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Mathematical Analysis and the Local Discontinuous Galerkin Method for Caputo–Hadamard Fractional Partial Differential Equation

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Abstract
In this paper, we study the Caputo–Hadamard fractional partial differential equation where the time derivative is the Caputo–Hadamard fractional derivative and the space derivative is the integer-order one. We first introduce a modified Laplace transform. Then using the newly defined Laplace transform and the well-known finite Fourier sine transform, we obtain the analytical solution to this kind of linear equation. Furthermore, we study the regularity and logarithmic decay of its solution. Since the equation has a time fractional derivative, its solution behaves a certain weak regularity at the initial time. We use the finite difference scheme on non-uniform meshes to approximate the time fractional derivative in order to guarantee the accuracy and use the local discontinuous Galerkin method (LDG) to approximate the spacial derivative. The fully discrete scheme is established and analyzed. A numerical example is displayed which support the theoretical analysis.

Keywords Caputo–Hadamard derivative · Regularity · Finite difference scheme on non-uniform meshes · Local discontinuous Galerkin method · Stability and convergence

Mathematics Subject Classification 26A33 · 35B65 · 65M12

1 Introduction

During the last few decades, many efforts have been done in the study of fractional calculus and fractional differential equations [14,15,25,27]. Up to now, there exist several kinds of fractional integrals and derivatives, like Riemann–Liouville, Caputo, Riesz, and Hadamard integrals and derivatives. However, it has been noticed that most of the work is devoted to the issues related to Riemann–Liouville, Caputo, and Riesz derivatives. Actually, the Hadamard

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derivative is also very worthy of in-depth study. There are at least two differences between the Hadamard derivative and Riemann–Liouville one. To be specific, the kernel of the integral appearing in the Hadamard derivative is the power of $(\log t - \log w)$, but the kernel takes the power of $(t-w)$ in the Riemann–Liouville one. On the other hand, the Hadamard derivative is viewed as a generalization of the operator $(t\frac{\mathrm{d}}{\mathrm{d}t})^n$, while the Riemann–Liouville derivative is considered as an extension of the classical operator $(\frac{\mathrm{d}}{\mathrm{d}t})^n$. Besides, the Hadamard derivative (also integral) starts at the initial time a which is bigger than zero, but the Riemann–Liouville derivative (also integral) often begins at the origin (or any other real number). Now it is known that the Hadamard type derivatives are used in some practical problems related to mechanics and engineering, e.g., both planar and three-dimensional elasticities, or the fracture analysis [1] and the Lomnitz logarithmic creep law of special substances, e.g., igneous rock [9,23]. For more details about Hadamard fractional derivative or integral, the readers can refer to [1,7,8,10,12–15] and the references cited therein.

In the following, several definitions of Hadamard fractional calculus are introduced.

Definition 1.1 [10,14,15] The Hadamard fractional integral of a given function $f(t)$ with order $\alpha > 0$ is defined by

$$ {}_H\mathrm{D}_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{w}\right)^{\alpha-1} f(w)\frac{\mathrm{d}w}{w},\ t > a > 0. \tag{1.1} $$

Definition 1.2 [10,14,15] The Hadamard fractional derivative of a given function $f(t)$ with order $\alpha > 0$ is defined by

$$ \begin{aligned} {}_H\mathrm{D}_{a,t}^{\alpha} f(t) &= \delta^n \left[{}_H\mathrm{D}_{a,t}^{-(n-\alpha)} f(t) \right] \\ &= \frac{1}{\Gamma(n-\alpha)}\delta^n \int_a^t \left(\log\frac{t}{w}\right)^{n-\alpha-1} f(w)\frac{\mathrm{d}w}{w},\ t > a > 0, \end{aligned} \tag{1.2} $$

where $\delta = t\frac{\mathrm{d}}{\mathrm{d}t}$, $n-1 < \alpha < n \in \mathbb{Z}^+$.

It can be shown that for a fixed t, $\lim\limits_{\alpha\to(n-1)^+} {}_H\mathrm{D}_{a,t}^{\alpha} f(t) = \delta^{n-1} f(t)$ and $\lim\limits_{\alpha\to n^-} {}_H\mathrm{D}_{a,t}^{\alpha} f(t) = \delta^n f(t)$ if $\delta^n f(t)$ is continuous and $\delta^{n+1} f(t)$ is integrable in $[a,t]$ $(t > a > 0)$ or refer to [24] with $\mu = 0$. Definition 1.2 seems not to be easily used in applied science and engineering. This can be seen from the following initial value problem. The initial value problem (i.e, Cauchy problem) with Hadamard derivative of order $\alpha \in (0,1)$ is given as follows,

$$ \begin{cases} {}_H\mathrm{D}_{a,t}^{\alpha} u(t) = f(t,u),\ t > a > 0,\ \alpha \in (0,1), \\ {}_H\mathrm{D}_{a,t}^{\alpha-1} u(t)|_{t=a} = b_0, \end{cases} \tag{1.3} $$

or equivalently,

$$ \begin{cases} {}_H\mathrm{D}_{a,t}^{\alpha} u(t) = f(t,u),\ t > a > 0,\ \alpha \in (0,1), \\ \left(\log\frac{t}{a}\right)^{1-\alpha} u(t)|_{t=a} = \dfrac{b_0}{\Gamma(\alpha)}. \end{cases} \tag{1.4} $$

Obviously, $u(t)$ at the starting time can be ∞ so generally is not used in the real applications [24], just like the case of Riemann–Liouville derivative problems. In the real realms, we use an alternative as follows.

Definition 1.3 [12] The Caputo–Hadamard fractional derivative of a given function $f(t)$ with order α $(n-1<\alpha<n\in\mathbb{Z}^+)$ is defined by

$$\begin{aligned} {}_{CH}\mathrm{D}^{\alpha}_{a,t}f(t) &= {}_{H}\mathrm{D}^{-(n-\alpha)}_{a,t}[\delta^n f(t)] \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \left(\log\frac{t}{w}\right)^{n-\alpha-1}\delta^n f(w)\frac{\mathrm{d}w}{w},\ t>a>0. \end{aligned} \tag{1.5}$$

If we use the Caputo–Hadamard derivative in problem (1.3), then the initial value condition is given as $u(t)|_{t=a}=u_a$ for $\alpha\in(0,1)$. This can be also seen from the following equational model (1.7). As to the relationship between Hadamard derivative and Caputo–Hadamard derivative, one can see the following formula,

$${}_{H}\mathrm{D}^{\alpha}_{a,t}f(t)=\sum_{k=0}^{n-1}\frac{\delta^k f(t)}{\Gamma(k+1-\alpha)}\left(\log\frac{t}{a}\right)^{k-\alpha}+{}_{CH}\mathrm{D}^{\alpha}_{a,t}f(t),\ n-1<\alpha<n\in\mathbb{Z}^+. \tag{1.6}$$

Refer to [13,15] for details.

In this paper, we study the Caputo–Hadamard fractional partial differential equation of the following form,

$$\begin{cases} {}_{CH}\mathrm{D}^{\alpha}_{a,t}u(x,t)-\Delta u(x,t)=f(x,t),\ t>a>0,\ x\in\Omega=(0,L),\ \alpha\in(0,1),\\ u(x,a)=u_a(x),\ x\in\Omega,\\ u|_{x\in\partial\Omega}=0,\ t>a>0, \end{cases} \tag{1.7}$$

where the source term $f(x,t)$ and the initial data $u_a(x)$ are given functions.

Because the involved fractional derivative starts at the initial time a which is bigger than zero, we can not directly use the standard Laplace transform. We shall define a modified Laplace transform in Sect. 2, where the corresponding properties are simply displayed. In addition, the finite Fourier sine transform is introduced. In Sect. 3, we apply the modified Laplace transform and the finite Fourier sine transform to derive the analytical solution of Eq. (1.7), where the regularity of the derived solution is carefully studied, and where the logarithmic decay of its solution is analyzed too. In Sect. 4, we present a finite difference method on non-uniform meshes to approximate the time derivative and use the LDG method to approximate the spacial derivative. Then a fully discrete scheme is established and studied. In the next section, a numerical example is provided which supports the theoretical results. Finally, a short conclusion is given in the last section.

2 Preliminaries

According to the definitions of Hadamard integral and derivative, we can not find their Laplace transforms due to the initial time starting at $a>0$. We need to define a new definition for the case with the starting time at $t=a>0$.

Definition 2.1 For a given function $f(t)$ defined on $[a,+\infty)$ $(a>0)$, the modified Laplace transform of $f(t)$ is defined as

$$\widetilde{f}(s)=\mathscr{L}_m\{f(t)\}=\int_a^\infty e^{-s\log\frac{t}{a}}f(t)\frac{\mathrm{d}t}{t},\ s\in\mathbb{C}. \tag{2.1}$$

The following theorem guarantees the existence of the modified Laplace transform of a given function $f(t)$ satisfying suitable conditions.

Theorem 2.1 *For a given function $f(t)$ defined on $[a, +\infty)$ $(a > 0)$, if*

(1) *$f(t)$ is continuous or piecewise continuous on every finite subinterval of $[a, +\infty)$,*
(2) *there exist a positive constant $M > 0$ and $\sigma > 0$ such that for a given large $T > a$*

$$|f(t)| \leq Mt^{\sigma}, \text{ when } t > T,$$

then the modified Laplace transform of $f(t)$ exists with $\mathrm{Re}(s) > \sigma$.

Proof The proof is almost the same as that of Lemma 2.2 in [19] so is omitted here. □

In [19], another kind of Laplace transform as well as the associate inverse transform was defined for Hadamard fractional calculus. In [11], a somewhat more general Laplace transform was introduced for a kind of generalized fractional derivatives but the corresponding inverse transform was not available. Here we continue to consider the possibility of the inverse transform of the modified Laplace transform (2.1). In effect, under the same condition the inverse modified Laplace transform exists too. Now we introduce the following definition.

Definition 2.2 The inverse modified Laplace transform of $\widetilde{f}(s)$ (in Definition 2.1) is given by

$$f(t) = \mathscr{L}_m^{-1}\{\widetilde{f}(s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\log\frac{t}{a}}\widetilde{f}(s)\mathrm{d}s,\ c > 0,\ i^2 = -1. \tag{2.2}$$

The following differential property can be proved by direct calculations.

Proposition 2.1 *If $\mathscr{L}_m\{f(t)\} = \widetilde{f}(s)$, then*

$$\mathscr{L}_m\{\delta^n f(t)\} = s^n\widetilde{f}(s) - \sum_{k=0}^{n-1} s^{n-k-1}\delta^k f(a),\ t > a > 0,\ n \in \mathbb{Z}^+.$$

Now we introduce the convolution and the corresponding property.

Definition 2.3 For given functions $f(t)$ and $g(t)$ defined on $[a, +\infty)$ $(a > 0)$, the integral $\int_a^t f(a\frac{t}{w})g(w)\frac{\mathrm{d}w}{w}$ is called the convolution of $f(t)$ and $g(t)$, i.e.,

$$f(t) * g(t) = (f * g)(t) = \int_a^t f\left(a\frac{t}{w}\right)g(w)\frac{\mathrm{d}w}{w}.$$

Proposition 2.2 (Convolution theorem) *If $\mathscr{L}_m\{f(t)\} = \widetilde{f}(s)$ and $\mathscr{L}_m\{g(t)\} = \widetilde{g}(s)$, then*

$$\mathscr{L}_m\{f(t) * g(t)\} = \mathscr{L}_m\{f(t)\}\mathscr{L}_m\{g(t)\} = \widetilde{f}(s)\widetilde{g}(s);$$

Or equivalently,

$$\mathscr{L}_m^{-1}\{\widetilde{f}(s)\widetilde{g}(s)\} = f(t) * g(t).$$

Proof Noticing that Definitions 2.1 and 2.3, and interchanging the order of integration, one gets

$$\mathscr{L}_m\{f(t) * g(t)\} = \int_a^{\infty} e^{-s\log\frac{t}{a}}\int_a^t f\left(a\frac{t}{w}\right)g(w)\frac{\mathrm{d}w}{w}\frac{\mathrm{d}t}{t}$$

$$
\begin{aligned}
&= \int_a^\infty \int_a^\infty e^{-s\log\frac{w\eta}{a^2}} f(\eta)\frac{a}{w\eta}\frac{w}{a}\mathrm{d}\eta g(w)\frac{\mathrm{d}w}{w} \\
&= \int_a^\infty \int_a^\infty e^{-s(\log\frac{w}{a}+\log\frac{\eta}{a})} f(\eta)\frac{\mathrm{d}\eta}{\eta} g(w)\frac{\mathrm{d}w}{w} \\
&= \mathscr{L}_m\{f(t)\}\mathscr{L}_m\{g(t)\} = \widetilde{f}(s)\widetilde{g}(s),
\end{aligned}
$$

where the change of variable $a\frac{t}{w} = \eta$ is utilized. □

Next, we present modified Laplace transforms of Hadamard integral and derivative, which will be useful in the coming section.

