



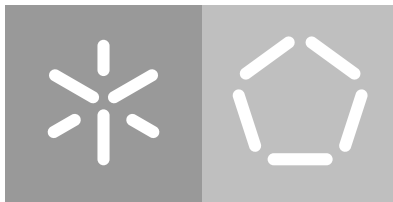
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Escola de Engenharia  
Departamento de Informática

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**Quantum Random Walks**  
**Second Part of Title**

**First Part of Subtitle**  
**Second part of Subtitle**

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**Luís Barbosa**  
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Write acknowledgements here

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## ABSTRACT

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Quantum computing is an emergent field that brings together quantum mechanics, computer science and information theory which promises improvements to classical algorithms such as simulation of quantum systems, cryptography, data base searching and many others. Quantum computing offers several different models for quantum walks, which have complexity advantages over their classical counterparts and many applications among which are element distinctness, searching problems, matrix product verification and hitting times in graphs. The present work offers a general theoretical overview, simulation and circuit implementation of the coined, staggered and continuous-time quantum walk models.

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## RESUMO

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Teste resumo

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## INTRODUCTION

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### 1.1 BRIEF HISTORY OF QUANTUM COMPUTING

The modern understanding of computer science was firstly announced by [Turing \(1936\)](#) where he developed the abstract concept of what is now called a *Turing machine*. These machines are the mathematical foundation of programmable computers, and Turing showed that there is a *Universal Turing Machine* that can be used to simulate any other Turing Machine. This means that if an algorithm can be executed in any piece of hardware, then there is a Universal Turing Machine that can accomplish the same task. This is known as the *Church-Turing thesis*, which connects the concept of what classes of algorithms can be run in some physical device with the mathematical framework of a Universal Turing Machine.

The paper published by Turing set in motion a series of events which led to the rapid advancement of electronic computers and computer science. One of the earliest theoretical models developed by John von Neumann (work later published in [von Neumann \(1993\)](#)), presented how to assemble all the necessary parts to create a computer with all the capabilities of a Universal Turing Machine. The true explosion of innovation in this field came after the invention of the transistor in 1947 by John Bardeen and Walter Brattain. The creation of the transistor led to an unprecedented growth quantified by [Moore \(1965\)](#) where he created *Moore's law*, stating that computer power will double with constant cost approximately every two years. Moore's law has roughly held true throughout the decades, by the ever increasing miniaturization of the transistor technology. However, conventional fabrication methods run into a problem of scale, as quantum effects begin to interfere more as the size of the devices becomes smaller.

[Feynman \(1959\)](#) recognized such a miniaturization was the way forward, and even predicted the problems quantum effects presented to a classical computer. With an amazing stroke of insight, Feynman imagined that these effects could be exploited given the right computational paradigm. Quantum computing begins to take form in later work developed by [Benioff \(1980\)](#), where the earliest quantum mechanical model of a computer was described. In this paper, Benioff showed that a computer working under the laws of quantum mechanics could be used to express a Schrödinger equation description of Turing machines. Shortly

after, Feynman (1982) pointed out that simulating quantum systems on classical computers is inefficient, and suggested using quantum computers for this purpose. Additional work in the following decade further explored this idea and showed that there are systems that quantum computers can simulate, which have no known efficient simulation on a classical computer, and even today this continues to be one of the most promising fields in quantum computing.

Driven by the work of Turing, Deutsch (1985) questioned if a stronger version of the Church-Turing thesis could be derived from the laws of physics. The strong Church-Turing thesis states that any algorithmic process can be simulated efficiently using a probabilistic Turing machine, and Deutsch was set to define some device that could efficiently simulate an arbitrary physical system. Whether Deutsch's formulation of a Universal Quantum Computer is sufficient for this function is still an open question, but what he accomplished was a challenge to the strong Church-Turing thesis by suggesting that there are tasks a quantum computer can accomplish efficiently that a probabilistic Turing machine cannot. Deutsch and Jozsa (1992) present an example of a quantum algorithm that is exponentially faster than a classical counterpart, the *Deutsch-Jozsa algorithm* that determines if a function is constant or balanced. Even though of little practical use, this is one of the first examples of possible advantages a quantum computer may have over a classical one.

Even though the Deutsch-Jozsa algorithm might not have real world applications, it led to further research on finding other such types of algorithms. Shor (1994a) showed that the problem of finding prime factors of an integer and the *discrete logarithm* problem can be efficiently solved by a quantum computer by an exponential factor. This brought a lot of interest to quantum computing, since both of these problems have real world applications and no efficient classical solution was/is known. Furthermore, most modern popular algorithms used for cryptography rely on the fact that the integer factorization or discrete logarithm problems are not efficiently solved. Since this is no longer the case, a new field has emerged called *Post-quantum cryptography*, whose purpose is to find suitable classical algorithms for cryptography that are not efficiently solved by quantum computing.

A more modest, but very relevant advantage was presented by Grover (1996) where he presented a quantum algorithm that promised to speed up unstructured database searches quadratically. Even though it's not an exponential improvement like Shor's algorithm, search-based algorithms are useful in many contexts, so even a "small" quadratic gain generated a lot of interest.

Contemporary to computer science, information theory is another field very relevant to this topic. Shannon (1948) revolutionized how communication and information are understood. In his paper, Shannon was interested in defining what resources are required to send information over a communication channel and how to reliably send that information mitigating the effects of noise. This led to the creation of the two fundamental theorems of

information theory. Firstly, Shannon's *noiseless channel coding theorem* specifies what resources are needed to store information sent from a source. Secondly, the *noisy channel coding theorem*, specifies how much information can be sent through a channel subject to noise. Even though Shannon's second theorem does not define any specific methodology to reduce noise, it sets an upper limit on how much noise can be mitigated through said methodology. These are known as *error-correcting codes* and research has developed better and better codes that get closer and closer to Shannon's limit, and they are used wherever there is need to store or transmit information.

Similar progress was made in quantum information theory. Schumacher (1995) developed a quantum version of Shannon's noiseless coding theorem, where he defined a *quantum bit* as a physical resource. There is no analogue for the second Shannon theorem, but that didn't stop the development of quantum error-correcting theory. For example, Calderbank and Shor (1996) and Steane (1996) proposed an important class of quantum error-correcting codes known as CSS.

Error-correcting was designed to protect quantum states, but another discovery by Bennett and Wiesner (1992) showed another interesting aspect about quantum information when transmitting classical information through a quantum channel. They explained how to send two classical bits of information using only one qubit, in a phenomenon known as *superdense coding*.

Another interesting application of quantum information is in the field of cryptography. Wiesner (1983) showed how quantum mechanics could be used to make sure that a information sent could not be interfered with without destroying it. Building on this work, Bennett and Brassard (1984) proposed a quantum key distribution protocol between sender and receiver that could not spied upon without notice. Many other protocols have since been proposed and experimental prototypes developed.

Finally, another interesting field within quantum computation is based on the concept of *distributed quantum computation*. Quantum clusters show promise since they require exponentially less communication to solve certain problems, such as modeling quantum systems, but are still in their infancy due to technical restrictions. There has been increasing international interest in taking advantage of these systems to build a *quantum internet* which promises better and safer transmission of information, but there are still many technological improvements to be made before this becomes a mainstream reality.

## 1.2 CLASSICAL AND QUANTUM RANDOM WALKS

The strong Church-Turing thesis, that states that any algorithmic process can be simulated efficiently using a Turing Machine, was challenged by Solovay and Strassen (1977) where they presented what is known as the *Solovay-Strassen primality test*. They showed that it

is possible to test whether a integer is prime or composite using a randomized algorithm. The implication is that, because of the randomness, the Solvay-Strassen primality test does not determine with certainty whether a integer is prime or composite, rather it computes that a number is *probably* prime or else *certainly* composite. This is of significance since no deterministic test for primality was known at the time (nor it is now), meaning that this was an example of a class of problems that could not be efficiently solved by a conventional deterministic Turing Machine.

This led to a modification of the Church-Turing thesis, now stating that any algorithm can be simulated efficiently using a *probabilistic* Turing machine. The discovery of more instances of such algorithms followed, [Motwani and Raghavan \(1995\)](#) and [Papadimitriou \(1994\)](#) show several problems that can be solved based on randomized algorithms. For example, the *Quicksort* algorithm, developed by [Hoare \(1961\)](#), has a high probability of finishing in  $O(n \log n)$ . In contrast to many deterministic algorithms that require  $O(n^2)$  time. They also show algorithms that take advantage of *Markov chains* and the *Monte Carlo method*. The volume of a convex body, proposed by [Dyer et al. \(1991\)](#), can be estimated by a randomized algorithm in polynomial time; the permanent of a nonnegative entry matrix can also be approximately calculated in probabilistic polynomial time as was shown by [Jerrum et al. \(2004\)](#) and the  $k$ -SAT and satisfiability with restrictions problem by [Schöning \(1999\)](#).

Random walks, as the name suggests, belong to this class of algorithms. [Pearson \(1905\)](#) coined the term random walk, and they can be described as path consisting of a succession of steps determined by a stochastic process, over a mathematical space. They are a special case of *Markov chains*, which are stochastic processes that assume discrete values and whose next state is dictated by a deterministic or random rule based only on the current state. This is a useful framework, since it can be used to explain the behaviour of systems across many fields, from the Brownian movement of particles moving through a gas, to the price of a fluctuating stock as shown by [Cootner \(1967\)](#).

The quantum analogue of the random walk was firstly developed by [Aharonov et al. \(1993\)](#), where they defined the *coined quantum random walk*. This model consists of a walker and a coin that determines the movement of the walker, which are both quantum systems where time is a discrete variable dictated by the successive quantum coin flips and shifts in position. [Nayak and Vishwanath \(2000\)](#) and [Aharonov et al. \(2001\)](#) presents the first analysis of the quantum walk on a graph described by a line. Further work by [Inui et al. \(2003\)](#) studies the behaviour of the walk on grids and [Aharonov et al. \(2001\)](#) on general regular graphs. The first algorithmic applications appear in the work of [Shenvi et al. \(2003\)](#) where they constructed a search problem based on the quantum random walk, and [Ambainis \(2007\)](#) applied it to the element distinction problem. On a more theoretical note, [Konno \(2002\)](#) demonstrated how the classical and quantum models of the random walk on the line differ, and [Grimmett et al. \(2003\)](#) generalized this to higher dimensions. [Lovett](#)



et al. (2010) demonstrated that any quantum algorithm can be reformulated as a discrete time quantum walk algorithm, effectively showing that this model can be used for universal quantum computation.

A different model for quantum random walks emerged from the work of Farhi and Gutmann (1998), where a different way of computing a search problem was presented. They showed that evolving a system in time between an initial and final Hamiltonian is analogous to the Grover algorithm, but continuous in time. Farhi et al. (2000) revised their work, now known as an adiabatic evolution, to solve boolean *sat* problems. Childs and Goldstone (2004) formulated a model of a quantum walk in terms of adiabatic evolution, known as *continuous time quantum walk* or *adiabatic quantum walk*. Aharonov et al. (2007) showed that any quantum algorithm can be efficiently simulated using adiabatic evolution, meaning that it is polynomially equivalent to the conventional quantum computation model. Further work by Childs (2009) showed that this model is indeed universal. The main difference between these two quantum walk models relies on the fact that in the discrete case, the system evolves with the flipping of a coin and subsequent movement of the walker, whilst in the adiabatic case the system evolves smoothly in time.

Yet another way of thinking about quantum walks was announced by Szegedy (2004), where he describes a discrete model based on Markov chain random walks. At the foundation of this model is the duplication of a graph, process by which a bipartite graph is created. Magniez et al. (2007) show how to use this walk for triangle detection in an undirected graph. Magniez et al. (2006) proposed a search problem, which takes advantage of ergodicity and symmetry properties of Markov chains, with quadratic gain compared to classical algorithms. Problems like element distinctness, matrix product verification and others were formulated within this framework by Santha (2008). Further work by Portugal (2015) established a connection between the coined and Szegedy's quantum walk, by defining a model that encompasses and expands the latter.

Patel et al. (2005) pointed that, at the time, there was confusion surrounding the scaling behaviour of discrete and adiabatic quantum walk algorithms. They argued that this was because the former model used a coin, which is an extra resource, and the latter didn't. So, in an attempt to resolve this confusion, they showed that a discrete time quantum walk could be constructed without the use of a coin. A new way of thinking about quantum walks came with the development of the methods behind the construction of evolution operators, by Falk (2013), introducing the concept of *tesselations*, based on local diffusion operators. Portugal et al. (2015) studied this model applied to a line graph, using the tessellation idea to construct the evolution operators. Further work by Portugal et al. (2016) formalized this approach naming it *staggered quantum walk*, and showed instances where the Szegedy quantum walk is equivalent. This suggests that this is a more general model, being able to describe other discrete time quantum walks. Portugal et al. (2017) delved deeper into this

topic, adding Hamiltonians to the model, and [Portugal and Fernandes \(2017\)](#) shows how this can be instantiated as a search problem in a grid. Finally, [Coutinho and Portugal \(2018\)](#) analyze how a continuous-time quantum walk can be casted into this discrete model and [Moqadam et al. \(2017\)](#) show a possible physical implementation of this walk.

### 1.3 STATE OF THE ART QUANTUM WALKS IMPLEMENTATIONS

One of the earliest works on a more computational approach to quantum random walks was by [Marquezino and Portugal \(2008\)](#), where they created a general simulator for discrete-time quantum walks on one- and two-dimensional lattices. They argued that this framework allowed researchers to focus more on the mathematical aspect of quantum walks, instead of the specific numerical implementations. Further work on these lattices was later presented by [Sawerwain and Gielera \(2010\)](#), where they studied the simulation of quantum walks by taking advantage of the GPU and CUDA technology. Another interesting program for simulating discrete-time quantum walks came with the work of [Berry et al. \(2011\)](#). This package allows for direct simulation of these walks, and visualisation of the time-evolution on arbitrary undirected graphs. It also allowed for plotting of continuous-time quantum walks, provided the data was provided externally. There are, however, direct simulators of continuous-time quantum walks, the earliest being by [Izaac and Wang \(2015\)](#). Their distributed memory software claims to be able to perform efficient simulation of multi-particle continuous-time quantum walk based systems, on *High Performance Computing* platforms. [Falloon et al. \(2017\)](#) provide a *Mathematica* package that implements a simulator of *Quantum Stochastic Walks*, which are a generalization of the continuous-time model. These walks incorporate both coherent and incoherent dynamics, which means that quantum stochastic walks can be instantiated as both quantum walks and classical random walks. What this paper then provides is a way of implementing quantum walks on directed graphs, opening the door to applications ranging from the capture of energy by photosynthetic protein complexes, to page ranking algorithms used by search engines. This package was ported to and expanded in the *Julia* programming language by [Glos et al. \(2018\)](#).

In order to truly harness the power of quantum computing, however, one must be able to perform these algorithms on quantum hardware. For this purpose, various implementations of quantum walks on quantum circuits have been proposed. [Douglas and Wang \(2009\)](#) pioneered this approach by developing efficient quantum circuits for discrete-time quantum walks on highly symmetric graphs, whose resources scale logarithmically with the size of the state space. [Shakeel \(2020\)](#) presented a new approach for building circuits for the discrete model, reducing resource requirement by using the quantum Fourier transform. For the continuous-time quantum walk, work by [Qiang et al. \(2016\)](#) presents efficient quantum circuits for the circulant graph class, and also an experimental implementation on a

photonic quantum processor. In the same year, [Loke and Wang \(2017a\)](#) showed how to build continuous-time quantum walk circuits for composite graphs, namely commuting graphs and Cartesian product of graphs. Considering the Szegedy quantum walk, [Chiang et al. \(2009\)](#) proposed an efficient method of creating quantum circuits for this model. In their work, they showed how to derive a quantum version of the arbitrary sparse classical random walk by approximating a *quantum update rule* with circuit complexity scaling linearly with the degree of sparseness of the structure. [Loke and Wang \(2017b\)](#) developed this method by showing that an efficient circuit for the Szegedy quantum walk can be constructed even if the structures are not sparse, given they possess translational symmetry in the columns of the transitional matrix. More specifically, they identified that the class of cyclic and bipartite graphs are compatible with this approach. Another interesting result in this paper was the creation of circuits that implement a quantum analogue of Google's *Page Rank* algorithm, in terms of Szegedy walks.

On the experimental side, there have been various implementations of quantum walks on quantum computers. On IBM's hardware, work by [Balu et al. \(2017\)](#) presents an efficient implementation of topological quantum walks where they ran the circuit on a five qubit computer over a 4 vertex lattice. [Georgopoulos and Zuliani \(2019\)](#) implements two instances of the discrete-time quantum walk. The first is based on the work of [Douglas and Wang \(2009\)](#) using generalized CNOT gates, and the second uses a rotational approach to the CNOT decomposition, which saves using an extra ancilla qubit register. They noted that IBM's simulator backend corresponded to the theoretical predictions, however the circuit for over three qubits was too much for the quantum hardware at that time. The paper presented by [Shakeel \(2020\)](#) also uses IBM's hardware to perform their formulation of the discrete-time quantum walk. For the staggered quantum walk model, work by [Acasiete et al. \(2020\)](#) obtains meaningful results when using the quantum computers to study the dynamics of this model on various graphs with 16 elements, requiring 4 qubits. They also modify these circuits to accommodate an oracle, showing that a spatial search algorithm could be performed on IBM's quantum computers for a search space of size 8, and with a bit of noise but still satisfactory for size 16.

#### 1.4 TEXT OVERVIEW AND CONTRIBUTIONS

The motivation behind the contents in this dissertation is to create an expanded overview on the topic of quantum random walks. To better contextualize the rest of this thesis, annex [A.1](#) presents the basic concepts and mathematical tools needed to study quantum walks.

Chapter [2](#) consists of the study of three major quantum random walk models, more specifically the discrete-time coined quantum walk, the continuous-time quantum walk and the staggered quantum walk. For each of these models, this work presents the theoretical

framework as well as Python implementations. In these simulations, the dynamics of the walks are analysed by changing various parameters, plotting the resulting probability distributions and seeing how these parameters alter the shape, propagation and other features of the quantum random walks.

Chapter 3 follows this approach, but now the structure where these walks take place is a complete graph and the goal is to find a marked element. For this purpose, the section about Grover's algorithm is used to introduce the notion of quantum searching problems. The following sections show how to change the various models of quantum walks to accommodate an oracle, thus performing an element search in time similar to Grover's algorithm.

Finally, chapter 4 is dedicated to constructing circuits for the models previously defined, using IBM's software *Qiskit* and their hardware. The first three sections are used to create circuits for the dynamics of the walks. The biggest contribution being the circulant graph approach to building diagonal operators for the continuous-time quantum walk, which can be easily translated to Qiskit circuits. Although work by [Qiang et al. \(2016\)](#) pioneered this approach, the work presented in this thesis aims to give a clear description on how to build these circuits in Qiskit and an original analysis on how the approximate quantum Fourier transform affects the accuracy of the results and the number of operations needed to perform the quantum walk. The last section shows how to introduce an oracle to the various circuits in order to perform a searching problem. For the continuous-time case, circulant graphs are again used in an original implementation of the searching problem making use of diagonal operators and the Suzuki-Trotter expansion.

In the context of this thesis, the following publications were created.

1. Jaime Santos, Bruno Chagas, Rodrigo Chaves. Quantum Walks on a Superconducting Quantum computer. (submitted)
2. Jaime Santos, Bruno Chagas. Implementing Grover's algorithm on a Superconducting Quantum Computer. (submitted)
3. Jaime Santos, Bruno Chagas. Searching with Continuous-Time Quantum Walks. (in preparation)

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## QUANTUM WALKS

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The structure of this chapter is as follows. The first section presents a brief introduction to random walks by reviewing the classical case. The following sections are dedicated to the study of several quantum random walk models and their advantages. This analysis is done by firstly describing the theoretical framework for these walks, and then simulating them in Python, with code that can be found on *Github* <sup>1</sup>.

Section 2.2 introduces the quantum case of random walks by analysing the dynamics of the coined model, making use of Python's plot capabilities in order to visualize the probability distributions associated with this algorithm. The quantum walk is generally said to be *quadratically* faster than the classical one, which is reflected on the behavior of the standard deviation associated with the probability distribution, and in this section a brief comparison of this metric is also presented. Section 2.3 is dedicated to the study of another instance of a discrete quantum walk, but where a coin is not needed. Instead, *tesselations* are used to construct the Hamiltonians of this algorithm, hence the name *staggered*, effectively reducing the associated Hilbert space. Here, plots are used to study the propagation of the walk, and the effects of altering available the parameters will also be analysed. Lastly, section 2.4 presents the continuous-time model of the quantum walk. Following the previous sections, the probability distributions will be plotted for the different parameters that can be altered. Finally, this chapter is closed with some final thoughts and remarks.

### 2.1 CLASSICAL RANDOM WALK

The term *random walk*, firstly introduced by [Pearson \(1905\)](#), is classically defined as a stochastic process that models the path a walker would take through a mathematical space, where each step made by the walker is random. This can be used to model systems such as a molecule displaying Brownian motion in a fluid, or even fluctuating stock prices as can be seen in [Sottinen \(2001\)](#).

The simplest instance of this walk is on an infinite discretely numbered line, whose mathematical space is composed of integer numbers. Here, the walker can only advance with

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<sup>1</sup> <https://github.com/JaimePSantos/QWSimulations>

equal probability in one of two directions, depending on the outcome of a random event such as tossing a coin. Starting from position  $x = 0$ , the walker moves to  $x = +1$  or  $x = -1$  with  $\frac{1}{2}$  probability after the first toss. On the second toss, the walker could be on  $x = \pm 2$  with  $\frac{1}{4}$  probability each, and on  $x = 0$  with  $\frac{1}{2}$ . Continuing this trend will result in a normal probability distribution around the origin, as can be seen in the Python coding of figure 1.

