SECOND-ORDER ERROR ANALYSIS FOR FRACTIONAL LAPLACIAN VIA RIESZ DERIVATIVES ON GRADED MESHES*

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Abstract. This is an example SIAM LATEX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

Key words. example, LATEX

MSC codes. ?????????????????

1. Introduction. For $\Omega = (0, 2T)$, $1 < \alpha < 2$,

$$(1.1) \qquad \begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x) & x \in \Omega, \\ u(x) = 0 & x \in \mathbb{R} \setminus \Omega, \end{cases}$$

where

$$(1.2) \qquad (-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = \frac{-\kappa_\alpha}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_\Omega |x-y|^{1-\alpha} u(y) dy$$

$$(1.3) \qquad \kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0$$

2. Preliminaries: Numeric scheme and main results.

2.1. Numeric Format.

$$(2.1) \qquad x_i = \begin{cases} T\left(\frac{i}{N}\right)^r & 0 \le i \le N, \\ 2T - T\left(\frac{2N-i}{N}\right)^r & N \le i \le 2N, \end{cases}$$

where $r \ge 1$. And let

$$(2.2) \qquad h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$(2.3) \qquad \phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

And then, define the piecewise linear interpolant of the true solution u to be

$$(2.4) \qquad \Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x)$$

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For convience, we denote

(2.5)
$$I^{2-\alpha}u(x) := \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x-y|^{1-\alpha}u(y)dy$$

and

(2.6)
$$D_h^2 u(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}u(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) u(x_i) + \frac{1}{h_{i+1}}u(x_{i+1}) \right)$$

Now, we discretise (1.1) by replacing $u(x)$ by a continuous piecewise linear function

(2.7)
$$u_h(x) := \sum_{j=1}^{2N-1} u_j \phi_j(x)$$

whose nodal values u_j are to be determined by collocation at each mesh point x_i for $i = 1, 2, ..., 2N-1$:

(2.8)
$$-\kappa_\alpha D_h^\alpha u_h(x_i) := -\kappa_\alpha D_h^2 I^{2-\alpha}u_h(x_i) = f(x_i) =: f_i$$

Here,

(2.9)
$$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i)\, u_j = \sum_{j=1}^{2N-1} a_{ij}\, u_j$$

where

(2.10)
$$a_{ij} = -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i) \quad \text{for} \quad i, j = 1, 2, ..., 2N-1$$

We have replaced $(-\Delta)^{\alpha/2}u(x_i) = f(x_i)$ in (1.1) by $-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i)$ in (2.8), with truncation error

(2.11)
$$\tau_i := -\kappa_\alpha \left(D_h^\alpha \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha}u(x_i) \right) \quad \text{for} \quad i = 1, 2, ..., 2N-1$$

where $-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^\alpha \phi_j(x_i)u(x_j) = \sum_{j=1}^{2N-1} a_{ij}u(x_j)$.

The discrete equation (2.8) can be written in matrix form

(2.12)
$$AU = F$$

where $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$, $U = (u_1, \cdots, u_{2N-1})^T$ is unknown and $F = (f_1, \cdots, f_{2N-1})^T$.

We can deduce a_{ij},

(2.13)
$$\begin{aligned} a_{ij} &= -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i) \\ &= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}\tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}}\tilde{a}_{i+1,j} \right) \end{aligned}$$

where

(2.14)
$$I^{2-\alpha}\Pi_h u(x_i) = \sum_{j=1}^{2N-1} I^{2-\alpha}\phi_j(x_i)u(x_j) = \sum_{j=1}^{2N-1} \tilde{a}_{ij}u(x_j)$$

and
(2.15)
$$\begin{aligned} \tilde{a}_{ij} &= I^{2-\alpha}\phi_j(x_i) \\ &= \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \end{aligned}$$

2.2. Regularity of the true solution. For any $\beta > 0$, we use the standard notation $C^\beta(\bar\Omega), C^\beta(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^\beta(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous [1, p. 52] with exponent β' in Ω, where $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on open set Ω.

For $x \in \Omega = (0, 2T)$, define

(2.16)
$$\delta(x) = \operatorname{dist}(x, \partial\Omega) = \begin{cases} x & 0 < x \leq T, \\ 2T - x & T < x < 2T, \end{cases}$$

and $\delta(x,y) = \min\{\delta(x), \delta(y)\}$. Then we have the following δ-dependent Hölder norms.

