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LECTURES ON COMPLEX MANIFOLDS*

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PREFACE

The theory of complex manifolds is a vast topic so any short introduction must of necessity be highly selective. My aim in these lectures has been to present, in a manner intelligible to physicists, the mathematics relevant to the compactification of superstrings. The principal aim being to present the basic facts relevant to the geometry and topology of Calabi-Yau spaces.

I assume a familiarity with complex analysis of one variable and with elementary real differential geometry. There are many good texts on differential geometry but a concise treatment similar in spirit to these lectures may be found in the relevant chapter of the book by Hawking and Ellis[1]. A more detailed reference is the two volume work by Kobayashi and Nomizu[2]. There are also many texts on complex manifolds. The book by Yano[3] employs the more concrete notation favoured by physicists. The books by Morrow and Kodaira[4] and Kodaira[5] are very good references. The latter being a particularly good account of the theory of deformations of complex structures. A concise and readable review of elementary homology and cohomology as well as the geometry of characteristic classes is provided by the article by Eguchi, Gilkey and Hansen[6]. Finally the book by Griffiths and Harris [7] is the definitive text on algebraic geometry.

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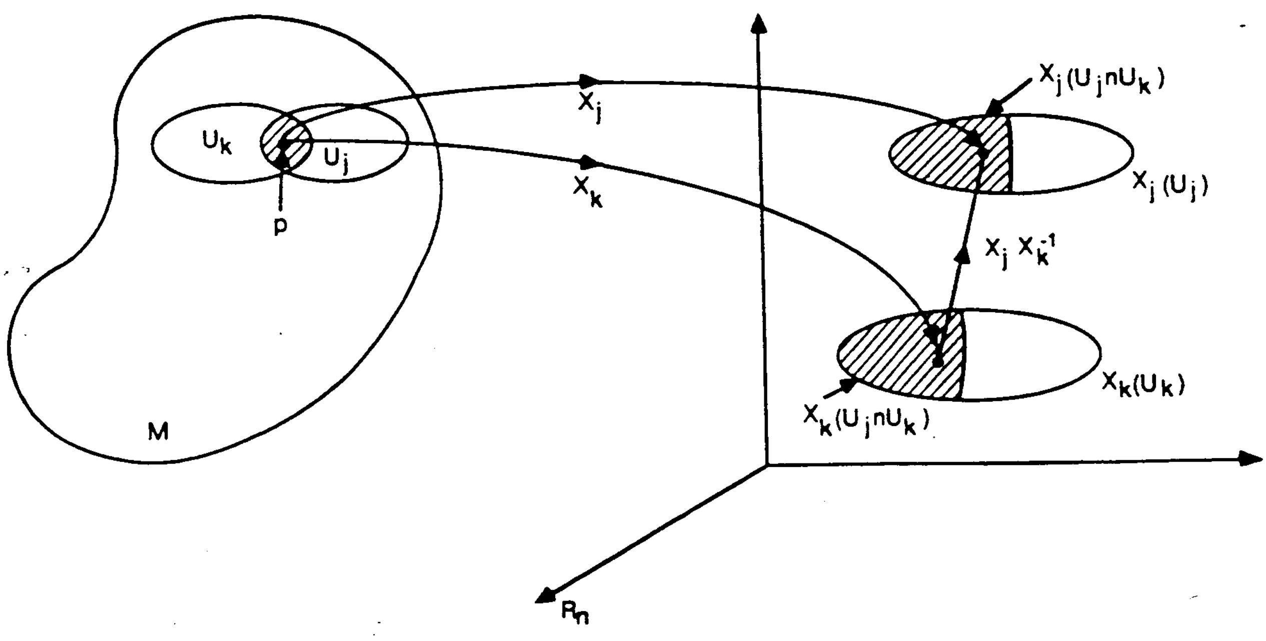
I. INTRODUCTION

We begin by recalling some facts about real manifolds. Heuristically a manifold is a space that is locally like R_n.

Definition: A C^r n-dimensional **manifold** M is a topological space together with an atlas, that is a collection of charts $(\mathbf{U}_j, x_j)$ where the $\mathbf{U}_j$ are open subsets of M and the x_j are one to one maps of the corresponding $\mathbf{U}_j$ to open subsets of R_n such that

(i) the $\mathbf{U}_j$ cover M *i.e.* $M = \cup_j \mathbf{U}_j$

(ii) if $\mathbf{U}_j \cap \mathbf{U}_k$ is non-empty then the map $x_j x_k^{-1}: x_k(\mathbf{U}_j \cap \mathbf{U}_k) \to x_j(\mathbf{U}_j \cap \mathbf{U}_k)$ is a C^r map of an open set in R_n to an open set in R_n



The content of the definition is that within each $\mathbf{U}_j$ we can choose coordinates x_j^m ($m = 1, \ldots, n$) such that where two coordinate patches $\mathbf{U}_j$ and $\mathbf{U}_k$ overlap the x_j^m are C^r functions of the x_k^m

$$x_j^m = f_{jk}^m(x_k). \tag{1.1}$$

Definition: A manifold M is **orientable** if there exists for M an atlas $\{\mathbf{U}_j, x_j\}$ such that for every non-empty $\mathbf{U}_j \cap \mathbf{U}_k$ the Jacobian $\frac{\partial(x_j)}{\partial(x_k)}$ is positive.

Definition: A manifold is **Hausdorff** if for any two points $p, q \epsilon M$ with $p \neq q$ there exist open sets V_p and V_q such that $p \epsilon V_p, q \epsilon V_q$ and $V_p \cap V_q = \emptyset$.

It is hard to picture spaces that are not Hausdorff but an example will be given shortly.

Definition: A manifold is **compact** if *every* atlas contains a finite refinement. That is if every atlas $\{\mathbf{U}_j, x_j\}_{j \epsilon J}$ contains a subatlas $\{\mathbf{U}_j, x_j\}_{j \epsilon K}$ which consists of only finitely many $\mathbf{U}_j$ but which still satisfies properties (i) and (ii) from Definition 1.

Loosely speaking, compactness corresponds to being closed or finite. The surface of a sphere is a compact space but R_n is non-compact. However, we have not yet introduced a metric so we do not yet know what "finite" means. Besides, the question of compactness can be subtle. For example, the closed interval $[0,1]$ of the real line is compact while the interval $[0,1)$ which omits the endpoint is non-compact. (Consider the atlas $\{\mathbf{U}_n\}$ consisting of the open sets $[0, 1-1/n)$, clearly there is no finite refinement.) Another example consists of the surface of revolution obtained by rotating the graph of e^{-x^2} about the x-axis, this yields a surface that is non-compact but which has a finite area.

Definition: A manifold is **paracompact** if every atlas contains a *countable* refinement.

Loosely speaking, this means that the manifold may be infinite in volume, but that the infinity is no worse than that of R_n.

Definition: Two manifolds are **diffeomorphic** if there is a differentiable and invertible map between them. Real manifolds are considered to be equivalent if they are diffeomorphic.

It is important not to confuse the concept of a manifold with that of a metric. A manifold that is endowed with a metric is a **Riemannian manifold** which is not what we have considered up to this point. We shall want to consider the possibility of a manifold admitting different metrics so it helps to keep the concepts separate. A two-sphere and the surface of an egg are diffeomorphic and hence equivalent as manifolds; however, if we think of them as embedded in R_3 then they have different metrics.

From now on, unless otherwise stated, all manifolds will be C^∞, paracompact and Hausdorff.

We come now to the definition of a complex manifold.

Definition: A **complex manifold** is a topological space M together with a *holomorphic* (*i.e.* analytic) atlas. That is a collection of charts $(\mathbf{U}_j, z_j)$ that are one to one maps of the corresponding $\mathbf{U}_j$ to C^n such that for every non-empty intersection $\mathbf{U}_j \cap \mathbf{U}_k$ the maps $z_j z_k^{-1}$ are holomorphic.

The crucial difference between the definition of a complex manifold and the definition of a real manifold is that the transition functions f_{jk} which relate the coordinates in overlapping coordinate patches U_j and U_k

$$z_j^\mu = f_{jk}^\mu(z_k) \tag{1.2}$$

are now required to be holomorphic rather than C^∞ . This means that the z_j^μ are functions of the z_k^μ but not of their complex conjugates $\bar{z}_k^\mu$. Since we can think of C^n as R^{2n} we see from the definition that every n-dimensional complex manifold is a 2n-dimensional real manifold.

Some examples will, perhaps, clarify some of the distinctions we have made:

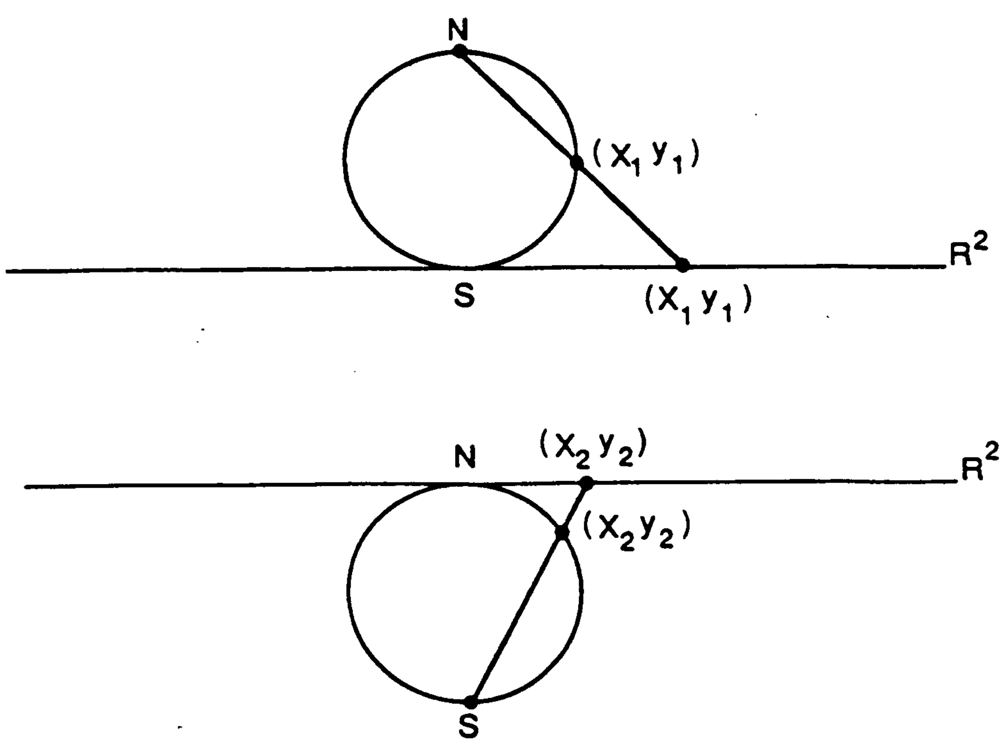
(1) R^n and C^n are respectively real and complex manifolds.

(2) The n-sphere S^n, which is the subset of R^{n+1} that satisfies the equation

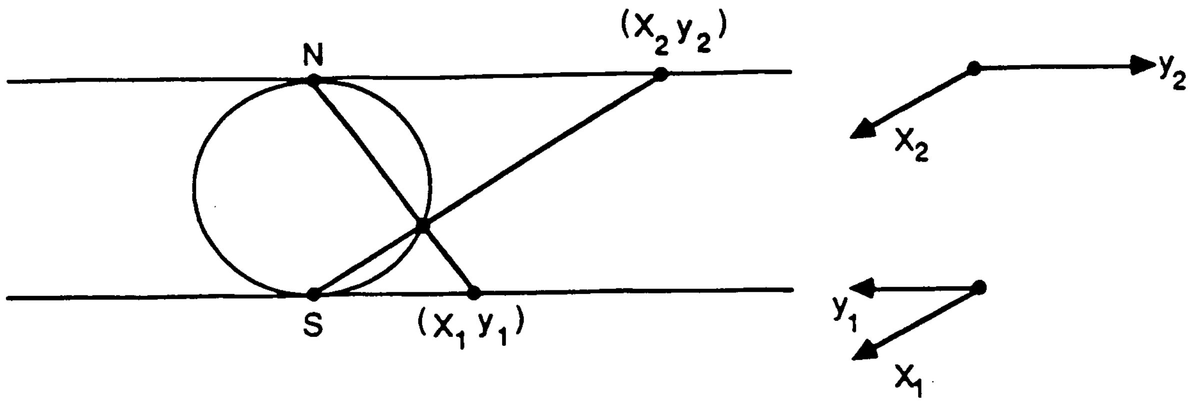
$$\sum_{i=1}^{n+1}(x^i)^2 = 1 \tag{1.3}$$

is a manifold. S^2, for example, is topologically distinct from R^2; this distinction is reflected in the fact that at least two coordinate patches are required to cover a two sphere.

The simplest way to achieve this is to project stereographically from the North and South poles. We obtain in this way two coordinate patches $U_1 = S^2\backslash N$ and $U_2 = S^2\backslash S$.



It is a simple matter to write down the transition functions on the overlap $\mathbf{U}_1 \cap \mathbf{U}_2 = S^2 \backslash (N \cup S)$



$$(x_1, y_1) = f_{12}(x_2, y_2) = \left(\frac{x_2}{x_2^2 + y_2^2} \quad , \quad \frac{-y_2}{x_2^2 + y_2^2}\right) \tag{1.4}$$

clearly the f_{12} are C^∞ functions. Note that if we set

$$z_1 = x_1 + iy_1 \tag{1.5}$$

and

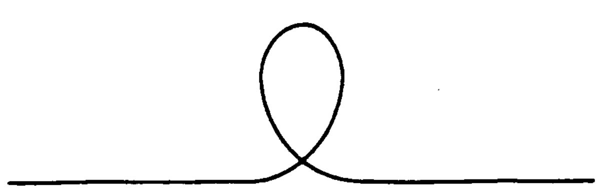
$$z_2 = x_2 + iy_2 \tag{1.6}$$

then we can rewrite the transition function in the form

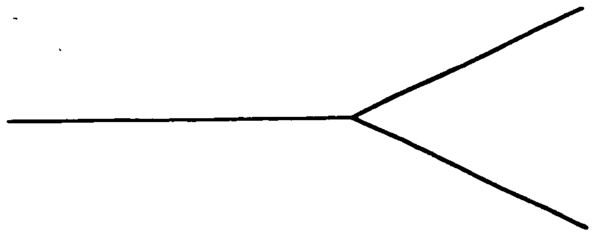
$$z_1 = f_{12}(z_2) = 1/z_2. \tag{1.7}$$

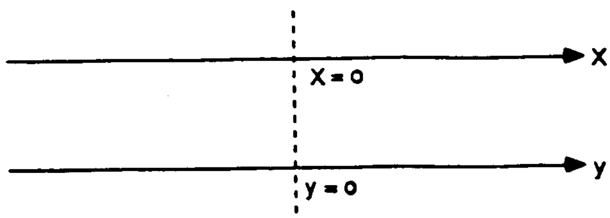
The transition functions are holomorphic functions on $\mathbf{U}_1 \cap \mathbf{U}_2$ since the points $z^1 = 0$ and $z^2 = 0$ have been excluded. Thus S^2 is also a complex manifold.

(3) This space is not a one-dimensional real manifold since there is no neighborhood of the crossover point that looks like a subset of R^1.



Consider this space





one's first reaction is to declare it not to be a manifold for the same reason as the previous case. However, one could escape this objection by constructing the manifold in the following way: take two lines as shown and identify points $x = y$ for $x, y < 0$ but not for $x, y \geq 0$. This construction produces a manifold. However the manifold is not Hausdorff because $x = 0$ is not the same point as $y = 0$ and yet there are no neighborhoods of $x = 0$ and $y = 0$ which do not intersect.

(4) **Complex projective space P_n.** This is the space of complex lines through the origin in C^{n+1}. More precisely, we take the space $C^{n+1}\backslash\{0\}$, that is the set$(z^1, z^2, \ldots, z^{n+1})$ where the z^i are not all zero, and identify

$$(z^1, \ldots, z^{n+1}) \approx \lambda(z^1, \ldots, z^{n+1}) \tag{1.8}$$

for any non-zero complex λ. We can take the sets

$$\mathbf{U}_j = \{z^j \neq 0\} \tag{1.9}$$

as coordinate neighborhoods and choose coordinates

$$\zeta_j^m = \frac{z^m}{z^j} \tag{1.10}$$

within each $\mathbf{U}_j$. On the overlap $\mathbf{U}_j \cap \mathbf{U}_k$ we have

$$\zeta_j^m = \frac{z^m}{z^j} = \frac{z^m}{z^k} \Big/ \frac{z^j}{z^k} = \frac{\zeta_k^m}{\zeta_k^j}. \tag{1.11}$$

ζ_j^m is a holomorphic function of ζ_k^m on the overlap since neither z^j nor z^k is zero there. P_n is a complex manifold of dimension n. It is a compactified form of C^n to which a hyperplane has been added at infinity. We shall come to know P_n well in what follows. P_1 is covered by two coordinate patches $\mathbf{U}_1$ and $\mathbf{U}_2$. On these patches we have coordinates

$$\zeta_1 = \frac{z^2}{z^1} \qquad \text{and} \qquad \zeta_2 = \frac{z^1}{z^2} \tag{1.12}$$

on the overlap $\mathbf{U}_1 \cap \mathbf{U}_2$

$$\zeta_1 = \frac{1}{\zeta_2} \tag{1.13}$$

and we see that P_1 is the Riemann sphere S^2.

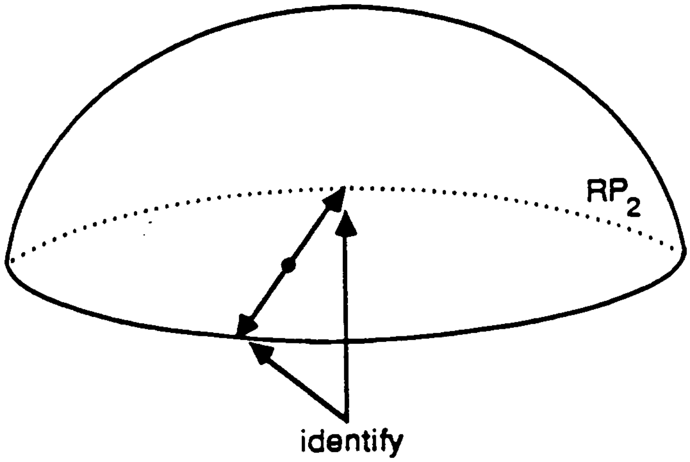
(5) Somewhat easier to visualize than P_n are the real projective spaces RP_n. These are the points of $R^{n+1}\backslash\{0\}$ identified if they differ by a scale

$$(x^1, x^2, \ldots, x^{n+1}) \approx \lambda(x^1, x^2, \ldots, x^{n+1}). \tag{1.14}$$

RP_n can be thought of as the set of straight lines through the origin in R^{n+1}. We can represent such a line by its points of intersection with the unit sphere. RP_2, for example, is a two sphere with opposite points identified

$$RP_2 = S^2/Z_2. \tag{1.15}$$

Alternatively we can think of it as a hemisphere with antipodal points of the equator identified.



RP_2 is compact. It is not orientable.

(6) Any orientable two-dimensional Riemannian manifold is a complex manifold.

It is well known that for such a space it is possible, in a neighborhood of any point, to choose coordinates such that the metric assumes the form

$$ds^2 = \lambda^2(x,y)(dx^2 + dy^2) \tag{1.16}$$

setting $z = x + iy$ we have

$$ds^2 = \lambda^2 dz d\bar{z} \tag{1.17}$$

for all z in some coordinate patch $\mathbf{U}_1$. If $\mathbf{U}_2$ is another coordinate patch that intersects $\mathbf{U}_1$ then we have also

$$ds^2 = \mu^2 dw d\bar{w} \quad , \quad w = u + iv \tag{1.18}$$

for some function μ. Moreover, since the manifold is orientable we shall assume the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ is positive. On the overlap $\mathbf{U}_1 \cap \mathbf{U}_2$ we have

$$\lambda^2 dz d\bar{z} = \mu^2 dw d\bar{w} \tag{1.19}$$

And also

$$dw = \frac{\partial w}{\partial z} dz + \frac{\partial w}{\partial \bar{z}} d\bar{z}. \tag{1.20}$$

Substituting into the relation above we find that

$$\frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial \bar{z}} = 0. \tag{1.21}$$

Thus w is either a holomorphic or an antiholomorphic function of z. If it were antiholomorphic then the Cauchy-Riemann equations (the real and imaginary parts of $\frac{\partial w}{\partial z} = 0$) would read

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{1.22}$$

Hence we would have

$$\frac{\partial(u,v)}{\partial(x,y)} = -\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial y}\right)^2 < 0 \tag{1.23}$$

which would contradict our assumption that the Jacobian is positive. Thus w is a holomorphic function of z and the manifold is a complex manifold.

Compact submanifolds of R^n (such as S^n) provide many examples of compact real manifolds. For application to physics we shall be especially interested in compact manifolds. However, the following theorem shows that the situation is very different with respect to C^n.

Theorem: A connected compact analytic submanifold of C^n is a point.

By an analytic submanifold is meant a submanifold defined by analytic equations $Z^j = Z^j(x)$ where the $Z^j, j = 1, \ldots, n$ are coordinates for C^n and the $x^m, m = 1, \ldots, k$ are local coordinates for the submanifold.

The proof rests on the maximum modulus principle. Recall from the theory of one complex variable that the modulus of a function f that is holomorphic in some open set U cannot have a maximum (or a minimum) at an interior point p of U unless f is a constant throughout U. The result extends to the case of several complex variables by applying the one variable result to the lines through p (a line, being a one-dimensional complex manifold is what is more usually referred to as the complex plane C).

Now suppose M is a complex manifold embedded in C^n. The coordinates Z^j of the embedding space are holomorphic functions on M. Since M is compact, each $|Z^j|$ must achieve a maximum somewhere on M hence each Z^j must be a constant. M is a point.

There are, however, compact complex manifolds that are submanifolds of P_n. P_n is compact (a fact that will be proved shortly) and all its complex submanifolds are compact. By a celebrated theorem of Chow (not proved here) any submanifold of P_n can be realized as the zero locus of a finite number of homogeneous polynomial equations. An example is the Fermat surface in P_4 defined by the equation

$$(z^1)^5 + (z^2)^5 + (z^3)^5 + (z^4)^5 + (z^5)^5 = 0. \tag{1.24}$$

Other examples of compact complex manifolds are quotient manifolds of C^n.

6) The complex torus: Let G be the group generated by translation by $2n$ complex vectors that are linearly independent over the reals

$$\begin{aligned} a_j &= (a_j^1, a_j^2, \ldots, a_j^n) \\ b_j &= (b_j^1, b_j^2, \ldots, b_j^n). \end{aligned} \tag{1.25}$$

Then C^n/G is a complex manifold.

There is an interesting example, due to Iwasawa, of a quotient manifold of C^3 that has a nonabelian fundamental group. It is usual to think of C^3 as the set (z^1, z^2, z^3), however think of it instead as the group of matrices of the form

$$\begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.26}$$

Let Δ be the discrete group of matrices of the above form for which (z_1, z_2, z_3) are complex integers (*i.e.* of the form $m + in$ with m and n integers). Then it is easy to check that C^3/Δ is a compact complex manifold. Its fundamental group is isomorphic to Δ which is nonabelian.

7) S^{2n} is *not* a complex manifold for $n > 1$. The proof of this fact for S^6 was a long standing problem which was settled only in 1969. By contrast $S^{2p+1} \times S^{2q+1}$ *is* a complex manifold. To see why, let us digress briefly to consider the Hopf fibration of S^{2p+1}.

S^{2p+1} is the set of points $(z^1, \ldots, z^{p+1})$ which satisfy the condition

$$\sum_{i=1}^{p+1} |z^i|^2 = 1. \tag{1.27}$$

However, we may also think of $(z^1, \ldots, z^{p+1})$ as the homogeneous coordinates of a point in P_p. Thus we have a map π from S^{2p+1} to P_p. This map is many-one since if $(z^1, \ldots, z^{p+1})$ is a point of S^{2p+1} then so is $e^{i\vartheta}(z^1, \ldots, z^{p+1})$ for any ϑ and this defines the same point in P_p. π projects an S^1 down onto each point in P_p. Thus S^{2p+1} is a fiber bundle over P_p in which each fiber is a circle. Note that this construction demonstrates that P_p is compact since it is the image of a compact space under a continuous map.

Returning to the case at hand we can perform the Hopf construction for both S^{2p+1} and S^{2q+1}

$$\pi\colon S^{2p+1} \times S^{2q+1} \longrightarrow P_p \times P_q. \tag{1.28}$$

Each fiber is now an $S^1 \times S^1 = T^2$. Both the base manifold and the fiber are complex manifolds so it is plausible that the fiber bundle is a complex manifold. Demonstrating that this is in fact the case involves writing down an atlas and checking that the transition functions are holomorphic. We leave this as an exercise.

