}(2) \times \mathfrak{so}(2) \times \mathfrak{so}(2)$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}) \oplus (-\mathbf{1}, -\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, -\mathbf{1}, -\mathbf{1})$ $\oplus (-\mathbf{1}, \mathbf{1}, -\mathbf{1}) \oplus$ c.c.

Table 0.1: Possible holonomy algebras and corresponding transformation of spinors in Riemannian manifolds.

Note that only for manifolds with $\mathfrak{su}(3)$ holonomy, precisely one spinor together with its CPT conjugate (in the 3+1-dimensional spacetime sense) is invariant under holonomy. Such a spinor, taken to be constant on \mathcal{M} , will also be

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covariantly constant on \mathcal{M} (being invariant under holonomy, the parallel transport is trivial) and will guarantee N=1 supersymmetry in M_{3+1} . We refrain from a detailed discussion of the other choices above; suffice it here to refer to the existing literature [199] and merely say that no other of the above choices has been found consistent with the various constraints of physical reality.

In the above discussion, by ‘holonomy group’ we have meant the restricted holonomy group^(L), which is in 1-1 correspondence with the *algebra* of the (full) holonomy group. If the manifold is multiply connected, the full and the restricted holonomy groups differ and this can have some interesting effects [192].

0.2.2 Ricci-flatness

In general, on a Kähler n -dimensional space, the holonomy algebra must be contained in $\mathfrak{u}(n)$. Also, the Riemann tensor simplifies :

$$R_{\mu\bar{\nu}\bar{\rho}}^{\bar{\sigma}} = \partial_\mu \Gamma_{\bar{\nu}\bar{\rho}}^{\bar{\sigma}} = R_{\mu\bar{\rho}\bar{\nu}}^{\bar{\sigma}}, \quad \Gamma_{\bar{\nu}\bar{\rho}}^{\bar{\sigma}} = g^{\tau\bar{\sigma}} \partial_{\bar{\nu}} g_{\tau\bar{\rho}}, \quad (0.2.3)$$

and the Ricci tensor becomes :

$$R_{\mu\bar{\nu}} \stackrel{\text{def}}{=} R_{\mu\bar{\sigma}\bar{\nu}}^{\bar{\sigma}} = R_{\mu\bar{\nu}\bar{\sigma}}^{\bar{\sigma}}. \quad (0.2.4)$$

Considering the Riemann tensor as a $\mathfrak{u}(n)$ -valued curvature (1,1)-form

$$\Theta_{\bar{\rho}}^{\bar{\sigma}} \stackrel{\text{def}}{=} dz^\mu \wedge dz^{\bar{\nu}} R_{\mu\bar{\nu}\bar{\rho}}^{\bar{\sigma}}, \quad (0.2.5)$$

we see, from Eq. (0.2.4), that the corresponding Ricci (1,1)-form

$$\Theta \stackrel{\text{def}}{=} \Theta_{\bar{\rho}}^{\bar{\rho}} \quad (0.2.6)$$

is the $\mathfrak{u}(1)$ -valued trace of the Riemann (1,1)-form. Recall that $\mathfrak{u}(n) \approx \mathfrak{su}(n) \times \mathfrak{u}(1)$, where $\mathfrak{u}(1)$ is generated by the trace of the $\mathfrak{u}(n)$ generators.

For the holonomy on \mathcal{M} to be $\mathfrak{su}(3)$ rather than the full $\mathfrak{u}(3)$, there must exist a metric for which the Ricci (1,1)-form and hence also the Ricci tensor vanishes. It would follow that this Ricci-flat metric should then be used in the Calabi-Yau σ -model action (0.1.2). However, quantum corrections will alter this choice. In general, the induced correction $\delta G_{\mu\bar{\nu}}$ is up to fourth-order in perturbations proportional to the Ricci tensor and so vanishes for our Ricci-flat choice of $G_{\mu\bar{\nu}}$. At the fourth order, $\delta G_{\mu\bar{\nu}}$ receives corrections that are proportional to

powers of the Riemann tensor and are non-vanishing [161]. Nevertheless, the compactification Ansatz can be modified so that $\delta G_{\mu\bar{\nu}} = 0$ at any desired order in (the σ -model) perturbation theory [196]. Two important lessons have been learned in this process :

1. The physically relevant metric $G_{\mu\bar{\nu}}$ for the action (0.1.2) is not the Ricci-flat metric on \mathcal{M} but can be obtained from it through a sequence of quantum corrections.
2. The cohomology class represented by the (1,1)-form

$$J \stackrel{\text{def}}{=} iG_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu \quad (0.2.7)$$

does not change in the quantum correction process. The fourth order correction is proportional to the double derivative of the Euler density and hence cohomologically trivial.

The above use of the Riemann and the Ricci (1,1)-form does not appear coordinate-independent. To settle this qualm, recall that the usual characteristic polynomial of an $n \times n$ matrix

$$\chi_\lambda[\Theta] \stackrel{\text{def}}{=} \det[\Theta - \lambda \mathbb{1}] = \sum_{k=0}^n (-)^k \lambda^k c_{n-k}[\Theta] \quad (0.2.8)$$

and that, $c_k[\Theta]$, the coefficients of λ^{n-k} are independent of the choice of basis. If in addition the components of $\Theta_i{}^j$ are closed (1,1)-forms, as is the case above, $c_k[\Theta]$ are closed (k,k) -forms and their integrals are topologically invariant. The c_k are collected in the total *Chern class*^(L)

$$c[\Theta] \stackrel{\text{def}}{=} \chi_{-1}\left[\frac{i}{2\pi}\Theta\right] = \det\left[\mathbb{1} + \frac{i}{2\pi}\Theta\right] = \mathbb{1} + c_1[\Theta] + c_2[\Theta] + \dots \quad (0.2.9)$$

The above Ricci-flatness condition therefore translates into the vanishing of the first Chern class, $c_1[\Theta] = c_1(\mathcal{M})$, which is coordinate-independent and in fact a topological characteristic. Indeed, Yau's proof of the Calabi conjecture guarantees that only the cohomology class of the Ricci two-form, rather than its explicit choice is the obstruction to finding a Ricci-flat Kähler metric.

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0.2.3 The Yang-Mills bundle (standard Ansatz)

So far, we have seen the requirement of N=1 local supersymmetry in spacetime to restrict the compactification space \mathcal{M} to be Ricci-flat. There are also further restrictions and they arise from requiring the effective model in 3+1-dimensional spacetime to be anomaly free. The discovery of Green and Schwarz [143], that anomaly freedom is guaranteed if the quantity

$$30R_{[MN|K}{}^L R_{|PQ]L}{}^K - F_{[MN}^\theta F_{PQ]}^\theta \quad (0.2.10)$$

vanishes in vacuum, was really the trigger for most of the subsequent interest in (super)string compactification. Here $R_{MNP}{}^Q$ is the Riemann tensor on the 9+1-dimensional spacetime and F_{MN}^θ is the $E_8 \times E_8$ (or $Spin(32)/\mathbb{Z}_2$) Yang-Mills field strength. Of course, prior to compactification, the 9+1-dimensional spacetime is flat so that $\langle R_{MNP}{}^Q \rangle = 0$ and consequently $\langle F_{MN}^\theta \rangle = 0$ too. When compactifying the Riemann tensor develops a background value and so must also the Yang-Mills field strength.

The simplest non-trivial way to satisfy $30\langle R \wedge R \rangle = \langle F \wedge F \rangle$ is to set the background spin connection equal to the background Yang-Mills connection. Since the compactification space has $\mathfrak{su}(3)$ holonomy, one seeks an $SU(3)$ sub-bundle in the $E_8 \times E_8$ Yang-Mills bundle. Now, the fibre of one E_8 factor of the Yang-Mills bundle decomposes as

$$248 \rightarrow (78, 1) \oplus (27, 3) \oplus (27^*, 3^*) \oplus (1, 8) \quad (0.2.11)$$

under the $E_6 \times SU(3)$ maximal subgroup. Group theoretically, $\text{Tr}(F \wedge F)$ is proportional of the second Dynkin index and since only the $SU(3)$ part of the connection acquires a background value, the E_6 representations merely give rise to a multiplicity. We straightforwardly have

$$2 \cdot 27 \cdot I_2(\mathbf{3}) + 1 \cdot I_2(\mathbf{8}) = 2 \cdot 27 \cdot \frac{1}{2} + 1 \cdot 3 = 30, \quad (0.2.12)$$

while the Riemann tensor is already in the $\mathbf{3} \oplus \mathbf{3}^*$ vector representation, so that the analogous Group-theoretical prefactor is $I_2(\mathbf{3} \oplus \mathbf{3}^*) = 2I_2(\mathbf{3}) = 1$. This exactly accounts for the extra factor of 30 in the expression (0.2.10).

In more geometrical terms, we find \mathcal{V} , an $SU(3)$ sub-bundle of the original $E_8 \times E_8$ Yang-Mills bundle, and identify it fibre by fibre with $T_{\mathcal{M}}$, the holomorphic tangent bundle of \mathcal{M} . This is done in a way that at every point of \mathcal{M} , the holomorphic tangent space is identified with the representation- $\mathbf{3}$ fibre of \mathcal{V} .



Let us mention that there are other possible ways to satisfy the vanishing of (0.2.10), where an $SU(n)$ sub-bundle of $E_8 \times E_8$ is non-vanishing on the background and $n > 3$. Some of these are constructed as holomorphic deformations of the standard Ansatz, but some involve quite elaborately constructed rank- n stable vector bundles over \mathcal{M} [112].

0.3 Massless Modes

Having determined the background values of the supergravity and super Yang-Mills connections, consider now fluctuations of these connections around the chosen background. While a connection itself is not an exterior form (it transforms inhomogeneously), the fluctuations are. Each of the connections therefore gives rise to 9+1-dimensional spacetime exterior forms which are then to be decomposed according to the compactification Ansatz. This gives a list of fields among which the massless modes (harmonic forms) are to be found.

0.3.1 Lorentz symmetry breaking

Upon replacing M_{9+1} with a product (locally) of M_{3+1} and \mathcal{M} , only the $SO(1, 3)$ subgroup of the original $SO(1, 9)$ *Lorentz group*^(L) provides still the usual Lorentz symmetry of the effective model. The various tensorial (spinorial) fields, such as the graviton (gravitino), ..., are rewritten as collections of fields that have definite transformation properties under $SO(1, 3)$. Since $SO(1, 3) \times SO(6)$ is a maximal subgroup of $SO(1, 9)$, the representation of $SO(6)$ can be used to classify the modes. Furthermore, since we decided to compactify on manifolds of $SU(3)$ holonomy, the states will be classified by the $SO(1, 3) \times SU(3)$ subgroup of $SO(9, 1)$.

The task is also simplified if we consider the respective helicity subgroups instead of the full Lorentz groups, whence

$$SO(1, 9) \longrightarrow SO(8) \supset SO(2) \times SO(6) \supset SO(2) \times SU(3). \quad (0.3.1)$$

The advantage here is that precisely the helicity groups (rather than the full Lorentz groups) describe physical degrees of freedom unambiguously.

Consider, for example, an 8-component Majorana-Weyl massless fermion Ψ in M_{9+1} . It satisfies the Dirac equation $\not{\nabla}_{(9+1)} \Psi = 0$, which can be rewritten as

$$\not{\nabla}_{(3+1)} \Psi(x, y) + \not{\nabla}_{(6)} \Psi(x, y) = 0, \quad (0.3.2)$$

0.3. Massless Modes

with x and y coordinates of M_{3+1} and \mathcal{M} , respectively. The ‘slash’ implies contraction with suitably chosen Dirac matrices on M_{3+1} and \mathcal{M} , as indicated by the subscripts*. Using compactness of the ‘internal’ space \mathcal{M} , we expand

$$\Psi(x, y) = \sum_A \psi_A(x) \eta^A(y), \quad (0.3.3)$$

where the η^A are analogous to the familiar spherical harmonics. Precisely the 0-modes $\dot{\eta}^a$ obey

$$\not{\nabla}_{(6)} \dot{\psi}_a(x) \dot{\eta}^a(y) = \dot{\psi}_a(x) (\not{\nabla}_{(6)} \dot{\eta}^a(y)) = 0 \quad (0.3.4)$$

and therefore yield

$$(\not{\nabla}_{(3+1)} \dot{\psi}_a(x)) \dot{\eta}^a(y) = 0, \quad \forall \dot{\eta}^a(y). \quad (0.3.5)$$

In such a decomposition, ψ_A transforms as a spinor of $SO(1, 3)$, while η^A are spinors of $SO(6)$. By supersymmetry, however, both will be accompanied by bosonic superpartners, ϕ and φ respectively, that transform as tensors of the respective groups. Massless spinors will be accompanied by massless bosons :

$$\Delta_{(3+1)} \dot{\phi}_a(x) = 0, \quad \Delta_{(6)} \dot{\varphi}^a(y) = 0. \quad (0.3.6)$$

The latter of these, for all the fields we shall have to consider, will turn out to be in 1-1 correspondence with certain harmonic forms on the ‘internal’ space \mathcal{M} . The correspondence of spinors to harmonic forms on a Calabi-Yau manifold can also be established by more direct means (see Ref. [94]).

The massless states of the heterotic string are those of the 9+1-dimensional spacetime, N=1 supergravity coupled to the supersymmetric $E_8 \times E_8$ Yang-Mills multiplet. We now discuss these two sectors in turn.

0.3.2 The supergravity multiplet

The supergravity multiplet may be represented by the following component fields

$$g_{MN}, B_{MN}, \Phi, \psi_M{}^A, \chi_A. \quad (0.3.7)$$

Here M, N are vectorial indices while A is spinorial; g_{MN} is symmetric while B_{MN} is skew-symmetric and $\psi_M{}^A$ satisfies

$$[\Gamma^M]_A{}^B \psi_M{}^A = 0, \quad (0.3.8)$$

* Ref. [22], p.365–367, explains a subtlety in this procedure and how to deal with it.

where $[\Gamma^M]$ are Dirac matrices. In general, the bosonic fields are non-vanishing in vacuum and we consider their fluctuations to find massless fields. Alternatively, one may start from fluctuations of the Zehnbein $E_M{}^N$ and its superpartner $E_M{}^A$, which are tensor coefficients of a tangent bundle valued and spin bundle valued 1-form, respectively.

Upon compactification, we distinguish vectors (co)tangent to M_{3+1} and the ‘internal’ space \mathcal{M} :

$$\begin{aligned} V^M &\rightarrow V^m \oplus V^\mu \oplus V^{\bar{\mu}}, \\ 8_V &\rightarrow (\pm 2, \mathbf{1}) \oplus (0, \mathbf{3}) \oplus (0, \mathbf{3}^*), \end{aligned} \quad (0.3.9)$$

where we have indicated the respective representations of $SO(8) \rightarrow SO(2) \times SU(3)$, using that. The Ψ^A spinors decompose:

$$\begin{aligned} \Psi^A &\rightarrow \psi^\alpha \eta^a \oplus \psi^\alpha \eta^0 \oplus \psi^{\dot{\alpha}} \eta_a \oplus \psi^{\dot{\alpha}} \eta_0, \\ 8_S &\rightarrow (+1, \mathbf{3}) \oplus (+1, \mathbf{1}) \oplus (-1, \mathbf{3}^*) \oplus (-1, \mathbf{1}), \end{aligned} \quad (0.3.10)$$

while the Ψ_A spinors decompose in the conjugate way. In the formulae (0.3.9) and (0.3.10), we have already implied expansions as in Eq. (0.3.3). Thus, e.g., ψ^α is a two-component left-handed Weyl spinor in M_{3+1} spacetime while η^a is a commuting spinor on the Calabi-Yau space \mathcal{M} and it is in 1–1 correspondence with tangent vectors [94].

Each component field of the supergravity multiplet is now decomposed according to Eqs. (0.3.9), (0.3.10). For example,

$$\delta g_{MN} \rightarrow \begin{pmatrix} \delta g_{mn} & \delta g_{m\nu} & \delta g_{m\bar{\nu}} \\ \delta g_{\mu n} & \delta g_{\mu\nu} & \delta g_{\mu\bar{\nu}} \\ \delta g_{\bar{\mu}n} & \delta g_{\bar{\mu}\nu} & \delta g_{\bar{\mu}\bar{\nu}} \end{pmatrix}. \quad (0.3.11)$$

The component δg_{mn} expands into scalars on \mathcal{M} so that the 0-mode of δg_{mn} (the M_{3+1} graviton) corresponds to the 1-dimensional $H^0(\mathcal{M}, \mathbb{C})$. Similarly, $i\delta g_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu$ is a (1,1)-form and the 0-modes correspond to elements of $H^1(\mathcal{M}, T_{\mathcal{M}}^*)$.

Having so decomposed the entire supergravity multiplet, we obtain several tensorial and spinorial quantities which combine into $N=1$ superfields. Among these, we are interested only in those that give rise to massless fields in 3+1-dimensional sense and the resulting list is given in the first part of Table 0.2.

0.3.3 The Yang-Mills multiplet

In compactifying, $E_8 \times E_8$ is broken to $E_6 \times SU(3) \times E_8$. Namely, once an $SU(3)$ sub-bundle of E_8 is identified with $T_{\mathcal{M}}$ and the corresponding connections are non-zero in vacuum, the physical gauge group will be the maximal subgroup of $E_8 \times E_8$ which commutes with this $SU(3)$. Eventually, further modification of the Ansatz will be needed to break E_6 to a physically more realistic subgroup thereof and we shall briefly discuss that below.

Combining the decomposition (0.2.11) for one factor in $E_8 \times E_8$ and (0.3.9)–(0.3.10) for both, a list of component fields is obtained. Expanding these on \mathcal{M} and keeping the 0-modes, we complete[†] the list of massless fields in the effective 3+1-dimensional model.

Superfield	Field Content (+h.c.)	Cohomology	Number ^a
$(2, 3/2)_\pm$	$\delta g_{mn}, (\psi_m{}^\alpha \eta^0)$	$H^0(\mathcal{M}, \mathbb{C})$	$b_{0,0} = 1$
$(1/2, 0)_+ + h.c.$	$\delta \Phi, \delta B_{mn}, (\chi^{\dot{\alpha}} \eta_0)$	$H^0(\mathcal{M}, \mathbb{C})$	$b_{0,0} = 1$
$(1/2, 0)_+ + h.c.$	$\delta g_{\mu\bar{\nu}}, \delta B_{\mu\bar{\nu}}, (\psi^{\dot{\alpha}} \eta_{\bar{\mu}a})$	$H^1(\mathcal{M}, T_{\mathcal{M}}^*)$	$b_{1,1} = b_2$
$(1/2, 0)_+ + h.c.$	$\delta g_{\mu\nu}, \delta g_{\bar{\mu}\bar{\nu}}, (\psi^\alpha \eta_{\bar{\mu}}{}^a)$	$H^1(\mathcal{M}, T_{\mathcal{M}})$	$b_{2,1}$
$(1, 1/2)_\pm$	$\delta A_m^{(78,1)}, (\lambda^\alpha {}^{(78)} \eta^0 {}^{(1)})$	$H^0(\mathcal{M}, \mathbb{C})$	$b_{0,0} = 1$
$(1/2, 0)_+ + h.c.$	$\delta A_{\bar{\mu}}^{(27,3)}, (\lambda^\alpha {}^{(27)} \eta^0 {}^{(3)})$	$H^1(\mathcal{M}, T_{\mathcal{M}}^*)$	$b_{1,1} = b_2$
$(1/2, 0)_+ + h.c.$	$\delta A_{\bar{\mu}}^{(27^*,3^*)}, (\lambda^\alpha {}^{(27^*)} \eta^0 {}^{(3^*)})$	$H^1(\mathcal{M}, T_{\mathcal{M}})$	$b_{2,1}$
$(1/2, 0)_\pm$	$\delta A_{\bar{\mu}}^{(1,8)}, (\lambda^\alpha {}^{(1)} \eta^0 {}^{(8)})$	$H^1(\mathcal{M}, \text{End } T_{\mathcal{M}})$	$b(\text{End } T_{\mathcal{M}})$
$(1, 1/2)_\pm$	$\delta A_m^{(248)}, (\lambda^\alpha {}^{(248)} \eta^0 {}^{(1)})$	$H^0(\mathcal{M}, \mathbb{C})$	$b_{0,0} = 1$

^aSee text for the identifications.

Table 0.2: The list of massless superfields upon compactification.

In Table 0.2, we have used the *Dolbeault Theorem*^(L),

$$H^q(\mathcal{M}, \wedge^p T_{\mathcal{M}}^* \otimes \mathcal{V}) = H^{p,q}(\mathcal{M}, \mathcal{V}), \quad (0.3.12)$$

[†]Explicit examples show that there occur additional 0-modes when \mathcal{M} possesses special symmetries [139,140,76]; special treatment is required when \mathcal{M} is singular [119,170,172].

for any (sheaf of holomorphic germs of a) holomorphic vector bundle \mathcal{V} over \mathcal{M} . In addition, since a Calabi-Yau manifold is equipped with a projectively unique (that is, defined up to an overall constant) covariantly constant holomorphic 3-form Ω , the mapping

$$\mathcal{T}_{\mathcal{M}} \xrightarrow{\cdot\Omega} \wedge^2 \mathcal{T}_{\mathcal{M}}^* \quad (0.3.13)$$

is an isomorphism, and so is

$$H^q(\mathcal{M}, \mathcal{T}_{\mathcal{M}} \otimes \mathcal{V}) \xrightarrow{\cdot\Omega} H^q(\mathcal{M}, \wedge^2 \mathcal{T}_{\mathcal{M}}^* \otimes \mathcal{V}). \quad (0.3.14)$$

In particular, $H^q(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \approx H^{2,q}(\mathcal{M})$.

It is rather useful to note that the various cohomology groups occurring in Table 0.2 are closely related to the various structures on the Calabi-Yau space. It is standard that elements of $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ span local deformations of the complex structure of \mathcal{M} . Since the Kähler class itself belongs to $H^{1,1}(\mathcal{M})$, its variations are spanned by elements of $H^{1,1}(\mathcal{M})$. Finally, elements of $H^1(\mathcal{M}, \text{End } \mathcal{T}_{\mathcal{M}})$ span local deformations of the complex structure of $\mathcal{T}_{\mathcal{M}}$.

0.4 Holes in the Vacuum

Quite independently of the considerations so far, the massless spectrum can be changed in a very important way if the compact space \mathcal{M} is multiply connected. Consider a closed loop Γ in \mathcal{M} and the associated line integral

$$\Phi_{\Gamma} \stackrel{\text{def}}{=} \oint_{\Gamma} (dy^{\bar{\mu}} A_{\bar{\mu}}^{(78)}(y) + h.c.). \quad (0.4.1)$$

If Γ is not a contractible loop, it is not the boundary of a surface and this line-integral cannot be related to a surface-integral over the field strength. Thus, while the field strength vanishes in vacuum, the gauge-potential $A_m^{(78)}$ need not and the quantity Φ_{Γ} may acquire a non-vanishing vacuum expectation value (VEV).

This value necessarily breaks the E_6 symmetry since only those generators of E_6 which commute with Φ_{Γ} will remain gauge *symmetries*. So far, the situation is identical to the case described in p.70–72 of Ref. [47], with Φ_{Γ} playing the rôle of a Higgs multiplet. Indeed, using the Dynkin basis, each component of a representation of E_6 is labeled by a 6-plet of integers $[\rho_1, \dots, \rho_6]$. Denoting the component of Φ_{Γ} which does not vanish in vacuum by $\langle \Phi_{\Gamma} \rangle = [a, \dots, f]$, we can compute the mass-eigenvalue for each of the 78 gauge potentials and also for the

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$27-27^*$ component pairs [47]. Consider R , an arbitrary representation of E_6 . Then

$$M_{[\rho^1, \dots, \rho^6]} \propto (a\rho^1 + b\rho^2 + \dots + f\rho^6) \quad (0.4.2)$$

is the mass-eigenvalue of the element of R , denoted by $[\rho^1, \dots, \rho^6]$. Inserting the Dynkin labels for the elements of the gauge 78-plet or $27-27^*$ pairs of matter we obtain their masses, all proportional to the compactification mass-scale which, essentially, is the inverse (average) size of \mathcal{M} .

It is then possible to determine the gauge symmetry spanned by the gauge potentials which remained massless or, alternatively, determine the vacuum configuration $[a, \dots, f]$ so that a certain subgroup of E_6 to remain massless. Thereafter, the eigenvalues of the

$$(27^*s | \langle \Phi_{\Gamma} \rangle | 27s) \quad (0.4.3)$$

mass matrix are determined by contracting $[a, \dots, f]$ with the weights in the 27 -representation. If say, $b_{2,1} > b_{1,1}$, we have $\frac{1}{2}\chi_E$ 27 's and $b_{1,1}$ $27-27^*$ pairs. Among the $b_{1,1}$ pairs of 27 -plets, those with vanishing mass-eigenvalue will remain massless in addition to $\frac{1}{2}\chi_E = (b_{2,1} - b_{1,1})$ complete 27 s. More precisely, that is the net count; once E_6 is broken, the various components of the $\frac{1}{2}\chi_E$ “complete” 27 s need not have come from the same original 27 s.

If Φ_{Γ} were a usual Higgs multiplet, we could choose the vacuum configuration $[a, \dots, f]$ freely. In the present case, however, there is an important restraint stemming from the fact that Γ is non-contractible but that, for compact projective spaces with $SU(3)$ holonomy rather than a subgroup thereof, multiple connectedness must be of finite order [23,79] (see Theorem 4.1). So, the path-ordered exponential

$$U_{\Gamma} \stackrel{\text{def}}{=} \mathcal{P} \exp \left\{ \oint_{\Gamma} (dx^m A_m^{(78)} + h.c.) \right\} \quad (0.4.4)$$

is an element of the *finite* structure group, $U_{\Gamma} \in \pi_1(\mathcal{M})$, and $(U_{\Gamma})^n = 1$ for some $n \in \mathbb{Z}_+$. Among other, this implies that the coefficients a, \dots, f in the configuration of $\langle \Phi_{\Gamma} \rangle$ must be understood n . In the mass formulae (0.4.2) and (0.4.3), this may create additional zeroes.

0.5 Yukawa Couplings

Once we have obtained the spectrum of massless fields for the effective 3+1-dimensional model, it is natural to inquire about their interactions. One part is certainly the gauge interactions with the gauge bosons left massless in the compactifications process and is simply through the standard minimal coupling, given the various charges as derived from the initial E_6 .

However, the matter (super)fields Φ_i also interact directly and the corresponding terms must appear in the effective superpotential. As well known, this basic property of supersymmetric theories provides an important restriction for the possible interactions, since the superpotential can depend only on *chiral*^(L) *superfields*^(L) (its Hermitian conjugate, the anti-superpotential, of course depends on anti-chiral superfields and in a Hermitian conjugate way). Since the masses of the fields in the effective low-energy models are not directly related to the compactification scale, there can be no mass-term in the superpotential at the compactification scale and its general form is

$$W|_{at M_C} = \sum_{n=3}^{\infty} \frac{\kappa_{i_1 \dots i_n}}{M_C^{(n-3)}} \prod_{k=1}^n \Phi_{i_k}(x; \theta), \quad (0.5.1)$$

where M_C is the compactification mass-scale. The leading such terms come directly from the 9+1-dimensional theory and can be deduced by studying the 9+1-dimensional action, more precisely, the coupling [220]

$$\int d^{10}X \sqrt{-g_{(10)}} \text{Tr} \left\{ \bar{\lambda}(X) \cdot \mathcal{A}(X) \cdot \lambda(X) \right\}. \quad (0.5.2)$$

Decomposing the E_8 gauge vector and gaugino as described above, we obtain terms of the generic form

$$\int d^4x \sqrt{-g_{(4)}} \left[\kappa_{ijk} \psi^i(x) \phi^k(x) \psi'^k(x) \right] \quad (0.5.3)$$

$$\kappa_{ijk} \stackrel{\text{def}}{=} \text{Tr} \int_M \eta_i(y) \cdot \varphi_j(y) \cdot \eta'_k(y), \quad (0.5.4)$$

where the indices i, j, k merely enumerate the respective 0-modes. The quantity in the square brackets is a so called Yukawa coupling and is manifestly of physics interest.

By E_6 gauge invariance, there are only four types of such Yukawa couplings :

0.5. Yukawa Couplings

- The $(27^*)^3$ couplings. The 27^* 's accompany $dz^\mu dz^{\bar{\nu}} e_{(A)\mu\bar{\nu}}$'s, i.e., elements of $H^1(\mathcal{M}, T_{\mathcal{M}}^*) = H^{1,1}(\mathcal{M})$. Using this property, this Yukawa coupling can be evaluated as*

$$\begin{aligned} \dot{\kappa}_{ABC} &\stackrel{\text{def}}{=} \int_M e_{(A)} \wedge e_{(B)} \wedge e_{(C)}, \\ &= \int_M (\Omega^{\bar{\mu}\bar{\nu}\bar{\rho}} e_{(A)\sigma\bar{\mu}} e_{(B)\tau\bar{\nu}} e_{(C)\lambda\bar{\rho}} \bar{\Omega}^{\sigma\tau\lambda}). \end{aligned} \quad (0.5.5)$$

Since there are no harmonic (2,0)- or (0,2)-forms on a manifold with $SU(n)$ holonomy (for $n \neq 2$), it follows that $H^{1,1}(\mathcal{M}) = H^2(\mathcal{M}) = H^2(\mathcal{M}, \mathbb{Z}) \approx H_4(\mathcal{M}, \mathbb{Z})$, where the last relation is an isomorphism since both H^2 and H_4 are dual to H^4 (see *duality*^(L)). This will make it possible to compute $\dot{\kappa}_{ABC}$, in favorable cases, from intersections of 4-cycles in \mathcal{M} . More formally, this Yukawa coupling realizes the mapping

$$\int : \wedge^3 H^{1,1}(\mathcal{M}) \rightarrow \mathbb{C}. \quad (0.5.6)$$

- The 27^3 couplings. Since the α^{th} 27 -plet accompanies $dz^{\bar{\mu}} \varphi_{(\alpha)\bar{\mu}}^\rho \partial_\rho$'s, i.e., the α^{th} element of $H^1(\mathcal{M}, T_{\mathcal{M}})$, this Yukawa coupling is

$$\begin{aligned} \kappa_{\alpha\beta\gamma} &\stackrel{\text{def}}{=} \int_M \Omega \wedge (\varphi_{(\alpha)} \wedge \varphi_{(\beta)} \wedge \varphi_{(\gamma)}) \cdot \Omega, \\ &= \int_M (\Omega^{\bar{\mu}\bar{\nu}\bar{\rho}} \varphi_{(\alpha)\bar{\mu}}^\sigma \varphi_{(\beta)\bar{\nu}}^\tau \varphi_{(\gamma)\bar{\rho}}^\lambda \Omega_{\sigma\tau\lambda}). \end{aligned} \quad (0.5.7)$$

Here $\Omega = \Omega_{\lambda\mu\nu} dz^\lambda \wedge dz^\mu \wedge dz^\nu$ is the covariantly constant holomorphic 3-form. More formally, this Yukawa coupling realizes the mapping

$$\int \Omega \wedge (\) \cdot \Omega : \wedge^3 H^1(\mathcal{M}, T_{\mathcal{M}}) \rightarrow \mathbb{C}, \quad (0.5.8)$$

which is coordinate independent because $c_1 = 0$ and so $\wedge^3 T_{\mathcal{M}} \cdot \Omega \approx \mathbb{C}$.

- The 1^3 couplings. Since the matter 1 's accompany $dz^{\bar{\mu}} s_{(k)\bar{\mu}}^\rho dz^\sigma \partial_\rho$'s, i.e., elements of $H^1(\mathcal{M}, \text{End } T_{\mathcal{M}}) \approx H^1(\mathcal{M}, T_{\mathcal{M}}^* \otimes T_{\mathcal{M}})$, this Yukawa coupling is

$$\begin{aligned} \dot{\kappa}_{ijk} &\stackrel{\text{def}}{=} \int_M \Omega \wedge \text{Tr}(s_{(i)} \wedge s_{(j)} \wedge s_{(k)}), \\ &= \int_M (\Omega^{\bar{\mu}\bar{\nu}\bar{\rho}} s_{(i)\bar{\mu}}^\rho s_{(j)\bar{\nu}}^\sigma s_{(k)\bar{\rho}}^\tau). \end{aligned} \quad (0.5.9)$$

*The little circle atop κ_{ABC} is to remind one that this expression will acquire quantum corrections.

More formally, this Yukawa coupling realizes the mapping

$$\int \Omega \wedge : \wedge^3 H^1(\mathcal{M}, \text{End } T_{\mathcal{M}}) \rightarrow \mathbb{C} , \quad (0.5.10)$$

because $\text{Tr}(\lambda^3) \in \mathbb{C}$ is an invariant for any matrix λ , with λ^3 denoting the usual matrix multiplication.

4. The mixed couplings. Using the expressions as above, this Yukawa coupling is

$$\begin{aligned} \dot{\kappa}_{\alpha j C} &\stackrel{\text{def}}{=} \int_{\mathcal{M}} \Omega \wedge \text{Tr}(\varphi_{(\alpha)} \wedge s_{(j)} \wedge e_{(C)}) , \\ &= \int_{\mathcal{M}} (\Omega^{\bar{\mu} \bar{\nu} \bar{\rho}} \varphi_{(\alpha)}{}^{\sigma} s_{(j)}{}^{\tau} {}_{\sigma} e_{(C)}{}_{\tau \bar{\rho}}) . \end{aligned} \quad (0.5.11)$$

More formally, this Yukawa coupling realizes the mapping

$$\int \Omega \wedge : H^1(\mathcal{M}, T_{\mathcal{M}}) \wedge H^1(\mathcal{M}, \text{End } T_{\mathcal{M}}) \wedge H^1(\mathcal{M}, T_{\mathcal{M}}^*) \rightarrow \mathbb{C} , \quad (0.5.12)$$

since $\text{End } T_{\mathcal{M}}$ is the traceless part of $T_{\mathcal{M}}^* \otimes T_{\mathcal{M}}$, which of course naturally contracts with $T_{\mathcal{M}}$ from the first factor and $T_{\mathcal{M}}^*$ from the third factor.

Remark.

Since e_A , φ_α , s_i and Ω are all $\bar{\partial}$ -closed (annihilated by $\bar{\partial}$), it is easy to show by partial integration that the value of the integrals is unchanged under the shift such as $s \rightarrow s + \bar{\partial}\alpha$. The couplings (0.5.7), (0.5.5), (0.5.9) and (0.5.11) therefore depend only on the cohomology classes represented by the e_A , φ_α and s_i .

0.6 Instanton Retouch

The various supersymmetries which have been required have rather important consequences regarding the σ -model/string corrections.

0.6.1 A simple non-renormalization theorem

A relatively general non-renormalization theorem (selection rule) for the superpotential in the effective low-energy theory can be derived using the chirality of the superpotential and the abelian infinitesimal symmetries related to the field B_{MN} in (0.3.7).

0.6. Instanton Retouch

In a supersymmetric compactification, there are no quantum corrections to the superpotential of the effective low-energy model to all finite orders in the σ -model and string perturbation.

To prove this (see Ref. [240]), recall first that the original heterotic string theory in 9+1-dimensional Minkowski spacetime contained the 2-form $B = B_{MN} dX^M dX^N$, which occurs only through its field-strength, $H = dB$. Manifestly, this implies the symmetry $B \rightarrow B + d\Lambda$ with any 1-form Λ . Upon compactification, this symmetry is inherited by the resulting model to all orders in the σ -model perturbation theory—as can be verified by explicit computation [232].

Consider now the components which are cotangent to the internal manifold, $B_{\mu\bar{\nu}}$. Since the superpotential does not contain derivative couplings, it can depend on $B_{\mu\bar{\nu}}$ only through its occurrence in a factor $e^{-S_{\mathcal{M}}}$, where

$$S_{\mathcal{M}} = \dots + \left(\int_{\Sigma} d^2\sigma \epsilon^{ab} B_{\mu\bar{\nu}} (\partial_a Z^{\mu}(\sigma)) (\partial_b \bar{Z}^{\bar{\nu}}(\sigma)) \right) , \quad \text{Image}(\Sigma \rightarrow \mathcal{M}) \quad (0.6.1)$$

which is non-vanishing only for non-trivial maps $Z^{\mu} : \Sigma \rightarrow \mathcal{M}$, i.e., for world-sheet instantons. Owing to supersymmetry, the components of $B_{\mu\bar{\nu}}$ occur as imaginary parts of the scalar components of a chiral superfields where the real parts are provided by corresponding components of the metric, $G_{\mu\bar{\nu}}$, on the internal space. Among the latter, the dilatational mode, r , corresponds to the overall size of the internal manifold : r^3 is the volume of the internal space in units of the string tension. $r^{-1/2}$ is then the coupling constant of the σ -model. Similarly, the component of the 2-form B which is (co)tangent to the Minkowski spacetime is paired up with the dilaton, Φ , into the complex scalar component field of a chiral superfield.

Since the superpotential is chiral, it must be a holomorphic function of $G + iB$. As it is independent of B to all orders in perturbation theory, it must also be independent of G and so also of the σ -model perturbation coupling constant, r , and of the string coupling constant, the dilaton. \square

In general, superpotential terms may be corrected only through world-sheet instanton effects and stringy non-perturbative effects.

0.6.2 world-sheet instantons

As the only possible source of quantum corrections to the superpotential terms (save for the string non-perturbative effects, which are beyond the present understanding), we now consider the effect of world-sheet non-perturbative effects.

Recall that, in the σ -model action (0.1.2), the field combinations $B + iG$ and $B - iG$ couple to different derivatives of Z, \bar{Z} . One argues that in contributions to the superpotential and *assuming* supersymmetry not to be broken by quantum corrections, only maps for which

$$\bar{\partial}Z^\mu(\sigma) = 0 \quad (0.6.2)$$

can contribute [111]. These are the holomorphic instantons, i.e., classical configurations of holomorphic Z^μ , inequivalent to the ‘tree-level’ Z^μ which is constant on Σ . In other words, they arise as homotopically non-trivial holomorphic maps $\Sigma \rightarrow \mathcal{M}$.

Remark.

The *assumption* of unbroken supersymmetry can be relaxed if a non-perturbative criterion for validity of supersymmetry can be evaluated favorably. One such criterion is the Witten index, which equals the Euler characteristic of the (total) space-time; if non-zero, supersymmetry will not be broken and the non-renormalization follows [237].

Now, the world-sheet being a Riemann surface, it is mapped into some complex curve in the internal space \mathcal{M} under Z . Since the harmonic $(1,1)$ -forms are dual to 2-cycles (which may be thought of as formal sums of certain 2-dimensional subspaces) in \mathcal{M} , holomorphic instantons will necessarily interact with $(1,1)$ -forms and will affect any result related to $(1,1)$ -forms. Odd-dimensional subspaces, on the other hand, can only be interacted with through their boundaries or the spaces that they themselves bound. Thus, results that depend only on odd-dimensional *cycles* in \mathcal{M} cannot be affected by holomorphic instantons and are in fact, by the above perturbative non-renormalization theorem, exact up to stringy non-perturbative effects. Since there are no 1-cycles or 5-cycles on Calabi-Yau 3-folds, this result is restricted to any numerical characteristic of $H_3(\mathcal{M})$.

In particular, we have that *triple- $(2,1)$ Yukawa couplings cannot be renormalized*. (This can also be verified by explicit computation; see Ref. [113].)

Moreover, the world-sheet is naturally mapped to 2-cycles, which are dual to $(1,1)$ -cohomology. To any basis for the latter, there corresponds a basis for the 2-cycles, hence a ‘natural’ parametrization for the instantons. Many features of their effects can therefore be obtained without any actual evaluation [114], aside from the fact that the evaluation also can be performed, at least in principle. The

0.7. VEVs for Free

$(1,1)^3$ Yukawa coupling (0.5.5) is therefore seen to be corrected as

$$\begin{aligned} \kappa_{ABC} &\stackrel{\text{def}}{=} \int_{\mathcal{M}} \omega_{(A)} \wedge \omega_{(B)} \wedge \omega_{(C)} \\ &+ \sum_L e^{-S_{\mathcal{M}}[J;L]} \left(\int_L \omega_{(A)} \right) \left(\int_L \omega_{(B)} \right) \left(\int_L \omega_{(C)} \right), \end{aligned} \quad (0.6.3)$$

where L ’s label the instanton configuration and one needs to sum (integrate) over all such possible configurations. J represents the choice of the Kähler class on \mathcal{M} and $S_{\mathcal{M}}[J;L]$ is simply the integral over L of the pull-back of J .

Remark.

The action, of course, contains the supersymmetric completion, but care has to be taken with 0-modes of the various fields, as those do not come in Fermi-Bose pairs. In view of this, it is usually assumed that (in most physically relevant calculations) the non-zero modes (which are Fermi-Bose paired) cancel one another’s contribution and only the 0-modes (the Fermi-Bose un-paired ones) do contribute. This assumption permits us to restrict to the un-paired 0-modes, which are usually finite in number.

0.7 VEVs for Free

Suppose that in a particular compactification, a component of some $\mathbf{27}$ and a corresponding component of some $\mathbf{27}^*$ have no self-interactions; more precisely, that their interaction terms vanish when all other fields are set to zero. In such a situation, the vacuum expectation values (VEVs) $\langle \mathbf{27} \rangle$ and $\langle \mathbf{27}^* \rangle$ are free parameters of the theory. They are also called “flat directions” in the field space.

However, $\langle \mathbf{27} \rangle, \langle \mathbf{27}^* \rangle \neq 0$ implies a breakdown of the E_6 Yang-Mills symmetry and if there is one such pair of fields, it is well known from spontaneous symmetry breaking analyses (see Ref. [47,44]) that the resulting theory will have $SO(10)$ Yang-Mills invariance. With another pair of such fields $SU(5)$ may be achieved, but care has to be taken, since the second pair of fields has to have a free VEV after the first two fields are shifted by their VEVs. In fact, in order to obtain $E_6 \rightarrow SU(3)^C \times SU(2)_W \times U(1)_Y$ and the matter spectrum anywhere near acceptable, it is likely that both a non-contractible gauge flux (see § 0.4) and two pairs of $\langle \mathbf{27} \rangle, \langle \mathbf{27}^* \rangle$ have to be generated*. With a non-abelian structure group

*Clearly, once the Yukawa couplings are known, the analysis of the matter potential will predict if some of the scalars develops a non-zero VEV or if some of the VEVs are free.

$\pi_1(\mathcal{M})$, non-commuting gauge flux configurations are possible and a single pair $\langle \mathbf{27}, \langle \mathbf{27}^* \rangle \rangle$ (or even none at all) may suffice.

Similarly, some of the chargeless fields might acquire a free VEV. Now, giving a non-zero VEV to a modulus field accompanying (2,1)-forms or another one accompanying (1,1)-forms corresponds, respectively, to shifting the complex structure or the Kähler class of the internal space. Since elements of $H^1(\mathcal{M}, \text{End } T_{\mathcal{M}})$ span linear deformations of the holomorphic tangent bundle, $T_{\mathcal{M}}$, we expect that turning on a VEV for a **1** will correspond to a deformation of this structure. We will return to this issue later.

0.8 The Wish List (Seven Veils)

Given a particular Calabi-Yau manifold, we need to supply the following information for characterizing the effective particle physics model :

1. $\frac{1}{2}\chi_E$: the net number of generations predicted; preferably, three.
2. $b_{1,1}$ and $b_{2,1}$: both the number of generations and the number of mirror-generations independently, not just $\frac{1}{2}\chi_E = (b_{1,1} - b_{2,1})$. For phenomenological reasons, one needs at least $\min(b_{1,1}, b_{2,1}) = 2$, but not much bigger. (Generally, proliferation of particles can easily have devastating effects on the low-energy model.)
3. $\dim H^1(\mathcal{M}, \text{End } T_{\mathcal{M}})$: the number of light chargeless matter fields.
4. $\pi_1(\mathcal{M})$: through non-contractible gauge flux lines, it measures the ability to break E_6 to a more sensible Yang-Mills group. \mathbb{Z}_2 is “not enough”, at least $\pi_1 = \mathbb{Z}_3$ is required for potentially successful phenomenology.
5. The $\langle \mathbf{27}^3 \rangle, \langle \mathbf{27}^{*3} \rangle, \langle \mathbf{1}^3 \rangle, \langle \mathbf{27} \cdot \mathbf{27}^* \cdot \mathbf{1} \rangle$ Yukawa (and also higher) couplings as given in Eq. (0.5.1).
6. The normalization matrices, i.e., the Zamolodchikov metric for the kinetic terms of the **27s**, **27***s and **1s**. To lowest order, this equals the Weil-Petersson metric on the space of complex structures, space of (complexified) Kähler classes and complex structures for $T_{\mathcal{M}}$.
7. σ -model/string corrections to the results in 3., 5. and 6.

0.8. The Wish List (Seven Veils)

This is indeed a rather taxing schedule and in the explicitly analyzed cases only the first few tasks have been completed. Indeed, we are at serious loss of ammunition and tactics for a *complete* assessment of the last two issues in the general case. In geometrical terms, the normalization matrices for the kinetic terms are components of a Zamolodchikov-type metric on the dual parameter space. This however does not provide a simple computational tool and exact results are known only in some simple cases. Also, the above non-renormalization theorems do protect some of the results from σ -model/string corrections but, unfortunately, rather little is known outside the shielding scope of these theorems; see however Ref. [219,89,59,60,88].

With best of hopes for future development in the areas pertaining to the last points (to which we will return), we concentrate on seeking and studying Calabi-Yau beasts for which the armament and spells provided herein suffice as far as stripping off the first four veils and getting a grip on the other three. The advanced Reader will hopefully enjoy learning some deeper and further secrets of the priesthood, for which the listed literature is recommended; those divulged herein reflect my own winding voyage and resulting opinion—may you learn from my mistakes.



Several methods of creating Calabi-Yau manifolds have been employed in the past few years. Roughly speaking, they come in two varieties.

In the first one, a Calabi-Yau manifold is realized as the space of solutions to a system of algebraic equations and is thereby embedded in some bigger and better understood space in which the equations are formulated; we may refer to such methods as ‘algebraic’. In the second one, some space of complex three dimensions is being modified and possibly glued to another such construction; we may call these ‘surgical’, stretching somewhat the usual meaning in mathematics.

Needless to say, the two methods are often combined to bring out the best in each. Algebraic techniques typically rely on polynomial algebra and are thus rather familiar. Algebraic and also some surgical methods yield access to numerical characteristics through straightforward albeit possibly tedious computations. On the other hand, certain techniques of surgery are on occasion indispensable but their effect on numerical characteristics is sometimes less well understood. We begin with describing the most often used algebraic techniques and discuss the various surgical techniques as suitable opportunities arise.

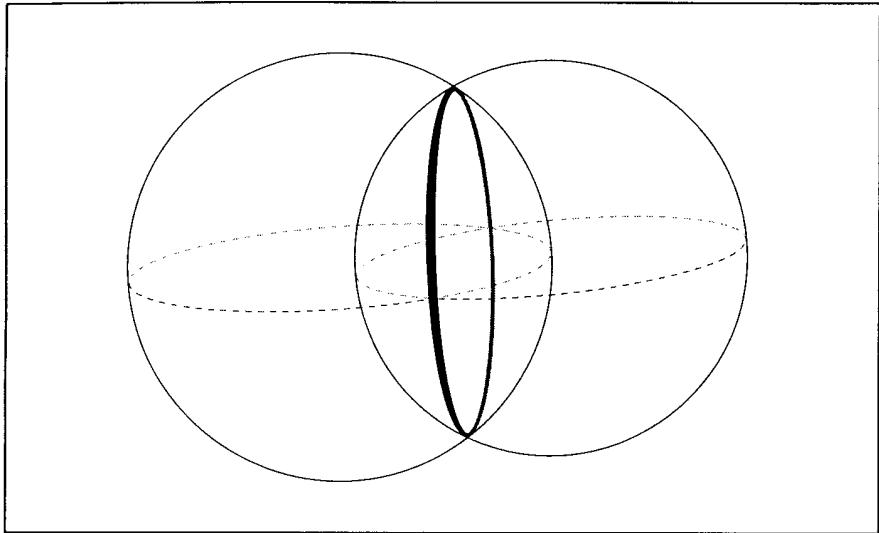


Figure 0.1: A circle as a complete intersection of two spheres.

Given a system of K constraint equations $P^a(w) = 0$ in the embedding space \mathcal{X} , it is useful to consider the system of hypersurfaces X_a each of which is defined as the zero-set of a single constraint. The space of common solutions is evidently $\mathcal{M} = \cap_a X_a \subset \mathcal{X}$. In an important class of such constructions, the hypersurfaces meet *transversely*. That is, at every point of \mathcal{M} , the gradients of $\{X_a\}$ span a non-degenerate system. Such an \mathcal{M} is a *complete intersection*^(L) and have $\dim \mathcal{M} = \dim \mathcal{X} - K$. For a toy example, we may describe a circle as the complete intersection of two spheres (each defined by a quadratic equation) in \mathbb{R}^3 , as in Fig 0.1. Both exceptional cases, a) when the spheres meet in a single point and b) when they coincide, fail our transversality test and are considered singular. Complete intersections are perhaps the easiest of the constructions and will serve to discuss many of the techniques. Subsequently, we shall also encounter other types of algebraic constructions, where manifestly $\dim \mathcal{M} > \dim \mathcal{X} - K$. I am not aware of a general classification of such constructions or of a particular type of geometry universally suitable for their analysis and will therefore only be able to present some examples... But now, onward!

Part I

Constructions

Chapter 1

Complex Kindergarten

Notice!

Practically everything in this volume concerns complex spaces, fibre *bundles*^(L) and *sheaves*^(L) over them and other complex and *holomorphic*^(L) derived structures. The same will be true of *cohomology groups*^(L) also, unless otherwise is explicitly stated. Finally, as usual in algebraic geometry, one works over a given algebraically closed field \mathbb{k} and all dimensions refer to that field. Owing to the usual physics application, by default, we have in mind $\mathbb{k} = \mathbb{C}$ throughout this volume.

In this chapter we construct Calabi-Yau 3-folds as intersections of hypersurfaces in a complex projective space, and study some of the properties of such 3-folds. We begin with a review of the properties of complex projective spaces.

1.1 Complex Manifolds and Holomorphic Vector Bundles

A manifold is smooth if local differentiability has an invariant meaning although we may need to choose a connection to define covariant derivatives. Similarly, on a complex manifold, local holomorphicity has an invariant meaning. Thus a holomorphic vector bundle is a smooth vector bundle for which one knows which local *sections*^(L) are holomorphic : since the transition functions are required to be holomorphic, the holomorphicity of the section is independent of the choice of coordinate neighborhoods and corresponding coordinates.

Most of the time, we will be discussing algebraic varieties, defined as the zero-set of a system of polynomials in certain holomorphic variables x^a . By choosing

the x^a to be coordinates on a complex space to begin with and using holomorphic polynomials, the algebraic variety is a complex space by construction.

A crucial difference between the smooth and holomorphic categories is detected by the number of *global* sections. A smooth vector bundle over a smooth manifold always has infinitely many linearly independent global sections. In contrast, a holomorphic vector bundle over a compact complex manifold has always a finite number of independent global holomorphic sections,— sometimes only of the 0-section. In other words, the holomorphic category is much more rigid. For one thing, in the smooth category, there always exist “bump-functions” so that smooth manifolds can be cut and pasted at will. Surgery in the holomorphic category is always much more restricted and may well be impossible on occasion.

Another important difference is that in the holomorphic category it is necessary to distinguish between sub-bundles and quotient bundles. Given a holomorphic vector bundle \mathcal{B} over a complex manifold \mathcal{M} which has a holomorphic sub-bundle \mathcal{A} , we can always form the quotient bundle \mathcal{B}/\mathcal{A} , which is spanned by holomorphic vectors of \mathcal{B} regarded equivalent if they differ by a vector of \mathcal{A} . However, \mathcal{B}/\mathcal{A} is almost never a holomorphic sub-bundle of \mathcal{B} . One can certainly choose (in many ways) a Hermitian metric on \mathcal{B} and define \mathcal{C} to be the Hermitian complement of $\mathcal{A} \subset \mathcal{B} = \mathcal{A} \oplus \mathcal{C}$. However, \mathcal{C} is almost never a holomorphic bundle over \mathcal{M} : using the metric to define it introduces antiholomorphic variables in a non-trivial way.

The simplest vector bundles over a compact complex manifold \mathcal{M} have unit rank, are called holomorphic line bundles and are of particular importance for us. Their fibre is isomorphic to \mathbb{C}^1 . Under the (fibre-wise) tensor product operation, these bundles form a group known as the Picard group of \mathcal{M} . Now, given any holomorphic vector bundle \mathcal{B} over \mathcal{M} of rank k (the fibres of \mathcal{B} are all \mathbb{C}^k -like) and any holomorphic sub-bundle \mathcal{A} , one can always form the respective determinant bundles $\det \mathcal{B}$ and $\det \mathcal{A}$ and these satisfy the identity

$$\det(\mathcal{B}) = \det(\mathcal{A}) \otimes \det(\mathcal{B}/\mathcal{A}). \quad (1.1.1)$$

All three determinant bundles are line bundles over X and are therefore easier to deal with than the original vector bundles.

Suppose X is a complex manifold, \mathcal{L} a holomorphic line bundle over X and f a non-trivial global holomorphic section of \mathcal{L} . We can define a subspace $\mathcal{M} \subset X$ to consist of all points $x \in X$ for which $f(x) = 0$. Regarding f as a mapping that

1.1. Complex Manifolds and Holomorphic Vector Bundles

takes all points $x \in \mathcal{M} \subset X$ to 0, we write $\mathcal{M} = f^{-1}(0)$ and call \mathcal{M} a hypersurface in X .

Reminder.

The usual gradient of $f(x)$ provides a normal to the hypersurface \mathcal{M} and has an invariant meaning. To see this, choose Γ_μ to be a local connection. Then $\nabla_\mu f = \partial_\mu f + \Gamma_\mu \cdot f = \partial_\mu f$ at $x \in \mathcal{M}$, since $\Gamma_\mu(x) \cdot f(x) = 0$ wherever $f(x) = 0$.

Now, \mathcal{M} fails to be a complex *submanifold* if f is singular, i.e., if there is at least one point where both f and all its gradients $\partial_\mu f$ vanish. One might worry if there exist non-singular choices of f . To that end, we have the following version of Bertini’s theorem :

Theorem 1.1 (Bertini) *Let X denote a compact complex manifold and \mathcal{L} a holomorphic line bundle over X . Suppose that, at every point $x \in X$, at least one section of \mathcal{L} is non-zero. Then a generic (= almost every) section f of \mathcal{L} defines a non-singular hypersurface* $\mathcal{M}_f \stackrel{\text{def}}{=} f^{-1}(0)$.*

To be precise, the qualification “generic” means “all except for a subset of strictly smaller dimension”. Also, note that the submanifolds \mathcal{M}_f so defined are parametrized by the choice of the section f .

For a complex manifold \mathcal{M} of complex dimension n , we write $\mathcal{T}_{\mathcal{M}}$ for the holomorphic tangent bundle of \mathcal{M} and $\mathcal{T}_{\mathcal{M}}^*$ for its holomorphic complex dual, the cotangent bundle (which is *not* its complex conjugate, $\bar{\mathcal{T}}_{\mathcal{M}}!$). $\mathcal{T}_{\mathcal{M}}^*$ is also identified with the bundle of holomorphic 1-forms, $\Omega_{\mathcal{M}}^1$. The bundle of holomorphic n -forms (also called holomorphic volume forms) is the determinant bundle of $\mathcal{T}_{\mathcal{M}}^*$,

$$\Omega_{\mathcal{M}}^n = \wedge^n \Omega_{\mathcal{M}}^1 = \det \mathcal{T}_{\mathcal{M}}^* \stackrel{\text{def}}{=} \mathcal{K}_{\mathcal{M}}, \quad (1.1.2)$$

and is also called the canonical bundle.

If a compact complex manifold \mathcal{M} admits a holomorphic volume form Ω which is non-singular and has no zeros, then Ω must be a constant over \mathcal{M} and the canonical bundle must be trivial so that all possible holomorphic volume forms are merely constant complex multiples of each other

$$\Omega' = \lambda \Omega, \quad \lambda \text{ constant on } \mathcal{M}. \quad (1.1.3)$$

*However, as Philip Candelas likes to point out, most examples that you in fact write down—turn out to be singular. The veracity of this statement is an increasing and swiftly saturating function of the number of variables involved.

This book is devoted, for the most part, to the construction and study of such 3-folds as three dimensional algebraic varieties and we will soon turn to the construction of simplest such spaces. First however, we find it useful to review a number of general properties of such spaces.

1.2 Calabi-Yau 3-Fold Specials

Some rather important properties of Calabi-Yau 3-folds are described in terms of their *cohomology groups*^(L). The Reader is probably most familiar with the De Rham cohomology groups, which are defined as the quotient spaces of closed modulo exact exterior forms. That is, elements of $H_{\text{DR}}^r(\mathcal{M})$ are *classes* of exterior r -forms ω such that

$$d\omega = 0, \quad \omega \simeq \omega' = \omega + d\alpha. \quad (1.2.1)$$

On a complex manifold, the exterior derivative may be decomposed into the holomorphic and the anti-holomorphic part, $d = \partial + \bar{\partial}$; also, any exterior r -form may be decomposed into (p, q) -forms such that $p + q = r$. The $\bar{\partial}$ - (also known as Dolbeault) cohomology groups $H_{\bar{\partial}}^{p,q}(\mathcal{M})$ are then defined in precise analogy with the De Rham cohomology. The *Hodge numbers*^(L) $b_{p,q} \stackrel{\text{def}}{=} \dim H_{\bar{\partial}}^{p,q}(\mathcal{M})$ are topological invariants. More generally, for a vector bundle \mathcal{V} over \mathcal{M} , $H^q(\mathcal{M}, \mathcal{V})$ denotes the cohomology group of \mathcal{V} -valued q -forms; $H_{\bar{\partial}}^{p,q}(\mathcal{M}) = H^q(\mathcal{M}, \wedge^p T_{\mathcal{M}}^*)$.

Now, given the nowhere vanishing and projectively unique (that is, unique up to an overall complex non-zero scaling factor) holomorphic 3-form Ω on our 3-fold \mathcal{M} , we have the isomorphism

$$\cdot \Omega : T_{\mathcal{M}} \xrightarrow{\sim} \wedge^2 T_{\mathcal{M}}^*, \quad (1.2.2)$$

and therefore also

$$\cdot \Omega : H^q(\mathcal{M}, T_{\mathcal{M}}) \xrightarrow{\sim} H^{2,q}(\mathcal{M}). \quad (1.2.3)$$

In other words, to every holomorphic tangent vector v^μ , we may assign a projectively unique holomorphic 2-form $\omega_{\nu\rho} \stackrel{\text{def}}{=} v^\mu \Omega_{\mu\nu\rho}$. The 3-form Ω is in fact the (trivial) section of the (trivial) canonical bundle $\mathcal{K}_{\mathcal{M}} \approx \mathbb{C}_{\mathcal{M}}$.

In view of this, the usual Serre duality formula

$$H^q(\mathcal{M}, \mathcal{V})^* \approx H^{3-q}(\mathcal{M}, \mathcal{V}^* \otimes \mathcal{K}_{\mathcal{M}}) \quad (1.2.4)$$

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simplifies in that “ $\otimes \mathcal{K}_{\mathcal{M}}$ ” may be omitted. For example,

$$H^q(\mathcal{M}, T_{\mathcal{M}})^* \approx H^{3-q}(\mathcal{M}, T_{\mathcal{M}}^*), \quad (1.2.5)$$

which, together with Eq. (1.2.3), implies

$$H^{2,q}(\mathcal{M})^* \approx H^{3-q}(\mathcal{M}, T_{\mathcal{M}}^*) \equiv H^{1,3-q}(\mathcal{M}). \quad (1.2.6)$$

Furthermore, since $H^{3,0}(\mathcal{M}) \approx \mathbb{C}$, there is a new duality, implying

$$H^{p,0}(\mathcal{M})^* \approx H^{3-p,0}(\mathcal{M}). \quad (1.2.7)$$

In a more systematic manner, we have the following relations (see also the Lexicon entries) :

Hodge star duality is a property of smooth compact spaces and is guaranteed by the existence and uniqueness of the standard volume-form*. It implies

$$b_{p,q} = b_{n-q,n-p}. \quad (1.2.8)$$

Complex conjugation is a symmetry of Kähler manifolds. To every harmonic (p, q) -form, it associates a harmonic (q, p) -form, so that

$$b_{p,q} = b_{q,p}. \quad (1.2.9)$$

Holomorphic duality is a special feature of Calabi-Yau spaces and is guaranteed by the existence and uniqueness of the holomorphic volume-form, Ω (because $b_{3,0} = 1$). Given a $(0, q)$ -class α , there is a unique $(0, 3-q)$ -class β such that $\int_{\mathcal{M}} \alpha \wedge \beta \wedge \Omega = 1$. Thus

$$b_{0,q} = b_{0,3-q}, \quad b_{p,0} = b_{3-p,0}. \quad (1.2.10)$$

*It should perhaps be pointed out that all the manifolds we shall consider have a single connected piece and hence for complex 3-folds, $b_{3,3} = 1$.

Considering Calabi-Yau manifolds in three complex dimensions, so that $b_{p,q} \equiv 0$ if $p+q > 6$, the use of the above relations reduces the number of independent Hodge numbers :

$$\begin{array}{ccccccccc} & b_{0,0} & & & & 1 & & & \\ & b_{1,0} & b_{0,1} & & & b_{1,0} & b_{1,0} & & \\ & b_{2,0} & b_{1,1} & b_{0,2} & & b_{1,0} & b_{1,1} & b_{1,0} & \\ 1 & b_{2,1} & b_{1,2} & b_{0,3} & \longrightarrow & 1 & b_{2,1} & b_{2,1} & 1 \\ & b_{3,1} & b_{2,2} & b_{1,3} & & b_{1,0} & b_{1,1} & b_{1,0} & \\ & b_{3,2} & b_{2,3} & & & b_{1,0} & b_{1,0} & & \\ & 1 & & & & 1 & & & \end{array} \quad (1.2.11)$$

Finally, to determine $b_{1,0}$, the following result may be used.

Theorem 1.2 (Bochner) *For any harmonic (De Rham) s -form ω , let*

$$F(\omega) := R_m^n \omega_{[nr_2 \dots r_s]} \omega^{[mr_2 \dots r_s]} + \frac{s-1}{2} R_m{}^n{}_p{}^q \omega_{[nqr_3 \dots r_s]} \omega^{[mpr_3 \dots r_s]}. \quad (1.2.12)$$

Then, if $F(\omega)$ is positive semi-definite, ω is covariantly constant.

Here, R_m^n and $R_m{}^n{}_p{}^q$ are the Ricci and the Riemann tensors, respectively.

Choosing the Ricci-flat metric on the Calabi-Yau 3-fold, R_m^n vanishes, so $F(\omega) \equiv 0$ for 1-forms. Therefore, for ω to be harmonic, it also has to be covariantly constant. But ω transforms as $\mathbf{3} \oplus \mathbf{3}^*$ under the $SU(3)$ holonomy and so transforms non-trivially under parallel transport and cannot be covariantly constant. Therefore, $b_1 = 0$ and so $b_{1,0} = b_{0,1} = 0$ and we have just derived the general form of the so-called Hodge diamond for Calabi-Yau 3-folds :

$$\begin{array}{ccccccccc} & & & & & 1 & & & \\ & & & & & 0 & 0 & & \\ & & & & & 0 & b_{1,1} & 0 & \\ & & & & & b_{2,1} & b_{2,1} & 1 & \\ 1 & & & & & 0 & b_{1,1} & 0 & \\ & & & & & 0 & 0 & & \\ & & & & & 1 & & & \end{array} \quad (1.2.13)$$

Remark.

For Calabi-Yau spaces of two or more complex dimensions, the Hodge diamond has

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the same basic form. The “outer” rim of Hodge numbers is completely determined : 1’s in the corners and 0’s otherwise, while the “inside” Hodge numbers may vary. In two complex dimensions there is only the $K3$ surface with $b_{1,1} = 20$; in higher dimensions more and more variant Hodge numbers appear.

Part of our goal is to determine both $b_{1,1}$ and $b_{2,1}$ for given constructions. In fact, we know that in general

$$\chi_E(\mathcal{M}) = \sum_{r=0}^{\dim \mathcal{M}} (-)^r b_r, \quad b_r = \sum_{p=0}^r b_{p,r-p}. \quad (1.2.14)$$

In view of our general result for the Hodge diamond of Calabi-Yau manifolds, it then follows that

$$\chi_E(\mathcal{M}) = 2(b_{2,1} - b_{1,1}) \quad (1.2.15)$$

for all Calabi-Yau 3-folds. If the Euler characteristic is easily obtained as in the subsequent constructions, it suffices to determine independently only one of the two variant Hodge numbers.

Remark.

There is another curious property of Calabi-Yau 3-folds. It has been observed that Calabi-Yau manifolds tend to come in mirror pairs, $(\mathcal{M}, \mathcal{W})$ such that

$$H^{2,1}(\mathcal{W}) \cong_Q H^{1,1}(\mathcal{M}), \quad H^{1,1}(\mathcal{W}) \cong_Q H^{2,1}(\mathcal{M}). \quad (1.2.16)$$

Here, “ \cong_Q ” denotes an equivalence of the ring structures generated, with the exact (quantum) Yukawa couplings as the ring structure constants. For a number of Calabi-Yau 3-folds, such a mirror-pair can be constructed explicitly. Related to this is the fact that general properties of superconformal field theories guarantee analogous mirror-models [117]. Often, both the original compactification model and its mirror-model have a geometric interpretation as compactification on a Calabi-Yau manifold and its respective mirror-pair. However, there do exist Calabi-Yau 3-folds for which no mirror-space has been found even though a compactification mirror-model (so far with no geometric interpretation) does exist.

1.3 The Quintic Hypersurface in \mathbb{P}^4

Perhaps the simplest and certainly the most popular Calabi-Yau manifold is the quintic hypersurface, i.e., the space of solutions to a quintic holomorphic constraint in \mathbb{P}^4 . We shall denote the family of such hypersurfaces (parametrized by the choices of the quintic constraint) by [4][5]. To see how the choice of the degree of the polynomial comes about, consider a general homogeneous, holomorphic polynomial of degree q in the homogeneous coordinates of \mathbb{P}^n :

$$f(z) \stackrel{\text{def}}{=} f_{i_1 \dots i_q} z^{i_1} \cdots z^{i_q}. \quad (1.3.1)$$

(Summation over repeated indices is implied through $i_k = 0, \dots, n_k$.)

Since a homogeneous polynomial can be interpreted as a section of an appropriate line bundle it follows from Bertini's theorem that for generic f , the hypersurface of solutions to $f(z) = 0$, in \mathbb{P}^n , is non-singular. Moreover, it is a Kähler manifold, inheriting this property from the embedding space \mathbb{P}^n . Of course it has dimension $n-1$.

A relatively simple non-singular example is provided by taking

$$F(z) = \sum_{i=0}^n (z^i)^q, \quad (1.3.2)$$

the so-called Fermat polynomial. Let $\mathfrak{F} \stackrel{\text{def}}{=} F^{-1}(0)$ be the hypersurface of solutions to $F(z) = 0$ in \mathbb{P}^n . Now, $dF(z) = q \sum_{i=0}^n dz^i (z^i)^{q-1}$ and the only solution to $dF = 0$ is $z^i = 0, \forall i$, which is not in \mathbb{P}^n by definition. So, since $F(z) = 0$ and $dF(z) = 0$ have no common solution, the gradient to the hypersurface \mathfrak{F} is everywhere well defined and the hypersurface is smooth.

For \mathfrak{F} to be a Calabi-Yau manifold, it has to admit a metric for which the Ricci tensor vanishes. Now, on a Kähler manifold we may without loss of generality restrict to Hermitian metrics and Riemannian (torsion-free) connections. We may take, e.g., the Fubini-Study metric $\mathbf{g}_{(FS)}$ on the \mathbb{P}^n and the induced metric on the hypersurface, i.e., the restriction of $\mathbf{g}_{(FS)}$. Straightforward but tedious calculation yields

$$\mathbf{R}|_{\mathfrak{F}} = [q - (n+1)] \mathbf{g}_{(FS)}|_{\mathfrak{F}}, \quad (1.3.3)$$

up to total derivatives. Here $\mathbf{R}|_{\mathfrak{F}}$ is the restriction to the hypersurface of the Ricci tensor computed from $\mathbf{g}_{(FS)}$. Note that the choice of precisely $q = n+1$, hence a quintic in \mathbb{P}^4 , essentially turns $\mathbf{R}|_{\mathfrak{F}}$ into a total derivative which is sufficient to show

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that a (different) metric does exist for which the Ricci tensor actually vanishes. Indeed, Yau's proof of Calabi's conjecture guarantees that the cohomology class of the Ricci two-form rather than the two-form itself is the only obstruction to constructing a Ricci-flat Kähler metric.

For future application, it will be useful to reformulate the latter result in terms of the Riemann curvature two-form Θ

$$\Theta_{\kappa}{}^{\lambda} = \frac{i}{2\pi} dz^{\mu} \wedge dz^{\bar{\nu}} R_{\mu\bar{\nu}\kappa}{}^{\lambda} \quad (1.3.4)$$

and the cohomology classes of the coefficient forms in the characteristic polynomial of Θ . Also, underlying most of our analysis is the relation between $T_{\mathcal{M}}$ and $T_{\mathcal{X}}$, the holomorphic tangent bundles of the hypersurface \mathcal{M} and of the embedding space \mathcal{X} , respectively, and the holomorphic line bundle \mathcal{E} over \mathcal{X} , which corresponds to $f(z)$. More precisely, polynomials of the same homogeneity as $f(z)$ above can be interpreted as sections of a holomorphic line bundle \mathcal{E} . In the case of quintics over \mathbb{P}^4 , \mathcal{E} is $\mathcal{O}_{\mathbb{P}^4}(5)$, the fifth tensor power of the *hyperplane bundle*^(L) over \mathbb{P}^4 . In general, a homogeneous polynomial of degree k in the homogeneous coordinates of \mathbb{P}^n supplies a holomorphic section of $\mathcal{O}_{\mathbb{P}^n}(k)$.

1.3.1 The normal bundle : adjunction formulae

Consider $T_{\mathcal{M}}$, the holomorphic tangent bundle of \mathcal{M} . It is manifestly a sub-bundle of $T_{\mathbb{P}^n}$, the corresponding quotient bundle is called the *normal bundle* of $\mathcal{M} \hookrightarrow \mathbb{P}^n$,

$$\mathcal{N}_{\mathbb{P}^n/\mathcal{M}} \stackrel{\text{def}}{=} \left\{ T_{\mathbb{P}^n}|_{\mathcal{M}} / T_{\mathcal{M}} \right\}. \quad (1.3.5)$$

Let us choose a connection Γ_i on $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(k)$ and represent a tangent vector, $X \in T_{\mathbb{P}^n}$, at a point of \mathbb{P}^n by $X^i \nabla_i$. The covariant gradient

$$\nabla_X f = X^i (f_{,i} + \Gamma_i f) \stackrel{\text{def}}{=} f_X \quad (1.3.6)$$

associates to each tangent vector X a quintic f_X , i.e., defines a complex linear mapping from the tangent bundle of \mathbb{P}^n to \mathcal{E} . Moreover, if we restrict to points of $\mathcal{M} \subset \mathbb{P}^n$, this mapping is holomorphic and independent of the choice of Γ_i since $f = 0$ and so $\Gamma_i \cdot f = 0$ on \mathcal{M} .

Now, this map yields zero whenever X is tangential to \mathcal{M} since f vanishes everywhere on \mathcal{M} and so there is no gradient 'along' \mathcal{M} vanishes also. In other words, the map annihilates precisely the $T_{\mathcal{M}}$ sub-bundle of $T_{\mathbb{P}^n}$, so that $\ker(\nabla f) =$

\mathcal{T}_M . Note that, around M , f can be used as a local coordinate in X and transversal to M since it vanishes at M and is nonzero away from M . Rescaling suitably $X' = \lambda X$, for $\lambda \neq 0$, the map $\nabla_{\lambda X} f$ will cover all of this local range of f so that, locally, $\nabla_X f$ indeed maps onto the appropriate fibre of E . In fact, the map is onto also globally, since it is holomorphic and $\text{rank } E = \dim T_X - \dim T_M$.

Adjunction formula 1. This is encoded in the short *exact sequence*^(L)

$$0 \rightarrow \mathcal{T}_M \xrightarrow{i} \mathcal{T}_{\mathbb{P}^n}|_M \xrightarrow{\nabla f} E|_M \rightarrow 0, \quad (1.3.7)$$

and serves to establish the identification

$$E|_M = \left\{ \mathcal{T}_{\mathbb{P}^n}|_M / \mathcal{T}_M \right\}. \quad (1.3.8)$$

From this it follows by Eq. (1.3.5) that

$$E|_M = \mathcal{N}_{\mathbb{P}^n/M}. \quad (1.3.9)$$

We refer to Eq. (1.3.9) as the Adjunction Formula 1 (see p. 146 of Ref. [23]).

Adjunction formula 2. Given the short exact sequence such as (1.3.7), using formula (1.1.1) we infer

$$\det \mathcal{T}_{\mathbb{P}^n}|_M = \det \mathcal{T}_M \otimes \det E|_M. \quad (1.3.10)$$

Using now the definition of the canonical bundle and that $\det E = E$ since E is a line bundle, we have

$$\mathcal{K}_M = (\mathcal{K}_{\mathbb{P}^n} \otimes E^*)|_M. \quad (1.3.11)$$

We refer to Eq. (1.3.11) as the Adjunction Formula 2 (see p. 147 of Ref. [23]).

1.3.2 Chern classes

We recall that for a complex vector bundle \mathcal{V} over a complex manifold M , we have the *Chern classes*^(L)

$$c_r(\mathcal{V}) \in H^{2r}(M), \quad r = 1, \dots, \min(\text{rank } \mathcal{V}, \dim M), \quad (1.3.12)$$

and the total Chern class

$$c(\mathcal{V}) = 1 + \sum_r c_r(\mathcal{V}). \quad (1.3.13)$$

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These represent the cohomology classes of the coefficient forms which are found in the expansion of the characteristic polynomial

$$\det [\mathbf{1} + \lambda \mathbf{F}] = 1 + \lambda \text{Tr}[\mathbf{F}] + \lambda^2 (\text{Tr}[\mathbf{F} \wedge \mathbf{F}] - 2[\text{Tr} \mathbf{F}]^2) + \dots \quad (1.3.14)$$

where \mathbf{F} is a suitably normalized curvature two-form on \mathcal{V} . In the case $\mathcal{V} = T_X$, i.e., when $\mathbf{F} = \Theta$ is the (suitably normalized) Riemann two-form, we write $c_i(X)$ or simply c_i if no confusion can arise.

Chern classes $\{c_r(\mathcal{V})\}$ are respectively represented by closed (r, r) -forms on M , with which we freely identify them in the sequel, and satisfy three important identities:

$$c_1(\mathcal{V} \otimes \mathcal{L}) = c_1(\mathcal{V}) + k c_1(\mathcal{L}') \quad (1.3.15)$$

for any rank- k vector bundle \mathcal{V} and any line bundle \mathcal{L} .

$$c(\mathcal{V}) = c(\mathcal{V}') \wedge c(\mathcal{V}'') \quad (1.3.16)$$

whenever

$$0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow \mathcal{V}'' \rightarrow 0 \quad (1.3.17)$$

is a short exact sequence (this includes the case $\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}''$). Both identities follow from the use of the determinant in the definition; compare also with Eq. (1.1.1). Also, if \mathbf{F} is the curvature matrix of a connection in a complex vector bundle \mathcal{V} , the dual connection in \mathcal{V}^* then has curvature matrix $-\mathbf{F}$ and so

$$c_r(\mathcal{V}^*) = (-)^r c_r(\mathcal{V}) \quad \text{no summation on } r. \quad (1.3.18)$$

Remark.

The Chern classes are invariant under smooth deformations, i.e., in the differentiable category. Hence, to compute them, we can ignore the requirement of holomorphicity and simplify the bundle in question. In particular, we may *smoothly* deform a rank- k (holomorphic) vector bundle \mathcal{V} into a k -fold direct sum of line bundles, $\bigoplus_{i=1}^k \mathcal{L}_i$. Then,

$$c(\mathcal{V}) = \sum_{i=1}^k c(\mathcal{L}_i) = \prod_{i=1}^k (1 + \ell_i), \quad (1.3.19)$$

where ℓ_i is the first Chern class of \mathcal{L}_i (being of rank-1, \mathcal{L}_i have no higher Chern classes). This is called the *splitting*^(L) principle and can be used for computing the coarsest properties of most vector bundles that we shall encounter.

Now, using the splitting principle $c[\mathcal{V}] = c[\bigoplus_i \mathcal{L}_i]$ and Eqs. (1.3.15)–(1.3.18), it follows that

$$c_1[\det \mathcal{V}] = c_1[\mathcal{V}]. \quad (1.3.20)$$

In particular, if $c_1[\mathcal{T}_{\mathcal{M}}] = 0$ then also $c_1[\det \mathcal{T}_{\mathcal{M}}] = c_1[\det \mathcal{T}_{\mathcal{M}}^*] = 0$ and since the canonical bundle, $\mathcal{K}_{\mathcal{M}} = \det \mathcal{T}_{\mathcal{M}}^*$ is a line bundle, its total Chern class is $c[\mathcal{K}_{\mathcal{M}}] = 1$. In other words, $\mathcal{K}_{\mathcal{M}}$ is holomorphically trivial; $\mathcal{K}_{\mathcal{M}} \approx \mathbb{C}_{\mathcal{M}}$.

Using the splitting principle, the total Chern class of a complex projective space is obtained [23,19]

$$c(\mathbb{P}^n) = (1 + J)^{n+1}, \quad (1.3.21)$$

where $J = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ and is the cohomology class of the curvature two-form of the hyperplane bundle over \mathbb{P}^n . With $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(q)$, we have

$$c(\mathcal{E}) = 1 + qJ \quad (1.3.22)$$

by Eq. (1.3.15). Moreover, from the sequence (1.3.7) and (1.3.16),

$$c(\mathcal{M}) = c(\mathbb{P}^n)/c(\mathcal{E}) = \frac{(1 + J)^{n+1}}{(1 + qJ)}. \quad (1.3.23)$$

This somewhat symbolic expression acquires its full meaning when expanded in symbolic Taylor series where products of J are wedge-products so that the series terminates at the $J^{\dim \mathcal{M}}$ term, since all forms, i.e., cohomology classes in question are here restricted to \mathcal{M} .

The result (1.3.3) is now recovered in the case $r = 1$ of

$$c_r = \left[\sum_{k=0}^r \binom{n+1}{k} (-q)^{r-k} \right] J^r, \quad (1.3.24)$$

so that

$$c_1 = [(n+1) - q] J. \quad (1.3.25)$$

To conclude, Ricci-flatness is equivalent to the vanishing of the first Chern class and so we choose $q = n + 1$ for the degree- q hypersurface in \mathbb{P}^n to be a Calabi-Yau manifold \mathcal{M} . Being interested in $\dim \mathcal{M} = 3$, the choice of a quintic hypersurface in \mathbb{P}^4 ensues.

In addition to recovering (1.3.3), Eq. (1.3.24) offers some more information. Since c_p are closed (p, p) -forms, it is quite obvious that $(c_1 \wedge c_1 \wedge c_1)$, $(c_1 \wedge c_2)$ and (c_3) are all closed $(3, 3)$ -forms and may be integrated over \mathcal{M} to give the Chern numbers

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C_1 , C_2 and C_3 , respectively. By Stokes' formula, these numbers are independent of the choice of the particular differential forms c_r representing the Chern classes and are topological in nature. In particular, the Gauss-Bonnet formula III (see p. 307 of Ref. [19] or p. 415–416 of Ref. [23]) tells that $C_3(\mathcal{M}) = \chi_E(\mathcal{M})$. Having chosen $c_1(\mathcal{M}) = 0$, C_3 is in fact the only non-vanishing Chern number.

To evaluate C_3 , we need to integrate over \mathcal{M} and for that we invoke the following general result about integration over complex submanifolds:

Theorem 1.3 *Let \mathcal{M} be a complex submanifold of \mathcal{X} , $\dim \mathcal{M} = n$, $\dim \mathcal{X} = n+K$, with normal bundle \mathcal{N} , $\text{rank } \mathcal{N} = K$. Then there is a closed (K, K) -form μ on \mathcal{X} whose restriction to \mathcal{M} represents the top Chern class of the normal bundle, $c_K(\mathcal{N})$. Moreover, if ω is any closed (n, n) -form on \mathcal{X} , then*

$$\int_{\mathcal{M}} \omega = \int_{\mathcal{X}} \mu \wedge \omega. \quad (1.3.26)$$

That is, the closed form μ restricts integration from \mathcal{X} to \mathcal{M} like a delta function.

Remark.

If $\mathcal{N}_{\mathcal{X}/\mathcal{M}}$ is the restriction to \mathcal{M} of some bundle over \mathcal{X} , as in the present application, $\mathcal{N}_{\mathcal{X}/\mathcal{M}} = \mathcal{E}|_{\mathcal{M}}$, then the existence of μ is straightforward; $\mu = c_k[\mathcal{E}]$, for $k = \text{rank } \mathcal{E}$, so $\mu = qJ$ for $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(q)$. In general, μ represents the Poincaré dual of the homology class $[\mathcal{M}] \in H_{2m}(\mathcal{X})$.

We normalize (the first equality is known as the *Wirtinger theorem*)

$$\text{Vol}(\mathbb{P}^n) = \frac{1}{n!} \int_{\mathbb{P}^n} J^n = 1, \quad (1.3.27)$$

and compute, for the quintic in \mathbb{P}^4 ($\chi_E = C_3$ for a 3-fold)

$$\begin{aligned} \chi_E(\mathcal{M}) &= \int_{\mathcal{M}} c_3(\mathcal{M}), \\ &= \int_{\mathbb{P}^4} [5J] \wedge [-1 \cdot 125 + 5 \cdot 25 - 10 \cdot 5 + 10] J^3 = -200. \end{aligned} \quad (1.3.28)$$

1.4 Four More Easy Pieces

Having introduced Chern classes, we now pay a visit to the remaining four complete intersection Calabi-Yau manifolds in a single \mathbb{P}^n . We now have that

$$\mathcal{E} = \bigoplus_{a=1}^K \mathcal{E}_a, \quad q_a \stackrel{\text{def}}{=} \deg(\mathcal{E}_a), \quad (1.4.1)$$

where each \mathcal{E}_a , $a = 1, \dots, K$, is the bundle corresponding to a degree- q_a constraint polynomial $f^a(z)$, $z \in \mathbb{P}^n$. The direct sum, \mathcal{E} , therefore corresponds to the system of constraint polynomials. Each hypersurface in \mathbb{P}^n has dimension $n-1$, so the intersection of k hypersurfaces has dimension $n-k$. Upon using up all K constraints, we want to obtain for their intersection $\dim M = 3$, whence $K = n-3$.

We may use Bertini's theorem iteratively to conclude that, in this case also, the generic choice of constraints leads to a non-singular complex manifold. We simply order the constraints and interpret each succeeding constraint as a holomorphic section of an appropriate line bundle, restricted to the intersection of hypersurfaces defined by the previous constraints. The hypotheses of Bertini's theorem are satisfied at each stage.

However, to verify for a particular choice of constraints that the resulting intersection of hypersurfaces is smooth, it is necessary to check that there is no point at which all the constraints vanish together with the wedge product of their gradients

$$df^1(z) \wedge \cdots \wedge df^K(z). \quad (1.4.2)$$

This wedge product is the holomorphic volume form of the normal bundle and for it to vanish, either some of the gradients vanish or some (linear combinations) of them become parallel; clearly, this would detect a singularity at the resulting intersection. As in the case of a single constraint, this condition is independent of the coordinates used to check it.

Although at the time we derived the Adjunction formula (1.3.9), \mathcal{E} denoted a single line bundle, the argument is equally valid in this more general situation. Moreover, the basic results on Chern classes listed above lead easily to the generalization of the Chern polynomial formula (1.3.23),

$$c(\mathcal{M}) = \frac{(1+J)^{n+1}}{\prod_{a=1}^K (1+q_a J)}, \quad (1.4.3)$$

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and we obtain straightforwardly

$$c_1(\mathcal{M}) = n + 1 - \sum_{a=1}^K q_a. \quad (1.4.4)$$

For Ricci-flatness, i.e., vanishing first Chern class, we demand

$$\sum_{a=1}^K q_a = n + 1, \quad (1.4.5)$$

since J is a positive (1,1)-form. This yields precisely five solutions :

1. The quintic in \mathbb{P}^4 , denoted [4||5];
2. The intersection of two cubics in \mathbb{P}^5 , [5||33];
3. The intersection of a quadric and a quartic in \mathbb{P}^5 , [5||24];
4. The intersection of two quadrics and a cubic in \mathbb{P}^6 , [6||223];
5. The intersection of four quadrics in \mathbb{P}^7 , [7||2222].

We have here adopted the somewhat cryptic ‘bra-ket’ notation of Ref. [168] : the left-most, ‘bra’ entry is the dimension of \mathbb{P}^n and the remaining, ‘ket’ entries are the degrees of homogeneity of the constraint polynomials.

1.4.1 The Euler numbers

Expanding the generalized Chern polynomial formula, we obtain for these five manifolds

$$c(\mathcal{M}) = 1 + \frac{1}{2} \left[\left(\sum_{a=1}^K q_a^2 \right) - (n+1) \right] J^2 - \frac{1}{3} \left[\left(\sum_{a=1}^K q_a^3 \right) - (n+1) \right] J^3, \quad (1.4.6)$$

where we have used the $c_1 = 0$ requirement (1.4.5).

To evaluate the Euler characteristics, we begin by observing that

$$c_K(\mathcal{E}) = \prod_{a=1}^K \deg(\mathcal{E}_a J) = \left(\prod_{a=1}^K \deg \mathcal{E}_a \right) J^K, \quad (1.4.7)$$

and then invoke Eq. (1.3.26) to obtain

$$\int_{\mathcal{M}} (\) = \int_{\mathbb{P}^n} (J^K \prod_{a=1}^K q_a) \wedge (\), \quad (1.4.8)$$

with which we pull the integration over the submanifold \mathcal{M} back to the ambient space. Hence,

$$\begin{aligned} \chi_E[4||5] &= 5[\tfrac{1}{3}(5 - 125)] &= -200, \\ \chi_E[5||33] &= 3\cdot 3[\tfrac{1}{3}(6 - 27 - 27)] &= -144, \\ \chi_E[5||24] &= 2\cdot 4[\tfrac{1}{3}(6 - 8 - 64)] &= -176, \\ \chi_E[6||223] &= 2\cdot 2\cdot 3[\tfrac{1}{3}(7 - 8 - 8 - 27)] &= -144, \\ \chi_E[7||2222] &= 2\cdot 2\cdot 2\cdot 2[\tfrac{1}{3}(8 - 8 - 8 - 8 - 8)] &= -128. \end{aligned} \quad (1.4.9)$$

1.5 Characteristic Classes

In § 1.3.2, we have seen an example of what is known as characteristic classes. For future reference, we now briefly review some other examples; more details are found in Ref. [29].

We recall that the total Chern class of a rank- k vector bundle \mathcal{V} over a manifold \mathcal{M} is

$$c[\mathcal{V}] = \det [\mathbf{1} + \mathbf{F}], \quad (1.5.1)$$

where \mathbf{F} is the (suitably normalized) curvature $(1,1)$ -form of \mathcal{V} . Using the splitting principle (1.3.19),

$$c[\mathcal{V}] = \sum_{r=1}^k c_r[\mathcal{V}] = \prod_{i=1}^k (1 + \ell_i), \quad (1.5.2)$$

where

$$c_r[\mathcal{V}] = \sum_{i_1 < i_2 < \dots < i_r} \underbrace{\ell_{i_1} \wedge \ell_{i_2} \wedge \dots \wedge \ell_{i_r}}_{r \text{ times}} \quad (1.5.3)$$

is the r^{th} Chern class of \mathcal{V} and $c_r[\mathcal{V}] \equiv 0$ for $r > \dim \mathcal{M}$. ℓ_i is the first Chern class of the i^{th} line bundle \mathcal{L}_i into a sum of which we have deformed \mathcal{V} (even though $\mathcal{V} \neq \bigoplus_i \mathcal{L}_i$ holomorphically!).

Using the ℓ_i , one can define the *Chern character*^(L) to be

$$\text{ch}[\mathcal{V}] \stackrel{\text{def}}{=} \sum_{i=1}^k e^{\ell_i}. \quad (1.5.4)$$

It follows that

$$\text{ch}[\mathcal{V} \oplus \mathcal{V}'] = \text{ch}[\mathcal{V}] + \text{ch}[\mathcal{V}'], \quad (1.5.5)$$

$$\text{ch}[\mathcal{V} \otimes \mathcal{V}'] = \text{ch}[\mathcal{V}] \wedge \text{ch}[\mathcal{V}']. \quad (1.5.6)$$

1.5. Characteristic Classes

In general,

$$\text{ch}[\mathcal{V}] = k + \sum_{r=1}^k \text{ch}_r[\mathcal{V}] = k + c_1 + \tfrac{1}{2}(c_1^2 - 2c_2) + \tfrac{1}{6}(c_1^3 - 3c_1c_2 + c_3) + \dots \quad (1.5.7)$$

where $\text{ch}_r[\mathcal{V}] = \tfrac{1}{r!} S_r[\mathcal{V}]$. The Chern classes c_j and the symmetric polynomials S_r (see *invariant polynomials*^(L)) are related through the Newton formulae

$$\sum_{j=0}^r (-)^j c_j[\mathcal{V}] \wedge S_{r-j}[\mathcal{V}] = 0, \quad S_0 = k. \quad (1.5.8)$$

Again using the ℓ_i , one can define the *Todd class* as

$$\text{td}[\mathcal{V}] \stackrel{\text{def}}{=} \prod_{i=1}^k \frac{\ell_i}{1 - e^{-\ell_i}}. \quad (1.5.9)$$

It follows that

$$\text{td}[\mathcal{V} \oplus \mathcal{V}'] = \text{td}[\mathcal{V}] \wedge \text{td}[\mathcal{V}']. \quad (1.5.10)$$

In general,

$$\text{td}[\mathcal{V}] = 1 + \sum_{r=1}^{\infty} \text{td}_r[\mathcal{V}], \quad (1.5.11)$$

$$= 1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}[c_2 + c_1^2] + \tfrac{1}{24}[c_1c_2] + \dots \quad (1.5.12)$$

Furthermore, for any almost complex n -fold, we have also the *Pontrjagin classes* $p_r \in H^{4r}(\mathcal{M}, \mathbb{Z})$,

$$p \stackrel{\text{def}}{=} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-)^r p_r = \sum_{i,j=0}^n (-)^i c_i \wedge c_j, \quad (1.5.13)$$

so that*

$$p_1 = c_1^2 - 2c_2, \quad p_2 = c_2^2 - 2c_1c_3 + 2c_4, \dots \quad (1.5.14)$$

We will also find useful the Hirzebruch *L*-polynomial

$$L[\mathcal{V}] \stackrel{\text{def}}{=} \prod_{i=1}^k \frac{\ell_i}{\tanh(\ell_i)}, \quad (1.5.15)$$

*By a characteristic class of a manifold, we mean the characteristic class of its tangent bundle, which is customary to omit in notation if no confusion can arise.

for which the expression $(\mathcal{V} \rightarrow \mathcal{T}_M)$

$$L = \frac{1}{3}p_1 + \frac{1}{3^2 \cdot 5}(7p_2 - p_1^2) + \frac{1}{3^3 \cdot 5 \cdot 7}(62p_3 - 13p_2p_1 + 2p_1^3) + \dots \quad (1.5.16)$$

will suffice for our purposes; for further details, see Ref. [29].

Finally, for completeness, we give the so-called A -roof polynomial,

$$\hat{A}[\mathcal{V}] \stackrel{\text{def}}{=} \prod_{i=1}^k \frac{\frac{1}{2}\ell_i}{\sinh(\frac{1}{2}\ell_i)}, \quad (1.5.17)$$

$$= -\frac{2}{3}p_1 - \frac{2}{3^2 \cdot 5}(4p_2 - 7p_1^2) - \frac{2}{3^3 \cdot 5 \cdot 7}(16p_3 - 44p_2p_1 + 31p_1^3) + \dots \quad (1.5.18)$$

where the second row refers to $\mathcal{V} = \mathcal{T}_M$. \hat{A} is associated with the index of the Dirac operator.

1.6 The Lefschetz Hyperplane Theorem

In certain favorable cases, and certainly in the five examples above, $b_{1,1} = b_2$ can be computed using a generalization of the Lefschetz hyperplane theorem which requires only ‘rather topological’ data.

We quote here the Lefschetz hyperplane theorem in the form proven by R. Bott [10] :

Theorem 1.4 (Lefschetz-Bott) *Let \mathcal{L} be a positive holomorphic line bundle over an $n+1$ -dimensional complex compact manifold X and let λ be a holomorphic section of \mathcal{L} . Then :*

$$\begin{aligned} \pi_q(X) &\approx \pi_q(\lambda^{-1}(0)) & 0 \leq q < n = \dim \lambda^{-1}(0), \\ \pi_n(X) &\xrightarrow{j} \pi_n(\lambda^{-1}(0)) & \text{is onto.} \end{aligned} \quad (1.6.1)$$

In particular, we then have

Corollary 1.1

$$\begin{aligned} H_q(X, \mathbb{Z}) &\approx H_q(\lambda^{-1}(0), \mathbb{Z}) & 0 \leq q < n = \dim \lambda^{-1}(0), \\ H_n(X, \mathbb{Z}) &\xrightarrow{j} H_n(\lambda^{-1}(0), \mathbb{Z}) & \text{is onto.} \end{aligned} \quad (1.6.2)$$

Moreover, if $b_n(X) = b_n(\lambda^{-1}(0))$ the map j is also an isomorphism.

Complete intersection	Label	χ_E	$b_{1,1}$	$b_{2,1}$
a quintic in \mathbb{P}^4	[4 5]	-200	1	101
a quartic and a quadric in \mathbb{P}^5	[5 4 2]	-176	1	89
two cubics in \mathbb{P}^5	[5 3 3]	-144	1	73
a cubic and two quadrics in \mathbb{P}^6	[6 3 2 2]	-144	1	73
four quadrics in \mathbb{P}^7	[7 2 2 2 2]	-128	1	65

Table 1.1: The five standard Calabi-Yau complete intersections in \mathbb{P}^n and some of their numerical characteristics.

The application to the five cases above is straightforward. In each case, we have one or more polynomials, corresponding to a line bundle or a direct sum of line bundles. Each of these line bundles is positive since it is a positive power of the hyperplane bundle, which itself is positive.

Now we proceed by iteration : Take any of the line bundles to begin with and consider the hypersurface in \mathbb{P}^n defined as the zero-set of a polynomial corresponding to that line bundle. Since the k^{th} line bundle is positive over \mathbb{P}^n , it is also positive over the subspace which is the complete intersection of the previous $k-1$ hypersurfaces and so Lefschetz hyperplane theorem applies again and guarantees $H^{p,q}(\mathcal{M}_k, \mathbb{Z}) = H^{p,q}(\mathbb{P}^n, \mathbb{Z})$ for $p+q \neq (n-k)$. Here \mathcal{M}_k denotes the complete intersection of the first k hypersurfaces; clearly, $\mathcal{M} = \mathcal{M}_{n-3}$. Upon including all $(n-3)$ constraints, we obtain

$$H^{p,q}(\mathcal{M}, \mathbb{Z}) = H^{p,q}(\mathbb{P}^n, \mathbb{Z}), \quad p+q \neq 3. \quad (1.6.3)$$

This certainly agrees with our general form of the Hodge diamond and moreover tells that $b_{1,1} = 1$ for all five above cases. From the above relation (1.2.15) we have that

$$b_{2,1} = b_{1,1} - \frac{1}{2}\chi_E = 1 - \frac{1}{2}\chi_E \quad (1.6.4)$$

for the five Calabi-Yau manifolds as summarized in Table 1.1.

1.7 Onward into the Jungle !

We shall soon generalize the five simple constructions of this chapter in several different aspects. In the next chapter, we consider complete intersections in a product of several complex projective spaces rather than a single one. In later chapters, we shall consider complete intersections in (products of) other complex spaces but also constructions which rely on various surgical techniques.

In favorable cases of more general complete intersections, a simple iterative application of the Lefschetz hyperplane theorem will provide $b_{1,1}$ and, through the independent computation of χ_E , the only other variant Hodge number, $b_{2,1}$. Unfortunately, more often than not—in the general case—this algorithm will fail to work and we will need to consider different techniques.

We shall then spend considerable time in describing the application of spectral sequences which yield a much more complete description of any particular cohomology group on any complete intersection in any product of so-called ‘flag spaces’ (of which complex projective spaces are the simplest cases). We shall also discuss the effect of dividing out group actions, i.e., passing to the quotient with respect to the holomorphic action of some symmetry group. All this builds up towards computing the certain overlap integrals which are, in corresponding compactification models, interpreted as couplings among the various light fields.

Finally, Calabi-Yau spaces in general may be endowed with various choices of the complex structure and Kähler class and the physics interpretation of such a model does depend on these choices. In the latter part of the book, we shall discuss the geometry of the space of complex structures and the complexified Kähler cone.

Chapter 2

Complete Intersections in Products of Projective Spaces

The five constructions described in the previous chapter may be thought of as toy models. In this chapter, we study a large number of examples—each defined as a complete intersection of hypersurfaces in a product of projective spaces.

2.1 Construction

Extending the ideas of the previous chapter, we now consider embedding Calabi-Yau manifolds into a general product

$$\mathcal{X} \stackrel{\text{def}}{=} \mathbb{P}_1^{n_1} \times \cdots \times \mathbb{P}_m^{n_m}. \quad (2.1.1)$$

One may consider generalizing this strategy further by including other spaces in the product. On one hand, weighted (anisotropic, so to speak) complex projective spaces provide a rich family of constructions; already as transverse hypersurfaces in a single weighted \mathbb{P}^4 , there are some 6000 Calabi-Yau 3-folds [96]. Including on the other hand Grassmannians or, even more so, general flag spaces generalizes the embedding space \mathcal{X} in yet another way. Furthermore, embeddings other than complete intersections can also be discussed (see § 3.5).

We shall return to some of these options later on; for now, we focus on complete intersection Calabi-Yau 3-folds. Each of these is well characterized by a so-called *configuration* and we now turn to describe these.

2.1.1 Configurations

A *complete intersection manifold* is defined to be a manifold \mathcal{M} embedded as complete intersection of hypersurfaces

$$\mathcal{M} \stackrel{\text{def}}{=} X^1 \cap \cdots \cap X^K \hookrightarrow \mathcal{X} \quad (2.1.2)$$

in the embedding space \mathcal{X} . Each hypersurface X^a is defined as the zero locus of a suitably chosen holomorphic polynomial :

$$X^a : \xi^a(z_{(1)}, \dots, z_{(m)}) = 0, \quad a = 1, \dots, K, \quad (2.1.3)$$

which is homogeneous of degree q_a^r with respect to the homogeneous coordinates $\vec{z}_{(r)}$ of $\mathbb{P}_r^{n_r}$.

Each homogeneous polynomial ξ^a may also be thought of as a holomorphic section of the line bundle

$$\mathcal{E}_a \stackrel{\text{def}}{=} \bigotimes_r \mathcal{O}_r(q_a^r). \quad (2.1.4)$$

Here $\mathcal{O}_r(1)$ denotes the hyperplane (line) bundle over $\mathbb{P}_r^{n_r}$. In other words, the bundle of hyperplanes each of which is defined by a linear constraint $\lambda_\mu z_{(r)}^\mu = 0$, where $z_{(r)}^\mu$ are homogeneous coordinates on $\mathbb{P}_r^{n_r}$ and $\lambda_\mu \in \mathbb{C}$ parametrize the hyperplane.



Definitions and notation. It will be convenient to define a *configuration matrix* as a pair $[n||q]$ of an m -dimensional positive integer valued column vector n_r and an $m \times K$ -dimensional non-negative integer valued matrix q_a^r . We shall denote it by

$$\left[\begin{array}{c|ccc} n_1 & q_1^1 & \cdots & q_K^1 \\ \vdots & \vdots & \ddots & \vdots \\ n_m & q_1^m & \cdots & q_K^m \end{array} \right]. \quad (2.1.5)$$

A similar notation,

$$\begin{aligned} \mathbb{P}^{n_1} & \left[\begin{array}{ccc} q_1^1 & \cdots & q_K^1 \end{array} \right] \\ \vdots & \vdots \\ \mathbb{P}^{n_m} & \left[\begin{array}{ccc} q_1^m & \cdots & q_K^m \end{array} \right] \end{aligned} \quad (2.1.6)$$

is also found in the literature. While the embedding space is a product of only complex projective spaces, the former notation is sufficient and we adhere to it.

2.1. Construction

Otherwise, we shall list the factors of \mathcal{X} explicitly, akin to the latter variant. More generally, the *configuration* will be denoted as the pair $[\mathcal{X}||\mathcal{E}]$, where \mathcal{E} is the vector bundle over \mathcal{X} the sections of which form the defining system of constraints for the embedded 3-fold \mathcal{M} .

A configuration matrix specifies the embedding space \mathcal{X} and the degrees of homogeneity of the defining polynomials, but does not specify the polynomials otherwise. In particular, e.g., the configuration matrix of the Tian-Yau manifold

$$\left[\begin{array}{c|cc} 3 & 3 & 0 & 1 \\ 3 & 0 & 3 & 1 \end{array} \right], \quad (2.1.7)$$

represents the system

$$f_{abc}x^a x^b x^c = 0, \quad g_{\alpha\beta\gamma}y^\alpha y^\beta y^\gamma = 0, \quad h_{\alpha\alpha}x^\alpha y^\alpha = 0, \quad (2.1.8)$$

but does not specify the coefficients f_{abc} , $g_{\alpha\beta\gamma}$ and $h_{\alpha\alpha}$. Such coefficients occur in every constraint and remain undetermined by the configuration matrix.

As we shall discuss below, for each configuration, almost every choice of the coefficients in the defining polynomials leads to a smooth manifold and the space spanned by the coefficients has a single, connected component. Two different sets of coefficients (i.e., points in this space) correspond to two complete intersections which are in general different as complex manifolds. The space of these coefficients is therefore a parameter space for the complete intersection manifolds.

By a *configuration*, we shall mean the family of all complete intersections (whether manifolds or singular) defined by a system of polynomial constraints which are represented by a configuration matrix and parametrized by the space of coefficients. This collection of varieties forms a deformation class. We represent the configuration by the configuration matrix and for a particular *member* of the configuration, a complete intersection space \mathcal{M} , we write

$$\mathcal{M} \in \left[\begin{array}{c|ccc} n_1 & q_1^1 & \cdots & q_K^1 \\ \vdots & \vdots & \ddots & \vdots \\ n_m & q_1^m & \cdots & q_K^m \end{array} \right] \quad (2.1.9)$$

and say that \mathcal{M} belongs to the configuration. Clearly,

$$\dim \mathcal{M} = \dim \mathcal{X} - K, \quad \dim \mathcal{X} = \sum_{r=1}^m n_r. \quad (2.1.10)$$

Consider a configuration where the \mathbf{q} matrix is block-diagonal

$$[\mathbf{n} \parallel \mathbf{q}] = \begin{bmatrix} \mathbf{n}' & \mathbf{q}' & \mathbf{0} \\ \mathbf{n}'' & \mathbf{0} & \mathbf{q}'' \end{bmatrix}. \quad (2.1.11)$$

It is rather obvious that $[\mathbf{n} \parallel \mathbf{q}]$ describes a class of product spaces of the form $\mathcal{M} = \mathcal{M}' \times \mathcal{M}''$ and where

$$\mathcal{M}' \in [\mathbf{n}' \parallel \mathbf{q}'] , \quad \mathcal{M}'' \in [\mathbf{n}'' \parallel \mathbf{q}''] . \quad (2.1.12)$$

A configuration $[\mathbf{n} \parallel \mathbf{q}]$ is called *decomposable* if it can be brought into a block-diagonal form by permuting its rows and the columns of \mathbf{q} ; otherwise we call it *indecomposable*.

Finally, note that a linear constraint in \mathbb{P}^n defines a $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ and is therefore uninteresting. Consequently, we require the degree of a constraint polynomial to be at least two if it depends on one $\mathbb{P}_r^{n_r}$ only. This is easily seen to be equivalent to the condition

$$\sum_{r=1}^m q_a^r \geq 2 , \quad a = 1, \dots, K \quad (2.1.13)$$

and we require all configuration matrices to satisfy this.

Ricci-flatness. Since the Chern polynomial is determined by the dimensions of the \mathbb{P}^n factors in X and the degrees of homogeneity of the constraints, we can write a general formula for each configuration. Straightforwardly, $\mathcal{T}_{\mathcal{M}}$ is a sub-bundle of \mathcal{T}_X , the holomorphic tangent bundle of the embedding space and an analogue of the *Adjunction formula*^(L) 2, (1.3.7), implies the exactness of

$$0 \rightarrow \mathcal{T}_{\mathcal{M}} \xrightarrow{i} \mathcal{T}_X|_{\mathcal{M}} \xrightarrow{\phi} \mathcal{E}|_{\mathcal{M}} \rightarrow 0 , \quad (2.1.14)$$

From this short exact sequence, the determinant formula (1.3.16) yields

$$\begin{aligned} c[X \parallel \mathcal{E}] &= c[X]/c[\mathcal{E}]_-, \\ c[\mathbf{n} \parallel \mathbf{q}] &= \frac{\prod_{r=1}^m (1 + J_r)^{n_r+1}}{\prod_{a=1}^K (1 + \sum_{s=1}^m q_a^s J_s)} . \end{aligned} \quad (2.1.15)$$

On expansion, the first Chern class is

$$c_1[\mathbf{n} \parallel \mathbf{q}] = \sum_{r=1}^m \left(n_r + 1 - \sum_{a=1}^K q_a^r \right) J_r . \quad (2.1.16)$$

2.1. Construction

Since each J_r is a positive (1,1)-form over $\mathbb{P}_r^{n_r}$, the only way to make the first Chern class of the configuration to vanish is to set

$$\sum_{a=1}^K q_a^r = n_r + 1 , \quad \forall r = 1, \dots, m . \quad (2.1.17)$$

This ensures that each smooth member of the configuration $[\mathbf{n} \parallel \mathbf{q}]$ admits a Ricci-flat Kähler metric (this can be suitably generalized for singular members).



Often, we shall need to consider slightly less restricted configurations, for which the

$$\sum_{a=1}^K q_a^r \leq n_r + 1 , \quad r = 1, \dots, m . \quad (2.1.18)$$

condition holds instead. If the inequality is strict for at least one r , a member \mathcal{M} of such a configuration has a non-trivial anti-canonical bundle, $\mathcal{K}_{\mathcal{M}}^*$, which is generated by global sections. Generalizing earlier results,

$$\mathcal{K}_{\mathcal{M}}^* = \wedge^3 \mathcal{T}_{\mathcal{M}} = \bigotimes_{r=1}^m \mathcal{O}_r(\sum_{a=1}^K q_a^r) . \quad (2.1.19)$$

Configurations for which Eq. (2.1.18) holds, with the inequality strict for at least one r , represent complete intersections that do not admit Ricci-flat metrics. Rather, the Ricci tensor is non-negative : it is positive along the $\mathbb{P}_r^{n_r}$ for which the inequality in Eq. (2.1.18) is strict, and vanishes (modulo total derivatives) along the $\mathbb{P}_r^{n_r}$ for which Eq. (2.1.18) becomes an equality. In general, we will adhere to the taxonomy of configurations as presented in Table 2.1. To con-

Configuration	Condition on $[\mathbf{n} \parallel \mathbf{q}]$
Calabi-Yau	$\sum_{a=1}^K q_a^r = n_r + 1 , \quad r = 1, \dots, m$
Almost-Ample	$\sum_{a=1}^K q_a^r \leq n_r + 1 , \quad r = 1, \dots, m$ (" < " for at least one r)
Ample	$\sum_{a=1}^K q_a^r < n_r + 1 , \quad r = 1, \dots, m$

Table 2.1: A partial taxonomy of configurations.

form with existing nomenclature for two and three dimensions, respectively, we

may say “(almost) del Pezzo” and “(almost) Fano” instead of “(almost) ample”. Clearly, with $n = \dim \mathcal{M}$,

$$H^0(\mathcal{M}, \mathcal{K}_{\mathcal{M}}) = H^{n,0}(\mathcal{M}) \approx \begin{cases} \mathbb{C} & \mathcal{M} \text{ is Calabi-Yau ,} \\ 0 & \mathcal{M} \text{ is almost ample;} \end{cases} \quad (2.1.20)$$

precisely Calabi-Yau n -folds have a nowhere vanishing holomorphic $(n,0)$ -form.

Of course, there exist configurations for which $[n_r + 1 - \sum_{a=1}^K q_a^r]$ is positive for some r , negative for some other r and possibly zero for some third r . As opposed to ample configurations, these might be called *limp*. On the other hand, if $[n_r + 1 - \sum_{a=1}^K q_a^r] \leq 0$, the configuration might be called *almost scant* (*scant* if the inequivalence is strict). When the canonical bundle $\mathcal{K}_{\mathcal{M}}$ is positive and non-trivial, $\dim H^0(\mathcal{M}, \mathcal{K}_{\mathcal{M}}) > 1$, there are more than one $(n,0)$ -forms and, in general, each of them vanishes somewhere on \mathcal{M} .

2.1.2 Diagrams

We shall find it useful to introduce a diagrammatic rendition of the configuration matrices representing the deformation classes of complete intersection embeddings.

For every \mathbb{P}^n factor in \mathcal{X} we introduce a hollow circle with $n+1$ legs. For every constraint we introduce a dot. We connect some (possibly all) of the free legs into the dots. The requirement (2.1.13) obviously translates into the rule that at least two legs be connected into every dot.

Decomposable configurations correspond to disconnected diagrams with more than one connected part while indecomposable configurations have diagrams that are fully connected. Unless stated otherwise, we consider only indecomposable configurations and therefore only connected diagrams.

The dimension of the members of the configuration represented by a diagram is $\ell - m - K$, where ℓ is the number of legs, m is the number of circles and K is the number of dots. We shall also refer to $(\ell - m - K)$ as the dimension of the diagram; it is always non-negative since $\ell \geq 2m$ and $\ell \geq 2K$.

A diagram represents a Calabi-Yau configuration (and we call it a Calabi-Yau diagram) precisely if there are no loose legs. If there is at least one loose leg, the diagram corresponds to an almost ample configuration (and we call it almost ample). If every hollow circle has at least one leg loose, the diagram corresponds to an ample configuration (and we call the diagram ample also).

2.1. Construction

It will also be useful to distinguish the following two special types of diagrams (configurations) :

Decomposing dots. A diagram is called *one-dot-decomposable*, denoted 1DD, if it breaks in two upon the deletion of some dot.

Proposition 2.1 *The complement of a decomposing dot in a diagram of dimension n has two components which represent (possibly singular) varieties of positive dimensions, which add up to $n+1$.*

☞ Prove this Proposition. ↩

We index a decomposing dot by these dimensions and refer to it as an x - y -dot, where x and y are the dimensions of the two components of the complement of the decomposing dot. When the whole diagram corresponds to 3-folds, there can only be decomposing 1-3- or 2-2-dots.

In a 1DD diagram with a decomposing x - y -dot, the decomposing dot corresponds to the defining constraint of a hypersurface in the product of two spaces of dimensions x and y , respectively.

Decomposing legs. A diagram is called *one-leg-decomposable*, denoted 1LD, if it breaks in two upon the deletion of a leg. Clearly, every 1LD diagram is also 1DD, since the decomposing leg connects to a dot, the removal of which also disconnects the diagram.

The complement of a decomposing leg in a diagram representing an n -fold has two components containing, respectively, the dot and the hollow circle incident with the deleted leg. We shall call these respectively as the dot-complement and the circle-complement of the leg in question. We write ℓ_d , m_d and K_d for the number of legs, circles and dots in the dot-complement and ℓ_c , m_c and K_c for the corresponding numbers in the circle-complement. We index the leg by $(\ell_d - m_d - K_d)$.

