The Radon and X-Ray Transforms

Research Training Group
Inverse Problems and Partial Differential Equations
Department of Mathematics, University of Washington

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Contents

Part I

Lectures

Chapter 1

A (tentative) brief outline of the lectures

# of lectures	Dates	topic
1		Introduction: meaning and history of X-ray tomography and Radon transform.
2		FT and F. series (The idea of harmonic analysis, Plancherel, Paley-Wiener, sampling, Sobolev spaces)
2		Properties of X-ray/Radon transform (invariances, projection-slice theorem, mapping theorems)
3-2		Inversion formulas and stability
2		Range and support
2		WF sets, singularity detection, local tomography
2-3		Generalizations: attenuated X-ray, Funk transform, hyperbolic X-ray (with applications to EIT)

Chapter 2

Computed tomography and Radon transform

2.1 Mathematical imaging

- Image processing (usually studied at engineering departments)
- Image understanding (belongs to the realm of artificial intelligence)
- Image reconstruction (the topic of tomography)

2.2 Idea of computed tomography (CT). Inverse problems

Tomography: from Greek slice ($\tau o\mu o\sigma$) and to write ($\gamma\rho\alpha\psi\epsilon\tau\epsilon$). It attempts to find the internal structure of a non-transparent object by sending some signals (waves, radiation) through it. Electromagnetic waves of various frequencies (radio and microwaves, visual light, X-rays, γ-rays) and acoustic waves are common.

In **Computed Tomography**, the image is not obtained directly from the measurements (like in the usual X-ray pictures), but rather is the result of an intricate mathematical reconstruction from the measured data.

Tomography is an **Inverse Problem**, where the unknown parameters of a system need to be estimated from the know reaction of the system to external signals.

A typical kind of inverse problems is the **recovery of coefficients of a differential equations** on a domain from some information about its solutions at the domain's boundary.

Another example of an inverse problem: the famous Mark Kac's problem "**Can one hear the shape of the drum?**"

2.3 Applications

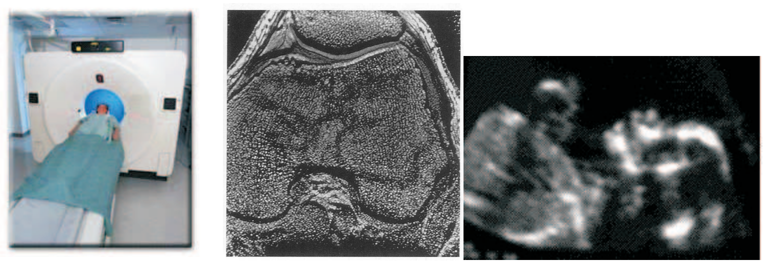


Figure 2.1: **Medicine**: diagnostics, CT guided surgery, CT guided radiotherapy.

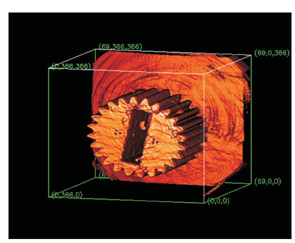


Figure 2.2: **Industry**: non-destructive testing

Figure 2.3: **Homeland security**: A detector gate (left). ... (right)

Fusion reactors: plasma diagnostics in Tokamaks.
Radar in defence and **sonar** in oceanography.

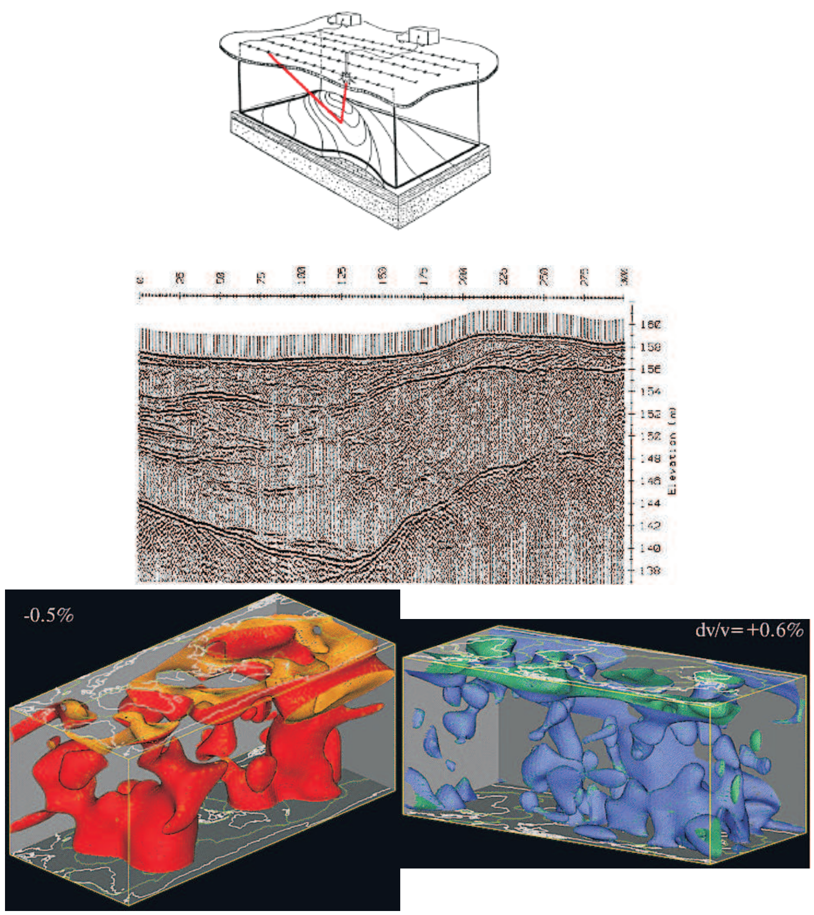


Figure 2.4: **Geology/geophysics/seismology**: oil prospecting, deep Earth imaging, earthquakes prediction.

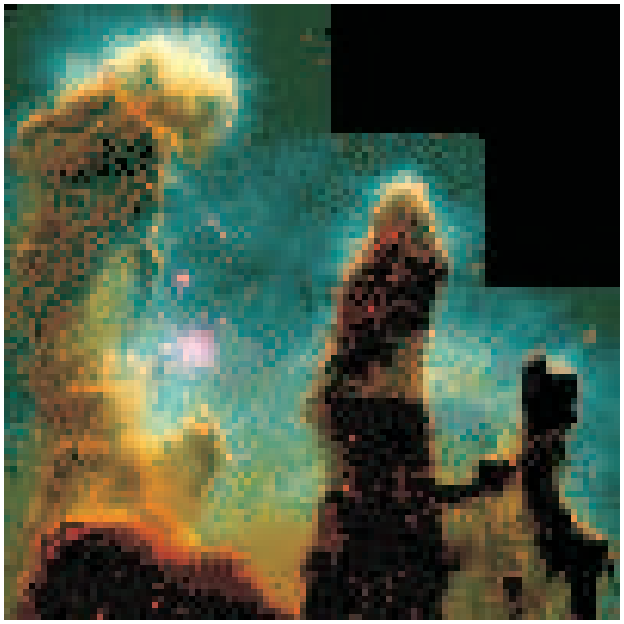


Figure 2.5: **Astronomy**

Figure 2.6: **Archeology**:Tutanchamun has a CT scan.

Chapter 3

History

• 1895 Röntgen discovers new radiation, which he calls X-ray. Receives Nobel Prize in 1901.
• 1905-1906 Lorenz solves "Radon transform" inversion in $3D$.
• 1917 Radon publishes his $2D$ reconstruction.
• 1925 Ehrenfest solves the n-dimensional problem.
• 1936 Cramer and Wold solve the reconstruction problem in statistics recovering a probability distribution from its marginal distributions.
• 1936 Eddington recovers the distribution of star velocities from their radial components.
• 1956 Bracewell solves inverse problem of radio astronomy.
• 1958 Korenblyum (Ukraine) develops the first X-ray scanner for medical purposes.
• 1963 Cormack (South Africa, then USA) implements tomographic reconstructions for an X-ray scanner.
• 1969 Hounsfield builds an X-ray scanner. The first medical scanner (with Ambrose) in 1972.
• 1979 Hounsfield and Cormack receive Nobel Prize in medicine.

3.1 General types of CT

- **Transmission**: the radiation transverses the body and is detected emerging "on the other side." Example: standard clinical X-ray CT.

- **Reflection**: the radiation bounces back and is detected where it was emitted. Examples: some instances of the ultrasound and geophysics

Allan MacLeod Cormack (1924–1998)

Sir Godfrey Hounsfield (1919–2004)

Johann Radon (1887–1956;

Wilhelm Conrad Röntgen (1845–1923;

imaging.

- **Emission**: the radiation is emitted inside the body and is detected emerging outside. Examples: clinical SPECT (single photon emission tomography) and PET (positron emission tomography), plasma diagnostics, nuclear reactors testing, detection of illicit nuclear materials.

3.2 What kind of mathematics is involved?

Anything you might want: Fourier (harmonic) analysis, differential equations, geometry (integral, differential, algebraic), complex analysis (including several variables), microlocal analysis, group representation theory, discrete mathematics, probability theory and statistics, numerical analysis.

3.3 Some major modalities of CT (Computed Tomography)

- **X-ray** CT is the most commonly used version.
- **SPECT** (Single Photon Emission Tomography)
- **PET** (Positron Emission Tomography)
- **MRI** (Magnetic Resonance Imaging, based upon the Nuclear Magnetic Resonance effect)
- **Ultrasound** Tomography
- **Optical** Tomography
- **Electrical impedance** Tomography

And MANY more:

Thermoacoustic, photoacoustic, ultrasound modulated optical, acousto-electric, magneto-acoustic, elastography, electron tomography, radar and sonar, Internet tomography, discrete tomography, ...

3.4 What features one should pay attention to?

• **Contrast**: variation between tissues in their response to radiation.
• **Resolution**: size of distinguishable details.

- **Uniqueness** of determining the unknown quantity.
- **Inversion** methods (formulas, algorithms).
- **Stability** of inversion. **Ill-posed problems!**
- **Incomplete data** effects.
- **Range** conditions.

3.5 PDE classification

Most imaging methods reduce mathematically to determining coefficients of a PDE from boundary data. The features we discussed in the previous section are closely related to the type of the PDE involved. The main types arising are:

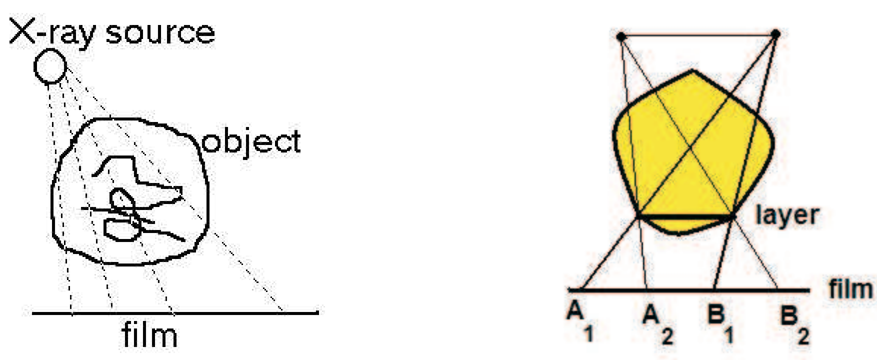
- **Transport equation** (X-ray CT, SPECT, PET)
- **Elliptic equations** (OT, EIT)
- **Wave equation** (TAT/PAT, SAR, Ultrasound imaging)

Lately, new breeds of tomographic techniques have been arising, the so-called hybrid (or coupled physics) methods, where one can relay upon some additional **internal** information.