8) Finally, we wish to make a simple but important observation. Two manifolds can be different if regarded as complex manifolds and yet be diffeomorphic and hence equivalent as real manifolds. As an example consider two two-dimensional tori

$$\begin{aligned} T_1 &= \{(x,y) \mid (x,y) \approx (x+1,y) \approx (x,y+1)\} \\ T_2 &= \{(\xi,\eta) \mid (\xi,\eta) \approx (\xi+1,\eta) \approx (\xi,\eta+2)\}. \end{aligned} \tag{1.29}$$

These manifolds are diffeomorphic as real manifolds since

$$(\xi,\eta) = (x, 2y) \tag{1.30}$$

defines a C^∞ map between them. Note, however, that if we set $z = x + iy$ and $\zeta = \xi + i\eta$ then we cannot write ζ as a holomorphic function of z. We have

$$\zeta = \frac{3}{2}z - \frac{1}{2}\bar{z} \tag{1.31}$$

and it is not possible to eliminate the $\bar{z}$.

II. DIFFERENTIAL GEOMETRY

This section is a telegraphic review of (mostly real) differential geometry and the exterior calculus. These concepts will be basic to our understanding of the geometry and topology of complex manifolds when we return to them in §IV.

Tangent and Cotangent Spaces

A vector is understood abstractly as the tangent vector to a curve λ: $x^m = x^m(t)$ at a point p.

More formally a vector is a linear differential operator that acts on scalar functions.

$$Vf = \left(\frac{\partial f}{\partial t}\right)_\lambda . \tag{2.1}$$

In the coordinate system x^m

$$Vf = \frac{dx^m}{dt}\frac{\partial f}{\partial x^m} \quad \text{or} \quad V = \frac{dx^m}{dt}\frac{\partial}{\partial x^m}. \tag{2.2}$$

The **tangent space** at $p, T_p(M)$ is the space of all vectors defined at p. A basis for $T_p(M)$ is $\{\frac{\partial}{\partial x^m}\}$.

An arbitrary vector $V \epsilon T_p(M)$ can be written in the form

$$V = V^m \frac{\partial}{\partial x^m} \tag{2.3}$$

and the coefficients V^m are the components of V in the basis $\{\frac{\partial}{\partial x^m}\}$. The utility of this definition is that with respect to a new coordinate system $x^{n'} = x^{n'}(x^m)$ we have by use of the chain rule,

$$V^m \frac{\partial}{\partial x^m} = V^m \frac{\partial x^{n'}}{\partial x^m}\frac{\partial}{\partial x^{n'}} = V^{n'}\frac{\partial}{\partial x^{n'}} \tag{2.4}$$

and we recognize the familiar transformation rule for the transformation of vector components

$$V^{n\prime} = \frac{\partial x^{n'}}{\partial x^m} V^m. \tag{2.5}$$

The **cotangent space** $T_p^*(M)$ is the vector space that is dual to $T_p(M)$. We denote by $\{dx^m\}$ the base of covectors that is dual to $\{\frac{\partial}{\partial x^m}\}$ that is, if we denote the inner product between $T_p(M)$ and $T_p^*(M)$ by angular brackets, the $\{dx^m\}$ satisfy the relation

$$\left\langle dx^m, \frac{\partial}{\partial x^n}\right\rangle = \delta^m_n . \tag{2.6}$$

The general $u \epsilon T_p^*(M)$ has the form

$$u = u_m dx^m \tag{2.7}$$

the u_m then automatically have the appropriate transformation law under a change of coordinates.

Tensor Fields

A tensor of type (k, ℓ) is defined by a straightforward extension of the above

$$T = T^{m_1 \ldots m_k}{}_{n_1 \ldots n_\ell} dx^{n_1} \otimes \ldots \otimes dx^{n_\ell} \otimes \frac{\partial}{\partial x^{m_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{m_k}}. \tag{2.8}$$

Metric

The **metric** is a positive definite (we will not here consider spaces of indefinite metric) inner product on $T_p(M)$. Given two vectors X and Y in $T_p(M)$ we write this inner product as $g(X, Y)$. The metric is bilinear in its arguments, thus the inner product is defined by giving the values of the inner products between basis elements

$$g\left(\frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^n}\right) = g_{mn}. \tag{2.9}$$

This equation defines g_{mn}. Thus

$$g(X, Y) = g\left(X^m \frac{\partial}{\partial x^m}, Y^n \frac{\partial}{\partial x^n}\right) = g_{mn} X^m Y^n. \tag{2.10}$$

fact, we can write

$$g = g_{mn} dx^m \otimes dx^n. \tag{2.11}$$

Differential forms

In differential geometry and in topology, a distinguished role is played by tensors that are totally skew symmetric because of their interpretation as elements of area and of volume. Take x and y to be cartesian coordinates in the plane and define

$$dx \wedge dy = \frac{1}{2}(dx \otimes dy - dy \otimes dx) = -dy \wedge dx. \tag{2.12}$$

In polar coordinates

$$x = r \cos \vartheta \qquad y = r \sin \vartheta \tag{2.13}$$

we have

$$\begin{aligned} dx \wedge dy &= (\cos\vartheta dr - r\sin\vartheta d\vartheta) \wedge (\sin\vartheta dr + r\cos\vartheta d\vartheta) \\ &= r\cos^2\vartheta dr \wedge d\vartheta - r\sin^2\vartheta d\vartheta \wedge dr \\ &= r dr \wedge d\vartheta. \end{aligned} \tag{2.14}$$

We recognize this as the transformation law for area elements. Like Molière's *bourgeois gentil homme* we are unaware that we speak forms daily. $dx \wedge dy$ is what we mean when we use $dxdy$ to denote the area element in a two-dimensional integral. The calculus of differential forms is an elaboration of this simple fact.

A p-form is a totally skew symmetric covariant tensor of rank p. Let $\Lambda^p(x)$ be the set of p forms at x and let $C^\infty(\Lambda^p)$ be the space of smooth p-forms. $\Lambda^p(x)$ is a vector space and has a basis

$$\{dx^{m_1} \wedge dx^{m_2} \wedge \ldots \wedge dx^{m_p}\}\ ,\ m_1 < m_2 < \ldots < m_p \tag{2.15}$$

where

$$\begin{aligned} dx^{m_1} \wedge \ldots \wedge dx^{m_p} = \frac{1}{p!}\{&\text{sum of even permutations of } dx^{m_1} \otimes \ldots \otimes dx^{m_p} \\ &- \text{sum of odd permutations}\}. \end{aligned} \tag{2.16}$$

A zero form is a function and a one form is a covariant vector. The dimension of Λ^p is $n!/p!(n-p)!$ which is also the dimension of Λ^{n-p}. p cannot assume a value greater than n since in the basis element $dx^{m_1}_{\wedge \ldots \wedge} dx^{m_p}$ at least one factor would be repeated and the basis element would vanish. Our convention for the components of a p-form α_p is that

$$\alpha_p = \frac{1}{p!}\alpha_{m_1 m_2 \ldots m_p} dx^{m_1} \wedge dx^{m_2} \wedge \ldots \wedge dx^{m_p} \tag{2.17}$$

with $\alpha_{m_1 \cdots m_p}$ skew symmetric.

We can use the wedge product to combine a p-form with a q-form to yield a $p+q$ form $\alpha_p \wedge \beta_q$

$$\alpha_p \wedge \beta_q = \frac{1}{p!q!}\alpha_{m_1 \ldots m_p}\beta_{n_1 \ldots n_q} dx^{m_1} \wedge \ldots \wedge dx^{m_p} \wedge dx^{n_1} \wedge \ldots \wedge dx^{n_q}. \tag{2.18}$$

We see that

$$\alpha_p \wedge \beta_q = (-1)^{pq}\beta_q \wedge \alpha_p. \tag{2.19}$$

Exterior Derivative

Exterior differentiation is essentially the process of taking the curl of a skew symmetric tensor. More formally the exterior derivative d is a map from the space of p-forms to the space of $p+1$ forms

$$d\colon C^\infty(\Lambda^p) \to C^\infty(\Lambda^{p+1}). \tag{2.20}$$

On zero forms

$$d\alpha = \frac{\partial \alpha}{\partial x^m} dx^m. \tag{2.21}$$

On one forms

$$d(\alpha_n dx^n) = \frac{\partial \alpha_n}{\partial x^m} dx^m \wedge dx^n. \tag{2.22}$$

On two forms

$$d(\alpha_{nr} dx^n \wedge dx^r) = \frac{\partial \alpha_{nr}}{\partial x^m} dx^m \wedge dx^n \wedge dx^r. \tag{2.23}$$

etc.

The convention is that the new dx goes in front. Note firstly that these expressions are covariant because we may replace the partial derivatives by covariant derivatives since the Christoffel symbols will cancel out owing to the complete skew-symmetry of the basis forms. Note secondly the simple but important fact that when iterated the exterior derivative gives zero

$$d^2 = 0. \tag{2.24}$$

For example, starting with a zero form

$$dd\alpha = d\left(\frac{\partial \alpha}{\partial x^n} dx^n\right) = \frac{\partial^2 \alpha}{\partial x^m \partial x^n} dx^m \wedge dx^n \tag{2.25}$$

which vanishes by the antisymmetry of the basis elements.

A simple example which illustrates these considerations is provided by the exterior calculus on R_3. As representative differential forms we take

$$\alpha_0 = f, \tag{2.26}$$

$$\alpha_1 = u_1 dx^1 + u_2 dx^2 + u_3 dx^3, \tag{2.27}$$

There are three independent basis two forms in three dimensions $dx^2 \wedge dx^3, dx^3 \wedge dx^1, dx^1 \wedge dx^2$

$$\alpha_2 = w_1 dx^2 \wedge dx^3 + w_2 dx^3 \wedge dx^1 + w_3 dx^1 \wedge dx^2, \tag{2.28}$$

$$\alpha_3 = h dx^1 \wedge dx^2 \wedge dx^3. \tag{2.29}$$

In the following operations we recognize quantities that are familiar from vector calculus

$$\alpha_1 \wedge \alpha_2 = (u_1 w_1 + u_2 w_2 + u_3 w_3) dx^1 \wedge dx^2 \wedge dx^3 \tag{2.30}$$

$$\begin{aligned} d\alpha_1 &= \frac{\partial u_k}{\partial x^\ell} dx^\ell \wedge dx^k \\ &= \varepsilon_{ijk} \frac{\partial u_k}{\partial x^j} \left(\frac{1}{2} \varepsilon_{i\ell m} dx^\ell \wedge dx^m\right) \end{aligned} \tag{2.31}$$

$$d\alpha_2 = \left(\frac{\partial w_1}{\partial x^1} + \frac{\partial w_2}{\partial x^2} + \frac{\partial w_3}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3. \tag{2.32}$$

Stoke's Theorem

In the language of forms, Stoke's theorem may be stated in a very concise manner: If M_p is a p-dimensional manifold with boundary ∂M_p we have

$$\int_{M_p} d\alpha_{p-1} = \int_{\partial M_p} \alpha_{p-1}. \tag{2.33}$$

In this form Stoke's theorem contains many familiar results. Consider the relation for $p = 1, 2, 3$ in turn.

$p = 1$; M is a line, $\partial M = \{a, b\}$, say, consists of the endpoints and α is a zero form so we have

$$\int_a^b df = f(b) - f(a) \tag{2.34}$$

$p = 2$

$$\int_M \left(\frac{\partial A_n}{\partial x^m} - \frac{\partial A_m}{\partial x^n}\right) dx^m \wedge dx^n = \int_{\partial M} A_m dx^m \tag{2.35}$$

$p = 3$

$$\int_M \left(\frac{\partial w_1}{\partial x^1} + \frac{\partial w_2}{\partial x^2} + \frac{\partial w_3}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3 = \int_{\partial M} (w_1 dx^2 \wedge dx^3 + w_2 dx^3 \wedge dx^1 + w_3 dx^1 \wedge dx^2). \tag{2.36}$$

We see that Stoke's theorem also contains Gauss' theorem as well as the fundamental theorem of differential calculus.

Hodge $*$

We have seen that, in three dimensions, exterior multiplication leads to both vector and scalar products. This is due to the fact that both one forms and two forms correspond to vectors, this in turn is due to the fact that $3 = 2 + 1$ which is of course special to three dimensions. To generalize this concept we introduce the notion of duality which is formalized by the Hodge$*$. The Hodge$*$ is a map from p forms to $n - p$ forms

$$*: C^\infty(\Lambda^p) \to C^\infty(\Lambda^{n-p}) \tag{2.37}$$

defined by its action on basis elements

$$\begin{aligned} &*(dx^{m_1} \wedge \ldots \wedge dx^{m_p}) \\ &= \frac{1}{(n-p)!} g^{\frac{1}{2}} g^{m_1 k_1} \ldots g^{m_p k_p} \varepsilon_{k_1 \ldots k_p k_{p+1} \ldots k_n} dx^{k_{p+1}} \wedge \ldots \wedge dx^{k_n}. \end{aligned} \tag{2.38}$$

Exercise: Show that $**\omega_p = (-1)^{p(n-p)}\omega_p$. ◊

With the aid of $*$ we can define an inner product on the space of real forms

$$(\alpha_p, \beta_p) = \int \alpha_p \wedge *\beta_p. \tag{2.39}$$

It is easy to show that the inner product is symmetric *i.e.* that $(\alpha_p, \beta_p) = (\beta_p, \alpha_p)$ and also that

$$(\alpha_p, \beta_p) = \frac{1}{p!} \int \alpha_{m_1 \cdots m_p} \beta^{m_1 \ldots m_p} g^{\frac{1}{2}} dx^1 \wedge \ldots \wedge dx^n. \tag{2.40}$$

Given an inner product we are in a position to define the adjoint, $d^\dagger$, of the exterior derivative

$$d^\dagger : C^\infty(\Lambda^p) \to C^\infty(\Lambda^{p-1}) \tag{2.41}$$

such that

$$(\alpha_p, d\beta_{p-1}) = (d^\dagger \alpha_p, \beta_{p-1}). \tag{2.42}$$

e start with $(\alpha_p, d\beta_{p-1})$ and integrate by parts

$$\begin{aligned}(\alpha, d\beta) &= \int d\beta \wedge *\alpha \\ &= \int \{d(\beta \wedge *\alpha) - (-1)^{p-1} \beta \wedge d*\alpha\}.\end{aligned} \tag{2.43}$$

We will assume that M is compact and has no boundary, $\partial M = 0$, then the first term under the integral vanishes by Stoke's theorem. Hence

$$\begin{aligned}(\alpha, d\beta) &= (-1)^p \int \beta \wedge d*\alpha \\ &= (-1)^{p+p(n-p)} \int \beta \wedge *(*d*\alpha).\end{aligned} \tag{2.44}$$

om the last equality we identify $d^\dagger$ as given by

$$d^\dagger = (-1)^{p(n-p+1)} *d*. \tag{2.45}$$

For n even, and all p, this reads

$$d^\dagger = *d*, \qquad n \text{ even} \tag{2.46}$$

while for n odd

$$d^\dagger = (-1)^p *d*, \quad n \text{ odd} \tag{2.47}$$

Exercise: Show that, for a form

$$\omega = \frac{1}{p!} \omega_{m_1 \ldots m_p} dx^{m_1} \wedge \ldots \wedge dx^{m_p}$$

$d^\dagger\omega$ has components given by

$$d^\dagger\omega = -\frac{1}{(p-1)!}\nabla^k \omega_{km_2\ldots m_p} dx^{m_2}\wedge\ldots\wedge dx^{m_p}.$$

$\diamondsuit$

The adjoint $d^\dagger$ shares with d the important property that its square is zero

$$d^\dagger d^\dagger = *d**d* = (-1)^{p(n-p)} * d^2 * = 0. \tag{2.48}$$

Hodge deRham Operator

The Hodge-deRham operator is a natural second order differential operator that acts on forms and generalizes the concept of the Laplacian. In fact, it is often referred to as the Laplacian in the mathematical literature even though it is not in general equal to the covariant Laplacian

$$\Delta: C^\infty(\Lambda^p)\to C^\infty(\Lambda^p) \tag{2.49}$$

$$\Delta = dd^\dagger + d^\dagger d. \tag{2.50}$$

Exercise: Show from the definition that

$$(\Delta\omega)_{m_1\ldots m_p} = -\nabla^k\nabla_k\omega_{m_1\ldots m_p} - pR_{k[m_1}\omega^k{}_{m_2\ldots m_p]} - \frac{1}{2}p(p-1)R_{jk[m_1m_2}\omega^{jk}{}_{m_3\ldots m_p]}.$$

For the conventions regarding the Riemann tensor, see §4. $\diamondsuit$

A p-form is said to be **harmonic** if it is annihilated by Δ

$$\Delta\omega = 0 \tag{2.51}$$

from the definition of Δ this is equivalent to

$$(dd^\dagger + d^\dagger d)\omega = 0 \tag{2.52}$$

and hence

$$(\omega, (dd^\dagger + d^\dagger d)\omega) = 0 \tag{2.53}$$

$$(d^\dagger\omega, d^\dagger\omega) + (d\omega, d\omega) = 0. \tag{2.54}$$

Since the inner product is positive definite we see that a form is harmonic if and only if it is both **closed**

$$dw = 0 \tag{2.55}$$

and **co-closed**

$$d^{\dagger} w = 0. \tag{2.56}$$

Hodge's Theorem

Hodge has shown that for a compact manifold without boundary that has a positive definite metric p-form admits a *unique* decomposition into harmonic, **exact** and **co-exact** parts

$$\omega = \alpha + d\beta + d^{\dagger}\gamma. \tag{2.57}$$

To prove uniqueness is easy, for if some form ω were to admit two decompositions of this type, their difference would also be of this type. Thus, it suffices to show that if

$$0 = \alpha + d\beta + d^{\dagger}\gamma \tag{2.58}$$

for some β, γ and harmonic α then each term separately vanishes. To this end operate on the equation with d. We find

$$\begin{aligned} dd^{\dagger}\gamma &= 0 \\ (\gamma, dd^{\dagger}\gamma) &= 0 \\ (d^{\dagger}\gamma, d^{\dagger}\gamma) &= 0 \\ d^{\dagger}\gamma &= 0. \end{aligned} \tag{2.59}$$

Similarly we find $d\beta = 0$ and hence also $\alpha = 0$.

A distinguished role in topology is played by closed forms. By reasoning analogous to that above it is easy to see that the co-exact part of such a form must vanish. A closed form can always be written as

$$\omega = \alpha + d\beta \tag{2.60}$$

with α harmonic. An exact form is always closed since $dd\beta = 0$. It is of interest to inquire under what circumstances the converse is true *i.e.* when is a closed form exact? Since a harmonic form is never exact this is equivalent to asking when a closed form has nonzero harmonic part. This question is perhaps unfamiliar because, as is well-known, in R_n a closed form is always exact $d\omega = 0$ implies there exists a β such that $\omega = d\beta$. On other manifolds this is not true in general. The significance of harmonic forms is due to the fact that *locally* any manifold looks like a subset of R_n. Thus, given a closed form, ω is always possible within any one coordinate patch to find a β such that $\omega = d\beta$. There is no guarantee, however, that the β's so defined will transform properly on the overlap regions. In general, the β's will not fit together to give a globally defined tensor field. Thus, the existence of harmonic forms is related to the *global* properties of the manifold.

A simple example of a manifold which admits non-trival harmonic forms is T_2.

Let us consider the Hodge decomposition of one-forms for T_2. We observe that there are two harmonic one-forms dx and dy. Clearly dx and dy are zero modes of Δ which for this case is just the negative of the ordinary Laplacian. Note also that dx and dy (despite the notation) are not exact since x and y are not well-behaved functions on the torus (x and y cannot be taken to be both single valued and continuous).

Consider an arbitrary smooth one-form on T_2

$$\omega = u(x,y)dx + v(x,y)dy \tag{2.61}$$

since $u(x,y)$ and $v(x,y)$ must be periodic functions of their arguments ω admits a Fourier decomposition

$$\omega = \sum_{m,n=0}^{\infty} (u_{mn}dx + v_{mn}dy)e^{i(mx+ny)}. \tag{2.62}$$

Exercise: Show that the Hodge decomposition of ω takes the form

$$\begin{aligned}\omega &= u_{00}dx + v_{00}dy \\ &+ d\{-i\sum{}' \frac{(mu_{mn} + nv_{mn})}{m^2+n^2}e^{i(mx+ny)}\} \\ &+ d^{\dagger}\{-i\sum{}' \frac{(nu_{mn} - mv_{mn})}{m^2+n^2}e^{i(mx+ny)}dx\wedge dy\}\end{aligned}$$

where $\sum'$ denotes that the term with $m = n = 0$ is omitted. Note that this demonstrates that there are precisely two linearly independent harmonic forms on T_2. ◇

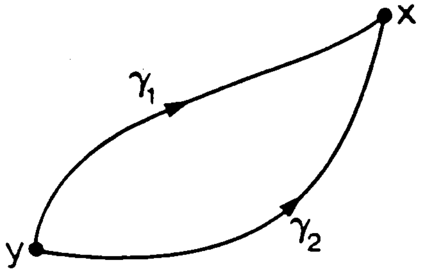
The example shows us that the topology of T_2 is different from that of R_2. Can we say more precisely what the existence of the harmonic forms is due to? We have seen that given a closed form ω we can always find a β such that *locally*

$$\omega = d\beta. \tag{2.63}$$

For simplicity let us suppose that ω is a one-form and recall how β is constructed for the case of R^n. We set

$$\beta(x) = \int_y^x \omega_m(\xi)d\xi^m \tag{2.64}$$

the integral being taken along some path from a base point y to the point x. If this construction is sensible i.e. independent of the path chosen from y to x then clearly we have $\omega_m = \partial_m\beta$. For the case of R^n the construction is sensible since the difference between taking the two paths γ_1 and γ_2, as in the figure, gives zero since a closed contour in R^n is always the boundary ∂M of some region M.



$$\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\partial M} \omega = \int_M d\omega = 0. \tag{2.65}$$

This is what fails for the torus. The curves $x = x_0$ and $y = y_0$, for example, are closed contours which however are not the boundaries of any regions.

T_2 admits two harmonic forms because there are two basic curves which are not boundaries. *Basic* here refers to the following. There are many curves on T_2 that are not boundaries. Both z and z', say, in the figure have this property but we can consider z and z' to be equivalent since the curve $z - z'$ is a boundary. The key observation is that harmonic forms are related to equivalence classes of surfaces that have no boundary with two such surfaces being considered to be equivalent if they differ by a surface that is a boundary. The study of Homology and Cohomology is an elaboration of this simple but essential fact.

III. HOMOLOGY AND COHOMOLOGY

This section is a very brief review of elementary results in homology and cohomology which will be needed later. We will follow, in part, the article by Eguchi, Gilkey and Hanson[6] to which the reader is refered for a more thorough account.

Let M be a smooth connected manifold.

Definition: A **p − chain** a_p is a sum

$$a_p = \sum_i c_k N_k \tag{3.1}$$

where the N_k are smooth p-dimensional oriented submanifolds of M. It helps to think of a_p as something that can be integrated over. We will understand an integral over a_p to be given by

$$\int_{\sum c_k N_k} = \sum_k c_k \int_{N_k} . \tag{3.2}$$

Different types of chain can be contemplated according to the nature of the coefficients c_k. If the coefficients are real we have a **real chain**, if they are complex we have a **complex chain**, if they are integers we speak of an **integral chain** and so on.

∂ is the operation of associating with a manifold M its oriented boundary ∂M. An essential fact is that the boundary of a boundary is zero.

$$\partial\partial M = 0 \tag{3.3}$$

(For example, the boundary of a disc is a circle and the boundary of a circle is empty.)

We can extend the ∂ operation to chains by linearity in the obvious way

$$\partial a_p = \sum_i c_k \partial N_k \tag{3.4}$$

Thus, ∂a_p is a $p-1$ chain.