Proposition 2.2 *The index of a decomposing leg in a diagram representing an n -fold may be $0, \dots, n-1$.*

☞ Prove this Proposition. ↩

Lemma 2.1 Every Calabi-Yau diagram representing an n -fold with a decomposing $(n-1)$ -leg is equivalent to another one with no decomposing $(n-1)$ -leg which is obtained by replacing the circle-complement of the decomposing $(n-1)$ -leg with a single circle and the decomposing leg with two legs. Diagrammatically :

$$\text{Diagram showing two configurations connected by } \cong: \quad \text{Diagram 1} \cong \text{Diagram 2} . \quad (2.1.21)$$

Proof: Consider the circle-complement of the decomposing $n-1$ -leg together with the leg. This is one of the two components of the complement of the dot incident with the $n-1$ -leg; call it F . By Proposition 2.1, F is one of the two connected components of the complement of the dot incident with the decomposing leg and represents (possibly singular) varieties of positive dimension d . We next have

$$\begin{aligned} n &= \ell - m - K, \\ &= (\ell_d - m_d - K_d) + 1 + (\ell_c - m_c - K_c), \\ &= (n-1) + (\ell_F - m_F - K_F), \end{aligned} \quad (2.1.22)$$

since $\ell_d - m_d - K_d = n-1$ is used to index the leg. It follows that

$$d_F = \ell_F - m_F - K_F = 1. \quad (2.1.23)$$

Since the complete diagram had no loose legs, F has precisely one—the decomposing one. So, F is an almost ample 1-fold and can only be a \mathbb{P}^1 . \square

Whenever possible, we use Lemma 2.1 to prune the diagrams and simplify the configurations. We also have the following easy corollary.

Corollary 2.1 The dot-complement of a 0-leg can without loss of generality be reduced to

$$\text{Diagram showing a dot connected to two circles.} \quad (2.1.24)$$

Therefore

$$\text{Diagram showing a dot connected to two circles. The left circle has a small circle inside it, labeled 0. Diagram 2: A Calabi-Yau diagram with a central dot connected to two circles. The left circle has a small circle inside it, and the right circle has a small circle inside it.} \cong \text{Diagram 2} . \quad (2.1.25)$$

Prove this Corollary. \blacksquare

2.1. Construction

Remark.

Note that, by swapping dots and circles, a Calabi-Yau diagram becomes another Calabi-Yau diagram. This corresponds to transposing the matrix \mathbf{q} in $[\mathbf{n} \parallel \mathbf{q}]$ and changing \mathbf{n} as appropriate for a Calabi-Yau configuration.

2.1.3 The Euler characteristic

Having obtained the total Chern class (2.1.15) for a generic configuration,

$$c[\mathbf{n} \parallel \mathbf{q}] = c_1^r J_r + c_2^{rs} J_r J_s + c_3^{rst} J_r J_s J_t , \quad (2.1.26)$$

we can list the three Chern classes :

$$c_1^r = \left[n_r + 1 - \sum_{a=1}^K q_a^r \right] = 0 , \quad (2.1.27)$$

$$c_2^{rs} = \frac{1}{2} \left[-\delta^{rs} (n_r + 1) + \sum_{a=1}^K q_a^r q_a^s + c_1^r c_1^s \right] , \quad (2.1.27)$$

$$c_3^{rst} = \frac{1}{6} \left[\delta^{rst} (n_r + 1) - \sum_{a=1}^K q_a^r q_a^s q_a^t - c_2^{rs} c_1^t + c_1^r c_1^s c_1^t \right] , \quad (2.1.28)$$

where the parenthesis around indices denotes symmetrization. Note the simplification which occurs for Calabi-Yau configurations.

The Euler characteristic is evaluated by finding the coefficient of $\Lambda_{r=1}^m J_r^{n_r}$, the volume form of X , in

$$\left[\left(\sum_{r,s,t=1}^m c_3^{rst} J_r J_s J_t \right) \cdot \bigwedge_{a=1}^K \left(\sum_{p=1}^m q_a^p J_p \right) \right] . \quad (2.1.29)$$

Imposing $c_1^r = 0$ for Calabi-Yau complete intersections, we obtain

$$\chi_E[\mathbf{n} \parallel \mathbf{q}] = \left[\sum_{r,s,t=1}^m \frac{1}{3} \left(\delta^{rst} (n_r + 1) - \sum_{a=1}^K q_a^r q_a^s q_a^t \right) J_r J_s J_t \cdot \bigwedge_{a=1}^K \left(\sum_{p=1}^m q_a^p J_p \right) \right]_{\text{top}} , \quad (2.1.30)$$

where the subscript “top” means the coefficient of $\prod_{r=1}^m J_r^{n_r}$. This expression gets out of hand rather quickly for large configurations but is fortunately well suited for mechanized evaluation. Note that the $c_1^r = 0$ condition also implies that $(n_r+1) \leq \sum_a (q_a^r)^3$, whence $\chi_E[\mathbf{n} \parallel \mathbf{q}] \leq 0$.

The argument of §1.3.1 from the previous chapter can be repeated with only minor modifications :

$$\mathcal{N}_{X/\mathcal{M}} = \mathcal{E} \stackrel{\text{def}}{=} \bigoplus_{a=1}^K \mathcal{E}_a = \bigoplus_{a=1}^K \left(\bigotimes_{r=1}^m \mathcal{O}_r(q_a^r) \right) \quad (2.1.31)$$

is the normal bundle of the embedding $\xi: \mathcal{M} \hookrightarrow X$ where $\xi \stackrel{\text{def}}{=} \bigoplus_a \xi_a$ is the collection of defining constraint polynomials and so a section of \mathcal{E} and $\mathcal{M} = \xi^{-1}(0)$.

2.2 Bertini Variations

The method of constructing Calabi-Yau 3-folds described so far yields a very rich laboratory* and it is natural to inquire if these 3-folds are smooth.

2.2.1 A simple warm-up example

We wish to prove that there exist choices of defining polynomials $\{\xi^a\}$ in $X = \prod_r \mathbb{P}_r^{n_r}$ with degrees prescribed by some configuration matrix (2.1.5) and such the subspace $\mathcal{M} \hookrightarrow X$ where $\xi^a = 0$, $a = 1, \dots, K$, is smooth. This amounts to verifying that the subspace \mathcal{M} has a system of linearly independent ($\dim X - \dim \mathcal{M}$) gradients at every point. In other words, we require

$$d^K \mathcal{N}_{X/\mathcal{M}} = \bigwedge_{a=1}^K d\xi^a(z) \neq 0, \quad \xi^a(z) = 0, \quad z \in \mathcal{M}. \quad (2.2.1)$$

Take, e.g., the configuration [4|5] of quintics in \mathbb{P}^4 . In view of Euler's theorem on homogeneous functions

$$z^a(\partial_a \xi) = 5\xi, \quad z^a \in \mathbb{P}^4, \quad (2.2.2)$$

the five derivatives in $d\xi = dz^a(\partial_a \xi)$ are not independent of ξ . Requiring the derivatives $d\xi$ to vanish where the defining polynomial ξ does imposes only four additional constraints which are independent for a general choice of ξ . This adds up to a system of five independent constraints in \mathbb{P}^4 , is overdetermined and has no solution in general. Precisely this is the essence of the general argument; to gain familiarity with this, we shall briefly re-examine the present example and then formulate the general theorem.



*Clearly, spaces of other dimensions or other types of canonical bundles are constructed as easily; see Table 2.1 on page 51 and subsequent discussion.

2.2. Bertini Variations

The most general homogeneous quintic in \mathbb{P}^4 can be written as

$$\xi(z; f) = f_{abcde} z^a z^b z^c z^d z^e \quad (2.2.3)$$

and has $\binom{4+5}{5} = 126$ coefficients f_{abcde} which must not vanish simultaneously. By homogeneity of $\xi(z; f)$, we may rescale all f_{abcde} 's by an arbitrary complex number, so that $\{f_{abcde}\}$ span a \mathbb{P}^{125} . Furthermore, (projective) linear coordinate reparametrizations of the embedding space \mathbb{P}^4 , $z^a \rightarrow \lambda_a^{a'} z^{a'}$, cannot change the locus of $\xi(z; f) = 0$. Therefore, we consider

$$f_{abcde} \simeq \lambda_a^{a'} \lambda_b^{b'} \lambda_c^{c'} \lambda_d^{d'} \lambda_e^{e'} f_{a'b'c'd'e'}, \quad \lambda_a^{a'} \in \mathrm{PGL}(5, \mathbb{C}). \quad (2.2.4)$$

Thus, the effective parameter space is

$$\mathfrak{M} \stackrel{\text{def}}{=} \left\{ \{f_{abcde}\} / \mathrm{PGL}(5, \mathbb{C}) \right\}, \quad (2.2.5)$$

since $\mathrm{PGL}(5, \mathbb{C})$ is the group of linear coordinate reparametrizations of the embedding space \mathbb{P}^4 . Clearly, $\dim \mathfrak{M} = 101$.

More accurately, the action of the (projective) general linear group of linear transformations, $z^a \rightarrow \lambda_a^{a'} z^{a'}$, has been *lifted* in Eq. (2.2.4) to the parameter space $\{f_{abcde}\}$ and is manifestly non-linear. The structure of the quotient with respect to the equivalence (2.2.4) is therefore quite complicated and it is straightforward that the resulting orbit space is badly singular and rather difficult to study.

At any particular point, say $z = (1, 0, 0, 0, 0) \in \mathbb{P}^4$, the condition $\xi(z; f) = 0$ may now be viewed as a *linear* constraint on \mathfrak{M} , defining a hyperplane $\mathfrak{M}_z \subset \mathfrak{M}$ of codimension^(L) one ($\dim \mathfrak{M} - \dim \mathfrak{M}_z = 1$). Similarly, the five constraints[†] $\partial_a \xi(z; f) = 0$ describe the intersection \mathfrak{M}_z^\sharp of five hyperplanes, so $\dim \mathfrak{M} - \dim \mathfrak{M}_z^\sharp = 5$. Since $z^a \partial_a \xi(z; f) = 5\xi(z; f)$ by homogeneity, we know that $\mathfrak{M}_z^\sharp \subset \mathfrak{M}_z$.

Sweeping now z through all of \mathbb{P}^4 , the hyperplane \mathfrak{M}_z moves about in \mathfrak{M} and in fact covers it all. To see this, note that for each $f_{abcde} \in \mathfrak{M}$, $\xi(z; f) = 0$ does have a solution in \mathbb{P}^4 . Likewise, sweeping z through all of \mathbb{P}^4 , the intersection of hyperplanes \mathfrak{M}_z^\sharp sweeps out a four parameter family $\mathfrak{M}^\sharp \subset \mathfrak{M}$. Assuming that the five hyperplanes $\partial_a \xi(z; f) = 0$ were in general position, we have that

$$\dim \mathfrak{M}^\sharp \leq \dim \mathfrak{M}_z^\sharp + \dim \mathbb{P}^4 \quad (2.2.6)$$

[†]Recall that we need no connection for the gradients of ξ_a at the subspace, since $\xi_a = 0$ and $\nabla_a \xi = \partial_a \xi + \Gamma_a \xi = \partial_a \xi$.

and therefore

$$\dim \mathfrak{M} - \dim \mathfrak{M}^\sharp \geq \dim \mathfrak{M} - (\dim \mathfrak{M}_z^\sharp + \dim \mathbb{P}^4) = 5 - 4 = 1. \quad (2.2.7)$$

Thus, \mathfrak{M}^\sharp is of strictly smaller complex dimension than \mathfrak{M} .

For the varieties belonging to the above configuration, we can draw some important conclusions :

1. The space of smooth varieties $\mathfrak{M} - \mathfrak{M}^\sharp$ is dense and open. In other words, most choices of $f_{abcde} \in \mathfrak{M}$ correspond to smooth subspaces $\mathcal{M} \subset \mathcal{X}$.
2. Since $\dim_{\mathbb{R}} \mathfrak{M} - \dim_{\mathbb{R}} \mathfrak{M}^\sharp = 2$, for every point in $t^\sharp \in \mathfrak{M}^\sharp$, there exists at least one complex disk $\mathcal{D} \subset \mathfrak{M}$ such that only $t^\sharp \in \mathcal{D}$ corresponds to a singular $\mathcal{M} \subset \mathcal{X}$; all other points in \mathcal{D} correspond to smooth \mathcal{M} 's.
3. Any two smooth subspaces $\mathcal{M} \subset \mathcal{X}$ may be deformed into each other continuously, ranging over smooth subspaces only.
4. A configuration defines a deformations class of *embeddings* $\mathcal{M} \hookrightarrow \mathcal{X}$, only a set of positive *codimension*^(L) of which consists of singular forms of \mathcal{M} .

Note that the deformation space of the abstract complex manifold \mathcal{M} may well be larger than this deformation space if its embedding.

2.2.2 Every configuration contains smooth varieties

It should be clear that none of the above four conclusions depend on the particular choice of the configuration (and had nothing to do with requiring the first Chern class to vanish), wherefore one expects them to hold for any complete intersection.

Indeed, the following general theorem is easily generalized to fit our purpose.

Theorem 2.1 (Bertini) *A generic element of a linear system is smooth away from the base locus of the system.*

The base locus of the system consists of points in the embedding space where *all* the admissible (systems of) defining equations vanish. It must be noted that not infrequently, a system is smooth also at the base locus—however, that must be checked case by case.

As the Reader may be unfamiliar with the notion of the base locus and since Bertini's theorem can be applied in many different variations, it is worth taking a

2.2. Bertini Variations

closer look at it. Consider the case of a *generic* quintic in \mathbb{P}^4 . Writing it in the form (2.2.3), it is easy to count 126 parameters, of which 101 are independent as we showed above. The base locus for this *generic* quintic consists of the set of points in \mathbb{P}^4 where the quintic vanishes for any choice of f_{abcde} . Equivalently, there is 101 independent quintic polynomials and the base locus is where they all vanish simultaneously—clearly an empty set. So the quintic (2.2.3) in \mathbb{P}^4 is smooth except (perhaps) for some special values of f_{abcde} .

The situation changes drastically if we require our quintic to have some special form, usually dictated by some symmetry. To see this, let us restrict to some specially symmetric quintics. Consider symmetries with the diagonal action

$$\pi_\omega^{[d_0, \dots, d_4]} : z^a \rightarrow \omega^{d_a} z^a, \quad a = 0, \dots, 4. \quad (2.2.8)$$

Let us restrict to $\mathbb{Z}'_5 \ltimes \mathbb{Z}_5^3$ -symmetric quintics where the cyclic permutation symmetry \mathbb{Z}'_5 acts by swapping the coordinates z^a and \mathbb{Z}_5^3 is generated by $\pi_\alpha^{[1,0,0,0,4]}$, $\pi_\beta^{[0,1,0,0,4]}$ and $\pi_\gamma^{[0,0,1,0,4]}$ where α, β, γ are primitive fifth roots of 1. The only two admissible quintic polynomials under these restrictions are

$$F(z) \stackrel{\text{def}}{=} \sum_{a=1}^5 (z^a)^5, \quad \Psi(z) \stackrel{\text{def}}{=} \prod_{a=0}^4 z^a. \quad (2.2.9)$$

The base locus of $\mathbb{Z}'_5 \ltimes \mathbb{Z}_5^3$ -symmetric quintics is then the subspace of \mathbb{P}^4 where $F(z) = 0 = \Psi(z)$, which is a set of five quintic 2-folds intersecting in ten quintic curves which in turn meet in ten points[†]. Bertini's theorem tells that the quintic hypersurface

$$F(z) - 5\psi \Psi(z) = 0 \quad (2.2.10)$$

may be singular only (1) at the base locus and arbitrary ψ or (2) away from the base locus but only for some special ψ . Explicit analysis of the system of gradients shows that such quintics are in fact smooth at the base locus, for general ψ . The only singularities occur at $\psi^5 = 1$ and $\psi = \infty$.

Changing our symmetry requirements by adding an *R*-symmetry such as $\pi_R^{[1,0,0,0,0]}$, only $F(z)$ becomes admissible. Alternatively, requiring a $\mathbb{Z}'_5 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_3^3)$ symmetry, where the additional \mathbb{Z}_2 acts by partial reflection : $(z^1, z^2) \rightarrow (-z^1, -z^2)$, permits only $\Psi(z)$. In both cases with only one polynomial at hand, we cannot move away from the base locus and Bertini's theorem tells nothing. Explicit analysis of the system of gradients shows that $F(z) = 0$ is smooth everywhere, while $\Psi(z) = 0$ is (rather badly) singular.



[†]Hint : $\Psi(z)$ vanishes where at least one $z^a = 0$ and so consists of five \mathbb{P}^3 's, intersecting in ten \mathbb{P}^2 's which meet in ten \mathbb{P}^1 's which in turn have five points in common.

Since the set of choices of constraints leading to non-singular Calabi-Yau manifolds is an open connected subset of all possible choices of constraints for a given configuration, we have

Corollary 2.2 *All manifolds belonging to a given configuration are equivalent by deformation.*

By common abuse of language, we will say that a system of defining equations is smooth if it defines a smooth subspace. The above corollary then says that all choices of smooth defining equations corresponding to a single configuration matrix can be varied one into another continuously and moreover without ever forcing the defining equations to become singular.

2.3 Finiteness of the Family

We have seen that every configuration with $\dim \mathcal{X} - K = n$ and which satisfies Eq. (2.1.17) represents a (partial) deformation class of smooth Calabi-Yau n -folds, which are therefore topologically equivalent. The question of how many complete intersection Calabi-Yau 3-folds exist then reduces to finding out how many configuration matrices there are.

2.3.1 Infinitely many configurations

A simple algorithm for this task is as follows. It is clear that two configuration matrices $[\mathbf{n} \parallel \mathbf{q}]$ that differ by a permutation of rows or a permutation of the columns of \mathbf{q} describe the same configuration. To obtain all possible configuration matrices for a particular choice of its size (m , the number of \mathbb{P}^n factors in \mathcal{X} and K , the number of constraints) one has to

1. List all possible positive-integer m -vectors \mathbf{n} ($n_i \geq n_j$ if $i > j$), such that $\sum_{r=1}^m n_r = K + 3$.
2. For each \mathbf{n} , list all matrices \mathbf{q} which satisfy Eqs. (2.1.17) and (2.1.13).

The first listing is simple. The second is simplified if one regards the r^{th} row of the matrix \mathbf{q} as a K -digit number Q^r and order \mathbf{q} so that $Q^r \geq Q^s$ if $r > s$. Starting with the smallest Q^r 's satisfying Eq. (2.1.17), row by row and keeping the hierarchy, we find the next number which satisfies Eq. (2.1.17) by shifting

$$[\dots, q_a^r, q_{a+1}^r, \dots] \rightarrow [\dots, (q_a^r + 1), (q_{a+1}^r - 1), \dots]. \quad (2.3.1)$$

For any given \mathbf{n} , the iteration clearly terminates and the algorithm is very well suited for machine execution. Unfortunately, however, the complete algorithm would never terminate since the list of \mathbf{n} 's is unbounded, as we now show.

Consider for the moment only embeddings in products of $m \mathbb{P}^1$'s. We clearly need to impose $K = m - 3$ constraints. Let us start with

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \cong \text{Diagram} \quad (2.3.2)$$

Add now a new \mathbb{P}^1 to the embedding space and replace the single constraint with two to obtain, say,

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix} \cong \text{Diagram} \quad (2.3.3)$$

In Ref. [85] such a procedure was called ‘splitting’. We can now split further, by adding one more \mathbb{P}^1 and replacing either of the constraints by two new constraints in the above fashion. This can clearly be repeated *ad infinitum*.

Having obtained such an ‘augmented’ configuration matrix however does not mean that the manifolds belonging to it are not equivalent up to deformation to the manifolds that belong to the original configuration.

2.3.2 Sufficiency of finitely many configurations

Indeed we now show that after a certain upper limit, no new configuration matrix corresponds to manifolds that are not equivalent by deformation to some already obtained.

Our proof was triggered by the observation that all configurations, after the upper limit, necessarily contain constraints that are effective only in a product of two \mathbb{P}^1 's and are bilinear. So, consider this sub-configuration,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cong \text{Diagram} \quad (2.3.4)$$

on its own. It defines a deformation class of complex 1-dimensional spaces. We compute

$$\begin{aligned} c \left[\begin{matrix} 1 & | & 1 \\ 1 & | & 1 \end{matrix} \right] &= \frac{(1+2J_x)(1+2J_y)}{(1+J_x+J_y)} = 1 + J_x + J_y , \\ \chi_E \left[\begin{matrix} 1 & | & 1 \\ 1 & | & 1 \end{matrix} \right] &= \int_{\mathbb{P}_x^1} \int_{\mathbb{P}_y^1} (J_x + J_y) \wedge (J_x + J_y) = 2 , \end{aligned} \quad (2.3.5)$$

and so it must be a \mathbb{P}^1 . Therefore,

$$\left[\begin{matrix} 1 & | & 1 \\ 1 & | & 1 \end{matrix} \right] \equiv \mathbb{P}^1 , \quad \text{i.e.} \quad \text{---o---} \bullet \cong \text{>o} . \quad (2.3.6)$$

Any configuration (not necessarily Calabi-Yau) that contains this sub-configuration can be reduced by substituting a single \mathbb{P}^1 :

$$\left[\begin{matrix} \mathbf{n} & | & \mathbf{0} & \mathbf{q} \\ 1 & | & 1 & \mathbf{a} \\ 1 & | & 1 & \mathbf{b} \end{matrix} \right] \cong \left[\begin{matrix} \mathbf{n} & | & \mathbf{q} \\ 1 & | & \mathbf{a+b} \end{matrix} \right] , \quad (2.3.7)$$

where \mathbf{a} and \mathbf{b} are some integer row-vectors. Because of this, we shall in addition to Eq. (2.1.13) require that constraints that involve only two \mathbb{P}^1 's, say \mathbb{P}_r^1 and \mathbb{P}_s^1 , satisfy a stronger condition

$$q_a^r + q_a^s \geq 3 , \quad \forall a . \quad (2.3.8)$$

Theorem 2.2 *Finitely many configurations suffice to represent all topologically distinct complete intersection Calabi-Yau 3-folds.*

Proof: [145] To show that there are finitely many minimal configurations, let us consider

$$\mathcal{X} \stackrel{\text{def}}{=} \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_s \times \mathbb{P}_1^{n_1} \times \cdots \times \mathbb{P}_p^{n_p} , \quad n_r \neq 1 , \quad \forall r = 1, \dots, p . \quad (2.3.9)$$

For future convenience we also define

$$\alpha = \sum_{r=1}^p (n_r - 1) , \quad \Rightarrow \quad p \leq \alpha , \quad (2.3.10)$$

where the inequality follows since $n_r \geq 2$.

The number of constraints necessary to obtain a variety of complex dimension three is

$$s + \left(\sum_{r=1}^p \right) - 3 = s + p + \alpha - 3 \quad (2.3.11)$$

2.3. Finiteness of the Family

and the total degree of all constraints is

$$2s + \left(\sum_{r=1}^p (n_r + 1) \right) = 2s + 2p + \alpha . \quad (2.3.12)$$

In order to satisfy Eq. (2.1.17). Since the constraints are at least of degree two, it follows that

$$2s + 2p + \alpha \geq 2(2 + p + \alpha - 3) , \quad \Rightarrow \quad \alpha \leq 6 , \quad (2.3.13)$$

and hence an upper limit

$$p \leq \alpha \leq 6 . \quad (2.3.14)$$

By the same argument that led to Eq. (2.3.8), we note that if a constraint is bilinear in two factors one of which is a \mathbb{P}^1 , the other factor must be one of $\mathbb{P}_r^{n_r}$, $n_r \neq 1$. Let the number of such constraints be β . Note that β is also the total degree of these constraints in coordinates of the $s \mathbb{P}^1$'s and also in those of the $p \mathbb{P}_r^{n_r}$'s. For constraints of degree > 2 , let γ denote the total degree (summed over constraints) in the coordinates of \mathbb{P}^1 's only. Then obviously, $\beta + \gamma = 2s$ to ensure Eq. (2.1.17) for the \mathbb{P}^1 's.

Since the bilinear constraints must involve a $\mathbb{P}_r^{n_r}$ also, $\beta \leq 2p + \alpha$. This follows since $2p + \alpha$ is the total degree of $\mathbb{P}_r^{n_r}$'s and we have, at most, one \mathbb{P}^1 for each $\mathbb{P}_r^{n_r}$. Furthermore, note that

$$\sum_{\text{all constr.}} (\text{degrees} - 2) = (2s + 2p + \alpha) - 2(s + p + \alpha - 3) = 6 - \alpha \quad (2.3.15)$$

is the number of constraints with degree > 2 , if they are of degree precisely three.

On the other hand, for biggest γ , these constraints should involve only \mathbb{P}^1 's and so

$$\gamma \leq 3(6 - \alpha) . \quad (2.3.16)$$

Since $\beta + \gamma = 2s$, it follows that the number of \mathbb{P}^1 's is limited too,

$$s \leq 9 - (\alpha - p) \leq 9 , \quad (2.3.17)$$

where the last inequality follows from the relation (2.3.14).

Since both the numbers of \mathbb{P}^1 's and the number of $\mathbb{P}_r^{n_r}$'s ($n_r \neq 1$) is bounded from above by relations (2.3.14) and (2.3.17), respectively, the number of possible minimal configurations is finite. In fact, such lists have been assembled [58,85,131] and are contained within some 97,000 different configurations, with over 6000 distinct topological types of Calabi-Yau 3-folds [164]. \square

2.3.3 Identities

The number of possibilities for minimal configurations are further reduced by identities like (2.3.6) and we now turn to discuss some of them. The basic form of the relations we seek is

$$\left[\begin{array}{c|ccc} n & P & R & 0 \\ n' & 0 & R' & Q' \end{array} \right] \simeq \left[\begin{array}{c|ccc} n & P & R & 0 \\ n'' & 0 & R'' & Q'' \end{array} \right], \quad (2.3.18)$$

where $P \dots Q''$ are integer matrices. The equivalence of the two configurations means that some of the manifolds belonging to one are equivalent up to deformation to the manifolds that belong to the other. If it can moreover be shown that the deformation class of manifolds belonging to the first configuration is isomorphic to the deformation class of manifolds belonging to the second, we will say that the configurations are *isomorphic* and write “ \approx ” instead of “ \simeq ”. More often, however, one configuration will turn out to be a subset of the other (in the sense of the deformation classes they respectively define).

While there certainly may exist relations among pairs of configurations that do not fall in the above category, the relation (2.3.18) offers the advantage of studying the sufficient condition

$$([n'\|Q'], [R']) \simeq ([n''\|Q''], [R'']). \quad (2.3.19)$$

Here $[n'\|Q']$ and $[n''\|Q'']$ ought to be equivalent configurations, corresponding to varieties of dimension N_Q . In addition, the bundles $[R']$ and $[R'']$ also have to be shown equivalent, i.e., the polynomials over the manifolds belonging to the two respective configurations have to be equivalent too, since that is how the hypersurface in (2.3.18) is defined.



Note that, for the two configurations in the relation (2.3.18) to be Calabi-Yau, the configurations in (2.3.19) must be almost ample. Moreover, let the matrices R , R' , R'' have K_R columns. By a simple generalization of Proposition 2.1, it is obvious that if the configuration $[n\|P]$ corresponds to varieties of dimension N_P , we have that

$$N_P + N_Q - K_R = 3. \quad (2.3.20)$$

Among these, the $N_Q = 1, 2$ and 3 cases are of particular interest, since (2.3.18) is then implied by relations among different embeddings of ample or almost ample varieties of complex dimension 1, 2 and 3 respectively and these have been classified (almost) completely. We now discuss these in turn.

2.3. Finiteness of the Family

1-fold identities. Complex 1-dimensional varieties, algebraic curves, are completely classified by their Euler characteristic. Moreover, for Eq. (2.1.17) to hold (or, for that matter, Eq. (2.1.18) for almost ample and ample 3-folds), the sub-configuration of algebraic curves occurring in relation (2.3.18) must be \mathbb{P}^1 's. We have just proven

Lemma 2.2 *If the sub-configurations*

$$[n'\|Q'] \quad \text{and} \quad [n''\|Q''] \quad (2.3.21)$$

represent 1-dimensional varieties,

$$\left[\begin{array}{c|ccc} n & P & R & 0 \\ n' & 0 & R' & Q' \end{array} \right]_{\text{Calabi-Yau}} \simeq \left[\begin{array}{c|ccc} n & P & R & 0 \\ n'' & 0 & R'' & Q'' \end{array} \right]_{\text{Calabi-Yau}}. \quad (2.3.22)$$

The earlier result, Lemma 2.1, is subsumed in scope by Lemma 2.2; the former is however quite easier to apply in practice. Also, according to Ref. [85],

$$[2\|2], \quad \left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \end{array} \right] \quad \text{and} \quad \left[\begin{array}{c|c} 1 & 2 \\ 1 & 1 \end{array} \right] \quad (2.3.23)$$

are all the different embeddings of \mathbb{P}^1 's that do occur in the computer listing complete intersection Calabi-Yau 3-folds. Their equivalence is, in fact, described already by Lemma 2.1.

2-fold identities. Very much like 1-folds, the topological types of almost ample (del Pezzo) 2-folds are classified by the Euler characteristic—except for $\chi_E = 4$, the so-called Hirzebruch surfaces. These latter 2-folds come in two homotopy (and therefore topological) types which need to be distinguished. As with 1-folds, one lists all possible almost ample 2-fold configurations and finds the smallest one for each Euler characteristic $\chi_E \neq 4$; for $\chi_E = 4$, the smallest configurations representing the two topologically distinct Hirzebruch surfaces are $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbf{F}_1 = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix} \right]$. The so obtained list of configurations then implies a number of identities among almost ample 2-fold configurations (see § 3.1 for more details on almost ample 2-folds, the distinction between the Hirzebruch surfaces and § 3.1.2 for a yet finer subtlety in these identities).

3-fold identities. As expected, the situation with almost ample 3-folds is more complicated. The ample (Fano) 3-folds were classified by Iskovskikh (for $b_2 = 1$) [32] and Mori and Mukai (for $b_2 > 1$) [40]; there are 18 and 87, respectively, topological types of these. Semi-ample 3-folds which are not ample have not been classified yet. Listing all almost ample 3-fold configurations is quite comparable in number to listing all Calabi-Yau 3-fold configurations and has not been done either.

The Hodge numbers (see § 2.4.1 and Chapter A for techniques) provide at least a necessary condition to check conjectured identities. Perhaps not surprisingly, only some of those suspected from listing Calabi-Yau 3-fold configurations [85] turn out to be correct; unfortunately, little more has been worked out on this issue so far.

n -fold identities. The situation with almost ample n -fold configurations is of course even less well understood. Suffice it here to state that the techniques for computation of the Hodge numbers provide at least some necessary criteria which can be used to verify any given conjectured identity.

2.4 $b_{1,1}$ in Favorable Configurations

Since the Calabi-Yau complete intersections we are studying are of the form $\mathcal{M} = \bigcap_{a=1}^K \xi_a^{-1}(0)$, we try to apply Lefschetz hyperplane theorem (see § 1.6) iteratively, including the factors $\xi_a^{-1}(0)$ one by one.

2.4.1 A simple iterating technique

Let us define

$$D_a \stackrel{\text{def}}{=} \{ r \mid q_a^r > 0 \}. \quad (2.4.1)$$

We call a configuration *favorable* if the constraints can be ordered so that

$$\text{for } k' < k, \quad \text{either } D_{k'} \cap D_k = \emptyset, \quad \text{or } D_{k'} \subseteq D_k. \quad (2.4.2)$$

Furthermore, let

$$\mathcal{X}_a = \prod_{\substack{r=1 \\ q_a^r > 0}}^m \mathbb{P}_r^{n_r}, \quad \mathcal{M}_a = \bigcap_{b < a} \xi_b^{-1}(0). \quad (2.4.3)$$

Note that, in the notation of Eq. (2.1.4), the line bundle \mathcal{E}_a is positive over \mathcal{X}_a . However, \mathcal{E}_a^{-1} is not in general positive over \mathcal{M}_a ; ι_a is the embedding $\mathcal{M}_a \hookrightarrow$

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\mathcal{X}_a . This implies certain limitations on the application of Lefschetz hyperplane theorem which will be shown below.

Theorem 2.3 *A Calabi-Yau manifold belonging to a favorable configuration*

(i) is simply connected;

(ii) has the $b_{1,1}$ at least equal to the number of $\mathbb{P}_r^{n_r}$ factors in the embedding space—if the configuration is also minimal.

Remark.

The minimality requirement, as needed in (ii), is effectively achieved through the use of Lemma 2.1 and Corollary 2.1, or Lemma 2.2; see Ref. [148].

The proof in Ref. [145] simply follows the ordering of constraints as above and applies the Lefschetz hyperplane theorem at every step.

2.4.2 Application and limitations

To clarify the technique and its limitations, we give some examples in detail.

1. Consider first $\mathcal{M} \in \left[\begin{smallmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \end{smallmatrix} \right]$, for which $\chi_E = -120$. Both constraints depend non-trivially on homogeneous coordinates of both \mathbb{P}^3 and \mathbb{P}^2 , i.e., are sections of line bundles that are positive over both factors in $\mathcal{X} = \mathbb{P}^3 \times \mathbb{P}^2$.

Firstly, using the Künneth formula

$$b_p(X \times Y) = \sum_{q=0}^p b_{p-q}(X)b_q(Y) \quad (2.4.4)$$

and the fact that $b_p(\mathbb{P}^n) = \begin{cases} 1 & p \text{ even} \\ 0 & p \text{ odd} \end{cases}$, we obtain

$$b_p(\mathcal{X}) = \begin{cases} 1, 2, 3, 3, 2, 1, & \text{for } p = 0, 2, 4, 6, 8, 10; \\ 0, & \text{for } p \text{ odd.} \end{cases} \quad (2.4.5)$$

Applying one constraint on the (5-dimensional) \mathcal{X} one gets a (4-dimensional) submanifold for which b_p , $0 \leq p < 4$, are the same as those for \mathcal{X} . Applying the other constraint yields \mathcal{M} for which b_p equals those of \mathcal{X} for $0 \leq p < 3$ and so $b_0 = 1$, $b_1 = 0$ and $b_2 = b_{1,1} = 2$. Using $\chi_E = -120$ it follows that $b_{2,1} = 62$ and we have the complete Hodge diamond.

2. For our next example, we take a slightly more complicated configuration :

$$\mathcal{M} \in \left[\begin{array}{c|cc} 2 & 2 & 1 \\ 2 & 2 & 1 \\ \hline 1 & 0 & 2 \end{array} \right] \quad \cong \quad \text{Diagram showing a configuration of points and lines in } \mathbb{P}^3, \quad (2.4.6)$$

for which $\chi_E = -96$. Here the first constraint is independent of the coordinates of the third factor in X , the \mathbb{P}^1 , and Lefschetz hyperplane theorem would not apply were we to consider the second constraint first. We therefore consider the sub-configuration

$$\left[\begin{array}{c|cc} 2 & 2 \\ 2 & 2 \\ \hline \end{array} \right] \quad \cong \quad \text{Diagram showing a configuration of points and lines in } \mathbb{P}^3, \quad (2.4.7)$$

This defines a 3-dimensional hypersurface (\mathcal{M}_1) embedded in $X_1 \stackrel{\text{def}}{=} \mathbb{P}^2 \times \mathbb{P}^2$. The Betti numbers of \mathcal{M}_1 are equal, by Lefschetz hyperplane theorem, to those of X_1 for $p = 0, 1, 2$ and are 1, 0, 2, respectively.

Next we impose the remaining constraint, which depends non-trivially on both \mathcal{M}_1 and \mathbb{P}^1 and embeds \mathcal{M} in $\mathcal{M}_1 \times \mathbb{P}^1$ as a hypersurface. Using the Künneth formula,

$$b_p(\mathcal{M}_1 \times \mathbb{P}^1) = 1, 0, 3, \quad p = 0, 1, 2. \quad (2.4.8)$$

Since the constraint is non-trivial over the entire embedding space $\mathcal{M}_1 \times \mathbb{P}^1$, Lefschetz hyperplane theorem applies again, equating the Betti numbers of \mathcal{M} with those of $\mathcal{M}_1 \times \mathbb{P}^1$ up to, but not including the “middle dimension”, i.e.,

$$b_p(\mathcal{M}) = 1, 0, 3, \quad p = 0, 1, 2. \quad (2.4.9)$$

Using $\chi_E = -96$, we have that $b_{2,1} = 51$.

3. Up to now, b_2 was equal to the number of factors in X , but not so in

$$\mathcal{M} \in \left[\begin{array}{c|cc} 3 & 3 & 1 \\ 1 & 0 & 2 \\ \hline 1 & 0 & 2 \end{array} \right] \quad \cong \quad \text{Diagram showing a configuration of points and lines in } \mathbb{P}^3, \quad (2.4.10)$$

for which $\chi_E = -48$. Namely, the first constraint is trivial over the $\mathbb{P}^1 \times \mathbb{P}^1$ factor of the embedding space and we must begin with considering the sub-configuration $[3\|3]$. This is a deformation class of cubic hypersurfaces (\mathcal{M}_1) in \mathbb{P}^3 . The constraint being non-trivial, Lefschetz hyperplane theorem applies and equates $b_p(\mathcal{M}_1)$ with $b_p(\mathbb{P}^3)$ but only up to the middle dimension,

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i.e., for $p = 0, 1$. As regards $b_2(\mathcal{M}_1)$, Lefschetz hyperplane theorem only states that $b_2(\mathcal{M}_1) \geq b_2(\mathbb{P}^3)$.

In such an occasion, we use

$$\begin{aligned} \chi_E(\mathcal{M}_1) &= \int_{[3\|3]} \frac{(1+\eta)^4}{(1+3\eta)} = \int_{\mathbb{P}^3} (3\eta) \wedge (1+\eta+3\eta^2), \\ &= 3 \cdot 3 \int_{\mathbb{P}^3} \eta^3 = 9. \end{aligned} \quad (2.4.11)$$

Since $\chi_E(\mathcal{M}_1) = \sum_{p=0}^2 (-1)^p b_p(\mathcal{M}_1) = 9$, $b_{\dim \mathcal{M}_1-p}(\mathcal{M}_1) = b_p(\mathcal{M}_1)$, it follows that $b_2(\mathcal{M}_1) = 7$.

Now we are ready to consider the second constraint acting in $\mathcal{M}_1 \times \mathbb{P}^1 \times \mathbb{P}^1$. By Künneth,

$$b_p(\mathcal{M}_1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 1, 0, 9, \quad p = 0, 1, 2. \quad (2.4.12)$$

Being non-trivial in both \mathcal{M}_1 and $\mathbb{P}^1 \times \mathbb{P}^1$, Lefschetz hyperplane theorem applies and equates $b_p(\mathcal{M})$ with $b_p(\mathcal{M}_1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ up to $p = 2$ and in particular, $b_2(\mathcal{M}) = 9$. Since $\chi_E(\mathcal{M}) = -48$, $b_{2,1}(\mathcal{M}) = 33$.

4. Finally, we give an example where theorem 2.3 does not guarantee the applicability of Lefschetz hyperplane theorem :

$$\mathcal{M} \in \left[\begin{array}{c|cc} 2 & 0 & 3 \\ 2 & 2 & 1 \\ \hline 1 & 2 & 0 \end{array} \right] \quad \cong \quad \text{Diagram showing a configuration of points and lines in } \mathbb{P}^3, \quad (2.4.13)$$

for which $\chi_E = -36$. The correct result here is $b_2 = 9$ and we will give all the details of this computation later.

Our present technique is not guaranteed to apply since the configuration is not favorable, i.e., the constraints cannot be ordered so as to satisfy Eq. (2.4.2).

If one starts with the first constraint, being non-trivial only over the latter two factors of the embedding space, we consider the sub-configuration $\left[\begin{smallmatrix} 2 & 2 \\ 1 & 2 \end{smallmatrix} \right]$. This describes complex surfaces (\mathcal{M}_1) embedded in $X_1 = \mathbb{P}^2 \times \mathbb{P}^1$. By Künneth,

$$b_p(X_1) = 1, 0, 2, \quad p = 0, 1, 2. \quad (2.4.14)$$

Lefschetz hyperplane theorem applies and equates $b_p(\mathcal{M}_1)$ with $b_p(X_1)$ but only up to the middle dimension, i.e., for $p = 0, 1$. We again compute

$\chi_E(\mathcal{M}_1) = 10$ using the Chern polynomial and thus have

$$\chi_E(\mathcal{M}_1) = \sum_r (-1)^r b_r(\mathcal{M}_1) = 10 , \quad \Rightarrow \quad b_2(\mathcal{M}_1) = 8 . \quad (2.4.15)$$

Now, the second configuration is not non-trivial over the entire $\mathbb{P}^2 \times \mathcal{M}_1$, as it is trivial over the \mathbb{P}^1 factor of the embedding space of \mathcal{M}_1 . Nevertheless, we shall show in § 3.1 that the line bundle—of which this constraint is a section—is positive, even though not manifestly so. By Künneth,

$$b_p(\mathbb{P}^2 \times \mathcal{M}_1) = 1, 0, 9 , \quad p = 0, 1, 2 . \quad (2.4.16)$$

Taking for granted at the moment that Lefschetz hyperplane theorem applies, we have the Betti numbers of $\mathbb{P}^2 \times \mathcal{M}_1$ equated to those of \mathcal{M} up through b_2 :

$$b_2(\mathcal{M}) = 1, 0, 9 , \quad p = 0, 1, 2 . \quad (2.4.17)$$

As a cautionary remark, suffice it to note here that were we to employ the constraints in the reversed order and ignore the question of applicability, the wrong result $b_2(\mathcal{M}) = 3$ would ensue.

2.4.3 Redundancy of inherited (2,2)-forms

We have seen two examples in which b_2 of the complete intersection threefold is greater than that of the embedding space. In fact, for these, the generators of the 2-dimensional homology of the submanifold were inherited from the embedding space (cf. Corollary 1.1). Generally, additional cycles (here, 2-cycles) of the submanifold, which have not been inherited from the embedding manifold are usually called “vanishing cycles”. It appears natural to inquire if it is possible that b_2 of the members of a configuration could be less than that of the embedding space.

The above possibility may now be rephrased as the question whether there exist configurations in which some of the (independent) 2-cycles from the embedding manifold do not descent as independent cycles on the submanifold. Note that the appearance of additional, vanishing cycles may mask this effect so that b_2 of the complete intersection submanifold is nevertheless greater or equal to that of the embedding manifold. All configurations for which not all of the 2-dimensional homology of the embedding manifold descends to the complete intersection Calabi-Yau submanifold can easily be modified (see Lemma 2.1) into equivalent configurations where there is no such defect [148].

2.5 Embedding Characteristics

In addition to the Hodge numbers, there exist certain quantities which characterize the embedding of the Calabi-Yau manifold in $X = \mathbb{P}_1^{n_1} \times \cdots \times \mathbb{P}_m^{n_m}$ and which can be computed with relative ease [95].

Consider a compact complex n -fold \mathcal{M} . Recall that the Euler characteristic is the index of the De Rham exterior derivative

$$\chi_E = \sum_r (-1)^r \dim H_{\text{DR}}^r(\mathcal{M}) = \int_{\mathcal{M}} c_n . \quad (2.5.1)$$

where the second equality is the statement of the Gauss-Bonnet formula. Similarly, on complex manifolds, there is the index of the $\bar{\partial}$ -derivative (acting on the $(0, q)$ -cohomology) :

$$\chi^h = \sum_q (-1)^q \dim H_{\bar{\partial}}^{(0,q)}(\mathcal{M}) = \int_{\mathcal{M}} \text{td}_n , \quad (2.5.2)$$

also called the *holomorphic Euler characteristic* or the *arithmetic genus*. Also, there exists the *Hirzebruch signature*

$$\tau_H = \sum_{p,q} (-1)^q \dim H_{\bar{\partial}}^{(p,q)}(\mathcal{M}) = \int_{\mathcal{M}} L_n , \quad (2.5.3)$$

and the index of the Dirac operator, which is the *A-roof genus*

$$\iota_D = \int_{\mathcal{M}} \hat{A}_n . \quad (2.5.4)$$

It is not hard to see that the last three quantities vanish for a Calabi-Yau 3-fold and thus appear to be of little use. However, given a vector bundle over \mathcal{M} , we may consider the related “twisted indices”

$$\chi^h[\mathcal{V}] \stackrel{\text{def}}{=} \int_{\mathcal{M}} \text{td}(\mathcal{M}) \wedge \text{ch}[\mathcal{V}] , \quad (2.5.5)$$

$$\tau_H[\mathcal{V}] \stackrel{\text{def}}{=} \int_{\mathcal{M}} L(\mathcal{M}) \wedge \text{ch}^2[\mathcal{V}] , \quad (2.5.6)$$

$$\iota_D[\mathcal{V}] \stackrel{\text{def}}{=} \int_{\mathcal{M}} \hat{A}(\mathcal{M}) \wedge \text{ch}^2[\mathcal{V}] . \quad (2.5.7)$$

Here, $\text{ch}^2[\mathcal{V}]$ denotes the Chern character calculated from the double curvature 2-form, $2F$, of \mathcal{V} .

In general, no suitable vector bundle \mathcal{V} intrinsic to our Calabi-Yau complete intersection 3-fold \mathcal{M} is readily available for computation of the above indices. On the other hand, there are simple candidate vector bundles over the embedding space \mathcal{X} , which naturally restrict to vector bundles over $\mathcal{M} \subset \mathcal{X}$. These are of the general form

$$\mathcal{V}(\ell\|\mathbf{k}) \stackrel{\text{def}}{=} \bigoplus_{r,a} (\mathcal{T}_r)^{\ell_r} \otimes \mathcal{O}_r(k_r^a) \Big|_{\mathcal{M}}, \quad (2.5.8)$$

where \mathcal{T}_r is the holomorphic tangent bundle of $\mathbb{P}_r^{n_r}$ and $\mathcal{O}_r(k_r^a)$ is the line bundle corresponding to degree- k_r^a polynomials over $\mathbb{P}_r^{n_r}$. Since all these are restrictions of bundles over the embedding space \mathcal{X} to the \mathbb{P}^n factors, the computation of these “twisted indices” is a straightforward generalization of the computation of the Euler characteristic.

Such “twisted indices” characterize the embedding $\mathcal{M} \hookrightarrow \mathcal{X}$ rather than the abstract manifold \mathcal{M} . Nevertheless, they may indeed be useful on a number of occasions. For example in Ref. [58,95], divisibility properties of certain such “twisted indices” were used to determine which configurations define Calabi-Yau 3-folds that admit a freely acting symmetry D such that the quotient with respect to D would have $\chi_E = -6$.



The large collection of constructions of Calabi-Yau 3-folds as complete intersections in complex projective spaces appears to be merely the tip of an iceberg. It also appears that it is a fairly representative collection, in the sense that it exhibits phenomena which are quite generic for Calabi-Yau 3-folds and which we shall study in the sequel. At the same time, the analysis of such 3-folds is always straightforward—given the powerful tools of spectral sequences which we shall describe later—and often quite easy so that even more elementary and better known techniques suffice.

Chapter 3

Some More General Embeddings

In this chapter, we consider some simple generalizations of the complete intersection constructions in Chapter 2, still avoiding singular models.

3.1 Intersections in Products of Surfaces

One way to generalize the constructions of Chapter 2 is of course to consider complete intersections in products of spaces other than \mathbb{P}^n 's. Relatively simple additional factors in embedding spaces are provided by algebraic surfaces.

To formulate our complete intersections as in products of complex projective spaces, we require of these surfaces to have a non-trivial anti-canonical bundle with at least one non-vanishing section at every point of the surface. These sections will be used to construct the system of defining constraints. It turns out that all such surfaces in fact can be represented as complete intersections in products of complex projective spaces, so that this attempt does not enrich our bestiary by truly new examples. Nevertheless, the exercise proves useful as this alternative point of view will provide some new means of computation in Chapter 8.

3.1.1 Almost del Pezzo surfaces

An *almost del Pezzo surface*⁽¹⁾, \mathcal{S} , has a non-trivial anti-canonical bundle, $\mathcal{K}_{\mathcal{S}}^*$, which has at least one non-zero section at any point of \mathcal{S} . We require this with the forethought of constructing smooth 3-folds as complete intersections in products which include almost del Pezzo surfaces. Namely, if $\mathcal{K}_{\mathcal{S}}^*$ is non-trivial and has at least one non-zero section at every point, it has to have at least two sections. This then essentially allows a suitable generalization of Bertini's theorem which

will guarantee the existence of smooth complete intersections in products of such surfaces.

The alert Reader will have noticed that, on a surface \mathcal{S} , each section of course vanishes along a curve and that two curves in a surface generically do meet at an isolated number of points. However, a limited number of such points can be *blown up*^(L) (a) without making the anti-canonical bundle $\mathcal{K}_{\mathcal{S}}^*$ non-positive and (b) leaving at every point one section non-zero. This is seen as follows. Let ϕ and ϕ' be two sections the zero-sets of which are two curves, C and C' , respectively. Let $C \cap C' = \{p_i\} \subset \mathcal{S}$, which is where ϕ and ϕ' both vanish. In blowing these points up, we replace each of them with an *exceptional divisor* E_i of codimension 1. Now the intersection points $C \cap E_i$ and $C' \cap E_i$ are determined by the angle at which C and C' passed through p_i before it was blown up, so, unless (the zero-sets of) two sections are tangential at p_i , they will not vanish simultaneously after p_i is blown up. If they were tangential at p_i , two blow-ups would have been needed to disconnect them and so on.

Since the anti-canonical bundle is nontrivial, so is the canonical one. However, since the anti-canonical bundle has holomorphic sections its dual cannot have, so that $H^0(\mathcal{S}, \mathcal{K}_{\mathcal{S}}) = H^{2,0}(\mathcal{S}) = 0$. Almost del Pezzo surfaces are also *regular*, i.e., they have no harmonic (1,0)-forms and the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & & 0 & 0 & \\ & 0 & b_{1,1} & 0 & \Rightarrow b_{1,1} = \chi_E - 2 \\ & 0 & 0 & & \\ & & 1 & & \end{array} \quad (3.1.1)$$

Using the two different expressions for both the Euler characteristic (2.5.1) and the Hirzebruch signature (2.5.3), we obtain :

$$\begin{aligned} 2 + b_{1,1}(\mathcal{S}) &= \chi_E(\mathcal{S}) = C_2(\mathcal{S}), \\ 2 - b_{1,1}(\mathcal{S}) &= \tau_H(\mathcal{S}) = \frac{1}{3}(C_1^2(\mathcal{S}) - 2C_2(\mathcal{S})), \end{aligned} \quad (3.1.2)$$

with C_1^2 and C_2 denoting the Chern numbers. Therefore

$$C_1^2(\mathcal{S}) + C_2(\mathcal{S}) = 12. \quad (3.1.3)$$

This also follows from the above Hodge diamond, using Eqs. (1.5.12) and (2.5.2) and is known as an example of the *Noether formula*^(L) [23]

$$C_1^2(\mathcal{S}) + C_2(\mathcal{S}) = 12\chi^h(\mathcal{S}). \quad (3.1.4)$$

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We note also the relation to the canonical bundle, $\mathcal{K}_{\mathcal{S}}$:

$$C_1^2(\mathcal{S}) = \mathcal{K}_{\mathcal{S}} \cdot \mathcal{K}_{\mathcal{S}}. \quad (3.1.5)$$

The dot-product occurring here is called the intersection number and is defined in general as follows. Given two line bundles \mathcal{L} and \mathcal{L}' over a surface \mathcal{S} , we define

$$\mathcal{L} \cdot \mathcal{L}' \stackrel{\text{def}}{=} \int_{\mathcal{S}} c_1[\mathcal{L}] \wedge c_1[\mathcal{L}']. \quad (3.1.6)$$

Eq. (3.1.5) is now easily seen to follow from Eq. (1.3.20) and the above definition (3.1.6).



We know that $\mathcal{K}_{\mathcal{S}}^*$ must have at least two (non-trivial, global and holomorphic) sections. Consequently, the m -fold tensor power of $\mathcal{K}_{\mathcal{S}}^*$ is also non-trivial and has (global, holomorphic) sections. Then $\mathcal{K}_{\mathcal{S}}^{\otimes m}$ is also non-trivial but cannot have sections, so all *plurigenera*^(L) vanish :

$$\mathfrak{P}_m(\mathcal{S}) \stackrel{\text{def}}{=} H^0(\mathcal{S}, \mathcal{K}_{\mathcal{S}}^{\otimes m}) = 0, \quad m \geq 0. \quad (3.1.7)$$

From this result and regularity ($b_{1,0} = 0$) it follows by the Castelnuovo–Enriques theorem [23] that \mathcal{S} are *rational*^(L). Thereupon, in view of the general classification theorems of complex surfaces, we know that \mathcal{S} can only be one of the following :

1. \mathbb{P}^2 with finitely many (ρ , possibly $\rho = 0$) points blown up;
2. a Hirzebruch (rational, ruled) surface \mathbf{F}_n .

Remark.

The almost del Pezzo surfaces of $\chi_E = 4$ are known as Hirzebruch surfaces \mathbf{F}_n . They are distinguished from each other by the exceptional divisor (with self-intersection $-n$) which they contain. Note also that the Euler characteristic of a ρ -fold blow-up of \mathbb{P}^2 is

$$\chi_E(\widetilde{\mathbb{P}^2}_{p_1 \dots p_\rho}) = \chi_E(\mathbb{P}^2) + \rho = 3 + \rho. \quad (3.1.8)$$



Restricting the range of n in the above list is rather easy :

Proposition 3.1 *No almost del Pezzo surface has a line of self-intersection less than -2 .*

Proof: Let \mathcal{S} be an almost del Pezzo surface with a line $L(\approx \mathbb{P}^1) \subset \mathcal{S}$ with self-intersection $-n < 0$. We know that the holomorphic tangent bundle of a line $L \approx \mathbb{P}^1$ is $\mathcal{T}_L = \mathcal{O}_L(2)$. So, $\mathcal{K}_L^* = \det \mathcal{T}_L = \mathcal{T}_L = \mathcal{O}_L(2)$ since $\text{rank } \mathcal{T}_L = 1$. On the other hand, the normal bundle of $L \subset \mathcal{S}$ must be $\mathcal{N}_{\mathcal{S}/L} = \mathcal{O}_L(-n)$ for the self-intersection of L to be $-n$. But then

$$\mathcal{K}_{\mathcal{S}}^*|_L = \det \mathcal{T}_{\mathcal{S}}|_L = \mathcal{K}_L^* \otimes \mathcal{N}_{\mathcal{S}/L} = \mathcal{O}_L(2-n), \quad (3.1.9)$$

and could not have sections if $n > 2$. \square

We now turn to restricting the number of blow-ups in $\mathcal{S} = \widetilde{\mathbb{P}}^2_{p_1 \dots p_8}$.

Proposition 3.2 *Let $\tilde{\mathcal{S}}_p$ be the blow-up of \mathcal{S} at $p \in \mathcal{S}$. The sections of the anti-canonical bundle of $\tilde{\mathcal{S}}_p$ are precisely the sections of $\mathcal{K}_{\mathcal{S}}^*$ which vanish at p .*

Proof: Let (z_1, z_2) be local coordinates on \mathcal{S} at p and let $\zeta_1 \stackrel{\text{def}}{=} z_1$, $\zeta_2 \stackrel{\text{def}}{=} z_2/z_1$ whenever $z_1 \neq 0$. In the limit where $\zeta_2 = \text{const.}$ while $\zeta_1 \rightarrow 0$, ζ_2 parametrizes the \mathbb{P}^1 into which p is blown up. Now, the left-hand-side of

$$dz_1 \wedge dz_2 = \zeta_1(d\zeta_1 \wedge d\zeta_2) \quad (3.1.10)$$

is a section of $\mathcal{K}_{\mathcal{S}}$, while the right-hand-side is a section of $\mathcal{K}_{\tilde{\mathcal{S}}_p}$.

Away from the point which is blown up, the surfaces $\mathcal{S} - p$ and $\tilde{\mathcal{S}} - \mathbb{P}^1$ are the same. Then, for a section ϕ of $\mathcal{K}_{\mathcal{S}-p}^* = \mathcal{K}_{\tilde{\mathcal{S}}_p-\mathbb{P}^1}^*$,

$$\langle \phi | dz_1 \wedge dz_2 \rangle = \zeta_1 \langle \phi | d\zeta_1 \wedge d\zeta_2 \rangle. \quad (3.1.11)$$

So, if ϕ extends over $\tilde{\mathcal{S}}_p$, we have

$$\lim_{\substack{\zeta_1 \rightarrow 0 \\ \zeta_2 \text{ fixed}}} \langle \phi | dz_1 \wedge dz_2 \rangle = 0, \quad (3.1.12)$$

independently of ζ_2 . Therefore :

$$\lim_{(z_1, z_2) \rightarrow (0,0)} \langle \phi | dz_1 \wedge dz_2 \rangle = 0, \quad (3.1.13)$$

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and ϕ extends to \mathcal{S} , taking the value 0 at p .

Conversely, if ϕ is a section of the anti-canonical bundle of \mathcal{S} , vanishing at p , we can extend ϕ over $\tilde{\mathcal{S}}_p$ by setting :

$$\langle \phi | d\zeta_1 \wedge d\zeta_2 \rangle|_{\text{at } (0, \zeta_2)} = \lim_{\substack{\zeta_1 \rightarrow 0 \\ (\zeta_2/\zeta_1) \text{ fixed}}} \frac{1}{\zeta_1} \langle \phi | dz_1 \wedge dz_2 \rangle. \quad (3.1.14)$$

\square

Proposition 3.3 *No almost del Pezzo surface has $\chi_E = 11$.*

Proof: If \mathcal{S} were a del Pezzo surface with $\chi_E = 11$, it would have to be $\widetilde{\mathbb{P}}^2_{p_1 \dots p_8}$, an 8-fold blow-up of \mathbb{P}^2 . The anti-canonical bundle of a blow-up of \mathbb{P}^2 is induced from the anti-canonical bundle of \mathbb{P}^2 , which is $\mathcal{O}_{\mathbb{P}^2}(3)$ and has 10 independent sections. Since sections of $\mathcal{K}_{\mathcal{S}}^*$ are (the extensions of) those of $\mathcal{K}_{\mathbb{P}^2}^*$ which vanish at p_1, \dots, p_8 , we need to count the number of cubic polynomials over \mathbb{P}^2 which simultaneously vanish at 8 (generically chosen) points. It is not hard to verify that only two of ten cubics will vanish at all given 8 points and that they will also vanish at a ninth point. If this ninth point is not blown up, all sections of $\mathcal{K}_{\mathcal{S}}^*$ vanish there and \mathcal{S} is not almost del Pezzo. If it is blown up, $\chi_E(\mathcal{S}) = 12$. \square

It is now not hard to see that there can be no almost del Pezzo surface \mathcal{S} with $\chi_E > 12$ either; essentially, when $\chi_E > 12$, $\mathcal{K}_{\mathcal{S}}^*$ becomes non-positive and has no sections. Complete details of this and the previous proofs are found in Ref. [147].

3.1.2 Jumping deformations

Deformations are usually thought of as continuous variations of the complex structure. That this is not necessarily quite the complete story is demonstrated by the following simple example. Besides serving to illustrate what might be called jumping deformations, our simple example here presents a rather generic construction of models which incorporate such deformations.

The almost ample 2-fold configuration $\left[\begin{smallmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix} \right]$ corresponds to a system of defining equations

$$\mathcal{S}_\varepsilon \in \left[\begin{smallmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix} \right] : \quad \left\{ \begin{array}{l} z_0 w_0 + z_1 w_1 = 0, \\ z_2 w_0 + [\sum_{i=0}^2 a_i z_i + \varepsilon z_3] w_1 = 0, \end{array} \right. \quad (3.1.15)$$

where we have, without loss of generality, chosen (z_0, z_1, z_2, z_3) and (w_0, w_1) for local homogeneous coordinates on $\mathbb{P}^3 \times \mathbb{P}^1$. The surface \mathcal{S}_ε of common solution to the two Eqs. (3.1.15) in $\mathbb{P}^3 \times \mathbb{P}^1$ is non-singular except in three special cases : (1) $a_1 = 0$, (2) $a_2 = 0$ and (3) $a_0 = -a_1 a_2 \neq 0$. So, except for these special cases, \mathcal{S}_ε is smooth, has $\chi_\varepsilon = 4$ and so must be one of the Hirzebruch surfaces : \mathbf{F}_n , $n = 0, 1, 2$.

- When $\varepsilon \neq 0$, the equation :

$$\Delta_{\varepsilon \neq 0}(z_i) \stackrel{\text{def}}{=} \det \begin{bmatrix} z_0 & z_1 \\ z_2 & (\sum_{i=0}^2 a_i z_i) + \varepsilon z_3 \end{bmatrix}_{\varepsilon \neq 0} = 0 \quad (3.1.16)$$

embeds $\mathcal{S}_{\varepsilon \neq 0}$ in \mathbb{P}^3 as a smooth quadric surface since $d\Delta_{\varepsilon \neq 0} \neq 0$ over \mathbb{P}^3 and therefore $\mathcal{S}_{\varepsilon \neq 0} \approx \mathbb{P}^1 \times \mathbb{P}^1 = \mathbf{F}_0$; this is an example of *splitting*^(L).

• However, for the special value $\varepsilon = 0$, both $\Delta_{\varepsilon=0}$ and $d\Delta_{\varepsilon=0}$ vanish at the point $(0, 0, 0, 1) \in \mathbb{P}^3$, so that the surface of solutions to $\Delta_{\varepsilon=0} = 0$ in \mathbb{P}^3 is singular at $(0, 0, 0, 1)$ although the surface of common solution to Eqs. (3.1.15), $\mathcal{S}_{\varepsilon=0} \subset \mathbb{P}^3 \times \mathbb{P}^1$ is smooth. In fact, over $(0, 0, 0, 1) \in \mathbb{P}^3$, the surface $\mathcal{S}_{\varepsilon=0}$ contains an entire copy of \mathbb{P}^1_w since with $z_0, z_1, z_2, \varepsilon = 0$, both w_0, w_1 remain undetermined. Therefore, $\mathcal{S}_{\varepsilon=0}$ is the blow-up of the singular space $z_0(\sum_i a_i z_i) = z_1 z_2$ at the singularity $z_0, z_1, z_2 = 0$.

The above construction makes it manifest that $\mathcal{S}_{\varepsilon \neq 0} = \mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ is a deformation (parametrized by ε) of $\mathcal{S}_{\varepsilon=0}$, whence they must be of the same homotopy type. Then it must be that $\mathcal{S}_{\varepsilon=0} = \mathbf{F}_2$. Note however that \mathcal{S}_ε is rigid for both $\varepsilon \neq 0$ and $\varepsilon = 0$. In other words, all \mathcal{S}_ε , $\varepsilon \neq 0$ and arbitrary a_i , are equivalent up to linear reparametrizations. Similarly are all $\mathcal{S}_{\varepsilon=0}$ equivalent. However, $\mathcal{S}_{\varepsilon=0}$ contains a line of self-intersection -2 , while $\mathcal{S}_{\varepsilon \neq 0}$ does not. It follows that the moduli space of the surfaces \mathcal{S}_ε belonging to $\left[\begin{smallmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix} \right]$ consists of two infinitesimally close but distinct points—one for \mathbf{F}_0 and one for \mathbf{F}_2 .

Similar situations are also found for $\chi_\varepsilon > 4$. For example,

$$\tilde{\mathbb{P}}_{p_1, p_2, p_3}^2 \in \left[\begin{array}{c|cc} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array} \right] \quad (3.1.17)$$

precisely if the three points are not collinear [147]. To include also the non-generic case when the three points are collinear, one embeds

$$\tilde{\mathbb{P}}_{p_1, p_2, p_3}^2 \in \left[\begin{array}{cccc} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right] . \quad (3.1.18)$$

3.1. Intersections in Products of Surfaces

A simple analysis of the defining equations verifies that the latter configuration contains surfaces each of which is a blow-up of a singular surface contained in the former configuration. As in the previous, $\chi_\varepsilon = 4$ case, however, the surfaces contained in both configurations are indeed deformations of this exceptional surface and they both have the same homotopy type.



Up to this subtlety, then, the list of almost del Pezzo surfaces is exhausted by \mathbb{P}^2 and the members of the following nine almost ample 2-fold configurations :

$$\begin{aligned} \mathbf{F}_0 &= \left[\begin{array}{c|c} 1 \\ 1 \\ 1 \end{array} \right]_4 , & \mathbf{F}_1 &= \left[\begin{array}{c|c} 2 & 1 \\ 1 & 1 \end{array} \right]_4 , & \left[\begin{array}{c|cc} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right]_5 , \\ \left[\begin{array}{c|cc} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array} \right]_6 , & \left[\begin{array}{c|cc} 2 & 2 \\ 1 & 1 \end{array} \right]_7 , & [4|2\ 2]_8 , \\ [3|3]_9 , & \left[\begin{array}{c|cc} 2 & 2 \\ 1 & 2 \end{array} \right]_{10} , & \left[\begin{array}{c|cc} 2 & 3 \\ 1 & 1 \end{array} \right]_{12} , \end{aligned} \quad (3.1.19)$$

where the subscripts denote the Euler characteristics.

It is perhaps less than obvious that $\left[\begin{smallmatrix} 2 & 2 \\ 1 & 2 \end{smallmatrix} \right]$ represents double covers of \mathbb{P}^2 branched over a smooth quartic curve. As this type of analysis will be recurring, let us take a closer look at it. Denoting the homogeneous coordinates $x_i \in \mathbb{P}^2$ and $z_0, z_1 \in \mathbb{P}^1$, we write

$$S \in \left[\begin{array}{c|c} 2 & 2 \\ 1 & 2 \end{array} \right] : Q_{00}(x) z_0^2 + 2Q_{01}(x) z_0 z_1 + Q_{11}(x) z_1^2 = 0 . \quad (3.1.20)$$

Now, for a generic point of $x \in \mathbb{P}^2$, this is a generic quadratic equation on \mathbb{P}^1 , the solution of which is two points. So, almost every point in \mathbb{P}^2 is doubled in S . Of course, when the discriminant $\Delta \stackrel{\text{def}}{=} (Q_{01}^2 - Q_{00}Q_{11})$ vanishes, the quadratic equation degenerates into a perfect square and the two solutions in \mathbb{P}^1 coincide. This happens at the quartic curve $Q \subset \mathbb{P}^2$, defined by $\Delta(x) = 0$. So, S is in fact a double cover of \mathbb{P}^2 , branched however over the quartic $Q \subset \mathbb{P}^2$. Finally, we note that Q is smooth for a generic choice of $Q_{ij}(x)$ because its singularities occur precisely where $d\Delta(x) = 0$, which is easily seen to require all three quadratics $Q_{ij}(x)$ to vanish. However, this is three independent constraints in \mathbb{P}^2 and so cannot have common solution in general. It turns out that also every smooth quartic in \mathbb{P}^2 can be represented as the discriminant of some quadric [230] and so the configuration (3.1.20) includes all double covers of \mathbb{P}^2 branched over smooth quartics.

3.1.3 Calabi-Yau hypersurfaces in $\mathcal{S} \times \mathcal{S}'$

Using these ten surfaces, we can now form arbitrary products and then consider Calabi-Yau complete intersections of hypersurfaces. Since all of these are covered as complete intersections in complex projective spaces, we forego a complete analysis here. Suffice it here merely to review the family of Calabi-Yau 3-folds obtained as hypersurfaces in products of some two almost del Pezzo surfaces [147].

One naively expects $\frac{10 \cdot 11}{2} = 55$ different Calabi-Yau 3-fold hypersurfaces, one for each $\mathcal{S} \times \mathcal{S}'$ pair. This however is not quite true. Note first of all that the almost del Pezzo surfaces with $3 \leq \chi_E \leq 10$ are in fact ample; only the $\chi_E = 12$ surface is not. We here state

Theorem 3.1 *Let \mathcal{S} and \mathcal{S}' be two del Pezzo surfaces (that is, almost del Pezzo surfaces with $\chi_E = 3, \dots, 10$). Then there are (a) $\frac{9 \cdot 10}{2} = 45$ distinct Calabi-Yau 3-folds, with however (b) $\frac{8 \cdot 9}{2} = 36$ distinct Hodge numbers :*

$$\chi_E \begin{bmatrix} \mathcal{S} & | & \mathcal{K}_{\mathcal{S}}^* \\ \mathcal{S}' & | & \mathcal{K}_{\mathcal{S}'}^* \end{bmatrix} = -2[12 - \chi_E(\mathcal{S})][12 - \chi_E(\mathcal{S}')], \quad (3.1.21)$$

$$b_{1,1} \begin{bmatrix} \mathcal{S} & | & \mathcal{K}_{\mathcal{S}}^* \\ \mathcal{S}' & | & \mathcal{K}_{\mathcal{S}'}^* \end{bmatrix} = b_{1,1}(\mathcal{S}) + b_{1,1}(\mathcal{S}'). \quad (3.1.22)$$

Proof: (a) This is almost clear by construction. We only need to show that replacing \mathbf{F}_0 with \mathbf{F}_1 in the embedding spaces yields a different Calabi-Yau 3-fold; we defer this until Chapter 8. The statement about the Euler characteristic in part (b) follows by direct computation. For the statement about the Hodge numbers, we need the following Lemma.

Lemma 3.1 *The anti-canonical bundle of a del Pezzo surface is positive.*

Proof: By our earlier result, del Pezzo surfaces are the almost del Pezzo surfaces with $\chi_E = 3, \dots, 10$. The statement is straightforward for the cases $\chi_E = 3, \dots, 9$ and is easily seen as follows. The anti-canonical bundle for all surfaces represented in (3.1.19) is obtained as the restriction to \mathcal{S} of

$$\bigotimes_{r=1}^m \mathcal{O}_r(\hat{q}_r), \quad \hat{q}_r \stackrel{\text{def}}{=} n_r + 1 - \sum_{a=1}^K q_r^a. \quad (3.1.23)$$

3.1. Intersections in Products of Surfaces

Except for the $\chi_E = 10$ case, $\hat{q}_r > 0$ for all factors, $r = 1, \dots, m$ and so $\mathcal{K}_{\mathcal{S}}^*$ is positive.

Consider now the case of $\begin{bmatrix} 2 & | & 2 \\ 1 & | & 2 \end{bmatrix}$, that is, the family of double covers of \mathbb{P}^2 branched over a smooth quartic curve in \mathbb{P}^2 . $\mathcal{K}_{\mathcal{S}}^*$ is then induced from $\mathcal{K}_{\mathbb{P}^2}^* = \mathcal{O}_{\mathbb{P}^2}(3)$ by the covering map. Since the hyperplane bundle is positive, the Chern class of $\mathcal{K}_{\mathcal{S}}^*$ has a representative, Θ , which is positive except at the branching curve where it is positive semi-definite and positive along the curve. Let \mathfrak{b} be a smooth function on \mathcal{S} which, near the branching set is the square of the distance from the branching set in some smooth metric. Then $\partial\bar{\partial}\mathfrak{b}$ is exact and positive semi-definite along the branching set and positive in directions transverse to it. For sufficiently small $\varepsilon > 0$, $(\Theta - i\varepsilon\partial\bar{\partial}\mathfrak{b})$ is a positive representative of the first Chern class of $\mathcal{K}_{\mathcal{S}}^*$. \square

Having proved that $\mathcal{K}_{\mathcal{S}}^*$ is positive for all del Pezzo surfaces, the Lefschetz hyperplane theorem (see § 1.6) may be used to derive the statement (3.1.22) about the Hodge numbers $b_{1,1}$. Since $b_{2,1} = b_{1,1} - \frac{1}{2}\chi_E$ for a Calabi-Yau 3-fold, the more general statement b) follows. \square



We now remain with the Calabi-Yau hypersurfaces embedded in $\mathcal{S} \times \mathcal{Z}$, where \mathcal{S} is any almost del Pezzo surface and \mathcal{Z} is one with $\chi_E = 12$. For these, we need the following

Lemma 3.2 *For every almost ample surface \mathcal{Z} with $\chi_E = 12$: $\begin{bmatrix} \mathcal{Z} & | & \mathcal{K}_{\mathcal{Z}}^* \\ \mathbb{P}^1 & | & \mathcal{O}(1) \end{bmatrix} = \mathcal{Z}$.*

Proof: Let $f : \mathcal{Z} \rightarrow \mathbb{P}^1$ be a map to an auxiliary $\mathbb{P}^1_{\mathcal{Z}}$. The anti-canonical bundle of \mathcal{Z} has only two sections and so is induced from $\mathcal{O}_{\mathbb{P}^1_{\mathcal{Z}}}(1)$ by the map f and we write $f : \mathcal{K}_{\mathcal{Z}}^* \xrightarrow{\sim} \mathcal{O}_{\mathcal{Z}}(1)$. Then we have

$$\begin{array}{ccc} \mathcal{Z} & \times & \mathbb{P}^1 \\ \downarrow f & & \downarrow \text{Id} \\ \mathbb{P}^1_{\mathcal{Z}} & \times & \mathbb{P}^1 \end{array} \quad (3.1.24)$$

Write the product map as $\mathfrak{f} = (f, \text{Id})$, and $\mathcal{O}_{\mathcal{Z}}(1)$ and $\mathcal{O}(1)$ for the hyperplane bundles of \mathcal{Z} and \mathbb{P}^1 respectively. Pulling back to $\mathcal{Z} \times \mathbb{P}^1$, we have that $\mathfrak{f}^*(\mathcal{O}_{\mathcal{Z}}(1), \mathcal{O}(1)) = \mathcal{K}_{\mathcal{Z}}^* \otimes \mathcal{O}(1)$. So, given a section ϕ of $\mathcal{O}_{\mathcal{Z}}(1) \otimes \mathcal{O}(1)$, $\varphi \stackrel{\text{def}}{=} \mathfrak{f}^*(\phi)$, is the pull-back section of $\mathcal{K}_{\mathcal{Z}}^* \otimes \mathcal{O}(1)$, so that the hypersurface defined as* $\varphi^{-1}(0)$

*Recall that $\varphi^{-1}(0)$ denotes the hypersurface defined by $\varphi = 0$.

is the same as $f^{-1}(\phi^{-1}(0))$. Without loss of generality, \mathbb{P}^1_Z and \mathbb{P}^1 can be identified, so that $\phi^{-1}(0) = \text{diag}(\mathbb{P}^1_Z \otimes \mathbb{P}^1)$, whereby :

$$f^{-1}(\phi^{-1}(0)) = \{(x, y) : f(x) = y\} = \{(x, f(x))\} \approx \mathcal{Z}, \quad (3.1.25)$$

with x denoting the points of \mathcal{Z} and y the points of \mathbb{P}^1 . \square

Corollary 3.1 For any almost del Pezzo surface S and \mathcal{Z} one with $\chi_E(\mathcal{Z}) = 12$, there is an almost del Pezzo surface \mathcal{Z}' with $\chi_E = 12$ such that :

$$\begin{bmatrix} S & \mathcal{K}_S^* \\ \mathcal{Z} & \mathcal{K}_{\mathcal{Z}}^* \end{bmatrix} = \begin{bmatrix} \mathcal{Z}' & \mathcal{K}_{\mathcal{Z}'}^* \\ \mathcal{Z} & \mathcal{K}_{\mathcal{Z}}^* \end{bmatrix}. \quad (3.1.26)$$

Prove this. \clubsuit

Since the $\chi_E = 12$ almost del Pezzo surfaces are not positive, the Lefschetz hyperplane theorem is not applicable to the Calabi-Yau manifolds considered in Corollary 3.1, which are all represented by the configuration

$$\begin{bmatrix} 2 & 3 & 0 \\ 2 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad (3.1.27)$$

the smooth members of which have $\chi_E = 0$. As we will discuss in more detail in Chapter 6, this can be viewed as a family of fibered spaces, where \mathbb{P}^1 is the base and the fibres are products of (possibly singular) tori; it follows that $b_{1,1} = 19$ for smooth such Calabi-Yau 3-folds [212].

We shall later return to Calabi-Yau hypersurfaces in products of surfaces to exemplify some handy computational techniques.

3.2 Complete Intersections in Products Involving Almost Fano 3-folds

A further generalization of possible embedding spaces is provided by including (products of) almost Fano 3-folds. Unlike almost del Pezzo surfaces, many almost Fano 3-folds cannot themselves be represented as a complete intersection in some product of complex projective spaces. It is therefore possible that this addition to the set of considered embedding spaces provides genuinely new Calabi-Yau 3-folds.

3.2.1 Almost Fano 3-folds—an introduction

Similarly to the treatment in § 3.1, we define an almost Fano 3-fold to be an algebraic 3-fold \mathfrak{F} , which has a non-trivial anti-canonical bundle $\mathcal{K}_{\mathfrak{F}}^*$ with at least one non-zero section at every point of \mathfrak{F} .

The Hodge diamond of an almost Fano 3-folds is*

$$\begin{array}{ccccc} & & 1 & & \\ & & 0 & 0 & \\ & 0 & b_{1,1} & 0 & \\ 0 & b_{2,1} & b_{2,1} & 0 & \implies \chi_E = 2(b_{1,1} + 1 - b_{2,1}). \\ 0 & b_{1,1} & 0 & & \\ 0 & 0 & & & \\ & & 1 & & \end{array} \quad (3.2.1)$$

We have used *Hodge star duality*^(L) and complex conjugation. We see that there are two variant Hodge numbers, rather similarly to Calabi-Yau spaces. As with almost del Pezzo surfaces, the plurigenera vanish $\mathfrak{P}_m \stackrel{\text{def}}{=} H^0(\mathfrak{F}, \mathcal{K}_{\mathfrak{F}}^{\otimes m}) = 0$ for all $m > 0$. Now all three Chern numbers C_1^3 , $C_1 \cdot C_2$ and $C_3 = \chi_E$ are non-vanishing. However, we have the following simple result.

Lemma 3.3 $C_1 \cdot C_2 = f_{\mathfrak{F}}(c_1 \wedge c_2) = 24$ for all almost Fano 3-folds \mathfrak{F} .

Proof: This follows from the above Hodge diamond 3.2.1, using Eqs. (1.5.12) and (2.5.2); to illustrate a useful idea however, another proof is included.

We embed a model of the K3 surface in \mathfrak{F} as a hypersurface, using a section of the anti-canonical bundle $\mathcal{K}_{\mathfrak{F}}^*$; then $\mathcal{N}_{\mathfrak{F}/\text{K3}} = \mathcal{K}_{\mathfrak{F}}^*$. Writing $c(\mathfrak{F}) = 1 + f_1 + f_2 + f_3$ and using Eq. (1.3.20), we know that $c_1[\mathcal{N}_{\mathfrak{F}/\text{K3}}] = f_1$. Thus,

$$c(\text{K3}) = \frac{c[\mathfrak{F}]}{c[\mathcal{N}_{\mathfrak{F}/\text{K3}}]} = 1 + f_2. \quad (3.2.2)$$

Therefore

$$24 = \chi_E(\text{K3}) = \int_{\text{K3}} f_2 = \int_{\mathfrak{F}} c_1[\mathcal{N}_{\mathfrak{F}/\text{K3}}] \wedge f_2 = \int_{\mathfrak{F}} (f_1 \wedge f_2). \quad (3.2.3)$$

*This is easy to establish using the Kodaira-Nakano Theorem 3.3, presented below. \square

In addition, there is another numerical characteristic of complex varieties, called the *index*. It is the maximal integer $r > 0$, for which there exists a line bundle \mathcal{L} over our almost Fano 3-fold \mathfrak{F} such that $\mathcal{L}^{\otimes r} \approx \mathcal{K}_{\mathfrak{F}}^*$. In analogy, we define the *multiindex* $\vec{r} = (r_1, \dots, r_k)$ where r_a are maximal integers for which $\mathcal{K}_{\mathfrak{F}}^* \approx \bigotimes_{a=1}^k \mathcal{L}_a^{\otimes r_a}$ and \mathcal{L}_a are independent line bundles. For example, $r(\mathbb{P}^n) = n+1$ because $\mathcal{K}_{\mathbb{P}^n} \approx \mathcal{O}_{\mathbb{P}^n}(n+1)$ and $\vec{r}(\mathbb{P}^m \times \mathbb{P}^n) = (m+1, n+1)$ since $\mathcal{K}_{\mathbb{P}^m \times \mathbb{P}^n} \approx \mathcal{O}_{\mathbb{P}^m}(m+1) \otimes \mathcal{O}_{\mathbb{P}^n}(n+1)$. Note however that $r(\mathbb{P}^m \times \mathbb{P}^n) = \gcd(m+1, n+1)$. Given these “root” line bundles \mathcal{L} , i.e., \mathcal{L}_a used to define the index r and the multiindex \vec{r} , we can also define the *degrees*

$$d \stackrel{\text{def}}{=} \mathcal{L}^3 \stackrel{\text{def}}{=} \int_{\mathfrak{F}} c_1[\mathcal{L}]^3 , \quad (3.2.4)$$

$$d_{abc} \stackrel{\text{def}}{=} \mathcal{L}_a \cdot \mathcal{L}_b \cdot \mathcal{L}_c \stackrel{\text{def}}{=} \int_{\mathfrak{F}} c_1[\mathcal{L}_a] \wedge c_1[\mathcal{L}_b] \wedge c_1[\mathcal{L}_c] . \quad (3.2.5)$$

Unfortunately, all these characteristics, $C_1^3, b_{1,1}, b_{2,1}$ and in addition r, \vec{r}, d, d_{abc} , are not enough to label all the Fano 3-folds unambiguously (see Ref. [32] for the 18 Fano 3-folds with $b_{1,1} = 1$ and Ref. [40] for the 87 ones with $2 \leq b_{1,1} \leq 10$). Allowing also non-ample almost ample 3-folds of course only complicates the classification; this remains an open subject.

3.2.2 Some general theorems

A little thought reveals that the tricks we had used to derive Eq. (3.1.3) and here Lemma 3.3 indicate a more general framework. Indeed, we quote from Ref. [163], p.149, without proof :

Theorem 3.2 (Riemann-Roch-Hirzebruch) *Let X be an algebraic manifold of dimension n and let \mathcal{L} be a holomorphic line bundle over X , with $c_1[\mathcal{L}] = \lambda$. The cohomology groups $H^q(X, \mathcal{L})$ are finite dimensional and the (generalized) Euler-Poincaré characteristic is*

$$\sum_{q=0}^n (-)^q H^q(X, \mathcal{L}) \stackrel{\text{def}}{=} \chi_E(X, \mathcal{L}) = \int_X \text{ch}[\mathcal{L}] \wedge \text{td}(X) \stackrel{\text{def}}{=} \int_X \left[e^{\lambda} \prod_{i=1}^n \frac{\ell_i}{1 - e^{-\ell_i}} \right] , \quad (3.2.6)$$

where

$$c(X) = 1 + c_1 + c_2 + \dots + c_n \stackrel{\text{def}}{=} \prod_{i=1}^n (1 + \ell_i) . \quad (1.3.19')$$

Note that integration over X picks out terms which are of degree n in $\{\lambda, \ell_i\}$. This is the celebrated Riemann-Roch theorem, generalized by Hirzebruch to manifolds X of arbitrary dimension.

Since the relation between ℓ_i and c_i is somewhat involved, we also quote (Ref. [163], p.5) the expressions for cases $\dim X = 1, 2$ and 3 , respectively, as indicated by the subscript of X :

$$\chi_E(X_{(1)}, \mathcal{L}) = \int_X \left(\lambda + \frac{1}{2} c_1 \right) , \quad (3.2.7)$$

$$\chi_E(X_{(2)}, \mathcal{L}) = \int_X \left(\frac{1}{2} (\lambda^2 + \lambda c_1) + \frac{1}{12} (c_1^2 + c_2) \right) , \quad (3.2.8)$$

$$\chi_E(X_{(3)}, \mathcal{L}) = \int_X \left(\frac{1}{6} \lambda^3 + \frac{1}{4} \lambda^2 c_1 + \frac{1}{12} \lambda (c_1^2 + c_2) + \frac{1}{24} c_1 c_2 \right) . \quad (3.2.9)$$

Armed with this theorem, the Reader should have no difficulty recovering Eq. (3.1.3) and Lemma 3.3, by choosing $\mathcal{L} = \mathcal{O}_X$, the trivial holomorphic line bundle. Then, quite straightforwardly, $\chi_E(X, \mathcal{O}) \equiv \chi^h(X) = \dim H^0(X) = 1$ for connected almost ample X .

Another very useful result is the *Kodaira-Nakano vanishing theorem*, which we quote (Ref. [23], p.154) without proof :

Theorem 3.3 (Kodaira-Nakano) *If \mathcal{L} is a positive holomorphic line bundle over the compact complex algebraic manifold X of dimension n , then*

$$H^q(X, \mathcal{L} \otimes \wedge^p T_X^*) = 0 \quad \text{for } p+q > n . \quad (3.2.10)$$

A useful fact to recall when using this theorem is that $\wedge^n T_X^* \stackrel{\text{def}}{=} \mathcal{K}_X$ is the canonical line bundle. In particular :

⇒ Derive that

$$\mathcal{L}^* > 0 \implies H^q(X, \mathcal{L}) = 0 , \quad q \neq n ; \quad (3.2.11)$$

$$\mathcal{L} \otimes \mathcal{K}_X^* > 0 \implies H^q(X, \mathcal{L}) = 0 , \quad q \neq 0 . \quad (3.2.12)$$

It is now straightforward to compute the number of sections of the m^{th} power of the anti-canonical bundle (anti-plurigenera), when the anti-canonical bundle is positive :

$$\dim S = 2 : \mathfrak{P}^m(S) = \frac{1}{2} m(m+1) C_1^2 + 1 , \quad (3.2.13)$$

$$\dim \mathfrak{F} = 3 : \mathfrak{P}^m(\mathfrak{F}) = \frac{1}{12} m(m+1)(2m+1) C_1^3 + (2m+1) . \quad (3.2.14)$$

Note that $m(m+1) \in 2\mathbb{Z}$ and $m(m+1)(2m+1) \in 6\mathbb{Z}$, so that the obvious integrality of \mathfrak{P}^m enforces $C_1^3(\mathfrak{F}) \in 2\mathbb{Z}$. This leads to

Definition. The integer $g \stackrel{\text{def}}{=} 1 + \frac{1}{2}C_1^n$ is called the genus of an n -fold.

(This agrees with $n = 1$, $g = 1 + \frac{1}{2}\chi_E$ for Riemann surfaces.)

Now, sections of \mathcal{K}_X^* may be used to construct defining polynomials. While $C_1^n > 0$, \mathcal{K}_X^* is positive, the Kodaira-Nakano theorem guarantees that $H^q(S, \mathcal{K}_S^*)$ is non-zero only for $q = 0$, and $\chi_E(\mathcal{X}_{(n)}, \mathcal{K}_S^*) = \mathbb{P}^1(\mathcal{X}_{(n)})$. However, when $C_1^n = 0$, \mathcal{K}_X^* is not positive, the Kodaira-Nakano theorem does not apply, $H^q(S, \mathcal{K}_S^*)$ is non-zero not only for $q=0$, so $\chi_E(\mathcal{X}_{(n)}, \mathcal{K}_S^*) \neq \mathbb{P}^1(\mathcal{X}_{(n)})$. For the almost del Pezzo surface with $C_1^2 = 0$, we have $\mathbb{P}^1(S) = 1$, but there does exist another section; see Propositions 3.2 and 3.3. For the almost Fano 3-folds with $C_1^3 = 0$, $\mathbb{P}^1(\mathfrak{F}) = 3$, but $\dim H^0(S, \mathcal{K}_S^*)$ is only 2-dimensional! The \mathcal{K}_X^* -valued cohomology will be calculated in § 9.6; suffice it here to observe that for all almost- but non-ample complete intersections of the type (3.2.23), $\left[\begin{smallmatrix} X & | & c_1(X) \\ \mathbb{P}^1 & | & 1 \end{smallmatrix} \right]$, the anti-canonical bundle is (the pull-back of) $\mathcal{O}_{\mathbb{P}^1}(1)$, and so has precisely two sections. In any case, note that we do not really need to know a polynomial representation in terms of some suitable coordinates; many computations can be completed at a rather formal level.

3.2.3 Fano 3-folds

Recall that with almost ample *surfaces*, there was only one addition to the list of ample surfaces and all of them were represented by complete intersections in products of complex projective spaces (3.1.19). With almost Fano 3-folds, there are more news.

Standard complete intersection Fano 3-folds. For one thing, as mentioned by Iskovskih, seven of the 18 Fano 3-folds with $b_2 = 1$ can be represented as complete intersections in a \mathbb{P}^n and are here given in Table 3.1. Of course, complete intersections in products of more than one \mathbb{P}^n would have $b_2 > 1$. So, with 11 Fano 3-folds with $b_2 = 1$ which are *not* complete intersections in a projective space, their inclusion as factors in the embedding space already enriches our menagerie.

Fano 3-fold branched coverings. A complete description of all 105 Fano 3-folds would take us far afield. Let us just note that some of them are easily described as multiple covers of \mathbb{P}^3 , branched over certain surfaces in \mathbb{P}^3 . This type of construction will be utilized in § 5.5 to produce Calabi-Yau 3-folds. Here we recall the discussion on page 79, shifted now into the context of 3-folds.

C_1^3	χ_E	$b_{2,1}$	g	r	d	Description
64	4	0	33	4	1	\mathbb{P}^3 ;
54	4	0	28	3	2	$[4 2]$: a quadric in \mathbb{P}^4 —also the double cover of \mathbb{P}^3 , branched over $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$;
24	-6	5	13	2	3	$[4 3]$: a cubic in \mathbb{P}^4 —also the triple cover of \mathbb{P}^3 , branched over a smooth cubic in \mathbb{P}^3 ;
32	0	2	17	2	4	$[5 2 2]$: the intersection of two quadrics in \mathbb{P}^4 ;
4	-56	30	3	1	4	$[4 4]$: a quartic in \mathbb{P}^4 —also the quadruple cover of \mathbb{P}^3 , branched over $K3 \subset \mathbb{P}^3$;
6	-36	20	4	1	6	$[5 3 2]$: the intersection of a quadric and a cubic in \mathbb{P}^5 ;
8	-24	14	5	1	8	$[6 2 2 3]$: the intersection of three quadrics in \mathbb{P}^6 ;

Table 3.1: Fano 3-fold complete intersections in \mathbb{P}^n .

Consider the simple equation

$$z^k + B(x) = 0, \quad B(x) \stackrel{\text{def}}{=} \sum_{i=0}^n (x^i)^{kl}, \quad (3.2.15)$$

where $x \in \mathbb{P}^n$; $B(x)$ may be deformed into any other degree- kl polynomial, of course. While $B(x) \neq 0$, $z = -\sqrt[k]{B(x)}$ are k distinct solutions per point $x \in \mathbb{P}^n$; at $B(x) = 0$, $z=0$ is the only solution for z . The space of solutions thereby becomes a k -fold cover of \mathbb{P}^n , branched over the hypersurface $\mathcal{B} \subset \mathbb{P}^n$, defined by $B(x)=0$.

Theorem 3.4 Let $\varpi: \bar{\mathcal{X}}_{\mathcal{B}}^{(k)} \rightarrow \mathcal{X}$ denote the k -fold cover of \mathcal{X} , branched over $\mathcal{B} \subset \mathcal{X}$, where \mathcal{B} is defined as the zero-set of a section of a line bundle $\mathcal{L}^{\otimes k}$. Then

$$\chi_E(\bar{\mathcal{X}}_{\mathcal{B}}^{(k)}) = k\chi_E(\mathcal{X}) - (k-1)\chi_E(\mathcal{B}), \quad (3.2.16)$$

$$\mathcal{K}_{\bar{\mathcal{X}}_{\mathcal{B}}^{(k)}}^* = \varpi^*(\mathcal{K}_{\mathcal{X}}^* \otimes \mathcal{L}^{*\otimes(k-1)}). \quad (3.2.17)$$

Sketch of proof: The first result is clear; $\bar{\mathcal{X}}_{\mathcal{B}}^{(k)}$ is constructed by (a) carving \mathcal{B} out of \mathcal{X} , (b) copying $(\mathcal{X}-\mathcal{B})$ k times and then (c) gluing a single copy of \mathcal{B} back. Since the Euler characteristic is additive under surgery[†], we have

$$\chi_E(\bar{\mathcal{X}}_{\mathcal{B}}^{(k)}) = k[\chi_E(\mathcal{X}) - \chi_E(\mathcal{B})] + \chi_E(\mathcal{B}). \quad (3.2.18)$$

[†]This can be shown, for example, by triangulation.

For a proof of the second statement, the Reader is referred to Ref. [41], p.109, or Ref. [31], Theorem 5.5 on p.202, which give the “ramification formula”

$$\mathcal{K}_{\overline{\mathcal{X}}^{(k)}} = \varpi^*(\mathcal{K}_X) \otimes \mathcal{L}(\mathcal{B}_\varpi), \quad (3.2.19)$$

where $\mathcal{L}(\mathcal{B}_\varpi)$ is the line bundle corresponding to the *branching locus*, \mathcal{B}_ϖ . Here we need the following

Definition. *The branching locus of the covering map ϖ , denoted \mathcal{B}_ϖ , is the divisor obtained from $\mathcal{B} \subset X$ by counting the points of \mathcal{B} with multiplicity $k-1$.*

Since the multiplicity of $\varpi(p)$, $\mu[\varpi(p)]$, has an invariant meaning for all $p \in \overline{\mathcal{X}}_B^{(k)}$, we count the points with multiplicity $(\mu[\varpi(p)]-1)$: only those in \mathcal{B} remain, and with multiplicity $k-1$, as defined for \mathcal{B}_ϖ . But then \mathcal{B}_ϖ corresponds to $\mathcal{L}(\mathcal{B}_\varpi) = \mathcal{L}^{\otimes(k-1)}$ —rather than $\mathcal{L}^{\otimes k}$ —and the second statement follows. \square

Using this theorem, it is not hard to construct various multiple branched covers of the Fano 3-folds, keeping the anti-canonical bundle under control. We will use this to construct Calabi-Yau 3-folds as suitably branched multiple covers, but can clearly also construct many Fano and many almost (but not) Fano 3-folds. In fact, the proof of Lemma 3.1 is related to Theorem 3.4; the $(1,1)$ -form $\partial\bar{\partial}\psi$ corresponds to $\mathcal{L}(\mathcal{B}_\varpi)$.

Looking back at Eq. (3.2.15), we see that for $l=1$, z can be made into the $(n+2)^{th}$ homogeneous coordinate of a \mathbb{P}^{n+1} , in addition to the $n+1$ x 's. Thus, these branched multiple covers straightforwardly become identified with simple hypersurfaces in \mathbb{P}^{n+1} . Now, for $l > 1$, the z and the x 's scale differently, so they cannot span a simple \mathbb{P}^{n+1} . However, they can span a *weighted complex projective space*, $\mathbb{P}_{(1,\dots,1,l)}^{n+1}$ (see Chapter 5), where

$$(x^0 : \dots : x^n : z) \cong (\lambda x^0 : \dots : \lambda x^n : \lambda^l z). \quad (3.2.20)$$

Eq. (3.2.15) is invariant with respect to such a scaling transformation. So, for $k=2$ and $l=2, 3$, respectively, we obtain

$$[\mathbb{P}_{(1:1:1:1:2)}^4 \parallel 4] \cong [\overline{\mathbb{P}}_{[3][4]}^3]^2, \quad [\mathbb{P}_{(1:1:1:1:3)}^4 \parallel 6] \cong [\overline{\mathbb{P}}_{[3][6]}^3]^2, \quad (3.2.21)$$

double covers of \mathbb{P}^3 , branched over a quartic and over a sextic surface in \mathbb{P}^3 , with $\mathcal{K}^* = \mathcal{O}_{\mathbb{P}^3}(2)$ and $= \mathcal{O}_{\mathbb{P}^3}(1)$, by Theorem 3.4.

Other Fano 3-folds. In addition, there are Fano 3-folds which are a little more complicated to describe. Suffice it here to mention three Fano 3-folds of $b_{1,1} = 1$ the description of which involves Grassmannians. Just like one uses the components of a complex $(n+1)$ -vector for the homogeneous coordinates of \mathbb{P}^n , k complex n -vectors $[\vec{z}_i]$ are needed to provide coordinates for the Grassmannian $G_{(k,n)}$ of k -planes through the origin of \mathbb{C}^n . Recall that $\dim G_{(k,n)} = k(n-k)$, which is easily seen by noting that a coordinate patch is chosen, up to a permutation of columns, as

$$\begin{pmatrix} \vec{z}_1 \\ \vec{z}_2 \\ \vdots \\ \vec{z}_k \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \zeta_{1,k+1} & \cdots & \zeta_{1,n} \\ 0 & 1 & 0 & \cdots & 0 & \zeta_{2,k+1} & \cdots & \zeta_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \zeta_{k,k+1} & \cdots & \zeta_{k,n} \end{pmatrix}, \quad (3.2.22)$$

leaving $k(n-k)$ local coordinates. Now, regarding the $\binom{n}{k}$ quantities $Z_{[j_1, \dots, j_k]} \stackrel{\text{def}}{=} e^{i_1 \dots i_k} z_{i_1, j_1} \dots z_{i_k, j_k}$ as homogeneous coordinates of a $\mathbb{P}^{\binom{n}{k}-1}$ defines the *Plücker embedding* $G_{(k,n)} \xrightarrow{\sim} \mathbb{P}^{\binom{n}{k}-1}$.

So, for example, $G_{(2,5)} \xrightarrow{\sim} \mathbb{P}^9$ embeds this 6-dimensional Grassmannian in \mathbb{P}^9 . Intersecting it with three hyperplanes in \mathbb{P}^9 yields a 3-fold which is ample [32]. Similarly, Fano 3-folds are also obtained by intersecting $G_{(2,5)} \xrightarrow{\sim} \mathbb{P}^9$, two hyperplanes and a quadric hypersurface in \mathbb{P}^9 or intersecting $G_{(2,6)} \xrightarrow{\sim} \mathbb{P}^{14}$ with five hyperplanes in \mathbb{P}^{14} . By construction, the hyperplane bundle on the Plücker $\mathbb{P}^{\binom{n}{k}-1}$ equals the determinant of the dual of the rank- k *universal sub-bundle* ${}^{(L)}$ —on Grassmannians, the analogue of $\mathcal{O}_{\mathbb{P}^n}(1)$; see Eq. (3.3.16). Therefore, these three Fano 3-folds can be described as complete intersections in Grassmannians, as presented in Table 3.2. We will return to these below.

Complementing these simple constructions, Ref. [32] and also Ref. [40] present several rather more involved cases, the description of which is beyond our present scope.

3.2.4 Almost but not Fano 3-folds

On the other hand, from any almost Fano 3-fold, we can easily construct another one which surely is not Fano, that is, it's non-ample. We now describe this. Let \mathfrak{F} be an almost Fano 3-fold with $c(\mathfrak{F}) = 1 + f_1 + f_2 + f_3$. Consider then $\mathfrak{F} \times \mathbb{P}^1$ and define a hypersurface using a section of $\mathcal{K}_{\mathfrak{F}}^* \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. Stretching the configuration

C_1^3	χ_E	$b_{2,1}$	g	r	d	Description
40	4	0	21	2	5	The intersection of three hyperplanes and the Grassmannian $G_{(2,5)} \subset \mathbb{P}^9$, also $[G_{(2,5)}][3\mathcal{L}]$;
10	-6	5	6	1	10	The intersection of two hyperplanes, a quadric hypersurface and the Grassmannian $G_{(2,5)} \subset \mathbb{P}^9$; also $[G_{(2,5)}][2\mathcal{L} \oplus \mathcal{L}^{\otimes 2}]$;
14	-16	10	8	1	14	The intersection of five hyperplanes and the Grassmannian $G_{(2,5)} \subset \mathbb{P}^9$, also $[G_{(2,6)}][5\mathcal{L}]$;

$\mathcal{L} \stackrel{\text{def}}{=} \det \mathcal{S}^*$, see Eq. (3.3.16); also, $3\mathcal{L} \stackrel{\text{def}}{=} \mathcal{L} \oplus \mathcal{L} \oplus \mathcal{L}$.

Table 3.2: Fano 3-fold complete intersections in Grassmannians.

notation in an obvious way, we have

$$\begin{aligned} c(\tilde{\mathfrak{F}}) &= c\left[\begin{array}{c|c} \tilde{\mathfrak{F}} & \mathfrak{f}_1 \\ \hline \mathbb{P}^1 & J \end{array}\right] = \frac{(1 + \mathfrak{f}_1 + \mathfrak{f}_2 + \mathfrak{f}_3)(1 + 2J)}{(1 + \mathfrak{f}_1 + J)} \\ &= 1 + [J] + [\mathfrak{f}_2 + \mathfrak{f}_1 J] + [(\mathfrak{f}_3 - \mathfrak{f}_1 \mathfrak{f}_2) + (\mathfrak{f}_2 - \mathfrak{f}_1^2)J], \end{aligned} \quad (3.2.23)$$

where J represents $\mathcal{O}_{\mathbb{P}^1}(1)$. Straightforwardly,

1. $C_1^3(\tilde{\mathfrak{F}}) = 0$;
2. $\chi_E(\tilde{\mathfrak{F}}) = \chi_E(\mathfrak{F}) - C_1^3(\mathfrak{F})$;
3. $b_{1,1}(\tilde{\mathfrak{F}}) = b_{1,1}(\mathfrak{F}) + 1$ if \mathfrak{F} is in fact ample.

The condition in the last result is related to the fact (2.3.6) and says that simple iteration of this construction yields nothing new.

The class of almost Fano 3-folds we have just defined has a rather simple and more intrinsic geometric description. Consider the defining equation for $\tilde{\mathfrak{F}}$:

$$K_0(z)x^0 + K_1(z)x^1 = 0, \quad z \in \tilde{\mathfrak{F}}, \quad x \in \mathbb{P}^1; \quad (3.2.24)$$

the $K_i(z)$ are sections of $\mathcal{K}_{\mathfrak{F}}^*$, as is dictated by $\mathfrak{f}_1 = c_1(\mathfrak{F})$. For a generic point $z \in \tilde{\mathfrak{F}}$, the above equation is simply a linear equation in two variables, x^0 and x^1 and has a single solution. At the special curve $C \subset \tilde{\mathfrak{F}}$, where $K_0(z) = 0 = K_1(z)$, the variables $x \in \mathbb{P}^1$ are not constrained at all. Thus, $\tilde{\mathfrak{F}}$ is isomorphic to \mathfrak{F} except

3.2. Complete Intersections in Products Involving Almost Fano 3-Folds

along $C \subset \tilde{\mathfrak{F}}$, where each point of C is replaced by a copy of \mathbb{P}^1 . We say that $\widetilde{\mathfrak{F}}_C$ is a *blow-up*^(L) of \mathfrak{F} along $C \subset \mathfrak{F}$.

So, there are also non-ample almost Fano 3-folds extending the lists of Iskovskikh and Mori and Mukai. A simple example is provided by

$$\begin{aligned} \widetilde{\mathbb{P}^3} \in \left[\begin{array}{c|c} 3 & 4 \\ \hline 1 & 1 \end{array} \right], \quad \chi_E(\widetilde{\mathbb{P}^3}) &= -60, \quad b_{1,1}(\widetilde{\mathbb{P}^3}) = 2, \\ C_1^3(\widetilde{\mathbb{P}^3}) &= 0, \quad b_{2,1}(\widetilde{\mathbb{P}^3}) = 33. \end{aligned} \quad (3.2.25)$$

Prove that this is a blow-up of \mathbb{P}^3 along a curve of genus $g = 34$. [Hint: the center of the blow-up is a curve which is an intersection of two quartic surfaces (why?). Also, remember that for curves, $\chi_E = 2(2 - g)$.] \spadesuit

A classification of almost Fano 3-folds will have to be rather more involved than that of the ample subset. For example, consider the set of complete intersections

$$\begin{aligned} \widetilde{\mathfrak{F}} \in \left[\begin{array}{c|c} 3 & k \\ \hline 1 & d \end{array} \right], \quad \chi_E(\widetilde{\mathfrak{F}}) &= 2[(6 - 4k + k^2) + 2d(1 - k)^3], \\ C_1^3(\widetilde{\mathfrak{F}}) &= 2(4 - k)^2(2d + 3k - 2dk); \end{aligned} \quad (3.2.26)$$

also, $b_{1,1}(\widetilde{\mathfrak{F}}) = 2$ and $b_{2,1}(\widetilde{\mathfrak{F}}) = (1 - k)(3 - 3k + k^2 + 2d(1 - k)^2)$. For $k = 1$, $\chi_E = 6$ and $C_1^3 = 54$ regardless of the value of d . Clearly, the spaces with various different $d \in \mathbb{Z}_+$ are not the same : the index $r_{d=1} = 1$ while $r_{d=2} = 3$. Also, for $d = 1$, $\widetilde{\mathfrak{F}}$ is a blow-up of \mathbb{P}^3 along a line $\mathbb{P}^1 \subset \mathbb{P}^3$. However, for $d = 2$, $\widetilde{\mathfrak{F}}$ is a double cover of \mathbb{P}^3 , branched over a quartic surface which is singular at a point and where $\widetilde{\mathfrak{F}}$ contains a \mathbb{P}^1 in place of the singularity. Of course, only the complete intersections with $d \leq 2$ and $k \leq 4$ are almost ample (and Calabi-Yau if $d = 2$, $k = 4$). \spadesuit

Prove this. \spadesuit

3.2.5 Calabi-Yau from almost Fano 3-folds. I

One can of course consider Calabi-Yau complete intersections in spaces which contain almost Fano 3-fold factors and the simplest such constructions are simple hypersurfaces in the product $\mathfrak{F} \times \mathbb{P}^1$.

Writing again $c(\tilde{\mathfrak{F}}) = 1 + \mathfrak{f}_1 + \mathfrak{f}_2 + \mathfrak{f}_3$, we consider

$$\begin{aligned} c(\mathcal{M}) = c\left[\begin{array}{c|c} \tilde{\mathfrak{F}} & \mathfrak{f}_1 \\ \hline \mathbb{P}^1 & 2J \end{array}\right] &= \frac{(1 + \mathfrak{f}_1 + \mathfrak{f}_2 + \mathfrak{f}_3)(1 + 2J)}{(1 + \mathfrak{f}_1 + 2J)} \\ &= 1 + [\mathfrak{f}_2 + 2\mathfrak{f}_1 J] + [(\mathfrak{f}_3 - \mathfrak{f}_1 \mathfrak{f}_2) - 2\mathfrak{f}_1^2 J]. \end{aligned} \quad (3.2.27)$$

It follows that

1. $\chi_E(\mathcal{M}) = 2\chi_E(\mathfrak{F}) - 2C_1^3(\mathfrak{F}) - 48$;
2. $b_{1,1}(\mathcal{M}) = b_{1,1}(\mathfrak{F}) + 1$ if \mathfrak{F} is in fact ample.

Referring to the fact (2.3.6), the condition in the last result is sufficient, not necessary; there may be cases in which it could be relaxed.

Using these results and the data in Tables 3.1 and 3.2, a short list of Calabi-Yau manifolds of the type (3.2.27) are obtained. The three Calabi-Yau manifolds obtained in this manner from the Fano 3-folds in Table 3.2 are not related to complete intersections in products of complex projective spaces in any simple way; we find $\chi_E = -120$, -80 and -108 and $b_{1,1} = 2$ for all three.



Just as the classification of almost del Pezzo surfaces may serve to derive identities between various almost ample 2-fold configurations[†], a future classification of almost Fano 3-folds can be used to derive identities between almost ample 3-fold configurations. Such identities can then be used to derive identities between various complete intersection constructions of Calabi-Yau 3-folds (and clearly, other varieties of the Reader's fancy) much the same as was done in proving Lemma 2.2.

Such identities are quite useful when looking for new Calabi-Yau 3-folds. For example, the class of non-ample almost Fano 3-folds $\tilde{\mathfrak{F}}$ defined in Eq. (3.2.23) will not yield any new Calabi-Yau 3-folds. To see this, note firstly that the anti-canonical bundle of $\tilde{\mathfrak{F}}$ is completely induced from \mathbb{P}^1 and is essentially $\mathcal{O}_{\mathbb{P}^1}(1)$. In the diagrammatic language of Chapter 2, we have that

$$\tilde{\mathfrak{F}} \approx \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (3.2.28)$$

Therefore, thought of as a subdiagram of a Calabi-Yau diagram, $\tilde{\mathfrak{F}}$ perfectly fits our Lemma 2.1. We can therefore conclude that

$$\left[\begin{array}{c|cc} \mathfrak{F} & 0 & f_1 \\ \mathbb{P}^1 & J & J \\ \hline \text{whatever} & 0 & 0 \end{array} \right]_{\text{Calabi-Yau}} \cong \left[\begin{array}{c|c} \mathfrak{F} & f_1 \\ \mathbb{P}^1 & 2J \end{array} \right]. \quad (3.2.29)$$

[†]We now, of course, extend the meaning of a configuration to include products of complex projective spaces, almost del Pezzo surfaces and also almost Fano 3-folds.

3.3. Kähler C-Spaces

In other words, the (deformation class of) Calabi-Yau 3-folds obtained as a complete intersection in however complicated embedding space with $\tilde{\mathfrak{F}}$ as one of the factors, can just as well be obtained as a simple hypersurface in $\mathfrak{F} \times \mathbb{P}^1$!

While we have just made the class of non-ample almost Fano 3-folds $\tilde{\mathfrak{F}}$ obsolete for constructing *new* Calabi-Yau spaces, there are many other non-ample almost Fano 3-folds the inclusion of which in considered embedding spaces may yield essentially new Calabi-Yau 3-folds. For instance, in the notation of § 3.2.3, the list of *splits*^(L)

$$\tilde{\mathfrak{F}}^{(n)} \in \left[\begin{array}{c|ccc} \mathfrak{F} & a_1 & \cdots & a_n \\ \mathbb{P}^n & J & \cdots & J \\ \hline & & & \end{array} \right] \quad \sum_{i=1}^n a_i = f_1, \quad n > 1, \quad (3.2.30)$$

provides non-ample almost Fano 3-folds for which the Calabi-Yau hypersurfaces in $\tilde{\mathfrak{F}}^{(n)} \times \mathbb{P}^1$ are not as trivially related to those in $\mathfrak{F} \times \mathbb{P}^1$ as when $n = 1$.

3.3 Kähler C-Spaces

In Chapter 2, we studied complete intersections in products of complex projective spaces. It appears natural to generalize this to products of arbitrary simply connected compact homogeneous spaces, which were classified and called *C-spaces* by Wang [231]. Baston and Eastwood [5] give a construction of Kähler (indeed, projective) such spaces which will be quite more familiar to most physicists, generalizing the notion of (complex) projective spaces, Grassmannians, ... into *generalized flag varieties*; we will generally denote them by \mathbb{F} . After some general definitions and remarks about such spaces, we will consider here only the simple case of single (Calabi-Yau) hypersurface and restrict moreover to the case of $b_2(\mathbb{F}) = 1$, as in Ref. [180].

3.3.1 Kähler C-spaces as flag spaces

General theory [231] says that every *C-space* \mathbb{F} , that is, simply connected compact homogeneous complex space is homeomorphic to a torus-bundle over a product of certain coset spaces $\{G/H\}$. A torus-bundle means that tori are being fibered over the base space, whence simply connected *C-spaces* are just the $\{G/H\}$. For all the coset spaces in this construction, G is a compact simple Lie group and H is a regular semi-simple subgroup.

A little Lie algebra and group theory reminder.

As customary, for a semi-simple Lie group G , \mathfrak{g} will denote the algebra spanned by

the generators \mathfrak{t} . For a complex(ified) \mathfrak{g} , the generators may be partitioned into* $\mathfrak{h} \oplus \mathfrak{r}_+ \oplus \mathfrak{r}_-$, where \mathfrak{h} are the Cartan generators and \mathfrak{r}_\pm are the positive (negative) roots, that is, \mathfrak{h} are the charge operators and \mathfrak{r}_\pm are the raising (lowering) operators.

The prototype or *standard Borel subalgebra* is $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{r}_+$. For example, for $\mathfrak{sl}(n, \mathbb{C})$, \mathfrak{h} consists of all the diagonal and \mathfrak{r}_+ of all strictly upper triangular $n \times n$ matrices. All Borel subalgebras are G -conjugate to \mathfrak{b} . A *parabolic subalgebra* \mathfrak{p} then satisfies $\mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$. Schematically, \mathfrak{p} consists some negative roots (from \mathfrak{r}_-) in addition to all elements in $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{r}_+$.

By exponentiation, these subalgebras give rise to subgroups B and P of the Lie group $G_{\mathbb{C}}$; all Borel (parabolic) subgroups of $G_{\mathbb{C}}$ are equivalent by $G_{\mathbb{C}}$ -conjugation.

For a given complex semi-simple Lie group $G_{\mathbb{C}}$ and some parabolic subgroup P thereof, one defines the homogeneous coset space $\mathbb{F}_{[P \subset G]} = G_{\mathbb{C}}/P$ and we state here without proof that it is simply connected, compact, has a natural complex structure induced from the Hermitian structure on $G_{\mathbb{C}}$ and is in fact a Kähler manifold of complex dimension $\frac{1}{2}(\dim G_{\mathbb{C}} - \dim P)$. Often, there exists a maximal regular subgroup $H \subset G$ such that $G/H \approx G_{\mathbb{C}}/P$; in fact, the simplest C -space is $\mathbb{P}^n = \{\frac{U(n+1)}{U(1) \times U(n)}\}$.