Chapter 4

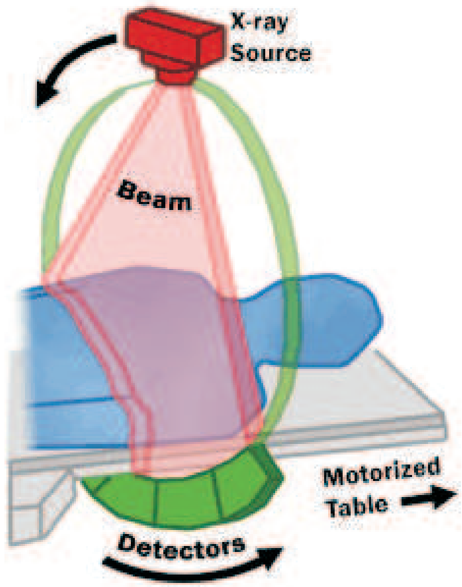
"Standard" CT. X-ray and Radon transform

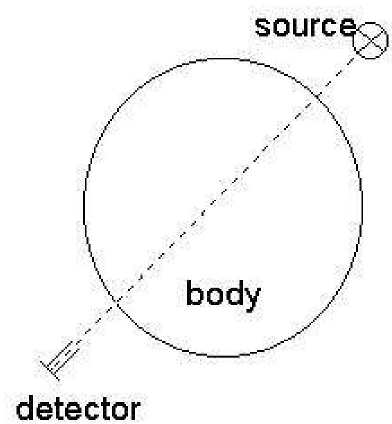
4.1 X-ray projection imaging: X-ray pictures and "Old tomography"



X-ray picture and old tomography

4.2 X-ray CT





Computed X-ray tomography. Cone beam geometry and parallel beam geometry.

4.3 Beer's law and Radon transform

4.3.1 Beer's law

The relative drop of intensity at a distance Δx at the location x is

$$\frac{\Delta I}{I} = -\mu(x)\Delta x$$

where $\mu(x)$ is the **attenuation coefficient** of the tissue at x.
This function μ is the tomogram we are looking for!

This leads to the ODE

$$\frac{dI}{dx} = -\mu(x)I$$

Thus, if the initial intensity is I_0 and after traversing the line L the intensity at the detector is I_1, then $I_1 = I_0 e^{-\int\limits_L \mu(x)dx}$, or

$$\int\limits_L \mu(x)dx = \log\frac{I_0}{I_1}.$$

(**We'll see later a deeper explanation through the radiative transfer equation.**)

We thus know all (theoretically, in practice just a finite number of) line integrals of $\mu(x)$.

Q.: **Can one recover a (piece-wise smooth) function $f(x)$ of two variables from all its line integrals?**

A.: **Yes** (under some reasonable conditions).

Definition 1. ***Radon (X-ray) transform*** *maps a function $f(x)$ in $2D$ into the set of all its line integrals:*

$$f(x) \mapsto Rf = g(l) := \int\limits_{l} f(x)ds.$$

Divergent beam (or ray) transform

$$(Df)(a,\omega) := \int\limits_{0}^{\infty} f(a + t\omega)dt \ \ with\ a, \omega \in \mathbb{R}^2, |\omega| = 1,$$

is another incarnation of an essentially the same operation. Here a is the source location and ω is the direction vector of the beam.

X-ray CT problem boils down to inversion of the Radon transform in $2D$.

This is **Integral Geometry**, i.e. studying functions and other object from their integral rather than local (differential) properties.

In $2D$ **X-ray and Radon transforms** mean the same.

4.3.2 3D case

In $3D$, **X-ray transform** still produces line integrals

$$f(x) \mapsto Pf = g(l) := \int\limits_{l} f(x)ds, \quad \text{where } l \text{ is a line,}$$

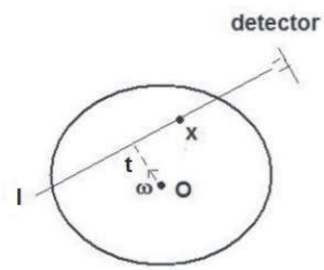
while **Radon transform** uses planar integrals:

$$f(x) \mapsto Rf = g(\Pi) := \int\limits_{\Pi} f(x)d\sigma, \quad \text{where } \Pi \text{ is a plane.}$$

In $3D$, the X-ray data is overdetermined (the space of lines in 3D is four-dimensional). The Radon data, however, is determined in all dimensions.

More options are available in higher dimensions: k-plane transforms, $1 \leq k \leq n-1$.

4.3.3 Parameterizations of the Radon transform:



$$(Rf)(t,\omega) = g(t,\omega) := \int\limits_{x\cdot\omega=t} f(x)dx = \int\limits_{-\infty}^{\infty} f(t\omega + s\omega^{\perp})ds,$$

where $\omega^{\perp}$ is the counterclockwise rotation through 90^o of $\omega = (\omega_1, \omega_2)$: $\omega^{\perp} = (-\omega_2, \omega_1)$. Notice $g(t,\omega) = g(-t,-\omega)$.

Another useful, albeit overdetermined, parametrization (to be discussed later) is analogous to the divergent beam transform one:

$$f(x) \mapsto F(a,b) := \int\limits_{-\infty}^{\infty} f(a+tb)dt \text{ with } a,b \in \mathbb{R}^2.$$

(Its natural multi-dimensional analog looks the same.)

4.3.4 Sinograms

A **sinogram** is the density plot of the Radon transform of a function, see Fig. 4.1

Exercise 2. *Why is it called a sinogram?*
Hint: *What is the Radon transform of a small single dot?*

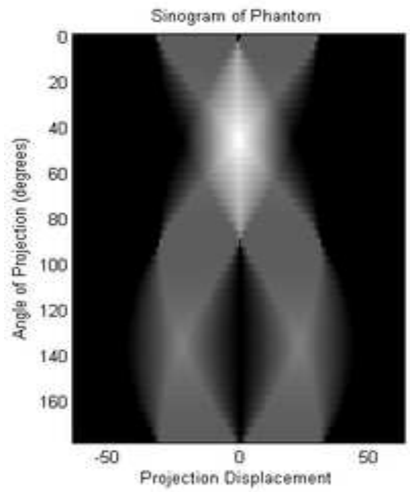


Figure 4.1: Sinogram (density plot of the Radon transform) of two diagonally touching squares.

4.4 Properties of Radon (= X-ray) transform in $2D$

Here we explore the basic properties of the Radon transform on the plane. We will not deal here with precise conditions on the functions involved, assuming them being "sufficiently nice" (whatever this means).

4.4.1 Notations:

S^1- the circle of unit vectors ω.

$T = S^1 \times \mathbb{R}$ - cylinder of points (t, ω) labeling the lines: $x \cdot \omega = t$.

Points (t, ω) and $(-t, -\omega)$ need to be identified, which turns T into the (infinitely wide) Möbius strip.

Ω - unit disk $|x| < 1$ on the plane. We will usually assume that our objects (functions) are located (supported) inside Ω.

Functions $f(x)$ to be recovered will be either defined in the whole space, or supported inside Ω only. The common spaces needed will be $L_2(\mathbb{R}^2)$ of square integrable functions on the plane with the inner product

$$(f_1, f_2) = \int\limits_{\mathbb{R}^2} f_1(x)\overline{f_2(x)}dx,$$

similar space $L_2(T)$ on the cylinder with

$$(g_1, g_2) = \int\limits_{T} g_1(t,\omega)\overline{g_2(t,\omega)}d\omega ds,$$

and the Schwartz space $\mathcal{S}$ of infinitely differentiable functions on the plane, which decay with all their derivatives faster than any power of $|x|$ at infinity: for any α and $N > 0$

$$|\frac{\partial^\alpha f(x)}{\partial x^\alpha}| \leq C_N(1+|x|)^{-N}.$$

Analogous Schwartz space can be defined on the cylinder T, where decay is understood with respect to the linear variable s. We will also denote by $C_0^\infty(\mathbb{R}^2)$ the space of smooth compactly supported functions on the plane, and a similar space on T

4.4.2 Radon transform in 2D

$$f(x) \mapsto g(t,\omega) := Rf(t,\omega) := \int\limits_{x\cdot\omega=t} f(x)ds = \int\limits_{-\infty}^{\infty} f(t\omega + s\omega^\perp)ds,$$

where $\omega^\perp$ is the counterclockwise rotation through 90^o of $\omega = (\omega_1, \omega_2)$: $\omega^\perp = (-\omega_2, \omega_1)$.

4.4.3 Shift - invariance and Fourier transform

Let $a \in \mathbb{R}^2$ and $T_a f(x) = f(x+a)$. Then

$$RT_a f(t,\omega) = \int\limits_{x\cdot\omega=t} f(x+a)dx$$
$$= \int\limits_{y\cdot\omega=s+a\cdot\omega} f(y)dy = Rf(\omega, s+a\cdot\omega) = T_{a\cdot\omega}Rf(t,\omega).$$

The resulting equality $RT_a f(t,\omega) = T_{a\cdot\omega} Rf(t,\omega)$ shows invariance of the Radon transform with respect to shifts.

Thus, **Fourier transform** must be useful when studying the Radon transform.

4.4.4 Rotation - invariance and Fourier series

Let A be a 2×2 rotation matrix and $M_A f(x) = f(Ax)$. Then

$$\begin{aligned} RM_A f(t,\omega) &= \int\limits_{x\cdot\omega=t} f(Ax)dx = \int\limits_{A^{-1}y\cdot\omega=s} f(y)dy \\ &= \int\limits_{y\cdot A\omega=s} f(y)dy = Rf(A\omega,s) = M_A Rf(t,\omega). \end{aligned}$$

The equality $RM_A = M_A R$ shows invariance of the Radon transform with respect to rotations.

Thus, **Fourier series** must be useful when studying the Radon transform.

4.4.5 Dilation - invariance and Mellin transform

Let $a > 0$ be a positive number and D_a be the radial dilation operator $D_a f(x) = f(ax)$. A straightforward calculation reveals a commutation relation between the Radon transform and dilations:

$$(RD_a f)(t,\omega) = \int\limits_{x\cdot\omega=t} f(ax)dx = \frac{1}{a}\int\limits_{y\cdot\omega=at} f(y)dy = \frac{1}{a}D_a(Rf)(t,\omega),$$

where in the last expression the dilation D_a is applied with respect to the scalar variable t only.

This quasi-invariance with respect to dilations shows that **Mellin transform** must be useful when studying Radon transform.

Although dilations and shifts might seem to be different, one knows a simple transform $t \in \mathbb{R}^+ \mapsto x = \ln t \in \mathbb{R}$, which establishes an isomorphism between the half-line $\mathbb{R}^+$ as a group with respect to multiplication and the real axis $\mathbb{R}$ with respect to addition. This mapping clearly translates dilation invariance into shift invariance. Thus, one understands that there must be a transform on functions defined on the half-line that corresponds to the Fourier transform on the whole axis. This is the **Mellin transform**.