Definition: A **cycle** is a chain with no boundary *i.e.* it is a chain z_p such that

$$\partial z_p = 0 \tag{3.5}$$

Definition: Let Z_p be the set of cycles and let B_p be the set of boundaries, more precisely let B_p be the set of p-chains which are the boundaries of $(p+1)$-chains

$$B_p = \{a_p \mid a_p = \partial a_{p+1}\} \tag{3.6}$$

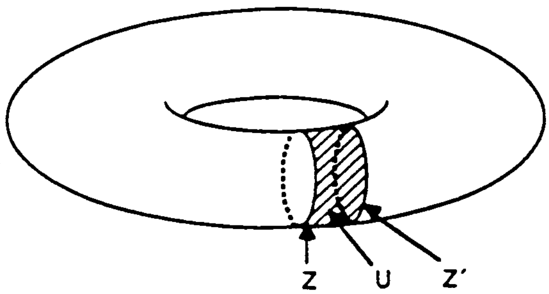
The **simplicial homology** of M is the quotient set

$$H_p = Z_p / B_p \tag{3.7}$$

H_p is the set of p-cycles with two cycles considered to be equivalent if they differ by a boundary *i.e.*

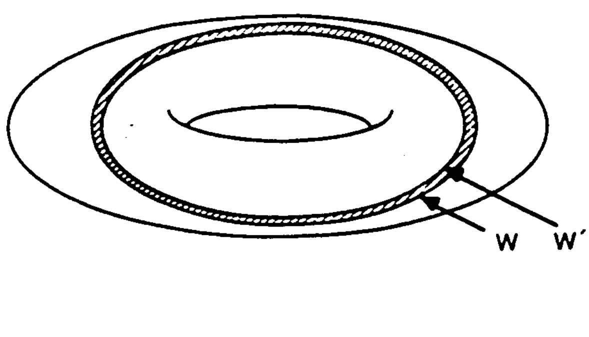
$$a_p \sim a_p + \partial c_{p+1} \tag{3.8}$$

Let us return to the torus T_2 to illustrate these considerations. In the figure we have drawn the torus embedded in R_3

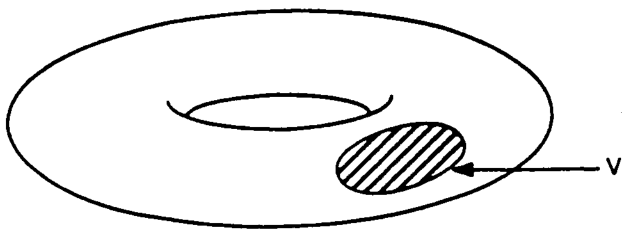


z and z' are cycles because they have no boundary and are not themselves boundaries. z and z' are considered equivalent in H_p because they differ by the boundary of U.

Similar considerations apply to the cycles w and w'.

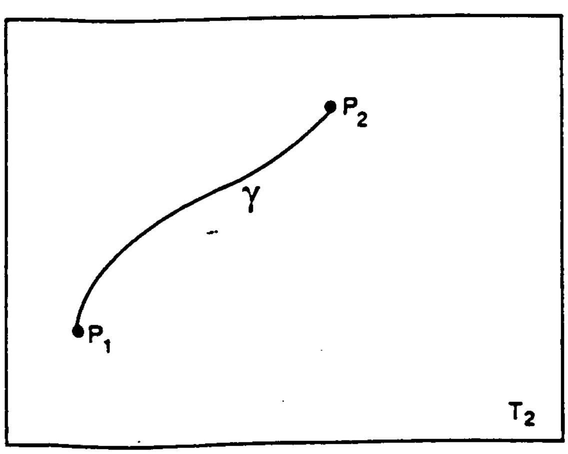


This figure illustrates a cycle u which is "*trivial*" because it is a boundary.



For the torus H_p has two linearly independent elements which we can think of as z and w.

As an example, let us calculate the homology groups $H_p(p = 0, 1, 2)$ for T_2. Consider first H_0. Zero chains are points, and points have no boundary so a zero-chain is also a zero cycle. Furthermore, any two points form the boundary of a curve.



$P_1 - P_2 = \partial\gamma$

Thus, if the manifold is connected, as it is in this case, all points are equivalent. H_0 consists of multiples of some representative point p. If our chains are real then H_0 is equivalent to the set of real numbers. We may write

$$H_0 \cong R. \tag{3.9}$$

H_1 consists of two independent cycles which we have called z and w; any element of H_1 can be expressed as a linear combination $\lambda z + \mu w$ which may be identified with the pair

(λ, μ). Thus,

$$H_1 \cong R \oplus R \tag{3.10}$$

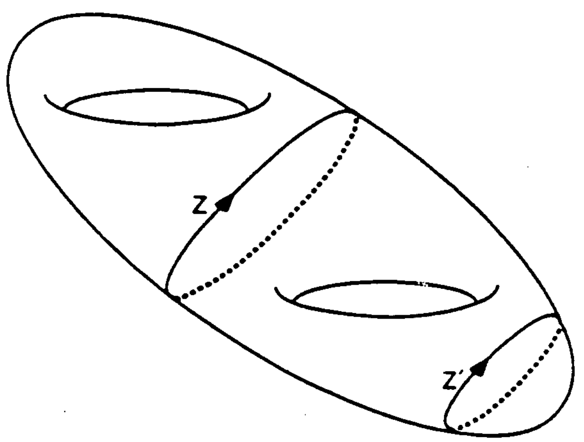
The only two-chain without boundary is T_2 itself and H_2 consists of arbitrary multiples of this so we have

$$H_2 \cong R \tag{3.11}$$

H_2 is the same as H_0; this is required by duality as will become clear later.

Had we been given the homology groups $H_p(T_2, R)$ we would have learnt from $H_0(T_2, R) = R$ that the torus is connected (more generally dim H_0 is the number of connected components of the manifold) and from $H_1(T_2, R) = R \oplus R$ that there are two independent one-cycles.

T_2 furnishes, in some ways too simple, an example the two cycles z and w which are not homologous to zero are also not homotopic to zero. It is important, however, not to confuse homology with homotopy since, in general, two manifolds may be homologous without being homotopic. The figure gives the standard example:



z and z' are homologous since $z - z'$ bounds the region between them. However, they are not homotopic since z cannot be deformed into z' without breaking and reconnecting. On the other hand, two manifolds that are homotopic are homologous since we can deform one manifold into the other.

If G is a field say $G = R, C, Z_2$, then the homology group $H_p(M, G)$ is a vector space over G.

We turn now to cohomology.

Definition: Let Z^p be the set of closed p-forms

$$Z^p = \{\omega_p \mid d\omega_p = 0\} \tag{3.12}$$

and let B_p be the set of exact p-forms

$$B^p = \{\nu_p \mid \nu_p = d\beta_{p-1}\} \tag{3.13}$$

We define the **de Rham cohomology groups** to the quotient of Z^p by B^p

$$H^p = Z^p / B^p \tag{3.14}$$

H^p is the set of closed p-forms where two members are considered to be equivalent if they differ by an exact form

$$\omega_p \simeq \omega_p + d\beta_{p-1} \tag{3.15}$$

The space H^0 is special because there are no (-1)-forms. H^0 is the space of constant functions hence $\dim H^0$ is the number of connected components of the manifold.

We have now defined two sets of vector spaces the de Rham cohomology groups consisting of equivalence classes of closed forms and the previously defined homology groups H_p whose elements are equivalence classes of surfaces with no boundary. deRham has proved two classic theorems which show that the two vector spaces H_p and H^p are dual to each other (and hence isomorphic).

In order to make the correspondence between the homology groups and the cohomology groups let us introduce a product $\pi(z_p, \omega_p)$ defined for any closed form ω_p and any cycle $z_p \in H_p$ by

$$\pi(z_p, \omega_p) = \int_{z_p} \omega_p. \tag{3.16}$$

$\pi(z_p, \omega_p)$ is called a **period**. Note that π is in fact defined on $H_p \otimes H^p$ since we can replace z_p and ω_p by any member of their respective equivalence classes.

$$\begin{aligned} \int_{z_p + \partial a_{p-1}} (\omega_p + d\alpha_{p-1}) &= \int_{z_p} \omega_p + \int_{z_p} d\alpha_{p-1} \\ &\quad + \int_{\partial a_{p-1}} \omega_p + \int_{\partial a_{p-1}} d\alpha_{p-1} \\ &= \int_{z_p} \omega_p \end{aligned} \tag{3.17}$$

$$\int_{z_i} \omega_i = \bar{a}_i^{\,j} \quad \Rightarrow \quad \int_{z_i} \omega_j (\bar{a}^{-1})^j_{\ k} = \delta_i^{\,k}$$

since the last three integrals in the first equality vanish in virtue of Stoke's theorem.

deRham's Theorems

Theorem: Given a basis $\{z_i\}$ for H_p and any set of periods $\nu_i, i = 1, \ldots, \dim H_p$ there exists a closed p-form ω such that $\pi(z_i, \omega) = \nu_i$

Theorem: If all the periods of a p-form vanish then, ω is exact.

If $\{z_i\}$ is a basis for H_p and $\{\omega_j\}$ a basis for H^p then given these theorems we see that the period matrix $\pi(z_i, \omega^j)$ is invertible.

Example: The homology of S^n

$$\begin{aligned} \dim H_0 &= 1, && \text{since } \mathrm{S}^n \text{ is connected} \\ \dim H_n &= 1, && \text{since } \mathrm{H}^n \text{ is represented by the permutation tensor} \\ \dim H_n &= 0, && \text{for } 1 \leq p \leq n-1 \end{aligned}$$

Exercise: Show this directly from the equation $\Delta\omega = 0$ by consideration of $\int \omega \Delta \omega$. ◊

The importance of harmonic forms is that each cohomology class contains precisely one harmonic form so for many purposes we may think of the harmonic form as representing the cohomology class. We know that each cohomology class contains at least one harmonic form in virtue of the Hodge decomposition (the harmonic form is zero if the same cohomology class is trivial). To see that each cohomology class contains at most one harmonic form, we simply suppose that some cohomology class contains two α and β, say. Since α and β are in the same cohomology class they differ by an exact form

$$\alpha - \beta = d\gamma \tag{3.18}$$

ut this implies that $d^\dagger d\gamma$ must vanish and hence also that $(\gamma, d^\dagger d\gamma)$ vanishes. It follows that $d\gamma$ must vanish and that α and β must be equal.

As was remarked previously the exterior derivative of a form does not involve the metric. However,

$$d^\dagger \omega_p = -\frac{1}{(p-1)!} \nabla^k \omega_{k m_2 \ldots m_p} dx^{m_2} \wedge \ldots \wedge dx^{m_p} \tag{3.19}$$

manifestly does depend on the choice of metric. Thus a cohomology class is a topological invariant (since a form that is closed for one metric is closed for all metrics). Although its harmonic representative will depend on (and hence vary with) the metric. Stated somewhat differently, suppose we are given a manifold M, a cohomology group H and two different metrics g_{mn} and g'_{mn} then the harmonic representatives of H, ω and ω' say, will be different for the two metrics. However, since both ω and ω' belong to H they will always differ by an exact form.

Perhaps the most basic topological numbers that can be associated with a manifold are the **Betti numbers**

$$b_p = \dim H^p. \tag{3.20}$$

b_p is the number of linearly independent harmonic p-forms which is equal to the number of irreducible p-cycles.

The **Euler characteristic** can be defined as the alternating sum of the Betti-numbers

$$\chi = \sum_{p=0}^{n} (-1)^p b_p \tag{3.21}$$

Examples

(i) S^n: We learn from the previous exercise that $b_0 = b_n = 1$ and $b_p = 0$ for $1 \leq p \leq n-1$. It follows that

$$\chi(S^n) = \begin{cases} 2, & \text{n even} \\ 0, & \text{n odd.} \end{cases} \tag{3.22}$$

(ii) T^n: A basis for the space of harmonic p-forms is furnished by

$$dx^{m_1} \wedge dx^{m_2} \wedge \ldots \wedge dx^{m_p}, \qquad m_1 < m_2 < \ldots < m_p \tag{3.23}$$

the number of such forms is $n!/(n-p)!p!$ hence

$b_0 = 1,\ b_1 = 2,\ b_2 = 1$

$$\chi(T^n) = \sum_{p=0}^{n} (-1)^p \frac{n!}{(n-p)!p!} = (1-1)^n = 0. \tag{3.24}$$

Product Manifolds

If α_1 is harmonic on a manifold M_1 and α_2 is harmonic on a manifold M_2 then $\alpha_1 \wedge \alpha_2$ is harmonic on $M_1 \times M_2$

$$d(\alpha_1 \wedge \alpha_2) = d\alpha_1 \wedge \alpha_2 \pm \alpha_1 \wedge d\alpha_2 = 0 \tag{3.25}$$

$$d^\dagger(\alpha_1 \wedge \alpha_2) = 0. \tag{3.26}$$

The set of all such forms $\alpha_p \wedge \beta_q$ with $p + q = k$ furnishes a basis for the harmonic k-forms on $M_1 \times M_2$ hence

$$b_k(M_1 \times M_2) = \sum_{p+q=k} b_p(M_1) b_q(M_2) \tag{3.27}$$

This is the **Künneth formula.**

The Euler characteristic has the useful property that it is multiplicative for product manifolds

$$\chi(M_1 \times M_2) = \sum_{k=0}^{n_1+n_2} \sum_{p+q=k} (-1)^{p+q} b_p(M_1) b_q(M_2) \tag{3.28}$$

$$= \sum_{p=0}^{n_1} (-1)^p b_p(M_1) \sum_{q=0}^{n_2} (-1)^q b_q(M_2),$$

hence

$$\chi(M_1 \times M_2) = \chi(M_1)\chi(M_2). \tag{3.29}$$

Poincaré Duality

A p-form ω is harmonic if and only if $d\omega = d^\dagger\omega = 0$. We also know that

$$d^\dagger\omega = (-1)^{p(n-p+1)} {*d*\omega} \tag{3.30}$$

hence

$$*d^\dagger\omega = (-1)^p d{*\omega} \tag{3.31}$$

and

$$*d\omega = (-1)^{p(n-p+1)} d^\dagger{*\omega}. \tag{3.32}$$

Since $*$ is invertible ($** = \pm 1$) we learn that ω is harmonic if and only if $*\omega$ is harmonic. It follows that the spaces H^p and H^{n-p} are isomorphic. Thus,

$$b_p = b_{n-p} \tag{3.33}$$

Consequently, χ vanishes for spaces of odd dimension.

As a consequence of deRham's Theorems we have the following result

Theorem: Given any p-cycle a there exists an $(n-p)$-form α, the **Poincaré dual** of a, such that

$$\int_a \omega = \int_M \alpha \wedge \omega \tag{3.34}$$

for any closed p-form ω.

We know from deRham's Theorems that, given a basis $\{z^i\}$ for H_p there exists a dual basis $\{\omega_i\}$ for H^p such that

$$\int_{z^i} \omega_j = \delta^i{}_j. \tag{3.35}$$

We will now show that there also exists a basis $\{\omega^i\}$ for H^{n-p} with the property that

$$\int_M \omega^i \wedge \omega_j = \delta^i{}_j. \tag{3.36}$$

Let $\{\nu^i\}$ be some basis for H^{n-p} and set

$$\int_M \nu^i \wedge \omega_j = m^i{}_j. \tag{3.37}$$

Clearly it is sufficient to show that $m^i{}_j$ is invertible since then $\{(m^{-1})^i{}_j \nu^j\}$ is the desired basis. Let us suppose that, on the contrary, $m^i{}_j$ is not invertible. Then there exists a nonzero vector k^j which is a null vector of $m^i{}_j$

$$m^i{}_j k^j = 0. \tag{3.38}$$

Setting $k = k^j \omega_j$ we have

$$\int_M \nu^i \wedge k = 0 \quad , \quad i = 1, \ldots, b_p. \tag{3.39}$$

But $*k$ is in H^{n-p} and is therefore a linear combination of the ν^i. Thus

$$(k,k) = \int_M *k \wedge k = 0 \tag{3.40}$$

which is impossible if k does not vanish.

Returning now to the original problem let

$$a = a_i z^i \quad \text{and} \quad \omega = \nu^i \omega_i \tag{3.41}$$

and define

$$\alpha = a_i \omega^i \tag{3.42}$$

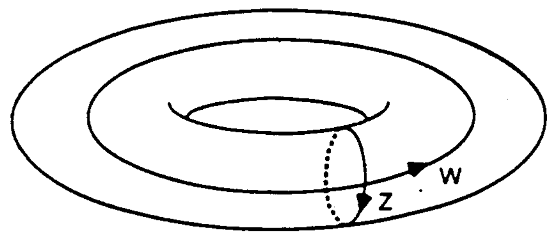
then (3.34) is satisfied in virtue of the fact that both the left and right hand sides of the equation are equal to $a_i \nu^i$.

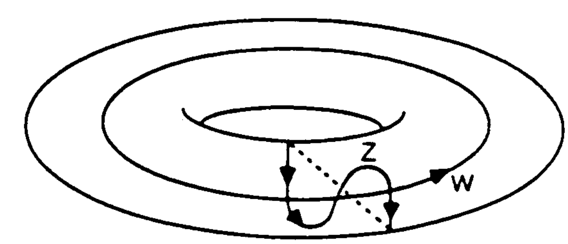
Since ω is closed α is defined up to an exact form. It is sometimes useful to think of α as differing from an exact form by a distribution whose support is concentrated on the cycle a.

Intersection Numbers

Let us return to T^2 and the two basic cycles z and w.

Two cycles that are homologous to z and w, respectively, have the property that they always intersect at least once. The figure shows that it is possible for the two cycles to intersect more than once.





However, by adopting a sign convention we can count two of the intersections as positive and one as negative so that the net intersection number is one for any two cycles that are cohomologous to z and w.

Initially we shall consider only cycles that have transverse intersection.

Definition: Two cycles a and b have **transverse intersection** if at each point p that they have in common the tangent spaces $T_p(a), T_p(b)$ have no vector in common.

Let M be an n-dimensional manifold and let a and b be two cycles in M of complimentary dimension (*i.e.* such that $\dim a + \dim b = n$) which intersect transversely. At a point $p \in a \cap b, T_p(a)$ admits an oriented basis $(u_1, \ldots u_k)$, say, and $T_p(b)$ admits an oriented basis $(v_{k+1}, \ldots, v_n)$. If the combined set $(u_1, \ldots u_k, v_{k+1}, \ldots, v_n)$ forms an oriented basis for $T_p(M)$ then we define $i_p(a, \mathrm{b})$, the intersection number of a and b at p, to be $+1$. If they do not form an oriented basis then we define $i_p(a, b)$ to be -1.

Definition: The **intersection number**, $\#(a, b)$, of a and b is defined by

$$\#(a, b) = \sum_{p \in a \cap b} i_p(a, b). \tag{3.43}$$

As is easily seen, the intersection number $\#(a, b)$ vanishes if either a or b is a boundary. It follows that $\#(a, b)$ depends only on the homology classes of a and b. The intersection number can also be expressed in terms of the Poincaré duals α and β of a and b.

$$\#(a, b) = \int_M \alpha \wedge \beta \tag{3.44}$$

which expresses the fact that we may think of α and β as being basically distributions that are concentrated on a and b.

IV. RIEMANNIAN MANIFOLDS

This section is a review of the formalism of curvature expressed in the language of differential forms. It is included largely to establish conventions.

Let $e^a, a = 1, \ldots, n$, be a basis of one-forms

$$e^a = e^a{}_m dx^m \tag{4.1}$$

and let e_a be the reciprocal basis of vectors

$$e_a = e_a{}^m \frac{\partial}{\partial x^m} \tag{4.2}$$

such that

$$< e^a, e_b >= \delta^a_b \tag{4.3}$$

or equivalently

$$(e_a{}^m) = (e^a{}_m)^{-1} \tag{4.4}$$

For a physicist the formalism of curvature begins with a discussion of covariant differentiation. Recall that the connection coefficients Γ^a_{bc} are defined by the relation

$$\nabla_b e_c = \Gamma_{bc}{}^a e_a \tag{4.5}$$

where ∇_b is short for $e_b{}^m \nabla_m$. (If the vectors e_a correspond to a coordinate basis then the connection coefficients are the more usual connection coefficients $\Gamma_{nr}^{\ \ m}$.)

Equivalent to this is the relation

$$\Gamma_{bc}{}^a =< e^a, \nabla_b e_c >= - < \nabla_b e^a, e_c > \tag{4.6}$$

it follows that the connection coefficients could be equivalently defined by the action of the covariant derivative on a basis of one-forms

$$\nabla_b e^a = -\Gamma_{bc}{}^a e^c. \tag{4.7}$$

Writing out the derivative explicitly and contracting with $e^b{}_m$ we find

$$\partial_m e^a{}_n - \Gamma_{mn}{}^r e^a{}_r = -\Gamma_{bc}{}^a e^b{}_m e^c{}_n. \tag{4.8}$$

Contracting this equation with $dx^m \wedge dx^n$ and rearranging gives

$$de^a + \Gamma_{bc}{}^a e^b \wedge e^c = \Gamma_{mn}{}^r e^a{}_r dx^m \wedge dx^n. \tag{4.9}$$

Recall that a connection can be chosen in many ways. It is usual to restrict this freedom by demanding (i) that the connection be metric compatible *i.e.* that the covariant derivative of the metric vanish and (ii) that the connection coefficients $\Gamma_{mn}{}^r$ be symmetric in their lower indices. These two requirements uniquely determine the connection to be the Christoffel connection $\{{}_{mn}^{\ r}\}$. In the following, however, demanding the symmetry of the connection coefficients turns out not to be useful so we shall not impose this requirement. Hence we shall allow the possibility that the connection have non-vanishing torsion

$$T_{mn}{}^{r} = 2\Gamma_{[mn]}{}^{r}. \tag{4.10}$$

We shall, however, continue to demand that the connection be metric compatible. We rewrite equation (4.9) in the form

$$de^{a} + \omega^{a}{}_{c} \wedge e^{c} = T^{a} \tag{4.11}$$

where

$$\omega^{a}{}_{c} = \Gamma_{bc}{}^{a} e^{b} \tag{4.12}$$

are the **connection one-forms** and

$$\begin{aligned} T^{a} &= \frac{1}{2} T_{bc}{}^{a} e^{b} \wedge e^{c} \\ &= \frac{1}{2} T_{mn}{}^{r} e^{a}{}_{r} dx^{m} \wedge dx^{n} \end{aligned} \tag{4.13}$$

is the **torsion two-form.**

Exercise : Verify that $\omega^{a}{}_{b} = (e^{a}{}_{k} \nabla_{m} e_{b}{}^{k}) dx^{m}$. ◇

The **curvature tensor** is defined such that for an arbitrary vector V^{k},

$$[\nabla_{m}, \nabla_{n}] V^{k} + T_{mn}{}^{\ell} \nabla_{\ell} V^{k} = R_{mn}{}^{k}{}_{\ell} V^{\ell}. \tag{4.14}$$

By writing out the covariant derivatives we have the explicit expression

$$R_{mn}{}^{k}{}_{\ell} = \partial_{m} \Gamma_{n\ell}{}^{k} - \partial_{n} \Gamma_{m\ell}{}^{k} + \Gamma_{mr}{}^{k} \Gamma_{n\ell}{}^{r} - \Gamma_{nr}{}^{k} \Gamma_{m\ell}{}^{r}. \tag{4.15}$$

The **curvature two – form** $R^{a}{}_{b}$ is defined in terms of the curvature tensor by the expression

$$\begin{aligned} R^{a}{}_{b} &= \frac{1}{2} R_{mn}{}^{k}{}_{\ell} e^{a}{}_{k} e_{b}{}^{\ell} dx^{m} \wedge dx^{n} \\ &= \left(\partial_{m} \Gamma_{n\ell}{}^{k} + \Gamma_{mr}{}^{k} \Gamma_{n\ell}{}^{r} \right) e^{a}{}_{k} e_{b}{}^{\ell} dx^{m} \wedge dx^{n}. \end{aligned} \tag{4.16}$$

A little algebra shows that this relation may be rewritten in the form

$$R^{a}{}_{b} = d\omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b}. \tag{4.17}$$

In the mathematical literature it is usual to adopt (4.11) and (4.17) as definitions. Note that care is required to avoid confusion between the curvature two-form and the Ricci tensor. The curvature two-form satisfies a differential identity that follows directly from (4.17). It is easy to check that

$$dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0 \tag{4.18}$$

this is known as the **Bianchi identity**.

Exercise: Check that this is equivalent to the differential Bianchi identity

$$R_{mn[pq;r]} = 0.$$

◇

e^a and $R^a{}_b$ are examples of forms that carry tangent space indices. It is useful to define a covariant derivative that acts on such objects.

If ν^a is a set of differential forms labelled by a tangent space index a, we define

$$D\nu^a = d\nu^a + \omega^a{}_b \wedge \nu^b. \tag{4.19}$$

If $\nu^a{}_b$ is a set of differential forms labelled by tangent space indices a and b then we define

$$D\nu^a{}_b = d\nu^a{}_b + \omega^a{}_c \wedge \nu^c{}_b - \omega^c{}_b \wedge \nu^a{}_c. \tag{4.20}$$

The definition extends to objects with any number of upper and lower indices in a familiar way by including a term with a positive sign for every upper index and a term with a negative sign for every lower index. With this notation we find, by definition of $\omega^a{}_b$,

$$De^a = T^a \tag{4.21}$$

while the Bianchi identity becomes

$$DR^a{}_b = 0. \tag{4.22}$$

Exercise : Show that under a change of frame

$$e^a \to e'^a = \Phi^a{}_b e^b$$

the new connection is related to the previous one by

$$\omega'^a{}_b = \Phi^a{}_c \omega^c{}_d (\Phi^{-1})^d{}_b + \Phi^a{}_c (d\Phi^{-1})^c{}_b$$

and the new curvature two-form is given by

$$R'^a{}_b = \Phi^a{}_c (\Phi^{-1})^d{}_b R^c{}_d.$$

Show also that the covariant derivative of a matrix of forms $\nu^a{}_b$ transforms according to the rule

$$(D\nu)'^a{}_b = \Phi^a{}_c (D\nu)^c{}_d (\Phi^{-1})^d{}_b.$$

◇

V. COMPLEX STRUCTURES

Let M be an n-dimensional complex manifold and let $\{z^\mu\}$ be local coordinates on a coordinate neighborhood U. We define a mixed tensor $J_m{}^n$ by

$$J = idz^\mu \otimes \frac{\partial}{\partial z^\mu} - idz^{\bar\mu} \otimes \frac{\partial}{\partial z^{\bar\mu}} \tag{5.1}$$

(where $z^{\bar\mu}$ is shorthand for $\overline{z}^\mu$). In the following we shall refer to complex coordinates $z^\mu, \mu = 1, \ldots, n$ and to real coordinates $x^m, m = 1, \ldots, 2n$. The tensor $J_m{}^n$ is called an **almost complex structure**.

