SMD Blatt 1

$\boxed{1}$ $f(v) = N \cdot \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_BT}\right) \cdot 4\pi v^2$

Bestimmung von $N$:

$$\int_0^\infty f(v)\,dv \overset{!}{=} 1$$

$$\int_0^\infty f(v)\,dv = N \cdot \left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot 4\pi \int_0^\infty \exp\left(-\frac{mv^2}{2k_BT}\right) v^2\,dv$$

Substitution: $a := \frac{mv^2}{2k_BT} \Rightarrow \frac{da}{dv} = \frac{mv}{k_BT} \Leftrightarrow dv = \frac{k_BT}{mv}\,da$

$$v = \sqrt{\frac{2k_BTa}{m}}$$

$$\int_0^\infty f(v)\,dv = N\left(\frac{m}{2\pi k_BT}\right)^{3/2} \cdot 4\pi \int_0^\infty \exp(-a)\frac{k_BT}{m}\sqrt{\frac{2k_BTa}{m}}\,da$$

$$= N\left(\frac{m}{2\pi k_BT}\right)^{3/2} \cdot 4\pi \left(\frac{k_BT}{m}\right)^{3/2} \cdot \sqrt{2} \int_0^\infty \sqrt{a}\exp(-a)\,da$$

$$= \frac{2N}{\sqrt{\pi}} \int_0^\infty \sqrt{a}\exp(-a)\,da = \frac{2N}{\sqrt{\pi}} \cdot \Gamma\left(\frac{3}{2}\right) = \frac{2N}{\sqrt{\pi}} \cdot \frac{1}{2}\sqrt{\pi} = N \overset{!}{=} 1$$

$$\Rightarrow N = 1$$

a) Bestimme das Maximum von $f(v)$.

$$\frac{df(v)}{dv} = \left(\frac{m}{2\pi k_BT}\right)^{3/2} \cdot 4\pi\left(2v \cdot \exp\left(-\frac{mv^2}{2k_BT}\right) - v^2 \cdot \frac{mv}{k_BT}\exp\left(-\frac{mv^2}{2k_BT}\right)\right)$$

$$= \underbrace{4\pi\left(\frac{m}{2\pi k_BT}\right)^{3/2}}_{\text{nicht abhängig von } v.} \underbrace{\exp\left(-\frac{mv^2}{2k_BT}\right)}_{\neq 0 \ \forall v \in \mathbb{R}} \cdot \left(2v - \frac{m}{k_BT}v^3\right) \overset{!}{=} 0$$

$$\Rightarrow v\left(2 - \frac{m}{k_BT}v^2\right) \overset{!}{=} 0$$

$\Rightarrow v_1 = 0$, $v_{2,3} = \pm\sqrt{\frac{k_BT}{m} \cdot 2}$ (aber nur "+", da keine $v < 0$)

$$\frac{d^2 f(v)}{dv^2} = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \frac{d}{dv}\left(\exp\left(-\frac{mv^2}{2k_BT}\right)\left(2v - \frac{m}{k_BT}v^3\right)\right)$$

$$= 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot \left[\left(-\frac{mv}{k_BT}\right)\exp\left(-\frac{mv^2}{2k_BT}\right)\left(2v - \frac{m}{k_BT}v^3\right) + \exp\left(-\frac{mv^2}{2k_BT}\right)\left(2 - \frac{3m}{k_BT}v^2\right)\right]$$

$$\frac{d^2 f}{dv^2}(v=0) = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot 2\exp\left(-\frac{mv^2}{2k_BT}\right) > 0$$

$\Rightarrow$ Minimum bei $v_1 = 0$

$\rightsquigarrow$ Maximum bei $v_2 = \sqrt{\frac{2k_BT}{m}}$

$\Rightarrow$ Die wahrscheinlichste Geschwindigkeit ist $\sqrt{\frac{2k_BT}{m}}$.

b)

$$\langle v \rangle = \int_0^\infty v f(v)\, dv = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot 4\pi \int_0^\infty \exp\left(-\frac{mv^2}{2k_BT}\right) v^3\, dv$$