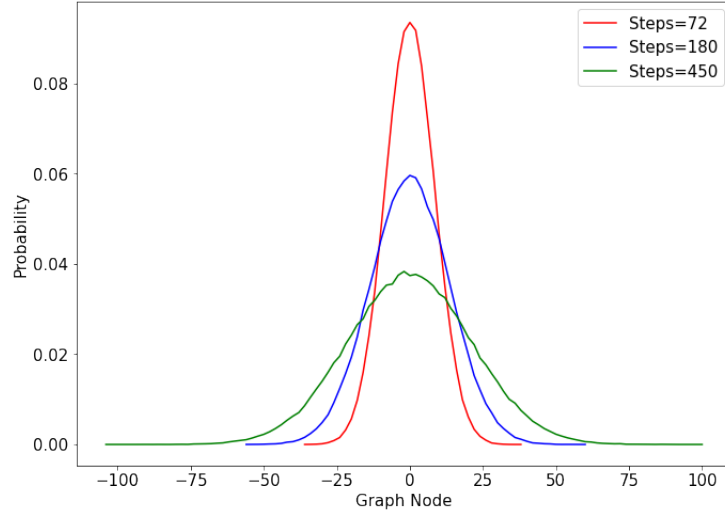


Figure 1: Probability distribution for the classical random walk on a line, after 72, 180 and 450, with starting position on vertex 0.

The number of iterations directly affects how far the walker can reach. As the number of steps increases, the height of each curve at the starting position decreases and the width of the curve increases. This relationship can be captured by the *position standard deviation*, and Portugal (2018) shows that the standard deviation is

$$\sigma(t) = \sqrt{t}. \quad (1)$$

In other words, equation 1 represents the rate at which a walker moves away from the origin.

Note that this algorithm can be abstracted to graphs of higher dimensions. For example, in a two dimensional lattice, a walker would be transversing a plane with integer coordinates, choosing one of four directions in every intersection. Notably, Pólya (1921) proved that a walker in a two dimensional lattice will almost surely return to the origin at some point. However, the probability of returning to the origin decreases as the number of dimensions increases, as shown by Montroll (1956) and Finch (2003). It is worth noting that a random walk, over a graph whose nodes are weighed and directed, is analogous to a *discrete-time Markov chain*<sup>2</sup>.

<sup>2</sup> A Markov chain can be described as a sequence of stochastic events where the the probability of each event depends only on the state of the previous event.

## 2.2 COINED QUANTUM WALK

In the quantum case, the walker is a quantum system whose position on a discretely numbered line, is described by a vector  $|x\rangle$  in Hilbert Space. The next position of the system will depend, in part, of a unitary operator, which can be viewed as a quantum coin. The analogy is, if the coin is tossed and rolls "heads", for example, the system transitions to position  $|x + 1\rangle$ , otherwise it advances to  $|x - 1\rangle$ . From a physical perspective, this coin can be the spin of an electron or the chirality of a particle, for example, and the outcome of measuring these properties decides whether the walker moves left or right. The coin is a unitary operator defined as

$$\begin{cases} C |0\rangle |x\rangle = a |0\rangle |x\rangle + b |1\rangle |x\rangle \\ C |1\rangle |x\rangle = c |0\rangle |x\rangle + d |1\rangle |x\rangle, \end{cases} \quad (2)$$

where  $a, b, c$  and  $d$  are the amplitudes associated with each outcome of the coin toss. One of the most commonly used coins is the unbiased coin, also known as Hadamard operator

$$H = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (3)$$

which will be the one used in this example.

The Hilbert space of the system is  $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_P$ , where  $\mathcal{H}_C$  is the two-dimensional Hilbert space associated with the coin and  $\mathcal{H}_P$  is the Hilbert space of the walker.

The transition from  $|x\rangle$  to either  $|x + 1\rangle$  or  $|x - 1\rangle$  must be described by a unitary operator, the *shift operator*

$$\begin{cases} \mathcal{S} |0\rangle |x\rangle = |0\rangle |x + 1\rangle \\ \mathcal{S} |1\rangle |x\rangle = |1\rangle |x - 1\rangle, \end{cases} \quad (4)$$

that can also be described by

$$S = |0\rangle \langle 0| \otimes \sum_{x=-\infty}^{x=\infty} |x + 1\rangle \langle x| + |1\rangle \langle 1| \otimes \sum_{x=-\infty}^{x=\infty} |x - 1\rangle \langle x|. \quad (5)$$

It follows that the operator that describes the dynamics of the quantum walk will be given by

$$U = S(C \otimes I) = S(H \otimes I). \quad (6)$$

Consider a quantum system located at  $|x = 0\rangle$  with coin state  $|0\rangle$ , for  $t = 0$ . It's state will be described by

$$|\Psi(0)\rangle = |0\rangle |x = 0\rangle. \quad (7)$$

After  $t$  steps

$$|\Psi(t)\rangle = U^t |\Psi(0)\rangle, \quad (8)$$

more explicitly

$$|\Psi(0)\rangle \xrightarrow{U} |\Psi(1)\rangle \xrightarrow{U} |\Psi(2)\rangle \xrightarrow{U} (\dots) \xrightarrow{U} |\Psi(t)\rangle. \quad (9)$$

In summary, the coined quantum walk algorithm consists on applying the coin operator followed by the shift operator a certain number of times. Iterating this twice, evolves the system to the following respective states

$$|\Psi(1)\rangle = \frac{|0\rangle |x = -1\rangle + |1\rangle |x = 1\rangle}{\sqrt{2}} \quad (10)$$

$$|\Psi(2)\rangle = \frac{|0\rangle |x = -2\rangle + |1\rangle |x = 0\rangle + |0\rangle |x = 0\rangle - |1\rangle |x = 2\rangle}{2} \quad (11)$$

If one were to measure the system after the first application of  $\mathcal{U}$ , it would be expected to see the walker at  $x = 1$  with probability  $P(x) = \frac{1}{2}$ , and at  $x = -1$  with  $P(x) = \frac{1}{2}$  aswell. Measure the system  $t$  times, after each application of  $\mathcal{U}$ , and the result is a binomial probability distribution similar to the one in 1. The conclusion is that repetitive measurement of a coined quantum walk system reduces to the classical case, which means that any desired quantum behaviour is lost.

It is possible, however, to make use of the quantum correlations between different positions to generate constructive or destructive interference, by applying the Hadamard and shift operators successively without intermediary measurements. The consequences of interference between states become very apparent after only 3 iterations

$$|\Psi(3)\rangle = \frac{|1\rangle |x = -3\rangle - |0\rangle |x = -1\rangle + 2(|0\rangle + |1\rangle) |x = 1\rangle + |0\rangle |x = 3\rangle}{2\sqrt{2}}. \quad (12)$$

Even though an unbiased coin was used, this state is not symmetric around the origin and the probability distributions will not be centered in the origin. Moreover, [Portugal \(2018\)](#) shows that the standard deviation will be

$$\sigma(t) \approx 0.54t. \quad (13)$$

This means that the standard deviation for the coined quantum walk grows linearly in time, unlike the classical case which grows with  $\sqrt{t}$ , as was seen in equation 1. The implication is that the quantum walk displays *ballistic* behaviour, as is reviewed in [Venegas-Andraca \(2012\)](#). This behaviour is usually defined in the context of a moving free particle with unit velocity in a single direction, which is expected to be found at  $x = t$  after  $t$  steps. The velocity of a walker in a Hadamard quantum walk is approximately half of the free particle example,



which is still a quadratic improvement over the classical random walk. Figure 2 shows the comparison between the evolution of the classical and the quantum standard deviation.

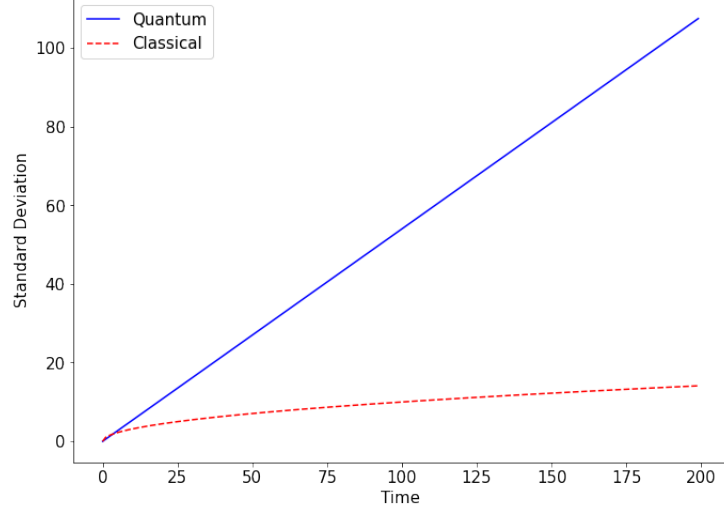


Figure 2: Standard deviation after 200 steps for the quantum walk in blue, and the classical random walk in red.

This quadratic gain implies exponentially faster hitting times in certain graphs, as shown by Childs et al. (2002), meaning improvements to problems that require transversing graphs. Ambainis (2007) also shows advantages of the coined quantum walk model in element distinctness problems, and Childs and Goldstone (2004) show advantages in spatial search problems, which will be studied in a later chapter.

In order to study this distribution, a simulation of the coined quantum walk was coded in *Python*. Figure 3 is the result of using the Hadamard coin and the initial condition in equation 7, for varying numbers of steps. Analyzing the plot, it is noticeable that the distributions are asymmetric. The probability of finding the walker on the right-hand side is much larger than on the left, with a peak around  $x \approx \frac{t}{\sqrt{2}}$ . Regardless of number of steps, this peak is always present (albeit in varying positions), which is to say that the walker can always be found moving in a uniform fashion away from the origin, consistent with ballistic behaviour.

Another interesting case study is to find if this behaviour is preserved for a symmetric distribution around the origin. For this purpose, one must first understand where the asymmetry comes from. The Hadamard operator flips the sign of state  $|1\rangle$ , hence more terms are cancelled when the coin state is  $|1\rangle$ . Since  $|0\rangle$  was defined to induce movement to the right, the result is as shown in 3. Following this logic, it would be expected that an initial condition

$$|\Psi(0)\rangle = |1\rangle |x = 0\rangle, \quad (14)$$

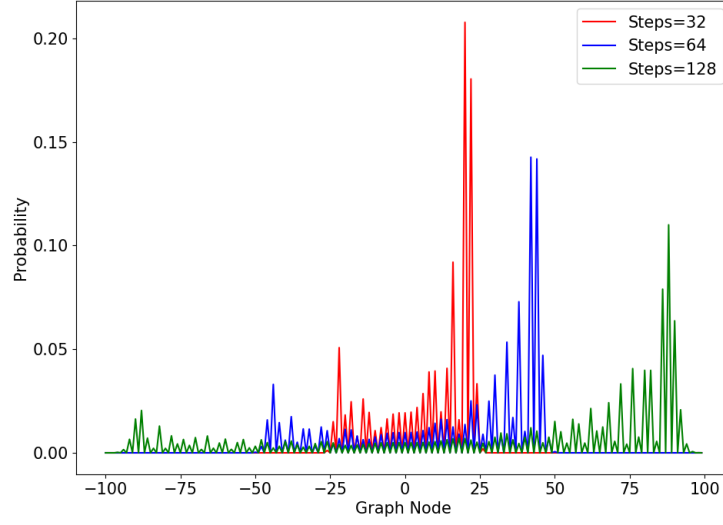


Figure 3: Probability distribution for the coined quantum walk on a line, after 32, 64 and 128 steps, with initial condition  $|\Psi(0)\rangle = |0\rangle |x = 0\rangle$  and the Hadamard coin.

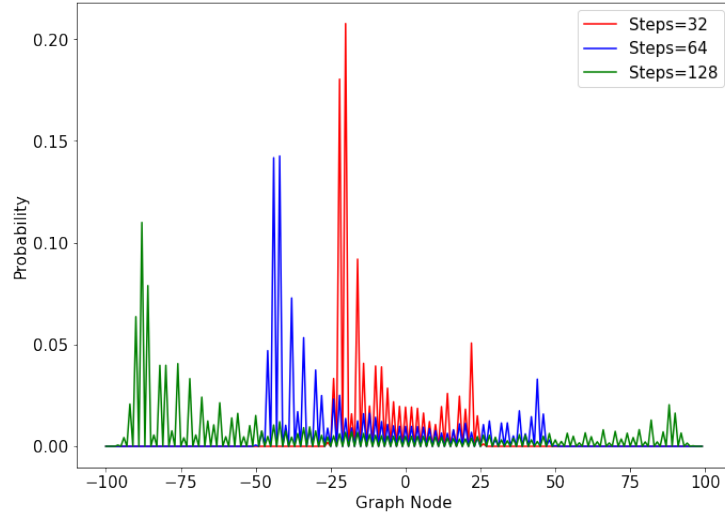


Figure 4: Probability distribution for the coined quantum walk on a line, after 32, 64 and 128 steps, with initial condition  $|\Psi(0)\rangle = -|1\rangle |x = 0\rangle$  and the Hadamard coin.

would result in more cancellations when the coin state is  $|0\rangle$ , thus the walker would be more likely found in the left-hand side of the graph. This is indeed what happens, as figure 4 is a mirror image of figure 3. The walker still moves away from the origin with ballistic behaviour, but in opposite direction. The peaks behave in a similar fashion, being instead found at  $x \approx -\frac{t}{\sqrt{2}}$ .

In order to obtain a symmetrical distribution, one must superpose the state in equation 7 with the state in equation 14. However, in order to not cancel terms before the calculation of the probability distribution, one must multiply state  $|1\rangle$  with the imaginary unit,  $i$

$$|\Psi(0)\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} |x=0\rangle. \quad (15)$$

This works because the entries of the Hadamard operator are real numbers. Terms with

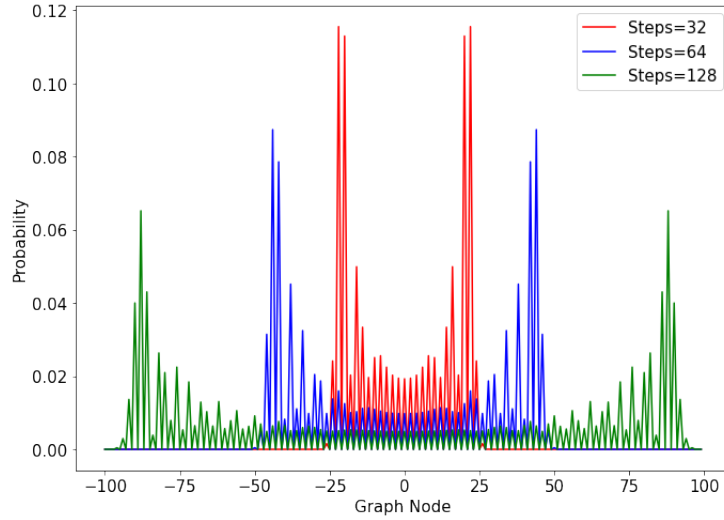


Figure 5: Probability distribution for the coined quantum walk on a line, after 32, 64 and 128 steps, with initial condition  $|\Psi(0)\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}} |x=0\rangle$  and the Hadamard coin.

the imaginary unit will not cancel out with terms without it, thus the walk can proceed to both left and right, as it is shown in figure 5. The probability distribution is now symmetric and it is spread over the range  $[-\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}]$  with peaks around  $x \approx \pm \frac{t}{\sqrt{2}}$ . This means that if the position of the walker was measured at the end, it would be equally probable to find him either in the left side or the right side of the graph, which is not possible in a classical diffusive motion.

All of the previous examples are in sharp contrast with the classical random walk distribution in figure 1. There, maximum probability is reached at  $x = 0$  since there are approximately equal steps in both directions. Furthermore, the further the vertex is away from the origin, the less likely the walker is to be found there. However, in the quantum case, the walker is more likely to be found away from the origin as the number of steps increases. More specifically, the walk spreads quadratically faster than the classical counterpart.

This is but one model of a quantum random walk. As it will be seen in further sections, there are other approaches to creating both discrete and continuous quantum walk models that do not use a coin.

## 2.3 STAGGERED QUANTUM WALK

The staggered quantum walk (SQW) case aims to spread a transition probability to neighboring vertices with discrete time steps. The notion of adjacency comes from cliques<sup>3</sup>, and the initial stage of this walk consists in partitioning the graph in several different cliques. This is called tessellation, and it is defined as the division of the set of vertices into disjoint cliques. An element of a tessellation  $\mathcal{T}$  is called a polygon, and it's only valid if all of its vertices belong to the clique in  $\mathcal{T}$ . The set of polygons of each tessellation must cover all vertices of the graph, and the set of tessellations  $(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$  must cover all edges.

These definitions allow the construction of operators  $H_1, H_2, \dots, H_k$  that will be used to propagate the probability amplitude locally, in each polygon. The state associated to each polygon is

$$|u_j^k\rangle = \frac{1}{\sqrt{|\alpha_j^k|}} \sum_{l \in \alpha_j^k} |l\rangle \quad (16)$$

where  $\alpha_j^k$  is the  $j^{th}$  polygon in the  $k^{th}$  tessellation.

The unitary and Hermitian operator  $H_k$ , associated to each tessellation is defined in [Portugal et al. \(2017\)](#) as

$$H_k = 2 \sum_{j=1}^p |u_j^k\rangle \langle u_j^k| - I. \quad (17)$$

Solving the time-independent Schrodinger equation for this Hamiltonian gives the evolution operator

$$U = e^{i\theta_k H_k} \dots e^{i\theta_2 H_2} e^{i\theta_1 H_1} \quad (18)$$

where

$$e^{i\theta_k H_k} = \cos(\theta_k)I + i \sin(\theta_k)H_k \quad (19)$$

since  $H_k^2 = I$ , meaning that the Hamiltonian is a projection operator that, when expanded in a Taylor series, generates a local operator.

The simplest use case of this quantum walk model is the one-dimensional lattice, where the minimum tessellations are

$$\mathcal{T}_\alpha = \{\{2x, 2x+1\} : x \in \mathbb{Z}\}, \quad (20)$$

$$\mathcal{T}_\beta = \{\{2x+1, 2x+2\} : x \in \mathbb{Z}\}. \quad (21)$$

Each element of the tessellation has a corresponding state, as can be seen in figure 6, and

<sup>3</sup> A clique is defined as the subset of vertices of an undirected graph such that every two distinct vertices in each clique are adjacent.

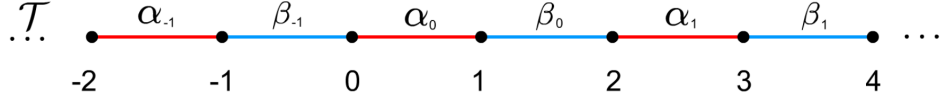


Figure 6: Tessellation of a line graph.

the uniform superposition of these states is

$$|\alpha_x\rangle = \frac{|2x\rangle + |2x+1\rangle}{\sqrt{2}} \quad (22)$$

$$|\beta_x\rangle = \frac{|2x+1\rangle + |2x+2\rangle}{\sqrt{2}} \quad (23)$$

One can now define Hamiltonians  $H_\alpha$  and  $H_\beta$  as

$$H_\alpha = 2 \sum_{x=-\infty}^{+\infty} |\alpha_x\rangle \langle \alpha_x| - I \quad (24)$$

$$H_\beta = 2 \sum_{x=-\infty}^{+\infty} |\beta_x\rangle \langle \beta_x| - I \quad (25)$$

The Hamiltonian evolution operator reduces to

$$U = e^{i\theta H_\beta} e^{i\theta H_\alpha} \quad (26)$$

and applying it to an initial condition  $|\Psi(0)\rangle$  results in the time evolution operator

$$U |\Psi(t)\rangle = U^t |\Psi(0)\rangle \quad (27)$$

Having defined the time evolution operator, the walk is ready to be coded with a certain initial condition and  $\theta$  value, to better understand how the probability distribution spreads through time. For the first case study, the initial condition will be a uniform superposition of states  $|0\rangle$  and  $|1\rangle$  and the  $\theta$  value will be varied in order to understand how this parameter impacts the walk, as can be seen in figure 7. The overall structure of the probability distribution is very similar for the different values of  $\theta$ , the difference being that the walker is more likely to be found further away from the origin as the angle increases.

Another interesting case study is to see how the initial condition affects the dynamics of the system, and figure 8 shows the results of plotting the quantum walk for initial conditions  $|\Psi(0)\rangle = |0\rangle$  and  $|\Psi(0)\rangle = |1\rangle$ . Similarly to the coined case, each initial condition results in asymmetric probability distributions,  $|\Psi(0)\rangle = |0\rangle$  leads to a peak in the left-hand side while condition  $|\Psi(0)\rangle = |1\rangle$  results in a peak in the right-hand side. As was shown in figure

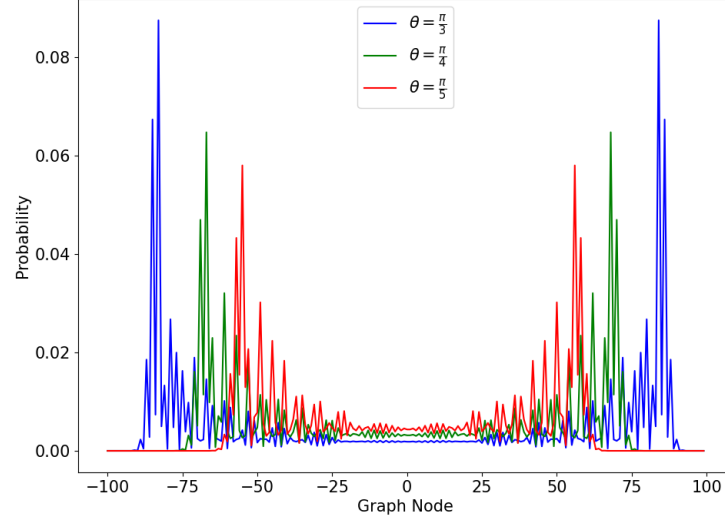


Figure 7: Probability distribution for the staggered quantum walk on a line after 50 steps, with initial condition  $|\Psi(0)\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ , for multiple values of  $\theta$ .