Theorem : J enjoys the following properties:

(i) J is a tensor *i.e.* the right-hand side of the definition is independent of choice of coordinates.

(ii) J is real.

(iii) $J_m{}^n J_n{}^p = -\delta_m^p$.

To prove (i) we simply recall from the definition of a complex manifold that wherever there is a non-empty overlap $U \cap V$, the respective coordinates z^μ , w^ν are analytic functions of each other. Thus by the chain rule

$$dz^\mu \otimes \frac{\partial}{\partial z^\mu} = \frac{\partial z^\mu}{\partial w^\nu}\frac{\partial w^\rho}{\partial z^\mu} dw^\nu \otimes \frac{\partial}{\partial w^\rho} = dw^\mu \otimes \frac{\partial}{\partial w^\mu} \tag{5.2}$$

hence

$$J = idw^\mu \otimes \frac{\partial}{\partial w^\mu} - idw^{\bar\mu} \otimes \frac{\partial}{\partial w^{\bar\mu}}. \tag{5.3}$$

Property (ii) is trivial since $\bar{J} = J$ but can be shown also by taking $z^\mu = x^\mu + iy^\mu$ so

$$J = dx^\mu \otimes \frac{\partial}{\partial y^\mu} - dy^\mu \otimes \frac{\partial}{\partial x^\mu} \tag{5.4}$$

which is manifestly real.

(iii) follows from the fact that

$$J_\mu{}^\nu = i\delta_\mu{}^\nu \ , \ J_{\bar\mu}{}^{\bar\nu} = -i\delta_{\bar\mu}{}^{\bar\nu} \ , \ J_\mu{}^{\bar\nu} = 0 \ , \ J_{\bar\mu}{}^\nu = 0. \tag{5.5}$$

Thus in a complex basis

$$J_\cdot{}^\cdot = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{5.6}$$

while in the real basis (x^μ, y^μ)

$$J_\cdot^{\cdot} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.7}$$

In either basis we see that J squares to minus the identity.

The burden of the theorem is that every complex manifold admits a *globally defined* tensor which squares to minus the identity. Local existance is of course trivial since any $2n$-dimensional real manifold looks locally like C^n.

With the aid of the almost complex structure $J_m{}^n$ we are able to define two projection tensors

$$P_m{}^n = \frac{1}{2}\left(\delta_m{}^n - iJ_m{}^n\right), \quad Q_m{}^n = \frac{1}{2}\left(\delta_m{}^n + iJ_m{}^n\right) \tag{5.8}$$

It is straightforward to verify, using a matrix notation, that

$$P^2 = P, \qquad Q^2 = Q, \qquad PQ = 0, \qquad P + Q = 1. \tag{5.9}$$

In a complex basis we have

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.10}$$

and

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.11}$$

The significance of $P_\cdot^{\cdot}$ and $Q_\cdot^{\cdot}$ is that they can be used to project out the holomorphic and antiholomorphic components of tensors. Consider, for example, a one-form

$$P_m{}^n u_n dx^m \tag{5.12}$$

we find that in terms of complex coordinates

$$P_m{}^n u_n dx^m = u_\mu dz^\mu. \tag{5.13}$$

Definition : A complex manifold is **Hermitean** if it is endowed with a metric of the form

$$dx^2 = g_{\mu\bar\nu} dz^\mu dz^{\bar\nu}. \tag{5.14}$$

The most general metric has also pure parts with two holomorphic and two antiholomorphic indices

$$ds^2 = g_{\mu\bar\nu} dz^\mu dz^{\bar\nu} + g_{\mu\nu} dz^\mu dz^\nu + g_{\bar\mu\bar\nu} dz^{\bar\mu} dz^{\bar\nu}. \tag{5.15}$$

Hermiticity is a *restriction on the metric and not on the manifold.* For if a manifold admits any metric h_{mn} then it also admits the metric

$$g_{mn} = \frac{1}{2}\left(h_{mn} + J_m{}^k J_n{}^\ell h_{k\ell}\right) \tag{5.16}$$

[illegible] p. 235ff
Kähler metric $g = \frac{1}{2}\left(g_{i\bar j}\, \lambda^i \otimes \bar\lambda^j + g_{\bar i j}\, \bar\lambda^i \otimes \lambda^j\right)$
$g_{i\bar j} = g_{\bar j i}, \quad \bar g_{i\bar j} = g_{j\bar i}$

Kähler form
$\Phi = -\frac{i}{2} g_{i\bar j}\left(\lambda^i \otimes \bar\lambda^j - \bar\lambda^j \otimes \lambda^i\right)$
$\equiv \frac{i}{2} g_{i\bar j}\, \lambda^i \wedge \bar\lambda^j$

which is positive definite if h_{mn} is and is moreover hermitean.

If g_{mn} is a hermitean metric it satisfies

$$g_{mn} = J_m{}^k J_n{}^\ell g_{k\ell}. \tag{5.17}$$

By multiplying this equation by $J_k{}^m$ and writing

$$J_{mn} = J_m{}^k g_{kn} \tag{5.18}$$

we find that J_{kn} is skew symmetric

$$J_{mn} = -J_{nm}. \tag{5.19}$$

Thus on a hermitean manifold the almost complex structure defines a natural two-form.

Almost Complex Manifolds

We have seen that a complex manifold always admits a globally defined mixed tensor $J_m{}^n$ which squares to minus the identity. It is natural to inquire to what extent there is a converse. If a real $2n$-dimensional manifold M admits a globally defined tensor $J_m{}^n$ which squares to minus the identity is M then necessarily a complex manifold?

Definition : If a real manifold M admits a globally defined mixed tensor $J_m{}^n$ with the property

$$J_m{}^n J_n{}^k = -\delta_m{}^k \tag{5.20}$$

then M is an **almost complex manifold.**

Definition : If an addition the metric of the manifold has the property that

$$g_{mn} = J_m{}^k J_n{}^\ell g_{k\ell} \tag{5.21}$$

which is equivalent to

$$J_{mn} = -J_{nm} \tag{5.22}$$

then the metric is hermitean with respect to J and M is an **almost hermitean manifold.**

Even if M is only an almost complex manifold we can still define the projection tensors

$$P_m{}^n = \frac{1}{2}(\delta_m{}^n - iJ_m{}^n), \quad Q_m{}^n = \frac{1}{2}(\delta_m{}^n + iJ_m{}^n). \tag{5.23}$$

These tensors permit a refinement of the exterior calculus. Given, say, a three-form ω_{mnr} we can define projected components

$$\begin{aligned}
\omega^{(3,0)}_{mnr} &= P_m{}^i P_n{}^j P_r{}^k \omega_{ijk} \\
\omega^{(2,1)}_{mnr} &= 3P_m{}^i P_n{}^j Q_r{}^k \omega_{ijk} \\
\omega^{(1,2)}_{mnr} &= 3P_m{}^i Q^j_n Q^k_r \omega_{ijk} \\
\omega^{(0,3)}_{mnr} &= Q_m{}^i Q^j_n Q^k_r \omega_{ijk}
\end{aligned} \tag{5.24}$$

and the following decomposition of ω_{mnr}, according to type, is unique

$$\omega_{mnr} = \omega^{(3,0)}_{mnr} + \omega^{(2,1)}_{mnr} + \omega^{(1,2)}_{mnr} + \omega^{(0,3)}_{mnr}. \tag{5.25}$$

A k-form can be decomposed in an analogous way

$$\omega = \sum_{p+q=k} \omega^{(p,q)}. \tag{5.26}$$

A form of definite type (p,q) is called a (p,q)-form. As is straightforward to verify that the exterior derivative acting on a (p,q)-form yields a linear combination of four forms of different type

$$d\omega^{(p,q)} = (d\omega)^{(p-1,q+2)} + (d\omega)^{(p,q+1)} + (d\omega)^{(p+1,q)} + (d\omega)^{(p+2,q-1)}. \tag{5.27}$$

If J is in fact a complex structure then, the two terms at the ends of this expression $d\omega^{(p-1,q+2)}$ and $d\omega^{(p+2,q-1)}$ are absent.

We define operators ∂ and $\bar\partial$ by

$$\begin{aligned} \partial\omega^{(p,q)} &= (d\omega)^{(p+1,q)} \\ \bar\partial\omega^{(p,q)} &= (d\omega)^{(p,q+1)} \end{aligned} \tag{5.28}$$

and we can think of ∂ and $\bar\partial$ as the $(1,0)$ and $(0,1)$ parts of d.

Let us return now to our question, when is an almost complex structure a complex structure? Consider the projection

$$\theta^m = P_n{}^m dx^n \tag{5.29}$$

of the coordinate differentials. If M were a complex manifold then the θ^m would be expressible in terms of complex coordinates z^μ in the form

$$\theta^m = \theta_\mu{}^m dz^\mu. \tag{5.30}$$

Our question divides into a local question (i) do there exist locally coordinates z^μ such that

$$P_n{}^m dx^n = \theta_\mu{}^m dz^\mu \ ? \tag{5.31}$$

and a global question (ii) given a covering of M by local coordinate neighborhoods such that (i) is true, are the coordinates z and w, say, in two coordinate patches that have non-empty intersection holomorphic functions of each other?

To answer these questions it is useful to introduce the **Niejenhuis tensor**

$$N_{ij}{}^k = J_{[i}{}^k{}_{;j]} - J_{[i}{}^p J_{j]}{}^q J_p{}^k{}_{;q}. \tag{5.32}$$

Note that the Niejenhuis tensor does not depend on the choice of metric since the Christoffel connections cancel so, in fact, the tensor could have been defined with partial derivatives.

Theorem : (*)

(i) An almost complex structure $J_m^{\ n}$ is a complex structure if and only if the associated Niejenhuis tensor vanishes.

(ii) If the Niejenhuis tensor vanishes then there exists a holomorphic atlas such that

$$J_\mu{}^\nu = i\delta_\mu{}^\nu \quad , \quad J_{\bar\mu}{}^{\bar\nu} = -i\delta_{\bar\mu}{}^{\bar\nu} \quad , \quad J_\mu{}^{\bar\nu} = 0 \quad , \quad J_{\bar\mu}{}^\nu = 0. \tag{5.33}$$

Let us seek a set of complex coordinates $z^\mu = z^\mu(x)$. Then

$$dz^\mu = \frac{\partial z^\mu}{\partial x^j} dx^j \tag{5.34}$$

inserting unity in the guise of $(P_\cdot^{\ \cdot} + Q_\cdot^{\ \cdot})$ on right-hand side of this equation we have

$$dz^\mu = \frac{\partial z^\mu}{\partial x^j} P_k^{\ j} dx^k + \frac{\partial z^\mu}{\partial x^j} Q_k^{\ j} dx^k. \tag{5.35}$$

The left-hand side of this equation is a $(1,0)$-form as is the first term on the right hand side. The second term on the right-hand side is a $(0,1)$-form and so must vanish.

$$\frac{\partial z^\mu}{\partial x^j} Q_k{}^j dx^k = 0. \tag{5.36}$$

We regard this equation as a differential equation for the complex coordinates z^μ. A necessary and sufficient condition for this equation to be integrable is that the equation

$$\frac{\partial z^\mu}{\partial x^j} Q_k{}^j = 0 \tag{5.37}$$

should be integrable. Acting on this equation with $Q_m{}^\ell \frac{\partial}{\partial x^\ell}$ we find

$$\frac{\partial^2 z^\mu}{\partial x^j \partial x^\ell} Q_k{}^j Q_m{}^\ell + \frac{\partial z^\mu}{\partial x^j} Q_{k,\ell}^{\ j} Q_m{}^\ell = 0 \tag{5.38}$$

and taking the skew-symmetric part with respect to k and m the integrability condition becomes

$$\frac{\partial z^\mu}{\partial x^j} Q_{[k,|\ell|}^{\ j} Q_{m]}^{\ \ell} = 0. \tag{5.39}$$

Now again insert unity in the guise of $P_\cdot^{\ \cdot} + Q_\cdot^{\ \cdot}$ and use (5.37) again

$$\frac{\partial z^\mu}{\partial x^j} P_i{}^j Q_{[k,|\ell|}^{\ i} Q_{m]}^{\ \ell} = 0 \tag{5.40}$$

(*) The proof presented here only works in the real analytic case (cf. Chern or Kobayashi–Nomizu, Vol II, App 8). In the C^∞ case it is difficult (Newlander–Nirenberg)

This equation overconstrains $\frac{\partial z^\mu}{\partial \bar{z}^j}$ unless

$$P_i{}^j Q_{[k,|\ell|}^{\;i} Q_{m]}^{\;\ell} = 0. \tag{5.41}$$

Finally, a little algebra shows that both the real and imaginary parts of this equation are equivalent to the condition

$$N_{km}{}^j = 0. \tag{5.42}$$

To summarize so far: complex coordinates z^μ exist locally if and only if the Niejenhuis tensor vanishes. It is necessary also to show that where two coordinate neighborhoods overlap that the respective complex coordinates are holomorphic functions of each other. Let us suppose that z^μ and w^ν are two such sets of coordinates. A consequence of the vanishing of the Niejenhuis tensor is that the projection $P_n dx^n$ is a $(1,0)$-form. Thus for some functions $\theta^m{}_\mu$ and $\varphi^m{}_\mu$

$$\theta^m{}_\mu dz^\mu = \varphi^m{}_\mu dw^\mu. \tag{5.43}$$

It follows that $\frac{\partial w^\mu}{\partial \bar{z}^\sigma}$ vanishes.

As for part (ii) of the theorem we return to (5.37) which we may write in the form

$$\frac{\partial z^\mu}{\partial x^j} + iJ_j{}^k \frac{\partial z^\mu}{\partial x^k} = 0. \tag{5.44}$$

Note that in this form it is evident that (5.37) is just an n-dimensional version of the Cauchy-Riemann equations. On contracting the equation with $dx^j \otimes \frac{\partial}{\partial z^\mu}$, we have

$$J_j{}^k dx^j \frac{\partial z^\mu}{\partial x^k} \frac{\partial}{\partial z^\mu} = i dz^\mu \frac{\partial}{\partial z^\mu}. \tag{5.45}$$

If we complex conjugate the equation and subtract we find

$$J_j{}^k dx^j \left(\frac{\partial z^\mu}{\partial x^k} \frac{\partial}{\partial z^\mu} + \frac{\partial z^{\bar\mu}}{\partial x^k} \frac{\partial}{\partial z^{\bar\mu}} \right) = i dz^\mu \frac{\partial}{\partial z^\mu} - i dz^{\bar\mu} \frac{\partial}{\partial z^{\bar\mu}} \tag{5.46}$$

or

$$J_j^k dx^j \frac{\partial}{\partial x^k} = i dz^\mu \frac{\partial}{\partial z^\mu} - i dz^{\bar\mu} \frac{\partial}{\partial z^{\bar\mu}} \tag{5.47}$$

which is the statement that was to be proved.

The most celebrated example of an almost complex manifold which is not a complex manifold is S^6. On S^6 there is a naturally defined almost complex structure $J_m{}^n$ related to the octonians. The almost complex structure squares to minus the identity but its Niejenhuis tensor does not vanish. Proving that S^6 is not a complex manifold was a long standing problem because, although it is easy to show that the almost complex structure that derives from the octonians is non-integrable *i.e.* it has a non-zero Niejenhuis tensor, it is much harder to prove that there does not exist some other almost complex structure which is integrable.

Exercise: Adopting (5.28) as the definition of ∂ for an almost complex manifold show directly that the condition for ∂ to square to zero is that the Niejenhuis tensor vanishes.$\diamond$

VI. COVARIANT DERIVATES AND CURVATURE TENSORS ON A HERMITEAN MANIFOLD

Recall that the Christoffel connection $\{{}^{r}_{mn}\}$ is uniquely determined by two requirements. The first is that the metric be covariantly constant and the second is that the connection be symmetric. When dealing with a complex manifold it is natural to require instead that the complex structure $J_m{}^n$ be covariantly constant as well as the metric. These two conditions do not fully specify the connection. A unique connection is singled out by also placing a requirement on the torsion $\Gamma_{[mn]}{}^r$.

Theorem : On a hermitean manifold there is a unique connection called the **hermitean connection** with the properties:

(i) The covariant derivative of the metric vanishes

(ii) The covariant derivative of the complex structure vanishes

(iii) The torsion $\Gamma_{[mn]}{}^r$ is pure in its lower indices.

If g_{mn} and $J_m{}^n$ are covariantly constant, then so are the projection tensors $P_m{}^n$ and $Q_m{}^n$. Consider the equation

$$\nabla_k P_m{}^n = 0. \tag{6.1}$$

The content of this equation is exhausted by taking (m,n) equal to (μ,ν) and $(\mu,\bar{\nu})$. The first of these choices does not yield any information while the second leads to the condition

$$\Gamma_{k\mu}{}^{\bar{\nu}} = 0 \tag{6.2}$$

where k can be replaced by either κ or $\bar{\kappa}$. This leads to the conditions

$$\Gamma_{\kappa\mu}{}^{\bar{\nu}} = 0, \qquad \Gamma_{\kappa\bar{\mu}}{}^{\nu} = 0 \tag{6.3}$$

Consideration of the derivative of $Q_m{}^n$ leads to nothing new. Condition (iii) provides further information on the mixed components of the connection since $\Gamma_{[\kappa\bar{\mu}]}{}^{\nu}$ vanishes we have also

$$\Gamma_{\bar{\mu}\kappa}{}^{\nu} = 0 \tag{6.4}$$

The relations we have together with their complex conjugates imply the vanishing of all the mixed components of the connection. In other words, we learn that the hermitean connection is pure in its indices.

To show existence and uniqueness, we solve for the connection in terms of the derivatives of the metric. Consider the equation

$$\nabla_m g_{nr} = \partial_m g_{nr} - \Gamma_{mn}{}^k g_{kr} - \Gamma_{mr}{}^k g_{nk} = 0 \tag{6.5}$$

and take $(m,n,r) = (\mu,\nu,\bar{\rho})$, we find

$$\partial_\mu g_{\nu\bar{\rho}} - \Gamma_{\mu\nu}{}^{\kappa} g_{\kappa\bar{\rho}} = 0 \tag{6.6}$$

and hence

$$\Gamma_{\mu\nu}{}^{\kappa} = g^{\kappa\bar{\rho}}\partial_\mu g_{\nu\bar{\rho}}. \tag{6.7}$$

It is straightforward to check that the vanishing of $\nabla_m J_{nr}$ leads to the same relation.

Exercise : Let $\Gamma_{mn}{}^k$ be an arbitrary connection on a complex manifold. Show directly from the transformation properties of $\Gamma_{mn}{}^k$ under a holomorphic change of coordinates that the mixed components of the connection are tensors. ◊

The fact that the hermitean connection is pure in its indices and has the simple form (6.7) leads to a great simplification in the structure of the Riemann tensor. In fact we shall see that the only nonzero components are those that are mixed in both the first and last pairs of indices *i.e.* the only nonvanishing components are of the form

$$R_{\mu\bar{\nu}\rho\bar{\sigma}} \quad , \quad R_{\bar{\nu}\mu\rho\bar{\sigma}} \quad , \quad R_{\mu\bar{\nu}\bar{\sigma}\rho} \quad , \quad R_{\bar{\nu}\mu\bar{\sigma}\rho}. \tag{6.8}$$

To demonstrate this recall the explicit expression (4.15) for the Riemann tensor in terms of the connection

$$R_{mn}{}^k{}_\ell = \partial_m\Gamma_{n\ell}{}^k - \partial_n\Gamma_{m\ell}{}^k + \Gamma_{mr}{}^k\Gamma_{n\ell}{}^r - \Gamma_{nr}{}^k\Gamma_{m\ell}{}^r. \tag{6.9}$$

First take (k,ℓ) equal to $(\bar{\kappa},\lambda)$ we see that every term on the right-hand side vanishes owing to the fact that the connection is pure in its indices. Thus †

$$R_{mn\kappa\lambda} = 0 \quad , \quad conj. \tag{6.10}$$

Thus in order for the Riemann tensor to be nonvanishing the last pair of indices must be mixed. Consider now the components of the form $R_{\mu\nu}{}^{\bar{\kappa}}{}_{\bar{\lambda}}$ these also vanish in virtue of the fact that the connection is pure in its indices. Finally consider the components

$$R_{\mu\nu}{}^{\kappa}{}_{\lambda} = \partial_\mu\Gamma_{\nu\lambda}{}^{\kappa} + \Gamma_{\mu\rho}{}^{\kappa}\Gamma_{\nu\lambda}{}^{\rho} - (\mu \leftrightarrow \nu) \tag{6.11}$$

these are seen to vanish in virtue of relation (6.7). By again invoking the fact that the connection is pure in its indices we find the following expression for the non-vanishing components of the Riemann tensor

$$R_{\mu\bar{\nu}\bar{\rho}}{}^{\bar{\sigma}} = -R_{\bar{\nu}\mu\bar{\rho}}{}^{\bar{\sigma}} = \partial_\mu\Gamma_{\bar{\nu}\bar{\rho}}{}^{\bar{\sigma}}. \tag{6.12}$$

Associated with the Riemann tensor is the **Ricci-form**

$$\begin{aligned}\mathcal{R} &= \frac{1}{4}R_{mnk\ell}J^{k\ell}dx^m \wedge dx^n \\ &= iR_{\mu\bar{\nu}\bar{\rho}}{}^{\bar{\rho}}dz^\mu \wedge dz^{\bar{\nu}} \\ &= i\partial\bar{\partial}\log g^{\frac{1}{2}}\end{aligned} \tag{6.13}$$

† In the following we write conj. to denote that there is a similar quantity or relation which may be obtained by complex conjugation.

the last equality following from (6.12). Since $\partial\bar{\partial} = -\frac{1}{2}d(\partial - \bar{\partial})$ we see that the Ricci-form is always closed

$$d\mathcal{R} = 0. \tag{6.14}$$

At first it might appear that we have shown that $\mathcal{R}$ is also exact but this is not so in general since $g^{\frac{1}{2}}$ and hence $(\partial - \bar{\partial})\log g^{\frac{1}{2}}$ is not a coordinate scalar. Thus (6.13) should be regarded as a relation that holds within each coordinate patch $\mathbf{U}_j$.

Exercise: Show directly that $\mathcal{R}$ is globally defined even though $\log g^{\frac{1}{2}}$ is not. $\diamond$

The Ricci form defines a cohomology class

$$c_1 = \left[\frac{1}{2\pi}\mathcal{R}\right] \tag{6.15}$$

called the **first Chern class** and the factor of $\frac{1}{2\pi}$ reflects a convention concerning the general definition of the higher Chern classes to which we shall return later. The importance of c_1 is that it is an analytic invariant, that is, it is invariant under smooth changes of the complex structure of the manifold. Consider the effect of a small variation $g_{mn} \to g_{mn} + \delta g_{mn}$ in the metric of M. From (6.13) and the relation

$$\delta g^{\frac{1}{2}} = \frac{1}{2} g^{\frac{1}{2}} g^{mn} \delta g_{mn} \tag{6.16}$$

we have

$$\delta\mathcal{R} = i\partial\bar{\partial}(g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}) = -\frac{i}{2}d[(\partial - \bar{\partial})g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}]. \tag{6.17}$$

Now $g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$ *is* a coordinate scalar so the last equality shows that $\delta\mathcal{R}$ is exact even though $\mathcal{R}$ itself may not be. Thus a smooth variation of the metric changes $\mathcal{R}$ but not c_1.

VII. KÄHLER MANIFOLDS

We have seen that a hermitean manifold is endowed with a natural two-form

$$J = \frac{1}{2} J_{mn} dx^m \wedge dx^n. \tag{7.1}$$

Definition : A hermitean manifold is **Kähler** if the natural two-form is closed

$$dJ = 0. \tag{7.2}$$

On a Kähler manifold J is often called the **Kähler-form**.

All one-dimensional complex manifolds are Kähler since dJ would be a three-form and hence vanishes.