Lemma 2.1 *Let $n-1<\alpha<n\in\mathbb{Z}^+$. Then the following equalities hold:*

$$\mathscr{L}_m\{{}_H\mathrm{D}_{a,t}^{-\alpha} f(t)\} = s^{-\alpha}\mathscr{L}_m\{f(t)\}, \tag{2.3}$$

$$\mathscr{L}_m\{{}_H\mathrm{D}_{a,t}^{\alpha} f(t)\} = s^{\alpha}\mathscr{L}_m\{f(t)\} - \sum_{k=0}^{n-1} s^{n-k-1}\left[\delta^k\ {}_H\mathrm{D}_{a,t}^{-(n-\alpha)} f(t)\right]\Big|_{t=a}, \tag{2.4}$$

$$\mathscr{L}_m\{{}_{CH}\mathrm{D}_{a,t}^{\alpha} f(t)\} = s^{\alpha}\mathscr{L}_m\{f(t)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1}\delta^k f(a). \tag{2.5}$$

Proof In view of Definitions 1.1 and 2.3, Hadamard integral can be rewritten as

$${}_H\mathrm{D}_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} * f(t).$$

By Proposition 2.2, there holds

$$\mathscr{L}_m\{{}_H\mathrm{D}_{a,t}^{-\alpha} f(t)\} = \mathscr{L}_m\left\{\frac{1}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} * f(t)\right\} = s^{-\alpha}\mathscr{L}_m\{f(t)\},$$

which means equality (2.3).

To prove (2.4), it follows from Definition 1.2 and Proposition 2.1 that

$$
\begin{aligned}
\mathscr{L}_m\{{}_H\mathrm{D}_{a,t}^{\alpha} f(t)\} &= \mathscr{L}_m\{\delta^n\ {}_H\mathrm{D}_{a^+}^{-(n-\alpha)} f(t)\} \\
&= s^{\alpha}\mathscr{L}_m\{f(t)\} - \sum_{k=0}^{n-1} s^{n-k-1}\left[\delta^k\ {}_H\mathrm{D}_{a,t}^{-(n-\alpha)} f(t)\right]\Big|_{t=a}.
\end{aligned}
$$

For (2.5), from Definition 1.5, equality (2.3), and Proposition 2.1, one has

$$
\begin{aligned}
\mathscr{L}_m\{{}_{CH}\mathrm{D}_{a,t}^{\alpha} f(t)\} &= \mathscr{L}_m\{{}_H\mathrm{D}_{a,t}^{-(n-\alpha)}\delta^n f(t)\} = s^{\alpha-n}\mathscr{L}_m\{\delta^n f(t)\} \\
&= s^{\alpha}\mathscr{L}_m\{f(t)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1}\delta^k f(a).
\end{aligned}
$$

The proof is completed. □

Now we give modified Laplace transform of Mittag–Leffler function. By means of the following formula [25]

$$\int_0^\infty e^{-st} t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm\lambda t^\alpha)\mathrm{d}t = \frac{k!s^{\alpha-\beta}}{(s^\alpha\mp\lambda)^{k+1}},\ \mathrm{Re}(s) > |\lambda|^{\frac{1}{\alpha}},$$

applying the change of variable $t = \log \frac{w}{a}$ gives

$$\int_a^\infty e^{-s\log\frac{w}{a}} \left(\log\frac{w}{a}\right)^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}\left(\pm\lambda\left(\log\frac{w}{a}\right)^\alpha\right)\frac{\mathrm{d}w}{w} = \frac{k!s^{\alpha-\beta}}{(s^\alpha\mp\lambda)^{k+1}},\ \mathrm{Re}(s) > |\lambda|^{\frac{1}{\alpha}}. \tag{2.6}$$

Finally, we present an application in solving fractional differential equation.

Example 2.1 Solve the following linear fractional differential equation

$$\begin{cases} {}_{CH}\mathrm{D}_{a,t}^\alpha y(t) + \lambda y(t) = f(t),\ t > a > 0,\ \lambda > 0,\ n-1 < \alpha < n \in \mathbb{Z}^+, \\ \delta^k y(t)|_{t=a} = y_k(a),\ k = 0, 1, \ldots, n-1. \end{cases} \tag{2.7}$$

Let $\mathscr{L}_m\{y(t)\} = \widetilde{y}(s)$ and $\mathscr{L}_m\{f(t)\} = \widetilde{f}(s)$. Then using the modified Laplace transform on the both sides of Eq. (2.7) yields

$$\widetilde{y}(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^\alpha+\lambda} y_k(a) + \frac{1}{s^\alpha+\lambda}\widetilde{f}(s).$$

Therefore it follows from the inverse modified Laplace transform that

$$\begin{aligned} y(t) = &\sum_{k=0}^{n-1}\left(\log\frac{t}{a}\right)^k E_{\alpha,k+1}\left(-\lambda\left(\log\frac{t}{a}\right)^\alpha\right) y_k(a) \\ &+ \int_a^t \left(\log\frac{t}{w}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\log\frac{t}{w}\right)^\alpha\right) f(w)\frac{\mathrm{d}w}{w}. \end{aligned}$$

Besides, we introduce the finite Fourier sine transform which will be used later.

Definition 2.4 [5] If $f(x)$ is a continuous or piecewise continuous function on a finite interval $0 < x < L$, then the finite Fourier sine transform of $f(x)$ can be defined by

$$\widehat{f_s}(n) = \mathscr{F}_s\{f(x); n\} = \int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x,\ n = 1, 2, 3, \ldots \tag{2.8}$$

The inverse Fourier sine transform is given by

$$f(x) = \mathscr{F}_s^{-1}\{\widehat{f_s}(n); x\} = \frac{2}{L}\sum_{n=1}^\infty \widehat{f_s}(n)\sin\left(\frac{n\pi x}{L}\right). \tag{2.9}$$

Proposition 2.3 [5] *If* $\mathscr{F}_s\{f(x); n\} = \widehat{f_s}(n)$, *then*

$$\mathscr{F}_s\{f''(x); n\} = -\left(\frac{n\pi}{L}\right)^2 \widehat{f_s}(n) + \left(\frac{n\pi}{L}\right)[f(0) + (-1)^{n+1} f(L)]. \tag{2.10}$$

3 Regularity and Logarithmic Decay of the Solution to FPDE (1.7)

In this section, we first find the analytical solution to FPDE (1.7) by using the modified Laplace transform and the finite Fourier sine transform.

Applying the modified Laplace transform with respect to t on the both sides of Eq. (1.7) and invoking (2.5) give

$$s^\alpha\widetilde{u}(x,s) - s^{\alpha-1}u_a(x) - \Delta\widetilde{u}(x,s) = \widetilde{f}(x,s). \tag{3.1}$$

Using the finite Fourier sine transform with respect to x in Eq. (3.1), equality (2.10), and the boundary conditions in Eq. (1.7) yields

$$\widehat{\widetilde{u}}(n,s)=\frac{s^{\alpha-1}}{s^{\alpha}+\left(\frac{n\pi}{L}\right)^{2}}\widehat{u}_{a}(n)+\frac{1}{s^{\alpha}+\left(\frac{n\pi}{L}\right)^{2}}\widehat{\widetilde{f}}(n,s).$$

By means of the inverse modified Laplace transform and formula (2.6), and the inverse Fourier sine transform, it holds that

$$\begin{aligned}u(x,t)&=\frac{2}{L}\sum_{n=1}^{\infty}E_{\alpha,1}\left(-\left(\frac{n\pi}{L}\right)^{2}\left(\log\frac{t}{a}\right)^{\alpha}\right)\widehat{u}_{a}(n)\sin\left(\frac{n\pi x}{L}\right)\\&\quad+\frac{2}{L}\sum_{n=1}^{\infty}\int_{a}^{t}\left(\log\frac{w}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\left(\frac{n\pi}{L}\right)^{2}\left(\log\frac{w}{a}\right)^{\alpha}\right)\widehat{f}(n,a\frac{t}{w})\frac{\mathrm{d}w}{w}\sin\left(\frac{n\pi x}{L}\right)\\&=\frac{2}{L}\sum_{n=1}^{\infty}E_{\alpha,1}\left(-\left(\frac{n\pi}{L}\right)^{2}\left(\log\frac{t}{a}\right)^{\alpha}\right)\int_{0}^{L}u_{a}(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\sin\left(\frac{n\pi x}{L}\right)\\&\quad+\frac{2}{L}\sum_{n=1}^{\infty}\int_{a}^{t}\left(\log\frac{w}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\left(\frac{n\pi}{L}\right)^{2}\left(\log\frac{w}{a}\right)^{\alpha}\right)\\&\quad\times\int_{0}^{L}f(x,a\frac{t}{w})\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\frac{\mathrm{d}w}{w}\sin\left(\frac{n\pi x}{L}\right).\end{aligned}\tag{3.2}$$

Here $E_{\alpha,1}$ and $E_{\alpha,\alpha}$ are the Mittag–Leffler functions which are introduced below.

Definition 3.1 [14] Let $\alpha>0$ and $\beta\in\mathbb{C}$. The two-parameter Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+\beta)},\ z\in\mathbb{C}.$$

$E_{\alpha,1}$ is often simply written as E_{α}. $E_{1}(x)$ is just the exponential function, i.e., $E_{1}(x)=e^{x}$. For the sake of convenience, we denote

$$\lambda_{n}=\left(\frac{n\pi}{L}\right)^{2},\quad\varphi_{n}=\sin\left(\frac{n\pi x}{L}\right),$$

and L^{2} inner product and norm as follows

$$(\phi,\psi)=\int_{\Omega}\phi(x)\psi(x)\mathrm{d}x,\quad||\phi||=\sqrt{(\phi,\phi)}.$$

Then the analytic solution (3.2) of Eq. (1.7) can be simplified as

$$\begin{aligned}u(x,t)&=\frac{2}{L}\sum_{n=1}^{\infty}E_{\alpha,1}\left(-\lambda_{n}\left(\log\frac{t}{a}\right)^{\alpha}\right)(u_{a},\varphi_{n})\varphi_{n}\\&\quad+\frac{2}{L}\sum_{n=1}^{\infty}\int_{a}^{t}\left(\log\frac{w}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{n}\left(\log\frac{w}{a}\right)^{\alpha}\right)f_{n}\left(a\frac{t}{w}\right)\frac{\mathrm{d}w}{w}\varphi_{n},\end{aligned}\tag{3.3}$$

where $f_{n}(a\frac{t}{w})=(f(\cdot,a\frac{t}{w}),\varphi_{n})$.

For $\nu \geq 0$, define the space

$$\dot{H}^{\nu}(\Omega) = \left\{ g \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{\nu} |(g, \varphi_n)|^2 < \infty \right\}$$

endowed with the norm

$$||g||_{\dot{H}^{\nu}} = \left(\sum_{n=1}^{\infty} \lambda_n^{\nu} |(g, \varphi_n)|^2 \right)^{\frac{1}{2}}.$$

Obviously, $\dot{H}^0(\Omega) = L^2(\Omega)$, the norm $||g||_{\dot{H}^1}$ is equivalent to the norm in $H_0^1(\Omega)$.

Lemma 3.1 [25] *If* $\alpha < 2$, $\beta \in \mathbb{R}$ *and* $\max\{\frac{\pi}{2}\alpha, \frac{\pi}{2}\} < \mu < \min\{\pi, \pi\alpha\}$, *then*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, \ \mu \leq |\arg(z)| \leq \pi, \ |z| \geq 0,$$

where C *is a real constant.*

Lemma 3.2 *Let* $0 < \alpha < 1$. *The following formulas hold:*

$$\delta^k E_{\alpha,1}\left(-\lambda \left(\log \frac{t}{a}\right)^{\alpha}\right) = -\lambda \left(\log \frac{t}{a}\right)^{\alpha-k} E_{\alpha,\alpha-k+1}\left(-\lambda \left(\log \frac{t}{a}\right)^{\alpha}\right), \tag{3.4}$$

$$\begin{aligned} &\delta^k \left(-\lambda \left(\log \frac{t}{a}\right)^{\beta-1} E_{\alpha,\beta}\left(-\lambda \left(\log \frac{t}{a}\right)^{\alpha}\right)\right) \\ &= -\lambda \left(\log \frac{t}{a}\right)^{\beta-k-1} E_{\alpha,\beta-k}\left(-\lambda \left(\log \frac{t}{a}\right)^{\alpha}\right), \end{aligned} \tag{3.5}$$

$${}_{CH}\mathrm{D}_{a,t}^{\alpha} E_{\alpha,1}\left(-\lambda \left(\log \frac{t}{a}\right)^{\alpha}\right) = -\lambda E_{\alpha,1}\left(-\lambda \left(\log \frac{t}{a}\right)^{\alpha}\right), \tag{3.6}$$

where $\lambda > 0$ *and* $k \in \mathbb{Z}^+$.