Exercise 3. *Derive the formulas for the direct and inverse Mellin transforms on $\mathbb{R}^+$ by changing variables $x \mapsto t = e^x$ in the Fourier transform on $\mathbb{R}$.*

4.4.6 Relations with Fourier transform. Projection-slice theorem

The $2D$ Fourier transform of a function $f(x)$ on $\mathbb{R}^2$ will be denoted by $\widetilde{f}(\xi)$:

$$\widetilde{f}(\xi) := \int f(x)e^{-i\xi\cdot x}dx. \tag{4.1}$$

The $1D$ Fourier transform of a function $g(t)$ on $\mathbb{R}$ will be denoted by $\widehat{g}(\sigma)$:

$$\widehat{g}(\sigma) := \int g(t)e^{-i\sigma t}dt. \tag{4.2}$$

The same notation will be applied to functions $g(t, \omega)$:

$$\widehat{g}(\sigma, \omega) := \int g(t, \omega)e^{-i\sigma t}dt. \tag{4.3}$$

The next statement (called **projection-slice**, **Fourier-slice**, or **central slice formula**) is central for studying the X-ray and Radon transforms.

Theorem 4. *Under appropriate conditions on a function $f(x)$ on $\mathbb{R}^2$ (e.g., being in L_2 suffices), the following relation holds:*

$$\widehat{Rf}(\sigma, \omega) = \widetilde{f}(\sigma\omega). \tag{4.4}$$

Before proving this theorem, we just notice that it says that taking $1D$ Fourier transform of the Radon transform Rf of a function f on the plane, one recovers the $2D$ Fourier transform of f (albeit in polar coordinates). Thus, one immediately gets the following consequences concerning uniqueness of reconstruction of f from Rf and inversion formulas:

Corollary 5.

1. ***Uniqueness of reconstruction:*** *If f is in L_2 and $Rf = 0$ almost everywhere, then $f = 0$ almost everywhere (**uniqueness**).*

2. ***An inversion procedure:*** *Function f can be recovered from its Radon transform Rf by the following formula (**Fourier inversion**):*

$$f = \left(\mathcal{F}_{2,x\to\xi}\right)^{-1} \mathcal{F}_{1,t\to\sigma} Rf. \tag{4.5}$$

Here $\mathcal{F}_j$ is the j-dimensional Fourier transform between the functions of variables indicated in the subscript.

Proof of Theorem 4. Let us write the Radon transform of f as follows:

$$Rf(t,\omega) = \int\limits_{-\infty}^{\infty} f(t\omega + s\omega^{\perp}) ds.$$

Then

$$\widehat{Rf}(\sigma,\omega) = \int dt \int ds f(t\omega + s\omega^{\perp}) e^{-i\sigma t} = \int\limits_{\mathbb{R}^2} f(x) e^{-i\sigma x\cdot\omega} dx = \widetilde{f}(\sigma\omega). \quad (4.6)$$

$\square$

4.4.7 X-ray/ Radon transform as a mapping between function spaces

Consider the weighted space $L_2(S^1 \times [-1,1], (1-s^2)^{-1/2})$ that consists of functions on the finite cylinder $S^1 \times [-1,1]$ that have finite weighted L_2 norm

$$\int\limits_{-1}^{1} \int\limits_{S^1} |g(t,\omega)|^2 \frac{d\omega ds}{\sqrt{1-s^2}}.$$

Theorem 6. *The Radon transform operator R is linear and maps $L_2(\Omega)$ continuously into $L_2(S^1 \times [-1,1], (1-t^2)^{-1/2})$.*

Linearity is obvious.

Consider a function $f \in L_2(\omega)$ (i.e., it is square integrable and supported in the unit disk). Then

$$|Rf(t,\omega)|^2 = |\int\limits_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\omega + s\omega^{\perp}) ds|^2$$

Consider the function

$$\chi(s) = \begin{cases} 1 \text{ when } |s| \le \sqrt{1-t^2} \\ 0 \text{ otherwise} \end{cases}.$$

Then the last integral can be rewritten as

$$\int_{-\infty}^{\infty} \chi(s) f(t\omega + s\omega^{\perp}) ds.$$

Thus, using Cauchy-Schwartz inequality $|\int \chi f|^2 \leq \int \chi^2 \int f^2$, one gets

$$|Rf(t,\omega)|^2 \leq 2\sqrt{1-t^2} \int_{-\infty}^{\infty} |f(t\omega + s\omega^{\perp})|^2 ds$$

Now dividing by $\sqrt{1-t^2}$ both sides and integrating with respect to ω and s, one gets the required inequality. Q.E.D.

Notice that R is also continuous as a mapping into the larger space $L_2(S^1 \times [-1,1])$ without a weight. This is a weaker statement than the Theorem above.

This theorem might create the wrong impression that Radon transform does not change the smoothness class of a function. In fact, as we will see later, it does make functions smoother.

4.4.8 Backprojection

Backprojection is the dual operator $R^\sharp : L_2(T) \mapsto L_2(\mathbb{R}^2)$ to $R : L_2 \mapsto L_2$, i.e. such that

$$(Rf, g)_{L_2(T)} = (f, R^\sharp g)_{L_2(\mathbb{R}^2)}.$$

A simple calculation starting with the left hand side and changing the order of integration should reveal what $R^\sharp$ is:

$$\begin{aligned}
(Rf, g)_{L_2(T)} &= \int_{S^1} d\omega \int_{-\infty}^{\infty} dt Rf(t,\omega) g(t,\omega) \\
&= \int_{S^1} d\omega \int_{-\infty}^{\infty} dt \int_{x\cdot\omega = t} f(x) dx g(t,\omega) \\
&= \int_{S^1} d\omega \int_{\mathbb{R}^2} dx f(x) g(x\cdot\omega,\omega) \\
&= \int_{\mathbb{R}^2} f(x) \left(\int_{S^1} g(x\cdot\omega,\omega) d\omega \right) dx
\end{aligned}$$

Thus,

$$R^\sharp g(x) = \int\limits_{S^1} g(x\cdot\omega,\omega)d\omega.$$

Geometrically, to get the value of $R^\sharp g$ at a point x, one chooses a line passing through x, which implies that the parameters of this line are $(x\cdot\omega,\omega)$, picks the corresponding measured data $g(x\cdot\omega,\omega)$, and then averages over all lines passing through x.

Another straightforward calculation leads to the value of the composition $R^\sharp R$. Indeed,

$$\begin{aligned} R^\sharp Rf(x) &= \int\limits_{S^1} Rf(x\cdot\omega,\omega)d\omega \\ &= \int\limits_{S^1}\int\limits_{-\infty}^{\infty} f((x\cdot\omega)\omega + t\omega^\perp)dtd\omega. \end{aligned}$$

Thus, we integrate $f(x)$ over each line passing through x and then integrate over the angle. This is almost like the polar integration with the pole at x, except two things: 1) the integral is doubled, since we integrate over the whole lines rather than polar rays; 2) the radius factor needed for the polar integration is missing. So, if we take these issues into the account and introduce the radial factor, we end up with

$$R^\sharp Rf(x) = \int \frac{2f(y)}{|x-y|}dy = \frac{2}{|x|} * f(x). \tag{4.7}$$

Thus, backprojecting the Radon data, one gets a blurred version of the original image $f(x)$. We will learn how to de-blur it later. The name "backprojection" can be explained by the following simple interpretation of its action. Imagine that whenever a detector is hit by a photon coming from a direction L, the detectors "projects it back," or geometrically draws the line L. As the result, one gets a web of lines, density of which can be understood as $R^\# g$. In other words, a single point source will produce the overlap of a bunch of lines passing through it (Fig. 4.2). This is the same as to say that $R^\#\delta = \frac{2}{|x|}$, and thus instead of the δ-pick, one gets its blurred version $2/|x|$.

4.5 Inversion formulas

An explicit inversion formula can be obtained by using the projection-slice formula (4.4) and Fourier inversion formula (4.5). Indeed, passing from carte-

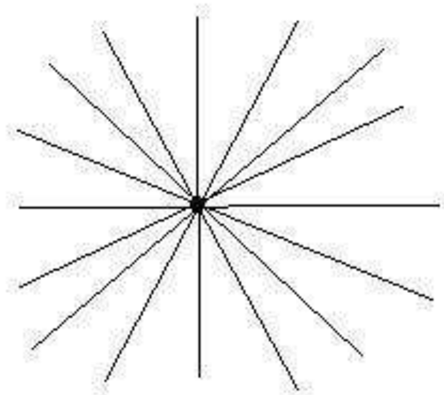


Figure 4.2: Backprojection of a point source

sian coordinates ξ to the polar ones (σ, ω) (where $\xi = \sigma\omega$), we obtain

$$f(x) = \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2} \widetilde{f}(\xi) e^{ix\cdot\xi} d\xi \tag{4.8}$$

$$= \frac{1}{(2\pi)^2} \int\limits_{S^1} \int\limits_{0}^{\infty} \widetilde{f}(\sigma\omega) e^{i\sigma x\cdot\omega} \sigma d\sigma d\omega \tag{4.9}$$

$$= \frac{1}{2} \frac{1}{(2\pi)^2} \int\limits_{S^1} \int\limits_{-\infty}^{\infty} \widetilde{f}(\sigma\omega) e^{i\sigma x\cdot\omega} |\sigma| d\sigma d\omega, \tag{4.10}$$

where we used that $\widetilde{f}((-\sigma)(-\omega)) = \widetilde{f}(\sigma\omega)$.