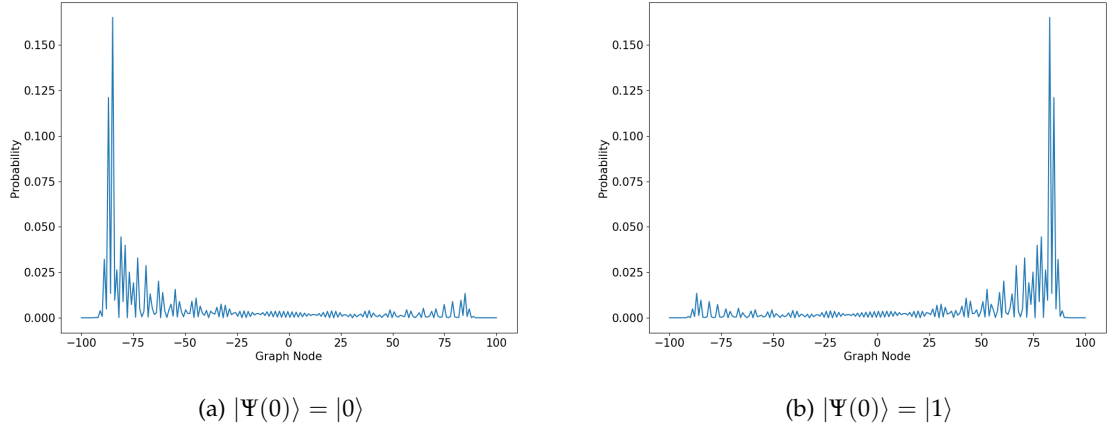


Figure 8: Probability distributions for the staggered quantum walk on a line after 50 steps, for different initial conditions.

7, the uniform superposition of both these conditions results in a symmetric probability distribution.

So far, only discrete-time quantum walks have been shown. The next section presents the continuous-time quantum walk model, where time is not discretized and whose Hilbert space is proportional to the space of the walker.

## 2.4 CONTINUOUS-TIME QUANTUM WALK

The continuous-time random walk model on a graph is a Markov process where transitions have a fixed probability per unit time,  $\gamma$ , of moving to adjacent vertices, firstly introduced by [Montroll and Weiss \(1997\)](#). Consider a graph  $G$  with  $N$  vertices and no self-loops, this walk can be defined by the linear differential equation that describes the probability of jumping to a connected vertex in any given time

$$\frac{dp_i(t)}{dt} = \gamma \sum_j L_{ij} p_j(t), \quad (28)$$

where  $L$  is the Laplacian defined as  $L = A - D$ , and  $p_j(t)$  is the time dependent probability associated with each vertex transition.  $A$  is the adjacency matrix that represents each vertex connection, given by

$$A_{ij} = \begin{cases} 1, & \text{if } (i, j) \in G \\ 0, & \text{otherwise,} \end{cases} \quad (29)$$

and  $D$  is the diagonal matrix  $D_{jj} = \deg(j)$  corresponding to the degree<sup>4</sup> of vertex  $j$ .

In the continuous-time quantum walk model (CTQW), the vertices are quantum states that form the basis for the Hilbert space. The continuous-time quantum walk model will also be described by a differential equation, the Schrödinger equation

$$i\hbar \frac{d|\Psi(t)\rangle}{dt} = \hat{H} |\Psi(t)\rangle, \quad (30)$$

where  $\hat{H} = -\gamma L$  is the Hamiltonian of the system. More explicitly,

$$\hat{H}_{ij} = \begin{cases} \deg(j)\gamma, & \text{if } i = j; \\ -\gamma, & \text{if } i \neq j \text{ and adjacent;} \\ 0, & \text{if } i \neq j \text{ and not adjacent.} \end{cases} \quad (31)$$

A general state of a system  $|\Psi(t)\rangle$  can be written as a function of it's complex amplitudes

$$q_i = \langle i | \Psi(t) \rangle, \quad (32)$$

which means 30 can be rewritten as

$$i\hbar \frac{dq_i(t)}{dt} = \sum_j \hat{H}_{ij} q_j(t). \quad (33)$$

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<sup>4</sup> The degree of a vertex refers to the number of edges that it is connected to.

Comparing equations 33 and 28, the Laplacian is replaced with the Hamiltonian, and the probabilities with amplitudes. One of the main differences is the complex phase  $i$ , which will result in a very different behaviour. Setting  $\hbar = 1$  and solving the differential equation results in the evolution operator of this walk

$$U(t) = e^{-iHt} = e^{i(-\gamma L)t} = e^{-i\gamma(A+D)t}, \quad (34)$$

In the regular graph case, where  $D$  is simply the degree of the whole graph multiplied by the identity matrix,  $A$  and  $D$  will commute, meaning that the evolution operator can be written in terms of the adjacency matrix

$$U(t) = e^{-i\gamma At + i\gamma Dt} = e^{-i\gamma At} e^{i\gamma Dt} = \phi(t) e^{-i\gamma At} \quad (35)$$

since the degree matrix becomes a global phase. Applying this operator to an initial condition  $\Psi(0)$ , will give the state of the system at a time  $t$

$$|\Psi(t)\rangle = U(t) |\Psi(0)\rangle. \quad (36)$$

Considering a uni-dimensional quantum system, each vertex will have at most 2 other neighboring vertices, reducing equation 31 to

$$\hat{H}_{ij} = \begin{cases} 2\gamma, & \text{if } i = j; \\ -\gamma, & \text{if } i \neq j \text{ and adjacent;} \\ 0, & \text{if } i \neq j \text{ and not adjacent.} \end{cases} \quad (37)$$

For a more detailed visualization, this quantum walk model was coded in python and figure 9 was obtained setting the transition rate to  $\gamma = \frac{1}{2\sqrt{2}}$  the initial condition to  $|\Psi(0)\rangle = |0\rangle$ . A brief look at figure 9 reveals several similarities to previous models. The property of having two peaks away from the origin and low probability near the origin is present across all the quantum walks, however in the continuous case a symmetric initial condition is not needed. In the staggered quantum walk model, the propagation of the walk could be altered by changing the values of  $\theta$ , whereas in this case different values of  $\gamma$  and time will influence the probability distribution.

Altering the initial condition will also differ in the continuous-time example. For example, setting the initial condition to the balanced superposition of states  $|0\rangle$  and  $|1\rangle$  has no effect on the overall pattern of the probability distribution as can be seen in figure 10. Both peaks still are still present and at the same distance from the origin, with intermediate amplitudes being attenuated relative to figure 9. This behavior is in contrast with the previous discrete-time cases, where a change in the initial condition would dictate the number of peaks and where they would appear. Figure 10 also shows the influence of the transition rate  $\gamma$ . As



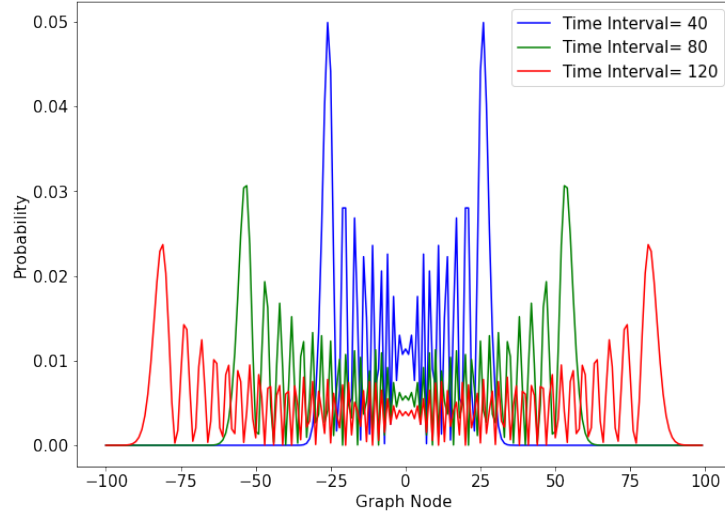


Figure 9: Probability distribution for the continuous-time quantum walk on a line, at  $t = 40, 80$  and  $120$ , with initial condition  $|\Psi(0)\rangle = |0\rangle$  and  $\gamma = \frac{1}{2\sqrt{2}}$ .

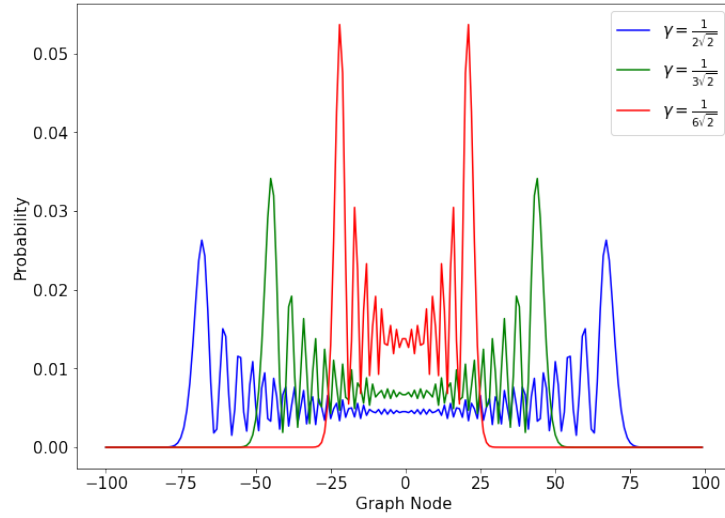


Figure 10: Probability distribution for the continuous-time quantum walk on a line, at  $t = 100$ , with initial condition  $|\Psi(0)\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ , for multiple values of  $\gamma$ .

would be expected from equation 35, the effects are very similar to altering time, since both parameters are multiplying in the exponential.

In conclusion, the purpose of this chapter was to present a helpful overview of several different models of the quantum walk. The probability distributions are very distinct from the classical case, but relatively similar to each other. The bigger differences between the models come from the size of the associated Hilbert space. The staggered and continuous-time quantum walks have Hilbert spaces proportional to the space of the walker, whereas the coined quantum walk also requires a space for the coin. It might seem unimportant now,

but it will play a major role when these models are translated into circuits, since the size of the Hilbert space will have a direct relation with the number of qubits required to perform the walk, which are a scarce resource for *Noisy intermediate-scale quantum* (NISQ) computers. Considering the case of the coined quantum walk on a simple line graph, only one extra qubit will be needed but, as will be seen in the next chapter, the search problem is optimal when performed over a complete graph, which in the coined case will double the number of required qubits.

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## SEARCHING PROBLEMS

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Following the previous chapters structure, here the study of the search problem using quantum walks will be presented. Section 3.1 introduces the basics of the search problem by presenting the theoretical framework of Grover's algorithm, followed by a complexity analysis together with Python plots for a better illustration. Different numbers of marked elements will be shown and, by the end of the section, it should be clear that Grover's algorithm is optimal for searching, as was shown by Zalka (1999).

Subsequent sections are dedicated for the quantum walk instance of this problem. In section 3.2, the coined quantum walk is defined for the search problem, which implies the introduction of an oracle. Here, instead of a line, a complete graph is used, which will increase the space of the search to  $2N$ , due to the connected nature of this graph and the need of a coin. Section 3.3 presents the staggered quantum walk version of the search problem. Again, the notion of cliques and tessellations is used instead of a coin, and the complete graph is again considered. The oracle is again defined for this walk, which makes it quite similar to Grover's algorithm. However, since it is possible to alter the parameter  $\theta$  and the structure over which the search is performed, this algorithm is known to be a more general case of Grover's. Like the coined quantum walk, the staggered model is discrete in time but, since a coin is not used, the space associated with it scales only with the size of the graph, meaning its implementation on a NISQ computer will be more feasible. Finally, section ?? closes this chapter with the search instance of the continuous-time quantum walk. Similarly to the staggered quantum walk, both the structure where the search is performed and parameter  $\gamma$  can be controlled. However, since time is not discrete in this instance, the probability distribution associated with this walk will be slightly different, and it will later be seen that this results in a circuit that does not scale up with time.

### 3.1 GROVER'S ALGORITHM

Searching through an unstructured database is a task classically achieved by exhaustively evaluating every element in the database. Assume there exists a black box (oracle) that can be asked to find out if two elements are equal. Since we're looking for a specific element in

a database of size  $N$ , we'd have to query the oracle on average  $\frac{N}{2}$  times, or in the worst case  $N$  times.

Grover's algorithm, presented in Grover (1996), comes as a quantum alternative to this type of problems, taking advantage of superposition by increasing desirable states' amplitudes through a process called *amplitude amplification*. This method has a quadratic gain over the classical counterpart Boyer et al. (1998), being able to find a target element in expected time  $\mathcal{O}(\sqrt{N})$ .

The inner workings of the black box will now be expanded upon. Instead of directly evaluating the elements, the searching indexes will be considered, and the number of elements is defined as  $N = 2^n$ ,  $n$  being a positive integer. The next step is to define a function  $f : \{0, 1, \dots, N-1\}$  that returns 1 when evaluating the desired (marked) element and 0 otherwise. Since this function is to be applied to a quantum system, a unitary operator can be built such as

$$\mathcal{O} |x\rangle |i\rangle = |x\rangle |i \oplus f(x)\rangle. \quad (38)$$

where  $|x\rangle$  is the index register,  $\oplus$  is the binary sum operation and  $|i\rangle$  is a qubit that is flipped if  $f(x) = 1$ .

The action of the oracle on state  $|0\rangle$  will be

$$\mathcal{O} |x\rangle |0\rangle = \begin{cases} |x_0\rangle |1\rangle, & \text{if } x = x_0 \\ |x\rangle |0\rangle, & \text{otherwise.} \end{cases} \quad (39)$$

where  $x_0$  is the marked element. More generically,  $\mathcal{O}$  can be written as

$$\mathcal{O} |x\rangle = (-1)^{f(x)} |x\rangle. \quad (40)$$

This offers a bit of insight into the oracle, it *marks* the solutions to the search problem by applying a phase shift to the solutions. The question now is, what is the procedure that determines a solution  $x_0$  using  $\mathcal{O}$  the minimum number of times? The answer lies in the amplitude amplification section of Grover's search, starting with the creation of a uniform superposition

$$|\Psi_0\rangle = H^{\otimes n} |x\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle \quad (41)$$

where  $H^{\otimes n}$  is the *Hadamard* operator applied to an arbitrary number of states.

If one was to measure  $|x\rangle$  at this point, the superposition would collapse to any of the base states with the same probability  $\frac{1}{N} = \frac{1}{2^n}$ , which means that on average, we'd need to try  $N = 2^n$  times to guess the correct item. This is where amplitude amplification comes into effect, by means of a second unitary operator

$$\mathcal{D} = (2 |\Psi_0\rangle \langle \Psi_0| - I) = H^{\otimes n} (2 |0\rangle \langle 0| - I) H^{\otimes n} \quad (42)$$

This operator applies a conditional phase shift, with every computational basis state except  $|0\rangle$  receiving a phase shift. This can also be described as the *inversion about the mean*, for a state of arbitrary amplitudes

$$|\phi\rangle = \sum_{k=0}^{N-1} \alpha_k |k\rangle \quad (43)$$

the action of  $\mathcal{D}$  on state  $\phi$  will be

$$\mathcal{D}|\phi\rangle = \sum_{k=0}^{N-1} (-\alpha_k + 2\langle\alpha\rangle) |k\rangle \quad (44)$$

where  $\langle\alpha\rangle$  is the average of  $\alpha_k$

$$\langle\alpha\rangle = \frac{1}{N} \sum_{k=0}^{N-1} \alpha_k \quad (45)$$

The evolution operator that performs one step of the algorithm is then

$$\mathcal{U} = \mathcal{D}\mathcal{O} \quad (46)$$

and after  $t$  steps the state of the system is

$$|\Psi(t)\rangle = \mathcal{U}^t |\Psi_0\rangle. \quad (47)$$

### 3.1.1 One marked element

The optimal number of steps is, as aforementioned, proportional to  $\sqrt{N}$ . More precisely, if there's only one solution, maximum probability can be reached in *approximately*  $\frac{\pi}{4}\sqrt{N}$  iterations. In order to show that this is the case, an iteration will be formally defined here as the process that transforms the state

$$|\Psi(k, l)\rangle = k|i_0\rangle + \sum_{i \neq i_0} l|i\rangle \quad (48)$$

into state  $|\Psi(\frac{N-2}{N}k + \frac{2(N-1)}{N}l, \frac{N-2}{N}l - \frac{2}{N}k)\rangle$ . Amplitudes  $l$  and  $k$  are real numbers that satisfy  $k^2 + (N-1)l^2 = 1$ . Running  $m$  iterations over state  $|\Psi_0\rangle$  will eventually lead to state  $|\Psi_j\rangle = |\Psi(k_j, l_j)\rangle$  after the  $j$ -th iteration, where  $k_0 = l_0 = \frac{1}{\sqrt{N}}$  and

$$\begin{cases} k_{j+1} = \frac{N-2}{N}k_j + \frac{2(N-1)}{N}l_j \\ l_{j+1} = \frac{N-2}{N}l_j + \frac{2}{N}k_j. \end{cases} \quad (49)$$

After the last iteration, the system will be in state  $|\Psi_m\rangle$  with a certain amplitude. If that amplitude corresponds to the marked element  $x_0$ , then it is said that the algorithm was successful.

Grover (1996) proves that there exists a value of  $m < \sqrt{2N}$  such that the probability of success is at least  $\frac{1}{2}$ . However the probability of success does not linearly increase with the number of iterations, in fact for  $m = \sqrt{2N}$  the system will succeed less than 1 in 10 times. Boyer et al. (1998) argues that an explicit value of  $m$  is needed, and it is achieved by finding a closed form formula for  $k_j$  and  $l_j$ . The first step is to define an angle  $\theta$  so that  $\sin^2 \theta = \frac{1}{N}$ , and equation 49 will become

$$\begin{cases} k_{j+1} = \sin((2j+1)\theta) \\ l_{j+1} = \frac{1}{\sqrt{N-1}} \cos((2j+1)\theta). \end{cases} \quad (50)$$

In order to maximize the probability of success, one must find a value of  $m$  so that  $k_m = 1$  and  $l_m$  is as close to 0 as possible. The value of  $k$  after  $m$  iterations will be at its maximum when  $\sin((2m+1)\theta) = \frac{\pi}{2}$ , and solving the trigonometric equation leads to a value of  $m = \frac{\pi-2\theta}{4\theta}$ . Conversely,  $l_{\tilde{m}} = 0$  when  $\tilde{m} = \frac{\pi-2\theta}{4\theta}$  for an integer number of  $\tilde{m}$ . Setting  $m$  to the nearest lower integer of  $\frac{\pi}{4\theta}$  will lead to

$$|m - \tilde{m}| \leq \frac{1}{2} \iff |(2m+1)\theta - (2\tilde{m}-1)\theta| \leq \frac{\pi}{2}. \quad (51)$$

By definition,  $(2\tilde{m}+1)\theta = \frac{\pi}{2}$  which means that  $|\cos((2m+1)\theta)| \leq |\sin \theta|$ . The probability

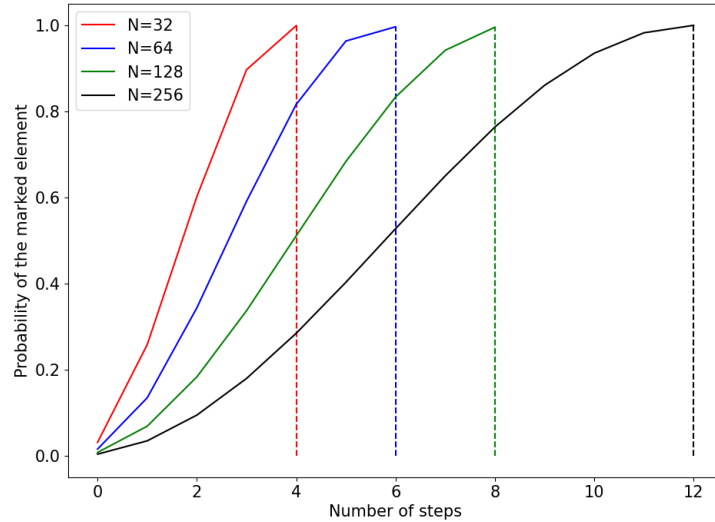


Figure 11: Probability of one marked element in the Grover search, as a function of the number of steps, for  $N = 32, 64, 128$  and  $256$ .

of failure after  $m$  iterations can then be written as

$$(N-1)l_m^2 = \cos^2((2m+1)\theta) \leq \sin^2 \theta = \frac{1}{N}. \quad (52)$$

Failure decreases as the number of elements increases. The run time of the algorithm will be

$$m \leq \frac{\pi}{4\theta} \leq \frac{\pi}{4} \sqrt{N} \quad (53)$$

since  $\theta \geq \sin \theta = \frac{1}{\sqrt{N}}$ . This means that, for a large  $N$ , the number of iterations that maximizes the probability of success will be very close to  $\frac{\pi}{4} \sqrt{N}$ .

Figure 11 was obtained by coding the appropriate operators as to simulate the system presented in equation 47. The unitary evolution operator was applied approximately  $\frac{\pi}{4} \sqrt{N}$  times and the amplitudes associated with those states were stored as a probability distribution. Filtering the probability of the marked element and plotting it against the number of steps, shows that the maximum is indeed reached after the said number of iterations. It also shows that the maximum probability for  $N = 16$  is lower than for  $N = 128$ , which makes sense since the the probability of success is maximized for larger values of  $N$ .

### 3.1.2 Multiple marked elements

When there's more than one element marked by the oracle, the number of iterations to achieve maximum probability changes. In fact, the latter part of this section will be used to discuss the case where one single iteration of this algorithm is enough to achieve maximum probability.