Proof Since both the Mittag–Leffler function $E_{\alpha,\beta}(x)$ $(x \in \mathbb{R})$ and the function $\left(\log \frac{t}{a}\right)^{\alpha}$ $(t > a > 0)$ are real analytic, so is $E_{\alpha,\beta}\left(-\lambda \left(\log \frac{t}{a}\right)^{\alpha}\right)$ in $t > a > 0$. By differentiating it term by term, one has

$$\begin{aligned} \delta^k E_{\alpha,1}\left(-\lambda \left(\log \frac{t}{a}\right)^{\alpha}\right) &= \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{\Gamma(\alpha m+1)} \frac{\Gamma(\alpha m+1)}{\Gamma(\alpha m-k+1)} \left(\log \frac{t}{a}\right)^{\alpha m-k} \\ &= \sum_{m=1}^{\infty} \frac{(-\lambda)^m \left(\log \frac{t}{a}\right)^{\alpha(m-1)+\alpha-k}}{\Gamma(\alpha(m-1)+\alpha-k+1)} \\ &= -\lambda \left(\log \frac{t}{a}\right)^{\alpha-k} E_{\alpha,\alpha-k+1}\left(-\lambda \left(\log \frac{t}{a}\right)^{\alpha}\right). \end{aligned}$$

One can likewise derive (3.5) and (3.6). The proof is thus completed. □

We are now ready to study regularity of the solution to Eq. (1.7). We will prove that the infinite series solution $u(x,t)$ given by (3.3) is indeed a classical solution of Eq. (1.7) (i.e., it is at least twice differentiable function with respect to the spatial variable and α-differentiable with respect to the time variable [21,28]), that is, $u(x,t)$ satisfies Eq. (1.7) in a pointwise sense. The generic constant C may depend on T in the following discussions.

Theorem 3.1 *Let* $0 < \alpha < 1$, $0 < \epsilon \ll 1$, $k \in \mathbb{Z}^+ \cup \{0\}$, *and* $T > a$. *Suppose that* $u_a \in \dot{H}^{\frac{2k+3+\epsilon}{2}}(\Omega)$, $f(\cdot,t) \in \dot{H}^{\frac{2k+1+\epsilon}{2}}(\Omega)$, *and* $\delta^{k-l} f(\cdot,t) \in \dot{H}^{\frac{1+\epsilon}{2}}(\Omega)$ $(l = 0, 1, \ldots, k-1)$ *for an arbitrary* $t \in (a, T]$ *with*

$$||f(\cdot,t)||_{\dot{H}^{\frac{2k+1+\epsilon}{2}}} \le C, \quad ||\delta^{k-l} f(\cdot,t)||_{\dot{H}^{\frac{1+\epsilon}{2}}} \le C,\ l = 0, 1, 2, \cdots, k-1,$$

for all $t \in (a, T]$ *and some constant* C *independent of* t. *Then Eq.* (1.7) *has a unique solution* $u(x,t)$ *in a pointwise sense, and there exists a constant* C *such that for all* $(x,t) \in \overline{\Omega} \times (a, T]$,

$$\left|\frac{\partial^k u(x,t)}{\partial x^k}\right| \le C, \tag{3.7}$$

$$|\delta^k u(x,t)| \le C\left(1 + \left(\log\frac{t}{a}\right)^{\alpha-k}\right). \tag{3.8}$$

In addition, if $u_a \in \dot{H}^{\frac{2k+5+\epsilon}{2}}(\Omega)$ *and* $f(\cdot,t) \in \dot{H}^{\frac{2k+5+\epsilon}{2}}(\Omega)$ *for each* $t \in (a, T]$ *with* $||f(\cdot,t)||_{\dot{H}^{\frac{2k+5+\epsilon}{2}}} \le C$ *for all* $t \in (a, T]$, *then for each* $(x,t) \in \overline{\Omega} \times (a, T]$, *there holds*

$$\left|\frac{\partial^k}{\partial x^k}\, {}_{CH}\mathrm{D}^{\alpha}_{a,t} u(x,t)\right| \le C. \tag{3.9}$$

Proof We first prove the infinite series solution $u(x,t)$ given by (3.3) is a classical solution of Eq. (1.7). According to conditions $u_a \in \dot{H}^{\frac{3+\epsilon}{2}}(\Omega)$, $f(\cdot,t) \in \dot{H}^{\frac{1+\epsilon}{2}}(\Omega)$ for each $t \in (a, T]$ with $||f(\cdot,t)||_{\dot{H}^{\frac{1+\epsilon}{2}}} \le C$, Lemma 3.1 and Cauchy–Schwarz inequality, it follows from the right hand side of Eq. (3.3) that

$$\begin{aligned}
&\left|\frac{2}{L}\sum_{n=1}^{\infty} E_{\alpha,1}\left(-\lambda_n\left(\log\frac{t}{a}\right)^{\alpha}\right)(u_a,\varphi_n)\varphi_n\right| \le C\sum_{n=1}^{\infty}|(u_a,\varphi_n)| \\
&\le C\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\frac{1+\epsilon}{2}}}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\lambda_n^{\frac{1+\epsilon}{2}}|(u_a,\varphi_n)|^2\right)^{\frac{1}{2}} \le C||u_a||_{\dot{H}^{\frac{1+\epsilon}{2}}},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{2}{L}\sum_{n=1}^{\infty}\int_a^t \left(\log\frac{w}{a}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n\left(\log\frac{w}{a}\right)^{\alpha}\right) f_n\left(a\frac{t}{w}\right)\frac{\mathrm{d}w}{w}\varphi_n\right| \\
&\le C\int_a^t\left(\log\frac{w}{a}\right)^{\alpha-1}\sum_{n=1}^{\infty}\left|f_n\left(a\frac{t}{w}\right)\right|\frac{\mathrm{d}w}{w} \\
&\le C\int_a^t\left(\log\frac{w}{a}\right)^{\alpha-1}\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\frac{1+\epsilon}{2}}}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\lambda_n^{\frac{1+\epsilon}{2}}\left|f_n\left(a\frac{t}{w}\right)\right|^2\right)^{\frac{1}{2}}\frac{\mathrm{d}w}{w} \\
&\le C\left(\log\frac{t}{a}\right)^{\alpha}.
\end{aligned}$$

Therefore the series expressed by (3.3) is absolutely and uniformly convergent on $(x,t) \in \overline{\Omega} \times [a, T]$ and defines a function that we call $v(x,t)$ at present. Hence one has, for $(x,t) \in \Omega \times (a, T]$,

$$|v(x,t)| \le C. \tag{3.10}$$

Furthermore, differentiating the series in (3.3) term by term with respect to x gives, for $k = 1, 2, \ldots$,

$$\frac{2}{L}\sum_{n=1}^{\infty} E_{\alpha,1}\left(-\lambda_n\left(\log\frac{t}{a}\right)^{\alpha}\right)(u_a,\varphi_n)\lambda_n^{\frac{k}{2}}\sin\left(\frac{\pi}{2}k+\frac{n\pi x}{L}\right)$$
$$+\frac{2}{L}\sum_{n=1}^{\infty}\int_a^t\left(\log\frac{w}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n\left(\log\frac{w}{a}\right)^{\alpha}\right)f_n\left(a\frac{t}{w}\right)\frac{\mathrm{d}w}{w}\lambda_n^{\frac{k}{2}}\sin\left(\frac{\pi}{2}k+\frac{n\pi x}{L}\right). \tag{3.11}$$

By the assumption $u_a \in \dot{H}^{\frac{2k+3+\epsilon}{2}}(\Omega)$, $f(\cdot,t) \in \dot{H}^{\frac{2k+1+\epsilon}{2}}(\Omega)$ for each $t \in (a,T]$ with $||f(\cdot,t)||_{\dot{H}^{\frac{2k+1+\epsilon}{2}}} \leq C$, it yields that, for $k = 1, 2, \ldots$,

$$\left|\frac{2}{L}\sum_{n=1}^{\infty} E_{\alpha,1}\left(-\lambda_n\left(\log\frac{t}{a}\right)^{\alpha}\right)(u_a,\varphi_n)\lambda_n^{\frac{k}{2}}\sin\left(\frac{\pi}{2}k+\frac{n\pi x}{L}\right)\right|$$
$$\leq C\sum_{n=1}^{\infty}\lambda_n^{\frac{k}{2}}|(u_a,\varphi_n)| \leq C\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\frac{1+\epsilon}{2}}}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\lambda_n^{k+\frac{1+\epsilon}{2}}|(u_a,\varphi_n)|^2\right)^{\frac{1}{2}} \leq C,$$

and

$$\left|\frac{2}{L}\sum_{n=1}^{\infty}\int_a^t\left(\log\frac{w}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n\left(\log\frac{w}{a}\right)^{\alpha}\right)f_n\left(a\frac{t}{w}\right)\frac{\mathrm{d}w}{w}\lambda_n^{\frac{k}{2}}\sin\left(\frac{\pi}{2}k+\frac{n\pi x}{L}\right)\right|$$
$$\leq C\int_a^t\left(\log\frac{w}{a}\right)^{\alpha-1}\sum_{n=1}^{\infty}\lambda_n^{\frac{k}{2}}\left|f_n\left(a\frac{t}{w}\right)\right|\frac{\mathrm{d}w}{w}$$
$$\leq C\int_a^t\left(\log\frac{w}{a}\right)^{\alpha-1}\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\frac{1+\epsilon}{2}}}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\lambda_n^{k+\frac{1+\epsilon}{2}}\left|f_n\left(a\frac{t}{w}\right)\right|^2\right)^{\frac{1}{2}}\frac{\mathrm{d}w}{w} \leq C.$$

Thus the series (3.11) is absolutely and uniformly convergent on $(x,t) \in \overline{\Omega}\times[a,T]$ for each $k = 1, 2, \ldots$. As a consequence, we show that for $(x,t) \in \Omega\times(a,T]$,

$$\left|\frac{\partial^k v(x,t)}{\partial x^k}\right| \leq C,\ k = 1, 2, \ldots. \tag{3.12}$$

By virtue of Example 2.1, one deduces that

$${}_{CH}\mathrm{D}^{\alpha}_{a,t}\left(\int_a^t\left(\log\frac{w}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n\left(\log\frac{w}{a}\right)^{\alpha}\right)f_n\left(a\frac{t}{w}\right)\frac{\mathrm{d}w}{w}\right)$$
$$= -\lambda_n\int_a^t\left(\log\frac{w}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n\left(\log\frac{w}{a}\right)^{\alpha}\right)f_n\left(a\frac{t}{w}\right)\frac{\mathrm{d}w}{w}+f_n(t).$$

Applying ${}_{CH}\mathrm{D}^{\alpha}_{a,t}$ to (3.3) term by term, and noticing that Lemma 3.2 and the above equality, one has

$$\frac{2}{L}\sum_{n=1}^{\infty}(-\lambda_n)E_{\alpha,1}\left(-\lambda_n\left(\log\frac{t}{a}\right)^{\alpha}\right)(u_a,\varphi_n)\varphi_n$$

$$+\frac{2}{L}\sum_{n=1}^{\infty}(-\lambda_n)\int_a^t\left(\log\frac{w}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n\left(\log\frac{w}{a}\right)^{\alpha}\right)f_n\left(a\frac{t}{w}\right)\frac{\mathrm{d}w}{w}\varphi_n$$
$$+\frac{2}{L}\sum_{n=1}^{\infty}f_n(t)\varphi_n. \tag{3.13}$$

Employing the assumption $u_a\in\dot{H}^{\frac{5+\epsilon}{2}}(\Omega)$ and $f(\cdot,t)\in\dot{H}^{\frac{5+\epsilon}{2}}(\Omega)$ for each $t\in(a,T]$ with $||f(\cdot,t)||_{\dot{H}^{\frac{5+\epsilon}{2}}}\le C$ for all $t\in(a,T]$, and Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}&\left|\frac{2}{L}\sum_{n=1}^{\infty}(-\lambda_n)E_{\alpha,1}\left(-\lambda_n\left(\log\frac{t}{a}\right)^{\alpha}\right)(u_a,\varphi_n)\varphi_n\right|\\&\quad\le C\sum_{n=1}^{\infty}\lambda_n|(u_a,\varphi_n)|\le C||u_a||_{\dot{H}^{\frac{5+\epsilon}{2}}},\end{aligned}$$

$$\begin{aligned}&\left|\frac{2}{L}\sum_{n=1}^{\infty}(-\lambda_n)\int_a^t\left(\log\frac{w}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n\left(\log\frac{w}{a}\right)^{\alpha}\right)f_n\left(a\frac{t}{w}\right)\frac{\mathrm{d}w}{w}\varphi_n\right|\\&\le C\int_a^t\left(\log\frac{w}{a}\right)^{\alpha-1}\sum_{n=1}^{\infty}\lambda_n\left|f_n\left(a\frac{t}{w}\right)\right|\frac{\mathrm{d}w}{w}\\&\le C\int_a^t\left(\log\frac{w}{a}\right)^{\alpha-1}\left|\left|f\left(\cdot,a\frac{t}{w}\right)\right|\right|_{\dot{H}^{\frac{5+\epsilon}{2}}}\frac{\mathrm{d}w}{w}\\&\le C\left(\log\frac{T}{a}\right)^{\alpha},\end{aligned}$$

and

$$\left|\frac{2}{L}\sum_{n=1}^{\infty}f_n(t)\varphi_n\right|\le C\sum_{n=1}^{\infty}|f_n(t)|\le C||f(\cdot,w)||_{\dot{H}^{\frac{1+\epsilon}{2}}}.$$

Together with these estimates, we derive that for each fixed $t\in(a,T]$, the series (3.13) is absolutely and uniformly convergent on $(x,t)\in\overline{\Omega}\times[a+\varepsilon,T]$ $(\forall\varepsilon>0)$. Thus ${}_{CH}\mathrm{D}^{\alpha}_{a,t}v(x,t)$ exists for $(x,t)\in\overline{\Omega}\times(a,T]$ and $v(x,t)$ is the solution of Eq. (1.7). Similar to [20], we can prove that the maximum principle holds for the solution of Eq. (1.7), which implies the solution to Eq. (1.7) is unique, that is, $v(x,t)=u(x,t)$. Hence, the function $u(x,t)$ given by (3.3) is the solution of Eq. (1.7) in the classical sense. Moreover, combining (3.10) and (3.12) yields (3.7).

