QWORLD

Friday, October 22, 2021

Quantum operators on the unit circle

Quantum state

$\mathbb{R}$ a single qubit

quantum state $\rightarrow \begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2 = x|0\rangle + y|1\rangle = x\begin{pmatrix}1\\0\end{pmatrix}+y\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix}+\begin{pmatrix}0\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$

$|x|^2+|y|^2=1$

$|x|^2$: the probability of being in state $|0\rangle$

$|y|^2$: the probability of being in state $|1\rangle$.

Unit circle

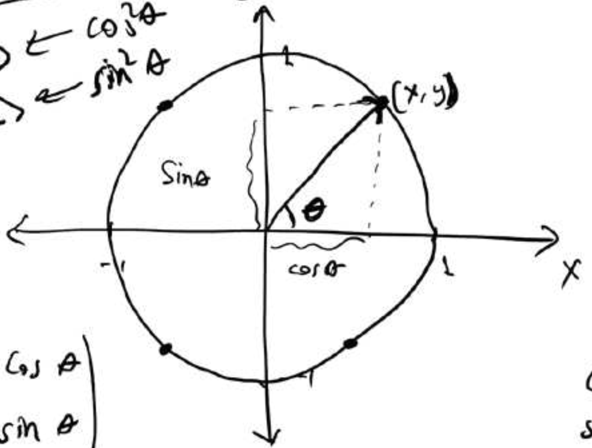
2D, a single qubit, all quantum states in the form of $\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2$

$|x|^2+|y|^2=1$

probabilities

$|0\rangle \leftarrow \cos^2\theta$

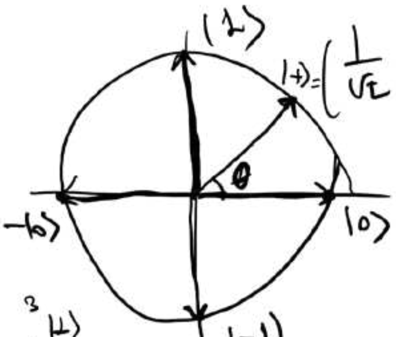
$|1\rangle \leftarrow \sin^2\theta$



$|u\rangle = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$

$\begin{pmatrix}x\\y\end{pmatrix}$: $\cos\theta$, $\sin\theta$, a single parameter to represent any quantum state on the circle.

Operators on the unit circle

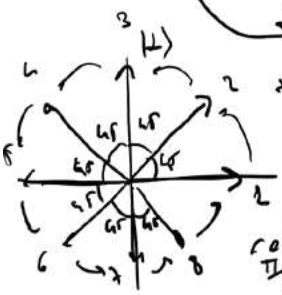


$|0\rangle \xrightarrow{A} |+\rangle \xrightarrow{A} |0\rangle$

$|+\rangle \xrightarrow{A} |1\rangle$

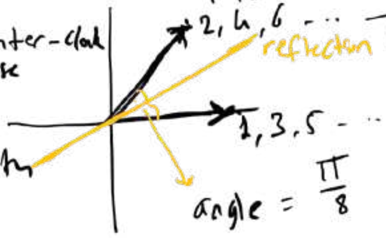
Rotate with 45 degree

$|0\rangle \xrightarrow{H} |+\rangle \xrightarrow{H} |0\rangle \xrightarrow{H} |+\rangle \xrightarrow{H} |0\rangle$