Firstly, one must define a set  $A$  that is composed of all the marked elements and set  $B$  of the remaining. The state from equation 48 will become

$$|\Psi(k, l)\rangle = \sum_{i \in A} k |i\rangle + \sum_{x \in B} l |x\rangle. \quad (54)$$

Assuming  $t$  marked elements, iterating over this state will result in

$$\left| \Psi\left(\frac{N-2t}{N}k + \frac{2(N-1)}{N}l, \frac{N-2}{N}l - \frac{2}{N}k\right) \right\rangle. \quad (55)$$

Choosing an angle  $\theta$  such that  $\sin^2 \theta = \frac{t}{N}$ , allows the definition of the amplitudes associated with the states after  $j$  iterations

$$\begin{cases} k_j = \frac{1}{\sqrt{t}} \sin((2j+1)\theta) \\ l_j = \frac{1}{\sqrt{N-t}} \cos((2j+1)\theta). \end{cases} \quad (56)$$

Similarly to the one solution case, it can be shown that setting the number of iterations

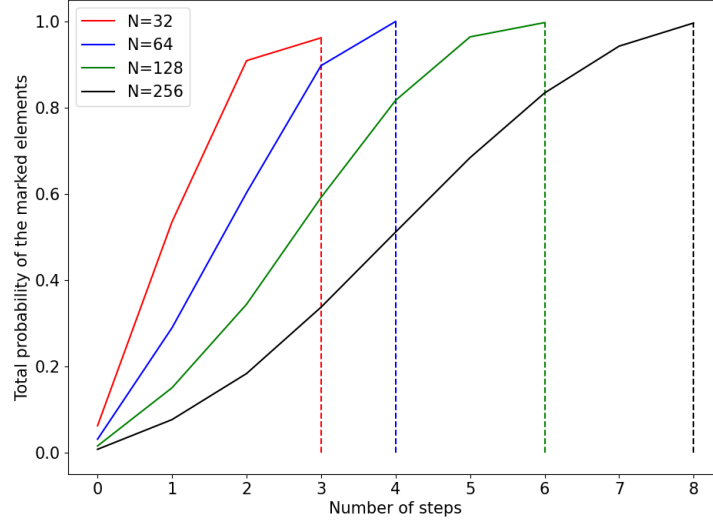


Figure 12: Probability of two marked elements in the Grover search, as a function of the number of steps, for  $N = 32, 64, 128$  and  $256$ .

$m$ , to the nearest lower integer of  $\frac{\pi}{4\theta}$  will result in a probability of failure  $(N - t)l_m^2 \leq \frac{t}{N}$ . Because  $\theta \geq \sin \theta = \sqrt{\frac{t}{N}}$ , then

$$m \leq \frac{\pi}{4\theta} \leq \frac{\pi}{4} \sqrt{\frac{N}{t}}. \quad (57)$$

From a more practical perspective, if one were to mark two elements of a 64 element set, maximum probability is expected to be reached in approximately 4 steps, since  $\lfloor \frac{\pi}{4} \sqrt{\frac{64}{2}} \rfloor = 4$ . Likewise, for  $N = 256$ , the number of iterations is rounded to 8, which is plotted along several other values of  $N$  in figure 12. The y-axis is now the sum total probability of the marked elements and the x-axis represents the range of steps that spans from 0 to  $\lfloor \frac{\pi}{4} \sqrt{\frac{N}{2}} \rfloor$  for each  $N$ . Again, the probability of success approaches 1 as  $N$  increases. However, comparing to figure 11, the number of iterations that maximize probability is lower for each  $N$ , in agreement with equation 57. Work by Zalka (1999) shows that the time complexity associated with Grover's algorithm is optimal, and that parallelization of quantum searching should not yield better results.



## 3.1.3 Single-Shot Grover

An interesting case arises when the number of marked elements is set to  $t = \frac{N}{4}$ , because

$$\sin^2 \theta = \frac{\frac{N}{4}}{N} = \frac{1}{4} \iff \sin \theta = \frac{1}{2} \iff \theta = \frac{\pi}{6}. \quad (58)$$

Note that there are an infinite number of negative and positive solutions, but equation 58

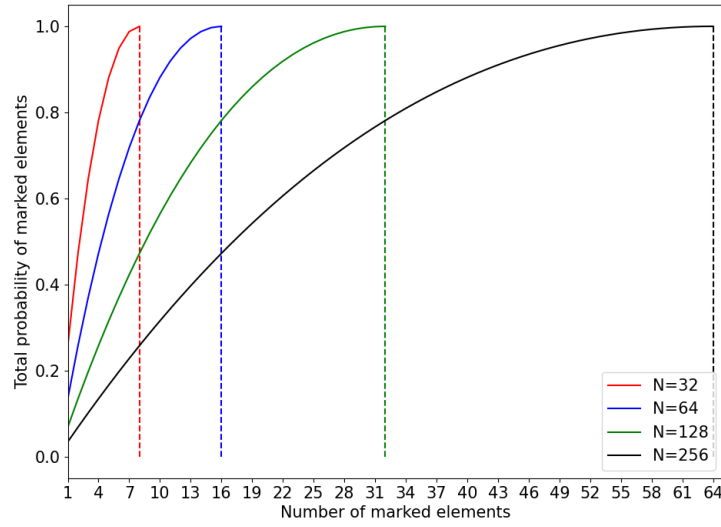


Figure 13: Total probability of marked elements in the Grover search, as a function of the number of marked elements, for 1 step, with  $N = 32, 64, 128$  and  $256$ .

reflects the only relevant one in this context. As a consequence, amplitudes associated with state  $|\Psi(k_1, l_1)\rangle$  become

$$\begin{cases} k_1 = \frac{1}{\sqrt{t}} \sin((2+1)\theta) = \frac{1}{\sqrt{\frac{N}{4}}} \sin(3\frac{\pi}{6}) = \frac{2}{\sqrt{N}} \\ l_1 = \frac{1}{\sqrt{N-t}} \cos((2+1)\theta) = \frac{1}{\sqrt{N-\frac{N}{4}}} \cos(3\frac{\pi}{6}) = 0. \end{cases} \quad (59)$$

These results show that the amplitudes associated with the marked states double in relation to  $|\Psi_0\rangle$  and the remaining states disappear after only one iteration. This behaviour can be seen in figure 13, where the total probability of marked elements reaches 1 after a single shot of the Grover procedure, once the number of marked elements is  $\frac{1}{4}$  of the total elements.

The next sections will present the search problem in the context of quantum walks. They are generalizations of Grover's algorithm, but implemented implemented in graphs.

## 3.2 COINED QUANTUM WALK

In classical computation, a *spatial search problem* focuses on finding marked points in a finite region of space. Defining this region with graphs is fairly straightforward, the vertices of the graph are the search space, and the edges define what transitions are possible through the search space. As was previously mentioned in section 3.1, exhaustively searching through an unstructured space, by means of a classical random walk for example, would mean that in the worst case, one would have to take as many steps to find the marked points as there are vertices in the graph. Quantum computing provides an alternative to this complexity through Grover's algorithm, and applying some of his ideas to the coined quantum walk not only allows a quantum counterpart to the random walk search, but also further insight into the algorithm itself.

Following Portugal (2018)'s definition, a good first step is to borrow the diffusion from Grover's algorithm and invert the sign of the state corresponding to the marked vertex while leaving unmarked vertices unchanged. This is done through the following operator

$$\mathcal{O} = I - 2 \sum_{x \in M} |x\rangle \langle x| \quad (60)$$

where  $M$  is the set of marked vertices and  $\mathcal{O}$  is an analogue to Grover's oracle. For one marked vertex, this oracle can be written as

$$\mathcal{O} = I - 2 |0\rangle \langle 0| \quad (61)$$

Notice that there is no loss of generality by choosing the marked vertex as 0, since the labelling of the vertices is arbitrary.

The next step is to combine the evolution operator from the coined quantum walk model with the oracle

$$U' = U\mathcal{O} \quad (62)$$

Similarly to the simple coined case, the walker starts at  $|\Psi(0)\rangle$  and evolves following the rules of an unitary operator  $U$  followed by the sign inversion of marked vertices. The walker's state after an arbitrary number of steps will be

$$\Psi(t) = (U')^t |\Psi(0)\rangle. \quad (63)$$

For a better understanding of the search problem in the coined quantum walk model, consider a graph where all the vertices are connected and each vertex has a loop that allows transitions to itself. The next step is to label the edges using notation  $\{(v, v'), v \geq 0 \wedge v' \leq$

$N - 1\}$  where  $N$  is the total number of vertices and  $(v, v')$  are the position and coin value, respectively. The shift operator, now called *flip-flop* shift operator, is

$$S |v1\rangle |v2\rangle = |v2\rangle |v1\rangle. \quad (64)$$

The coin operator is defined as

$$C = I_N \otimes G \quad (65)$$

where

$$G = 2 |D\rangle \langle D| - I \quad (66)$$

is the Grover coin, with  $|D\rangle$  being the diagonal state of the coin space.

Marking an element in a complete graph is done through the following oracle

$$\mathcal{O}' = \mathcal{O} \otimes I = (I_N - 2 |0\rangle \langle 0|) \otimes I_N = I_{N^2} - 2 \sum_v |0\rangle |v\rangle \langle 0| \langle v|, \quad (67)$$

that can be seen as an operator that marks all edges leaving 0. Recalling 62, now that all the operators are defined, the modified evolution operator can then be written as

$$U' = S(I \otimes G)\mathcal{O}' = S(I \otimes G)\mathcal{O} \otimes I = S(\mathcal{O} \otimes G), \quad (68)$$

and the state of the system will evolve according to equation 63.

As was shown in Portugal (2018), maximum probability of the marked vertex is achieved after  $\frac{\pi}{2}\sqrt{N}$  steps. Figure 14 is the result of coding and plotting the evolution of this probability distribution, for graphs of varying sizes. It shows that the probability is close to one at *approximately* the predicted ideal steps, because of the discrete nature of the walk. The probability distributions have a stair-like shape, because transitions in this model only occur on even numbered time steps, because of how the unmodified evolution operator was constructed.

The following section is dedicated to the study of the search problem using the staggered quantum walk model. The algorithm is still discrete, however since it does not use a coin, its Hilbert space will be much smaller. In the coined case, due to how the coin and shift operators were defined, for every qubit that represents the space of the walker, another qubit will be needed for the coin. This means that for a  $N$  qubit walk,  $2N$  qubits will be needed. Therefore, the staggered quantum walk will be better suited for running in a NISQ computer.

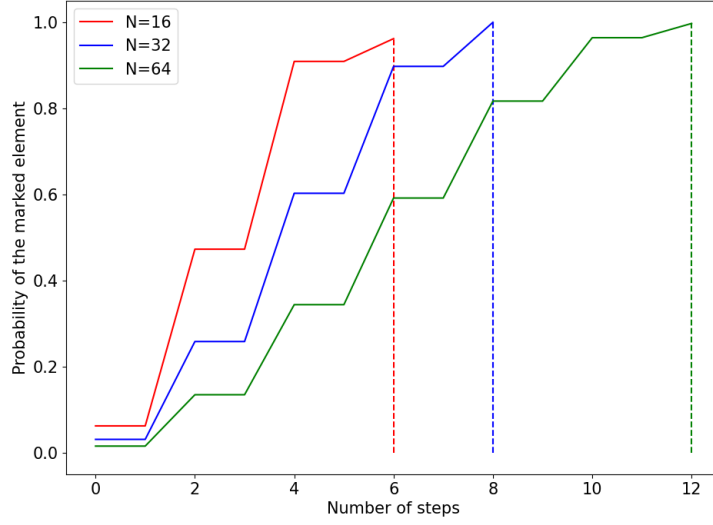


Figure 14: Probability of one marked element in the coined quantum walk search, as a function of the number of steps, for complete graphs of size  $N = 16, 32$  and  $64$ .

### 3.3 STAGGERED QUANTUM WALK

The process for defining the search problem in this model is similar to the coined quantum walk case. The oracle still inverts the sign of a certain state and amplifies it, and the system's state will still be described by equation 63. However, instead of using a coin, the staggered model takes advantage of the notions of cliques and tessellations, as was shown in chapter 2.3, which means the unmodified evolution operator has to be defined for an undirected complete graph.

As was previously seen, a complete graph is defined as a simple undirected graph where each pair of distinct vertices is connected by a unique edge. This is a special case, because this is the only connected graph that can be covered by a single tessellation, due to the fact that the graph is its own clique. The minimum tessellations required to cover this structures are defined by the one clique that encompasses all  $N$  nodes of the graph

$$\mathcal{T}_\alpha = \{\{0, 1, 2, \dots, N-1\}\}. \quad (69)$$

The associated polygon can then be described as the balanced superposition of all the nodes in the graph

$$|\alpha\rangle = \frac{1}{\sqrt{N}} \sum_{v=0}^{N-1} |v\rangle. \quad (70)$$

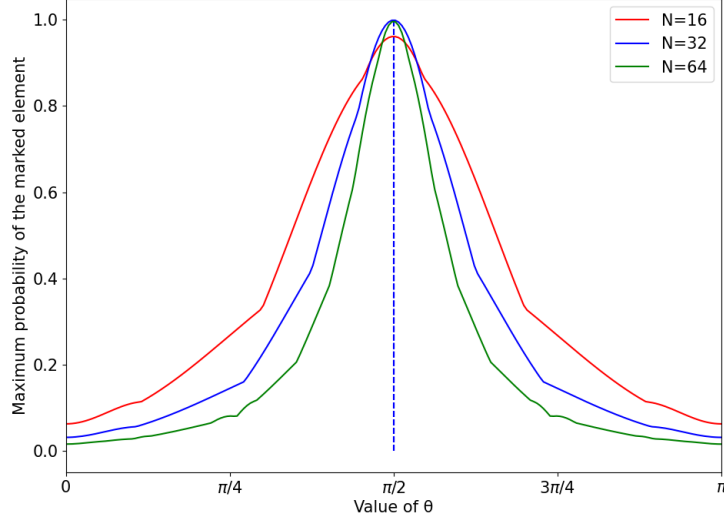


Figure 15: Maximum probability of the marked element as a function of the  $\theta$  value plotted from 0 to  $\pi$ , for complete graphs of size  $N = 64, 128$  and  $256$ .

The Hamiltonian, as defined in 17, is

$$H_\alpha = 2 \sum_0^1 |\alpha\rangle \langle \alpha| - I = 2 |\alpha_0\rangle \langle \alpha_0| - I \quad (71)$$

The unmodified evolution operator from equation 18

$$U = e^{i\theta_k H_k} \dots e^{i\theta_2 H_2} e^{i\theta_1 H_1} \quad (72)$$

reduces to the single Hamiltonian case

$$U = e^{i\theta H_\alpha}. \quad (73)$$

The choice of the  $\theta$  value is an important one, since maximum probability is achieved at  $\theta = \frac{\pi}{2}$ , as shown in figure 15.

Since  $H_\alpha^2 = I$ , equation 73 can be rewritten as

$$U = e^{-i\frac{\pi}{2} H_\alpha} = \cos \frac{\pi}{2} I + i \sin \frac{\pi}{2} H_\alpha = i H_\alpha = i(2 |\alpha_0\rangle \langle \alpha_0| - I). \quad (74)$$

Having defined the the evolution operator associated with the complete graph, the next step is to use the oracle

$$\mathcal{O} = I_N - 2 |0\rangle \langle 0|, \quad (75)$$

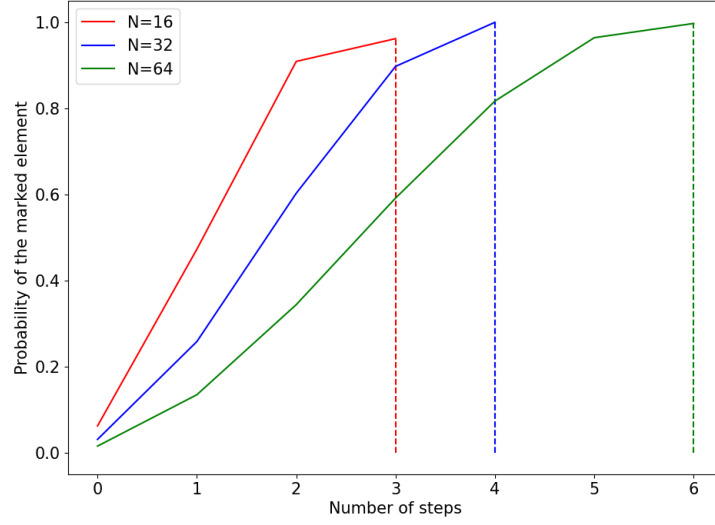


Figure 16: Probability of one marked element in the staggered quantum walk search, as a function of the number of steps, for complete graphs of size  $N = 16, 32$  and  $64$ .

to create the modified evolution operator associated with the search

$$U' = U\mathcal{O}. \quad (76)$$

The walk achieves the same result as Grover's algorithm after  $\frac{\pi}{4}\sqrt{N}$  steps, as shown in figure 16. This plot also shows that the probabilities converge to 1 as  $N$  increases. Because time is discretized, deviations to the ideal number of steps will matter less for bigger values of  $N$ . Unlike Grover's algorithm, however, the parameter  $\theta$  can be changed in order to alter how many iterations are required to achieve maximum probability of the marked element. Combined with the fact that one has control over which structure this search problem is performed on, the staggered quantum walk is a more general algorithm when compared to Grover's search, both being equivalent when a complete graph is considered and  $\theta = \frac{\pi}{2}$ .

Finally, the next section will present the search problem using the continuous-time quantum walk model. Since time is not discretized, a finer control over the evolution is possible. This will have significant impact when translating the algorithm to a quantum circuit, since it will be seen that circuit depth will not scale with an increase in time.

## 3.4 CONTINUOUS-TIME QUANTUM WALK

As was previously seen, the continuous-time quantum walk model is defined by an evolution operator obtained by solving Schrödinger's equation

$$U(t) = e^{-iHt}. \quad (77)$$

The search problem requires introducing an oracle to the Hamiltonian, that will mark an arbitrary vertex  $m$

$$H' = -\gamma L - \sum_{m \in M} |m\rangle \langle m|, \quad (78)$$

where  $M$  is the set of marked vertices. Since the complete graph is a regular graph, the operator can be rewritten in terms of the adjacency matrix plus the marked elements. Considering the case where only element  $|0\rangle$  is marked,

$$U'(t) = e^{iH't} = e^{i(-\gamma L - |0\rangle \langle 0|)t} = e^{i(-\gamma A + \gamma D - |0\rangle \langle 0|)t} = e^{-i\gamma(A + |0\rangle \langle 0|)t + i\gamma Dt}. \quad (79)$$

The degree matrix is again  $D = dI$ , which means it will commute with  $A + |0\rangle \langle 0|$  and become a global phase

$$U'(t) = e^{-i\gamma(A + |0\rangle \langle 0|)t} e^{i\gamma Dt} = \phi(t) e^{-i\gamma(A + |0\rangle \langle 0|)t}. \quad (80)$$

As was show by [Childs and Goldstone \(2004\)](#), the value of  $\gamma$  is crucial for the success of the search. As  $\gamma$  increases, the contribution of the marked element in the Hamiltonian decreases, and as  $\gamma$  approaches 0 the contribution of the adjacency matrix decreases. To find the optimum value, the Hamiltonian can be rewritten by adding multiples of the identity matrix to the adjacency matrix

$$H' = -\gamma(A + NI) - |0\rangle \langle 0| = -\gamma N |s\rangle \langle s| - |0\rangle \langle 0|, \quad (81)$$

where  $|s\rangle = \frac{1}{\sqrt{N}} \sum_i |i\rangle$ . Now it is obvious that, for  $\gamma = \frac{1}{N}$ , the Hamiltonian becomes  $H = -|s\rangle \langle s| - |0\rangle \langle 0|$ . Its eigenstates are proportional to  $|s\rangle \pm |w\rangle$  and eigenvalues are  $-1 - \frac{1}{\sqrt{N}}$  and  $-1 + \frac{1}{\sqrt{N}}$ , respectively. This means that the evolution rotates from the state of balanced superposition to the marked vertex state in time  $\frac{\pi}{\Delta E} = \frac{\pi}{2} \sqrt{N}$  which is, as shown by [Farhi et al. \(2000\)](#), equivalent to Grover's algorithm.

Plotting  $\Delta E$  as a function of  $\gamma N$ , as can be seen in figure 17, has a minimum at  $\gamma N = 1$ . The difference between the largest eigenvalue and second largest, plotted in the y-axis, is the smallest for a value of  $\gamma N = 1 \implies \gamma = \frac{1}{N}$ , which will correspond to the maximum probability for the marked vertex, in optimal steps.

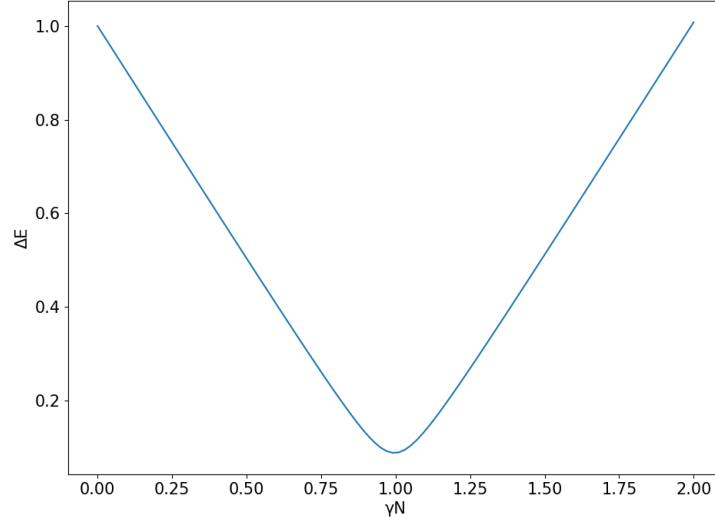


Figure 17: Value of the difference between the largest eigenvalue and the second largest plotted as a function of  $\gamma N$ , for  $N = 512$ .