Next we give the proof of (3.8). Acting the δ-derivative on the series (3.3) term by term with respect to t and using Lemma 3.2 yield that, for $k=1,2,\ldots,$

$$\begin{aligned}&\frac{2}{L}\sum_{n=1}^{\infty}\left(-\lambda_n\left(\log\frac{t}{a}\right)^{\alpha-k}\right)E_{\alpha,\alpha-k+1}\left(-\lambda_n\left(\log\frac{t}{a}\right)^{\alpha}\right)(u_a,\varphi_n)\varphi_n\\&+\frac{2}{L}\sum_{n=1}^{\infty}\int_a^t\left(\log\frac{w}{a}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n\left(\log\frac{w}{a}\right)^{\alpha}\right)\delta^k f_n\left(a\frac{t}{w}\right)\frac{\mathrm{d}w}{w}\varphi_n\\&+\frac{2}{L}\sum_{l=1}^{k}\sum_{n=1}^{\infty}\left(\log\frac{t}{a}\right)^{\alpha-l}E_{\alpha,\alpha-l+1}\left(-\lambda_n\left(\log\frac{t}{a}\right)^{\alpha}\right)\delta^{k-l}f_n(a)\varphi_n.\end{aligned}\tag{3.14}$$

From the conditions $u_a \in \dot{H}^{\frac{2k+3+\epsilon}{2}}(\Omega)$, $f(\cdot,t) \in \dot{H}^{\frac{2k+1+\epsilon}{2}}(\Omega)$, and $\delta^{k-l} f(\cdot,t) \in \dot{H}^{\frac{1+\epsilon}{2}}(\Omega)$ $(l = 0, 1, \ldots, k-1)$ for an arbitrary $t \in (a, T]$ with

$$||f(\cdot,t)||_{\dot{H}^{\frac{2k+1+\epsilon}{2}}} \le C, \quad ||\delta^{k-l} f(\cdot,t)||_{\dot{H}^{\frac{1+\epsilon}{2}}} \le C,\ l = 0, 1, 2, \ldots, k-1,$$

for all $t \in (a, T]$, it holds that, for $k = 1, 2, \ldots$,

$$\begin{aligned}
&\left| \frac{2}{L} \sum_{n=1}^{\infty} \left(-\lambda_n \left(\log \frac{t}{a} \right)^{\alpha-k} \right) E_{\alpha,\alpha-k+1} \left(-\lambda_n \left(\log \frac{t}{a} \right)^{\alpha} \right) (u_a, \varphi_n) \varphi_n \right| \\
&\quad \le C \left(\log \frac{t}{a} \right)^{\alpha-k} \sum_{n=1}^{\infty} \lambda_n |(u_a, \varphi_n)| \le C \left(\log \frac{t}{a} \right)^{\alpha-k}, \qquad (3.15)
\end{aligned}$$

$$\begin{aligned}
&\left| \frac{2}{L} \sum_{n=1}^{\infty} \int_a^t \left(\log \frac{w}{a} \right)^{\alpha-1} E_{\alpha,\alpha} \left(-\lambda_n \left(\log \frac{w}{a} \right)^{\alpha} \right) \delta^k f_n \left(a \frac{t}{w} \right) \frac{\mathrm{d}w}{w} \varphi_n \right| \\
&\quad \le C \sum_{n=1}^{\infty} \int_a^t \left(\log \frac{w}{a} \right)^{\alpha-1} \left| \delta^k f_n \left(a \frac{t}{w} \right) \right| \frac{\mathrm{d}w}{w} \\
&\quad \le C \int_a^t \left(\log \frac{w}{a} \right)^{\alpha-1} \left\| \delta^k f(\cdot, a \frac{t}{w}) \right\|_{\dot{H}^{\frac{1+\epsilon}{2}}} \frac{\mathrm{d}w}{w} \le C \left(\log \frac{t}{a} \right)^{\alpha-k}, \qquad (3.16)
\end{aligned}$$

and

$$\begin{aligned}
&\left| \frac{2}{L} \sum_{l=1}^{k} \sum_{n=1}^{\infty} \left(\log \frac{t}{a} \right)^{\alpha-l} E_{\alpha,\alpha-l+1} \left(-\lambda_n \left(\log \frac{t}{a} \right)^{\alpha} \right) \delta^{k-l} f_n(a) \varphi_n \right| \\
&\quad \le C \sum_{l=1}^{k} \sum_{n=1}^{\infty} \left(\log \frac{t}{a} \right)^{\alpha-l} |\delta^{k-l} f_n(a)| \\
&\quad \le C \sum_{l=1}^{k} \left(\log \frac{t}{a} \right)^{\alpha-l} \left\| \delta^{k-l} f(\cdot, a) \right\|_{\dot{H}^{\frac{1+\epsilon}{2}}} \le C \left(\log \frac{t}{a} \right)^{\alpha-k}. \qquad (3.17)
\end{aligned}$$

Based on (3.14)–(3.17), we obtain that for each fixed $t \in (a, T]$, the series (3.14) is absolutely and uniformly convergent on $(x, t) \in \overline{\Omega} \times [a + \varepsilon, T]$ $(\forall \varepsilon > 0)$ for each $k = 1, 2, \ldots$. Using (3.7) for $k = 0$ and considering the above discussion yield (3.8).

Finally, we show (3.9). Applying $\frac{\partial^k}{\partial x^k}$ to (3.13) term by term, and one has, for $k = 0, 1, 2, \ldots$,

$$\begin{aligned}
&\frac{2}{L} \sum_{n=1}^{\infty} (-\lambda_n) E_{\alpha,1} \left(-\lambda_n \left(\log \frac{t}{a} \right)^{\alpha} \right) (u_a, \varphi_n) \lambda_n^{\frac{k}{2}} \sin \left(\frac{\pi}{2} k + \frac{n\pi x}{L} \right) \\
&\quad + \frac{2}{L} \sum_{n=1}^{\infty} (-\lambda_n) \int_a^t \left(\log \frac{w}{a} \right)^{\alpha-1} E_{\alpha,\alpha} \left(-\lambda_n \left(\log \frac{w}{a} \right)^{\alpha} \right) f_n \left(a \frac{t}{w} \right) \frac{\mathrm{d}w}{w} \lambda_n^{\frac{k}{2}} \sin \left(\frac{\pi}{2} k + \frac{n\pi x}{L} \right) \\
&\quad + \frac{2}{L} \sum_{n=1}^{\infty} f_n(t) \lambda_n^{\frac{k}{2}} \sin \left(\frac{\pi}{2} k + \frac{n\pi x}{L} \right). \qquad (3.18)
\end{aligned}$$

Taking into account the assumption $u_a \in \dot{H}^{\frac{2k+5+\epsilon}{2}}(\Omega)$ and $f(\cdot,t) \in \dot{H}^{\frac{2k+5+\epsilon}{2}}(\Omega)$ for each $t \in (a,T]$ with $||f(\cdot,t)||_{\dot{H}^{\frac{2k+5+\epsilon}{2}}} \le C$ for all $t \in (a,T]$, and using the same technique as used in (3.13), we immediately know that (3.9) is true. The proof of the theorem is thus completed. □

Remark 3.1 From the proof of Theorem 3.1, we can find that we only need $u_a \in \dot{H}^{\frac{2k+1+\epsilon}{2}}$ in (3.7) and $u_a \in \dot{H}^{\frac{5+\epsilon}{2}}$ in (3.8).

Remark 3.2 Inequality (3.8) shows that the solution to Eq. (1.7), together with its δ-derivatives, has logarithmic asymptotics.

Remark 3.3 Stynes et al. [26] derived a similar theorem for Caputo-type fractional partial differential equation where the time fractional derivative is in the sense of Caputo and the space derivative is the integer-order one. They established the bound on the time derivative with $\left|\frac{\partial^\ell u}{\partial t^\ell}(x,t)\right| \le C(1+t^{\alpha-\ell})$ $(0<\alpha<1, \ell=0,1,2)$, see also (2.9b) of Theorem 2.1 in [26], while we give the bound with respect to time derivative $|\delta^\ell u(x,t)| \le C\left(1+\left(\log\frac{t}{a}\right)^{\alpha-\ell}\right)$ $(0<\alpha<1, \ell=0,1,2,\ldots)$ due to the fact that Caputo–Hadamard derivative is used in this paper.

The following theorem shows the logarithmic decay estimate of the analytical solution (3.3).

Theorem 3.2 *Let $0<\alpha<1$ and $0<\epsilon\ll 1$. Assume that $u_a \in L^2(\Omega)$ and $f(\cdot,t) \in \dot{H}^{\frac{1+\epsilon}{2}}(\Omega)$ with*

$$||f(\cdot,t)||_{\dot{H}^{\frac{1+\epsilon}{2}}} \le C\left(1+\log\frac{t}{a}\right)^{-\gamma}, \ \alpha<\gamma<1.$$

Then for the solution (3.3) of Eq. (1.7), there exists a constant C such that

$$|u(x,t)| \le C\left(\log\frac{t}{a}\right)^{-\min\{\alpha,\gamma-\alpha\}}, \ t\to\infty.$$

Proof According to expression (3.3), Lemma 3.1, Cauchy–Schwarz inequality, and the assumption, it holds that

$$\begin{aligned}
|u(x,t)| \le{}& \frac{2}{L}\sum_{n=1}^{\infty}\left|E_{\alpha,1}\left(-\lambda_n\left(\log\frac{t}{a}\right)^{\alpha}\right)\right||(u_a,\varphi_n)||\varphi_n| \\
&+\frac{2}{L}\sum_{n=1}^{\infty}\int_a^t\left(\log\frac{t}{w}\right)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-\lambda_n\left(\log\frac{t}{w}\right)^{\alpha}\right)\right||f_n(w)|\frac{\mathrm{d}w}{w}|\varphi_n| \\
\le{}& \sum_{n=1}^{\infty}\frac{C}{1+\lambda_n\left(\log\frac{t}{a}\right)^{\alpha}}|(u_a,\varphi_n)| + C\sum_{n=1}^{\infty}\int_a^t\left(\log\frac{t}{w}\right)^{\alpha-1}|(f(\cdot,w),\varphi_n)|\frac{\mathrm{d}w}{w} \\
\le{}& C\left(\log\frac{t}{a}\right)^{-\alpha}\sum_{n=1}^{\infty}\lambda_n^{-1}|(u_a,\varphi_n)| + C\int_a^t\left(\log\frac{t}{w}\right)^{\alpha-1}||f(\cdot,w)||_{\dot{H}^{\frac{1+\epsilon}{2}}}\frac{\mathrm{d}w}{w}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\left(\log\frac{t}{a}\right)^{-\alpha}||u_a|| + C\int_a^t \left(\log\frac{t}{w}\right)^{\alpha-1}\left(1+\log\frac{w}{a}\right)^{-\gamma}\frac{\mathrm{d}w}{w}\\
&\leq C\left(\log\frac{t}{a}\right)^{-\alpha} + C\left(\log\frac{t}{a}\right)^{\alpha-\gamma} \leq C\left(\log\frac{t}{a}\right)^{-\min\{\alpha,\gamma-\alpha\}}.
\end{aligned}
$$

The theorem is thus proved. □

As shown in Theorem 3.1, the solution of Eq. (1.7) very likely has a weak regularity at $t = a$, that is, $u(x, t)$ exists but $\delta^k u(x, t)$ blows up as $t \to a^+$ for $\alpha \in (0, 1)$ and $k = 1, 2, \ldots$. When seeking numerical solution to Eq. (1.7), the finite difference method on non-uniform meshes needs using. In the following section, we study this case.