Counter-clockwise

rotation with $\frac{\pi}{4}$ radian



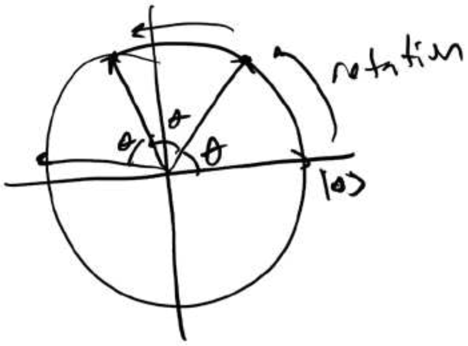
angle $= \frac{\pi}{8}$

reflection

mirroring

1	2
3	4
5	6
⋮	⋮

Rotations



angle	rotations
0	
	1st
θ	
	2nd
2θ	
	3rd
3θ	
	4th
4θ	
⋮	⋮

$\Rightarrow$ length preserving

$\Downarrow$

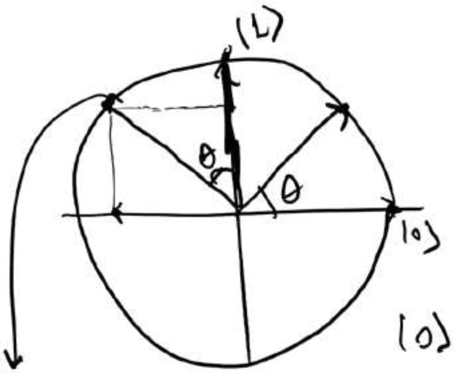
quantum operators.

$$R(\theta) = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$R(\theta)|0\rangle \Rightarrow$ the first column

$R(\theta)|1\rangle \Rightarrow$ the second column

Rotation matrix



$$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$R(\theta)|0\rangle \rightarrow$ the first column of the matrix

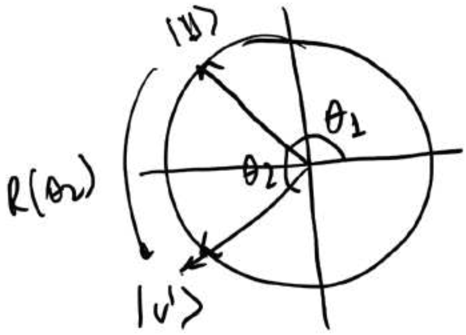
$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$|0\rangle \rightarrow \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$|1\rangle \rightarrow \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

generic form of rotation on the unit circle.