Now we use the projection-slice formula (4.4) to get

$$\begin{aligned} f(x) &= \tfrac{1}{2}\tfrac{1}{(2\pi)^2} \int\limits_{S^1} \int\limits_{-\infty}^{\infty} \widehat{g}(\sigma, \omega) e^{i\sigma x\cdot\omega} |\sigma| d\sigma d\omega \\ &= \tfrac{1}{4\pi} \int\limits_{S^1} d\omega \left(\tfrac{1}{2\pi} \int\limits_{\mathbb{R}} \widehat{g}(\sigma, \omega) e^{i\sigma t} |\sigma| d\sigma \right) |_{t=x\cdot\omega} \end{aligned} \tag{4.11}$$

Looking at the expression in parentheses, one recognizes the inverse $1D$ Fourier transform applied in variable σ to $\widehat{g}(\sigma, \omega)|\sigma|$. If it were just $\widehat{g}(\sigma, \omega)$, the result would be $g(t, \omega)$. What is the role of the factor $|\sigma|$? If it were $i\sigma$, we would get $\frac{\partial g(t,\omega)}{\partial t}$. Representing $|\sigma| = -i\mathrm{sgn}\sigma \times i\sigma$, where

$$\mathrm{sgn}x = \begin{cases} 1 \text{ when } x > 0 \\ -1 \text{ when } x < 0 \end{cases},$$

one sees that one has in the parenthesis the function

$$H\frac{\partial g}{\partial t}(t,\omega).$$

Here we denoted by H the **Hilbert transform** that acts on a function $u(t)$ as follows:

$$Hu(t) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \widehat{u}(\sigma)e^{i\sigma t}(-i\mathrm{sgn}\sigma)d\sigma.$$

This transform happens to be well known and important and can be rewritten without using Fourier transform as follows:

$$Hu(t) = \frac{1}{\pi}p.v.\int\limits_{-\infty}^{\infty}\frac{u(s)}{t-s}ds. \tag{4.12}$$

The improper integral in this formula diverges at the point $s = t$. It has to be understood in the **principal value** ($p.v.$) sense, i.e. as

$$\lim_{\epsilon\downarrow 0}\left(\int\limits_{-\infty}^{t-\epsilon}\cdots + \int\limits_{t+\epsilon}^{\infty}\cdots\right),$$

or, equivalently, as

$$-\lim_{\epsilon\downarrow 0}\int\limits_{\epsilon}^{\infty}\frac{u(t+s)-u(t-s)}{t-s}ds.$$

So,

$$Hu(t) = -\frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int\limits_{\epsilon}^{\infty}\frac{u(t+s)-u(t-s)}{t-s}ds. \tag{4.13}$$

Finally, we look at the last operation to be performed in (4.11):

$$u(t,\omega) \to \int_{S^1} u(x\cdot\omega,\omega)d\omega$$

to recognize in it the backprojection $R^{\#}u(x)$. Thus, we get the celebrated

FILTERED BACKPROJECTION (FBP) FORMULA:

$$f = \frac{1}{4\pi} R^{\#} H \frac{d}{dt} (Rf) . \tag{4.14}$$

Hurray!!

Here the **filtration** part is $H\frac{d}{dt}$ and the backprojection is $R^{\#}$, which explains the name of the formula.

Remark 7.

1. *The filtration is responsible for removing the blur, which would have occurred if just backprojection were used.*
2. *The filtration can also be done* ***after*** *the backprojection:*
$$f = \frac{1}{4\pi} \Lambda R^{\#} (Rf) , \tag{4.15}$$
where $\Lambda = \sqrt{-\Delta}$ is the ***Calderon operator****. This is the so called* ***ρ-filtered backprojection****.*
3. *One can also do a partial filtering before, and partially after the backprojection. In order to do this, one needs to introduce the* ***Riesz potential*** *operator I^{α}, acting on functions defined on $\mathbb{R}^n$, when $\alpha < n$:*
$$\widetilde{I^{\alpha} f}(\xi) := |\xi|^{-\alpha} \tilde{f}(\xi). \tag{4.16}$$
Then the following series of inversion formulas holds in 2D:
$$f = \frac{1}{4\pi} I^{-\alpha} R^{\#} I^{\alpha-1} (Rf) , \alpha < 2. \tag{4.17}$$
Then $\alpha = 0$ corresponds to the filtered backprojection and $\alpha = 1$ - to the ρ-filtered one.
4. *In dimension n analogous series of inversion formulas hold for the Radon (not the X-ray) transform:*
$$f = \frac{1}{4\pi} I^{-\alpha} R^{\#} I^{\alpha-n+1} (Rf) , \alpha < n. \tag{4.18}$$
5. *The projection slice formula, as we have already mentioned, leads to what is called Fourier inversion formula (4.5).*

4.5.1 It works!!!

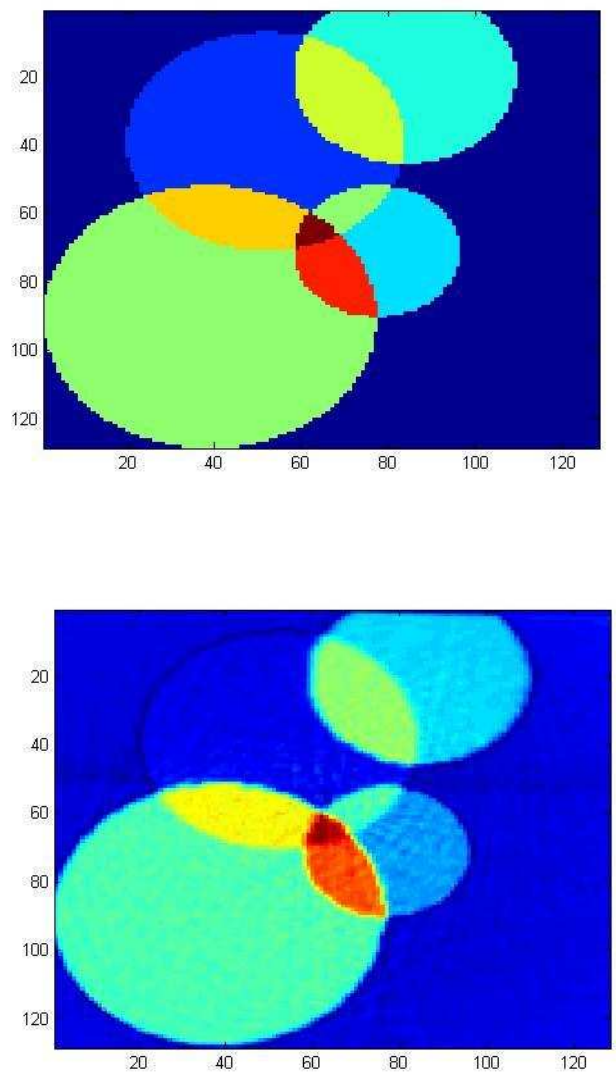


Figure 4.3: A phantom (top) and its reconstruction (bottom)

4.5.2 A word of caution: left inversion versus inversion

One might be surprised that there are several different inversion formulas (4.17). Why would one need several inversions for the same operator? Don't they all give the same? The answer is a resounding "no". What's up here?

As we already know, the Radon transform has zero kernel on "reasonably good" (e.g., compactly supported and piece-wise continuous) functions. Thus, such f can be uniquely reconstructed from Rf. However, the word "inversion" that we used carries some danger of misunderstanding. Namely, absence of kernel is NOT necessarily invertibility even for finite matrices.

Exercise 8. *1. Prove that if A is a square matrix with zero kernel, it is invertible, i.e. exists $B := A^{-1}$ such that $BA = I, AB = I$.*

2. Prove that if the kernel of any (not necessarily square) matrix A is zero, then there exists a **left inverse** *B such that $BA = I$. Show that if the matrix is not square, but rectangular, this is* **not a right inverse***, i.e. $AB \neq I$. Prove that the left inverse is NOT uniquely defined.*

3. Prove that a matrix with zero kernel is square (and thus invertible) if and only if its range coincides with the whole ambient space.

This exercise shows that absence of the kernel implies unique recovery of f, but the reconstruction can be done by <u>different</u> **left inverses**. It also shows that unique inversion arises only if additionally we know that the range of the matrix is the whole space. Thus, knowledge of range of Radon transform is germane to its inversion. We will tackle the range later and see that it is far from being the whole space.

Certainly, all left inverses coincide on any element from the range of the operator. However, they act differently on functions outside the range. Since one never has precise data, the chance that the data given by a scanner belongs to the range, is zero. And then, on the data with errors different left inverses do give different results. This again emphasizes the importance of knowing the range.

4.5.3 Non-uniqueness

Well, one should not get too elated about our uniqueness of inversion theorem. It requires us to know the line integrals of a function for **all** lines. Clearly, we can measure by the scanner only finitely many such integrals.

Then clearly there cannot be any uniqueness: a function from an infinite dimensional space cannot be recovered from the values of finitely many linear functionals. It seems that the situation is even worse. Namely,

Theorem 9. *[249] Let $\omega_1, \dots, \omega_k$ be a finite set of unit vectors and $K \subset \mathbb{R}^2$ be a compact, and $f \in C_0^\infty(K)$. Let $K_0 \subset U \subset K$, where $K_)$ is a compact and U is open. Then there exists $f_0 \in C_0^\infty(K)$, such that $f_0 = f$ on K_0 and $Pf_0(\cdot, \omega_j) = 0, j = 1, \dots, k$.*

This seems to be rather devastating: even allowing all values of the linear variable t, but restricting to a finite number of projections (projection directions), one gets non-uniqueness. Indeed, it is known that the Fourier transform of any function f "invisible" under the k projections, has he following representation in polar coordinates:

$$\tilde{f}(\sigma\omega) = \sum_{m>k} i^m \sigma^{-1} J_{m+1}(\sigma) q_m(\omega),$$

Where J_m is the Bessel function of order m of the first kind and q_m is a polynomial of degree m. Thus, due to the known behavior of Bessel functions, the Fourier transform $\tilde{(f)}(\xi)$ is mostly concentrated in the set where $|\xi| > k$, and thus the values inside the ball $|\xi| < k$ should be determined rather reliably.

This gives us the following rule of thumb: a function is reliably determined from its k Radon projections, if 1) it is *a-priori* expected that $|\tilde{f}(\xi)|$ is small for $|\xi| > b > 0$ (this is called essential b-band limitedness of f), for some $b < k$, and 2) the reconstruction algorithm produces a function which is also essentially b- band limited.

4.6 Stability of inversion

We have the Radon transform operator R that we would like to invert, i.e. determine the function f by its transform $g = Rf$. By stability one means the situation when small errors in the data g lead to small errors in the reconstructed function f. In other words, one would like to have an estimate of the following kind:

$$\|f\| \leq C|||g|||, \tag{4.19}$$

where $\| \dots \|$ and $||| \dots |||$ are some norms in the spaces of the originals f and data g respectively. The larger the norm $||| \dots |||$ needs to be taken to satisfy this (e.g., it might need to involve more derivatives of g), the less stable the problem becomes. Sometimes, the problem is so unstable, that no reasonable norm would work.

Inversion of the Radon transform is only mildly unstable. Namely, the following result holds:

Theorem 10. *Let f be supported inside of a given ball B. Then*

$$\frac{1}{C_{B,s}}\|Rf\|_{H^{s+1/2}} \leq \|f\|_{H^s} \leq C_{B,s}\|Rf\|_{H^{s+1/2}}$$

for some $C_{B,s} > 0$. Here H^s denotes the Sobolev space of order s.

The right hand side inequality is not hard, while the left hand one requires somewhat more work to prove. Let us show the right inequality for $s = 0$. Then we have

$$\|f\|^2_{L^2} = const\|\widetilde{f}\|^2_{L^2} = const \int |\widetilde{f}(\xi)|^2 d\xi.$$

Switching to polar coordinates $\xi = \sigma\omega$ gives

$$\begin{aligned} &\|f\|^2_{L^2} = const \textstyle\int |\widetilde{f}(\sigma\omega)|^2|\sigma| d\sigma d\omega \\ &= const \textstyle\int \left(|\widehat{Rf}(\sigma,\omega)||\sigma|^{1/2}\right)^2 d\sigma d\omega \leq |||g|||^2_{H^{1/2}}. \end{aligned}$$

This claim follows from (4.11), where $|\sigma|$ term is responsible for gaining the "1/2 of a derivative" in the norm of Rf. In other words, Rf is "1/2 of a derivative smoother than f."