More generally, we will write $N = \sum_{f=1}^F n_f$ and

$$\mathbb{F}_{[n_1, \dots, n_F]} \stackrel{\text{def}}{=} \left\{ \frac{U(N)}{U(n_1) \times \dots \times U(n_F)} \right\}; \quad (3.3.1)$$

Baston and Eastwood show that specifying the inclusion $P \subset G_{\mathbb{C}}$ characterizes $\mathbb{F}_{[P \subset G]}$. Next, we note that

$$\mathbb{F}_{[n_1, \dots, n_F]} \approx \{V_{n_1} \subset V_{(n_1+n_2)} \subset \dots \subset V_N = V\} \quad (3.3.2)$$

where V_n is an n -dimensional linear subspace of V . A particular choice of such nested subspaces is called a *flag* and the space of such flags, identifiable with the quotient $\mathbb{F}_{[n]}$, is called the *flag variety*. The notion of a *generalized flag variety* of Baston and Eastwood stems from extending this to general quotients $\{G_{\mathbb{C}}/P\}$ of compact complex semi-simple Lie groups by parabolic subgroups, thus covering Wang's classification of Kähler C -spaces.

*We forego a mathematically rigorous exposition here, chapters 2 and 3 of Ref. [5] provide all the necessary details. Note however that without \mathfrak{g} being complex(ified), \mathfrak{r}_+ and \mathfrak{r}_- would not be possible to split asunder; write $G_{\mathbb{C}}$ to remind of this.

3.3. Kähler C -Spaces

An important simplification occurs if we require in addition that $b_2(\mathbb{F}) = 1$; such C -spaces were called *simple* in Ref. [180]. Then $\mathbb{F}_{[n]} = G_{\mathbb{C}}/P$, where P is a *maximal* parabolic subgroup of the simple complex Lie group $G_{\mathbb{C}}$. The list of such spaces is of course infinite since the list of Lie groups is, providing us therefore with an enormous arena for constructing Calabi-Yau 3-folds (and other varieties at the Reader's preference). Nevertheless, Ref. [180] reports that a generalization of the proof presented in § 2.3 ensures that there is a finite number of Calabi-Yau 3-fold hypersurfaces in products of simple Kähler C -spaces.

3.3.2 Homogeneous bundles over flag varieties

In general, a holomorphic vector bundle over some complex base space X is a holomorphically varying family of vector spaces parametrized by the points of X . That is, to every point in X we associate a vector space (fibre) and require that as we move from point to point in X , the corresponding vector spaces move into each other in holomorphic manner.

Take now $X = \{G_{\mathbb{C}}/P\}$, think of it as the space of P -orbits in $G_{\mathbb{C}}$ and consider a vector bundle \mathcal{V} over $G_{\mathbb{C}}$, with fibres V_g at every $g \in G_{\mathbb{C}}$. For \mathcal{V} to qualify as a vector bundle over $\{G_{\mathbb{C}}/P\}$, we recall that two different points in $G_{\mathbb{C}}$, g and g' , are considered the same point in $\{G_{\mathbb{C}}/P\}$ if they are related by the action of P , $g = p^{-1}gp$. But, if g and g' are considered the same, then V_g and $V_{p^{-1}gp}$ must be identified also. For the latter to make sense, the fibres V_g must admit an action of P , $V_{p^{-1}gp} = p(V_g)$, and are therefore labeled by representations of P . Such vector bundles are homogeneous and, conversely, all homogeneous vector bundles can be obtained this way. There might be other holomorphic vector bundles on $\{G_{\mathbb{C}}/P\}$ which do not arise this way, but we will not have use for them.

So, homogeneous holomorphic vector bundles over $\mathbb{F}_{[P \subset G]} = \{G_{\mathbb{C}}/P\}$ are labeled by representations of P and we recall that a representation can be written as a direct sum of *irreducible* ones; accordingly, we need to focus on irreducible homogeneous holomorphic vector bundles. As each irreducible representation is uniquely labeled by its highest weight, this information suffices also to distinguish the vector bundles of our interest. Amongst the several different notations [6,53], the Young tableaux are perhaps best known to physicists and we adhere to them. These become a little cumbersome for $SO(n)$, $Sp(n)$ and the exceptional Lie groups, but Ref. [125] provides all required information in detail. Alternatively, one may use the Dynkin labels $[d_1, \dots, d_r]$, where r is the rank of the group or even the Dynkin diagrams themselves as in Ref. [5].

We will quite exclusively deal with unitary groups $G = U(n)$, which allows the simpler notation used here and we adopt the following conventions. We write (b_1, \dots, b_n) , where $b_r \leq b_{r+1}$ is the number of boxes in the r^{th} row of the tableau, the bottom row being $r = 1$. For $b_r > 0$ (< 0), the boxes are strung to the right (left) of the ‘spine’. In tensorial notation, $(-1, 0, \dots, 0)$ labels a covariant vector, v_μ , while $(0, \dots, 0, 1) \sim v^\mu$ is contravariant. All other representations are obtained from these by multiplication and then decomposition into a direct sum of irreducible components through symmetrizing, anti-symmetrizing and taking traces with the invariant tensor, the Kronecker symbol, δ_μ^ν . Note that the Levi-Civita alternating symbol $\epsilon^{\mu_1 \dots \mu_n}$ is *not invariant*; it transforms with a phase

$$\epsilon^{\mu_1 \dots \mu_n} \longmapsto \det(\Lambda) \epsilon^{\mu_1 \dots \mu_n}, \quad \Lambda \in U(n). \quad (3.3.3)$$

Since locally $U(n) \approx U(1) \times SU(n)$, the phase belongs to this $U(1)$ and it follows that the homogeneous holomorphic vector bundle the fibres of which transform like $\epsilon^{\mu_1 \dots \mu_n}$ is holomorphically—but not homogeneously—trivial. So, when discussing bundles over $\mathbb{F}_{[n_1, \dots, n_F]}$ which were defined in Eq. (3.3.1), we will be careful not to cancel the common $U(1)$ factor and will not pass to $\{\frac{SU(N)}{S[U(n_1) \times \dots \times U(n_F)]}\}$. Not forgetting this $U(1)$ phase will prove useful later, especially when comparing with other, more stringy ways of analysis.

As mentioned above, specifying the subgroup $H = \prod_{f=1}^F U(n_f)$ of $G = U(N)$ suffices to characterize $\mathbb{F}_{[n]}$ and we note that, because of Eq. (3.3.2) the order of the factors $U(n_f)$ is *essential*. Since a representation of such a Lie group H is obtained as a tensor product of representations of the factor $U(n_f)$ ’s, we write

$$(a_1, \dots, a_{n_1}|b_1, \dots, b_{n_2}|\dots|d_1, \dots, d_{n_F}) \quad (3.3.4)$$

for the corresponding Young tableau or, to save some space, stack the partitions atop each other.

Examples.

- For $\mathbb{P}^n = \{\frac{U(n+1)}{U(1) \times U(n)}\}$, we have that

$$\mathcal{T}_{\mathbb{P}^n} \sim (-1|0, \dots, 0, 1), \quad \mathcal{T}_{\mathbb{P}^n}^* \sim (1|-1, 0, \dots, 0), \quad (3.3.5)$$

are the holomorphic tangent and cotangent bundles, respectively. The bundle corresponding to degree- k polynomials is

$$\mathcal{S}_{\mathbb{P}^n}^{* \otimes k} = \mathcal{O}_{\mathbb{P}^n}(k) \sim (-k|0, \dots, 0). \quad (3.3.6)$$

So, for example,

$$(-1|0, \dots, 0, 1) \otimes (1|-1, 0, \dots, 0) = (0|0, \dots, 0) \oplus (0|-1, 0, \dots, 0, 1) \quad (3.3.7)$$

corresponds to $\mathcal{T}_{\mathbb{P}^n} \otimes \mathcal{T}_{\mathbb{P}^n}^* = \mathcal{O}_{\mathbb{P}^n} \oplus \text{End } \mathcal{T}_{\mathbb{P}^n}$, where the latter is the bundle of traceless endomorphisms of the holomorphic tangent bundle. Also,

$$\wedge^2 \mathcal{T}_{\mathbb{P}^n} \sim \wedge^2 (-1|0, \dots, 0, 1) = (-2|0, \dots, 0, 1, 1), \quad (3.3.8)$$

and so on.

- On the 6-dimensional Grassmannian $G_{(2,5)} = \mathbb{F}_{[2,3]} = \{\frac{U(5)}{U(2) \times U(3)}\}$,

$$\mathcal{T}_{\mathbb{F}_{[2,3]}} \sim (-1, 0|0, 0, 1), \quad \mathcal{T}_{\mathbb{F}_{[2,3]}}^* \sim (0, 1|-1, 0, 0), \quad \mathcal{S}_{\mathbb{F}_{[2,3]}} \sim (-1, 0|0, 0, 0),$$

$$\wedge^2 \mathcal{T}_{\mathbb{F}_{[2,3]}} \sim (-1, -1|0, 0, 2) \oplus (-2, 0|0, 1, 1), \quad \mathcal{K}_{\mathbb{F}_{[2,3]}}^* \sim (-3, -3|2, 2, 2). \quad (3.3.9)$$

As regards the first two examples in Table 3.2, we now see that $\mathcal{L} = \det \mathcal{S}^* = (1, 1|0, 0, 0)$. The anti-canonical bundle of $[G_{(2,5)}] \# 3\mathcal{L}$ is (with $k\mathcal{L} = \oplus^k \mathcal{L}$)

$$(-3, -3|2, 2, 2) \otimes \det [3(1, 1|0, 0, 0)] = (0, 0|2, 2, 2), \quad (3.3.10)$$

which is non-trivial and has 50 sections. For $[G_{(2,5)}] \# 2\mathcal{L} \oplus \mathcal{L}^{\otimes 2}$, $\mathcal{K}^* = (1, 1|2, 2, 2)$ and similarly, for $[G_{(2,6)}] \# 5\mathcal{L}$, $\mathcal{K}^* = (1, 1|2, 2, 2, 2)$.

3.3.3 The (anti)canonical bundle

The last few results present an immediate application of the preceding information about bundles over flag varieties $\mathbb{F}_{[PCG]}$: we can now determine the canonical bundle of any complete intersection submanifold in products of flag varieties.

Generalizing the complete intersections in (products of) complex projective spaces, we now consider an arbitrary product of flag varieties $X \stackrel{\text{def}}{=} \prod_r \mathbb{F}^{(r)}$ for the embedding space. Clearly, vector bundles over X are simply the products of vector bundles over the factors. We again consider complete intersection spaces \mathcal{M} , defined just as in Eq. (2.1.3), as the common zero-set of a system of constraints, $\xi^a = 0$, where ξ^a is a *sufficiently generic* section of \mathcal{V}_a . Note, however that if $\text{rank } \mathcal{V}_a > 1$, the corresponding $\xi^a = 0$ is itself a system of constraints; by ‘sufficiently generic’, we will imply that these constraints are independent. This then also implies that

$$\mathcal{N}_{X/\mathcal{M}} = \bigoplus_{a=1}^k \mathcal{V}_a, \quad \dim \mathcal{M} = \dim X - \text{rank}(\bigoplus_{a=1}^k \mathcal{V}_a). \quad (3.3.11)$$

Then, there is again a short exact sequence such as (1.3.7) and

$$\det(\mathcal{T}_M) = \det(\mathcal{T}_X) \otimes \det\left(\bigoplus_a \mathcal{V}_a\right) = \mathcal{K}_X^* \otimes \bigotimes_a \det(\mathcal{V}_a). \quad (3.3.12)$$

As we are interested in a characteristic class property that is, in the ampleness of the anti-canonical bundle, we may in also apply the splitting principle and think of the system of vector bundles as a system of (systems of) line bundles.

This is clearly the straightforward generalization of the formula (2.1.16). However, this time we need a little more information. Surely, $\otimes_a \det(\mathcal{V}_a)$ is given easily in terms of a Kronecker product of Young tableaux (or any other mnemonic tool for multiplying representations of Lie groups), as these bundles were open to our choice. All we need to know is that these bundles have sections; as we shall see later, they do have sections whenever the symbol $(\dots | \dots | \dots)$ labels a valid representation of $U(N)$, that is whenever all integers in $(\dots | \dots | \dots)$ are non-decreasing from left to right.

However, \mathcal{K}_X and so \mathcal{T}_X must be determined independently; the price to pay for the simplicity of this technique is that for flag spaces of the type (3.3.1) with more than two factors in the denominator, the \mathcal{T}_X is no longer irreducible and it of course becomes more and more complicated. \mathcal{T}_X becomes an *extension*^(L) of two or more irreducible homogeneous bundles \mathcal{T}_i . Luckily, for the purpose of characteristic class computations, we may replace \mathcal{T}_X with the direct sum (although not holomorphically!) of \mathcal{T}_i . The determinant formula will of course hold also holomorphically. So we remain ‘only’ with the task to determine \mathcal{T}_i .

3.3.4 Bundles for free

While discussing the simplest C -spaces, the \mathbb{P}^n ’s, the relation (1.3.20), i.e., that $\mathcal{K}_{\mathbb{P}^n} \approx \mathcal{O}_{\mathbb{P}^n}(-n-1)$ was simple to derive. The analogous is also possible for Grassmannians, although the derivation is a little more involved. The reason for this is that the total Chern class of \mathbb{P}^n was a simple power series in a single variable, while the total Chern class of a Grassmannian will have more algebraically independent generators (see Ref. [9], p.292–294). Clearly, this gets more and more out of hand as we allow more and more ‘stripes’, n_f , in the flags of our flag varieties $\mathbb{F}_{[n_1, \dots, n_F]}$.

Let us reexamine the relation (1.3.20) in a little more detail so as to be able to generalize for all $\mathbb{F}_{[n]}$ ’s. We may think of \mathbb{P}^n as the space of all lines $L \approx \mathbb{C}^1$

3.3. Kähler C -Spaces

through the origin of \mathbb{C}^{n+1} . Each line L is defined as the zero-set of some linear polynomial, $\ell(x)$ over \mathbb{C}^{n+1} . Of all tangent vectors on \mathbb{C}^{n+1} , those written as

$$\ell^i(x) \frac{\partial}{\partial x^i}, \quad \ell^i(x) \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)), \quad (3.3.13)$$

will span $T_{\mathbb{P}^n}$ if taken modulo the homoteties, that is $x^i \frac{\partial}{\partial x^i} \approx 0$. We have just obtained the so-called Euler short exact sequence :

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \rightarrow T_{\mathbb{P}^n} \rightarrow 0. \quad (3.3.14)$$

From a related point of view, at every point $p \in \mathbb{P}^n$, vectors of $T_{\mathbb{P}^n}$ act by mapping $p \in \mathbb{P}^n$ into a nearby point. In other words, the line $L \subset \mathbb{C}^{n+1}$ is being rotated around the origin into another line, whence[†] $T_{\mathbb{P}^n} = \text{Hom}(L, \mathbb{C}^{n+1}/L)$, i.e., $T_{\mathbb{P}^n} \otimes L = \{\mathbb{C}^{n+1}/L\}$. This relation is equivalent to the exactness of the short sequence, which we write as :

$$0 \rightarrow \mathcal{S} \rightarrow \mathbb{C}^N \rightarrow Q \rightarrow 0, \quad Q = T_{\mathbb{F}_{[n]}} \otimes \mathcal{S}, \quad (3.3.15)$$

where in our present case, $\mathbb{F}_{[n]} = \mathbb{F}_{[1,n]} = \mathbb{P}^n$, $N = n+1$ and $\mathcal{S} = L$. Of course, \mathbb{C}^N is here considered as the trivial rank- N bundle over $\mathbb{F}_{[n]} = \mathbb{P}^n$, $\mathbb{P}^n \times \mathbb{C}^N$, while \mathcal{S} is the *universal sub-bundle* of $\mathbb{F}_{[n]} \times \mathbb{C}^N$,

$$\mathcal{S} = \{(S_p, x) \in \mathbb{F}_{[n]} \times \mathbb{C}^N : x \in S_p\}. \quad (3.3.16)$$

The quotient bundle $Q = \{\mathbb{C}^N/\mathcal{S}\}$ is called the *universal quotient bundle* and the sequence (3.3.15) is called the *tautological sequence*. A little thought reveals that $L = \mathcal{O}_{\mathbb{P}^n}(-1)$, whence the exact sequence (3.3.14) can also be obtained from Seq. (3.3.15) through tensoring the latter with $L^* = \mathcal{O}_{\mathbb{P}^n}(1)$.

Note : since $L^* = \mathcal{O}_{\mathbb{P}^n}(1) \sim (-1|0, \dots, 0)$ in the above notation, $T_{\mathbb{P}^n} \sim (-1|0, \dots, 0, 1)$.

Next we need to reinterpret \mathcal{S} for more complicated cases. For a Grassmannian $G_{(k,N)} = \mathbb{F}_{[k,(N-k)]}$, the universal sub-bundle is $\mathcal{S} = \{(H_p, x) \in G_{(k,N)} \times \mathbb{C}^N : x \in H_p\}$, where H_p is the k -plane through the origin of \mathbb{C}^N corresponding to the point $p \in G_{(k,N)}$, since $G_{(k,N)}$ is the space of k -planes through the origin of \mathbb{C}^N . Clearly, this interpretation needs to be iterated inclusion after inclusion for $\mathbb{F}_{[n_1, \dots, n_F]}$, recalling the relation (3.3.2). As a consequence, $T_{\mathbb{F}_{[n]}}$ will be given in

[†]See the Hom(\mathcal{A}, \mathcal{B}) entry in the Lexicon.

terms of a sequence of sequences and will fail to be given as a single irreducible factor. Nevertheless, the above formalism of homogeneous bundles over simply connected C -spaces, which we have adopted from Eastwood [18], deals with this easily. Suffice it here to give some examples; the general pattern will be clear :

$$\mathcal{T}_{\mathbb{F}_{[n_1, n_2]}} \sim (-1, 0, \dots, 0|0, \dots, 0, 1) ; \quad (3.3.17)$$

$$\mathcal{T}_{\mathbb{F}_{[n_1, n_2, n_3]}} \sim (-1, 0, \dots, 0|0, \dots, 0|0, \dots, 0, 1) + \begin{pmatrix} (-1, 0, \dots, 0|0, \dots, 0, 1|0, \dots, 0) \\ (0, \dots, 0|-1, 0, \dots, 0|0, \dots, 0, 1) \end{pmatrix} ; \quad (3.3.18)$$

$$\begin{aligned} \mathcal{T}_{\mathbb{F}_{[n_1, n_2, n_3, n_4]}} \sim & (-1, 0, \dots, 0|0, \dots, 0|0, \dots, 0, 1) + \begin{pmatrix} (-1, 0, \dots, 0|0, \dots, 0|0, \dots, 0, 1|0, \dots, 0) \\ (0, \dots, 0|-1, 0, \dots, 0|0, \dots, 0|0, \dots, 0, 1) \end{pmatrix} \\ & + \begin{pmatrix} (-1, 0, \dots, 0|0, \dots, 0, 1|0, \dots, 0|0, \dots, 0) \\ (0, \dots, 0|-1, 0, \dots, 0|0, \dots, 0, 1|0, \dots, 0) \\ (0, \dots, 0|0, \dots, 0|-1, 0, \dots, 0|0, \dots, 0, 1) \end{pmatrix}, \end{aligned} \quad (3.3.19)$$

and so on. Summands of a direct sum of vector bundles were stacked vertically to save some space. The symbol ‘+’ here means ‘ \oplus ’ in the C^∞ category, but not holomorphically; see *exact sequence*^(L). The various brackets in the above examples represent the irreducible summands \mathcal{T}_i of $\mathcal{T}_{\mathbb{F}_{[m]}}$.

3.4 Calabi-Yau Submanifolds in C -Spaces

Because $\mathcal{K}_X = \det T_X^*$, once we have the irreducible summands of T_X , \mathcal{K}_X is easy to obtain using that

$$\mathcal{K}_{\mathbb{F}_{[n_1, \dots, n_F]}} \approx \det \left(\bigoplus_{i=1}^{(F)} \mathcal{T}_i^* \right) = \bigotimes_{i=1}^{(F)} \det (\mathcal{T}_i^*) , \quad (3.4.1)$$

and that

$$\det \underbrace{(0, \dots, 0, 1)}_p \mid \underbrace{(-1, 0, \dots, 0)}_q = (q, \dots, q| -p, \dots, -p) . \quad (3.4.2)$$

We now have all we need to verify for any particular case the (non)triviality of the canonical bundle of the complete intersection (*if non-empty*) in an arbitrary product of flag spaces.

3.4.1 Some examples

For $\mathcal{M} \in [5|4 2]$, this becomes : $\mathcal{V}_1 \sim (-4|0, 0, 0, 0, 0)$ and $\mathcal{V}_2 \sim (-2|0, 0, 0, 0, 0)$ are both line bundles, corresponding to a quartic and a quadric constraint.

$$\begin{aligned} \mathcal{K}_{\mathcal{M}} &= \det \mathcal{T}_{\mathbb{P}^5}^* \otimes \mathcal{V}_1 \otimes \mathcal{V}_2 \\ &= \det(1|-1, 0, 0, 0, 0) \otimes (-4|0, 0, 0, 0, 0) \otimes (-2|0, 0, 0, 0, 0) \\ &= (5|-1, -1, -1, -1) \otimes (-6|0, 0, 0, 0, 0) \\ &= (-1|-1, -1, -1, -1) = (1|1, 1, 1, 1)^{-1} . \end{aligned} \quad (3.4.3)$$

Note that $\mathcal{K}_{\mathcal{M}}$ is holomorphically trivial, although not homogeneously so; its fibres transform by a phase, as the $\epsilon_{i_1 \dots i_6}$ of \mathbb{P}^5 .

Consider now the 4-dimensional Grassmannian $G_{(2,4)} = \mathbb{F}_{[2,2]}$. $\mathcal{T}_{\mathbb{F}_{[2,2]}}^* \sim (0, 1|-1, 0)$, so $\mathcal{K}_{\mathbb{F}_{[2,2]}} \sim (2, 2|-2, -2)$. To make the canonical bundle on a 3-dimensional subspace \mathcal{M} holomorphically trivial, we need to define \mathcal{M} as the zero-set of a section of the line bundle $\mathcal{L} \sim (-2, -2|2, 2)$, since then $\mathcal{K}_{\mathcal{M}} = \mathcal{K}_{\mathbb{F}_{[2,2]}} \otimes \mathcal{L} \sim (0, 0|0, 0)$, which is holomorphically and in fact also homogeneously trivial; \mathcal{M} is therefore a Calabi-Yau hypersurface in the Grassmannian $G_{(2,4)}$.

As another example of this kind, consider embedding a Calabi-Yau 3-fold in the 5-dimensional flag space $\mathbb{F}_{[1,2,1]}$. We have

$$\mathcal{T}_{\mathbb{F}_{[1,2,1]}} \sim (-1|00|1) + \begin{pmatrix} (-1|01|0) \\ (0|-10|1) \end{pmatrix} , \quad \Rightarrow \quad \mathcal{K}_{\mathbb{F}_{[1,2,1]}} \sim (3|00|-3) . \quad (3.4.4)$$

Prove this. \clubsuit

For a 3-fold $\mathcal{M} \hookrightarrow \mathbb{F}_{[1,2,1]}$, we need two constraints and these may be chosen as sections of $(-3|00|0) \oplus (0|00|3)$ or of $(-2|00|1) \oplus (-1|00|2)$. This produces two (deformation classes of) Calabi-Yau manifolds; it can be shown moreover that these can also be embedded via the configurations

$$\left[\begin{array}{c|cc} 3 & 3 & 0 & 1 \\ 3 & 0 & 3 & 1 \end{array} \right] \quad \text{and} \quad \left[\begin{array}{c|cc} 3 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{array} \right], \quad (3.4.5)$$

respectively, since $\mathbb{F}_{[1,2,1]}$ is isomorphic to a generic member of $\left[\begin{array}{c|c} 3 & 1 \\ 3 & 1 \end{array} \right]$, defined by $h_{\alpha\beta}x^\alpha y^\beta = 0$. Note, however, that special (singular) hypersurfaces in $\left[\begin{array}{c|c} 3 & 1 \\ 3 & 1 \end{array} \right]$, when $\text{rank}[h_{\alpha\beta}] < 4$, are not isomorphic to $\mathbb{F}_{[1,2,1]}$. Since the cubics can be arranged to avoid the singularity of the hypersurface, there are smooth members of $\left[\begin{array}{c|cc} 3 & 3 & 0 & 1 \\ 3 & 0 & 3 & 1 \end{array} \right]$ which cannot be embedded in $\mathbb{F}_{[1,2,1]}$ (see Ref. [121,76]).

As a final example of this kind, consider embedding $\mathcal{M} \hookrightarrow \mathbb{F}_{[2,3]} = G_{(2,5)}$, $\mathcal{K}_{\mathbb{F}_{[2,3]}} \sim (3,3|-2,-2,-,2)$. This Grassmannian is 6-dimensional, so we need three locally independent constraints. We may choose two of them to be the components of a 2-vector section of the rank-2 vector bundle $\mathcal{S}^* \sim (-1,0|000)$. Since $\det(-1,0|000) = (-1,-1|000)$, we must choose for the remaining line bundle $\mathcal{L} \sim (-2,-2|2,2,2)$ so that

$$\mathcal{K}_\mathcal{M} \sim (3,3|-2,-2,-,2) \otimes (-1,-1|000) \otimes (-2,-2|2,2,2) = (00|000). \quad (3.4.6)$$

Note that \mathcal{L} is holomorphically equivalent to $(\det \mathcal{S}^*)^{\otimes 4}$:

$$\mathcal{L} = (\det \mathcal{S}^*)^{\otimes 4} \otimes (2,2|2,2,2). \quad (3.4.7)$$

So, using a 2-vector section of \mathcal{S}^* and a section of $(\det \mathcal{S}^*)^{\otimes 4}$, we define another Calabi-Yau 3-fold, \mathcal{M}' with $\mathcal{K}_{\mathcal{M}'} \sim (-2,-2|-2,-2,-2)$. For appropriate choices of the defining sections, \mathcal{M} and \mathcal{M}' are isomorphic as complex manifolds, although as submanifolds of the homogeneous space $\mathbb{F}_{[2,3]}$, $\mathcal{M} \not\cong \mathcal{M}'$: $\mathcal{K}_\mathcal{M}$ is trivial under overall rescalings of $\mathbb{F}_{[2,3]}$ while $\mathcal{K}_{\mathcal{M}'}$ is not*.

We will return later to 3-folds embedded in products of Kähler simply connected compact homogeneous spaces, when we turn to discuss computations of cohomology groups. The interested Reader should by now be well equipped for some preliminary independent exploration in the wilderness of this uncharted wealth of new constructions and—quite expectedly—some hitherto unknown beasts.

*It does not seem to be clear at this time whether such a distinction is in fact relevant in the immediate physics application.

3.5 Pfaffian Constraint Systems

Considering the constructions discussed so far, the Reader may wonder if embeddings other than complete intersections can be analyzed. Somewhat as a curiosity at the moment, we now turn to a simple construction of a Calabi-Yau 3-fold which is not a complete intersection [103].

3.5.1 A Pfaffian 3-fold

Start with an skew-symmetric 7×7 matrix of linear polynomials $\mathbf{A}(x) \stackrel{\text{def}}{=} a_{[ij]}(x)$, where $(x^0:\dots:x^6) \in \mathbb{P}^6$. Let \mathbf{A}_k , $k = 0, \dots, 6$, denote the seven principal 6×6 minors

$$\mathbf{A}_k \stackrel{\text{def}}{=} \{a_{[ij]}(x) : i, j \neq k\}. \quad (3.5.1)$$

Then $\det(\mathbf{A}_k)$ are seven sextic polynomials over \mathbb{P}^6 and since the \mathbf{A}_k are skew-symmetric,

$$\det(\mathbf{A}_k) = f_k(x)^2, \quad f_k(x) = \text{Pf}[\mathbf{A}_k], \quad k = 0, \dots, 6. \quad (3.5.2)$$

$\text{Pf}[\mathbf{A}_k]$ denotes the *Pfaffian* of \mathbf{A}_k . Clearly, this provides a system $\{f_k(x)\}$ of seven rather special cubic polynomials over \mathbb{P}^6 .

We define now our prospective 3-fold \mathfrak{X} as the vanishing set of all seven cubics $f_k(x)$:

$$\mathfrak{X} \stackrel{\text{def}}{=} \{x \in \mathbb{P}^6 \mid f_k(x) = 0, k = 0, \dots, 6\} \subset \mathbb{P}^6. \quad (3.5.3)$$

This clearly cannot be a complete intersection, for if it were— \mathfrak{X} would have been -1 -dimensional. In fact, for a general choice of the initial 7×7 matrix $a_{[ij]}(x)$, we shall see that \mathfrak{X} is in fact a smooth Calabi-Yau 3-fold.

To begin with, we want to show that \mathfrak{X} is a 3-fold. To this end, note that since \mathbf{A} is skew-symmetric, it must be of even rank, that is of rank 0, 2, 4 or 6.

Now, at a generic point $p \in \mathbb{P}^6$, we may choose local coordinates so that

$$\mathbf{A}(p) = \left(\begin{array}{c|c|c} \mathbb{O} & \mathbb{1}_3 & \mathbb{0} \\ \hline -\mathbb{1}_3 & \mathbb{O} & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{0} & 0 \end{array} \right), \quad (3.5.4)$$

where the matrix is divided into blocks of 3–3–1 elements on each side; $\mathbb{1}_3$ denotes the 3×3 identity matrix. Then $\text{rank}(\mathbf{A}) = 6$ for all generic $p \in \mathbb{P}^6$ and not all the cubics in the system $\{f_k(x)\}$ can vanish. So, \mathfrak{X} is a proper subspace of \mathbb{P}^6 .

Next, at a sub-generic point of \mathbb{P}^6 , local coordinates may be chosen so that

$$\mathbf{A} = \left(\begin{array}{c|c|ccc} \mathbb{O} & \mathbb{1}_2 & 0 & 0 & 0 \\ \hline -\mathbb{1}_2 & \mathbb{O} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & a_{[56]} & a_{[57]} \\ 0 & 0 & -a_{[56]} & 0 & a_{[67]} \\ 0 & 0 & -a_{[57]} & -a_{[67]} & 0 \end{array} \right) . \quad (3.5.5)$$

So, to obtain $\text{rank}(\mathbf{A}) = 4$, we need to impose three locally independent conditions, $a_{[56]}(x) = a_{[57]}(x) = a_{[67]}(x) = 0$. This implies that the dimension of \mathfrak{X} is not more than 3. That it is also precisely 3 at each point of \mathfrak{X} , we need to check the possibilities of $\text{rank } \mathbf{A} = 0, 2$. By a similar argument, to bring the rank of \mathbf{A} down to 2, we would need to impose 10 locally independent conditions, that is, given that the initial 21 linear polynomials $a_{[ij]}(x)$ were chosen generically. On \mathbb{P}^6 , this is four constraints too many. Finally, to achieve $\text{rank } \mathbf{A} = 0$, we would need to impose 21 locally independent conditions.

Alternatively, locally at $p \in \mathbb{P}^6$, the choice of coordinates as in Eq. (3.5.4) implies that $f_i(x) \equiv 0$ for $i = 1, \dots, 6$ and $f_7(x) = 1 \neq 0$, whence $p \notin \mathfrak{X}$. Similarly, Eq. (3.5.5) implies that locally $f_i(x) \equiv 0$ for $i = 1, \dots, 4$ and $f_5 = a_{[67]}, f_6 = a_{[57]}, f_7 = a_{[56]}$. So, at \mathfrak{X} where $f_k(x) = 0$, for $k = 1, \dots, 7$, $\text{rank } \mathbf{A} = 4$.

Thus, we have proved that $\text{rank } \mathbf{A}(x) = 4$ at the subspace $\mathfrak{X} \subset \mathbb{P}^6$, implying that $\dim \mathfrak{X} = 3$. Analyzing the system of gradients in a similar fashion, it should be clear that \mathfrak{X} will also be smooth for a generic choice of the initial matrix $\mathbf{A}(x)$.

To amuse the Reader, we pause to show that the similar construction with a 3×3 skew-symmetric matrix of linear polynomials $\mathbf{B}(x)$ in \mathbb{P}^2 , yields \emptyset . Since a skew-symmetric 3×3 matrix has precisely three independent entries, by choosing coordinates on \mathbb{P}^2 , we have that

$$\mathbf{B}(x) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} . \quad (3.5.6)$$

Then $\mathbf{B}_k = \{x\tau_2, -y\tau_2, z\tau_2\}$, where $\tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Clearly,

$$\text{Pf}[\mathbf{B}_k] = \{x, y, z\}, \quad k = 1, 2, 3 . \quad (3.5.7)$$

So, in fact, the system $\text{Pf}[\mathbf{B}_k] = 0$ is immediately seen to have no solutions and this lower dimensional analogue of \mathfrak{X} is empty. However, we find it instructive to show the analogue of the above analysis. In the patch $z \neq 0$, we choose local coordinates

$\xi = x/z, \eta = y/z$ and set $\zeta = z/z = 1$. At the point $p = (1:0:0) \in \mathbb{P}^2$, we indeed have $\mathbf{B}(p) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, as above, and $\text{rank } \mathbf{B}(x) \leq 2$. In fact, $\text{rank } \mathbf{B}(x) = 2$ precisely since, in analogy to the case (3.5.5), for $\text{rank } \mathbf{B}(x) < 2$ we need $x = y = z = 0$ which cannot happen on \mathbb{P}^2 .

>Show that the analogous construction based on a 5×5 skew-symmetric matrix of linear polynomials in \mathbb{P}^4 provides a Pfaffian embedding of the torus $T^2 \hookrightarrow \mathbb{P}^4$. \clubsuit

3.5.2 Triviality of the canonical bundle

To verify that the 3-fold \mathfrak{X} constructed above is Calabi-Yau, we would need to verify that it is Ricci-flat. Equivalently, we would need to show that the first Chern class vanishes, that is, that the canonical bundle is holomorphically trivial.

To see why this might be so, recall that we started from the skew-symmetric matrix \mathbf{A} , with 21 different non-zero entries. However, these were chosen to be linear polynomials over \mathbb{P}^6 , that is, sections of $\mathcal{O}_{\mathbb{P}^6}(1)$. Since there are only seven global homogeneous coordinates, not more than seven entries of the matrix \mathbf{A} can be linearly independent. Furthermore, the defining functions, $f_k(x)$, were found as Pfaffians of the principal minors. Now, although the normal bundle is not spanned by these defining functions globally (there are too many of them), *locally* some three (combinations) of these seven defining functions do span the normal space. The determinant of the normal bundle therefore does correspond to a product of the independent elements of \mathbf{A} , with some constant coefficient. If \mathbf{A} is chosen generically, then it has seven independent elements and $\det \mathcal{N}_{\mathbb{P}^6/\mathfrak{X}} = \mathcal{O}_{\mathbb{P}^6}(7)$, so that $\mathcal{K}_{\mathfrak{X}} = \mathcal{K}_{\mathbb{P}^6} \otimes \det \mathcal{N}_{\mathbb{P}^6/\mathfrak{X}} \approx \mathbb{C}$.

It turns out that certain techniques which we will discuss later allow an analysis of such a relation in quite more detail and in a rather more general framework. We therefore return to the proof of $\mathcal{K}_{\mathfrak{X}} \approx \mathbb{C}$ in § 9.5.

Chapter 4

Group Actions, Quotients and Singularities

Most Calabi-Yau manifolds constructed by the methods reviewed so far are inappropriate for realistic compactification, since they are simply connected. Recalling the wish list in § 0.8 and the discussion in § 0.4 and 0.7, we see that—most likely—we need our Calabi-Yau 3-fold to be multiply connected.

4.1 Free Holomorphic Actions

The simplest method of constructing multiply connected varieties is to construct a variety \mathcal{Y} which admits a *free action*^(L) of some group G and then pass to the quotient space $\mathcal{M} \stackrel{\text{def}}{=} \{\mathcal{Y}/G\}$. In fact, this is possible for all \mathcal{M} with $c_1 = 0$ [79,7] :

Theorem 4.1 (Bogomolov) *Let \mathcal{M} be a Kähler n -fold with $b_{1,0} = b_{0,1} = k$ and $\mathcal{K}_{\mathcal{M}} \approx \mathbb{C}$. Then $\mathcal{Y} \times T^k$ is a finite cover of \mathcal{M} , where \mathcal{Y} is simply connected, $\mathcal{K}_{\mathcal{Y}} \approx \mathbb{C}$ and T^k is a complex k -dimensional torus.*

Clearly, $\dim \mathcal{Y} = \dim \mathcal{M} - k$ meaning that \mathcal{Y} is a point if $\dim \mathcal{M} = k$. Most of the time, we concern ourselves with the $k = 0$ case (which ensures $SU(n)$ holonomy) and so, in principle, the multiply connected \mathcal{M} can always be obtained by constructing the simply connected covering space \mathcal{Y} first.

There are however several requirements on the action of G , which come from our goal to construct complex (in fact Kähler) and Ricci-flat 3-folds.

4.1. Free Holomorphic Actions

4.1.1 Holomorphic actions

The first of these requirements, namely that $\mathcal{M} = \{\mathcal{Y}/G\}$ is a complex space is easily fulfilled if \mathcal{Y} is itself complex *and the action of G is holomorphic*. The complex structure of \mathcal{Y} will then be inherited by \mathcal{M} . Using the foregoing ideas to construct \mathcal{Y} as embedded in a (product of) homogeneous space(s), $\mathcal{Y} \hookrightarrow X$, and to preserve the homogeneity of the defining equations (so that these make sense), the action of G must preserve this homogeneity. So, unless the defining constraints are very special (and then typically singular), we expect the action of G to be linear in terms of coordinates on the embedding space X . For future reference, we write

$$G : x^\mu \mapsto g^\mu_\nu x^\nu, \quad g^\mu_\nu \text{ is constant.} \quad (4.1.1)$$

Now, Calabi-Yau n -folds ($n > 1$) admit no continuous holomorphic symmetries*, so one seeks a finite group G with a holomorphic and free action on \mathcal{Y} .

4.1.2 Multiple connectedness and $c_1 = 0$

If the holomorphic n -form Ω on \mathcal{Y} is invariant under the free action of G on \mathcal{Y} , then $G^*(\Omega)$ is a nowhere zero holomorphic n -form on $\mathcal{M} = \{\mathcal{Y}/G\}$, so \mathcal{M} is manifestly a Calabi-Yau manifold. This, however, provides no new condition :

Lemma 4.1 *Let \mathcal{Y} be an n -fold with $b_{0,q} = 1$ if $q \equiv 0 \pmod{n}$ and $b_{0,q} = 0$ otherwise. If $G : \mathcal{Y} \rightarrow \mathcal{Y}$ has a free action, the holomorphic n -form Ω transforms as $G^* : \Omega \mapsto (-)^{n+1}\Omega$.*

This is in fact an easy consequence of the holomorphic *Lefschetz fixed point formula*^(L). So, for Calabi-Yau 3-folds, the invariance of the holomorphic 3-form Ω with respect to a freely acting symmetry is automatic. The converse, however, is not true; it is not sufficient to check the invariance of Ω as the contributions on the right hand side of Eq. (L.37) can cancel out.

As an elementary consequence of the definition, the structure group $\pi_1(\mathcal{M})$ equals G if \mathcal{Y} is simply connected. In other words, for every element $\alpha \in G$,

*If they did, there would exist nowhere zero holomorphic Killing vector fields. Contracting this with the nowhere zero holomorphic n -form, there would also exist harmonic $(n-1)$ -forms, which contradicts our general condition for Calabi-Yau n -folds: $b_{n-1,0} = 0$ unless $n = 1$; recall Eq. (1.2.13).

there will exist a non-contractible loop $\Gamma_\alpha \subset \mathcal{M}$ such that if $\alpha^k = \mathbf{1}$ then $\circlearrowleft^k \Gamma_\alpha$ is contractible, that is, looping k times along Γ_α produces a contractible closed path. Moreover, Γ_α is *homotopic*^(L) to Γ_β precisely if α and β are in the same conjugacy class of G . Of course, if \mathcal{Y} was multiply connected already, then passing to the quotient by the free action of G increases the multiple connectedness, so to speak.

For a freely acting symmetry group $G : \mathcal{Y} \rightarrow \mathcal{Y}$, we have that various (holomorphic) bundles \mathcal{V} over \mathcal{Y} give rise to corresponding bundles over $\mathcal{M} = \{\mathcal{Y}/G\}$ if the fibres V_x and $V_{g(x)}$ can be made to agree. That is, if $V_{g(x)} = g(V_x) \approx V_x$ (holomorphically). By their construction, this will automatically be true of T_y and T_y^* , giving rise to T_M and T_M^* , respectively. So, in particular, $H^{p,q}(\mathcal{M})$ is obtained from $H^{p,q}(\mathcal{Y})$ by projecting out all elements which are not invariant under G ; the same is true of $H^q(\mathcal{M}, \text{End } T_M)$. Of course, to carry this through in practice, we need to know the action of the symmetry group on the various cohomology groups and we will return to this issue later.

4.1.3 Numerical characteristics

Certain quantities, however, behave much simpler in the process $\mathcal{Y} \xrightarrow{G} \mathcal{M}$. From the very definition of the Euler characteristic as an integral (over the manifold) of the Euler density (which is a product of curvature forms), it follows that

$$\chi_E(\mathcal{M}) = \chi_E(\mathcal{Y}/G) = \frac{\chi_E(\mathcal{Y})}{|G|}, \quad (4.1.2)$$

where $|G|$ denotes the order of G ; for example, $|\mathbb{Z}_n| = n$. By the same token, the indices^(L) of the other three classical complexes^(L)— χ^h , τ_h and \hat{A} —have the same divisibility property [see Eqs. (2.5.1)–(2.5.4)] and provide some necessary conditions for the action of a symmetry to be free.

Now, $\tau_h(\mathcal{X})$ and $\hat{A}(\mathcal{X})$ vanish identically when $\dim_{\mathbb{C}} \mathcal{X} \neq 0 \bmod 2$ and χ^h vanishes also if \mathcal{X} is in addition Calabi-Yau[†]. So, for Calabi-Yau 3-folds, only the divisibility of the Euler characteristics remains a non-trivial condition for the action of a symmetry to be free.

However, one can modify these indices as shown in Eqs. (2.5.5)–(2.5.7), ‘twisting’ them by a bundle \mathcal{V} over the n -fold in question. In general, it is difficult to compute such twisted indices with \mathcal{V} a bundle intrinsic to the Calabi-Yau n -fold \mathcal{Y} . On the other hand, when \mathcal{Y} is given as a submanifold of some better

[†]Just recall Eqs. (2.5.1)–(2.5.4) and the expressions (1.5.12), (1.5.16), (1.5.18).

4.2 Fixed Points, Singularities and Smoothings

understood embedding space \mathcal{X} , we may twist the indices by $T_x^{\otimes k} \otimes \mathcal{N}_{x/\mathcal{Y}}^{\otimes \ell}$, for arbitrary k, ℓ . It is important to note that such indices are characteristics of the embedding $\mathcal{Y} \hookrightarrow \mathcal{X}$, not the n -fold \mathcal{Y} by itself.

Furthermore, as we study properties of characteristic classes, we may use the *splitting principle* (1.3.19) freely to decompose $\mathcal{N}_{x/\mathcal{Y}}$ into line bundles \mathcal{L}_i , even if not holomorphically, and then twist by the various products $\bigotimes_{\text{some } i} \mathcal{L}_i$. In doing so, of course, one has to restrict to those products which admit a proper action of the candidate symmetry.

Such twisted indices were explicitly computed for the nearly 8000 complete intersection Calabi-Yau 3-folds in products of complex projective spaces [95]. Their divisibility properties were sufficient to isolate a handful of models which may admit a freely acting symmetry, G , such that χ_E is properly divisible by $|G|$, with remainder -6 , that is, yielding 3 generations of Standard Model particles, as desired. Several complete intersections thus pass all such twisted index tests and also some case by case study; in contrast, only one deformation class of multiply connected $\chi_E = -6$ Calabi-Yau 3-folds has been constructed as a free quotient.

As the twisted index tests are necessary rather than sufficient conditions, it is still not clear if there exists only one complete intersection in products of complex projective spaces which is a proper multiple cover of a multiply connected $|\chi_E| = 6$ Calabi-Yau 3-fold. No such analysis has been made for embeddings in spaces containing factors other than complex projective spaces.

4.2 Fixed Points, Singularities and Smoothings

Freely acting symmetries are of course rather more difficult to find than symmetries which have fixed point sets. So, in lieu of more smooth quotient models (of potential physical interest), we turn to quotient models which involve non-free acting symmetries.

4.2.1 Which fixed point produce singularities?

When passing to the quotient by a symmetry $G : \mathcal{Y} \rightarrow \mathcal{Y}$ which fixes some points, $\dot{p}_i \in \mathcal{Y}$, we expect $\mathcal{M} = \{\mathcal{Y}/G\}$ to be singular at each $p_i = G(\dot{p}_i)$. If the fixed points \dot{p}_i are isolated, there is a local coordinate neighborhood $U_i \approx \mathbb{C}^n$ with \dot{p}_i at the origin of U_i and $\dot{p}_i \notin U_j$ for all $j \neq i$. Then U_i folds up under the division by the action of G as the cone $\{\mathbb{C}^n/G\}$, leaving $p_i = G(\dot{p}_i)$ at the (singular) vertex.

More correctly, however, if the symmetry $G : \mathbb{C}^n \rightarrow \mathbb{C}^n$ fixes (only) the origin, $\{\mathbb{C}^n/G\}$ is singular at the origin—if $n > 1$. In each patch of a 1-fold, containing a ‘singular’ point, one re-defines $z \mapsto (z - z_i)^k$ where z_i are the ‘singular’ loci and k is a suitable integer. The resulting coordinate patches are smooth and the transition functions are holomorphic : the singularity was fake.

In general, essentially the same argument shows that if a (finite!) symmetry $G : \mathcal{Y} \rightarrow \mathcal{Y}$ fixes (among others) a set $\mathcal{S} \subset \mathcal{Y}$ of codimension 1, then the quotient $\{\mathcal{Y}/G\}$ is not singular at \mathcal{S} . As we are mostly interested in the case $\dim \mathcal{Y} = 3$, we will have to consider only the cases when the fixed point sets (and so the singular sets in the quotient space) have dimension 0 and/or 1.



More precisely, one finds the fixed point for each element $g_\alpha \in G$. The union of these is the fixed point set of the symmetry group G and it may be decomposed, for $\dim \mathcal{Y} = 3$, as

$$\text{f.p.}(G : \mathcal{Y}) = \left\{ \bigcup_i p_i \right\} \cup \left\{ \bigcup_j C_j \right\}, \quad (4.2.1)$$

where p_i are the isolated fixed points and C_j are the fixed curves. This decomposition is done in such a way that there is no overlap. To this end, (a) whenever two fixed curves intersect, $C'_1 \cap C'_2 = p$, the above decomposition lists p as a fixed point and $C_1 \stackrel{\text{def}}{=} (C'_1 - p)$ and $C_2 \stackrel{\text{def}}{=} (C'_2 - p)$ as two separate (punctured) curves. Similarly, (b) if a point q is fixed by $g_\alpha \in G$ and lies on a curve C'_3 which is fixed by $g_\beta \in G$ for $\alpha \neq \beta$, the above decomposition lists q as a fixed point and $C_3 \stackrel{\text{def}}{=} (C'_3 - q)$ as a (punctured) fixed curve. Once such special points have been separated, the C_j are said to be of *pure dimension 1*.

In passing to the quotient $\mathcal{M} = \{\mathcal{Y}/G\}$, the disjoint union on the right hand side of Eq. (4.2.1) lists the irreducible components of the singular set

$$\text{Sing}(\mathcal{M}) = \left\{ \bigcup_i p_i \right\} \cup \left\{ \bigcup_j C_j \right\}, \quad G(p_i) = p_i, \quad G(C_j) = C_j. \quad (4.2.2)$$

The decomposition of the singularity in Eq. (4.2.2) is useful in determining the possible desingularization(s) of the quotient space \mathcal{M} . For isolated singular points, we study the singularity locally, choosing a suitable \mathbb{C}^3 -like coordinate chart U_i which places the singular point p_i at the origin and contains no other singularity. For the singular curves C_j , we choose a tubular neighborhood “built”

from \mathbb{C}^2 -like discs $D(x_j)$ such that x_j is the origin of $D(x_j)$ and x_j sweep out C_j . Note that this would not be possible if the fixed point set (4.2.1) had not been disjoined as it was*.

As it is, one may smooth each component of the singularity separately and then make sure that the resulting patches do fit together. Generally, such resolutions will be possible: If the fixed set contains a curve C' and a point p on it, and if $C' - p$ is fixed by $g_\alpha \in G$ while $g_\beta \in G$ fixes p , then $g_\beta = (g_\alpha)^k$ for some integer k . This induces a relation between the possible resolutions of p and $C' - p$, respectively, which can be used to patch everything back together.

4.2.2 Smoothings in general ...

The general idea, of course, is to replace the singular point $p \in \mathcal{M}$ with something bigger—the *exceptional set*, E . Such sets can typically be regarded as formal sums of non-contractible subspaces of the resolved space $\widehat{\mathcal{M}}$; that is, we smooth \mathcal{M} into $\widehat{\mathcal{M}}$ by “stretching” the singular points into some bigger-than-zero dimensional exceptional sets.

Elementary logic dictates that, in a real 6-dimensional space, an isolated singular point may be smoothed essentially in three different ways : by replacing it with an exceptional set, E , of (real) dimension 2, 3 or 4.

The cases $\dim_{\mathbb{R}} E = 1$ and $\dim_{\mathbb{R}} E = 5$ are related through Poincaré duality. Introducing (real) 1-dimensional exceptional sets, which essentially must be non-contractible circles, would imply that $H_1(\widehat{\mathcal{M}})$ —and therefore also $H^1(\widehat{\mathcal{M}})$ —would become non-zero, the holonomy of $\widehat{\mathcal{M}}$ would have to be a proper subgroup of $SU(3)$, implying either $N=2$ or no supersymmetry, both of which do not seem to have physical application (see Chapter 0).

Now, consider the case when a \mathcal{M} has a single singular point, p , which is smoothed by replacing it with some (real) 3-dimensional E . Simple local analysis shows that the non-compact manifolds $\mathcal{M} - p$ and $\widehat{\mathcal{M}} - E$ are distinct as complex manifolds, although they are diffeomorphic. Thus, smoothing in $\dim_{\mathbb{R}} E = 3$ deforms the complex structure; we refer to this generally as *deformation*.

We remain with cases $\dim_{\mathbb{R}} E = 2$ and $\dim_{\mathbb{R}} E = 4$; that is $\dim_{\mathbb{C}} E = 1$ and $\dim_{\mathbb{C}} E = 2$, respectively—assuming that the exceptional sets are in fact complex

*The gastronomically inclined Reader will find this rather familiar: preparation of many a culinary delight involves disjoining, separate and not infrequently rather elaborate treatment and only then putting everything back together.

subsets of the smoothed complex 3-dimensional manifold $\widehat{\mathcal{M}}$. Because of this essentially, $\mathcal{M}-p$ and $\widehat{\mathcal{M}}-E$ are the same complex (non-compact) manifold. One says: *resolving in codimension 2 and 1*, respectively. Of course, a combination of these two is possible; as usual however, we discuss them separately. Quite generally, the $\dim_{\mathbb{C}} E = 2$ case is called *blowing up*, while the $\dim_{\mathbb{C}} E = 1$ case is called *small resolution*.

Blowing up and *small resolutions* of complex 3-folds share some similarities, but also feature some important differences, as we will see below.

4.2.3 Divisors and line bundles

We have so far discussed (holomorphic) line bundles \mathcal{L} over complex n -folds X and some elementary facts about \mathcal{L} -valued cohomology—especially sections of \mathcal{L} , i.e., elements of $H^0(X, \mathcal{L})$. Now we come to the homological analogue of \mathcal{L} .

Definition. On a complex (not necessarily compact) n -fold \mathcal{M} , a locally finite linear combination $D \stackrel{\text{def}}{=} \sum_i a_i \mathcal{Y}_i$ of irreducible analytic hypersurfaces \mathcal{Y}_i of \mathcal{M} is called a divisor.

An “analytic hypersurface” means that $\dim \mathcal{Y}_i = n-1$ and that every \mathcal{Y}_i is (locally at every point) given as the zero-set of a single holomorphic function. “Locally finite” means that for any $p \in \mathcal{M}$, there exists a neighborhood of p meeting only a finite number of the \mathcal{Y}_i ’s appearing in D ; if \mathcal{M} is compact, this simply means that the defining sum for D is finite. A divisor is *effective* if $a_i \geq 0$ for all i .

Now, every holomorphic function $g(x)$ which vanishes at a point $p \in \mathcal{Y}_i$ is locally a holomorphic multiple of (some power of) the defining function of \mathcal{Y}_i ; at p , $g(x) = h(x) \cdot f^a(x)$, h holomorphic. The exponent a turns out to be independent of the point, characterizes \mathcal{Y}_i and is called the *order of g along \mathcal{Y}_i* ; $a = \text{ord}_{\mathcal{Y}_i}(g)$. Clearly,

$$\text{ord}_{\mathcal{Y}_i}(gh) = \text{ord}_{\mathcal{Y}_i}(g) + \text{ord}_{\mathcal{Y}_i}(h). \quad (4.2.3)$$

Also, $\text{ord}_{\mathcal{Y}_i}(g/h) = \text{ord}_{\mathcal{Y}_i}(g) - \text{ord}_{\mathcal{Y}_i}(h)$ for the meromorphic function g/h . With this, to each meromorphic function, $\mu(x)$, we can associate a divisor

$$\mu(x) = \frac{g(x)}{h(x)} \iff (\mu) \stackrel{\text{def}}{=} \sum_{\mathcal{Y} \subset \mathcal{M}} \text{ord}_{\mathcal{Y}}(\mu) \mathcal{Y}. \quad (4.2.4)$$

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Similarly, $(\mu)_0 \stackrel{\text{def}}{=} \sum_{\mathcal{Y} \subset \mathcal{M}} \text{ord}_{\mathcal{Y}}(g)$ is the divisor of zeros and $(\mu)_{\infty} \stackrel{\text{def}}{=} \sum_{\mathcal{Y} \subset \mathcal{M}} \text{ord}_{\mathcal{Y}}(h)$ is the divisor of poles of μ ; $(\mu) = (\mu)_0 - (\mu)_{\infty}$. Loosely then, divisors correspond to meromorphic sections of certain line bundles (more accurately, of rank-1 sheaves^(L)).

As a consequence of the definition, the set of divisors in \mathcal{M} form an additive group, denoted $\text{Div}(\mathcal{M})$. Consider two simple divisors D_1 and D_2 , each of which consists of a suitable irreducible hypersurface. Then D_i is the zero-set of a (perhaps meromorphic) section of some line bundle \mathcal{L}_i , $i = 1, 2$. The zero-set of the product section is the union of the two hypersurfaces, that is the divisor $D_1 + D_2$. The product section is, of course, a section of the tensor product line bundle $\mathcal{L}_1 \otimes \mathcal{L}_2$. This and the foregoing discussion of (μ) indicates the correspondence

$$\begin{array}{ccc} \text{Div}(\mathcal{M}) & \stackrel{\text{def}}{=} & (\{\text{divisors in } \mathcal{M}\}, +) \\ \downarrow & & \downarrow \\ \text{Pic}(\mathcal{M}) & \stackrel{\text{def}}{=} & (\{\text{line bundles over } \mathcal{M}\}, \otimes) \end{array} \quad (4.2.5)$$

This correspondence is in fact an isomorphism, as can be seen from the more precise description in Ref. [23], p.129–134. Suffice it here just to note : $[D]$ stands for the line bundle associated to the divisor D ; two divisors are *equivalent*, $D \sim D'$, if $[D] = [D']$. This permits us to freely toggle between line bundle and divisor parlance.

For any complex n -fold X , the canonical line bundle $\mathcal{K}_X = \det T_X^*$ comes for free and associated to it is the *canonical divisor* K_X . In view of the above identification, these are often used interchangeably.

4.3 Blowing Up

This is a very general process, in which a point p in a complex n -fold X is replaced by a (complex) $n-1$ -dimensional exceptional set E . The result of blowing up, the *blow-up* \tilde{X} , is such that $X-p$ and $\tilde{X}-E$ are equal as complex manifolds. One writes $\sigma : \tilde{X} \rightarrow X$ for the process indicating that σ is a proper holomorphic map except at $E \subset \tilde{X}$, which is being crushed to the point $p \in X$.

4.3.1 The template

As a simple example which will be useful to keep in mind, we recall the complex projective space. As discussed in § 3.3.4, over \mathbb{P}^n we have the universal line bundle

$$\mathcal{O}_{\mathbb{P}^n}(-1) = \mathcal{L} \stackrel{\text{def}}{=} \{(x, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : v^{[i]} x^j = 0\} \quad (4.3.1)$$

where $(v^0, \dots, v^n) \in \mathbb{C}^{n+1}$, $(x^0 : \dots : x^n) \in \mathbb{P}^n$. The right hand side of the defining equality specifies the total space of \mathcal{L} , which is isomorphic to \mathbb{C}^{n+1} away from the origin. At the origin, however, the $v^{[i]}x^{[j]} = 0$ conditions places no restriction on the x 's, so that $(0, \dots, 0) \in \mathbb{C}^{n+1}$ is replaced by all of \mathbb{P}^n . The origin of \mathbb{C}^{n+1} is said to have been blown up; the total space of the universal line bundle \mathcal{L} is the blow-up of \mathbb{C}^{n+1} at the origin and is denoted—following the general notation—by $\widetilde{\mathbb{C}^{n+1}}$. The copy of the \mathbb{P}^n which replaces the origin is the *exceptional divisor* of the blow-up.

Note that the origin of \mathbb{C}^{n+1} was a smooth point. So, by blowing up a smooth point* p in an n -fold X , we mean considering a \mathbb{C}^n -like neighborhood centered at p and replacing it with $\widetilde{\mathbb{C}^n}$, the total space of $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. With this in mind, blowing up is essentially a local operation—indeed, one of the very few surgical techniques available in the holomorphic category.

We note here without proof that

Theorem 4.2 *If $\sigma : \widetilde{X}_p \rightarrow X$ is the blow-up of the n -fold X at a smooth point $p \in X$ and $E \subset \widetilde{X}_p$ the resulting exceptional divisor, then*

$$\mathcal{K}_{\widetilde{X}_p} = \sigma^* \mathcal{K}_X \otimes [E]^{\otimes(n-1)}. \quad (4.3.2)$$

Here we stick to the multiplicative notation as appropriate for line bundles; this statement is often seen written in the additive notation, akin to divisors (see Ref. [23], p.187, to which we also refer for a proof).

Now, consider the total space of the k^{th} tensor power of \mathcal{L} . In terms of coordinate patches and for simplicity for $\mathcal{O}_{\mathbb{P}^1}(-1)$ vs. $\mathcal{O}_{\mathbb{P}^1}(-k)$,

patch	$\mathcal{O}_{\mathbb{P}^1}(-1)$		$\mathcal{O}_{\mathbb{P}^1}(-k)$	
	base	fibre	base	fibre
$(x^0 \neq 0) : \xi_{(0)} = x^1/x^0$	$V_{(0)} = v x_0$		$\xi_{(0)} = x^1/x^0$	$W_{(0)} = w (x_0)^k$
$(x^1 \neq 0) : \xi_{(1)} = x^0/x^1$	$V_{(1)} = v x_1$		$\xi_{(1)} = x^0/x^1$	$W_{(1)} = w (x_1)^k$

(4.3.3)

Comparing the two bundles, we see that we can go from $\mathcal{O}_{\mathbb{P}^1}(-1)$ to $\mathcal{O}_{\mathbb{P}^1}(-k)$ simply by replacing $V_{(i)} \mapsto V_{(i)}^k = v^k (x_i)^k$. For $V_{(i)}^k = W_{(i)}$ and so $v^k = w$ to be 1–1, we have to identify $(V_{(0)}, V_{(1)})$ with $(e^{(2i\pi/k)}V_{(0)}, e^{(2i\pi/k)}V_{(1)})$ away from the

*§ 3.1 shows that—except for the Hirzebruch surfaces—almost del Pezzo surfaces are blow-ups of \mathbb{P}^2 at a number of smooth points; hence the exceptional divisors $E_i \approx \mathbb{P}^1$.

4.3. Blowing Up

origin of \mathbb{C}^2 , that is, away from $\mathbb{P}^1 \subset \mathcal{O}_{\mathbb{P}^1}(-k)$. In other words, away from the “origin”, the total space of $\mathcal{O}_{\mathbb{P}^1}(-k)$ is isomorphic to $\{\mathbb{C}^2/\mathbb{Z}_k\}$ rather than \mathbb{C}^2 .

More generally, away from the “origin”, the total space of $\mathcal{O}_{\mathbb{P}^{n-1}}(-k)$ is isomorphic to $\{\mathbb{C}^n/\mathbb{Z}_k\}$ rather than \mathbb{C}^n . Therefore, a $\{\mathbb{C}^n/\mathbb{Z}_k\}$ -like singular neighborhood in an n -fold X can be replaced by the smooth total space of $\mathcal{O}_{\mathbb{P}^{n-1}}(-k)$ and thereby the \mathbb{Z}_k quotient singularity at the origin is replaced by a \mathbb{P}^{n-1} .

Of course, the result of the above theorem needs to be corrected now; it is not hard to see that the canonical bundle changes into

$$\widetilde{\mathcal{K}_{(X/\mathbb{Z}_k)_p}} = \sigma^* \mathcal{K}_X \otimes [E]^{\otimes(n-k)}. \quad (4.3.4)$$

This is an example of holomorphic local surgery. However, this *local* process induces *global* changes in that the canonical (global, holomorphic) line bundle, that is, the first Chern class is being altered. Also, this affects a number of numerical characteristics such as the Euler characteristic, the Hodge numbers...

So if we started from a Calabi-Yau space \mathcal{Y} , where $\mathcal{K}_{\mathcal{Y}} \approx \mathbb{C}$, Eq. (4.3.4) indicates that each \mathbb{Z}_3 quotient singularity in $\mathcal{M} = \{\mathcal{Y}/\mathbb{Z}_3\}$ can be blown up yielding $\widetilde{\mathcal{K}_{\mathcal{M}}} \approx \mathbb{C}$. So, one might ask what can one do with other \mathbb{Z}_k singularities?

4.3.2 Blowing up quotient singular curves

In the case \mathbb{C}^2/G , the situation is quite simple. Let G be a finite subgroup of $SU(2)$ with the action that fixes only the origin. The quotient singularities $\{\mathbb{C}^2/G\}$ can then all be resolved by successive blow-ups.

A detailed analysis would take us far afield; instead, the interested Reader is advised to consult Ref. [1] as a general reference. Suffice it here to describe briefly the case of abelian $G \approx \mathbb{Z}_r$ when the technique of so-called *toric resolutions* is applicable [42,20]. In suitable local coordinates, consider the action

$$\mathbf{g} : (x, y) \longmapsto (\omega^a x, \omega^b y), \quad \omega \stackrel{\text{def}}{=} e^{(2i\pi/r)}. \quad (4.3.5)$$

It is useful to define the lattice of exponents invariant under this action of \mathbb{Z}_r :

$$\mathfrak{E}_g \stackrel{\text{def}}{=} \{(\xi, \eta) \in \mathbb{Z}^2 : a\xi + b\eta \equiv 0 \pmod{r}\}. \quad (4.3.6)$$

Without loss of generality, we may take $(a, b) = (1, r-1)$, so that $\xi \equiv \eta \pmod{r}$. A suitable integral basis in \mathfrak{E}_g is given by

$$e_1 = (r, 0), \quad e_2 = (1-r, 1), \quad (4.3.7)$$

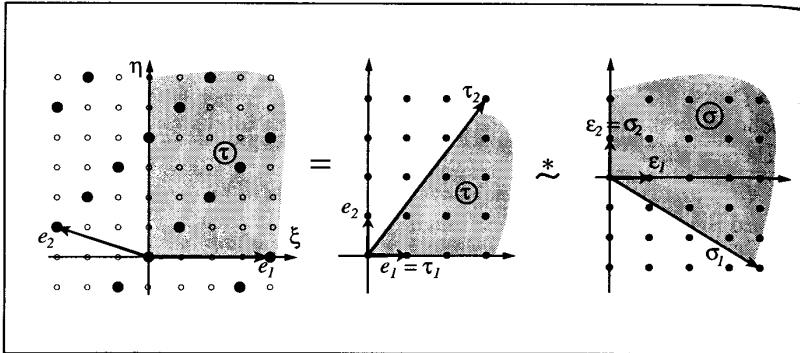


Figure 4.1: The invariant lattice of exponents \mathfrak{E}_g , the cone of positive exponents, τ and its dual, σ .

and sketched in Figure 4.1. In the basis (e_1, e_2) , the real cone τ , which “fills” the integral cone of positive exponents, has an opening smaller than $\frac{\pi}{2}$. The real cone τ may be taken to be spanned by the vectors $\tau_1 = (1, 0)$ and $\tau_2 = (r-1, r)$, given in terms of the basis elements e_i ; write $\tau = \begin{pmatrix} 1 & 0 \\ r-1 & r \end{pmatrix}$.

The dual cone is defined as

$$\sigma \stackrel{\text{def}}{=} \{a_1 \varepsilon_1 + a_2 \varepsilon_2 : a_i \in \mathbb{R}, a_i \geq 0 \ \forall i\}, \quad (4.3.8)$$

has an opening greater than $\frac{\pi}{2}$ and may be taken to be spanned by the vectors $\sigma_1 = \begin{pmatrix} r \\ 1-r \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; write $\sigma = \begin{pmatrix} r & 0 \\ 1-r & 1 \end{pmatrix}$. Clearly, $\sigma = r\tau^{-1}$.

Given a collection of dual cones σ_i , one for each coordinate patch, it is possible to reconstruct the space the singularities of which these cones describe [42, 20]. The fact that $\det(\tau) = \det(\sigma) = r$ indicates that these cones describe a single coordinate patch with a \mathbb{Z}_r quotient singularity at the origin. Therefore, to obtain a smooth model we simply need to divide σ into several subcones σ_k , in such a way that $\det(\sigma_k) = 1$, for each k . To that end, for $k = 1, \dots, r$, we define

$$\begin{aligned} r\sigma_k &\stackrel{\text{def}}{=} k \begin{pmatrix} r & 0 \\ (1-r)+(r-k) & 0 \end{pmatrix} + (k-1) \begin{pmatrix} 0 & r \\ 0 & (1-r)+(r-k) \end{pmatrix} \\ &= r \begin{pmatrix} k & (k-1) \\ (1-k) & (2-k) \end{pmatrix}. \end{aligned} \quad (4.3.9)$$

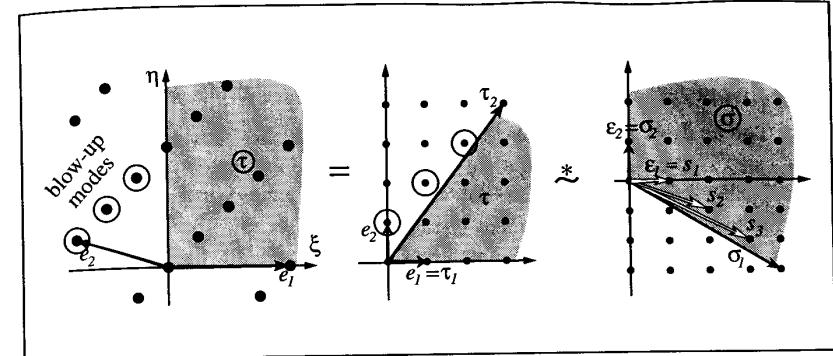


Figure 4.2: Three (toric) renditions of the blow-up of $\mathbb{C}^2/\mathbb{Z}_r$.

$\det(\sigma_k) = 1$ follows straightforwardly, so the $r-1$ new vectors $\vec{s}_k \stackrel{\text{def}}{=} \begin{pmatrix} k \\ 1-k \end{pmatrix}$, for $k = 1, \dots, (r-1)$, divide the σ -cone into r subcones each of which corresponds to a smooth coordinate patch. Accordingly, the set $\{\vec{s}_k\}_{k=1, \dots, (r-1)}$ corresponds to the exceptional set, which has $r-1$ components : $E \stackrel{\text{def}}{=} \{E_k\}_{k=1, \dots, (r-1)}$, $E_k \approx \mathbb{P}^1$. For a \mathbb{Z}_r -fixed curve C in a 3-fold, we blow up each point in the \mathbb{C}^2 -like transversal neighborhood. Remember, however, that the whole exceptional set is then $E \times C$.

Pulling the \vec{s}_k back to the original integral exponent lattice \mathfrak{E}_g , we see that the exceptional divisors E_k correspond to \mathbb{Z}_r -invariant rational monomials $\{(x^{k-r}y^k)\}_{k=1, \dots, (r-1)}$. Note the similarity to the parametrization used in the proof of Proposition 3.2, concerning blow-ups of \mathbb{P}^2 . Figure 4.2 shows the exceptional set E from three different (toric) vantage points.

Let us just note that the intersection matrix (E_i, E_j) equals minus the Cartan matrix of the A_{r-1} Lie algebra (the Coxeter element of the Weyl group of A_{r-1} generates a \mathbb{Z}_r). In fact, the analogous is true also of other discrete groups : the exceptional lines in the blow-up of \mathbb{C}^2/G are related to the D_r , E_6 , E_7 and E_8 Lie algebra for G the dihedral, tetrahedral, cubic and icosahedral groups, respectively [1].

4.3.3 Blowing up isolated quotient singular points

Even a semi-comprehensive treatment of isolated 3-fold quotient singularities is quite beyond our scope; the Reader is referred to Ref. [46]. We will just sketch the blowing up of \mathbb{C}^3/G for abelian G , where again toric methods prove effective [42, 20,206,104,188,57].

The analysis is very similar to that of the \mathbb{C}^2/G case. Given the action

$$\mathbf{g} : (x, y, z) \mapsto (\omega^a x, \omega^b y, \omega^c z), \quad \omega \stackrel{\text{def}}{=} e^{(2i\pi/r)}, \quad (4.3.10)$$

we start from the integral exponent lattice

$$\mathfrak{E}_{\mathbf{g}} \stackrel{\text{def}}{=} \{(\xi, \eta, \zeta) \in \mathbb{Z}^3 : a\xi + b\eta + c\zeta \equiv 0 \pmod{r}\}. \quad (4.3.11)$$

Again, without loss of generality, we take $(a, b, c) = (1, b, c)$ and end up with

$$\sigma \stackrel{\text{def}}{=} \begin{pmatrix} r & 0 & 0 \\ -b & 1 & 0 \\ -c & 0 & 1 \end{pmatrix}. \quad (4.3.12)$$

σ is now a 3-dimensional triangular cone in \mathbb{R}^3 , which we may conveniently represent by its base. As in the 2-dimensional case, $\det(\sigma) = r$ indicates that σ describes a single coordinate patch with a \mathbb{Z}_r quotient singularity at the origin. The division of σ into subcones \vec{s}_k with $\det(\sigma_k) = 1$ is then conveniently described by a sub-triangulation of the triangular base of σ . Generalizing Eq. (4.3.9), the new vertices of this sub-triangulation are found in the form

$$\vec{s}_k(\alpha_k, \beta_k, \gamma_k) \stackrel{\text{def}}{=} \frac{1}{\alpha_k + \beta_k + \gamma_k} \left[\alpha_k \begin{pmatrix} r \\ -b \\ -c \end{pmatrix} + \beta_k \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \gamma_k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right], \quad (4.3.13)$$

such that $\alpha_k, \beta_k, \gamma_k$ and also the components of \vec{s}_k are integral (the aid of a computer is quite welcome here). The loci of \vec{s}_k in the base of the cone specify the vertices of the desired sub-triangulation. As in the 2-dimensional case, these vectors (vertices) correspond to exceptional divisors E_k (codimension-1).

To complete the sub-triangulation, we introduce edges which join the vertices (representing plaquettes). Now, the plaquette spanned between \vec{s}_k and \vec{s}_ℓ represents the intersection $E_k \cap E_\ell$. These exceptional curves (codimension-2) are a novel feature of the 3-dimensional case.

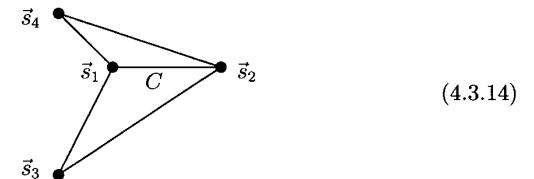
Generally, if $C \subset \mathcal{M}$ is a curve in a 3-fold, then $\det(\mathcal{N}_{\mathcal{M}/C}) = \mathcal{K}_{\mathcal{M}}|_C \otimes \mathcal{K}_C^*$ and so if \mathcal{M} is Calabi-Yau, $\det(\mathcal{N}_{\mathcal{M}/C}) = \mathcal{K}_C^* = T_C^*$. The latter equality holds since $\dim C = 1$. Now

4.3. Blowing Up

Theorem 4.3 (Grothendieck) All holomorphic vector bundles over \mathbb{P}^1 are isomorphic to the direct sum of line bundles $\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(\ell_i)$.

This in fact is the only (non-trivial) case when the splitting principle (1.3.19) is valid also holomorphically. The pattern (ℓ_1, \dots, ℓ_k) is called the *splitting type* of the rank- k vector bundle over \mathbb{P}^1 .

For the exceptional lines $C \approx \mathbb{P}^1$ occurring in toric resolutions, the splitting type of the normal bundle $\mathcal{N}_{\widetilde{\mathcal{M}}/C}$ can be determined from the sub-fan



We have that the ℓ_i in $\mathcal{N}_{\widetilde{\mathcal{M}}/C} = \mathcal{O}_C(\ell_1) \oplus \mathcal{O}_C(\ell_2)$ satisfy the (overdetermined) vectorial equation

$$\ell_1 \vec{s}_1 + \ell_2 \vec{s}_2 + \vec{s}_3 + \vec{s}_4 = 0. \quad (4.3.15)$$

Now, if our $C \subset \widetilde{\mathcal{M}}$ is a line and $\widetilde{\mathcal{M}}$ is Calabi-Yau, then also $\ell_2 = \ell_1 - 2$, since for $C \approx \mathbb{P}^1$, $T_{\mathbb{P}^1}^* = \mathcal{O}_{\mathbb{P}^1}(2)$.

◻

Blowing up, in general, introduces an exceptional set which contains both surfaces (codimension 1) and curves (codimension-2). Thus, we see that it affects both the canonical divisor—the effect of which is given in Eq. (4.3.4)—and the Kähler class. The latter happens because for a harmonic $(1, 1)$ -form J to be a Kähler form, we need its positivity, i.e., that

$$\int_C J > 0, \quad \int_S J^2 > 0, \quad \int_M J^3 > 0, \quad (4.3.16)$$

for all curves C and all surfaces S in the manifold \mathcal{M} . From another point of view, recall that

$$H_4(\mathcal{M}, \mathbb{Z}) \stackrel{*}{\sim} H_2(\mathcal{M}, \mathbb{Z}) \stackrel{*}{\sim} H^2(\mathcal{M}, \mathbb{Z}) \Rightarrow H_4(\mathcal{M}, \mathbb{Z}) \approx H^2(\mathcal{M}, \mathbb{Z}) \quad (4.3.17)$$

(see *duality*^(L)) and that $H^2(\mathcal{M}, \mathbb{Z}) \otimes \mathbb{C} \approx H^{1,1}(\mathcal{M})$ since there are no $(2,0)$ - or $(0,2)$ -forms. Since blowing up introduces new elements in H_4 , these reflect in

new harmonic (1,1)-forms and thereby lead to new variations of the Kähler class. So, one naturally inquires if the toric resolutions of singularities can distort the Kähler form ‘beyond repair’, so that the resulting space—although smooth and Calabi-Yau—may no longer admit any Kähler form.

In fact, given the triangular base and the ‘new’ vertices \vec{s}_k , there may well be more than one sub-triangulation. All of these are valid resolutions of the singularity, but not all of them may turn out to be Kähler. (Note that this is *not* a local issue; each coordinate patch may support a Kähler form, but it may be that these cannot be extended throughout the compact manifold.) For the case of a simple blow-up, $\sigma : \widehat{X}_p \rightarrow X$, where the exceptional set is a single \mathbb{P}^2 , it is straightforward to show that the blown up space is again Kähler (see Section II.7 of Ref. [27]). Moreover, the Kähler form on \widehat{X}_p can be obtained from that on $X - p$ and the Eguchi-Hanson metric [19] on the exceptional \mathbb{P}^2 . It then follows that a toric resolution leads to a Kähler manifold if it is a sequence of blow-ups. In toric terms, the resolution is Kähler if the sub-triangulation of the triangular base of σ can be obtained by successive sub-triangulations for each of the \vec{s}_k , one at a time.

4.4 Small Resolution

Unlike blowing up and toric resolutions, small resolutions always occur purely in codimension-2 and are thus a novelty with respect to the (algebraic or otherwise) geometry of curves and surfaces.

Consider a pair of 3-folds, \widehat{X} and X , such that X has some isolated singular points p_i and \widehat{X} is smooth and is a small resolution of X . Let E_i denote the (complex 1-dimensional!) exceptional set introduced to smooth out the p_i . As usual then, the non-compact complex manifolds $(X - \{p_i\})$ and $(\widehat{X} - \{E_i\})$ are equal. This means that the canonical divisor K_X (complex 2-dimensional!) of X equals $K_{\widehat{X}}$. Thus, small resolutions do not change c_1 and if we started from a singular Calabi-Yau space, its small resolution is also Calabi-Yau.

The language of divisors allows us to conveniently generalize the notion of Ricci-flatness to singular spaces. While the condition $c_1 = 0$ is independent of the specific choice of the metric, and the triviality of the canonical bundle refers to no metric at all, these concepts make sense only if the underlying manifold is smooth. The triviality of the canonical divisor, however, allows the underlying space to be singular, although not very pathologically so. It is at present quite unclear just how singular spaces one should be prepared to consider—neither from the mathematical,

4.4. Small Resolution

nor from the physical point of view. Certainly, however, hyperquotient singularities which are essentially quotient and hypersurface singularities perhaps atop each other [46] should be allowed from the physical point of view [170]. This is the rationale for the present selection of topics regarding singularities.

One of the simplest occasions for small resolutions is not a quotient singularity, but a *hypersurface singularity*. That is, we describe the singular point by specifying its local neighborhood as the solution of a single constraint in \mathbb{C}^4 . Of course, for the resulting space to be singular, we require the four gradients to have a common zero—and precisely one—if we discuss isolated hypersurface singular points. By a simple change of coordinates, the singular point is typically arranged to be at the origin.

4.4.1 A node : the complex story

The simplest isolated hypersurface singular point in a 3-fold is called a *node*, or a *double point*. It occurs at the vertex of the cone in \mathbb{C}^4 , defined by the simple equation $xy = zt$.

Let $\psi \stackrel{\text{def}}{=} (xy - zt)$ be the defining polynomial of the cone containing the node. Clearly, $d\psi = 0$ precisely at the origin, were the node is. To obtain the (complex) base of this cone, we simply projectivize, interpreting $(x : y : z : t) \in \mathbb{P}^3$. $\psi = 0$ is then a non-degenerate quadric in \mathbb{P}^3 , since the only degenerate point, $(0, 0, 0, 0) \in \mathbb{C}^4$ is not in \mathbb{P}^3 . As discussed above, a smooth quadric in \mathbb{P}^3 equals $\mathbb{P}^1 \times \mathbb{P}^1$ and the cone has \mathbb{C}^1 -like generators sweeping over a $\mathbb{P}^1 \times \mathbb{P}^1$ base.

It should be obvious that the node can be resolved most simply by replacing it with a copy of a \mathbb{P}^1 [2]. This happens entirely in codimension 2, as promised. Since the base is a product of two \mathbb{P}^1 's, we have a topological ambiguity in this process; typically, we will deal with 3-folds with many nodes and at each of them this ambiguity shows up. Thus, in principle, a 3-fold with N nodes has 2^N distinct small resolutions.

It is sometimes useful to think of the small resolution as a *halfway blow-up*. Indeed, in the present case, if we blew the vertex of the quadratic cone up, the exceptional divisor would have been the $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ surface. One or the other small resolution is then obtained by collapsing one of the two \mathbb{P}^1 's in \mathbf{F}_0 . This situation sometimes occurs in the course of toric resolutions and is then described in the diagram in Figure 4.3. The process by which we pass from one small resolution of a node to the blow-up and then to the other small resolution is called a *flop* and is indicated in the diagram.

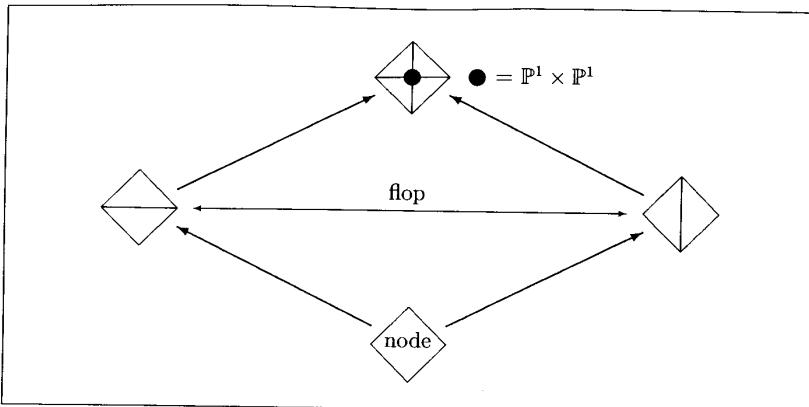


Figure 4.3: The blow-up and two distinct small resolutions of a node.

4.4.2 A node : the real story

Alternatively, we may change coordinates into $\psi = \sum_{A=1}^4 (\zeta^A)^2$, think of ζ^A as a four-vector and write this equation as $\vec{\zeta}^2 = 0$. To find the base of this real cone, separate $\vec{\zeta}$ into its real and imaginary parts $\vec{\zeta} = \vec{\xi} + i\vec{\eta}$, whence $\psi = 0$ becomes the pair

$$\vec{\xi}^2 - \vec{\eta}^2 = 0, \quad \vec{\xi} \cdot \vec{\eta} = 0. \quad (4.4.1)$$

The base is the intersection of the cone with a 7-sphere centered at the origin of $\mathbb{C}^4 \approx \mathbb{R}^8$. We take the sphere to have radius $r\sqrt{2}$ so that $\vec{\xi}^2 + \vec{\eta}^2 = 2r^2$. The base is thus described by the equations

$$\vec{\xi}^2 = \vec{\eta}^2 = r^2, \quad \vec{\xi} \cdot \vec{\eta} = 0. \quad (4.4.2)$$

The space of ξ 's is an S^3 . For each point of this S^3 , say $\vec{\xi} = (r, 0, 0, 0)$, $\vec{\xi} \cdot \vec{\eta} = 0$ restricts $\vec{\eta}$ to be the 3-vector $\vec{\eta} = (0, \eta^2, \eta^3, \eta^4)$. $\vec{\eta}^2 = r^2$ then defines an S^2 . Thus the base of the cone is a fiber bundle with base S^3 and fiber S^2 . Now, all bundles over S^3 are trivial* so the base must in fact be the product $S^2 \times S^3$; the generators are copies of the real semi-axis, $\mathbb{R}_{\geq 0}$.

*To see this, choose two B^3 -like neighborhoods to cover S^3 . Their overlap is a “thick S^2 ”, that is $S^2 \times I$ and this is where the transition functions of any bundle have to live. Now if G is the structure group of the vector bundle, the transition functions map $(S^2 \times I) \rightarrow G$ and are therefore elements of $\pi_2(G)$. Finally, it is a classic result that $\pi_2(G) = 0$ for any Lie group G [12].

4.4. Small Resolution

Defining Polynomial	Lie Algebra	Small Resolution
$x^{k+1} + y^2 + z^2 + t^2$	A_k	$\infty \cdots \circ$
$x^{k-1} + xy^2 + z^2 + t^2$	D_k	$\infty \cdots \circ \circ$
$x^4 + y^3 + z^2 + t^2$	E_6	$\circ \circ \circ \circ \circ$
$x^3y + y^3 + z^2 + t^2$	E_7	$\circ \circ \circ \circ \circ \circ$
$x^5 + y^3 + z^2 + t^2$	E_8	$\circ \circ \circ \circ \circ \circ \circ$

Table 4.1: Simple hypersurface singularities and their small resolutions.

The fact that the base of the real cone with a node at the vertex has a base which is a product $S^2 \times S^3$ can be related to the fact that S^3 is parallelizable. Consequently, the same is true only in two more cases: a nodal 1-fold and a nodal 7-fold. In the former case, the base is $S^0 \times S^1$, that is, two circles—as is well known. The complex 7-dimensional nodal cone has $S^6 \times S^7$ for its real base.

Thus, besides replacing the vertex of the cone with an S^2 , we can also replace it with an S^3 . As discussed earlier, this amounts to deformation as the exceptional set is (real) 3-dimensional. Since there are two \mathbb{P}^1 's in the base of the complex cone, one may inquire if there are in fact two different choices of S^3 's also, fibering the phase of the generator of the complex cone over one or the other \mathbb{P}^1 . It is straightforward to show that these two S^3 's can in fact be rotated nicely one into another, so that there is no topological ambiguity in the deformation of a node [92]. We will return to this below.

4.4.3 Other small-resolvable singularities

Without much ado, Table 4.1 just lists the generalizations of the nodal defining polynomial and their small resolutions. These again correspond to a list of (some) Lie algebras in the sense that their small resolution involves a chain of \mathbb{P}^1 's which touch one another as described by the Dynkin diagram of the corresponding Lie algebra. Of course, a node corresponds to A_1 . All these sport the same (in)affection to the canonical divisor : K_X is completely untouched in the process of small resolution.

The Kähler class is however disturbed by small resolutions. In fact, if X has a single node, then the small resolution cannot be Kähler. To see this, recall that

the base of the real cone with the node at the apex is $S^3 \times S^2$. Now the $\mathbb{R}_{\geq 0}$ -like generators of the cone and the S^2 from the base form a (real) B^3 -like cap, which is completed elsewhere in the manifold into a 3-cycle—in fact, precisely the 3-cycle, $c^{(3)}$, which is dual to the S^3 in the base of the cone (prove this). When the apex of the cone is replaced by a $E \approx \mathbb{P}^1 = S^2$, the 3-cycle $c^{(3)}$ has been punctured and has become a 3-chain, with the exceptional \mathbb{P}^1 as the boundary. Now any candidate Kähler form, J , would have to satisfy the relations (4.3.16). However, $\int_E J$ would have to vanish by Stokes' theorem, because E is a boundary.

If there are several nodes and they are replaced by S^2 's, these S^2 's are components of the boundary of at least one such 3-chain. Now Stokes' theorem requires that the integral of J vanishes over the linear combination of S^2 's which constitute the boundary of each such 3-chain. But in this boundary, the S^2 may occur with just such relative orientations that the total integral vanishes although the separate contributions do not. In such a case, the candidate J is not obstructed to be a Kähler form. It is therefore certain that, amongst 2^N small resolutions of a 3-fold with N nodes, only very few *may* admit a Kähler form and it is in general difficult to determine which are—if any.



We refer to Ref. [1] and Ref. [46] for the general analysis of these singularities and details of their resolutions. Suffice it here just to note that they are of the more general type called *compound Du Val singularities*^(L), written cDV. The general cDV defining equation is

$$F(x, y, z, t) = f(x, y, z) + t g(x, y, z, t), \quad (4.4.3)$$

where $f(x, y, z) = 0$ defines the Du Val singularities in 2-folds and $g(x, y, z, t)$ is an arbitrary (holomorphic) polynomial. The list of Du Val polynomials $f(x, y, z)$ may be read off the Table 4.1, upon setting $t = 0$. For Du Val singularities in 2-folds, the diagrams in Table 4.1 denote the respective blow-ups. Thus, cDV singularities in 3-folds may be viewed as a one-parameter (t) deformation of the Du Val singularities. Alternatively, a cDV singularity p^\sharp appears as a Du Val singularity in the hyperplane section through p . Because of the relation to the Lie algebras of the A , D and E type, the cDV singularities are also called ADE-singularities.

4.5 Numerical Characteristics

It is now not hard to determine the Euler characteristic of a resolution of a quotient $\mathcal{M} = \{\mathcal{Y}/G\}$, where the symmetry $G : \mathcal{Y} \rightarrow \mathcal{Y}$ has a fixed set, $F \subset \mathcal{Y}$:

$$\chi_E(\widetilde{\mathcal{M}}) = \frac{\chi_E(\mathcal{Y}) - \chi_E(F')}{|G|} + \chi_E(E), \quad (4.5.1)$$

where $\widetilde{\mathcal{M}}$ is the blow-up or small resolution of \mathcal{M} (as appropriate), E is the exceptional set into which F has been resolved and F' is the fixed point set, taken however with some multiplicities. More precisely, let \mathbf{g}_i denote the element of G which fixes the component $F_i \subset F$ and let $|\mathbf{g}_i|$ denote the order of \mathbf{g}_i (which must divide $|G|$ or be equal to it). Then

$$\chi_E(F') = \sum_i \frac{|G|}{|\mathbf{g}_i|} \chi_E(F_i). \quad (4.5.2)$$

Another rendition of the above formula is then

$$\chi_E(\widetilde{\mathcal{M}}) = \frac{\chi_E(\mathcal{Y})}{|G|} - \sum_i \frac{\chi_E(F_i)}{|\mathbf{g}_i|} + \sum_k \chi_E(E_k), \quad (4.5.3)$$

where the sum on i runs over all irreducible components of the fixed point set (see Eq. (4.2.1)) and k labels the irreducible components of the exceptional set.

As another important issue, it should be clear that all resolutions considered in this chapter leave the structure group as it was. This is not hard to show; one has to analyze the multiple connectedness of the neighborhood where the singular point has been cut out and the also the structure group of the exceptional divisor used for smoothing. In all the cases considered above, no change is being introduced to π_1 . For explicit calculations to this end in some models, we refer to Ref. [188].

With the detailed information presented in the analysis of toric resolutions or small resolution, we have the complete information on new 2-cycles and new 4-cycles which have been introduced in the smoothing process. This, in principle, provides the dimensions of $H_2(\mathcal{M})$ and $H_4(\mathcal{M})$, and therefore also $b_{1,1}$ and $b_{2,1}$. One only has to be cautious to take into account various homology relations between the exceptional sets; this may in fact become a global problem, as we will see later.

Chapter 5

Embeddings in Weighted Projective Spaces

In Chapter 3, we have studied some generalizations of complete intersections in products of *complex projective spaces*^(L), namely complete intersections in products including now also *almost del Pezzo surfaces*^(L), *almost Fano 3-folds*^(L) and (generalized) *flag spaces*^(L).

5.1 Weighted Projective Spaces

Already the simple *complex projective spaces*^(L), \mathbb{P}^n , may be generalized in another important fashion as follows. Recall that in \mathbb{P}^n , there exist global homogeneous coordinates z_i , $i = 0, \dots, n$, which are sections of the *hyperplane bundle*^(L) $\mathcal{O}_{\mathbb{P}^n}(1)$ and obey the projectivity relation*

$$(z_0 : \dots : z_n) \cong (\lambda z_0 : \dots : \lambda z_n) = \lambda(z_0 : \dots : z_n). \quad (5.1.1)$$

The fact that this scaling is isotropic suggests an immediate generalization; modifying the projectivity relation into

$$(z_0 : \dots : z_n) \cong (\lambda^{w_0} z_0 : \dots : \lambda^{w_n} z_n) \quad (5.1.2)$$

defines a *weighted complex projective space* with weights $(w_0 : \dots : w_n)$. We will write $\mathbb{P}_{(w_0 : \dots : w_n)}^n$ or $\mathbb{P}_{\mathbf{w}}^n$ for the space spanned by the $n+1$ quasi-homogeneous coordinates z_i , taken modulo the equivalence (5.1.2).

*Throughout this Chapter, for ease of notation, we subscript coordinates as if they were co-variant rather than contra-variant.

5.1. Weighted Projective Spaces

Notation.

As with (quasi)homogeneous coordinates of a (weighted) projective space, only the relative ratios of the weights are really of consequence. To remind the Reader of this, we write $(z_0 : \dots : z_n)$ and $(w_0 : \dots : w_n)$ for the projective space, $\mathbb{P}_{(w_0 : \dots : w_n)}^n$. In contrast, we write (z_0, \dots, z_n) and (w_0, \dots, w_n) for the affine space $\mathbb{C}_{(w_0 : \dots : w_n)}^{n+1}$; of course, $\mathbb{P}_{(w_0 : \dots : w_n)}^n = \{\mathbb{C}_{(w_0 : \dots : w_n)}^{n+1} - (0, \dots, 0)\} / \sim$, where “ \sim ” denotes the equivalence relation (5.1.2). We also write \mathbf{w} for both $(w_0 : \dots : w_n)$ and (w_0, \dots, w_n) : the context should clarify which of the two is meant.

5.1.1 Some basics

Now, it is clear from the projectivity relation (5.1.2) that if all weights are divisible by an integer, $w_i = kw'_i$, by redefining λ , $\mathbb{P}_{\mathbf{w}}^n \approx \mathbb{P}_{\mathbf{w}'}^n$. This is already encompassed in our notation $\mathbf{w} = (w_0 : \dots : w_n)$. However, there are further identities among weighted complex projective spaces and to this effect we quote from Ref. [16], without proof

Lemma 5.1 (Delorme) *Let $\mathbf{w} = (w_0 : \dots : w_n)$ and*

$$\begin{aligned} d_i &\stackrel{\text{def}}{=} \gcd(w_0, \dots, w_{i-1}, w_{i+1}, \dots, w_n), \\ m_i &\stackrel{\text{def}}{=} \text{lcm}(d_0, \dots, d_{i-1}, d_{i+1}, \dots, d_n). \end{aligned} \quad (5.1.3)$$

Then

$$\mathbb{P}_{\mathbf{w}}^n \approx \mathbb{P}_{\mathbf{w}'}^n, \quad w'_i \stackrel{\text{def}}{=} (w_i/m_i), \quad i = 0, \dots, n. \quad (5.1.4)$$

Note that for each i , m_i is divisible by w_i , d_i and m_i are coprime and $d_i m_i = \text{lcm}(d_0, \dots, d_n)$ (no summation on i).

Some easy examples are $\mathbb{P}_{(1:2:2)}^3 \approx \mathbb{P}^3$ (the isotropic projective space), $\mathbb{P}_{(1:2:4:4)}^4 \approx \mathbb{P}_{(1:1:2:2)}^4$, $\mathbb{P}_{(2:3:6:12:18)}^5 \approx \mathbb{P}_{(1:1:1:2:3)}^5$ and so on. In particular, $\mathbb{P}_{\mathbf{w}}^1 \approx \mathbb{P}^1$ for any \mathbf{w} . This clearly cuts down on the number of $\mathbb{P}_{\mathbf{w}}^n$'s that need to be considered.

Definition. A weighted projective space, $\mathbb{P}_{\mathbf{w}}^n$, is called well-formed if each n out of $n+1$ weights are coprime, that is when

$$\gcd(w_0, \dots, w_{i-1}, w_{i+1}, \dots, w_n) = 1, \quad i = 0, \dots, n. \quad (5.1.5)$$

In view of the above Lemma, we need to consider only well-formed $\mathbb{P}_{\mathbf{w}}^n$'s.



There is an important aspect in which $\mathbb{P}_{\mathbf{w}}^n$ (with $\mathbf{w} \neq \vec{1}$) differs from \mathbb{P}^n : most weighted projective spaces are singular. Consider the simple weighted projective space, $\mathbb{P}_{(1:1:1:2)}^4$. Then

$$(z_0 : z_1 : z_2 : z_3 : z_4) \cong (\lambda z_0 : \lambda z_1 : \lambda z_2 : \lambda z_3 : \lambda^2 z_4). \quad (5.1.6)$$

In the neighborhood of $(0 : 0 : 0 : 0 : 1)$ and for $\lambda = -1$, we have

$$(z_0 : z_1 : z_2 : z_3 : 1) \cong (-z_0 : -z_1 : -z_2 : -z_3 : 1). \quad (5.1.7)$$

Therefore, this coordinate patch has a \mathbb{Z}_2 -quotient singular point. Similarly, $\mathbb{P}_{(1:1:1:2:2)}^5$ has a \mathbb{Z}_2 -quotient singular line, parametrized by $(0 : 0 : 0 : z_4 : z_5)$. From their definition, we have [16,110] :

Corollary 5.1 *The singular set of a well-formed $\mathbb{P}_{\mathbf{w}}^n$ has codimension at least 2 (dimension at most $n-2$) and consists of cyclic quotient singularities.*

Suffice it here just to note that weighted projective spaces are also projective, that is, they can be embedded in some big enough \mathbb{P}^N . This also implies that they admit a Kähler class, represented by a Kähler form which degenerates only at the singular points of $\mathbb{P}_{\mathbf{w}}^n$. As a matter of fact, one can even use the well-known Fubini-Study type Kähler potential

$$\Phi_{\text{F.S.}}(z_i, \bar{z}_i) = \log \|z\|^2 \stackrel{\text{def}}{=} \log \left[\sum_{i=0}^n |z_i|^{2q_i} \right], \quad q_i \stackrel{\text{def}}{=} \frac{\text{lcm}\{w_j\}_{j=0,\dots,n}}{w_i} \quad (5.1.8)$$

much the same as one would do with the Fubini-Study Kähler potential for the (isotropic, $w_i = 1$) projective space.

5.1.2 Canonical stuff

Being so similar to (isotropic) projective spaces, one surmises that the “naturally bundled hardware” (see section 3.3.4) of homogeneous spaces should come for free with weighted (quasi-homogeneous) spaces also. Indeed, recall that the homogeneous coordinates on \mathbb{P}^n are all sections of the same bundle, $\mathcal{O}_{\mathbb{P}^n}(1)$, and so have the same weight— $w_i = 1$. Clearly then, for $\mathbb{P}_{\mathbf{w}}^n$, the coordinate z_i of weight w_i is a section of the bundle $\mathcal{O}_{\mathbb{P}_{\mathbf{w}}^n}(w_i)$. With this, we have the weighted analogue of the Euler sequence^(L) (Adjunction formula 1)

$$0 \longrightarrow \mathbb{C} \longrightarrow \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}_{\mathbf{w}}^n}(w_i) \longrightarrow T_{\mathbb{P}_{\mathbf{w}}^n} \longrightarrow 0, \quad (5.1.9)$$

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which is equivalent to the relation $T_{\mathbb{P}_{\mathbf{w}}^n} = \{\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}_{\mathbf{w}}^n}(w_i)\}/\mathbb{C}$. Using the determinant formula (1.1.1) we easily obtain the Adjunction formula 2

$$c(\mathbb{P}_{(w_0,\dots,w_n)}^n) = \prod_{i=0}^n (1 + w_i J). \quad (5.1.10)$$

Furthermore, a polynomial $f_a(z)$ in the quasi-homogeneous coordinates of $\mathbb{P}_{\mathbf{w}}^n$ of total weight q_a is a section of the line bundle $\mathcal{O}_{\mathbb{P}_{\mathbf{w}}^n}(q_a)$ and we therefore have a straightforward generalization of Eq. (2.1.15) :

$$c[X \parallel \mathcal{E}] = \frac{\prod_{r=1}^m \prod_{i=0}^{n_r} (1 + w_i^{(r)} J_r)}{\prod_{a=1}^K (1 + \sum_{s=1}^m q_a^s J_s)}. \quad (5.1.11)$$

Where X is a product of m weighted projective spaces, labeled by r , the weights of which are $w_i^{(r)}$. \mathcal{E} is the direct sum of line bundles

$$\mathcal{E} = \bigoplus_{a=1}^K \mathcal{E}_a, \quad \mathcal{E}_a = \bigotimes_{r=1}^m \mathcal{O}_{\mathbb{P}_{\mathbf{w}^{(r)}}^n}(q_a^r). \quad (5.1.12)$$

As we discuss below, even the generic members of such configurations are singular. Thus, the analogue of the vanishing first Chern class (2.1.17), which now reads

$$\sum_{i=0}^{n_r} w_i^{(r)} = \sum_{a=1}^K q_a^r, \quad r = 1, \dots, m, \quad (5.1.13)$$

says that the (probably singular) space $\mathcal{M} \in [X \parallel \mathcal{E}]$ has trivial canonical divisor, $K_{\mathcal{M}} \approx 0$. The next task is therefore to determine the singularity types expected to be found in \mathcal{M} and to see if they can be resolved in such a way as to (1) preserve the triviality of the canonical divisor and (2) preserve the projectivity (the possibility to embed in some big enough \mathbb{P}^N). The triviality of the canonical divisor implies that the canonical bundle over the nonsingular (and non-compact) part of \mathcal{M} is trivial. The Calabi-Yau desingularization can therefore be seen as one that enables the trivial canonical bundle to extend over the exceptional set used to smooth out the singularity.



Clearly, the Euler characteristic computed from the above expression for $c[X \parallel \mathcal{E}]$ and a straightforward generalization of Eq. (2.1.30) refers to the generic member of this configuration and we accordingly write $\chi_{\mathcal{E}}[X \parallel \mathcal{E}]$. Since the generic member will most certainly inherit some of the singularity of the embedding

weighted space X , we consider its resolution $[X/\mathcal{E}]^\sim$ instead, and χ_E will surely be affected in this process. Assuming that all singularities are of cyclic quotient type, using Eq. (4.5.3), the Euler characteristic becomes

$$\chi_E[X/\mathcal{E}]^\sim = \chi_E[X/\mathcal{E}] - \sum_i \frac{\chi_E(F_i)}{|g_i|} + \sum_k \chi_E(E_k), \quad (5.1.14)$$

where F_i are the irreducible components of the singular set fixed locally in X by g_i and E_k the irreducible components of the exceptional set introduced to smooth the singularity.



Consider now a space, \mathcal{M} , embedded in a (product of) weighted projective space(s), X , by means of a (system of) defining constraint(s), just as we did in the isotropic case, in Chapter 2. \mathcal{M} may turn out to be singular in two different ways : by having a non-transversal defining (system of) constraint(s) and/or by meeting the singular set of the embedding space.

5.2 Acquired Singularities

Besides the hereditary singularities described below, the defining equations of our subspace $\mathcal{M} \hookrightarrow X$ may lack transversality. Of course, this can always happen because of a bad choice of polynomials on the human part. More seriously, however, a sufficiently generic choice of defining polynomials may just not be possible : because of the different weights, there are generally fewer monomials available at any given degree.

Now, a glance at Eq. (5.1.13) tells that, as compared to embeddings in (isotropic) projective spaces, embeddings in weighted projective spaces require defining polynomials of higher degree. One might hope that this will allow a sufficient number of different polynomials. As a simple example, to obtain a Calabi-Yau hypersurface in \mathbb{P}^4 , we need a (smooth) quintic; a Calabi-Yau hypersurface in $\mathbb{P}_{(2:3:5:7:11)}^4$ is of (total) degree 28. Nevertheless, while there are 126 quintic monomials on \mathbb{P}^4 , there are only 53 monomials of weight 28 on $\mathbb{P}_{(2:3:5:7:11)}^4$. However, the question still remains whether the generic linear combination of these 53 monomials is actually transversal.

The answer turns out to be negative, as can be shown remembering *Bertini's Theorem*^(L) and the definition of the base locus. By Bertini, a generic element of

5.2. Acquired Singularities

a linear system is smooth away from the base locus; so we first need to determine the latter. The base locus is the common zero-set of all monomials from which the linear system is constructed. Note that this system will contain the simple monomials z_0^{14} and z_3^4 , these two vanish when $z_0 = z_3 = 0$, so that the base locus must be in the \mathbb{P}_w^2 parametrized by $(0 : z_1 : z_2 : 0 : z_4)$. Straightforwardly, there are only four monomials of total weight 28 in these three variables :

$$z_1 z_2^5, \quad z_1^2 z_4^2, \quad z_1^4 z_2 z_4, \quad z_1^6 z_2^2. \quad (5.2.1)$$

All of these will vanish if either $z_1 = 0$ or $z_2 = z_4 = 0$ and so the base locus consists of the \mathbb{P}^1 parametrized by $(0 : 0 : z_2 : 0 : z_4)$ and the point $(0 : 1 : 0 : 0 : 0)$. Thus, the most general polynomial vanishes at these points; it is easy to verify that it must be singular only at $(0 : 0 : 0 : 0 : 1)$ and $(0 : 1 : 0 : 0 : 0)$.

Although these turn out not to be simple singular points, it is quite possible that a non-transversal (complete intersection of) hypersurface(s) in a (product of) weighted projective space(s) contains at worst cyclic quotient and/or cDV hypersurface (a.k.a. hyperquotient) singularities.



For the rest of this section, we turn to the general problem of characterizing transversal (quasi-)homogeneous polynomials in n (weighted) variables. Recall that transversality means that the gradients of the polynomial vanish only at the origin.

For a few variables, such as $n \leq 3$, this problem has been analyzed in great detail and solutions can be found in the literature; for example Chapter 13 of the first volume by Arnold, Gusein-Zade and Varchenko [1] presents a complete discussion. Following this, we state here without proof :

Theorem 5.1 (Arnold) *Quasi-homogeneous polynomials $\phi(x, y, z)$ which contain either of the following five groups of three monomials are transverse for arbitrary positive a, b, c :*

$$\begin{aligned} &\{x^a, y^b, z^c\}, \quad \{x^a, y^b, z^c y\}, \quad \{x^a y, y^b z, z^c x\}, \\ &\{x^a, y^b z, z^c y\}, \quad \{x^a, y^b z, z^c x\}. \end{aligned} \quad (5.2.2)$$

Also :

Theorem 5.2 (Arnold) Quasi-homogeneous polynomials $\phi(x, y, z)$ which contain either of the following two groups of four monomials are transverse for positive a, b, c and almost all ε :

$$\begin{aligned} \{x^a, y^b x, z^c x, \varepsilon y^p z^q\}, & \quad (a-1) \mid \text{lcm}(b, c) \\ \{x^a y, y^b x, z^c x, \varepsilon y^p z^q\}, & \quad [(a-1) \gcd(b, c)] \mid (b-1)c. \end{aligned} \quad (5.2.3)$$

The conditions on the right merely assure quasi-homogeneity*, for which, of course, p, q have to be chosen suitably.

Now, following this pattern, it is not hard to compile a list of groups of monomials in more than three variable, such that a generic linear combination of the monomials from each one group at a time provides a transverse polynomial. Such a straightforward extension of Arnold's list for the case of five variables can be found in Ref. [96][†]. It is by far not clear, how could such a procedure be proven to be complete. In any case, however, we enclose some remarks that might prove helpful in explorations of this kind.

Consider the first of the two cases in Theorem 5.2 (without loss of generality, we may consider the non-zero coefficients to be unity) :

$$\phi(x, y, z) = x^a + y^b x + z^c x + \varepsilon y^p z^q. \quad (5.2.4)$$

For $\varepsilon = 0$, this becomes $\phi_{\varepsilon=0} = x(x^{a-1} + y^b + z^c)$ which is singular when $x = 0$ and $(y^b + z^c) = 0$, that is, $y = (-1)^{1/b} z^{c/b}$. Alternatively, we say that $\phi_{\varepsilon=0}$ vanishes there to second order, having in mind its Taylor expansion around the singular set. To change this polynomial into one which would be singular only at the origin, we need to add a monomial which does not vanish at the singular set of $\phi_{\varepsilon=0}$; thereby, we will have moved the zero away from the singular set. Indeed, by adding the monomial $y^p z^q$ with a non-zero (generic) coefficient, ε , we obtain $\phi_{\varepsilon \neq 0}$, which is transversal.

This "trick" can clearly be applied to polynomials in more variables also, the additional variables can just 'tag along', added to the polynomial in the form of pure powers :

$$\phi(x, y, z, t) = x^a + y^b x + z^c x + \varepsilon y^p z^q + t^d \quad (5.2.5)$$

*Recall, $x|y$ means that y is divisible by x , i.e., that $(y/x) \in \mathbb{Z}$.

[†]The Authors of Ref. [96] also restrict themselves to monomials in no more than two of the five variables and so omit a number of transversal polynomials; some of these are given here.

is certainly transverse. Moreover, we can also set $\varepsilon = t$, leading to

$$\phi(x, y, z, t) = x^a + y^b x + z^c x + t y^p z^q + t^d, \quad (5.2.6)$$

which has been omitted in Ref. [96].

The monomials $t y^p z^q$ are rather different from those used in the above two theorems of Arnold's and the straightforward extensions in Ref. [96]. Those not only had at most two variables, but were of degree either $(a, 0, \dots, 0)$ or $(a, 1, 0, \dots, 0)$. Note the latter of these two, say 'elementary', types can be obtained from a suitable power of a single variable by a *linear* change of variables. The monomials $y^p z^q$ and $t y^p z^q$, however, cannot be obtained in such a way except $x^p y^q$ with $p = 1$ or $q = 1$. Simultaneously, note that the number of monomials needed to make the polynomials in Theorem 5.2 and also our example 5.2.6 is one more than the number of variables. It would appear that such 'less elementary' monomials are required to occur in addition to the 'elementary ones' of which there is always as many as there are variables.

Clearly, with five variables, the list diversifies even more and in fact no complete classification or otherwise general characterization seems to be known.

5.3 Hereditary Singularities

Given that most weighted projective spaces have finite quotient (orbifold-like) singularities, hypersurfaces and intersections therein are likely to meet these singular sets and inherit singular points, curves, and so on. It is therefore important to know just how badly singular spaces should one expect in our $\mathcal{M} \hookrightarrow \mathcal{X}$.

It is gratifying to know that there is a just absolute bound as to how singular \mathcal{M} may become. For simplicity, consider a complete intersection of K polynomials $f_a(z)$ in a single \mathbb{P}_w^{K+3} . Following Ref. [110], we define

$$m(p) \stackrel{\text{def}}{=} \#\{i : p|w_i\}, \quad k(p) \stackrel{\text{def}}{=} \#\{a : p|w(f_a(z))\}, \quad (5.3.1)$$

where $w(f_a)$ is the total weight of $f_a(z)$ and p is any prime.

Theorem 5.3 Let $\mathcal{M} = \bigcap_{a=1}^K f_a^{-1}(0) \subset \mathbb{P}_w^{K+n}$, for a generic choice of the defining polynomials $f_a(z)$. Then $(\dim \mathcal{M}) = n$

$$\text{Sing}(\mathcal{M}) = \mathcal{M} \cap \text{Sing}(\mathbb{P}_w^{K+n}), \quad \text{and} \quad \dim(\text{Sing}(\mathcal{M})) \leq n-2 \quad (5.3.2)$$

precisely if $k(p) + n - m(p) - 1 \geq 0$ for any prime p . Also,

$$\dim(\text{Sing}(\mathcal{M})) \leq m(p) + n - k(p) - 4 . \quad (5.3.3)$$

Clearly, the above criterion needs to be checked only for $p \leq \max(w_i, q_a)$.

Thus, when defining a (Calabi-Yau or not) 3-fold as a complete intersection in a weighted projective space, generic such models have—at worst—isolated cyclic quotient singular points and/or curves. Precisely the resolutions of these singularities have been discussed in the previous Chapter and the formula (5.1.14) is correct. We are now ready to construct new smooth 3-folds in a much larger set of available embedding spaces; for comparison, Chapter 1 listed precisely five Calabi-Yau 3-folds constructed as complete intersections of hypersurfaces in a single (isotropic) \mathbb{P}^n . By contrast, even as simple transversal hypersurfaces in single $\mathbb{P}_{\mathbf{w}}^4$'s (for various choices of \mathbf{w}), Ref. [96] constructs some 6000 Calabi-Yau 3-folds.

Now, the bounding value from Theorem 5.3, $[m(p) - k(p) - 1]$, can in fact become as large as 2; take for example $[\mathbb{P}_{(1:1:3:3:3)}^4 \parallel 11]$: $m(3) = 3$, $k(3) = 0$, while for all other primes p , $[m(p) - k(p) - 1] = -1$. The bound indicates that the degree-11 Calabi-Yau 3-folds in $[\mathbb{P}_{(1:1:3:3:3)}^4 \parallel 11]$ may inherit all of the singular surface $\mathbb{P}_{(3:3:3)}^3 \subset \mathbb{P}_{(1:1:3:3:3)}^4$. Note however that since 11 is not divisible by 3, any cubic $C(z_2, z_3, z_4)$ must in fact be multiplied by a quadric $Q(z_0, z_1)$. Therefore, the “best bet” for our defining equation is something like

$$\phi(z) \stackrel{\text{def}}{=} z_0^{11} + z_1^{11} + \sum_i Q_i(z_0, z_1) C_i(z_2, z_3, z_4) = 0 , \quad (5.3.4)$$

and obviously $d\phi(z) = 0$ when $z_0 = z_1 = 0$. In fact, general genericity arguments (of the type we used when discussing Bertini's theorem) lead us to expect that

$$\dim(\mathcal{M} \cap \text{Sing}) < 2 \quad (5.3.5)$$

if \mathcal{M} is a *transversal* intersection. For single hypersurfaces in $X = \mathbb{P}_{\mathbf{w}}^4$, this has been proven in Ref. [96]. For X a product of weighted projective spaces and \mathcal{M} a complete intersection of several hypersurfaces, we still expect the same result, a little thought and perhaps a few experiments will assure the Reader.