The fact that J is closed has profound consequences

$$\begin{aligned} dJ &= \partial J + \bar{\partial} J \\ &= i\partial_\kappa g_{\mu\bar{\nu}} dz^\kappa \wedge dz^\mu \wedge dz^{\bar{\nu}} - i\partial_{\bar{\rho}} g_{\mu\bar{\nu}} dz^\mu \wedge dz^{\bar{\rho}} \wedge dz^{\bar{\nu}}. \end{aligned} \tag{7.3}$$

The $(2,1)$ and $(1,2)$ parts of dJ must vanish separately, hence

$$\begin{aligned} \partial_\kappa g_{\mu\bar{\nu}} &= \partial_\mu g_{\kappa\bar{\nu}} \\ \partial_{\bar{\rho}} g_{\mu\bar{\nu}} &= \partial_{\bar{\nu}} g_{\mu\bar{\rho}}. \end{aligned} \tag{7.4}$$

We learn that on each coordinate neighborhood $\mathbf{U}_j$ there exists a real scalar φ_j, known as the **Kähler potential**, such that on $\mathbf{U}_j$

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} \varphi_j, \tag{7.5}$$

hence on $\mathbf{U}_j$

$$J = i\partial\bar{\partial}\varphi_j. \tag{7.6}$$

This equation is just a restatement of (7.4). It is important to note that the scalars φ_j cannot fit together on the overlaps $\mathbf{U}_j \cap \mathbf{U}_k$ to give a globally defined function if M is compact. The metric is, of course, globally defined. What happens is that on some non-trivial overlap $\mathbf{U}_j \cap \mathbf{U}_k$ the two Kähler potentials φ_j and φ_k are related by

$$\varphi_j = \varphi_k + f_{jk}(z) \tag{7.7}$$

with $f_{jk}(z)$ a holomorphic function. To see why this is so recall that

$$\partial\bar{\partial} = -\frac{1}{2} d(\partial - \bar{\partial}) \tag{7.8}$$

so that

$$J = -\frac{1}{2} d[(\partial - \bar{\partial})\varphi_j] \tag{7.9}$$

which, of course, ensures that $dJ = 0$. If $(\partial - \bar{\partial})\varphi_j$ were globally defined, J would also be exact. However, J cannot be exact. This is most easily seen by considering the identity

$$\begin{aligned}\underbrace{J \wedge J \wedge \ldots \wedge J}_{\text{n factors}} &= i^n g_{\mu_1 \bar{\nu}_1} \cdots g_{\mu_n \bar{\nu}_n} dz^{\mu_1} dz^{\bar{\nu}_1} \ldots dz^{\mu_n} dz^{\bar{\nu}_n} \\ &= i^n \epsilon^{\mu_1 \ldots \mu_n} \epsilon^{\bar{\nu}_1 \ldots \bar{\nu}_n} g_{\mu_1 \bar{\nu}_1} \cdots g_{\mu_n \bar{\nu}_n} dz^1 dz^{\bar{1}} \ldots dz^n dz^{\bar{n}} \\ &= i^n n! det(g_{\mu\bar{\nu}}) dz^1 dz^{\bar{1}} \ldots dz^n dz^{\bar{n}}.\end{aligned} \tag{7.10}$$

Now the metric has the form

$$g_{mn} = \begin{pmatrix} 0 & g_{\mu\bar{\nu}} \\ g_{\rho\bar{\sigma}} & 0 \end{pmatrix} \tag{7.11}$$

and $g_{\mu\bar{\nu}}$ is moreover hermitean

$$\overline{g_{\mu\bar{\nu}}} = g_{\nu\bar{\mu}} \tag{7.12}$$

it follows that $detg_{\mu\bar{\nu}}$ is proportional to $g^{\frac{1}{2}}$, the square root of $det(g_{mn})$.

We see that n-fold product $J \wedge \ldots \wedge J$ is proportional to the volume form. Thus, the integral

$$\int J \wedge \ldots \wedge J$$

is proportional to the volume of M. If J were exact, $J = dA$, we could replace one of the J's by dA and invoke Stoke's theorem to show that the integral is zero.

In virtue of the identities (7.4) we see that the hermitean connection whose only non-zero components are

$$\Gamma^{\kappa}_{\mu\nu} = g^{\kappa\bar{\sigma}} \partial_\mu g_{\nu\bar{\sigma}} \quad , \ conj. \tag{7.13}$$

is symmetric in its lower indices. Thus, for a Kähler manifold, the hermitean connection coincides with the Christoffel connection. A defining property of the hermitean connection is that J_{mn} has vanishing covariant derivatives. J_{mn} is covariantly constant. In particular, it has vanishing divergence

$$d^{\dagger} J = 0. \tag{7.14}$$

Thus J is harmonic.

The prime example of a Kähler manifold is P_n. As before, we may choose coordinate neighborhoods $\mathbf{U}_j$ such that one of the homogeneous coordinates z^j is non-zero and we can again use the

$$\zeta_j^m = \frac{z^m}{z^j} \tag{7.15}$$

as coordinates on $\mathbf{U}_j$.

Set

$$\varphi_j = \log\left(\sum_{m=1}^{n+1} |\zeta_j^m|^2\right). \tag{7.16}$$

On an overlap $\mathbf{U}_j \cap \mathbf{U}_k$ we have

$$\zeta_j^m = \frac{\zeta^m{}_k}{\zeta^j{}_k} \tag{7.17}$$

and hence

$$\varphi_j = \varphi_k - \log \zeta^j{}_k - \log \left(\overline{\zeta^j{}_k}\right) \tag{7.18}$$

this being so

$$\partial\bar{\partial}\varphi_j = \partial\bar{\partial}\varphi_k. \tag{7.19}$$

On P_n we choose a metric (the Fubini-Study metric)

$$g_{m\bar{n}} = \partial_m \partial_{\bar{n}} \varphi_j. \tag{7.20}$$

This is well-defined in virtue of the above and induces the Kähler form

$$J = i\partial\bar{\partial}\varphi_j. \tag{7.21}$$

In order to complete the demonstration that P_n is Kähler, it remains only to check that the metric we have introduced is positive definite.

Exercise : Show that on $\mathbf{U}_j$ the metric that follows from (7.16) takes the explicit form

$$g_{m\bar{n}} = \frac{1}{\sigma}\left(\delta_{m\bar{n}} - \frac{\zeta_m \zeta_{\bar{n}}}{\sigma}\right)$$

where

$$\sigma = 1 + \zeta^k \zeta_k$$

and we introduce a convention that will be useful later by writing $\zeta^m = \zeta_{\bar{m}}$ and $\zeta^{\bar{m}} = \zeta_m$. Check that this metric is positive definite. ◇

An example of a complex manifold that is not Kähler is $S^{2p+1} \times S^{2q+1}$ with $q > 1$ to exclude the case of a torus. A Kähler manifold has $b_2 \geq 1$ since the Kähler form is harmonic. However for the case of $S^{2p+1} \times S^{2q+1}$ b_2 is zero. This follows from the fact that a harmonic two-form on $S^{2p+1} \times S^{2q+1}$ would admit a decomposition into a sum of forms of the type

$$\alpha_2 \quad , \quad \alpha_1 \wedge \beta_1 \quad , \quad \beta_2$$

with α_2 a harmonic two-form on the first factor, α_1 and β_1 harmonic one-forms on their respective factors and β_2 a harmonic two-form on the second factor. By our earlier observations about the cohomology of spheres we know that α_2, β_1 and β_2 do not exist. Thus $b_2 = 0$.

Another complex manifold which is not Kähler is the Iwasawa manifold from Section 1. The Iwasawa manifold is subset of C_3 such that the points (z_1, z_2, z_3) and (z_1', z_2', z_3') are identified whenever

$$\begin{pmatrix} 1 & z_1' & z_2' \\ 0 & 1 & z_3' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m_1 & m_2 \\ 0 & 1 & m_3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.22}$$

for any set of complex integers $m_j = k_j + il_j$. Explicitly the identifications are

$$\begin{aligned} z_1' &= z_1 + m_1 \\ z_2' &= z_2 + m_1 z_3 + m_2 \, . \\ z_3' &= z_3 + m_3 \end{aligned} \tag{7.23}$$

An arbitrary point, thought of as a point in R_6, can always be brought by one of these identifications to a point in the unit hypercube. We can think of this manifold as a twisted torus. It is easy to see that there cannot exist a Kähler form for this manifold since a putative Kähler form would have to be

$$dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3 \tag{7.24}$$

which is not respected by the identifications.

The properties that the curvature tensor enjoys on a Kähler manifold follow from those on a Hermitean manifold. Additional simplifications follow from the fact that since the torsion now vanishes, the curvature tensor enjoys the familiar symmetries.

$$R_{mnrs} = R_{rsmn} \tag{7.25}$$

$$R_{m[nrs]} = 0 \tag{7.26}$$

Writing out the later identity with the indices $(\mu\bar{\nu}\rho\bar{\sigma})$ we find that $R_{\mu\bar{\nu}\rho\bar{\sigma}}$ is symmetric in its holomorphic and antiholomorphic indices

$$R_{\mu\bar{\nu}\rho\bar{\sigma}} = R_{\rho\bar{\nu}\mu\bar{\sigma}} = R_{\mu\bar{\sigma}\rho\bar{\nu}} \tag{7.27}$$

The Ricci tensor

$$R_{mn} = R^k{}_{mkn} \tag{7.28}$$

has only mixed components

$$R_{\mu\bar{\nu}} = g^{\lambda\bar{\kappa}} R_{\bar{\kappa}\mu\lambda\bar{\nu}} = -g^{\lambda\bar{\kappa}} R_{\mu\bar{\nu}\lambda\bar{\kappa}} \tag{7.29}$$

the last equality following from (7.27). From (6.13) we identify the components of the Ricci form as being those of the Ricci tensor, whence the name.

As we have seen, the Ricci form defines the first Chern class c_1. The utility of c_1 can be appreciated by considering the following question. Suppose there is given a Kähler manifold M; under what circumstances does M admit a Ricci-flat metric? Suppose g_{mn} is any metric on M and suppose there exists for M a Ricci-flat metric g'_{mn}. The Ricci-forms of g_{mn} and g'_{mn} both belong to c_1

$$\mathcal{R}(g) = \mathcal{R}(g') + dA \tag{7.30}$$

but, by hypothesis, g'_{mn} is a Ricci-flat metric so $R(g')$ vanishes, therefore $R(g)$ must be exact and c_1 must be the trivial cohomology class

$$c_1 = 0 \tag{7.31}$$

This is a necessary condition for M to admit a Ricci-flat metric.

Exercise : Show that the Ricci-form for P_n with the Fubini-Study metric satisfies

$$\mathcal{R} = -(n+1)J$$

where J is the Kähler form. $\diamond$

The exercise shows that the Ricci form for P_n with the Fubini-Study metric is harmonic so it is certainly not exact. c_1 is not trivial, hence P_n cannot admit a Ricci-flat metric. The necessity of the vanishing of c_1 for M to admit a Ricci-flat metric was noticed by Calabi who conjectured that c_1 is the only topological obstruction. In other words, if c_1 vanishes then there exists for M a Ricci-flat metric. This conjecture was a long standing problem until it was proved by Yau. We shall not attempt a proof here and will merely be content to state Yau's theorem.

Theorem: Given a complex manifold with $c_1 = 0$ and any Kähler metric g_{mn} with Kähler form J, then there exists a unique Ricci-flat metric g'_{mn} whose Kähler J' is in the same cohomology class as J. [illegible]

We have seen that the curvature tensor of a Kähler manifold has a remarkably simple structure. This structure and the role played by the Ricci-form are best understood in terms of the holonomy group.

The **holonomy group** $\mathcal{H}$ associated with a d-dimensional manifold M (not necessarily complex) is defined in the following way. Pick a point p of M and consider the effect of parallely transporting a vector $VJI6$ from p back to p around a closed curve C.

In general, the vector V' obtained by this process will no longer coincide with V. Let us suppose that C is parametrized by a proper length parameter s that takes values in the range [0,t] with $x(0) = x(t) = p$ then $V'^m = V^m(t)$ is obtained by solving the differential equation

parallel transport
$$\frac{dx^n}{ds}\nabla_n V^m(s) = 0 \tag{7.32}$$

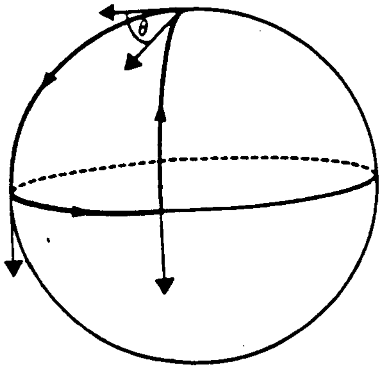
subject to the boundary condition that $V^m(0) = V^m$. Being a linear differential equation of first order the solution depends linearly on the boundary condition, hence

$$V'^m = S^m{}_n V^n \tag{7.33}$$

with $S^m{}_n$ a tensor that depends on M and on C but not on V^n. Since parallel transport preserves length the components $S^a{}_b$ of the tensor in an orthonormal frame form an orthogonal matrix. In this way a rotation matrix $S(C)$ can be associated with each curve

C that starts and ends at p. The set of all such $S(C)$ for all possible curves C form the holonomy group $\mathcal{H}$ of M. It is necessarily a subgroup of $O(d)$ and it is easy to show that $\mathcal{H}$ is independent of the point p that is chosen.

The holonomy group gives information about how curved the space is. A flat manifold has as its holonomy group just the trivial group consisting of the identity element only. S^2 has $O(2)$ as its holonomy group as can be seen by choosing the basepoint p to be the north pole and considering curves C(θ) that are spherical triangles that subtend an angle θ with the equator. After parallel transport around C(θ) the vector is rotated through an angle θ as in the figure.



More generally, the holomy group of S^d is $SO(d)$ for $d \geq 2$.

The holonomy group of an n-dimensional Kähler manifold is contained in $U(n)$ which is of course a subgroup of $O(2n)$. To see this note that since the connection coefficients are pure, a vector v that has only holomorphic components is carried into another such vector after parallel transport around a closed curve. Thus, if we choose a basis of vectors at p such that

$$e_\alpha = e_\alpha{}^\mu \frac{\partial}{\partial z^\mu} \tag{7.34}$$

$$g(e_\alpha, e_{\bar\beta}) = \delta_{\alpha\bar\beta} \tag{7.35}$$

then under parallel transport around a closed curve C the e_α will change according to

$$e_\alpha \to S_\alpha{}^\beta e_\beta \tag{7.36}$$

and eqn.(7.35) tells us that $(S_\alpha{}^\beta)$ is a unitary matrix.

Consider now the change induced in a vector V^k by parallel transport around an infinitesimal rectangle of area δa^{mn} with edges that are parallel to the vectors $\frac{\partial}{\partial x^m}$ and $\frac{\partial}{\partial x^n}$. It is a standard result that

$$V'^k = V^k + \delta a^{mn} R_{mn}{}^k{}_\ell V^\ell. \tag{7.37}$$

The matrices

$$\delta^k{}_\ell + \delta a^{mn} R_{mn}{}^k{}_\ell \tag{7.38}$$

are the elements of $\mathcal{H}$ that are infinitesimally close to the identity. We know that these matrices transform a vector which has only holomorphic components into another one such. Thus $R_{mn}{}^{k}{}_{\ell}$ is pure in the indices (k,ℓ). Lowering k, the conclusion is that $R_{mnk\ell}$ mixed in (k,ℓ). When this is combined with the symmetries of the curvature tensor we recover the fact that the only nonzero components of the curvature tensor are those of the form $R_{\mu\bar{\nu}\rho\bar{\sigma}}$.

The matrix $\delta a^{mn} R_{mn}{}^{k}{}_{\ell}$ is in the Lie algebra of $U(n)$. In a neighborhood of the identity

$$U(n) \cong SU(n) \times U(1) \tag{7.39}$$

the $U(1)$ being generated by the trace

$$\delta a^{mn} R_{mn}{}^{k}{}_{k} = -4\delta a^{\mu\bar{\nu}} R_{\mu\bar{\nu}} \tag{7.40}$$

the equality following from (7.29). Thus the $U(1)$ part of the holonomy is generated by the Ricci tensor. So if the manifold is both Kähler and Ricci-flat then the holonomy group is contained in $SU(n)$. Certainly this statement is true for the **local holonomy group,** which is the subgroup of the holonomy group associated with paths C that may be continuously shrunk to a point, however if the manifold is multiply connected then there are paths that cannot be continuously shrunk to a point. The statement that the holonomy group of a Ricci-flat Kähler manifold is contained in $SU(n)$ even when the manifold is multiply connected is nevertheless true though the proof is more involved. We will return to complete the proof later.

Exercise: Show, by explicit construction of the operators in terms of covariant derivatives and Riemann tensors, that on a Kähler manifold

$$\partial\partial^{\dagger} + \partial^{\dagger}\partial = \bar{\partial}\bar{\partial}^{\dagger} + \bar{\partial}^{\dagger}\bar{\partial} = \frac{1}{2}(dd^{\dagger} + d^{\dagger}d).$$

This result shows that deRham cohomology and $\bar{\partial}$-cohomology are equivalent. ◊

VIII CALABI-YAU MANIFOLDS

Definition: A Calabi-Yau manifold is a compact Kähler manifold of vanishing first Chern class.

The following is a very useful preliminary result.

Theorem: A Calabi-Yau manifold with nonzero Euler number has $b_{10} = 0$.

We have

$$b_{10} = b_{01} = \frac{1}{2} b_1 \tag{8.1}$$

and b_1 is a topological invariant so it suffices to establish the result for the Ricci-flat metric. Let $u_m dx^m$ be a harmonic one-form. The explicit form of the Hodge-deRham equation for one-forms is

$$-\nabla^k \nabla_k u_m + R_m{}^n u_n = 0. \tag{8.2}$$

Since the Ricci-form vanishes we have

$$\nabla^k \nabla_k u_m = 0. \tag{8.3}$$

We premultiply this equation by u_m integrate over M and integrate by parts to obtain

$$\int_M (\nabla_m u_n)(\nabla^m u^n) g^{\frac{1}{2}} d^6 x = 0. \tag{8.4}$$

Thus u_m is covariantly constant

$$\nabla_m u_n = 0. \tag{8.5}$$

A result that we do not prove here is that a vector field on a manifold of Euler number χ has at least $|\chi|$ zeros. Thus u^m must have a zero and being covariantly constant it must vanish identically.

A remarkable property of Calabi- Yau manifolds is the fact that they admit gauge covariantly constant spinors. We state this in the form of a theorem.

Theorem: A Calabi-Yau manifold admits a globally defined pair of spinors $\zeta, \bar{\zeta}$ of opposite chirality and a globally defined one-form $A = A_m dx^m$ such that

$$\begin{aligned} (\nabla_m - \frac{i}{2} A_m)\zeta &= 0 \\ (\nabla_m + \frac{i}{2} A_m)\bar{\zeta} &= 0 \end{aligned} \tag{8.6}$$

with A a potential for the Ricci-form

$$\mathcal{R} = dA. \tag{8.7}$$

Here $\bar{\zeta}$ is the complex conjugate (not a Dirac conjugate) of ζ.

The theorem is really an observation concerning the fact that the holonomy group of a Calabi-Yau manifold is the product $SU(n) \otimes U(1)$ and the covariant derivative is a gauge potential with respect to this group, the gauge potential being the spin connection. This is seen explicitly by recalling that the covariant derivative acts on an arbitrary spinor according to the rule

$$\nabla_m \eta = \partial_m \eta - \frac{1}{4}\omega_m{}^a{}_b \gamma_a{}^b \eta \tag{8.8}$$

where $\omega_m{}^a{}_b$ are the components of the connection one-form and $\gamma^{ab} = \gamma^{[a}\gamma^{b]}$ is a product of Dirac matrices. Note also that a spinor in six dimensions has eight components which transform in the $\mathbf{4} \oplus \bar{\mathbf{4}}$ of $O(6) \approx SU(4)$. The $\mathbf{4}$ and the $\bar{\mathbf{4}}$ have opposite chiralities. The $\mathbf{4}$ of $SU(4)$ decomposes with respect to $SU(3) \otimes U(1)$ as

$$\mathbf{4} = \mathbf{3}^{\mathbf{1}} \oplus \mathbf{1}^{-\mathbf{3}}. \tag{8.9}$$

Thus there is an $SU(3)$ singlet which is charged with respect to the $U(1)$. In (8.6) ζ is the $SU(3)$ singlet and A_m cancels the $U(1)$ part of the gauge potential contained in the covariant derivative so that the combined quantity

$$D_m = \nabla_m - \frac{i}{2}A_m \tag{8.10}$$

is the $SU(3)$ gauge covariant derivative. Equation (8.7) expresses the fact that the Ricci-form is the field strength of the $U(1)$ part of the holonomy. All that is required is to check the normalization of A_m. We will do this presently. First however we need to develop some of the properties of ζ and of the Dirac algebra.

On a six-dimensional manifold we may take the Dirac matrices γ_m , $m = 1, \ldots, 6$ to be hermitean and imaginary and the matrix γ that determines the chirality

$$\gamma = \frac{i}{6!}\epsilon_{mnpqrs}\gamma^m\gamma^n\gamma^p\gamma^q\gamma^r\gamma^s \tag{8.11}$$

is then also hermitean and imaginary. We choose ζ such that

$$\gamma\zeta = +\zeta. \tag{8.12}$$

Now $\zeta^\dagger\zeta$ is a constant on M in virtue of (8.6) so we will assume ζ to be normalized such that

$$\zeta^\dagger\zeta = 1. \tag{8.13}$$

Define

$$J_m{}^n = -i\zeta^\dagger \gamma_m{}^n \zeta \tag{8.14}$$

we see from (8.6) that $J_m{}^n$ is covariantly constant. It is straightforward also to show by means of Fierz identities that

$$J_m{}^n J_n{}^p = -\delta_m{}^p. \tag{8.15}$$

Thus we identify J_{mn} as the Kähler form of M.

Now the Dirac algebra on a Kähler manifold takes the form

$$\begin{gathered}\{\gamma^\mu, \gamma^\nu\} = 0 \quad , \quad \{\gamma^{\bar\mu}, \gamma^{\bar\nu}\} = 0 \\ \{\gamma^\mu, \gamma^{\bar\nu}\} = 2g^{\mu\bar\nu}.\end{gathered} \tag{8.16}$$

The Dirac algebra is the algebra of fermionic raising and lowering operators. We may think of the γ^μ, say, as raising operators. We shall now show that ζ is the state of highest weight and is annihilated by the γ^μ. Consider (8.14) which we write as

$$\begin{aligned} ig_{\mu\bar\nu} &= -\zeta^\dagger \gamma_{\mu\bar\nu}\zeta \\ &= -i\zeta^\dagger(\gamma_\mu\gamma_{\bar\nu} - g_{\mu\bar\nu})\zeta. \end{aligned} \tag{8.17}$$

It follows that

$$\begin{aligned} 0 &= \zeta^\dagger \gamma_\mu \gamma_{\bar\nu}\zeta \\ &= (\gamma_{\bar\mu}\zeta)^\dagger(\gamma_{\bar\nu}\zeta) \end{aligned} \tag{8.18}$$

by considering the case $\bar\mu = \bar\nu$ we see this is impossible unless

$$\gamma_{\bar\nu}\zeta = 0 \tag{8.19}$$

or equivalently

$$\gamma^\mu \zeta = 0. \tag{8.20}$$

We may act on ζ with lowering operators. Since the γ_μ anticommute among themselves we have

$$\begin{array}{ll} \zeta & \mathbf{1}^{-3} \\ \gamma_\mu\zeta & \bar{\mathbf{3}}^{-1} \\ \gamma_{\mu\nu}\zeta & \mathbf{3}^{1} \\ \gamma_{\mu\nu\rho}\zeta & \mathbf{1}^{3} \end{array} \tag{8.21}$$

a total of eight states. In the second column we give the $SU(3) \otimes U(1)$ representations in which these spinors transform. Note that these spinors are linearly independent since they transform differently under the holonomy group. As there are eight states an arbitrary spinor η can be decomposed in the form

$$\eta = \omega^{(0,0)}\zeta + \omega^{(0,1)}{}_{\bar\mu}\gamma^{\bar\mu}\zeta + \omega^{(0,2)}{}_{\bar\mu\bar\nu}\gamma^{\bar\mu\bar\nu}\zeta + \omega^{(0,3)}{}_{\bar\mu\bar\nu\bar\rho}\gamma^{\bar\mu\bar\nu\bar\rho}\zeta. \tag{8.22}$$

The coefficients $\omega^{(0,q)}{}_{\bar\mu_1\ldots\bar\mu_q}$ are naturally $(0,q)$ forms. Thus a spinor may be identified with $(0,q)$- forms $q=0,\ldots,3$. This can be used to show that spinors exist on Calabi-Yau manifolds *i.e.* that each Calabi-Yau manifold is a spin manifold.