4 A Fully Discrete Scheme

In this section, a finite difference scheme on non-uniform meshes is used to approximate the time fractional derivative and the spatial derivative is discreted by the LDG method. A full discrete scheme to solve Eq. (1.7) is established. The stability and error estimate are also presented. The generic constant C in this section may depend on α, L, a, and T but is always independent of the meshes. Let us start by recalling some notations and a lemma which will be used later on.

4.1 Notations and Lemma

We first divide the interval $\overline{\Omega} = [0, L]$ into $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = L$ with $N \in \mathbb{Z}^+$, define the cell $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ for $1 \leq j \leq N$. Denote the cell center $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$ and the cell length $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ and the length of the largest cell $h = \max_{1\leq j\leq N} h_j$.

We denote by $u^-_{j+\frac{1}{2}}$ and $u^+_{j+\frac{1}{2}}$ the values of u at the node $x_{j+\frac{1}{2}}$, from the left cell I_j and from the right cell I_{j+1}, respectively. We choose the following piecewise polynomials space as the finite element space

$$
V_h \equiv V_h^k = \left\{v \in L^2(\Omega) : v|_{I_j} \in P^k(I_j), 1 \leq j \leq N\right\}, \ k = 0, 1, 2, \ldots, \tag{4.1}
$$

where $P^k(I_j)$ denotes a set of polynomials of the degree up to k in each cell I_j.

To prove the error estimate, we need the following projections in the finite element space V_h. Given a function $q \in H^1(\Omega)$, the standard L^2 projection is defined by, for each j,

$$
\int_{I_j} (\mathscr{P}_h q - q)v_h \mathrm{d}x = 0, \ \forall\, v_h \in V_h. \tag{4.2}
$$

Moreover, Gauss-Radau projections $\mathscr{P}_h^{\pm}$ are given by [3],

$$
\int_{I_j} (\mathscr{P}_h^+ q - q)v_h \mathrm{d}x = 0, \ \forall\, v_h \in P^{k-1}(I_j), \quad (\mathscr{P}_h^+ q)^+_{j-\frac{1}{2}} = q(x^+_{j-\frac{1}{2}}), \tag{4.3}
$$

$$
\int_{I_j} (\mathscr{P}_h^- q - q)v_h \mathrm{d}x = 0, \ \forall\, v_h \in P^{k-1}(I_j), \quad (\mathscr{P}_h^- q)^-_{j+\frac{1}{2}} = q(x^-_{j+\frac{1}{2}}). \tag{4.4}
$$

Lemma 4.1 [2] *Let* $\eta = \mathscr{P}_h q - q$ *or* $\eta = \mathscr{P}_h^{\pm} q - q$*. Then there holds*

$$||\eta|| \le Ch^{k+1}.$$

4.2 Non-uniform Meshes

From (3.8) in Theorem 3.1, $u(x,t)$ can be continuous on the considered domain. But $\delta u(x,t)$ and $\delta^2 u(x,t)$ have weak regularities at $t = a$. In order to increase the error order in time direction, we need use the non-uniform meshes.

4.2.1 Time Discretization

For a given positive number $T > a$, we divide the interval $[a, T]$ into M subintervals with $a = t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots < t_M = T$ and

$$t_n = a\left(\frac{T}{a}\right)^{(n/M)^r}, \; r \ge 1, \; n = 0, 1, \ldots, M.$$

Correspondingly, the interval $[\log a, \log T]$ is also divided into $\log a = \log t_0 < \log t_1 < \cdots < \log t_{n-1} < \log t_n < \cdots < \log t_M = \log T$ with

$$\log t_n = \log a + \left(\log \frac{T}{a}\right)\left(\frac{n}{M}\right)^r.$$

It is evident that

$$\log t_{n+1} - \log t_n = \left(\log \frac{T}{a}\right) M^{-r}[(n+1)^r - n^r] \le C\left(\log \frac{T}{a}\right) M^{-r} n^{r-1}. \tag{4.5}$$

Obviously, this division is non-uniform. Since $|\delta u(x,t)| \le C(1 + (\log \frac{t}{a})^{\alpha-1})$ and $|\delta^2 u(x,t)| \le C(1 + (\log \frac{t}{a})^{\alpha-2})$ (see Theorem 3.1), such a division for $[\log a, \log T]$ is very efficient.

At $t = t_n$, $1 \le n \le M$, it holds that

$$\begin{aligned}
{}_{CH}\mathrm{D}^{\alpha}_{a,t} u(x,t)\Big|_{t=t_n} &= \frac{1}{\Gamma(1-\alpha)} \int_a^{t_n} \left(\log \frac{t_n}{w}\right)^{-\alpha} \delta u(x,w) \frac{\mathrm{d}w}{w} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left(\log \frac{t_n}{w}\right)^{-\alpha} \delta u(x,w) \frac{\mathrm{d}w}{w} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{i=0}^{n-1} \frac{u(x,t_{i+1}) - u(x,t_i)}{\log \frac{t_{i+1}}{t_i}} \int_{t_i}^{t_{i+1}} \left(\log \frac{t_n}{w}\right)^{-\alpha} \frac{\mathrm{d}w}{w} + \Upsilon^n \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} \frac{u(x,t_{i+1}) - u(x,t_i)}{\log \frac{t_{i+1}}{t_i}} \left(\left(\log \frac{t_n}{t_i}\right)^{1-\alpha} - \left(\log \frac{t_n}{t_{i+1}}\right)^{1-\alpha}\right) + \Upsilon^n \\
&= \frac{1}{\Gamma(2-\alpha)} \left(b_{n,1} u(x,t_n) - b_{n,n} u(x,t_0) - \sum_{i=1}^{n-1} (b_{n,i} - b_{n,i+1}) u(x,t_{n-i})\right) + \Upsilon^n \\
&:= \Lambda^{\alpha}_{\log} u(x,t_n) + \Upsilon^n,
\end{aligned} \tag{4.6}$$

where the coefficients are given by

$$b_{n,i} = \frac{\left(\log \frac{t_n}{t_{n-i}}\right)^{1-\alpha} - \left(\log \frac{t_n}{t_{n-i+1}}\right)^{1-\alpha}}{\log \frac{t_{n-i+1}}{t_{n-i}}}, \ i = 1, 2, \ldots, n, \tag{4.7}$$

and the local truncation error is given by

$$\Upsilon^n = \frac{1}{\Gamma(1-\alpha)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left(\log \frac{t_n}{w}\right)^{-\alpha} \left(\frac{u(x, t_{i+1}) - u(x, t_i)}{\log \frac{t_{i+1}}{t_i}} - \delta u(x, w)\right) \frac{\mathrm{d}w}{w}. \tag{4.8}$$

Remark 4.1 The scheme

$$ {}_{CH}\mathrm{D}_{a,t}^{\alpha} u(x,t)\Big|_{t=t_n} \approx \Lambda_{\log}^{\alpha} u(x, t_n)$$

is just the non-uniform L1 scheme for this Caputo–Hadamard derivative. The usual L1 scheme for Caputo–Hadamard derivative is constructed in [8].

Lemma 4.2 *For $0 < \alpha < 1$, the coefficients $b_{n,i}$ $(1 \le i \le n, 1 \le n \le M)$ in (4.7) satisfy*

$$b_{n,1} > b_{n,2} > \cdots > b_{n,i} > b_{n,i+1} > \cdots > b_{n,n} > 0,$$

and

$$(1-\alpha)\left(\log \frac{t_n}{t_{n-i}}\right)^{-\alpha} \le b_{n,i} \le (1-\alpha)\left(\log \frac{t_n}{t_{n-i+1}}\right)^{-\alpha}.$$

Proof This is a direct result of the mean value theorem, so is omitted here. □

Lemma 4.3 *Let $0 < \alpha < 1$. If $u(x,t)$ is continuous with $t \in [a, T]$, and $|\delta u(x,t)| \le C(1 + (\log \frac{t}{a})^{\alpha-1})$ and $|\delta^2 u(x,t)| \le C(1 + (\log \frac{t}{a})^{\alpha-2})$, then the local truncation error Υ^n $(1 \le n \le M)$ in (4.8) satisfies*

$$|\Upsilon^n| \le Cn^{-\min\{2-\alpha,\, r\alpha\}}. \tag{4.9}$$

Proof According to (4.8), one has

$$\begin{aligned}\Upsilon^n &= \frac{1}{\Gamma(1-\alpha)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left(\log \frac{t_n}{w}\right)^{-\alpha} \left(\frac{u(x, t_{i+1}) - u(x, t_i)}{\log \frac{t_{i+1}}{t_i}} - \delta u(x, w)\right) \frac{\mathrm{d}w}{w} \\ &:= \sum_{i=0}^{n-1} Q_{ni}.\end{aligned}$$

Therefore,

$$|\Upsilon^n| \le \sum_{i=0}^{n-1} |Q_{ni}|. \tag{4.10}$$

For $1 \le i < n-1$, an integration by part implies

$$\begin{aligned}Q_{ni} &= \frac{1}{\Gamma(1-\alpha)} \int_{t_i}^{t_{i+1}} \left(\log \frac{t_n}{w}\right)^{-\alpha} \left(\frac{u(x, t_{i+1}) - u(x, t_i)}{\log \frac{t_{i+1}}{t_i}} - \delta u(x, w)\right) \frac{\mathrm{d}w}{w} \\ &= \frac{1}{\Gamma(1-\alpha)} \int_{t_i}^{t_{i+1}} \left(\log \frac{t_n}{w}\right)^{-\alpha} \mathrm{d}\Big(\phi(x,w) - \psi(x,w)\Big)\end{aligned}$$

$$= \frac{-\alpha}{\Gamma(1-\alpha)} \int_{t_i}^{t_{i+1}} \left(\log \frac{t_n}{w}\right)^{-\alpha-1} \Big(\phi(x,w) - \psi(x,w)\Big) \frac{\mathrm{d}w}{w},$$

where

$$\phi(x,w) = \frac{u(x,t_{i+1}) - u(x,t_i)}{\log \frac{t_{i+1}}{t_i}} \log \frac{w}{t_i}, \quad \psi(x,w) = u(x,w) - u(x,t_i).$$

Applying the mean value theorem for integral yields

$$Q_{ni} = \frac{-\alpha}{\Gamma(1-\alpha)} \Big(\phi(x,\xi_1) - \psi(x,\xi_1)\Big) \int_{t_i}^{t_{i+1}} \left(\log \frac{t_n}{w}\right)^{-\alpha-1} \frac{\mathrm{d}w}{w}, \ \xi_1 \in (t_i, t_{i+1}).$$

In addition, using the mean value theorem, there holds

$$\phi(x,\xi_1) - \psi(x,\xi_1) = \log \frac{\xi_1}{t_i} \Big(\delta u(x,\xi_2) - \delta u(x,\xi_3)\Big) = \log \frac{\xi_1}{t_i} \log \frac{\xi_2}{\xi_3} \delta^2 u(x,\xi_4),$$

where $\xi_2, \ \xi_3, \ \xi_4 \in (t_i, t_{i+1})$.