Rotating a quantum state

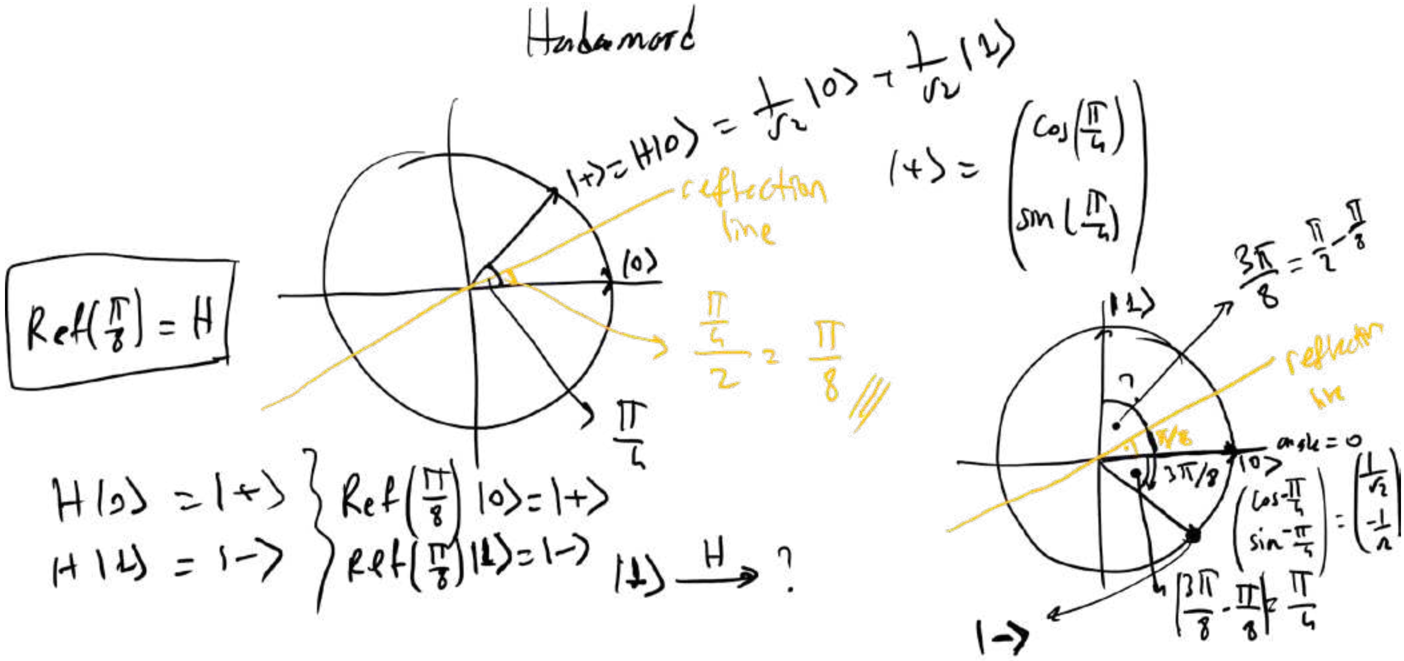


$$|v'\rangle = R(\theta_2) \cdot |v\rangle$$

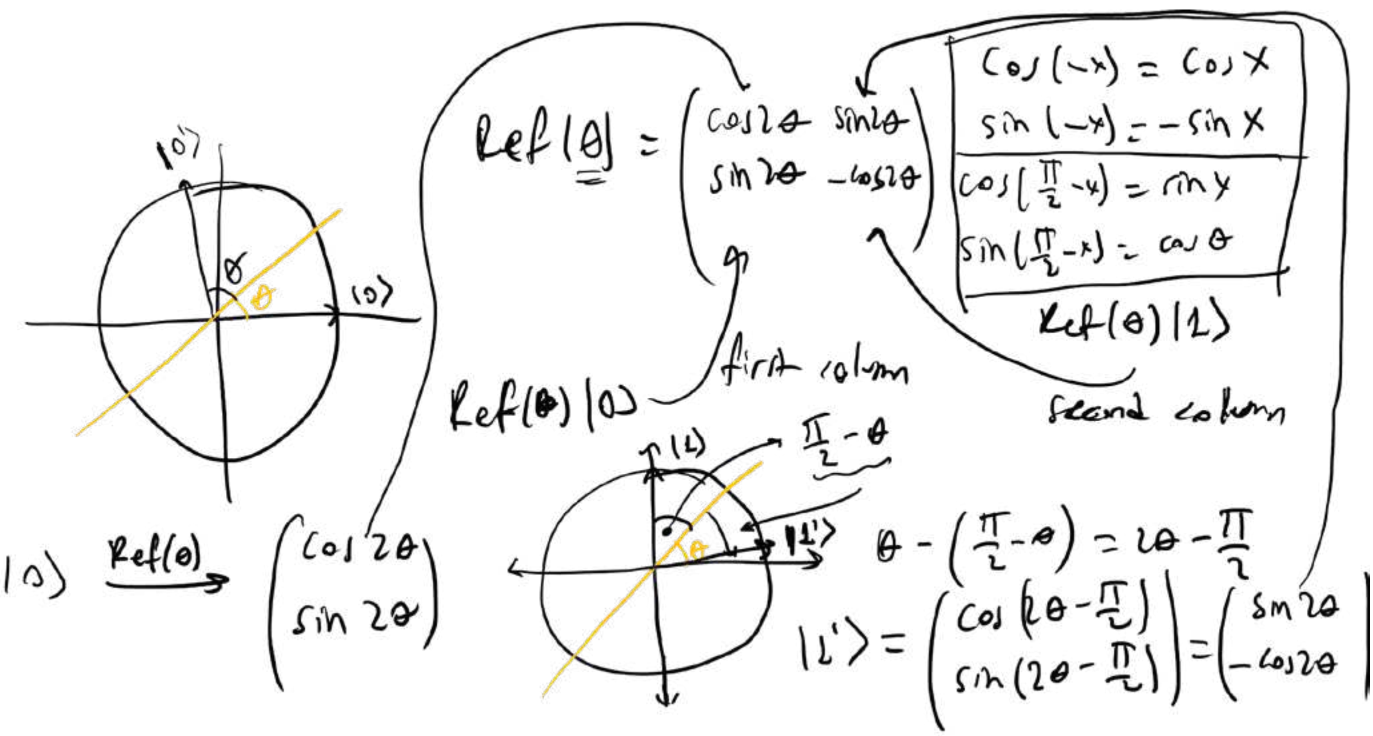
$$|v'\rangle = \begin{pmatrix} \cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) \end{pmatrix}$$