Let us return now to equation (8.6) in order to deduce (8.7). To this end we operate on (8.6) with ∇_k and skew on k and m and use the relation

$$[\nabla_k, \nabla_m]\zeta = -\frac{1}{4}R_{kmpq}\gamma^{pq}\zeta \tag{8.23}$$

we obtain the relation

$$R_{kmpq}\gamma^{pq}\zeta + 2i\nabla_{[k}A_{m]}\zeta = 0. \tag{8.24}$$

We now contract this relation with γ^m and use the identity

$$\begin{aligned}\gamma^m\gamma^{pq} =& \frac{1}{2}(\{\gamma^{pq},\gamma^m\} - [\gamma^{pq},\gamma^m]) \\ =& \frac{1}{2}(2\gamma^{mpq} + 4g^{m[p}\gamma^{q]}).\end{aligned} \tag{8.25}$$

The quantity $R_{kmpq}\gamma^{mpq}$ vanishes in virtue of the undifferentiated Bianchi identity. We are left with

$$(R_{km} - i\nabla_{[k}A_{m]})\gamma^m\zeta = 0. \tag{8.26}$$

We now exhaust the content of this equation by taking k equal to κ and $\bar\kappa$ in turn and using the fact that the three spinors $\gamma^{\bar\mu}\zeta$ are linearly independent. We find

$$\begin{aligned}R_{\kappa\bar\mu} =& -2i\nabla_{[\kappa}A_{\bar\mu]} \\ \nabla_{[\kappa}A_{\bar\mu]} =& 0.\end{aligned} \tag{8.27}$$

Contracting with $dx^\kappa \wedge dx^{\bar\mu}$ gives

$$\mathcal{R} = \partial(A_{\bar\mu}dx^{\bar\mu}) + \bar\partial(A_\mu dx^\mu) \tag{8.28}$$

and

$$\bar\partial(A_{\bar\mu}dx^{\bar\mu}) = \partial(A_\mu dx^\mu) = 0 \tag{8.29}$$

this relation together with the one above are eqivalent to (8.7).

We are now in a position to establish a very useful theorem.

Theorem:

(i) A compact Kähler manifold has vanishing first Chern class if and only if the manifold admits a nowhere vanishing holomorphic three-form. That is a (3,0)-form

$$\Omega = \frac{1}{3!}\Omega_{\mu\nu\rho}(x)dx^\mu \wedge dx^\nu \wedge dx^\rho \tag{8.30}$$

whose components are moreover holomorphic functions of position.

(ii) Ω is harmonic.

(iii) The holomorphic three-form is covariantly constant in the Ricci-flat metric.

First let us suppose that a holomorphic three-form exists on M. We set

$$\|\Omega\|^2 = \frac{1}{3!}\Omega_{\kappa\mu\nu}\bar{\Omega}^{\kappa\mu\nu}. \tag{8.31}$$

Since the holomorphic indices take three values $\Omega_{\kappa\mu\nu}$ must be proportional to the permutation symbol $\epsilon_{\kappa\mu\nu}$. So within each coordinate patch we may write

$$\Omega_{\kappa\mu\nu}(x) = f(x)\epsilon_{\kappa\mu\nu} \tag{8.32}$$

with $f(x)$ a nowhere vanishing holomorphic function. Complex conjugating and raising the indices we find

$$\begin{aligned} \bar{\Omega}^{\kappa\mu\nu} &= \bar{f}\epsilon_{\bar{\rho}\bar{\sigma}\bar{\tau}}g^{\kappa\bar{\rho}}g^{\mu\bar{\sigma}}g^{\nu\bar{\tau}} \\ &= g^{-\frac{1}{2}}\bar{f}\epsilon^{\kappa\mu\nu} \end{aligned} \tag{8.33}$$

where g again denotes $det(g_{mn})$. Hence

$$g^{\frac{1}{2}} = \frac{|f|^2}{\|\Omega\|^2} \tag{8.34}$$

and by (6.13) we have

$$\mathcal{R} = i\partial\bar{\partial}\log g^{\frac{1}{2}} = -i\partial\bar{\partial}\log(\|\Omega\|^2). \tag{8.35}$$

Since $\log(\|\Omega\|^2)$ is a coordinate scalar which by hypothesis is globally defined we learn that $\mathcal{R}$ is exact and hence that $c_1(M)$ vanishes.

We give two proofs of the converse. The first utilizes the properties of the gauge covariantly constant spinors. This has the advantage of being intuitively clear but is not very direct. The second proof is based on Yau's Theorem and is much more direct but requires a result in Čech cohomology.

I. If c_1 vanishes then we know from our previous discussion that there exists a spinor ζ satisfying (8.6) and (8.7). We also know from (8.29) that $A_{\bar{\mu}}dx^{\bar{\mu}}$ is a $\bar{\partial}$-closed (0,1)-form. We have shown that if the Euler number of a Calabi-Yau manifold is nonzero then the Hodge number b_{10} vanishes. Thus if the Euler number does not vanish there exists a scalar $\bar{\alpha}$ such that

$$A_{\bar{\mu}} = \bar{\partial}\bar{\alpha}. \tag{8.36}$$

We set

$$\Omega_{mnr} = e^{-i\bar{\alpha}}\zeta^T\gamma_{mnr}\zeta \tag{8.37}$$

and

$$\Omega = \frac{1}{3!}\Omega_{mnr}dx^m \wedge dx^n \wedge dx^r. \tag{8.38}$$

It is easy to show, in virtue of (8.20) that the only nonvanishing components of Ω are the $\Omega_{\mu\nu\rho}$, moreover they are nowhere zero since it may be shown by means of a Fierz rearrangement that

$$||\Omega||^2 = 27e^{i(\bar{\alpha}-\alpha)}. \tag{8.39}$$

In virtue of (8.36) and (8.37) we have

$$\bar{\partial}\Omega = 0, \tag{8.40}$$

also since Ω is a (3,0)-form $\partial\Omega$ must vanish since otherwise it would be a (4,0)-form. Thus

$$d\Omega = 0. \tag{8.41}$$

It is also easy to see that Ω is co-closed.

$$\begin{aligned} d^{\dagger}\Omega &= -\frac{1}{2}\nabla^m\Omega_{mnr}dx^n \wedge dx^r \\ &= -\frac{1}{2}g^{\mu\bar{\kappa}}(\partial_{\bar{\kappa}}\Omega_{\mu\nu\rho})dx^\nu \wedge dx^\rho \\ &= 0 \end{aligned} \tag{8.42}$$

II. Yau's theorem assures us that every Calabi-Yau manifold admits a metric $g_{\mu\bar{\nu}}$ satisfying

$$\partial_\mu\partial_{\bar{\nu}} \log g^{\frac{1}{2}} = 0 \tag{8.43}$$

in other words there exist functions f_j defined on each chart $\mathbf{U}_j$ such that

$$g_j^{\frac{1}{2}} = |f_j^2| \tag{8.44}$$

moreover each f_j must be nonvanishing on $\mathbf{U}_j$ since otherwise the metric would be singular. The idea of the proof is to show that it is possible to choose a set of phases θ_j in such a way that the quantity

$$e^{-i\theta_j}f_j(x)dx_j^1 \wedge dx_j^2 \wedge dx_j^3 \tag{8.45}$$

is independent of j and is in fact equal to the holomorphic three-form.

We need to study the transformation properties of the f_j. From the transformation rules

$$g_i^{\frac{1}{2}}\left|\frac{\partial(x_i)}{\partial(x_j)}\right|^2 = g_j^{\frac{1}{2}}. \tag{8.46}$$

Taking this together with (8.44) we make a separation of variables argument

$$\frac{f_i \frac{\partial(x_i)}{\partial(x_j)}}{f_j} = \overline{\left(\frac{f_j}{f_i \frac{\partial(x_i)}{\partial(x_j)}}\right)} = e^{i\theta_{ij}}. \tag{8.47}$$

The first quantity is a function of the x^μ while the second is a function of the $x^{\bar{\mu}}$. These can be equal only if they are both equal to a constant which we have written as $e^{i\theta_{ij}}$. It is immediate that the θ_{ij} are real. By considering also the inverse transfomation we have

$$\theta_{ij} = -\theta_{ji}. \tag{8.48}$$

It is important that the θ_{ij} are a set of *constants* associated with the nonempty overlaps $\mathbf{U}_i \cap \mathbf{U}_j$. We may consider also nonempty triple overlaps $\mathbf{U}_i \cap \mathbf{U}_j \cap \mathbf{U}_k$. By considering succesive transformations we find that

$$\theta_{ij} + \theta_{jk} + \theta_{ki} = 0. \tag{8.49}$$

Now in the language of Čech cohomology θ_{ij} is a one-cochain and is the analogue of a one-form (the counting is that a zero-cochain is a set of constants θ_j defined on the $\mathbf{U}_j$ and is the counterpart of a zero-form *i.e.* a function). Relation (8.49) is the cocycle condition analogous to the statement that the corresponding form is closed. Čech cohomology is equivalent to deRham cohomology so if $b_1 = 0$ the cocycle condition can be satisfied only if θ_{ij} is in fact a coboundary, *i.e.* the analogue of an exact form,

$$\theta_{ij} = \theta_i - \theta_j \tag{8.50}$$

for some choice of constants θ_i associated with each $\mathbf{U}_i$. From (8.47) we have the transformation law

$$e^{-i\theta_i} f_i \frac{\partial(x_i)}{\partial(x_j)} = e^{-i\theta_j} f_j. \tag{8.51}$$

So

$$\Omega = e^{-i\theta_j} f_j(x) dx_j^1 \wedge dx_j^2 \wedge dx_j^3 \tag{8.52}$$

is independent of the coordinates used in its definition.

By repeating the arguments of the previous proof one may show that Ω is harmonic. It remains only to show that Ω is covariantly constant in the Ricci-flat metric. Note that

$$\nabla_{\bar{\kappa}} \Omega_{\mu\nu\rho}(x) = 0 \tag{8.53}$$

since f_j is holomorphic. Consider therefore

$$\nabla_\kappa \Omega_{\mu\nu\rho} = \partial_\kappa \Omega_{\mu\nu\rho} - 3\Gamma_{\kappa[\mu}{}^\sigma \Omega_{\nu\rho]\sigma}. \tag{8.54}$$

Now $\Gamma_{\kappa[\mu}{}^{\sigma}\Omega_{\nu\rho]\sigma}$, being skew in (μ,ν,ρ), must be proportional to $\Omega_{\mu\nu\rho}$. In fact

$$\begin{aligned}\Gamma_{\kappa[\mu}{}^{\sigma}\Omega_{\nu\rho]\sigma} &= \frac{\bar{\Omega}^{\xi\eta\zeta}}{3!\|\Omega\|^2}\Gamma_{\kappa\xi}{}^{\sigma}\Omega_{\eta\zeta\sigma}\Omega_{\mu\nu\rho}\\ &= \frac{1}{3}\Gamma_{\kappa\sigma}{}^{\sigma}\Omega_{\mu\nu\rho}\\ &= \frac{1}{3}\partial_\kappa(\log g^{\frac{1}{2}})\Omega_{\mu\nu\rho}.\end{aligned} \tag{8.55}$$

Thus

$$\begin{aligned}\nabla_\kappa\Omega_{\mu\nu\rho} &= \left(\partial_\kappa f - f\partial_\kappa \log(|f|^2)\right)\epsilon_{\mu\nu\rho}\\ &= 0.\end{aligned} \tag{8.56}$$

In passing we note a useful identity. Although it might appear improbable $\bar{\Omega}^{\mu\nu\rho}/\|\Omega\|^2$ is a nonsingular holomorphic tensor field

$$\bar{\partial}\left(\frac{\bar{\Omega}^{\mu\nu\rho}}{\|\Omega\|^2}\right) = 0. \tag{8.57}$$

This is a simple consequence of (8.33) and (8.34) from which we have

$$\frac{\bar{\Omega}^{\mu\nu\rho}}{\|\Omega\|^2} = \frac{1}{f}\epsilon^{\mu\nu\rho} \tag{8.58}$$

with f nonvanishing.

Hodge Numbers of a Calabi-Yau Manifold

We have just shown that a Calabi-Yau manifold always admits a holomorphic three-form and that this form is harmonic. Thus $b_{30} \geq 1$. It follows that b_{30} is precisely one. For another harmonic (3,0)-form $\tilde{\Omega}$, also being proportional to $dx^1 \wedge dx^2 \wedge dx^3$, must satisfy the relation

$$\tilde{\Omega}(x) = h(x)\Omega(x) \tag{8.59}$$

with h a nonsingular function.

Now

$$d\tilde{\Omega} = \bar{\partial}h \wedge \Omega \tag{8.60}$$

hence

$$\bar{\partial}h = 0. \tag{8.61}$$

We learn that h is holomorphic on M. But it follows from the maximum modulus principle as in §1 that a globally defined holomorphic function is a constant. Thus $\tilde{\Omega}$ is simply a constant multiple of Ω.

Theorem: A Calabi-Yau manifold has $b_{20} = b_{10}$.

Let w be a harmonic (2,0)-form and set

$$v_{\bar\kappa} = \frac{1}{2}\bar\Omega_{\bar\kappa\bar\mu\bar\nu} w^{\bar\mu\bar\nu} \tag{8.62}$$

then

$$w^{\bar\rho\bar\sigma} = \frac{\Omega^{\bar\kappa\bar\rho\bar\sigma}}{\|\Omega\|^2} v_{\bar\kappa}. \tag{8.63}$$

From (8.62) we have

$$\nabla^{\bar\kappa} v_{\bar\kappa} = \frac{1}{2}\bar\Omega^{\kappa\mu\nu}\nabla_\kappa w_{\mu\nu} = 0 \tag{8.64}$$

and from (8.63) and the complex conjugate of (8.38) we have

$$\nabla_{\bar\rho} w^{\bar\rho\bar\sigma} = \frac{\Omega^{\bar\kappa\bar\rho\bar\sigma}}{\|\Omega\|^2}\nabla_{\bar\rho} v_{\bar\kappa} = 0. \tag{8.65}$$

We see that the (0,1)-form v is harmonic with respect to $\bar\partial$ if and only if w is harmonic with respect to ∂. However it was shown in §7 that a form is harmonic with respect to d if and only if it is harmonic with respect to $\bar\partial$ or equivalently if and only if it is harmonic with respect to ∂. Thus w is harmonic if and only if v is. Hence $b_{20} = b_{10}$. More generally for an n-dimensional manifold we have $b_{p0} = b_{n-p,0}$.

It is customary to display the Hodge numbers of a complex manifold in an array called the **Hodge diamond**

$$\begin{array}{ccccccc} & & & b_{00} & & & \\ & & b_{10} & & b_{01} & & \\ & b_{20} & & b_{11} & & b_{02} & \\ b_{30} & & b_{21} & & b_{12} & & b_{03} \\ & b_{31} & & b_{22} & & b_{13} & \\ & & b_{23} & & b_{32} & & \\ & & & b_{33} & & & \end{array} \tag{8.66}$$

The diamond has a number of symmetries. Complex conjugation gives $b_{pq} = b_{qp}$ so the diamond is symmetric under reflection in the vertical axis. Poincaré duality implies $b_{pq} = b_{n-q,n-p}$ which makes the diamond symmetric also under reflection in a horizontal axis. Finally, as mentioned above, the existence of a holomorphic n-form implies $b_{p0} = b_{n-p,0}$.

For a Calabi-Yau manifold the Hodge diamond takes the form (at least for $\chi \neq 0$)

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & b_{11} & & 0 & \\ 1 & & b_{21} & & b_{12} & & 1 \\ & 0 & & b_{11} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{8.67}$$

Thus a three-dimensional Calabi-Yau manifold has $b_{11}$and b_{21} as its only undetermined Hodge numbers. These can take many values subject, of course, to the restriction that $b_{11} \geq 1$ since the Kähler form is a (1,1)-form.

The Euler number of a Calabi-Yau manifold is related to b_{11} and b_{21} in a simple way

$$\begin{aligned} \chi &= 2b_0 - 2b_1 + 2b_2 - b_3 \\ &= 2 + 2b_{11} - (2 + 2b_{21}) . \\ &= 2(b_{11} - b_{21}) \end{aligned} \tag{8.68}$$

Spinors On Ricci-Flat Manifolds

A number of further properties of spinors may be established for the case that the Calabi-Yau manifold is endowed with its Ricci-flat metric.

The first of these is that the gauge covariantly constant spinor ζ may now be taken to be covariantly constant. Recall that ζ satifies the equation

$$\nabla_m \zeta - \frac{i}{2} A_m \zeta = 0 \tag{8.69}$$

where now

$$dA = 0. \tag{8.70}$$

If we assume that b_1 vanishes then A must be exact

$$A = da \tag{8.71}$$

for some globally defined scalar a. Thus (8.6) becomes

$$\nabla_m (\zeta e^{-\frac{i}{2}a}) = 0. \tag{8.72}$$

Henceforth we shall assume that ζ has been redefined so as to absorb the phase factor.

The expression for the holomorphic three-form simplifies to

$$\Omega_{mnr} = \zeta^T \gamma_{mnr} \zeta. \tag{8.73}$$

We have seen that an arbitrary spinor can be expressed in terms of (0,q)-forms $0 \leq q \leq 3$.

$$\eta = \sum_{q=0}^{3} \frac{(-1)^q}{q! 2^{\frac{q}{2}}} \omega_{\bar{\mu}_1 \ldots \bar{\mu}_q} \gamma^{\bar{\mu}_1 \ldots \bar{\mu}_q} \zeta \tag{8.74}$$

This expression enables us to extend the action of the operators $\bar{\partial}$ and $\bar{\partial}^\dagger$ to spinors.

We define

$$\bar{\partial}\eta = -\frac{1}{\sqrt{2}} \gamma^{\bar{\mu}} \nabla_{\bar{\mu}} \eta \tag{8.75}$$

and

$$\bar{\partial}^\dagger \eta = -\frac{1}{\sqrt{2}} \gamma^{\mu} \nabla_{\mu} \eta. \tag{8.76}$$

Note first that $\bar{\partial}$ and $\bar{\partial}^\dagger$ square to zero. For example

$$\bar{\partial}\bar{\partial}\eta = \frac{1}{4} \gamma^{\bar{\mu}\bar{\nu}} [\nabla_{\bar{\mu}}, \nabla_{\bar{\nu}}] \eta = 0 \tag{8.77}$$

since the Riemann tensor is zero when its first two components take antiholomorphic values.

Exercise: Show that if η corresponds to a given set of forms $\{\omega^{(0,q)}\}$ which we may denote by

$$\eta \sim \{\omega^{(0,q)}\}$$

then

$$\bar{\partial}\eta \sim \{\bar{\partial}\omega^{(0,q)}\} \qquad \text{and} \qquad \bar{\partial}^\dagger \eta \sim \{\bar{\partial}^\dagger \omega^{(0,q)}\}.$$ ◇

Observe that the Dirac operator is given by

$$\gamma^m \nabla_m \eta = -\sqrt{2}(\bar{\partial} + \bar{\partial}^\dagger)\eta. \tag{8.78}$$

If η is a zero mode of the Dirac operator then, in an obvious notation,

$$\begin{aligned} ((\bar{\partial} + \bar{\partial}^\dagger)\eta, (\bar{\partial} + \bar{\partial}^\dagger)\eta) &= 0 \\ (\eta, (\bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}))\eta &= 0. \end{aligned} \tag{8.79}$$

Thus η is a zero mode of the Dirac operator if and only if the forms $\{\omega^{(0,q)}\}$, corresponding to η are harmonic. By referring to the Hodge diamond we see that the only nonzero harmonic $(0,q)$-forms are those for which $q=0$ and $q=3$. Thus the only zero modes are ζ and $\bar{\zeta}$. The situation is more interesting, however, if we enquire about zero modes of the Dirac operator acting on spinors η_μ that have a holomorphic index. The formalism that has been developed applies equally well to this case except that η_μ is now related to one-forms

$$\eta_\mu = \sum_{q=0}^{3} \frac{(-1)^q}{q! 2^{\frac{q}{2}}} \omega_{\mu\bar{\nu}_1\ldots\bar{\nu}_q} \gamma^{\bar{\nu}_1\ldots\bar{\nu}_q} \zeta. \tag{8.80}$$

The solutions of the equations

$$\gamma^n \nabla_n \eta_\mu = 0 \tag{8.81}$$

correspond therefore to harmonic $(1,q)$-forms. Reference to (8.67) shows there to be b_{11} solutions corresponding to (1,1)-forms and b_{21} solutions corresponding to (2,1)-forms. These have opposite chirality and correspond in the compactification of strings to families and antifamilies of particles[8,9].

Global Holonomy of Ricci-Flat Manifolds

Finally this is an appropriate point to complete the argument that the holonomy group of a Ricci-flat Kähler manifold is $SU(3)$ even if the manifold is multiply connected. The argument depends a key fact which we shall merely quote. It is a corollary to the Gromoll-Meyer Theorem [10] to the effect that any multiply connected, compact, Ricci-flat, manifold $\mathcal{M}$ is the quotient of a simply connected manifold $\mathcal{M}_0$ by a finite isometry group $\mathcal{G}$ that acts without fixed points

$$\mathcal{M} = \mathcal{M}_0/\mathcal{G}. \tag{8.82}$$

Since the covering manifold is simply connected and Ricci-flat, as well as Kähler if $\mathcal{M}$ is, we know that its holonomy group is contained in $SU(3)$ by our previous reasoning.

Consider now the holomorphic three-form Ω at a point p' of $\mathcal{M}_0$. This projects down to a (3,0)-form at the corresponding point p of $\mathcal{M}$. Let C be a non-contractible loop in $\mathcal{M}$ that begins and ends at p and let $\Phi_\kappa{}^\nu$ be the parallel propagation matrix corresponding to C so that after parallely propagating Ω around C we have

$$\Omega_{\kappa\lambda\mu} \longrightarrow \Phi_\kappa{}^\nu \Phi_\lambda{}^\rho \Phi_\mu{}^\sigma \Omega_{\nu\rho\sigma} = (\det\Phi)\Omega_{\kappa\lambda\mu}. \tag{8.83}$$

The question is whether $\det\Phi$ is the identity. In $\mathcal{M}_0$ the loop C corresponds to an open curve connecting two points p' and p'' which both project down to p. Thus the question is equivalently whether the results of projecting Ω down from p' and p'' agree. But they must agree since otherwise $\mathcal{M}$ and$\mathcal{M}_0$ could not both be Calabi-Yau spaces.

IX EXAMPLES OF CALABI-YAU MANIFOLDS

At the time of writing there is no systematic way to construct or classify all Calabi-Yau spaces. It is possible however to construct examples of Calabi-Yau manifolds by means of special techniques that succeed in favourable circumstances. In this section we review two of these techniques, the construction of Calabi-Yau manifolds as hypersurfaces in projective spaces and as 'blow ups' of orbifolds.

The reason that one seeks to construct Calabi-Yau manifolds as submanifolds of P_n is that one may not easily construct them in the way that one might naively wish, as submanifolds of C_n. The reason for this is that one wishes to construct Kähler manifolds. Since C_n is, of course, Kähler submanifolds described by holomorphic equations are guaranteed also to be Kähler. The catch is that we know by the maximum modulus principle that none of these analytic submanifolds is compact. It is for this reason that one turns to projective spaces. P_n is both compact and Kähler so its analytic submanifolds also have these properties. There is also a theorem due to Chow [11], not proved here, to the effect that analytic submanifolds of projective spaces may be described as the zero locus of a finite number of polynomials of the homogeneous coordinates.

We shall discuss only submanifolds $\mathcal{M}$ that are complete intersections of N polynomials p^α, $\alpha = 1, ..., N$ in a product of projective spaces of total dimension N+3. We shall refer to these submanifolds as CI manifolds. By a **complete intersection** is meant that the N-form

$$\Psi = dp^1 \wedge dp^2 \wedge ... \wedge dp^N \tag{9.1}$$

does not vanish on $\mathcal{M}$. This condition guards against the polynomials describing a surface with cusps or nodes. Ψ describes the N directions normal to $\mathcal{M}$. If $\mathcal{M}$ is smooth then Ψ cannot vanish. If Ψ were to vanish at a point p of $\mathcal{M}$ then this would imply that $\mathcal{M}$ did not have well defined normal directions at p and so could not be smooth. The assumption that Ψ does not vanish is quite restrictive. Of course one expects that giving N equations in an N+3 dimensional space will describe a 3-dimensional manifold locally. The restrictive assumption is that they should in fact do so globally. Perhaps the simplest example of a manifold that cannot be described as a complete intersection of polynomials is the Segré embedding

$$P_1 \times P_2 \hookrightarrow P_5. \tag{9.2}$$

Let x^i, i=1,2 and y^m, m=1,2,3 be homogeneous coordinates for P_1 and P_2 respectively and set

$$t^{im} := x^i y^m \tag{9.3}$$

which we regard as homogeneous coordinates in P_5. The embedded submanifold is described by the equations

$$t^{im}t^{jn} - t^{in}t^{jm} = 0. \tag{9.4}$$

Nontrivial equations are obtained if $i \neq j$ and $m \neq n$. There are three such equations corresponding to taking (i,j)=(1,2) and (m,n) to be one of the three choices (1,2),(2,3),(3,1)

$$\begin{aligned} p^1 &:= t^{11}t^{22} - t^{12}t^{21} = 0 \\ p^2 &:= t^{12}t^{23} - t^{13}t^{22} = 0 \\ p^3 &:= t^{13}t^{21} - t^{11}t^{23} = 0. \end{aligned} \tag{9.5}$$

These three equations together describe a smooth *three*-dimensional manifold embedded in a *five*-dimensional manifold. Moreover it is not possible to describe the submanifold as a complete intersection with just two of the polynomials p^1 and p^2, say, since for example for the embedded P_1 described by the equations

$$y^2 = y^3 = 0 \tag{9.6}$$

we have

$$t^{12} = t^{13} = t^{22} = t^{23} = 0 \tag{9.7}$$

so the polynomials p^1 and p^2 both vanish there and so does dp^2 and hence also $dp^1 \wedge dp^2$. There is always a choice of two of the polynomials p^1, p^2, p^3 which describes the embedded submanifold in a neighborhood of any point but it is not possible to specify it globally with just two of the three.