= Analog zu Teil 1: Best. von N, selbe Substitution ... $= \frac{2}{\sqrt{\pi}} \cdot \sqrt{\frac{2k_BT}{m}} \int_0^\infty a \exp(-a)\, da$

da $v^3$ im Integral stand, nicht $v^2$ wie in Teil 1

$$\langle v \rangle = \sqrt{\frac{8k_BT}{\pi m}} \cdot \underbrace{\Gamma(2)}_{=1!=1} = \sqrt{\frac{8k_BT}{\pi m}}$$

d)

$$v_{max} = \sqrt{\frac{2k_BT}{m}}$$

$$f(v_{max}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m}{2k_BT} \cdot \frac{2k_BT}{m}\right) \cdot 4\pi \cdot \frac{2k_BT}{m}$$

$$= 4\sqrt{\frac{m}{2\pi k_B T}} \exp(-1)$$

$\Rightarrow \frac{1}{2} f(v_{max}) = f(v') = \sqrt{\frac{2m}{\pi k_B T}} \exp(-1) \longrightarrow$ Bestimme $v'$:

$$\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{mv'^2}{2k_BT}\right) 4\pi v'^2 = \sqrt{\frac{2m}{\pi k_B T}} \exp(-1)$$

$$\Leftrightarrow \quad \exp\left(-\frac{mv'^2}{2k_BT}\right) v'^2 = \frac{2k_BT}{m} \exp(-1)$$

1 e) $\sigma_v = \sqrt{Var(v)} = \sqrt{E(v^2) - (E(v))^2}$

aus 1b): $E(v) = \sqrt{\frac{8k_BT}{\pi m}}$

Berechne $E(v^2)$:

$$E(v^2) = \left(\frac{m}{2\pi k_BT}\right)^{3/2} \cdot 4\pi \int_0^\infty \exp\left(-\frac{mv^2}{2k_BT}\right) v^4 \, dv$$

Substitution: $b := \frac{mv^2}{2k_BT}$ $\quad dv = \frac{k_BT}{mv} db$ $\quad v = \sqrt{\frac{2k_BTb}{m}}$

$$E(v^2) = \left(\frac{m}{2\pi k_BT}\right)^{3/2} \cdot 4\pi \int_0^\infty \exp(-b) \cdot \frac{k_BT}{m} \cdot \sqrt{\frac{2k_BTb}{m}} \, db$$

$$= \frac{4k_BT}{\sqrt{\pi} \cdot m} \int_0^\infty b^{3/2} \exp(-b) \, db = \frac{4k_BT}{\sqrt{\pi}\, m} \cdot \underbrace{\Gamma\left(\frac{5}{2}\right)}_{= \frac{3}{4}\sqrt{\pi}}$$

$$= \frac{3k_BT}{m}$$

$$\Rightarrow \sigma_v = \sqrt{\frac{3k_BT}{m} - \frac{8k_BT}{\pi m}} = \sqrt{\frac{k_BT}{\pi m}(3\pi - 8)}$$

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$\boxed{3}$ $W_r, W_b \in \{1,2,3,4,5,6\}$, $P(W_b = i) = P(W_r = i) = \frac{1}{6} \; \forall i \in \{1,\dots,6\}$

a) $W_b + W_r = 9 =: W_g$

$$P(W_b + W_r = 9) = P(W_g | W_r = 3) + P(W_g | W_b = 3) + P(W_g | W_r = 4) + P(W_g | W_b = 4)$$

$= 4 \cdot \frac{1}{6} \cdot \frac{1}{6}$ (Die Würfel sind nicht gekoppelt)

$= \frac{1}{9}$

b) $W_b + W_r \geq 9$

$$P(W_b + W_r \geq 9) = P(W_g) + P(W_b = 5 | W_r = 5) + P(W_b = 5 | W_r = 6)$$
$$+ P(W_b = 6 | W_r = 5) + P(W_b = 6 | W_r = 6)$$
$$= \frac{1}{9} + 4 \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{2}{9}$$

c) $P(W_{c)}) = P(W_b = 4 \mid W_r = 5) + P(W_r = 5 \mid W_b = 4) = 2 \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{18}$

d) $P(W_{d)}) = P(W_r = 4 \mid W_b = 5) = \frac{1}{36}$

e) $P(W_{e)}) = P(W_r + W_b = 9 \mid W_r = 4) = P(W_b = 5 \mid W_r = 4) = \frac{1}{36}$

f) $P(W_{f)}) = P(W_{e)}) + P(W_r = 6 \mid W_b = 4) = \frac{2}{36} = \frac{1}{18}$

g) $P(W_{g)}) = P(W_b = 5 \mid W_r = 4) = \frac{1}{36}$