This "smoothing" is responsible for some instability of the reconstruction. Indeed, the inversion formula involves multiplication of the Fourier transform $\widehat{g}(\sigma,\omega)$ of the data by the **filter** $|\sigma|$. If the data has a small, but fast oscillating error, this error will make a small contribution located at high values of σ. Due to the growing factor $|\sigma|$, the contribution of this error will be large. The situation would have been worse if the filter needed grew as a higher power of σ, or even worse, exponentially. In the latter case of **exponentially unstable problems**, only a very blurred (low frequency) version of f could be reconstructed.

4.7 Fourier series and Cormack inversion formulas

We have discovered the rotational invariance of the Radon transform, which indicates that Fourier series expansions might be useful. Let $f(x)$ be a function on $\mathbb{R}^2$. We can write it in polar coordinates as $f(r,\theta)$ and then expand into the Fourier series with respect to the polar angle θ:

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)e^{in\theta}.$$

Analogously, $g(t,\omega) = Rf(t,\omega)$ can be expanded into the Fourier series with respect to the polar angle ϕ, where $\omega = (\cos\phi, \sin\phi)$:

$$g(t,\phi) = \sum_{n=-\infty}^{\infty} g_n(t)e^{in\phi}.$$

Exercise 11. *Prove that if $g = Rf$, then g_n depends on f_n only. Do this using rotational invariance only, without computations.*

Then one can find direct formulas relating these Fourier coefficients. Indeed, according to the previous exercise,

$$g_n(t)e^{in\phi} = R(f_n(r)e^{in\theta}).$$

Thus, it is sufficient to compute only the case when $\phi = 0$ (i.e., "vertical" line L).

Exercise 12. *Do this calculation to show that*

$$g_n(t) = 2\int_t^{\infty} f_n(r)\cos n \arccos\left(\frac{t}{r}\right)\frac{rdr}{\sqrt{r^2-t^2}}. \tag{4.20}$$

The expression $T_n(x) := \cos n \arccos x$ is known to be a polynomial, called the nth Tchebychev polynomial of the first kind. Thus,

$$g_n(t) = 2\int_t^{\infty} f_n(r)T_n\left(\frac{t}{r}\right)\frac{rdr}{\sqrt{r^2-t^2}}. \tag{4.21}$$

Thus, inversion of the Radon transform R reduces to the inversion of the sequence of integral transforms (4.20), which are called **transforms of Abel type**. Fortunately, they can be inverted explicitly, which was done by A. Cormack (see, e.g., [193]). The inversion uses another important invariance consideration:

Exercise 13. *Find out how the Abel transform (4.20)-(4.21) commutes with the dilation $f(r) \to f(ar)$.*

4.8 Range conditions for the Radon transform

As we have already discussed, knowledge that the range of R is the whole space (whatever this means) is needed to know that the inversion procedure is unique. As we will see, the range of R in appropriate function spaces is in fact very small, of infinite co-dimension, and so there exists a huge variety of non-equivalent (for imperfect data) inversion procedures.

Let $f(x)$ be a locally integrable function on the plane such that it decays at infinity faster than any power of $|x|$. Then clearly $Rf(t, \omega)$ decays when $t \to \infty$ faster than any power of $|t|$.

Here are the two **range conditions** for a function $g(t, \omega)$ to be the Radon transform of a function $f(x)$ of that class:

1. **Evenness** $g(t, \omega) = g(-t, -\omega)$

2. **Moment conditions** (also called Cavalieri conditions or Helgason-Ludwig conditions): for any integer $k \geq 0$, the kth moment

$$G_k(\omega) := \int_{-\infty}^{\infty} t^k g(t, \omega) dt$$

is the restriction to the unit circle S^1 of a homogeneous polynomial of degree k with respect to ω.

The evenness is straightforward.

The moment conditions follow from the direct calculation that gives

$$G_k(\omega) = \int_{\mathbb{R}^2} (x \cdot \omega)^k f(x) dx.$$

Since for any vector x, the function $(x \cdot \omega)^k$ of ω is linear and homogeneous, we conclude that $(x \cdot \omega)^k$ is a homogeneous polynomial of degree k with respect to ω. Coefficients of this polynomial depend on x as a parameter. Clearly, integrating with respect to x we still get a homogeneous polynomial of degree k.

Notice that without sufficient decay of f, the moments cannot be defined.

As it turns out, the evenness and moment conditions happen to be also sufficient:

Theorem 14. *Let $g(t, \omega)$ be a smooth and compactly supported function. It can be represented as Rf for a smooth and compactly supported function f if and only if g is even and satisfies moment conditions.*

A similar statement holds for the case of the Schwartz space $\mathcal{S}$ of smooth fast decaying functions.

The important implication of this range description is that there exist many different left "inverse" operators $R^{(-1)}$ such that $R^{(-1)}R = I$. One can argue that if $g = Rf$, then all these left inverses should act the same on g. Although this is correct, the fact of life is that g will always be measured with errors, and thus will NOT belong to the range of R. Outside the range, though, different left inverses act differently.

The range conditions for the Radon transform have a rather simple interpretation. Consider a smooth function $F(x)$ on the plane. Then it defines in polar coordinates (r, θ) a smooth function $G(r, \theta) := F(r\theta)$. Let us now have an arbitrary smooth function $G(r, \theta)$ of $r \in \mathbb{R}$ and the unit vector $\theta \in S^1$. We wonder whether it defines a smooth function $F(r\theta) := G(r, \theta)$ on the plane. In other words, how are smoothness in cartesian and polar coordinates related?

Exercise 15. *1. Prove that such a smooth F is defined if and only if G is even: $G(-r, -\theta) = G(r, \theta)$, and each expression*

$$\frac{\partial^k g}{\partial r^k}(0, \theta), k = 0, 1, 2, \ldots$$

extends to a homogeneous polynomial of degree k on the plane.

2. Show that the range conditions for R in Fourier domain coincide with the conditions in 1.

4.9 Support theorem

The proof of the following "hole" theorem, due to S. Helgason, can be found in [114].

Theorem 16. *Let K be a convex compact set on the plane $\mathbb{R}^2$ and a continuous function $f(x)$ decay at infinity faster than any power of $|x|$ (i.e., $|x|^k|f(x)|$ is bounded for any k). Then, if $\int\limits_L f(x)dx = 0$ for any line L not intersecting K, then $f(x) = 0$ outside K (in other words, the support of f is in K).*

Remark 17. *It is interesting to note that if we assume that $f(x)$ decays as $|x|^{-k}$ for a* **fixed** *value of k, no matter how large, the statement of the theorem is no longer true, as the next exercise (which requires some knowledge of complex analysis) shows.*

Exercise 18. *Consider $\mathbb{R}^2$ as the complex plane $\mathbb{C}$, where $z = x + iy$. We then pick a large integer value of k and define the function $\phi(x, y)$ that is equal to z^{-k} when $z = x + iy$ is outside the unit disk at the origin, and is smooth everywhere (in particular, inside the disk). Prove that, for a sufficiently large k, the integral of this function along any line L not intersecting the disk is equal to zero.*

Chapter 5

A survey of Fourier transform and harmonic analysis

5.1 An idea of harmonic analysis

We will try to provide here a crude cartoon of the idea of what is called **harmonic (or Fourier) analysis**. The reader might enjoy reading the wonderful historical survey "Harmonic analysis as exploitation of symmetry" by G. Mackey [174].

Let A be an $n \times n$ matrix and e be its eigenvector corresponding to the eigenvalue λ:

$$Ae = \lambda e.$$

Theorem 19. *Assume that λ is a simple eigenvalue, i.e. it has unique (up to a scalar multiple) eigenvector e. Let matrix B commute with A. Then e must be an eigenvector of B as well.*

Proof. $ABe = BAe = B(\lambda e) = \lambda Be$, so Be (as well as e) is an eigenvector of A corresponding to λ. Since λ is simple, Be must differ from e by a scalar factor only: $Be = \mu e$. QED

Corollary 20. *If A has a basis of eigenvectors and all the eigenvalues are simple, then any matrix B that commutes with A is diagonal in this basis.*

The idea of harmonic analysis: **If B that you are interested in commutes with a matrix A with simple eigenvalues, then choosing the basis of eigenvectors of A simplifies (i.e. diagonalizes) B.**

Remark 21. *Notice that when λ is not simple (i.e., has multiplicity), the conclusion is incorrect: while Be still is an eigenvector that corresponds to λ, it does not have to be proportional to e. In other words, B can "move around" eigenvectors corresponding to the same eigenvalue λ of A.*

Example 22. *Let $A = diag(2,2,3)$, i.e.*

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

then any matrix B of the form

$$A = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

(where stars denote arbitrary numbers) commutes with A, but is not necessarily diagonal. The reason is the multiplicity of $\lambda = 2$.

This problem of multiplicity (which is rather common) can be alleviated by the following observation, which gives a more general harmonic analysis principle:

Theorem 23. *Let $A_1, ..., A_m$ be $n \times n$ matrices and $e_1, ..., e_n$ be a basis such that all its vectors are eigenvectors of* ***all*** *the matrices A_j (i.e., $A_j e_i = \lambda_{ij} e_i$). Assume that for each two of these vectors there is a matrix among A_j whose eigenvalues corresponding to these two vectors are distinct. Let B be a matrix that commutes with all matrices A_j. Then B is diagonal in the basis $\{e_i\}$.*

Exercise 24. *Prove this statement.*

Example 25.

$$A_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},\ A_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix},\ A_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Show that these matrices satisfy the conditions of the above theorem. Prove that any B commuting with all of them is diagonal.

Exercise 26. *In the example above matrices A_j commute with each other. Does it always have to be the case under the conditions of the Theorem above?*

Main idea of harmonic analysis extended: **If B that you are interested in commutes with a family of matrices A_j satisfying the conditions of Theorem 23, then choosing the basis of joint eigenvectors of A_j simplifies (i.e. diagonalizes) B.**

5.2 Fourier series expansions

Although we consider here only functions of one variable, the considerations can be easily extended to Fourier series for periodic functions in several variables (e.g., [259]).

Space (infinite-dimensional) $H = L_2[-\pi/h, \pi/h]$ of functions $f(x)$ on $[-\pi/h, \pi/h]$ whose square $|f(x)|^2$ is integrable. **Hermitian metric** (analog of the dot-product) on this space

$$(f, g) = \int_{-\pi/h}^{\pi/h} f(x)\overline{g(x)}dx, ||f||^2 = \int_{-\pi/h}^{\pi/h} |f(x)|^2 \, dx.$$

(See [137][sections 32, 33]).

Consider the sequence of functions $e_k = \frac{1}{\sqrt{2\pi}} \exp(ikhx)$, $k = 0, \pm1, \pm2, ...$

Lemma 27.

$$(e_k, e_j) = \frac{1}{h}\delta_{ij},$$

where $\delta_{ij} = \begin{cases} 1 & \textit{when } i = j \\ 0 & \textit{otherwise} \end{cases}$ *is the Kronecker's delta.*

Exercise 28. *Prove the lemma.*

Theorem 29. *Functions*

$$\sqrt{\frac{h}{2\pi}} e^{ikhx}, \; k = 0, \pm1, \pm2, ...$$

form an ortho-normal basis of $L_2[-\pi/h, \pi/h]$.

