T-1

Случайная величина распределена равномерно на $[0, \theta]$. По выборке объёма n найдены оценки параметра θ:

$\tilde{\theta}_1 = 2\bar{x}$; $\tilde{\theta}_2 = n x_{min}$; $\tilde{\theta}_3 = x_{max}$;

$$\tilde{\theta}_4 = \left(x_1 + \frac{\sum\limits_{k=2}^{n} x_k}{(n-1)} \right)$$

а) $\varphi \sim R(0, \theta)$ θ – неизв. парам.

$\theta > 0$ – вероятн. модель

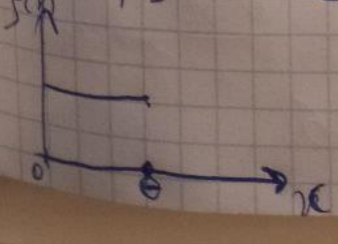
Пусть $\vec{x}_n$ – выборка объёма n

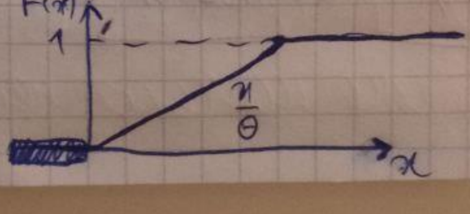
Числовые хар-ки φ:

$$M[\varphi] = \int\limits_{-\infty}^{+\infty} x\,p(x)\,dx = \int\limits_0^{\theta} \frac{x}{\theta}\,dx = \frac{x^2}{2\theta}\Big|_0^{\theta} = \frac{\theta}{2}$$

$$M\varphi^2 = \int\limits_0^{\theta} x^2 p(x)\,dx = \int\limits_0^{\theta} \frac{x^2}{\theta}\,dx = \frac{x^3}{3\theta}\Big|_0^{\theta} = \frac{\theta^2}{3}$$

$$D[\varphi] = M[\varphi^2] - (M\varphi)^2 = \frac{1}{12}\theta^2$$





$\tilde{\theta}_1 = 2\bar{x} = 2 \cdot \frac{1}{n} \sum_{i=1}^{n} x_i$

• Несмещённость:

Определение несмещенности:

$\forall \theta > 0 \; M(\tilde{\theta}_1) = \theta$

$M\left[\frac{2}{n}\sum_{i=1}^{n} x_i\right] = \frac{2}{n} \cdot n M[x_i] = 2M[x_i] =$

$= \{$т.к. x_i - одинаково распред.$\} = 2M[\varphi] =$

$= 2 \cdot \frac{\theta}{2} = \theta \Rightarrow \underbrace{\tilde{\theta}_1 \text{ - несмещенная}}$

• Состоятельность:

ДУ:

$\left.\begin{array}{l} \tilde{\theta} \text{ - несмещенная} \\ D[\tilde{\theta}] \xrightarrow[\substack{n\to\infty \\ \forall\theta\in\Theta}]{} 0 \end{array}\right\} \Rightarrow \tilde{\theta}$ - состоят.

$D[\tilde{\theta}_1] = D\left[\frac{2}{n}\sum_{i=1}^{n} x_i\right] = \frac{4}{n^2} D\left[\sum_{i=1}^{n} x_i\right] \overset{\text{т.к. независ.}}{=}$

$= \frac{4}{n^2} \cdot n D[x_i] \overset{\text{т.к. одинак. распр.}}{=} \frac{4}{n} \cdot D[\varphi] = \frac{4}{n} \cdot \frac{\theta^2}{12} = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0$

$\Rightarrow \underbrace{\tilde{\theta}_1 \text{ - состоят.}}$

$\tilde{\theta}_2 = \min(x_i)$

· Несмещенность

$$M[\theta_2] = \int_{-\infty}^{+\infty} \cancel{x p(x) dx}\; y\,\varphi(y)\,dy$$

$$\Phi(y) = 1-(1-F(y))^n$$

$$\Phi'(y) = \varphi(y) = n(1-\underbrace{F(y)}_{\frac{y}{\theta}})^{n-1}\underbrace{p(y)}_{\frac{1}{\theta}\{[0,\theta]\}} =$$

$$= n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}$$

$$M[\theta_2] = \int_0^{\theta} n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}\,y\,dy = \left\{t = 1-\frac{y}{\theta}\right\} =$$

$$= \int_1^{0} n t^{n-1}(1-t)\,\theta\,dt = \int_0^1 n\theta t^{n-1}\,dt -$$

$$- \int_0^1 n\theta t^n\,dt = n\theta\left(\frac{t^n}{n}\Big|_0^1 - \frac{t^{n+1}}{n+1}\Big|_0^1\right) =$$

$$= n\theta\left(\frac{1}{n} - \frac{1}{n+1}\right) = n\theta\left(\frac{1}{n(n+1)}\right) = \frac{\theta}{n+1} \Rightarrow$$