$$\begin{pmatrix} \quad \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}$$

Hadamard



Reflection



Trigonometric identities

$\cos -x = \cos x$

$$\cos\left(2\theta - \frac{\pi}{2}\right) = \cos\left(-\left(\frac{\pi}{2} - 2\theta\right)\right) = \cos\left(\frac{\pi}{2} - 2\theta\right) = \sin(2\theta)$$

$$\sin\left(2\theta - \frac{\pi}{2}\right) = \sin\left(-\left(\frac{\pi}{2} - 2\theta\right)\right) = -\sin\left(\frac{\pi}{2} - 2\theta\right) = -\cos(2\theta)$$

$\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$\sin\left(\frac{\pi}{2} - x\right) = \cos\theta$

Rotation and reflection

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

rotation angle (counter-clockwise direction)

$$Ref(\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

the angle of the reflection line with x-axis

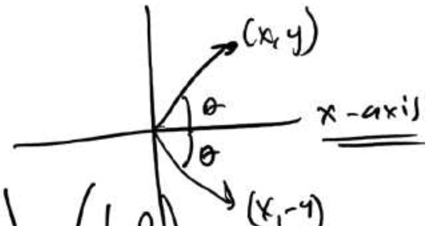
Z and I operators

rotation or reflection

θ?

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Ref(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

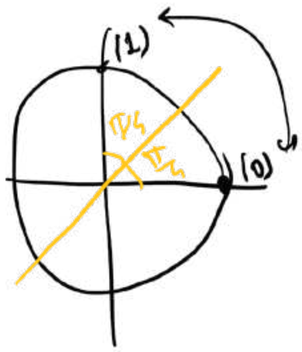


$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

NOT operator

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- reflection along the y-axis $Ref(\frac{\pi}{2})$?
- $Ref(\frac{\pi}{4})$.
-



$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

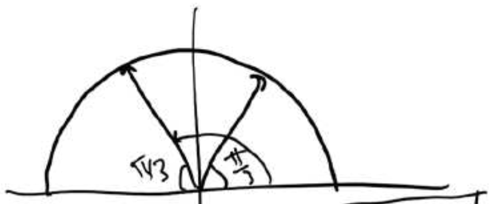
$2\theta = \frac{\pi}{2}$ $\qquad \cos(\frac{\pi}{2}) = 0$

$\theta = \frac{\pi}{4}$ $\qquad \sin(\frac{\pi}{2}) = 1$

Tomography

100 identical copies of the same qubit

$$|u\rangle = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \qquad 0 < \theta < \frac{\pi}{2}$$



$$\boxed{\theta \Rightarrow \begin{array}{l} \theta_1 = \frac{\pi}{3} \\ \theta_2 = \frac{2\pi}{3} \end{array}}$$

Measure All

$\downarrow$

25 : '0' , 75 : '1'

$pr('0') = \cos^2\theta$

$pr('1') = \sin^2\theta$

$$\boxed{100 \cdot \cos^2\theta \approx 25}$$

$100 \cdot \cos^2\theta = 25 \leftarrow$ expected value of '0' to be observed after measuring 100 copies.

$\theta = ?$

$$\cos^2\theta = \frac{25}{100}$$

$$\cos^2\theta = \frac{1}{4}$$

$$\boxed{\cos\theta = \frac{1}{2} \text{ or } \cos\theta = -\frac{1}{2}}$$