Orthogonality and normalization have just been proven. Completeness (i.e. that there is no function orthogonal to all these exponents) is proven in Fourier analysis books (e.g., [137] section 34).

Definition 30. *For any function $f(x) \in L_2[-\pi/h, \pi/h]$ its* ***Fourier coefficients*** *are defined as*

$$\widehat{f}(k) = (f, \frac{1}{\sqrt{2\pi}} e^{ikhx}) = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} f(x) e^{-ikhx} dx. \tag{5.1}$$

Fourier analysis*: take a function $f(x) \in L_2[-\pi/h, \pi/h]$ and produce its Fourier coefficients $\widehat{f}(k)$.*

Fourier synthesis (Fourier series expansion)*: for any $f(x) \in L_2[-\pi/h, \pi/h]$*

$$f(x) = \frac{h}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{ikhx}. \tag{5.2}$$

Fourier series expansion represents any function $f(x)$ on $[-\pi/h, \pi/h]$, or alternatively any $2\pi/h$-periodic function on $\mathbb{R}$, as a sum of harmonic oscillations (sinusoidal waves).

Theorem 31. *(Plancherel's or Parseval's)*

$$\int_{-\pi/h}^{\pi/h} |f(x)|^2 \, dx = h \sum |f_k|^2 \, .$$

Exercise 32. *prove this theorem*

5.3 Properties of Fourier series expansions

Remark 33. *Notice an important thing: The domain of the original function is bounded (the segment $[-\pi/h, \pi/h]$), and the domain of the Fourier transform $\widehat{f}(k)$ is discrete. Relevance of this observation will be clearer later.*

Remark 34. *All functions e^{ikhx} with integer k are $\frac{2\pi}{h}$-periodic. Hence, it is natural to consider the sum in (5.2) as a $\frac{2\pi}{h}$-periodic function too.*

Considering the function $f(x)$ as $\frac{2\pi}{h}$-periodic, we can talk about its values on the whole axis.

Exercise 35. *Prove that the Fourier coefficients of a $\frac{2\pi}{h}$-periodic function $f(x)$ can be computed as*

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{a}^{a+2\pi/h} f(x)e^{-ikhx}dx$$

*for **any** value of a.*

Remark 36. *Convergence in (5.2) in general does not hold in point-wise sense, but as L_2-convergence (i.e., in an average sense).*

Namely, the following convergence results hold:

Theorem 37. *(see, e.g. [137] Sect. 34) If $f \in L_2[-\pi/h, \pi/h]$ and $f_N(x)$ is the N-th partial sum of (5.2), then*

$$\int\limits_{-\pi/h}^{\pi/h} |f(x) - f_N(x)|^2 \, dx \underset{N\to\infty}{\longrightarrow} 0.$$

Theorem 38. *(e.g., [137] Sect. 15-17, Tolstov Ch1, Sect. 10)*

1. *If $f(x) \in C[-\pi/h, \pi/h]$, has bounded first derivative, and is periodic (i.e., $f(-\pi/h) = f(\pi/h)$), then*

$$\lim_{N\to\infty} f_N(x) = f(x)$$

for all $x \in [-\pi/h, \pi/h]$.

2. *If $f(x)$ like in the first part of the theorem, except of a finite number of finite discontinuities, then*

$$\lim_{N\to\infty} f_N(x) = \begin{cases} \frac{f(x+0)+f(x-0)}{2} & \text{when } x \in (-\pi/h, \pi/h) \\ \frac{f(-\pi/h+0)+f(\pi/h-0)}{2} & \text{when } x = \pm\pi/h \end{cases}.$$

5.4 Smoothness of $f(x) \Leftrightarrow$ decay of Fourier coefficients f_k.

In the theorem below, the word "periodic" applied to a function f on $[-\frac{\pi}{h}, \frac{\pi}{h}]$ means $f(-\frac{\pi}{h}) = f(\frac{\pi}{h})$.

Theorem 39.

1. *If $f(x)$ is continuous (in fact, $f \in L_1$ suffices), then*
$$\left|\widehat{f}(k)\right| \leq const.$$
2. *If $f(x)$ is periodic and has a continuous first derivative, then*
$$\left|\widehat{f}(k)\right| \leq \frac{const}{|k|} \text{ for } k \neq 0.$$
3. *If $f(x)$ has n continuous derivatives and the first $n-1$ of them are periodic, then*
$$\left|\widehat{f}(k)\right| \leq \frac{const}{|k|^n} \text{ for } k \neq 0.$$
4. *If*
$$\left|\widehat{f}(k)\right| \leq \frac{const}{|k|^\alpha} \text{ for some } \alpha > 1 \text{ and } k \neq 0,$$
then $f(x)$ is continuous and periodic.
5. *If*
$$\left|\widehat{f}(k)\right| \leq \frac{const}{|k|^\alpha} \text{ for some } \alpha > 2 \text{ and } k \neq 0,$$
then $f(x)$ is continuous, periodic, and once continuously differentiable.

Exercise 40. *Prove this theorem.*

Exercise 41. *Find necessary and sufficient conditions on the Fourier coefficients for a function $f(x)$ to be real.*

Exercise 42. *Find the Fourier coefficients of the function $f(x) = x$ on $[-\pi, \pi]$.*

Exercise 43. *Let on $[-\pi, \pi]$*
$$f(x) = \begin{cases} x + \pi & \text{on } [-\pi, 0] \\ -x + \pi & \text{on } [0, \pi] \end{cases}.$$
Find the Fourier coefficients.

5.5 Relations with shifts and derivatives

Functions on $[-\frac{\pi}{h}, \frac{\pi}{h}]$ will be extended $\frac{2\pi}{h}$-periodically to $\mathbb{R}$. Then we can **shift** (translate) them:

$$(T_t f)(x) := f(x+t).$$

Exercise 44.

- *Prove that for functions $e_k = e^{ikhx}$, one has $(T_t e_k)(x) = \lambda_{k,t} e_k(x)$ for some constant $\lambda_{k,t}$. Find $\lambda_{k,t}$.*

- *Prove that if a **continuous** function $e(x)$ on $\mathbb{R}$ satisfies $T_t e = \lambda_t e_k$ for all $t \in \mathbb{R}$ and some numbers λ_t, then $e(x) = Ce^{\mu x}$ for some $\mu \in \mathbb{C}$ and a constant C.*
  Prove that if such $e(x)$ is $2\pi/h$ periodic, then $e(x) = Ce_k(x)$ for some $k \in \mathbb{Z}$.
  These statements are not necessarily true if e is discontinuous.

- *Let B be a linear operator acting on functions of x, such that B commutes with shifts (i.e., $BT_t = T_t B$ for all $t \in \mathbb{R}$). If Be_k is continuous, then $Be_k = \beta_k e_k$ for some constant β_k, which is called the **Fourier multiplier** corresponding to B. In other words, B is diagonal in the basis $\{e_k\}$.*

- *Prove that $\frac{d^l}{dx^l}$ commutes with T_t for any t and find the corresponding Fourier multipliers.*

- *Check when the operator of multiplication by a given function $g(x)$ commutes with the shifts.*

So, if there is a linear transformation B commuting with all shifts T_t, then it has exponents as eigenvectors. I.e., $Be_k = \beta_k e_k$ for some numbers β_k depending on B. In particular, action of B on any function f is easy to write down in terms of the Fourier expansion: if $f(x) = \sum_k f_k e_k$, then $Bf = \sum_k \beta_k f_k e_k$. The common examples of such operations are differentiation and convolution (considered in the next section).

5.6 Product-convolution relations

Definition 45. *Convolution $f*g$ of two $\frac{2\pi}{h}$-periodic functions (say, belonging to $L_1[-\pi/h, \pi/h]$) is defined as*

$$f * g(x) = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} f(y)g(x-y)dy.$$

Exercise 46.

1. *Prove that convolution is commutative.*

2. *Prove that in the definition of convolution one can integrate over arbitrary segment of the length of the period.*

3. *Prove that (under appropriate smoothness conditions)*

$$\frac{d}{dx}(f * g) = \left(\frac{df}{dx} * g\right) = \left(f * \frac{dg}{dx}\right)$$

4. *Prove that convolution commutes with shifts:*

$$T_t(f * g) = (T_t f * g)$$

5. *Prove that* $(f * g)_k = \sqrt{2\pi} f_k g_k$

5.7 Convolution on $\mathbb{R}^n$

Definition 47. *Convolution of two (sufficiently fast decaying, so the integral converges) functions on $\mathbb{R}^n$ is*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy.$$

Exercise 48. *Prove the following* ***properties of convolution****:*

1. ***Linearity****:* $f(x) * (ag_1(x) + bg_2(x)) = a(f * g_1) + b(f * g_2)$

2. ***Commutativity****:* $f * g = g * f$.

3. ***Commuting with shifts***: *If* $(T_a f)(x) := f(x+a)$, *then* $f * (T_a g) = T_a(f * g)$.

4. ***Commuting with differentiation***: *If* $f \in L^1_{loc}(\mathbb{R}^n)$ *and* g *is smooth and compactly supported, then* $f * g$ *is smooth and* $\frac{d^l}{dx^l}(f * g) = (f * \frac{d^l g}{dx^l})$.

5. *Convolution operation has* ***no unity*** *element, i.e., there is no function* $i(x)$ *such that* $i * f = f$ *for all functions* f *(make reasonable assumptions on the functions* f *in order for this to make sense). (If you know distributions (Section* **??***), there* ***is*** *a distribution, namely Dirac's* δ-*function (see Section* **??***), with this property.))*

One can make precise[1] the following statement (converse to statement 3 of the Exercise):

"Theorem": Any *linear operator* A *mapping functions of* $x \in \mathbb{R}^n$ *into functions of* $x \in \mathbb{R}^n$ *and commuting with shifts, i.e.,* $AT_a = T_a A$ *for all* $a \in \mathbb{R}^n$, *is a convolution, i.e.* $Af = f * g$ *for some* g.

5.8 Fourier transform

Definition 49. *Fourier transform of a function* $f(x)$ *on* $\mathbb{R}$ *is defined as*

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) := \int\limits_{-\infty}^{\infty} f(x) e^{-i\xi x} dx. \tag{5.3}$$

In $\mathbb{R}^n$,

$$\widehat{f}(\xi) = \int e^{-ix\cdot\xi} f(x) dx,$$

where $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$.

This is well defined when $f(x)$ decays sufficiently fast, e.g. $f \in L_1(\mathbb{R}^n)$. If $f \in L_2(\mathbb{R}^n)$, then the definition should be carefully adjusted (e.g., [259]).

Theorem 50. *(**Plancherel's Theorem/Parseval's identity**). The following identity holds:*

$$\int\limits_{\mathbb{R}^n} |f(x)|^2 dx = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right|^2 d\xi. \tag{5.4}$$

[1] considering objects more general than functions

I.e., operator $(2\pi)^{-n/2}\mathcal{F} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ is isometric.

Exercise 51.

- *Prove that the adjoint operator to* $\mathcal{F}$ *is given by*

$$(\mathcal{F}^* g)(x) = \int\limits_{\mathbb{R}^n} g(\xi) e^{i\xi\cdot x} d\xi. \tag{5.5}$$

- *Prove that* $(2\pi)^{-n}\mathcal{F}^*$ *is inverse to* $\mathcal{F}$. **Hint:** *Use isometric property of* $(2\pi)^{-n/2}\mathcal{F}$.