DEFINITION 2.1 (δ-dependent Hölder norms [2]). *Let $\beta = k+\beta' > 0$ and $\sigma \geq -\beta$, with k integer and $\beta' \in (0, 1]$. For $w \in C^\beta(\Omega) = C^{k,\beta'}(\Omega)$, define the seminorm*

$$|w|_\beta^{(\sigma)} = \sup_{x,y\in\Omega} \left(\delta(x,y)^{\beta+\sigma} \frac{|w^{(k)}(x) - w^{(k)}(y)|}{|x-y|^{\beta'}} \right).$$

For $\sigma > -1$, we also define the norm $\|\cdot\|_\beta^{(\sigma)}$ as follows: in case that $\sigma \geq 0$,

$$\|w\|_\beta^{(\sigma)} = \sum_{l=0}^{k} \sup_{x\in\Omega} \left(\delta(x)^{l+\sigma} |w^{(l)}(x)| \right) + |w|_\beta^{(\sigma)},$$

while for $-1 < \sigma < 0$,

$$\|w\|_\beta^{(\sigma)} = \|w\|_{C^{-\sigma}(\bar\Omega)} + \sum_{l=1}^{k} \sup_{x\in\Omega} \left(\delta(x)^{l+\sigma} |D^l w(x)| \right) + |w|_\beta^{(\sigma)}.$$

LEMMA 2.2. *[2, pp. 276-277] Assume $f \in L^\infty(\Omega)$. Let u be a solution of (1.1). Then, $u \in C^{\alpha/2}(\mathbb{R})$ and $u/\delta^{\alpha/2} \in C^\sigma(\bar\Omega)$ for some $\sigma \in (0, 1 - \alpha/2)$, with*

$$\|u\|_{C^{\alpha/2}(\mathbb{R})} \leq C\|f\|_{L^\infty(\Omega)} \quad and \quad \|u/\delta^{\alpha/2}\|_{C^\sigma(\bar\Omega)} \leq C\|f\|_{L^\infty(\Omega)},$$

for some positive constant $C = C(\Omega, \alpha)$.

In particular, if $f \in L^\infty(\Omega)$, then

(2.17)
$$|u(x)| \leq C\delta(x)^{\alpha/2} \quad \text{for all } x \in \Omega.$$

LEMMA 2.3. *[2, Proposition 1.4] Let Ω be a bounded domain, and $\beta > 0$ be such that neither β nor $\beta + \alpha$ is an integer. Let $f \in C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R})$ be a solution of (1.1). Then, $u \in C^{\beta+\alpha}(\Omega)$ and*

(2.18)
$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right),$$

for some positive constant $C = C(\Omega, \alpha, \beta)$.

By defination of δ-dependent Hölder norms, we have following result obviusly.

83   LEMMA 2.4. *Let $\beta = 4 - \alpha + \gamma$ with $0 < \gamma < \alpha - 1$. Assume that $f \in C^\beta(\Omega)$ be such*
84   *that $\|f\|_\beta^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R})$ be a solution of (1.1). Then for $l = 0, 1, 2, 3, 4$*

85   (2.19) $$|u^{(l)}(x)| \leq C\delta(x)^{\alpha/2 - l}.$$

86   *where $C = C(\Omega, \alpha, \beta, f)$.*

87   *Proof.* Our hypotheses imply that $2 < \beta < 3$, and $4 < \beta + \alpha < 5$. By Lemma 2.3,
88   we have

89                $$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)}\right).$$

90   And by Definition 2.1

91            $$\sum_{l=1}^{4} \sup_{x \in \Omega} \left(\delta(x)^{l-\alpha/2}|w^{(l)}(x)|\right) \leq C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)}\right),$$

92   which is desired result $l = 1, 2, 3, 4$. The case $l = 0$ is covered by (2.17).   □

93   LEMMA 2.5. *Let $\beta = 4 - \alpha + \gamma$ with $0 < \gamma < \alpha - 1$. Assume that $f \in C^\beta(\Omega)$ be*
94   *such that $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

95   (2.20) $$|f^{(l)}(x)| \leq C\delta(x)^{-l-\alpha/2}.$$

96   *where $C = C(\Omega, \alpha, \beta, f)$.*

97   *Proof.* By Definition 2.1, with $2 < \beta < 3$

                                                                            □

98            $$\sum_{l=0}^{2} \sup_{x \in \Omega} \left(\delta(x)^{l+\alpha/2}|f^{(l)}(x)|\right) \leq \|f\|_\beta^{(\alpha/2)}.$$

99   And in this paper bellow, without special instructions, we allways assume that
100   $\beta = 4 - \alpha + \gamma$ with $0 < \gamma < \alpha - 1$, $f \in C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and solution
101   $u \in C^{\alpha/2}(\mathbb{R})$.