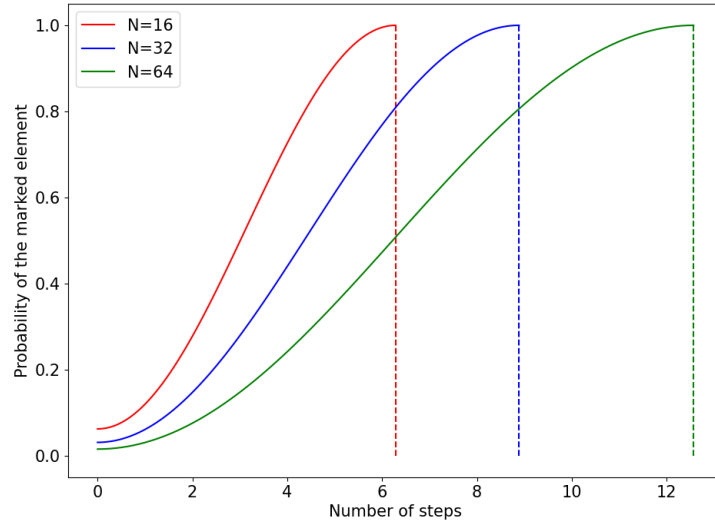


Figure 18: Probability of one marked element in the continuous quantum walk search, as a function of the number of steps, for complete graphs of size  $N = 16, 32$  and  $64$ .

Figure 18 shows the evolution of the probability of the marked vertex in time, which is continuous in this model. In contrast with previous models, the distributions are smooth and reach exactly one, since the walk is allowed to evolve to exactly the ideal time steps.



Again, this walk will require only as many qubits as needed to represent the space of the walker. Similarly to the staggered case, this model allows the adjustment for the  $\gamma$  parameter, that will affect the number of ideal steps, and also allows the choice of structure over which to perform the search, making it more general than the Grover algorithm as well. Another interesting property is that, because time is treated as a continuous variable, the circuit associated will not scale in time. For the case without search, the circuit will be relatively small and somewhat resistant to noise. When considering the search problem, however, the introduction of the oracle and the dependency of the Suzuki-Trotter expansion will render the circuit less than ideal for NISQ implementation, as will be seen in the coming chapter.

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## IMPLEMENTATIONS AND APPLICATIONS

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This chapter is dedicated to the construction of quantum walk circuits using IBM's software, *Qiskit*, whose code is presented in the Github repository<sup>1</sup>. It is composed of two parts, sections 4.1 through 4.3 present the circuits for the dynamics of each previously studied quantum walk model and the chapter is closed with section 4.4, where the search problem for each model is implemented.

The first part of this chapter begins with section 4.1, where the coined quantum walk implementation based on the work of Douglas and Wang (2009) can be found. Here, the line graph is again considered, whose shift operation is composed of increment and decrement gates constructed with generalized CNOT gates. As was seen in previous chapters, this model requires an extra qubit for the space of the coin that, when combined with all the operations required to implement the generalized CNOT gates, results in a circuit far too deep to be implemented in current NISQ computers, as was concluded at the end of this section. The staggered quantum walk circuit is presented in the following section, whose construction was based on the work of Acasiete et al. (2020). This implementation still uses the notion of the increment and decrement operations however, since a coin was not required, the resulting circuit becomes much more NISQ-friendly. The first part is then concluded with section 4.3, where the continuous-time quantum walk circuit is implemented. Here, the best results for the dynamics of the quantum walk were obtained, due to the circulant graph approach that, unlike the previous discrete models, does not require extra iterations of the algorithm to represent time. Work by Qiang et al. (2016) firstly presented the circulant graph definition of this model limited to the complete graph case, and the final part of this section greatly expands on this work by means of statistical analysis of the impact of the approximate quantum Fourier transform on a large collection of circulant graphs.

Finally, section 4.4 closes this chapter with the creation of circuits for the search problem using quantum walks. It starts with the introduction of quantum searching by means of the circuit associated with Grover's algorithm, followed by the coined quantum walk, which produces the worst results due to the requirement of  $2n$  qubits and swap gates for the representation of the complete graph. Surprisingly, the best results were achieved for

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<sup>1</sup> <https://github.com/JaimePSantos/QWQiskit>

the staggered quantum walk search problem, since it produced the circuit with the least operations. The continuous-time quantum walk search was expected to produce the best results in this section also, but the introduction of the oracle together with the requirement of the Suzuki-Trotter expansion resulted in a circuit somewhat impacted by noise.

#### 4.1 COINED

Consider the example of a quantum walker on a discretely numbered cycle. It was seen that the evolution operator associated with such a system is, as was defined in equation 6

$$U = S(C \otimes I), \quad (82)$$

where  $S$  is a shift operator, defined in equation 5 as

$$S = |0\rangle\langle 0| \otimes \sum_{x=-\infty}^{x=\infty} |x+1\rangle\langle x| + |1\rangle\langle 1| \otimes \sum_{x=-\infty}^{x=\infty} |x-1\rangle\langle x|, \quad (83)$$

that increments or decrements the position of the walker according to the coin operator  $C$ .

Previously, this system was simulated in Python by coding its equations. Now, the focus is to study and implement a quantum circuit based on the work presented by [Douglas and Wang \(2009\)](#). This approach relies on multi-controlled CNOT gates in order to shift the state of the walker by  $+1$  or  $-1$ , each with a probability associated with the chosen coin, as can be seen in figure 19. The generalized CNOT gates act on the vertex states as a cyclic

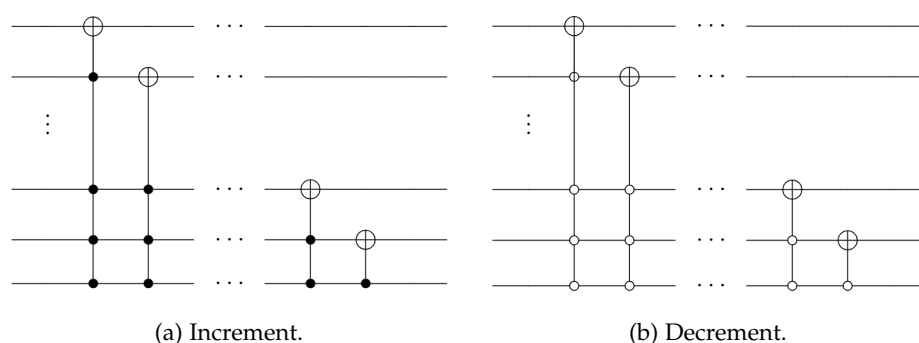


Figure 19: General circuits of the components of the shift operator for the coined quantum walk.

permutator, where each vertex state is mapped to an adjacent state. This can be seen as the walker moving left or right, in the line graph example.

The coin operator will simply be a Hadamard gate acting on a single qubit. For a graph of size  $N = 8$ , for example,  $n = 3$  qubits are required to encode each vertex, and an extra qubit for the coin.

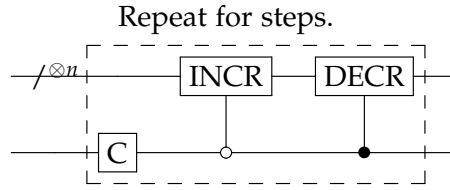
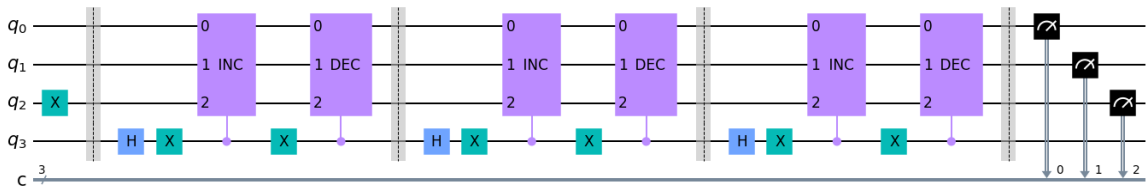


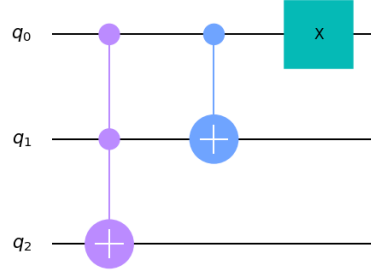
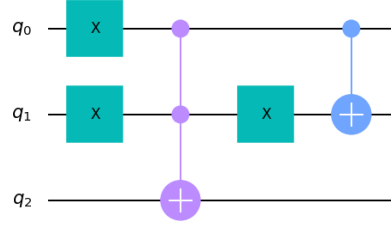
Figure 20: General circuit for the coined quantum walk.

The general circuit for the coined quantum walk will then be as shown in figure 20. Note that this circuit limits the number of graph nodes to powers of 2, and an arbitrary implementation of  $2^n$  nodes requires  $n + 1$  qubits. However, it is possible to have any number of nodes, given that the proper correction is made, as was shown in the work of [Douglas and Wang \(2009\)](#). The method used for this correction is called *Gray Code Ordering* proposed by [Slepoy \(2006\)](#), whereby a certain arrangement of CNOT gates results in control states only differing by a single bit.

Figure 21: Qiskit circuit for the coined quantum walk, for a line graph of size  $N = 8$  and initial condition  $|\psi_0\rangle = |4\rangle$ , with 3 steps and the Hadamard coin.

This circuit was implemented in Qiskit, as can be seen in figure 21. In this example, the increment and decrement sequence was applied three times on a graph of size  $2^3 = 8$  nodes. The starting position of the walker was set to  $\Psi(0) = |4\rangle$ , and the Hadamard coin was used. The first block after the barrier is the sequence of operations that will increment the state of the walker, as is shown in figure 22. The generalized CNOT gates are implemented using Qiskits *mcx* function, which implements the decomposition presented in lemma 7.5 of [Barenco et al. \(1995\)](#) to create CNOT gates up to four controls, and the aforementioned Gray Code for more generalized instances of the gate.

The next set of operations perform the decrement of the state of the walker, which is just an increment block with it's controls negated, as is shown in figure 23. The rest of the circuit is just the repetition of these operations as a function of the number of steps required.

Figure 22: Qiskit circuit of the increment operator for a line graph of size  $N = 8$ .Figure 23: Qiskit circuit of the decrement operator for a line graph of size  $N = 8$ .

Lastly, the circuit is measured and the results can be seen in figure 24. These results can be verified by calculating the time evolution of the wave function associated with the system

$$|\Psi(0)\rangle = |4\rangle, \quad (84)$$

$$|\Psi(1)\rangle = \frac{|0\rangle |x=3\rangle + |1\rangle |x=5\rangle}{\sqrt{2}}, \quad (85)$$

$$|\Psi(2)\rangle = \frac{|0\rangle |x=2\rangle + |1\rangle |x=4\rangle + |0\rangle |x=4\rangle - |1\rangle |x=6\rangle}{2}, \quad (86)$$

$$|\Psi(3)\rangle = \frac{|1\rangle |x=1\rangle - |0\rangle |x=3\rangle + 2(|0\rangle + |1\rangle) |x=5\rangle + |0\rangle |x=7\rangle}{2\sqrt{2}}. \quad (87)$$

Taking the modulus squared of the amplitudes associated with the states, confirms that the probability distribution associated with the *Qasm* simulator, presented in figure 24 as the blue bar plot, is correct. However, the results obtained from running in IBM's *Toronto* backend are not satisfactory. This is because of the size of the circuit, the coined quantum walk model besides requiring an extra qubit for the coin, also needs a very large number of CNOT gates, which have an average associated error of  $1.284e - 2$  in this specific backend.

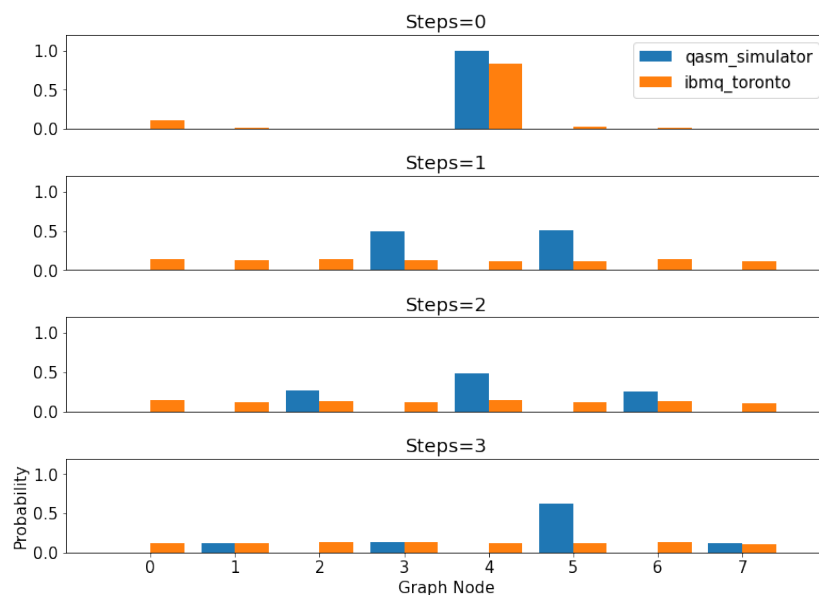


Figure 24: Probability distributions of the coined quantum walk for several steps in a line graph of size  $N = 8$ . The blue bar plot represents a circuit run in the QASM simulator, and the orange bar plot on IBM's Toronto backend.

For a better error analysis, one can calculate the fidelity between the ideal distribution  $p(x)$  and the experimental  $q(x)$  using the formula

$$F(p, q) = \sum_{x=0}^{N-1} \sqrt{p(x)q(x)} \quad (88)$$

where  $x$  is the vertex. For 0 steps, the obtained fidelity is approximately 0.91 which is very high, because the circuit is simply the initial condition, and when the circuit is increased to 1 step the fidelity lowers to 0.49, due to the increase of CNOTS. However, for 2 and 3 steps, the fidelity increases again to approximately 0.63 due to the fact that the probability distribution in the simulator becomes more spread out with the increase of steps, meaning that the random uniform distribution obtained from the noisy experiments ran in IBM's backend will be closer to them than a highly localized distribution like in the case of 1 step.

In order to reduce the effects introduced by noise, the next chapter will present the circuit for the alternative discrete model, the staggered quantum walk. Besides not needing an extra qubit for the coin, this model also does not require as many CNOT gates for each iteration.

## 4.2 STAGGERED

As was seen in section 2.3, the elements of each tessellation of a discretely numbered cycle can be described by states

$$|\alpha_x\rangle = \frac{|2x\rangle + |2x+1\rangle}{\sqrt{2}}, \quad (89)$$

$$|\beta_x\rangle = \frac{|2x+1\rangle + |2x+2\rangle}{\sqrt{2}}. \quad (90)$$

These states allow the construction of the Hamiltonians

$$H_\alpha = 2 \sum_{x=-\infty}^{+\infty} |\alpha_x\rangle \langle \alpha_x| - I, \quad (91)$$

$$H_\beta = 2 \sum_{x=-\infty}^{+\infty} |\beta_x\rangle \langle \beta_x| - I, \quad (92)$$

as was seen in equations 24 and 25.

Following the implementation presented in the work of [Acasiete et al. \(2020\)](#), these operators can be rewritten in matrix form

$$H_\alpha = I \otimes X, \quad (93)$$

$$H_\beta = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & H_\alpha & \vdots \\ 1 & \cdots & 0 \end{pmatrix}, \quad (94)$$

which are very useful representations for the construction of the circuit.

As was shown in equation 26, the unitary evolution operator is

$$U = e^{i\theta H_\beta} e^{i\theta H_\alpha} = U_\beta U_\alpha, \quad (95)$$

and knowing that

$$R_x(\theta) = e^{\frac{-i\theta X}{2}}, \quad (96)$$

then each of the evolution operators associated with the different tessellation Hamiltonians will be

$$U_\alpha = I \otimes R_x(\theta), \quad (97)$$

$$U_\beta = \begin{pmatrix} \cos \theta & \cdots & -i \sin \theta \\ \vdots & U_\alpha & \vdots \\ -i \sin \theta & \cdots & \cos \theta \end{pmatrix}. \quad (98)$$

Notice that  $U_\beta$  is simply a permutation of  $U_\alpha$ , therefore it can be rewritten as

$$U_\beta = P^{-1} U_\alpha P, \quad (99)$$

where  $P = \sum_x |x+1\rangle \langle x|$ .

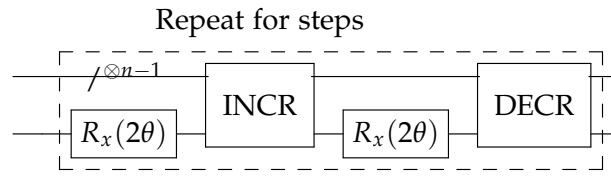


Figure 25: General circuit for the staggered quantum walk.

Remember from equation 83 and figure 19 that these permutation operators can be implemented as increment and decrement gates, defined in the work of Douglas and Wang (2009). Therefore, the circuit for the staggered quantum walk on the line can be constructed as is shown in figure 25.

The next step is to implement the circuit in Qiskit, in order to test it in a real quantum computer. Here, the walk will take place in a cyclic graph with 8 elements, meaning 3 qubits will be required, as is shown in figure 26. The circuit starts with a Pauli-X gate in the

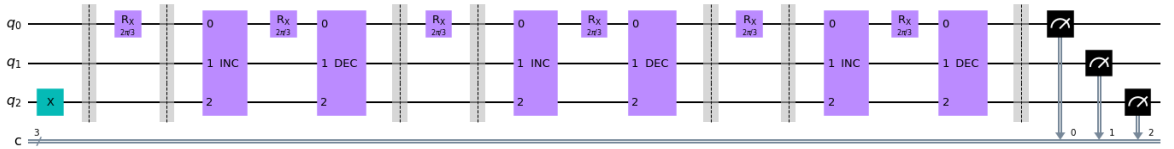


Figure 26: Qiskit circuit for the staggered quantum walk, for a line graph of size  $N = 8$  and initial condition  $|\psi_0\rangle = |4\rangle$ , with 3 steps.

third qubit so that  $|\psi_0\rangle = |4\rangle$ . The following operation is a rotation in the X basis, where  $\theta = \frac{\pi}{3}$ , since it was seen in figure 7 that this value of  $\theta$  maximizes the propagation of the walk. Finally,  $U_\beta$  is applied, making use of the increment and decrement gates defined in figures 22 and 23. Note that now, because this model does not require a coin, these gates do not need to be controlled, meaning that the largest multi-controlled NOT gate will not be needed, making it more NISQ-friendly. This procedure is repeated 3 times, and the resulting probability distributions after measurement can be seen in figure 27.



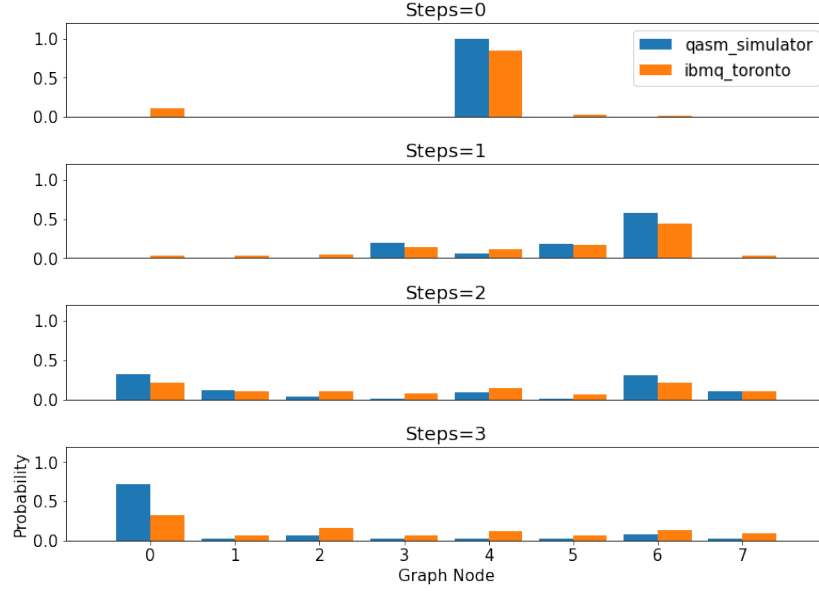


Figure 27: Probability distributions of the staggered quantum walk for several steps in a line graph of size  $N = 8$ . The blue bar plot represents a circuit run in the QASM simulator, and the orange bar plot on IBM's Toronto backend.

Analyzing the figure, it is clear that this model is much better suited for running in current quantum hardware. Even though the probability distribution was somewhat affected by noise, the dynamics of the walk are relatively unaffected by noise in the Toronto backend experiment. The highest fidelity was achieved for 2 steps, with a value of approximately 0.95, and the remaining steps ranging from 0.91 to 0.92. Again, it may seem counter-intuitive that a higher fidelity is achieved for a larger circuit, compared to 0 steps for example, but this is again due to the balance of circuit size and spread of the probability distribution.

Nevertheless, because this discrete model requires ever more operations with the increase of steps, it will eventually become intractable for NISQ technology. For this reason, the next model studied in this work is one where the circuit will be constant in time.

### 4.3 CONTINUOUS

As was seen in section 2.4, the unitary evolution operator of this model is defined as

$$U(t) = e^{-iHt} = e^{i(-\gamma L)t} = e^{-i\gamma(A+D)t}. \quad (100)$$

Considering a regular graph, this operator can be rewritten as

$$U(t) = \phi(t)e^{-i\gamma(A)t}, \quad (101)$$

where  $\phi(t)$  is a global phase and  $A$  is the adjacency matrix associated with the graph.