Taking into account (3.8) and (4.5), for $1 \le i < n-1$, one has

$$\begin{aligned}
|Q_{ni}| &\le \frac{\alpha}{\Gamma(1-\alpha)} \left|\log \frac{\xi_1}{t_i}\right| \left|\log \frac{\xi_2}{\xi_3}\right| \left(\max_{w \in [t_i, t_{i+1}]} |\delta^2 u(x,w)|\right) \int_{t_i}^{t_{i+1}} \left(\log \frac{t_n}{w}\right)^{-\alpha-1} \frac{\mathrm{d}w}{w} \\
&\le C \left(\log \frac{t_{i+1}}{t_i}\right)^2 \left(\log \frac{t_i}{a}\right)^{\alpha-2} \left(\log \frac{t_n}{t_{i+1}}\right)^{-\alpha-1} \log \frac{t_{i+1}}{t_i} \\
&\le C \left(\log \frac{T}{a}\right)^3 M^{-3r} i^{3r-3} \left(\log \frac{T}{a}\right)^{-3} M^{3r} i^{r(\alpha-2)} [n^r - (i+1)^r]^{-\alpha-1} \\
&= C i^{r(\alpha+1)-3} [n^r - (i+1)^r]^{-\alpha-1}.
\end{aligned}$$

As a consequence [26],

$$\sum_{i=1}^{[\frac{n}{2}]-1} |Q_{ni}| \le C \sum_{i=1}^{[\frac{n}{2}]-1} i^{r(\alpha+1)-3} n^{r(-\alpha-1)} \le \begin{cases} Cn^{-r(\alpha+1)}, \ r(\alpha+1) < 2, \\ Cn^{-2} \log n, \ r(\alpha+1) = 2, \\ Cn^{-2}, \ r(\alpha+1) > 2. \end{cases} \tag{4.11}$$

For $[\frac{n}{2}] \le i < n-1$, applying (3.8) and (4.5) leads to

$$\begin{aligned}
|Q_{ni}| &\le C \left(\log \frac{t_{i+1}}{t_i}\right)^2 \left(\max_{w \in [t_i, t_{i+1}]} |\delta^2 u(x,w)|\right) \int_{t_i}^{t_{i+1}} \left(\log \frac{t_n}{w}\right)^{-\alpha-1} \frac{\mathrm{d}w}{w} \\
&\le C \left(\left(\log \frac{T}{a}\right) M^{-r} i^{r-1}\right)^2 \left(\log \frac{t_i}{a}\right)^{\alpha-2} \int_{t_i}^{t_{i+1}} \left(\log \frac{t_n}{w}\right)^{-\alpha-1} \frac{\mathrm{d}w}{w} \\
&\le C \left(\log \frac{T}{a}\right)^{\alpha} M^{-r\alpha} n^{r\alpha-2} \int_{t_i}^{t_{i+1}} \left(\log \frac{t_n}{w}\right)^{-\alpha-1} \frac{\mathrm{d}w}{w}.
\end{aligned}$$

Henceforth,

$$\begin{aligned}
\sum_{i=[\frac{n}{2}]}^{n-2} |Q_{ni}| &\le C \left(\log \frac{T}{a}\right)^{\alpha} M^{-r\alpha} n^{r\alpha-2} \int_{t_{[\frac{n}{2}]}}^{t_{n-1}} \left(\log \frac{t_n}{w}\right)^{-\alpha-1} \frac{\mathrm{d}w}{w} \\
&\le C \left(\log \frac{T}{a}\right)^{\alpha} M^{-r\alpha} n^{r\alpha-2} \left(\log \frac{t_n}{t_{n-1}}\right)^{-\alpha} \le Cn^{-(2-\alpha)}.
\end{aligned} \tag{4.12}$$

Next we investigate the estimate of Q_{n0} and $Q_{n,n-1}$. If $n = 1$, then

$$Q_{10} = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_1} \left(\log \frac{t_1}{w}\right)^{-\alpha} \left(\frac{u(x,t_1) - u(x,t_0)}{\log \frac{t_1}{t_0}} - \delta u(x,w) \right) \frac{\mathrm{d}w}{w}$$

$$= \frac{\left(\log \frac{t_1}{a}\right)^{-\alpha}}{\Gamma(2-\alpha)} \Big(u(x,t_1) - u(x,a) \Big) - \frac{1}{\Gamma(1-\alpha)} \int_a^{t_1} \left(\log \frac{t_1}{w}\right)^{-\alpha} \delta u(x,w) \frac{\mathrm{d}w}{w}.$$

It follows from (3.8) that

$$\left| \frac{\left(\log \frac{t_1}{a}\right)^{-\alpha}}{\Gamma(2-\alpha)} \Big(u(x,t_1) - u(x,a) \Big) \right| = \left| \frac{\left(\log \frac{t_1}{a}\right)^{-\alpha}}{\Gamma(2-\alpha)} \int_a^{t_1} \delta u(x,w) \frac{\mathrm{d}w}{w} \right|$$
$$\leq \frac{C \left(\log \frac{t_1}{a}\right)^{-\alpha}}{\Gamma(2-\alpha)} \int_a^{t_1} \left(\log \frac{w}{a}\right)^{\alpha-1} \frac{\mathrm{d}w}{w} \leq C,$$

and

$$\left| \frac{1}{\Gamma(1-\alpha)} \int_a^{t_1} \left(\log \frac{t_1}{w}\right)^{-\alpha} \delta u(x,w) \frac{\mathrm{d}w}{w} \right|$$
$$\leq \frac{C}{\Gamma(1-\alpha)} \int_a^{t_1} \left(\log \frac{t_1}{w}\right)^{-\alpha} \left(\log \frac{w}{a}\right)^{\alpha-1} \frac{\mathrm{d}w}{w} \leq C.$$

Hence one has

$$|Q_{10}| \leq C. \tag{4.13}$$

If $n > 1$, then

$$Q_{n0} = \frac{\left(\log \frac{t_1}{a}\right)^{-1}}{\Gamma(2-\alpha)} \Big(u(x,t_1) - u(x,a) \Big) \left(\left(\log \frac{t_n}{a}\right)^{1-\alpha} - \left(\log \frac{t_n}{t_1}\right)^{1-\alpha} \right)$$
$$- \frac{1}{\Gamma(1-\alpha)} \int_a^{t_1} \left(\log \frac{t_n}{w}\right)^{-\alpha} \delta u(x,w) \frac{\mathrm{d}w}{w}.$$

Using (3.8) again gives

$$\left| \frac{\left(\log \frac{t_1}{a}\right)^{-1}}{\Gamma(2-\alpha)} \Big(u(x,t_1) - u(x,a) \Big) \left(\left(\log \frac{t_n}{a}\right)^{1-\alpha} - \left(\log \frac{t_n}{t_1}\right)^{1-\alpha} \right) \right|$$
$$\leq \frac{C \left(\log \frac{t_1}{a}\right)^{-1}}{\Gamma(2-\alpha)} \left(\left(\log \frac{t_n}{a}\right)^{1-\alpha} - \left(\log \frac{t_n}{t_1}\right)^{1-\alpha} \right) \int_a^{t_1} \left(\log \frac{w}{a}\right)^{\alpha-1} \frac{\mathrm{d}w}{w}$$
$$\leq C \left(\log \frac{t_n}{t_1}\right)^{-\alpha} \left(\log \frac{t_1}{a}\right)^{\alpha},$$

and

$$\left| \frac{1}{\Gamma(1-\alpha)} \int_a^{t_1} \left(\log \frac{t_n}{w}\right)^{-\alpha} \delta u(x,w) \frac{\mathrm{d}w}{w} \right| \leq \frac{\left(\log \frac{t_n}{t_1}\right)^{-\alpha}}{\Gamma(1-\alpha)} \int_a^{t_1} |\delta u(x,w)| \frac{\mathrm{d}w}{w}$$
$$\leq \frac{C \left(\log \frac{t_n}{t_1}\right)^{-\alpha}}{\Gamma(1-\alpha)} \int_a^{t_1} \left(\log \frac{w}{a}\right)^{\alpha-1} \frac{\mathrm{d}w}{w} \leq C \left(\log \frac{t_n}{t_1}\right)^{-\alpha} \left(\log \frac{t_1}{a}\right)^{\alpha}.$$

Consequently,

$$|Q_{n0}| \le C\left(\log\frac{t_n}{t_1}\right)^{-\alpha}\left(\log\frac{t_1}{a}\right)^{\alpha} = C(n^r-1)^{-\alpha} \le Cn^{-r\alpha}. \tag{4.14}$$

Combining (4.13) and (4.14) yields

$$|Q_{n0}| \le Cn^{-r\alpha},\ n \ge 1. \tag{4.15}$$

Finally, we estimate $Q_{n,n-1}$. By the mean value theorem and (3.8), there exist $\xi_5,\ \xi_6 \in (t_{n-1}, t_n)$ such that

$$\begin{aligned}
|Q_{n,n-1}| &\le \frac{1}{\Gamma(1-\alpha)}\int_{t_{n-1}}^{t_n}\left(\log\frac{t_n}{w}\right)^{-\alpha}\left|\frac{u(x,t_n)-u(x,t_{n-1})}{\log\frac{t_n}{t_{n-1}}} - \delta u(x,w)\right|\frac{\mathrm{d}w}{w}\\
&= \frac{1}{\Gamma(1-\alpha)}\int_{t_{n-1}}^{t_n}\left(\log\frac{t_n}{w}\right)^{-\alpha}|\delta u(x,\xi_5)-\delta u(x,w)|\frac{\mathrm{d}w}{w}\\
&= \frac{1}{\Gamma(1-\alpha)}\int_{t_{n-1}}^{t_n}\left(\log\frac{t_n}{w}\right)^{-\alpha}\left|\log\frac{\xi_5}{w}\right||\delta^2 u(x,\xi_6)|\frac{\mathrm{d}w}{w}\\
&\le \frac{\log\frac{t_n}{t_{n-1}}}{\Gamma(1-\alpha)}\left(\max_{w\in[t_{n-1},t_n]}|\delta^2 u(x,w)|\right)\int_{t_{n-1}}^{t_n}\left(\log\frac{t_n}{w}\right)^{-\alpha}\frac{\mathrm{d}w}{w}\\
&\le \frac{C\log\frac{t_n}{t_{n-1}}}{\Gamma(2-\alpha)}\left(\log\frac{t_{n-1}}{a}\right)^{\alpha-2}\left(\log\frac{t_n}{t_{n-1}}\right)^{1-\alpha}\\
&= C[n^r-(n-1)^r]^{2-\alpha}(n-1)^{r(\alpha-2)} \le Cn^{-(2-\alpha)}.
\end{aligned} \tag{4.16}$$

In (4.11), it is obvious that $n^{-r(\alpha+1)} \le n^{-r\alpha}$ when $r(\alpha+1) < 2$; we can also derive $n^{-2}\log n \le Cn^{-r\alpha}$ if $r(\alpha+1) = 2$; and it is easy to find $n^{-2} \le n^{-(2-\alpha)}$ in the case of $r(\alpha+1) > 2$. Thus, combining the bounds (4.11), (4.12), (4.15), and (4.16), we have

$$|\Upsilon^n| \le \sum_{i=0}^{n-1}|Q_{ni}| \le Cn^{-\min\{2-\alpha,\,r\alpha\}}.$$

The proof is thus completed. □

We now study the fully discrete scheme to Eq. (1.7). Using the LDG method [4,6,22,29], we introduce an auxiliary variable $p = u_x$, then Eq. (1.7) is rewritten in the following lower-order system

$$\begin{cases} {}_{CH}\mathrm{D}^{\alpha}_{a,t}u - p_x = f,\\ p = u_x. \end{cases} \tag{4.17}$$

Take the test function $v, w \in L^2(\Omega)$, the weak form of (4.17) at $t = t_n$ reads

$$\begin{cases} ({}_{CH}\mathrm{D}^{\alpha}_{a,t}u^n, v) + (p^n, v_x) - \displaystyle\sum_{j=1}^{N}\left((p^n v^-)_{j+\frac{1}{2}} - (p^n v^+)_{j-\frac{1}{2}}\right) = (f^n, v),\\ (p^n, w) + (u^n, w_x) - \displaystyle\sum_{j=1}^{N}\left((u^n w^-)_{j+\frac{1}{2}} - (u^n w^+)_{j-\frac{1}{2}}\right) = 0, \end{cases} \tag{4.18}$$

where $u^n = u(x, t_n)$.

We replace the solution u^n, p^n in (4.18) with its approximation u_h^n, $p_h^n \in V_h$, respectively. Then the fully discrete scheme is defined as follows: find u_h^n, $p_h^n \in V_h$ such that for v_h, $w_h \in V_h$,

$$\begin{cases} (\Lambda_{\log}^{\alpha} u_h^n, v_h) + \big(p_h^n, (v_h)_x\big) - \sum_{j=1}^{N} \Big((\widehat{p}_h^n v_h^-)_{j+\frac{1}{2}} - (\widehat{p}_h^n v_h^+)_{j-\frac{1}{2}} \Big) = (f^n, v_h), \\ (p_h^n, w_h) + \big(u_h^n, (w_h)_x\big) - \sum_{j=1}^{N} \Big((\widehat{u}_h^n w_h^-)_{j+\frac{1}{2}} - (\widehat{u}_h^n w_h^+)_{j-\frac{1}{2}} \Big) = 0, \end{cases} \tag{4.19}$$

where the “hat” terms are the so-called “numerical fluxes” [16–18,30], they can be taken in the following form

$$\widehat{u}_h^n = (u_h^n)^-, \quad \widehat{p}_h^n = (p_h^n)^+. \tag{4.20}$$

We remark that the choice for the fluxes (4.20) is not unique.

To obtain stability and convergence for the fully discrete scheme (4.19), the following assertion is needed.

Lemma 4.4 *If $\beta \le r\alpha$, then one has*

$$\left(\log \frac{t_n}{t_{n-1}}\right)^{\alpha} \sum_{l=1}^{n} l^{-\beta} \theta_{n,l} \le \frac{M^{-\beta}}{1-\alpha} \left(\log \frac{T}{a}\right)^{\alpha}, \ n = 1, 2, \dots, M,$$

where $\theta_{n,n} = 1$, $\theta_{n,l} = \sum_{\rho=1}^{n-l} \left(\log \frac{t_{n-\rho}}{t_{n-\rho-1}}\right)^{\alpha} (b_{n,\rho} - b_{n,\rho+1})\theta_{n-\rho,l}$, $l = 1, 2, \dots, n-1$.

Proof The proof is almost the same as that of Lemma 4.3 in [26], so is omitted here. □

4.2.2 Numerical Stability

Now we investigate the stability of the fully discrete scheme (4.19).