We thus get the **Fourier inversion formula**:

$$f(x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \widehat{f}(\xi) e^{i\xi\cdot x} d\xi. \tag{5.6}$$

Exercise 52. *Show that this inversion formula for* $n = 1$ *can be obtained formally as a limit when* $h \to 0$ *from (5.2).*

Remark 53. *The Fourier inversion is almost the same as the direct Fourier transform, one only needs to flip the sign of the independent variable and introduce the constant factor* $(2\pi)^{-n}$*:*

$$\left(\mathcal{F}^{-1} g\right)(x) = (2\pi)^{-n} \left(\mathcal{F} g\right)(-x).$$

Fourier transform $f \mapsto \widehat{f}$ provides **Fourier analysis** of a function, i.e., finds the amplitudes with which different oscillating exponents enter the function. Fourier inversion $\widehat{f} \mapsto f$ provides **Fourier synthesis**, synthesizing the function back from these amplitudes.

5.9 Properties of FT

Exercise 54. *Prove the following properties of the Fourier transform in* $\mathbb{R}^n$*:*

1. **Dilation invariance:** *Let* $f_r(x) = f(rx)$. *Then* $\widehat{f_r}(\xi) = r^{-n}\widehat{f}(r^{-1}\xi)$
2. **Homogeneity preservation:** *If* $f(x)$ *is homogeneous of order* a, *then its Fourier transform* $\widehat{f}(\xi)$ *is homogeneous of order* $-a-n$, *where* n *is the number of independent variables.*

3. **Shift invariance:** $\widehat{T_y f}(\xi) = e^{i\xi\cdot y}\widehat{f}(\xi)$.

4. **Rotational invariance:** *If A is a **rotation** in $\mathbb{R}^n$, then $\widehat{f(Ax)}(\xi) = \widehat{f(x)}(A\xi)$. (The formula is somewhat more complicated when A is an arbitrary invertible linear transformation, not necessarily a rotation (orthogonal matrix). Work this case out.)*

5. $\widehat{\frac{\partial^{|\alpha|} f}{dx^\alpha}}(\xi) = (i\xi)^\alpha \widehat{f}(\xi)$, *where* $(\xi)^\alpha = (\xi_1)^{\alpha_1}(\xi_2)^{\alpha_2}\dots(\xi_n)^{\alpha_n}$.

6. *Find a formula for $\widehat{x^l f}(\xi)$ in terms of $\widehat{f}(\xi)$ (**Hint**: use the previous question and the remark above).*

7. *Show that the following relation between convolution and Fourier transform holds:*
$$\widehat{f * g} = \widehat{f}\widehat{g}.$$

8. *Analogously,* $\widehat{fg} = \frac{1}{(2\pi)^n}\widehat{f} * \widehat{g}$.

9. ***Parseval identity:*** $\int f\,\widehat{g}dx = \int \widehat{f}\,g dx$

5.10 Some common functions

Let us recall some common functions, which we have considered before:

Box function
$$\Pi(x) = \begin{cases} 1 & \text{when } |x| \le 0.5 \\ 0 & \text{otherwise} \end{cases}$$

Gaussian function
$$G(x) = e^{-\frac{x^2}{2}}$$

Cardinal sine function, or $\sin cx$
$$\sin cx = \begin{cases} \frac{\sin x}{x} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}$$

Normal density with the mean μ and standard deviation σ
$$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Exercise 55. *Find the Fourier transform of the box function.*

5.11 Fourier transform of the Gaussian

Check that the Gaussian $G(x)$ satisfies the differential equation

$$\frac{dG}{dx} + xG = 0.$$

Exercise 56.

- *Prove that the Fourier transform of the Gaussian satisfies the same equation:*
$$\frac{d\widehat{G}}{d\xi} + \xi\widehat{G} = 0. \tag{5.7}$$

- *Prove that the Fourier transform of the Gaussian is the Gaussian (with an extra constant factor):*
$$\widehat{G}(\xi) = \sqrt{2\pi}e^{-\frac{\xi^2}{2}}$$

  Hint: *Use (5.7) and Plancherel's theorem.*

5.12 Paley-Wiener theorem

Several versions of the **Paley-Wiener theorem**, which describe Fourier transforms of various classes of functions, are combined here ($\Im\xi$ denotes the imaginary part of a complex vector ξ):

Theorem 57.

1. *Fourier transform $\widehat{f}(\xi)$ of a function $f \in C_0^\infty(\mathbb{R}^n)$ supported in the ball $\{x \mid |x| \leq A\}$ is an entire function in $\mathbb{C}^n$ and satisfies for any $N > 0$ the estimate:*
$$|\widehat{f}(\xi)| \leq C_N(1 + |\xi|)^{-N}e^{A|\Im\xi|.} \tag{5.8}$$
   The converse statement also holds: any entire function with such estimates is the Fourier transform of a smooth function supported in that ball.

2. *Fourier transform* $\widehat{f}(\xi)$ *of a function* $f \in L_2(\mathbb{R}^n)$ *supported in the ball* $\{x \mid |x| \leq A\}$ *is an entire function in* $\mathbb{C}^n$*, which is square integrable along* $\mathbb{R}^n$ *and satisfies in* $\mathbb{C}^n$ *the estimate*

$$|\widehat{f}(\xi)| \leq Ce^{A|\xi|}. \tag{5.9}$$

   The converse statement also holds.

3. *A function* f *belongs to the Schwartz class* $\mathcal{S}(\mathbb{R}^n)$ *if and only if its Fourier transform* $\widehat{f}(\xi)$ *belongs to the Schwartz class.*

5.13 Smoothness and decay of Fourier transform

As for the Fourier series, smoothness of a function is tied to the decay of its Fourier transform (although exact theorems must be stated rather carefully). An example:

Exercise 58. *Prove:*

1. *If* $f \in L_1(\mathbb{R})$*, then* $\widehat{f}$ *is bounded (in fact, even continuous and tending to zero at infinity).*

2. *If* $f, f' \in L_1(\mathbb{R})$*, then* $\left|\widehat{f}(\xi)\right| \leq C(1+|\xi|)^{-1}$.

3. *If* $f, f', f'', ..., f^{(n)} \in L_1(\mathbb{R})$*, then* $\left|\widehat{f}(\xi)\right| \leq C(1+|\xi|)^{-n}$.

4. *If* $\left|\widehat{f}(\xi)\right| \leq C(1+|\xi|)^{-\alpha}, \alpha > 1$*, then* f *is bounded and continuous.*

5. *If* $\left|\widehat{f}(\xi)\right| \leq C(1+|\xi|)^{-\alpha}, \alpha > 2$*, then* f *has a bounded and continuous derivative.*

6. *If for a function* f *on* $\mathbb{R}^n$, $|\widehat{f}(\xi)| \leq C_N(1+|\xi|)^{-N}$ *for all* $N > 0$ *(i.e.,* $\widehat{f}$ *decays faster than any power), then* $f \in C^\infty(\mathbb{R}^n)$.

5.14 Smoothing

If one can guarantee fast decay of the Fourier transform of a function, then the function is smooth. This is the basis of standard smoothing procedures. Namely, suppose function $f(x)$ is not smooth (e.g., $f \in L_2(\mathbb{R})$ only). Assume that we have another function $w(x)$ whose Fourier transform $\widehat{w}$ is smooth and very fast decaying (for instance, even supported on a finite interval). Then taking convolution $f * w$, we get a smooth function. The reason is that $\widehat{f * w}$ coincides (up to a constant factor) with $\widehat{f}\widehat{w}$, which decays due to the decay of $\widehat{w}$. In other words, multiplication of $\widehat{f}$ by $\widehat{w}$ "filters out" high frequencies ξ, making the original function smoother. This is why $\widehat{w}$ is often called a **filter**, or a **window function** (the window that allows certain frequencies through), while w is called a **mollifier**. There are quite a few window functions used in practice. The simplest one is the box function $\Pi(\xi)$ (the rectangular window). It has the disadvantage that it is not continuous, hence after the convolution the function will not decay fast, and one has to deal with long "tails." One also uses Gaussian filters, where the window function is the Gaussian $G_a(\xi) = \exp(-a\xi^2)$. There are many more commonly used filters.

Can one make the smoothed (**mollified**) function $f * w$ close to the original one? We cannot make it equal to f, since there is no identity element for the convolution. So, the question is whether one can find an approximate identity under the convolution, i.e. a sequence of functions w_n such that $w_n * f \to f$ for a reasonable class of functions f and a reasonable notion of convergence. This can be done. The simplest way of constructing approximate identities is the following:

Theorem 59. *Let $w(x)$ be smooth, supported on $[-1, 1]$, and such that $\int\limits_{\mathbb{R}} w(x) = 1$. Define $w_n(x) = nw(nx)$. Then for any continuous function $f(x)$ on $\mathbb{R}$ the convolutions $f_n = w_n * f$ converge when $n \to \infty$ to f, where convergence is uniform on any finite interval.*

Analogous statements hold for different classes of functions, for instance for L_1-functions (then convergence also need to be understood in L_1-sense).

5.15 Sobolev spaces

The relations between smoothness of a function and decay of its Fourier transform is seen best in the **Sobolev spaces** of functions.

Definition 60. *(**Sobolev spaces of a positive integer order**) Let $k \geq 0$ be an integer. A function f belongs to the **Sobolev space** $H^k(\mathbb{R}^n)$ if $D^\alpha f \in L^2(\mathbb{R}^n)$ for any $|\alpha| \leq k$ and*

$$\|f\|_{H^k}^2 := \int_{\mathbb{R}^n} \sum_{|\alpha|\leq k} |D^\alpha f(x)|^2 dx.$$

Here the derivatives $D^\alpha f$ are understood in the distributional sense.

The following statement can be proven using the properties of Fourier transform:

Theorem 61. *$f \in H^k(\mathbb{R}^n)$ if and only if*

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \langle \xi \rangle^{2k} d\xi < \infty$$

(here $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$). The norm $\|f\|_{H^k}$ is equivalent to

$$\left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \langle \xi \rangle^{2k} d\xi \right)^{1/2}.$$

Now one can define **Sobolev spaces of arbitrary (not necessarily integer and positive) order**:

Definition 62. *A function f belongs to the **Sobolev space** $H^s(\mathbb{R}^n), s \in \mathbb{R}$ if*

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \langle \xi \rangle^{2s} d\xi < \infty.$$

The norm is defined as

$$\|f\|_{H^s}^2 := \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \langle \xi \rangle^{2s} d\xi.$$

5.16 Sampling

Let $f(x)$ be a function on $\mathbb{R}$. *Sampling* of f consists of evaluating this function at a sequence of points $\{x_j\}_{j=-\infty}^{\infty}$ (for example, $x_j = jh$ with a fixed step h). The questions that arise are: Can the function be uniquely recovered from these values? If not, how precisely can it be recovered?