Here, the study will focus on the circuit implementation of this walk in a cyclic graph, and a different approach was used to define the adjacency matrix. This approach relies on the concept of a circulant graph, which are a class of graphs defined by a circulant matrix such that

$$A = \begin{pmatrix} c_0 & c_{N-1} & \cdots & c_3 & c_2 \\ c_1 & c_0 & c_{N-1} & & c_3 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{N-2} & & \ddots & \ddots & c_{N-1} \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{pmatrix}. \quad (102)$$

In order to generate the proper circulant graphs, restrictions on this matrix are needed. Firstly,  $c_0 = 0$ , since self-loops are not part of the structure. Secondly, the matrix must be symmetric, therefore  $c_{n-j} = c_j$ .

These matrices can be fully described by the first column of the matrix

$$v_1 = [c_0, c_1, \dots, c_{N-2}, c_{N-1}]^T \quad (103)$$

with a discrete convolution operator performing cyclic permutations of  $c$ , on each column, known as the *deque* operator. For example,

$$Dv_1 = [c_{N-1}, c_0, \dots, c_{N-3}, c_{N-2}]^T = v_2. \quad (104)$$

More specifically, for the cycle case

$$Dv_1 = D[0, 1, 0, \dots, 0, 1]^T = [1, 0, 1, 0, \dots, 0, 0]^T = v_2. \quad (105)$$

The eigenvalues of a circulant matrix can be found by

$$\lambda_p = c_0 + \sum_{q=1}^{N-1} c_{N-q} \omega^{pq}, \quad (106)$$

and the eigenvectors are

$$|\varphi_p\rangle = \frac{1}{\sqrt{N}} \sum_{q=0}^{N-1} \omega^{pq}. \quad (107)$$

This given, it is possible to construct an operator that diagonalizes the circulant matrix through the eigenvectors, which is useful for constructing the circuit. For this purpose, the Quantum Fourier Transform can be used and it is defined as

$$F = \frac{1}{\sqrt{N}} \sum_{p,q} \omega^{pq} |p\rangle \langle q|. \quad (108)$$

The adjacency matrix of a circulant graph can then be diagonalized such that

$$A = F^\dagger \Lambda F, \quad (109)$$

where  $\Lambda$  is a diagonal operator that encodes the eigenvalues

$$\Lambda = \sum_j \lambda_j |j\rangle \langle j|. \quad (110)$$

The unitary operator of the walk can then be rewritten as

$$U = F^\dagger e^{i\gamma\Lambda t} F \quad (111)$$

where

$$e^{i\gamma\Lambda t} = \sum_j e^{i\gamma\lambda_j t} |j\rangle \langle j|. \quad (112)$$

This representation is very easily translatable to a quantum circuit, as can be seen in figure 28.

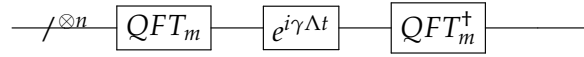


Figure 28: General circuit for the continuous-time quantum walk.

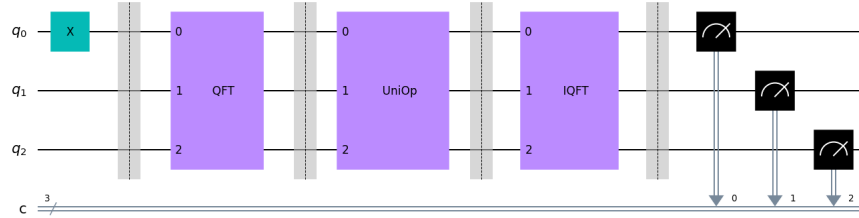


Figure 29: Qiskit circuit for the continuous-time quantum walk, for a line graph of size  $N = 8$  and initial condition  $|\psi_0\rangle = |4\rangle$ , for time  $t$ .

The circuit can now be constructed making use of the *diagonal* function provided by Qiskit, which decomposes diagonal operators based on the method presented in theorem 7 of Shende et al. (2006). The other tool used was the quantum Fourier transform (QFT), also provided by the Qiskit package and defined in section B.1. Figure 29 shows the implementation of the circuit for 3 qubits or  $2^3 = 8$  graph nodes and  $t = 3$ .

The quantum Fourier transform circuit is well known and presented in figure 30. The inverse QFT is similarly constructed with the negation of the rotations associated with the QFT.

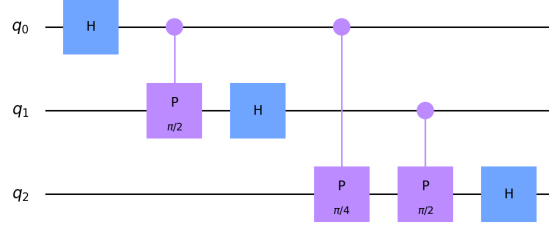


Figure 30: Qiskit circuit of the quantum Fourier transform for a line graph of size  $N = 8$ .

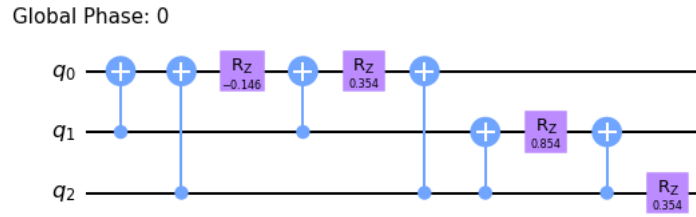


Figure 31: Qiskit circuit of the diagonal operator associated with the adjacency matrix, for a line graph of size  $N = 8$ .

The circuit associated with the diagonal operator is shown in figure 31. Furthermore equation 111 says that time is simply a constant inside the exponential, which means that the diagonal operator's circuit will not need extra operations when increasing time, it will only need different rotations and differ in global phase. This is an advantage when comparing to previous discrete models, where each extra step required another increment and decrement gates.

Finally, the circuit was measured, and the resulting probability distributions can be seen in figure 32. The dynamics of the walk, when ran on the Toronto backend, is closer to the simulation than the previous examples. As expected, since the size of the circuit does not scale with time, a fidelity of approximately 0.97 was achieved for  $t = 3$ . For  $t = 0, 1, 2$ , the respective fidelities were 0.96, 0.90, 0.92, which are small improvements when compared to the staggered quantum walk, but that stay relatively constant with increasing time, making this type of circuit well suited for studying the dynamics of the continuous-time quantum walk.

#### Further Experiments

Another improvement can be made to this model by use of the approximate quantum Fourier transform (AQFT). The AQFT, proposed by Coppersmith (2002), is achieved through the modification of the regular quantum Fourier transform circuit, as defined in annex

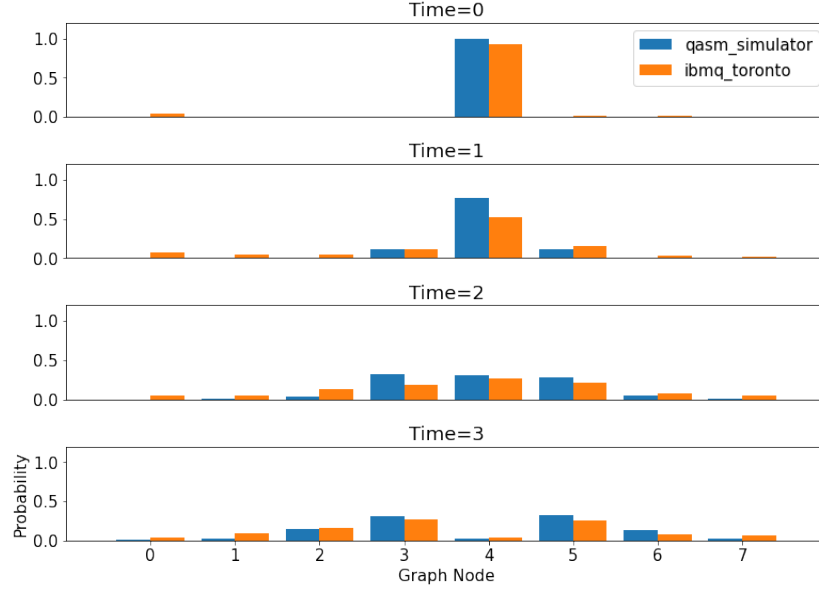


Figure 32: Probability distributions of the continuous-time quantum walk for several steps in a line graph of size  $N = 8$ . The blue bar plot represents a circuit run in the QASM simulator, and the orange bar plot on IBM’s Toronto backend.

[B.1](#), by removing the phase-shift operations between the most distant qubits. This can be implemented in Qiskit by providing the QFT function the approximation degree.

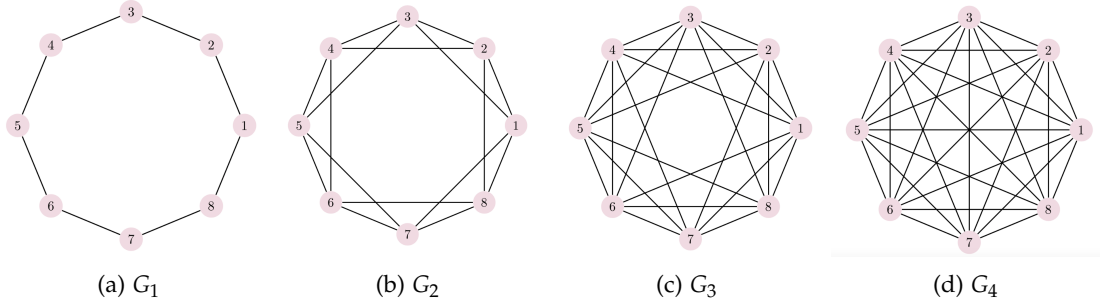
Each experiment is performed 10 times, with 3000 shots each, in order to extract substantial statistical data, using the confidence interval of 95%. The average fidelity between the ideal  $p(x, t)$  and the experimental  $q(x, t)$  distributions is again calculated using

$$F(p, q) = \frac{1}{10} \sum_{i=1}^{10} \sum_{x=0}^{N-1} \sqrt{p(x, t)q(x, t)}, \quad (113)$$

which is a simple modification of equation 88.

For this section, not only is the cyclic graph considered, but also a considerable number of other circulant graph, as is shown in figure 33. The numbering of the graphs follows the rule that  $G_k$  will have entries  $c_k, \dots, c_1 = 1$  and  $c_{n-k}, \dots, c_{n-1} = 1$ , and the remaining elements are 0. This way, it is possible to systematically construct circulant graphs varying from sparse to dense.

Starting with the smaller  $N = 2^2 = 4$  case, two non-isomorphic circulant graphs are possible to construct.  $G_1$  corresponds to the cycle graph, and  $G_2$  to the complete graph. Table 1 shows the achieved average fidelities, which were calculated with equation 113. Comparing to the complete graph case presented in [Qiang et al. \(2016\)](#), where they obtained a fidelity of  $0.967 \pm 0.003$ , it is possible to see that the implementation presented here slightly outperforms it in terms of fidelity.

Figure 33: Circulant graphs  $G_k$  for  $N = 8$  elements.

$m \backslash G$	$G_1$	$G_2$
0	$0.98 \pm 0.01$	$0.99 \pm 0.01$
1	$0.98 \pm 0.02$	$0.993 \pm 0.006$

Table 1: Fidelity of quantum state with  $N=4$ , backend *Toronto*, and  $t=1$ .

For the  $N = 2^3 = 8$  case, table 2 shows that state fidelity is greater as graph connectivity increases, and as  $m$  increases. This is due to the fact that a greater  $m$  implies a smaller circuit, but also an increase in error. However, graphs have less distinct eigenvalues as they are more connected, which means that a higher degree of approximation of the QFT will generally introduce less error, while keeping circuit depth lower.

$m \backslash G$	$G_1$	$G_2$	$G_3$	$G_4$
0	$0.80 \pm 0.01$	$0.92 \pm 0.02$	$0.968 \pm 0.007$	$0.965 \pm 0.006$
1	$0.894 \pm 0.007$	$0.95 \pm 0.01$	$0.98 \pm 0.01$	$0.973 \pm 0.008$
2	$0.852 \pm 0.009$	$0.955 \pm 0.004$	$0.985 \pm 0.003$	$0.990 \pm 0.002$

Table 2: Fidelity of quantum state with  $N=8$ , backend *Toronto*, and  $t=1$ .

Finally, table 3 presents the  $N = 2^4 = 16$  case. Here, the behaviour is similar to the previous case up to graph  $G_5$ , meaning that higher graph connectivity and larger  $m$  will result in higher fidelity. However, graphs  $G_6$ ,  $G_7$  and  $G_8$ , even though are highly connected and have relatively low depth, present lower fidelity. This seems to contradict the results

$m \backslash G$	$G_1$	$G_2$	$G_3$	$G_4$	$G_5$	$G_6$	$G_7$	$G_8$
0	$0.47 \pm 0.03$	$0.61 \pm 0.02$	$0.78 \pm 0.02$	$0.86 \pm 0.01$	$0.86 \pm 0.01$	$0.70 \pm 0.04$	$0.54 \pm 0.03$	$0.49 \pm 0.04$
1	$0.50 \pm 0.03$	$0.63 \pm 0.03$	$0.79 \pm 0.03$	$0.87 \pm 0.02$	$0.85 \pm 0.03$	$0.70 \pm 0.03$	$0.55 \pm 0.05$	$0.50 \pm 0.04$
2	$0.55 \pm 0.03$	$0.71 \pm 0.03$	$0.83 \pm 0.02$	$0.90 \pm 0.01$	$0.89 \pm 0.02$	$0.75 \pm 0.02$	$0.62 \pm 0.04$	$0.59 \pm 0.06$
3	$0.60 \pm 0.03$	$0.70 \pm 0.02$	$0.85 \pm 0.01$	$0.92 \pm 0.01$	$0.91 \pm 0.01$	$0.80 \pm 0.03$	$0.71 \pm 0.04$	$0.69 \pm 0.04$

Table 3: Fidelity of quantum state with  $N=16$ , backend *Toronto*, and  $t=1$ .

in table 2, however this behavior can be explained due to the fact that the probability distribution of the dynamics of the walk on these structures is highly concentrated in a small number for vertices. Considering that the size of the circuit starts to push the limits of NISQ

computers, it is expected that the spreading out of the probability distribution due to the effects of noise produces a lower fidelity. Nonetheless, the increase of  $m$  still has a positive impact on the fidelity of these circuits, which means that the reduction in circuit size will indeed produce better results.

#### 4.4 SEARCH PROBLEMS WITH QISKIT

##### 4.4.1 Grover

As was previously seen in section 3.1, Grover's algorithm is a quantum alternative to unstructured search problems. Consider the case of finding element  $x_0$  out of an unordered list of size  $N$ . Worst case scenario, a classical algorithm would need to check every element of the list, requiring  $N$  steps.

The first stage of Grover's algorithm is to create an uniform superposition of all states in the system

$$|\Psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle. \quad (114)$$

The next stage is the application of the Grover iteration process, which starts with an oracle that adds a negative phase to the solution states

$$\mathcal{O} |x\rangle = (-1)^{f(x)} |x\rangle. \quad (115)$$

This operator can be seen as an identity matrix with negative entries corresponding to the solution states, and the operator can be rewritten as

$$\mathcal{O} = I - 2 \sum_{m \in M} |m\rangle \langle m|. \quad (116)$$

where  $I$  is the identity matrix and  $M$  is a set of solutions where  $f(m) = 1$ , otherwise it's 0. The matrix associated with this operator will be

$$\mathcal{O} = \begin{pmatrix} (-1)^{f(0)} & 0 & \dots & 0 \\ 0 & (-1)^{f(1)} & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \dots & (-1)^{f(N-1)} \end{pmatrix}. \quad (117)$$

The second part of the iteration is an amplitude amplification process by means of the diffusion operator

$$\mathcal{D} = (2 |\Psi_0\rangle \langle \Psi_0| - I) = H^{\otimes n} (2 |0\rangle \langle 0| - I) H^{\otimes n}. \quad (118)$$

The unitary operator that describes the Grover iteration process will then be

$$\mathcal{U} = \mathcal{D}\mathcal{O}. \quad (119)$$

As was previously discussed, this iteration process will be done several times, depending on the number of elements. Optimal probability of success finding a single solution will be reached after  $\lfloor \frac{\pi}{4}\sqrt{N} \rfloor$  steps, and  $\lfloor \frac{\pi}{4}\sqrt{\frac{N}{K}} \rfloor$  for  $K$  solutions, which is a quadratic gain when compared to the classical case. The general Grover circuit can then be constructed as is shown in figure 34.

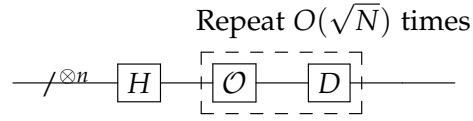


Figure 34: General circuit for the Grover search.

Consider the 3 qubit case, where  $N = 8$  and solution state  $|4\rangle$ . The optimal number of iterations is approximately 2, and figure 35 is the circuit for 3 iterations implemented in Qiskit.

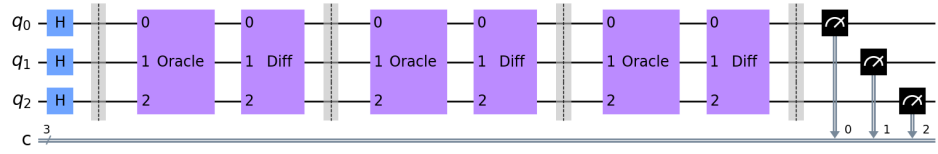


Figure 35: Qiskit circuit for the Grover algorithm, for a search size of  $N = 8$  and 3 steps.

The system starts with the creation of an uniform superposition state, which means applying Hadamard gates to each qubit. Immediately following the barrier, the first operator

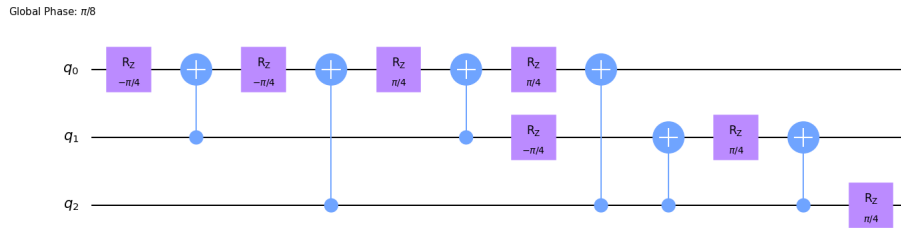


Figure 36: Qiskit circuit of the diagonal oracle operator in search space of size  $N = 8$ , for marked element  $|m\rangle = |4\rangle$ .

of the iteration process is the oracle, which is shown in figure 36. Because the oracle operator



is simply the identity matrix with negative entries corresponding to the solution states, it can be simply translated into a circuit by means of the *diagonal* function in Qiskit.

The last part of the iteration is the diffusion operator, whose circuit is shown in figure 37. Comparing equations 116 and 118, it is easy to see why figures 36 and 37 are so similar.

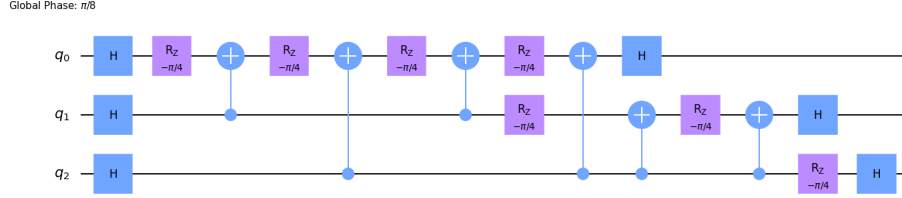


Figure 37: Qiskit circuit of the diagonal Grover diffusion operator in search space of size  $N = 8$ .

The diffusion circuit will simply be the oracle circuit for state  $|0\rangle$  in between Hadamard operations.

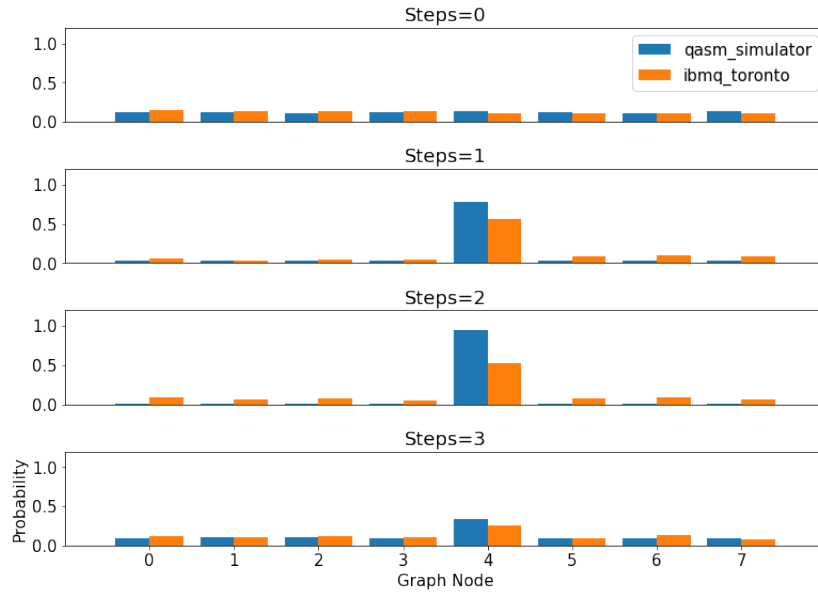


Figure 38: Probability distributions of the Grover search algorithm for several steps, in a search space of size  $N = 8$ . The blue bar plot represents a circuit run in the QASM simulator, and the orange bar plot on IBM's Toronto backend.

The results of measurement are shown in figure 38. As was expected, maximum probability for the marked element was reached after 2 iterations, on the simulator, and it decreases in subsequent steps. However, the experimental result from the Toronto backend present maximum probability for the marked element for 1 step, with a fidelity of 0.96. The optimal number of steps, in contrast, have a lower fidelity of 0.89. This is because as the number of steps increases, so does circuit depth, meaning that the circuit does not achieve the maximum probability after 2 steps due to the effects introduced by noise.