On the other hand, if \mathcal{M} does contain a singular surface—it is a consequence of non-transversality, as in the $[\mathbb{P}_{(1:1:3:3:3)}^4 \parallel 11]$ case, rather than the intersection with the singular set of $\mathbb{P}_{(1:1:3:3:3)}^4$. We will come back to this in § 5.4.1.

5.4 Weighted Calabi-Yau Complete Intersections

The foregoing discussion clearly indicates that complete intersections in embedding spaces which include weighted projective factors are on one hand a straightforward generalization of the constructions considered so far. On the other hand, they typically describe singular spaces. We then inquire if it is always possible to resolve these singularities in such a way that the resolution is Ricci-flat.

In answer to this, we quote from Ref. [158] :

Theorem 5.4 *For a quasismooth weighted complete intersection 2- or 3-fold X in a product of weighted projective spaces, the minimal desingularization \widetilde{X} of X is a smooth Calabi-Yau manifold as long as Eq. (5.1.13) is satisfied.*

Quasismooth here means that the complete intersection is transversal, so that the only singularities are those inherited from intersecting the singular set of the (weighted) embedding spaces. An extension of this theorem to include mildly non-transversal cases would be desirable, but remains for now an open question.



Let us mention also that similarly to weighted projective spaces, one can consider weighted Grassmannians and furthermore, weighted flag spaces, generalizing even more the embeddings considered thus far. We leave the exploration of these uncharted territories to the adventurous Reader and proceed with some simple examples which relate to our earlier study.

5.4.1 Some simple (smooth) samples

We recall the two simple constructions mentioned in § 3.2.3 :

$$[\mathbb{P}_{(1:1:1:1:2)}^4 \parallel 4] \cong \overline{\mathbb{P}}_{[3|4]}^2 , \quad [\mathbb{P}_{(1:1:1:1:3)}^4 \parallel 6] \cong \overline{\mathbb{P}}_{[3|6]}^2 . \quad (3.2.21')$$

As we have computed there, the anti-canonical bundles are $\mathcal{O}_{\mathbb{P}^3}(2)$ and $\mathcal{O}_{\mathbb{P}^3}(1)$, respectively. Consulting again Theorem 3.4, we see that we can obtain in rather similar fashion two Calabi-Yau 3-folds

$$[\mathbb{P}_{(1:1:1:1:2)}^4 \parallel 6] \cong \overline{\mathbb{P}}_{[3|6]}^3 , \quad [\mathbb{P}_{(1:1:1:1:4)}^4 \parallel 8] \cong \overline{\mathbb{P}}_{[3|8]}^2 . \quad (5.4.1)$$

On one hand, these are a triple cover of \mathbb{P}^3 , branched over a smooth sextic surface and a double cover of \mathbb{P}^3 branched over a smooth octic surface, respectively. At the

same time, they can be viewed as simple hypersurfaces in the weighted projective space $\mathbb{P}_{(1:1:1:1:2)}^4$ and $\mathbb{P}_{(1:1:1:1:4)}^4$, respectively.



Now we are ready for yet another description of such spaces (not necessarily Calabi-Yau). Compare the isotropic projectivity relation of a usual projective space and the anisotropic projectivity relation of a weighted projective space, both of which are represented both by

$$(z_0 : \dots : z_n) \cong (\lambda^{w_0} z_0 : \dots : \lambda^{w_n} z_n) \quad (5.4.2)$$

in the sense that $w = \vec{1}$ characterizes \mathbb{P}^n , while $w \neq \vec{1}$ specifies \mathbb{P}_w^n . Quite obviously, we can map

$$\mathbb{P}^n(x_0 : \dots : x_n) \xrightarrow{x_i \mapsto z_i = x_i^{w_i}} \mathbb{P}_w^n(z_0 : \dots : z_n). \quad (5.4.3)$$

This however makes sense only if we identify $x_i \cong \alpha_{(i)}^{w_i} x_i$, where $\alpha_{(i)}$ is a w_i primitive root of unity. Otherwise, the map would not be one-to-one, or in other words, the inverse would fail to be single-valued. This means that

$$\mathbb{P}_w^n = \frac{\mathbb{P}^n}{\prod_{i=0}^n \mathbb{Z}_{w_i}}. \quad (5.4.4)$$

Let us write $\varpi_w : \mathbb{P}^n \rightarrow \mathbb{P}_w^n$ for the quotient map.



With this in mind, we now re-consider the first (family of) 3-fold(s) described by Eqs. (5.4.1), writing $\phi(z) = 0$ for the defining equation. We now have the ‘commutative diagram’*

$$\begin{array}{ccc} \mathbb{P}^4(x_0 : \dots : x_4) & \xrightarrow{\phi(x)=0} & [\mathbb{P}^4 \parallel 6] \\ \downarrow z_i = x_i^{w_i} & & \downarrow \varpi_{(1:1:1:1:2)} \\ \mathbb{P}_{(1:1:1:1:2)}^4(z_0 : \dots : z_4) & \xrightarrow{\phi(z)=0} & [\mathbb{P}_w^4 \parallel 6] \cong \overline{\mathbb{P}}^{3(3)}_{[3:6]} \end{array} \quad (5.4.5)$$

So, a Calabi-Yau space embedded as a sextic hypersurface in the weighted $\mathbb{P}_{(1:1:1:1:2)}^4$, alternatively a double cover of \mathbb{P}^3 , branched over a smooth sextic surface in \mathbb{P}^3 .

*A ‘commutative diagram’ is quite simply, exactly what it says: there’s more than one way to get from ‘object’ A to ‘object’ B, but ‘all roads lead to Rome’; you end up with the same result no matter which way you go about it.

can also be described as a finite quotient (by ϖ_w) of a sextic in an ordinary (isotropic) \mathbb{P}^4 ! Note that the anti-canonical bundle of $[4][6]$ is $\mathcal{O}_{\mathbb{P}^4}(-1)$, that is, it is negative. In other words, the canonical bundle is $\mathcal{K}_{[4][6]} = \mathcal{O}_{\mathbb{P}^4}(1)$ and has five sections, rather than the unique holomorphic 3-form for Calabi-Yau manifolds. Indeed, according to the tentative taxonomy suggested in Table 9.1, the sextic in \mathbb{P}^4 is sort-of opposite to an ample 3-fold and is there called *scant*, just like in § 2.1.2, after Table 2.1.



Now, we have seen that there are just four hypersurfaces in \mathbb{P}^4 with negative canonical bundle, one (the quintic) with a trivial canonical bundle and there are infinitely many hypersurfaces with positive canonical bundle. The right most vertical arrow in the diagram (5.4.5) therefore indicates a process which could—in principle—produce Calabi-Yau 3-folds from any (almost)scant 3-fold. Let us therefore examine this a little closer.

To that end, note that the action of ϖ_w is non-trivial only on the fifth coordinate, x_4 , which is essentially being squared. So the intersection of the $x_4 = 0$ hyperplane in \mathbb{P}^4 with the sextic defining equation $\phi(x) = 0$ specifies a surface which then belongs to a configuration $[4][1, 6]$. At that surface, ϖ_w maps one-to-one, while away from it, it is a two-to-one map. That is, we have a \mathbb{Z}_2 action with a codimension-1 fixed set; as discussed in § 4.2.1, such a fixed set does not lead to a singularity; a simple local change of variables suffices to eliminate this (apparent) coordinate singularity. However, since neighboring points to this surface are covered twice, unlike the surface, we do have a branching set, \mathcal{B}_{ϖ_w} , at this surface and the “ramification formula” (3.2.19) is applicable; we only need to specify $\mathcal{L}(\mathcal{B}_{\varpi_w})$, the bundle which defines the branching locus.

As discussed in the sketch of proof of Theorem 3.4, the branching divisor for a k -fold covering is taken with a $k-1$ -fold multiplicity. Here, we have $k = 2$, so the branching divisor \mathcal{B}_{ϖ_w} is taken at face value, with no multiplicity. Furthermore, since it belongs to the configuration $[4][1, 6]$ and is a surface on a 3-fold which belongs to $[4][6]$, i.e., the sextic defining equation is in common, it should be obvious that $\mathcal{L}(\mathcal{B}) = \mathcal{O}_{\mathbb{P}^4}(1)$ and so

$$\mathcal{O}_{\mathbb{P}^4}(-1) = \mathcal{K}_{[\mathbb{P}^4 \parallel 6]}^* \sim \mathcal{K}_{[\mathbb{P}_w^4 \parallel 6]}^* \otimes \mathcal{L}^*(\mathcal{B}_{\varpi_w}) = \mathcal{K}_{[\mathbb{P}_w^4 \parallel 6]}^* \otimes \mathcal{O}_{\mathbb{P}^4}(-1), \quad (5.4.6)$$

from which it follows that $\mathcal{K}_{[\mathbb{P}_w^4 \parallel 6]}^* \approx \mathbb{C}$. In a way, this just manifests what has perhaps been obvious to the erudite Reader from the beginning—that the process

of passing to a finite quotient is in a way inverse to constructing finite covering spaces, where the fixed points of the former correspond to the branching sets of the latter.

- ☞ Calculate $\chi_E[\mathbb{P}^4 \parallel 6]$ and relate it to $\chi_E[\mathbb{P}_w^4 \parallel 6]$ and Eqs. (3.2.16) and (4.5.3). ↗
- ☞ Repeat the above analysis for $[\mathbb{P}_{(1:1:1:1:4)}^4 \parallel 8]$ (caution : $k = 4$ here). ↗



In view of the recent discussion, we look back at the example defined in Eq. (5.3.4), a degree-11 hypersurface in $\mathbb{P}_{(1:1:3:3:3)}^4$. By pulling it back to the covering hypersurface in (the isotropic) \mathbb{P}^4 , following the diagram (5.4.5), the defining equation becomes

$$\phi(x) \stackrel{\text{def}}{=} x_0^{11} + x_1^{11} + \sum_i Q_i(x_0, x_1) C_i(x_2^3, x_3^3, x_4^3) = 0, \quad (5.4.7)$$

which is clearly singular at the projective surface $(0 : 0 : x_2 : x_3 : x_4) \subset \mathbb{P}^4$. The fact that $\phi(x)$ is a singular choice in the scant configuration $[\mathbb{P}^4 \parallel 11]$ is the “true cause” for the singularity of the example (5.3.4) although the singular locus of course coincides with $\text{Sing}(\mathbb{P}_{(1:1:3:3:3)}^4)$.

We conclude that if the multiple cover of the type described by the diagram (5.4.5) is smooth, the weighted complete intersection is singular only by intersecting the singularity set of the (weighted) embedding space. The converse is, however, not true as can be seen from the following very simple example.

$$\{z_0^2 + z_0 z_1^2 + z_2^4 + z_3^4 + z_4^4\} \subset [\mathbb{P}_{(2:1:1:1:1)}^4 \parallel 4] \quad (5.4.8)$$

is smooth while its cover,

$$\{x_0^4 + x_0^2 x_1^2 + x_2^4 + x_3^4 + x_4^4\} \subset [\mathbb{P}^4 \parallel 4] \quad (5.4.9)$$

is singular at $(0 : 1 : 0 : 0 : 0)$. Although weighted complete intersections can be analyzed in their own terms, it is often useful to employ the relations indicated in the diagram (5.4.5).

This situation is not at all unlike that of Riemann surfaces : there is one with negative $(g-1)$, one with vanishing $(g-1)$ and then an infinite number of those with positive $(g-1)$. The Riemann surfaces with higher genus can be described as branched covers of Riemann surfaces with lower genus simply by passing to a

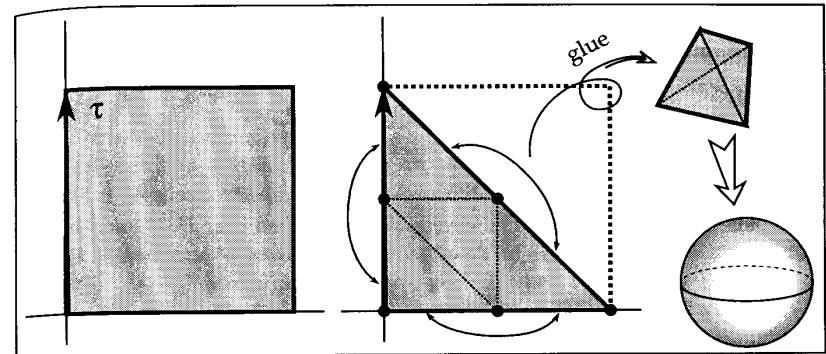


Figure 5.1: The fundamental domain of a \mathbb{Z}_2 -symmetric torus, its \mathbb{Z}_2 quotient and another view thereof. The second transition only requires local coordinate changes at the vertices.

finite quotient. One of the simplest examples is provided by the \mathbb{Z}_2 -symmetric torus, $T_{\mathbb{Z}_2}^2$, the \mathbb{Z}_2 quotient of which is a sphere; see Fig. 5.1.



It is quite straightforward in principle, although formidable in notation, to extend these relations to arbitrary complete intersections in products including weighted homogeneous spaces. In a way though, because of this relation between complete intersections in weighted projective spaces and finite quotients of intersections in (isotropic) homogeneous spaces, it may appear that the whole excursion regarding weighted spaces is unnecessary. While it is true that these various constructions are related by ‘commutative diagrams’ and so it ultimately does not matter which way one goes about it, one or the other route might prove useful in some particular computations. For instance, hypersurfaces in weighted \mathbb{P}_w^4 turn out to be in a very simple correspondence with certain supersymmetric field theories and allow in many cases exact quantum calculations (we will return to this briefly later; see also Ref. [190,184,227,152], for example).

5.5 Calabi-Yau from Almost Fano 3-Folds. II

Recall again the relation $[\mathbb{P}^4_{(1:1:1:1:4)} \parallel 8] \cong \overline{\mathbb{P}}^{(2)}_{[3|6]}$ and the simple defining equation

$$z_2 + B(x) = 0, \quad B(x) \stackrel{\text{def}}{=} \sum_{i=0}^3 (x_i)^8, \quad (5.5.1)$$

like in Eq. (3.2.15), except that now we chose the weights and the degree of the hypersurface to obtain a Calabi-Yau 3-fold. As already discussed in § 5.4.1, manifolds of this type are double covers of \mathbb{P}^3 branched over \mathcal{B} , a smooth octic surface. Note that \mathcal{B} can also be described as the zero-set of a section of $\mathcal{K}_{\mathbb{P}^3}^{*\otimes 2}$.

5.5.1 An easy corollary

Now, quite more generally, given any almost Fano 3-fold, \mathfrak{F} , we can take its double cover branched over the zero-set, \mathcal{B} , of a generic section of $\mathcal{K}_{\mathfrak{F}}^{*\otimes 2}$ —we obtain a Calabi-Yau 3-fold, $\mathcal{M} \stackrel{\text{def}}{=} \overline{\mathfrak{F}}^{(2)}_{\mathcal{B}}$. The Euler characteristic of such a Calabi-Yau 3-fold is straightforwardly given by Eq. (3.2.16) in terms of $\chi_E(\mathfrak{F})$ and $\chi_E(\mathcal{B})$. Since the branching surface was there already in the almost Fano 3-fold, no 4-cycle or 2-cycle has been added in this construction and since $H_4(\mathcal{M}) \cong H_2(\mathcal{M}) \cong H^2(\mathcal{M})$, $b_{1,1}(\mathcal{M}) = b_{1,1}(\mathfrak{F})$ (neither \mathcal{M} nor \mathfrak{F} have any (2,0)- or (0,2)-forms, so that $(H^2 \otimes \mathbb{C}) \cong H^{1,1}$).

In fact, even more generally,

Corollary 5.2 *Let \mathfrak{F} be any almost Fano 3-fold with index r and q equal to 1 or some positive integer that divides r . Then the $(q+1)$ -fold covering of \mathfrak{F} , branched over a generic surface $\mathcal{B} \in [\mathfrak{F} \parallel \mathcal{K}_{\mathfrak{F}}^{*\otimes (q+1)/q}]$ is a Calabi-Yau 3-fold \mathcal{M} with Euler characteristic*

$$\chi_E(\mathcal{M}) = (q+1)\chi_E(\mathfrak{F}) - q\chi_E(\mathcal{B}). \quad (5.5.2)$$

Proof: To begin with, recall that by definition of the index (see page 84), there exists a line bundle \mathcal{L} such that $\mathcal{L}^{\otimes r} \approx \mathcal{K}_{\mathfrak{F}}^*$, where r is the index. This means that $\mathcal{K}_{\mathfrak{F}}^{*\otimes (q+1)/q} \approx \mathcal{L}^{\otimes (q+1)(r/q)}$, which is a well defined holomorphic bundle since $(r/q) \in \mathbb{Z}$ and so $\mathcal{B} \in [\mathfrak{F} \parallel \mathcal{L}^{\otimes (q+1)(r/q)}]$ is a well defined surface. The statement now follows as a straightforward application of Theorem 3.4, which is why this result is labeled as a corollary. \square

It is amusing to note that, since the index of \mathbb{P}^3 is 4, a (suitably branched) double, triple or five-fold cover will be a Calabi-Yau 3-fold. The first two of these have been identified in Eq. (5.4.1). A little thought reveals that the third one is the best known Calabi-Yau space, the quintic in \mathbb{P}^5 .

5.5.2 Coverings, complete intersections and resolutions

The main reason for postponing this result until here is an important relation between the Calabi-Yau manifolds constructed in Corollary 5.2 and those constructed in § 3.2.5, which however involves small resolutions.

For simplicity, compare

$$\mathcal{M} \in \left[\begin{matrix} 3 & \parallel & 4 \\ 1 & & 2 \end{matrix} \right] \quad \text{vs.} \quad \mathcal{M}' \in \overline{\mathbb{P}}^{(2)}_{[3|8]}. \quad (5.5.3)$$

For the record, $\chi_E(\mathcal{M}) = -168$, while $\chi_E(\mathcal{M}') = -296$; also $b_{1,1}(\mathcal{M}) = 2$ while $b_{1,1}(\mathcal{M}') = 1$. The relation becomes apparent when we consider the defining equation for the configuration on the left. Quite like in the 2-fold case considered on page 79, we write

$$\mathcal{M} : \quad Q_{00}(x)z_0^2 + 2Q_{01}(x)z_0z_1 + Q_{11}(x)z_1^2 = 0, \quad (5.5.4)$$

where the $Q_{ij}(x)$ are generic quartics over \mathbb{P}^3 . For a generic point $x \in \mathbb{P}^3$, the defining equation is a non-degenerate quadric and has therefore two distinct solutions. \mathcal{M} therefore covers the generic part of \mathbb{P}^3 twice. However, over some sub-generic points $x \in \mathbb{P}^3$, the discriminant

$$\Delta(x) \stackrel{\text{def}}{=} 2(Q_{00}(x)Q_{11}(x) - Q_{01}(x)^2) \quad (5.5.5)$$

of the quadric equation vanishes; this being a single octic equation in \mathbb{P}^3 , this defines an octic surface, \mathcal{B} . So, \mathcal{M} is a double cover of \mathbb{P}^3 , branched over the octic surface* $\mathcal{B} \in \mathbb{P}^3$. This appears identical to \mathcal{M}' , so how can they have so different numerical characteristics?

The answer lies in the form of $\Delta(x)$: the surface \mathcal{B} , defined by $\Delta(x) = 0$ is in fact singular. Indeed, $d\Delta = 0$ whenever all three $Q_{ij}(x) = 0$. These being three quartic equations in \mathbb{P}^3 , this is *bound* to happen at $4 \cdot 4 \cdot 4 = 64$ points. But, From Bertini's theorem, we know that the generic \mathcal{M} is smooth (and we have assumed a generic choice of the $Q_{ij}(x)$'s)!? Again, a second look at the quadratic equation—for the case when all three $Q_{ij} = 0$ —tells that the variables of the \mathbb{P}^1 are completely unconstrained.

Finally, then, we have a complete description of \mathcal{M} : it is a double cover of \mathbb{P}^3 , branched over a *singular* octic surface and the 64 singular points of which

*Note that, when thought of as a special subset of the zero-set of the quadratic equation for \mathcal{M} , the points of the branching set come with multiplicity 2!

have been small-resolved into a \mathbb{P}^1 each. Similarly, \mathcal{M}' is a double cover of \mathbb{P}^3 , branched however over a smooth octic surface. Note that

$$\chi_E(\mathcal{M}) - \chi_E(\mathcal{M}') = 64\chi_E(\mathbb{P}^1). \quad (5.5.6)$$

Clearly, any almost Fano 3-fold could have been used in place of \mathbb{P}^3 in the relation (5.5.3).

Chapter 6

Fibered Products

After having reviewed a large number of constructions of Calabi-Yau 3-folds, we here briefly mention one more class of constructions—not quite unrelated to those discussed before. The geometric description however will be somewhat novel, which is our primary reason to examine them.

6.1 Fibering Elliptic Curves

To begin with, we describe a simple, 1-dimensional model rather than a Calabi-Yau 3-fold. Consider the family of cubics in \mathbb{P}^2

$$\phi_\alpha(x, y, z) \stackrel{\text{def}}{=} x^3 + y^3 + z^3 - 3\alpha xy z = 0. \quad (6.1.1)$$

Unless α is chosen to satisfy $\alpha^3 = 1$ (or in the limit $\alpha \rightarrow \infty$), this defines a smooth complex 1-dimensional torus, also called a smooth elliptic curve, as a hypersurface in \mathbb{P}^2 . As the parameter α is varied, the resulting torus varies and we have a family of elliptic curves, parametrized by α .

In a way, the whole picture is rather similar to the general image one has about vector bundles, in that α of course spans a complex plane and we have associated to every point of that plane an elliptic curve, that is, a torus. Moreover, as is clear from the defining equation, these tori vary holomorphically with α and we have a holomorphic fibre space of which the α -plane is the base and the particular torus parametrized by a given value of α is the *fibre* at α . The present situation is however markedly different from the standard notion of a bundle : at special values of α , the fibre is singular and we are unable to construct appropriate transition functions.



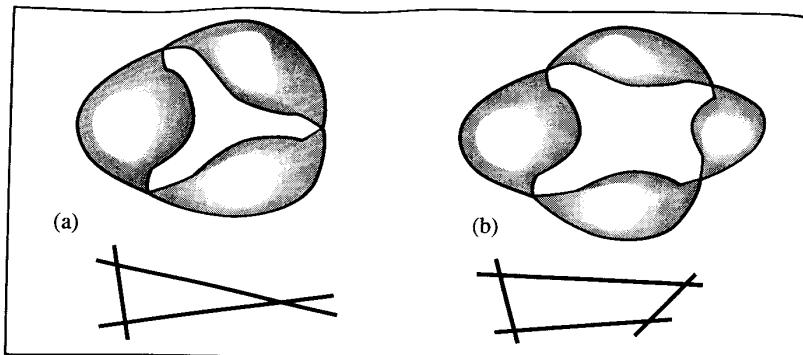


Figure 6.1: Degenerations of elliptic curves : (a) cubic, (b) bi-quadratic.

On first impulse, one would expect that therefore the total (complex 2-dimensional) space of this fibration (base and fibres together), while obviously complex space, is singular where the singular tori are. But, is that so?

First of all, the tori *are* singular when $\alpha^3 = 1$, as can be verified since

$$\frac{1}{3}d\phi_{\alpha=1} = (x^2 - yz)dx + (y^2 - xz)dy + (z^2 - xy)dz \quad (6.1.2)$$

which vanishes at the $(1:1:1)$, $(1:\omega:\omega^2)$ and $(1:\omega^2:\omega)$ points of \mathbb{P}^2 ; $\omega = e^{2i\pi/3}$. These cubic elliptic curves degenerate into a connected union of three lines (\mathbb{P}^1 's) each two of which touch at a point—the three singular points; see Figure 6.1, (a).

However, to determine the singularity of the total space, we have to stop treating α as a constant parameter and regard it as a coordinate instead. This changes the total differential of $\phi_\alpha(x, y, z)$ into

$$\frac{1}{3}d\phi = (x^2 - \alpha yz)dx + (y^2 - \alpha xz)dy + (z^2 - \alpha xy)dz - (xyz)d\alpha, \quad (6.1.3)$$

which is, quite clearly, non-vanishing at any of the singular points on the singular tori. Instead, at those points, $d\phi$ is purely in the α direction.

The above equation may appear a little awkward as it is homogeneous in the x, y, z variables but not in α ; this we remedy easily by declaring $\alpha = (v/u)$ and multiplying through with u (assumed non-zero here), to obtain

$$\phi(x, y, z; u, v) \stackrel{\text{def}}{=} u(x^3 + y^3 + z^3) - 3v(xyz) = 0. \quad (6.1.4)$$

6.1. Fibering Elliptic Curves

By adding the point at $\alpha = \infty$, that is $u = 0$, we obtain the standard compactification of the α -plane (\mathbb{C}) into the $(u:v)$ projective space, \mathbb{P}^1 . Of course, the torus at $u = 0$ is also singular, again at three points in \mathbb{P}^2 : $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$. It can be readily checked that the total differential, with respect to x, y, z , but also u, v is non-zero everywhere.



On a second thought, Eq. (6.1.4) clearly describes a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$, of bi-degree $(3,1)$ and so corresponds to the configuration $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. A quick glance back to section § 3.1 and the list (3.1.19) reminds us that the generic compact surfaces which belong to this configuration are almost del Pezzo surfaces of $\chi_E = 12$. That is, the total space of elliptic curves fibered linearly over \mathbb{P}^1 is in fact this almost ample surface (which we recall from § 3.1 is the only one which is *not* ample). Note that the elliptic curves which provide the fibres are themselves embedded in a del Pezzo surface (here quite simply \mathbb{P}^2) by means of a section of its anti-canonical bundle; $\mathcal{K}_{\mathbb{P}^3}^* \approx \mathcal{O}_{\mathbb{P}^3}(3)$ —hence cubics.

The thought of replacing \mathbb{P}^2 with another del Pezzo surface comes readily and, indeed, any surface belonging to any of the configurations in the list (3.1.19) could be used*. Of course, all the so obtained almost del Pezzo surfaces will have $\chi_E = 12$, as that is the only one which is not ample†. Yet, there are some interesting differences.

Consider, as compared to Eq. (6.1.4) the fibre space again over \mathbb{P}^1 , but this time with the fibre being given not as a cubic in \mathbb{P}^2 , but rather as a bi-quadratic in $\mathbb{P}^1 \times \mathbb{P}^1$. The defining equation is now

$$A(x^0 : x^1 ; y^0 : y^1)u + B(x^0 : x^1 ; y^0 : y^1)v = 0, \quad (6.1.5)$$

where A and B are quadratic in the first pair of arguments and also in the second pair. A quick analysis of this says that there are three values of (u/v) where the bi-quadratic elliptic curve is singular and, at each of those values of (u/v) , one finds that the bi-quadratic curve degenerates into a connected union of four lines (\mathbb{P}^1 's), as in Figure 6.1, (b).

*Remember that all del Pezzo surfaces can be represented as complete intersections in a product of complex projective spaces.

†This fibration representation of the $\chi_E = 12$ almost del Pezzo surfaces lets us see explicitly why they cannot be ample : the canonical bundle is trivial along all (except perhaps finitely many) fibres.

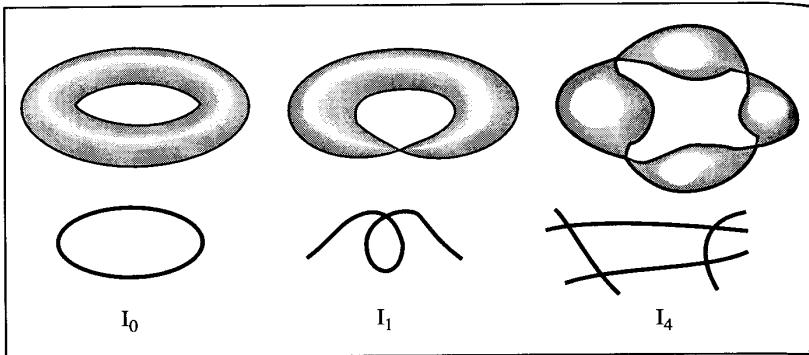


Figure 6.2: The simplest types of elliptic fibres : I_0 —non-degenerate, I_1 —irreducible and I_n — n -reducible.

In fact, general theory (see Section V.7 of Ref. [4]) shows that elliptic curves can be fibered over \mathbb{P}^1 much more generally than the above two cases. In particular, there are fibrations in which there occur fibres of type I_n , that is, where the elliptic curve had degenerated into n copies of \mathbb{P}^1 in a sort of a circular chain, with n singular points where two \mathbb{P}^1 's meet (see Figure 6.2). Our above examples thus have fibres of types I_3 and I_4 , respectively.



By the way, just to dispel some possible misconception about the cubics in \mathbb{P}^2 , consider

$$\phi'(x, y, z; u, v) \stackrel{\text{def}}{=} u(xy z) - v(\tfrac{1}{3}(x^3 + y^3 + z^3) - x^2y + y^2x) = 0. \quad (6.1.6)$$

Unlike the symmetric case above, this fibration has seven singular tori :

1. $u = 1, v = 0$: a single fibre with three singular points, at $(1:0:0), (0:1:0)$ and $(0:0:1)$;
2. $u = 2\omega, v = 1$: at $\omega \in \{1, e^{2i\pi/3}, e^{4i\pi/3}\}$, there are three fibres, with two singularities each, at $(1: \mp i:\omega^2(\pm i-1))$;
3. $u = 4^{1/3}\omega, v = 1$: at $\omega \in \{1, e^{2i\pi/3}, e^{4i\pi/3}\}$, there are three fibres, with one singularity each, at $(1:1:-2\omega^2 2^{1/3})$.

6.1. Fibering Elliptic Curves

Thus, by breaking the symmetry of the most popular cubic (6.1.4), it is possible to obtain several different elliptic curves, including here I_1 and I_2 . Still, rather few of the possible I_n 's can be realized in such an explicit way.



Since we have obtained an almost del Pezzo surface, one naturally comes to think of fibering elliptic curves in a way that would produce a Calabi-Yau surface, that is, a model of K3. Looking at the configuration which describes the total space of the above fibrations as a complete intersection, we see that we can simply consider, for example,

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} : \quad A(x, y, z) u^2 + 2B(x, y, z) u v + C(x, y, z) v^2 = 0. \quad (6.1.7)$$

Nothing really changes in the geometrical interpretation, except that the elliptic curve races *quadratically* over the base \mathbb{P}^1 instead of linearly as in the above examples. The Reader may wonder why we have written the defining equation in terms of three, rather than just two summands as in the linear case. It turns out that without having all three terms—as a generic quadratic equation in u, v has them—the defined surface is singular.

Prove this.

Unlike in the above less-than-maximally symmetric example (6.1.6), we are now *forced* to introduce a term which will break at least some of the many symmetries of (6.1.4). Consequently, some of the singular fibers again ‘decompose’ into I_n , $n < 3$.



Suppose, for the moment, that instead of these non-trivial fibrations, we were considering a simple product $T \times \mathbb{P}^1$. The Euler characteristic is multiplicative, so

$$\chi_E(T \times \mathbb{P}^1) = \chi_E(T) \cdot \chi_E(\mathbb{P}^1) = 0 \cdot 2 = 0. \quad (6.1.8)$$

This would pertain to be true even if the fibration were non-trivial but all fibres were smooth. Instead, in the above examples, we know that a number of fibres has become singular—now we see that comparing the Euler characteristic of the total space of the fibration with the trivial product in fact tells the number of singular fibres, counting multiplicities. In the fibered cubic case, $\chi_E \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = 12$

and there were four fibres with three singular points each. In the other example, there were three fibres with four singular points each—again twelve, as predicted.

This calculation of course seems trivial here, but the general principle holds for higher dimensions; the difference between the Euler characteristic of the total space of a fibration and the Euler characteristic of the trivial product counts the singular fibres, with multiplicities counted. More precisely, of course,

$$\chi_E(\text{non-trivial}) - \chi_E(\text{trivial}) = \sum_i \chi_E(\text{singularity}_i), \quad (6.1.9)$$

where “trivial” and “non-trivial” refer to the fibrations and “singularity” is the fibre-wise singularity, i.e., while keeping the locus in the base fixed (treat $\alpha = (v/u)$ again as a parameter).

6.2 Fibering Elliptic Surfaces

Now, just as the foregoing showed that the almost del Pezzo surface (which is not ample) may be regarded as a fibration in which the fibres have vanishing first Chern class and the base is \mathbb{P}^1 , could it be that almost Fano 3-folds can be regarded as a similar fibration?

Indeed, the answer is yes. In fact, understanding the example (6.1.4) in this way, it is not hard to see that Eq. (3.2.23) :

$$\tilde{\mathfrak{F}} \in \left[\begin{array}{c|c} \mathfrak{F} & f_1 \\ \hline \mathbb{P}^1 & J \end{array} \right], \quad (6.2.1)$$

represents just that. The defining equation being

$$A(x)u + B(x)v = 0, \quad x \in \mathfrak{F}, \quad (u:v) \in \mathbb{P}^1, \quad (6.2.2)$$

at any given point of \mathbb{P}^1 , we have a particular equation in variables over \mathfrak{F} . As dictated by the configuration, $A(x)$ and $B(x)$ are sections of a line bundle of which the first Chern class equals $f_1 = c_1(\mathfrak{F})$, i.e., they are sections of the anti-canonical bundle. Consequently, at every point of \mathbb{P}^1 , we have a surface with $c_1 = 0$, which can only be a model of K3; it is embedded as a hypersurface in \mathfrak{F}^* .

*The Reader unsettled with such generality may replace \mathbb{P}^3 for the almost Fano threefold \mathfrak{F} and consequently have $\mathcal{O}_{\mathbb{P}^3}(4)$ for the anti-canonical bundle; the K3 surface is then simply one of the well known quartics in \mathbb{P}^3 .

Just as with the elliptic curves, these elliptic surfaces (K3's) vary holomorphically over the base \mathbb{P}^1 and form a fibered space. Again some special points of \mathbb{P}^1 carry singular K3's, as can be revealed from the Euler characteristic computation; $\chi_E(\tilde{\mathfrak{F}})$ is given, as a consequence of Eq. (3.2.23), in terms of numerical characteristics of \mathfrak{F} and so

$$2 \cdot 24 - \chi_E(\mathfrak{F}) + C_1^3(\mathfrak{F}) \quad (6.2.3)$$

is the number of fibre-wise singularities (the sign changes with the dimension). \(\ddot{\epsilon}\)

Instead of fibering a surface over a \mathbb{P}^1 , can we form a fibered space using for the base a surface, and if so, what type of surface do we need so that the total space would turn out to be a Calabi-Yau manifold?

Note that in all the previous cases, we used a copy of \mathbb{P}^1 as the base, which is the only almost ample curve (in fact it is ample). We thus expect that (almost) ampleness is required for the base space. There are of course more than one almost ample surfaces, so there will be more possible constructions. Indeed, W.-W. Sung [221] proves that our intuition has not failed us; we quote the main theorem without proof as a proper rendition would take us too deep into technical details.

Theorem 6.1 *For \mathcal{M} a smooth Calabi-Yau (projective) 3-fold and let $\varpi : \mathcal{M} \rightarrow S$ be a proper, surjective holomorphic map with connected fibres, onto a smooth compact complex surface S . Then, S is an almost del Pezzo surface.*

So, indeed, for a Calabi-Yau 3-fold to be represented as a fibration over a surface S , it must be that S is one of those defined in § 3.1.

6.3 Fibering Double Elliptic Curves

Finally, we come to the closing of the part of this volume where the primary concern was how to construct (families of) Calabi-Yau 3-folds and carry through some of the most rudimentary analysis. To this end, we consider double elliptic curve fibrations studied by C. Schoen [212] and which also relates to the complete intersections discussed in Chapter 2.

Following Schoen, we consider a class of Calabi-Yau 3-folds which may be constructed from a singular bi-cubic hypersurface in $\mathbb{P}_1^2 \times \mathbb{P}_2^2$, defined by

$$\mathcal{M}_S : A_{10}(x_{(1)})A_{21}(x_{(2)}) - A_{11}(x_{(1)})A_{20}(x_{(2)}) = 0 , \quad (6.3.1)$$

where $A_{ij}(x_{(i)})$ are cubic polynomials in the homogeneous coordinates $x_{(i)} \in \mathbb{P}_i^2$, $i = 1, 2$. At any particular point in $\mathbb{P}_1^2 \times \mathbb{P}_2^2$, we may use the four variables A_{ij} as local holomorphic coordinates and note that the defining equation becomes $xy - zt = 0$ or, after a linear change of variables, $\sum_{i=1}^4 \zeta_i^2 = 0$. This is precisely the equation for the simplest of all hypersurface singularities, the node (see Table 4.1); the location of the singularity is at the origin, $x = y = z = t = 0$, that is, at $A_{ij} = 0$. These are four cubic equations, two in each \mathbb{P}^2 and, for a generic choice of the A_{ij} , they have $(3 \cdot 3)(3 \cdot 3) = 91$ nodes for the space of solutions.

Thus, \mathcal{M}_S is a (purely) nodal Calabi-Yau 3-fold. From the discussion in § 4.4, we know that each node can be resolved into a \mathbb{P}^1 in two distinct ways, so that there are 2^{81} (!) small resolutions of this singular 3-fold. What is not obvious is whether any of these is Kähler. It turns out—yes, and it is moreover embeddable as a smooth complete intersection mentioned in (3.1.27),

$$\mathring{\mathcal{M}}_S \in \left[\begin{array}{c|cc} 2 & 3 & 0 \\ 2 & 0 & 3 \\ \hline 1 & 1 & 1 \end{array} \right] . \quad (3.1.27')$$

To prove this, consider the generic system of defining equations in this configuration :

$$\begin{aligned} A_{10}(x_{(1)})y^0 + A_{11}(x_{(1)})y^1 &= 0 , \\ A_{20}(x_{(2)})y^0 + A_{21}(x_{(2)})y^1 &= 0 . \end{aligned} \quad (6.3.2)$$

Regard this as a system of two linear homogeneous equations in two variables, y^0 and y^1 , which must not vanish simultaneously. The solvability condition of course is the vanishing of the determinant of the system, which is exactly Eq. (6.3.1). This relates the Calabi-Yau manifold $\mathring{\mathcal{M}}_S$ and the Calabi-Yau nodal 3-fold \mathcal{M}_S .

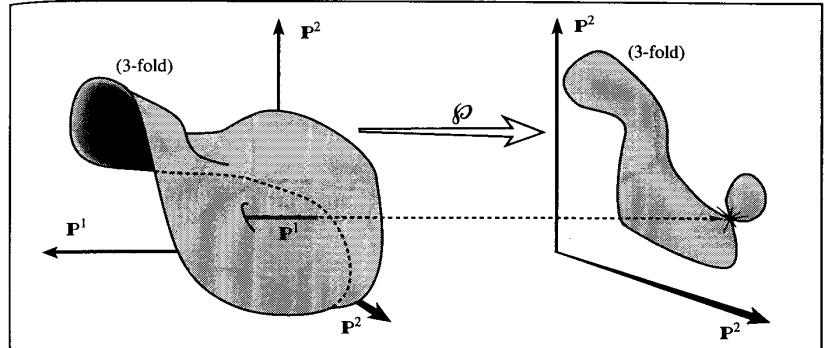


Figure 6.3: A rough sketch of the projection $\mathring{\mathcal{M}}_S \xrightarrow{\varphi} \mathcal{M}$.

In other words, $\mathring{\mathcal{M}}_S$ is embedded in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^1$. If we project along the \mathbb{P}^1 factor (eliminate the y^i variables), we obtain the projection on $\mathbb{P}^2 \times \mathbb{P}^2$, which is precisely \mathcal{M}_S as defined in Eq. (6.3.1)

$$\mathcal{M}_S = \varphi(\mathring{\mathcal{M}}_S) . \quad (6.3.3)$$

On a more local level, the situation is as follows. \mathcal{M}_S is singular only when all four A_{ij} vanish. On the other hand, $\mathring{\mathcal{M}}_S$ is not singular when all four A_{ij} vanish; rather, it contains a full copy of the embedding \mathbb{P}^1 as the variables y^i are not constrained at all. One says that the inverse image of each node in \mathcal{M}_S , under φ , is a \mathbb{P}^1 in $\mathring{\mathcal{M}}_S$ (see Figure 6.3). Thus, $\mathring{\mathcal{M}}_S$ is a small resolution of \mathcal{M}_S !

From the nodal Calabi-Yau 3-fold (6.3.2), Schoen then constructs a collection of Calabi-Yau manifolds by varying the coefficients A_{ij} so that the generic member is \mathcal{M}_S while less generic members involve special choices of $A_{ij}(x_{(i)})$.

Note that, for a given point in \mathbb{P}^1 , Eqs. (3.1.27) define two separate cubics in a product of two \mathbb{P}^2 's, that is, a product of two elliptic curves parametrized by the variables y^i . Similarly to the fibrations above, we have that $\mathring{\mathcal{M}}_S$ is a double elliptic fibration over \mathbb{P}^1 . To describe the rest of this collection, note that as these elliptic curves vary along the \mathbb{P}^1 (see Fig. 6.4), at certain points of the \mathbb{P}^1 , these tori degenerate having a number of their “small” circles pinched to points. Of course, for a generic choice of $A_{ij}(x_{(i)})$, the two tori do not degenerate simultaneously and the resulting fibre space is a smooth manifold, \mathcal{M}_S .

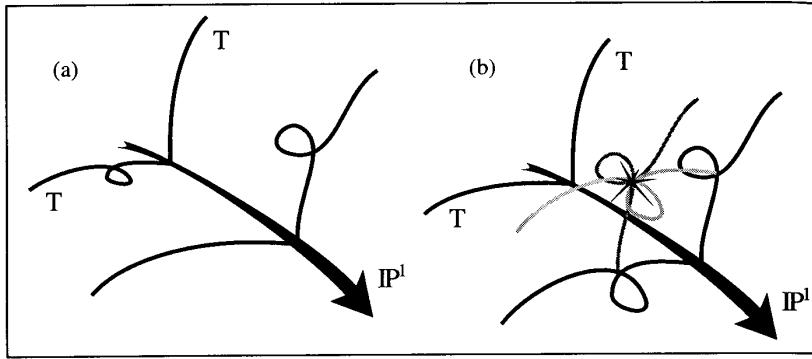


Figure 6.4: Double elliptic fibrations : (a) smooth; (b) singular.

Choosing special cubics for which the two elliptic curves do degenerate (but no worse than nodal) simultaneously at some special points of \mathbb{P}^1 gives rise to nodal limits of \mathcal{M}_S , classifying them in terms of the classification of singular elliptic surfaces. The nodes of nearly all of these conifolds can be repaired by small resolution leading to other Kähler manifolds [212]; we state here without proof

Lemma 6.1 Let \mathcal{M} denote the double elliptic fibration over \mathbb{P}^1 with fibres $I_{m,n} \stackrel{\text{def}}{=} I_m \times I_n$, where I_n is an elliptic curve with n nodes. Let S and S' denote the set of points in \mathbb{P}^1 where, respectively, one and the other elliptic curve degenerate.

- (0) \mathcal{M} is singular only over $S \cap S'$;
- (i) if \mathcal{M} contains only $I_{0,0}$, it is smooth and Kähler;
- (ii) if \mathcal{M} contains only $I_{m,m}$, it has a smooth and Kähler resolution;
- (iii) if \mathcal{M} contains a fibre $I_{m,1}$ and also $I_{p,0}$ or $I_{0,q}$, with $m, p, q \neq 0$, it has no smooth and Kähler resolution.

Note : the reason for the negative result in (iii) is essentially the phenomenon described in § 4.4.3.

This provides a collection of Calabi-Yau manifolds, with many values $0 < \chi_E \leq +168$, as a list of small resolutions of various nodal deformations of $\varphi(\mathcal{M}_S)$.

Part II

Cohomology

Chapter 7

(Co)homology Basics

The reader is most probably at least acquainted with the basic notion of cohomology, differential forms and we forego introducing these in detail. Should need arise, some of the basic texts on this subject should be consulted, starting perhaps with the relevant sections from the physicist's favorite Ref. [19], Ref. [25,52] and then carrying on with Ref. [9] ...

We will however, review the barest essentials of some (co)homology theories which are ultimately those of most physical relevance, in my own personal opinion.

7.1 Cohomology $1 \wedge 1$

7.1.1 Some generalities

Generally, we think of a cohomology group as an abelian additive group of certain objects for which we can specify two integers called degrees* so that the total cohomology group takes the form

$$H^{*,*} = \bigoplus_{p,q} H^{p,q}, \quad (7.1.1)$$

where p and q take some integer values. One can think of the (p,q) -pair as a *bi-grading* of the total cohomology group $H^{*,*}$. Usually, $H^{p,q}$ vanishes identically outside of some finite range of p and q .

*This is also often called 'dimension' which may lead to silly conflicts in nomenclature and possible confusion, so we try to stick to 'degree'. Also, we immediately deal with (p,q) -cohomology, having in mind the application throughout this volume.

The elements of these cohomology groups can generally be obtained from a collection of objects called ‘forms’, α . These also have a well-defined degree, so we speak of (p, q) -forms (which vanish identically outside of some finite range of p and q) and generally denote their space by $A^{p,q}$. Bundled with these $A^{p,q}$ comes a nilpotent operator ∂ , called a differential of type $(1, 0)$, which acts on these forms and so maps

$$\partial : A^{p,q} \longmapsto A^{p+1,q}, \quad \partial \circ \partial = 0. \quad (7.1.2)$$

The conjugate operator, $\bar{\partial}$, sends $A^{p,q}$ to $A^{p,q+1}$ and is similarly nilpotent. Now, we may define the $\bar{\partial}$ -cohomology[†] as

$$H_{\bar{\partial}}^{p,q} \stackrel{\text{def}}{=} \left\{ \{ \alpha \in A^{p,q} : \bar{\partial}\alpha = 0 \} / \{ \bar{\partial}\eta, \eta \in A^{p,q-1} \} \right\}. \quad (7.1.3)$$

That is, the elements of $H_{\bar{\partial}}^{p,q}$ are identified with classes of elements in $A^{p,q}$, where two α ’s are considered equivalent if they differ by something that is annihilated by $\bar{\partial}$. Because of the nilpotency of $\bar{\partial}$, $\alpha \cong \alpha + \bar{\partial}\eta$ (see also *Hodge decomposition*^(L) of forms).

Finally, if we are given a Hermitian scalar product $\langle \alpha, \beta \rangle$, we can define the adjoint operator $\bar{\partial}^\dagger$ with respect to this Hermitian product by

$$\langle \bar{\partial}^\dagger \alpha, \beta \rangle \stackrel{\text{def}}{=} \langle \alpha, \bar{\partial}\beta \rangle, \quad \forall \alpha, \beta. \quad (7.1.4)$$

The analogous goes for ∂^\dagger . Then

$$\Delta_{\bar{\partial}} \stackrel{\text{def}}{=} \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial} \quad (7.1.5)$$

is called the $\bar{\partial}$ -Laplacian and the ∂ -Laplacian is defined analogously; clearly, the two are related by complex conjugation and if that is a symmetry, $\Delta_{\bar{\partial}} = \Delta_\partial$. The forms annihilated by the Laplacian are called “harmonic”; there generally exist natural isomorphisms between the cohomology groups defined as above and the space of harmonic forms.

With these general facts, the definition of some particular cohomology theory boils down to selecting a suitable collection of ‘forms’, $A^{p,q}$, and finding an appropriate (conjugate pair of) nilpotent operator(s) for the above definition.



[†]Clearly, nothing stops us from defining the ∂ -cohomology either. The above notation is just aimed to remind of the Dolbeault cohomology, which refers to the antiholomorphic derivative, $\bar{\partial}$. Complex conjugation relates these two cohomology groups and provides an isomorphism if it is a symmetry.

7.1. Cohomology $1 \wedge 1$

In our standard situation, the ‘ (p, q) -forms’ are defined over some complex n -fold X and may be represented in some local coordinate chart by

$$\omega_{\mu_1 \dots \mu_p \bar{\nu}_1 \dots \bar{\nu}_q}(z, \bar{z}) dz^{\mu_1} \wedge \dots \wedge dz^{\mu_p} \wedge d\bar{z}^{\bar{\nu}_1} \wedge \dots \wedge d\bar{z}^{\bar{\nu}_q}. \quad (7.1.6)$$

Here $\omega_{\mu_1 \dots \mu_p \bar{\nu}_1 \dots \bar{\nu}_q}(z, \bar{z})$ is the *tensor coefficient* of the (p, q) -form the components of which are local functions of the coordinates z and \bar{z} , and ‘ \wedge ’ is the usual antisymmetric product of differentials.

The skew symmetry of ‘ \wedge ’ is not at all a frivolous novelty of differential geometry; it is as old as the notion of surface integration. For, if we perform the change of coordinates $(x, y) \mapsto (y, x)$, the integration measure changes as $dx dy \mapsto -dy dx$ owing to the Jacobian of the transformation.

The differential ∂ is, of course, then just the wedge product from left with $dz^\mu \partial_\mu$, where ∂_μ denotes partial derivation with respect to z^μ ; the analogous holds for $\bar{\partial}$.

In $N=2$ supersymmetric models in 2-dimensional spacetime, $A^{p,q}$ are field-function(al)s p -linear in the spinor field ψ , q -linear in the Hermitian conjugate, $\bar{\psi}$ and more or less of arbitrary dependence on the bosonic fields. The spinorial derivatives, D_\pm and \bar{D}_\pm are all nilpotent and satisfy

$$D_\pm \bar{D}_\pm + \bar{D}_\pm D_\pm = \partial_{\pm\pm}, \quad D_\pm \bar{D}_\mp + \bar{D}_\mp D_\pm = 0. \quad (7.1.7)$$

With the usual scalar product in the field space, $\bar{D}_\pm = D_\pm^\dagger$ and the *first order* light-cone partial derivatives ∂_{++} and ∂_{--} play the rôle of the Laplacians.

Another, structurally almost identical, example of cohomology which is found in field theory is the BRST-cohomology. There however, all usual BRST operators (differentials) anti-commute, so that the “Laplacians” are zero. There are also many rather differently defined *cohomology groups*^(L) in ‘standard mathematics’, but we leave that to the said references.

7.1.2 Cohomology ring

By definition, we have addition between forms, with respect to which they form an abelian group. Forms can be multiplied by scalars (elements of the *base field*, $\mathbf{k} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \dots$). Forms can also be “ \mathcal{V} -valued” if they map (in some suitable sense) into \mathcal{V} , where \mathcal{V} may be a vector *bundle*^(L) or a *sheaf*^(L) (or some other suitable structure); by default, if no \mathcal{V} is specified, the base field \mathbf{k} is understood.

In addition to this structure, as a rather important characteristic of all cohomology groups, they give rise to a *ring*^(L) structure. Namely, we can also multiply two forms and obtain a third one and this multiplication satisfies the axioms of the ring structure.

The simplest and in many ways the natural *choice* is, of course, the \wedge product which takes^t

$$\begin{aligned} \wedge : A^{p,q} \otimes A^{r,s} &\rightarrow A^{p+r, q+s}, \\ H^{p,q} \otimes H^{r,s} &\rightarrow H^{p+r, q+s}. \end{aligned} \quad (7.1.8)$$

In particular, note that the subset $\bigoplus_p H^{p,p}$ (and more generally, $\bigoplus_p A^{p,p}$) is closed under \wedge . In general, we can write

$$\omega_i \wedge \omega_j = C_{ij}^k \omega_k \quad (7.1.9)$$

where the labels i, j, k range over all (harmonic) forms; the C_{ij}^k are the *structure constants* and characterize the ring structure. One should however be aware that there could (and often do) exist other possible choices, which may involve more structure but appear more ‘natural’ from some particular point of view.



For complex 3-folds, the wedge product of two harmonic (1,1)-forms produces a harmonic (2,2)-form. On 3-folds, $H^{1,1}$ is dual to $H^{2,2}$. That is, the wedge product of harmonic (1,1)-forms and harmonic (2,2)-forms produces harmonic (3,3)-forms, the integral of which over the 3-fold \mathcal{M} depends only on the cohomology class of the forms involved. So, with any basis $\{e_A\}$ for $H^{1,1}$ and a basis $\{e^A\}$ for $H^{2,2}$ such that $\int_{\mathcal{M}} e_A \wedge e^B = \delta_A^B$, we can write

$$e_A \wedge e_B = \kappa_{ABC} e^C, \quad \text{i.e. } \kappa_{ABC} \stackrel{\text{def}}{=} \int_{\mathcal{M}} e_A \wedge e_B \wedge e_C, \quad (7.1.10)$$

which is precisely the expression of the tree-level Yukawa couplings in Eq. (0.5.5). Therefore, the world sheet instanton corrections to the Yukawa couplings in Eq. (0.6.3) provide a correction to the ring structure on $H^{1,1}$; this new structure could therefore be called the *quantum ring structure*, implying the name *classical ring structure* for the one defined by Eq. (0.5.5).

^tNote that a product of two harmonic forms need not be harmonic, i.e., a product of two non-trivial elements of a cohomology group need not be also non-trivial. For example, on Calabi-Yau 3-folds, a product of a harmonic (2,1)-form and a harmonic (1,1)-form is clearly a (3,2)-form, of which there are no nontrivial harmonic ones.

7.1. Cohomology $1 \wedge 1$

This quantum ring structure depends on the tangle of holomorphic images, L , of \mathbb{P}^1 in the 3-fold and is weighted exponentially by $-S[J; L]$, the 2-dimensional action evaluated on L with J the chosen Kähler class. Therefore, the “smallest” such curves dominate the correction while “large” ones may be neglected^s. In a situation where all such curves are large, the quantum ring structure attains the classical limit, as should be the case. Conversely, the quantum ring structure may be regarded as a (quantum) deformation of the classical one. Thus, we may define the “quantum cohomology product” by requiring that the structure constant in

$$e_A \diamond_{J,L} e_B = \kappa_{ABC} e^C, \quad (7.1.11)$$

be the quantum Yukawa coupling

$$\kappa_{ABC} \stackrel{\text{def}}{=} \hat{\kappa}_{ABC} + \sum_L e^{-S_{\mathcal{M}}[J; L]} \left(\int_L \omega_A \right) \left(\int_L \omega_B \right) \left(\int_L \omega_C \right). \quad (0.6.3')$$



For a Calabi-Yau n -fold \mathcal{M} , the degree- n cohomology can also be given a ring structure which depends on the (projectively) unique element $\Omega \in H^{n,0}(\mathcal{M})$. By means of Eq. (0.3.13), $H^{n-1,1}(\mathcal{M})$ may be identified with the holomorphic tangent bundle valued cohomology $H^1(\mathcal{M}, T_{\mathcal{M}})$. With a basis $\{\varphi_{\alpha}^{\mu}\}$ for $H^1(\mathcal{M}, T_{\mathcal{M}})$ and a basis $\{\varphi_{\mu}^{\alpha}\}$ for $H^2(\mathcal{M}, T_{\mathcal{M}}^*) = H^{1,2}(\mathcal{M})$ such that $\int_{\mathcal{M}} \varphi_{\alpha}^{\mu} \wedge \varphi_{\mu}^{\beta} = \delta_{\alpha}^{\beta}$, we can write

$$(\varphi_{\alpha}^{\mu} \wedge \varphi_{\beta}^{\nu}) \Omega_{\mu\nu\rho} = \kappa_{\alpha\beta\gamma} \varphi_{\rho}^{\gamma}, \quad \text{i.e. } \kappa_{\alpha\beta\gamma} = \int_{\mathcal{M}} \Omega \wedge (\varphi_{\alpha}^{\mu} \wedge \varphi_{\beta}^{\nu} \wedge \varphi_{\gamma}^{\rho}) \Omega_{\mu\nu\rho}, \quad (7.1.12)$$

which is precisely the expression of the Yukawa couplings in Eq. (0.5.7). These Yukawa couplings receive no quantum corrections (see § 0.6) and so the classical ring structure is the same as the quantum one. Through this roundabout procedure, we have defined a product “ \diamond_{Ω} ” :

$$\begin{array}{ccc} H^1(\mathcal{M}, T_{\mathcal{M}}) \otimes H^1(\mathcal{M}, T_{\mathcal{M}}) & \xrightarrow{\wedge, \Omega} & H^2(\mathcal{M}, T_{\mathcal{M}}^*) \\ \downarrow \Omega & \downarrow \Omega & \downarrow \text{Dolbeault} \\ H^{2,1}(\mathcal{M}) \otimes H^{2,1}(\mathcal{M}) & \xrightarrow{\circ_{\Omega}} & H^{1,2}(\mathcal{M}) \end{array} \quad (7.1.13)$$

which certainly is *not* the wedge product. In fact, the usual wedge product is mostly zero on the degree-3 cohomology.

^sThe “size” of such a curve, L , is measured by integrating over L the pull-back of the chosen Kähler class on the 3-fold; this is, essentially, $S_{\mathcal{M}}[J; L] = \int_{L \hookrightarrow \mathcal{M}} \|dz\|_J^2 = i \int_L J$.

7.2 Homology $1 \cap 1$

Here, we establish some notation and review some less well known facts about homology computations in Kähler manifolds.

Let \mathcal{X} be a Kähler manifold of complex dimension n and let \mathcal{Y} be a complex submanifold of complex dimension k then \mathcal{Y} represents a homology class, $[\mathcal{Y}] \in H_{2k}(\mathcal{X})$. Note that elements of $H_0(\mathcal{Y}, \mathbb{k})$ may be identified with corresponding elements of the field \mathbb{k} . In particular, for example, integers n are identified with elements of $H_0(\mathcal{X}, \mathbb{Z})$ and in the case $n \geq 0$ correspond to n points of \mathcal{X} .

If \mathcal{Y} and \mathcal{Y}' are complex submanifolds of \mathcal{X} with complex dimensions respectively k and k' , we say \mathcal{Y} and \mathcal{Y}' intersect *transversely* if $T_p(\mathcal{Y}) + T_p(\mathcal{Y}') = T_p(\mathcal{X})$ at every point p of their intersection. In other words, transversality implies that, at every point of intersection, the tangent space of \mathcal{X} is spanned by vectors tangent to \mathcal{Y} and those tangent to \mathcal{Y}' . Now, if $k + k' < n$, this requires \mathcal{Y} and \mathcal{Y}' to be disjoint. This may be understood intuitively in the sense that for $k + k' < n$ \mathcal{Y} and \mathcal{Y}' may be deformed and ‘moved away’* within \mathcal{X} until they have no point in common. If \mathcal{Y} and \mathcal{Y}' intersect transversely then $[\mathcal{Y} \cap \mathcal{Y}'] \in H_{2k+2k'-2n}(\mathcal{X})$. Note that the *codimension*⁽²⁾ (with respect to \mathcal{X}) is additive :

$$\begin{aligned}\text{codim}(\mathcal{Y} \cap \mathcal{Y}') &= (n - (k + k' - n)) \\ &= (n - k) + (n - k') = \text{codim}(\mathcal{Y}) + \text{codim}(\mathcal{Y}') .\end{aligned}\quad (7.2.1)$$

This intersection product can be generalized to a bilinear product of homology classes whether or not they can be represented by submanifolds so that we have

$$[\mathcal{Y}] \cap [\mathcal{Y}'] = [\mathcal{Y} \cap \mathcal{Y}'] .\quad (7.2.2)$$

Note also in this context that, besides being a submanifold of \mathcal{X} as are \mathcal{Y} and \mathcal{Y}' , $\mathcal{Y} \cap \mathcal{Y}'$ is in fact a submanifold of both \mathcal{Y} and \mathcal{Y}' and so determines elements of $H_{2k+2k'-2n}(\mathcal{Y})$ and $H_{2k+2k'-2n}(\mathcal{Y}')$. The homomorphism

$$H_m(\mathcal{X}) \xrightarrow{\cap [\mathcal{Y}]} H_{m+2k-2n}(\mathcal{Y})\quad (7.2.3)$$

There is no obstruction to doing this smoothly but it is not necessarily possible if one requires that the move also be complex analytic. Fortunately, the computation of $\&ABC$ does not require such holomorphicity; as elements of $H_(\mathcal{X})$, $[\mathcal{Y}]$ and $[\mathcal{Y}']$ may be equivalent even though they are distinct as complex submanifolds of \mathcal{X} .

7.2. Homology $1 \cap 1$

also generalizes to the full homology of \mathcal{X} regardless of representability as submanifolds. In this context, we have the identity

$$(A \cap_X [\mathcal{Y}]) \cap_Y (B \cap_X [\mathcal{Y}]) = (A \cap_X B) \cap_X [\mathcal{Y}] .\quad (7.2.4)$$

Similarly, if \mathcal{X} and \mathcal{X}' are two Kähler manifolds of dimension n and n' respectively, and \mathcal{Y} and \mathcal{Y}' are submanifolds respectively of \mathcal{X} and \mathcal{X}' of respective dimensions k and k' , then $[\mathcal{Y} \times \mathcal{Y}'] \in H_{2k+2k'}(\mathcal{X} \times \mathcal{X}')$. Again this external product generalizes to all homology classes regardless of representability so that we have

$$[\mathcal{Y}] \times [\mathcal{Y}'] = [\mathcal{Y} \times \mathcal{Y}'] .\quad (7.2.5)$$

The two ‘ \times ’-products are related by the identity

$$(A \times B) \cap (C \times D) = (-1)^{\dim(B) \cdot \dim(C)} (A \cap C) \times (B \cap D) .\quad (7.2.6)$$

Now let $\mathcal{X} \stackrel{\text{def}}{=} \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, where $n_r = \dim \mathcal{X}_r$ and $N = \dim \mathcal{X}$. Denote $x_r \in H^2(\mathcal{X}_r, \mathbb{Z})$ and let $\chi^r \in H_{(2n_r-2)}(\mathcal{X}_r, \mathbb{Z})$ correspond to x_r so that

$$x_r[\chi^s] \stackrel{\text{def}}{=} \int_{\chi^s} x_r = \delta_r^s .\quad (7.2.7)$$

Write

$$\hat{\chi}^r \stackrel{\text{def}}{=} [\mathcal{X}_1] \times \cdots \times [\mathcal{X}_{r-1}] \times \chi^r \times [\mathcal{X}_{r+1}] \times \cdots \times [\mathcal{X}_m] \in H_{(2N-2)}(\mathcal{X}, \mathbb{Z})\quad (7.2.8)$$

for the homology element in \mathcal{X} which projects to $\chi^r \in H_{(2n_r-2)}(\mathcal{X}_r, \mathbb{Z})$ under $\pi_r: \mathcal{X} \rightarrow \mathcal{X}_r$. If \mathcal{Y} is a hypersurface in \mathcal{X} defined as the zero set of a section of a line bundle with Chern class $\sum_{r=1}^m \pi_r^*(\chi^r)$, we then have that

$$H_{(2N-2)}(\mathcal{X}, \mathbb{Z}) \ni [\mathcal{Y}] = \sum_{r=1}^m \hat{\chi}^r .\quad (7.2.9)$$

If \mathcal{Y} is an intersection of K hypersurfaces, then $[\mathcal{Y}] \in H_{(2N-2K)}(\mathcal{X}, \mathbb{Z})$ is the intersection product of terms of the form of the right hand side of Eq. (7.2.9).

7.3 The Kähler Package

In Chapter 0, we have identified the massless fields in superstring compactification with certain harmonic forms (cohomology classes) on the compactifying space \mathcal{M} . While \mathcal{M} is smooth, we may use any of the standard cohomology theories in explicit computations, as standard theorems ensure that all these rather differently defined cohomologies yield isomorphic results. On singular space, however, all Hell seems to break loose; the different cohomology theories easily yield rather different results, so—which one does the physics of compactification choose?

Based on general properties of field theories and in particular, (2,2)-supersymmetric 2-dimensional ones, we expect several important characteristics of the cohomology which is to correspond to the massless fields. Consulting Chapter 0 for the dictionary between field theory and cohomology, we see that the prospective cohomology must feature :

1. *Hodge decomposition*^(L) : $H^r(\mathcal{M}, \mathbb{C}) \approx \bigoplus_{p+q=r} H^{p,q}(\mathcal{M})$, with the additional requirement of
2. *Complex conjugation*^(L) : $H^{p,q}(\mathcal{M}) = \overline{H^{q,p}(\mathcal{M})}$; this follows directly from *CPT-conjugation* in the field theory.
3. *Poincaré duality*^(L), that is, *Hodge star duality*^(L), so that, in particular $H^{1,1} \xrightarrow{*} H^{2,2}$.
4. *Lefschetz Hard Theorem*^(L), which amounts to a decomposition of forms into irreducible representations of the holonomy group.
5. *Künneth formula*^(L), which says that in a product of two spaces, harmonic forms are products of a harmonic form from one space and a harmonic form from the other space, with their degrees added.

These features are all parts of what is called the “Kähler package” [15]. We therefore inquire if there is a cohomology which maintains the above features even when the underlying space singularizes (and we keep in mind that at least hypersurface and finite quotient singularities should be included).

Note that the standard (co)homology theories experience difficulties even with the simplest of the singularities. For example, consider the complex torus with one of its “small” cycles pinched to a point. The 1-cycle represented by the small circles therefore becomes trivial as it manifestly can be contracted to a point.

The 1-cycle represented by the “big” circle cannot be contracted to a point and so remains non-trivial. The standard homology of the pinched torus is therefore

$$H_q(T_{\text{pinched}}, \mathbb{k}) = \mathbb{k}, \quad q = 0, 1, 2, \quad \mathbb{k} = \mathbb{Z}, \mathbb{Q}, \dots \quad (7.3.1)$$

If the $H_* \xrightarrow{*} H^*$ duality holds, H^1 is odd-dimensional and there can be no Hodge decomposition! If we simply excise the singular point, the “big” circle becomes an open interval and thus contractible, but now the “small” circle is non-contractible; in fact, the space is a cylinder and we are back to an odd-dimensional H^1 and so no Hodge decomposition.

7.3.1 L^2 -cohomology

The solution to the above requirements turns out to exist and moreover be defined by the very axioms of field theory. Namely, the 2-dimensional field theory which is used to describe the propagation of (super)strings through spacetime—including the compactification space—requires *all physical states to be square-integrable*.

The cohomology theory which takes incorporates square-integrability is called simply L^2 -cohomology and denoted $H_{(2)}^{p,q}$; it has been developed mainly by Cheeger [15]. The standard definition is changed in that candidate (p, q) -forms are now required to be square-integrable with respect to a chosen norm. This in general means that some of the usual cohomology classes of (p, q) -forms do not exist in $H_{(2)}^{p,q}$. On the other hand, in the equivalence relation $\omega \cong \omega + \bar{\partial}\eta$, both ω and η must be square-integrable. This restricts the equivalence relations so that, if $\omega - \omega' = \bar{\partial}\eta$ where η is not square-integrable, ω and ω' are not equivalent in $H_{(2)}^{p,q}$. This makes it possible for $H_{(2)}^{*,*}$ to be bigger than $H^{*,*}$. Clearly, both effects are typically present and $H_{(2)}^{p,q}$ can be rather different from $H^{p,q}$.

Most importantly however, $H_{(2)}^{p,q}(\mathcal{M}) = H^{p,q}(\mathcal{M})$ on any smooth manifold \mathcal{M} , and on a rather large class of singular spaces, $H_{(2)}^{p,q}$ remains “well-behaved” and still possesses the above listed properties.



By definition, $H_{(2)}^{p,q}$ does depend on the norm chosen on the space. This dependence is however rather weak. Given two norms $N[\omega]$ and $N'[\omega]$ on the space of forms, the L^2 -cohomology computed using one will equal the cohomology computed with the other norm precisely if there exists a constant $K \in \mathbb{R}$ such that

$$\frac{1}{K} N[\omega] < N'[\omega] < K N[\omega]. \quad (7.3.2)$$

This defines the universality class of norms for which $H_{(2)}^{p,q}$ is constant and therefore provides a computational trick : Typically, the desired norm (metric) is too complicated or even unknown. It may be possible, however, to find another norm in the same universality class such that the required computation becomes easier.

7.3.2 Intersection (co)homology

Just as in all classical cohomology theories, the explicit computation of $H_{(2)}^{p,q}$ may be a very difficult task. In the classical case, rescue often comes in the form of some homology theory, in that the desired cohomology group can be related to some homology group (by a duality relation or even an isomorphism), which in turn is easy to compute (see Chapter 8).

There exists a homology theory, called *middle perversity* intersection homology theory* [26,34], denoted $IH_*(X)$, which has been proven to satisfy the *De Rham duality^(L)* pairing with $H_{(2)}^{p,q}(X)$ for a class of singular spaces $X \hookrightarrow \mathbb{P}^N$ [15], where $H_{(2)}^{p,q}(X)$ refers to the restriction to X of the Fubini-Study metric on the \mathbb{P}^N . In particular, this class includes spaces with isolated conical singularities—precisely those which have occurred in Chapters 4–6.

Let us just mention though that IH_* may be defined for a very broad class of so-called *pseudomanifolds*.

Definition. A pseudomanifold X of dimension n is a space which admits a stratification

$$X = X_n \supset X_{n-2} \supset \cdots \supset X_1 \supset X_0 , \quad (7.3.3)$$

such that $(X_n - X_{n-2})$ is an oriented dense n -dimensional manifold and $(X_k - X_{k-1})$ for $k \leq n-2$ is a k -dimensional manifold along which the normal structure of X is trivial.

Akin to n -folds, we will also say “pseudo- n -fold”. The last requirement roughly means that for each point $p \in (X_k - X_{k-1})$, there exists a distinguished neighborhood $N_{p \in X}$ which looks like an open cone the base of which is again nicely stratified. In other words, an n -dimensional pseudomanifold (recall that all dimensions are complex) looks like a collection of manifolds of all dimensions not bigger than n and not $(n-1)$, which are glued together in an orderly fashion.

*So, ... not only physicists indulge in silly nomenclature; the ‘middle perversity’ adjective is usually omitted, though.

7.3. The Kähler Package

Because of the ‘emptiness’ requirement in $n-1$ dimensions, pseudomanifolds are really different from ordinary manifolds for $n > 2$.

There is no *a priori* reason why such object would be banned from string theory, and it would seem worthwhile trying to extend the current understanding of what a “Calabi-Yau 3-fold” means, so as to include pseudomanifolds also. We leave this to the adventurous reader and turn now to some results which can be of help in explicit computations.

7.3.3 Some helpful facts

CW complexes. In many cases of interest, the desired homology groups of a space can be determined by iterating a simple procedure, called *attaching cells* [9]. By an n -cell, one simply means a real n -dimensional ball together with its boundary. This then is glued to another space (not necessarily of the same dimension!) in a way prescribed by a “function” f . With $C^{(n)}$ denoting the n -cell, we write $Y = X \cup_f C^{(n)}$, meaning that the space Y is obtained by attaching $C^{(n)}$ to X .

The pet example for this process is the relation

$$S^n = \{\text{point}\} \cup C^{(n)} . \quad (7.3.4)$$

That is, we obtain the n -sphere by attaching a bounded n -ball to a point (see Figure 7.1). Usually, one thinks of making an n -sphere from \mathbb{R}^n by attaching the point at infinity and ‘attaching cells’ seems sort-of the same but the other way around : we first add the (S^{n-1} -like) boundary to \mathbb{R}^n , producing $C^{(n)}$ and then glue all of the boundary together, to the point “at infinity”. (Since all of the boundary is being mapped to a single point, the specification f is not really a function.)

The following is an important property of this procedure.

Theorem 7.1 Attaching an n -cell to a space X does not alter the homotopy in dimension strictly less than $n-1$, but may kill elements in $\pi_{n-1}(X)$; more precisely, the inclusion $X \hookrightarrow X \cup C^{(n)}$ induces isomorphisms

$$\pi_q(X) \xrightarrow{\sim} \pi_q(X \cup C^{(n)}) , \quad q < n-1 , \quad (7.3.5)$$

and a surjection

$$\pi_{n-1}(X) \longrightarrow \pi_{n-1}(X \cup C^{(n)}) . \quad (7.3.6)$$

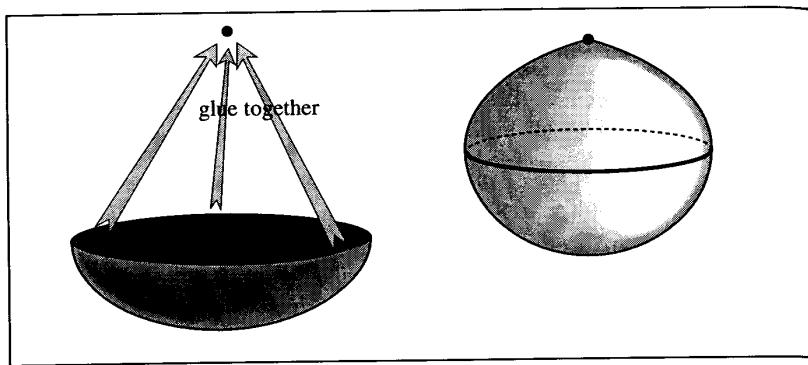


Figure 7.1: Building a sphere by attaching a cell.

Compare this with the *Lefschetz Hyperplane Theorem*^(L) (as proven by Bott).

Definition. A CW complex is a space Y built up from a collection of points by the successive attaching of cells, where the cells are attached in the order of increasing dimensions.

The cells of dimension at most n in a CW complex Y together comprise the n -skeleton of Y .

Clearly, every triangulizable space and also every manifold is a CW complex. The well known example is

$$\mathbb{P}^n = \{\text{point}\} \cup \mathbb{C}^{(2)} \cup \mathbb{C}^{(4)} \cup \dots \cup \mathbb{C}^{(2n)}. \quad (7.3.7)$$

Or, the other way around : \mathbb{P}^n may be thought of as defined recursively

Definition. (recursive) \mathbb{P}^n is a \mathbb{C}^n with a \mathbb{P}^{n-1} glued at the “infinity” and where \mathbb{P}^0 is defined to be a point.

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Relating the homotopy and homology groups, we have the

Theorem 7.2 (Hurewicz) If X is a simply connected path-connected CW complex, the first non-zero homotopy and homology groups are equal. That is, if $\pi_k = 0$ for $k < n$, then $\pi_n \approx \oplus_{p+q=n} H^{p,q}$.

Also worth knowing is that π_k are all abelian groups for $k > 1$ ($H^{p,q}$ are all abelian). For further relations, we refer to Ref. [9].

Computing L^2 and intersection (co)homology. For all the cases considered in this volume, the following few results will suffice to obtain the intersection homology and the L^2 -cohomology from the usual (classical) (co)homology.

Theorem 7.3 Let X be a n -fold with a single isolated singularity, χ . Then

$$IH_k(X) = \begin{cases} H_k(X) & k > n, \\ \text{Im}[H_n(X - \chi) \rightarrow H_n(X)] & k = n, \\ H_k(X - \chi) & k < n. \end{cases} \quad (7.3.8)$$

The map in the middle dimension simply means the following : (1) discard all would-be n -cycles from $H_n(X)$ which contract to χ and (2) discard all n -cycles which break open upon excising χ . Note that $IH_k(X)$ need not be calculated for $k < n$ at all—Poincaré duality relates it to $IH_{2n-k}(X)$.

As a simple example, note that on the torus with one of the “small” circles pinched to a point, the “small” circles are no longer 1-cycles since they contract to the (singular) point. As part 2, excise the singularity, thereby turning the “big” circles into open intervals, which are contractible and so have to be discarded. We remain with

$$IH_q(T_\bowtie, \mathbb{k}) = \begin{cases} \mathbb{k} & q = 0, 2, \\ 0 & q = 1, \end{cases} \quad \mathbb{k} = \mathbb{Z}, \mathbb{Q}, \dots \quad (7.3.9)$$

In other words, $IH_*(T_\bowtie)$ equals $H_*(S^2)$. And more generally, for a Riemann surface of genus g and with k handles pinched at a “small” circle, the intersection homology equals that of a Riemann surface of genus $g-k$. Now, in (super)string theory, when the world-sheet of the (super)string pinches in this fashion, the dynamics effectively replaces it with a lower genus sheet (we refer to Ref. [22] for details of this process).

Of course, in the preceding example, the intersection homology and the L^2 -cohomology do satisfy the formal duality requirements and we safely conclude that (super)strings speak the L^2 -cohomology, at least as far as the dynamics of the world-sheet goes.

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The next result is somewhat trivial for curves and surfaces and becomes really non-trivial when the dimension is at least 3.

Definition. Given a singular complex n -fold X and a non-singular complex n -fold \mathcal{Y} , such that $(\mathcal{Y} - E) \xrightarrow{f} (X - \chi)$ is an isomorphism and $E \xrightarrow{f} \chi$ is surjective^(L), f is called a small map if

$$\text{codim}_{\mathbb{C}}(x \in X) > \dim_{\mathbb{C}}(f^{-1}(x) = E). \quad (7.3.10)$$

This means that in curves and surfaces, a small map is finite, that is, each point $x \in X$ must be replaced by finitely many copies of x in \mathcal{Y} . In 3- and 4-folds, a small map is finite, except perhaps for a number of isolated points which may be replaced at most by curves. In 5- and 6-folds, a small map is finite, except perhaps for a number of isolated curves C_i , each of which may be replaced at most by a surface E_i ; each E_i is obtained by replacing each point on C_i by a curve so that E_i is a fibered space over C_i .

Actually, both X and \mathcal{Y} may be allowed to be singular, in that \mathcal{Y} is a partial resolution of X . This also provides a general definition : *small resolution* is a small map $f : \mathcal{Y} \rightarrow X$ which provides at least a partial resolution of singularities in X .

Since divisors and, in particular, the canonical divisor are all in (complex) codimension-1, the foregoing yields

Theorem 7.4 (Small Map) If $f : \mathcal{Y} \rightarrow X$ provides a small resolution of the singular 3-fold X , then $IH_*(X) = H_*(\mathcal{Y})$ and $K_X = K_{\mathcal{Y}}$.

7.4 The Poincaré Polynomial

As a collective notation for the cohomology of a manifold \mathcal{M} , one has*

Definition. The Poincaré series of a manifold \mathcal{M} is defined to be

$$P_t(\mathcal{M}) \stackrel{\text{def}}{=} \sum_r t^r \dim H^r(\mathcal{M}) K. \quad (7.4.1)$$

*For a fuller account on Poincaré series, we refer the reader to Ref. [9] and Ref. [3,4]. Here we only list some possibly useful results.

7.4. The Poincaré Polynomial

Of course, all finite dimensional manifolds have $H^r(\mathcal{M}) = 0$ for $r < 0$ and for $r > \dim \mathcal{M}$, so that the Poincaré series truncates to a finite polynomial.

As a few important examples, we list

$$P_t(\mathbb{P}^n) = \frac{(1 - t^{2n})}{(1 - t^2)}, \quad (7.4.2)$$

$$P_t(G_{(k,n)}) = \frac{\prod_{i=n-k+1}^n (1 - t^{2i})}{\prod_{i=1}^k (1 - t^{2k})}, \quad (7.4.3)$$

$$P_t(\mathbb{F}_{[n_1, \dots, n_f]}) = \frac{\prod_{i=1}^n (1 - t^{2i})}{\prod_{r=1}^f \prod_{j=1}^{n_r} (1 - t^{2j})}, \quad (7.4.4)$$

which are the Poincaré series of the complex projective space, the Grassmannian and the generalized flag space, respectively. Recall that

$$\mathbb{F}_{[n_1, \dots, n_f]} \stackrel{\text{def}}{=} \frac{U(N)}{U(n_1) \times \dots \times U(n_f)}, \quad N = \sum_{r=1}^f n_r \quad (7.4.5)$$

is the generalized (unitary) flag space, so that $\mathbb{P}^n = \mathbb{F}_{[1,n]}$ and $G_{(k,n)} = \mathbb{F}_{[k,n-k]}$.

Of course, one can similarly define the \mathcal{V} -valued Poincaré series, where the cohomology $H^r(\mathcal{M})$ is replaced in the above definition with the \mathcal{V} -valued cohomology.

■

In a more general context, one can define the Poincaré series of a graded algebra $\mathfrak{A} = \bigoplus_i \mathfrak{A}_i$, as

$$P_t(\mathfrak{A}) \stackrel{\text{def}}{=} \sum_i t^i \dim(\mathfrak{A}_i). \quad (7.4.6)$$

Useful examples of this kind are the Poincaré series of polynomial rings. If x is a single variable of degree d , then

$$P_t(\mathbb{C}[x]) = 1 + t^d + t^{2d} + \dots = \frac{1}{1 - t^d}, \quad (7.4.7)$$

where $\mathbb{C}[x]$ denotes the polynomial ring generated by x (of course, simple monomials), with complex coefficients. Owing to the well known properties of polynomials, we have that for several independent variables x_i , of degree d_i ,

$$\prod_i \mathbb{C}[x_i] = \mathbb{C}[\{x_i\}], \implies P_t(\mathbb{C}[\{x_i\}]) = \prod_i \frac{1}{1 - t^{d_i}}. \quad (7.4.8)$$

If a single relation, I , of degree q is imposed on the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$, we have

$$P_t(\mathbb{C}[x_1, \dots, x_n]/I) = P_t(\mathbb{C}[x_1, \dots, x_n]) \cdot (1 - t^q) = \frac{(1 - t^q)}{\prod_{i=1}^n (1 - t^{d_i})}. \quad (7.4.9)$$

It would be desirable to extend this result to the case of several relations. There, however, we need the following

Definition. Let \mathcal{R} be a ring. A non-zero element $z \in \mathcal{R}$ is called a zero-divisor if there exists another non-zero element $a \in \mathcal{R}$ such that $az = 0$ in \mathcal{R} .

A sequence of elements r_1, \dots, r_k is regular if

1. r_1 is not a zero-divisor in \mathcal{R} and
2. for each $i > 1$, the image of r_i in $\{\mathcal{R}/(r_1, \dots, r_{i-1})\}$ is not a zero-divisor.

Roughly, the elements of a regular sequence are non-zero and are supposed not to annihilate their successors.

Then we have the following results [9]

Lemma 7.1 If r_1, \dots, r_k, a and r_1, \dots, r_k, b are both regular sequences, then so is r_1, \dots, r_k, ab .

And finally,

Theorem 7.5 Let \mathcal{R} be a graded ring over a field \mathbb{k} and r_1, \dots, r_k a regular sequence of elements $r_i \in \mathcal{R}$, with degrees q_1, \dots, q_k . Then

1. any permutation of r_1, \dots, r_k is also regular;
2. $P_t(\{\mathcal{R}/(r_1, \dots, r_k)\}) = P_t(\mathcal{R}) \cdot \prod_{i=1}^k (1 - t^{q_i})$.

These results will have an obvious application in Chapter C, where we will see that, for all complete intersections \mathcal{M} in products of flag spaces, the cohomology on \mathcal{M} can be given the structure of a quotient ring $\mathcal{R} = \{\mathbb{C}[x_1, \dots, x_k] / \mathfrak{I}\}$, where $\mathbb{C}[x_1, \dots, x_k]$ is the polynomial ring on the embedding space, generated by homogeneous coordinates and \mathfrak{I} is an ideal, generated by gradients of the defining constraints, modulo perhaps some relations among those.

Chapter 8

Topological Triple Couplings

In the previous chapters we have constructed Calabi-Yau 3-folds in several different ways, including complete intersections of hypersurfaces in products of complex projective spaces and hypersurfaces in a products of two compact complex surfaces. Three such constructions,

$$\begin{aligned} \mathcal{M}_{-18} &\in \left[\begin{array}{c|ccc} 3 & 3 & 0 & 1 \\ 3 & 0 & 3 & 1 \end{array} \right]_{-18}^{14}, \\ \mathcal{M}_{-54} &\in \left[\begin{array}{c|cc} 3 & 3 & 1 \\ 2 & 0 & 3 \end{array} \right]_{-54}^8, \\ \mathcal{M}_{-162} &\in \left[\begin{array}{c|c} 2 & 3 \\ 2 & 3 \end{array} \right]_{-162}^2, \end{aligned} \quad (8.0.1)$$

may in fact provide phenomenologically acceptable models (see § 0.8 for the requirements), and have been analyzed in the literature. It is interesting to note that relatively few multiply connected Calabi-Yau 3-folds of $|\chi_E| = 6$ are known and only members of these configurations have so far been found physically interesting.

The Euler characteristic χ_E of such manifolds is easily computed from Eq. (2.1.29) and is given, respectively, in the subscripts above. The Hodge numbers $b_{p,q}$ are then obtained using the Lefschetz hyperplane theorem to yield $b_{1,1}$ as indicated by the superscript above. The three 3-folds (8.0.1) are hypersurfaces in products of two surfaces : $\Sigma \times \Sigma$, $\Sigma \times \mathbb{P}^2$ and $\mathbb{P}^2 \times \mathbb{P}^2$ respectively, where $\Sigma \in [3|3]$ is a generic cubic surface in \mathbb{P}^3 . This structure turns out to allow a rather detailed description of the (1,1)-cohomology and we now turn to it.

8.1 The Structure of $H^{1,1}(\mathcal{M})$

We will now be interested in evaluating the triple product integral (0.5.5) :

$$\hat{\kappa}_{ABC} \stackrel{\text{def}}{=} \int_{\mathcal{M}} e_A \wedge e_B \wedge e_C, \quad e_A \in H^{1,1}(\mathcal{M}).$$

For all Calabi-Yau n -folds ($n \neq 2$) $b_{2,0} = b_{0,2} = 0$ so that $H^{1,1}(\mathcal{M}) \approx H^2(\mathcal{M}, \mathbb{C})$. By the same argument, we can choose an integral basis and work with $H^2(\mathcal{M}, \mathbb{Z})$; that is, any integral harmonic 2-form must also be a $(1,1)$ -form.

Remark.

On K3, the Calabi-Yau 2-fold, $H^{2,0} \approx H^{0,2} \approx \mathbb{C}$, so $H^{1,1}$ is a codimension-2 subset of $H^2(\mathbb{C})$ and so has lots of room to miss $H^2(\mathbb{Z}) \subset H^2(\mathbb{C})$ completely (just think of a real line in \mathbb{R}^2 with irrational slope, missing thus all non-zero integral points). In fact, if we do not require projectivity (the possibility of embedding in some \mathbb{P}^n) and define Calabi-Yau 2-folds as simply connected compact complex 2-space with trivial canonical bundle, a generic K3 is not projective and $H^{1,1} \cap H^2(\mathbb{Z}) = 0$.

Now, $H^2(\mathcal{M}, \mathbb{Z})$ is naturally isomorphic to $H_4(\mathcal{M}, \mathbb{Z})$ since both are dual to $H^4(\mathcal{M}, \mathbb{Z})$ (see *duality*^(L)). Clearly, this natural identification of $H^2(\mathcal{M})$ with $H_4(\mathcal{M})$ is equally valid over \mathbb{Q}, \mathbb{R} or \mathbb{C} as well. Under this identification, the triple products (0.5.5) correspond to triple intersections,

$$\hat{\kappa}_{ABC} = \hat{\kappa}(A, B, C) = [A] \cap [B] \cap [C], \quad \text{where } e_A \cdot [B] \stackrel{\text{def}}{=} \int_B e_A = \delta_A^B, \quad (8.1.1)$$

and which are often easily computed and yield intersection numbers. In particular, when \mathcal{M} is an n -fold (Calabi-Yau or not), given as a complete intersection in a product of manifolds whose homology and intersection structure we know, we can use the ideas presented in § 7.2 to compute all n -fold intersection products on \mathcal{M} of classes of the form $\hat{x}^r \cap [\mathcal{M}]$.

8.1.1 Wall's classification theorem

There is another numerical invariant of $H^{1,1}(\mathcal{M})$, which has so far not surfaced in the physical application but plays an important rôle in classification. It is

$$p_1[\omega] \stackrel{\text{def}}{=} \int_{\mathcal{M}} p_1 \wedge \omega = -2 \int_{\mathcal{M}} c_2 \wedge \omega, \quad \omega \in H^{1,1}(\mathcal{M}). \quad (8.1.2)$$

We have used that $c_1(\mathcal{M}) = 0$, so that the first Pontrjagin class $p_1 = c_1^2 - 2c_2 \in H^4(\mathcal{M})$ simplifies. Computed for elements of $H^2(\mathcal{M}, \mathbb{Z})$, the cubic form $\hat{\kappa}(, ,)$

8.1. The Structure of $H^{1,1}(\mathcal{M})$

and linear form $p_1[]$ are together with the Hodge numbers $b_{1,1}$ and $b_{2,1}$ sufficient for classification of Calabi-Yau manifolds up to homotopy type [50].

More precisely, we have

Theorem 8.1 (Wall) *The homotopy types of complex compact 3-folds are classified by the following numerical characteristics : $b_{p,q}$, $\hat{\kappa}_{ABC}$ and $p_1[\omega(A)]$.*

Clearly, two cubic forms $\hat{\kappa}(, ,)$ and $\hat{\kappa}'(, ,)$ are equivalent if there is a change of bases under which $\hat{\kappa}(, ,) \rightarrow \hat{\kappa}'(, ,)$; simultaneously, we must also have $p_1[] \rightarrow p_1'[]$.

Remark.

Generally, the linear form $p_1[]$ on H^2 of some (not necessarily Calabi-Yau) 3-fold \mathcal{M} is given in terms of the cubic form $\hat{\kappa}(, ,)$ and $c_1(\mathcal{M})$. To see this, restrict to a surface $S \subset \mathcal{M}$; write $c(\mathcal{M}|_S) = 1 + c_1 + c_2$ and $c(S) = 1 + s_1 + s_2$. Then $c[\mathcal{N}_{\mathcal{M}/S}] = 1 + [c_1 - s_1]$ and so

$$c(S) = \frac{c(\mathcal{M}|_S)}{c[\mathcal{N}_{\mathcal{M}/S}]} \iff c_2 = s_2 - s_1^2 + c_1 \wedge s_1. \quad (8.1.3)$$

Integrating over S and using Eq. (3.1.3), it is not hard to verify that

$$p_1[S] = 4\hat{\kappa}(S, S, S) - 24\chi^h(S) + \hat{\kappa}(K_{\mathcal{M}}, K_{\mathcal{M}}, S) - 2\hat{\kappa}(K_{\mathcal{M}}, S, S), \quad (8.1.4)$$

where $K_{\mathcal{M}}$ is the *canonical divisor*^(L) in \mathcal{M} . For a Calabi-Yau 3-fold \mathcal{M} , this simplifies since $c_1(\mathcal{M}) = 0$, and the last two terms drop out.

In order for two Calabi-Yau manifolds \mathcal{M} and \mathcal{M}' to have the same homotopy type it is necessary that they have the same Hodge numbers and that there is an isomorphism $f: H_4(\mathcal{M}, \mathbb{Z}) \rightarrow H_4(\mathcal{M}', \mathbb{Z})$ such that :

$$\hat{\kappa}(x, y, z) = \hat{\kappa}(f(x), f(y), f(z)), \quad p_1[x] = p_1[f(x)]. \quad (8.1.5)$$

This criterion can be difficult to use in practice both because it is often hard to find an explicit integral basis for the homology, and also because it is generally far from obvious whether or not two cubic forms are equivalent up to a change of basis. For this reason it is useful to define some readily computable numerical invariants of the pair, $(\hat{\kappa}(, ,), p_1[])$ which will serve to distinguish homotopy types.



The simplest divisibility invariants are

$$\begin{aligned} d_1 &= \gcd\{\hat{\kappa}(x, y, z)\}, \\ d_2 &= \gcd\{\hat{\kappa}(x, y, y)\}, \\ d_3 &= \gcd\{\hat{\kappa}(x, x, x)\}, \end{aligned} \quad (8.1.6)$$

where x, y and z range all over $H_4(\mathcal{M}, \mathbb{Z})$. It suffices to compute respectively

$$\begin{aligned} d_1 &= \gcd\{\hat{\kappa}(a, b, c)\}, \\ d_2 &= \gcd\{\hat{\kappa}(a, a, c), 2\hat{\kappa}(a, b, c)\}, \\ d_3 &= \gcd\{\hat{\kappa}(a, a, a), 3(\hat{\kappa}(a, a, c) \pm \hat{\kappa}(a, c, c)), 6\hat{\kappa}(a, b, c)\}, \end{aligned} \quad (8.1.7)$$

where a, b and c range over any integral basis of $H_4(\mathcal{M})$. We note the evident fact that d_1 divides d_2 , which in turn divides d_3 .

Similarly, we have $d_p = \gcd\{p_1[x]\} = \gcd\{p_1[a]\}$.

Next, we consider the symmetric quadrilinear form obtained by symmetrizing

$$\langle x, y, z, w \rangle \stackrel{\text{def}}{=} (\hat{\kappa}(x, y, z)p_1[w] + \text{cyclic permutations}).$$

Clearly there is an analogous hierarchy of divisibility invariants for $\langle \cdot, \cdot, \cdot, \cdot \rangle$:

$$\begin{aligned} d_4 &= \gcd\{\langle a, b, c, d \rangle\}, \\ d_5 &= \gcd\{\langle a, a, c, d \rangle, 2\langle a, b, c, d \rangle\}, \\ d_6 &= \gcd\{\langle a, a, a, d \rangle, 3(\langle a, a, c, d \rangle \pm \langle a, c, c, d \rangle), 6\langle a, b, c, d \rangle\}, \\ d_7 &= \gcd\{\langle a, a, a, a \rangle, 2(2\langle a, a, a, d \rangle \pm 3\langle a, a, d, d \rangle \pm 2\langle a, c, c, d \rangle), \\ &\quad 12(\langle a, a, c, d \rangle \pm \langle a, c, c, d \rangle \pm \langle a, c, d, d \rangle), 24\langle a, b, c, d \rangle\}. \end{aligned} \quad (8.1.8)$$

Additionally, we can interpret $\langle \cdot, \cdot, \cdot, \cdot \rangle$ as a quadratic form $\mathcal{Q}(\cdot, \cdot)$ on the symmetric square of $H_4(\mathcal{M}, \mathbb{Z})$. The rank and the signature of $\mathcal{Q}(\cdot, \cdot)$ are invariants over the real homology of \mathcal{M} . Further, let $d_{\mathcal{Q}}$ denote the determinant of \mathcal{Q} and note that, under a change of basis $x_i \rightarrow \Lambda_i^k x_j$, for $x_i \in H_4(\mathcal{M}, \mathbb{k})$ the symmetric ‘square’ transforms as :

$$x_i x_j \rightarrow \Lambda_i^k \Lambda_j^\ell x_k x_\ell, \quad \Lambda \in \mathrm{GL}(b_{1,1}, \mathbb{k}), \quad (8.1.9)$$

whence

$$d_{\mathcal{Q}} \rightarrow (\det \Lambda)^{2(b_{1,1}+1)} d_{\mathcal{Q}}. \quad (8.1.10)$$

If we have an integral basis for $H_4(\mathcal{M})$, Λ must be an integral linear transformation and so of unit determinant. Then, $d_{\mathcal{Q}}$ is strictly invariant. However, even if

8.2. Favorable Configurations, Again

we compute \mathcal{Q} over $H_4(\mathcal{M}, \mathbb{Q})$, it retains certain invariance. $d_{\mathcal{Q}}$ is invariant up to multiplicative $2(b_{1,1}+1)^{\text{th}}$ powers of rational numbers and we may reduce $d_{\mathcal{Q}}$ by such powers; write $d_{\mathcal{Q}} \cong_{\mathbb{Q}} d_{\mathcal{Q}}^{(\text{reduced})}$. Just like the formula for the Euler characteristic (2.1.29), the divisibility properties are well suited for machine computation, although of course d_1, d_2 and d_3 are often easily calculated by hand.

8.2 Favorable Configurations, Again

Consider again the set of complete intersection varieties in products of complex projective spaces as defined in Chapter 2. In the notation (2.1.5) :

$$\mathcal{M} \in [\mathcal{X} \parallel \mathcal{E}] \equiv \left[\begin{array}{c|cccc} n_1 & q_1^1 & \cdots & q_K^1 \\ \vdots & \ddots & \ddots & \vdots \\ n_m & q_1^m & \cdots & q_K^m \end{array} \right], \quad (8.2.1)$$

where \mathcal{E} denotes the direct sum of K line bundles \mathcal{E}_a over \mathcal{X} , the sections of which, ξ_a , define $\mathcal{M}: \xi_a = 0$.

We recall that a generic $\mathcal{M} \in [\mathcal{X} \parallel \mathcal{E}]$, corresponding to a generic choice of polynomials, is a Kähler manifold of dimension $\dim \mathcal{M} = \dim \mathcal{X} - K$. The Euler characteristic, χ_E , of such a smooth n -fold is computed as the coefficient in the formal expansion in J_r , the generators of $H^{1,1}(\mathbb{P}_r^{n_r}, \mathbb{Z})$:

$$\left[\frac{\prod_{r=1}^m (1 + J_r)^{n_r+1}}{\prod_{a=1}^K (1 + \sum_{r=1}^m q_a^r J_r)} \right] \cdot \prod_{a=1}^K \left(\sum_{r=1}^m q_a^r J_r \right) = \dots + \chi_E(\mathcal{M}) \cdot \prod_{r=1}^m (J_r)^{n_r}. \quad (8.2.2)$$

The quantity in the square brackets is $c(\mathcal{M})$ whence the series is truncated to an order- n ($= \dim \mathcal{M}$) polynomial in J_r ; also, we have that $(J_r)^{n_r+1} \equiv 0$. In the extreme case $\dim \mathcal{M} = 0$, i.e., when \mathcal{M} is discrete, $c(\mathcal{M}) = 1$ and $\chi_E(\mathcal{M})$ reduces to the coefficient of $\prod_{r=1}^m (J_r)^{n_r}$ in $\prod_{a=1}^K (\sum_{r=1}^m q_a^r J_r)$.

The first Pontrjagin class $p_1 = c_1^2 - 2c_2$ is computed straightforwardly and for Calabi-Yau 3-folds ($c_1 = 0$) :

$$p_1 = -2c_2 = \sum_{r,s=1}^m [(n_r + 1)\delta^{r,s} - \sum_{a=1}^K q_a^r q_a^s] J_r J_s. \quad (8.2.3)$$

8.2.1 Computing integrals by counting

In the sector of $H^{1,1}(\mathcal{M})$ generated by the Kähler forms of the $\mathbb{P}_r^{n_r}$ factors in \mathcal{X} , $\hat{\kappa}(, ,)$ and $p_1[]$ may be computed as follows :

1. The Kähler form $J_r \in H^{1,1}(\mathbb{P}_r^{n_r})$ may be represented by a generic hyperplane in $X_r \subset \mathbb{P}_r^{n_r}$ (corresponds to a linear constraint on $\mathbb{P}_r^{n_r}$ and independent of the other factors in \mathcal{X}); we denote such a constraint by X_r as well.
2. The triple product (Yukawa coupling) $\hat{\kappa}(X_r, X_s, X_t)$ is evaluated as

$$\hat{\kappa}(X_r, X_s, X_t) = \chi_E[X \parallel \mathcal{E} X_r X_s X_t], \quad (8.2.4)$$

enforcing the three constraints X_r , X_s and X_t in addition to the defining constraints of \mathcal{M} . This makes $[X \parallel \mathcal{E} X_r X_s X_t]$ into a collection of points and $\hat{\kappa}(X_r, X_s, X_t)$ is their number. (For an n -fold, one considers the n -fold intersection.)

3. The $p_1[X_r]$ evaluation is computed as in Eq. (8.2.2), replacing the term in the square brackets with $p_1[X_r]$.

The foregoing is sufficient to compute all $(1,1)^3$ couplings and p_1 -evaluations whenever $H^2(\mathcal{M})$ is given completely by a restriction from $H^2(\mathcal{X})$. As we show in Chapter 9, the kernel of this restriction is

$$\ker [H^{1,1}(\mathcal{X}) \rightarrow H^{1,1}(\mathcal{M})] = H^1(\mathcal{M}, \mathcal{E}^*), \quad \mathcal{E}^* = \bigoplus_{a=1}^K \mathcal{E}_a^*, \quad (8.2.5)$$

and if non-zero, indicates that not all generators J_r are independent when restricted to $H^{1,1}(\mathcal{M})$. The cohomology group on the left hand side can be computed with the techniques discussed below. However, if $H^1(\mathcal{M}, \mathcal{E}^*)$ is non-zero, the embedding of \mathcal{M} may be modified using Lemma 2.2, in such a way that $H^1(\mathcal{M}, \mathcal{E}^*)$ becomes zero. Thus we incur no loss of generality by considering embeddings for which the restriction $H^{1,1}(\mathcal{X}) \rightarrow H^{1,1}(\mathcal{M})$ is 1-1, that is, all the generators J_r restrict to independent elements of $H^{1,1}(\mathcal{M})$.

On the other side, there might be $(1,1)$ -forms on \mathcal{M} which do not stem from J_r ; the cycles corresponding to such forms are usually called “vanishing cycles”. However, if $b_2(\mathcal{X}) = b_2(\mathcal{M})$, the restriction $H^{1,1}(\mathcal{X}) \rightarrow H^{1,1}(\mathcal{M})$ is an isomorphism over the complex numbers and hence over the reals and rationals as well, so, we can choose $\{J_r\}_{r=1,\dots,m}$ as a basis for $H^{1,1}(\mathcal{M}, \mathbb{Q})$.

8.3. Some Examples

In a number of cases, we can verify that $H^{1,1}(\mathcal{X}) \rightarrow H^{1,1}(\mathcal{M})$ is actually an isomorphism also over the integers by relying on Bott’s proof [10] of the Lefschetz hyperplane theorem (see § 1.6). As described in § 2.4, one applies this theorem iteratively [145] provided positivity of the line bundles \mathcal{E}_a is satisfied in every iteration.

8.3 Some Examples

As a quick application of our computations to homotopy classification and to clarify the technique, we include a few simple examples.

8.3.1 Simple hypersurfaces

Consider $\mathcal{M} \in [5|33]_{-144}^1$ vs. $\mathcal{M}' \in [6|322]_{-144}^1$. In both cases $b_{1,1} = 1$ and $b_{1,2} = 73$ so that their Hodge diamonds are identical. The Lefschetz hyperplane theorem applies in both cases and implies that the only element of the $(1,1)$ -cohomology on the Calabi-Yau manifold is the pullback of J , the Kähler form of the ambient space \mathcal{X} . Moreover, J generates the *integral* cohomology on the Calabi-Yau manifold and a generic element of this cohomology is $X = kJ$, an integral multiple of J . Thus we have :

$$\begin{aligned} [5|33]_{-144}^1 : \quad \hat{\kappa}(X, X, X) &= 9k^3, & p_1[X] &= -12k, \\ [6|322]_{-144}^1 : \quad \hat{\kappa}(X, X, X) &= 12k^3, & p_1[X] &= -10k. \end{aligned} \quad (8.3.1)$$

It is quite obvious that there is no linear transformation of $H^2(\mathcal{M})$ which would bring $\hat{\kappa}(X, X, X)$ and $p_1[X]$ of \mathcal{M} in the form of those of \mathcal{M}' . Therefore, they are distinct Calabi-Yau manifolds. In point of fact, \mathcal{M} and \mathcal{M}' differ in *all* of the invariants discussed in § 8.1.1 except, of course, the rank and signature of $\mathcal{Q}(,)$.

8.3.2 Three complete intersections

To illustrate further the application of the divisibility invariants of $\hat{\kappa}(, ,)$ and $p_1[]$, we present the detailed computations for three related examples.

Consider the three configurations :

$$\mathcal{A} \in \left[\begin{array}{c|ccc} 2 & 2 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{array} \right]_{-96}^3, \quad \mathcal{B} \in \left[\begin{array}{c|ccc} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{array} \right]_{-96}^3, \quad \mathcal{C} \in \left[\begin{array}{c|ccc} 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{array} \right]_{-96}^3. \quad (8.3.2)$$

In all three cases, Lefschetz hyperplane theorem can be applied to guarantee that $H^2(\mathcal{A}, \mathbb{Z})$, $H^2(\mathcal{B}, \mathbb{Z})$ and $H^2(\mathcal{C}, \mathbb{Z})$ respectively are generated by the restriction of the J_r .

For \mathcal{A} , we compute $\hat{\kappa}(\ , \ , \)$ as follows :

$$A, B, C \in H_4(\mathcal{A}, \mathbb{Z}), \quad A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (8.3.3)$$

so that we have :

$$\begin{aligned} \hat{\kappa}(A, A, A) &= \chi_E \left[\begin{smallmatrix} 2 & | & 2 & 1 & 0 & 1 & 1 & 1 \\ 2 & | & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & | & 1 & 1 & 1 & 0 & 0 & 0 \end{smallmatrix} \right] = 0, & \hat{\kappa}(A, A, B) &= \chi_E \left[\begin{smallmatrix} 2 & | & 2 & 1 & 0 & 1 & 1 & 0 \\ 2 & | & 1 & 1 & 1 & 0 & 0 & 1 \\ 2 & | & 1 & 1 & 1 & 0 & 0 & 0 \end{smallmatrix} \right] = 3, \\ \hat{\kappa}(A, A, C) &= \chi_E \left[\begin{smallmatrix} 2 & | & 2 & 1 & 0 & 1 & 1 & 0 \\ 2 & | & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & | & 1 & 1 & 1 & 0 & 0 & 1 \end{smallmatrix} \right] = 3, & \hat{\kappa}(A, B, B) &= \chi_E \left[\begin{smallmatrix} 2 & | & 2 & 1 & 0 & 1 & 0 & 0 \\ 2 & | & 1 & 1 & 1 & 0 & 1 & 1 \\ 2 & | & 1 & 1 & 1 & 0 & 0 & 0 \end{smallmatrix} \right] = 3, \\ \hat{\kappa}(A, B, C) &= \chi_E \left[\begin{smallmatrix} 2 & | & 2 & 1 & 0 & 1 & 0 & 0 \\ 2 & | & 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & | & 1 & 1 & 1 & 0 & 0 & 1 \end{smallmatrix} \right] = 6, & \hat{\kappa}(A, C, C) &= \chi_E \left[\begin{smallmatrix} 2 & | & 2 & 1 & 0 & 1 & 0 & 0 \\ 2 & | & 1 & 1 & 1 & 0 & 1 & 1 \\ 2 & | & 1 & 1 & 1 & 0 & 1 & 1 \end{smallmatrix} \right] = 3, \\ \hat{\kappa}(B, B, B) &= \chi_E \left[\begin{smallmatrix} 2 & | & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & | & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & | & 1 & 1 & 1 & 0 & 0 & 0 \end{smallmatrix} \right] = 0, & \hat{\kappa}(B, B, C) &= \chi_E \left[\begin{smallmatrix} 2 & | & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & | & 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & | & 1 & 1 & 1 & 0 & 0 & 1 \end{smallmatrix} \right] = 2, \\ \hat{\kappa}(B, C, C) &= \chi_E \left[\begin{smallmatrix} 2 & | & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & | & 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & | & 1 & 1 & 1 & 0 & 1 & 1 \end{smallmatrix} \right] = 2, & \hat{\kappa}(C, C, C) &= \chi_E \left[\begin{smallmatrix} 2 & | & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & | & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & | & 1 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix} \right] = 0, \end{aligned}$$

and of course the permutations of these.

Using Eq. (8.2.3) we have, denoting $X_r = A, B, C$, for $r = 1, 2, 3$:

$$p_1 = \sum_{r,s=1}^3 (3\delta^{r,s} - \sum_{a=1}^3 q_a^r q_a^s) X_r X_s = -2[A^2 + 3AB + 3AC + 3BC]. \quad (8.3.4)$$

Some of the homotopy invariants introduced above are $d_1 = 1$, $d_2 = 1$, $d_3 = 6$, $d_p = 72$ and $d_Q = -2^{30} 3^{14} \cong_{\mathbb{Q}} -2^6 3^6$.

By analogous choice of basis and computation in the case of \mathcal{B} , we obtain the non-vanishing $\hat{\kappa}(\ , \ , \)$ couplings :

$$\begin{aligned} \hat{\kappa}(A, A, B) &= 4, & \hat{\kappa}(A, A, C) &= 2, & \hat{\kappa}(A, B, B) &= 4, \\ \hat{\kappa}(A, B, C) &= 4, & \hat{\kappa}(B, B, C) &= 2, \end{aligned} \quad (8.3.5)$$

and the first Pontrjagin class :

$$p_1 = -2[A^2 + B^2 + 5AB + 2AC + 2BC]. \quad (8.3.6)$$

8.4. Hypersurfaces in Products of Surfaces, Again

Now we have $d_1 = 2$, $d_2 = 2$, $d_3 = 6$, $d_p = 24$ and $d_Q = -2^{32} 3^{10} \cong_{\mathbb{Q}} -3^2$.

Finally, in the case of \mathcal{C} , we obtain the non-vanishing $\hat{\kappa}(\ , \ , \)$ couplings :

$$\begin{aligned} \hat{\kappa}(A, A, B) &= 4, & \hat{\kappa}(A, A, C) &= 2, & \hat{\kappa}(A, B, B) &= 2, \\ \hat{\kappa}(A, B, C) &= 5, & \hat{\kappa}(B, B, C) &= 2, \end{aligned} \quad (8.3.7)$$

and the first Pontrjagin class :

$$p_1 = -2[A^2 + B^2 + 4AB + 4AC + 2BC]. \quad (8.3.8)$$

This time $d_1 = 1$, $d_2 = 2$, $d_3 = 6$, $d_p = 24$ and $d_Q = -2^{18} 3^{12} 13^2 \cong_{\mathbb{Q}} -2^2 3^4 13^2$.

Comparing the divisibility properties of $\hat{\kappa}(X, X, X)$ and $p_1[X]$ shows that these three Calabi-Yau manifolds belong in fact to three distinct homotopy types. While d_Q alone suffices to distinguish all three, it is also harder to compute; d_1 and d_2 are of course easier to compute and provide the same resolution in the present case.

Compute d_4, d_5, d_6 and d_7 . \clubsuit

8.4 Hypersurfaces in Products of Surfaces, Again

Let \mathcal{M} be a Calabi-Yau manifold which is the zero locus of a section of the anti-canonical bundle $\mathcal{K}_{\mathcal{S}}^* \otimes \mathcal{K}_{\mathcal{S}'}^*$ of the product of two almost del Pezzo surfaces, \mathcal{S} and \mathcal{S}' . By Eq. (7.2.9), we have

$$[\mathcal{M}] = K_{\mathcal{S}} \times [\mathcal{S}'] + [\mathcal{S}] \times K_{\mathcal{S}'}, \quad (8.4.1)$$

where $K_{\mathcal{S}} \in H_2(\mathcal{S})$ is dual to $c_1(\mathcal{S}) \in H^{1,1}(\mathcal{S})$. Since $c_1(\mathcal{S}) = c_1[K_{\mathcal{S}}]$, the homology class $K_{\mathcal{S}}$ is called the *canonical class*^(t). \mathcal{M} being a hypersurface in $\mathcal{S} \times \mathcal{S}'$, $H_4(\mathcal{M})$ is generated by terms of the form $(x \times \mathcal{S}') \cap [\mathcal{M}]$ and $(\mathcal{S} \times x') \cap [\mathcal{M}]$, where x and x' denote elements of $H_2(\mathcal{S})$ and $H_2(\mathcal{S}')$ respectively. By Eq. (7.2.4), the computation of triple intersection products of these terms is equivalent to the computation of quadruple products in $\mathcal{S} \times \mathcal{S}'$ with $[\mathcal{M}]$ as the fourth factor :

$$\bigcap_{i=1}^3 (x_i \times \mathcal{S}') \cap [\mathcal{M}] = ((x_1 \cap x_2 \cap x_3) \times [\mathcal{S}']) \cap [\mathcal{M}] = 0, \quad (8.4.2)$$

$$\bigcap_{i=1}^2 (x_1 \times \mathcal{S}') \cap (\mathcal{S} \times x') \cap [\mathcal{M}] = (x_1 \cap x_2) \times (x' \cap K_{\mathcal{S}'}) , \quad (8.4.3)$$

with corresponding modifications for two primed terms and one unprimed term.