Given that the submanifold $\mathcal{M}$ is compact and Kähler the question becomes whether the polynomials may be chosen such that $\mathcal{M}$ has vanishing first Chern class. This can be answered by computing the Chern polynomial of $\mathcal{M}$. However we prefer to follow another construction [12] which is perhaps more intuitive. This makes use of the theorem that the first Chern class of a three-dimensional compact Kähler manifold vanishes if and only if M admits a globally defined and nowhere vanishing holomorphic three-form. The utility of the theorem is that one can attempt a direct construction of Ω.

Consider a complete intersection of N polynomials p^α, $\alpha = 1, ..., N$ in a single projective space P_{N+3}. The natural quantities that we have at our disposal are the homogeneous coordinates z^A, A=1,...,N+4, their differentials dz^A and the permutation symbol with N+4 indices

$$\epsilon_{A_1 A_2 .... A_{N+4}}. \tag{9.8}$$

The metric $G_{A\overline{B}}$ of the embedding space cannot be used since it has an antiholomorphic index. There is only one differential form that can be constructed from these quantities it is an (N+3)-form

$$\mu = \epsilon_{A_1 A_2 .... A_{N+4}} z^{A_1} dz^{A_2} \wedge dz^{A_3} \wedge .... \wedge dz^{A_{N+4}}. \tag{9.9}$$

Now z^A and λz^A are the same point of P_{N+3} so μ is not even a well definded quantity. Under the scaling

$$z^A \longrightarrow \lambda z^A \tag{9.10}$$

it is not invariant, in fact it transforms as

$$\mu \longrightarrow \lambda^{N+4}\mu. \tag{9.11}$$

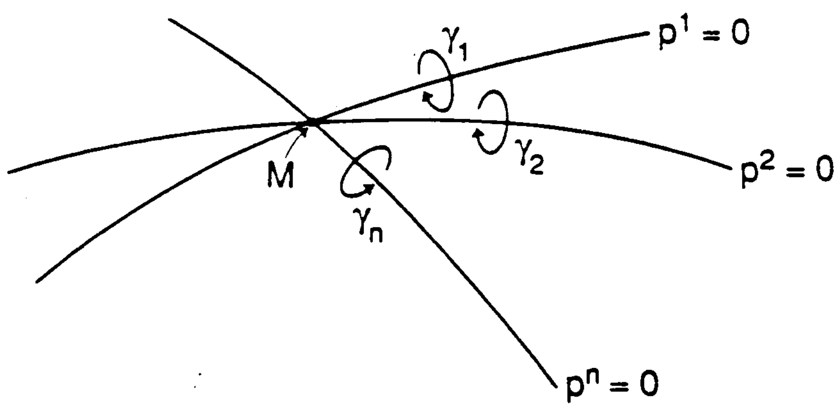
We can however construct a well defined 3-form μ by proceeding in the following way. We divide by the product of the N polynomials p^α thereby obtaining a form

$$\nu = \frac{\mu}{p^1 p^2 .... p^N} \tag{9.12}$$

that has poles on the curves $p^\alpha = 0$ and then integrate around an N-dimensional contour

$$\Gamma_N = \gamma_1 \times \gamma_2 \times .... \times \gamma_N \tag{9.13}$$

which is the Cartesian product of N small circles each of radius δ that wind around the N curves $p^\alpha = 0$ as in the figure.



We obtain in this way a 3-form

$$\lim_{\delta \to 0} \int_{\Gamma_N} \frac{\mu}{p^1 p^2 ... p^N} = (2\pi i)^N \Omega \tag{9.14}$$

which is holomorphic by construction and is in fact nowhere zero though to show this directly requires more work. The important point is that from μ we can construct Ω

provided only the meromorphic form ν is well defined. Under a scaling each p^α will transform as

$$p^\alpha \longrightarrow \lambda^{deg(\alpha)} p^\alpha \tag{9.15}$$

where $deg(\alpha)$ is the homogeneity degree of p^α. Thus (9.12) is well defined if

$$\sum_{\alpha=1}^{N} deg(\alpha) = N + 4. \tag{9.16}$$

This then is the condition that $\mathcal{M}$ has vanishing first Chern class. There are not as many possibilities here as might at first be thought. A linear equation in a projective space P_n is, by choice of coordinates, equivalent to the equation

$$z^{n+1} = 0 \tag{9.17}$$

the remaining n homogeneous coordinates define P_{n-1}. So to avoid a redundant description we require

$$deg(\alpha) \geq 2\ ; \tag{9.18}$$

it follows that

$$N + 4 = \sum_{\alpha=1}^{N} deg(\alpha) \geq 2N \tag{9.19}$$

i.e. that

$$4 \geq N. \tag{9.20}$$

This leads to five manifolds

$$\begin{gathered} P_4[5]_{-200} \\ P_5[3,3]_{-144}, \qquad P_5[2,4]_{-176} \\ P_6[2,2,3]_{-144} \\ P_7[2,2,2,2]_{-128} \end{gathered} \tag{9.21}$$

where the integers within brackets are the degrees of the corresponding polynomials and the number appended to the brackets is the Euler number of the manifold which may be calculated from the Chern polynomial.

Tian and Yau [13] have presented an interesting manifold of Euler number -18. The manifold is realized in $P_3 \times P_3$ by three polynomial equations

$$\begin{aligned}
\sum_{A=0}^{3} x^A y^A &= 0 & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\sum_{A=0}^{3} (x^A)^3 &= 0 & \begin{bmatrix} 3 \\ 0 \end{bmatrix} \\
\sum_{A=0}^{3} (y^A)^3 &= 0 & \begin{bmatrix} 0 \\ 3 \end{bmatrix}
\end{aligned} \tag{9.22}$$

where x^A and y^B denote the homogeneous coordinates of the two P_3's. As a real manifold the manifold is fully specified by the degree vectors that give the degrees of each of the polynomials in the variables of each projective space. These are written in the right hand column of equation (9.22). To establish notation we denote this manifold by a matrix whose columns are the degree vectors:

$$\begin{array}{c} P_3 \\ P_3 \end{array} \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}_{-18} \tag{9.23}$$

The Tian-Yau manifold is of interest because it admits a freely acting Z_3 symmetry which may be realized by cyclically permuting the first three homogeneous coordinates of each space and multiplying the fourth by cube roots of unity

$$(x^1, x^2, x^3, x^4) \times (y^1, y^2, y^3, y^4) \longrightarrow (x^2, x^3, x^1, \omega x^4) \times (y^2, y^3, y^1, \omega^2 y^4). \tag{9.24}$$

The group action on the manifold is fixed point free so the quotient manifold, which has points identified under the action of the group, is also a smooth Calabi-Yau manifold which has Euler number -6, corresponding to three generations of particles. The Tian-Yau manifold suggests a generalization to a large class of spaces. We will consider transverse intersections of N polynomials p^α in products of projective spaces such that the total dimension of the ambient manifold is N+3. These spaces are specified by a degree matrix that gives the degree of each polynomial in the variables of each projective space

$$\begin{array}{c} P_1 \\ P_1 \\ \vdots \\ P_1 \\ P_{n_1} \\ P_{n_2} \\ \vdots \\ P_{n_F} \end{array} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{f1} & a_{f2} & \dots & a_{fN} \\ b_{11} & b_{12} & \dots & b_{1N} \\ b_{21} & b_{22} & \dots & b_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ b_{F1} & b_{F2} & \dots & b_{FN} \end{bmatrix} \tag{9.25}$$

It is convenient to separate the P_1 factors explicitly so we suppose there to be f P_1 factors and then a further F factors of spaces of dimension at least two. In order for the submanifold, $\mathcal{M}$, to be three dimensional we have

$$\sum_{k=1}^{F} n_k + f - N = 3. \tag{9.26}$$

We can construct the holomorphic three form for $\mathcal{M}$ by generalizing our previous construction. We now have coordinates z_j^A, $A = 1, ..., n_j + 1$ for each factor space and for each factor space we may construct an n_j-form

$$\mu_j = \epsilon_{A_1 A_2 .... A_{n_j+1}} z^{A_1} dz^{A_2} \wedge ..... \wedge dz^{A_{n_j+1}}. \tag{9.27}$$

The product of all these,

$$\mu = \prod_{j=1}^{f+F} \mu_j \tag{9.28}$$

is an (N+3)-form in virtue of (9.26). We may seek to construct Ω by dividing μ by the product of the p^α and integrating over a contour Γ_N as in the previous case. This construction is sensible provided

$$\frac{\mu}{p^1 ... p^N} \tag{9.29}$$

is well defined on the ambient space. The coordinates of each projective space may be scaled separately

$$z_j^A \longrightarrow \lambda_j z_j^A. \tag{9.30}$$

So (9.29) is invariant if

$$\sum_{\alpha=1}^{N} deg_j(\alpha) = n_j + 1. \tag{9.31}$$

This is the condition that (9.25) represents a manifold with vanishing first Chern class and we shall refer to a matrix that satisfies this condition together with (9.26) as a Calabi-Yau (CY) matrix. In order to avoid redundancy we exclude the possibility of polynomials that are linear in the coordinates of one space and independent of other coordinates since this would serve only to reduce the dimension of the relevant factor space. This amounts to demanding that

$$\sum_{j=1}^{f+F} deg_j(\alpha) \geq 2. \tag{9.32}$$

As in the previous case of only one polynomial this inequality greatly restricts the possibilities. If we sum eqn(9.31) over j and employ (9.32) we find

$$\sum_{j=1}^{f+F}(n_j+1) = \sum_{\alpha=1}^{N}\sum_{j=1}^{f+F} deg_j(\alpha) \geq 2N \tag{9.33}$$

which, in virtue of (9.26), becomes

$$F+f+3 \geq N. \tag{9.34}$$

Consider the quantity $\sum_{i=1}^{F}(n_i-1)$ which, for want of a better name, we shall refer to as the overdimension. By (9.26) we have

$$\sum_{j=1}^{F}(n_j-1) = N+3-f-F \tag{9.35}$$

and hence by (9.34)

$$\sum_{j=1}^{F}(n_j-1) \leq 6. \tag{9.36}$$

Each of the spaces of dimension greater than one contributes at least one to the sum so we see that there are at most six spaces that are not $P_1's$

$$F \leq 6. \tag{9.37}$$

The number of $P_1's$ on the other hand is, so far, unrestricted. To place a bound on f Green and Hübsch noted recently [14] that certain other constraints result in a redundant description. As explained in Section III a bilinear constraint acting in $P_1 \times P_1$ is equivalent to a P_1. In a block notation we have

$$\begin{array}{c} P_1 \\ P_1 \\ X \end{array}\begin{bmatrix} 1 & a \\ 1 & b \\ 0 & M \end{bmatrix} \quad = \quad \begin{array}{c} P_1 \\ X \end{array}\begin{bmatrix} a+b \\ M \end{bmatrix} \tag{9.38}$$

where X denotes any product of projective spaces, a and b are row vectors, and M is a matrix. Also a quadratic constraint acting in a P_1 should involve the variables of another factor also since otherwise the matrix is a block product

$$\begin{array}{c} P_1 \\ X \end{array}\begin{bmatrix} 2 & 0 \\ 0 & M \end{bmatrix} \tag{9.39}$$

corresponding to a product manifold. We exclude product manifolds as trivial cases since one of the factors of a product $\mathcal{M}_1 \times \mathcal{M}_2$ must be one dimensional and hence a torus. The remaining factor is either itself a torus or the K3 surface. By forbidding configurations corresponding to (9.39) and the left hand side of (9.38) Green and Hübsch show that

$$f \leq 9. \tag{9.40}$$

This shows that the number of manifolds that can be constructed this way is finite. The possible ambient spaces satisfying the constraints (9.37) and (9.40) are listed in Table I. By means of a computer it is possible to compile an exhaustive list of almost 8,000 matrices that satisfy the constraints [15]. The Euler numbers that arise in this way are displayed in Table II. We display also in diagramatic form a few of the manifolds of the CICY list. The diagrams are due to Green and Hübsch who denote a projective space by a circle and an equation by a dot. The degree of the equation in the variables of each projective space is recorded in the number of lines that connect the dot to the circle. Thus the dimension of each projective space is one less than the number of lines emanating from the respective circle. In this notation the Tian-Yau manifold is

It is difficult to know how many matrices in the list correspond to distinct manifolds. Certainly there are many identities similar to (9.38). Green and Hübsch [16] have given an algorithm based on the method of spectral sequences which permits the individual Hodge numbers b_{11} and b_{12} to be calculated. There are 250 sets of values of the pair (b_{11}, b_{21}) [17] which provides a lower bound to the number of distinct manifolds in the list.

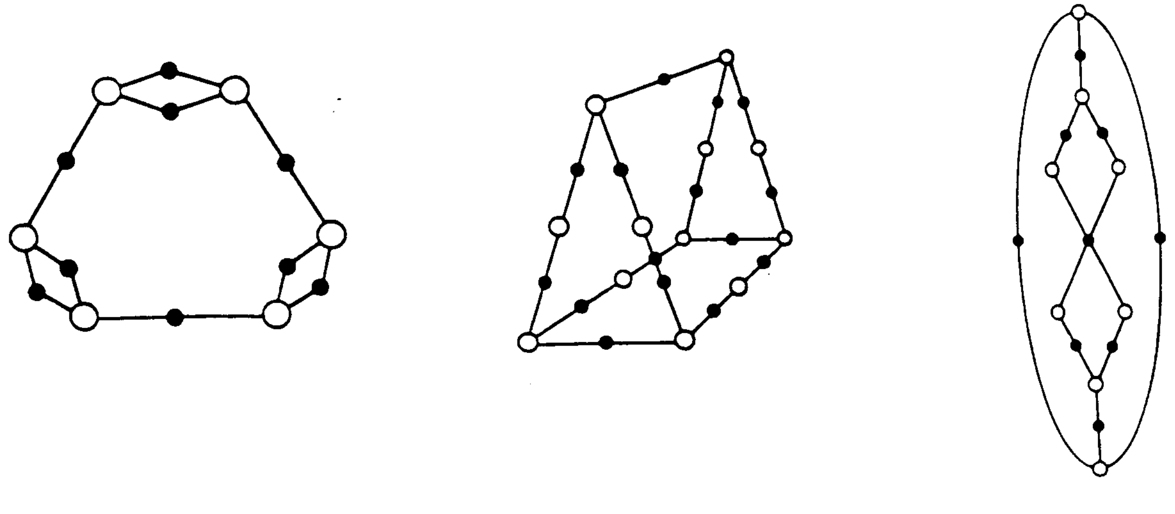


Table I: Ambient Spaces

Space	g	f_{max}	Exnb	Number
$(P_1)^f P_7$	–	4	0	5
$(P_1)^f P_4^2$	–	5	0	6
$(P_1)^f P_3 P_5$	–	5	0	6
$(P_1)^f (P_2)^g P_6$	$0 \rightarrow 1$	5	$1-g$	12
$(P_1)^f (P_3)^3$	–	6	0	7
$(P_1)^f (P_2)^g P_3 P_4$	$0 \rightarrow 1$	6	$1-g$	14
$(P_1)^f (P_2)^g P_5$	$0 \rightarrow 2$	6	$2-g$	21
$(P_1)^f (P_2)^g (P_3)^2$	$0 \rightarrow 2$	7	$2-g$	24
$(P_1)^f (P_2)^g P_4$	$0 \rightarrow 3$	7	$3-g$	32
$(P_1)^f (P_2)^g P_3$	$0 \rightarrow 4$	8	$4-g$	44
$(P_1)^f (P_2)^g$	$0 \rightarrow 6$	9	$6-g$	64

In this table we show all possible ambient spaces. These 235 ambient manifolds fall into classes according to the number of P_1- and P_2-factors. The third column gives the excess number

$$exnb := \sum_{i=1}^{f+F} (n_i + 1) - 2N$$

which vanishes when all the columns sum to two, the minimum value which avoids trivial constraints. A large exnb usually means that there are many ways to construct inequivalent degree matrices for a given ambient space. The minimum number of P_1 factors is zero except for

$$\begin{aligned} (P_1)^f &\quad \text{where } f_{min} = 4 \\ (P_1)^f P_2 &\quad \text{where } f_{min} = 2\,. \\ (P_1)^f P_3 &\quad \text{where } f_{min} = 1 \end{aligned}$$

Table II: Euler numbers

0				
*	−42	−82	*	−162
−4	−44	−84	−124	*
*	−46	−86	−126	*
−8	−48	−88	−128	−168
*	−50	−90	*	*
−12	−52	−92	−132	*
−14	−54	−94	*	*
−16	−56	−96	*	−176
−18	−58	−98	−138	*
−20	−60	−100	−140	*
−22	−62	−102	*	*
−24	−64	−104	−144	*
−26	−66	−106	*	*
−28	−68	−108	−148	*
−30	−70	−110	−150	*
−32	−72	−112	*	*
−34	−74	−114	*	*
−36	−76	−116	*	*
−38	−78	*	*	*
−40	−80	−120	*	−200

The Hodge numbers do not exhaust the topological information available. There is considerable information available in the numbers

$$\mu_{ijk} = \int_{\mathcal{M}} \omega_i \wedge \omega_j \wedge \omega_k$$
$$\nu_i = \int_{\mathcal{M}} c_2 \wedge \omega_i \tag{9.41}$$

where the ω_i are the harmonic $(1,1)$-forms and c_2 represents the second Chern class (see §10). These numbers are topological *i.e.* they do not involve the complex structure in virtue of two facts. (i) The Hodge number b_{20} vanishes so $H^{(1,1)}(\mathcal{M}) \cong H^2(\mathcal{M})$ and (ii) the Pontrjagin class p^1 is related to the Chern classes by

$$p^1 = c_1^2 - 2c_2. \tag{9.42}$$

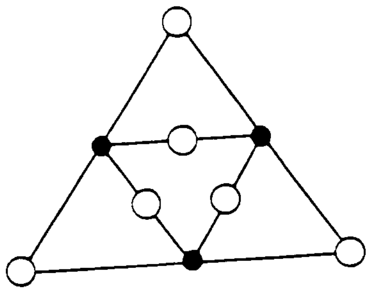
Since c_1 vanishes, c_2 and p^1 are proportional and p^1 is defined for a real manifold independent of any complex structure.

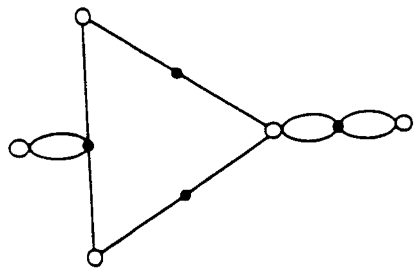
A theorem of Wall [18] shows that the data (9.41), together with b_3, classify simply connected, real, six-manifolds. The classification of Calabi-Yau manifolds is more complicated since not every real six-manifold is a Calabi-Yau manifold and a real manifold may admit distinct complex structures in such a way that two such may not be continuously deformed into each other. Nevertheless it seems that the data (9.41) will play an important role in this classification.

A prime motivation for the compilation of the list was to seek additional examples of Calabi-Yau manifolds corresponding to three generations of particles. All the matrices in the CICY list have negative Euler number so these would be manifolds corresponding to Euler number -6. No spaces with Euler number -6 appear in the list. However there do appear a large number with Euler numbers that are divisible by 6, suggesting that, as with Yau's manifold it may be possible to find in the list a manifold $\mathcal{M}$ with Euler number -6k which admits a freely acting discrete isometry group $\mathcal{G}$ of order k. For such a manifold the quotient

$$\mathcal{M}_1 = \mathcal{M}/\mathcal{G} \tag{9.43}$$

would be a Calabi-Yau manifold with Euler number -6. Surprisingly one may show that, apart from the Tian-Yau manifold and the two manifolds of Euler number -48 of the figure, no manifold in the list can admit a freely acting group of the requisite order that acts linearly on the coordinates of the embedding space [19]. It has been shown that if these two spaces of Euler number -48 admit a freely acting group of order eight then this group must be $Z_2 \times Z_2 \times Z_2$, despite considerable ingenuity no such group action has been found.





Recently Tian and Yau[20] have given a procedure which, given a Calabi-Yau manifold, yields other Calabi-Yau manifolds with the same Euler number as the given one. This procedure yields many new examples of manifolds with Euler number -6. The new manifolds have the same values for the Hodge numbers as the original manifold and have the same intersection numbers μ_{ijk}. The ν_i are however different, which shows that the new manifolds are not diffeomorphic to the original one. C.Schoen[21] has also presented a method of constructing a large class of Calabi-Yau manifolds. In particular this construction yields manifolds with every even Euler number between -8 and 92 and includes a manifold with Euler number 6.

The Eguchi-Hansen Geometry

The Eguchi-Hansen geometry, EH_n is a non-compact Ricci-flat geometry which is useful for repairing the singularities of orbifolds. It is of additional interest because it can be constructed explicitly. Our discussion follows that of Strominger and Witten [9].

Let us seek a Ricci-flat Kähler metric of the form

$$ds^2 = A(\sigma)dz^\mu dz_\mu + B(\sigma)z_\mu dz^\mu z_{\bar\nu} dz^{\bar\nu} \tag{9.44}$$

where we adopt the convention that

$$z_\mu = z^{\bar\mu} \quad , \quad z_{\bar\mu} = z^\mu \tag{9.45}$$

as in an exercise in §7 and

$$\sigma = z_\mu z^\mu = z_{\bar\mu} z^{\bar\mu}. \tag{9.46}$$

First we impose the Kähler condition. We have

$$\partial_\rho g_{\mu\bar\nu} = A' z_\rho \delta_{\mu\bar\nu} + B z_\mu \delta_{\rho\bar\nu} + B' z_\rho z_\mu z_{\bar\nu}. \tag{9.47}$$

For the metric to be Kähler this must be symmetric in ρ and μ. This requires

$$B = A'. \tag{9.48}$$

The condition that the metric be Ricci-flat is that

$$\partial_\mu \partial_{\bar\nu} \log g^{\frac{1}{2}} = 0. \tag{9.49}$$

Now apart from other considerations the determinant of the metric is a function of σ only and for an arbitrary such function we have

$$\partial_\mu \partial_{\bar\nu} F(\sigma) = F''(\sigma) z_\mu z_{\bar\nu} + F'(\sigma)\delta_{\mu\bar\nu}. \tag{9.50}$$

The only solution to the equation

$$\partial_\mu \partial_{\bar\nu} F(\sigma) = 0 \tag{9.51}$$

is therefore

$$F = const. \tag{9.52}$$

Since we may rescale the metric by a constant the Ricci-flatness condition (9.49) is equivalent to the condition

$$\det(g_{\mu\bar\nu}) = 1. \tag{9.53}$$

Now

$$\begin{aligned} \det(g_{\mu\bar\nu}) =& \frac{1}{n!}\epsilon^{\mu_1\ldots\mu_n}\epsilon^{\bar\nu_1\ldots\bar\nu_n}(A\delta_{\mu_1\bar\nu_1} + A' z_{\mu_1} z_{\bar\nu_1})\ldots(A\delta_{\mu_n\bar\nu_n} + A' z_{\mu_n} z_{\bar\nu_n}) \\ =& A^{n-1}(A + \sigma A') \end{aligned} \tag{9.54}$$

Substituting into (9.53) and multiplying by $n\sigma^{n-1}$ we have

$$\frac{d}{d\sigma}((\sigma A)^n) = n\sigma^{n-1}. \tag{9.55}$$

Thus

$$A(\sigma) = \sigma^{-1}(c + \sigma^n)^{\frac{1}{n}} \quad , \quad B(\sigma) = -\sigma^{-2}(c+\sigma^n)^{\frac{1}{n}-1} \tag{9.56}$$

and the metric is

$$g_{\mu\bar\nu} = \sigma^{-1}(c+\sigma^n)^{\frac{1}{n}}\left\{\delta_{\mu\bar\nu} - \frac{c z_\mu z_{\bar\nu}}{\sigma(c+\sigma^n)}\right\}. \tag{9.57}$$

We shall take the constant c to be positive since otherwise the manifold has a singularity where $\sigma^n = -c$. As $\sigma \to \infty$ we find $g_{\mu\bar\nu} \to \delta_{\mu\bar\nu}$ so the metric is asymptotically flat. c is a parameter that measures the region in which the curvature is concentrated. We can effectively set $c = 1$ by scaling the coordinates $z^\mu \to c^{\frac{1}{n}} z^\mu$. The metric becomes

$$g_{\mu\bar\nu} = \sigma^{-1}(1+\sigma^n)^{\frac{1}{n}}\left\{\delta_{\mu\bar\nu} - \frac{z_\mu z_{\bar\nu}}{\sigma(1+\sigma^n)}\right\}. \tag{9.58}$$

We need to examine the singularity of the metric at $\sigma = 0$. We will show that it is merely a coordinate singularity which may be removed by a proper choice of coordinates.