$\Rightarrow \tilde{\theta}_2$ – смещённая

$\tilde{\theta}_2' = (n+1)\tilde{\theta}_2$ – несмещенная

• Состоятельность:

$$M[\tilde{\theta}_2^2] = \int_0^{\theta} n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}y^2\,dy =$$

$$= \left\{t = 1-\frac{y}{\theta}\right\} = \int_1^0 n\,t^{n-1}\theta^2(1-t)^2\,dt =$$

$$= n\theta^2\left[\int_0^1 \left(t^{n-1}-2t^n+t^{n+1}\right)dt\right] =$$

$$= n\theta^2\left[\frac{1}{n} - 2\frac{1}{n+1} + \frac{1}{n+2}\right] = n\theta^2\left[\frac{(n+1)(n+2)-2n(n+2)+n(n+1)}{(n+1)(n+2)}\right] = \theta^2\,\frac{n^2+3n+2-2n^2-4n+n^2+n}{(n+1)(n+2)} = \frac{2\theta^2}{(n+1)(n+2)}$$

$$D[\tilde{\theta}_2] = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \theta^2\left[\frac{n}{(n+1)^2(n+2)}\right] \xrightarrow[n\to\infty]{} 0$$

$$\left.\begin{array}{l} D[\tilde{\theta}_2] \xrightarrow[n\to\infty]{} 0 \\ \tilde{\theta}_2 \text{ – смещённая} \end{array}\right\} \Rightarrow \text{Th. про } \tilde{\theta}_2 \text{ нич. неизв.}$$

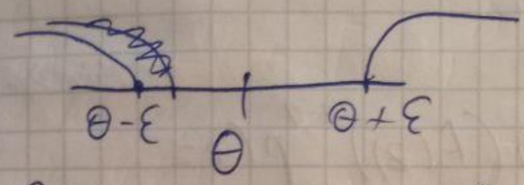
$D[\tilde{\theta}_2']$

$$D[\tilde{\theta}_2'] = (n+1)^2 D[\tilde{\theta}_2] = \frac{\theta^2 n}{n+2} \underset{n\to\infty}{\not\to} 0 \Rightarrow$$

$\Rightarrow$ тоже ничего неизв.

..Исследуем по определению

$$\forall\theta>0\ \forall\varepsilon>0\ P(|\tilde{\theta}_2-\theta|>\varepsilon)\xrightarrow[n\to\infty]{}0$$



$\tilde{\theta}_2'$:

$$P(|\tilde{\theta}_2'-\theta|\geq\varepsilon)\geq P(\tilde{\theta}_2'\geq\theta+\varepsilon)=$$

$$=P((n+1)x_{min}\geq\theta+\varepsilon)=$$

$$=P\left(x_{min}\geq\frac{\theta+\varepsilon}{n+1}\right)=1-\underbrace{P\left(x_{min}<\frac{\theta+\varepsilon}{n+1}\right)}_{\Phi\left(\frac{\theta+\varepsilon}{n+1}\right)}=$$

$$=1-\left(1-\left(1-F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n\right)=$$

$$=\left(1-\left(\frac{\theta+\varepsilon}{\theta(n+1)}\right)\right)^n\xrightarrow[n\to+\infty]{}e^{\frac{\theta+\varepsilon}{\theta}}>0\Rightarrow\tilde{\theta}_2'\ \text{несостоятельная}$$

$\tilde{\theta}_2$: $P(|\tilde{\theta}_2-\theta|\geq\varepsilon)\xrightarrow[n\to\infty]{}0$

$\forall\varepsilon>0$
$\forall\theta>0$: $P(\tilde{\theta}_2<\theta-\varepsilon)+P(\underset{x_{min}}{\tilde{\theta}_2}\cancel{\geq}>\theta+\varepsilon)$

$$P(x_{min}<\theta-\varepsilon)=\Phi(\theta-\varepsilon)=1-\left(1-\frac{\theta-\varepsilon}{\theta}\right)^n=$$

$$=1-\left(\frac{\varepsilon}{\theta}\right)^n\xrightarrow[n\to\infty]{}\overset{1}{-\infty}\neq0\Rightarrow\tilde{\theta}_2\ \text{не состоятельная}$$

$\tilde{\Theta}_3 = ~~max(~~ x_{max}$

• ~~Смещенность~~ смещенность?