For Wall's classification theorem 8.1, it is necessary to provide $p_1[x^i]$, the elements $x^i \in H^2(\mathcal{M}, \mathbb{Z})$ integrated against the first Pontrjagin class of \mathcal{M} . We thus compute the Chern class of $\mathcal{M} \hookrightarrow \mathcal{S} \times \mathcal{S}'$, related to the Chern classes of \mathcal{S} and \mathcal{S}' :

$$\begin{aligned} c \left[\begin{array}{c|c} \mathcal{S} & c_1 \\ \mathcal{S}' & c'_1 \end{array} \right] &= \frac{c(\mathcal{S}) \cdot c(\mathcal{S}')}{(1 + c_1 + c'_1)} = \frac{(1 + c_1 + c_2) \cdot (1 + c'_1 + c'_2)}{(1 + c_1 + c'_1)}, \\ &= 1 + [c_2 + c_1 c'_1 + c'_2] - [c_1 c_2 + c_1 c'_1 (c_1 + c'_1) + c'_1 c'_2]. \end{aligned} \quad (8.4.4)$$

Using that the homology class corresponding to $c_2(\mathcal{S})$ may be identified with $\chi_E(\mathcal{S})$, the homology class corresponding to $c_2(\mathcal{M})$ is

$$(\chi_E(\mathcal{S}) \times [\mathcal{S}'] + K_{\mathcal{S}} \times K_{\mathcal{S}'} + [\mathcal{S}] \times \chi_E(\mathcal{S}')) \cap [\mathcal{M}] . \quad (8.4.5)$$

Using Eq. (8.4.1) and Eq. (3.1.3) yields

$$c_2(\mathcal{M}) \stackrel{*}{\sim} 12(K_{\mathcal{S}} \times 1 + 1 \times K_{\mathcal{S}'}) \quad (8.4.6)$$

as a homology class on $\mathcal{S} \times \mathcal{S}'$. If we now write x^i indifferently for an element of $H^2(\mathcal{S})$ and also for the corresponding element of $H^2(\mathcal{M})$, and x_i for the corresponding element of $H_2(\mathcal{S})$, we have

$$\int_{\mathcal{M}} c_2 \wedge x^i = 12(K_{\mathcal{S}} \cap x_i) . \quad (8.4.7)$$

Hence

$$p_1[x^i] = \int_{\mathcal{M}} p_1 \wedge x^i = -24(K_{\mathcal{S}} \cap x_i) , \quad (8.4.8)$$

and similarly for elements of $H^2(\mathcal{S}')$.

8.5 The Homotopy Type of Surfaces

In § 3.1, we have considered Calabi-Yau 3-folds embedded as a hypersurface in the product of some two almost del Pezzo surfaces. Among these, we distinguish the class of *generic* almost del Pezzo surfaces which includes $\mathbf{F}_0 \stackrel{\text{def}}{=} \mathbb{P}^1 \times \mathbb{P}^1$ and blow-ups of \mathbb{P}^2 at 0, ..., 7 or 9, distinct points in general position (see Proposition 6 of Ref. [147] for exact conditions).

Since the Calabi-Yau manifolds obtained as hypersurfaces in a product of two generic *vs.* non-generic almost del Pezzo surfaces are diffeomorphic, we shall analyze only the generic case in this section. Then we discuss the application of our analysis in the non-generic case and conclude with some 'stringy' issues.

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To begin with, we will compute the intersection matrix (x, y) and $c_1[x]$ for all generic almost del Pezzo surfaces \mathcal{S} , for $x, y \in H_2(\mathcal{S}, \mathbb{Z})$. We write (x, y) for the homology intersection, and $c_1[x]$ denotes the integral of $c_1(\mathcal{S})$ over a cycle representing x or, equivalently, $c_1[x] = (K_{\mathcal{S}}, x)$. Note that $(x, y) \in H_0(\mathcal{S}, \mathbb{Z})$ is an integer and can be computed as

$$(x, y) = \chi_E\{x \cap y\} \quad (8.5.1)$$

provided the cycles representing the homology classes x and y can be chosen to be transversely intersecting smooth curves.

8.5.1 The homology of \mathbf{F}_0

The most natural basis for $H_2(\mathbf{F}_0, \mathbb{Z})$, $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ (see § 8.6), consists of copies of the \mathbb{P}^1 factors themselves. Since a hyperplane in \mathbb{P}^1 is simply a point, we can obtain a copy of one factor \mathbb{P}^1 by imposing a linear constraint on the other one and have :

$$a \stackrel{\text{def}}{=} \mathbb{P}^1 \times \{\cdot\} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad b \stackrel{\text{def}}{=} \{\cdot\} \times \mathbb{P}^1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \quad (8.5.2)$$

The intersection number of two generators is computed as :

$$\begin{aligned} (a, a) &:= \chi_E \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 0, & (a, b) &:= \chi_E \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = 1, & (b, b) &:= \chi_E \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = 0, \\ && \hline & a & 0 & 1 \\ & b & 1 & 0 \end{aligned} \quad (8.5.3)$$

Note that the intersection matrix for \mathbf{F}_0 is of Type II, i.e., for a general element $x \in H_2(\mathbf{F}_0)$, $x = \alpha a + \beta b$,

$$(x, x) = 2\alpha\beta \in 2\mathbb{Z}, \quad \alpha, \beta \in \mathbb{Z}. \quad (8.5.4)$$

In other words, \mathbf{F}_0 contains only curves of even self-intersection.

The anti-canonical class of \mathbb{P}^1 is the square of its hyperplane bundle and corresponds to generic quadratic polynomials in the homogeneous coordinates of \mathbb{P}^1 . The homology class dual to the Chern class of the anti-canonical bundle of \mathbb{P}^1 is $2 \in H_0(\mathbb{P}^1)$, i.e., two points of the 0-locus of a quadric equation. It follows from Eq. (7.2.9) that the canonical class of \mathbf{F}_0 is given by

$$K_{\mathbf{F}_0} = 2a + 2b, \quad \Rightarrow \quad c_1[a] = 2, \quad c_1[b] = 2. \quad (8.5.5)$$

8.5.2 Successive blow-ups of \mathbb{P}^2

For the generic almost del Pezzo surfaces which are ρ -fold ($\rho = b_{1,1}-1$) iterated blow-ups of \mathbb{P}^2 , the homology classes $\{E_i\}_{1 \leq i \leq r}$, corresponding to the exceptional divisors (lines) into which the ρ points were blown up, offer a natural choice of basis elements of $H_2(S, \mathbb{Z})$. These are disjoint and each has self-intersection -1 .

For the remaining basis element of $H_2(S, \mathbb{Z})$ we choose the class E_0 carried by a generic line on \mathbb{P}^2 , which misses all the points which were blown up. Thus we have :

$$\begin{array}{c|cc} (\cdot, \cdot) & E_0 & E_j \\ \hline E_0 & 1 & 0 \\ E_i & 0 & -\delta_{i,j} \end{array} \quad (8.5.6)$$

Note that, by the first line of (8.5.6), all these intersection matrices are of Type I, that is they do contain curves with an odd self-intersection.

Remark.

The fact that their intersection matrices are of different types serves to distinguish between the homotopy types of $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbf{F}_1 = \tilde{\mathbb{P}}^2_{p_1}$, i.e., \mathbb{P}^2 blown up at one point although they have the same Hodge numbers.

In this basis, the divisor K_S corresponding to the anti-canonical class is [23] :

$$K_S = 3E_0 - \sum_{i=1}^{\rho} E_i, \quad \Rightarrow \quad c_1[E_0] = 3, \quad c_1[E_i] = 1, \quad i = 1, \dots, \rho. \quad (8.5.7)$$

For all almost del Pezzo surfaces the intersection matrices have rank $b_{1,1}$ and signature $(1, b_{1,1}-1)$ (i.e., one plus sign and $(b_{1,1}-1)$ minus signs).

8.5.3 Topological Yukawa couplings computed

For the Calabi-Yau manifolds embedded as hypersurfaces in the product of two del Pezzo surfaces, the following cases arise :

1. \mathcal{M} is a hypersurface in $\mathbf{F}_0 \times \mathbf{F}_0$. In this case we have $\{a', b', a'', b''\}$ as a basis for $H^{1,1}(\mathcal{M}) \approx H_4(\mathcal{M})$, and the non-vanishing triple couplings are :

$$\begin{aligned} \hat{\kappa}(a', b', a'') &= 2, & \hat{\kappa}(a', b', b'') &= 2, \\ \hat{\kappa}(a', a'', b'') &= 2, & \hat{\kappa}(b', a'', b'') &= 2. \end{aligned} \quad (8.5.8)$$

We readily compute $d_1 = 2$, $d_2 = 4$, $d_3 = 12$ and $d_Q = 48$.

2. \mathcal{M} is a hypersurface in $\mathbf{F}_0 \times S$ where S is a ρ -fold iterated blow-up of \mathbb{P}^2 where $0 \leq \rho \leq 7$, i.e., $3 \leq \chi_E(S) \leq 10$. In this case we have $\{a, b, E_i\}$, $i = 0, \dots, \rho$, as a basis for $H_4(\mathcal{M}) \approx H^{1,1}(\mathcal{M})$, and the non-vanishing triple couplings are :

$$\begin{aligned} \hat{\kappa}(a, b, E_0) &= 3, & \hat{\kappa}(a, b, E_i) &= 1, \\ \hat{\kappa}(a, E_0, E_0) &= 2, & \hat{\kappa}(a, E_i, E_j) &= -2\delta_{i,j}, \\ \hat{\kappa}(b, E_0, E_0) &= 2, & \hat{\kappa}(b, E_i, E_j) &= -2\delta_{i,j}. \end{aligned} \quad (8.5.9)$$

This time $d_1 = 1$, $d_2 = 2$, $d_3 = 6$ and $d_Q = 24$.

3. \mathcal{M} is a hypersurface in $S \times S'$ where S and S' is a ρ -fold and a ρ' -fold iterated blow-up of \mathbb{P}^2 where $0 \leq \rho, \rho' \leq 7$, i.e., $3 \leq \chi_E(S), \chi_E(S') \leq 10$. In this case we have $\{E_a, E'_\alpha\}$, $a = 0, \dots, \rho$ and $\alpha = 0, \dots, \rho'$, as a basis for $H_4(\mathcal{M}) \approx H^{1,1}(\mathcal{M})$, and the non-vanishing triple couplings are :

$$\begin{aligned} \hat{\kappa}(E_0, E_0, E'_0) &= 3, & \hat{\kappa}(E_0, E'_0, E'_0) &= 3, \\ \hat{\kappa}(E_0, E_0, E'_\alpha) &= 1, & \hat{\kappa}(E_a, E'_0, E'_0) &= 1, \\ \hat{\kappa}(E_a, E_b, E'_0) &= -3\delta_{a,b}, & \hat{\kappa}(E_0, E'_\alpha, E'_\beta) &= -3\delta_{\alpha,\beta}, \\ \hat{\kappa}(E_a, E_b, E'_\gamma) &= -\delta_{a,b}, & \hat{\kappa}(E_c, E'_\alpha, E'_\beta) &= -\delta_{\alpha,\beta}. \end{aligned} \quad (8.5.10)$$

Now $d_1 = 1$, $d_2 = 1$, $d_3 = 6$ and $d_Q = 24$.

Remark.

The computation of d_2 shows that for S any almost del Pezzo surface with $\chi_E \leq 10$, Calabi-Yau hypersurfaces in $\mathbf{F}_0 \times S$ are of a different homotopy type from those in $\mathbf{F}_1 \times S$, although they have the same Hodge numbers. This completes the homotopy classification of Calabi-Yau hypersurfaces in products of almost del Pezzo surfaces.



Finally, let \mathcal{Z} denote an iterated 9-fold blow-up of \mathbb{P}^2 , $\mathcal{Z} = \tilde{\mathbb{P}}_{p_1, \dots, p_9}^2$ and $\chi_E(\mathcal{Z}) = 12$. It has been proven (Chapter 3, also Ref. [147]) that for any almost del Pezzo surface S , the Calabi-Yau manifold \mathcal{M} embedded in $\mathcal{Z} \times S$ has $\chi_E(\mathcal{M}) = 0$ and is biholomorphic to one embedded in $\mathcal{Z} \times \mathcal{Z}'$, where \mathcal{Z}' is a almost del Pezzo surface with $\chi_E(\mathcal{Z}') = 12$, and we focus now on the latter embedding.

Firstly, $b_{1,1}(\mathcal{M}) = 19$ (see Ref. [212]) and the set of 20 elements

$$\{E_0, \dots, E_9, E'_0, \dots, E'_9\}, \quad (8.5.11)$$

which would follow from the construction above, is redundant. Indeed, using the above results, it can be shown that $\kappa(x, y, z) \equiv 0 \forall y, z \in H^2(\mathcal{M})$ precisely if x is a multiple of

$$I = 3(E_0 - E'_0) - \left(\sum_{a=1}^9 E_a - \sum_{\alpha=1}^9 E'_{\alpha} \right). \quad (8.5.12)$$

Therefore $I \cong 0 \in H_4(\mathcal{M})$, leaving only 19 non-trivial independent generators.

Secondly, all the formulae (8.5.10) of case 3. are valid modulo, of course, the relation (8.5.12). In order to show that the triple products and Pontrjagin class satisfy the corresponding divisibility condition, it would however be necessary to know that $H^2(\mathcal{Z} \times \mathcal{Z}', \mathbb{Z})$ restricts onto $H^2(\mathcal{M}, \mathbb{Z})$. As $\mathcal{K}_{\mathcal{Z}}^*$ is not positive, the Lefschetz hyperplane theorem cannot be used. The triple product computation shows that the restriction is onto over the rationals, so that d_Q , *modulo multiplicative 40th powers of integers*, is still a valid and rather precise invariant.

☞ Calculate d_Q . ☚

8.6 Non-Generic Surfaces

The results of § 8.5 were derived for generic almost del Pezzo surfaces and we discuss now how to apply our analysis including non-generic models.

8.6.1 Hirzebruch surfaces

It was demonstrated in § 3.1.2 that $\mathbf{F}_0 \approx \mathbb{P}^1 \times \mathbb{P}^1$ is a deformation of \mathbf{F}_2 and that they both may be constructed as members of the configuration $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (see Eqs. (3.1.15) and the related discussion). For generators of H_2 , we choose $A := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, so that :

$$\begin{aligned} (A, A) &= \chi_E \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} = 2, & (\ , \) \mid \begin{array}{cc} A & B \\ \hline A & 2 & 1 \\ B & 1 & 0 \end{array} \\ (A, B) &= \chi_E \begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} = 1, \\ (B, B) &= \chi_E \begin{bmatrix} 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = 0. \end{aligned} \quad (8.6.1)$$

The anti-canonical class is now represented by $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (check : $c_1(\begin{bmatrix} 3 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}) = 0$) so that :

$$c_1[A] = \chi_E \begin{bmatrix} 3 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} = 4, \quad c_1[B] = \chi_E \begin{bmatrix} 3 & 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} = 2. \quad (8.6.2)$$

Note that this computation makes no homotopy distinction between \mathbf{F}_0 and \mathbf{F}_2 , as should be the case since the former is a deformation of the latter.

By the Lefschetz hyperplane theorem (see § 1.6), $\{A, B\}$ generate $H_2(\mathcal{Z}, \mathbb{Z})$ and note that :

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \quad \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1, \quad (8.6.3)$$

so that one can freely choose between the $\{a, b\}$ and the $\{A, B\}$ basis; we have used the former for convenience.

8.6.2 Non-generic blow-ups of \mathbb{P}^2

In § 8.5.2 we have assumed that $\tilde{\mathbb{P}}_{p_1, \dots, p_\rho}^2$ was constructed by blowing up ρ points in general position on \mathbb{P}^2 . More generally, if any p_i is on the exceptional line obtained by blowing up another point p_j , $j \neq i$, we have

$$(E_j, E_j) = -2, \quad (E_i, E_i) = -1, \quad (E_i, E_j) = +1. \quad (8.6.4)$$

To recover the basis of Subsection 8.5.2, we replace E_j by $\hat{E}_j := E_j - E_i$ and have

$$(\hat{E}_j, \hat{E}_j) = -1, \quad (E_i, E_i) = -1, \quad (E_i, \hat{E}_j) = 0, \quad (8.6.5)$$

which clearly generalizes for any other non-generic almost del Pezzo surface of this type and (dropping the hats) we recover the basis with the intersection matrix (8.5.6).

We also recall that

$$\tilde{\mathbb{P}}_{p_1, p_2, p_3}^2 \in \left[\begin{array}{c|cc} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array} \right] \quad (8.6.6)$$

precisely if the three points are not collinear. To include the non-generic case when the three points are collinear, one embeds

$$\tilde{\mathbb{P}}_{p_1, p_2, p_3}^2 \in \left[\begin{array}{c|ccc} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right]. \quad (8.6.7)$$

Interestingly enough, $b_2(\tilde{\mathbb{P}}_{p_1, p_2, p_3}^2) = 4$, and explicit computation verifies that $H_2(\tilde{\mathbb{P}}_{p_1, p_2, p_3}^2)$ is completely spanned by (the pull-backs of) the four Kähler forms. This could not have been predicted using the Lefschetz hyperplane theorem—because it does not apply.

8.7 Physics Application

In order to employ the above formulae and derive certain physical observables several further steps of analysis are needed and we now turn briefly to discuss these.

8.7.1 Normalization matrix of the kinetic term

The lowest order term in the superpotential of the effective low-energy theory that involves only E_6 27^* -plet superfields is :

$$d_{IJK} \kappa_{ABC} \bar{\Psi}_A^I(x) \bar{\Psi}_B^J(x) \bar{\Psi}_C^K(x), \quad (8.7.1)$$

where d_{IJK} is the E_6 $[27^* \otimes 27^* \otimes 27^*]$ Clebsch-Gordan coefficient and κ_{ijk} is (the point-field limit) Yukawa coupling (0.5.5).

The kinetic term for these superfields is however :

$$\sum_{A, \bar{B}, I} N_{A\bar{B}} (\bar{\Psi}_B^I(x))^\dagger \bar{\Psi}_A^I(x), \quad (8.7.2)$$

where the normalization matrix $N_{A\bar{B}}$ is (in the point-field limit) given [218] by the inner product :

$$N_{A\bar{B}} \equiv N(e_A, e_B) = \frac{1}{2} \int_M e_A \wedge (*e)_{\bar{B}}, \quad e_A \in H^2(M), \quad (8.7.3)$$

and $(*e)_{\bar{B}} \in H^4(M)$ is the dual of e_B . For the triple couplings $\kappa(, ,)$ to have usual interpretation as Yukawa couplings, the superfields $\bar{\Psi}_A^I(x)$ have to be (re)normalized by “ $(N_{AB})^{-1/2}$ ” so that :

$$\sum_{A, B, \bar{I}} N_{A\bar{B}} (\bar{\Psi}_B^I(x))^\dagger \bar{\Psi}_A^I \rightarrow \sum_{A, \bar{I}} (\tilde{\Psi}_A^{\bar{I}})^\dagger \tilde{\Psi}_A^{\bar{I}}. \quad (8.7.4)$$

Denoting the Kähler class of M by J , $N_{A\bar{B}}$ may be expressed as [218] :

$$N_{A\bar{B}} = \frac{3}{8} \frac{\kappa(e_A, J, J) \kappa(e_B, J, J)}{\kappa(J, J, J)} - \frac{1}{4} \kappa(e_A, e_B, J). \quad (8.7.5)$$

Using the above results for $\kappa(, ,)$, this normalization matrix can be obtained straightforwardly. As should be manifest from Eq. (8.7.5), $N_{A\bar{B}}$ depends on the choice of the cohomology class of the Kähler form

$$J = \sum_{i=1}^{b_{1,1}-1} v^A e_A, \quad e_A \in H^{1,1}(M), \quad v^A \in \mathbb{R}_+, \quad (8.7.6)$$

where the coefficients v^A are restricted so that J is a positive (1,1)-form and $\kappa(J, J, J) \in \mathbb{R}_+$ since this is, up to a positive numerical coefficient, the volume of the Calabi-Yau manifold.

As an illustration, consider M defined as a hypersurface in $X = \mathbf{F}_0 \times \mathbf{F}_0$. We choose the Kähler class of M to be represented by the sum of the first Chern classes of the two surfaces F_0 in X . Using Eq. (8.5.8) and Eq. (8.7.5) we now obtain

$2^5 \cdot N(, ,)$	a	b	a'	b'
a	9	1	1	1
b	1	9	1	1
a'	1	1	9	1
b'	1	1	1	9

$$[N^{-1/2}]_{A\bar{B}} = \frac{1}{3} \begin{pmatrix} \alpha & \beta & \beta & \beta \\ \beta & \alpha & \beta & \beta \\ \beta & \beta & \alpha & \beta \\ \beta & \beta & \beta & \alpha \end{pmatrix}, \quad (8.7.7)$$

where $\alpha = \frac{1}{2}(4\sqrt{6}-11)\sqrt{5+2\sqrt{2}}$ and $\beta = \frac{1}{2}\sqrt{5+2\sqrt{6}}$. This is obtained using the fact that matrices of the above form generate a simple finite-dimensional algebra (the Cayley-Hamilton theorem). The four E_6 27^* -plet superfields representing the “mirror” particles in a low-energy model compactified on \mathcal{M} are represented by four linearly independent combinations of a , b , a' and b' and normalized by $“(N_{A\bar{B}})^{-1/2}”$. This procedure is model-dependent since the choice of particular linear combinations of the basis elements that represent physical fields depends on the symmetries of the defining equations and we refer the interested Reader to the first article in Ref. [114] for further details in a particular model.

8.7.2 Higher order corrections

The value of $\hat{\kappa}(, ,)$ and therefore also $N(, ,)$ will receive corrections from quantum effects. While corrections to $\hat{\kappa}(, ,)$ are governed by certain non-renormalization theorems [111], it is not clear if $N(, ,)$ receives quantum corrections other than those induced, through Eq. (8.7.5), by corrections to $\hat{\kappa}(, ,)$.

In particular, $\hat{\kappa}(, ,)$ will receive corrections from world-sheet instantons. Such an instanton is represented by a non-trivial holomorphic map of the world-sheet into the “internal” Calabi-Yau manifold \mathcal{M} and is therefore represented by embeddings of non-trivial holomorphic curves in $L \subset \mathcal{M}$. By definition then, $[L]$ is an element of $H_2(\mathcal{M})$.

The contribution to the full Yukawa coupling $\kappa(e_A, e_B, e_C)$, Eq. (0.6.3), of a world-sheet instanton represented by the holomorphic curve $L \subset \mathcal{M}$ is :

$$\delta_L \hat{\kappa}(e_A, e_B, e_C) = \Delta e^{-M_{Pl}/M_C} \cdot (\int_L e_A)(\int_L e_B)(\int_L e_C), \quad e_A \in H^2(\mathcal{M}), \quad (8.7.8)$$

where M_{Pl} and M_C are the Planck mass and the compactification mass-scale and Δ is a factor independent of e_A . Denoting by e^A , the four dimensional homology class corresponding to e_A , we have

$$\int_L e_A = [L] \cap e^A. \quad (8.7.9)$$

Therefore $\delta_L \hat{\kappa}(e_A, e_B, e_C)$ can be computed by a straightforward application of our results presented in § 8.2–8.6, once $[L]$ has been identified. Needless to say, one has to add up/integrate all the contributions of all world-sheet instanton configurations L . In a particular model, for a particular Calabi-Yau manifold, the specific choice of the defining equations may simplify this task considerably

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and one may be able to derive phenomenologically relevant facts without actually having to complete these computations.

The issue of higher-order corrections to the normalization matrix $N_{A\bar{B}}$ is not understood as well as those for $\hat{\kappa}(, ,)$. Nevertheless, since $N_{A\bar{B}}$ is a function of $\hat{\kappa}(, ,)$, there will at least be the correction induced by $\delta_L \hat{\kappa}(, ,)$:

$$\begin{aligned} \delta_L N_{A\bar{B}} &= \delta \hat{\kappa}_{A\bar{B}} \cdot \hat{\kappa} - \hat{\kappa}_{A\bar{B}} \cdot \delta \hat{\kappa} - \frac{3}{2}[\hat{\kappa}_A \cdot \delta \hat{\kappa}_B + \delta \hat{\kappa}_A \cdot \hat{\kappa}_B + \hat{\kappa}_A \cdot \hat{\kappa}_B \cdot (\delta \hat{\kappa}/\hat{\kappa})] + \dots \\ \delta \hat{\kappa} &\equiv \Delta e^{-M_{Pl}/M_C} (\int_L J)^3, \\ \delta \hat{\kappa}_A &\equiv \Delta e^{-M_{Pl}/M_C} (\int_L e_A) \cdot (\int_L J)^2, \\ \delta \hat{\kappa}_{AB} &\equiv \Delta e^{-M_{Pl}/M_C} (\int_L e_A) \cdot (\int_L e_B) \cdot (\int_L J). \end{aligned} \quad (8.7.10)$$

We have assumed here that M_C is sufficiently smaller than M_{Pl} so that an expansion in $\Delta e^{-M_{Pl}/M_C}$ is justified.

Chapter 9

(Co)homological Algebra

We now turn to discuss some techniques of cohomological algebra which can be used to compute a variety of cohomological data for complete intersections, but also in some more general situations.

9.1 A Short Exact Sequence

The basic idea was introduced already in (1.3.7) for the simple case of hypersurfaces in \mathbb{P}^n and then generalized in Chapter 2.

If X is a complex manifold of dimension n and \mathcal{M} an m -dimensional submanifold, the *normal bundle*^(L), $\mathcal{N}_{X/\mathcal{M}}$, of \mathcal{M} in X is *defined* to be the quotient bundle $\{\mathcal{T}_X|_{\mathcal{M}}/\mathcal{T}_{\mathcal{M}}\}$, where $\mathcal{T}_{\mathcal{M}}$ and \mathcal{T}_X denote the (holomorphic) tangent bundles of \mathcal{M} and X respectively. Let \mathcal{E} be a holomorphic bundle over X and ξ a holomorphic section of \mathcal{E} such that \mathcal{M} is the zero locus of ξ , i.e., $\xi^{-1}(0) = \mathcal{M} \hookrightarrow X$. Then we have the map $\nabla\xi : \mathcal{T}_X|_{\mathcal{M}} \rightarrow \mathcal{E}|_{\mathcal{M}}$, which is independent of any choice of a connection (as discussed in § 1.3.1) and is a holomorphic linear mapping of bundles over \mathcal{M} . As in § 1.3.1, $\ker \nabla\xi = \mathcal{T}_{\mathcal{M}}$. If, in addition, $\nabla\xi$ is onto, i.e., covers all of $\mathcal{E}|_{\mathcal{M}}$, we have the short *exact sequence*^(L) :

$$0 \rightarrow \mathcal{T}_{\mathcal{M}} \xrightarrow{i} \mathcal{T}_X|_{\mathcal{M}} \xrightarrow{\nabla\xi} \mathcal{E}|_{\mathcal{M}} \rightarrow 0 \quad (9.1.1)$$

which allows us to identify $\mathcal{E}|_{\mathcal{M}}$ with $\mathcal{N}_{X/\mathcal{M}}$.

This is precisely the situation in the case of non-singular complete intersections, as exemplified in Chapters 1, 2 and 3.

9.1.1 The accompanying cohomology sequence

An important consequence of the exactness of the short sequence (9.1.1) is that it induces another exact sequence, one which links $H^*(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ with $H^*(\mathcal{M}, \mathcal{T}_X)$ and $H^*(\mathcal{M}, \mathcal{E})$:

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) & \xrightarrow{i_0} & H^0(\mathcal{M}, \mathcal{T}_X) & \xrightarrow{j_0} & H^0(\mathcal{M}, \mathcal{E}) \\ & & \xrightarrow{\delta_0} & & \xrightarrow{i_1} & & \xrightarrow{\delta_1} \\ & & H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) & & H^1(\mathcal{M}, \mathcal{T}_X) & \xrightarrow{j_1} & H^1(\mathcal{M}, \mathcal{E}) \\ & & \xrightarrow{\delta_1} & & \xrightarrow{i_2} & & \xrightarrow{\delta_2} \\ & & H^2(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) & & \dots & & \dots \\ & & & & & & \xrightarrow{j_m} \\ & & & & & & H^m(\mathcal{M}, \mathcal{E}) & \rightarrow 0. \end{array} \quad (9.1.2)$$

Oft quoted as an ‘elementary fact’ and presented in this still somewhat unusual notation for many physicists, this may just hide the simplicity of the idea and we pause to explain it in some detail. The maps i_q are the cohomology embeddings which may be written in a local coordinate system as

$$i_q : dz^{\mu_1} \dots dz^{\mu_q} \omega_{\mu_1 \dots \mu_q}{}^\alpha \mapsto dz^{\mu_1} \dots dz^{\mu_q} \omega_{\mu_1 \dots \mu_q}{}^a \Big|_{a=\alpha}, \quad (9.1.3)$$

where a and α are holomorphic tangent indices on X and \mathcal{M} , respectively, and the μ 's are cotangent indices on \mathcal{M} . Similarly, the maps j_q project the tangent index so to speak of a \mathcal{T}_X -valued q -form on \mathcal{M} to the normal space directions.

9.1.2 The dimension changing map

The action of the dimension changing maps δ_q is a little trickier. For a rigorous account, we refer the Reader to Ref. [163], p.28; suffice it here to give a somewhat heuristic description. Consider, for example, \mathcal{E} -valued 0-forms. To construct a 1-form, we take two such 0-forms at nearby points, $\xi(z)$ and $\xi(z + \delta z)$. They are valued, respectively in the normal vectors \hat{n}_z and $\hat{n}_{z+\delta z}$. Their difference points in the direction $(\hat{n}_z - \hat{n}'_z)$, where \hat{n}'_z is $\hat{n}_{z+\delta z}$, parallel-transported back to z . To first order in δz , $(\hat{n}_z - \hat{n}'_z) \propto \hat{t}_z$, where $\hat{t}_z \in T_z(\mathcal{M})$. Therefore,

$$dz^\mu \omega_\mu \stackrel{\text{def}}{=} dz^\mu \lim_{\delta z \rightarrow 0} \frac{\xi(z) - \xi(z + \delta z)}{\delta z^\mu} \quad (9.1.4)$$

is $\mathcal{T}_{\mathcal{M}}$ -valued and belongs to $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$. This is straightforwardly generalized to all δ_q . A little more work remains to show that the *long exact cohomology sequence* (9.1.2) is exact, i.e., that for the maps δ_q just constructed, $\ker(\delta_q) = \text{Im}(j_q)$ and $\ker(i_{q+1}) = \text{Im}(\delta_q)$. The Reader should have no trouble filling in these details, bearing in mind that the forms being mapped are harmonic and are hence equivalent to zero if exact.

The main utility of the long exact cohomology sequence (9.1.2) is that it links $H^q(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ —typically the unknown cohomology group—to $H^q(\mathcal{M}, \mathcal{T}_X)$ and $H^q(\mathcal{M}, \mathcal{E})$, which are valued in bundles over the embedding space X , which is usually better known. However, $H^q(\mathcal{M}, \mathcal{T}_X)$ and $H^q(\mathcal{M}, \mathcal{E})$ are cohomology groups on \mathcal{M} , involve the restriction of \mathcal{T}_X and \mathcal{E} to \mathcal{M} and are in general not simply equal to $H^q(X, \mathcal{T}_X)$ and $H^q(X, \mathcal{E})$, respectively.

Indeed, the general strategy for computing the cohomology on a submanifold \mathcal{M} , valued in a bundle \mathcal{V} intrinsic to the submanifold proceeds as follows : (1) relate \mathcal{V} to (restrictions of) bundles \mathcal{E}_i over the embedding space X , (2) determine the \mathcal{E}_i -valued cohomology on \mathcal{M} as a restriction of certain cohomology groups on X . Then (3) compute all required cohomology on X and (4) work backwards, from this data, to determine the \mathcal{V} -valued cohomology on \mathcal{M} . This general scenario for such computations involves some very powerful machinery, for which will need a little digression on *sheaves*^(L).

9.2 Sheaves

A *presheaf* \mathfrak{S} over a manifold X assigns to each open subset U of X , a set (actually a vector space in all the examples we shall consider) \mathfrak{S}_U . We require that $\mathfrak{S}_\emptyset = 0$. Also, to each inclusion of open sets $U \subseteq V$, a presheaf assigns a map (for us a homomorphism) $\mathbf{r}_U^V : \mathfrak{S}_V \rightarrow \mathfrak{S}_U$. These maps can generally be interpreted as restrictions and must satisfy

$$\mathbf{r}_U^V \circ \mathbf{r}_V^W = \mathbf{r}_U^W, \quad U \subseteq V \subseteq W \quad (9.2.1)$$

and, clearly, \mathbf{r}_U^U must be the identity. For \mathfrak{S} to be moreover a *sheaf*^(L), we need in addition that

1. if U is an open set and $\{V_i\}$ an open cover of U —then $\mathfrak{s} = 0$ is the only element $\mathfrak{s} \in \mathfrak{S}_U$ for which

$$\mathbf{r}_{V_i}^U(\mathfrak{s}) = 0, \quad \forall i; \quad (9.2.2)$$

2. if U is an open set, $\{V_i\}$ an open cover of U and there are elements over each neighborhood, $\mathfrak{s}_i \in \mathfrak{S}_{V_i}$, which equal over all pairwise overlaps, $(V_i \cap V_j)$, then there is a unique $\mathfrak{s} \in \mathfrak{S}_U$ such that $\mathfrak{s}_i = \mathbf{r}_{V_i}^U(\mathfrak{s})$.

The examples we will need are, in order of increasing generality:

- (A) Let X be a complex manifold and $U \subseteq V$ two open neighborhoods as above. $\mathcal{O}_X(U)$ is defined to be the ring of holomorphic functions on U . Also, with $\{U_i\}$ an open covering of X , $\mathcal{O}_X = \bigcup_i \mathcal{O}_X(U_i)$ is the *structure sheaf*^(L).
- (B) Let \mathcal{E} be a holomorphic vector bundle over the complex manifold \mathcal{W} . We define $\mathcal{E}(U)$ to be the space of holomorphic sections of $\mathcal{E}|_U$ and the sheaf* \mathcal{E} to be the union over some open covering.
- (C) Let \mathcal{M} be a submanifold of the complex manifold X and let \mathcal{E} be a holomorphic vector bundle over \mathcal{M} . Since \mathcal{M} is a submanifold in X , \mathcal{E} clearly has measure-zero support in X and cannot be used to define a vector bundle over X . However, we define $\mathcal{E}_{\mathcal{M}}$ to be the sheaf over X which assigns to each open $U \subseteq X$ the space of holomorphic sections of $\mathcal{E}|_{(U \cap \mathcal{M})}$. In this respect, sheaves are ‘looser’ than vector bundles and this is one of the reasons for introducing them.

In all cases, \mathbf{r}_U^V is the appropriate restriction map.

If \mathfrak{S} and \mathfrak{S}' are sheaves over X , a map of sheaves, $f : \mathfrak{S} \rightarrow \mathfrak{S}'$ assigns to each open U a map $f_U : \mathfrak{S}_U \rightarrow \mathfrak{S}'_U$ which satisfy the obvious compatibility condition

$$f_U \circ \mathbf{r}_U^{(\mathfrak{S}')V} = \mathbf{r}_U^{(\mathfrak{S})V} \circ f_V. \quad (9.2.3)$$

The definition of exactness for a sequence of linear maps of linear sheaves over X is not precisely what one would assume; a sequence of maps of sheaves is called exact if it is exact over sufficiently small open subsets; to emphasize this, we may write *sheaf-exact*. In particular, an exact sequence of holomorphic vector bundles induces an exact sequence of the corresponding sheaves, but the reverse is not true in general.

There exists a cohomology theory of sheaves which assigns to a manifold X and a linear sheaf \mathfrak{S} over X cohomology groups $H^q(X, \mathfrak{S})$, $q \geq 0$. What we need to know about sheaf cohomology is summarized in the following

Theorem 9.1 *i) If \mathcal{E} is a holomorphic vector bundle over X , the sheaf cohomology groups $H^q(X, \mathcal{E})$ may be identified with the $\bar{\partial}$ -cohomology of X with coefficients† in \mathcal{E} .*

*Most of the time, we will make no notational distinction between a vector bundle and the associated sheaf; the choice should be clear from the context.

†These cohomology groups may also be identified with the space of harmonic \mathcal{E} -valued $(0, q)$ -forms which has a natural (additive) Abelian group structure.

ii) With X , \mathcal{M} and \mathcal{E} as in example (C) above, $H^q(X, \mathcal{E}_\mathcal{M}) = H^q(\mathcal{M}, \mathcal{E})$.

iii) If $0 \rightarrow \mathfrak{S} \xrightarrow{f} \mathfrak{S}' \xrightarrow{g} \mathfrak{S}'' \rightarrow 0$ is a short exact sequence of sheaves over X , then the long cohomology sequence :

$$0 \rightarrow \dots \xrightarrow{\delta_{q-1}} H^q(X, \mathfrak{S}) \xrightarrow{f_q} H^q(X, \mathfrak{S}') \xrightarrow{g_q} H^q(X, \mathfrak{S}'') \xrightarrow{\delta_q} \dots \rightarrow 0 \quad (9.2.4)$$

is exact. The maps f_q and g_q are induced from f and g and δ_q is constructed as usual (see § 9.1).

Proofs of these statements can be found in Refs. [23], [24], [27], [29]...

9.3 The Koszul Complex

One of the steps in our Programme is to obtain (harmonic) forms on the submanifold \mathcal{M} from (harmonic) forms on the embedding space X . To this end, we will have to ask how to obtain functions on \mathcal{M} from those on X , that is, to study the restriction $\mathcal{O}_X \xrightarrow{\varrho} \mathcal{O}_\mathcal{M}$.

9.3.1 The Koszul resolution

Consider, for example, a complete intersection $\mathcal{M} \subset X$ defined as the zero-set of a system of 3 sections, $f(x) = g(x) = h(x) = 0$, where $f(x)$ is a section of the line bundle \mathcal{E}_f , $g(x)$ a section of \mathcal{E}_g and so on. Let us write $\xi = (f, g, h)$, i.e., $\xi^a = f, g, h$ for $a = 1, 2, 3$ and $\mathcal{E} = \mathcal{E}_f \oplus \mathcal{E}_g \oplus \mathcal{E}_h$, accordingly.

Clearly, for any function $\vartheta(x) \in \mathcal{O}_X(U)$ we can simply restrict the range to $x \in \mathcal{M} \cap U$, but this is not sufficient. Because $\xi^a = 0$ on \mathcal{M} , $\vartheta(x)$ and $\vartheta(x) + \sum_{a=1}^3 \beta_a \xi^a$ become indistinguishable when restricted to \mathcal{M} . So, it is the equivalence classes of functions on X —modulo multiples of the defining sections—which are identified as functions on \mathcal{M} . Now, ξ^a are sections of non-trivial line bundles while ϑ are just plain (local) holomorphic functions (when extended globally, they must be constant since X is compact). Consequently, the products $\beta_1 \xi^1, \beta_2 \xi^2$ and so on must be plain functions also, whence β_a must correspond to bundles dual to \mathcal{E}_a .

Typically, the ξ^a are polynomials* of degree $q(a)$, $\xi^a(x) = \xi_{i_1 \dots i_{q(a)}}^a x^{i_1} \dots x^{i_{q(a)}}$, and are represented by the tensor coefficients $\xi_{i_1 \dots i_{q(a)}}^a$. Then, β^a are represented

*These may be polynomials in the homogeneous coordinates of the projective embedding space or simply some suitable variables, representing local coordinates with perhaps some redundancy.

9.3. The Koszul Complex

by contravariant tensor coefficients, $\beta_a^{i_1 \dots i_{q(a)}}$, so that

$$\langle \beta_a | \xi^a \rangle = \sum_{i_1, \dots, i_{q(a)}} \beta_a^{i_1 \dots i_{q(a)}} \xi_{i_1 \dots i_{q(a)}}^a \in \mathcal{O}_X, \quad \text{no sum on } a. \quad (9.3.1)$$

The general case requires only more complicated notation, but is otherwise precisely the same. Let us write ϕ^* , γ^* and η^* for some generic formal representatives of \mathcal{E}_f^* , \mathcal{E}_g^* and \mathcal{E}_h^* , respectively, and $\langle \phi | f \rangle$ for the natural contraction (inner product) of ϕ^* with f .

It follows that

$$\mathcal{E}^* \xrightarrow{\xi} \mathcal{O}_X \xrightarrow{\varrho} \mathcal{O}_\mathcal{M} \rightarrow 0 \quad (9.3.2)$$

is sheaf-exact, that is, the sheaf of holomorphic functions on \mathcal{M} , $\mathcal{O}_\mathcal{M}$, may be regarded as a quotient of \mathcal{O}_X modulo suitable multiples of ξ . Moreover, away from $\mathcal{M} \subset X$ where $\xi \neq 0$, the map provided by contraction with $\xi = (f, g, h)$ covers all of \mathcal{O}_X . Indeed : varying ϕ^* , γ^* and η^* over all variables which yield a scalar when contracted with f , g and h , respectively, $\sum_{a=1}^3 \langle \beta_a | \xi^a \rangle$ easily covers all of $\mathcal{O}_{X/\mathcal{M}}$. Therefore, on $(X - \mathcal{M})$, the map ξ has no cokernel, ϱ vanishes and so does $\mathcal{O}_\mathcal{M}$. On the other hand, at $\mathcal{M} \subset X$ the triple of sections, ξ , vanishes and the map ξ is null. Hence ϱ identifies $\mathcal{O}_\mathcal{M}$ with $\mathcal{O}_X|_\mathcal{M}$, the restriction of \mathcal{O}_X to \mathcal{M} —precisely as desired.

To complete the above sequence to the left, note that the map ξ has a kernel. In fact, all complex linear combinations of

$$\begin{pmatrix} -\langle \gamma | g \rangle \phi^* \\ \langle \phi | f \rangle \gamma^* \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \langle \eta | h \rangle \phi^* \\ 0 \\ -\langle \phi | f \rangle \eta^* \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -\langle \eta | h \rangle \gamma^* \\ \langle \gamma | g \rangle \eta^* \end{pmatrix} \quad (9.3.3)$$

in \mathcal{E}^* , are (trivially) annihilated in the contraction with ξ ; such triples span $\ker(\mathcal{E}^* \xrightarrow{\xi} \mathcal{O}_X)$. The very same elements of \mathcal{E}^* are also obtained from

$$\wedge^2 \mathcal{E}^* = \mathcal{E}_f^* \wedge \mathcal{E}_g^* \oplus \mathcal{E}_f^* \wedge \mathcal{E}_h^* \oplus \mathcal{E}_g^* \wedge \mathcal{E}_h^*. \quad (9.3.4)$$

as the image of the mapping through ξ . To see this, note that $\wedge^2 \mathcal{E}^*$ is represented by linear combinations of

$$\begin{pmatrix} 0 & \phi^* \gamma^* & 0 \\ -\phi^* \gamma^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -\phi^* \eta^* \\ 0 & 0 & 0 \\ \phi^* \eta^* & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma^* \eta^* \\ 0 & -\gamma^* \eta^* & 0 \end{pmatrix} \quad (9.3.5)$$

(the antisymmetry of the wedge product, \wedge , accounts for the alternating signs). Then, for example,

$$(f, g, h) \cdot \begin{pmatrix} 0 & \phi^* \gamma^* & 0 \\ -\phi^* \gamma^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\langle \gamma | g \rangle \phi^* \\ \langle \phi | f \rangle \gamma^* \\ 0 \end{pmatrix}^T \quad (9.3.6)$$

Iterating this procedure, we obtain the *Koszul complex*^(L), K_* , represented for a general complete intersection of K hypersurfaces by the sheaf-exact sequence, also called the *resolution* of \mathcal{O}_M ,

$$0 \rightarrow \wedge^K \mathcal{E}^* \xrightarrow{\xi} \dots \xrightarrow{\xi} \wedge^2 \mathcal{E}^* \xrightarrow{\xi} \mathcal{E}^* \xrightarrow{\xi} \mathcal{O}_X \xrightarrow{\rho} \mathcal{O}_M \rightarrow 0. \quad (9.3.7)$$

This sequence relates holomorphic functions (scalars) on M (\mathcal{O}_M) to quantities defined entirely on X ; the analogous will also be true if we tensor the whole sequence with a vector bundle over X , such as T_X, \mathcal{E}, \dots

Clearly, had we chosen an r -tuple of sections which do not vanish simultaneously over X , the complete intersection is empty, $\mathcal{O}_M = 0$ and the resulting sequence is exact even as a sequence of vector bundles.

Given the exact sequence 9.3.7, we are tempted to write a long cohomology sequence formally analogous to Seq. (9.1.2) as derived from the short exact sequence (9.1.1). However, looking back at the derivation of δ_q in § 9.1.2, we see that it crucially depended on the fact that the exact sequence (9.1.1) had precisely three non-trivial terms. When $K \neq 1$, Seq. (9.3.7) is longer than three terms and it will soon be clear that no suitable dimension changing map could possibly be found in general.

9.3.2 One hypersurface at a time

We aim to describe the cohomology groups $H^q(M, \mathcal{O}_M)$ in terms of $H^q(X, \wedge^k \mathcal{E}^*)$ —a task to be generalized for $H^q(M, \mathfrak{S})$, where \mathfrak{S} is a sheaf over X —in terms of certain cohomology data entirely over X . A simple trick to circumvent the non-shortness of Seq. (9.3.7) is to consider one hypersurface at a time instead of restricting to the complete intersection at once. To simplify the notation, let's take M to be an intersection of just two hypersurfaces, \mathcal{Y} and \mathcal{Z} in X , defined respectively as $f(x) = 0$ and $g(x) = 0$.

We then first restrict to \mathcal{Y} , for example, using the short sheaf-exact sequence (a length-1 Koszul complex)

$$0 \rightarrow \mathcal{E}_f^* \xrightarrow{f} \mathcal{O}_X \rightarrow \mathcal{O}_{\mathcal{Y}} \rightarrow 0. \quad (9.3.8)$$

9.3. The Koszul Complex

From this we have the long exact cohomology sequence

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{E}_f^*) &\xrightarrow{f} H^0(X) \xrightarrow{\rho_y} H^0(\mathcal{Y}) \xrightarrow{\delta_0} \\ &\xrightarrow{\delta_0} H^1(X, \mathcal{E}_f^*) \xrightarrow{f} \dots \xrightarrow{\rho_y} H^4(\mathcal{Y}) \xrightarrow{\delta_4} \\ &\xrightarrow{\delta_4} H^5(X, \mathcal{E}_f^*) \xrightarrow{f} H^5(X) \xrightarrow{\rho_y} 0, \end{aligned} \quad (9.3.9)$$

which determines $H^q(\mathcal{Y})$ in terms of $H^0(X)$ and $H^0(X, \mathcal{E}_f^*)$.

Now embed $(M = \mathcal{Y} \cap \mathcal{Z}) \hookrightarrow \mathcal{Y}$, that is, we impose the $g(x) = 0$ constraint in addition to $f(x) = 0$ which was used to define $\mathcal{Y} \hookrightarrow X$. This being a simple hypersurface, we again have a short sheaf-exact sequence

$$0 \rightarrow \mathcal{E}_g^*|_{\mathcal{Y}} \xrightarrow{f} \mathcal{O}_{\mathcal{Y}} \xrightarrow{\rho_z} \mathcal{O}_M \rightarrow 0. \quad (9.3.10)$$

From this we have the long exact cohomology sequence

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{Y}, \mathcal{E}_g^*) &\xrightarrow{g} H^0(\mathcal{Y}) \xrightarrow{\rho_z} H^0(M) \xrightarrow{\delta_0} \\ &\xrightarrow{\delta_0} H^1(\mathcal{Y}, \mathcal{E}_g^*) \xrightarrow{g} \dots \xrightarrow{\rho_z} H^3(M) \xrightarrow{\delta_3} \\ &\xrightarrow{\delta_3} H^4(\mathcal{Y}, \mathcal{E}_g^*) \xrightarrow{g} H^4(\mathcal{Y}) \xrightarrow{\rho_z} 0, \end{aligned} \quad (9.3.11)$$

which determines $H^q(M)$ in terms of $H^0(\mathcal{Y})$ and $H^0(\mathcal{Y}, \mathcal{E}_g^*)$. Of these last two cohomology groups, the former has been determined just above, while the latter we determine using Seq. (9.3.8), tensored however with \mathcal{E}_g^*

$$0 \rightarrow (\mathcal{E}_f \otimes \mathcal{E}_g)^* \xrightarrow{f} \mathcal{E}_g^* \rightarrow \mathcal{E}_g^*|_{\mathcal{Y}} \rightarrow 0. \quad (9.3.12)$$

and the associated long exact cohomology sequence

$$\begin{aligned} 0 \rightarrow H^0(X, (\mathcal{E}_f \otimes \mathcal{E}_g)^*) &\xrightarrow{f} H^0(X, \mathcal{E}_g^*) \xrightarrow{\rho_y} H^0(\mathcal{Y}, \mathcal{E}_g^*) \xrightarrow{\delta_0} \\ &\xrightarrow{\delta_0} H^1(X, (\mathcal{E}_f \otimes \mathcal{E}_g)^*) \xrightarrow{f} \dots \xrightarrow{\rho_y} H^4(\mathcal{Y}, \mathcal{E}_g^*) \xrightarrow{\delta_4} \\ &\xrightarrow{\delta_4} H^5(X, (\mathcal{E}_f \otimes \mathcal{E}_g)^*) \xrightarrow{f} H^5(X, \mathcal{E}_g^*) \xrightarrow{\rho_y} 0. \end{aligned} \quad (9.3.13)$$

The end result of this calculation is that $H^q(M)$ is related, through three long exact cohomology sequences (9.3.9), (9.3.11) and (9.3.13) and a number of intermediary results, to $H^q(X)$, $H^q(X, \mathcal{E}_f^*)$, $H^q(X, \mathcal{E}_g^*)$ and $H^q(X, (\mathcal{E}_f \otimes \mathcal{E}_g)^*)$.

This can straightforwardly be done for every complete intersection, regardless of the number of constraints although, of course, the procedure becomes more and more involved with each new constraint.

As the foregoing cavalcade might have lost the Reader unfamiliar with exact sequences, let us sum up this procedure in terms of the following diagram

$$\begin{array}{ccccccc}
 & 0 & 0 & 0 & & & \\
 & \downarrow & \downarrow & \downarrow & & & \\
 0 \rightarrow & (\mathcal{E}_f \otimes \mathcal{E}_g)^* & \xrightarrow{f} & \mathcal{E}_g^* & \xrightarrow{\rho_y} & \mathcal{E}_g^*|_{\mathcal{M}} & \rightarrow 0 \\
 & \downarrow g & & \downarrow g & & \downarrow g & \\
 0 \rightarrow & \mathcal{E}_f^* & \xrightarrow{f} & \mathcal{O}_X & \xrightarrow{\rho_y} & \mathcal{O}_{\mathcal{M}} & \rightarrow 0 \\
 & \downarrow e_z & & \downarrow e_z & & \downarrow e_z & \\
 0 \rightarrow & \mathcal{E}_f^*|_Z & \xrightarrow{f} & \mathcal{O}_Z & \xrightarrow{\rho_y} & \mathcal{O}_{\mathcal{M}} & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & 0 & 0 & & &
 \end{array} \tag{9.3.14}$$

where all row-sequences and all column-sequences are exact. The $\mathcal{O}_{\mathcal{M}}$ -valued cohomology on \mathcal{M} (lower right corner) is related to the cohomology on X , valued in the bundles inside the dotted rectangle. Quite clearly, Seq. (9.3.7), for $K = 2$, may be thought of as the diagonal of the diagram (9.3.14) since $\wedge^2 \mathcal{E}^* = (\mathcal{E}_f \otimes \mathcal{E}_g)^*$ and $\mathcal{E} = \mathcal{E}_f \oplus \mathcal{E}_g$. Indeed,

Lemma 9.1 A 3×3 square diagram of short exact sequences as in (9.3.14) implies a 4-term exact sequence as in (9.3.7) for $K = 2$.

is a standard result and is not hard to generalize beyond $K = 2$; for $K = 3$, in place of (9.3.14), we have a $3 \times 3 \times 3$ cube of short exact sequences which implies the exactness of a 5-term exact sequence obtained as the ‘body-diagonal’ of the cube of short exact sequences, and so on.

Markedly, whatever the length (K) of the Koszul complex, the exactness of the sequence (9.3.7) is equivalent to saying that the structure sheaf of the submanifold \mathcal{M} is given as a quotient, $\mathcal{O}_{\mathcal{M}} = \{\mathcal{O}_X/\mathfrak{I}_K\}$, where \mathfrak{I}_K might be called the Koszul ideal^(u). Furthermore, in the simplest case $K = 1$ and $\mathfrak{I}_K = \mathcal{E}^*$ for $K = 2$, $\mathfrak{I}_K = \{\mathcal{E}^* / \wedge^2 \mathcal{E}^*\}$; when $K = 3$,

$$\mathfrak{I}_K = \left\{ \frac{\mathcal{E}^*}{\{\wedge^2 \mathcal{E}^* / \wedge^3 \mathcal{E}^*\}} \right\}, \tag{9.3.15}$$

and so on.

The Seq. (9.3.7) may be seen as a diagonally collapsed version of (hyper)cubic diagrams like (9.3.14) where some of the ‘intermediate’ bundles (sheaves) have been dropped. It is plausible to expect that a similar collapsed version of the tangle of intertwined long exact cohomology sequences (9.3.9), (9.3.11) and (9.3.13) also exists, where no ‘intermediate’ (and often otherwise useless) cohomology group is ever computed. There indeed does exist such a general tool, called a spectral sequence, and we now turn to describe it.

9.3.3 The Koszul spectral sequence

Given the exact sequence 9.3.7, we are tempted to write a long cohomology sequence formally analogous to Seq. (9.1.2) as derived from the short exact sequence (9.1.1). The corresponding cohomology groups can again be arranged in analogy to (9.1.2)

$$\begin{array}{cccccc|c}
 \wedge^K \mathcal{E}^* & \dots & \wedge^2 \mathcal{E}^* & \mathcal{E}^* & \mathcal{O}_W & \mathcal{O}_{\mathcal{M}} & \\
 \hline
 H^0(X, \wedge^K \mathcal{E}^*) & \dots & H^0(X, \wedge^2 \mathcal{E}^*) & H^0(X, \mathcal{E}^*) & H^0(X) & H^0(\mathcal{M}) & \\
 H^1(X, \wedge^K \mathcal{E}^*) & \dots & H^1(X, \wedge^2 \mathcal{E}^*) & H^1(X, \mathcal{E}^*) & H^1(X) & H^1(\mathcal{M}) & \\
 \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\
 & & & & & &
 \end{array} \tag{9.3.16}$$

but there is no analogue of δ_q . Instead, there will now be maps (‘differentials’ d_r) that act $r + 1$ steps to the right and r steps up the chart[†]. The horizontal, single-step maps, acting as in (9.1.2), are the $r = 0$ case thereof. From the action of these maps, the cohomology in the right-most column is related to the cohomology groups which appear in the lower left quadrant of (9.3.16), somewhat as a summary of the circuitous procedure of § 9.3.2. The precise explanation of how this machinery works in full generality requires some technical details which are beyond the scope of this volume; these can be found in Ref. [23], p.438–445, for example.

[†]That all maps must act in such a ‘knight tour’ like fashion on a spectral sequence chart can be proven in complete generality [23]. In our case, we just note that the differentials of our spectral sequence are induced from the vertical and horizontal maps in the diagram (9.3.14) and can be traced that way.

It may be clarifying to actually write down the intertwined six long exact cohomology sequences accompanying each of the nine short exact sequences in the diagram (9.3.14). The three horizontal ones of course extend horizontally, for $q = 0, \dots, 5$, while the vertical ones extend downwards, wrap around and keep crossing the horizontal ones for every q . It is not hard to draw a model of this tangle of sequences on a cylindrically wrapped sheet of paper and recover the action of the maps in the Koszul spectral sequence for $K = 2$.

In practice, one starts from (9.3.16), called the 0^{th} level of the spectral sequence; the entries in the lower left quadrant are usually labeled as $E_0^{q,k} = H^q(X, \wedge^k \mathcal{E}^*)$. We have the maps induced straightforwardly from the contraction with ξ in the Koszul complex : $E_0^{q,k} \xrightarrow{\text{d}_0} E_0^{q,k-1}$. We then pass to the d_0 -cohomology,

$$E_1^{q,k} = \left\{ (\ker(d_0) \cap E_0^{q,k}) / d_0(E_0^{q,k+1}) \right\}. \quad (9.3.17)$$

To deal with the higher ‘differentials’, d_r , we determine their action

$$d_r : E_r^{q,k} \longrightarrow E_r^{q-r, k-r-1} \quad (9.3.18)$$

and pass to the cohomology quotients

$$E_{r+1}^{q,k} \stackrel{\text{def}}{=} \left\{ (\ker(d_r) \cap E_r^{q,k}) / d_r(E_r^{q+r, k+r+1}) \right\}, \quad (9.3.19)$$

order by order in r . It should be obvious that this sequence of approximations converges for some $r \leq K$; when $r > K$, even for the entries in the left-most column, the maps would point out of the lower left quadrant of (9.3.16).

Now, contributions to $H^q(\mathcal{M})$ consist only of d_r -closed (modulo d_r -exact) forms, for all r . In practice, we advance order by order in r canceling out all pairs (α, β) which satisfy $d_r \alpha = \beta$ for some r . What remains, spans $E_\infty^{q,k}$ and abuts to $H^q(\mathcal{M})$ according to

$$\sum_{k=0}^K E_\infty^{q,k} \implies H^q(\mathcal{M}). \quad (9.3.20)$$

The collection of such contributions does not form a (holomorphic) direct sum in $H^q(\mathcal{M})$ and, as in § 3.3.4, we shall use the ‘+’ symbol instead of ‘ \oplus ’.

The astute Reader will have noticed that all computations which employ the long exact cohomology sequences (9.1.2), and even more so the spectral sequences (9.3.16), rest upon the ability to discern the action of several maps. This

9.4. The Bott-Borel-Weil Theorem

otherwise hopeless task[†] is however straightforwardly accomplished using the coset representation of flag spaces and the tensor algebra associated with representations of Lie groups.

9.4 The Bott-Borel-Weil Theorem

9.4.1 Homogeneous cohomology over flag varieties

One of the main advantages of considering such generalized flag spaces is that all cohomology of all homogeneous vector bundles over them can be given by means of an amusingly simple [18] (see also Ref. [5] for generalizations) algorithmic variant of the Bott-Borel-Weil theorem :

Theorem 9.2 (Bott-Borel-Weil) *Let \mathbb{F} be a flag space as in Eqs. (3.3.1)–(3.3.2) and $\mathcal{V} \sim (a_1, \dots, a_{n_1} | \dots | b_1, \dots, b_{n_F})$ a holomorphic homogeneous vector bundle over \mathbb{F} . Then :*

- (1) *Homogeneous vector bundles \mathcal{V} over \mathbb{F} are in 1-1 correspondence with the $U(n_1) \times \dots \times U(n_F)$ representations.*
- (2) *The $H^q(X, \mathcal{V})$ is non-zero for at most one value of q , in which case it furnishes an irreducible representation of $U(N)$, $H^q(X, \mathcal{V}) \approx (c_1, \dots, c_N) \mathbb{C}^N$.*
- (3) *$(a_1, \dots, a_{n_1} | \dots | b_1, \dots, b_{n_F})$ determines (c_1, \dots, c_N) according to the following algorithm :*

1. *Add the sequence $1, \dots, N$ to the entries in $(a_1, \dots, a_{n_1} | \dots | b_1, \dots, b_{n_F})$.*
2. *If any two entries in the result of Step 1 are equal, all cohomology vanishes; otherwise proceed.*
3. *Swap the minimum number ($= q$) of neighboring entries required to produce a strictly increasing sequence.*
4. *Subtract the sequence $1, \dots, N$ from the result of 3, to obtain (c_1, c_2, \dots, c_N) .*

In more general flag spaces, where G is an arbitrary Lie group and P and arbitrary parabolic subgroup thereof, the same algorithm applies, with the only distinction that instead of counting ‘swaps’ in Step 3, we consider the action of the Weyl subgroup of G ; in fact, ‘swaps’ are the elementary transformations of the permutation group S_n which, in turn, is the Weyl subgroup of $U(n)$. We

[†]Of course, in fortunate cases, the $E_0^{q,k}$ chart is sufficiently sparse, so that either all differentials vanish and $E_0^{q,k} = E_\infty^{q,k}$ or their action is otherwise elementary to deduce.

refer the Reader to Ref. [5] for a more ‘serious’ wording of this theorem and a rather simple proof. Note that the theorem provides not only the dimensions of the cohomology groups, but in fact the transformation property with respect to G ; that will provide one of the most general and powerful computational tools in this neck of the woods.

9.4.2 A general viewpoint

The cohomological computation described here is a combination of (1) the spectral sequence accompanying the Koszul resolution, (2) the Bott-Borel-Weil theorem for embedding spaces which are products of flag spaces and (3) sequences derived like (1.3.7) which relate bundles over the submanifold to bundles over the embedding space. It exhibits one of the goals in algebraic geometry : solving differential equations through algebraic means. Indeed, we see from Table 0.2, that we need $H^q(\mathcal{M}, T_{\mathcal{M}}^*)$ and $H^1(\mathcal{M}, \text{End } T_{\mathcal{M}})$, that is, we need forms like $\omega = dz^\mu dz^\nu \omega_{\mu\nu}$ which are 0-modes of the Laplacian on \mathcal{M} .

The purely algebraic technique which we describe in this couple of Chapters and to which we refer simply as ‘Koszul spectral sequence’ describes the solutions of these Laplace’s differential equations on \mathcal{M} without any differential geometry. In a way, this works just like the Laplace transform which converts a system of differential equations into a system of algebraic equations. Essentially, the calculations become manipulations of systems of algebraic equations where the variables and the parameters are certain (irreducible) tensors.

From the point of view of the physics application, the manifold itself is not really required to be known. Rather, it is the rings generated by the cohomology groups $H^1(\mathcal{M}, T_{\mathcal{M}})$, $H^1(\mathcal{M}, T_{\mathcal{M}}^*)$ and $H^1(\mathcal{M}, \text{End } T_{\mathcal{M}})$ which are physically relevant, although this information largely determines the underlying compactification manifold \mathcal{M} . As usual in algebraic geometry, one will be interested in the specifics of the polynomial ring over \mathcal{M} and its image in the interesting cohomology groups. Many of the answers can be formulated in terms of certain subrings of the residue class ring $\{\mathbb{C}[x]/\mathfrak{I}\}$, where $\mathbb{C}[x]$ is the ring of (complex) polynomials in some suitable variables such as coordinates and \mathfrak{I} is some *ideal*^(L).

9.5 Calabi-Yau or Not?

Knowing the resolution sequence of the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of a submanifold $\mathcal{M} \subset X$ allows one to immediately tell the type of the canonical bundle $\mathcal{K}_{\mathcal{M}}$, that is, whether \mathcal{M} is Calabi-Yau, almost ample, …

Consider, for simplicity, a smooth complete intersection of a quartic and a quadric hypersurface in \mathbb{P}^5 , $\mathcal{M} \in [5|42]$. The resolution, written in the notation of § 9.4, is

$$0 \rightarrow (6|00000) \xrightarrow{(4|00000)} (2|00000) \rightarrow \mathcal{O}_{\mathcal{M}} \rightarrow 0; \quad (9.5.1)$$

again, we stack summands in a direct sum atop each other. Using Theorem 9.2, it is straightforward to fill in the lower left quadrant of the 0th level of the spectral sequence as in (9.3.16), associated to this resolution. The Reader should have no trouble verifying that only the top right hand and bottom left hand entries, $E_0^{0,0}$ and $E_0^{5,2}$ respectively, in the lower left quadrant are non-zero. Moreover, these cohomology groups are 1-dimensional, $\approx \mathbb{C}^1$. It is then straightforward that all differentials, d_r , vanish and $E_0^{q,k} = E_\infty^{q,k}$, for all k, q .

Through (9.3.20), we have that $\dim H^0(\mathcal{M}) = 1 = \dim H^3(\mathcal{M})$. The former of these results simply says that \mathcal{M} is a connected manifold. The latter, however, says that the anti-holomorphic (harmonic) 3-form is projectively unique since

$$H^3(\mathcal{M}) = H^0(\mathcal{M}, \wedge^3 T_{\mathcal{M}}^*) = H^0(\mathcal{M}, \mathcal{K}_{\mathcal{M}}), \quad (9.5.2)$$

implying that $\mathcal{K}_{\mathcal{M}} \approx \mathcal{O}_{\mathcal{M}}$ is the trivial line bundle over \mathcal{M} . That Seq. (9.3.7) always results in such a simple spectral sequence for Calabi-Yau complete intersections in products of complex projective spaces was proven in Ref. [146].

Using the Kodaira-Nakano Theorem 3.3, it is not hard to prove this quite more generally. The proof rests on a few simple observations. Firstly, \mathcal{O}_X clearly has $H^q(X, \mathcal{O}_X) \neq 0$ only for $q = 0$. Second, the map ξ in Seq. (9.3.7) is realized as a contraction with a chosen section of the bundle \mathcal{E} over X , whence \mathcal{E} must be non-trivial and positive for otherwise it would have no non-trivial global holomorphic sections. Then, however, $\wedge^k \mathcal{E}^*$ is non-trivial and more and more negative as k is increased from 1 to K . Thus, $H^q(X, \wedge^k \mathcal{E}^*) = 0$ except perhaps for $q = \dim X$.

So then, what we need for \mathcal{M} to be Calabi-Yau is that $\wedge^K \mathcal{E}^* \approx \mathcal{K}_X$. Then $\wedge^k \mathcal{E}^* \otimes \mathcal{K}_X^*$ is (almost) positive for $k < K$ and all cohomology in the lower left

quadrant of the chart like (9.3.16) vanishes except for the top right and bottom left corners, which are moreover 1-dimensional.

It is straightforward to see that the preceding argument in fact did not require \mathcal{M} to be a complete intersection. Indeed, we recall the simple construction in § 3.5.1. The Pfaffian 3-fold \mathfrak{X} was defined as the zero-set of seven Pfaffian cubic polynomials $f_k(x)$ over \mathbb{P}^6 . This tells us that the resolution sequence ends as

$$\mathcal{O}_{\mathbb{P}^6}(-3)^{\oplus 7} \xrightarrow{\hat{f}} \mathcal{O}_{\mathbb{P}^6} \xrightarrow{\varrho} \mathcal{O}_{\mathfrak{X}} \rightarrow 0, \quad (9.5.3)$$

where the map \hat{f} obtained by multiplication with the 7-vector of Pfaffians $f_k(x)$, $k = 1, \dots, 7$. Given that the seven cubics $f_k(x)$ were obtained as the Pfaffians of the principal minors of a single skew-symmetric matrix, \mathbf{A} , a little thought reveals that the kernel of the map \hat{f} is also obtained as the image of multiplication by the matrix \mathbf{A} . That map again has a kernel, easily verified to be again obtained as the image of multiplication by (the transpose of) the 7-vector $f_k(x)$.

The result is the resolution

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^6}(-7) \xrightarrow{f'} \mathcal{O}_{\mathbb{P}^6}(-4)^{\oplus 7} \xrightarrow{\mathbf{A}} \mathcal{O}_{\mathbb{P}^6}(-3)^{\oplus 7} \xrightarrow{\hat{f}} \mathcal{O}_{\mathbb{P}^6} \xrightarrow{\varrho} \mathcal{O}_{\mathfrak{X}} \rightarrow 0. \quad (9.5.4)$$

Indeed, the left-most bundle in the sequence is the canonical bundle of \mathbb{P}^6 and it is a simple exercise to verify that

$$\dim H^q(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = \begin{cases} 1 & q = 0, 3, \\ 0 & \text{otherwise,} \end{cases} \quad (9.5.5)$$

precisely as required for \mathfrak{X} to be a Calabi-Yau 3-fold. In full detail, we obtain

q	$\mathcal{O}_{\mathbb{P}^6}(-7)$	$\mathcal{O}_{\mathbb{P}^6}(-4)^{\oplus 7}$	$\mathcal{O}_{\mathbb{P}^6}(-3)^{\oplus 7}$	$\mathcal{O}_{\mathbb{P}^6}$	$\mathcal{O}_{\mathfrak{X}}$
0:	0	0	0	\mathbb{C}	$H^0(\mathfrak{X}) \approx \mathbb{C}$
1:	0	0	0	0	$H^1(\mathfrak{X}) = 0$
2:	0	0	0	0	$H^2(\mathfrak{X}) = 0$
3:	0	0	0	0	$H^3(\mathfrak{X}) \approx \mathbb{C}$
4:	0	0	0	0	$\equiv 0$
5:	0	0	0	0	$\equiv 0$
6:	\mathbb{C}	0	0	0	$\equiv 0$

Since all $E_0^{q,k}$ vanish except $E_0^{0,0}$ and $E_0^{6,3}$, which are 1-dimensional. The dotted arrows represent the relation (9.3.20) and provide us with the required cohomology of a Calabi-Yau 3-fold.

9.5. Calabi-Yau or Not?

Clearly, if the left-most bundle turned out to be $\mathcal{O}_{\mathbb{P}^6}(-k)$ with $k < 7$, \mathfrak{X} would have been ample; with $k > 7$, \mathfrak{X} would have been scant (see p. 52 for our taxonomy). More generally, let \mathcal{E}_K^* be the left-most bundle in the resolution of $\mathcal{O}_{\mathcal{M}}$, for the submanifold $\mathcal{M} \subset \mathcal{X}$. The same will of course be said of a

Submanifold \mathcal{M}	Condition on \mathcal{E}_K ($\mathcal{E}_0^* \stackrel{\text{def}}{=} \mathcal{O}_{\mathcal{X}}$)
Ample	$\mathcal{E}_K \otimes \mathcal{K}_{\mathcal{X}} < 0$
Almost Ample	$\mathcal{E}_K \otimes \mathcal{K}_{\mathcal{X}} \leq 0$ (" <" along at least some $\mathcal{Y} \subset \mathcal{X}$)
Calabi-Yau	$\mathcal{E}_K \otimes \mathcal{K}_{\mathcal{X}} = 0$ ($\approx \mathbb{C}$)
Almost Scant	$\mathcal{E}_K \otimes \mathcal{K}_{\mathcal{X}} \geq 0$ (" >" along at least some $\mathcal{Y} \subset \mathcal{X}$)
Scant	$\mathcal{E}_K \otimes \mathcal{K}_{\mathcal{X}} > 0$
Limp	$\mathcal{E}_K \otimes \mathcal{K}_{\mathcal{X}} \neq 0$ (" >" along at least some $\mathcal{Y} \subset \mathcal{X}$, " <" along at least some $\mathcal{Z} \subset \mathcal{X}$)

\mathcal{E}_K^* is the left-most bundle in the resolution of $\mathcal{O}_{\mathcal{M}}$.

Table 9.1: A partial taxonomy of submanifolds.

configuration^(L) $[\mathcal{X}||\mathcal{E}]$, where \mathcal{X} is the embedding space and \mathcal{E} is the positive bundle over \mathcal{X} the sections of which are used as defining constraints for $\mathcal{M} \in [\mathcal{X}||\mathcal{E}]$. (For a complete intersection \mathcal{M} , the left-most bundle in the resolution of $\mathcal{O}_{\mathcal{M}}$ is $\det \mathcal{E}^*$.)

Recall that for a complete intersection \mathcal{M} , the left-most bundle in the resolution (9.3.7) is always the determinant of the dual of the bundle the sections of which we have used to define \mathcal{M} . In the present example, this is not so because \mathfrak{X} is not a complete intersection, but may instead be thought of as a generalization thereof. The resolution sequences of both the complete intersections and the Pfaffian 3-fold \mathfrak{X} (and also other similar constructions) do have a common property : in their resolution sequence ($\mathcal{E}_0^* \stackrel{\text{def}}{=} \mathcal{O}_{\mathcal{X}}$)

$$0 \rightarrow \mathcal{E}_K^* \rightarrow \mathcal{E}_{K-1}^* \rightarrow \cdots \rightarrow \mathcal{E}_1^* \rightarrow \mathcal{E}_0^* \rightarrow \mathcal{O}_{\mathcal{M}} \rightarrow 0, \quad (9.5.7)$$

we have that

$$\mathcal{E}_k^* \otimes \mathcal{E}_{K-k}^* \approx \mathcal{K}_X, \quad k = 0, \dots, K. \quad (9.5.8)$$

The above criterion for the triviality of the canonical bundle, \mathcal{K}_M , (equivalently : the vanishing first Chern class, Ricci-flatness, existence of nowhere vanishing holomorphic 3-form, and so on) remains valid generally :

Theorem 9.3 A 3-dimensional compact submanifold $M \subset X$ of a compact manifold X is a Calabi-Yau 3-fold if the structure sheaf \mathcal{O}_M fits into a sheaf-exact resolution sequence

$$0 \rightarrow \mathcal{E}_K^* \rightarrow \mathcal{E}_{K-1}^* \rightarrow \dots \rightarrow \mathcal{E}_1^* \rightarrow \mathcal{E}_0^* \rightarrow \mathcal{O}_M \rightarrow 0, \quad (9.5.9)$$

where $\mathcal{E}_0^* = \mathcal{O}_X$ and $\mathcal{E}_K^* = \mathcal{K}_X$.

Remark.

By requirements of a resolution sequence, all \mathcal{E}_k are bundles over X . \mathcal{O}_M defines a sheaf over X extending by zero; that is, at $M \subset X$, \mathcal{O}_M is the sheaf of germs of holomorphic functions over M , while over $X-M$, \mathcal{O}_M is defined to be zero.

9.6 Plurigenera

With the foregoing technology, we are now in position to complete the calculation of $\mathcal{K}_X^{*\otimes m}$ -valued cohomology for arbitrary complete intersections in products of flag-varieties, without having to rely on the Kodaira-Nakano theorem 3.3 and the positivity of the anti-canonical bundle \mathcal{K}_X^* . To this end, we multiply each term in the resolution (9.3.7) by $\mathcal{K}_X^{*\otimes m}$ and use the associated spectral sequence to determine the $\mathcal{K}_X^{*\otimes m}$ -valued cohomology.

Consider then an almost-Fano 3-fold $\mathfrak{F} \in \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$. As discussed at the end of § 3.2.2, the anti-canonical bundle is $\mathcal{K}_{\mathfrak{F}}^* \approx \mathcal{O}_{\mathbb{P}^1}(1)$, and so is not positive; it is null along \mathbb{P}^3 . That is why the Kodaira-Nakano theorem does not apply and the cohomology $H^q(\mathfrak{F}, \mathcal{K}_{\mathfrak{F}}^*)$ is not restricted to $q = 0$ only. The Riemann-Roch-Hirzebruch theorem 3.2 provides the alternating sum of the dimensions of the cohomology groups, but not each of those individually. The above described application of exact and spectral sequences however is straightforward to apply and does completely determine the $\mathcal{K}_{\mathfrak{F}}^{*\otimes m}$ -valued cohomology for any m . The

9.6. Plurigenera

almost-Fano 3-fold \mathfrak{F} actually being a simple hypersurface the resolution is a short exact sequence, but we will utilize the spectral sequence format, as in (9.5.6), so as to indicate how the more involved complete intersections would be dealt with.

As \mathfrak{F} is embedded in $\mathbb{P}^3 \times \mathbb{P}^1$, all relevant bundles will be products of bundles over \mathbb{P}^3 and those over \mathbb{P}^1 ; we stack the corresponding Young tableaux (see § 3.3.2) one above the other. Thus, $\mathcal{O}_{\mathbb{P}^3}(4) \otimes \mathcal{O}_{\mathbb{P}^1}(1) = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and $\mathcal{K}_{\mathfrak{F}}^{*\otimes m} \approx \mathcal{O}_{\mathbb{P}^1}(m) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -m & 0 & 0 & 0 \end{pmatrix}$. The resolution sequence is

$$0 \rightarrow \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \rightarrow \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1} \rightarrow \mathcal{O}_{\mathfrak{F}} \rightarrow 0. \quad (9.6.1)$$

Multiplied by $\mathcal{K}_{\mathfrak{F}}^{*\otimes m} \approx \begin{pmatrix} 0 & 0 & 0 & 0 \\ -m & 0 & 0 & 0 \end{pmatrix}$, this becomes

$$0 \rightarrow \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1-m & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ -m & 0 & 0 & 0 \end{pmatrix} \rightarrow \mathcal{K}_{\mathfrak{F}}^{*\otimes m} \rightarrow 0. \quad (9.6.2)$$

Stacking the cohomology under the corresponding bundles in this sequence, we obtain

q	$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 1-m & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -m & 0 & 0 & 0 \end{pmatrix}$	$\mathcal{K}_{\mathfrak{F}}^{*\otimes m}$
0:	0	$(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ -m & 0 & 0 & 0 \end{smallmatrix})^1$	$(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ -m & 0 & 0 & 0 \end{smallmatrix})_{m+1}^1$
1:	0	0	$\Rightarrow 0$
2:	0	0	$(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1-m & 0 & 0 & 0 \end{smallmatrix})_m^1$
3:	$(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1-m & 0 & 0 & 0 \end{smallmatrix})_m^1$	0	$\Rightarrow 0$
4:	0	0	$\equiv 0$

Adding up the dimensions, we obtain

$$\begin{aligned} \mathfrak{P}^m(\mathfrak{F}) &\stackrel{\text{def}}{=} \sum_{q=0}^3 (-1)^q \dim H^q(\mathfrak{F}, \mathcal{K}_{\mathfrak{F}}^{*\otimes m}) \\ &= (m+1) + 0 + m + 0 = 2m+1, \end{aligned} \quad (9.6.4)$$

in complete agreement with Eq. (3.2.14), for $C_1^3 = 0$.

☞ Calculate the anti-canonical bundle-valued cohomology for analogous almost-ample n-folds, where \mathbb{P}^3 is replaced by \mathbb{P}^n , and the defining equation has bi-degree $(n+1, 1)$ rather than $(4, 1)$. ↗

A.1 Computing Tangent Bundle Valued Forms

As discussed in § 9.1, the tangent bundle of the submanifold, $T_{\mathcal{M}}$, is related to the tangent bundle of the embedding space T_X and the bundle \mathcal{E} a (system of) section(s) of which has been used to define $\mathcal{M} \hookrightarrow X$. These bundles and their cohomology over X are supposed to be known—and surely is easy to obtain whenever we can use the Bott-Borel-Weil theorem 9.2.

A.1.1 Representing the normal bundle

The sequence (9.1.1) is exact and specifies $T_{\mathcal{M}}$ completely if the map $\nabla\xi : T_X|_{\mathcal{M}} \rightarrow \mathcal{E}|_{\mathcal{M}}$ is onto, that is, if $\mathcal{E}|_{\mathcal{M}}$ may be identified with $N_{X/\mathcal{M}}$ (compare with the adjunction formula 1^(L)).

Such is the case precisely for non-singular complete intersections, as exemplified in Chapters 1, 2 and 3, but is not so for the Pfaffian Calabi-Yau 3-fold discussed in § 3.5. In general, we have in mind a subspace $\mathcal{M} \subset X$ defined as the zero-set of a (system of) section(s) ξ of a bundle \mathcal{E} over X . Then locally some combination of the defining section(s) spans the normal space and we expect the normal bundle to be a sub-bundle of \mathcal{E} . Then we write

$$0 \rightarrow T_{\mathcal{M}} \xrightarrow{i} T_X|_{\mathcal{M}} \xrightarrow{\nabla\xi} \mathcal{E}|_{\mathcal{M}} \rightarrow \mathcal{F} \rightarrow 0 , \quad (\text{A.1.1})$$

where $\mathcal{F} \stackrel{\text{def}}{=} \text{cok}[T_X|_{\mathcal{M}} \xrightarrow{\nabla\xi} \mathcal{E}|_{\mathcal{M}}]$ so that the sequence is exact. However, unless we can determine \mathcal{F} independently, the relation given as Seq. (A.1.1) is of little practical use. Of course, Seq. (A.1.1) may extend even further to the right, with a sequence $\mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \dots$ replacing the single \mathcal{F} .

For simplicity and as it still applies to a large number of constructions, we restrict our further discussion to complete intersections. There Seq. (A.1.1) simplifies in that $\mathcal{F} = 0$ and leads to the long exact cohomology sequence (9.1.2). If $\mathcal{F} \neq 0$, we have to construct a corresponding spectral sequence instead of Seq. (9.1.2) and the computation becomes quite more involved.

A.1.2 An example

For a compact complex n -fold \mathcal{M} embedded in a compact complex space X as a complete intersection, the computation proceeds as follows : (1) The $H^q(\mathcal{M}, \mathcal{E})$ and $H^q(\mathcal{M}, T_X)$ cohomology groups are obtained from the spectral sequence constructed from the resolution (9.3.7) tensored by \mathcal{E} and by T_X , respectively. (2) This data is then filled into Seq. (9.1.2), determining $H^q(\mathcal{M}, T_{\mathcal{M}})$, as required.

Chapter A

Tangent Bundle Valued Cohomology

As we have seen in Table 0.2, in a compactification on a Calabi-Yau 3-fold \mathcal{M} , elements of $H^1(\mathcal{M}, T_{\mathcal{M}})$ represent generations of particles in the Standard model (together with some additional particles completing E_6 27-plets and the 27 superpartners), while elements of $H^1(\mathcal{M}, T_{\mathcal{M}}^*)$ represent generations of their mirror particles. We will shortly discuss the general method for computing these and other related cohomology groups.

Recall that $H^1(\mathcal{M}, T_{\mathcal{M}}^*) = H^{1,1}(\mathcal{M})$ is dual to $H^{2,2}(\mathcal{M})$ and that

$$H^{2,2}(\mathcal{M}) = H^2(\mathcal{M}, \wedge^2 T_{\mathcal{M}}^*) \approx H^2(\mathcal{M}, \mathcal{K}_{\mathcal{M}} \otimes T_{\mathcal{M}}) . \quad (\text{A.0.1})$$

In view of this, we may proceed either by computing $H^1(\mathcal{M}, T_{\mathcal{M}})$ and $H^1(\mathcal{M}, T_{\mathcal{M}}^*)$, or—if \mathcal{M} is Calabi-Yau, so $\mathcal{K}_{\mathcal{M}} \approx \mathbb{C}$ —equivalently relate $H^1(\mathcal{M}, T_{\mathcal{M}}^*)$ to (the complex conjugate of) $H^2(\mathcal{M}, T_{\mathcal{M}})$ and compute only the $T_{\mathcal{M}}$ -valued cohomology or only the $T_{\mathcal{M}}^*$ -valued one. Since neither of the computations requires the manifold \mathcal{M} to be Calabi-Yau or a 3-fold, we will describe briefly the computation both of $H^q(\mathcal{M}, T_{\mathcal{M}})$ and of $H^q(\mathcal{M}, T_{\mathcal{M}}^*)$, and also of some related cohomology groups. For Calabi-Yau n -folds, we will have the relation

$$H^q(\mathcal{M}, T_{\mathcal{M}}) \approx H^q(\mathcal{M}, \wedge^{(n-1)} T_{\mathcal{M}}^*) = H^{n-1,q}(\mathcal{M}) , \quad \dim \mathcal{M} = n , \quad (\text{A.0.2})$$

and also that $H^0(\mathcal{M}, T_{\mathcal{M}})$ and $H^n(\mathcal{M}, T_{\mathcal{M}})$ have to vanish since $b_{n-1,0} = 0 = b_{n-1,n}$ on any Calabi-Yau n -fold (see § 1.2).

Consider, for example, the complete intersection (8.0.1)

$$\mathcal{M}_{-54} \in \left[\begin{array}{c|cc} 3 & 3 & 1 \\ 2 & 0 & 3 \end{array} \right]_{-54}^8. \quad (\text{A.1.2})$$

We choose this simple case both because of its possible physical importance [210] and because several of the previously studied techniques apply (see § 1.6, § 3.1).

Let us denote the defining polynomials by

$$f(x) \stackrel{\text{def}}{=} f_{abc} x^a x^b x^c, \quad g(x, y) \stackrel{\text{def}}{=} g_{\alpha\beta\gamma} x^\alpha y^\beta y^\gamma, \quad (\text{A.1.3})$$

where $x \in \mathbb{P}^3$ and $y \in \mathbb{P}^2$. Here $f(x)$ and $g(x, y)$ are sections of $\mathcal{E}_f = \left(\begin{smallmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)$ and $\mathcal{E}_g = \left(\begin{smallmatrix} -1 & 0 & 0 & 0 \\ -3 & 0 & 0 \end{smallmatrix} \right)$, respectively, so \mathcal{E} is the rank-two direct sum of these two line bundles. The two lines in this notation for bundles over $X = \mathbb{P}^3 \times \mathbb{P}^2$ refer to representations of $U(1) \times U(3)$ and $U(1) \times U(2)$, corresponding to the restriction of a bundle to \mathbb{P}^3 and \mathbb{P}^2 , respectively. The tangent bundle on X is given as

$$\mathcal{T}_X = [\mathcal{T}_x = \left(\begin{smallmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \right)] \oplus [\mathcal{T}_y = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix} \right)]. \quad (\text{A.1.4})$$

The cohomology groups on \mathcal{M} valued in each of these irreducible bundles are computed by using a corresponding spectral sequence. To this end, note that the resolution (9.3.7) now becomes

$$0 \rightarrow \left(\begin{smallmatrix} 4 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{smallmatrix} \right) \xrightarrow{f} \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{smallmatrix} \right) \xrightarrow{g} \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix} \right) \rightarrow \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix} \right)_{\mathcal{M}} \rightarrow 0. \quad (\text{A.1.5})$$

The arrows labeled by f and g denote maps induced by contraction with the tensors f_{abc} and $g_{\alpha\beta\gamma}$, respectively.

Tensoring by $\mathcal{E}_f = \left(\begin{smallmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)$, the resolution sequence (A.1.5) yields

$$\begin{array}{ccc|c} q & \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{smallmatrix} \right) & \left(\begin{smallmatrix} -2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{smallmatrix} \right) & \mathcal{E}_f|_{\mathcal{M}} \\ & \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix} \right) & \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix} \right) & \\ \hline 0: & 0 & 0 & \ker(f) \\ 1: & 0 & \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix} \right)_1 \xrightarrow{f} \left(\begin{smallmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix} \right)_1 \xrightarrow{g} \left(\begin{smallmatrix} -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{smallmatrix} \right)_1^{10} + \text{cok}(f) \\ 2: & 0 & \left(\begin{smallmatrix} -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{smallmatrix} \right)_1^{10} & \Rightarrow 0 \\ 3: & 0 & 0 & \Rightarrow 0 \\ 4: & 0 & 0 & \equiv 0 \\ 5: & 0 & 0 & \equiv 0 \end{array} \quad (\text{A.1.6})$$

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The map f —induced from the one in the Koszul complex in the upper left quadrant of the above chart—is of maximum rank and yields the only non-vanishing differential in this spectral sequence.

This is completely general; consider the resolution sequence (9.5.7), tensored by any bundle \mathcal{V} over the embedding space X . All differential maps d_r acting in the SW (lower left) quadrant of

$$E_r^{q,k}(\mathcal{V}) \xrightarrow{d_r} E_r^{q-r,k-r-1}(\mathcal{V}) \quad (\text{A.1.7})$$

are induced from the corresponding maps $E_k^* \otimes \mathcal{V} \rightarrow E_{k-r-1}^* \otimes \mathcal{V}$ between the irreducible summands in the resolution sequence.

In this simple case, the map

$$f : \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)_1^1 \rightarrow \left(\begin{smallmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)_1^{20} \quad (\text{A.1.8})$$

is represented by mapping an arbitrary complex number $\vartheta \in \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)$ to the (additive, abelian) group of rank-3 covariant symmetric tensors, simply by multiplying ϑ with $f_{abc} : (\vartheta f_{abc}) \in \left(\begin{smallmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)$. Clearly, ϑ is not annihilated in doing this, whence $\ker(f) = 0$ here. On the other hand,

$$\text{cok} \left[\left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)_1^1 \xrightarrow{f} \left(\begin{smallmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)_1^{20} \right] = \left\{ \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)_1^{20} / f \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)_1^1 \right\} \quad (\text{A.1.9})$$

is 19-dimensional and is represented by symmetric rank-3 tensors, modulo arbitrary complex multiples of f_{abc} , $\text{cok}(f) \approx \{\phi_{(abc)} \cong \phi_{(abc)} + \vartheta f_{abc}\}$.

The contribution $\left(\begin{smallmatrix} -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{smallmatrix} \right)_1^{10}$ is represented by symmetric rank-2 symmetric tensors $\varphi_{(ab)} \in (-2000)$, multiplied by an overall factor of $\epsilon^{\alpha\beta\gamma}$, representing the (111) tableau.

Similarly, tensoring by $\mathcal{T}_x = \left(\begin{smallmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \right)$, the resolution sequence (A.1.5) yields

$$\begin{array}{ccc|c} q & \left(\begin{smallmatrix} 3 & 0 & 0 & 1 \\ 3 & 0 & 0 \end{smallmatrix} \right) & \left(\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 3 & 0 & 0 \end{smallmatrix} \right) & \mathcal{T}_x|_{\mathcal{M}} \\ & \left(\begin{smallmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \right) & \left(\begin{smallmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \right) & \\ \hline 0: & 0 & 0 & \left(\begin{smallmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{smallmatrix} \right)_1^{15} \\ 1: & 0 & 0 & \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix} \right)_1 \\ 2: & 0 & \left(\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{smallmatrix} \right)_1^4 & \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix} \right)_1 \\ 3: & 0 & 0 & \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix} \right)_1 \\ 4: & \left(\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{smallmatrix} \right)_1^4 & 0 & 0 \\ 5: & 0 & 0 & 0 \end{array} \quad \begin{array}{l} \xrightarrow{\quad} H^0(\mathcal{M}, \mathcal{T}_x) \\ \xrightarrow{\quad} H^1(\mathcal{M}, \mathcal{T}_x) \\ \xrightarrow{\quad} H^2(\mathcal{M}, \mathcal{T}_x) \\ \xrightarrow{\quad} \Rightarrow 0 \\ \equiv 0 \\ \equiv 0 \end{array} \quad (\text{A.1.10})$$

There clearly are no effective differential maps, so $E_0^{*,*} = E_\infty^{*,*}$ and there is a non-zero contribution to $H^q(\mathcal{M}, \mathcal{T}_x)$ for $q = 0, 1, 2$, while $H^3(\mathcal{M}, \mathcal{T}_x) = 0$.

Compute $H^q(\mathcal{M}, \mathcal{E}_g)$ and $H^q(\mathcal{M}, \mathcal{T}_y)$. \clubsuit

Through the *Künneth formula*^(L), all the information for Seq. (9.1.2) is now ready; in with it. Charting the long exact cohomology sequence under Seq. (9.1.1), we have

$$\begin{array}{ccccc} \mathcal{T}_{\mathcal{M}} & \xrightarrow{i} & \mathcal{T}_{\mathcal{X}}|_{\mathcal{M}} & \xrightarrow{f,g} & \mathcal{E}|_{\mathcal{M}} \\ H^0(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) & \xrightarrow{i_0} & \left(\begin{smallmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{smallmatrix} \right)_1^{15} & \xrightarrow{\quad f \quad} & \left[\left(\begin{smallmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix} \right)_1^{20} / f \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix} \right)_1 \right] \\ & & \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{smallmatrix} \right)_8^1 & \xrightarrow{\quad g \quad} & \left[\left(\begin{smallmatrix} -1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 1 \end{smallmatrix} \right)_0^{10} / g \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix} \right)_1 \right] \\ H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) & \xrightarrow{i_1} & \left(\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix} \right)_1^4 & \xrightarrow{\quad f \quad} & \left(\begin{smallmatrix} -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{smallmatrix} \right)_1^{10} \\ H^2(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) & \xrightarrow{i_2} & 2 \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix} \right)_1^1 & & 0 \\ H^3(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) & \xrightarrow{i_3} & 0 & & 0 \end{array} \quad (\text{A.1.11})$$

Clearly, $H^3(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ vanishes. On general grounds (§ 1.2), we also know that $H^0(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) = 0$, but this can be computed here explicitly.

Relying on the exactness of the long cohomology sequence, we see that it is sufficient to discern the action of the maps labeled by f and g to determine $H^q(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ completely. In particular, denoting the combined triple map by j_0 , and the f -map in $q = 1$ by j_1 , we have

$$H^0(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \approx \ker(j_0), \quad (\text{A.1.12})$$

$$H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \approx \ker(j_1) + \text{cok}(j_0), \quad (\text{A.1.13})$$

$$H^2(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \approx 2 \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix} \right)_1 + \text{cok}(j_1). \quad (\text{A.1.14})$$

To be pedantic, recall that ‘+’ denotes *extension*^(L), not a holomorphic direct sum although this does not seem to have surfaced in any of the physically relevant computations.

Most notably, determining the action of maps in a long exact cohomology sequence (or more generally, in the spectral sequences) is *in general* a tough problem; consequently, so is the computation of $H^q(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$. However, if the Bott-Borel-Weil theorem 9.2 can be applied as we do it here and as can be done for all complete intersections in products of generalized flag spaces, the computation becomes an easy (albeit possibly lengthy) exercise in linear algebra. Moreover, as

A.1. Computing Tangent Bundle Valued Forms

we will discuss at the end of this volume, it also links into some modern techniques in quantum field theories.

Now, the $q = 1$ single map corresponds to

$$j_1 : f_{abc}(\epsilon^{\alpha\beta\gamma} \lambda^c) \mapsto (\epsilon^{\alpha\beta\gamma} \theta_{(ab)}). \quad (\text{A.1.15})$$

Clearly, no component of the 4-vector λ^c is annihilated, so $\ker j_1 = 0$. However,

$$\text{cok}(j_1) \approx \{ \epsilon^{\alpha\beta\gamma} \theta_{(ab)} \cong \epsilon^{\alpha\beta\gamma} \theta_{(ab)} + (f_{abc} \epsilon^{\alpha\beta\gamma} \lambda^c) \}. \quad (\text{A.1.16})$$

and so $H^2(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \approx H^{2,2}(\mathcal{M})$ is the extension

$$H^2(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) = \{ (*J_x), (*J_y) \} + \{ \epsilon^{\alpha\beta\gamma} \theta_{(ab)} \cong \epsilon^{\alpha\beta\gamma} \theta_{(ab)} + (f_{abc} \epsilon^{\alpha\beta\gamma} \lambda^c) \}. \quad (\text{A.1.17})$$

By $(*J_x)$ and $(*J_y)$ we of course mean the (2,2)-classes dual to the pullbacks to \mathcal{M} of the Kähler (1,1)-classes of \mathbb{P}^3 and \mathbb{P}^2 , respectively. Straightforwardly then,

$$H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}^*) = \{ J_x, J_y \} + \{ \theta^{(ab)} \epsilon^{cdef} : f_{abc} \theta^{(ab)} = 0 \}. \quad (\text{A.1.18})$$

We have used here that

$$(a, b, \dots, d) \xrightarrow{\text{Serre-duality}} (1-d, \dots, 1-b, 1-a), \quad (\text{A.1.19})$$

because $\mathcal{K}_{\mathbb{P}^n}^* = (1|1, \dots, 1)$. (See also *exact sequence*^(L).)

At $q = 0$, we have the combined map

$$j_0 : \begin{array}{ccc} \left(\begin{smallmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{smallmatrix} \right)_1^{15} & \xrightarrow{\quad f \quad} & \left[\left(\begin{smallmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix} \right)_1^{20} / f \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix} \right)_1 \right] \\ \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{smallmatrix} \right)_8^1 & \xrightarrow{\quad g \quad} & \left[\left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{smallmatrix} \right)_8^{10} / g \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix} \right)_1 \right] \end{array}, \quad (\text{A.1.20})$$

represented by

$$\begin{array}{ccc} \{ \lambda_a^b \} & \xrightarrow{\quad f \quad} & \{ (\lambda_{(a}^d f_{bc)d} \} \\ \{ \lambda_\alpha^\beta \} & \xrightarrow{\quad g \quad} & \{ (\lambda_a^d g_{d\alpha\beta\gamma}) + (g_{a\delta(\alpha\beta} \lambda_{\gamma)\delta}) \} \end{array} . \quad (\text{A.1.21})$$

>Show that $H^0(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) = \ker(j_0) = 0$. (Caution: in addition to showing that no element of the domain is annihilated in contraction with f_{abc} and $g_{a\alpha\beta\gamma}$, it must be shown that only zero is mapped to the null element(s) of the target, which is a *quotient*.) \clubsuit

The cokernel is simple to express upon noticing that, for example, $f\left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$ and $f\left(\begin{smallmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$ may be combined into f acting on an arbitrary (not traceless) 4×4 matrix. Since $\ker(j_1) = 0$, $H^1(\mathcal{M}, T_{\mathcal{M}}) = \text{cok}(j_0)$ and so

$$H^1(\mathcal{M}, T_{\mathcal{M}}) = \left\{ \begin{array}{l} \{\phi_{(abc)} \cong \phi_{(abc)} + (\lambda_{(a}{}^d f_{bc)d})\} \\ \{\varphi_{a(\alpha\beta\gamma)} \cong \varphi_{a(\alpha\beta\gamma)} + (\lambda_{(a}{}^d g_{d\alpha\beta\gamma}) + (g_{a\delta(\alpha\beta} \lambda_{\gamma)\delta})\} \end{array} \right\} \quad (\text{A.1.22})$$

where $\lambda_a{}^b$ and $\lambda_{\alpha}{}^{\beta}$ are now arbitrary invertible 4×4 and 3×3 complex matrices.

A.2 Computing Cotangent Bundle Valued Forms

To compute $H^q(\mathcal{M}, T_{\mathcal{M}}^*)$, we consider the dual of Seq. (9.1.1),

$$0 \rightarrow \mathcal{E}^*|_{\mathcal{M}} \xrightarrow{\nabla\xi} T_{\mathcal{M}}^* \xrightarrow{i} T_{\mathcal{M}}^* \rightarrow 0. \quad (\text{A.2.1})$$

In view of *Serre duality*^(L), this is of course an unnecessary exercise if \mathcal{M} is Calabi-Yau, since then

$$H^q(\mathcal{M}, T_{\mathcal{M}}^*)^* \approx H^{n-q}(\mathcal{M}, T_{\mathcal{M}} \otimes \mathcal{K}_{\mathcal{M}}) \approx H^{n-q}(\mathcal{M}, T_{\mathcal{M}}). \quad (\text{A.2.2})$$

In the general case however, when $\mathcal{K}_{\mathcal{M}} \not\approx \mathbb{C}$, the computation of $H^q(\mathcal{M}, T_{\mathcal{M}}^*)$ provides the *Hodge numbers*^(L) and the standard (p, q) -cohomology ring. On the other hand, $H^q(\mathcal{M}, T_{\mathcal{M}})$ provides information on deformations of the complex structure : $H^1(\mathcal{M}, T_{\mathcal{M}})$ spans the local deformations.

To determine $H^q(\mathcal{M}, T_{\mathcal{M}}^*)$ from the long exact cohomology sequence associated to Seq. (A.2.1), we need to fill in the blanks for $H^q(\mathcal{M}, T_{\mathcal{X}}^*)$ and $H^q(\mathcal{M}, \mathcal{E}^*)$. This information (see Table A.1) is obtained from the resolution (A.1.5), upon tensoring with the irreducible components of

$$T_{\mathcal{X}}^* = \left(\begin{smallmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{smallmatrix}\right), \quad \mathcal{E}^* = \left(\begin{smallmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{smallmatrix}\right). \quad (\text{A.2.3})$$

A.2. Computing Cotangent Bundle Valued Forms

Bundle	Non-Zero Cohomology on \mathcal{M}
$\left(\begin{smallmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$	$H^1 = \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)_1^1, \quad H^2 = \left(\begin{smallmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)_1^4, \quad H^3 = \left(\begin{smallmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right)_1^{15}$
$\left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{smallmatrix}\right)$	$H^1 = \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)_1^1, \quad H^3 = \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{smallmatrix}\right)_8^1$
$\left(\begin{smallmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$	$H^2 = \left(\begin{smallmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)_1^{10}, \quad H^3 = [\ker\left(\begin{smallmatrix} 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right)_1^{20} \xrightarrow{f} \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right)_1^1]_{19}$
$\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{smallmatrix}\right)$	$H^3 = [\ker\left(\begin{smallmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 4 \end{smallmatrix}\right)_8^1 \xrightarrow{g} \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right)_1^1]_{39}$

Table A.1: The non-zero cohomology on \mathcal{M} , required for $H^q(\mathcal{M}, T_{\mathcal{M}}^*)$.

Verify the results in Table A.1.

With this information in the long exact cohomology sequence accompanying Seq. (A.2.1), we have

$$\begin{array}{ccccccc} \mathcal{E}^*|_{\mathcal{M}} & \xrightarrow{f,g} & T_{\mathcal{X}}^*|_{\mathcal{M}} & \xrightarrow{i} & T_{\mathcal{M}}^* & & \\ \hline 0 & & 0 & \xrightarrow{i_0} & H^0(\mathcal{M}, T_{\mathcal{M}}^*) & & \\ 0 & & 2\left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)_1^1 & \xrightarrow{i_1} & H^1(\mathcal{M}, T_{\mathcal{M}}^*) & & \\ \left(\begin{smallmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)_1^{10} & \xrightarrow{f} & \left(\begin{smallmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)_1^4 & \xrightarrow{i_2} & H^2(\mathcal{M}, T_{\mathcal{M}}^*) & & (\text{A.2.4}) \\ \left[\ker\left(\begin{smallmatrix} 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right)_1^{20} \xrightarrow{f} \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right)_1^1\right] & \xrightarrow{f} & \left(\begin{smallmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right)_1^{15} & \xrightarrow{i_0} & H^3(\mathcal{M}, T_{\mathcal{M}}^*) & & \\ \left[\ker\left(\begin{smallmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 4 \end{smallmatrix}\right)_8^1 \xrightarrow{g} \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right)_1^1\right] & \xrightarrow{g} & \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{smallmatrix}\right)_8^1 & & & & \end{array}$$

Comparing with (A.1.11), we see Serre duality at work. We can again analyze the action of each of the indicated maps $j_q : H^q(\mathcal{M}, \mathcal{E}^*) \xrightarrow{f,g} H^q(\mathcal{M}, T_{\mathcal{X}}^*)$, just as we did for $H^q(\mathcal{M}, T_{\mathcal{M}})$.

It is easy to re-derive Eq. (A.1.18) from (A.2.4). Also,

$$H^2(\mathcal{M}, T_{\mathcal{M}}^*) = \left\{ \begin{array}{l} \{\epsilon^{a_1 a_2 a_3 a_4} \epsilon^{a_1 a_2 a_3} \phi^{(bcd)} : f_{abc} \phi^{(bcd)} = 0\} \\ \{\epsilon^{a_1 a_2 a_3 a_4} \epsilon^{a_1 a_2 a_3} \varphi^{b(\beta\gamma\delta)} : g_{b\alpha\beta\gamma} \varphi^{b(\beta\gamma\delta)} = 0\} \end{array} \right\} \quad (\text{A.2.5})$$

where the map

$$\left[\ker\left(\begin{smallmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 4 \end{smallmatrix}\right)_8^1 \xrightarrow{g} \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right)_1^1\right] \xrightarrow{g} \left(\begin{smallmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right)_1^{15} \quad (\text{A.2.6})$$

was chosen ineffective while the other two are effective. This freedom is exactly the counterpart of the freedom we have in choosing representatives for the quotient (A.1.22). Clearly, there are many different but equivalent choices which must be born in mind in calculations with cohomology groups.

A.3 Relation to Polynomial Deformation

Tracing the calculation in the preceding sections, we see that there is always a contribution to $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ essentially of the form

$$\left\{ H^0(\mathcal{X}, \mathcal{E}) / [H^1(\mathcal{X}, \mathcal{E}) \oplus H^0(\mathcal{X}, \mathcal{T}_{\mathcal{X}})] \right\}. \quad (\text{A.3.1})$$

Now, elements of $H^0(\mathcal{X}, \mathcal{E})$ are just the sections of \mathcal{E} , usually polynomials of the homogeneity of the defining constraints. As we have seen above, the quotient with respect to $H^1(\mathcal{X}, \mathcal{E})$ means taking such sections modulo additive complex multiplets of the defining sections. The quotient with respect to $H^0(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$ is precisely the quotient with respect to linear reparametrizations.

In full generality, consider a configuration $[\mathcal{X} \parallel \mathcal{E}]$ where \mathcal{X} is the embedding space and \mathcal{E} is the bundle the sections of which are used to define $\mathcal{M} \in [\mathcal{X} \parallel \mathcal{E}]$ (not necessarily as a complete intersection). Then we define the space of ‘polynomial deformations’ to be

$$\mathfrak{d}[\mathcal{X} \parallel \mathcal{E}] \stackrel{\text{def}}{=} \left\{ \frac{H^0(\mathcal{X}, \mathcal{E})}{\text{Aut}(\mathcal{X}) \times \text{Aut}(\mathcal{E})} \right\}, \quad (\text{A.3.2})$$

where $\text{Aut}(\mathcal{X})$ is the group of continuous automorphisms^(L) (symmetries) of \mathcal{X} and $\text{Aut}(\mathcal{E})$ is the group of continuous automorphisms of \mathcal{E} . $\text{Aut}(\mathcal{X})$ is usually just the group of coordinate reparametrizations of \mathcal{X} . $\text{Aut}(\mathcal{E})$, on the other hand, may on occasion become pretty involved. Even in a simple example, $\mathcal{M} \in [5 \parallel 24]$ where $\mathcal{E} = \mathcal{O}(2) \oplus \mathcal{O}(4)$, $\text{Aut}(\mathcal{E})$ is generated by adding a complex multiple of the two defining polynomials to quadric and quartic monomials, respectively, but also by adding a quadric multiple of the first defining polynomial (the quadric) to quartic monomials. Without the spectral sequence to systematize all possible maps, accounting for all these can get out of hand rather quickly.

Depending on the relation between \mathcal{E} and the normal bundle $\mathcal{N}_{\mathcal{X}/\mathcal{M}}$ (A.1.1), the space of ‘polynomial deformations’ (A.3.2) is related to $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$. For complete intersections, $\mathcal{E} = \mathcal{N}_{\mathcal{X}/\mathcal{M}}$ and in favorable cases moreover, $\mathfrak{d}[\mathcal{X} \parallel \mathcal{E}]$ is equal to $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ (see below).

A.3.1 Polynomial deformations

Clearly, arbitrary polynomials (more generally, sections) of the homogeneity of the defining polynomials (sections)—modulo the above ideal—may be considered as effective deformations of the defining polynomials (sections). So, a subset

A.3. Relation to Polynomial Deformation

(possibly all) of $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ may be spanned by effective deformations of the defining polynomials; this was derived in Ref. [84] using standard tensor calculus. Now, general theory [35] shows that $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ —for any complex manifold—spans the local deformations of the complex structure on \mathcal{M} . On the other hand, effective deformations of the defining constraints span the local deformations of the embedding $\mathcal{M} \hookrightarrow \mathcal{X}$. In fortunate cases, these two different notions of deformation coincide and polynomial deformations in fact do span $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ completely and with no redundancy.

Having gone through a generally valid calculation of $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$, we see that polynomial deformations may in general be both incomplete and redundant. More precisely, the above cohomology computation involves

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \downarrow & \downarrow & \downarrow \\ H^0(\mathcal{X}, \mathcal{T}_{\mathcal{X}}) & H^0(\mathcal{X}, \mathcal{E}) & H^1(\mathcal{X}, \mathcal{T}_{\mathcal{X}}) \\ \downarrow \varrho_0^{\mathcal{T}_{\mathcal{X}}} & \downarrow \varrho_0^{\mathcal{E}} & \downarrow \varrho_1^{\mathcal{T}_{\mathcal{X}}} \\ H^0(\mathcal{M}, \mathcal{T}_{\mathcal{X}}) \rightarrow H^0(\mathcal{M}, \mathcal{E}) \rightarrow H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \xrightarrow{i_0} H^1(\mathcal{M}, \mathcal{T}_{\mathcal{X}}) \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & \vdots \end{array} \quad (\text{A.3.3})$$

Clearly, the restrictions $\varrho_0^{\mathcal{T}_{\mathcal{X}}}$ and $\varrho_0^{\mathcal{E}}$ are in general *not* 1–1, that is, there will in general be elements of $H^0(\mathcal{M}, \mathcal{T}_{\mathcal{X}})$ and $H^0(\mathcal{M}, \mathcal{E})$ which do not arise as restrictions of $H^0(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$ and $H^0(\mathcal{X}, \mathcal{E})$, respectively. So, by approximating

$$\left\{ H^0(\mathcal{M}, \mathcal{E}) / H^0(\mathcal{M}, \mathcal{T}_{\mathcal{X}}) \right\} \rightsquigarrow \left\{ H^0(\mathcal{X}, \mathcal{E}) / H^0(\mathcal{X}, \mathcal{T}_{\mathcal{X}}) \right\}, \quad (\text{A.3.4})$$

we may be undercounting the ‘numerator’ or the ‘denominator’, or even both of them. Finally, *in principle*, there is no reason for the map i_0 to be null. So, some elements of $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ may have nothing to do with elements of the quotients (A.3.4), but could instead be the preimages under i_0 of certain elements in $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{X}})$. (Caution, again: $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{X}})$ cannot be simply replaced by $H^1(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$, the restriction $\varrho_1^{\mathcal{T}_{\mathcal{X}}}$ need not be 1–1.)

It is, however, an *experimental* fact, observed upon a case by case computation, that for complete intersection Calabi-Yau 3-folds in products of complex projective spaces* i_0 does vanish [148]. It does not appear to be known why this

*Clearly, only a finite list was actually explicitly searched. In view of certain identities subsumed here in Lemma 2.2, a list of about 8000 configurations^(L) contains all topological types [85] and this is the list which has been checked.

is so and how generally this is true.

Even with $i_0 = 0$, however, much room remains for polynomial deformations to fail in providing a correct representation of $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$. On the other hand, if it can be ascertained that $\varrho_0^{\mathcal{T}_{\mathcal{X}}}$ and $\varrho_0^{\mathcal{E}}$ in fact are injective maps^(L), they are also isomorphisms and the polynomial deformation interpretation gives us a handy computational tool. It is indeed a felicitous twist of Fate that polynomial deformations correctly represent $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ for the three manifolds (8.0.1)—those which attracted the most physicists' attention.

For complete intersections in products of complete projective spaces, a simple criterion for redundancy of $\mathfrak{d}[\mathcal{X} \parallel \mathcal{E}]$ has been derived [146] :

1. If there is no decomposing 1-leg (see 1-leg-decomposable^(L)) in the diagram of $[\mathcal{X} \parallel \mathcal{E}]$, then $E_0^{k+1,k}(\mathcal{T}_{\mathcal{X}}) = 0$ and $E_0^{q,k}(\mathcal{T}_{\mathcal{X}}) = E_{\infty}^{q,k}(\mathcal{T}_{\mathcal{X}})$. So,

$$H^0(\mathcal{M}, \mathcal{T}_{\mathcal{X}}) = \sum_{k=0}^K E_0^{k,k}(\mathcal{T}_{\mathcal{X}}), \quad \text{and} \quad H^2(\mathcal{M}, \mathcal{T}_{\mathcal{X}}) = \sum_{k=0}^K E_0^{k+2,k}(\mathcal{T}_{\mathcal{X}}), \quad (\text{A.3.5})$$

while $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{X}}) = 0 = H^3(\mathcal{M}, \mathcal{T}_{\mathcal{X}})$.

2. If, moreover, there is a decomposing 0-leg, $H^0(\mathcal{M}, \mathcal{T}_{\mathcal{X}})$ has a contribution other than $E_0^{0,0}(\mathcal{T}_{\mathcal{X}}) = H^0(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$ which cannot be canceled out since $E_0^{k+1,k}(\mathcal{T}_{\mathcal{X}}) = 0$ and so $\mathfrak{d}[\mathcal{X} \parallel \mathcal{E}]$ is ineffective. That is, there are equivalence relations (and non-trivial elements also, as it turns out) amongst elements of $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ which are not accounted for by Eq. (A.3.2).

There is unfortunately no such simple “selection rule” for the \mathcal{E} -valued cohomology. Similarly, when the embedding space contains factors other than complex projective spaces, the above criteria become substantially more complicated.

For the manifold (A.1.2), the polynomial deformations are spanned by

$$\begin{bmatrix} \phi_{(abc)} x^a x^b x^c \\ \varphi_{a(\alpha\beta\gamma)} x^a y^{\alpha} y^{\beta} y^{\gamma} \end{bmatrix} \cong \begin{bmatrix} \phi_{(abc)} x^a x^b x^c \\ \varphi_{a(\alpha\beta\gamma)} x^a y^{\alpha} y^{\beta} y^{\gamma} \end{bmatrix} + (x^a \lambda_a{}^b \partial_b + y^{\alpha} \lambda_{\alpha}{}^{\beta} \partial_{\beta}) \begin{bmatrix} f(x) \\ g(x, y) \end{bmatrix}. \quad (\text{A.3.6})$$

It is easy to identify this with the quotient (A.1.22), through contracting the latter with x 's and y 's. For example,

$$\phi_{(111)} \rightarrow (x^1)^3, \quad \phi_{(123)} \rightarrow x^1 x^2 x^3, \quad \text{and so on...} \quad (\text{A.3.7})$$

A.3. Relation to Polynomial Deformation

The relationship between the tensor coefficients such as $\phi_{(abc)}$ and monomials is another type of duality^(L), in that for every component $\phi_{(abc)}$, for some fixed a, b, c , there is a unique monomial $x^a x^b x^c$ which contracts with $\phi_{(abc)}$ into $\phi(x)$.

Notation:

We adhere to the standard relativity notation in which coordinates are contravariant vectors (their components having upper indices), at the risk of clumsiness when it comes to powers of coordinates. Keeping consistently to this convention essentially identifies the duality between (totally symmetric) tensor coefficients vs. coordinate monomials with the Einstein summation convention and clarifies much of the calculations as usual in tensor calculus.