Theorem 4.1 *The fully discrete scheme* (4.19) *with the flux* (4.20) *is stable, that is,*

$$||u_h^n||^2 \le 2||u_h^0||^2 + 2\left(\Gamma(1-\alpha)\left(\log \frac{T}{a}\right)^{\alpha}\right)^2 \max_{1\le \ell \le M} ||f^{\ell}||^2, \ 1 \le n \le M. \tag{4.21}$$

Proof Choosing the test functions $v_h = u_h^n$, $w_h = p_h^n$ and adding the two equations in (4.19), one gets

$$(\Lambda_{\log}^{\alpha} u_h^n, u_h^n) + ||p_h^n||^2 = (f^n, u_h^n).$$

Using (4.6) gives

$$\begin{aligned} \frac{b_{n,1}}{\Gamma(2-\alpha)} ||u_h^n||^2 \le & \frac{b_{n,n}}{\Gamma(2-\alpha)} (u_h^0, u_h^n) \\ & + \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{n-1} (b_{n,i} - b_{n,i+1})(u_h^{n-i}, u_h^n) + (f^n, u_h^n). \end{aligned}$$

By the Cauchy–Schwarz inequality, there holds

$$\frac{b_{n,1}}{\Gamma(2-\alpha)}||u_h^n||^2 \leq \frac{1}{2\Gamma(2-\alpha)}\sum_{i=1}^{n-1}(b_{n,i}-b_{n,i+1})||u_h^{n-i}||^2 + \frac{1}{2\Gamma(2-\alpha)}b_{n,1}||u_h^n||^2 + \frac{b_{n,n}}{\Gamma(2-\alpha)}||u_h^0||^2 + \frac{\Gamma(2-\alpha)}{b_{n,n}}||f^n||^2.$$

Hence,

$$b_{n,1}||u_h^n||^2 \leq 2b_{n,n}||u_h^0||^2 + \frac{2\big(\Gamma(2-\alpha)\big)^2}{b_{n,n}}||f^n||^2 + \sum_{i=1}^{n-1}(b_{n,i}-b_{n,i+1})||u_h^{n-i}||^2.$$

Noticing Lemma 4.2 and $b_{n,1} = \left(\log \frac{t_n}{t_{n-1}}\right)^{-\alpha}$, one has

$$||u_h^n||^2 \leq \left(\log \frac{t_n}{t_{n-1}}\right)^{\alpha}\Bigg[2b_{n,n}||u_h^0||^2 + \frac{2\big(\Gamma(2-\alpha)\big)^2}{1-\alpha}\left(\log\frac{T}{a}\right)^{\alpha}||f^n||^2 + \sum_{i=1}^{n-1}(b_{n,i}-b_{n,i+1})||u_h^{n-i}||^2\Bigg]. \tag{4.22}$$

Next we show the following inequality holds

$$||u_h^n||^2 \leq 2||u_h^0||^2 + \frac{2\big(\Gamma(2-\alpha)\big)^2}{1-\alpha}\left(\log\frac{T}{a}\right)^{\alpha}\left(\log \frac{t_n}{t_{n-1}}\right)^{\alpha}\sum_{i=1}^{n}\theta_{n,i}||f^i||^2. \tag{4.23}$$

If $n = 1$, (4.22) leads to

$$\begin{aligned}||u_h^1||^2 &\leq \left(\log\frac{t_1}{a}\right)^{\alpha}\left[2b_{1,1}||u_h^0||^2 + \frac{2\big(\Gamma(2-\alpha)\big)^2}{1-\alpha}\left(\log\frac{T}{a}\right)^{\alpha}||f^1||^2\right]\\ &= 2||u_h^0||^2 + \frac{2\big(\Gamma(2-\alpha)\big)^2}{1-\alpha}\left(\log\frac{T}{a}\right)^{\alpha}\left(\log\frac{t_1}{a}\right)^{\alpha}\theta_{1,1}||f^1||^2,\end{aligned}$$

which is obviously true.

Assume that (4.23) remains valid for $\ell = 2, 3, \ldots, n-1$, we need to prove (4.23) holds for $\ell = n$. It follows from (4.22) that

$$\begin{aligned}||u_h^n||^2 &\leq \left(\log \frac{t_n}{t_{n-1}}\right)^{\alpha}\Bigg[2b_{n,n}||u_h^0||^2 + \frac{2\big(\Gamma(2-\alpha)\big)^2}{1-\alpha}\left(\log\frac{T}{a}\right)^{\alpha}||f^n||^2\\ &\quad + \sum_{i=1}^{n-1}(b_{n,i}-b_{n,i+1})\\ &\quad \left(2||u_h^0||^2 + \frac{2\big(\Gamma(2-\alpha)\big)^2}{1-\alpha}\left(\log\frac{T}{a}\right)^{\alpha}\left(\log\frac{t_{n-i}}{t_{n-i-1}}\right)^{\alpha}\sum_{\rho=1}^{n-i}\theta_{n-i,\rho}||f^{\rho}||^2\right)\Bigg]\\ &= \left(\log \frac{t_n}{t_{n-1}}\right)^{\alpha}\Bigg[2b_{n,1}||u_h^0||^2 + \frac{2\big(\Gamma(2-\alpha)\big)^2}{1-\alpha}\left(\log\frac{T}{a}\right)^{\alpha}||f^n||^2\\ &\quad + \frac{2\big(\Gamma(2-\alpha)\big)^2}{1-\alpha}\left(\log\frac{T}{a}\right)^{\alpha}\sum_{\rho=1}^{n-1}||f^{\rho}||^2\sum_{i=1}^{n-\rho}\left(\log\frac{t_{n-i}}{t_{n-i-1}}\right)^{\alpha}(b_{n,i}-b_{n,i+1})\theta_{n-i,\rho}\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= \left(\log \frac{t_n}{t_{n-1}}\right)^{\alpha} \Bigg[2b_{n,1}||u_h^0||^2 + \frac{2\big(\Gamma(2-\alpha)\big)^2}{1-\alpha} \left(\log \frac{T}{a}\right)^{\alpha} \theta_{n,n} ||f^n||^2 \\
&\quad + \frac{2\big(\Gamma(2-\alpha)\big)^2}{1-\alpha} \left(\log \frac{T}{a}\right)^{\alpha} \sum_{\rho=1}^{n-1} \theta_{n,\rho} ||f^{\rho}||^2 \Bigg] \\
&= 2||u_h^0||^2 + \frac{2\big(\Gamma(2-\alpha)\big)^2}{1-\alpha} \left(\log \frac{T}{a}\right)^{\alpha} \left(\log \frac{t_n}{t_{n-1}}\right)^{\alpha} \sum_{i=1}^{n} \theta_{n,i} ||f^i||^2.
\end{aligned}
$$

It does hold for $\ell = n$. So (4.23) is true for all n, indeed.

Taking $\beta = 0$ in Lemma 4.4 and using (4.23), one gets

$$
||u_h^n||^2 \le 2||u_h^0||^2 + 2\left(\Gamma(1-\alpha)\left(\log \frac{T}{a}\right)^{\alpha}\right)^2 \max_{1\le \ell \le M} ||f^{\ell}||^2, \ 1 \le n \le M.
$$

All this completes the proof. □

4.2.3 Error Estimate

This subsection devotes to the error estimate of the fully discrete scheme (4.19).

Theorem 4.2 *Suppose that u^n is the exact solution to Eq. (1.7) and u_h^n is the numerical solution to the fully discrete scheme (4.19) with the flux (4.20). Then one has the following estimate*

$$
||u^n - u_h^n|| \le C(h^{k+1} + M^{-\min\{r\alpha, 2-\alpha\}}), \ 1 \le n \le M, \tag{4.24}
$$

where C is a constant independent of M and h.

Proof Set

$$
e_u^n = u^n - u_h^n = (u^n - \mathscr{P}_h^- u^n) + (\mathscr{P}_h^- u^n - u_h^n) := \eta_u^n + \xi_u^n, \tag{4.25}
$$
$$
e_p^n = p^n - p_h^n = (p^n - \mathscr{P}_h^+ p^n) + (\mathscr{P}_h^+ p^n - p_h^n) := \eta_p^n + \xi_p^n. \tag{4.26}
$$

Subtracting equation (4.19) from (4.18), the resulting error equation is given as follows:

$$
\begin{cases}
({}_{CH}\mathrm{D}_{a,t}^{\alpha} u^n - \Lambda_{\log}^{\alpha} u_h^n, v_h) + \big(e_p^n, (v_h)_x\big) - \displaystyle\sum_{j=1}^{N} \Big(\big((e_p^n)^+ v_h^-\big)_{j+\frac{1}{2}} - \big((e_p^n)^+ v_h^+\big)_{j-\frac{1}{2}} \Big) = 0, \\
(e_p^n, w_h) + \big(e_u^n, (w_h)_x\big) - \displaystyle\sum_{j=1}^{N} \Big(\big((e_u^n)^- w_h^-\big)_{j+\frac{1}{2}} - \big((e_u^n)^- w_h^+\big)_{j-\frac{1}{2}} \Big) = 0.
\end{cases}
\tag{4.27}
$$

It follows from (4.25), (4.26), and the truncation error Υ^n that

$$\begin{cases} (\Lambda_{\log}^{\alpha}\xi_u^n, v_h) + \left(\xi_p^n, (v_h)_x\right) - \sum_{j=1}^{N}\left(\left((\xi_p^n)^+ v_h^-\right)_{j+\frac{1}{2}} - \left((\xi_p^n)^+ v_h^+\right)_{j-\frac{1}{2}}\right) \\ \quad = (\Upsilon^n, v_h) - (\Lambda_{\log}^{\alpha}\eta_u^n, v_h) - \left(\eta_p^n, (v_h)_x\right) + \sum_{j=1}^{N}\left(\left((\eta_p^n)^+ v_h^-\right)_{j+\frac{1}{2}} - \left((\eta_p^n)^+ v_h^+\right)_{j-\frac{1}{2}}\right), \\ (\xi_p^n, w_h) + \left(\xi_u^n, (w_h)_x\right) - \sum_{j=1}^{N}\left(\left((\xi_u^n)^- w_h^-\right)_{j+\frac{1}{2}} - \left((\xi_u^n)^- w_h^+\right)_{j-\frac{1}{2}}\right) \\ \quad = -(\eta_p^n, w_h) - \left(\eta_u^n, (w_h)_x\right) + \sum_{j=1}^{N}\left(\left((\eta_u^n)^- w_h^-\right)_{j+\frac{1}{2}} - \left((\eta_u^n)^- w_h^+\right)_{j-\frac{1}{2}}\right). \end{cases} \tag{4.28}$$

Noticing that the projection properties (4.3) and (4.4), equation (4.28) can be reduced to

$$\begin{cases} (\Lambda_{\log}^{\alpha}\xi_u^n, v_h) + \left(\xi_p^n, (v_h)_x\right) - \sum_{j=1}^{N}\left(\left((\xi_p^n)^+ v_h^-\right)_{j+\frac{1}{2}} - \left((\xi_p^n)^+ v_h^+\right)_{j-\frac{1}{2}}\right) \\ \quad = (\Upsilon^n, v_h) - (\Lambda_{\log}^{\alpha}\eta_u^n, v_h), \\ (\xi_p^n, w_h) + \left(\xi_u^n, (w_h)_x\right) - \sum_{j=1}^{N}\left(\left((\xi_u^n)^- w_h^-\right)_{j+\frac{1}{2}} - \left((\xi_u^n)^- w_h^+\right)_{j-\frac{1}{2}}\right) \\ \quad = -(\eta_p^n, w_h). \end{cases} \tag{4.29}$$

Choosing the test functions $v_h = \xi_u^n$ and $w_h = \xi_p^n$ yields

$$(\Lambda_{\log}^{\alpha}\xi_u^n, \xi_u^n) + ||\xi_p^n||^2 = (\Upsilon^n, \xi_u^n) - (\Lambda_{\log}^{\alpha}\eta_u^n, \xi_u^n) - (\eta_p^n, \xi_p^n). \tag{4.30}$$

Applying (4.6) and the Cauchy-Schwarz inequality to (4.30) gives

$$\begin{aligned} ||\xi_u^n||^2 \le \left(\log\frac{t_n}{t_{n-1}}\right)^{\alpha}\Bigg[&\frac{2\big(\Gamma(2-\alpha)\big)^2}{b_{n,n}}||\Upsilon^n||^2 + \frac{2\big(\Gamma(2-\alpha)\big)^2}{b_{n,n}}||\Lambda_{\log}^{\alpha}\eta_u^n||^2 \\ &+ \frac{\Gamma(2-\alpha)}{2}||\eta_p^n||^2 + \sum_{i=1}^{n-1}(b_{n,i} - b_{n,i+1})||\xi_u^{n-i}||^2\Bigg], \end{aligned}$$

that is,

$$\begin{aligned} ||\xi_u^n||^2 \le \left(\log\frac{t_n}{t_{n-1}}\right)^{\alpha}\Bigg[&\frac{2\big(\Gamma(2-\alpha)\big)^2}{1-\alpha}\left(\log\frac{t_n}{a}\right)^{\alpha}||\Upsilon^n||^2 \\ &+ \frac{2\big(\Gamma(2-\alpha)\big)^2}{1-\alpha}\left(\log\frac{t_n}{a}\right)^{\alpha}||\Lambda_{\log}^{\alpha}\eta_u^n||^2 \\ &+ \frac{\Gamma(2-\alpha)}{2}||\eta_p^n||^2 + \sum_{i=1}^{n-1}(b_{n,i} - b_{n,i+1})||\xi_u^{n-i}||^2\Bigg]. \end{aligned} \tag{4.31}$$