Despite the effects introduced by noise, the results are satisfactory when considering the properties of NISQ computers. The following sections will present the search problem adapted to several quantum walk models.

#### 4.4.2 Coined

Following the structure of section 3.2, this chapter aims to expand the coined quantum walk model by incorporating concepts like the oracle and diffusion operator of the Grover search, with the purpose of generating a quantum circuit that performs this algorithm.

The modified unitary evolution operator is

$$U' = S(\mathcal{O} \otimes G), \quad (120)$$

as was defined in equation 68, where  $S$  is the flip-flop shift operator,  $\mathcal{O}$  is the oracle operator and  $G$  is the Grover diffusion as a coin operator.

Consider the case of a complete graph, where every vertex is adjacent to one another. The general quantum circuit to implement this, as shown in figure 39, will require  $n$  qubits to represent the state of the walker and  $n$  qubits for the state of the coin. The shift operator was constructed based on the work of Douglas and Wang (2009), where the state of the walker is flip-flopped with the state of the coin by means of the swap operation.

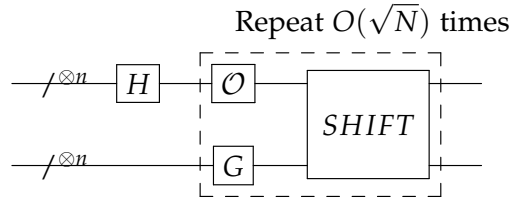


Figure 39: General circuit for the search problem using the coined quantum walk model.

This was implemented in Qiskit, for a graph of size  $N = 2^3 = 8$ , which means 6 qubits will be required. For the case of one marked element, the number of iterations that maximizes the amplitude of the solution state is  $\lfloor \frac{\pi}{2} \sqrt{N} \rfloor$ , and figure 40 shows the circuit for 5 iterations of the walk.

The circuit starts in a uniform superposition of the states corresponding to the vertices of the graph, and the first step of the iteration is the oracle. This operator will flip the amplitude of the vertex state  $|4\rangle$ , and can be translated to a circuit making use of Qiskit's *diagonal*, as is shown in figure 41. It is the same oracle as in the Grover search of figure 36, but in the coined quantum walk model it is only applied to the states associated with the position of the walker. The states associated with the coin space of the walk will be transformed according to Grover's diffusion of figure 37, as is seen in figure 42.

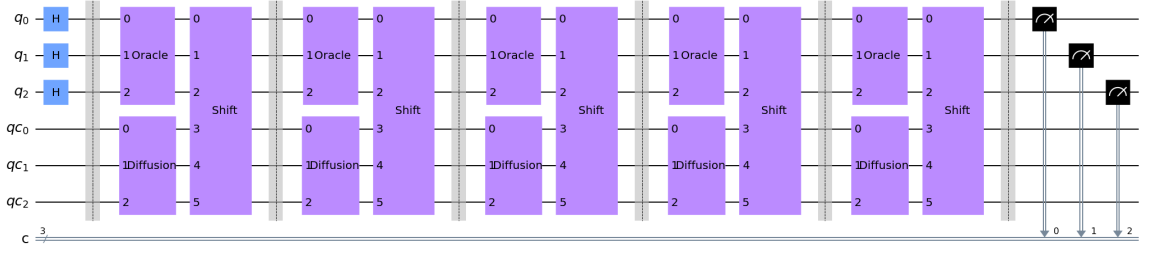


Figure 40: Qiskit circuit for the search problem using the coined quantum walk model, for a complete graph of size  $N = 8$  and with 5 steps.

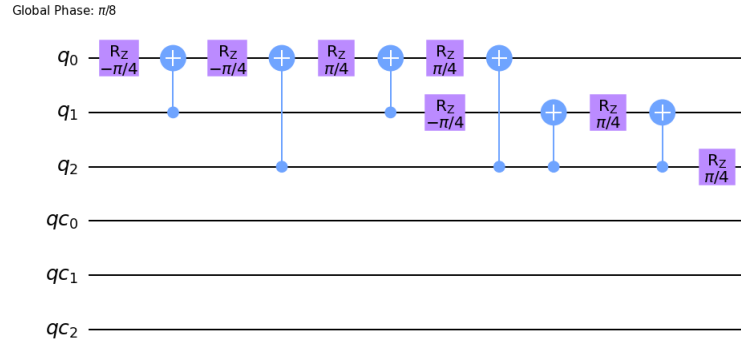


Figure 41: Qiskit circuit of the diagonal oracle operator in the coined quantum walk search problem, for a complete graph of size  $N = 8$ , with marked element  $|m\rangle = |4\rangle$ .

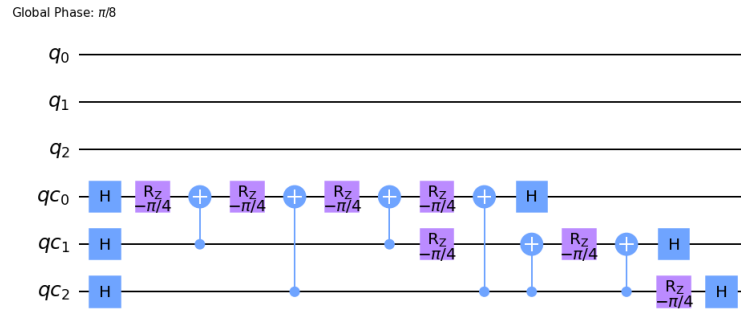


Figure 42: Qiskit circuit of the diagonal diffusion operator in the coined quantum walk search problem, for a complete graph of size  $N = 8$ .

The final part of the iteration is the shift operator, as can be seen in figure 43. The flip-flop shift operator was defined in equation 64 as

$$S |v1\rangle |v2\rangle = |v2\rangle |v1\rangle, \quad (121)$$

where  $|v1\rangle$  represents the position of the walker and  $|v2\rangle$  is the state of the coin. Making use of the swap gate, this operator can be implemented as in figure 43.

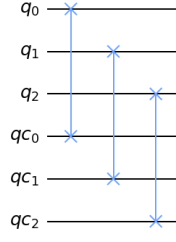


Figure 43: Qiskit circuit of the flip-flop shift operator in the coined quantum walk search problem, for a complete graph of  $N = 8$ .

Lastly, measurements were performed, and the results plotted in figure 44. Maximum probability of the marked element was reached after 4 steps in the simulator, and extra steps reduce said probability. The resulting probability distribution from the Toronto backend is again unsatisfactory for the coined quantum walk model, with fidelities ranging from 0.58 for 4 steps, to 0.75 for 2 steps. This is expected, since the complete graph representation requires  $N$  extra qubits for the coin, and swap operations which are decomposed into 3 CNOTs each. The optimal number of steps that maximizes the probability of the marked element is also a contributing factor to the size of the circuit, requiring more iterations to achieve the same probability when compared to Grover's search in the previous chapter.

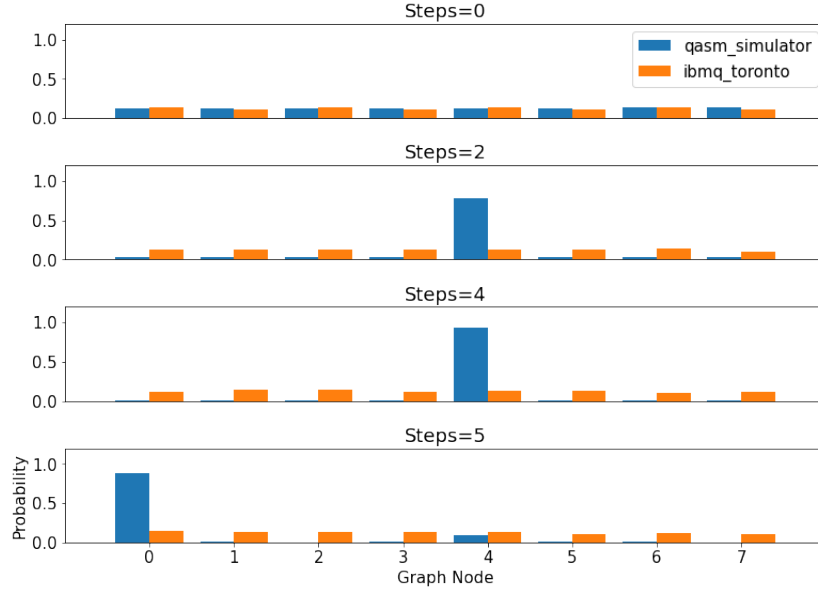


Figure 44: Probability distributions of the coined quantum walk search problem for several steps, in a complete graph of size  $N = 8$ . The blue bar plot represents a circuit run in the QASM simulator, and the orange bar plot on IBM's Toronto backend.

As was done previously, other models of the quantum walks that do not require coins or iterations will be studied in the following sections, in the context of the searching problem.

The staggered quantum walk, for example, should be able to present better results when ran in a NISQ computer, considering the smaller Hilbert space due to its coinless nature.

#### 4.4.3 Staggered

Recalling from section 3.3, the staggered quantum walk on a complete graph requires a single tessellation with associated polygon

$$|\alpha\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle. \quad (122)$$

The Hamiltonian will then be

$$H_\alpha = 2 \sum_0^1 |\alpha\rangle \langle \alpha| - I = H^{\otimes n} (2 |0\rangle \langle 0| - I) H^{\otimes n} = H^{\otimes n} \mathcal{O}_0 H^{\otimes n}, \quad (123)$$

which is equivalent to the Grover diffusion operator, meaning it can be implemented in a similar fashion.

The evolution operator for the staggered quantum walk on the complete graph can then be defined as

$$U = e^{i\theta H_\alpha} = e^{i\theta(H^{\otimes n} \mathcal{O}_0 H^{\otimes n})} = H^{\otimes n} e^{i\theta \mathcal{O}_0} H^{\otimes n}. \quad (124)$$

This is a very useful representation since the exponent part of the operator is a diagonal matrix, which means implementing the circuit in Qiskit is a straightforward task.

Now that the staggered quantum walk associated with the complete graph is defined, what remains is to add an oracle to the evolution operator as was done in equation 76,

$$U' = U\mathcal{O}, \quad (125)$$

where

$$\mathcal{O} = I_N - 2 \sum_{m \in M} |m\rangle \langle m|, \quad (126)$$

and  $M$  is the set of marked elements.

The general circuit for implementing the staggered quantum walk search problem in a complete graph will then be as shown in figure 45. Since only one tessellation is required,

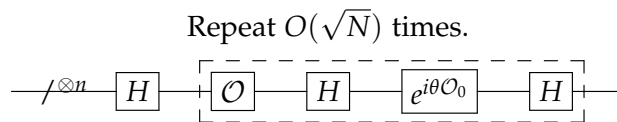


Figure 45: General circuit for the search problem using the staggered quantum walk model.

there is no need for the Suzuki-Trotter approximation. However, several iterations will be needed in order to achieve maximum probability for the marked vertex. Because the staggered quantum walk search on a complete graph is equivalent to Grover's algorithm, the optimum number of steps will also be  $\lfloor \frac{\pi}{4} \sqrt{\frac{N}{K}} \rfloor$ , where  $K$  is the number of solutions.

Consider the case of  $N = 8$  and one marked vertex,  $|m\rangle = |4\rangle$ . The number of steps that maximizes the probability of the marked element is  $\lfloor \frac{\pi}{4} \sqrt{\frac{8}{1}} \rfloor = 2$ . Translating to Qiskit,  $n = 3$  qubits will be needed and the circuit will be as in figure 46.

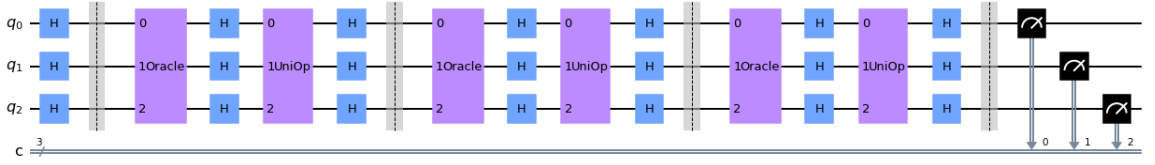


Figure 46: Qiskit circuit for the search problem using the staggered quantum walk model, for a complete graph of size  $N = 8$ , with 3 steps and a value of  $\theta = \frac{\pi}{2}$ .

Similar to previous examples, the circuit begins with the uniform superposition achieved through Hadamard gates. The next operation of the search problem is the oracle, which was implemented through the use of Qiskit's *diagonal* function that produces a circuit similar to the one in figure 36.

Next, an analogue of Grover's diffusion operator is applied, where the operation named *UniOp* is a diagonal matrix, easily translated to Qiskit as is shown in figure 47. This circuit

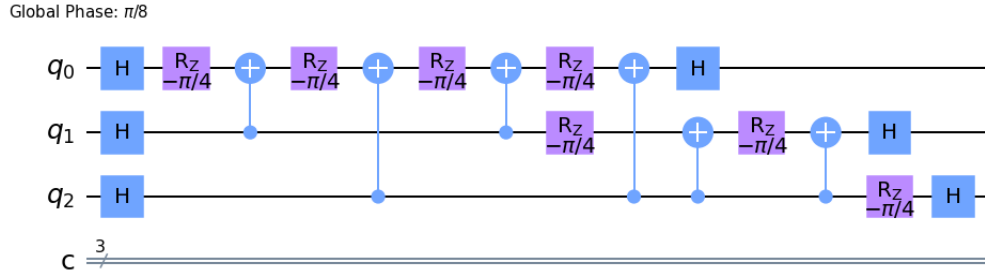


Figure 47: Qiskit circuit of the diagonal diffusion operator in the staggered quantum walk search problem, for a complete graph of size  $N = 8$  and a value of  $\theta = \frac{\pi}{2}$ .

is very similar to the one in figure 37, the difference being that in the staggered quantum walk search model one can control the value of  $\theta$ , as can be seen in equation 124, which influences how fast maximum probability of the marked element is achieved. Since Grover's algorithm is optimal, a value of  $\theta = \frac{\pi}{2}$  yields a diffusion circuit equal to the one in figure 37, implying that the staggered quantum walk is a more general model of quantum searching.

Finally, measurement is performed and the results for several steps of the walk are shown in figure 48. The circuit for each step of the walk was run both in the QASM simulator and

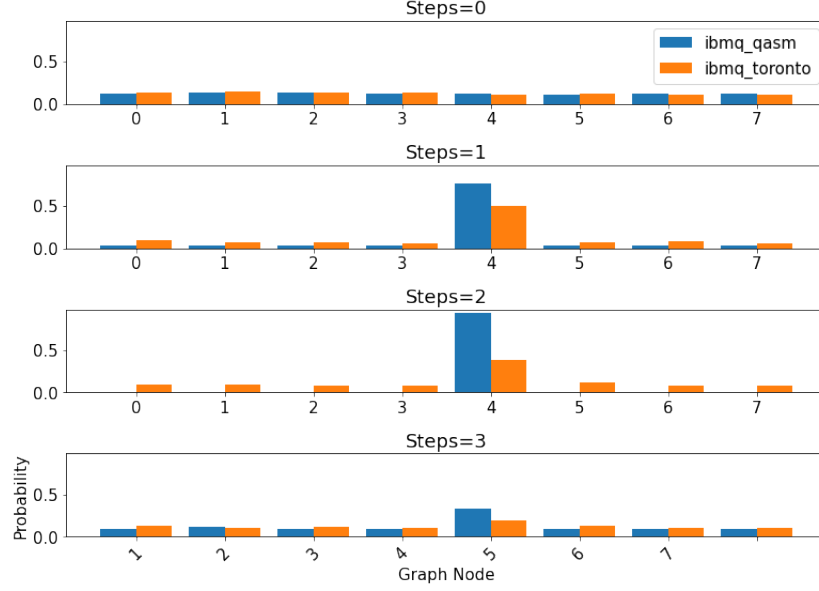


Figure 48: Probability distributions of the staggered quantum walk search problem for several steps, in a complete graph of size  $N = 8$ . The blue bar plot represents a circuit run in the QASM simulator, and the orange bar plot on IBM’s Toronto backend.

IBM’s backend named Toronto. The experiment was performed with 3000 shots in both cases, and the probability distributions show that this model is indeed more suited for a NISQ computer than the previous case. However, unlike the simulator, maximum probability of the marked vertex was achieved withing 1 step of the walk instead of the predicted 2 steps, as was the case of the Grover algorithm. Looking at the 1 step case in the simulator, one can see that the probability of vertex  $|4\rangle$  is very close to the maximum, while the circuit has about half of the operations of the 2 step case. This means it is not surprising that the smaller circuit produces higher results for the probability of the marked vertex. This can be further confirmed by the fidelities of each of the states, which are approximately 0.954 and 0.780 for 1 step and 2 steps, respectively, implying that the former circuit produces better results because of the number of operations, even though that the latter should theoretically yield the highest probability.

Even though these results are a great improvement to the search problem using the coined quantum walk, the staggered model is still discrete, meaning its circuit will increase in depth as the number of steps increases. This can be avoided by turning again to the continuous-time quantum walk, whose circuit remains constant with time. However, for the searching problem, the continuous-time model might not have the best results, because it will need iterations due to the Suzuki-Trotter approximation and it will also require extra

operations to implement the oracle, which was not the case in the pure dynamics example of section 4.3.

#### 4.4.4 Continuous

As was seen in section 3.4, the unitary operator associated with the continuous time quantum walk model can be modified as to mark an element for amplitude amplification

$$U'(t) = e^{iH't} = \phi(t)e^{-i(\gamma A + O)t}, \quad (127)$$

where  $\phi(t)$  is a global phase,  $A$  is the adjacency matrix and the oracle is defined as

$$O = \sum_{m \in M} |m\rangle \langle m|, \quad (128)$$

where  $M$  is the set of marked elements.

This section will focus on constructing and analyzing the circuit form of the continuous-time quantum walk search problem, which will be performed over a complete graph whose adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}. \quad (129)$$

which is simply a matrix with all entries set to 1, except the diagonal.

The first step is to borrow the diagonal definition of the adjacency matrix from equation 109

$$A = F^\dagger \Lambda F, \quad (130)$$

and use the Suzuki-Trotter expansion

$$e^{i(H_0 + H_1)t} = \lim_{r \rightarrow \infty} (e^{i\frac{H_0 t}{r}} e^{i\frac{H_1 t}{r}})^r, \quad (131)$$

to decompose the operator in equation 127

$$e^{i(\gamma A + O)t} = \lim_{r \rightarrow \infty} (F^\dagger e^{i\gamma \frac{\Lambda t}{r}} F e^{i\frac{O t}{r}})^r, \quad (132)$$

which can be easily translated into circuit form as in figure 49.

Consider the case of a graph of size  $N = 2^3 = 8$  and trotter number of  $r = 1$ . The corresponding Qiskit circuit is as shown in figure 49. The system starts out in an uniform



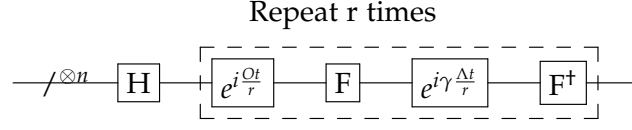


Figure 49: General circuit for the search problem using the continuous-time quantum walk model.

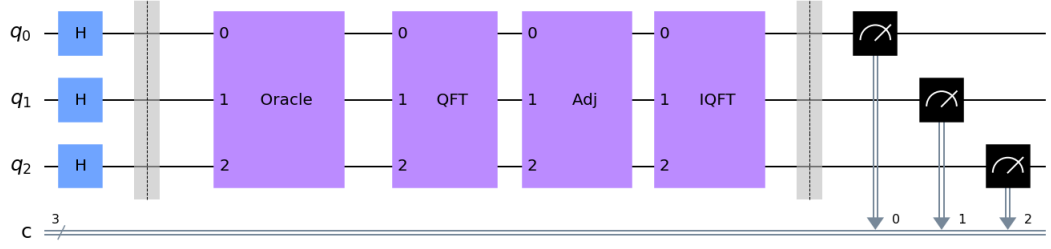


Figure 50: Qiskit circuit for the search problem using the continuous-time quantum walk model, for a complete graph of size  $N = 8$ , time  $t$  and a value of  $\gamma = \frac{1}{8}$ .

superposition followed by the application of the oracle operator as can be seen in figure 51. Note that the circuit was obtained by using Qiskit's *diagonal* function that takes the diagonal entries of the operator corresponding to the oracle that was defined in equation 132.

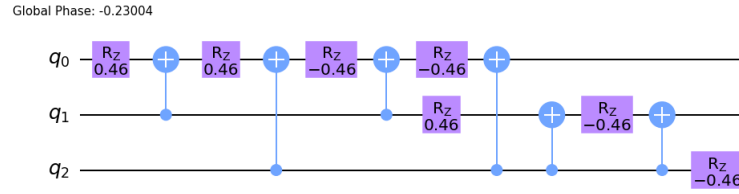


Figure 51: Qiskit circuit of the diagonal oracle operator in the continuous-time quantum walk search problem, for a complete graph of size  $N = 8$ , marked element  $|m\rangle = |4\rangle$  and time  $t = \frac{\pi}{2}\sqrt{8}$ .

The following operation will be the QFT, but because a complete graph is considered, the AQFT can be used with a degree of  $m = 2$ , which means the circuit will simply be Hadamard transforms, thus reducing circuit depth.

Lastly, the operator associated with the adjacency matrix is shown in figure 52. Since  $A$  is the diagonal adjacency matrix of a complete graph, it is easily implemented using the aforementioned *diagonal* function.