This relation shows that the cohomology computation discussed above may be considered as a generalization of the polynomial deformations—if nothing else than at least in that all types of tensors can be discussed not only the totally symmetric ones which correspond to coordinate monomials. Also, through contraction with coordinates, we can translate the generally valid results of the above cohomology computations into the polynomial deformation type language which is better known to the physics audience and is closely related to certain recent results of quantum field theory, as we will see at the end of this volume.

A.3.2 A residue formula

Given the tensor coefficient vs. coordinate monomial duality, all results obtained for polynomial deformations have an immediate translation into the tensor language. For example, consider the spectral sequence accompanying the resolution (A.1.5). We have

q	$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\mathcal{O}_{\mathcal{M}}$
0:	0	0	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}_1^1$	$\rightarrow H^0(\mathcal{M}, \mathbb{C})$
1:	0	0	0	$H^1(\mathcal{M}, \mathbb{C})$
2:	0	0	0	$H^2(\mathcal{M}, \mathbb{C})$
3:	0	0	0	$\rightarrow H^3(\mathcal{M}, \mathbb{C})$
4:	0	0	0	$\equiv 0$
5:	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_1^1$	0	0	$\equiv 0$

(A.3.8)

This provides a single generator of $H^0(\mathcal{M}, \mathbb{C})$, a scalar $\vartheta \in \binom{0}{0} \otimes \binom{0}{0}$. Similarly, there is a single generator for

$$H^3(\mathcal{M}, \mathbb{C}) = H^{0,3}(\mathcal{M}) \ni \epsilon^{abcd}\epsilon^{\alpha\beta\gamma} \in \binom{1}{1} \otimes \binom{1}{1} \otimes \binom{1}{1}. \quad (\text{A.3.9})$$

being contravariant themselves, these cannot be contracted with coordinates. However, $\epsilon^{abcd}\epsilon^{\alpha\beta\gamma}$ are clearly dual to $\epsilon_{abcd}\epsilon_{\alpha\beta\gamma}$, which in turn can be contracted with x 's and y 's. The ϵ -tensor densities being totally antisymmetric, there is only one possibility :

$$\epsilon^{abcd} \stackrel{*}{\sim} \epsilon_{abcd} \stackrel{*}{\sim} x^a dx^b dx^c dx^d. \quad (\text{A.3.10})$$

The double duality provides us with a natural identification

$$\epsilon^{abcd} \approx x^a dx^b dx^c dx^d \stackrel{\text{def}}{=} (xd^3x) \quad (\text{A.3.11})$$

since they are both dual to ϵ_{abcd} . (Similarly for $\epsilon^{\alpha\beta\gamma}$)

Recall now that standard tensor calculus methods yield [84]

$$H^{3,0}(\mathcal{M}) \ni \Omega \stackrel{\text{def}}{=} \oint_{\Gamma(f)} \oint_{\Gamma(g)} \frac{\epsilon_{abcd} x^a dx^b dx^c dx^d \epsilon_{\alpha\beta\gamma} y^\alpha dy^\beta dy^\gamma}{f(x)g(x,y)}. \quad (\text{A.3.12})$$

Here the contour integrals are along circuits $\Gamma(f)$ and $\Gamma(g)$ in $\mathbb{P}^3 \times \mathbb{P}^2$ and go around the hypersurfaces $f(x) = 0$ and $g(x, y) = 0$, respectively. The above double contour integral would vanish were it not for the division by $f(x)$ and $g(x, y)$. These create poles at the $f(x) = 0$ and $g(x, y) = 0$ hypersurfaces, rendering the double integral non-zero precisely on the common zero-set, that is on \mathcal{M} . Also, the double integral is needed to convert the holomorphic 5-form $[(xd^3x)(yd^2y)]$ into the holomorphic 3-form of Calabi-Yau manifolds[†]. The integrals are most easily evaluated using the residue formula.

Comparing Eqs. (A.3.12) and (A.3.9), we see that the duality between $H^{3,0}$ and $H^{0,3}$ reflects itself in the contravariancy of the tensor coefficients

$$\epsilon^{abcd}\epsilon^{\alpha\beta\gamma} \quad \text{vs.} \quad \epsilon_{abcd}\epsilon_{\alpha\beta\gamma}, \quad (\text{A.3.13})$$

respectively.

In general, therefore, there is an integral formula similar to Eq. (A.3.12) for all cohomology representatives obtained using the Koszul complex and spectral

[†]So, the criteria for Ricci-flatness in § 9.5 are in fact tailored to construct the non-zero holomorphic 3-form Ω via an integral formula like Eq. (A.3.12).

sequences. However, the tensor coefficients alone are usually sufficient to recover this integral expression completely.

One may ask if there are general formulae of the type (L.15) for arbitrary bundles over an n -fold X . In general, the answer is ‘no’. However, the situation is much more favorable for holomorphic tensor bundles, $T_s^r \stackrel{\text{def}}{=} (\otimes^r T_X \otimes \otimes^s T_X^*)$, because a local coordinate system x^μ at a point $p \in X$ determines a natural frame ∂_μ for the holomorphic tangent bundle, T_X , and dz^μ for the holomorphic cotangent bundle, T_X^* . At a point $p \in X$, a holomorphic vector field v can then be written as

$$v_p(x) = x^\mu v_\mu^\nu \partial_\nu + \text{higher order terms}, \quad (\text{A.3.14})$$

and let $v_p = [v_\mu^\nu]_p$. Under a local change of coordinates, v_p is determined up to conjugation. Therefore, an *invariant polynomial*^(L) $P^k(v_p)$ is an invariant of v and p and we expect this polynomial to carry global information. Indeed, we have the

Theorem A.1 (The Bott Residue Formula) *Let Θ be a curvature 2-form for the holomorphic tangent bundle on a complex manifold X and v a global vector field over X . Then*

$$\int_M P\left(\frac{i}{2\pi}\Theta\right) = \sum_{v(p)=0} \frac{P(v_p)}{\det(v_p)}, \quad (\text{A.3.15})$$

for any invariant polynomial $P(v)$ of degree $\dim X$.

Very much like the *Hopf Index Theorem*^(L), this allows us to replace certain integrals over X with finite sums over the set of zeros of a global (holomorphic) vector field. In conjunction with the *Lefschetz fixed-point formulae*^(L), it is often possible to further relate such integrals to traces of the action of certain symmetries of X over some cohomology on X .

A.4 Sequencing Sequences

In retrospect, the computation of $H^q(\mathcal{M}, T_{\mathcal{M}})$ and also of $H^q(\mathcal{M}, T_{\mathcal{M}}^*)$ described above involves computing several intermediate cohomology groups which we do not really need in the end. One wonders if it is possible to avoid that, somewhat like collapsing the square of short exact sequences into a spectral sequence described in § 9.3.2 and § 9.3.3.

Consider a (not necessarily Calabi-Yau) hypersurface $\mathcal{M} \subset X$, defined as the zero-set of a section $f(x)$ of a line bundle \mathcal{E} over X ; $\mathcal{M} = f^{-1}(0)$. The computation of $H^q(\mathcal{M}, T_{\mathcal{M}}^*)$, as described above, relies on the diagram

$$\begin{array}{ccccc} & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ (\mathcal{E}^*)^{\otimes 2} & \dashrightarrow & T_X^* \otimes \mathcal{E}^* & \dashrightarrow & \\ & \downarrow & & \downarrow & \\ \mathcal{E}^* & & T_X^* & & \\ & \downarrow & & \downarrow & \\ 0 \rightarrow & \mathcal{E}^*|_{\mathcal{M}} & \rightarrow & T_X^*|_{\mathcal{M}} & \rightarrow T_{\mathcal{M}}^* \rightarrow 0 \\ & \downarrow & & \downarrow & \\ & 0 & & 0 & \end{array} \quad (\text{A.4.1})$$

in which the column-sequences are sheaf-exact over X and the row-sequence is exact on \mathcal{M} . Superficially, this looks almost like the diagram (9.3.14), in that there is a mesh of long exact cohomology sequences in which we input the cohomology of the bundles framed by the dotted square. From these, we determine the cohomology of the bundles directly beneath the dotted square and this then leads to the required cohomology of the bundle at the lower right. Given the formal similarity *in usage*, one expects that that the diagram (A.4.1) can be collapsed to a single (sheaf-) exact sequence, something like

$$0 \rightarrow \mathcal{E}^{*\otimes 2} \rightarrow \bigoplus_{\mathcal{E}^*} T_X^* \xrightarrow{\ell} T_{\mathcal{M}}^* \rightarrow 0. \quad (\text{A.4.2})$$

This would then induce a spectral sequence just like (9.3.14) is collapsed to the exact sequence (9.3.7), which then induces the wanted spectral sequence.

There is, however, a major difference between the diagram (9.3.14) and the (A.4.1) one. Namely, while the former is naturally completed into a 3×3

A.4. Sequencing Sequences

square of short exact sequences, the diagram (A.4.1) cannot be complete in such a way. To begin with, there is no natural map $\mathcal{E}^* \rightarrow T_X^*$. Of course, restricted to \mathcal{M} , the map $(\nabla f)(x) : \mathcal{E}^*|_{\mathcal{M}} \rightarrow T_X^*|_{\mathcal{M}}$ is natural, since it is independent of the choice of the connection

$$(\nabla f)(x) = (df)(x) + (\Gamma \cdot f)(x) = df(x) \quad (\text{A.4.3})$$

on \mathcal{M} . Away from \mathcal{M} in X , however, ∇f indeed depends on the choice of the connection and so does not provide a well defined bundle map. Thus, there is no well defined map $\mathcal{E}^{*\otimes 2} \rightarrow T_X^* \otimes \mathcal{E}^*$ either and the sequence (A.4.2) fails to be exact in any sense.

However, when X is a Grassmannian $G_{(k,n)}$ of k -planes in \mathbb{C}^n (including the simpler case of projective spaces, $\mathbb{P}^n = G_{(1,n+1)}$), there is a straightforward remedy for this. The two off-diagonal bundles in the dotted rectangle of diagram (A.4.1) are connected by a short exact sequence, known as the *tautological sequence*^(L); tensoring the dual of Seq. (L.63) by $\mathcal{E}^* \otimes \mathcal{S}$, we have that*

$$0 \rightarrow T_X^* \otimes \mathcal{E}^* \rightarrow \mathcal{S}^{\otimes n} \otimes \mathcal{E}^* \rightarrow \mathcal{E}^* \rightarrow 0 \quad (\text{A.4.4})$$

is exact. This fits in the diagram (A.4.1), in the NE-SW (2 o'clock–8 o'clock) direction and induces the NW-SE (10 o'clock–4 o'clock) sheaf-exact sequence threaded in as shown below

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \dashrightarrow & (\mathcal{E}^*)^{\otimes 2} & \dashrightarrow & T_X^* \otimes \mathcal{E}^* & \dashrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{E}^* & & T_X^* & & \\ & & \downarrow & & \downarrow & & \\ 0 & \dashrightarrow & \mathcal{E}^*|_{\mathcal{M}} & \dashrightarrow & T_X^*|_{\mathcal{M}} & \dashrightarrow & T_{\mathcal{M}}^* \dashrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \quad (\text{A.4.5})$$

The exactness of the ‘diagonal’ (NW-SE) sequence relies on the general fact that, given the solid arrows as indicated in the diagram

$$\begin{array}{ccc} A & \xrightarrow{1-1} & B \\ \downarrow 1-1 & & \swarrow \text{onto} \\ C & \xrightarrow{\text{onto}} & B \end{array} \quad (\text{A.4.6})$$

*For $G_{(1,n+1)} = \mathbb{P}^n$, $\mathcal{S} = \mathcal{O}_{\mathbb{P}^n}(-1)$.

we can always recover the dotted map $A \rightarrow B$ to have no kernel and such that the diagram be commutative[†]. To see this, recall that if $B \xrightarrow{\alpha} C$ is onto then $C = \{B/\ker(\alpha)\}$. Now, the map $A \rightarrow C$ naturally implies a map to the numerator B of the quotient $C = \{B/\ker(\alpha)\}$. It can have no kernel, for if some non-zero element of A would be annihilated in $A \rightarrow B$, it would be mapped to $0 \in C$ by $B \rightarrow C$. From commutivity, this would mean that the same non-zero element is also annihilated in $A \rightarrow C$, which contradicts the assumption that $A \rightarrow C$ is 1-1 (has no kernel).

This explains the map $\mathcal{E}^{*\otimes 2} \xrightarrow{1-1} \mathcal{S}^{\otimes n} \otimes \mathcal{E}^*$ in the diagram (A.4.5), for $f(x) \neq 0$, that is, away from \mathcal{M} in \mathcal{X} . Along \mathcal{M} , the map $\mathcal{E}^{*\otimes 2} \rightarrow \mathcal{E}^*$ is zero and so is then $\mathcal{E}^{*\otimes 2} \xrightarrow{1-1} \mathcal{S}^{\otimes n} \otimes \mathcal{E}^*$. The rest of the ‘diagonal’ sequence follows by similar arguments and we obtain the sheaf-exact sequence

$$0 \longrightarrow \mathcal{E}^{*\otimes 2} \longrightarrow \mathcal{S}^{\otimes n} \otimes \mathcal{E}^* \longrightarrow \mathcal{T}_{G_{(k,n)}}^* \xrightarrow{\ell} \mathcal{T}_{\mathcal{M}}^* \longrightarrow 0, \quad (\text{A.4.7})$$

which induces a spectral sequence very much like Seq. (9.3.7) induces the spectral sequence with the 0th level (9.3.16).

This spectral sequence is then analyzed just as that in § 9.3.3 and, of course, yields the same result. The benefit lies in the fact that we need to compute the cohomology of only three bundles over \mathcal{X} , namely $\mathcal{E}^{*\otimes 2}$, $\mathcal{S}^{\otimes n} \otimes \mathcal{E}^*$ and $\mathcal{T}_{G_{(k,n)}}^*$, instead of four bundles over \mathcal{X} and two auxiliary ones over \mathcal{M} , as required in the computation of § A.1. The generalization to complete intersections and/or flag varieties with more than two ‘stripes’, $\mathbb{F}_{[n_1, \dots, n_f]}$ with $f > 2$, the result is less simple and we forego an analysis here.

Chapter B

Other Tangent Bundle Related Cohomology

There are many other cohomology groups, besides $H^q(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ and $H^q(\mathcal{M}, \mathcal{T}_{\mathcal{M}}^*)$, which we can easily access using the Koszul complex and related spectral sequence. Recalling the immediate physical application (see Table 0.2), we describe here the computation of $H^q(\mathcal{M}, \text{End } \mathcal{T}_{\mathcal{M}})$, as a simple and physically relevant case of the more general $H^q(\mathcal{M}, \mathcal{T}_{\mathcal{M}}^{\otimes r} \otimes \mathcal{T}_{\mathcal{M}}^{*\otimes s})$, for all of which the present techniques are valid.

B.1 Some General Facts

As noted in Chapter 0, by $\text{End } \mathcal{T}_{\mathcal{M}}$ (more generally, $\text{End } \mathcal{V}$) we mean the bundle of traceless *endomorphisms*^(L) of the holomorphic tangent bundle $\mathcal{T}_{\mathcal{M}}$ (arbitrary vector bundle \mathcal{V}). Given a local frame for $\mathcal{T}_{\mathcal{M}}$, $\varphi^\mu(x)$, $\text{End } \mathcal{T}_{\mathcal{M}}$ acts on it by simple matrix multiplication,

$$\vartheta^\mu{}_\nu(x) : \varphi^\nu(x) \longmapsto \varphi^\mu(x) = (\vartheta^\mu{}_\nu \varphi^\nu)(x), \quad (\text{B.1.1})$$

where $\vartheta^\alpha{}_\beta(x)$ is local and (Hermitian) matrix-valued. Clearly, $\vartheta^\mu{}_\nu \in \text{End } \mathcal{T}_{\mathcal{M}}$ belongs to the traceless part of $\mathcal{T}_{\mathcal{M}} \otimes \mathcal{T}_{\mathcal{M}}^*$, that is, $\text{End } \mathcal{T}_{\mathcal{M}}$ is a (holomorphic) sub-bundle of $\mathcal{T}_{\mathcal{M}} \otimes \mathcal{T}_{\mathcal{M}}^*$.

A little thought reveals that the short sequence of holomorphic bundles and appropriate bundle maps

$$0 \rightarrow \text{End } \mathcal{T}_{\mathcal{M}} \xrightarrow{i} (\mathcal{T}_{\mathcal{M}} \otimes \mathcal{T}_{\mathcal{M}}^*) \xrightarrow{\text{tr}} \text{Tr}[\mathcal{T}_{\mathcal{M}} \otimes \mathcal{T}_{\mathcal{M}}^*] \rightarrow 0 \quad (\text{B.1.2})$$

[†]A diagram is commutative if by starting at one end, from a particular element, we obtain the same element no matter which way we trace through the diagram.

is exact. Of course, the projection on the trace is onto and its *kernel*^(L) consists precisely of traceless endomorphisms, while the embedding i has no kernel and $\text{cok}(i) \stackrel{\text{def}}{=} \{(\mathcal{T}_M \otimes \mathcal{T}_M^*)/\text{End } \mathcal{T}_M\}$ may clearly be identified with $\text{Tr}[\mathcal{T}_M \otimes \mathcal{T}_M^*] \approx \mathbb{C}$. This suffices to express the $\text{End } \mathcal{T}_M$ -valued cohomology in terms of the $(\mathcal{T}_M \otimes \mathcal{T}_M^*)$ -valued and the \mathbb{C} -valued cohomology on M .

However, we also have a comfortable simplification in this case, since the bundle $\mathbb{C} \approx \text{Tr}[\mathcal{T}_M \otimes \mathcal{T}_M^*]$ may also be regarded as a sub-bundle of $\mathcal{T}_M \otimes \mathcal{T}_M^*$. In other words, there exists a map $\tau: \text{Tr}[\mathcal{T}_M \otimes \mathcal{T}_M^*] \rightarrow (\mathcal{T}_M \otimes \mathcal{T}_M^*)$ such that $\tau \circ j = \mathbf{1}_{(\mathcal{T}_M \otimes \mathcal{T}_M^*)}$ and $j \circ \tau = \mathbf{1}_{\text{Tr}[\mathcal{T}_M \otimes \mathcal{T}_M^*]}$. τ is simply (fibre-wise) multiplication with the identity matrix. Because of this, we have that in fact

$$(\mathcal{T}_M \otimes \mathcal{T}_M^*) = \text{End } \mathcal{T}_M \oplus \text{Tr}[\mathcal{T}_M \otimes \mathcal{T}_M^*] \quad (\text{B.1.3})$$

holomorphically, not just in the C^∞ category. This implies that

$$H^q(M, \mathcal{T}_M \otimes \mathcal{T}_M^*) = H^q(M, \text{End } \mathcal{T}_M) \oplus H^q(M, \mathbb{C}). \quad (\text{B.1.4})$$

A Split Exact Sequence:

Generally, let $0 \rightarrow \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \rightarrow 0$ be a short exact sequence of holomorphic bundles over some compact complex space X and let there also exists a bundle map $\mathcal{C} \xrightarrow{\gamma} \mathcal{B}$ such that $j \circ \gamma = \mathbf{1}_{\mathcal{C}}$ and $\gamma \circ i = \mathbf{1}_{\mathcal{B}}$. Then the short exact sequence is said to *split*, $\mathcal{B} = \mathcal{A} \oplus \mathcal{C}$ holomorphically and

$$H^q(X, \mathcal{B}) = H^q(X, \mathcal{A}) \oplus H^q(X, \mathcal{C}). \quad (\text{B.1.5})$$

Furthermore, note that $(\mathcal{T}_M \otimes \mathcal{T}_M^*)$ is self-adjoint; on a Calabi-Yau n -fold where $\mathcal{K}_M \approx \mathbb{C}$, *Serre duality*^(L) implies that $H^q(M, \mathcal{T}_M \otimes \mathcal{T}_M^*) \approx H^{n-q}(M, \mathcal{T}_M \otimes \mathcal{T}_M^*)$.

Now we recall a theorem, which is proven in Ref. [43] for bundles over \mathbb{P}^n but is in fact valid for any (para)compact complex manifold [226].

Definition. A holomorphic bundle \mathcal{V} over a compact Kähler n -fold X (with Kähler class J) is stable if

$$\frac{\int_X J^{n-1} \wedge c_1[\mathcal{V}]}{\text{rank } \mathcal{V}} > \frac{\int_X J^{n-1} \wedge c_1[\mathfrak{V}_i]}{\text{rank } \mathfrak{V}_i}, \quad (\text{B.1.6})$$

for all i , where \mathfrak{V}_i are subsheaves of the sheaf corresponding to \mathcal{V} .

B.2 Computing $H^q(M, \text{End } \mathcal{T}_M)$

In fortunate cases, it suffices to test the condition over sub-bundles. Also, stability may be reformulated more directly, just in terms of curvature 2-forms of \mathcal{V} and \mathfrak{V}_i . In fact, there may exist a more general and more robust definition of stability, but this will suffice for our present purposes.

Theorem B.1 If \mathcal{V} is a stable holomorphic bundle over a paracompact complex manifold X , then $H^0(X, \text{End } \mathcal{V}) = 0$.

In general, it is quite laborious to prove that a particular bundle is stable. However, Ref. [226] proves that the holomorphic tangent bundle of a regular Calabi-Yau n -fold ($b_{1,0} = 0$, see *Genus*^(L)) is stable, prompting the following

Corollary B.1 For a regular Calabi-Yau n -fold M ,

$$H^q(M, \mathcal{T}_M \otimes \mathcal{T}_M^*) \approx \begin{cases} \mathbb{C} & q = 0 \pmod{n} \\ H^q(M, \text{End } \mathcal{T}_M) & \text{otherwise.} \end{cases} \quad (\text{B.1.7})$$

For the currently interesting spaces, Calabi-Yau 3-folds, we can compute $H^1(M, \mathcal{T}_M \otimes \mathcal{T}_M^*)$ and then simply use that $H^1(M, \mathcal{T}_M \otimes \mathcal{T}_M^*) = H^1(M, \text{End } \mathcal{T}_M)$. In practice, we compute all $H^q(M, \mathcal{T}_M \otimes \mathcal{T}_M^*)$ and check the calculations against Corollary B.1 and Serre duality.

B.2 Computing $H^q(M, \text{End } \mathcal{T}_M)$

Before we engage in the Koszul complex/spectral sequence computation, let's consider the deformation theoretic alternative.

B.2.1 Deformation theoretic results

Comparing the computation $H^q(M, \text{End } \mathcal{T}_M)$ with that of $H^q(M, \mathcal{T}_M)$ in the previous chapter, one asks if the ‘polynomial deformation’ type of representation could be applied (at least in part) to represent $H^1(M, \text{End } \mathcal{T}_M)$. Roughly, in view of Eq. (B.1), if (some) \mathcal{T}_M -valued 1-forms can be represented by deformations of the defining polynomial, can (some) $\mathcal{T}_M \otimes \mathcal{T}_M^*$ -valued 1-forms can be represented by deformations of the *differentials* of the defining polynomial?

In other words, given the defining polynomials* $f(x)$ and $g(x, y)$ as specified in Eqs. (A.1.3), we consider their differentials

$$dx^a(f_{abc}x^bx^c), \quad dx^a(g_{a\alpha\beta\gamma}y^\alpha y^\beta x^\gamma), \quad dy^\alpha(g_{a\alpha\beta\gamma}x^a y^\beta x^\gamma). \quad (\text{B.2.1})$$

The variations of each of these can be cast in two irreducible types; for example

$$\delta dx^a(f_{abc}x^bx^c) = dx^a(\phi_{(abc)}x^bx^c) + dx^a(\varphi_{a(bc)}x^bx^c). \quad (\text{B.2.2})$$

The former, involving the totally symmetric tensor coefficient $\phi_{(abc)}$ is a total differential $dx^a(\phi_{(abc)}x^bx^c) = \frac{1}{3}d(\phi_{(abc)}x^ax^bx^c)$ and will not qualify for representing a harmonic form. On the other hand, $dx^a(\varphi_{a(bc)}x^bx^c)$ involves the mixed-symmetric tensor $\varphi_{a(bc)} \sim (-2, -1, 0, 0)$, which is symmetric in $b \leftrightarrow c$, but satisfies

$$\varphi_{a(bc)} + \varphi_{b(ac)} + \varphi_{c(ab)} = 0. \quad (\text{B.2.3})$$

Its components are obtained from

$$\phi_{a(bc)} \stackrel{\text{def}}{=} \phi_{a|(bc)} - \phi_{(abc)}, \quad (\text{B.2.4})$$

and

$$\phi_{a(bc)} = \frac{1}{3}(\phi_{a|(bc)} - \phi_{b|(ac)}) + \frac{1}{3}(\phi_{a|(bc)} - \phi_{c|(ba)}) \quad (\text{B.2.5})$$

where $\phi_{a|(bc)} \stackrel{\text{def}}{=} \frac{1}{2}(\phi_{a|b|c} + \phi_{a|c|b})$.

Notation:

In general, we use parentheses to denote symmetrization, square brackets to denote skew-symmetrization. The indices which do not participate in a symmetrization are separated off by vertical pipes.

Proceeding in this fashion with the available differentials of the defining polynomials $f(x)$ and $g(x, y)$, we obtain 20+32 deformations

$$dx^a(\varphi_{a(bc)}x^bx^c), \quad dy^\alpha(\varphi_{a\alpha(\beta\gamma)}x^a y^\beta x^\gamma). \quad (\text{B.2.6})$$

This easily saturates the lower bound $\dim H^1(\mathcal{M}, \text{End } T_{\mathcal{M}}) \geq \dim H^1(\mathcal{M}, T_{\mathcal{M}})$ derived in Ref. [168]†.

*We again consider the example presented in Chapter A rather than trying to formulate this in some general and probably bewildering notation. This will hopefully suffice to present the idea and the Reader will have no insurmountable problem in extending this to any other desired model.

†In fact, using the *Mirror map*^(L), a stronger bound would follow, namely that $\dim H^1(\mathcal{M}, \text{End } T_{\mathcal{M}}) \geq \dim H^1(\mathcal{M}, T_{\mathcal{M}}) + \dim H^1(\mathcal{M}, T_{\mathcal{M}}^*)$.

B.2.2 The general method

By tensoring Seq. (9.1.1) with $T_{\mathcal{M}}^*$ and the dual of Seq. (9.1.1) with \mathcal{E} and with T_X respectively, we obtain three short exact sequences that fit together into the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & (T_X \otimes \mathcal{E}^*)|_{\mathcal{M}} & & (\mathcal{E} \otimes \mathcal{E}^*)|_{\mathcal{M}} & & \\ & & \downarrow & & \downarrow & & \\ & & (T_X \otimes T_X^*)|_{\mathcal{M}} & & (\mathcal{E} \otimes T_X^*)|_{\mathcal{M}} & & \\ & & \downarrow & & \downarrow & & \\ 0 \rightarrow T_{\mathcal{M}} \otimes T_{\mathcal{M}}^* & \rightarrow & T_X|_{\mathcal{M}} \otimes T_{\mathcal{M}}^* & \rightarrow & \mathcal{E}|_{\mathcal{M}} \otimes T_{\mathcal{M}}^* & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \quad (\text{B.2.7})$$

with exact rows and columns. It relates the vector bundle $T_{\mathcal{M}} \otimes T_{\mathcal{M}}^*$ over \mathcal{M} —intrinsic to \mathcal{M} —to the restriction to \mathcal{M} of various vector bundles over the (rather better known) embedding space X .

All three short exact sequences appearing in the diagram (B.2.7) induce long exact cohomology sequences. These are tangled as dictated by the original diagram (B.2.7): one may imagine the horizontal one being crossed from above by the vertical ones once for each q at $H^q(\mathcal{M}, T_X \otimes T_{\mathcal{M}}^*)$ and at $H^q(\mathcal{M}, \mathcal{E} \otimes T_{\mathcal{M}}^*)$. Through these latter two sets of cohomology groups, we relate $H^*(\mathcal{M}, T_{\mathcal{M}} \otimes T_{\mathcal{M}}^*)$ to

$$H^q(\mathcal{M}, \mathcal{E} \otimes \mathcal{E}^*), \quad H^q(\mathcal{M}, \mathcal{E} \otimes T_X^*), \quad H^q(\mathcal{M}, T_X \otimes T_X^*). \quad (\text{B.2.8})$$

These three sets of cohomology groups are determined, in terms of cohomological data entirely on X , from the spectral sequences induced by Seq. (9.3.7) tensored by $\mathcal{E} \otimes \mathcal{E}^*$, $\mathcal{E} \otimes T_X^*$ and $T_X \otimes T_X^*$, respectively.

To complete the plan, we need to fill in the various cohomology groups—entirely over X —into the lower left quadrant of these spectral sequences [see the chart (9.3.16)]. Iterating the *Künneth formula*^(L)

$$H^q(X \times Y, \mathcal{V}) = \bigoplus_{r+s=q} H^r(X, \mathcal{V}|_X) \otimes H^s(Y, \mathcal{V}|_Y), \quad (\text{B.2.9})$$

these are given in terms of cohomology groups on the factors of the embedding space. For the flag space type factors, all cohomology is determined by the Bott-Borel-Weil theorem 9.2; more generally, one need not know *all* the cohomology

of a factor in \mathcal{X} —only those needed in the Koszul spectral sequences for (B.2.8), that is

$$\begin{aligned} H^q(\mathcal{M}, \wedge^k \mathcal{E}^* \otimes \mathcal{E} \otimes \mathcal{E}^*) , \quad H^q(\mathcal{M}, \wedge^k \mathcal{E}^* \otimes \mathcal{E} \otimes T_{\mathcal{X}}^*) , \\ H^q(\mathcal{M}, \wedge^k \mathcal{E}^* \otimes T_{\mathcal{X}} \otimes T_{\mathcal{X}}^*) , \end{aligned} \quad (\text{B.2.10})$$

for $k = 0, \dots, K$. Of course, to work the Koszul spectral sequence machinery, we also need to know the ‘differential’ maps d_r acting in (9.3.16). While this is the hardest part in a general application of spectral sequences, for embeddings in products of flag spaces $\{G/P\}$ —this is elementary linear algebra of G -tensors (see *flag spaces*^(L) and the Bott-Borel-Weil theorem 9.2).

By iterating the construction which resulted in diagram (B.2.7), we can straightforwardly obtain any $T_{\mathcal{M}}^{*\otimes r} \otimes T_{\mathcal{M}}^{*\otimes s}$ tensor bundle and whence the correspondingly valued cohomology on \mathcal{M} .

B.3 Simple Examples

To illustrate the foregoing description, we follow Ref. [121] and consider two simple configurations, both chosen from the list (8.0.1) in view of their physical application.

B.3.1 A bi-cubic Calabi-Yau 3-fold

To begin with, consider $\mathcal{M} \in \left[\begin{smallmatrix} 2 & 3 \\ 2 & 3 \end{smallmatrix} \right]^{2;83}_{-162}$, that is, the space of solutions of a bi-cubic constraint

$$f(x, y) \stackrel{\text{def}}{=} f_{abc\alpha\beta\gamma} x^a x^b x^c y^\alpha y^\beta y^\gamma = 0 . \quad (\text{B.3.1})$$

The defining polynomial $f(x, y)$ is a section of $\mathcal{E} = \left(\begin{smallmatrix} -3 & 0 & 0 \\ -3 & 0 & 0 \end{smallmatrix} \right)$, in the obvious notation for the tensor product $(-3|00)_x \otimes (-3|00)_y$. The tangent bundle is $T_{\mathcal{X}} = T_x \oplus T_y$, where $T_x = \left(\begin{smallmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \right)$ and $T_y = \left(\begin{smallmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \end{smallmatrix} \right)$. Therefore :

$$\mathcal{E} \otimes \mathcal{E}^* = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right) ,$$

$$\mathcal{E} \otimes T_x^* = \left(\begin{smallmatrix} -2 & -1 & 0 \\ -3 & 0 & 0 \end{smallmatrix} \right) \oplus \left(\begin{smallmatrix} -3 & 0 & 0 \\ -2 & -1 & 0 \end{smallmatrix} \right) ,$$

$$T_x \otimes \mathcal{E}^* = \left(\begin{smallmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \end{smallmatrix} \right) \oplus \left(\begin{smallmatrix} 3 & 0 & 0 \\ 2 & 0 & 1 \end{smallmatrix} \right) ,$$

$$T_x \otimes T_x^* = 2 \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right) \oplus \left(\begin{smallmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{smallmatrix} \right) \oplus \left(\begin{smallmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{smallmatrix} \right) \oplus \left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \right) \oplus \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix} \right) .$$

B.3. Simple Examples

Bundle	Cohomology on \mathcal{M}	Bundle	Cohomology on \mathcal{M}
$\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)$	$H^0 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)_1$, $H^3 = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix} \right)_1$		
$\left(\begin{smallmatrix} -2 & -1 & 0 \\ -3 & 0 & 0 \end{smallmatrix} \right)$	$H^0 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)_1 + \left(\begin{smallmatrix} -2 & -1 & 0 \\ -3 & 0 & 0 \end{smallmatrix} \right)_{10}^8$	$\left(\begin{smallmatrix} -3 & 0 & 0 \\ -2 & -1 & 0 \end{smallmatrix} \right)$	$H^0 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)_1 + \left(\begin{smallmatrix} -3 & 0 & 0 \\ -2 & -1 & 0 \end{smallmatrix} \right)_8^{10}$
$\left(\begin{smallmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \end{smallmatrix} \right)$	$H^3 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 4 \end{smallmatrix} \right)_{10}^8 + \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix} \right)_1^1$	$\left(\begin{smallmatrix} 3 & 0 & 0 \\ 2 & 0 & 1 \end{smallmatrix} \right)$	$H^3 = \left(\begin{smallmatrix} 1 & 1 & 4 \\ 1 & 2 & 3 \end{smallmatrix} \right)_8^{10} + \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix} \right)_1^1$
$\left(\begin{smallmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{smallmatrix} \right)$	$H^1 = \left(\begin{smallmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \right)_1^8$, $H^2 = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{smallmatrix} \right)_8^1$	$\left(\begin{smallmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{smallmatrix} \right)$	$H^1 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \end{smallmatrix} \right)_8^1$, $H^2 = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix} \right)_1^8$
$\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \end{smallmatrix} \right)$	All cohomology vanishes	$\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \right)$	All cohomology vanishes

Table B.1: The cohomology input required for $H^q(\mathcal{M}, \text{End } T_{\mathcal{M}})$, for the bi-cubic Calabi-Yau 3-fold in $\mathbb{P}^2 \times \mathbb{P}^2$.

Restricting the ambient cohomology to the submanifold. For each of these ten irreducible bundles, we employ the spectral sequence separately. As there is only one defining function, the Koszul resolution (9.3.7) collapses to a simple short exact sequence

$$0 \rightarrow \left(\begin{smallmatrix} 3 & 0 & 0 \\ 3 & 0 & 0 \end{smallmatrix} \right) \xrightarrow{f} \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right) \rightarrow \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right)_{\mathcal{M}} \rightarrow 0 \quad (\text{B.3.2})$$

and the spectral sequence is equivalent to the corresponding long exact cohomology sequence. Using the Bott-Borel-Weil theorem, we fill in the SW (lower left) quadrant of $E_0^{q,k}(\mathcal{V})$, setting \mathcal{V} equal, one by one, to each of the ten bundles above. Just by elementary $U(3)$ transformation properties, all possible actions of f in this spectral sequence vanish. That means that $E_0^{q,k}(\mathcal{V}) = E_0^{q,k}(\mathcal{V})$ and the results in Table B.1 are easily obtained. Of course, the superscripts and subscripts are the dimensions of the \mathbb{P}_x^2 - and \mathbb{P}_y^2 -factor cohomology groups and the total dimension is therefore the product. The symmetry obtained by interchanging the factors $\mathbb{P}_x^2 \leftrightarrow \mathbb{P}_y^2$ corresponds to swapping the columns in Table B.1. Also there is *Serre duality*:

$$H^q(\mathcal{M}, \mathcal{V}) = H^{3-q}(\mathcal{M}, \mathcal{V}^*)^* \otimes \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix} \right) . \quad (\text{B.3.3})$$

Using the information in Table B.1, the long exact cohomology sequence accompanying the right most column sequence of the diagram (B.2.7) implies that $H^1(\mathcal{M}, \mathcal{E} \otimes T_{\mathcal{M}}^*)$ and $H^3(\mathcal{M}, \mathcal{E} \otimes T_{\mathcal{M}}^*)$ vanish, that $H^2(\mathcal{M}, \mathcal{E} \otimes T_{\mathcal{M}}^*) = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix} \right)_1^1$ and

that $H^0(\mathcal{M}, \mathcal{E} \otimes \mathcal{T}_\mathcal{M}^*)$ equals the extension

$$\binom{0 \ 0 \ 0}{0 \ 0 \ 0}_1 + \left[\binom{-2 \ -1 \ 0}{-3 \ 0 \ 0}_{10}^8 \oplus \binom{-3 \ 0 \ 0}{-2 \ -1 \ 0}_8^{10} \right], \quad (\text{B.3.4})$$

in total, a vector space of dimension 161.

From the long exact cohomology sequence accompanying the central column sequence of the diagram (B.2.7), we obtain :

$$H^0(\mathcal{M}, \mathcal{T}_X \otimes \mathcal{T}_\mathcal{M}^*) = 2 \binom{0 \ 0 \ 0}{0 \ 0 \ 0}_1^1 \quad (\text{B.3.5})$$

$$H^1(\mathcal{M}, \mathcal{T}_X \otimes \mathcal{T}_\mathcal{M}^*) = \binom{-1 \ 0 \ 1}{0 \ 0 \ 0}_1^8 \oplus \binom{0 \ 0 \ 0}{-1 \ 0 \ 1}_8^1, \quad (\text{B.3.6})$$

and that

$$0 \rightarrow \binom{1 \ 1 \ 1}{0 \ 1 \ 2}_8^1 \rightarrow H^2(\mathcal{M}, \mathcal{T}_X \otimes \mathcal{T}_\mathcal{M}^*) \rightarrow \left[\binom{1 \ 2 \ 3}{1 \ 1 \ 4}_{10}^8 + 2 \binom{1 \ 1 \ 1}{1 \ 1 \ 1}_1^1 \right] \xrightarrow{j} \\ \xrightarrow{j} 2 \binom{1 \ 1 \ 1}{1 \ 1 \ 1}_1^1 \rightarrow H^3(\mathcal{M}, \mathcal{T}_X \otimes \mathcal{T}_\mathcal{M}^*) \rightarrow 0. \quad (\text{B.3.7})$$

As before, vertical stacks represent direct sums. It is not hard to verify (see Appendix A of Ref. [121] for the details) that the map j here acts in what is essentially the only possible way consistent with $U(3) \times U(3)$ tensor algebra—as the identity map $\binom{1 \ 1 \ 1}{1 \ 1 \ 1} \rightarrow \binom{1 \ 1 \ 1}{1 \ 1 \ 1}$. These contributions are therefore canceled out, whence $H^3(\mathcal{M}, \mathcal{T}_X \otimes \mathcal{T}_\mathcal{M}^*)$ vanishes and

$$H^2(\mathcal{M}, \mathcal{T}_X \otimes \mathcal{T}_\mathcal{M}^*) = \binom{1 \ 2 \ 3}{1 \ 1 \ 4}_{10}^8 + \binom{1 \ 1 \ 1}{0 \ 1 \ 2}_8^1 + \binom{1 \ 1 \ 1}{1 \ 1 \ 1}_1^1, \quad (\text{B.3.8})$$

a vector space of dimension 176.

If there were a non-vanishing map $\binom{1 \ 2 \ 3}{1 \ 1 \ 4} \xrightarrow{j} \binom{1 \ 1 \ 1}{1 \ 1 \ 1}$, it would essentially be induced from the map $\mathcal{T}_X^*|_{\mathcal{M}} \rightarrow \mathcal{E}|_{\mathcal{M}}$, that is from a contraction with $df(x, y)$. It follows that it would therefore be given as (up to overall factors $\epsilon^{abc}\epsilon^{\alpha\beta\gamma}$)

$$f_{abc\alpha\beta\gamma} \cdot g^{i(jk)(\mu\nu\rho)} \mapsto \lambda. \quad (\text{B.3.9})$$

By $U(3) \times U(3)$ covariance, there can be no such map (no way to contract the tensors on the left-hand side of the map (B.3.9) into a scalar).

B.3. Simple Examples

The $\text{End}\mathcal{T}_\mathcal{M}$ -valued cohomology on the submanifold. With the information obtained so far, we are ready for the base row of the diagram (B.2.7). The accompanying long exact cohomology sequence breaks into two exact sequences (dual to each other), the first of which is

$$0 \rightarrow H^0(\mathcal{M}, \mathcal{T}_\mathcal{M} \otimes \mathcal{T}_\mathcal{M}^*) \xrightarrow{\iota} 2 \binom{0 \ 0 \ 0}{0 \ 0 \ 0}_1 \xrightarrow{j} \left[\binom{0 \ 0 \ 0}{0 \ 0 \ 0} + \binom{-4 \ -1 \ 0}{-3 \ 0 \ 0} \right] \xrightarrow{\delta} \\ \xrightarrow{\delta} H^1(\mathcal{M}, \mathcal{T}_\mathcal{M} \otimes \mathcal{T}_\mathcal{M}^*) \xrightarrow{\iota} \binom{0 \ 0 \ 0}{-1 \ 0 \ 1} \xrightarrow{\delta} \binom{0 \ 0 \ 0}{-1 \ 0 \ 1} \rightarrow 0. \quad (\text{B.3.10})$$

To specify $H^0(\mathcal{M}, \mathcal{T}_\mathcal{M} \otimes \mathcal{T}_\mathcal{M}^*)$ and $H^1(\mathcal{M}, \mathcal{T}_\mathcal{M} \otimes \mathcal{T}_\mathcal{M}^*)$, all we need is to determine the action of the map j . Considering its domain and the target, the only possible action of j here—as in Seq. (B.3.7) too—is the identity. This identifies (and thus cancels out) two $\binom{0 \ 0 \ 0}{0 \ 0 \ 0}$ terms, leaving

$$H^0(\mathcal{M}, \mathcal{T}_\mathcal{M} \otimes \mathcal{T}_\mathcal{M}^*) = \binom{0 \ 0 \ 0}{0 \ 0 \ 0}_1^1, \quad (\text{B.3.11})$$

in agreement with Corollary (B.1). This leaves

$$H^1(\mathcal{M}, \mathcal{T}_\mathcal{M} \otimes \mathcal{T}_\mathcal{M}^*) = \binom{-1 \ 0 \ 1}{0 \ 0 \ 0}_1^8 + \binom{-2 \ -1 \ 0}{-3 \ 0 \ 0}_{10}^8 + \binom{-2 \ -1 \ 0}{-3 \ 0 \ 0}_8^{10}, \quad (\text{B.3.12})$$

a vector space of dimension 176, as found in Ref. [115]. The remaining part of the exact sequence yields

$$H^2(\mathcal{M}, \mathcal{T}_\mathcal{M} \otimes \mathcal{T}_\mathcal{M}^*) = \binom{1 \ 2 \ 3}{1 \ 1 \ 4}_{10}^8 + \binom{1 \ 1 \ 1}{0 \ 1 \ 2}_8^1 + \binom{1 \ 1 \ 1}{1 \ 1 \ 1}_1^1, \quad (\text{B.3.13})$$

in agreement with (B.3.3).

Beyond deformation theory. In this simple case, we used the $U(3) \times U(3)$ transformation properties in a rather elementary way, merely to obtain independently what can also be ascertained using Corollary B.1. The computations become more complicated in the next example and the transformation properties delivered by the Bott-Borel-Weil theorem will be absolutely essential. Now, the result (B.3.12) reveals more than the deformation theoretic discussion in § B.2.1.

The 8×10 dimensional contribution $\begin{pmatrix} -2 & -1 & 0 \\ -3 & 0 & 0 \end{pmatrix}$ is represented by $\vartheta_{a(bc)(\alpha\beta\gamma)}$; it is totally symmetric in α, β, γ , symmetric in b, c , but vanishes when a, b, c are symmetrized. Contracting with a dx , two x 's and three y 's, we obtain

$$dx^a \vartheta_{a(bc)(\alpha\beta\gamma)} x^b x^c y^\alpha y^\beta y^\gamma , \quad (\text{B.3.14})$$

precisely as foretold in § B.2.1. These are non-trivial variations of $d_x f(x, y)$. The other 80-dimensional contribution is obtained analogously, after $x \leftrightarrow y$.

However, the two 8-dimensional contributions do not have such an interpretation. They are simply two 3×3 , traceless Hermitian matrices $\lambda_a{}^b$ and $\lambda_a{}^\beta$. Tracing back through the calculations (see Table B.1), we see that the first one arises from the product $T_{\mathbb{P}_x^2} \otimes T_{\mathbb{P}_y^2}^*$. So, it acquires a natural interpretation as a type of linear reparametrization which maps $T_{\mathbb{P}_y^2}$ to $T_{\mathbb{P}_x^2}$. It is not clear from a strictly deformation theory point of view why such contributions should arise, whereas the foregoing computation based on the Koszul complex and spectral sequence provides it readily. In general, there are many more contributions, the computation becomes rather more tedious and may easily get out of (human) hand even though it is completely straightforward; perhaps a Reader adept at programming might try to develop an automated version of this.

B.3.2 Another example

Recall the Calabi-Yau 3-fold $\mathcal{M} \in \left[\begin{smallmatrix} 3 & | & 3 & 1 \\ 2 & || & 0 & 3 \\ & & | & -54 \end{smallmatrix} \right]^{8,35}$, defined as the zero set of the defining polynomials $f(x)$ and $g(x, y)$, as specified in Eqs. (A.1.3),

$$f(x) \stackrel{\text{def}}{=} f_{abc} x^a x^b x^c , \quad g(x, y) \stackrel{\text{def}}{=} g_{\alpha\beta\gamma} x^\alpha y^\beta y^\gamma , \quad (\text{B.3.15})$$

with $x \in \mathbb{P}^3$ and $y \in \mathbb{P}^2$. As noted before, f and g are sections of $\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{pmatrix}$, respectively and \mathcal{E} is now the rank-two direct sum of these two line bundles. The tangent bundle on X is given similarly as in the previous case,

$$\mathcal{T}_X = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} . \quad (\text{B.3.16})$$

We therefore have

$$\mathcal{E} \otimes \mathcal{E}^* = 2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} -2 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 2 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{pmatrix} ,$$

$$\mathcal{E} \otimes \mathcal{T}_X^* = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} -3 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} -1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 \end{pmatrix} ,$$

$$\mathcal{T}_X \otimes \mathcal{E}^* = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 3 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix} ,$$

$$\mathcal{T}_X \otimes \mathcal{T}_X^* = 2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} .$$

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B.3.3 Restricting the ambient cohomology to the submanifold

As in the previous example, we compute the cohomology groups on \mathcal{M} valued in each of these irreducible bundles by using the corresponding spectral sequence. Recall that the resolution sequence (9.3.7), for this complete intersection, becomes Seq. (A.1.5)

$$0 \rightarrow \begin{pmatrix} 4 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{f} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{g} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mathcal{M}} \rightarrow 0 . \quad (\text{A.1.5}')$$

The arrows labeled by f and g denote maps induced by contraction with the respective polynomials, i.e., tensor coefficients f_{abc} and $g_{\alpha\beta\gamma}$.

Unlike the previous example, not all spectral sequences now converge at the first level. The action of the spectral sequence differentials, all induced by f and g , must be determined. To show that this is nothing complicated, consider for example the spectral sequence for $\mathcal{E}_f \otimes \mathcal{T}_y^* = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$:

$q: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 4 & -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -3 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$	\xrightarrow{g}	$(\mathcal{E}_f \otimes \mathcal{T}_y^*) _{\mathcal{M}}$
0: 0	0	0	$\Rightarrow 0$	
1: 0	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_1^1$	$\xrightarrow{f} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_1^{20}$	see below	
2: 0	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}_8^{10}$	0	$\Rightarrow 0$	
3: 0	0	0	$\Rightarrow 0$	
4: 0	0	0	$\equiv 0$	
5: 0	0	0	$\equiv 0$	

The map labeled by f acts by taking

$$f_{abc} : \left[\lambda \in \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \mapsto \left[(\lambda f_{abc}) \in \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] , \quad (\text{B.3.17})$$

is clearly 1-1 and of maximum rank (=1), and yields the only non-vanishing differential in this spectral sequence. The corresponding chart of $E_2^{q,k}(\mathcal{E}_f \otimes \mathcal{T}_y^*)$, the next and final level, is therefore obtained by replacing $E_1^{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ with 0 and $E_1^{1,0} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ by its quotient with $f(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix})$. Therefore, the only non-zero $\mathcal{E}_f \otimes \mathcal{T}_y^*$ -valued cohomology on \mathcal{M} is

$$H^1(\mathcal{M}, \mathcal{E}_f \otimes \mathcal{T}_y^*) = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}_8^{10} + \left[\text{cok } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_1^1 \xrightarrow{f} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_1^{20} \right]_{19} . \quad (\text{B.3.18})$$

In this way we obtain all the cohomology on \mathcal{M} , valued in the various bundles listed above. The results are:

Bundle	Non-zero cohomology
$(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix})$	$H^0 = (\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix})_1^1$ $H^3 = (\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix})_1^1$
$(\begin{smallmatrix} -2 & 0 & 0 & 0 \\ 3 & 0 & 0 \end{smallmatrix})$	$H^1 = \ker (\begin{smallmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 4 \end{smallmatrix})_{10}^4 \xrightarrow{g} (\begin{smallmatrix} -2 & 0 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix})_1^{10}$
$(\begin{smallmatrix} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix})$	$H^0 = (\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix})_1^1 + (\begin{smallmatrix} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix})_1^{20}$ $H^1 = (\begin{smallmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix})_1^6$
$(\begin{smallmatrix} -3 & 0 & 0 & 0 \\ 1 & -1 & 0 \end{smallmatrix})$	$H^1 = (\begin{smallmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 2 \end{smallmatrix})_8^{10} + [\text{cok } (\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix})_1^1 \xrightarrow{f} (\begin{smallmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix})_1^{20}]$
$(\begin{smallmatrix} 0 & -1 & 0 & 0 \\ -3 & 0 & 0 \end{smallmatrix})$	$H^0 = (\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix})_1^1$ $H^1 = (\begin{smallmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 \end{smallmatrix})_1^4$
$(\begin{smallmatrix} -1 & 0 & 0 & 0 \\ -2 & -1 & 0 \end{smallmatrix})$	$H^0 = (\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix})_1^1 + (\begin{smallmatrix} -1 & 0 & 0 & 0 \\ -2 & -1 & 0 \end{smallmatrix})_8^4$
$(\begin{smallmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix})$	$H^1 = (\begin{smallmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 \end{smallmatrix})_1^4$ $H^2 = (\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix})_1^4$
$(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 \end{smallmatrix})$	All cohomology vanishes
$(\begin{smallmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 \end{smallmatrix})$	$H^1 = (\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 \end{smallmatrix})_8^4 + (\begin{smallmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix})_1^{15}$ $H^2 = (\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 \end{smallmatrix})_8^1$

where we have omitted entries that can be obtained using *Serre duality*^(L); in the present case, it yields

$$H^q(\mathcal{M}, \mathcal{V}) = H^{3-q}(\mathcal{M}, \mathcal{V}^*)^* \otimes (\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}). \quad (\text{B.3.19})$$

To proceed, the sequence (A.2.1) is tensored with each of \mathcal{E}_f , \mathcal{E}_g , \mathcal{T}_x and \mathcal{T}_y . This yields four separate short exact sequences, refining the two vertical exact sequences in the diagram (B.2.7). From the accompanying long exact cohomology sequences we then determine the $\mathcal{E}_i \otimes \mathcal{T}_M^*$ -valued and the $\mathcal{T}_i \otimes \mathcal{T}_M^*$ -valued cohomology on \mathcal{M} . Let's denote generically by j all maps in the Koszul spectral sequence and also all the maps induced from $\nabla\xi$ in Seq. (A.2.1) which represented as various possible contractions with (possibly products of) the defining tensors f_{abc} and $g_{\alpha\beta\gamma}$. This representation is again sufficient to locate and determine all non-vanishing maps between the cohomology groups. The resulting cohomology data is then plugged into the long exact cohomology sequence accompanying the short exact row sequence in the diagram (B.2.7).

❷ In the next couple of paragraphs, only the last few steps—the most complicated ones—are presented. Fill in the rest. ❸

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B.3.4 Endomorphism valued cohomology on the submanifold

In the base row short exact sequence in the diagram (B.2.7), we study the action of j for each q separately,

$$\cdots \rightarrow H^q(\mathcal{M}, \mathcal{T}_M \otimes \mathcal{T}_M^*) \rightarrow H^q(\mathcal{M}, \mathcal{T}_x \otimes \mathcal{T}_M^*) \xrightarrow{j} H^q(\mathcal{M}, \mathcal{E} \otimes \mathcal{T}_M^*) \rightarrow \cdots \quad (\text{B.3.20})$$

and determine thereby $H^q(\mathcal{M}, \mathcal{T}_M \otimes \mathcal{T}_M^*)$.

For $q = 0$, we have

$$2(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}) \xrightarrow{j} \left[(\begin{smallmatrix} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}) \oplus (\begin{smallmatrix} -1 & 0 & 0 & 0 \\ -2 & -1 & 0 \end{smallmatrix}) \oplus (\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}) \right]. \quad (\text{B.3.21})$$

As before, with Seq. (B.3.7), j here identifies the copy of $(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix})$ in the domain with the one in the target. This leaves us with $H^0(\mathcal{M}, \mathcal{T}_M \otimes \mathcal{T}_M^*) = (\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix})$, in agreement with Corollary B.1.

$$\text{cok } j = \left[(\begin{smallmatrix} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}) \oplus (\begin{smallmatrix} -1 & 0 & 0 & 0 \\ -2 & -1 & 0 \end{smallmatrix}) \right] \quad (\text{B.3.22})$$

is mapped via the connecting homomorphism to $H^1(\mathcal{M}, \mathcal{T}_M \otimes \mathcal{T}_M^*)$.

For $q = 1$, the stack of cohomology groups on both sides of the map j is rather longish (see Fig. B.1). The computation itself is however less complicated than the wiring diagram which Fig. B.1 resembles at first—we determine all mappings essentially by $U(4) \times U(3)$ covariance. In Fig. B.1, we have written any mapping derived from f or g as just f or g .

For example, to obtain the cokernel^(L)

$$\left[\text{cok } (\begin{smallmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 4 \end{smallmatrix})_{10}^4 \xrightarrow{g} \begin{array}{c} (\begin{smallmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix})_1^6 \\ (\begin{smallmatrix} -2 & 0 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix})_1^{10} \\ \xrightarrow{g} (\begin{smallmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 2 \end{smallmatrix})_8^{10} \end{array} \right]_{56}$$

note that the linear transformation

$$g : (\begin{smallmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 4 \end{smallmatrix})_{10}^4 \xrightarrow{\quad} \begin{array}{c} (\begin{smallmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix})_1^6 \\ (\begin{smallmatrix} -2 & 0 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix})_1^{10} \\ \nearrow (\begin{smallmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 2 \end{smallmatrix})_8^{10} \end{array} \quad (\text{B.3.23})$$

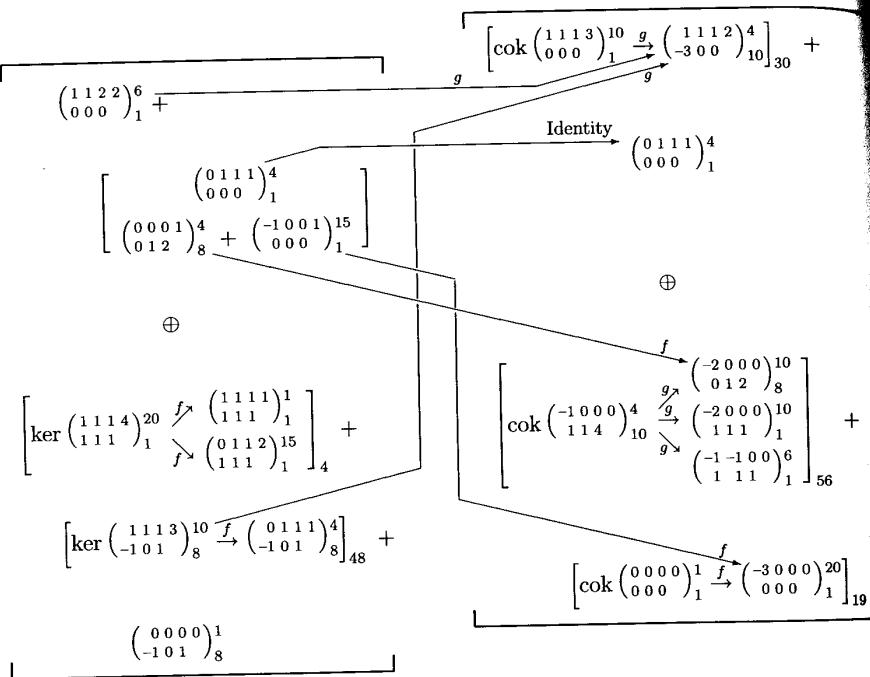


Figure B.1: The mapping $H^1(M, T_M \otimes T_M^*) \rightarrow H^1(M, E \otimes T_M^*)$.

is 1-1 (is injective, has no kernel) for generic choice of $g(x, y)$, that is, for generic $g_{\alpha\beta\gamma} = g_{\alpha\beta\gamma}$. We verify this here explicitly, but leave other such calculations as an exercise.

Firstly, the $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ Young tableau corresponds to a tensor $\lambda_a^{\alpha\beta\gamma} = \lambda_a^{(\alpha\beta\gamma)}$. The combined map can be represented as

$$g : \lambda_a^{(\alpha\beta\gamma)} \xrightarrow{\theta_{[ab]}} \theta_{(ab)} = \lambda_{(a}^{(\alpha\beta\gamma)} g_{b)\alpha\beta\gamma}, \quad (B.3.24)$$

$$\theta_{(ab)\delta\gamma} = (\delta_{\nu\delta}^{\gamma\mu} - \frac{1}{3}\delta_{\delta\nu}^{\gamma\mu})\lambda_{(a}^{(\alpha\beta\nu)} g_{b)\alpha\beta\mu}.$$

The combined kernel of these three maps then consists of tensors $\lambda_a^{(\alpha\beta\gamma)}$ which

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are annihilated by all three maps, that is, those non-zero components of $\lambda_a^{(\alpha\beta\gamma)}$ which satisfy the system of three linear tensorial equations

$$\lambda_{|a}^{\alpha\beta\gamma} g_{b)\alpha\beta\gamma} = 0 \quad 6 \text{ equations ,} \quad (B.3.25)$$

$$\lambda_{(a}^{\alpha\beta\gamma} g_{b)\alpha\beta\gamma} = 0 \quad 10 \text{ equations ,} \quad (B.3.26)$$

$$(\delta_{\nu\delta}^{\gamma\mu} - \frac{1}{3}\delta_{\delta\nu}^{\gamma\mu})\lambda_{(a}^{(\alpha\beta\nu)} g_{b)\alpha\beta\mu} = 0 \quad 80 \text{ equations ,} \quad (B.3.27)$$

corresponding to the three maps, respectively. (The $(\delta_{\nu\delta}^{\gamma\mu} - \frac{1}{3}\delta_{\delta\nu}^{\gamma\mu})$ prefactor projects the trace-free part.)

Now, these are 96 equations in only 40 unknowns and it is only reasonable to expect that all components of $\lambda_a^{\alpha\beta\gamma}$ are forced to vanish by these equations—for a generic choice of $g_{a\alpha\beta\gamma}$. Making a specific choice of $g_{a\alpha\beta\gamma}$ such as

$$g_{1111} = g_{2222} = g_{3333} = 1, \quad g_{0123} = \kappa \neq 0, \quad (B.3.28)$$

where all other $g_{a\alpha\beta\gamma} = 0$, it can be shown explicitly that the choice is ‘generic enough’, that is, all $\lambda_a^{\alpha\beta\gamma} = 0$ and there is no kernel.

Consider now the sub-generic case, when $g_{0123} = \kappa = 0$ [210]. It is not hard to show that the complete intersection is still smooth. However, the combined mapping (B.3.24) changes drastically. Since all non-zero components of $g_{a\alpha\beta\gamma}$ have $\alpha = \beta = \gamma$, the four components $\lambda_a^{(123)}$ cannot possibly occur non-trivially in the map (B.3.24), will not be set to zero and therefore span (part of) the kernel.

Find the full kernel of the combined map (B.3.24), with $g_{a\alpha\beta\gamma}$ as in Eq. (B.3.28), and when $\kappa = 0$. \clubsuit

On the other hand, the cokernel of the combined (generic) map is the space spanned by the 96 components $\{\theta_{[ab]}, \theta_{(ab)}, \theta_{(ab)\delta\gamma}\}$, taken however modulo the combined additive image of the 40 components of $\lambda_a^{\alpha\beta\gamma}$. That is—for a ‘generic enough’ choice of $f(x)$ and $g(x, y)$ —we can use the 40 independent components of the three different $(\lambda \cdot g)$ ’s to gauge away forty components of the θ ’s. As usual, representatives of this quotient are not uniquely defined and there are various (but equivalent) choices. Although it may become cumbersome to keep track of this, it will eventually be useful to toggle between such equivalent choices when it comes to compute some Yukawa couplings.

Find the full cokernel of the combined map (B.3.24), where $g_{a\alpha\beta\gamma}$ is given in Eq. (B.3.28) with $\kappa = 0$. \clubsuit

Finally, after completing several similar computations, $H^1(\mathcal{M}, \text{End } T_{\mathcal{M}})$ is obtained as the double extension $A + B + C$, where

$$A = \left[\ker \begin{pmatrix} 1 & 1 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix}_1^{20} \xrightarrow{f} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_1^1 \right]_4, \quad (\text{B.3.29})$$

$$B = \left[\ker \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}_1^6 \xrightarrow{g} \begin{pmatrix} 1 & 1 & 1 & 2 \\ -3 & 0 & 0 \end{pmatrix}_{10}^4 \right]_{24}, \quad (\text{B.3.30})$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}_8^1 + \left[\begin{pmatrix} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_1^{20} \oplus \begin{pmatrix} -1 & 0 & 0 & 0 \\ -2 & -1 & 0 \end{pmatrix}_8^4 \right]. \quad (\text{B.3.31})$$

$H^2(\mathcal{M}, \text{End } T_{\mathcal{M}})$ is given dually.

Explicit tensorial representation. The contribution A in (B.3.29) is rather simple. It consists of elements of

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix} \sim \theta^{(abc)} \epsilon^{ijkl} \epsilon^{\alpha\beta\gamma}, \quad (\text{B.3.32})$$

which are annihilated in the mapping, by f , into $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$. Up to overall factors of $\epsilon^{ijkl} \epsilon^{\alpha\beta\gamma}$, these Young tableaux correspond to scalars and traceless Hermitian matrices. The restrictions defining the kernel are easily enforced by

$$\begin{aligned} f_{abc} \theta^{(abc)} &= 0, \\ (\delta_{jd}^{ai} - \frac{1}{4} \delta_{dj}^{ai}) f_{abc} \theta^{(bcd)} &= 0. \end{aligned} \quad (\text{B.3.33})$$

Combining these two conditions in the obvious way, we have that

$$A \sim \{ (\theta^{(abc)} \epsilon^{ijkl} \epsilon^{\alpha\beta\gamma}) : f_{abc} \theta^{(bcd)} = 0 \}. \quad (\text{B.3.34})$$

Now, the contribution B (B.3.30) is mostly the dual of (B.3.24), which we have considered before. Omitting overall factors of ϵ^{ijkl} , we introduce irreducible $U(4) \times U(3)$ tensor variables $\theta^{[ab]}$, $\theta^{(ab)}$ and $\theta^{(ab)\alpha\beta\gamma}$ for the three Young tableaux on the left hand side of B , respectively. The kernel will be described by a subset of these variables which vanish upon the mapping in B . The last of these, $\theta^{(ab)\alpha\beta\gamma}$, must also be trace-free in α, β and is subject to the additional restriction

$$f_{abc} \theta^{(ab)\alpha\beta\gamma} = 0, \quad (\text{B.3.35})$$

B.3. Simple Examples

which describes the kernel of the bottom mapping in B . With a generically chosen f_{abc} , this produces 32 independent equations in 80 variables in $\theta^{(ab)\alpha\beta\gamma}$, leaving 48 of them free to parametrize the kernel. For specific choices, this has to be verified explicitly; the Fermat-type choice

$$f(x) = \sum_{a=0}^3 (x^a)^3, \quad \iff \quad f_{abc} = \begin{cases} 1 & a = b = c, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{B.3.36})$$

is ‘generic enough’ in this sense, that is, it provides a map of maximum rank.

The kernel of the combined mapping into $\begin{pmatrix} 0 & 0 & 0 & 1 \\ -3 & 0 & 0 \end{pmatrix}$ is described by

$$\phi_1 g_{b(\alpha\beta\gamma)} \theta^{[ab]} + \phi_2 g_{b(\alpha\beta\gamma)} \theta^{(ab)} + \phi_3 g_{b(\alpha\delta\gamma)} \theta^{(ab)\delta\beta} = 0. \quad (\text{B.3.37})$$

The parameters ϕ_i describe the triple mapping more precisely and are non-zero but otherwise arbitrary. Choosing $g_{a\alpha\beta\gamma}$ generically, this produces 40 independent relations among 6 + 10 + 48 components of our tensor variables, leaving 24 to span the combined kernel representing the contribution B .

Finally, the three Young tableaux in (B.3.31) correspond to

$$\theta^{\alpha\beta}, \quad \theta_{a(bc)}, \quad \theta_{a\alpha(\beta\gamma)}. \quad (\text{B.3.38})$$

The last two of these occur as non-trivial variations of

$$d_x f(x) = dx^a f_{abc} x^b x^c, \quad \text{and} \quad d_y g(x, y) = dy^\alpha g_{a\alpha\beta\gamma} x^a y^\beta y^\gamma, \quad (\text{B.3.39})$$

and cover the result of the polynomial deformation method; the remaining 36 components have been missed.

Special defining polynomials. Consider again the smooth but sub-generic case, when $g_{0123} = \kappa = 0$ [210]. Just as the combined mapping (B.3.24) suffers drastic changes, so does the final result for $H^1(\mathcal{M}, \text{End } T_{\mathcal{M}})$. In practice, we have to go back through the computations and carefully reconsider all instances of contraction with $g_{a\alpha\beta\gamma}$. As in (B.3.24), there will be cases where the kernel, and so also the cokernel of a map induced by contraction with $g_{a\alpha\beta\gamma}$, becomes larger in the limit $g_{0123} = \kappa \rightarrow 0$, so we expect new elements of $H^1(\mathcal{M}, \text{End } T_{\mathcal{M}})$.

There is however another, subtler change : the representatives themselves change and in fact discontinuously so in the limit $g_{0123} = \kappa \rightarrow 0$. In particular, choose [210]

$$f_{abc} = \begin{cases} 1 & a = b = c, \\ 0 & \text{otherwise,} \end{cases} \quad g_{a\alpha\beta\gamma} = \begin{cases} 1 & a = b = c, \\ \kappa & (a, \alpha, \beta, \gamma) = (0, 1, 2, 3), \\ 0 & \text{otherwise.} \end{cases} \quad (\text{B.3.40})$$

The contribution A in (B.3.34) now becomes

$$\{ \theta^{012}, \theta^{013}, \theta^{023}, \theta^{123} \}. \quad (\text{B.3.34}')$$

These and the contributions C , presented in (B.3.38), are unchanged in the limit $g_{0123} = \kappa \rightarrow 0$.

For the contribution B , specified in (B.3.30), a convenient basis is, in the limit $g_{0123} = \kappa \rightarrow 0$, provided by

$$\{ (\theta^{(ab)\gamma}\delta\epsilon^{ijkl}) : a \neq b, \alpha = \gamma, \theta^{(ab)\gamma}\gamma = 0 \}_{12}, \quad (\text{B.3.41})$$

$$\{ (\theta^{(\hat{a}\hat{b})\gamma}\delta\epsilon^{ijkl}) : \delta = \hat{a}, \hat{a}, \hat{b}, \gamma \text{ different} \}_6, \quad (\text{B.3.42})$$

$$\{ (\theta^{(\hat{b}\hat{b})\gamma}\delta\epsilon^{ijkl}) : \hat{b}, \gamma, \delta \text{ different} \}_6, \quad (\text{B.3.43})$$

where the caret on an index denotes that the index does not take the value 0. When we deform the 3-fold by switching on $g_{0123} = \kappa \neq 0$, the first 18 of these remain unchanged. However, the last six (B.3.43) become replaced with certain linear combinations, such as

$$\theta^{(01)^3}{}_2 \rightarrow (\theta^{(01)^3}{}_2 - (\frac{g_{2222}}{g_{0123}})\theta^{(21)^2}{}_1). \quad (\text{B.3.44})$$

Finally, there are also 20 elements of $H^1(\mathcal{M}, \text{End } \mathcal{T}_{\mathcal{M}})$ and of course 20 matching elements of $H^2(\mathcal{M}, \text{End } \mathcal{T}_{\mathcal{M}})$ which cannot be canceled out with the special choice of $g_{\alpha\beta\gamma}$ given in (B.3.40). However, as soon as $g_{0123} = \kappa \neq 0$, these 20 elements of $H^1(\mathcal{M}, \text{End } \mathcal{T}_{\mathcal{M}})$ are mapped to the matching 20 from $H^2(\mathcal{M}, \text{End } \mathcal{T}_{\mathcal{M}})$; the former 20 are no longer closed and the latter 20 become exact and so drop out of the $\text{End } \mathcal{T}_{\mathcal{M}}$ -valued cohomology. A suitable basis for these 20 elements of $H^1(\mathcal{M}, \text{End } \mathcal{T}_{\mathcal{M}})$ is

$$\begin{aligned} & \{ \theta^{(0b)}\epsilon^{ijkl} \}_4, & & \{ \theta^{(\hat{b}\hat{b})\gamma}\delta\epsilon^{ijkl} : \hat{b} = \delta \neq \gamma \}_6, \\ & \{ \theta_a^{(123)}\epsilon^{\mu\nu\rho} \}_4, & & \{ \theta_{\hat{a}}^{(\alpha\beta\gamma)}\epsilon^{\mu\nu\rho} : \hat{a} = \alpha = \beta \neq \gamma \}_6, \end{aligned} \quad (\text{B.3.45})$$

B.4 Cohomology and Moduli

General deformation theory of Kodaira and Spencer (see Ref. [35]) ensures that $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ spans the deformations of the complex structure of any compact complex manifold \mathcal{M} —locally near the original choice of the complex structure for \mathcal{M} . Consider now the manifold \mathcal{M} embedded in some compact complex manifold \mathcal{X} as the vanishing set of a (system of) section(s) of a bundle \mathcal{E} over \mathcal{X} ; that is, $\mathcal{M} \in [\mathcal{X}][\mathcal{E}]$.

By varying the defining equations of the manifold \mathcal{M} , embedded in \mathcal{X} , we span the ‘polynomial deformation’ space $\mathfrak{d}[\mathcal{X}][\mathcal{E}]$ of the embedding $\mathcal{M} \hookrightarrow \mathcal{X}$. The image of this shows up in $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ as the ‘polynomial deformation’ of the complex structure, as discussed above.

The results of the preceding Chapter show that $\dim H^q(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ and also $\dim H^q(\mathcal{M}, \mathcal{T}_{\mathcal{M}}^*) = b_{1,q}$ are independent of the choice of the complex structure, and in particular, the defining polynomials of \mathcal{M} —as long as \mathcal{M} is smooth. By contrast, we have seen that $H^q(\mathcal{M}, \text{End } \mathcal{T}_{\mathcal{M}})$ exhibits a more interesting behavior which is known as *semi-continuity*: $\dim H^1(\mathcal{M}, \text{End } \mathcal{T}_{\mathcal{M}})$ is minimal for generic choices of the complex structure, but increases in jumps as we specialize to less and less generic choices. These changes are accurately computed using the Koszul spectral sequence as described here.

So far, we have considered varying the complex structure of the (Calabi-Yau) 3-fold \mathcal{M} . The choice of the Kähler class J is another important datum on \mathcal{M} , relevant for (super)string compactification. There is no reason why $H^1(\mathcal{M}, \text{End } \mathcal{T}_{\mathcal{M}})$ should vary with the choice of the complex structure but be inert under variations of the choice of the Kähler class. In fact, the *mirror map*^(L) strongly suggests that $H^1(\mathcal{M}, \text{End } \mathcal{T}_{\mathcal{M}})$ depends on the choice $J \in H^{1,1}(\mathcal{M})$ in precisely the same way as it depends on the choice of the complex structure. Unfortunately, this dependence does not show up in the Koszul spectral sequence computation and cannot be controlled in this way.

Finally, we note that $H^1(\mathcal{M}, \text{End } \mathcal{T}_{\mathcal{M}})$ spans the deformations of the complex structure of the total space of $\mathcal{T}_{\mathcal{M}}$, the holomorphic tangent bundle of \mathcal{M} . Variations of this choice are also beyond the Koszul spectral sequence computation as described here.

Comparing the results of the Koszul spectral sequence with other, more stringy methods (some of which will be discussed at the end of this volume) suggests that the Koszul spectral sequence computation refers to (1) a specified choice

of the complex structure (as controlled by the choice of the defining equations), but (2) a generic choice of the Kähler class and the complex structure of $T_{\mathcal{M}}$. That is, certain other methods involve special choices of the Kähler class (metric) and also yield what appears to be a bigger dimensional $H^1(\mathcal{M}, \text{End } T_{\mathcal{M}})$. Similarly, certain properties of the ring structures generated by $H^1(\mathcal{M}, T_{\mathcal{M}})$, $H^1(\mathcal{M}, T_{\mathcal{M}}^*)$ and $H^1(\mathcal{M}, \text{End } T_{\mathcal{M}})$ seem to support this possibility. Clearly, however, the problem of describing the variation of these cohomology groups and the ring structures which they generate remains a rather new and largely uncharted territory in the jungle.

Chapter C

The (2, 1) Triple Couplings and Generalization

Continuing the discussion in Chapter 8, we now turn to the task of computing the $\mathbf{27}^3$ Yukawa couplings, as given in Eq. (0.5.7) :

$$\kappa_{\alpha\beta\gamma} \stackrel{\text{def}}{=} \int_{\mathcal{M}} \left(\Omega^{\bar{\mu}\bar{\nu}\bar{\rho}} \varphi_{(\alpha)}{}^{\bar{\mu}}{}^{\sigma} \varphi_{(\beta)}{}^{\bar{\nu}}{}^{\tau} \varphi_{(\gamma)}{}^{\bar{\rho}}{}^{\lambda} \Omega_{\sigma\tau\lambda} \right). \quad (0.5.7')$$

Of course, as it stands, this expression has little practical use, since we have next to no knowledge about the actual functional form of the integrand. In a way, we need to reformulate this integral formula into something more tractable.

This is not at all unlike the computation described in Chapter 8, where an integral was related to intersection numbers which were obtained by counting, or the spectral sequence computations presented in Chapters 9-B, where the solutions of some partial differential equations are described without ever solving them*. In particular examples, there may be more than one way of doing this and not infrequently, some of these “translations” of the original problem will offer a solution which is not quite complete in some aspect. Usually, however, different methods can be combined to complete the desired computations.

*One might say that most of the tricks of the trade boil down to integration without integrating. To this end, the general theory of residues comes in handy. We have seen one of its results in Eq. (A.3.12) and many of the subsequent facts about residue class (quotient) rings follow from this general theory; see Chapter 5 of Ref. [23] for a general discussion.

C.1 The Deformation Theoretic (2,1) Triple Couplings

In a good number of complete intersection type embeddings, there is a useful computational tool which turns the integral (0.5.7) into a product and which can often be evaluated by inspection [84].

It relies on two facts : (1) On a complex manifold \mathcal{M} , harmonic (2,1)-forms, or equivalently, elements of $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ parametrize locally the space of complex structures of \mathcal{M} . (2) For many complete intersections, the deformation space of the embedding $\mathcal{M} \hookrightarrow \mathcal{X}$, as defined in Eq. (A.3.2) is in fact isomorphic to the space of complex structure locally around any particular choice. The precise conditions when this happens were discussed in Chapter A. We now assume that these conditions are met.

The degree-3 cohomology group

$$H^3 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} \quad (\text{C.1.1})$$

is then completely parametrized and given the structure of a *residue class ring*,

$$\mathcal{R}_{[\mathcal{X}||\mathcal{E}]} \stackrel{\text{def}}{=} \left\{ \mathbb{C}[(\mathcal{X})] / \mathfrak{I}[\text{Aut}[\mathcal{X}||\mathcal{E}]] \right\}. \quad (\text{C.1.2})$$

$\mathbb{C}[(\mathcal{X})]$ is the ring of homogeneous polynomials over \mathcal{X} with complex coefficients, generated from polynomials of the degree of the defining constraints, $f_a(x) = 0$, so that there is a natural grading in multiples of $\deg(f_a)$. $\mathfrak{I}[\text{Aut}[\mathcal{X}||\mathcal{E}]]$ is the ideal generated by various symmetries of the configuration, such as linear reparametrizations of the embedding space, suitable rescalings of the defining constraints and so on. When polynomial deformations represent $H^{2,1}$ faithfully, the grading of \mathcal{R} corresponds to the decomposition (C.1.1) and so there is a unique equivalence class, sometimes a single monomial, $Q \in \mathcal{R}$, corresponding to elements of $H^{0,3}$. If m_{α} are elements of \mathcal{R} corresponding to $H^{2,1}$, then

$$m_{\alpha} \cdot m_{\beta} \cdot m_{\gamma} \equiv \kappa_{\alpha\beta\gamma} \cdot Q \pmod{\mathfrak{I}}, \quad (\text{C.1.3})$$

where \mathfrak{I} is standing for $\mathfrak{I}[\text{Aut}[\mathcal{X}||\mathcal{E}]]$. The so defined constant $\kappa_{\alpha\beta\gamma}$ must then be equal to the Yukawa coupling (0.5.7). Some numerical characteristics of \mathcal{R} can be obtained easily using the standard results given in § 7.4.

On *algebraic varieties*, which are generally defined as the zero-set of a polynomial constraint system, the residue class ring is one of the fundamental objects [27,38]. So much so, in fact, that the variety itself can be practically disregarded and one may proceed simply by manipulating \mathcal{R} and various related

C.1. The Deformation Theoretic (2,1) Triple Couplings

structures. The beautiful simplicity* of complete intersection spaces where the polynomial deformation technology does faithfully represent $H^{2,1}$ and the related Yukawa couplings implies that in these cases the residue class ring \mathcal{R} which is explicitly constructed from the space of ‘polynomial deformations’ coincides with the abstract such ring of the corresponding abstract algebraic variety.



To illustrate this technique, we use the result (A.3.6). The $m_{\alpha} \in \mathcal{R}$ which correspond to $H^{2,1}(\mathcal{M})$ of the complete intersection manifold (A.1.2) are

$$\begin{bmatrix} \phi_{(abc)} x^a x^b x^c \\ \varphi_{a(\alpha\beta\gamma)} x^a y^{\alpha} y^{\beta} y^{\gamma} \end{bmatrix} \cong \begin{bmatrix} \phi_{(abc)} x^a x^b x^c \\ \varphi_{a(\alpha\beta\gamma)} x^a y^{\alpha} y^{\beta} y^{\gamma} \end{bmatrix} + (x^a \lambda_a{}^b \partial_b + y^{\alpha} \lambda_{\alpha}{}^{\beta} \partial_{\beta}) \begin{bmatrix} f(x) \\ g(x,y) \end{bmatrix}. \quad (\text{A.3.6}')$$

where $f(x)$ and $g(x,y)$ are the defining equations of bi-degree (3,0) and (1,3), respectively, in coordinates $x \in \mathbb{P}^3$ and $y \in \mathbb{P}^2$. Note in fact how obvious the relationship between the monomial representatives and the tensorial representatives appears in Eq. (A.3.6') : we may regard $x^a x^b x^c$ as the relevant objects and $\phi_{(abc)}$ merely as coefficients, or we may think of the monomials as basis-elements and of $\phi_{(abc)}$ as the tensor components. Calculating with $\phi_{(abc)}$ is then just like representing a vector by its components : v^{μ} .

Roughly, the equivalence relation in (A.3.6) is like setting monomials equal to zero if they are proportional to the gradients of the defining equations. However, one has to be careful in doing so. Since the equivalence relations stem from the action of the linear reparametrization groups, the number of relations imposed using these generators must be precisely equal to the dimension of the reparametrization group. $\dim \text{Aut}[\mathcal{X}||\mathcal{E}] = 25$ in the above case ($\dim(PGL(n, \mathbb{C})) = n^2 - 1$, plus the overall rescaling of the defining equations). This is precisely accounted for by the two matrices $[\lambda_a{}^b]$ and $[\lambda_{\alpha}{}^{\beta}]$.

The various components of $[\lambda_a{}^b]$ and $[\lambda_{\alpha}{}^{\beta}]$ can then be chosen so as to eliminate one or another monomial from the list of candidates for a basis. This ambiguity leads to many different choices of bases, which are all of course equivalent. This is completely described by the Koszul complex. The example at hand corresponds to the mapping

$$\begin{array}{ccc} \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}_1^{15} & \xrightarrow{\begin{array}{l} f \\ g \end{array}} & \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}_1^{20} / \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}_1^1 \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}_8^1 & \xrightarrow{\begin{array}{l} f \\ g \end{array}} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}_8^1 / \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}_1^1 \end{array}, \quad (\text{A.1.20}')$$

*... in principle, although not always also in practice ...

the cokernel of which is identified as $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ and presented in (A.1.22). The horizontal maps correspond to using the λ_a^b and the λ_α^β matrix in ‘gauging away’ monomials from $\{\phi_{(abc)}x^ax^bx^c\}$ and $\{\phi_{a(\alpha\beta\gamma)}x^ay^\alpha y^\beta y^\gamma\}$, respectively. The slanted map represents using the λ_a^b matrix to ‘gauge away’ monomials from the second set. Therefore, the true ideal is generated by the horizontal maps modulo the effect of the slanted map. More explicitly, we find that the ideal is not generated by all the three non-zero gradients

$$[\partial_a f(x)], \quad [\partial_a g(x, y)], \quad [\partial_\alpha g(x, y)], \quad (\text{C.1.4})$$

independently, but rather by

$$\left\{ \frac{[\partial_a f(x)], [\partial_a g(x, y)], [\partial_\alpha g(x, y)]}{[\partial_a f(x) - \partial_a g(x, y)]} \right\}, \quad (\text{C.1.5})$$

where the condition on the generators of the ideal represents the action of the slanted map since it allows us to replace a relation generated by $\partial_a f(x)$ with another relation generated by $\partial_a g(x, y)$ in the action of the ideal on the tensors to the right of the combined map (A.1.20). This quotient ideal is equivalent to

$$\mathfrak{I}_K[(\partial_a f + \partial_a g), (\partial_\alpha g)], \quad (\text{C.1.6})$$

and which was called the *Koszul ideal* in (9.3.15).

To describe this residue class ring structure more succinctly, we should require the relations should be homogeneous and have $\deg(\partial_a f(x)) = \deg(\partial_a g(x, y))$. This implies that $2\deg(x^a) = 3\deg(y^\alpha)$, i.e., $\deg(x^a) = 3$, $\deg(y^\alpha) = 2$ and clearly also that $\deg(f(x)) = \deg(g(x, y)) = 9$. We record $\deg((\partial_a f - \partial_a g)) = 6$ and $\deg(\partial_\alpha g) = 7$. With this, the cohomology has just been given the structure of a *graded* quotient ring, for which the results in § 7.4 apply straightforwardly and we have

$$P_t(\mathbb{C}[x^a, y^\alpha]/\mathfrak{I}_K) = \frac{(1-t^6)^4 \cdot (1-t^7)^3}{(1-t^3)^4 \cdot (1-t^2)^3}, \quad (\text{C.1.7})$$

which gives $P_t = \dots + 35t^9 + \dots$, corresponding to $\dim H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) = 35$. In general, the result will appear as a sum of such contributions, each of which can be assigned a grade. This makes the full cohomology ring into a bi-graded ring, where one grading comes from the different degrees of the homogeneous coordinates in order for the Koszul ideal to be generated by homogeneous relations and the other grading labels the different contributions in formulae of the type (9.3.20).

C.2. A General Trace Formula

With the simple choice $f(x) \stackrel{\text{def}}{=} \sum_{a=0}^3 (x^a)^3$ and $g(x, y) \stackrel{\text{def}}{=} \sum_{\alpha=1}^3 x^\alpha (y^\alpha)^3$, we have that

$$\partial_a f(x) = 3(x^a)^2, \quad \partial_a g(x, y) = (y^\alpha)^3 \quad \text{and} \quad \partial_\alpha g(x, y) = 3x^\alpha (y^\alpha)^2. \quad (\text{C.1.8})$$

It is then not hard to show that

$$Q \propto (x^0 x^1 x^2 x^3)(x^1 y^1 x^2 y^2 x^3 y^3), \quad (\text{C.1.9})$$

or, more generally,

$$Q \propto (\det[f''(x)]) (\det[\ddot{g}(x, y)]), \quad (\text{C.1.10})$$

where primes denotes derivative with respect to x 's and dots derivatives with respect to y 's[†].

C.2 A General Trace Formula

We now wish to generalize the polynomial deformation formula for the Yukawa coupling [84] in a way that would let us compute Yukawa coupling when polynomial deformations do not represent $H^{2,1}(\mathcal{M})$ faithfully.

To that end we will (1) translate the polynomial deformation formula into the tensorial representations’ language and then also (2) provide a dual representation of the formula, partly for ease of computation but also because that will prove to have the most general form*.

C.2.1 The Yukawa coupling synopsis

On writing Eq. (C.1.10) explicitly out, we obtain the rather bewildering expression

$$Q = d_{a_1 a_2 a_3 a_4} x^{a_1} x^{a_2} x^{a_3} x^{a_4} d_{b_1 b_2 b_3 \beta_1 \beta_2 \beta_3} x^{b_1} x^{b_2} x^{b_3} y^{\beta_1} y^{\beta_2} y^{\beta_3}, \quad (\text{C.2.1})$$

where we have abbreviated

$$d_{a_1 a_2 a_3 a_4} \stackrel{\text{def}}{=} \varepsilon^{b_1 b_2 b_3 b_4} \varepsilon^{c_1 c_2 c_3 c_4} f_{a_1 b_1 c_1} f_{a_2 b_2 c_2} f_{a_3 b_3 c_3} f_{a_4 b_4 c_4} \quad (\text{C.2.2})$$

$$d_{b_1 b_2 b_3 \beta_1 \beta_2 \beta_3} \stackrel{\text{def}}{=} \varepsilon^{\alpha_1 \alpha_2 \alpha_3} \varepsilon^{\gamma_1 \gamma_2 \gamma_3} g_{b_1 \alpha_1 \beta_1 \gamma_1} g_{b_2 \alpha_2 \beta_2 \gamma_2} g_{b_3 \alpha_3 \beta_3 \gamma_3}, \quad (\text{C.2.3})$$

[†]The other contribution is proportional to $\det[f] = 0$; see Ref. [84], Eq. (4.34).

*I am grateful to Per Berglund for partly unpublished collaboration on these issues.

up to some uninteresting numerical factors.

Note that Q contains precisely two of both the ϵ^{abcd} - and the $\epsilon^{\alpha\beta\gamma}$ -tensors. Recall the residue formula (A.3.12),

$$H^{3,0}(\mathcal{M}) \ni \Omega \stackrel{\text{def}}{=} \oint_{\Gamma(f)} \oint_{\Gamma(g)} \frac{\epsilon_{abcd} x^a dx^b dx^c dx^d \epsilon_{\alpha\beta\gamma} y^\alpha dy^\beta dy^\gamma}{f(x)g(x,y)} . \quad (\text{A.3.12}')$$

Its dual, $\overline{\Omega} \in H^{0,3}(\mathcal{M})$, is therefore represented by $\epsilon^{abcd} \epsilon^{\alpha\beta\gamma}$, as was also obtained by the Koszul computation (A.3.9). So, Q in fact represents the square of the dual of the holomorphic 3-form, Ω . Since the Yukawa coupling definition (0.5.7) contains precisely two Ω 's,

$$(\Omega \cdot m_\alpha \cdot m_\beta \cdot m_\gamma \cdot \Omega) \mapsto (\Omega \cdot (\overline{\Omega} \kappa_{\alpha\beta\gamma} \overline{\Omega}) \cdot \Omega) \mapsto \kappa_{\alpha\beta\gamma} \quad (\text{C.2.4})$$

is the formal synopsis for the Yukawa coupling computation.



An alternative and both more general and more rigorous description of this can be given as follows (see also Ref. [224,193]). One observes that elements of $H^1(\mathcal{M}, T_{\mathcal{M}})$ act on elements of $H^{n-1,1}(\mathcal{M}) = H^1(\mathcal{M}, \wedge^{n-1} T_{\mathcal{M}}^*)$ essentially by the natural contraction between $T_{\mathcal{M}}$ and $T_{\mathcal{M}}^*$ and by the wedge product :

$$H^1(\mathcal{M}, T_{\mathcal{M}}) \ni \varphi : H^q(\mathcal{M}, \wedge^p T_{\mathcal{M}}^*) \longrightarrow H^{q+1}(\mathcal{M}, \wedge^{p-1} T_{\mathcal{M}}^*) . \quad (\text{C.2.5})$$

In meticulous tensorial notation, this reads

$$\begin{aligned} & (dz^{\bar{\mu}} \varphi_{\bar{\mu}}^\alpha \partial_a) \cdot (dz^{\nu_1} \cdots dz^{\nu_p} dz^{\bar{\rho}_1} \cdots dz^{\bar{\rho}_q} \omega_{\nu_1 \cdots \nu_p \bar{\rho}_1 \cdots \bar{\rho}_q}) \mapsto \\ & \mapsto \sum_i (-)^{i+p} \delta_a^{\nu_i} dz^{\nu_1} \cdots dz^{\nu_{i-1}} dz^{\nu_{i+1}} \cdots dz^{\nu_p} \wedge \\ & dz^{\bar{\mu}} dz^{\bar{\rho}_1} \cdots dz^{\bar{\rho}_q} (\varphi_{\bar{\mu}}^\alpha \omega_{\nu_1 \cdots \nu_{i_1} \alpha \nu_{i+1} \cdots \nu_p \bar{\rho}_1 \cdots \bar{\rho}_q}) \end{aligned} \quad (\text{C.2.6})$$

where we have omitted the wedge products for brevity. This provides a map

$$H^1(\mathcal{M}, T_{\mathcal{M}}) \longrightarrow \text{Hom}(H^q(\mathcal{M}, \wedge^p T_{\mathcal{M}}^*), H^{q+1}(\mathcal{M}, \wedge^{p-1} T_{\mathcal{M}}^*)) , \quad (\text{C.2.7})$$

and recall that generally $\text{Hom}(A, B) = B \otimes A^*$. Quite straightforwardly then,

$$\begin{aligned} \text{Sym}^n H^1(\mathcal{M}, T_{\mathcal{M}}) & \longrightarrow \text{Hom}(H^0(\mathcal{M}, \wedge^n T_{\mathcal{M}}^*), H^n(\mathcal{M}, \mathbb{C})) \\ & = \text{Hom}(H^{n,0}(\mathcal{M}), H^{0,n}(\mathcal{M})) . \end{aligned} \quad (\text{C.2.8})$$

C.2. A General Trace Formula

Since $H^{n,0}(\mathcal{M})$ is naturally the dual of $H^{0,n}(\mathcal{M})$,

$$\text{Sym}^n H^1(\mathcal{M}, T_{\mathcal{M}}) \longrightarrow (H^{0,n}(\mathcal{M}))^{\otimes 2} . \quad (\text{C.2.9})$$

This map is the Yukawa coupling (0.5.7).

Note :

Should $H^{n-k,k} = 0$ for $k < m$ (and by conjugation also for $k > n-m$), the map

$$\text{Sym}^{(n-m)} H^1(\mathcal{M}, T_{\mathcal{M}}) \longrightarrow (H^{0,n-m}(\mathcal{M}))^{\otimes 2} . \quad (\text{C.2.10})$$

still defines a (generalized) Yukawa coupling.



C.2.2 Tensorial translation

Following the gist of Eq. (C.1.3), we form a triple product of the tensor representatives, for our sample model, $\varphi_{(abc)}$ and $\varphi_{a(\alpha\beta\gamma)}$. Then multiply this with the square of the tensor representative of $\Omega \in H^{3,0}$, in the present case, $\epsilon^{abcd} \epsilon^{\alpha\beta\gamma}$. Finally, we massage the so obtained tensor using the relations from the ideal until the product is brought into the form of Q .

As the reduction modulo the ideal may become nerve-wrecking and time-consuming (without a clever computer program), we note that the same result may be obtained also in a *dual* computation, sometimes with more efficiency. To that end, consider the defining equations such as $f(x) = f_{abc} x^a x^b x^c$ and isolate their tensor coefficients, which we might call the defining tensors. Then simply swap their indices to obtain the dual defining tensors[†], such as

$$f^{(abc)} \stackrel{\sim}{\sim} f_{abc} , \quad g^{a(\alpha\beta\gamma)} \stackrel{\sim}{\sim} g_{a(\alpha\beta\gamma)} . \quad (\text{C.2.11})$$

If $f_a(x)$ denote some generic defining polynomials, the dual defining tensors will be denoted by f^a ; note the similarity to the variables used in § 9.3.1 to represent the duals of the line bundles of which the defining polynomials are sections.

Back to Eqs. (C.2.3) : the Yukawa coupling can now be written, with the aid of the dual defining tensors as

$$\kappa_{\alpha\beta\gamma} = \text{Tr}_{X,f^a} [\epsilon^2 \cdot t_\alpha \cdot t_\beta \cdot t_\gamma] , \quad (\text{C.2.12})$$

[†]The acronym DDT readily comes to mind, but we should prefer to avoid the pun.

where ϵ is the tensor representative of $\Omega \in H^{3,0}(\mathcal{M})$ and t_α are the tensor representatives of $H^{2,1}(\mathcal{M})$ obtained by the Koszul spectral sequence computation. By “ Tr_{X,f_a} ” we mean taking traces using the invariant tensors of the tensor algebra on X and the dual defining tensors. For example, the product

$$(\epsilon_{abcd}\epsilon_{\alpha\beta\gamma})(\epsilon_{efgh}\epsilon_{\delta\lambda\kappa})\phi_{(ijk)}\phi_{(lmn)}\varphi_{p(\mu\nu\rho)} \quad (\text{C.2.13})$$

can only be contracted (completely!) with four $f^{(\dots)}$'s and three $g^{(\dots)}$'s. Of course, there are several ways to do so, but they are related by the symmetry properties of the tensors involved; this contributes to the numerical factor which we have so far neglected. However, when comparing two or more Yukawa couplings, the relative numerical factor is of course the measure of the relative strength of the two couplings.

Clearly, this generalizes the monomials of the polynomial deformations, which correspond to totally symmetric tensors of the same degree as the defining equations, to essentially arbitrary tensors. Also, the formula depends only on the defining equations and as usual in tensor calculations, being a trace, it is coordinate independent. Note in particular that the number of the ϵ -tensors is precisely given in the definition of the Yukawa coupling and that their presence (and skew symmetry) restricts the possible contractions rather severely.

Of course, in models where polynomial deformations do faithfully represent $H^{2,1}$, the t 's (that is, the tensor coefficients such as $\phi_{(abc)}$ and $\varphi_{a(\alpha\beta\gamma)}$) and the coordinate monomials m 's may be used interchangeably. More generally, however, only the tensorial representation is going to represent $H^{2,1}$ faithfully and yield the Yukawa coupling for arbitrary complete intersections in products of flag spaces. For embeddings in more general spaces, the Koszul computation is of course valid, but it expresses the cohomology on the complete intersection in terms of the cohomology on the embedding space. At that point, of course, it all hinges upon how well we know the latter one, which is the primary reason for embedding in projective spaces and some of their simple generalizations—their cohomology is best known.



At first, Eq. (C.2.12) may seem *ad hoc*, but actually it is the only expression that has an invariant meaning and is constructed entirely from “invariant tensors”. Since $f_a(x)$ vanish on the manifold, their tensor coefficients are special in the sense that they characterize the complete intersection manifold. Just as the tensor

algebra on \mathbb{P}^n has the Kronecker tensor as the only invariant tensor and the Levi-Civita ϵ -tensor as an “almost invariant”, when we pass to a subspace \mathcal{M} , the tensor algebra on \mathcal{M} will be inherited from that of the embedding space, modified however by the inclusion of the new “invariant tensors”—those corresponding to the defining equations.

In the example which we used throughout the Chapter, we consider the Calabi-Yau manifold

$$\mathcal{M} \in \left[\begin{array}{c|cc} 3 & 3 & 1 \\ 2 & 0 & 3 \end{array} \right], \quad (\text{C.2.14})$$

that is, $\mathcal{M} \hookrightarrow \mathbb{P}^3 \times \mathbb{P}^2$ as the space of common solutions to a cubic constraint $f(x) = 0$ over \mathbb{P}^3 and a constraint $g(x,y) = 0$ of bi-degree $(1,3)$ in $\mathbb{P}^3 \times \mathbb{P}^2$. While the tensor algebra over \mathbb{P}^n is the $U(n+1)$ -tensor algebra since $\mathbb{P}^n = \left\{ \frac{U(n+1)}{U(1) \times U(n)} \right\}$, the only tensorial objects that carry information about the subspace $\mathcal{M} \subset \mathbb{P}^3 \times \mathbb{P}^2$ are the tensor coefficients of the defining polynomials $f(x)$ and $g(x,y)$.

C.3 The $(1,1)^3$ and Other Yukawa Couplings

Since the formula (C.2.12) is so general in view of the above explanation, we expect something similar to be valid also for other Yukawa couplings :

$$\hat{\kappa}_{ABC} = \int_{\mathcal{M}} e_A \wedge e_B \wedge e_C, \quad (\text{C.3.1})$$

$$\hat{\kappa}_{ijk} = \int_{\mathcal{M}} \Omega \wedge \text{Tr}(s_i \wedge s_j \wedge s_k), \quad (\text{C.3.2})$$

$$\hat{\kappa}_{\alpha jC} = \int_{\mathcal{M}} \Omega \wedge \text{Tr}(\varphi_\alpha \wedge s_j \wedge e_C). \quad (\text{C.3.3})$$

Indeed, this is almost true. However, in $\hat{\kappa}_{ABC}$, which we have already discussed in Chapter 8, and in $\hat{\kappa}_{\alpha jC}$ there is some degree of freedom. Namely, the pull-backs of the Kähler forms appear as simple scalars in the Koszul computation. When our complete intersection is embedded in a single projective space, there is a single Kähler form, which is by the Lefschetz hyperplane theorem inherited by the complete intersection and this degree of freedom is simply an overall multiplicative factor. However, when the embedding space has more than one factor, this becomes more involved.

This is seen by translating into the tensorial language the $\hat{\kappa}_{ABC}$ Yukawa coupling as computed in Chapter 8. For the model (C.2.14), the Koszul computation

provides a basis

$$H^1(\mathcal{M}, T_{\mathcal{M}}^*) = \{ J_x, J_y, \epsilon^{abcd} \vartheta^{(ef)} \}, \quad (\text{C.3.4})$$

where J_x and J_y are the (pull-backs of the) Kähler forms of \mathbb{P}^3 and \mathbb{P}^2 respectively. The trace-formula would allow the following couplings

$$(J_r, J_s, J_t), \quad (J_r, \epsilon^{0123} \vartheta^{(ab)}, \epsilon^{0123} \vartheta^{(cd)}) \quad (\text{C.3.5})$$

where $r, s, t = x, y$ unrestrictedly and a, b, c, d must all be different. The latter couplings are, in detail

$$J_r \cdot f_{a_1 b_1 c_1} f_{a_2 b_2 c_2} f_{a_3 b_3 c_3} f_{a_4 b_4 c_4} (\epsilon^{a_1 a_2 a_3 a_4} \vartheta^{(c_1 c_2)}) (\epsilon^{b_1 b_2 b_3 b_4} \vartheta^{(c_3 c_4)}) \quad (\text{C.3.6})$$

for specific choices of the superscripts c_i —as dictated by the chosen form of the defining polynomial $f(x)$ —and $r = x, y$. The various couplings for $r, s, t = x, y$ are all unrelated by any of the Koszul spectral sequence machinery as understood in the above Chapters.

On the other hand, the eight tensorial representatives of $H^1(\mathcal{M}, T_{\mathcal{M}}^*)$ are clearly in 1-1 correspondence with the 8-dimensional basis that we can define along the results in Eq. (8.5.10). In fact those results can be used directly, identifying [77] :

$$J_x = 3E_0 - \sum_{i=1}^6 E_i, \quad J_y = 3E'_0, \quad (\text{C.3.7})$$

and

$$\begin{aligned} \epsilon \phi^{(01)} &\longleftrightarrow -(E_1 + \omega^2 E_2 + \omega E_3), \\ \epsilon \phi^{(02)} &\longleftrightarrow (\omega^2 E_4 + \omega E_5 + E_6), \\ \epsilon \phi^{(03)} &\longleftrightarrow \bar{\gamma} [H + \omega^2(E_1 + E_2 + E_3) + \omega(E_4 + E_5 + E_6)], \\ \epsilon \phi^{(12)} &\longleftrightarrow \gamma [H + \omega(E_1 + E_2 + E_3) + \omega^2(E_4 + E_5 + E_6)], \\ \epsilon \phi^{(13)} &\longleftrightarrow (\omega E_4 + \omega^2 E_5 + E_6), \\ \epsilon \phi^{(23)} &\longleftrightarrow -(E_1 + \omega E_2 + \omega^2 E_3). \end{aligned} \quad (\text{C.3.8})$$

Upon this identification, half of the couplings in (C.3.5) vanishes and the remaining ones are all related in value to a single overall scale.

The vanishing of half of the couplings in (C.3.5) can be caused simply by *including* an additional requirement to the trace-formula (C.2.12). Note that J_x is in fact a double pull-back. First, it is inherited by the cubic surface in \mathbb{P}^3 and

C.3. The $(1,1)^3$ and Other Yukawa Couplings

then by the Calabi-Yau 3-fold. Because of the fact that it is a pull-back of a $(1,1)$ -form on the cubic 2-fold in \mathbb{P}^3 , we have that J_x^3 vanishes identically. Moreover, the $\epsilon^{0123} \vartheta^{(ab)}$'s also correspond to $(1,1)$ -forms inherited from this 2-fold, so that—upon this consideration—the only non-vanishing Yukawa couplings are

$$(J_x, J_x, J_y), \quad (J_x, J_y, J_y), \quad (J_y, \epsilon^{0123} \vartheta^{(ab)}, \epsilon^{0123} \vartheta^{(cd)}). \quad (\text{C.3.9})$$

Only the relative values within the third group of couplings can be determined by the structure so far revealed; the relative values with respect to the first two remain undetermined. Exactly the same freedom will occur in any other Yukawa coupling computation which involves a tensorial representative which in fact is a scalar. Generally, such representatives correspond to (pull-backs of) Kähler forms or other generators of the (r,r) -cohomology of the embedding space (flag spaces may contain higher such forms which are independent from the Kähler form).

Using also the representatives (B.3.34), (B.3.37) and (B.3.38), the couplings $\hat{\kappa}_{ijk}$ in Eq. (C.3.2) and $\hat{\kappa}_{\alpha jC}$ in (C.3.3) can be evaluated also, up to the same freedom in the $(1,1)$ -cohomology (for a similar calculation, see Ref. [77]).

In this respect, the Koszul spectral sequence computation as presented above manifests the dependence on the choice of complex structure, through the dependence on the defining polynomials. In addition, it does feature some degree of freedom controlled by a choice in the $(1,1)$ -cohomology (variations of the Kähler class). Also, it provides an evaluation of Yukawa couplings in the same tensorial language and therefore it allows a comparison of all four types of Yukawa couplings (0.5.7), (0.5.5), (0.5.9) and (0.5.11), which is much more than any other available technique. Finally, a complete and detailed identification between these calculations and the Landau-Ginzburg orbifold counterparts [159,227] seems to be possible [72].

C.4 Using Symmetries

Needless to say, when the space \mathcal{M} exhibits symmetries, these can be employed to simplify the computation of the Yukawa couplings.

Given the tensorial representation of $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$, $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}^*)$ and also of $H^1(\mathcal{M}, \text{End } \mathcal{T}_{\mathcal{M}})$, it is straightforward to deduce the lifted action of a symmetry $\varpi : \mathcal{M} \rightarrow \mathcal{M}$ to the cohomology. The Yukawa couplings must of course be invariant under any reparametrization of \mathcal{M} and so also with respect to ϖ . This usually annihilates a good deal of the Yukawa couplings and relates many of them. In a way, this is just an application of the Wigner-Eckart theorem, where several different physically measurable quantities $X^{(I)}$ can be “reduced” to a common quantity

$$X^{(I)} = (\text{Clebsh-Gordan})^{(I)} \cdot X_{\text{reduced}}, \quad (\text{C.4.1})$$

and the Clebsh-Gordan coefficients can be computed purely by symmetry considerations.

Also, one may consider constructing the complete intersection 3-fold one hypersurface at a time and computing the cohomology for every intermediate step. The advantage of doing so becomes apparent when we recall, once again, the example (C.2.14). Thinking of the Calabi-Yau 3-fold \mathcal{M} as a hypersurface in $\Sigma \times \mathbb{P}^2$, where Σ is the cubic surface in \mathbb{P}^3 , allows us to completely determine the required cohomology on Σ . Most of the properties of representatives of the $H^{*,*}(\Sigma)$ cohomology are retained as these representatives are inherited by $H^{*,*}(\mathcal{M})$. This can be traced by following the representatives carefully through the *Künneth formula*^(L) for $\Sigma \times \mathbb{P}^2$ and using the *Lefschetz Hyperplane Theorem*^(L) to deduce precisely which of these representatives are inherited by the 3-fold \mathcal{M} .

Finally, the *Lefschetz fixed point formulae*^(L) can be used, for one of the generators of the various symmetries at a time, to decompose the cohomology groups of curves and 2-folds according to representations of the symmetry groups. Combining these results, the same decomposition can then be achieved for any complete intersection 3-fold which can be represented as a complete intersection in products of curves and surfaces. One of the main parts of Ref. [153] is precisely such a computation, pertaining to a particularly simple and very symmetric member of the so-called Tian-Yau family of Calabi-Yau 3-folds, represented by the configuration

$$\left[\begin{matrix} 3 & \parallel & 3 & 0 & 1 \\ 3 & \parallel & 0 & 3 & 1 \end{matrix} \right]. \quad (\text{C.4.2})$$

C.4. Using Symmetries

We refer the interested Reader to consult Ref. [153] for details of this calculation. In view of the rather more general method of Koszul spectral sequences, one may regard this as an independent verification of computations. Alternatively, of course, one may not be interested in the complete answer, so a simpler computational method may be preferable.

Part III

Changelings

Chapter D

Parameter Spaces : From Afar

In most part of this volume, we have been discussing ways of constructing Calabi-Yau 3-folds (and in fact also spaces of other dimensions and types) and some of the most basic and general methods of computing the various numerical characteristics, guided in this latter task by a prospective physics application. In the process, it has become obvious that the number of topologically distinct Calabi-Yau 3-folds is rather immense. By their construction, they typically come in families and groups, and a general classification is beginning to take shape, albeit a very fuzzy one thus far.

D.1 The Programme

Before trying to describe what is known of these preliminary concepts of classification, it is only fair to say that the topic seems to be in an early stage of assembling a puzzle, without a templet picture and without a clue as to how many pieces there are. Some of the pieces however do fit together, rather nicely.

D.1.1 Identification numbers

Consulting any algebraic geometry text, one finds that curves are essentially classified by their genus (number of handles), which is equal to $(1 - \frac{1}{2}\chi_E)$. For surfaces, the situation becomes more complicated and one is advised first of all to compute the Kodaira dimension* [23,37] :

*The name “dimension” is in fact descriptive. There does exist an abstractly defined space, called however the Iitaka variety, the dimension of which equals the Kodaira dimension; for details, see p.247–248 of Ref. [37].

Definition. For a compact complex n -fold X with plurigenera^(L) \mathfrak{P}_m , the Kodaira dimension is defined to be

$-\infty$ (or -1) if $\mathfrak{P}_m = 0$ for all $m > 0$;

0 if $0 < \mathfrak{P}_m < \dim X$;

k if $m^{-k}\mathfrak{P}_m$ is bounded and $0 < k < n$ is the smallest such integer;

n all other cases.

There are no other types (☞ Prove this. ↪).

All Calabi-Yau n -folds have vanishing Kodaira dimension.



Next, (for vanishing Kodaira dimension) we are advised to check *regularity*, measured by $b_{1,0}$; this too vanishes for Calabi-Yau 3-folds (except if we agree to include complex 3-tori, the product of a complex torus and a K3 and the generalizations thereof, so called *abelian varieties*).

Finally, there are several numerical characteristics such as the Hodge numbers, which feature in Wall's classification Theorem 8.1, wherein a broad class of 3-folds, including Calabi-Yau ones, are classified up to diffeomorphism. We must caution here that a serious misunderstanding might arise between Calabi-Yau practitioners of mathematical *vs.* physics persuasion. A collection of valuable numerical characteristics with which to identify a Calabi-Yau model is supplied with the Yukawa couplings (see Chapters 0, 8 and C). However, there are "classical" and "quantum" Yukawa couplings, which are supposed to agree in a certain limit, but might be drastically different otherwise. In particular, the "classical" $(1,1)^3$ Yukawa couplings appear in Wall's theorem and are purely topological. The "quantum" $(1,1)^3$ Yukawa couplings include also a quantum correction term which depends on the mesh of rational curves in the manifold and the choice of the Kähler class and are therefore far from topological in the usual sense.

This being a bestiary *for physicists*, we will tip the balance in that direction when in a dilemma. Fortunately, I believe, there will always exist a limit in which the two points of view will agree and a bridge is possible.

D.1.2 Cartography or genealogy?

By simply naming examples and attaching a string of numerical characteristics not much more is done, however, than assembling a telephone directory. Although

D.1. The Programme

even such a simple-minded task is far from completion; computer lists of models have been compiled but it is unclear how to unambiguously tell one model from another since it is not in general known precisely what numerical characteristics identify a model.

A more unifying description would probably be universally agreed to be preferable. Amongst mathematicians, there are in general rather different points of view, ranging from "teleological" [216] to "biological" [202]. Physicists have of course an altogether different point of view, in that Calabi-Yau 3-folds occur only as a sector and are fitted into superstring theory models. Yet, perhaps surprisingly, some conclusions and conjectures will jibe.



First and foremost, Calabi-Yau spaces come bundled not only with a string of numerical characteristics (not all of which do we always know how to obtain), but also with a number of continuous parameters. Clearly, one generally tries to model these in the image of the far better understood *moduli spaces* of algebraic curves and surfaces.

As it has perhaps by now become obvious, by a Calabi-Yau 3-fold we *usually* mean a compact real 6-dimensional space endowed with a complex structure and a *complexified* Kähler class. Some examples in string theory suggest that the requirement of Kählerness might be relaxed although it is not clear what is precisely the broader class which is to replace this; the *Hodge decomposition*^(L) of cohomology is however supposed to persist in some suitable sense. Unless otherwise stated, we will assume that the Calabi-Yau 3-fold is Kähler. In addition, most constructions and analyses never really leave the holomorphic category and one is actually talking about algebraic varieties over the field of complex numbers.

Finally, the complexification of the Kähler class happens owing to requirements of supersymmetry; the action (0.1.2) which describes the propagation of *superstrings* contains not only the hermitian metric $G_{\mu\nu}$ on our 3-fold but also an antihermitian 2-form $B_{\mu\nu}$. Although they occur in the antihermitian combination $(B + iG)$ and the conjugate thereof, for certain purposes, the combinations $(B \pm G)$ might be used [93]. If $G_{\mu\nu}$ is a Kähler metric, $iG_{\mu\nu}dz^\mu dz^\nu$ represents the corresponding Kähler class, denoted J and so $\not{G} \stackrel{\text{def}}{=} (B - G) = (B + iJ)$, which is the complex variable usually taken to span the complexified Kähler cone.