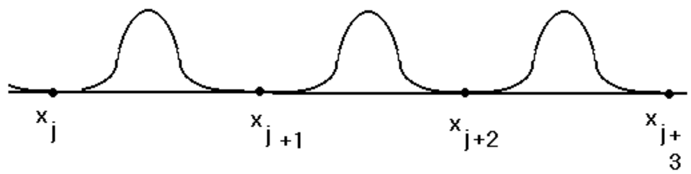


Figure 5.1: nnn

It is clear that in general one cannot recover an arbitrary smooth function from its values at a discrete set of points, since one can create an arbitrarily smooth function that is zero at any of these points, and not entirely zero:

Here is the important idea of sampling: in order for a non-zero function to be zero at a sequence jh, its Fourier transform must contain sufficiently high frequencies (otherwise the function does not change fast enough to be zero at two consecutive points and non-zero in between). This rule of thumb is made precise by the following Nyquist condition and sampling theorem.

Proposition 63. *If $h > \pi/b$, then there exist non-zero b-band-limited functions that vanish at the points $\{jh\}$. In other words, a b-band-limited function in general* ***cannot*** *be determined from its values at the points $\{jh\}$ with $h > \pi/b$.*

Proof is done by constructing an example. Let $h > \pi/b$, and hence $c = b - \pi/h > 0$. Consider any smooth (infinite differentiable) function $\zeta(\xi)$ supported in $[-c, c]$ and define $\psi(x) = \mathcal{F}^{-1}(\zeta)$. Then $\psi(x)$ is c-band-limited (since $\widehat{\psi} = \zeta$) and $\psi(x)$ decays faster than any power of x. Paley-Wiener theorem claims that ψ is analytically extendable for the whole complex plane $\mathbb{C}$ and satisfies there the estimate

$$|\psi(z)| \leq const\, e^{c|\,\mathrm{Im} z|}.$$

Now let $f(x) = \sin(\pi x/h)\psi(x)$. Then $f \in L_2(\mathbb{R})$, it extends analytically to the complex plane, and it satisfies the inequality

$$|f(z)| = |\sin(\pi z/b)|\,|\psi(z)| \leq const\, e^{\frac{\pi}{h}|\,\mathrm{Im} z|} e^{c|\,\mathrm{Im} z|} = const\, e^{h|\,\mathrm{Im} z|}. \quad (5.10)$$

We have used here the Euler's formula for the sine function:

$$\sin z = \sin(x+iy) = \frac{1}{2}\left(e^{ix-y} - e^{-ix+y}\right),$$

which implies right away that

$$|\sin z| \le e^{|y|} = e^{|\,\mathrm{Im} z|}.$$

Now Paley-Wiener theorem and (5.10) imply that f is b-band-limited. Since its values at all points kh with integer k are zeros, looking at these values one cannot distinguish $f(x)$ from zero.

The conclusion is that in order to be able to reconstruct a b-band-limited function, one needs to sample it at least with the step size $h = \pi/b$.

Definition 64. *Condition*

$$h \le \frac{\pi}{b} \tag{5.11}$$

is called the ***Nyquist condition****. A function sampled with a larger step is said to be* ***under-sampled****, and the one sampled with a smaller step is* ***over-sampled****.*

The question arises whether Nyquist condition is sufficient. The answer is given by the following famous sampling theorem:

Theorem 65. *(Whittacker-Kotel'nikov-Shannon) Let $h \le \frac{\pi}{b}$. Then*

1. *Any b-band-limited function $f(x)$ can be recovered from its values at the points kh as follows:*

$$f(x) = \sum_k f(kh)\sin c\left(\frac{\pi}{h}(x-kh)\right). \tag{5.12}$$

The series converges as a series of functions in $L_2(\mathbb{R})$.

2. *The Fourier transform of f can be obtained as follows:*

$$\widehat{f}(\xi) = \frac{h}{\sqrt{2\pi}}\sum_k f(kh)e^{-ikh\xi}. \tag{5.13}$$

3. If g is another b-band-limited function, then

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = h\sum_k f(kh)\overline{g(kh)}. \tag{5.14}$$

Proof. Consider the function $\widehat{f}(\xi)$ on $\mathbb{R}$. It is zero outside segment $[-\pi/h, \pi, h]$ (use the Nyquist condition and the fact that f is b-band-limited). Let us expand it into the Fourier series on this segment. It will give

$$\widehat{f}(\xi) = \frac{h}{\sqrt{2\pi}}\sum_k \left(\widehat{f}\right)^{\hat{}}_k e^{ikh\xi}.$$

It is clear (since $\widehat{f}$ is zero outside $[-\pi/h, \pi, h]$) that

$$\left(\widehat{f}\right)^{\hat{}}_k = f(-kh).$$

This gives (5.13).

In order to get (5.14), we use the Parseval's equality for Fourier transform

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi = \int_{-\pi/h}^{\pi/h} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi,$$

and for the last integral use the Parseval's equality for Fourier series:

$$\int_{-\pi/h}^{\pi/h} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi = h\sum_k \left(\widehat{f}\right)^{\hat{}}_k \overline{\left(\widehat{g}\right)^{\hat{}}_k} = h\sum_k f(kh)\overline{g(kh)}.$$

In order to obtain the sampling formula (5.12), we use the Fourier inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{ix\xi}d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\pi/h}^{\pi/h} \widehat{f}(\xi)e^{ix\xi}d\xi$$

and then plug (5.13) for $\widehat{f}(\xi)$:

$$f(x) = \frac{h}{2\pi}\sum_k f(kh)\int_{-\pi/h}^{\pi/h} e^{-ikh\xi}e^{ix\xi}d\xi.$$

Now direct integration leads to (5.12).

Exercise 66. *Look at the example of the cardinal sine function, its zeros, and its Fourier transform.. Why does not it provide a counterexample to the WKS sampling theorem when $h = \pi/b$?*

When $f \in L_2$, the series (5.12) converges in L_2-sense:

$$\left\| f - \sum_{|k|\le N} f(kh) \sin c(\frac{\pi}{h}(x-kh)) \right\|^2 \le C \sum_{|k|>N} |f(kh)|^2 \underset{N\to\infty}{\to} 0.$$

This, however, is not such a fast convergence. In the case when the function is over-sampled, i.e. $h < \pi/b$, one can significantly improve the speed of convergence.

Theorem 67. *Let f be b-band-limited, $h < \pi/b$, and γ be a smooth function such that $\gamma(x) = 0$ for $|x| > 1$ and $\int \gamma(x)dx = (2\pi)^{-1/2}$. Then*

$$f(x) = \sum_k f(kh)\widehat{\gamma}\left(\left(\frac{\pi}{h} - b\right)(x-kh)\right) \sin c(\frac{\pi}{h}(x-kh)). \tag{5.15}$$

Remark 68. *Notice that due to smoothness of γ, function $\widehat{\gamma}(x)$ decays at infinity very fast (faster than any power of $|x|$). This improves convergence of the series significantly.*

Proof goes as follows. Let us look at the expansion (5.12) again:

$$\widehat{f}(\xi) = \frac{h}{\sqrt{2\pi}} \sum_k f(kh)e^{-ikh\xi}. \tag{5.16}$$

It holds when $|\xi| \le \pi/h$. However, we know that the function $\widehat{f}(\xi)$ is supported on a smaller segment $[-b, b]$. The right hand side is the $2\pi/h$-periodic extension of the left one. Hence, the right hand side vanishes on $\left(\frac{\pi}{h} - b\right)$-neighborhoods of the points $\pm\frac{\pi}{h}$. This means that (5.16) holds on $[-\left(\frac{2\pi}{h} - b\right), \left(\frac{2\pi}{h} - b\right)]$ (which is wide than $[-\frac{\pi}{h}, \frac{\pi}{h}]$). Let us denote $\gamma_t(x) = \gamma(tx)$ and consider the function

$$w(\xi) = \left(\frac{\pi}{h} - b\right)^{-1} \sqrt{2\pi}\gamma_{\left(\frac{\pi}{h}-b\right)^{-1}} * \chi_{[-\frac{\pi}{h},\frac{\pi}{h}]}.$$

One checks by the definition of the convolution that $w = 1$ on $[-a, a]$ and is equal to zero outside $[-\left(\frac{2\pi}{h} - b\right), \left(\frac{2\pi}{h} - b\right)]$. This means that we can multiply

(5.16) by w and make it work on the whole axis:

$$\widehat{f}(\xi) = \frac{h}{\sqrt{2\pi}} \sum_k f(kh)w(\xi)e^{-ikh\xi},\ \xi \in \mathbb{R}.$$

Now taking inverse Fourier transform of this equality we end up with (5.15).

Exercise 69. *Write MATLAB scripts that evaluate the sums of* 10, 20, *and* 30 *terms of the Fourier series of the functions* $f(x) = x$ *and*

$$g(x) = \begin{cases} x + \pi & \text{on } [-\pi, 0] \\ -x + \pi & \text{on } [0, \pi] \end{cases}$$

on $[-\pi, \pi]$. *Graph them against the graphs of the original functions. Do you observe any phenomena discussed before? Do both functions display them?*

5.17 Mellin transform

Part II

Literature

Books and Surveys

Disclaimer: There are many other good sources, besides the ones mentioned below.

Fourier series

[267] for $1D$ and [259] for any dimension.

Fourier transform

A nice introduction to Fourier transform, distributions, and microlocal analysis is [262]. Several books by E. Stein provide Fourier analysis theory on different levels (e.g., [258] and the classics [259]). Körner's book [137] is a wonderful collection of essays, proofs, historical accounts, etc. concerning the Fourier analysis. Hörmander's volume [123] is a comprehensive (albeit very technical) account of Fourier analysis. The classical Natterer's book [193] shows how Fourier analysis works in tomography.

Sampling

See [193, Ch. III].

Harmonic analysis

A wonderful historical survey "Harmonic analysis as exploitation of symmetry" by G. Mackey [174]. See also some parts of [137].

Collections of articles on Inverse Problems and tomography

[29, 96, 104, 117, 119, 207, 235, 240, 274, 275, 280]

Integral geometry and Radon transform

Introductory texts of Radon transform and integral geometry [100, 114, 295].
More advanced math [75, 97, 115, 116, 131, 229, 245, 245].
Integral geometry in relation to tomography [140, 151, 193, 195, 207, 213].

Video lectures and slides on inverse problems and tomography

[142, 278, 279]

Tomography

An outdated, but still very useful collection of **non-technical surveys of physics**, mathematics, and challenges of various types of medical imaging [178]. See also [246].

Undergraduate texts [77, 82].

Classical **applied tilt texts** [118, 132].

Classical **texts on analysis of tomography** [193, 195].

More texts of this kind [79, 140, 175, 233, 249].

Emission tomography [85, 117, 140, 193, 195]
EIT (Electrical Impedance Tomography) [46, 62, 63]
MREIT, CDI, CDII (hybrid versions of EIT) [29, 244, 286]
Thermo-/Photoacoustics [4, 92, 128, 143, 146, 218, 222, 257, 271, 280, 285]
Hybrid methods [25, 141, 142, 153, 257, 279]
Synthetic aperture radar (SAR) [60, 61]

Inverse problems for PDEs

[129, 131, 272]

Microlocal analysis

Gentle introductions [198, 262].
Textbooks and lecture notes [93, 94, 180, 266].
More advanced [106, 107, 112, 123–125, 269]

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