Finally, the results of circuit measurement can be seen in figure 53. Even though circuit depth does not scale up with time, introducing the oracle operation to the continuous-time quantum walk model appears to make the circuit hard to run in a NISQ computer. Note that, theoretically, 2 iterations of the Suzuki-Trotter expansion are needed for maximum probability of the marked vertex to be achieved in optimal time. In practice, however, greater probability for this element is achieved when only 1 iteration is performed since the smaller

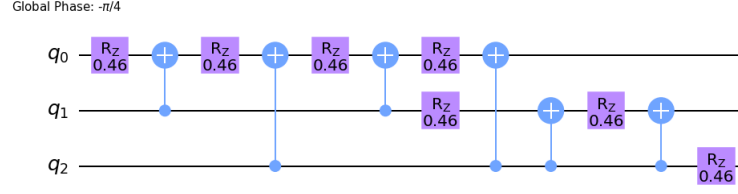


Figure 52: Qiskit circuit of the diagonal operator associated with the adjacency matrix in the continuous-time quantum walk search problem, for a complete graph of size  $N = 8$ , marked element  $|m\rangle = |4\rangle$ , time  $t = \frac{\pi}{2}\sqrt{8}$  and  $\gamma = \frac{1}{8}$ .

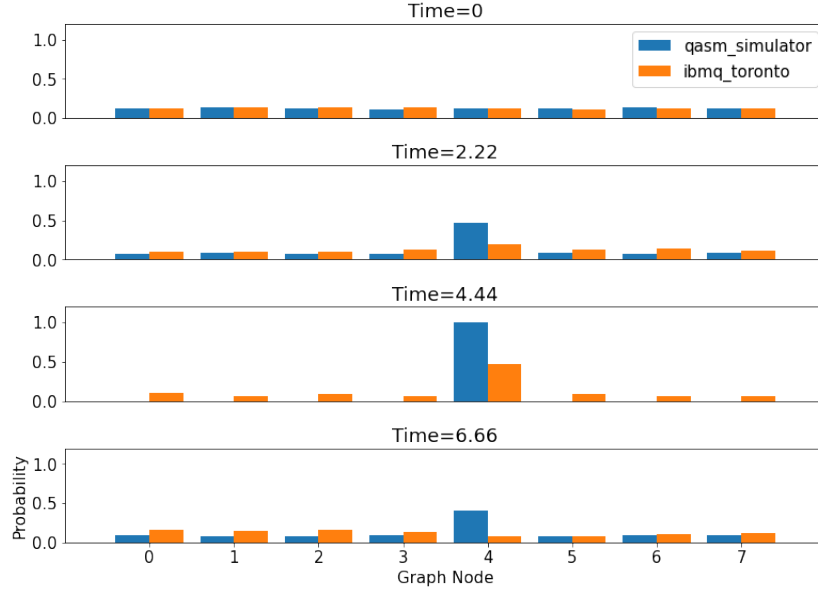


Figure 53: Probability distributions of the continuous-time quantum walk search problem for several time intervals, in a complete graph of size  $N = 8$ . The blue bar plot represents a circuit run in the QASM simulator, and the orange bar plot on IBM's Toronto backend.

circuit seems to produce a greater fidelity of 0.71, when compared to a fidelity of 0.59 for the 2 iteration case. The approximate quantum Fourier transform also has a positive impact, increasing fidelity from 0.60 when the full QFT circuit is used, to the aforementioned 0.71 when only Hadamard transformations are performed.

In conclusion, when comparing the several quantum walk models studied in previous sections, the coined quantum walk seems to be the least suited for running the search problem in NISQ computers, followed by the continuous-time model which has the advantage of not scaling with time, but produces a circuit slightly too large to achieve fidelities which rival the staggered quantum walk. The latter appears to perform the best because, even though its discrete nature requires several iterations, the circuit depth does not appear to introduce a drastic amount of noise in the  $N = 8$  case.

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## DISCUSSIONS AND CONCLUSION

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### 5.1 CONCLUSIONS

Chapter 1 begins with a brief historical overview of the origins of quantum computation, starting with the early days of computer science all the way to quantum walks. An overview of the state of the art encompassing both theoretical and circuit implementations of quantum walks and searching problems was presented in a further section, including the relevant literature to the work presented in this thesis. This introduction is then closed with an overview of the following chapters and main contributions.

The goal of the chapter 2 was to define the theoretical framework associated with the three quantum walk models studied in this thesis. This was done in the context of the quantum walk on the line, starting with a brief definition of the classical random walk, followed by the discrete models of the quantum walk, namely the coined model and the staggered model, and finishing with the continuous-time quantum walk. All of the models were simulated in Python, and various plots were created in order to analyse how the different parameters influence how the dynamics of each walk behave.

Chapter 3 was dedicated to the study of the previously defined quantum walks when applied to the searching problem. It starts with the definition of the Grover algorithm, where a complexity analysis was performed, followed by the quantum walk analogue. Here, the theoretical explanation for the various quantum walks on complete graphs was presented, combined with the introduction of the oracle. Similarly to the previous chapter, Python was used to simulate the algorithms and study how the different aspects of each model affect how the search problem evolves.

The main contributions of this thesis are presented in chapter 4. The first contribution comes in the form of a systematic way of creating Qiskit circuits continuous-time quantum walks for a myriad of circulant graphs, by way of the quantum Fourier transform and Qiskits *diagonal* function. Previous work by [Qiang et al. \(2016\)](#) uses the concept of circulant graphs for the implementation of the CTQW model on a complete graph for a small number of qubits, however the present work provides a systematic way of generating graphs for a much greater number of circulant graphs, for an arbitrary number of qubits, as well as

an analysis of how the approximate quantum Fourier transform can be used to reduce circuit depth for better NISQ computer results. Another contribution, that is not found on the literature, was the use of this method to implement search on circulant graphs using the continuous-time quantum walk. The results were not as satisfactory as the previous dynamics example, because of the increased circuit size due to the introduction of the oracle and Suzuki-Trotter iterations. This is still a work in progress, which will be further expanded upon in future publications. This chapter also contains a compilation of methods for creating circuits for the dynamics and search of the discrete-time quantum walk models, based mainly on the work of [Douglas and Wang \(2009\)](#) and [Acasiete et al. \(2020\)](#), which are not original contributions but hopefully provide a useful overview on how to implement these algorithms on current quantum hardware.

Based on the work presented in this thesis, future work for the study and implementation of quantum walks includes the following:

- Circuit implementation of the continuous-time quantum walk search problem using circulant graphs, and how the AQFT affects circuit depth and implementability on NISQ computers. This is the goal of the next publication regarding this thesis.
- Simulation, implementation and analysis of the searching problem for multiple marked elements using the staggered quantum walk model. The search problem with the SQW is not very well researched in the literature, meaning there are contributions to be made in this regard.
- Analysis through the creation of simulation software of the continuous-time quantum walk from the perspective of transport problems, such as location, perfect state transfer and standard deviation, and further analysis of the search problem for multiple marked elements.
- Development of a method for constructing staggered quantum walk circuits for a larger variety of graphs.
- Expansion of the circulant graph method for the CTQW in order to perform the walk over a larger class of graphs.

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## MATHEMATICAL FOUNDATIONS

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### A.1 POSTULATES OF QUANTUM MECHANICS

Quantum mechanics, firstly discovered in the decade of 1920, is a mathematical framework for developing physical theories. In this section, the goal is to provide the basic postulates of quantum mechanics, which are formalized through Dirac notation. Further reading on this notation includes the work of [Sakurai \(1994\)](#), and for an extensive review of quantum computation the book by [Nielsen and Chuang \(2011\)](#).

The first postulate defines where the processes of quantum mechanics take place. The *state* of a system describes its physical characteristics, so some rules are required for these mathematical objects to have a connection to the real world.

**Postulate 1 (State Space)** *Any isolated physical system has an associated Hilbert Space,  $\mathcal{H}$ , known as the state space. The state of the system is wholly described by its state vector  $|\psi\rangle \in \mathcal{H}$ . The physical system's degrees of freedom dictate the dimension of  $\mathcal{H}$*

Note that this postulate does not tell us the Hilbert space of any given physical system, nor does it tell us its state vector. It is generally hard to define the Hilbert space of an arbitrary system, which makes the work physicist have done developing certain theories, like quantum electrodynamics, even more remarkable.

Considering the computational basis  $\{|0\rangle, |1\rangle\}$ , the simplest quantum system, the *qubit*, can be defined as

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle, \quad (133)$$

where  $\alpha, \beta$  are complex numbers and

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (134)$$

Because Hilbert spaces are also vector spaces, linear combinations of these states are also allowed

$$|\psi\rangle = \sum_i \alpha_i |\varphi_i\rangle. \quad (135)$$

Since  $|\psi\rangle$  is required to be a unit vector,  $|\psi\rangle\langle\psi| = 1$  or, equivalently,

$$\sum_j |\alpha_j|^2 = 1. \quad (136)$$

These combinations are the main difference between a classical bit and a qubit, known as *superpositions*, in which the states are not in a definite value before measurement. This is a well studied quantum phenomenon that leads to constructive and destructive interference between states, which is an aspect of quantum computation many algorithms exploit.

The second postulate aims to describe how a quantum system evolves with time, and it can be formulated in the following way.

**Postulate 2 (Evolution)** *The time evolution of a closed quantum system is described by a unitary operator. Considering an initial condition  $|\psi_0\rangle$ , then for any time evolution of a closed quantum system, a unitary operator  $U$  exists such that  $|\psi_f\rangle = U|\psi_0\rangle$ .*

Just as the first postulate does not specify a Hilbert space, the evolution postulate does not state which unitary operators  $U$  describe an arbitrary physical system. What it does state is that the evolution of a closed quantum system follows those rules. More specifically, the second postulate describes the dynamics of such systems. In the case of a qubit, any  $U$  can be performed in a realistic system as long as  $UU^\dagger = U^\dagger U = I$ , which is another way of saying that  $U$  is unitary.

There are a few unitary operators very relevant to quantum computation, known as *Pauli matrices*, that are usually composed to create more general matrices. These are defined as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (137)$$

Postulate 2 tells us the relationship between the states of the system at two different times. An improved version of this postulate takes time as a continuous variable, stating that the temporal evolution of a closed quantum system can be described by the Schrodinger equation

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle, \quad (138)$$

where  $\hbar$  is the *Planck's constant* and  $H$  is a hermitian operator known as the *Hamiltonian* of the system. In principle, the Hamiltonian can be used to describe a system in it's entirety, however figuring out the Hamiltonian is generally a hard task.

So far only single quantum systems have been considered. The next postulate describes how one can create composite quantum systems made of smaller distinct systems.

**Postulate 3 (Composite Systems)** *The state space of a system composed of smaller sub-systems can be described by the tensor product of the individual state spaces  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . Moreover, if the first system's state is  $|\psi_1\rangle$  and the second is  $|\psi_2\rangle$  then the state of the composite system is  $|\psi_1\rangle \otimes |\psi_2\rangle$ .*

The tensor product is used because of the nature of superposition in quantum mechanics. A system composed of subsystems  $\{|\psi\rangle, |\varphi\rangle\}$  can be denoted as  $|\psi\rangle \otimes |\varphi\rangle$ . Applying the superposition principle, that tells us that any complex linear combination of states belonging to the system should also be allowed, the tensor product naturally follows.

The tensor notation can be written in a more compact way. Considering a system described by  $n$  component states

$$|\psi\rangle = |\psi_0\rangle \otimes |\psi_1\rangle \otimes \cdots \otimes |\psi_{n-1}\rangle, \quad (139)$$

it can be rewritten as

$$|\psi\rangle = |\psi_0\rangle |\psi_1\rangle \cdots |\psi_{n-1}\rangle, \quad (140)$$

and even further compacted to

$$|\psi\rangle = |\psi_0 \psi_1 \cdots \psi_{n-1}\rangle. \quad (141)$$

However, not all composite systems can be described as the tensor product of the component states. For example, state

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad (142)$$

cannot be broken down further. These subsystems are known to be *entangled*, and the state in equation 142 is called a *Bell state*. Bell states describe the set of 2 qubit states that present maximum entangled, and are used in many quantum applications like *quantum teleportation* and *superdense coding*.

As was aforementioned, closed quantum systems evolve unitarily in time. However, in order to do something useful with such a system, one must extract the classical information relating to the state of the system. This is achieved by the process of measurement, which implies some form of interaction with the system, thus making it no longer closed nor described by a unitary evolution.

**Postulate 4 (Measurement)** *Quantum measurements are described by a set,  $\{M_m\}$ , of measurement operators that act on the state space of the system, satisfying the completeness relation  $\sum_m M_m^\dagger M_m = I$ , where  $m$  refers to the measurement outcomes of the experiment. Considering a system with state  $|\psi\rangle$ , immediately before measurement, then the probability of event  $m$  is*

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle. \quad (143)$$

*The state of the system after the measurement will be*

$$|\psi'\rangle = \frac{1}{\sqrt{p(m)}} M_m |\psi\rangle. \quad (144)$$

Measurement can be done and interpreted in several different ways, however this section focuses on what is called *projective* measurement. A projective measurement is described by an Hermitian operator,  $M$ , known as an *observable*, with spectral decomposition

$$M = \sum_m m P_m \quad (145)$$

where  $P_m$  is an Hermitian projection operator with eigenvalue  $m$ . For example,  $P_1$  and  $P_2$  are projection operators that are orthogonal to each other and whose product is a zero matrix. A set of operators with these characteristics obey the completeness equation

$$\sum_i P_i = I. \quad (146)$$

The probability of outcome  $m$  associated with the measurement of state  $|\psi\rangle$  can be written as

$$p(m) = \langle \psi | P_m | \psi \rangle, \quad (147)$$

and knowing that  $m$  was the result of the measurement, the state of the quantum system is then

$$\frac{P_m}{\sqrt{p(m)}} |\psi\rangle. \quad (148)$$

An important case of projective measurement is when it is performed in the *computational basis* of a qubit. Given operators

$$M_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } M_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (149)$$

it is obvious that they are Hermitian and that they obey the completeness relation. Consider again the state  $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ . The probability of outcome 0 will be

$$p(0) = \langle \psi | M_0^\dagger M_0 | \psi \rangle = \langle \psi | M_0 | \psi \rangle = |\alpha|^2 \quad (150)$$

and outcome 1

$$p(1) = \langle \psi | M_1^\dagger M_1 | \psi \rangle = \langle \psi | M_1 | \psi \rangle = |\beta|^2. \quad (151)$$

For each of the outcomes, the state after measurement will be

$$\frac{M_0}{|\alpha|} |\psi\rangle = \frac{\alpha}{|\alpha|} |0\rangle, \quad (152)$$

$$\frac{M_1}{|\beta|} |\psi\rangle = \frac{\beta}{|\beta|} |1\rangle. \quad (153)$$

Projective measurement, in some ways, destroys the superposition of possible states. This is known as the *collapse of the wave function*.



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CIRCUITS

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## B.1 QUANTUM FOURIER TRANSFORM

As was seen in section ??, quantum computers can perform certain tasks more efficiently than classical computers. Another such example is the problem of finding the prime factorization of an  $n$ -bit integer, which the most efficient solution to date, proposed by Pollard et al. (1994), requires  $e^{O(n^{\frac{1}{3}} \log^{\frac{2}{3}} n)}$  operations. In contrast, a quantum algorithm proposed by Shor (1994b) accomplishes the same task in  $O((\log n)^2 (\log \log n) (\log \log \log n))$  operations, which is an exponential gain due to the efficiency of the quantum Fourier transform.

The quantum Fourier transform is an implementation of the discrete Fourier transform over amplitudes of quantum states. It offers no speed ups when used in computing Fourier transforms of classical data, since the amplitudes cannot be accessed directly by measurement. Moreover, there is no known generalized efficient way of preparing the initial state to be Fourier Transform. What this means is that the uses of the QFT are not in the straightforward way of calculating discrete Fourier transforms, but in the form of algorithms, such as *phase estimation*, that take advantage of its properties. This transform can be described as the following operation over an orthonormal basis  $|0\rangle, |1\rangle, \dots, |N-1\rangle$

$$QFT(|j\rangle) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i j k}{N}} |k\rangle, \quad (154)$$

where  $N = 2^n$ . With a little bit of algebra, this can be rewritten as a product

$$\begin{aligned} \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i j k}{2^n}} |k\rangle &= \frac{1}{\sqrt{N}} \sum_{k_1=0}^1 \dots \sum_{k_n=0}^1 e^{2\pi i j (\sum_{l=1}^n k_l 2^{-l})} |k_1 \dots k_n\rangle \\ &= \frac{1}{\sqrt{N}} \bigotimes_{l=1}^n \left( \sum_{k_l=0}^1 e^{2\pi i j k_l 2^{-l}} |k_l\rangle \right) \\ &= \frac{1}{\sqrt{N}} \bigotimes_{l=1}^n (|0\rangle + e^{2\pi i j 2^{-l}} |1\rangle) \end{aligned} \quad (155)$$

The quantum Fourier transform applied to a state as in equation 154 can then be rewritten as

$$QFT(|x_1, \dots, x_n\rangle) = \frac{(|0\rangle + e^{2\pi i 0.x_n} |1\rangle)(|0\rangle + e^{2\pi i 0.x_{n-1}x_n} |1\rangle) \dots (|0\rangle + e^{2\pi i 0.x_1x_2 \dots x_n} |1\rangle)}{2^{\frac{N}{2}}}, \quad (156)$$

where  $x = x_1 2^{n-1} + x_2 2^{n-2} + \dots + x_n 2^0$  and the notation  $0.x_1 x_{l+1} \dots x_n$  represents the binary fraction  $\frac{x_l}{2^{l^0}} + \frac{x_{l+1}}{2^1} \dots \frac{x_m}{2^{m-l+1}}$ . This is a very useful representation because it makes constructing an efficient circuit much simpler, as can be seen in figure 54. However, the circuit implementation of the QFT requires exponentially smaller phase-shift gates as the number of qubits increases. This can be somewhat mitigated by eliminating the smaller phase-shift gates at the cost of some accuracy, as was shown in Coppersmith (2002) where he defined the *approximate* quantum Fourier transform. This approximation requires only  $O(n \log n)$  gates, and work by Barenco et al. (1996) and Cheung (2004) established lower bounds for the probability of the approximate state accurately representing the state without approximation.

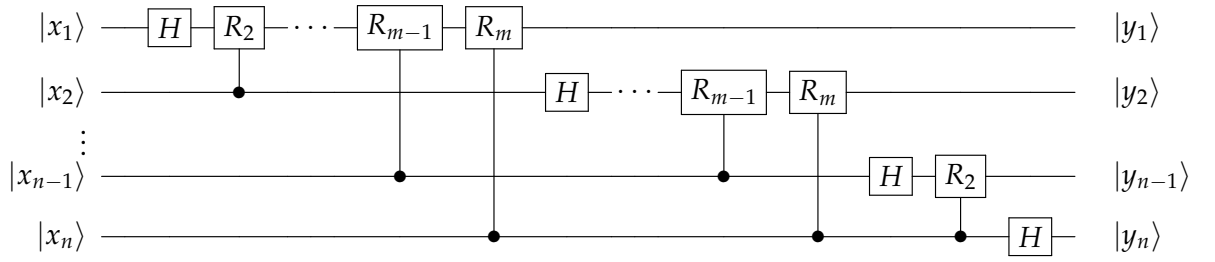


Figure 54: Temp

The rotation  $R_k$  in figure 54 is defined as the controlled version of

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^k}} \end{pmatrix}. \quad (157)$$

To verify that this circuit is the QFT, consider the state  $|x_1 \dots x_n\rangle$  as input. Applying the Hadamard gate on the first qubit produces the state

$$H |x_1 \dots x_n\rangle = \frac{1}{\sqrt{N}} (|0\rangle + e^{2\pi i 0.x_1} |1\rangle) |x_1 \dots x_n\rangle. \quad (158)$$

The next operation is the rotation  $R_2$ , controlled by the second qubit, resulting in state

$$\frac{1}{\sqrt{N}} (|0\rangle + e^{2\pi i 0.x_1 x_2} |1\rangle) |x_1 \dots x_n\rangle. \quad (159)$$

Applying the successive rotations up to  $R_n$  appends an extra bit to the phase of the first  $|1\rangle$ , ultimately becoming

$$\frac{1}{\sqrt{N}}(|0\rangle + e^{2\pi i 0.x_1 x_2 \dots x_n} |1\rangle) |x_1 \dots x_n\rangle. \quad (160)$$

A similar process is applied to the second qubit, and at the end of the rotations the state is

$$\frac{1}{\sqrt{N}}(|0\rangle + e^{2\pi i 0.x_1 x_2 \dots x_n} |1\rangle)(|0\rangle + e^{2\pi i 0.x_2 \dots x_n} |1\rangle) |x_1 \dots x_n\rangle, \quad (161)$$

and the successive application of this process to the remaining qubits results in state

$$\frac{1}{\sqrt{N}}(|0\rangle + e^{2\pi i 0.x_1 x_2 \dots x_n} |1\rangle)(|0\rangle + e^{2\pi i 0.x_2 \dots x_n} |1\rangle) \dots (|0\rangle + e^{2\pi i 0.x_n} |1\rangle) |x_1 \dots x_n\rangle, \quad (162)$$

confirming that this is indeed the Fourier transform derived in equation 156 minus the order of the qubits, which is reversed. It also shows that the QFT is unitary, since all operations in the circuit are unitary.

Counting the number of gates on the circuit, one can conclude that the first qubit will have 1 Hadamard gate followed by  $n - 1$  controlled rotations. The second qubit is another Hadamard followed by  $n - 2$  controlled rotations. After  $n$  qubits, the total number of gates will be  $\frac{n(n+1)}{2}$ . This means the circuit provides a  $O(n^2)$  algorithm, compared to the fastest classical algorithm, the *Fast Fourier Transform* which requires  $O(n2^n)$  operations. This is an exponential gain, which can be improved upon at the cost of accuracy, but it's not a replacement for calculating classical Fourier transforms for the aforementioned reasons.

## B.2 DIAGONAL FUNCTION