$$M[\tilde{\Theta}_3] = \int_{-\infty}^{+\infty} z\psi(z)dz = \int_0^\theta n\frac{z^n}{\theta^n}dz = \frac{n}{n+1}\theta$$

$$\Psi = (F(z))^n$$

$$\psi(z) = \Psi'(z) = n(F(z))^{n-1}p(z) =$$

$$= n\left(\frac{z}{\theta}\right)^{n-1}\frac{1}{\theta}\{(0,\theta)\}$$

$\tilde{\Theta}_3' = \frac{n+1}{n} x_{max}$ — несмещ. оценка

$$M[\tilde{\Theta}_3'] = \int_0^\theta n\frac{z^{n+1}}{\theta^n}dz = \frac{n}{n+2}\tilde{\theta}^2$$

• состоятельность:

$$D[\tilde{\Theta}_3] = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \theta^2\left[\frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2}\right] =$$

$$= \theta^2\left[\frac{n^3+2n^2+n-n^3-2n^2}{(n+2)(n+1)^2}\right] = \frac{n\theta^2}{(n+2)(n+1)^2} \xrightarrow[n\to\infty]{} 0$$

$$\left.\begin{array}{l} D[\tilde{\Theta}_3] \xrightarrow[n\to\infty]{} 0 \\ \tilde{\Theta}_3 \text{ — смещенная} \end{array}\right\} \Rightarrow \text{про } \tilde{\Theta}_3 \text{ ничего сказать нельзя}$$

θ_3' на сост:

$D[\tilde{\theta}_3'] = \frac{(n+1)^2}{n^2} D[\tilde{\theta}_3] = \frac{\theta^2}{4(n+2)} \xrightarrow[n\to\infty]{} 0 \Rightarrow$

$\Rightarrow \theta_3'$ - сост

$\tilde{\theta}_3$ на сост. по определ.:

$\forall \varepsilon > 0$
$\forall \theta > 0: P(|\tilde{\theta}_3 - \theta| \geq \varepsilon) = P(x_{max} < \theta - \varepsilon) +$

$+ P(x_{max} > \theta + \varepsilon) = (F(\theta - \varepsilon))^n$

$\varepsilon < \theta: \left(\frac{\theta - \varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 0$

$\varepsilon > \theta \quad (0)^n \xrightarrow[n\to\infty]{} 0 \Rightarrow$ состоят.

$\tilde{\theta}_4 = x_1 + \sum_{k=2}^{n} x_k / n-1$

• Несмещенность

$M[\tilde{\theta}_4] = M[x_1 + \frac{1}{n-1} \sum_{n=2}^{n} x_i] =$

$= M[x_1] + \frac{1}{n-1} \sum_{i=2}^{n} M[x_i] = \frac{\theta}{2} + \frac{\theta}{2} = \theta \Rightarrow$

$\Rightarrow \tilde{\theta}_4$ - несмещ.

• Состоятельность

$D[\hat{\theta}_4] = D[x_1 + \frac{1}{n-1} \sum_{i=2}^{n} x_i] =$

$= D[x_i \xi] + \frac{1}{(n-1)^2}(n-1) D\xi = \frac{\theta^2}{12} + \frac{\theta^2}{12(n-1)} =$

$= \frac{\theta^2}{12} \frac{n}{n-1} \not\xrightarrow[n\to\infty]{} 0$ $D\xi$ не работает

$\tilde{\theta}_4 \xrightarrow{P} \theta$

Св-во: $\xi_n \xrightarrow{P}, \eta_n \xrightarrow{P} \eta \Rightarrow \xi_n + \eta_n = \xi + \eta$

$x_1 \xrightarrow{P} x_1$

$\frac{1}{n-1} \sum_{i=2}^{n} x_i \rightarrow \left\{ \begin{array}{l} \text{ЗБЧ Хинчина} \\ \xi_1, \ldots, \xi_n \text{ одинак. распр.} \\ \text{и независимы} \\ \frac{1}{n} \sum_{i=1}^{n} \xi_i \xrightarrow{P} M[\xi] \end{array} \right\} \rightarrow M[\xi] =$

$= \frac{\theta}{2}$

$\tilde{\theta}_4 = x_1 + \frac{1}{n-1} \sum_{i=2}^{n} x_i \rightarrow x_1 + \frac{\theta}{2}$

б) $\tilde{\theta}_1 = 2\bar{x}$

$\tilde{\theta}_3 = \frac{n+1}{n} x_{max}$

$$D[\tilde{\theta}_1] = \frac{\theta^2}{3n}$$

$$D[\tilde{\theta}'_3] = \frac{\theta^2}{n(n+2)}$$

$$\forall \theta > 0: \frac{\theta^2}{n(n+2)} < \frac{\theta^2}{3n} \Rightarrow$$

$\Rightarrow \tilde{\theta}'_3$ — эффективнее $\tilde{\theta}_1$ $\forall n > 1$

T-3

Сл. вел. имеет экспоненц. распред.