102   **2.3. Main results.** Here we state our main results; the proof is deferred to
103   section 3 and section 4.
104   Let's denote $h = \frac{1}{N}$, we have

105   THEOREM 2.6 (Local Truncation Error). *If $u(x)$ is a solution of the equation*
106   *(1.1) where f satisfy the regular condition (??), then there exists $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$*
107   *and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$, such that the truncation error (2.11) satisfies*

108   (2.21) $$\begin{aligned} |\tau_i| := &|-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i)| \\ &\leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} \delta(x_i)^{-\alpha} + C_2(r-1)h^2(T - \delta(x_i) + h_N)^{1-\alpha} \end{aligned}$$

109

110   THEOREM 2.7 (Global Error). *The discrete equation (2.8) has sulotion and there*
111   *exists a positive constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$ such that the error be-*
112   *tween the numerial solution U with the exact solution $u(x_i)$ satisfies*

113   (2.22) $$\max_{1 \leq i \leq 2N-1} |u_i - u(x_i)| \leq Ch^{\min\{\frac{r\alpha}{2}, 2\}}$$

114   *That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.*

Remark 2.8. ...

3. Local Truncation Error. We shall first introduce some notations.

For convenience, we use the notation $\simeq$. That $x_1 \simeq y_1$, means that $c_1 x_1 \leq y_1 \leq C_1 x_1$ for some positive constants c_1 and C_1 that are independent of N.

And for $1 \leq j \leq 2N$, we define

(3.1)
$$y_j^\theta = (1 - \theta)x_{j-1} + \theta x_j, \quad \theta \in (0, 1)$$

Then we have

LEMMA 3.1. *For* $1 \leq i \leq 2N - 1$

(3.2)
$$h_i \simeq h_{i+1} \simeq h\delta(x_i)^{1-1/r}, \quad \delta(x_i) \simeq \delta(x_{i+1}) \simeq \delta(y_{i+1}^\theta)$$

Since $i^r - (i-1)^r \simeq i^{r-1}$, *for* $i \geq 1$, *where* $\theta \in (0, 1)$.

Meanwhile, let's define kernel functions

(3.3)
$$K_y(x) := \frac{|y - x|^{1-\alpha}}{\Gamma(2 - \alpha)}$$

3.1. Proof of Theorem 2.6. The truncation error of the discrete format can be written as

(3.4)
$$-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i) = -\kappa_\alpha (D_h^2 I^{2-\alpha} \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i))$$

$$= -\kappa_\alpha D_h^2 I^{2-\alpha} (\Pi_h u - u)(x_i) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i)$$

THEOREM 3.2. *There exits a constant* $C = C(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$ *such that*

(3.5)
$$\left| -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) \right| \leq Ch^2 \delta(x_i)^{-\alpha/2 - 2/r}$$

Proof. Since $f \in C^2(\Omega)$ and

(3.6)
$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha} u(x)) = f(x), \quad x \in \Omega,$$

we have $I^{2-\alpha} u \in C^4(\Omega)$. Therefore, using equation (A.2) of Lemma A.1, for $1 \leq i \leq 2N - 1$, we have

(3.7)
$$-\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i)$$

$$+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right)$$

By Lemma B.1, Lemma 2.5 and Lemma B.2, we get the result. $\qquad \square$

And now define

(3.8) $$R_i := D_h^2 I^{2-\alpha}(u - \Pi_h u)(x_i), \quad 1 \le i \le 2N - 1$$

We have some results about the estimate of R_i

THEOREM 3.3. *For $1 \le i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

(3.9) $$|R_i| \le \begin{cases} Ch^2 x_i^{-\alpha/2 - 2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

THEOREM 3.4. *For $N/2 \le i \le N$, there exists constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

(3.10) $$|R_i| \le C(r-1)h^2(T - x_i + h_N)^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

And for $N < i \le 2N - 1$, it is symmetric to the previous case.

Combine Theorem 3.2, Theorem 3.3 and Theorem 3.4, and for $1 \le i \le N$, we have

(3.11) $$h^2 x_i^{-\alpha/2 - 2/r} \le T^{\alpha/2 - 2/r} h^{\min\{\frac{r\alpha}{2}, 2\}} x_i^{-\alpha}$$

(3.12) $$h^{r\alpha/2+r} x_i^{-1-\alpha} \le T^{-1} h^{r\alpha/2} x_i^{-\alpha}$$

(3.13) $$h^r x_i^{-1}\ln(i) = T^{-1}\frac{\ln(i)}{i^r} \le T^{-1}, \quad h^r \ln(N) = \frac{\ln(N)}{N^r} \le 1$$

the proof of Theorem 2.6 completed.