For the first term of the right hand side of (4.31), it follows from Lemma 4.3 that

$$\left(\log \frac{t_n}{a}\right)^{\alpha} ||\Upsilon^n||^2 \leq C \left(\log \frac{T}{a}\right)^{\alpha} M^{-r\alpha} n^{-(2\min\{r\alpha, 2-\alpha\}-r\alpha)}. \tag{4.32}$$

To estimate the second term of the right hand side of (4.31), using (3.9) and Lemmas 4.1 and 4.3, one gets

$$\begin{aligned}\left(\log \frac{t_n}{a}\right)^{\alpha} ||\Lambda^{\alpha}_{\log}\eta^n_u||^2 &\leq \left(\log \frac{t_n}{a}\right)^{\alpha} C(h^{2k+2} + n^{-2\min\{r\alpha,2-\alpha\}}) \\ &\leq C \left(\log \frac{T}{a}\right)^{\alpha} (h^{2k+2} + M^{-r\alpha} n^{-(2\min\{r\alpha,2-\alpha\}-r\alpha)}).\end{aligned} \tag{4.33}$$

Combining (4.32) with (4.33) and using Lemma 4.1 lead to

$$\begin{aligned}||\xi^n_u||^2 \leq \left(\log \frac{t_n}{t_{n-1}}\right)^{\alpha} \Bigg[& C(h^{2k+2} + M^{-r\alpha} n^{-(2\min\{r\alpha,2-\alpha\}-r\alpha)}) \\ & + \sum_{i=1}^{n-1}(b_{n,i} - b_{n,i+1})||\xi^{n-i}_u||^2\Bigg].\end{aligned}$$

By the similar techniques used in the proof of Theorem 4.1 and Lemma 4.4, it holds that

$$\begin{aligned}||\xi^n_u||^2 &\leq C\left(\log \frac{t_n}{t_{n-1}}\right)^{\alpha} \sum_{i=1}^{n} \theta_{n,i}(h^{2k+2} + M^{-r\alpha} i^{-(2\min\{r\alpha,2-\alpha\}-r\alpha)}) \\ &\leq Ch^{2k+2}\left(\log \frac{t_n}{t_{n-1}}\right)^{\alpha} \sum_{i=1}^{n}\theta_{n,i} + CM^{-r\alpha}\left(\log \frac{t_n}{t_{n-1}}\right)^{\alpha} \sum_{i=1}^{n}\theta_{n,i} i^{-(2\min\{r\alpha,2-\alpha\}-r\alpha)} \\ &\leq Ch^{2k+2} + CM^{-r\alpha}M^{-(2\min\{r\alpha,2-\alpha\}-r\alpha)} \\ &\leq C(h^{2k+2} + M^{-2\min\{r\alpha,2-\alpha\}}).\end{aligned}$$

Consequently,

$$||\xi^n_u|| \leq C(h^{k+1} + M^{-\min\{r\alpha,2-\alpha\}}),$$

which together with the estimate of η^n_u in Lemma 4.1 yields the desired result. The proof is thus finished. □

Table 1 L^2-norm errors and temporal convergence orders for (5.1), $M = N$, $r = 1$

M	$\alpha = 0.4$	EOC	$\alpha = 0.6$	EOC	$\alpha = 0.8$	EOC
32	8.76E−03	–	5.64E−03	–	2.36E−03	–
64	7.41E−03	0.2400	4.37E−03	0.3675	1.65E−03	0.5128
128	6.32E−03	0.2312	6.32E−03	0.3750	1.12E−03	0.5691
256	5.37E−03	0.2333	2.56E−03	0.3980	7.21E−04	0.6291
512	4.56E−03	0.2381	1.90E−03	0.4274	4.50E−04	0.6806
1024	3.84E−03	0.2448	1.45E−03	0.4406	2.80E−04	0.6857

Table 2 L^2-norm errors and temporal convergence orders for (5.1), $M = N$, $r = \frac{1}{\alpha}$

M	$\alpha = 0.4$	EOC	$\alpha = 0.6$	EOC	$\alpha = 0.8$	EOC
32	2.47E−03	–	2.35E−03	–	1.53E−03	–
64	1.48E−03	0.7498	1.39E−03	0.7590	9.04E−04	0.7547
128	8.30E−04	0.8224	7.80E−04	0.8311	5.40E−04	0.7442
256	4.48E−04	0.8878	4.19E−04	0.8943	3.06E−04	0.8186
512	2.35E−04	0.9346	2.26E−04	0.8928	1.70E−04	0.8472
1024	1.20E−04	0.9640	1.18E−04	0.9322	9.18E−05	0.8904

Table 3 L^2-norm errors and temporal convergence orders for (5.1), $M = N$, $r = \frac{2-\alpha}{\alpha}$

M	$\alpha = 0.4$	EOC	$\alpha = 0.6$	EOC	$\alpha = 0.8$	EOC
32	1.38E−03	–	1.27E−03	–	1.10E−03	–
64	3.30E−04	2.0665	4.58E−04	1.4671	5.42E−04	1.0231
128	1.10E−04	1.5865	2.06E−04	1.1556	2.89E−04	0.9084
256	4.01E−05	1.4534	8.80E−05	1.2236	1.48E−04	0.9659
512	1.41E−05	1.5035	3.65E−05	1.2721	7.35E−05	1.0089
1024	4.89E−06	1.5304	1.47E−05	1.3126	3.56E−05	1.0464

5 Numerical Experiments

In the section, an example is provided to test and verify the theoretical analysis derived in Theorem 4.2.

Example 5.1 Consider the following Caputo–Hadamard FPDE with $0 < \alpha < 1$,

$$\begin{cases} {}_{CH}\mathrm{D}_{1,t}^{\alpha} u(x,t) - \Delta u(x,t) = f(x,t),\ t > 1,\ 0 < x < 1, \\ u(x,1) = 0,\ 0 < x < 1, \\ u(0,t) = u(1,t) = 0,\ t > 1, \end{cases} \tag{5.1}$$

where

$$\begin{aligned} f(x,t) = &\left[\Gamma(\alpha+1) + \frac{2}{\Gamma(3-\alpha)}(\log t)^{2-\alpha}\right]\sin(2\pi x) \\ &+ [(\log t)^{\alpha} + (\log t)^2] 4\pi^2 \sin(2\pi x). \end{aligned}$$

The exact solution of Eq. (5.1) is $u(x,t) = [(\log t)^{\alpha} + (\log t)^2]\sin(2\pi x)$, which displays the weak regularity of $\delta u(x,t)$ and $\delta^2 u(x,t)$ at $t = 1$. Its derivatives exactly agree with the bounds presented in Theorem 3.1. Tables 1, 2 and 3 exhibit the L^2-norm errors and the associated error orders of convergence (EOC) at $T = 2$ when $\alpha = 0.4, 0.6, 0.8$ and different parameters r, respectively. Here we take $M = N$ so that the temporal error dominates the results and $k = 1$ is chosen in the spatial direction. We can observe $O(M^{-\min\{r\alpha, 2-\alpha\}})$ temporal convergence order which supports the theoretical result obtained in Theorem 4.2.

6 Conclusion

This paper deals with the initial-boundary value problem for fractional partial differential equation with Caputo–Hadamard derivative. We define a modified Laplace transform which can easily get the analytical solution of Eq. (1.7). Regularity and logarithmic decay of the analytical solution are studied. Then we construct numerical schemes to approximate Caputo–Hadamard derivative by using finite difference scheme on non-uniform meshes, where the truncation error is obtained. Furthermore the fully discrete scheme for Eq. (1.7) is established. The stability and error estimate are showed for the scheme in L^2-norm, where the convergence order turn out to be $O(h^{k+1}+M^{-\min\{r\alpha,2-\alpha\}})$. Numerical example is displayed which support the theoretical analysis.

References

1. Ahmad, B., Alsaedi, A., Ntouyas, S.K., Tariboon, J.: Hadamard-Type Fractional Differential Equations. Springer, Switzerland (2017)
2. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amserdam (1978)
3. Castillo, P., Cockburn, B., Schötzau, D., Schwab, C.: Optimal a priori error estimates for the hp-version of the local discontinuous Galerkin method for convection–diffusion problems. Math. Comput. **71**, 455–478 (2002)
4. Cockburn, B., Shu, C.-W.: The local discontinuous Galerkin method for time-dependent convection–diffusion systems. SIAM J. Numer. Anal. **35**, 2440–2463 (1998)
5. Debnath, L., Bhatta, D.: Integral Transforms and Their Applications. Chapman and Hall/CRC, Boca Raton (2007)
6. Du, Y.W., Liu, Y., Li, H., Fang, Z.C., He, S.: Local discontinuous Galerkin method for a nonlinear time-fractional fourth-order partial differential equation. J. Comput. Phys. **344**, 108–126 (2017)
7. Gohar, M., Li, C.P., Yin, C.T.: On Caputo–Hadamard fractional differential equations. Int. J. Comput. Math. **97**, 1459–1483 (2020)
8. Gohar, M., Li, C.P., Li, Z.Q.: Finite difference methods for Caputo-Hadamard fractional differential equations. Mediterr. J. Math. **17**, 194 (2020)
9. Garra, R., Mainardi, F., Spada, G.: A generalization of the Lomnitz logarithmic creep law via Hadamard fractional calculus. Chaos Solitons Fractals **102**, 333–338 (2017)
10. Hadamard, J.: Essai sur létude des fonctions données par leur développement de Taylor. J. Math. Pures Appl. **8**, 101–186 (1892)
11. Jarad, F., Abdeljawad, T.: Generalized fractional derivatives and Laplace transform. Discrete Contin. Dyn. S **13**, 709–722 (2020)
12. Jarad, F., Abdeljawad, T., Baleanu, D.: Caputo-type modification of the Hadamard fractional derivatives. Adv. Differ. Equ. **2012**, 142 (2012)
13. Kilbas, A.A.: Hadamard-type fractional calculus. J. Korean Math. Soc. **38**, 1191–1204 (2001)
14. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
15. Li, C.P., Cai, M.: Theory and Numerical Approximations of Fractional Integrals and Derivatives. SIAM, Philadelphia (2019)
16. Li, C.P., Wang, Z.: The local discontinuous Galerkin finite element methods for Caputo-type partial differential equations: numerical analysis. Appl. Numer. Math. **140**, 1–22 (2019)
17. Li, C.P., Wang, Z.: The local discontinuous Galerkin finite element methods for Caputo-type partial differential equations: mathematical analysis. Appl. Numer. Math. **150**, 587–606 (2020)
18. Li, C.P., Wang, Z.: The discontinuous Galerkin finite element method for Caputo-type nonlinear conservation law. Math. Comput. Simulat. **169**, 51–73 (2020)
19. Li, C.P., Li, Z.Q.: Asymptotic behaviors of solution to Caputo–Hadamard fractional partial differential equation with fractional Laplacian. Int. J. Comput. Math. (2020). https://doi.org/10.1080/00207160.2020.1744574
20. Luchko, Y.: Maximum principle for the generalized time-fractional diffusion equation. J. Math. Anal. Appl. **351**, 218–223 (2009)
21. Luchko, Y.: Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation. Fract. Calc. Appl. Anal. **15**, 141–160 (2012)

22. Liu, Y., Yan, Y., Khan, M.: Discontinuous Galerkin time stepping method for solving linear space fractional partial differential equations. Appl. Numer. Math. **115**, 200–213 (2017)
23. Ma, L.: On the kinetics of Hadamard-type fractional defferential systems. Fract. Calc. Appl. Anal. **23**, 553–570 (2020)
24. Ma, L., Li, C.P.: On Hadamard fractional calculus. Fractals **25**, 1750033 (2017)
25. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
26. Stynes, M., O'Riordan, E., Gracia, J.L.: Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation. SIAM J. Numer. Anal. **55**, 1057–1079 (2017)
27. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon (1993)
28. Sakamoto, K., Yamamoto, M.: Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. J. Math. Anal. Appl. **382**, 426–447 (2011)
29. Wei, L.L., He, Y.N.: Analysis of a fully discrete local discontinuous Galerkin method for time-fractional fourth-order problems. Appl. Math. Model. **38**, 1511–1522 (2014)
30. Xu, Q.W., Hesthaven, J.S.: Discontinuous Galerkin method for fractional convection–diffusion equations. SIAM J. Numer. Anal. **52**, 405–423 (2014)