In any case, therefore, there are two parameter spaces associated to a (topological class of) Calabi-Yau 3-fold(s) \mathcal{M} , the space of complex structures, which

we denote \mathfrak{M} and the complexified space of Kähler classes, which we denote by \mathfrak{W} . Now, quite generally, one first chooses a complex structure and is only then able to talk about (1,1)-cohomology classes. This would imply that the two parameter spaces are related non-trivially, that is that even the definition of \mathfrak{W} depends on where in \mathfrak{M} one is. However, because there are no non-trivial (2,0)- and (0,2)-cohomology classes, all of the degree-2 cohomology, $H^2(\mathcal{M}, \mathbb{k})$, can be identified with the (1,1)-cohomology. This makes \mathfrak{W} fairly independent of \mathfrak{M} , although not absolutely so; at certain special regions of \mathfrak{M} , the space \mathfrak{W} does undergo changes, tied to certain subtle change in the 3-fold parametrized [235]. Also, more drastically, at special regions where the parametrized 3-fold undergoes certain degenerations, \mathfrak{W} changes rather drastically. We will give examples of this behavior below. In view of this, by the parameter space of a (topological class of a) Calabi-Yau 3-fold we will mean the pair $(\mathfrak{M}, \mathfrak{W})$, with their dependence implied.

It becomes obvious then that the parameter space $(\mathfrak{M}, \mathfrak{W})$ is a smooth complex space at its generic points but we expect some degree of roughness in lower-dimensional subsets. In particular, we certainly expect \mathcal{M} to have finite quotient singularities since there certainly exist Calabi-Yau 3-folds in every topological class which have rather special (finite) symmetries. This is very much alike the \mathbb{Z}_2 - and \mathbb{Z}_3 -symmetric points in the moduli space of the torus.



One striking result is that the parameter spaces of topologically distinct Calabi-Yau spaces do touch along certain special regions, which we loosely speaking call “boundary” although the dimension of these regions is at least one (complex!) dimension smaller than the generic part. Such ‘interface’ regions correspond to certain singular limits of the respective Calabi-Yau 3-folds. By including these limit points, the parameter spaces of a large number of simply connected Calabi-Yau spaces are explicitly shown to join together to form a connected ‘web’ [149]. This resonates strongly with Reid’s conjecture [203], but in its full span it of course remains a conjecture.

We will presently describe some of this *global* picture and will turn to some preliminary description of the local (and metric) properties of $(\mathfrak{M}, \mathfrak{W})$ in the next Chapter. The reader should bear in mind that this is still a very rapidly developing field and the present account is only meant as a rough sketch. It will transpire, I hope, that the conglomeration of Calabi-Yau 3-folds, tentatively and partially aligned by the web of Ref. [149], do seem to be organizable into an orderly

chart, which moreover appears to take a hierarchical (genealogical?) structure. Finally, I believe that this structure will play a rôle not only in the mathematical classification but will also have a physical interpretation, perhaps not too far from the ideas in Ref. [149,90].

D.2 Splitting : New Manifolds for Old

As described so far, Calabi-Yau spaces may be constructed in a variety of ways. The moduli spaces of complete intersections in products of complex projective spaces have been explicitly shown to be linked together [149,92]. This was done utilizing the process of *splitting* [85,149] and we now turn to describe this.

We first consider a simple example to illustrate how it is possible to move continuously from the moduli space of one Calabi-Yau manifold to the moduli space of another.

$$\check{\mathcal{M}} \in \left[\begin{array}{c|ccc} 5 & 4 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 \end{array} \right]_{x_E = -168}^{b_{1,1}=2} \quad (\text{D.2.1})$$

for which the defining equations, encoded by the columns of the matrix (D.2.1), may be without loss of generality be written as

$$\begin{aligned} Q(x) &= 0, \\ x^1y^1 + x^2y^2 &= 0, \\ x^3y^1 + x^4y^2 &= 0. \end{aligned} \quad (\text{D.2.2})$$

A relatively simple choice of transverse polynomials is obtained by choosing, in addition to the bi-linear constraints, $Q(x) = \sum_{a=1}^6 (x^a)^4$. We may also without loss of generality think of this example as described by the configuration

$$\check{\mathcal{M}} \in \left[\begin{array}{c|cc} \mathfrak{X} & 1 & 1 \\ \hline \mathbb{P}^1 & 1 & 1 \end{array} \right] , \quad \mathfrak{X} \in [5||4] , \quad (\text{D.2.3})$$

that is, \mathfrak{X} is the quartic 4-fold in \mathbb{P}^5 defined by $Q(x) = 0$.

For any particular $x \in \mathfrak{X}$, we are left with a system of two equations in two variables, y^i . Being homogeneous coordinates of a \mathbb{P}^1 , y^1 and y^2 cannot both vanish, it follows that the determinant of the coefficients must vanish :

$$\mathcal{C}(x) \stackrel{\text{def}}{=} x^1x^4 - x^2x^3 = 0 . \quad (\text{D.2.4})$$

The quadric hypersurface $\mathcal{M}^\sharp \subset \mathfrak{X}$, defined by $\mathcal{C}(x) = 0$, is clearly a Calabi-Yau 3-fold and may be regarded as the projection $\mathcal{M}^\sharp = \varphi(\check{\mathcal{M}})$ along the \mathbb{P}^1 in (D.2.2).

Alternatively, we may identify $\check{\mathcal{M}} = \lim_{R \rightarrow 0} \check{\mathcal{M}}$, where R is the average radius of the \mathbb{P}^1 . Clearly, being an intersection of the quartic $Q(x) = 0$ and the quadric $C(x) = 0$ in \mathbb{P}^5 , \mathcal{M}^\sharp is singular at $x^1, x^2, x^3, x^4 = 0$ as there $Q(x)$ and $C(x)$, but also $dC(x)$ vanish. Since $Q(x)$ is a (generic) quadric, this happens at four points. For the simple choice $Q(x) = \sum_{i=1}^6 (x^i)^4$, the singular points are at $(0 : 0 : 0 : 0 : \omega : 1)$, $\omega^4 = -1$. Note that singularities induced from such a projection occur at points and are nodes (recall § 4.4.1 and 4.4.2); this follows from the smoothness of $\check{\mathcal{M}}$ and Bertini's theorem, and can also easily be checked.

To smooth out \mathcal{M}^\sharp , we perturb the defining equation (D.2.4) of $\mathcal{M}^\sharp \subset \mathfrak{X}$ and obtain

$$C(x) = r^2(x). \quad (\text{D.2.5})$$

Here, $r^2(x)$ is a quadric perturbation, non-zero at the nodes of \mathcal{M}^\sharp , such as $r^2(x) = t(\sum_{a=1}^6 (x^a)^2)$. For any small $t \neq 0$, this defines a non-singular quadric \mathcal{M}^\flat_t in \mathfrak{X} in \mathbb{P}^5 .

Rather explicitly, the conifold* \mathcal{M}^\sharp occurs at a common point in the boundary of the relevant parameter spaces of non-singular 3-folds such as $\check{\mathcal{M}}$ and smooth 3-folds such as $\mathcal{M}^\flat_t \subset \mathfrak{X}$. Alternatively, the conifold \mathcal{M}^\sharp can be smoothed in two topologically distinct ways : by a *small resolution* into $\check{\mathcal{M}}$, or by a *deformation* into \mathcal{M}^\flat . That is,

$$\begin{array}{ccc} \check{\mathcal{M}} & & \check{\mathcal{M}} \\ \downarrow p & , & \uparrow \text{small resol.} \\ \mathcal{M}^\flat \xrightarrow{r^2 \rightarrow 0} \mathcal{M}^\sharp & \leftarrow \text{deform.} & \mathcal{M}^\sharp \end{array} \quad (\text{D.2.6})$$

Superficially, it may appear that the present situation is an isolated case. Not so; it exemplifies an ubiquitous process [149] which connects parameter spaces of a great many Calabi-Yau spaces and which—suitably generalized—may in fact link the moduli spaces of all simply connected Calabi-Yau manifolds. In this large class (described below) there always exist conifolds which occur as limit points (in the two different senses) in the two respective parameter spaces. These conifolds have isolated nodes, each described locally by an equation of the type (D.2.4) :

$$XY - UV = 0. \quad (\text{D.2.7})$$

*Following Ref. [92], we use this name for a space which has only isolated conical singularities. For all of these, the square-integrable cohomology (in, say, the projectively induced metric) is dual to intersection homology, which simplifies explicit calculations.

The quantities X, Y, U, V (generally polynomials in the homogeneous coordinates on the embedding \mathbb{P}^n 's) may be regarded as local coordinates, whence they describe a cone with apex at the origin, as discussed in § 4.4. Note that the conifold \mathcal{M}^\sharp is in fact defined by such a determinantal equation also *globally*, inside \mathfrak{X} .

D.2.1 Nodes : locally and what to do with them

To analyze the local properties of nodes, we consider an open neighborhood of $p^\sharp \in \mathcal{M}^\sharp \subset \mathfrak{X}$ around each node. The neighborhood is locally like \mathbb{C}^4 and the four polynomials X, Y, U, V serve as local coordinates. In § 4.4 we saw that this cone can be viewed either as a complex cone over $\mathbb{P}^1 \times \mathbb{P}^1$, or a real cone over $S^2 \times S^3$.

Next, we compare the two distinct ways of smoothing a node, by following the two processes $\mathcal{M}^\sharp \rightarrow \mathcal{M}^\flat$ and $\mathcal{M}^\sharp \rightarrow \check{\mathcal{M}}$, respectively.

To begin with, consider the deformation introduced in Eq. (D.2.5). Locally at the nodes, we can treat r^2 as a parameter independent of U, V, X, Y . This is easily justified by considering the above simple choices : For $r^2(x) = t(\sum_{a=1}^6 (x^a)^2)$ at the 4 nodes $(0, 0, e^{2i\pi(2k+1)/8}, 1)$ and with $k = 0, 1, 2, 3$, we have $r^2 = -t(1 \pm i)$, clearly non-zero for $t \neq 0$. A node $p^\sharp \in \mathcal{M}^\sharp$ then corresponds to $r = 0$ in

$$\vec{\xi}^2 = \vec{\eta}^2 = r^2, \quad \vec{\xi} \cdot \vec{\eta} = 0, \quad (\text{D.4.2}')$$

where a local and holomorphic change of variables has been performed. In the $\mathcal{M}^\sharp \rightarrow \mathcal{M}^\flat$ process, each node is being “inflated” into an S^3 described by

$$\vec{\xi}^2 = r^2; \quad \vec{\eta} = 0. \quad (\text{D.2.8})$$

In § 4.4, we have described a different way of smoothing, when each node p^\sharp becomes replaced by a \mathbb{P}^1 , aligned with one or the other \mathbb{P}^1 in the base of the complex cone. That this is indeed what happens when replacing \mathcal{M}^\sharp with $\check{\mathcal{M}}$, is easily seen by analyzing what happens to the y 's from Eq. (D.2.2). Since $x^1 x^4 - x^2 x^3 = 0$, the matrix

$$\begin{pmatrix} x^1 & x^2 \\ x^3 & x^4 \end{pmatrix} \quad (\text{D.2.9})$$

has rank 1, except at a node where $x^1, x^2, x^3, x^4 = 0$ and the rank is 0. Generically, therefore, the equations fix y^1/y^2 uniquely and hence specify a point in \mathbb{P}^1 . In contrast, y^1 and y^2 remain undetermined when $x^1, x^2, x^3, x^4 = 0$ and $\check{\mathcal{M}}$ contains a copy of the embedding \mathbb{P}^1 there. When we project along the embedding \mathbb{P}^1 , this entire copy of \mathbb{P}^1 is crushed to a node of \mathcal{M}^\sharp . This is depicted in Figure 6.3.

So, in the $\mathcal{M}^\sharp \rightarrow \check{\mathcal{M}}$ process, each node p^\sharp is resolved into a copy of $\mathbb{P}^1 = S^2$ generated by $y^1/y^2 = -x^2/x^1 = -x^4/x^3$. This indeed is the small resolution, the present simplicity of which is special to three dimensions [2]. These two processes are depicted schematically in Figure D.1

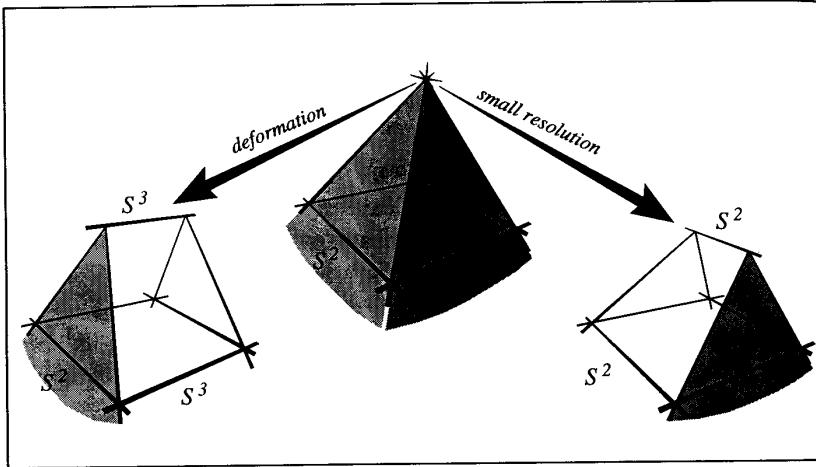


Figure D.1: Local transformation of cycles near a node under a smoothing deformation *vs.* a small resolution.

D.2.2 The flop

The careful reader may have complained to first solving $\vec{\xi}^2 = r^2$ from Eqs. (4.4.2) which produced the “ S_ξ^3 ”, spanned by the $\vec{\xi}$. Starting from $\vec{\eta}^2 = r^2$ instead, we find an “ S_η^3 ”. So, with which of these two S^3 ’s do we “align” the S^3 introduced in the deformation $\mathcal{M}^\sharp \rightarrow \mathcal{M}^\flat$? In fact, there is a continuous set of choices, parametrized by an angle :

$$\vec{\xi}(\theta) \stackrel{\text{def}}{=} \vec{\xi} \cos \theta + \vec{\eta} \sin \theta, \quad \vec{\eta}(\theta) \stackrel{\text{def}}{=} -\vec{\xi} \sin \theta + \vec{\eta} \cos \theta. \quad (\text{D.2.10})$$

Since $\vec{\xi}(\theta)$ and $\vec{\eta}(\theta)$ satisfy Eqs. (4.4.2), Eqs. (D.2.10) establish a (homotopy) equivalence between all these choices, including “ S_ξ^3 ” ($\theta = 0$) and “ S_η^3 ” ($\theta = \pi$).

As for small resolutions, however, no such equivalence exists and a topological ambiguity does persist, as described in general in § 4.4. Here, this ambiguity is seen explicitly. Each node is resolved into a \mathbb{P}^1 generated by $y^1/y^2 = -x^2/x^1 = -x^4/x^3$. Instead, a node could be resolved into a \mathbb{P}^1 parametrized by $y^1/y^2 = -x^3/x^1 = -x^4/x^2$ and this obviously requires the transposition

$$\begin{pmatrix} x^1 & x^2 \\ x^3 & x^4 \end{pmatrix} \rightarrow \begin{pmatrix} x^1 & x^3 \\ x^2 & x^4 \end{pmatrix}. \quad (\text{D.2.11})$$

This could be done at each of the nodes independently. In a way, the *global* construction $\varphi: \check{\mathcal{M}} \rightarrow \mathcal{M}^\sharp$ provides a “ready made” Kähler small resolution of the conifold \mathcal{M}^\sharp , answering positively the otherwise (in general) rather difficult question, whether any of the 2^N small resolutions are Kähler.

In the present example, we can flop all four exceptional \mathbb{P}^1 ’s, by relabeling $x^2 \leftrightarrow x^3$ *globally*. In principle, the result of the above transposition, if possible (especially in other examples, when the elements are other than linear in the coordinates), yields a new 3-fold : the global *flop* of $\check{\mathcal{M}}$, which we might denote by $\widehat{\mathcal{M}}$. In the present case, however, we can also relate the two by deformation : Bertini’s theorem tells us that the parameter space of smooth quartics is connected, so that, whatever our choice of the quartic $Q(x)$, $Q(x)|_{x^2 \leftrightarrow x^3}$ can be deformed back into $Q(x)$ smoothly.

D.2.3 Transvestite cycles

Let us now follow the change of some numerical characteristics, using the above analysis. In the process $\check{\mathcal{M}} \rightarrow \mathcal{M}^\flat$, we replace an S^2 by and S^3 for each of the N nodes in \mathcal{M}^\sharp . Knowing that $\chi_E(S^3) = 0$ and $\chi_E(S^2) = 2$, we have

$$\chi_E(\check{\mathcal{M}}) = \chi_E(\mathcal{M}^\flat) + 2N. \quad (\text{D.2.12})$$

Consider now the diagram in Fig. D.1. It represents the local transformation occurring at the neighborhood containing a minimal $S^3_t \subset \mathcal{M}^\flat$, which is being crushed into a node of \mathcal{M}^\sharp by deformation and then replaced by an $S^2 \subset \mathcal{M}$ by small resolution. In view of the general discussion of small resolutions in § 4.4, for all A, D, E type singularities the transformation is merely an iteration of the small resolution part.

Now, in \mathcal{M}^\flat , the (\mathbb{R}_+ -like) generators of the cone form a copy of B^3_t by passing through the points of the S^2 in the base at every fixed point of the minimal

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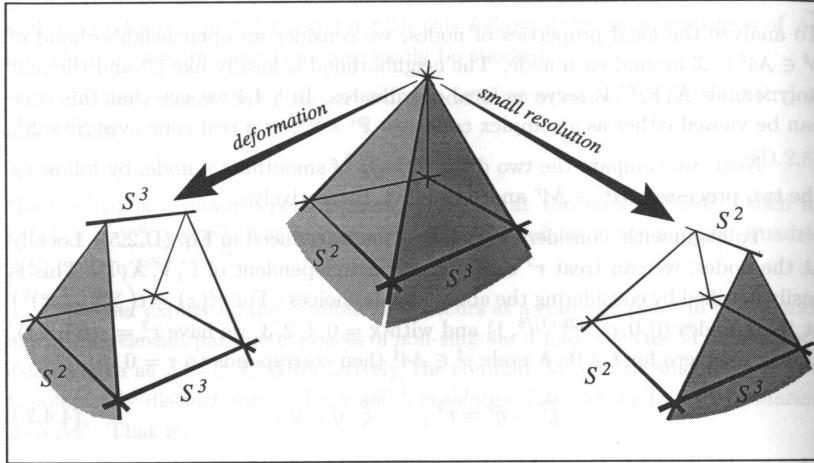


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In the present example, we can flop all four exceptional \mathbb{P}^1 ’s, by relabeling $x^2 \leftrightarrow x^3$ *globally*. In principle, the result of the above transposition, if possible (especially in other examples, when the elements are other than linear in the coordinates), yields a new 3-fold : the global *flop* of $\check{\mathcal{M}}$, which we might denote by $\tilde{\mathcal{M}}$. In the present case, however, we can also relate the two by deformation : Bertini’s theorem tells us that the parameter space of smooth quartics is connected, so that, whatever our choice of the quartic $Q(x)$, $Q(x)|_{x^2 \leftrightarrow x^3}$ can be deformed back into $Q(x)$ smoothly.

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Now, in \mathcal{M}^\flat , the (\mathbb{R}_+ -like) generators of the cone form a copy of B^3_t by passing through the points of the S^2 in the base at every fixed point of the minimal

S^3_t , drawn in Figure D.1 as the crest of the “tent”. This is suggested by the shaded area. This B^3_t is then completed, possibly into a cycle, elsewhere in \mathcal{M}^\flat , outside of this neighborhood. Note that the analogous construction obtained by collecting all (\mathbb{R}_+ -like) generators passing through the points of the S^3 in the base while keeping a point of the S^2 in the base fixed yields a “4-hoop” $\Pi^4 \approx S^3 \times \mathbb{R}_+$, the boundary of which is the minimal S^3_t . This clearly is not a cycle for it has a boundary, but will become important as we progress towards the right hand side of Figure D.1. Similarly, the S^2 in the boundary is not a cycle because it can be contracted (along the \mathbb{R}_+ -like generators of the cone) to a point (on S^3_t).

It should be obvious that precisely if B^3_t becomes completed into a 3-cycle, $C^{(3)}$, S^3_t is homotopically non-trivial and the cycle that S^3_t represents is dual to $C^{(3)}$. Each can be “propagated” along the other one and they meet in a single point although, of course, S^3_t may develop into some different 3-space as it is propagated along a representative of $C^{(3)}$. This is very much like in the case of Riemann surfaces.

Now deform \mathcal{M}^\flat into \mathcal{M}^\sharp : both S^3 and S^2 in the base can be contracted to the vertex of the cone and therefore fail to represent non-trivial 3- and 2-cycles respectively. However, nothing has happened to the cycle represented by the 3-space the portion of which we see in Figure D.1 as B^3_t ; if this was a non-trivial cycle in \mathcal{M}^\flat , it remains to be so in a ‘smooth homology’ of \mathcal{M}^\sharp . However, using Theorem 7.3, we see that it fails to be a non-trivial element of $IH_3(\mathcal{M}^\sharp)$, since it becomes an open interval when the singular point is removed and is therefore contractible (outside of this neighborhood). Now, as for the “4-hoop” Π^4 , we see that its boundary, S^3_t , has shrank to a point; this was the only local obstruction for it to be a portion of a representative of a 4-cycle. If there is no obstruction outside of this neighborhood, we have just obtained a new 4-cycle. Note that, also by Theorem 7.3, this 4-cycle survives as an element of $IH_4(\mathcal{M}^\sharp)$ because of the different treatment of $IH_3(\mathcal{M}^\sharp)$ and $IH_k(\mathcal{M}^\sharp)$ for $k \neq 3$. Also, there is a dual element of $IH_2(\mathcal{M}^\sharp)$ even though there is no contribution to the ‘smooth’ $H_2(\mathcal{M}^\sharp)$ from this neighborhood.

Finally, we pass to the small resolution, replacing the node with the minimal S^2_t , drawn as the crest of the “tent”-like neighborhood in the right hand side in Figure D.1. Now B^3_t develops a hole— S^2_t itself—and becomes a “3-hoop” $\Pi^3 \approx S^2 \times \mathbb{R}_+$, representing therefore the trivial 3-cycle. Each S^3 in the base is homologous to a point on S^2_t and hence homologically trivial also; there is no local 3-homology left. On the other hand, each S^2 in the base (forming a

D.2. Splitting : New Manifolds for Old

family parametrized by the S^3) is homologous to the S^2_t in the crest and is locally non-trivial as an element of $H_2(\mathcal{M})$. Similarly, the “4-hoop” Π^4 , the boundary of which, S^3_t , has been contracted into a point is possibly a portion of a representative of a non-trivial 4-cycle, $C^{(4)}$. Indeed, precisely if it is non-trivial—then S^2_t is a representative of a non-trivial 2-cycle and these two are dual to each other.

D.2.4 Three deformations of four nodes

When running the process Eq. (D.2.2)–(D.2.4)–(D.2.5) backwards, $\mathcal{M}^\flat \xrightarrow{t \rightarrow 0} \mathcal{M}^\sharp \xrightarrow{\psi^{-1}} \tilde{\mathcal{M}}$, we easily count four nodes in \mathcal{M}^\sharp as four solutions for (x_5/x_6) of $Q(x) = 0$ at $x_1 = x_2 = x_3 = x_4 = 0$. Therefore, there must be four $S^3_{t^\alpha}$'s in \mathcal{M}^\flat which are being crushed into nodes during the $\mathcal{M}^\flat \rightarrow \mathcal{M}^\sharp$ deformation. It is, however, an elementary fact that there are only three independent quadric polynomials in two variables. It follows that there must be precisely one relation between the four $S^3_{t^\alpha}$'s and can be represented as a 4-chain, Z^4 , the boundary of which is precisely the union of these four 3-spheres. Also, it cannot possibly be a boundary of something else, since then the boundary of a boundary would not be empty. So, the four minimal 3-spheres $S^3_{t^\alpha}$ are *the only obstruction* for the “4-chain” Z^4 to become a 4-cycle!

As we pass to \mathcal{M}^\sharp , it is obvious that Π^4 becomes a 4-cycle and the Theorem 7.3 guarantees that there is also a dual element in $IH_2(\mathcal{M}^\sharp)$, although an explicit representative is hard to find. When we pass to the small resolution, $\tilde{\mathcal{M}}$, essentially nothing happens to this 4-cycle. The introduction of the four $S^2_{t^\alpha}$, however provides explicit representatives for the 2-cycle dual to the 4-cycle represented by Π^4 (with the $S^3_{t^\alpha}$ boundaries contracted to points). Moreover, the four S^2 's are in fact homologous to each other by “propagation” along Π^4 . Fig. D.2 might perhaps clarify the situation.

Table D.1 displays the effect on the cohomology groups. In $k = 3$, the $3 + 3$ denote the three independent $S^3_{t^\alpha}$'s plus their three dual cycles, so that $0 + 3$ means that while the three $S^3_{t^\alpha}$'s have become trivial, their dual cycles are still nontrivial in the ‘smooth homology’ group $H_k(\mathcal{M}^\sharp)$. In $IH_k(\mathcal{M}^\sharp)$, however, neither of these 3-spaces represents a non-trivial cycle, whence the “0 + 0” entry. Throughout, $x = 1$ and $y = 174$ count cycles that do not depend on the four described neighborhoods.

The reader must have noted the fact that $IH_k(\mathcal{M}^\sharp) = H_k(\mathcal{M})$. Indeed, this is just an explicitly computed consequence of Theorem 7.4. Finally, because the

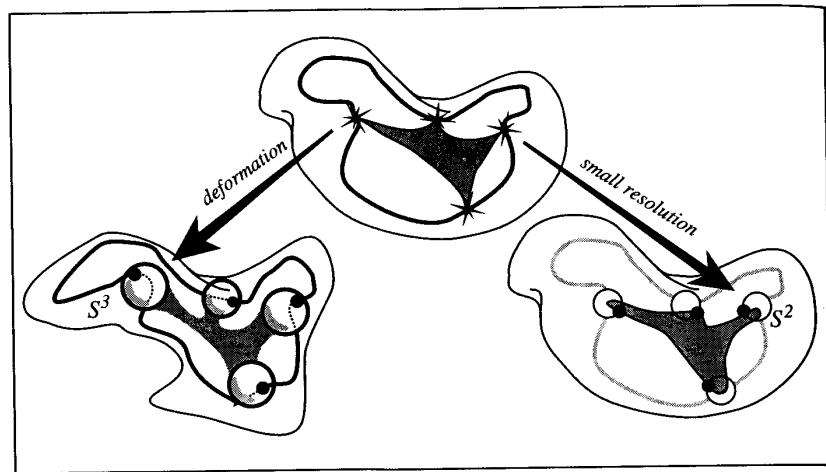


Figure D.2: Global transformation of cycles in a nodal (singular) deformation which is then followed by a small resolution.

L^2 -cohomology and the intersection homology are dual[†] for the case of conical singularities, the same relation will hold for cohomology : $H_{(2)}^{p,q}(\mathcal{M}^\sharp) = H^{p,q}(\check{\mathcal{M}})$. In view of Theorem 7.4, the same will be true if instead of nodes, other A, D, E -singularities had appeared.

D.2.5 101 deformations (of 125 nodes)

Consider now the Calabi-Yau manifold embedded in \mathbb{P}^4 by means of

$$\mathcal{Q}(x; \psi) \stackrel{\text{def}}{=} \sum_{i=1}^5 x_i^5 - 5\psi \prod_{i=1}^5 x_i = 0. \quad (\text{D.2.13})$$

At $\psi = 1$, this quintic has 125 nodes of the type $(x_1 : x_2 : x_3 : x_4 : x_5) = (\alpha : \beta : \gamma : \delta : \omega)$, where $\alpha^5 = \beta^5 = \gamma^5 = \delta^5 = \omega^5 = 1$. Because of projectivity, we may set $\omega = 1$; for the gradients of \mathcal{Q} to vanish (condition for singularity), it suffices to set $\alpha\beta\gamma\delta = 1$, which fixes, e.g., δ . We are thus left with three fifth roots of unity, α, β, γ , chosen freely—hence 125 nodes. One of these is (1:1:1:1:1).

[†]With a suitable class of norms (metrics) for the L^2 -cohomology.

D.3. A View into Moduli Spaces

k	$H_k(\mathcal{M}^\flat)$	$H_k(\mathcal{M}^\sharp)$	$IH_k(\mathcal{M}^\sharp)$	$H_k(\mathcal{M})$
0	1	1	1	1
1	0	0	0	0
2	x	x	$x + 1$	$x + 1$
3	$y + 3 + 3$	$y + 0 + 3$	$y + 0 + 0$	$y + 0 + 0$
4	x	$x + 1$	$x + 1$	$x + 1$
5	0	0	0	0
6	1	1	1	1

Table D.1: The effect of “splitting” on homology.

Now, similarly to the case above, we can deform the singular quintic $\mathcal{Q}(x; 1)$ by adding arbitrary quintic monomials. Trouble is, there are only 101 independent quintic monomials (modulo linear reparametrizations). On the other hand, there are 125 nodes, each being deformed into an S^3 . These 125 S^3 's however can represent only 101 independent cycles and there must exist precisely 24 relations between them, each represented by a 4-chain Z_a^4 , $a = 1, \dots, 24$. At the conifold, $\psi = 1$, these 4-chains become 4-cycles, $C_a^{(4)}$, and give rise to 24 new elements of IH_4 and also IH_2 , by duality. Indeed, Ref. [211] reports $\dim H^{1,1} = 25$ for the small resolution of this conifold with 125 nodes (1 generator for the Kähler class and 24 associated to $C_a^{(4)}$ -type cycles).

D.3 A View into Moduli Spaces

Towards our promise in the first section of this Chapter : we will reinterpret the processes (D.2.6) as operations on the respective parameter spaces. To that end, recall that as discussed above, Calabi-Yau compactification is specified by giving a Calabi-Yau manifold \mathcal{M} with a complex structure and (complexified) Kähler class. Thus, there are two distinct types of deformation of this structure; we first study, in turn, the parameter spaces for these.

D.3.1 Complex structure moduli

From the general discussion in sections 4.3 and 4.4, we know that $(\check{\mathcal{M}} - E) = (\mathcal{M}^\sharp - \{p^\sharp\})$, where E is the collection of 2-spheres into which the nodes p^\sharp have been resolved. As there are only finitely many nodes and small resolutions and

since the defining polynomial(s) of \mathcal{M}^\sharp depend(s) holomorphically on those defining $\check{\mathcal{M}}$, we can locally identify the parameters of $\check{\mathcal{M}}$ with those of \mathcal{M}^\sharp , that is, $\check{\mathcal{M}} \sim \mathcal{M}^\sharp$. In fact, we can identify

$$\check{\mathcal{M}} \xrightarrow{\sim} \mathcal{M}^\sharp = \lim_{t \rightarrow 0} \mathfrak{M}^b \subset \mathfrak{M}_+^b, \quad (\text{D.3.1})$$

where $\check{\mathcal{M}}$ and \mathcal{M}^\sharp parametrize the complex structure for $\check{\mathcal{M}}$ and \mathcal{M}^\sharp respectively and \mathfrak{M}_+^b parametrizes the complex structure of \mathcal{M}^b , including (in the ‘boundary’) conifold singularizations thereof. To orient the reader, we had $\dim \check{\mathcal{M}} = \dim \mathcal{M}^\sharp = 86$ while $\dim \mathfrak{M}^b = 89$ in (D.2.6). Locally, the map $\xrightarrow{\sim}$ may be treated as an identification. Whenever polynomial deformations represent $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ faithfully, these parameter spaces and maps can be constructed explicitly; although this is clearly not the case in general, the above relations can be generalized appropriately [149].

D.3.2 Kähler class moduli

At any complex structure (point $t \in \mathfrak{M}$), the Kähler $(1,1)$ -class $[J]$ determines, by Yau’s theorem, a positive Kähler metric $J = i g_{\mu\nu} dz^\mu \wedge d\bar{z}^\nu$. As J is a $(1,1)$ -form, it is determined as a linear combination of $(1,1)$ -forms $J = v^A e_A$ from a complete set and with real coefficients v^A .

For J to determine a positive metric (such a $(1,1)$ -form is said to be positive) the allowed values of v^A must be restricted and we may, in general, end up with an empty set. $b_{1,1} \geq 0$ is only a necessary, not a sufficient condition for a manifold to be Kähler; for a $(1,1)$ -form to qualify for a Kähler form, it must satisfy

$$\int_c J > 0, \quad \int_s J^2 > 0, \quad \int_M J^3 > 0. \quad (\text{4.3.16}')$$

It is a simple fact that if J_x and J_y are both positive, so is $v^x J_x + v^y J_y$ for v^x, v^y real and positive. The parameter space of the Kähler structure \mathfrak{RM} is a then cone in $\mathbb{R}^{b_{1,1}}$, called the Kähler cone. As discussed above, we include the antihermitian 2-form B by expanding $B = u^A e_A$, writing $\not{B} \stackrel{\text{def}}{=} (B + iJ) = (u^A + iv^A)e_A$ for the complexified Kähler form and \mathfrak{W} for the so complexified Kähler cone.

These characterizations of (the real part of) \mathfrak{W} depend in general on the choice of $t \in \mathfrak{M}$. We expect therefore the combined parameter space of Calabi-Yau manifolds to be a space fibered over the space of complex structures, with Kähler cones as fibers.

D.3. A View into Moduli Spaces

For many complete intersection (Calabi-Yau) manifolds \mathcal{M} embedded in products of flag spaces $\mathcal{X} = \mathbb{F}_{[n_1]} \times \cdots \times \mathbb{F}_{[n_m]}$, at least a subgroup of $H^{1,1}(\mathcal{M})$ will equal (an isomorphic image of) $H^{1,1}(\mathcal{X})$ (see chapters 2 and 3). The latter cohomology group has a basis consisting of the generators $J_{(r)} \in H^2(\mathbb{F}_{[n_r]}, \mathbb{Z})$. Let \check{e}_s denote the remaining, ‘vanishing generators’ of $H^{1,1}(\mathcal{M})$. Positive one forms are then given as $v^r J_{(r)} + v_0^s \check{e}_s$, with suitable coefficients v_0^s and positive and large enough v^r . When $J_{(r)}$ span all of $H^{1,1}(\mathcal{M})$, the combined parameter space is *locally* and at a generic choice a direct product*.

That the above emphasis on locality and genericity is not pedantic nitpicking is seen from a simple example tailored from the surfaces studied in § 3.1.2 and 8.6. Consider the Calabi-Yau 3-fold defined as

$$\mathcal{M}_\varepsilon \in \left[\begin{array}{c|ccc} 3 & 1 & 1 & 2 \\ \hline 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ \hline 0 & 0 & 2 \end{array} \right] : \quad \left\{ \begin{array}{l} z_0 w_0 + z_1 w_1 = 0, \\ z_2 w_0 + [\sum_{i=0}^2 a_i z_i + \varepsilon z_3] w_1 = 0, \\ Q(z, x, y) = 0. \end{array} \right. \quad (\text{D.3.2})$$

Here z, w, x, y are, respectively, homogeneous coordinates for $\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $Q(z, x, y)$ is a smooth degree-(2, 2, 2) polynomial in z, x, y . The Lefschetz hyperplane theorem guarantees that the (pull-backs of the) Kähler forms, J_z, J_w, J_x, J_y generate $H^{1,1}(\mathcal{M}_\varepsilon, \mathbb{Z})$ for all ε , including $\varepsilon = 0$. Also, $J(v^z, v^w, v^x, v^y) \stackrel{\text{def}}{=} v^z J_z + v^w J_w + v^x J_x + v^y J_y$ is Kähler if $v^z, v^w, v^x, v^y > 0$.

For $\varepsilon \neq 0$, we may use Eqs. (8.6.3) to replace $\{J_z, J_w\}$ with $\{a, b\}$ since $\left[\begin{smallmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix} \right]_{\varepsilon \neq 0} \approx \mathbb{P}^1 \times \mathbb{P}^1$. In terms of $\{a, b, J_x, J_y\}$, the real Kähler cone is given by the wedge $v^a, v^b, v^x, v^y > 0$, which implies $v^z, v^w, v^y > 0$ and $v^w > -v^z$! So, the $0 > v^w > -v^z$ region of $\mathfrak{RM}_{\varepsilon \neq 0}$ is not allowed in the $\varepsilon = 0$ case. The reason for this is that $\mathcal{M}_{\varepsilon=0}$ contains an elliptic surface (related to the line of self-intersection -2 in the surface $\left[\begin{smallmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix} \right]_{\varepsilon=0} = \mathbf{F}_2$) and $J(v^z, v^w, v^x, v^y)$, with $v^z, v^x, v^y > 0$ and $v^w > -v^z$, fails to be positive over this exceptional set unless $v^w > 0$. In Ref. [235], it was proven that this typifies the only process in which smooth but special deformations of the complex structure affect the (complexified) Kähler cone.



*In the general case and at a generic point in the parameter space, the integral cohomology still gives a canonical *local* product structure to the fibration over the space of complex structures and with the *real* $(1,1)$ -cohomology as the fibre. However, the location of the (positive) Kähler cone within this fibre might depend on the complex structure.

Roughly, the v^r 's may be identified with the sizes of the factors $\mathbb{F}_{[n_r]}$ in \mathcal{X} . In Eq. (D.2.1), we had $J(\mathcal{M}^\flat) = v^x J_x$ and \mathfrak{W}^\flat is the positive semi-axis $\{0 < v^x < \infty\}$ while $J(\check{\mathcal{M}}) = v^x J_x + v^y J_y$ so that $\check{\mathfrak{W}}$ is the first quadrant $\{0 < v^x, v^y < \infty\}$.

Note that \mathcal{M}^\sharp is singular and does not admit a Kähler class in the usual sense; however, we define a real cohomology class on \mathcal{M}^\sharp to be positive if its restriction to the complement of the nodes is represented by a positive $(1,1)$ -form the cube of which has a finite integral. We can therefore identify both \mathfrak{W}^\sharp and $\check{\mathfrak{W}}$ with the positive multiples of J_x and have

$$\check{\mathfrak{W}}_+ \supset (\lim_{\mathbb{P}^1 \rightarrow} \check{\mathfrak{W}}) \xleftarrow{\sim} \mathfrak{W}^\sharp = \mathfrak{W}^\flat. \quad (\text{D.3.3})$$

Here $\check{\mathfrak{W}}_+$ is the Kähler cone of $\check{\mathcal{M}}$ together with its boundaries. Also, the map “ $\xleftarrow{\sim}$ ” is a and identification between \mathfrak{W}^\sharp and the (corresponding subset of the) boundary $\lim_{\mathbb{P}^1 \rightarrow} \check{\mathfrak{W}}$.

The combined parameter space of \mathcal{M}^\sharp can be identified therefore with an interfacing region in the boundary of both the parameter space for $\check{\mathcal{M}}$ and that for \mathcal{M}^\flat . This fact motivated the suggestion [149,90,92] that the process (D.2.6) might acquire a physical interpretation as a phase transition : certain paths, passing through the parameter space of \mathcal{M}^\sharp , connect points of distinct parameter spaces which represent non-singular Calabi-Yau 3-folds of distinct homotopy types.

D.3.3 An odd example

To illustrate some properties of nodal singularizations, we turn to another example of a deformation class of smooth CICYs and some of its nodal limits for which no Kähler small resolution is known.

Let \mathcal{M} now be the quintic hypersurface in \mathbb{P}^4 :

$$I = x_5^3 \left(\sum_{i=1}^4 x_i^2 \right) + \sum_{i=1}^5 a_i x_i^5, \quad (\text{D.3.4})$$

where x_i are the homogeneous coordinates of \mathbb{P}^4 . On the $x_5 = 0$ hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$

$$\begin{aligned} I &= \sum_{i=1}^4 a_i x_i^5, \\ dI &= 5 \sum_{i=1}^4 dx_i \cdot a_i x_i^4. \end{aligned} \quad (\text{D.3.5})$$

D.3. A View into Moduli Spaces

For generic a_i , $I = 0 = dI$ only at $x_i = 0$, which is not in \mathbb{P}^4 , so the singular points of \mathcal{M} are all in the $x_5 \neq 0$ coordinate patch and we set $x_5 = 1$.

0. For a generic choice of a_i , $I = 0$ and $dI = 0$ have no common solution, so that the quintic hypersurface $I = 0$ in \mathbb{P}^4 is a manifold.

Let now $a_5 \rightarrow 0$. Then, $dI = 0$ implies $x_i(5a_i x_i^3 + 2) = 0$ and candidate singular points (p^\sharp) are parametrized as :

$$x_i = -\alpha_i \sqrt[3]{2/5a_i} \cdot \omega^{k_i},$$

where $\omega = e^{2i\pi/3}$, $k_i = 0, 1, 2$, $\alpha_i = 0, 1$, $i = 1, 2, 3, 4$. At these points

$$I(p^\sharp) = \frac{3}{5} \sqrt[3]{\frac{4}{25}} \left(\sum_{i=1}^4 \frac{\alpha_i \omega^{2k_i}}{a_i^{2/3}} \right)$$

and we have three cases :

1. $(0, 0, 0, 0, 1)$, i.e., $\alpha_i = 0$, $i = 1, 2, 3, 4$ satisfies both $I = 0$ and $dI = 0$ regardless of the choice of the a_i 's. There $\det I'' = 16 \neq 0$ and this singular point is a node.

Since $\omega^3 = 1$, other singularities (solutions of both $I = 0$ and $dI = 0$) are possible only if at least three of the four $a_i^{2/3}$ are equal. Thus, we have :

2. If $a_1 = a_2 = a_3 = a \neq a_4$, simultaneous solutions to $I = 0$ and $dI = 0$ are of the form $(\omega^{2k_1}, \omega^{2k_2}, \omega^{2k_3}, 0, -\sqrt[3]{5a/2})$, where k_1, k_2, k_3 are all different; there are 6 such points. All these are nodes since $\det I'' = -432 \neq 0$.

3. If $a_i = a$, $i = 1, 2, 3, 4$, singular points are found as in case 2., with obvious permutations, and now there are 24 nodes.

Starting from a non-singular quintic (case 0.) with $b_{2,1} = 101$, we have $\dim \mathfrak{W}^\flat = 101$. Sending $a_5 \rightarrow 0$, while keeping no three of the other four a_i equal, produces a nodal quintic in \mathbb{P}^4 which features a single node. Clearly, the space of complex structures for this conifold satisfies $\dim \mathfrak{W}_{(1)}^\sharp = 100$. Moreover, for every point $t^\sharp \in \mathfrak{W}_{(1)}^\sharp$, there is a \mathbb{C} -like disk in \mathfrak{W}^\flat , with the origin at t^\sharp . In this sense $\mathfrak{W}_{(1)}^\sharp$ lies *inside* the boundary of \mathfrak{W}^\flat .

Let $\mathfrak{W}_{(N)}$ denote the space of complex structures for quintic conifolds in \mathbb{P}^4 with at least N nodes (such as the examples above and those in Ref. [149,90,91]), and which is the N -fold self-intersection of $\mathfrak{W}_{(1)}^\sharp$. From the fact that the number of available smoothing deformations is less than or equal to the number of nodes, we conclude

Lemma D.1 Let \mathfrak{M} denote the space of complex structures for a Calabi-Yau 3-fold and let \mathfrak{M}^\sharp denote the singularity divisor in \mathfrak{M} . Then (1) the self-intersections of \mathfrak{M}^\sharp are not all transversal.

Let $\mathfrak{M}_{(\delta)}^{\sharp\sharp}$ be the set where \mathfrak{M}^\sharp self-intersects non-transversely, with $\delta = N - c$, N the number of components self-intersecting at a generic point of $\mathfrak{M}_{(\delta)}^{\sharp\sharp}$, and c the actual codimension $\mathfrak{M}_{(\delta)}^{\sharp\sharp}$ in \mathfrak{M}^\sharp . (2) Only the conifolds parametrized by $\mathfrak{M}_{(\delta)}^{\sharp\sharp}$ may have a Kähler small resolution and δ new (1,1)-cohomology classes.

Sketch of proof:

Point (1) is clear, although often ignored.

The codimension c can be easily calculated by counting the smoothing deformations locally at each point t_0 in $\mathfrak{M}_{(\delta)}^{\sharp\sharp}$, as has been done in the above example. N counts the number of nodes in the 3-fold corresponding to t_0 . As discussed above, their difference $\delta \geq 0$ implies that there are δ relations between the corresponding 3-cycles, which in turn implies the existence of δ ‘vanishing 4-cycles’ upon small resolution. These are isomorphic to δ new (1,1)-cohomology classes.

For conifolds which are parametrized by points of \mathfrak{M}^\sharp , but not in $\mathfrak{M}_{(\delta)}^{\sharp\sharp}$, there are no new (1,1)-cohomology classes, the candidate Kähler form is a linear combination of “old” (1,1)-classes and must be exact on the exceptional sets of the small resolution, hence null. As described above, however, when $\delta > 0$, there are new (1,1)-cohomology classes to allow positivity for the candidate Kähler class. \square

Although nodes are produced by (local) deformations, for the conifold to be a complete intersection, the global nature of the defining equations may enforce more than one node to occur. In the example just above, the nodes occur in odd-numbered groups of 1, 7 and 25.

D.4 Many Calabi-Yau Manifolds are Connected

Generalizations of the simple process (D.2.6) have been proven to link the parameter spaces of many large families of Calabi-Yau 3-folds [149]. We now present some of these generalizations.

D.4.1 Determinantal splitting

Complete intersection Calabi-Yau (CICY) manifolds were described in Chapter 2; each of them is defined by a system of transverse polynomial constraints $\{f_a = 0\}_{a=1\dots K}$ in $X = \mathbb{P}_1^{n_1} \times \dots \times \mathbb{P}_m^{n_m}$. Recall the *configuration matrix*, an integer-valued matrix of the general form :

$$[X \parallel \mathcal{E}] = \left[\begin{array}{c|cccc} n_1 & q_1^1 & \cdots & q_K^1 \\ \vdots & \vdots & \ddots & \vdots \\ n_m & q_1^m & \cdots & q_K^m \end{array} \right], \quad q_a^r \geq 0, \quad (\text{D.4.1})$$

where n_r is the dimension of $\mathbb{P}_r^{n_r}$, the r^{th} factor in X , and q_a^r is the degree of homogeneity of the a^{th} polynomial with respect to the homogeneous coordinates of $\mathbb{P}_r^{n_r}$. \mathcal{E} simply denotes the matrix of the degrees q_a^r .

A straightforward generalization of the relation

$$\left[\begin{smallmatrix} 5 & 4 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{smallmatrix} \right] \leftrightarrow [5|4, 2], \quad (\text{D.4.2})$$

is provided by any two configurations of the form

$$\left[\begin{smallmatrix} X & \mathcal{E} & a_1 & a_2 & \dots & a_{n+1} \\ n & 0 & 1 & 1 & \dots & 1 \end{smallmatrix} \right] \longleftrightarrow [X \parallel \mathcal{E}, \sum_{k=1}^{n+1} a_k]. \quad (\text{D.4.3})$$

The X and \mathcal{E} may be substituted as in Eq. (D.4.1) and a_i are column vectors. Owing to Eq. (2.1.17), if the l.h.s. configuration represents a Calabi-Yau 3-fold, so does the r.h.s. one. The relation “ \longleftrightarrow ” specifies that the parameter spaces of the two configurations have common limit points (parametrizing conifolds) and that the one on the l.h.s. has a strictly smaller dimension. The relation is self-evident upon writing out the last $n + 1$ equations which are linear in the $n + 1$ coordinates of \mathbb{P}^n and results in a determinant equation of degree $\sum_{k=1}^{n+1} a_k$. With the constraints represented by \mathcal{E} , this equation generically describes a conifold which may be smoothed to give a manifold belonging to the configuration on the right. We refer to this as a *determinantal contraction*, while passing from the

r.h.s. of Eq. (D.4.3) to the l.h.s. was called a *determinantal splitting* [149]. We also use these terms for the corresponding replacement of one moduli space by another.

By straightforward iteration of (D.4.3) we have, for example,

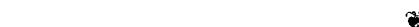
$$\begin{bmatrix} 5 \parallel 4 \ 2 \end{bmatrix} \leftrightarrow \begin{bmatrix} 5 \parallel 4 \ 1 \ 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 5 \parallel 3 \ 1 \ 1 \ 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 5 \parallel 2 \ 1 \ 1 \ 1 \ 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 5 \parallel 1 \ 1 \ 1 \ 1 \ 1 \ 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 \parallel 2 \end{bmatrix}. \quad (\text{D.4.4})$$

Indeed, we have [149]

Theorem D.1 *The parameter spaces of all Calabi-Yau complete intersections in products of complex projective spaces are connected.*

Proof: Starting any Calabi-Yau complete intersection with a configuration matrix $[\mathcal{X}|\mathcal{E}]$ as in (2.1.5), perform determinantal splittings iteratively, introducing only \mathbb{P}^1 's into the embedding space, in a way that finally leaves each constraint at most linear in at most one \mathbb{P}^{n_r} factor in \mathcal{X} with $n_r \neq 1$. Perform next determinantal contractions of all \mathbb{P}^{n_r} 's, $n_r \neq 1$. This leaves a Calabi-Yau complete intersection which is embedded in a product of only \mathbb{P}^1 's. Finish with all possible determinantal contractions of \mathbb{P}^1 's :

$\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$ is the result of the above algorithm, regardless of the initial Calabi-Yau complete intersection. Thus, every Calabi-Yau complete intersection in products of complex projective spaces connects to this one, and the statement follows. The moduli spaces of all such Calabi-Yau complete intersections therefore piece together, much the same as they did in the simple example above, and form a *web* of Calabi-Yau complete intersections. Figure D.3 illustrates this procedure. \square



Suffice it here to note that a similar systematic procedure connects into this same web all multiple branched covers of almost Fano 3-folds and fibered products presented in chapter 6 [149,92]. At the local level, all these procedures involve exactly the same process, a singularizing deformation followed by a small resolution (or the other way around). Compare this with Theorem 7.3 and also the well-understood notion of the “universal curve”, whereby moduli spaces of Riemann surfaces of different genera are pieced together in a hierarchical fashion.

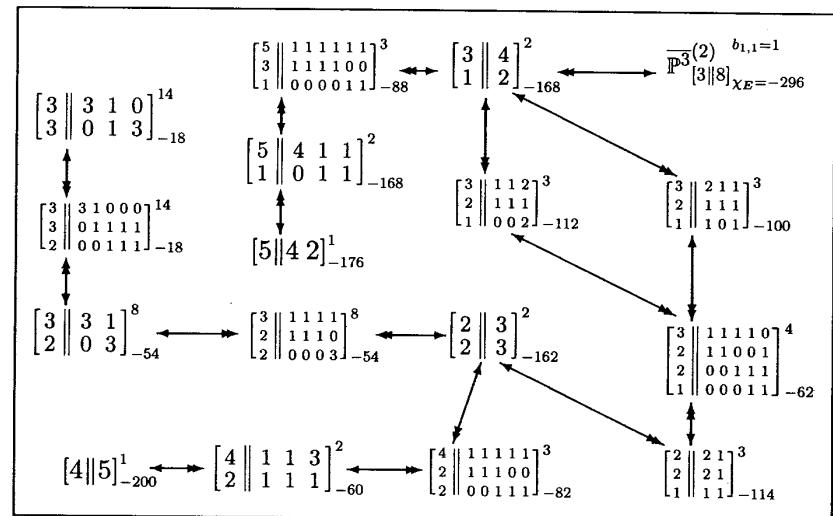


Figure D.3: A minute region of the web of connected Calabi-Yau manifolds.

Chapter E

Parameter Spaces : A Closer Look

E.1 Metric? What Metric?

The main result of the preceding Chapter was that the relevant parameter spaces of a large number of Calabi-Yau 3-folds piece together into a web, so that there exist (large) variations of parameters which connect (ultimately) all such 3-folds with one another. The typical situation is sketched in Figure E.1. One inquires next if the (shortest) distances between two topologically distinct 3-folds in this web are finite. To answer this, some appropriate metric must be chosen on the parameter space and we face several seemingly different options. Fortunately, these choices end up resulting in the same structure, indicating that a general theory, not unlike the Teichmüller theory on the moduli space of Riemann surfaces, is probably in store.

Recall that the combined parameter space is the a $(\mathfrak{M}, \mathfrak{W})$. Points in \mathfrak{M} parametrize the complex structure for \mathcal{M} , or equivalently, specify one harmonic 3-form to be the holomorphic 3-form Ω (determined up to a multiplicative constant). Points in \mathfrak{W} specify a complexified Kähler class $[\mathcal{G}]$, and by Yau's theorem a unique Ricci-flat Kähler metric corresponding to the imaginary part of $[\mathcal{G}]$.

E.1.1 Effective kinetic terms

As we have seen in Chapter 0, Calabi-Yau compactification of the Heterotic (super)string features $b_{2,1}$ massless (spacetime N=1) superfields $\Phi^\alpha(x)$, each containing a family of Standard Model particles, together with $b_{2,1}$ gauge-singlet “moduli superfields” (the spin-(1/2, 0) superfields in the first part of Table 0.2). These are associated with metric variations of the pure type,

$$\varphi_{\bar{\mu}}^\alpha \stackrel{\text{def}}{=} (g^{\alpha\bar{\nu}} \delta g_{\bar{\mu}\bar{\nu}}) \sim H^1(\mathcal{M}, T_{\mathcal{M}}) . \quad (\text{E.1.1})$$

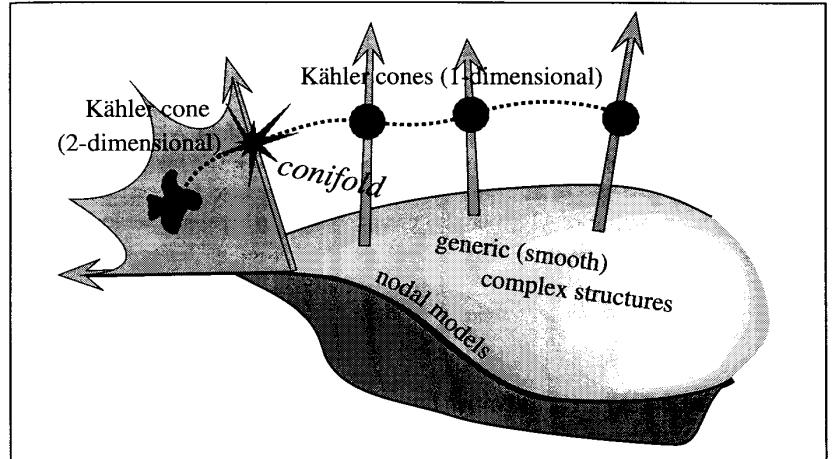


Figure E.1: A sketch of the parameter spaces of two Calabi-Yau 3-folds of different topology type, which connect at an interfacing region, corresponding to Calabi-Yau conifolds.

There also occur $b_{1,1}$ massless families $\Psi^A(x)$ of mirror-particles likewise accompanied by gauge-singlet “moduli superfields” and associated with metric variations of mixed type

$$e_{\bar{\mu}\alpha} \stackrel{\text{def}}{=} \delta g_{\alpha\bar{\mu}} \sim H^1(\mathcal{M}, T_{\mathcal{M}}^*) . \quad (\text{E.1.2})$$

Recall, $H^1(\mathcal{M}, T_{\mathcal{M}}^*) \approx H^{2,1}(\mathcal{M})$.

To lowest order, we obtain the effective Lagrangian density upon superspace integration of

$$\begin{aligned} \mathcal{L}_{\text{eff.}} = & \int d^2\theta d^2\bar{\theta} [N_{\alpha\bar{\beta}} \Phi^\alpha (\Phi^\beta)^\dagger + N_{AB} \Psi^A (\Psi^B)^\dagger] \\ & + \int d^2\theta [\kappa_{\alpha\beta\gamma} \Phi^\alpha \Phi^\beta \Phi^\gamma + \hat{\kappa}_{ABC} \Psi^A \Psi^B \Psi^C] + \text{c.c.} \end{aligned} \quad (\text{E.1.3})$$

Following Ref. [218], we find explicit expressions for these terms with respect to a basis $\{\varphi_\alpha\}_{\alpha=1\dots b_{2,1}}$ for $H^{(2,1)}(\mathcal{M})$ and $\{e_A\}_{A=1\dots b_{1,1}}$ for $H^{(1,1)}(\mathcal{M})$:

$$N_{\alpha\bar{\beta}} = i \int_{\mathcal{M}} \varphi_\alpha \wedge \bar{\varphi}_{\bar{\beta}} , \quad N_{AB} = \int_{\mathcal{M}} e_A \wedge *e_B . \quad (\text{E.1.4})$$

For discussions including also the gauge non-invariant massless superfields see Ref. [123,120].

E.1.2 The natural line element

From a relativist's point of view, we have a complex manifold \mathcal{M} with a (Ricci-flat or not) Hermitian metric, $g_{\mu\bar{\nu}}$ and we wish to describe the metric properties of the space of metrics. In addition, of course, we have also the anti-Hermitian 2-form B , the variation of which is also considered. Furthermore, we know from the analysis that resulted in Table 0.2, that we are dealing with the variations

$$\delta g_{\mu\nu}, \quad \delta g_{\bar{\mu}\bar{\nu}}, \quad \delta g_{\mu\bar{\nu}}, \quad \delta B_{\mu\bar{\nu}}. \quad (\text{E.1.5})$$

On the space locally spanned by these variables, we must choose a metric. To do so, we shall neglect : (1) derivative terms (these are systematically introduced by σ -model corrections), (2) a possible term $(g^{mn}\delta g_{mn})^2$, since $g^{mn}\delta B_{mn}$ vanishes, this term would spoil the symmetry between g_{mn} and B_{mn} , and (3) terms involving the $E_8 \times E_8$ Yang-Mills sector (for the inclusion of this see Ref. [123]). Within these restrictions, the most general metric is

$$\begin{aligned} ds^2 &= \frac{1}{V} \int_{\mathcal{M}} \|\delta g_{\mu\nu}\|^2 + \|\delta g_{\bar{\mu}\bar{\nu}} \pm \delta B_{\mu\bar{\nu}}\|^2, \\ &= \frac{2}{V} \int_{\mathcal{M}} g^{\kappa\bar{\mu}} g^{\lambda\bar{\nu}} (\delta g_{\kappa\lambda} \delta g_{\bar{\mu}\bar{\nu}} + (\delta g_{\kappa\bar{\nu}} \delta g_{\lambda\bar{\mu}} + \delta B_{\kappa\bar{\nu}} \delta B_{\lambda\bar{\mu}})) g^{\frac{1}{2}} d^6 x. \end{aligned} \quad (\text{E.1.6})$$

Somewhat surprisingly perhaps, this "metric on the space of metrics" is block-diagonal, with a separate block corresponding to variations of the complex structure and of the Kähler class, respectively. Note, however, that this is not a direct product space metric : while the (1,1)- and the (2,1)-sectors are *orthogonal*, they are not independent. The factor $g^{\kappa\bar{\mu}} g^{\lambda\bar{\nu}}$ depends on both parameters.

On general grounds, one would expect that the expressions (E.1.4) and (E.1.6) agree and we now turn to verifying that.

E.2 The (2,1)-Forms

Consider first the deformations of the complex structure (we follow Ref. [224]). Define

$$\varphi_{\alpha\kappa\lambda\bar{\mu}} \stackrel{\text{def}}{=} -\frac{1}{2} \Omega_{\kappa\lambda}{}^{\bar{\nu}} \frac{\partial g_{\mu\bar{\nu}}}{\partial t^\alpha}, \quad \varphi_\alpha \stackrel{\text{def}}{=} \frac{1}{2} \varphi_{\alpha\kappa\lambda\bar{\mu}} dx^\kappa \wedge dx^\lambda \wedge dx^{\bar{\mu}}. \quad (\text{E.2.1})$$

The parameters t^α , $\alpha = 1, \dots, b_{2,1}$ are chosen to be local coordinates in an open neighborhood of \mathfrak{M} . Each φ_α is a harmonic (2,1)-form. Inverting, we have

$$\delta g_{\mu\bar{\nu}} = -\frac{1}{\|\Omega\|^2} \bar{\Omega}_{\bar{\mu}}{}^{\rho\sigma} \varphi_{\alpha\rho\sigma\bar{\nu}} \delta t^\alpha, \quad (\text{E.2.2})$$

where $\|\Omega\|^2 \stackrel{\text{def}}{=} \frac{1}{3!} \Omega_{\mu\nu\rho} \bar{\Omega}^{\mu\nu\rho}$ and is constant on the Calabi-Yau 3-fold, but varies over \mathfrak{M} .

Derive

$$\mathbf{G}_{\alpha\bar{\beta}} \delta t^\alpha \delta t^{\bar{\beta}} = -\left(\frac{i \int_{\mathcal{M}} \varphi_\alpha \wedge \bar{\varphi}_{\bar{\beta}}}{i \int_{\mathcal{M}} \Omega \wedge \bar{\Omega}}\right) \delta t^\alpha \delta t^{\bar{\beta}}. \quad (\text{E.2.3})$$

This is called the Weil-Petersson metric [224] and coincides, up to the normalization by the (nonstandard) volume $i \int_{\mathcal{M}} \Omega \wedge \bar{\Omega}$, with the matrix $N_{\alpha\bar{\beta}}$ in Eq. (E.1.4).

We however continue with an observation which is attributed to K. Kodaira (see Ref. [224])

$$\frac{\partial \Omega}{\partial t^\alpha} = K_\alpha \Omega + \varphi_\alpha, \quad (\text{E.2.4})$$

where the K_α may depend on t^α but not on the coordinates of \mathcal{M} . Essentially, this can be seen as follows. Consider a pure (3,0)-form change under a variation of the complex structure, that is, varying the z 's into a linear combination of z 's and \bar{z} 's. To lowest order, only one of its three indices might change from holomorphic to antiholomorphic and the result is partly a (3,0)-form and partly a (2,1)-form. With a little care, it can also be shown that these terms are harmonic.

From Eqs. (E.2.3) and (E.2.4), it follows immediately that

$$\mathbf{G}_{\alpha\bar{\beta}} = -\frac{\partial}{\partial t^\alpha} \frac{\partial}{\partial t^{\bar{\beta}}} \log \left(i \int_{\mathcal{M}} \Omega \wedge \bar{\Omega} \right) \quad (\text{E.2.5})$$

and the space of complex structures is itself Kähler, with the logarithm of the nonstandard volume form, $i \int_{\mathcal{M}} \Omega \bar{\Omega}$, being the Kähler potential.



The holomorphic 3-form has so far shown up in several different ways as an essential marking, so we will describe this in a little more detail, following Ref. [92,87]. Recall that, after all, Ω is just an element of the real degree-3 cohomology. By choosing which of the available harmonic 3-forms is regarded as the purely holomorphic one, we have specified the complex structure. The available choices are then parametrized by expanding Ω over a basis of the dual 3-homology.

Let (A^a, B_b) , $a, b = 0, \dots, b_{2,1}$ be a canonical homology basis for $H_3(\mathcal{M}; \mathbb{Z})$ and let (α_a, β^b) be the dual cohomology basis so that

$$\int_{A^b} \alpha_a = \int_{\mathcal{M}} \alpha_a \wedge \beta^b = \delta_a^b, \quad \int_{B_a} \beta^b = \int_{\mathcal{M}} \beta^b \wedge \alpha_a = -\delta_a^b. \quad (\text{E.2.6})$$

Also,

$$z^a \stackrel{\text{def}}{=} \int_{A^a} \Omega, \quad \mathcal{G}_a \stackrel{\text{def}}{=} \int_{B_a} \Omega \quad (\text{E.2.7})$$

are called the *periods* of Ω . Bryant and Griffiths [82] show that, locally in the moduli (effective parameter) space, the complex structure of \mathcal{M} is entirely determined by a choice of the z^a , so that \mathcal{G}_a may be regarded (locally!) as a function of z . The z^a cannot vanish simultaneously. Since Ω is defined only up to an overall scaling $\Omega(z) \rightarrow \lambda \Omega(z)$, the same must be required for $z^a \rightarrow \lambda z^a$. We may therefore regard the z^a as projective coordinates for the complex structure and Ω as being homogeneous of degree 1 in these coordinates :

$$z^a \in \mathbb{P}^{b_{2,1}}, \quad \Omega(\lambda z) = \lambda \Omega(z). \quad (\text{E.2.8})$$

Finally, we may now expand Ω as

$$\Omega = z^a \alpha_a - \mathcal{G}_a(z) \beta^a. \quad (\text{E.2.9})$$

It easily follows that

$$e^{-K_{(2,1)}} = i \int_{\mathcal{M}} \Omega \wedge \bar{\Omega} = -2 \Im m(\bar{z}^a \mathcal{G}_a), \quad (\text{E.2.10})$$

where $K_{(2,1)} = -\log(i \int_{\mathcal{M}} \Omega \bar{\Omega})$ is the Kähler potential on \mathcal{M} . We see that the $\mathcal{G}_a(z)$ completely determine this metric. On the other hand, the $\mathcal{G}_a(z)$ themselves

E.2. The $(2,1)$ -Forms

are determined by integration from the Yukawa couplings (see below) up to an additive term $\frac{1}{2} \kappa_{ab} z^a z^b$ where κ_{ab} is a symmetric (locally) constant matrix. Recalling now that these Yukawa couplings receive no quantum correction, It follows that this metric structure on \mathcal{M} is exact up to this κ_{ab} .



Eq. (E.2.4) induces a number of vanishing relations such as

$$\Omega_{(1)} \stackrel{\text{def}}{=} idz^a \int_{\mathcal{M}} (\Omega \wedge \frac{\partial \Omega}{\partial z^a}) = 0. \quad (\text{E.2.11})$$

This implies that $2\mathcal{G}_a = \frac{\partial}{\partial z^a}(z^c \mathcal{G}_c)$ and so \mathcal{G}_a is (locally!) the gradient of a degree-2 homogeneous function

$$\mathcal{G}_a = \frac{\partial \mathcal{G}}{\partial z^a}, \quad \mathcal{G}(\lambda z) = \lambda^2 \mathcal{G}(z). \quad (\text{E.2.12})$$

Several other identities of similar form may be derived from $d\Omega_{(1)} = 0$ and

$$\Omega_{(2)} \stackrel{\text{def}}{=} idz^a \otimes dz^b \int_{\mathcal{M}} (\Omega \wedge \frac{\partial^2 \Omega}{\partial z^a \partial z^b}) = 0, \quad (\text{E.2.13})$$

but they do not restrict the z 's and the $\mathcal{G}(z)$ any further.

The third derivative of $\int_{\mathcal{M}} \Omega \wedge \Omega$ is the lowest non-zero one and relates to the Yukawa coupling between $(2,1)$ -forms [84]. Using Eq. (E.2.9) and Eq. (E.2.12) :

$$\frac{\partial^3}{\partial z^a \partial z^b \partial z^c} (i \int_{\mathcal{M}} \Omega \wedge \Omega) = i \int_{\mathcal{M}} (\Omega \wedge \frac{\partial^3 \Omega}{\partial z^a \partial z^b \partial z^c}) = i \frac{\partial^3 \mathcal{G}}{\partial z^a \partial z^b \partial z^c}, \quad (\text{E.2.14})$$

up to lower order terms (and vanishing at least in these coordinates). In general, therefore, this expression appears ill defined, as it seems to depend on the basis chosen to express it. Note, however, that we can easily introduce a covariant derivative from Eq. (E.2.4) :

$$D_\alpha \stackrel{\text{def}}{=} \frac{\partial}{\partial t^\alpha} - K_\alpha, \quad K_\alpha \stackrel{\text{def}}{=} -\frac{\partial}{\partial t^\alpha} \log(i \int_{\mathcal{M}} \Omega \wedge \Omega), \quad (\text{E.2.15})$$

which also defines φ_α as a proper vector. In the left hand side of formula (E.2.14), we may freely replace all partial derivatives with D_α , the end result will not change [219,87]. A little thought reveals that the Yukawa coupling is independent of any connection because of the vanishing relations of the type (E.2.11)

and (E.2.13). This is very much like the fact that gradients of a section are independent of any connection on the zero-locus of the section, which we have noted in the first Chapters.

Clearly not all values of z^a are allowed since we must have

$$(i \int_{\mathcal{M}} \Omega \wedge \bar{\Omega}) = -2 \Im m(\bar{z}^a \xi_a) > 0 . \quad (\text{E.2.16})$$

•

The vector space $H^{(3,0)} \oplus H^{(2,1)}$ is a linear subspace of $H^3(\mathcal{M})$ and varies holomorphically with z^a . A basis for $H^{(3,0)} \oplus H^{(2,1)}$ is then

$$\omega_a \stackrel{\text{def}}{=} \frac{\partial \Omega}{\partial z^a} = \alpha_a - \frac{\partial^2 G}{\partial z^a \partial z^b} \beta^b , \quad (\text{E.2.17})$$

in which the period matrix $\varpi \stackrel{\text{def}}{=} \left(\int_{A^b} \omega_a, \int_{B_b} \omega_a \right)$ simplifies :

$$\varpi = (1, \Xi) , \quad \Xi_{ab} = \frac{\partial^2 \xi}{\partial z^a \partial z^b} . \quad (\text{E.2.18})$$

Since $i \int_{\mathcal{M}} \omega_a \wedge \bar{\omega}_b = -2 \Im m \left(\frac{\partial^2 \xi}{\partial z^a \partial z^b} \right)$, we learn that $\Im m \Xi$ has signature $(b_{2,1}, 1)$ ($b_{2,1}$ plus signs and one minus). This is analogous to lying in the Siegel upper half plane, just as is the action of the symplectic group on Ξ to the action on period matrices of Riemann surfaces. With

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} , \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2b_{2,1} + 2; \mathbb{Z}) \quad (\text{E.2.19})$$

being the action on a basis for H_3 , we have

$$\begin{pmatrix} \bar{\partial} G \\ \bar{z} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \partial G \\ z \end{pmatrix} , \quad (\text{E.2.20})$$

from which it easily follows that

$$\tilde{\Xi} = (A\Xi + B)(C\Xi + D)^{-1} . \quad (\text{E.2.21})$$

E.3 The $(1,1)$ -Forms

Corresponding to Eq. (E.1.6), we have on $H^{(1,1)}$

$$G(\varrho, \sigma) = \frac{1}{2V} \int_{\mathcal{M}} \varrho_{\mu\nu} \sigma_{\rho\sigma} g^{\mu\bar{\sigma}} g^{\rho\bar{\nu}} g^{\frac{1}{2}} d^6 x = \frac{1}{2V} \int_{\mathcal{M}} \varrho \wedge * \sigma , \quad (\text{E.3.1})$$

for real $(1,1)$ -forms, ϱ and σ . Owing to the positivity of $g^{\mu\nu}$, $G(\varrho, \sigma)$ is positive. Up to normalization by the (now standard) volume $V = \frac{1}{n!} \int_{\mathcal{M}} J^n$, this is precisely the metric adopted in point-field theory [218,98]. Also, $G(\varrho, \sigma)$ may be rewritten entirely in terms of the (topological) cubic form $\hat{\kappa}(\varrho, \sigma, \tau) \stackrel{\text{def}}{=} \int \varrho \wedge \sigma \wedge \tau$,

$$G(\varrho, \sigma) = -\frac{3}{2} \left[\frac{\hat{\kappa}(\varrho, \sigma, J)}{\hat{\kappa}(J, J, J)} - \frac{3}{2} \frac{\hat{\kappa}(\varrho, J, J) \hat{\kappa}(\sigma, J, J)}{\hat{\kappa}^2(J, J, J)} \right] , \quad (\text{E.3.2})$$

where we have used that

$$*\sigma = -J \wedge \sigma + \frac{3}{2} \frac{\hat{\kappa}(\sigma, J, J)}{\hat{\kappa}(J, J, J)} J \wedge J . \quad (\text{E.3.3})$$

Since $H^{2,0} = H^{0,2} = 0$, we may choose the basis used in Eq. (E.1.4) to be integral and expand

$$g^a \stackrel{\text{def}}{=} (B + iJ) = w^A e_A , \quad w^A = u^A + iv^A \in \mathfrak{W} . \quad (\text{E.3.4})$$

Straightforwardly,

$$G_{AB} \stackrel{\text{def}}{=} G(e_A, e_B) = \frac{\partial}{\partial w^A} \frac{\partial}{\partial w^B} \log \hat{\kappa}(J, J, J) . \quad (\text{E.3.5})$$

Therefore, \mathcal{W} , the space of the w^A is also a Kähler manifold and the Kähler potential is the logarithm of the (now standard) volume of the Calabi-Yau manifold. If we define [87]

$$\mathcal{F}(w) \stackrel{\text{def}}{=} -\frac{1}{3!} \frac{\hat{\kappa}_{ABC} w^A w^B w^C}{w^0} , \quad (\text{E.3.6})$$

then

$$e^{-K_{(1,1)}} = \frac{4}{3} \int_{\mathcal{M}} J^3 = -2 \Im m(\bar{w}^I \mathcal{F}_I) , \quad (\text{E.3.7})$$

the similarity of which to Eq. (E.2.10) may appear quite startling.

Again, not all values of v^A are allowed, since we must ensure that

$$V = \hat{\kappa}_{ABC} v^A v^B v^C > 0 , \quad (\text{E.3.8})$$

this being the mildest of the three conditions (4.3.16), $\int_M J^3 > 0$. The other two, $\int_S J^2 > 0$ and $\int_C J > 0$ for every surface $S \subset M$ and every curve $C \subset M$ are harder to verify.

We can however obtain another restriction since Eq. (E.3.2) must be a positive definite metric. Upon rewriting,

$$\mathbf{G}(\varrho, \sigma) = -\frac{3}{2} \left\{ \frac{\dot{\kappa}(\varrho, \sigma, J)}{\dot{\kappa}(J, J, J)} - \frac{\dot{\kappa}(\varrho, J, J)\dot{\kappa}(\sigma, J, J)}{\dot{\kappa}^2(J, J, J)} \right\} + \frac{3}{4} \frac{\dot{\kappa}(\varrho, J, J)\dot{\kappa}(\sigma, J, J)}{\dot{\kappa}^2(J, J, J)}, \quad (\text{E.3.9})$$

we have just decomposed this metric into a the “ J -parallel” and “ J -orthogonal” part. That is, every $(1, 1)$ -form may be written as

$$\varrho = \varrho^\perp + \varrho^\parallel = \frac{1}{6}(\varrho_{mn} J^{mn})J + (\varrho - \frac{1}{6}\varrho_{mn} J^{mn}J), \quad (\text{E.3.10})$$

where ϱ^\parallel is covariantly constant if ϱ is harmonic. This is the Lefschetz decomposition $H^2 = (H_\parallel^2 \approx H^0) \oplus H_\perp^2$, where H_\perp^2 is the group of primitive two-forms.

In this notation, Eq. (E.3.9) decomposes the metric into $\mathbf{G}(\varrho^\perp, \sigma^\perp)$ and $\mathbf{G}(\varrho^\parallel, \sigma^\parallel)$ both of which must be positive definite. $\mathbf{G}(\varrho^\parallel, \sigma^\parallel)$ clearly is, but

$$\mathbf{G}(\varrho^\perp, \sigma^\perp) = -\frac{3}{2} \frac{\dot{\kappa}(\varrho^\perp, \sigma^\perp, J)}{\dot{\kappa}(J, J, J)} > 0 \quad (\text{E.3.11})$$

imposes a restriction. That is, $\dot{\kappa}_{AB}(v) \stackrel{\text{def}}{=} \dot{\kappa}_{ABC}v^A$ ought to be negative definite on H_\perp^2 but positive definite on H_\parallel^2 . Thus it must have signature $(1, b_{1,1} - 1)$ (one plus and the rest minus), which can change only if one of the eigenvalues passes through zero. The boundary of the allowed region is therefore given by

$$\det \dot{\kappa}_{AB}(v) = 0. \quad (\text{E.3.12})$$

The fundamental restriction on the parameters is of course that the Kähler form J be positive and this implies both positive volume and positive \mathbf{G}_{AB} .

E.3.1 Coordinate-free description of the metric

The foregoing can be summarized in more mathematical terms. We have observed that for a Calabi-Yau manifold M

$$H^{0,3}(M) \subseteq H^3(M, \mathbb{C}) = H^3(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \quad (\text{E.3.13})$$

E.3. The $(1, 1)$ -Forms

and that the integral periods of Ω determine (at least locally) the holomorphic structure on M .

Whereas positivity of J depends in principle on the choice of a complex structure for M , the reality of J does not, as it can be characterized in terms of the reality of the integral periods, via the identity

$$H^2(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} = H^2(M, \mathbb{R}) \subseteq H^2(M, \mathbb{C}) = H^{1,1}(M) \quad (\text{E.3.14})$$

independently of the holomorphic structure on M . For similar reasons, $\int_M J^3$ depends only on the cohomology class $[J]$ and is independent of the holomorphic structure on M . We note that if $[J]$ is positive, so is $\int_M J^3$, but that the converse does not hold in general. However, the set of positive classes is an open subset of the set of classes with positive cubes.

The exponentials of the Kähler potentials on \mathfrak{M} and \mathfrak{W} , $e^{-K_{(2,1)}} = i \int_M \Omega \wedge \bar{\Omega}$ and $e^{-K_{(1,1)}} = \frac{3}{4} \int_M J^3$, respectively, can both be interpreted as volumes for M (the latter one being standard). However, for the purpose of defining a metric on the metric moduli space we impose no normalization but allow Ω and J to vary independently. Moreover we can fix the integral periods of either Ω or J , thereby making it meaningful to vary one while fixing the other (locally, at a generic choice). The two subspaces so defined become thereby orthogonal and we define a metric on each.

The integral periods of J may be regarded as coordinates for the subspaces defined by holding Ω constant, and take the metric on such a subspace to be the Hessian of $\log(\int_M J^3)$. For the subspaces of constant J , we choose any moduli on which the periods of Ω can be chosen to depend holomorphically, and we define the metric to be the Kähler metric with the Kähler form $\partial\bar{\partial} \log(i \int_M \Omega \wedge \bar{\Omega})$. This is known as the Weil-Petersson metric for the (complex structure) moduli space of M , and is independent of all the choices involved. Note that Ω is undetermined up to multiplication by a holomorphic function $\Omega \rightarrow f(t)\Omega$ but this does not affect the metric basically because $\partial\bar{\partial} \log(f\bar{f}) = 0$.

E.4 Finiteness of Distances between Distinct Manifolds

In the foregoing, we have seen that the Weil-Petersson type metric is favored both from the mathematical and from the (point-field) physical vantage point of view. Section F.2 will reveal that it also appears in the full string theory, at the lowest non-zero order. We then wish to estimate the distances in the relevant parameter spaces for the process

$$\begin{array}{ccc} & \check{\mathcal{M}} & \\ & \downarrow \wp & \\ \mathcal{M}^b & \xrightarrow{r^2 \rightarrow 0} & \mathcal{M}^\sharp \end{array} \quad (\text{E.4.1})$$

In the space of complex structures of \mathcal{M}^b , the $\xrightarrow{r^2 \rightarrow 0}$ process traces a “horizontal path”, while a “vertical path” in the space of (complexified) Kähler classes of $\check{\mathcal{M}}$ is traced by \wp ; see Fig. E.1. We now discuss these in turn.

We need to show that the (exponentials of the) Kähler potentials on \mathfrak{M} and \mathfrak{W} , $(i \int_{\mathcal{M}} \Omega \bar{\Omega})$ and $\int_{\mathcal{M}} J^3$, respectively, may degenerate but mildly enough so that the distance to the conifold converges. This is automatic for $\int_{\mathcal{M}} J^3$, as the dependence on the parameters is completely polynomial, but we also expect significant contributions from world sheet instanton corrections. As discussed above, these dominate if their sizes are small—which is precisely the case when the Calabi-Yau 3-fold singularizes (the rational curves can tag onto the collapsing \mathbb{P}^1 's in the $\check{\mathcal{M}} \xrightarrow{\wp} \mathcal{M}^\sharp$ process). Theorem 7.4 ensures that the canonical divisor of $\check{\mathcal{M}}$ is being inherited by \mathcal{M}^\sharp , so the only uncertainty is indeed the behavior of $\int_{\mathcal{M}} J^3$. This remains an open problem although one expects, in view of the *Mirror Map* ${}^{(\mathcal{L})}$, that this is going to be as well behaved as $(i \int_{\mathcal{M}} \Omega \bar{\Omega})$.

E.4.1 Finiteness in the spaces of complex structure

Following Ref. [92], we present here a proof based on the monodromy properties of the A^a and B_a cycles.

The singularities which we wish to describe are isolated hypersurface singular points and we can consider a \mathbb{C}^1 -family of manifolds (see § D.3.3) \mathcal{M}_t^b given by hypersurfaces in some 4-fold and defined by

$$q + tp = 0, \quad t \in \mathbb{C}, \quad |t| < \rho, \quad (\text{E.4.2})$$

where $q = 0$ is a conifold and p does not vanish at the nodes of q . For any $0 \neq t < \rho$, \mathcal{M}_t^b is smooth and a symplectic basis $\{A^a, B_a\}$ for $H_3(\mathcal{M}_t^b, \mathbb{Z})$ may be

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chosen such that A^1, \dots, A^k span the subgroup of $H_3(\mathcal{M}_t^b, \mathbb{Z})$ generated by the 3-spheres discussed in § D.2.3. This is possible since none of these 3-spheres meets any of the others. Then $B_{k+1}, \dots, B_{b_{2,1}+1}$ do not meet these 3-spheres either and can be represented by cycles well removed from the nodes. B_1, \dots, B_k on the other hand, do meet these 3-spheres and each will include some of the 3-hoops, Π^3 , also discussed in § D.2.3.

Since $\{A^a, B_a\}$ is an integral basis it is locally constant as a function of t . However, if we make a complete circuit in the t -disk around the origin (which parametrizes the conifold), the basis will in general change. The $B_a(t)$ being simply integral cycles dual to the $A^a(t)$ and since $A^a(t) \cap A^b(t) = 0$, there is nothing to prevent the $B_a(t)$ from acquiring multiples of the $A^a(t)$. If we denote the monodromy operator by Σ , that is $\Sigma(A^a(t))$ and $\Sigma(B_a(t))$ are the results of transporting $A^a(t)$ and $B_a(t)$ around $t = 0$ in the t -disk, then we see that

$$\Sigma(A^a) = A^a, \quad \Sigma(B_a) = B_a + \sigma_{ab} A^b, \quad (\text{E.4.3})$$

where σ_{ab} is an integral matrix the entries of which vanish for a or $b > k$. From the fact that Σ respects the symplectic structure it follows that $\sigma_{ab} = \sigma_{ba}$. The erudite reader will note the similarity with the Dehn twists of Riemann surfaces.

From § E.2 we recall that the periods $z^a(t) = \int_{A^a} \Omega_t$ and $\xi_a(t) = \int_{B_a} \Omega_t$ are holomorphic if $t \neq 0$ and if Ω_t depends holomorphically on t and has the local representation as in § D.2.3. We now need to estimate these periods.

Changing coordinates suitably, the equation $\mathcal{C}(x) = r^2(x)$ from Eqs. (D.2.5) which defines \mathcal{M}^b near the node of \mathcal{M}^\sharp becomes

$$x^2 + y^2 + uv = t^2, \quad t^2 > 0. \quad (\text{E.4.4})$$

The quadratic expression on the left may be regarded as the ratio q/p , comparing to Eq. (E.4.2). S_t^3 is then given by x, y real and $v = \bar{u}$, while the 3-cap B_t^3 is given by $x > 0$, y imaginary and $v = -\bar{u}$.

The holomorphic 3-form now becomes

$$\Omega = f \frac{dx \wedge dy \wedge du}{u}, \quad (\text{E.4.5})$$

where f does depend on p , but is non-zero and bounded at each of the nodes. The integral

$$\int_{S_t^3} \Omega = \int_{x^2+y^2 \leq t^2} dx dy \int_{u\bar{u}=t^2-x^2-y^2} f \frac{du}{u} \sim t^2 \quad (\text{E.4.6})$$

vanishes quadratically as $t^2 \rightarrow 0$, since its absolute value is bounded by $2\pi^2 t^2 F$ where $|f| \leq F$.

Derive this.

On the other hand, by writing $y = i\gamma$ and R for the radius of the boundary of the neighborhood, we have

$$\int_{B_t^3} \Omega = \int_0^{\sqrt{(R^2+t^2)/2}} dx \int_0^{\sqrt{x^2-t^2}} d\gamma \int_{u\bar{u}=x^2-\gamma^2-t^2} f \frac{du}{u}. \quad (\text{E.4.7})$$

Just as in the previous case, the last integral reduces to $\int_0^{2\pi} if d\theta$, and the full expression manifestly remains bounded as $t^2 \rightarrow 0$. It is perhaps amusing to write out the first few terms :

$$\begin{aligned} \int_{B_t^3} \Omega &\sim \left| \int_0^{\sqrt{(R^2+t^2)/2}} dx \int_0^{\sqrt{x^2-t^2}} d\gamma \int_{u\bar{u}=x^2-\gamma^2-t^2} f \frac{du}{u} \right|, \\ &\sim f_0 \left| \left[x\sqrt{x^2-t^2} - t^2 \log \left(x + \sqrt{x^2-t^2} \right) \right]_0^{\sqrt{(R^2+t^2)/2}} \right|, \\ &\sim \left\{ \frac{1}{2} R^2 + \left| \log \left(-i\sqrt{2} \left(\frac{t}{R} \right) \right) \right| t^2 - \frac{1}{4R^2} t^4 + \frac{1}{8R^4} t^6 + \dots \right\}; \end{aligned} \quad (\text{E.4.8})$$

only the t^2 term has a non-constant coefficient.

This calculation shows that $\lim_{t \rightarrow 0} z^a(t) = 0$ for $1 \leq a \leq k$, and can be extended to show that all the periods are bounded. Now, $\mathcal{G}_a(t)$ is not single valued for $a \leq k$, because of the monodromy. However,

$$\hat{\mathcal{G}}_a(t) \stackrel{\text{def}}{=} \mathcal{G}_a(t) - \frac{1}{2\pi i} \sigma_{ab} z^b \log(t), \quad (\text{E.4.9})$$

is single-valued since action of the monodromy cancels between the two terms on the right hand side. From the single-valuedness and the boundedness, it follows that the singularity at $t = 0$ is removable so that $\hat{\mathcal{G}}_a(t)$ is also holomorphic at 0.

With these,

$$\begin{aligned} i \int_{M^k} \Omega \wedge \bar{\Omega} &= -2\Im m(\bar{z}^a(t) \xi_a(t)), \\ &= -2\Im m(\bar{z}^a(t) \hat{\xi}_a(t) + \frac{1}{2i\pi} \bar{z}^a(t) \sigma_{ab} z^b(t) \log(t)). \end{aligned} \quad (\text{E.4.10})$$

Since $z^a(0) = z^b(0)$ for $\sigma_{ab} \neq 0$, this has the form

$$a(t) + b(t)|t|^2 \log|t|, \quad (\text{E.4.11})$$

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where $a(t)$ and $b(t)$ are C^∞ functions and $a(0) \neq 0$. This being the asymptotic behavior of the Kähler potential for the Weil-Petersson metric (along the disk in the moduli space which is parametrized by t), it is easy to see that the metric diverges at worst logarithmically. Therefore, the distance to the origin remains finite even though the metric becomes singular at $t = 0^*$.

In fact, much worse asymptotic behavior of $i \int_M \Omega \wedge \bar{\Omega}$ would still lead to finite distance : even $i \int_M \Omega \wedge \bar{\Omega} \sim |F(t)|^2 \{a(t) + b(t)|t|^2 \log|t|\}$ with $F(t)$ analytic in the neighborhood of $t = 0$ but singular at $t = 0$. It is therefore natural to expect certain singularizations of Calabi-Yau manifolds, with singularities worse than nodes also to be at a finite distance from the smooth manifolds. Indeed, Ref. [144] proves finite distance for singularizations of Calabi-Yau manifolds which have at worst isolated singular points with conical neighborhoods defined by

$$x^{e/a} + y^{e/b} + u^{e/c} + v^{e/d} = 0, \quad (\text{E.4.12})$$

with all exponents integral, (x, y, u, v) local coordinates of weights (a, b, c, d) and $e < a + b + c + d$.

Eq. (E.4.12), in particular, straightforwardly subsumes the cases of A_k , E_6 and E_8 singularities and the D_k and E_7 singularities can be shown to be orbifolds of (E.4.12). Therefore, indeed all A, D, E -singularities are at finite distance in the moduli space, which is related to the fact that the canonical divisor behaves relatively tamely in this singularizations.

Note also that we have used cohomology theory to describe the moduli spaces locally—as indeed is the general technique. In view of our earlier discussion, we should really use the L^2 -cohomology[†]. This cohomology however jumps at the special regions of \mathfrak{M} which correspond to conifolds which do have a Kähler small resolution and makes the formation of the web of Ref. [203,149] practically unavoidable.

^{*}The reader unfamiliar with the type of computation presented here may benefit from consulting Ref. [92], where the result (E.4.11) has also been derived by straightforward but less rigorous computation also, largely following Ref. [91].

[†]From the physics point of view this is a must; from a mathematician's point of view, this is less decided. In view of the many nice properties which are preserved by the L^2 -cohomology and especially the intersection homology in spite of singularization, one would hope that mathematicians would agree to this choice also. It is a separate and important issue how to characterize the (physically) interesting metric with respect to which square-integrability is required.

Part IV

Concordance

Chapter F

A Prelude to Quantum Geometry

Having spent essentially all of this volume on various techniques and methods in (mostly algebraic) geometry, as adapted for dealing with Calabi-Yau 3-folds, we are now ready to look back at our initial aim—superstring compactification. At this point then a disclaimer is in order (if not already overdue). It is simply physically not possible to collect all the vigorously industrious study in this area, compress it into a volume that could be held in a single hand and fairly represent all results. Instead, a modest sample is presented, with the hopes that the reader will find them inviting and motivating. Meanwhile, developments march on, and the Reader is strongly advised to catch up with Refs. [241,174], [100,101,102], [178], [57,59,60,74,71,88] and the references therein and thereafter.

F.1 Some Generalities

To describe the propagation of the heterotic superstring, we need to deal with local (1,0)-supersymmetry on the world-sheet and we adopt the superspace formalism (for a general reference, see Ref. [21]).

F.1.1 Flat spacetime

Certainly the simplest example of some relevance is the case where the heterotic string propagates through flat spacetime, that is, 9+1-dimensional Minkowski spacetime [160]. The action for the underlying 2-dimensional field theory has two main parts, $S_{String}^{D=10}$ and S_{Gauge} . The former describes string propagation through spacetime

$$S_{String}^{D=10} = \int_{\Sigma} d^2\sigma d\zeta^- E^{-1} \left[i\frac{1}{2} \eta_{mn} (\nabla_+ X^m) (\nabla_- X^n) \right], \quad (F.1.1)$$

where E is the superdeterminant of the super-Zweibein. The off-shell string coordinate superfields are given by

$$\mathbf{X}^m(\zeta^+, \sigma, \tau) = X^m(\sigma, \tau) + \zeta^+ \psi_+^m(\sigma, \tau). \quad (\text{F.1.2})$$

Here, X^m map the world-sheet into the flat Minkowski spacetime, the latter being specified through $\eta_{\underline{m}\underline{n}}$. Imposing the equations of motion,

$$\mathbf{X}^m(\zeta^+, \sigma, \tau) = X^m(\sigma^{++}) + X^m(\sigma^{--}) + \zeta^+ \psi_+^m(\sigma^{++}). \quad (\text{F.1.3})$$

Thus, on-shell X^m contains both left- and right-moving modes while the equation of motion for each ψ_+^m implies that it is left-moving, i.e., independent of σ^{--} .

To propagate in a curved spacetime instead, the replacement

$$\eta_{\underline{m}\underline{n}} \rightarrow G_{(\underline{m}\underline{n})}(\mathbf{X}) + B_{[\underline{m}\underline{n}]}(\mathbf{X}) \quad (\text{F.1.4})$$

would be necessary, introducing interactions among the X^m and ψ_+^m , non-linear in X^m . In the effective (target spacetime) theory, the background (vacuum expectation) value of these $X(\sigma)$ -dependent coupling tensors determine the geometry of the spacetime in which the string propagates. Their fluctuations represent the various states stemming from the D=10 supergravity sector.

As classical field theories, the above action is perfectly consistent. For quantum field theories, several of the symmetries will in general be violated by anomalies. The standard field theory trick is of course to find collections of fields such that all anomaly contributions cancel out. To give ourselves more freedom in doing so, we add another type of terms to the action.

F.1.2 The Yang-Mills sector

The second part, S_{Gauge} , introduces the degrees of freedom usually referred to as ‘internal’ and thereby allows us to shift the burden of anomaly cancellation into something less conspicuous than, say, 26-dimensional Minkowski spacetime. Luckily, perhaps, this summand is (by far) not unique. Consider the fermionic representation

$$S_{Gauge}^{(32)} = -\frac{1}{2} \int_{\Sigma} d^2\sigma d\zeta^- E^{-1} [\eta_-^{\hat{I}} \delta_{\hat{I}\hat{J}} \nabla_+ \eta_-^{\hat{J}}], \quad (\text{F.1.5})$$

with $\hat{I} = 1, \dots, 32$, where only the lowest component field of each $\eta_-^{\hat{I}}$ propagates and which is enforced by the equations of motion to be right-moving, i.e.,

F.1. Some Generalities

independent of σ^{++} . A simple calculation tells that, together with $S_{String}^{D=10}$, this provides a consistent model, the $Spin(32)/\mathbb{Z}_2$ -heterotic string.

For the $Spin(32)/\mathbb{Z}_2$ model, these 32 fermionic fields $\eta_-^{\hat{I}}$ form the vector representation of $Spin(32)$. The 496 fermionic bilinear combinations

$$\eta_-^{[\hat{I}} \eta_-^{\hat{J}]} \quad (\text{F.1.6})$$

span the *complete* adjoint representation and generate the full group, so that σ -model coupling tensors can be added through minimal coupling

$$\delta_{\hat{I}\hat{J}} \nabla_+ \rightarrow \delta_{\hat{I}\hat{J}} \nabla_+ + A_{+[\hat{I}\hat{J}]}(\mathbf{X}), \quad (\text{F.1.7})$$

and generate $Spin(32)$. The world-sheet gauge (super)field $A_{+[\hat{I}\hat{J}]}$ is the so-called “pull-back” of the target space gauge field :

$$A_{+[\hat{I}\hat{J}]}(\mathbf{X}) = (\nabla_+ \mathbf{X}^m) A_{m[\hat{I}\hat{J}]}(\mathbf{X}). \quad (\text{F.1.8})$$

Finally, the passage $\eta_{\underline{m}\underline{n}} \rightarrow G_{(\underline{m}\underline{n})}(\mathbf{X}) + B_{[\underline{m}\underline{n}]}(\mathbf{X})$ need not be done for the full span of 9+1-spacetime dimensions, rather, we will want to leave 3+1-spacetime dimensions flat and curve only the others. Clearly, many other possibilities exist—from the 2-dimensional point of view—involving the usual palette of fields : scalars, spinors, and so on.

The partitioning of the ten spacetime coordinate fields into 3+1 flat ones and 6 curved ones suspiciously looks like compactification, but leaves two essential questions. (1) How do we know that the curved space is compact? (2) What other criteria are there to be satisfied? The answer to the latter one is that the criteria are essentially translations of the various conditions known from point field theory (with the known exception of modular invariance). As to the compactness, the actions so far described in fact do not know anything about it. However, recall that the fundamental object in quantum field theory is the partition function(al), which involves (path-)integration over the various fields used. The integration range is where the quantum field theory registers compactness and also other global information. In general, this will not be easily accessible. See Ref. [136] for more on this.

F.2 Calabi-Yau σ -Models in General

To simplify somewhat the notation, we temporarily restrict to the bosonic sector. Consider the action functional governing the propagation of strings in an “internal” space :

$$\mathcal{A} = -\frac{1}{2\pi\alpha'} \int_{\Sigma} \left\{ (G_{mn} + iB_{mn})(\bar{\partial}X^m(\sigma))(\partial X^n(\sigma)) + (G_{mn} - iB_{mn})(\partial X^m(\sigma))(\bar{\partial}X^n(\sigma)) \right\}.$$

On Calabi-Yau spaces, it is standard to choose a background Kähler metric $G_{\mu\nu}$ and set $B = 0$ in vacuum. The relevant σ -model describing a Calabi-Yau background action is then

$$\mathcal{A}_0 = -\frac{1}{\pi\alpha'} \int_{\Sigma} G_{\mu\nu}(Z, \bar{Z}) [(\bar{\partial}Z^\mu(\sigma))(\partial Z^\nu(\sigma)) + (\partial Z^\mu(\sigma))(\bar{\partial}Z^\nu(\sigma))] . \quad (\text{F.2.1})$$

To qualify for describing superstring vacua, the above action needs to be conformally invariant which is, up to four loop corrections, equivalent to the Ricci-flatness of $G_{\mu\nu}(Z, \bar{Z})$. At four loops, the condition acquires a correction proportional to the double derivative of the Euler density, but it is possible to redefine G and B suitably; this does not change their respective cohomology classes [196]. This being so we may as well concentrate on the Ricci-flat metric for simplicity.

Any choice of such a $G_{\mu\nu}(Z, \bar{Z})$ simply specifies a point in the relevant moduli space $(\mathfrak{M}, \mathfrak{W})$. To determine the metric on the moduli space at that point, we shall need to consider deformations* of this choice and evaluate second order effects. We skim through this here, following Ref. [93]; a more complete and careful field theoretic analysis is found in Ref. [120].

Defining the following variations of \mathcal{A}_0 :

$$\mathcal{A}_{(1,1)} \stackrel{\text{def}}{=} -\frac{1}{\pi\alpha'} \int_{\Sigma} [\mathcal{G}_{\mu\nu}^+(\bar{\partial}Z^\mu(\sigma))(\partial Z^\nu(\sigma)) + \mathcal{G}_{\mu\nu}^-(\partial Z^\mu(\sigma))(\bar{\partial}Z^\nu(\sigma))] , \quad (\text{F.2.2})$$

$$\mathcal{A}_{(2,1)} \stackrel{\text{def}}{=} -\frac{1}{\pi\alpha'} \int_{\Sigma} [\mathcal{G}_{\mu\nu}(\bar{\partial}Z^\mu(\sigma))(\partial Z^\nu(\sigma)) + \mathcal{G}_{\mu\nu}(\partial Z^\mu(\sigma))(\bar{\partial}Z^\nu(\sigma))] , \quad (\text{F.2.3})$$

*It is important to note that the deformations of the σ -model action (F.2.1) describing Calabi-Yau vacua of superstring theories may be parametrized through deformations of the complex structure and of the Kähler class.

F.2. Calabi-Yau σ -Models in General

$\mathcal{A}_0 + \mathcal{A}_{(1,1)} + \mathcal{A}_{(2,1)}$ is the most general expression for an action that still has the geometrical meaning of embedding the world-sheet into the target (Calabi-Yau) manifold.

For a conformally invariant theory, Zamolodchikov’s metric [243] is the (fourth order) residue of the irreducible two-point function $\langle \Phi_I(\sigma)\Phi_K(0) \rangle$ where the set $\{\Phi_I(\sigma)\}$ parametrizes the variations of the background action density. As always,

$$\lim_{\sigma \rightarrow 0} |\sigma|^4 \frac{\delta}{\delta J^I(\sigma)} \frac{\delta}{\delta J^K(0)} , \quad (\text{F.2.4})$$

where $J^I(\sigma)$ are suitable sources probing the variation $\Phi_I(\sigma)$. For $\mathcal{A}_0 + \mathcal{A}_{(1,1)} + \mathcal{A}_{(2,1)}$, we introduce the world-sheet sources through

$$\begin{aligned} \mathcal{G}_{\mu\nu}^+ &= J^{A+}(\sigma) \mathcal{G}_{A\mu\nu}^+(Z, \bar{Z}) , & \mathcal{G}_{\mu\nu}^- &= J^{A-}(\sigma) \mathcal{G}_{A\mu\nu}^-(Z, \bar{Z}) , \\ \mathcal{G}_{\mu\nu} &= T^\alpha(\sigma) \mathcal{G}_{\alpha\mu\nu}(Z, \bar{Z}) , & \mathcal{G}_{\mu\nu} &= T^{\bar{\alpha}}(\sigma) \mathcal{G}_{\bar{\alpha}\mu\nu}(Z, \bar{Z}) . \end{aligned} \quad (\text{F.2.5})$$

Of course, we have

$$\mathcal{G}_{\mu\nu}^\pm = \delta \mathcal{G}_{\mu\nu} \pm i \delta B_{\mu\nu} . \quad (\text{F.2.6})$$

We can evaluate the Zamolodchikov metric perturbatively, by expanding $Z^\mu = z^\mu + \tilde{Z}^\mu$, where z^μ is constant over the world-sheet and \tilde{Z}^μ are fluctuations around z^μ . Similarly, we expand all functions of Z^μ , for example, $G_{\mu\nu}(Z) = g_{\mu\nu}(z) + \dots$. Rather easily, this proves that the lowest non-zero contribution to the Zamolodchikov metric equals the Weil-Petersson type metric in Eq. (E.1.6).

The alert Reader might have noticed that $\overline{\mathcal{G}_{\mu\nu}^\pm \bar{\partial}Z^\mu \partial Z^\nu} = \mathcal{G}_{\mu\nu}^\pm \bar{\partial}Z^\mu \partial Z^\nu$. It therefore appears that another parametrization may be more appropriate for the complexified Kähler cone, one in which the two types of variations will be one another’s complex conjugate. Indeed, let

$$[\delta \mathcal{G}_{\mu\nu} \pm \delta B_{\mu\nu}] \stackrel{\text{def}}{=} (1 \pm i)[\mathcal{G}_{\mu\nu}^+ \pm i \mathcal{G}_{\mu\nu}^-] , \quad (\text{F.2.7})$$

and $W^A \mathcal{G}_{A\mu\nu} = [\delta \mathcal{G}_{\mu\nu} - \delta B_{\mu\nu}]$ and $W^{\bar{A}} \bar{\mathcal{G}}_{A\mu\nu} = [\delta \mathcal{G}_{\mu\nu} + \delta B_{\mu\nu}]$. Then, the constant mode of $W^A(\sigma)$ is exactly the parameter w^A introduced in Eq. (E.3.4). Furthermore, let us just state here that marginal operators as in $\mathcal{A}_{(1,1)}$ correspond to harmonic $(1,1)$ -forms and become twisted-chiral superfields when supersymmetrized. This then provides the field-theoretic version of the complexified Kähler cone and space of complex structures (see also Ref. [98]).

On the other hand, the constant mode of $T^\alpha(\sigma)$ may be identified with the parameter t^α introduced in (E.2.1) as it is. Note also that only a non-holomorphic

redefinition of coordinate fields can possibly bring $\mathcal{A}_0 + \mathcal{A}_{(2,1)}$ in the form of \mathcal{A}_0 . This clearly corresponds to deformations of the complex structure. Hence the correspondence with (2,1)-forms which in turn correspond to marginal modes in $\mathcal{G}_{\mu\nu}$. These become chiral superfields when supersymmetrized. It transpires that the parameter space relevant to the above σ -model coincides with the combined space of complex structures and the Kähler cone (complexified through the inclusion of $\delta B!$). In addition, it is precisely these geometrical data on the Calabi-Yau manifold that determine the essential physical observables, such as the Yukawa couplings, in the low-energy limit.

F.3 The Projective σ -Model

Just as Calabi-Yau spaces are being constructed from certain better known spaces, such as \mathbb{P}^n 's, σ -models can also be treated in a similar fashion.

F.3.1 Three projective actions

We start with the well known supersymmetric \mathbb{P}^n model in 2-dimensional space-time and (2,2)-supersymmetry. The classical action is

$$\begin{aligned}\mathcal{A}[\mathbb{P}^n] &\stackrel{\text{def}}{=} \int_{\Sigma} d^2\sigma d^2\zeta d^2\bar{\zeta} \left(\|X\|^2 e^{-V} + \frac{n+1}{2f} V \right), \\ \|X\|^2 &\stackrel{\text{def}}{=} \sum_{\mu=0}^n X_\mu^\dagger X^\mu.\end{aligned}\quad (\text{F.3.1})$$

It involves $n+1$ chiral superfields, X^μ , and a gauge superfield, V for which no kinetic term is introduced. V gauges a complexified and supersymmetric version of $U(1)$, which we denote by $\tilde{U}(1)$ and which includes $U(1)$, dilation and their supersymmetric counterpart, a chiral symmetry acting on the fermion component fields in X^μ . Indeed, the body of the classical configuration space (spanned by the unconstrained bosonic fields) is $[\mathbb{C}^{n+1} - \{0\}] / U(1, \mathbb{C}) = \mathbb{P}^n$. The $\tilde{U}(1)$ gauge transformation, also known as the Kähler symmetry, acts by

$$X \rightarrow e^\Theta X, \quad V \rightarrow V + \Theta + \Theta^\dagger, \quad (\text{F.3.2})$$

where Θ is an arbitrary chiral superfield and Θ^\dagger its antichiral Hermitian conjugate.

The equation of motion for V reads $V = \log(\frac{2f}{n+1} \|X\|^2)$. Using this to eliminate V is equivalent to path-integration over V and produces, up to an

F.3. The Projective σ -Model

uninteresting additive constant,

$$\begin{aligned}\mathcal{A}'[\mathbb{P}^n] &= \int_{\Sigma} d^2\sigma d^2\zeta d^2\bar{\zeta} K_{(FS)}(X^\dagger, X), \\ K_{(FS)}(X^\dagger, X) &= \frac{n+1}{2f} \log \|X\|^2,\end{aligned}\quad (\text{F.3.3})$$

where $\|X\|^2 = \sum_{\mu=0}^n X_\mu^\dagger X^\mu$ and $K_{(FS)}$ is the Fubini-Study Kähler potential (5.1.8). Since the action (F.3.1) is quadratic in X 's, it is also possible to integrate out the X 's, which results in [108] :

$$\mathcal{A}''[\mathbb{P}^n] \stackrel{\text{def}}{=} \frac{n+1}{4\pi} \int_{\Sigma} d^2\sigma \left(\int d\zeta^+ d\zeta^- S [\log(S/\mu) - 1] + \text{h.c.} \right) + \dots \quad (\text{F.3.4})$$

where the higher terms all involve $\int d^2\zeta d^2\bar{\zeta}$ -integrals.

F.3.2 Marginal operators

The superfields $S \stackrel{\text{def}}{=} D_+ \bar{D}_- V$ and its Hermitian conjugate are both twisted chiral superfields [134]. Since the leading terms in $\mathcal{A}''[\mathbb{P}^n]$ involve integration over only “half” of the superspace, these terms are protected by the usual non-renormalization theorems regarding (twisted) F -terms. So, while many D -terms supply only irrelevant operators [99], there do exist D -terms such as in Eq. (F.3.1) which have a marginal residue, represented in Eq. (F.3.4). That the kinetic terms considered here are not in the universality class represented by $K_0(X^\dagger, X) = X^\dagger X$ is most easily seen from the distinct behavior of the partition functionals. When defined with the kinetic term (F.3.3), a partition function clearly has an essential singularity at $X^\mu = 0$. Removing the origin from the field-space, however, changes its topology and is seen to distinguish the two universality classes.

A few remarks are in order before carrying on to related models. Firstly, the continuous gauge symmetry (F.3.2) is anomalous; while the classical actions (F.3.1) and (F.3.3) are invariant, the action (F.3.4) is not, except for the leading term S and its conjugate [108]. The “leading log” term $S \log(S/\mu)$ and its conjugate carry the $\tilde{U}(1)$ gauge-anomaly of the \mathbb{P}^n model; the numerical value is $(n+1)$ and equals the first Chern class of \mathbb{P}^n ; the $1/4\pi$ factor turns out to be merely a suitable normalization. Also, because of the $(n+1)$ prefactor, however, a \mathbb{Z}_{n+1} subgroup of (F.3.2) does remain a symmetry and its action has to be divided out. This will turn out to be the Lagrangian root of the GSO-type projection in the Landau-Ginzburg models.

Secondly, note that the equation of motion for \mathbf{V} and the definition of the superfield \mathbf{S} lets us re-interpret Eq. (F.3.4) in terms of the \mathbf{X} 's. To that end we record

$$\mathbf{S} \sim (D_+ \mathbf{X}^\mu) G_{\mu\bar{\nu}}^{(FS)} (\bar{D}_- \mathbf{X}^{\dagger\bar{\nu}}), \quad G_{\mu\bar{\nu}}^{(FS)} \stackrel{\text{def}}{=} \partial_\mu \partial_{\bar{\nu}} K_{(FS)}(\mathbf{X}, \mathbf{X}^\dagger). \quad (\text{F.3.5})$$

where $G_{\mu\bar{\nu}}^{(FS)}$ is the Fubini-Study metric. Clearly, the (anticommuting) fermionic derivatives act as the field theory generalization of the dz 's and $d\bar{z}$'s*. Hence, the quantity

$$(\mathbf{S}^{\text{“+”}} \mathbf{S}^\dagger) \stackrel{\text{def}}{=} \left(\int d\zeta^+ d\bar{\zeta}^- \mathbf{S} + \int d\zeta^- d\bar{\zeta}^+ \mathbf{S}^\dagger \right) \quad (\text{F.3.6})$$

has a natural interpretation as the (super)field theory generalization of the Fubini-Study Kähler $(1,1)$ -form; this supports the identification of the $\tilde{U}(1)$ -anomaly with the first Chern class. Note that the term (F.3.6), with the interpretation (F.3.5), also appears when the “twisted half” of the fermionic integral is explicitly performed in Eq. (F.3.3).

Finally, the coupling constant f from $\mathcal{A}[\mathbb{P}^n]$ has undergone a “dynamical transmutation”: in Eq. (F.3.4), it occurs through the renormalization ‘mass’ $\text{scale}^\dagger \mu = \text{const. } \varepsilon e^{2\pi/f}$, where ε is the dimensional regularization parameter. When describing a σ -model with $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}$ target space, the effective action (F.3.4) is replaced by a linear combination of such actions, one for each \mathbb{P}^{n_i} factor, which contains the linear combination $\sum_{i=1}^m (\text{const.} + (2\pi/f_i)) (\mathbf{S}_i^{\text{“+”}} \mathbf{S}_i^\dagger)$. Indeed, the Kähler class of a product of \mathbb{P}^n 's is a linear superposition of the individual Kähler classes.

*Indeed, the lowest component of the superfield $D_\pm \mathbf{X}^\mu$ is ψ_\pm^μ and has been identified as a formal analogue of dz ever since the early work on σ -models [97].

^{\dagger}In d -dimensional spacetime, μ has dimension $(d-2)/2$ and is actually dimensionless in $d = 2$.

F.4 Constrained Models

Many Kähler manifolds are constructed by embedding in \mathbb{P}^n 's and rather important examples are obtained as the (sub)space of common solutions to a system of constraints in a product of complex projective spaces. Most naturally, one imposes the constraints $P^a(\mathbf{X})=0$ by means of Lagrange multiplier fields on a system of \mathbb{P}^n models [184,136]. The corresponding action is $\mathcal{A}_{\text{kin.}} + \mathcal{A}_{\text{con.}}$, where

$$\mathcal{A}_{\text{kin.}} = \sum_{i=1}^m w^i \int_{\Sigma} d^2\sigma d^2\zeta d^2\bar{\zeta} \left(\|\mathbf{X}_i\|^2 e^{-\mathbf{V}_i} + \frac{n_i+1}{2f_i} \mathbf{V}_i \right), \quad (\text{F.4.1})$$

$$\mathcal{A}_{\text{con.}} = \sum_{a=1}^K \int_{\Sigma} d^2\sigma d^2\zeta \left(\mathbf{A}_a P^a(\mathbf{X}) + \text{h.c.} \right) \quad (\text{F.4.2})$$

and where \mathbf{X}_i^μ are coordinate superfields on \mathbb{P}^{n_i} . While this provides the exact field theory parallel of the construction of such Kähler spaces, its use in practical computations is limited, because of the non-linear couplings in (F.4.1) and since the Lagrange multiplier fields effectively introduce infinitely strong coupling (see however Ref. [169,170] for some applications).

Since the terms in $\mathcal{A}_{\text{con.}}$ are generally not quadratic in the \mathbf{X} 's, we cannot in general perform the $fD[\mathbf{X}]$ path-integration and obtain the constrained space analogues of Eqs. (F.3.4) and (F.3.5). On general grounds, we know that analogous relations must exist although an explicit evaluation eludes us. Clearly, the analogue of \mathbf{S} should then again be of the form (F.3.5), featuring, however the physical metric on the constrained space \mathcal{M} ,

$$\mathbf{S}_{\mathcal{M}} \sim (D_+ \mathbf{X}^\mu) G_{\mu\bar{\nu}}^{(\mathcal{M})} (\bar{D}_- \mathbf{X}^{\dagger\bar{\nu}}), \quad (\text{F.4.3})$$

where the indices μ and $\bar{\nu}$ have to be restricted so as to label directions locally (co)tangent to $\mathcal{M} \subset \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}$. This is most easily performed by inserting appropriate local projection operators of the form $\mathbb{1} - [\partial P]$, where the matrix of gradients $\partial_M P^a(\mathbf{X})$ and its conjugate serve as projections (locally on \mathcal{M}) from $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}$ to the directions transversal to \mathcal{M} .