First note that $\sigma = 0$ is the limit in which all the z^μ are zero. It is preferable to work with coordinates such that $\sigma = 0$ corresponds to the vanishing of just one of the coordinates.

Set

$$y = z^n \text{ and } y^i = \frac{z^i}{z^n}, \quad i = 1, \ldots, n-1. \tag{9.59}$$

and to save writing define also

$$\rho = 1 + y^i y_i. \tag{9.60}$$

The limit we shall take is $y \to 0$ with y^i constant. Substitution into (9.58) and expansion of the metric components to leading order in powers of $|y|$ yields

$$ds^2 \sim \frac{1}{\rho}\left\{dy_i dy^i - \frac{1}{\rho} y_i dy^i y_{\bar{j}} dy^{\bar{j}}\right\} + \rho^{n-1}|y|^{2n-2}\left\{\rho dy d\bar{y} + \left(\bar{y} dy y_{\bar{i}} dy^{\bar{i}} + y d\bar{y} y_i dy^i\right)\right\}. \tag{9.61}$$

In this form the metric is still singular at $y = 0$ since the coefficients of all the terms containing dy and $d\bar{y}$ vanish there, so at $y = 0$ the metric has a vanishing row and column and hence has no inverse. To cure this we make a further change of variables. Set

$$w = \frac{1}{n} y^n \text{ so that } y^{n-1} dy = dw. \tag{9.62}$$

In terms of (w, y^i) coordinates the metric becomes

$$ds^2 \sim \frac{1}{\rho}\left\{dy_i dy^i - \frac{1}{\rho} y_i dy^i y_{\bar{j}} dy^{\bar{j}}\right\} + \left\{\rho^n dw d\bar{w} + n\rho^{n-1}\left(\bar{w} dw y_{\bar{i}} dy^{\bar{i}} + w d\bar{w} y_i dy^i\right)\right\} \tag{9.63}$$

In these coordinates the metric is regular at $w = 0$. Note however two important facts. Firstly $w = 0$ corresponds to an $(n-1)$-dimensional submanifold and the metric of this submanifold is given by setting $dw = 0, d\bar{w} = 0$ in (9.63) and we recognize the resulting metric as the Fubini-Study metric. Thus $\sigma = 0$ corresponds in fact to a P_{n-1}. Secondly we see from the relations

$$w = \frac{1}{n}(z^n)^n \quad , \quad y^i = \frac{z^i}{z^n} \tag{9.64}$$

that the n points $\alpha^k z^\mu$, $k = 0, \ldots, n-1$, where $\alpha = e^{2\pi i/n}$ is an n'th root of unity determine the same values of (w, y^i). Since it is in the (w, y^i) coordinates that the manifold is nonsingular and these must be single valued functions of position we must identify the points

$$z^\mu \approx \alpha^k z^\mu \quad , \quad k = 1, \ldots, n-1. \tag{9.65}$$

Because of this Z_n identification EH_n is not asymptotically like R_n. The statement that the metric is asymptotically flat is only true locally. The surface corresponding to $\sigma = R^2$, with R a large constant, is not an S^{2n-1} but rather S^{2n-1}/Z_n.

Exercise: If $n = 3$ then $\sigma = 0$ corresponds to a P_2. There is a (1,1)-cohomology class dual to it. (i) By making an ansatz similar to (9.44) and demanding that the resulting form $\omega = \omega_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^{\bar{\nu}}$ be trace-free, or otherwise, find ω. (ii) Normalize ω by requiring that its integral over the hypersurface $\sigma = 0$ be unity. (iii) Show that

$$\int \omega \wedge \omega \wedge \omega = -9. \qquad \diamond$$

The Z_3 Orbifold

Consider the torus T_3 obtained by making in C^3 the identifications

$$z^k \approx z^k + 1 \approx z^k + e^{\pi i/3}. \tag{9.66}$$

On T_3 we impose the additional Z_3 identification

$$z^k \approx e^{2\pi i/3} z^k. \tag{9.67}$$

In each z^k-plane there are three fixed points

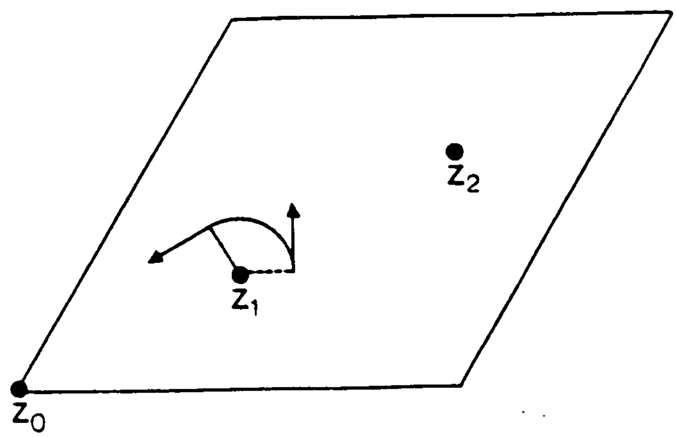
$$z_r = \frac{r}{3}\left(\frac{3}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{r}{\sqrt{3}} e^{i\pi/6} \quad , \quad r = 0, 1, 2, \tag{9.68}$$

corresponding to points for which

$$e^{2i\pi/3} z_r = z_r + r. \tag{9.69}$$

This construction gives the Z_3 orbifold. It is a singular manifold with δ-function curvature concentrated at the fixed points. To see this consider the curve C of radius ϵ of the figure. It is a closed curve in view of the identifications and has length $2\pi\epsilon/3$. This shows that there is curvature concentrated at the fixed point. We see also from the figure that after parallel transport around C the vector v becomes v' which is rotated by $2\pi/3$ with respect to v. The holonomy group of the orbifold is Z_3. The orbifold can be turned into a smooth manifold by the following process. Excise a small ball around each of the 27 fixed points. The boundary of each ball is S^5/Z_3 which is the same as the hypersurface $\sigma = R^2$ in EH_3. The interior of this hypersurface in EH_3 may be glued in in place of the excised balls in the orbifold. Since, as mentioned previously the Eguchi-Hansen metric has a size parameter that measures the region in which the curvature is concentrated, the join can be made arbitrarily smooth and it can be shown that a Calabi-Yau manifold results. The way in which the Z manifold was constructed permits us to calculat its Euler number in a simple manner. It may be shown that the Euler number of EH_n is n. The Euler number of a torus is zero and the Euler number of a ball is unity. In order to construct the Z manifold we took a torus, excised 27 balls, took a quotient by Z_3 and glued in 27 copies of EH_3. Hence the Euler number is

$$\chi = \frac{(0-27)}{3} + 27 \times 3 = 72. \tag{9.70}$$



It is instructive to compute the triple intersection numbers (9.41) for the smoothed manifold. This was first done by Strominger [22] whose treatment we follow. We begin by writing down the cohomolgy groups of Z. The Hodge numbers are as in (8.54). The group $H^{(1,0)}$, for example, is empty even though it is not empty for the torus T_3. For T_3 the group is spanned by the basis dz^k , $k = 1, 2, 3$. However these forms are not invariant under the replacement $z^k \to \alpha z^k$ with $\alpha = e^{2\pi i/3}$.

$H^{(1,1)}$: As a basis we take the 27 forms ω_A , $A = 1, \ldots, 27$, that correspond to the P_2's located at the centers of each EH_3 and the nine forms

$$\nu_{i\bar{j}} \sim dz^i \wedge dz^{\bar{j}}. \tag{9.70}$$

By this notation is meant that $\nu_{i\bar{j}}$ is asymptotically equal to $dz^i \wedge dz^{\bar{j}}$ far from the regions where the curvature is concentrated. Since the intersection numbers that are being calculated are topological we may suppose that the EH_3's are very small and that the $\nu_{i\bar{j}}$ is dual to the four-surface to which $dz^i \wedge dz^{\bar{j}}$ is dual on T_3. The dimension of $H^{(1,1)}$ is

$$b_{1,1} = 27 + 9 = 36. \tag{9.72}$$

$H^{(2,1)}$: This group is empty. Quantities such as $dz^i \wedge dz^j \wedge dz^{\bar{k}}$ and $dz^i \wedge \omega_A$ are not invariant under the Z_3 action. Thus $b_{2,1} = 0$.

As a check we have

$$\chi = 2(b_{1,1} - b_{2,1}) = 72 \tag{9.73}$$

in agreement with our previous assertion.

In this basis it is easy to see that the only intersection numbers that are not zero are

$$\int \omega_A \wedge \omega_A \wedge \omega_A = \kappa \tag{9.74}$$

and

$$\int \nu_{i\bar{l}} \wedge \nu_{j\bar{m}} \wedge \nu_{k\bar{n}} = \mu \epsilon_{ijk} \epsilon_{\bar{l}\bar{m}\bar{n}}. \tag{9.75}$$

Exercise: Make a model of the z^1 plane of the Z-orbifold by folding and taping a sheet of paper. Draw on this model two surfaces corresponding to the elements of $H^1(T_3)$. What happens to these surfaces when the orbifold is smoothed to give the Z-manifold? ◇

We have reviewed the simplest orbifold. Many other constructions are possible and these have been the object of considerable study owing to the fact that strings can propagate consistently on orbifold backgrounds[23]. It is an interesting problem to what extent it is possible to blow up the singularities of orbifolds to obtain smooth Calabi-Yau manifolds[24]. It has been argued on the basis of string theory[25] that this may always be possible for orbifolds for which the group $\mathcal{G}$ is abelian.

X. CHERN CLASSES

The Chern classes are cohomology classes that are analytic invariants of the manifold and are defined by polynomials of the curvature two-form. On a complex manifold it is natural to choose basis forms $(e^\alpha, e^{\bar\alpha})$, which are related by complex conjugation, such that e^α is a (1,0)-form

$$e^\alpha = e^\alpha{}_\mu dx^\mu. \tag{10.1}$$

Consider the connection $\omega^a{}_b$. In virtue of (4.21) we have

$$(\partial + \bar\partial)e^\alpha + \omega^\alpha{}_\beta \wedge e^\beta + \omega^\alpha{}_{\bar\beta} \wedge e^{\bar\beta} = 0. \tag{10.2}$$

This relation contains forms of type (2,0) , (1,1) and (0,2) and leads to the three equations

$$\begin{aligned} \partial e^\alpha + \omega^{(1,0)\alpha}{}_\beta \wedge e^\beta &= 0 \\ \bar\partial e^\alpha + \omega^{(0,1)\alpha}{}_\beta \wedge e^\beta + \omega^{(1,0)\alpha}{}_{\bar\beta} \wedge e^{\bar\beta} &= 0 \\ \omega^{(0,1)\alpha}{}_{\bar\beta} \wedge e^{\bar\beta} &= 0. \end{aligned} \tag{10.3}$$

We can now show that it is always possible to choose the mixed components $\omega^\alpha{}_{\bar\beta}$ to vanish. To this end note that taking $\omega^{(0,1)\alpha}{}_{\bar\beta}$ to vanish solves the third relation, while taking $\omega^{(1,0)\alpha}{}_{\bar\beta}$ to vanish leaves us with the relation

$$\bar\partial e^\alpha + \omega^{(0,1)\alpha}{}_\beta \wedge e^\beta = 0. \tag{10.4}$$

But this may always be solved by taking

$$\omega^{(0,1)\alpha}{}_\beta = \omega^{(0,1)\alpha}{}_{\beta\bar\gamma} e^{\bar\gamma} \tag{10.5}$$

in such a way that

$$\bar\partial e^\alpha = \omega^{(0,1)\alpha}{}_{\beta\bar\gamma} e^\beta \wedge e^{\bar\gamma}. \tag{10.6}$$

Clearly the first equation may also be solved in a similar manner. Thus $\omega^a{}_b$ may be chosen such that $\omega^\alpha{}_\beta$ and $\omega^{\bar\alpha}{}_{\bar\beta}$ are the only nonzero components. It follows immediately that the curvature two-form is also pure in its frame indices and

$$R^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta \quad , \quad \text{conj.} \tag{10.7}$$

Let now Θ be the matrix of two-forms whose components are $R^\alpha{}_\beta$

$$\Theta = (R^\alpha{}_\beta). \tag{10.8}$$

and define an **invariant polynomial** to be any polynomial in Θ which is invariant under all unitary frame transformations

$$e^\alpha \longrightarrow \Phi^\alpha{}_\beta e^\beta. \tag{10.9}$$

In other words an invariant polynomial is a sum of terms each of which has its frame indices fully contracted. Examples are

$$\Theta^\alpha{}_\alpha \ , \ \Theta^\alpha{}_\beta \wedge \Theta^\beta{}_\alpha \ , \ \Theta^\alpha{}_\beta \wedge \Theta^\beta{}_\gamma \wedge \Theta^\gamma{}_\alpha + \Theta^\alpha{}_\alpha \wedge \Theta^\beta{}_\beta \wedge \Theta^\gamma{}_\gamma. \tag{10.10}$$

Each invariant polynomial defines a cohomology class which is an analytic invariant. To show this we must show (i) that each invariant polynomial is closed and (ii) that under a variation of the connection the change in the invariant polynomial is exact. (i) follows easily in virtue of the Bianchi identity (4.22). Consider for example

$$\begin{aligned} d(\Theta^\alpha{}_\beta \wedge \Theta^\beta{}_\alpha) &= (D\Theta^\alpha{}_\beta) \wedge \Theta^\beta{}_\alpha + \Theta^\alpha{}_\beta \wedge (D\Theta^\beta{}_\alpha) \\ &= 0. \end{aligned} \tag{10.11}$$

Clearly d acting on any invariant polynomial gives a sum of terms each of which contains a $D\Theta^\alpha{}_\beta$ and so vanishes. With regard to (ii) consider a variation

$$\omega^\alpha{}_\beta \longrightarrow \omega^\alpha{}_\beta + \delta\omega^\alpha{}_\beta. \tag{10.12}$$

It is convenient to adopt a matrix notation and not write the frame indices explicitly. The corresponding variation in Θ is by (4.20) and (10.7)

$$\begin{aligned} \delta\Theta &= d(\delta\omega) + \delta\omega \wedge \omega + \omega \wedge \delta\omega \\ &= D(\delta\omega) \end{aligned} \tag{10.13}$$

Thus, for example,

$$\begin{aligned} \delta\,(tr(\Theta \wedge \Theta)) &= tr\,(D(\delta\omega) \wedge \Theta) + tr\,(\Theta \wedge D(\delta\omega)) \\ &= d\,(tr(\delta\omega \wedge \Theta) + tr(\Theta \wedge \delta\omega)). \end{aligned} \tag{10.14}$$

In a similar way any invariant polynomial varies by an amount that is exact

$$\delta F(\Theta) = d(\delta Q). \tag{10.15}$$

To establish the result for finite differences let ω and ω' be any two connections and let

$$\omega_t = (1-t)\omega + t\omega' \tag{10.16}$$

so that $\omega_0 = \omega$ and $\omega_1 = \omega'$. We have shown that

$$\frac{d}{dt} F(\Theta_t) = dQ_t \tag{10.17}$$

for some Q_t and integrating this relation we have

$$F(\Theta') - F(\Theta) = d\left(\int_0^1 dt Q_t\right). \tag{10.18}$$

A specific set of invariant polynomials are the **Chern polynomials** which are $2k$-forms $k = 0, \dots, n$, defined by a formal expansion of the **total Chern polynomial**

$$c = det\left(1 + \frac{i}{2\pi}\Theta\right) \tag{10.19}$$

in powers of the curvature

$$c = c_0 + c_1 + c_2 + ... + c_n. \tag{10.20}$$

For three dimensional manifolds for example these give rise to the following analytic invariants

$$\begin{aligned} \mathcal{C}_1 &= deg(\mathcal{M}) = \int_{\mathcal{M}} c_1^3(\mathcal{M}) \\ \mathcal{C}_2 &= \int_{M} c_1(\mathcal{M})c_2(\mathcal{M}) \\ \mathcal{C}_3 &= \chi = \int_{M} c_3(\mathcal{M}) \end{aligned} \tag{10.21}$$

$\mathcal{C}_1$ is often referred to as the degree of $\mathcal{M}$ and $\mathcal{C}_3$ is the Euler number. If the first Chern class, c_1, vanishes then only the Euler number is nontrivial.

Associated with the Chern polynomials is the Chern character ch(E) and the symmetric polynomials $S_k(E)$ defined for a vector bundle E by

$$\begin{aligned} ch(E) &= tr(e^x) \\ &= \sum \frac{1}{k!} S_k(E), \\ S_k(E) &= tr(x^k) \end{aligned} \tag{10.22}$$

where

$$x = \frac{i\Theta}{2\pi}. \tag{10.23}$$

The utility of the Chern character being that it behaves well under addition and multiplication of bundles

$$ch(E \oplus F) = ch(E) + ch(F) \tag{10.24}$$

$$ch(E \otimes F) = ch(E) \wedge ch(F). \tag{10.25}$$

Let λ_m , $m = 1, ..., n$ be the eigenvalues of the matrix x. Then

$$c = \prod_m (1+\lambda_m) = 1 + \sum_m \lambda_m + \sum_{m>n} \lambda_m \lambda_n + \sum_{m>n>r} \lambda_m \lambda_n \lambda_r + \ldots \tag{10.26}$$

and

$$S_k = \sum_m \lambda_m^k. \tag{10.27}$$

Thus there are relations between the Chern polynomials and the S_k which it is convenient to write in the form

$$\begin{aligned} c_1 &= S_1 \\ c_2 &= \frac{1}{2}(-S_2 + c_1^2) \\ c_3 &= \frac{1}{3}(S_3 - c_1^3 + 3c_1c_2) \\ c_4 &= \frac{1}{4}(-S_4 + c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2) \\ &\ etc. \end{aligned} \tag{10.28}$$

A convenient alternative way of generating these equations is from the Newton formulæ

$$S_k - c_1 S_{k-1} + \ \ldots \ + (-)^k c_k k = 0 \quad (k \geq 1). \tag{10.29}$$

In order to illustrate the utility of the formalism we shall find an explicit expression for the Euler number of a CICY manifold in terms of its degree matrix . For a manifold $\mathcal{M}$ embedded in an ambient space X by a complete intersection of hypersurfaces the tangent bundle to X is spanned by the tangent bundle to $\mathcal{M}$ and the normal bundle $\mathcal{N}$ to $\mathcal{M}$

$$TX = T\mathcal{M} \oplus \mathcal{N}, \tag{10.30}$$

so that

$$ch(T\mathcal{M}) = ch(TX) - ch(\mathcal{N}). \tag{10.31}$$

Note that the Chern classes on the tangent bundle to $\mathcal{M}$ are often referred to as the Chern classes of $\mathcal{M}$.

Let H be the Kähler form on P_n, i.e. the generator of the second cohomology group of P_n. If h is the pullback of H, i.e. the Kähler form of the induced metric on a hypersurface defined by the vanishing of a polynomial p, then the first Chern class of the normal bundle is

$$c_1(\mathcal{N}) = deg(p)h \tag{10.32}$$

since the normal bundle is the deg(p)'th tensor power of the pullback of the hyperplane bundle. For an ambient space

$$X = \prod_{i=1}^{F} P_{n_i} \tag{10.33}$$

(where in this context we do not deal separately with possible P_1 factors) subject to N polynomial constraints p^α the normal bundle is the direct sum

$$\mathcal{N} = \bigoplus_{\alpha=1}^{N} \mathcal{N}_\alpha \tag{10.34}$$

of the normal bundles to each constraint considered separately. To save writing we write $\xi(\alpha)$ for $c_1(\mathcal{N}_\alpha)$. It follows from (10.32) that this quantitiy is given by

$$\xi(\alpha) = \sum_{i=1}^{F} deg_i(\alpha) h_i. \tag{10.35}$$

By repeated application of (10.24) we find that

$$\begin{aligned} ch(TX) &= \sum_{i=1}^{F} [(n_i + 1)e^{h_i} - 1] \\ ch(\mathcal{N}) &= \sum_{\alpha=1}^{N} e^{\xi(\alpha)} \end{aligned} \tag{10.36}$$

The expression for $ch(TX)$ may be justified by noting that although the the tangent bundle of the P_n does not itself split the sum of this bundle with a trivial bundle $\mathcal{E}$ splits into n+1 hyperplane bundles†

$$\mathcal{E} \oplus TP_n = (n+1)\mathcal{O}(1). \tag{10.37}$$

† A less sophisticated statement of this fact is that a vector in TP_n can be thought of as a linear differential operator

$$v = v^A \frac{\partial}{\partial z^A}$$

that acts on functions of the homogeneous coordinates of P_n. Functions on P_n are the functions that have homogeneity degree zero in the z^A. Thus

$$z^A \frac{\partial}{\partial z^A} = 0$$

which reduces the number of independent basis vectors to n. It follows that we may consider a set v^A of homogeneity degree one to be a vector provided we understand v^A to be subject to the identification

$$v^A \simeq v^A + \rho z^A$$

for any ρ.

Hence, in virtue of (10.31) we find

$$S_k(T\mathcal{M}) = \sum_{i=1}^{F}(n_i+1)h_i^k - \sum_{\alpha=1}^{N}\xi^k(\alpha). \tag{10.38}$$

This relation in conjunction with recursive use of relations (10.28) determines all the Chern classes of a CI manifold.

In particular

$$\begin{aligned} c_1(\mathcal{M}) &= \sum_{i=1}^{F}(n_i+1)h_i - \sum_{\alpha=1}^{N}\xi(\alpha) \\ &= \sum_{i=1}^{F}\{n_i+1-\sum_{\alpha=1}^{N}deg_i(\alpha)\}h_i \end{aligned} \tag{10.39}$$

and we see that we recover in this way our previous criterion for the vanishing of the first Chern class.

In order to compute Chern numbers we must integrate products of the Chern classes over the manifold. This is accomplished by "lifting" the integral to the embedding space where the integration is trivial.

First "lift" the integrand to the ambient space, which in this case means remove all the pullbacks from the H_i that restricted the Kähler forms to $\mathcal{M}$. Next insert the Poincaré dual of $\mathcal{M}$

$$\eta_{\mathcal{M}} = \prod_{\alpha=1}^{N} c_1(\mathcal{N}_\alpha) \tag{10.40}$$

into the integrand. This form has the "delta-function" property that when we now integrate the resulting (N+3)-form over the ambient space X, the only contribution is from the sub-manifold $\mathcal{M}$, and we have that

$$\int_{\mathcal{M}} \tilde{\omega} = \int_X \omega \wedge \eta_{\mathcal{M}}. \tag{10.41}$$

Finally we use that the $H_i's$ are normalized so that

$$\int_{P_1} H_i = 1 \qquad (i=1,...,F). \tag{10.42}$$

Since also H_i^p vanishes for $p > n_i$ and $\omega \wedge \eta_{\mathcal{M}}$ is a form of order $\sum_{i=1}^{F} n_i$ it follows that

$$\omega \wedge \eta_{\mathcal{M}} = C\prod_{i=1}^{F} H_i^{n_i} \tag{10.43}$$

with C a numerical constant. C then is the value of the integral

$$\int_X \omega \wedge \eta_{\mathcal{M}} = C. \tag{10.44}$$

A simple example of the application of this formalism is afforded by the computation of the Euler number for a CICY manifold. For this case $c_1(\mathcal{M})$ vanishes so we see from (2.10) that $c_3 = (1/3)S_3$. Thus

$$\chi \prod_{i=1}^{F} H_i^{n_i} = \frac{1}{3}\left[\sum_{i=1}^{F}(n_i+1)H_i^3 - \sum_{\alpha=1}^{N}\xi^3(\alpha)\right]\prod_{\beta=1}^{N}\xi(\beta). \tag{10.45}$$

This expression is well adapted to rapid machine computation. It is also easy to see from this expression that the Euler number of a CICY manifold is nonpositive

$$\chi \leq 0. \tag{10.46}$$

Exercise: Check the Euler numbers of the CICY manifolds of §IX. $\diamond$

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