Having identified the $\tilde{U}(1)$ -anomaly $(n+1) \mathbf{S} \log(\mathbf{S}/\mu)$ with the first Chern class of \mathbb{P}^n , one expects that integration over the \mathbf{X} 's, for a degree- q polynomial constraint would reduce this anomaly to $(n+1-q) \mathbf{S} \log(\mathbf{S}/\mu)$. For example, if the constraint polynomial were linear, it would simply require a linear combination of \mathbf{X}^μ 's to vanish. Integrating the remaining n \mathbf{X}^μ 's would yield the decreased anomaly $n \mathbf{S} \log(\mathbf{S}/\mu)$, as expected.

In the general case, the explicit path-integration eludes us, but the anomaly contribution can be discerned by considering the transformation of the path-integral measure [130]. Alternatively, in particle physics parlance, integrating out the superfields $X_i^{\mu_i}$ allows us to interpret the A_a as bound states of the charge-conjugates of those X 's which appear in $P^a(\mathbf{X})$ (see Ref. [208] for a situation where such an interpretation is experimentally verified), so that the so “dressed” A_a 's will have charge* $-q_a^i$ with respect to V_i , where q_a^i is the degree of $P^a(\mathbf{X})$ with respect to the coordinate superfields of \mathbb{P}^{n_i} . There will also appear effective propagators for A_a 's and V_i 's (S_i 's), whence the anomaly 1-loop diagrams yield the coefficients $(n_i+1 - \sum_{a=1}^K q_a^i)$ in place of (n_i+1) in Eq. (F.3.4). This is in perfect agreement with our identification of the chiral (“Kähler”) gauge anomaly with the first Chern class of the constrained subspace \mathcal{M} .

F.5 The Calabi-Yau Case

For \mathcal{M} to be a Calabi-Yau space, the first Chern class of \mathcal{M} has to vanish, so

$$\sum_{a=1}^K \deg_{\mathbb{C}\mathbb{P}^{n_i}}(P^a(\mathbf{X})) = n_i+1, \quad i = 1, \dots, m. \quad (2.1.17)$$

Therefore, the “Kähler” $U(1)$ gauge anomaly of each V_i is completely canceled. However, the numerical pre-factor in the effective action analogous to $\mathcal{A}'[\mathbb{P}^n]$, Eq. (F.3.4), appears also to have been annihilated. Can it be that the anomaly cancellation efforts have been counter-productive, in that Eq. (2.1.17) not only ensures cancellation of the chiral gauge anomaly, but also kills the marginal terms in an effective action which represent the Kähler variations?

The answer clearly ought to be negative. In fact, it is not hard to see how that comes about. In any realistic situation, the couplings to (super)gravity should be included. When considering 2-dimensional field theories to describe Calabi-Yau compactification of heterotic strings, for example, the constrained (product of) \mathbb{P}^n model(s), Eqs. (F.4.1) and (F.4.2), should be coupled to (1,0)-supergravity [136]. Indeed, the (super)gravitational anomaly [56] is nothing but the well-known trace-anomaly, i.e., the central charge of the Virasoro algebra.

*Note that, before integrating the X^μ out, the Lagrange superfields A_a in \mathcal{A}_{con} must have no charge with respect to the gauge fields V_i ; for, if they had, they would have to interact with the V_i . In supersymmetric theories this would be possible only through kinetic terms for A_a 's which would contradict their rôle as Lagrange multiplier superfields.

F.5. The Calabi-Yau Case

But then, the locally (1,0)-supersymmetric version of $\mathcal{A}_{\text{kin}} + \mathcal{A}_{\text{con}}$ in fact must be anomalous and precisely so as to cancel the contributions of a flat σ -model representing the 4-dimensional spacetime and of the super-reparametrization ghosts. In 2-dimensional field theories, the chiral gauge-, spin-3/2 and gravitational anomalies are all carried by the same operator [56]. It follows that the effective action for Calabi-Yau compactification of heterotic strings is bound to contain the marginal terms

$$\frac{c}{4\pi} \int_{\Sigma} d^2\sigma \left(\int d\zeta^+ d\bar{\zeta}^- S_{CY} [\log(S_{CY}/\mu) - 1] + \text{h.c.} \right), \quad (\text{F.5.1})$$

where c is the central charge, $c = 9$ for (2,2)-supersymmetric σ -models with a complex 3-dimensional target space. The S_{CY} may be thought of, in analogy to Eq. (F.4.3), as

$$S_{CY} \sim (D_+ X^\mu) G_{\mu\nu}^{(CY)} (\bar{D}_- X^{\dagger\nu}), \quad (\text{F.5.2})$$

where the same remark applies as for Eq. (F.4.3) and where $G_{\mu\nu}^{(CY)}$ is the “repaired” combination of Fubini-Study metrics : From the linear combination

$$\sum_{i=1}^m (\text{const.} + (2\pi/f_i)) (S_i “+” S_i^\dagger), \quad (\text{F.5.3})$$

one constructs first the Ricci-flat metric following Yau [54], and then corrects this along the lines in Ref. [196] to restore (super)conformal invariance of the complete model order by order. Thus, S_{CY} is the correspondingly “repaired” linear combination of the S_i from the \mathbb{P}^n models. This again perfectly agrees with the identification of $(S_{CY} “+” S_{CY}^\dagger)$ with the Kähler class on the Calabi-Yau space and may be thought of as the result of integrating the true, superconformal D -term $\int d^2\zeta d^2\bar{\zeta} K_{CY}$ over the “twisted half” of the fermionic space. Note that even after dropping the anomalous $S \log S$ terms (after all, the anomaly eventually cancels out), the dependence on the f_i 's remains through the $\log \mu$ factor; the f_i 's in fact control the dependence of S_{CY} on the S_i .

That the trace-anomaly had to appear through a marginal operator should have been obvious since it is a physical observable and can not be phased away through renormalization. That the gravitational and the gauge anomalies appear through the same marginal operator is a special feature of 2-dimensional spacetime; Eq. (F.5.1) then appears merely as a “twisted chiral” re-write of the more standard expressions.

Finally, note that world-sheet instantons are expected to contribute to the terms (F.5.1). Determining these corrections appears to be a rather interesting

problem for future study, because of the following intriguing possibility : Assume that the fully corrected terms will, instead of just S_{CY} , contain a polynomial $Q(\mathbf{S})$ and that we may ignore the anomalous “ $\mathbf{S} \log \mathbf{S}$ ” terms. The resulting action would be strikingly similar to a Landau-Ginzburg model in which the variations of $Q(\mathbf{S})$ are the marginal (twisted) F -terms describing the Kähler variations of our Calabi-Yau σ -model with target space \mathcal{M} . Alternatively, it could be interpreted as the Landau-Ginzburg model for describing the complex structure variations of the *mirror Calabi-Yau σ-model*, one which has target space \mathcal{W} , such that

$$H^q(\mathcal{M}, T_{\mathcal{M}}) \cong_Q H^q(\mathcal{W}, T_{\mathcal{W}})^* \quad (\text{F.5.4})$$

is an equivalence of ring structures (exact quantum Yukawa couplings), not only the equality of the dimensions of the respective spaces.

F.6 Landau-Ginzburg Mean-Field Theories

Now, by choosing the ‘physical’ gauge in which $\mathbf{V}_i = 0$ (gauge fields in 2-dimensions are unphysical) the kinetic term becomes

$$\mathcal{A}_{\text{kin.}} = \sum_{i=1}^m w^i \int_{\Sigma} d^2\sigma d^2\zeta d^2\bar{\zeta} \|\mathbf{X}_i\|^2, \quad (\text{F.6.1})$$

which is just the standard Wess-Zumino type, in the generic universality class and irrelevant. In addition, we set $A_a = \lambda_a$ (*const.*). This second step is not really any loss of generality—if we send $\lambda_a \rightarrow \infty$ at the end of the calculation. With some finite and big λ_a , we simply have a very steep potential, so that the fields are predominantly hovering about the minimum of the potential, $P^a(\mathbf{X}) = 0$. Taking the limit $\lambda_a \rightarrow \infty$ then corresponds to narrowing the potential until it becomes a delta function(al), constraining the fields at $P^a(\mathbf{X}) = 0$. In fact, if we agree to focus on the IR sector, then it does not really matter whether the higher frequency modes of the fields are constrained or not; as long as the low-frequency modes are restricted to the subspace $P^a(\mathbf{X}) = 0$, global properties of the model will be successfully computed.

With this in mind, we set $\lambda_a \rightarrow 1$ and the action reduces to

$$\int_{\Sigma} d^2\sigma d^2\zeta d^2\bar{\zeta} K(\mathbf{X}, \mathbf{X}^\dagger) + \int_{\Sigma} d^2\sigma W(\mathbf{X}) + \text{h.c.}, \quad (\text{F.6.2})$$

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where $K(\mathbf{X}, \mathbf{X}^\dagger)$ is the Kähler potential into which the kinetic term finally renormalizes and

$$W[\mathbf{X}] = \sum_{a=1}^K P^a(\mathbf{X}) \quad (\text{F.6.3})$$

is the superpotential. In favorable circumstances, the so obtained model will have all the right properties with respect to the various anomalies and one proceeds to work with this.

However, there still exist superficial fields since without the non-trivial kinetic term we have lost the projectivity relation. For example, instead of having four local coordinates on \mathbb{P}^4 , there would be the five homogeneous coordinates amongst which we cannot distinguish. Also, we have “lost” the Lagrange multipliers. However, one can perform a holomorphic change of variables, which will bring out a monomial in front of every P^a . These monomials can then be thought of as a single variable, provided that the total Jacobian is field-independent*. The condition for that turns out to be the Ricci-flatness condition (2.1.17). The change of variables, however, is not single-valued, so that one needs to pass to an orbifold and it is that Landau-Ginzburg Field Theory which corresponds to the Calabi-Yau space that we have started from.

The cyclic symmetry group by the action of which the Field Theory had to be divided is nothing else than the maximal subgroup of the (product of) $U(1)$ symmetry(ies)—one for each \mathbb{P}^n factor in the embedding space—such that the constrained \mathbb{P}^n action is left invariant. (Since the Kähler potential is strictly invariant, so must the superpotential be.) Being a remnant of a gauge symmetry, the necessity to divide out its action is obvious. Some powerful techniques have recently been advanced [229,159,227,174], by which the Field Theory analogue of the (p, q) -cohomology can be determined, together with its ring structure and so the Yukawa couplings. The so obtained results strikingly resemble the Koszul spectral sequence. In particular, it is possible to show in case-by-case analysis that in both cases all the cohomology (untwisted and twisted, (c, c) - and (a, c) -sectors) is represented by elements of certain quotient rings, and where both the polynomial ring and the quotient ideal are *identical* [72]. The two methods are nevertheless rather different [241]; also, the Koszul spectral sequence is much more widely applicable while the Landau-Ginzburg orbifold analysis provides more detail.

*Supersymmetry of course helps a lot, by canceling out the contributions from every Bose-Fermi pair. Zero modes, however are not paired and do contribute non-trivially.

F.7 How Singular a Space Can Superstrings Thread?

Consider again a Calabi-Yau constrained \mathbb{P}^n model with the action (F.4.1), (F.4.2), where

$$X^\mu \stackrel{\text{def}}{=} X^\mu + \varsigma^+ \xi_+^\mu + \varsigma^- \xi_-^\mu + \varsigma^+ \varsigma^- X^\mu \quad (\text{F.7.1})$$

are the coordinate superfields and

$$\Lambda_a \stackrel{\text{def}}{=} \Lambda_a + \varsigma^+ \lambda_{+a} + \varsigma^- \lambda_{-a} + \varsigma^+ \varsigma^- L_a \quad (\text{F.7.2})$$

are the Lagrange superfields.

Path-integration over $\Lambda_a, \lambda_{\pm a}, L_a$ and X^μ yield delta-functionals which enforce algebraic field equations. Upon the fermionic integration, these are :

$$\int D[\Lambda_a] \Rightarrow 0 = P_{,\mu\nu}^\alpha \xi_+^\mu \xi_-^\nu - P_{,\mu}^\alpha X^\mu, \quad (\text{F.7.3})$$

$$\int D[\lambda_{\pm a}] \Rightarrow 0 = P_{,\mu}^\alpha \xi_{\pm}^\mu, \quad (\text{F.7.4})$$

$$\int D[L_a] \Rightarrow 0 = P^\alpha, \quad (\text{F.7.5})$$

$$\int D[X^\mu] \Rightarrow 0 = G_{\mu\nu} X^\nu - \Gamma_{\mu\nu\rho}^\alpha \xi_+^\nu \xi_-^\rho - \Lambda_a P_{,\mu}^\alpha, \quad (\text{F.7.6})$$

with $G_{\mu\nu} = K_{,\mu\nu}$ and $\Gamma_{\mu\nu\rho}^\alpha = K_{,\mu\nu\rho}$. Contracting Eq. (F.7.6) with $G^{\mu\rho} \bar{P}_{,\rho}^\alpha$, we obtain

$$\Lambda_a M^{a\bar{\alpha}} = \bar{P}_{,\bar{\mu}}^\alpha X^\bar{\mu} - \Gamma_{\bar{\mu}\bar{\rho}\bar{\sigma}}^\alpha \bar{P}_{,\bar{\mu}}^\alpha \xi_+^\bar{\rho} \xi_-^\bar{\sigma}, \quad (\text{F.7.7})$$

$$M^{a\bar{\alpha}} \stackrel{\text{def}}{=} (P_{,\mu}^\alpha G^{\mu\bar{\mu}} \bar{P}_{,\bar{\mu}}^\alpha), \quad (\text{F.7.8})$$

here and hereafter, $G^{\mu\bar{\mu}}$ is used to raise indices.

F.7.1 Smooth target space

$M^{a\bar{\alpha}}$ is easily seen to be invertible and actually positive definite on the constrained subspace, provided the matrix of gradients $[P_{,\mu}^\alpha(\mathbf{X})]$ is of rank K where $P^\alpha(\mathbf{X}) = 0$, i.e., provided the $\{P^\alpha(\mathbf{X}) = 0\}$ subspace is smooth. Then,

$$\Lambda_a = M_{a\bar{\alpha}} \bar{P}_{,\bar{\mu}}^\alpha X^\bar{\mu} - M_{a\bar{\alpha}} \bar{P}_{,\bar{\mu}}^\alpha \Gamma_{\bar{\mu}\bar{\rho}\bar{\sigma}}^\alpha \xi_+^\bar{\rho} \xi_-^\bar{\sigma}. \quad (\text{F.7.9})$$

Path-integration over X^μ results in a delta-function that enforces Eq. (F.7.9).

F.7. How Singular a Space Can Superstrings Thread?

As always, the matrix of gradients $[P_{,\mu}^\alpha]$ is a proper tensor and independent of any connection Γ_μ on the constrained subspace where $P^\alpha = 0$ since $P_{,\mu}^\alpha \stackrel{\text{def}}{=} P_{,\mu}^\alpha + \Gamma_\mu \cdot P^\alpha = P_{,\mu}^\alpha$. Furthermore, since the polynomials $P^\alpha(\mathbf{X})$ are locally transverse to the subspace $\mathcal{M} \hookrightarrow \mathcal{X}$, the matrix $[P_{,\mu}^\alpha(\mathbf{X})]$ projects onto the normal space to \mathcal{M} at X^μ ; this is the $\nabla_X f$ map of Eq. (1.3.6).

Upon elimination of the X 's and \bar{X} 's by means of their new equations of motion, the Lagrangian in (F.4.1) contains the four-fermion term

$$\left[R_{\mu\nu\rho\bar{\sigma}} + P_{,\mu;\rho}^\alpha M^{\alpha\bar{\alpha}} \bar{P}_{,\bar{\nu};\bar{\sigma}}^\alpha \right] \xi_+^\mu \xi_-^\rho \xi_+^\bar{\nu} \xi_-^\bar{\sigma}, \quad (\text{F.7.10})$$

where

$$R_{\mu\nu\rho\bar{\sigma}} \stackrel{\text{def}}{=} K_{,\mu\nu\bar{\rho}\bar{\sigma}} - \Gamma_{\mu\rho}^\kappa \Gamma_{\kappa\bar{\nu}\bar{\sigma}}, \quad P_{,\mu;\nu}^\alpha \stackrel{\text{def}}{=} P_{,\mu\nu}^\alpha + \Gamma_{\mu\nu}^\rho P_{,\rho}^\alpha, \quad (\text{F.7.11})$$

are the standard Riemann tensor on \mathcal{X} and the extrinsic curvature of the constrained subspace, respectively. The sum of the two contributions in Eq. (F.7.10) is indeed the induced Riemann tensor on the constrained subspace, to which the four-fermion term should couple.

F.7.2 Singular target space

We now prove the following.

Lemma F.1 *For a constrained σ -model to inherit the (2,2)-supersymmetry of the ambient σ -model and the possible singularities to be innocuous, it suffices that the constraints have nonzero Taylor series up to and including second order.*

Proof: Consider the case where $\text{rank}[P_{,\mu}^\alpha] < K$ at a finite number of points of the $\{P^\alpha = 0\}$ subspace \mathcal{M} , which is therefore singular at those points. Also, for ease of notation, we take $K = 1$. At a singular point, both P^1 and $P_{,\mu}^1$ vanish and there Eqs. (F.7.3) becomes

$$\int D[\Lambda_1] \Rightarrow 0 = P_{,\mu\nu}^1 \xi_+^\mu \xi_-^\nu, \quad (\text{F.7.12})$$

Eq. (F.7.5) is unchanged and restricts the string coordinates X^μ from \mathcal{X} to the 3-dimensional constrained hypersurface \mathcal{M} while Eq. (F.7.4) becomes trivial, $0 = 0$. However, the ξ_{\pm}^μ are now constrained by Eq. (F.7.12)—as long as the matrix $[P_{,\mu\nu}^1]$ does not vanish. It is easy to see that the ξ_{\pm}^μ now span a 3-dimensional

cone with the tip at the singularity, as they should, tangential to the constrained hypersurface spanned by the X^μ subject to $P^1(X) = 0$.

Note that, for each fixed a , the matrix of second derivatives $[P_{,\mu\nu}^a]$, the so-called Hessian, is independent of any connection at a singularity since both the polynomials P^a and the gradients $P_{,\mu}^a$ vanish there and so

$$P_{,\mu\nu}^a \stackrel{\text{def}}{=} P_{,\mu\nu}^a + \Gamma_{\mu,\nu} \cdot P^a + \Gamma_{\mu\nu}^\rho P_{,\rho}^a = P_{,\mu\nu}^a.$$

The rank of $[P_{,\mu\nu}^a]$ has therefore an invariant meaning. In case of more constraints, the matrices of second derivatives $P_{,\mu\nu}^a$ form a direct sum and the analogous conclusion follows immediately.

Note also that, at a singularity, Eq. (F.7.6) becomes

$$\int D[X^\mu] \Rightarrow 0 = G_{\mu\nu} X^\nu - \Gamma_{\mu\nu\sigma} \xi_+^\nu \xi_-^\sigma \quad (\text{F.7.13})$$

and decouples from Eqs. (F.7.4), (F.7.5) and (F.7.12). Using these to eliminate all of the X^μ 's at the singularity, the four-fermion term there is reduced to $R_{\mu\nu\sigma\tau} \xi_+^\mu \xi_-^\sigma \xi_+^\nu \xi_-^\tau$, with the ambient space Riemann tensor defined in (F.7.11) and the fermions subject to Eq. (F.7.12). At the singularity, the σ -model decouples from the extrinsic curvature term which occurred in Eq. (F.7.10) and would have been divergent since $M_{a\bar{a}}$ is the inverse of $M^{a\bar{a}}$ and the latter vanishes at any singularity. Writing $P^a = \phi^a + t\varphi^a$ with ϕ^a singular and φ^a smooth, divergent terms include t^{-1} and are seen to decouple in view of Eq. (F.7.12). \checkmark

We remark that all simple (A, D, E), modality-1 and modality-2 singular polynomials (see Ref. [1], p.158–160 of the first book) satisfy this criterion. To see this, take for example the A_k polynomial germ, $f(x) = x^{k+1}$. A germ represents all constraints in a \mathbb{C}^n -like neighborhood which can be brought, through a holomorphic change of local coordinates, to the form

$$f(x_1) + x_2^2 + \dots + x_n^2 = 0. \quad (\text{F.7.14})$$

They all exhibit the same type of singularity and can be smoothed in the same way. The inclusion of squares of all the local coordinates not involved in the germ is called Morsification. Since the germs for all simple, modality-1 and modality-2 singularities involve less than four local coordinates, their Morsification always contains at least one coordinate occurring as a square and the matrix of second derivatives has at least rank one. It is important to note that the omission of even

one local coordinate in Eq. (F.7.14) represents a *much worse* singularity than is indicated by the germ.

With the above result about supersymmetry and the fact that the σ -model decouples from the diverging extrinsic curvature, we conclude that isolated A, D, E type singularities are just as innocuous as orbifold singularities were well known to be [119].

LEXICON

The following is a collection of definitions and short explanations of various terms used in the text. Some of the entries in the Lexicon duplicate partly the information given in the text, however in a form which is more suitable for a quick reference. Terms which are typeset as for example ‘*ample*^(L)’ have their own entries elsewhere in this Lexicon.



- ▷ **0-Mode** : Originally, this derives from Fourier expansion $f(x) = \sum_{k=-\infty}^{+\infty} \phi_k e^{ikx}$, where it refers to ϕ_0 . More generally, other (integral or otherwise) transforms stand in place of the Fourier transform, and 0-modes refer to ‘zeros’ in the transform spectrum. As the latter typically includes energy, momentum and/or mass, these are also synonyms for ‘massless modes’; see for example § 0.3.
- ▷ **1-Dot-Decomposable** (1DD) : A diagram (see § 2.1.2) which disconnects upon deletion of a dot.
- ▷ **1-Leg-Decomposable** (1LD) : A diagram (see § 2.1.2) which disconnects upon deletion of a leg.
- ▷ **Adjunction Formulae** : Let X be a *manifold*^(L) with a holomorphic line *bundle*^(L) \mathcal{E} over X and let ϕ be a *section*^(L) of \mathcal{E} . We can define the corresponding submanifold, called *hypersurface*, $\mathcal{M}_\phi \subset X$ as the subspace of solutions to $\phi(x) = 0$, $x \in X$. If \mathcal{M} is smooth, then

$$\mathcal{N}_{X/\mathcal{M}} = \mathcal{E}|_{\mathcal{M}} \quad (\text{Adjunction Formula 1})$$

equates the *normal bundle*^(L) of $\mathcal{M} \hookrightarrow X$ with the restriction to \mathcal{M} of \mathcal{E} (see § 1.3.1 or Ref. [23], p. 146). Furthermore,

$$\mathcal{K}_{\mathcal{M}} = (\mathcal{K}_X \otimes \mathcal{E})|_{\mathcal{M}} \quad (\text{Adjunction Formula 2})$$

equates the *canonical bundle*^(L) on \mathcal{M} with the restriction to \mathcal{M} of the tensor product of the canonical bundle of X and the determinant of \mathcal{E} (see § 1.3.1 or Ref. [23], p. 146). If $\mathcal{M} \hookrightarrow X$ is a *complete intersection*^(L), defined as the common zero-set of a system of sections $\{\phi^a\}$, where each $\phi^a(x)$ is the section of a line bundle \mathcal{E}_a over X , the above formulae become

$$\mathcal{N}_{X/\mathcal{M}} = (\bigoplus_a \mathcal{E}_a)|_{\mathcal{M}}, \quad \mathcal{K}_{\mathcal{M}} = (\mathcal{K}_X \otimes \bigotimes_a \mathcal{E}_a)|_{\mathcal{M}}.$$

▷ **Algebraic Variety** : A (compact, complex) n -dimensional space which can be embedded in some \mathbb{P}^N , for large enough N . Called a *curve* for $n=1$, a *surface* for $n=2$, and (less often) a *solid* for $n=3$; *Riemann surface*^(L) is in fact a (complex) *curve*.

▷ **Almost Ample** : A holomorphic line *bundle*^(L) \mathcal{L} over a complex *manifold*^(L) \mathcal{M} is called almost ample if it is non-trivial and, at each $x \in \mathcal{M}$, admits a global holomorphic *section*^(L) which does not vanish there. In particular, \mathcal{L} must have enough global holomorphic sections so that at least one of them is non-zero at any given point $x \in \mathcal{M}$. The absolute minimum is two global holomorphic sections: their common zero-locus is in *codimension*^(L) 2 and may be blown-up; see § 3.1.1.

▷ **Almost Complex Structure** : If \mathcal{M} is an even dimensional *manifold*^(L), an almost complex structure on \mathcal{M} is an *endomorphism*^(L) J of the *tangent bundle*^(L) of \mathcal{M} satisfying $J^2 = -1$. It is important to realize that, unlike for a *complex structure*^(L), an almost complex structure J need not be integrable. Every complex manifold admits an almost complex structure, given in terms of local complex coordinates $\{z_k = x_k + iy_k\}$ by

$$J\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k}; \quad J\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k}. \quad (\text{L.1})$$

There may, however, exist almost complex structures which do not arise in this way.

▷ **Almost del Pezzo Surface** : A non-singular algebraic surface (compact, complex 2-dimensional *manifold*^(L)) \mathcal{S} is called almost del Pezzo if its anti-canonical bundle $\mathcal{K}_{\mathcal{S}}^*$ is almost ample. Synonyms : almost ample surface (2-fold).

▷ **Almost Fano 3-Fold** : A non-singular algebraic 3-fold (compact, complex 3-dimensional *manifold*^(L)) \mathcal{M} is called almost Fano if its anti-canonical bundle $\mathcal{K}_{\mathcal{M}}^*$ is almost ample. Synonyms : almost ample solid, almost ample 3-fold.

▷ **Almost Positive** : A bundle over X is almost positive if it is *positive*^(L) over X except over a special subset $S \subset X$ ($\dim S < \dim X$), where it is flat.

▷ **Ample** : A *holomorphic*^(L) line *bundle*^(L) \mathcal{L} over a complex manifold \mathcal{M} is called ample if at every point of \mathcal{M} there exist at least two *sections*^(L) of \mathcal{L} such that one vanishes, but its gradients and the other section are non-zero. More precisely, for every point $x \in \mathcal{M}$ and for every tangent vector $X = \xi^i \partial_i$ to \mathcal{M} at x , there must exist global *holomorphic*^(L) sections σ and τ of \mathcal{L} such that

$$\sigma(x) \neq 0; \quad (\text{almost ample})$$

$$\tau(x) = 0; \quad (X\tau)(x) = (\xi^i \partial_i \tau)(x) \neq 0.$$

Stated differently, \mathcal{L} is ample if the bundle of one-jets^(L) of local holomorphic sections of \mathcal{L} , i.e., the local Taylor series up to second order is generated at every point by global holomorphic sections.

▷ **Analytic** : A function $f(z)$ is analytic in a domain \mathcal{D} if it has a local Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)$ for all $z_0 \in \mathcal{D}$. This is true precisely if $f(z)$ is holomorphic in \mathcal{D} .

▷ **Automorphism** : An *endomorphism*^(L) which is also an *isomorphism*^(L), that is, a 1–1 onto mapping of a group G to itself which preserves the group structure.

▷ **Bertini's Theorem** (in its most general form) : *A generic element of a linear system is smooth away from the base locus of the system.*

Essentially, it means that a system of sufficiently generally chosen algebraic equations over a space X has a subspace \mathcal{M} of simultaneous solutions which is a smooth subspace of X except perhaps at the *base locus* of the system. The latter is the simultaneous zero-set in X of all polynomials the linear combinations of which form the given system. See § 1.1, § 1.4 and especially § 2.2.2 for application and clarifications.

▷ **Betti Numbers** : Given any topological space X of dimension n , the Betti numbers b_0, \dots, b_n are defined to be the dimensions of its integral, real or complex *homology*^(L) or *cohomology*^(L) groups. So,

$$b_q \stackrel{\text{def}}{=} \dim_{\mathbb{k}} H^q(X, \mathbb{k}) = \dim_{\mathbb{k}} H_q(X, \mathbb{k}), \quad \mathbb{k} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \dots \quad (\text{L.2})$$

▷ **Birational Map** : A pair of *Rational Maps*^(L) $\alpha : A \rightarrow B$ and $\beta : B \rightarrow A$, such that $\alpha \circ \beta = \text{Id}_B$ and $\beta \circ \alpha = \text{Id}_A$ —as rational maps.

▷ **Blow-Up** : A process by which a point p in a complex n -dimensional space X is replaced by a compact complex $(n-1)$ -dimensional space E in such a way that E is a complex subspace of the resulting blow-up, \tilde{X}_p . In other words, the space E must admit a complex 1-dimensional fibration such that the n -dimensional total space of this fibration fits in the ‘hole’ left in X when p was removed. Note that p could have been a smooth or a singular point of X ; its replacement, E is called the exceptional divisor of \tilde{X}_p .

Given a complex k -dimensional subspace $C \subset X$, at every point $p \in C \subset X$, we can stack along C local $(n-k)$ -dimensional ‘neighborhoods’ which are transversal to C and do not intersect. Each point on C can then be blown up in such a way that the $(n-k-1)$ -dimensional exceptional divisors E_p vary holomorphically

as we vary $p \in C$ and form the $(n-1)$ -dimensional exceptional divisor in \widetilde{X}_C . Usually, C is called the *center* of the blowing up process. Most notably, regardless of the dimension of C , the exceptional divisor has *codimension*^(L) 1 in X ; one says that blowing up is a codimension-1 process.

▷ **Bosons** : Particles and fields of integral spin, that is, transforming as integral representations of the *Lorentz group*^(L), and its covering *Spin*-group. These are then usual commuting (c-number-valued) functions or sections of appropriate bundles.

▷ **Bott-Borel-Weil Theorem** : Consider a flag space^(L)

$$X = \mathbb{F}_{[n_1| \dots | n_F]} \approx \frac{U(N)}{U(n_1) \times \dots \times U(n_F)}, \quad N = \sum_{i=1}^F n_i. \quad (\text{L.3})$$

(1) The homogeneous vector bundles over X are in 1-1 correspondence with the $U(n_1) \times \dots \times U(n_F)$ representations. We label irreducible such vector bundles by

$$\mathcal{V} = (a_1, \dots, a_{n_1} | \dots | b_1, \dots, b_{n_F}). \quad (\text{L.4})$$

(2) The $H^q(X, \mathcal{V})$ is non-zero for at most one value of q , in which case it furnishes an irreducible representation of $U(N)$, $H^q(X, \mathcal{V}) \approx (c_1, \dots, c_N) \mathbb{C}^N$.

(3) $(a_1, \dots, a_{n_1} | \dots | b_1, \dots, b_{n_F})$ determines (c_1, \dots, c_N) according to the following algorithm :

1. Add the sequence $1, \dots, N$ to the entries in $(a_1, \dots, a_{n_1} | \dots | b_1 \dots b_{n_F})$.
2. If any two entries in the result of Step 1 are equal, all cohomology vanishes; otherwise proceed.
3. Swap the minimum number ($= q$) of neighboring entries required to produce a strictly increasing sequence.
4. Subtract the sequence $1, \dots, N$ from the result of 3, to obtain (c_1, c_2, \dots, c_N) .

For more details, see Ref. [18] and for generalizations to other (generalized) flag spaces, see Ref. [5].

▷ **Bott Vanishing Theorem** : Several vanishing theorems are generally understood under this title and they are precursors of the more general Bott-Borel-Weil Theorem, so they may be viewed as special cases thereof. For example :

Theorem Consider $X = \mathbb{F}_{[1|n]} = \mathbb{P}^n$, and $\mathcal{V} = \mathcal{O}(k)$. Then

$$H^q(\mathbb{P}^n, \mathcal{O}(k)) = 0, \quad \begin{cases} q \neq 0 & \text{when } 0 \leq k, \\ \forall q & \text{when } -n \leq k < 0, \\ q \neq n & \text{when } k \leq -(n+1). \end{cases} \quad (\text{L.5})$$

Since $\mathcal{O}_{\mathbb{P}^n}(k) = (-k|0, \dots, 0)$, this easily follows from the Bott-Borel-Weil Theorem.

▷ **Branched Covering** : Consider a complex n -dimensional space X and $B \subset X$, a complex $(n-1)$ -dimensional subspace thereof. Then \mathcal{Y} is a p -fold ($p \in \mathbb{Z}_+$) cover of X if there exists a mapping $\pi : \mathcal{Y} \rightarrow X$ such that to every point $x \in X$, it associates p points $y_1(x), \dots, y_p(x) \in \mathcal{Y}$. If however, this is true except along $B \subset X$, where to every point $x_{(B)} \in B$ only a single point $y(x_{(B)}) \in \mathcal{Y}$ is associated, \mathcal{Y} is a p -fold cover of X , branched over (along) $B \subset X$. “Ramified cover” is a synonym.

▷ **Bundle** (Fibre~, Line~, Vector~) : A fibre bundle is a triple $(\mathcal{B}, \mathcal{E}, \varpi)$, where \mathcal{B} is called the base-space, \mathcal{E} the total space and $\varpi : \mathcal{E} \rightarrow \mathcal{B}$ is a projection.

At any point $x \in \mathcal{B}$, the inverse image of ϖ at x , i.e., the set of points $\{\xi \in \mathcal{E} : \varpi(\xi) = x\}$ form the *fibre* at x . Usually, all fibres are in a certain uniform (e.g., holomorphic) sense isomorphic to some vector space; we then have a (holomorphic) *vector bundle*. The simplest vector bundles over a compact complex manifold \mathcal{B} have unit rank, are called holomorphic *line bundles* and their fibre is isomorphic to \mathbb{C}^1 . Under the (fibre-wise) tensor product, these bundles form a group known as the *Picard group*^(L) of \mathcal{B} .

▷ **Canonical Bundle** : Given \mathcal{T}_M^* , the *holomorphic*^(L) *cotangent bundle*^(L) of a manifold M , we construct the determinantal holomorphic line bundle^(L) $\det \mathcal{T}_M^*$, which is called the canonical bundle and is denoted

$$\mathcal{K}_M \stackrel{\text{def}}{=} \det \mathcal{T}_M^* = \wedge^{\dim M} \mathcal{T}_M^*. \quad (\text{L.6})$$

▷ **Chern Character** : Given a vector bundle \mathcal{V} over a complex manifold X and its total Chern class, $c(\mathcal{V})$, we use the splitting principle :

$$c(\mathcal{V}) = \prod_{i=1}^k c(\mathcal{L}_i) = \prod_{i=1}^k (1 + \ell_i), \quad (\text{1.3.19})$$

to introduce $k = \text{rank } \mathcal{V}$ auxiliary line bundles and $\ell_i \stackrel{\text{def}}{=} c(\mathcal{L}_i)$. Then,

$$\text{ch}(\mathcal{V}) \stackrel{\text{def}}{=} \sum_{i=1}^k e^{\ell_i} = k + \sum_{r=1}^{\infty} \text{ch}_r[\mathcal{V}], \quad (\text{L.7})$$

is the *total Chern character*, given as a formal sum of the r^{th} Chern characters. Clearly, $\text{ch}_r[\mathcal{V}]$ is a closed (r, r) -form.

▷ **Chern Class** : Given a vector bundle \mathcal{V} over a complex manifold X and a closed curvature $(1,1)$ -form Θ on \mathcal{V} , we define

$$c[\Theta] \stackrel{\text{def}}{=} \det[\mathbf{1} + \frac{i}{2\pi} \Theta] = \mathbf{1} + c_1[\Theta] + c_2[\Theta] + \dots \quad (\text{0.2.9})$$

Clearly, each $c_q[\Theta]$ is a closed (q, q) -form and represents the q^{th} Chern class; $c[\Theta]$ represents the *total Chern class*.

- ▷ **Chern Numbers** : Given the Chern classes on a compact complex n -dimensional manifold X , the integrals

$$C_1^n \stackrel{\text{def}}{=} \int_X \wedge^n c_1, \quad C_1^{n-2} \cdot C_2 \stackrel{\text{def}}{=} \int_X \wedge^{n-2} c_1 \wedge c_2, \quad (\text{L.8})$$

and so on, are topological (deformation) invariants of X since $c_i = c_i(T_X)$ are closed. For 3-folds, we write just C_1 , C_2 and C_3 , understanding the integrals of $(\wedge^3 c_1)$, $(c_1 \wedge c_2)$ and (c_3) , respectively.

- ▷ **Chirality** : Originally, denoted the sign of the non-zero helicity (the projection of the spin along the linear momentum). If necessary, we adhere to the convention that positive helicity corresponds to left-handedness. Note that for precisely massless particles, helicity is Lorentz-invariant (massless particles move at the speed of light, so no observer can move past them to ‘flip’ the helicity). Related to this, a ‘chiral superfield’ denotes a *superfield*^(L) with two component fields : a complex scalar (spin-0) and a left-handed Weil spinor (spin-1/2, helicity+). In *superspace*^(L) formulation, this implies that a chiral superfield is independent of the left-handed spinorial coordinates and this property is in many ways analogous to *holomorphicity*^(L).

- ▷ **Closed Subset** : In a variety X , a subset $\mathcal{Y} \subset X$ consisting of all common zeros of finitely many (local, holomorphic) polynomials. In all cases we will consider, these polynomials are (local) realizations of certain global (holomorphic) sections.

- ▷ **Codimension** : Given a space X of dimension $\dim X = n$ and a subspace $\mathcal{Y} \subset X$ of dimension $\dim \mathcal{Y} = m$, the codimension of \mathcal{Y} in X is defined simply to be $(n-m)$. One then speaks of how deep (in codimension) \mathcal{Y} lies in X . By convention, the statement $\text{codim}(\mathcal{Y} \subset X) > \dim X$ means that \mathcal{Y} is an empty set.

- ▷ **Cohomology Group** (for some more details, see § 7.1, but there are classic books written on this subject, such as Ref. [9]; it would be foolish to attempt a self-contained review here) : There exist many different ways of defining cohomology groups : de Rham[~], Čech[~], simplicial[~], equivariant[~], Delbeault[~] (also known as $\bar{\partial}$ [~]), sheaf[~], . . . are all standard and well known in the mathematics literature. On a smooth manifold M , the different definitions lead to isomorphic results. Most of the time we will deal with the latter two mentioned for both of which we simply write $H^q(M, V)$, with V a vector bundle or a sheaf over M in which the elements of $H^q(M, V)$ take values. $H^q(M, V)$ is isomorphic to the space of harmonic V -valued q -forms, i.e., rank-(0, q) antisymmetric covariant tensors (no dz 's, g $d\bar{z}$'s) ω_q which take values in V and satisfy the second order differential equation $\Delta \omega_q = 0$.

If M is singular, the above-mentioned cohomology groups differ and are often (somewhat) ill-behaved. More recently, so-called intersection[~] and L^2 [~] were introduced by Goresky and MacPherson [26] and respectively Cheeger [15]. On smooth manifolds, these are isomorphic to the previously known ones; on a large class of singular spaces, however, the resulting cohomology groups are better behaved and retain many of the usual properties such as *Poincaré duality*^(L), *Hodge decomposition*^(L), *Complex conjugation*^(L), *Serre duality*^(L), *Lefschetz decomposition*^(L) (*Lefschetz Hard Theorem*^(L)), etc.

Somewhat confusingly perhaps, q is referred to as the dimension and is not to be confused with the dimension of the vector space $H^q(M, V)$ —loosely speaking—the number of linearly independent elements of $H^q(M, V)$. Since $H^q(M, V)$ is an abelian group under addition, the dimension of the vector space $H^q(M, V)$ may also be referred to as the rank of the cohomology group $H^q(M, V)$.

- ▷ **Cokernel** : Given two sets A and B and a map $A \xrightarrow{\gamma} B$, the cokernel of γ is defined to be the quotient $\{B/\gamma(A)\}$. In other words, the cokernel of the map $A \xrightarrow{\gamma} B$ may be regarded as spanned by elements of B , taken modulo $\gamma(a)$, for all $a \in A$. (For *sheaves*^(L), this must be corrected; see Ref. [23], p.36–37.) We will denote the cokernel of γ by $\text{cok}(\gamma)$; the notation $\text{coker}(\gamma)$ is also found in the literature.

- ▷ **Compact** : A Hausdorff^(L) space X is *compact* if for every open covering $\mathfrak{U} = \{U_i\}$, there is a finite sub-collection of open sets $\{U_{i_1}, \dots, U_{i_n}\}$ which covers X .

- ▷ **Complete Intersection** : A variety X of dimension m in \mathbb{P}^n is a *strict complete intersection* if the *Ideal of the variety*^(L), $I(X)$ can be generated by $n-m$ elements. X is a *set-theoretic complete intersection* if it can be realized as the intersection of some $n-m$ hypersurfaces^(L).

- ▷ **Complex** : A *complex* $(K^*, d) = \{K^0 \xrightarrow{d} K^1 \xrightarrow{d} \dots\}$ is a sequence of Abelian groups with differentials $d : K^r \rightarrow K^{r+1}$ satisfying $d \circ d = 0$. The *cohomology* of a complex is $H^*(K^*) = \bigoplus_{r \geq 0} H^r(K^*)$, with

$$H^r(K^*) \stackrel{\text{def}}{=} \left\{ [\ker d : K^r \rightarrow K^{r+1}] / dK^{r-1} \right\}. \quad (\text{L.9})$$

- ▷ **Complex Conjugation** (on the cohomology) : The requirement that $H^{p,q} = \overline{H^{q,p}}$; this is usually included in the *Hodge decomposition*^(L).

▷ **Complex Projective Space** : The complex projective space, $\mathbb{P}^n \stackrel{\text{def}}{=} \mathbb{P}(\mathbb{C}^{n+1})$ is the complex n -dimensional space of all complex lines (copies of \mathbb{C}^1) in \mathbb{C}^{n+1} , that is, a point in \mathbb{P}^n is a 1-dimensional linear subspace $L \subset \mathbb{C}^{n+1}$.

Explicitly, let z^0, \dots, z^n be coordinates on \mathbb{C}^{n+1} . Then $(z^0 : \dots : z^n)$ and $\lambda(z^0 : \dots : z^n)$ lie on the same complex line $L \subset \mathbb{C}^{n+1}$, parametrized by $\lambda \in \mathbb{C}$ and correspond to the same point of \mathbb{P}^n . In the coordinate patch where $z^i \neq 0$, suitable local coordinates are $\zeta_{(i)}^j \stackrel{\text{def}}{=} z^j/z^i$. The (one dimension redundant) coordinates z^i are called homogeneous and the local coordinates $\zeta_{(i)}^j$ are called affine.

Alternatively, \mathbb{P}^n can also be defined as the quotient

$$\frac{U(n+1)}{U(n) \times U(1)} \cong \frac{SU(n+1)}{S(U(n) \times U(1))}, \quad (\text{L.10})$$

where $U(n)$ denotes the Lie group of $n \times n$ unitary matrices while the prefix S denotes unimodularity. This fact is employed to a great extent in Chapters 9–B.

▷ **Complex Structure** : An integrable *almost complex structure*^(L). That is, an *endomorphism*^(L) J of the tangent bundle of an even-dimensional manifold, which squares to $-\mathbf{1}$ and in addition satisfies the integrability condition, the vanishing of the Nijenhuis tensor

$$N_{ij}{}^k \stackrel{\text{def}}{=} J_{[i}{}^p J_{j]}{}^q (\partial_p J_q{}^k) - \partial_{[i} J_{j]}{}^k. \quad (\text{L.11})$$

Note that this is independent of any connection.

▷ **Compound Du Val (cDV) Singularities** : A 3-fold hypersurface singularity $p^\sharp \in X$, which is locally given by

$$F(x, y, z, t) \stackrel{\text{def}}{=} f(x, y, z) + tg(x, y, z, t) = 0, \quad (\text{L.12})$$

in \mathbb{C}^4 , where g is an arbitrary polynomial and $f(x, y, z)$ is one of the following

A_k	$: x^{k+1} + y^2 + z^2 = 0,$	$(n \geq 1),$
D_k	$: x^{k-1} + y^2 z + z^2 = 0,$	$(n \geq 4),$
E_6	$: x^4 + y^3 + z^2 = 0,$	
E_7	$: x^3 y + y^3 + z^2 = 0,$	
E_8	$: x^5 + y^3 + z^2 = 0.$	

(L.13)

These are called ‘simple’, ‘rigid’ or Du Val 2-fold (surface) singularities and a cDV 3-fold singularity is 1-parameter deformation of some isolated *A-D-E* 2-fold singularity.

▷ **Configuration** : In our context, most generally, a configuration is a pair $[X||\mathcal{E}]$, where X is the *embedding space* and \mathcal{E} is a vector bundle over X (typically, a sum of line bundles). We say that a variety \mathcal{M} *belongs* to, or is a *member* of, a configuration if it can be realized as the zero-set of a section ϕ of \mathcal{E} and we write $\mathcal{M} = \phi^{-1}(0)$. Naturally, a configuration contains a deformation class of varieties, parametrized by the choice of the section ϕ .

▷ **Conifold** : A space which fails to be smooth at an isolated set of singular points, the neighborhood of each of which is isomorphic to a cone. This includes *hyperquotient* singularities [46], that is singular points which, loosely speaking, arise by mounting orbifold (finite quotient) and isolated hypersurface singularities atop each other.

▷ **Contraction** : Generally, it refers to crushing a subspace $E \subset X$ to a smaller subspace $e \subset X^\#$ in a certain ‘uniform’ way. Unless explicitly stated otherwise, we think of a collapse of a (real) 2-cycle $E \subset M$ to an A_k -, D_k - or E_k -singular point (L.13), where the 2-cycle consists of \mathbb{P}^1 's configured according to the Dynkin diagram of the respective Lie algebra. Then, the normal bundle of all these \mathbb{P}^1 's is $\mathcal{O}(-1) \otimes \mathcal{O}(-1)$ and they cannot be deformed. Since for 3-folds this happens in (complex) codimension 2, the canonical class is not affected and the singular space obtained upon the contraction inherits the holomorphic 3-form. See also the Small Map Theorem 7.4.

▷ **Contractible** : Can be diminished in dimension (usually, to a point). In doing so, care must be taken that the category of objects under scrutiny is not left: for example, non-contractibility of a subspace X in some space Y as a potential cycle (homology class representative) in $H_*(Y)$ implies that X cannot be “shrunk” continuously to a point and remaining throughout the process within Y . So, e.g., a ‘small circle’ is non-contractible within a (real) 2-torus, although all circles are contractible within, e.g., 2-spheres or \mathbb{R}^3 .

▷ **Deformation Space** : In general, given a (topological class of) real (compact, orientable) manifold(s) the choice of the complex structure will depend on a number of parameters. That is, we can deform the space so as to become a different complex manifold while remaining in the same topological class. The space spanned by such parameters is called the deformation space and is often rather straightforwardly parametrized by the choice of the embedding, for example.

▷ **De Rham Duality** : On a real (compact, orientable) differentiable manifold X , let $\alpha_{(r)}^i$ denote closed differential r -forms and $\gamma_i^{(r)}$ elements of $H_r(X)$ (the vector space of r -cycles modulo boundaries of finite $(r+1)$ -chains). Then the condition $\int_{\gamma_j^{(r)}} \alpha_{(r)}^i = \delta_j^i$ depends only on the respective (co)homology classes of $\gamma_i^{(r)}$ and

$\alpha_{(r)}^o$, associates pairwise a class of r -cycles to each class of r -forms and establishes that $H_{DR}^r(\mathcal{X})$ is dual to $H_r(\mathcal{X})$. Since definitions of cohomology groups other than De Rham's result in groups $H^r(\mathcal{X})$ which are similarly dual to $H_r(\mathcal{X})$, it follows that the De Rham cohomology groups are isomorphic to these other cohomology groups.

▷ **Diagram** : We find it sometimes useful to represent a complete intersection in a product of complex projective spaces by means of an organic chemistry-like diagram. For details, see § 2.1.2 and [95, 146, 148]. It is of course straightforward to generalize these to any other type of embedding spaces.

▷ **Discriminant Locus** : The subset \mathfrak{M}^\sharp of the space of *complex structures*^(L) \mathfrak{M} which parametrizes singular complex spaces. Generically, $\dim \mathfrak{M}^\sharp = \dim \mathfrak{M} - 1$. That is, \mathfrak{M}^\sharp is a *divisor*^(L) in \mathfrak{M} ; see § D.3.3.

▷ **Divisor** : On a (possibly non-compact) complex manifold \mathcal{M} of dimension n , a locally finite formal linear combination $D \stackrel{\text{def}}{=} \sum_i a_i S_i$ of irreducible analytic hypersurfaces of \mathcal{M} .

Here “locally finite” means that every point of \mathcal{M} has some local neighborhood which intersects only a finite number of S_i 's appearing in D ; if \mathcal{M} is compact, this means that the sum is finite. An “analytic hypersurface” can be given as the zero-set of some single holomorphic function in the local neighborhood of every point of \mathcal{M} .

▷ **Dolbeault Theorem** : $H^q(\mathcal{M}, \Lambda^p T_{\mathcal{M}}^*) \approx H_{\bar{\partial}}^{p,q}(\mathcal{M})$ for complex manifolds \mathcal{M} .

▷ **Double Solid** : A branched double cover^(L) of an Almost Fano 3-fold^(L) 3-fold. We discuss these in § 5.5; see also [14] and the second article in Ref. [149] for generalizations.

▷ **Duality** : A non-degenerate pairing of elements from A and B , such that $(a, b) \mapsto e$, where e is some special (naturally given) quantity. For example, in a real (compact, orientable) n -fold, to every closed class of r -form we can associate a class of $(n-r)$ -forms such that their product is cohomologous to the volume-form. We write $A \sim B$. It is a standard fact that if furthermore some $C \sim B$, then this double duality induces an isomorphism $A \approx C$. See *De Rham*^(L)~, *Hodge Star*^(L)~, *Poincaré*^(L)~, and *Serre*^(L)~.

▷ **Dualizing Sheaf** : On a smooth complex n -fold \mathcal{M} , the dualizing sheaf $\mathfrak{K}_{\mathcal{M}}$ is the sheaf^(L) corresponding to the canonical (line) bundle $\mathcal{K}_{\mathcal{M}} \stackrel{\text{def}}{=} \Lambda^n T_{\mathcal{M}}^*$. For a large class of singular spaces, $\mathcal{K}_{\mathcal{M}}$ is ill-defined at the singular points, but $\mathfrak{K}_{\mathcal{M}}$ is still well-defined and may be thought of as the generalization of $\mathcal{K}_{\mathcal{M}}$.

▷ **Endomorphism** : A mapping of a group G to itself which preserves its group structure. That is, f is an endomorphism if $f : G \rightarrow G$ and $f(ab) = f(a)f(b)$, for any $a, b \in G$. For a vector *bundle*^(L), this means that the group structure of each fibre is preserved, point by point over the base space.

▷ **Euler Number (Characteristic)** : It may be defined as the index of the exterior derivative operator, d , so that

$$\chi_E(\mathcal{M}) \stackrel{\text{def}}{=} \sum_{r=0}^{\dim \mathcal{M}} (-)^r b_r(\mathcal{M}) = \sum_{p,q=0}^{\dim \mathcal{M}} (-)^{p+q} b_{p,q}(\mathcal{M}), \quad (\text{L.14})$$

where $b_r(\mathcal{M}) \stackrel{\text{def}}{=} \dim H_{DR}^r(\mathcal{M})$ are the Betti numbers and $b_{p,q}(\mathcal{M}) \stackrel{\text{def}}{=} \dim H^{p,q}(\mathcal{M})$ are the *Hodge numbers*^(L). By the well known Gauss-Bonnet formula, we also have

$$\chi_E(\mathcal{M}) = \int_{\mathcal{M}} c_n(\mathcal{M}), \quad n = \dim \mathcal{M}. \quad (\text{L.15})$$

The closed (n, n) -form c_n is known as the n^{th} Chern class of \mathcal{M} . Suitable generalizations of these formulae exist for singular \mathcal{M} .

▷ **Euler Sequence** : On a complex projective space \mathbb{P}^n , the holomorphic tangent bundle is spanned by tangent vectors $\ell^i(z)\partial_i$, modulo the relation $z^i\partial_i \approx 0$ (overall scaling is trivial in \mathbb{P}^n), and where $\ell^i(z)$ are linear polynomials in the homogeneous coordinates. Being linear in z , each $\ell^i(z)$ corresponds to $\mathcal{O}_{\mathbb{P}^n}(1)$ and the above description of the tangent bundle as a quotient is equivalent to the short exact sequence^(L)

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \rightarrow T_{\mathbb{P}^n} \rightarrow 0, \quad (\text{L.16})$$

which is called the Euler sequence. See also the *tautological sequence*^(L).

▷ **Exact Sequence** : A sequence of spaces (bundles over some base space) and appropriate maps (preserving the structure of the spaces, bundles ...) such that $\text{Im}(\alpha_k) = \ker(\alpha_{k+1})$, (see *image*^(L) and *kernel*^(L)). Of interest are exact sequences that have a zero at either end, or even more importantly, at both ends. $0 \rightarrow A \xrightarrow{\alpha} B$ being exact means that α is 1-1; $B \xrightarrow{\beta} C \rightarrow 0$ being exact means that β is onto. The short exact sequence

$$X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3 \xrightarrow{\alpha_3} \dots \quad (\text{L.17})$$

such that $\text{Im}(\alpha_k) = \ker(\alpha_{k+1})$, (see *image*^(L) and *kernel*^(L)). Of interest are exact sequences that have a zero at either end, or even more importantly, at both ends. $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ being exact means that α is 1-1; $B \xrightarrow{\beta} C \rightarrow 0$ being exact means that β is onto. The short exact sequence

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0 \quad (\text{L.18})$$

means that $A \subseteq B$ and that $C = \{B/A\}$. B is then called the *extension* of A by C ; we will write $B = C + A$ (note the order; this use of ‘+’ is *not commutative!*) and

warn that when A, B, C are holomorphic bundles over some space, it is generally impossible to find a metric in which A and C will be orthogonal. So $B \neq C \oplus A$ holomorphically, although differentiably B is the direct sum of C and A .

The above short exact sequence is equivalent to $C = \{B/A\}$, and implies that $A = \ker(\beta)$ and that $C = \text{cok}(\alpha)$. Also, $0 \rightarrow C^* \xrightarrow{\beta} B^* \xrightarrow{\alpha} A^* \rightarrow 0$ is exact, $A^* = \{B^*/C^*\}$, $C^* = \ker(\alpha)$ and $A^* = \text{cok}(\beta)$.

▷ **Fermions** : Particles and fields of half-integral spin, that is, transforming as half-odd-integral representations of the *Spin*-group. As corresponding representations of the *Lorentz group*^(L), spinors are double-valued: they change sign upon a 2π -rotation. These are then anticommuting (a-number-valued) functions or sections of appropriate bundles.

▷ **Fibre Bundle** : See *bundle*^(L).

▷ **Finite Quotient Singularity** : The finiteness of the quotient in some X/G refers to the finiteness of the group (action) G . Locally, the singularity is \mathbb{C}^n/G . Best known are the easiest examples, where $G = \mathbb{Z}_m$ is a cyclic group; the singularity is then called a *cyclic quotient singularity*. Note that in the process of *passing to the quotient* $X \rightarrow X/G$, a finite number of points (an orbit of G) is being identified. By contrast, if G were continuous, so would be the orbits of G .

▷ **Flag Space** : A sequence of inclusions $X_1 \subset X_2 \subset \dots \subset X_n = X$, where $\dim X_i < \dim X_{i+1}$, is called a *flag*. A *flag space* then is the space of all flags in X with the $\dim X_i$ fixed by the given pattern. An inclusion $X_i \subset X_{i+1}$ is called a *stripe* of a flag. Projective spaces are the simplest flag spaces, having a single stripe and of dimension one : $\mathbb{P}^n = \{L \subset \mathbb{C}^n\}$, $L \approx \mathbb{C}^1$.

▷ **Fixed Point** : In a topological space X equipped with an (symmetry) map $g : X \rightarrow X$, a point $p \in X$ left invariant is a *fixed point*. If a collection of maps g_i forms a group G , p is a fixed point of G if it is fixed by at least one element of G other than the identity.

▷ **Free Action** (of a Group) : Let G be a group acting on a topological space $G : X \rightarrow X$ in some specified way. This action is called *free* if it has no fixed points in X . Note that this is a property of the action of the group on X , not the group itself. For example, \mathbb{Z}_2 acts freely on a circle by 180° -rotations, but the action by reflections about an axis of symmetry has two fixed points.

▷ **Free Sheaf** : A *sheaf*^(L) \mathfrak{S} over \mathcal{M} , which admits a *free resolution* in terms of a sequence of vector bundles $\{\mathcal{V}_q\}_{q \geq 0}$, over \mathcal{M} . That is, the sequence

$$0 \rightarrow \dots \rightarrow \mathcal{V}_{q+1} \rightarrow \mathcal{V}_q \rightarrow \dots \mathcal{V}_0 \rightarrow \mathfrak{S} \rightarrow 0 \quad (\text{L.19})$$

is sheaf-exact, that is, exact over a sufficiently fine open covering of \mathcal{M} (a collection of sufficiently small open sets which cover \mathcal{M} completely).

▷ **Fubini-Study Metric** : On a complex projective space \mathbb{P}^n , we have the homogeneous coordinates x^0, \dots, x^n and define the Fubini-Study (sometime also called ‘round’) metric as the Kähler metric

$$g_{\mu\nu}^{(FS)} \stackrel{\text{def}}{=} \partial_\mu \partial_\nu \log \left(\sum_{\mu=0}^n |x^\mu|^2 \right), \quad (\text{L.20})$$

where $\log(\sum_{\mu=0}^n |x^\mu|^2)$ is the Fubini-Study Kähler potential. $J = dx^\mu \wedge dx^\nu (ig_{\mu\nu}^{(FS)})$ is a closed (1,1)-form; it however is not exact owing to the branch cut in $\log \|x\|^2$ and its cohomology class in fact generates $H^{1,1}(\mathbb{P}^n)$.

▷ **Generic** : Loosely speaking, means ‘almost all’. More precisely, denotes the subspace of available choices such that its *complement* is of strictly smaller dimension. For example, a family of polynomials is parametrized by the coefficients and a ‘generic polynomial’ refers a choice corresponding to any point which is not in a subset of parameters of a dimension strictly smaller than the whole parameter space. Note that, in the complex (holomorphic) setting, this means that the (real) codimension of the ‘bad’ subset is at least 2.

▷ **Genus (Arithmetic \sim , Geometric \sim)** : For a complex n -fold X , $b_{p,q} \stackrel{\text{def}}{=} \dim H^{p,q}(X)$ are the *Hodge numbers*^(L). $b_{0,n}$ is called the *geometric genus*, $b_{0,1}$ is the *irregularity* and

$$\chi^h(X) \stackrel{\text{def}}{=} \sum_{q=0}^n (-1)^q \dim H^{0,q}(X). \quad (\text{L.21})$$

is the *arithmetic genus*, also called the *holomorphic Euler characteristic (number)*. See also *plurigenera*^(L).

▷ **Germ** (of sections of a sheaf) : See *Stalk (of a sheaf)*^(L).

▷ **Hausdorff** : A topological space X is *Hausdorff* if for any two points of X there are disjoint open sets each containing exactly one of the points.

▷ **Helicity (Sub-)Group** : The maximal subgroup of the proper orthochronous *Lorentz group*^(L), $SO(t, s)$ which leaves invariant a light-like vector (for massless particles) or a time-like vector (for massive particles); s and t ($s \geq t$) denote the number of space-like and time-like dimensions, respectively. So, the helicity (sub)group of massless particles is $SO(s-t)$, while for massive particles it is $SO(s)$. Clearly, if $s = t$, the helicity (sub)group for massless particles is finite. The concept and use of this subgroup of the Lorentz subgroup is due to E. Wigner, and is also called the ‘Wigner little group’.

▷ **Hessian** : The matrix of second derivatives of a function $f(x)$. It is independent of any connection at singular points, since there $f(x) = 0 = \partial f(x)$. The Hessian is therefore used to characterize singular points.

▷ **Hirzebruch Signature** : F. Hirzebruch defines [29] the signature or a manifold \mathcal{M} as

$$\tau_H(\mathcal{M}) \stackrel{\text{def}}{=} \sum_{q=0}^{\dim \mathcal{M}} (-)^q b_{p,q}(\mathcal{M}), \quad (\text{L.22})$$

where $b_{p,q}(\mathcal{M}) \stackrel{\text{def}}{=} \dim H_{\bar{\partial}}^{p,q}(\mathcal{M})$ are the Hodge numbers. He then proves

$$\tau_H(\mathcal{M}) = \int_{\mathcal{M}} L(\mathcal{M}), \quad (\text{L.23})$$

where $L(\mathcal{M})$ has is known as the Hirzebruch L polynomial as defined in § 1.5.

▷ **Hodge Decomposition¹** (of forms) : Every (p, q) -form ω may be decomposed as

$$\omega = \mathcal{H}(\omega) + \bar{\partial}(\bar{\partial}^\dagger G(\omega)) + \bar{\partial}^\dagger(\bar{\partial}G(\omega)) \quad (\text{L.24})$$

where $\mathcal{H}(\)$ is the projection on harmonic forms, $G(\)$ is the Green’s operator and $\bar{\partial}^\dagger$ is adjoint to $\bar{\partial}$, that is $\langle \alpha, \bar{\partial}\beta \rangle = \langle \bar{\partial}^\dagger\alpha, \beta \rangle$. This is equivalent to saying that, given a (p, q) -form η , the equation $\Delta\omega = \eta$ has a solution precisely if $\mathcal{H}(\eta) = 0$ and then $\omega = G(\eta)$.

▷ **Hodge Decomposition²** (of cohomology) : For a compact Kähler manifold \mathcal{M} , the complex cohomology satisfies

$$H_{\text{DR}}^r(\mathcal{M}, \mathbb{C}) \approx \bigoplus_{p+q=r} H_{\bar{\partial}}^{p,q}(\mathcal{M}), \quad H_{\bar{\partial}}^{p,q}(\mathcal{M}) = \overline{H_{\bar{\partial}}^{q,p}(\mathcal{M})}. \quad (\text{L.25})$$

▷ **Hodge Number** : $b_{p,q} \stackrel{\text{def}}{=} \dim H_{\bar{\partial}}^{p,q}(\mathcal{M})$, also denoted $h^{p,q}$, are the *Hodge numbers* of a complex manifold \mathcal{M} .

▷ **Hodge Star Duality** : On a real n -dimensional (compact, orientable) differentiable manifold X , we define the *Hodge star* operator by its action on a differential r -form $\alpha = a_{i_1 \dots i_r} dx^{i_1} \dots dx^{i_r}$:

$$\begin{aligned} * \alpha &= a_{j_1 \dots j_{n-r}}^* dx^{j_1} \wedge \dots \wedge dx^{j_{n-r}}, \\ a_{j_1 \dots j_{n-r}}^* &= \epsilon^{i_1 \dots i_r} {}_{j_1 \dots j_{n-r}} a_{i_1 \dots i_r}. \end{aligned} \quad (\text{L.26})$$

This clearly takes (closed or exact) r -forms into (closed or exact) $(n-r)$ -forms.

Since the scalar product $\int \alpha \wedge * \alpha \geq 0$ and is zero only if α is exact, it establishes that $H^r(X)$ and $H^{n-r}(X)$ are dual. Combining with the De Rham duality, we have the isomorphism $H^{n-r}(X) \approx H_r(X)$. Moreover, if X is complex n -dimensional, the Hodge star operator takes (p, q) -forms into $(n-q, n-p)$ -forms and we have that $H^{p,q}(X)$ and $H^{n-q, n-p}(X)$ are dual.

▷ **Holomorphic** : A function $f(z)$ is holomorphic in a domain $z \in \mathcal{D}$ if $\frac{\partial f}{\partial z_i} = 0$ everywhere in \mathcal{D} . Also, this is true precisely if $f(z)$ is analytic in \mathcal{D} .

▷ **Holomorphic Euler Number (Characteristic)** : See *genus*^(L).

▷ **Holonomy** : On a manifold X , consider parallel-transporting a vector \vec{v}_p tangent to X at $p \in X$, along a closed path back to p . The transformation required to bring the parallel-transported vector \vec{v}'_p back to the original \vec{v}_p is called the *holonomy*. The collection of such transformations, parametrized by all closed paths in X containing p , manifestly forms a group and is called the *holonomy group*; it is independent of the choice of $p \in X$. If the paths are restricted to be contractible, the resulting group is called the *restricted holonomy group*.

▷ **Hom**(\mathcal{A}, \mathcal{B}) : For two bundles \mathcal{A} and \mathcal{B} , the bundle of *homomorphism* $\mathcal{A} \rightarrow \mathcal{B}$, that is, maps that preserve the bundle structure. Clearly, $\text{Hom}(\mathcal{A}, \mathcal{B}) = \mathcal{A}^* \otimes \mathcal{B}$: when contracting $a \in \mathcal{A}$ with $a^* \otimes b$, a is replaced with b .

▷ **Homology** : Two subspaces \mathcal{Y} and \mathcal{Y}' of a manifold X are said to be homologous if $\mathcal{Y} \cup \mathcal{Y}'$ is the boundary of a subspace of X . For example, two ‘little’ circles on a torus are homologous because they are the boundaries of a cylindrical portion of the torus.

▷ **Homotopy** : A one (real) parameter family of continuous mappings $f_t : \mathcal{Y} \xrightarrow{\text{into}} X$, $0 \leq t \leq 1$, which is also continuous with respect to t . So, for two subspaces \mathcal{Y} and \mathcal{Y}' in a manifold X to be homologous, we need that (1) $\mathcal{Y} = f_0(Y)$ and $\mathcal{Y}' = f_1(Y)$ are images in X of the same space Y and that there is a family of images $\mathcal{Y}_t = f_t(Y)$ ‘interpolating’ continuously between \mathcal{Y} and \mathcal{Y}' . Loosely speaking, it must be possible to deform \mathcal{Y} into \mathcal{Y}' continuously and without breaking or tearing.

- **Hopf Index Theorem** : Let \mathcal{M} be a compact oriented n -fold and v a global C^∞ vector field on \mathcal{M} . A zero, p , of v is *non-degenerate* if it is isolated and

$$v(x) = x^i \lambda_i{}^j \partial_j + \dots \quad (\text{L.27})$$

in local coordinates around the zero of v , where λ_i^j is non-singular and the ellipses denote terms of at least quadratic in x . Integrating the vector field v to time t produces a flow $f_t : \mathcal{M} \rightarrow \mathcal{M}$, such that for $t > 0$ and sufficiently small, $\iota_{f_t}(p) = \iota_v(p)$ at every zero of v . However, since f_t is manifestly homotopic to the identity, f_t^* acts as the identity on the cohomology and we have

Theorem

$$\sum_{\substack{p \in \mathcal{M} \\ v(p)=0}} \iota_v = L(f_t) = \chi_E(\mathcal{M}) \stackrel{\text{def}}{=} \int_{\mathcal{M}} c_n(\mathcal{M}). \quad (\text{L.28})$$

See Eq. (L.35) and the *Lefschetz Fixed-Point Formulae*^(L).

- ▷ **Hyperplane** : A linear subspace of codimension 1. That is, given an n -fold X , a hyperplane $H \subset X$ is defined locally by $\ell(z) = 0$, where $\ell(z)$ is a single linear constraint.
 - ▷ **Hyperplane Bundle** : Over an n -fold X , the line $bundle^{(L)}$ the fibre of which, at $p \in X$, is the space of hyperplanes through that point. As hyperplanes are defined by linear constraints, the hyperplane bundle written as $\mathcal{O}(1)$ is also the (line) bundle the (global, holomorphic) sections of which are linear polynomials over X .
 - ▷ **Hypersurface** : Given a complex manifold X and a function (section of some line bundle) ϕ over X , we define the hypersurface $F \subset X$ as the subspace of X by $F \stackrel{\text{def}}{=} \{x \in X : \phi(x) = 0\}$. We also write, somewhat formally, $F = \phi^{-1}(0) \hookrightarrow X$. The (complex) $codimension^{(L)}$ of F in X is 1, that is, $\dim X = \dim F + 1$. It is a basic fact that $F \subset X$ carries an integral homology class, $[F] \in H_{\dim F}(X, \mathbb{Z})$, which is isomorphic to some $[f] \in H^2(X, \mathbb{Z})$ as both are $dual^{(L)}$ to some $[*F] \in H^{\dim F}(X, \mathbb{Z})$. This establishes a correspondence between the *Picard group*^(L) $\text{Pic}(X)$, $H^2(X, \mathbb{Z})$ and $H_{\dim F}(X, \mathbb{Z})$.
 - ▷ **Ideal** : A right (left) ideal \mathfrak{I} of a *ring*^(L) R is an additive subgroup of R such

- ▷ **Ideal of a Variety** : For a (projective) subvariety \mathcal{Y} of dimension m in a (projective) variety X of dimension n , the collection of all (homogeneous) functions which vanish precisely on \mathcal{Y} forms an *ideal*^(L) of the *ring*^(L) of all (homogeneous) functions on X and is denoted $I(\mathcal{Y} \subset X)$. If $X = \mathbb{P}^n$, it is omitted in conventional notation.

- ▷ **Image** : The image of a map $\pi: A \rightarrow B$ is defined as $\text{Im}(\pi) \stackrel{\text{def}}{=} \bigcup_{a \in A} \{\pi(a) = b \in B\}$. Naturally, $\text{Im}(\pi) \subseteq B$.

- **Implicit Function Theorem** : Let f_a be K local functions of $n > K$ local variables $x_i \in X$ which can be chose so that $\det\left(\frac{\partial f_a}{\partial x_i}\right) \neq 0$ for $a, i = 1, \dots, K$. Then there exist functions q_a , $a = 1, \dots, K$, such that

$$f_a(x) = 0, \quad \iff \quad x_a = q_a(x_{K+1}, \dots, x_n), \quad a = 1, \dots, K. \quad (\text{L.29})$$

Clearly, the $(n-K)$ local coordinates x_{K+1}, \dots, x_n remain for the subspace of X where $f_a(x) = 0, \forall a$.

- ▷ **Index** : The index of a map $\pi: A \rightarrow B$ is defined as $\iota(\pi) \stackrel{\text{def}}{=} (\ker(\pi) - \text{cok}(\pi))$. If π^\dagger is adjoint to π (with respect to some norm, $\langle b, \pi(a) \rangle = \langle \pi^\dagger(b), a \rangle$, for all $a \in A$, $b \in B$) then $\iota(\pi) = (\ker(\pi) - \ker(\pi^\dagger))$.

- ▷ **Injective Map** : Synonym for ‘a map is 1–1’. That is, a map $A \xrightarrow{f} B$ is injective (1–1) if there exists precisely one $b \in B$ for every $a \in A$ such that $b = f(a)$.

- ▷ **Invariant Polynomial** : For \mathbb{M}_n denote the space of $n \times n$ complex matrices, the (homogeneous) k -linear form

$$P^k : \underbrace{\mathbb{M}_m \times \cdots \times \mathbb{M}_n}_{k \text{ times}} \longrightarrow \mathbb{C} \quad (\text{L.30})$$

is called *invariant* if for any $A_1, \dots, A_k \in M_n$ and $g \in GL(n, \mathbb{C})$,

$$P^k(\mathbf{A}_1, \dots, \mathbf{A}_k) = P^k((g\mathbf{A}_1g^{-1}), \dots, (g\mathbf{A}_kg^{-1})). \quad (\text{L.31})$$

Then, $S_k(\mathbf{A}) = P^k(\mathbf{A}, \dots, \mathbf{A})$ are the *elementary symmetric polynomials*. It can be shown that $S_k(\mathbf{A})$ are unique; most naturally, they arise in the characteristic polynomial :

$$\det[\mathbf{A} - \lambda \mathbf{1}] = \sum_{k=0}^n \lambda^k S_{n-k}(\mathbf{A}) . \quad (\text{L.32})$$

▷ **Inverse Function Theorem** : Let U, V be two open sets in \mathbb{P}^n with $0 \in U$ and let $f: U \rightarrow V$ be a holomorphic map with the Jacobian $J(\frac{\partial f_i}{\partial z_j})$ non-singular at 0. Then f is one-to-one in a neighborhood of 0, and f^{-1} is holomorphic at $f(0)$.

▷ **Jet** (in particular, k -jet) : In a local neighborhood around a point p in a manifold \mathcal{M} and for a local function (or section of some bundle) in that neighborhood, the k -jet is its Taylor series around p up to and including the k^{th} order terms. Statements like ‘non-zero k -jet’ make sense globally provided that the function (section of a bundle) is globally defined and means that at every point p of the manifold \mathcal{M} , the local k -jet is non-zero.

▷ **Kähler Class** : The cohomology class defined by the Kähler metric, on a Kähler manifold \mathcal{M} , as follows. Given a Kähler metric $g_{\mu\nu}$, $J \stackrel{\text{def}}{=} i g_{\mu\nu} dz^\mu \wedge d\bar{z}^\nu$ is the associated Kähler (1,1)-form representing the Kähler (cohomology) class. Clearly, J is self-conjugate (real) and *positive* in the sense that $g_{\mu\nu}$ is (in any local coordinate chart) a Hermitian and positive matrix. A coordinate independent equivalent statement is that $\int_C J > 0$, $\int_S J \wedge J > 0 \dots \int J^{\dim \mathcal{M}} > 0$, where C, S, \dots are submanifolds of \mathcal{M} of complex dimension 1,2,...

▷ **Kähler Cone** : The space of choices of the *Kähler class*^(L). It has the structure of a cone because if J and J' are two Kähler classes, so is $J + J'$ and so is αJ for $\alpha \in \mathbb{R}_+$. Guided by string theory, we often extend the Kähler class by considering $(B + iJ) = w^A e_A$ instead of the Kähler form J , where B is a harmonic (1,1)-class, $\{e_A\}$ form a basis of $H^{1,1}(\mathcal{M})$ and w^A are $b_{1,1}$ complex parameters. The space spanned by w^A is referred to as the complexified Kähler cone.

▷ **Kähler Manifold** : A complex manifold which *admits* a Kähler metric^(L).

▷ **Kähler Metric** : A metric $g_{\mu\nu}$ which locally Euclidean up to second order terms. Associated to a Hermitian metric is the (1,1)-form $J \stackrel{\text{def}}{=} (i g_{\mu\nu} dz^\mu \wedge d\bar{z}^\nu)$; a Hermitian metric is Kähler precisely if the associated (1,1)-form is closed. The cohomology class of J is called the Kähler class. Locally, $J = \partial\bar{\partial}\Phi$, where Φ is the Kähler potential.

▷ **Kernel** : The kernel of a map $\pi: A \rightarrow B$ is defined as $\ker(\pi) \stackrel{\text{def}}{=} \{a \in A : \pi(a) = 0 \in B\}$, the subset of A which is annihilated by π .

▷ **Kodaira Embedding Theorem** : A compact complex manifold \mathcal{M} is an algebraic variety—is embeddable in a projective space—precisely if it has a closed, positive (1,1)-form ω the cohomology class of which is rational.

▷ **Kodaira-Nakano Vanishing Theorem** : If \mathcal{L} is a positive holomorphic line bundle over the compact complex algebraic manifold X of dimension n , then

$$H^q(X, \mathcal{L} \otimes \wedge^p \mathcal{T}_X) = 0 \quad \text{for } p+q > n.$$

A useful fact to recall when using this theorem is that $\wedge^n \mathcal{T}_X \stackrel{\text{def}}{=} \mathcal{K}_X$ is the canonical line bundle, so $\wedge^p \mathcal{T}_X = \mathcal{K}_X \otimes \wedge^{(p-n)} \mathcal{T}_X$.

▷ **Kodaira-Serre Duality** : See *Serre Duality*^(L).

▷ **Koszul Complex** : In complete generality, this is defined in Ref. [38], p.127–131. For the purposes of this volume, the following description will suffice. Let $\mathcal{E} = \bigoplus_a \mathcal{L}_a^K$, where \mathcal{L}_a are line bundles over a complex manifold X and have global holomorphic sections, which we denote by ξ_a . Let $\xi = \bigoplus_a \xi_a$ and let \mathcal{M} be the zero-set of ξ . Then, over $(X - \mathcal{M})$, the sequence of bundles and mappings

$$0 \rightarrow \wedge^K \mathcal{E}^* \xrightarrow{\wedge \xi} \dots \xrightarrow{\wedge \xi} \wedge^2 \mathcal{E}^* \xrightarrow{\wedge \xi} \mathcal{E}^* \xrightarrow{\wedge \xi} \mathcal{O}_X \rightarrow 0. \quad (\text{L.33})$$

is exact; \mathcal{O}_X denotes the trivial (constant) bundle over X . To show exactness, note that ξ is non-zero over $(X - \mathcal{M})$ and that $\wedge \xi$ results in contraction with one factor in $\wedge^k \mathcal{E}$. Owing to the skew symmetry of this product, $(\wedge \xi)^2 = 0$ and the above sequence is the *Koszul complex*. See Chapter 9 for more information.

▷ **Künneth Formula** : Given two complex compact spaces, X and \mathcal{Y} , and a vector bundle^(L) (*sheaf*^(L)) \mathcal{V} over the product space $X \times \mathcal{Y}$, we have that

$$H^{p,q}(X \times \mathcal{Y}) \approx \bigoplus_{\substack{r+u=p \\ s+v=q}} (H^{r,s}(X, \mathcal{V}|_X) \otimes H^{r,s}(\mathcal{Y}, \mathcal{V}|_{\mathcal{Y}})) \quad (\text{L.34})$$

Note that the *holonomy group*^(L) of the product $X \times \mathcal{Y}$ is just the product of the holonomy groups of the factor spaces. Given an action of the factor holonomy groups on \mathcal{V} and the naturally induced action on the \mathcal{V} -valued cohomology, the Künneth formula can be refined further, into a direct sum of (products of) representations of the factor holonomy groups. Clearly, a further decomposition may be obtained if there is a symmetry acting on $X \times \mathcal{Y}$ and \mathcal{V} .

▷ **Lefschetz Fixed-Point Formulae** : Let \mathcal{M} and $G: \mathcal{M} \rightarrow \mathcal{M}$ a symmetry with fixed points p_i ; write G^* stand for the action on the cohomology on \mathcal{M} . Locally at the i^{th} fixed point, p_i , let the action be given as $G: z_{(i)}^\mu \rightarrow g_{(i)}{}^\mu_\nu z_{(i)}^\nu$ and write $\iota_{\mathbf{g}}(p_i) \stackrel{\text{def}}{=} \text{sign det}(\mathbf{g}_{(i)} - \mathbf{1})$, where $\mathbf{g}_{(i)} \stackrel{\text{def}}{=} [g_{(i)}{}^\mu_\nu]$. Then

$$L(G) \stackrel{\text{def}}{=} \sum_r (-1)^r \text{Tr}[G^*|_{H_{\text{DR}}^r(\mathcal{M})}] = \sum_{\mathbf{g}(p)=p} \iota_{\mathbf{g}}(p) \quad (\text{L.35})$$

is the *Lefschetz fixed point formula*. Using the *Hopf Index Theorem*^(L), it may be rewritten as

$$L(f) = \sum_{\mathbf{g}(\mathcal{N})=\mathcal{N}} \chi_E(\mathcal{N}). \quad (\text{L.36})$$

On a complex manifold \mathcal{M} ,

$$L^h(G) \stackrel{\text{def}}{=} \sum_q (-)^q \text{Tr}[G^*|_{H^{0,q}(\mathcal{M})}] = \sum_{\mathbf{g}(p)=p} [\mathbf{1} - \mathbf{g}(p)]^{-1} \quad (\text{L.37})$$

is called the *holomorphic Lefschetz fixed point formula*.

▷ **Lefschetz Hard Theorem** : Given the Kähler class J on a compact Kähler n -fold X , the map $(\wedge J)^k : H^{p,q} \rightarrow H^{p+k, q+k}$ is an isomorphism. Writing J^\dagger for the adjoint of the Kähler form and $J^\dagger.$ for the adjoint of $\wedge J$,

$$H_\perp^{p,q} \stackrel{\text{def}}{=} \ker[(J^\dagger.)^k : H^{p-k, q-k} \rightarrow H^{p,q}] \quad (\text{L.38})$$

is the primitive cohomology, that is, the J -traceless cohomology. Then

$$H^{p,q} = \bigoplus_{k=0}^{\min(p,q)} [J^k \wedge H_\perp^{p-k, q-k}] \quad (\text{L.39})$$

is called the *Lefschetz decomposition*.

▷ **Lefschetz Hyperplane Theorem** (as proven by Bott) : Let \mathcal{L} be a positive holomorphic line bundle over an $n+1$ -dimensional complex compact manifold X and let λ be a holomorphic section of \mathcal{L} . Then :

$$\begin{aligned} \pi_q(X) &\approx \pi_q(\lambda^{-1}(0)) & 0 \leq q < n = \dim \lambda^{-1}(0), \\ \pi_n(X) &\xrightarrow{j} \pi_n(\lambda^{-1}(0)) & \text{is onto.} \end{aligned} \quad (\text{L.40})$$

▷ **Lorentz Group** : The (connected component of the) group of linear transformations of the tangent space of spacetime. In usual 4-dimensional spacetime, this is $SO(1, 3)$; for the 10-dimensional spacetime, this is $SO(1, 9)$. In both cases, a pseudo-Riemannian metric is assumed to exist in spacetime, with signature $(1, 3)$ and $(1, 9)$ respectively. Of interest are also the covering, so-called *Spin*-groups, denoted $Spin(1, 3) \approx SL(2, \mathbb{C})$ and $Spin(1, 9)$.

▷ **Manifold** : An everywhere smooth space. More precisely, a real (complex) n -dimensional manifold \mathcal{M} is defined to have a finite set of subsets $\{\mathcal{U}_i \subset M\}$ so that the following is true :

1. Each $p \in M$ belongs to at least one U_i .
2. For each i , there is a 1-1, onto, map $\psi_i : U_i \rightarrow O_i$, where O_i is an open neighborhood \mathbb{R}^n (\mathbb{C}^n).
3. For any $U_i \cap U_j \neq \emptyset$, the composition of maps $\psi_i \circ \psi_j^{-1}$ is C^∞ .

See *compact*^(L), *Hausdorff*^(L) and *separable*^(L).

▷ **Minimal Configuration** : A *configuration*^(L) the members of which do not belong to a smaller configuration (see Chapter 2).

▷ **Mirror Map** : Given a Calabi-Yau 3-fold \mathcal{M} , the *mirror map* associates to it another Calabi-Yau 3-fold, \mathcal{W} , such that

$$\bigoplus_{p+q=3} H^{p,q}(\mathcal{M}) \cong_Q \bigoplus_{p=q} H^{p,q}(\mathcal{W}), \quad (\text{L.41})$$

and ‘ \cong_Q ’ denotes an equivalence of the ring structures as determined by the *exact quantum Yukawa couplings*. In other words, the exact value (with all quantum corrections) of all possible n -tuple product integrals computed on the right hand side equal those computed likewise on the left hand side. Straightforwardly, $H^{p,q}(\mathcal{M}) = H^{3-p,q}(\mathcal{W})$ and so $\chi_E(\mathcal{M}) = -\chi_E(\mathcal{W})$. More generally for n -folds, $H^{p,q}(\mathcal{M}) = H^{n-p,q}(\mathcal{W})$ and so $\chi_E(\mathcal{M}) = (-)^n \chi_E(\mathcal{W})$.

▷ **Moduli Space** : The effective parameter space. Concretely, for the complex n -folds possibly endowed with a Kähler class, the moduli space consists of the (effective) space of *complex structures*^(L), at every point of which there is a *Kähler cone*^(L), a space of choices of the *Kähler class*^(L). Clearly, as the complex structures are being varied, the associated Kähler cones vary into each other in a uniform way, so that the total space has the structure of a fibration. See also *Kähler cone*^(L).

▷ **n -Fold** : Unless otherwise noted, throughout this volume, a compact complex n -dimensional space, possibly singular at some relatively nicely defined subset (isolated points, curves which themselves are not too badly singular and so on).

▷ **Node** : Also called a ‘double point’, is the mildest hypersurface singularity. That is, the hypersurface $\phi(x) = 0$ ($x \in \mathbb{C}^n$) has a node at $x = 0$ if all gradients vanish but the Hessian is non-singular. Up to a change of coordinates then, $\phi(x) = \sum_i x_i^2$; a node is therefore an A_1 singularity in the classification of Ref. [1].

▷ **Noether Formula** : For surfaces, $C_1^2 + C_2 = 12\chi^h$, where $C_1^2 = \int c_1^2$ and $C_2 = \int c_2$ are the Chern numbers^(L) and χ^h is the holomorphic Euler characteristic^(L).

▷ **Normal Bundle** : Given a submanifold \mathcal{M} of a manifold X , at every point $p \in \mathcal{M}$, we define the normal space N_p to be spanned by the tangent vectors of X , considering however two vectors equivalent if they differ by a vector tangent to \mathcal{M} . In other words,

$$N_p \stackrel{\text{def}}{=} \{T_p(X)/T_p(\mathcal{M})\}, \quad \mathcal{N}_{X/\mathcal{M}} \stackrel{\text{def}}{=} \bigcup_{p \in \mathcal{M}} N_p = \{T_X/T_{\mathcal{M}}\} \quad (\text{L.42})$$

are, respectively, the normal space at $p \in \mathcal{M}$ and the normal bundle of the embedding $\mathcal{M} \hookrightarrow X$. Taking the union with p ranging all over \mathcal{M} , we also obtain $\mathcal{T}_{\mathcal{M}} = \bigcup_{p \in \mathcal{M}} T_p(\mathcal{M})$, $T_X = \bigcup_{p \in \mathcal{M}} T_p(X)|_{\mathcal{M}}$. See also exact sequence^(L).

▷ **Orbifold** : In general, a space which is smooth except at a finite number of isolated points, at each of which the space is locally of the form \mathbb{C}^n/D , where D is a finite group.

▷ **Periods** : Given an exterior r -form α and let $\{\gamma_i\}$ be a basis of r -cycles, $\varpi_i(\alpha) \stackrel{\text{def}}{=} \int_{\gamma_i} \alpha$ are the periods of α . If all periods of an exterior r -form vanish (sweeping γ_i through all of H_r), the form is exact.

▷ **Period Matrix** : Given a basis of exterior r -forms $\{\alpha\}$ and a basis of r -cycles $\{\gamma_i\}$, $\varpi_{(r)}{}^i_j \stackrel{\text{def}}{=} \int_{\gamma_j} \alpha^i$ is the period matrix. Of special interest, in a complex n -dimensional manifold, is the period matrix $\varpi_{(n)}$.

▷ **Picard Group** : Holomorphic line bundles over a compact complex manifold X form, under the (fibre-wise) tensor product, a multiplicative group called the Picard group, denoted $\text{Pic}(X)$.

▷ **Picard Number** : The dimension of the Picard group^(L) $\text{Pic}(X)$. Also equal to $\dim(H^{1,1}(X) \cap H^2(X, \mathbb{Z}))$; see Ref. [23], p. 456-7.

▷ **Plurigenera** : Let $\mathcal{K}_X = \det \mathcal{T}_X^*$ be the canonical bundle over X and let $\mathcal{K}_X^{\otimes n}$ denote the n^{th} tensor power line bundle over X . Then $\mathfrak{P}_n \stackrel{\text{def}}{=} \dim H^0(X, \mathcal{K}_X^{\otimes n})$, the number of holomorphic sections of $\mathcal{K}_X^{\otimes n}$ is the n^{th} plurigenus of X . By analogy (and contravariantly): $\mathfrak{P}^m \stackrel{\text{def}}{=} \dim H^0(X, \mathcal{K}_X^{*\otimes n})$.

▷ **Poincaré Duality** : On a compact oriented real n -dimensional manifold X , the intersection pairing

$$H_k(X, \mathbb{Z}) \times H_{n-k}(X, \mathbb{Z}) \longrightarrow \mathbb{Z} \quad (\text{L.43})$$

is unimodular; i.e., any linear functional $H_{n-k}(X, \mathbb{Z}) \rightarrow \mathbb{Z}$ is expressible as the intersection with some class $\alpha \in H_k(X, \mathbb{Z})$ and any class $\alpha \in H_k(X, \mathbb{Z})$ having intersection number 0 with all classes in $H_{n-k}(X, \mathbb{Z})$ is a torsion class.

It is easy to see that this is (De Rham) dual to the Hodge star duality.

▷ **Positive Bundle** : A bundle^(L) over X which admits a curvature 2-form Θ which is everywhere on X locally represented by a positive (Hermitian) matrix. A positive first Chern class^(L) implies positivity of the bundle, but is not necessary; see Lemma 3.1.

▷ **Preimage** : The preimage (inverse-image) of an element $b \in B$ under a map $\pi: A \rightarrow B$ is defined as $\pi^{-1}(b) \stackrel{\text{def}}{=} \{a \in A : \pi(a) = b\}$. Naturally, $\pi^{-1}(b) \subseteq A$.

▷ **Ramified Covering** : See Branched Covering^(L).

▷ **Rank of a Mapping** : Given a mapping $\pi: A \rightarrow B$, the rank of π is defined to be the dimension of the image, $\dim[\pi(A) \in B]$.

▷ **Rank of a Vector Bundle** : The dimension of the fibre. Recall that a vector bundle \mathcal{V} over the manifold X may be constructed by attaching a copy of the vector space V_x to each point $x \in X$ in such a way that this vector space varies smoothly (differentially, holomorphically ...) as we sweep the point x over the base space X . V_x is called the fibre at $x \in X$.

▷ **Rational Function** : A rational (or meromorphic) function on an open set $U \subset \mathcal{M}$ is given locally as the quotient of two holomorphic functions. So, for some open covering $\mathfrak{U} = \{U_i\}$, $f|_{U_i} = g_i/h_i$, where g_i and h_i have no common (holomorphic) factor and $g_i h_j = g_j h_i$ over $(U_i \cap U_j)$. A rational function is not a function in the usual sense, even if ‘∞’ is included as a value; f is simply not defined where both g_i and h_i vanish.

▷ **Rational Curve** : An algebraic (embeddable in some \mathbb{P}^n) curve which is birationally isomorphic to \mathbb{P}^1 (see Birational Map^(L)).

▷ **Rational Map** : A rational (or meromorphic) map of a complex manifold \mathcal{M} to a projective space \mathbb{P}^n is a map

$$f: z \rightarrow [1, f_1(z), \dots, f_n(z)], \quad z \in \mathcal{M}, \quad (\text{L.44})$$

given by n global meromorphic functions on \mathcal{M} . A *rational map* $f : \mathcal{M} \rightarrow \mathcal{N}$ to an algebraic variety $\mathcal{N} \subset \mathbb{P}^n$ is a rational map to \mathbb{P}^n , such that its *image*^(L) lies in $\mathcal{N} \subset \mathbb{P}^n$.

Caution : a rational map $f : \mathcal{M} \rightarrow \mathcal{N}$ need not be defined everywhere but is instead given by a *holomorphic map*

$$f_h : (\mathcal{M} - \mathcal{E}) \rightarrow \mathcal{N}, \quad (L.45)$$

where \mathcal{E} is the exceptional set (where the rational map fails to be defined) and which has *codimension*^(L) 2 in \mathcal{M} .

▷ **Rational Surface** : An algebraic (embeddable in some \mathbb{P}^n) surface which is birationally isomorphic to \mathbb{P}^2 . (see *Birational Map*^(L)).

▷ **Residue Class Ring** : See *Ideal*^(L).

▷ **Resolution¹** (of a singularity) : Given a singular model X (a singular point with its local complex neighborhood or a complex n -fold with singularities), its resolution is a smooth model Y together with a map $\sigma : Y \rightarrow X$, such that $(X - \text{Sing } X) = (Y - \mathcal{E})$ as (non-compact) complex manifolds. \mathcal{E} here denotes the (collection of) exceptional set(s) into which the singularities of X have been resolved. When Y is a smooth complex 3-fold (with no boundaries), \mathcal{E} consists of complex 1- and/or 2-dimensional subspaces of Y .

▷ **Resolution²** (of a sheaf) : A *sheaf*^(L) \mathfrak{S} over a manifold X is said to have a resolution if it fits into a sequence

$$0 \rightarrow \mathcal{V}_K \rightarrow \mathcal{V}_{K-1} \rightarrow \cdots \rightarrow \mathcal{V}_1 \rightarrow \mathcal{V}_0 \xrightarrow{\ell} \mathfrak{S} \rightarrow 0 \quad (L.46)$$

where \mathcal{V}_k are fibre bundles over X and the sequence is *sheaf-exact*, i.e., is an *exact sequence*^(L) for a sufficiently fine open cover of X . Equivalently, this sequence is exact with \mathfrak{S} set to 0 over the generic part of X , and being non-zero only over a special subset of X .

▷ **Riemann-Roch-Hirzebruch Theorem** : Let X be an algebraic manifold of dimension n and let \mathcal{L} be a holomorphic line bundle over X , with $c_1[\mathcal{L}] = \lambda$. The cohomology groups $H^q(X, \mathcal{L})$ are finite dimensional and the (generalized) Euler-Poincaré characteristic is

$$\sum_{q=0}^n H^q(X, \mathcal{L}) \stackrel{\text{def}}{=} \chi_E(X, \mathcal{L}) = \int_X \left[e^\lambda \prod_{i=1}^n \frac{\ell_i}{1 - e^{-\ell_i}} \right], \quad (3.2.6)$$

where

$$c(X) = 1 + c_1 + c_2 + \dots + c_n = \prod_{i=1}^n (1 + \ell_i). \quad (1.3.19')$$

Note that integration over X picks out terms which are of degree n in λ, ℓ_i .

▷ **Riemann Tensor and Form** : Writing

$$\nabla_m v^k \stackrel{\text{def}}{=} \partial_m v^k + \Gamma_{mn}{}^k v^n \quad (L.47)$$

for a covariant derivative (connection), the Riemann curvature tensor is defined by

$$[\nabla_m, \nabla_n] v^l \stackrel{\text{def}}{=} T_{mn}{}^k \nabla_k v^l + R_{mnk}{}^l v^k, \quad (L.48)$$

where $T_{mn}{}^k$ is the *torsion* tensor. The

$$\Psi_k{}^l \stackrel{\text{def}}{=} (dx^m \Gamma_{mk}{}^l) \quad \text{and} \quad \Theta_k{}^l \stackrel{\text{def}}{=} (dx^m \wedge dx^n R_{mnk}{}^l) \quad (L.49)$$

are the (matrix-valued) *torsion 1-form* and *curvature 2-form*, respectively. For a *Riemannian* ∇_m , $T_{mn}{}^k = 0$ and

$$R_{mnp}{}^q g_{rq} = R_{pm}{}^q g_{nq}, \quad \text{and} \quad R_{m[np}{}^q g_{r]q} = 0. \quad (L.50)$$

A typical choice for a Riemannian connection involves the Cristoffel connection

$$\Gamma_{mn}{}^k \stackrel{\text{def}}{=} \frac{1}{2} g^{kl} (\partial_m g_{nl} + \partial_n g_{ml} - \partial_l g_{mn}). \quad (L.51)$$

On a Hermitian manifold, $g_{\mu\nu} = 0 = g_{\bar{\mu}\bar{\nu}}$ and so $\Gamma_{\mu\nu}{}^\sigma, R_{\mu\bar{\nu}\rho}{}^\sigma$, (and their complex conjugates) are the only non-zero components of the connection and Riemann tensor.

▷ **Ricci Tensor and Form** : On a Hermitian manifold (complex manifold with a Hermitian metric), the Ricci tensor, $R_{mnp}{}^n$, becomes

$$R_{\mu\bar{\nu}} \stackrel{\text{def}}{=} g^{\pi\bar{\theta}} R_{\mu\bar{\theta}\pi\bar{\nu}} = -g^{\pi\bar{\theta}} R_{\mu\pi\bar{\nu}\bar{\theta}} = -R_{\mu\bar{\nu}\rho}{}^\rho \quad (L.52)$$

and defines the Ricci curvature two-form

$$i R_{\mu\bar{\nu}\rho}{}^\rho dz^\mu \wedge d\bar{z}^\nu = i \partial \bar{\partial} \log \sqrt{\det g}, \quad (L.53)$$

which is both ∂ - and $\bar{\partial}$ -closed but not exact because of the logarithm. It is also the trace of the Riemann curvature two-form $\Theta_\rho{}^\sigma \stackrel{\text{def}}{=} i R_{\mu\bar{\nu}\rho}{}^\sigma dz^\mu \wedge d\bar{z}^\nu$.

▷ **Ring** : An non-empty set R endowed with two binary operations, ‘+’ and ‘·’ such that (1) for any $a, b \in R$, $(a+b), (a \cdot b) \in R$; (2) $(a+b) = (b+a)$; (3) $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$; (4) $a \cdot (b+c) = (a \cdot b)+(a \cdot c)$ and $(a+b) \cdot c = (a \cdot c)+(b \cdot c)$; (5) for every $a, b \in R$, there is a unique $c \in R$ such that $a+c = b$. Note, in particular, that ‘multiplicative unity’ ($1 \in R$) or ‘division’ need not be defined. If there exists $1 \in R$, R is a *unitary ring*. ‘1’ must be distinct from ‘0’ unless $R = \{0\}$.

▷ **Section** : A (holomorphic) *section* of a vector *bundle*^(L) ($\mathcal{B}, \mathbf{E}, \varpi$) over $U \subseteq \mathcal{B}$ is a C^∞ (holomorphic) map $\sigma : U \rightarrow \mathbf{E}$ such that $\sigma(x) \in E_x$ for all $x \in U$. That is, while the map ϖ projects the total space of the bundle \mathbf{E} to the base manifold \mathcal{B} , a section takes a point $x \in \mathcal{B}$ as the argument and produces a point in the fibre E_x over that point. Clearly, a section is *global* if it can be extended over all of \mathcal{B} . A complete set of sections $\{\sigma_i(x)\}$ which provide a basis for E_x for all $x \in U \subseteq \mathcal{B}$ is called at *frame*.

▷ **Segré Embedding** : The embedding of the product of two *complex projective spaces*^(L) $\mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{(m+1)(n+1)-1}$ given by sending the coordinates

$$(x_a; y_i) \mapsto z_{ai}, \quad a = 0, \dots, m, \quad i = 0, \dots, n \quad (\text{L.54})$$

and imposing the obvious constraints

$$z_{ai}z_{bj} - z_{aj}z_{bi} = 0, \quad (\text{that is, } x_a y_i x_b y_j = x_a y_j x_b y_i). \quad (\text{L.55})$$

Note that $\mathbb{P}^m \times \mathbb{P}^n$ has *codimension*^(L) mn inside $\mathbb{P}^{(m+1)(n+1)-1}$, while there are $\binom{m+1}{2} \binom{n+1}{2}$ constraints. Clearly, unless $m = n = 1$, there are too many constraints for them to be independent and the Segré embeddings are not *complete intersections*^(L).

▷ **Separable** : A topological space is *separable* if it admits a countable basis for topology.

▷ **Serre Duality** :

For a compact complex n -fold \mathcal{M} and a holomorphic vector bundle \mathcal{V} over it,

1. $H^n(\mathcal{M}, \mathcal{K}_{\mathcal{M}}) = H^{n,n}(\mathcal{M}) \xrightarrow{f_{\mathcal{M}}} \mathbb{C}$ is an isomorphism, and
2. $H^q(\mathcal{M}, \mathcal{V})^* \approx H^{n-q}(\mathcal{M}, \mathcal{V}^* \otimes \mathcal{K}_{\mathcal{M}})$.

▷ **Sheaf** : Over a topological space X , a *sheaf* \mathfrak{S} assigns (a) for every open neighborhood $U \subset X$ an abelian group (in most examples, under addition) \mathfrak{S}_U and (b) for every inclusion $V \subseteq U$ of open neighborhoods provides a *restriction map* $r_V^U : \mathfrak{S}_U \rightarrow \mathfrak{S}_V$, subject to the conditions

1. $\mathfrak{S}_\emptyset = 0$;
2. r_U^U is the identity;
3. if $U \subseteq V \subseteq W$ are open sets, $r_U^W = r_U^V \circ r_V^W$;
4. if U is an open set, $\{V_i\}$ an open cover of U and $s \in \mathfrak{S}_U$ —then

$$r_{V_i}^U(s) = 0, \quad \forall i, \quad \iff \quad s = 0; \quad (\text{L.56})$$

5. if U is an open set, $\{V_i\}$ an open cover of U and there are elements $s_i \in \mathfrak{S}_{V_i}$ such that

$$r_{(V_i \cap V_j)}^{V_i}(s_i) = r_{(V_j \cap V_i)}^{V_j}(s_j), \quad \forall i, j, \quad (\text{L.57})$$

holds true—then there is a unique $s \in \mathfrak{S}_U$ such that $s_i = r_{V_i}^U(s)$.

Note : if the last two conditions are not necessarily satisfied, \mathfrak{S} is called a *presheaf*. We may define a *(pre)sheaf with values in a category* \mathfrak{C} by replacing ‘abelian group’ with ‘object of \mathfrak{C} ’ and ‘map’ with ‘morphism of \mathfrak{C} ’. For example, in addition to the above structure we often use a multiplication between the elements of \mathfrak{S}_U , hence the corresponding sheaf of rings.

▷ **Spectral Sequence** : A sequence $\{E_r, d_r\}_{r \geq 0}$ of bi-graded groups $E_r \stackrel{\text{def}}{=} \bigoplus_{i,j \geq 0} E_r^{i,j}$ together with the *differentials*

$$d_r : E_r^{i,j} \rightarrow E_r^{i-r, j-r-1}, \quad d_r^2 = 0, \quad (\text{L.58})$$

such that $E_{r+1}^{i,j}$ consists of the d_r -closed elements of $E_r^{i,j}$, taken modulo d_r -exact terms. A spectral sequence is *finite* if $E_r = E_{r+1} = \dots$ for $r \geq r_0$; this limit group is called E_∞ and one says that the spectral sequence *converges* to E_∞ . Clearly, the steps E_r , $r \leq r_0$, may be considered as successive approximations for E_∞ .

Note that, while E_{r+1} is completely determined in terms of E_r and d_r , the ‘next level’ differential is not. Thus, the main utility of spectral sequences is in situation when the arrays $E_r^{i,j}$, for each r , are sparse enough so that a detailed knowledge of the differentials is not necessary to pass to the next level, $r+1$. Alternatively, as in Chapters 9, A and B, it is sometimes possible to discern the action of the differentials from some general principles and reach E_∞ with complete certainty.

▷ **Splitting**¹ (for exact sequences) : Given a short *exact sequence*^(L)

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

and in addition a map $\gamma : C \rightarrow B$ such that $\beta \circ \gamma = \text{Id}_B$ and $\gamma \circ \beta = \text{Id}_C$, then $B = A \oplus C$ and the sequence is said to split.

▷ **Splitting**² (for configurations) : Given a *configuration*^(L) $[X \parallel \mathcal{E}]$, a *split* is another configuration of the type $\left[\begin{smallmatrix} X & \\ \mathbb{P}^n & \parallel \mathcal{E}_1 \cdots \mathcal{E}_{n+1} \\ & \mathcal{O}(1) \cdots \mathcal{O}(1) \end{smallmatrix} \right]$, such that $\mathcal{E} = \bigotimes_{r=1}^{n+1} \mathcal{E}_r$. The former configuration is called the *contraction* of the latter one. Their members are related as follows. Given a variety belonging to the latter configuration, it is defined by a system of constraints which may be regarded as a system of linear equations in the $n+1$ homogeneous variables of \mathbb{P}^1 . Passing to the determinant of this system, a (singular if $\dim > 2$) member of the former configuration is obtained.

▷ **Stalk** (of a Sheaf) : In a *sheaf*^(L) \mathfrak{S} over X , the stalk \mathfrak{S}_p at the point $p \in X$ is defined as the direct limit of \mathfrak{S}_{U_i} , for all open sets $U_i \supset U_{i+1} \ni p$, via the restriction map. Elements $s_p \in \mathfrak{S}_p$ of a stalk are called *germs of sections* of \mathfrak{S} at the point $p \in X$.

▷ **Structure Sheaf** : The *sheaf*^(L) of germs of holomorphic functions over a complex manifold X , that is, the sheaf the *stalks*^(L) of which are just complex numbers. In other words, for every open neighborhood $U \subset X$, it associates to U local holomorphic functions on U .

▷ **Superfield** : A field Φ over superspace spanned by $(x; \theta)$, where x are commuting and θ are anticommuting coordinates, is defined by the formal expansion

$$\Phi(x; \theta) \stackrel{\text{def}}{=} \phi(x) + \theta \cdot \psi(x) + \theta \cdot \theta \cdot F(x) + \dots$$

where we have suppressed the indices labeling the θ 's. Owing to the relation $\theta^i \theta^j + \theta^j \theta^i = 0$, the above products are skew-symmetric and the series terminates. The fields $\phi(x), \psi(x), F(x) \dots$ are referred to as the ‘component fields’ of $\Phi(x; \theta)$. If there are several different coordinates θ^i , a superfield which is independent of some of them is generally referred as chiral. Since all $\Phi(x; \theta)$ are polynomials in θ , the notion of chirality is practically synonymous to holomorphicity.

▷ **Superspace** : For our purposes, superspace may be thought of as a supersymmetric extension of ordinary space in that it is spanned by both commuting and anticommuting coordinates.

(A) In applications for supersymmetry, the commuting spacetime coordinates are transforming as a vector under the *Lorentz group*^(L), while the anticommuting coordinates furnish N copies of the spinor representation; N is referred to as the extendedness of the superspace. 2-dimensional spacetime is exceptional in that the helicity subgroup of the Lorentz group finite. As a consequence, the right-extendedness may differ from the left-extendedness, so one speaks of (p, q) -superspace which is spanned by a commuting $SO(1, 1)$ -vector coordinate, p anticommuting left-handed spinor coordinates and q anticommuting right-handed spinor coordinates. Moreover, the spinors can be chosen to be of the Weyl-Majorana type.

(B) In applications for the Becchi-Rouet-Stora-Tyupin (BRST) symmetry, the commuting spacetime coordinates are again spanning a vector representation of the Lorentz group while the anticommuting coordinates are taken to furnish N copies of the trivial (scalar) representation. One may combine pairs of the anticommuting coordinates into complex pairs, and the associated holomorphic and antiholomorphic translations give rise to BRST and anti-BRST symmetries.

▷ **Supersymmetry** : A continuous symmetry of a system such that: (1) *fermions*^(L) and *bosons*^(L) are mapped into each other, and (2) the composition of two super-transformations generates (ordinary) spacetime translation. Generators of supersymmetry: Q and Q^\dagger (the adjoint), and also the super-shift parameters ϵ and $\bar{\epsilon}$, must then be fermionic: spin-1/2 anticommuting quantities. Then

$$\{ \epsilon \cdot Q, \bar{\epsilon} \cdot Q^\dagger \} = (\epsilon \cdot \bar{\epsilon}) \cdot P . \quad (\text{L.59})$$

Since ϵ and $\bar{\epsilon}$ are anticommuting, this is equivalent to

$$\{ Q_\alpha, Q_\beta^\dagger \} = \gamma_{\alpha\beta}^\mu P_\mu , \quad (\text{L.60})$$

where $\gamma_{\alpha\beta}^\mu$, $\mu = 0, 1, 2, 3$ are the Dirac matrices and P_μ the 4-vector components of energy-momentum. Notice the similarity with the algebra of the exterior differentials:

$$\{ d, d^\dagger \} = \Delta , \quad (\text{L.61})$$

with all other anticommutators vanishing. Supersymmetry may be regarded as a translation in *superspace*^(L), in the fermionic direction. However, the nontrivial relation between such ‘fermionic’ translations and ordinary ‘bosonic’ translations represented by Eq. (L.60) is essential. Without it, the transformation (and symmetry) is merely a fermionic symmetry—the BRST and anti-BRST symmetries are examples of such (see *superspace*^(L)).

▷ **Surjective Map** : Synonym for ‘a map is onto’. That is, a map $A \xrightarrow{f} B$ is surjective (onto) if there exists an $a \in A$ for every $b \in B$ such that $b = f(a)$.

▷ **Tangent Bundle** : Given a manifold \mathcal{M} , consider all the curves passing through a point $p \in \mathcal{M}$. The tangent space at p , $T_p(\mathcal{M})$, is then defined as the vector space spanned by the tangents to the curves through p , that is, by $\ell^i(x) \frac{\partial}{\partial x^i}$, where $\ell^i(x)$ are linear combinations of the local coordinates x^i . The tangent bundle is then defined as the union thereof, $\mathcal{T}_{\mathcal{M}} = \bigcup_{p \in \mathcal{M}} T_p(\mathcal{M})$. When \mathcal{M} is a complex manifold, $\mathcal{T}_{\mathcal{M}}$ will denote the holomorphic tangent bundle, spanned locally by $\ell^i(z) \partial_i$, where $\ell^i(z)$ are holomorphic and linear in z^i and $\partial_i = \frac{\partial}{\partial z^i}$.

▷ **Tautological Sequence** : Consider the Grassmannian $G_{(k,n)}$, that is the space of k -planes H through the origin of \mathbb{C}^n . Let \mathbb{C}^n denote the trivial bundle $G_{(k,n)} \times \mathbb{C}^n$ over $G_{(k,n)}$ and define its *universal sub-bundle*

$$\mathcal{S} \stackrel{\text{def}}{=} \{ (H, x) \in G_{(k,n)} \times \mathbb{C}^n : x \in H \} \quad (\text{L.62})$$

Defining the universal quotient bundle $Q = \{\mathbb{C}^n / \mathcal{S}\}$, we have a short *exact sequence*^(L)

$$0 \rightarrow \mathcal{S} \rightarrow \mathbb{C}^n \rightarrow Q \rightarrow 0 , \quad (\text{L.63})$$

which is called the *tautological sequence*. Know also that $\mathcal{T}_{G_{(k,n)}} = \mathcal{S}^* \otimes \mathcal{Q}$; see Euler sequence^(L) too. On $\mathbb{P}^n = G_{(1,n+1)}$, $\mathcal{S} = \mathcal{O}_{\mathbb{P}^n}(-1)$.

▷ **Technique of Exact and Spectral Sequences (TESS)** : General reference for a computation of cohomology groups $H^*(\mathcal{M}, \mathcal{V})$ in the following situation : (1) \mathcal{M} is a submanifold of X and the resolution of the structure sheaf $\mathcal{O}_{\mathcal{M}}$ in terms of a sheaf-exact sequence^(L) of bundles over X is known. (2) The bundle \mathcal{V} over \mathcal{M} is related to (the restriction to \mathcal{M} of) certain bundles (sheaves) over X in terms of exact sequences over \mathcal{M} . See Chapters 9, A and B for examples.

▷ **Threefold** : See n -fold.

▷ **Transversal** : Two subspaces \mathcal{Y} and \mathcal{Y}' of X , with dimensions k , k' and n , respectively, intersect transversely if $\dim(N_p(\mathcal{Y}) + N_p(\mathcal{Y}')) = (n-k)+(n-k')$ at every point of the intersection, where $N_p(\mathcal{Y})$ is the normal space of $\mathcal{Y} \subset X$. If \mathcal{Y} and \mathcal{Y}' are defined as by $\phi(x) = 0$ and $\phi'(x) = 0$, respectively, then $N_p(\mathcal{Y})$ is spanned by the local gradients $\partial_i \phi(p)$, $p \in \mathcal{Y}$ and similarly for \mathcal{Y}' . The gradients are independent of any connection since $\nabla \phi = \partial \phi + \Gamma \cdot \phi = \partial \phi$ at \mathcal{Y} , where $\phi = \Gamma \cdot \phi = 0$.

▷ **Twofold** : See n -fold.

▷ **Universal Sub-bundle** : See tautological sequence.

▷ **Variety** : A affine (projective) variety is an irreducible zero-set of a finite collection of (homogeneous) polynomials on \mathbb{C}^n (\mathbb{P}^n). A nonempty subset \mathcal{Y} of a topological space X is *irreducible* if it cannot be expressed as the union $(\mathcal{Y}_1 \cup \mathcal{Y}_2)$ of two proper subsets each of which closed in \mathcal{Y} . (The empty set is not considered to be irreducible.) For a more detailed definition of a variety and related analysis, see Ref. [41,27,31,38].

▷ **Vector Bundle** : See *bundle*^(L).

▷ **Veronese Map** : The embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$ by means of sections of $\mathcal{O}_{\mathbb{P}^n}(d)$. For example,

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^n : (x:y) \mapsto (x^n : x^{(n-1)}y : \dots : y^n), \quad (\text{L.64})$$

$$\mathbb{P}^2 \hookrightarrow \mathbb{P}^5 : (x:y:z) \mapsto (x^2 : xy : xz : y^2 : yz : z^2). \quad (\text{L.65})$$

▷ **Whitney Product Formula** : For two vector bundles \mathcal{V} and \mathcal{V}' over a compact complex space X , total Chern classes are multiplicative

$$c[\mathcal{V} \oplus \mathcal{V}'] = c[\mathcal{V}] \wedge c[\mathcal{V}']. \quad (\text{L.66})$$

(It suffices if ‘ \oplus ’ denotes a direct sum differentiably.)

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The list of books, reviews and research articles compiled here is meant to be a helpful guide through the literature rather than a complete catalogue. I apologize to the many researchers whose work is underrepresented here; omissions were made only to bring the list to a tractable length. The bibliographic items are listed alphabetically, by the first author. Many of the items contain several related references; these are ordered according to date of publication.

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