We prove Theorem 3.3 and Theorem 3.4 in next subsections.

3.2. Outlines and Mesh Transport Functions. For convience, let's denote

DEFINITION 3.5.

(3.14) $$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y))\frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy, \quad i = 0, \cdots, 2N, \; j = 1, \cdots, 2N$$

Also, we denote vertical difference quotients of T_{ij}

(3.15) $$\begin{aligned} V_{ij} &= \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_i}T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right)T_{i,j} + \frac{1}{h_{i+1}}T_{i+1,j}\right) \\ &= \int_{x_{i-1}}^{x_i} (u(y) - \Pi_h u(y))D_h^2 K_y(x_i)dy \end{aligned}$$

And skew difference quotients of T_{ij}

(3.16) $$S_{ij} = \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_i}T_{i-1,j-1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right)T_{i,j} + \frac{1}{h_{i+1}}T_{i+1,j+1}\right)$$

then $R_i = \sum_{j=1}^{2N} V_{ij}$.

Our main idea is to depart R_i by V_{ij} and S_{ij}. For $3 \le i < N/2$, let's denote $k = \lceil \frac{i}{2} \rceil$, and take some suitable integer m, then

$$
\begin{aligned}
R_i &= \sum_{j=1}^{2N} V_{ij} \\
&= \sum_{j=1}^{k-1} V_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\
&\quad + \sum_{j=k+1}^{m-1} S_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,m} + T_{i-1,m-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,m} \right) \\
&\quad + \sum_{j=m+1}^{2N} V_{ij} \\
&= I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}
\tag{3.17}
$$

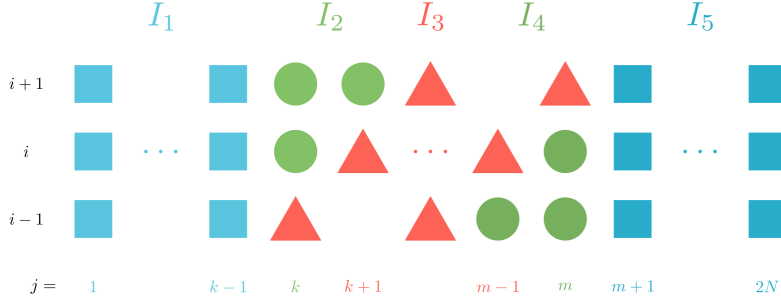FIG. 1. *The departure of R_i for $i \ge 3$*

and discuss $i = 1, 2$ separately, where

$$
R_1 = \sum_{i=1}^{3} V_{1,j} + \sum_{i=4}^{N} V_{i,j}, \quad R_2 = \sum_{i=1}^{4} V_{1,j} + \sum_{i=5}^{N} V_{i,j}
\tag{3.18}
$$

The difficulty for esitmating S_{ij} is that $T_{i-1,j-1}, T_{i,j}$ and $T_{i+1,j+1}$ have different integral region. We first make them normalized.

LEMMA 3.6. *For $y \in (x_{j-1}, x_j)$, we can rewrite $y = y_j^\theta$, from (3.14), and Lemma* A.2,

$$
\begin{aligned}
T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
&= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\
&= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
&\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta
\end{aligned}
\tag{3.19}
$$

where $\eta_{j1}^\theta \in (x_{j-1}, y_j^\theta), \eta_{j2}^\theta \in (y_j^\theta, x_j)$.

175   Since j changes with i at indices of elements in S_{ij} by (3.16), we create some
176 functions satisfy the property.

177

178   DEFINITION 3.7 (Mesh Transport Functions). *For $1 \leq i, j \leq 2N - 1$.*

179 (3.20) $$y_{i,j}(x) = \begin{cases} (x^{1/r} + Z_{j-i})^r & i < N, j < N \\ \dfrac{x^{1/r} - Z_i}{Z_1} h_N + x_N & i < N, j = N \\ 2T - (Z_{2N-(j-i)} - x^{1/r})^r & i < N, j > N \\ \left(\dfrac{Z_1}{h_N}(x - x_N) + Z_j \right)^r & i = N, j < N \\ x, & i = N, j = N \\ 2T - \left(\dfrac{Z_1}{h_N}(2T - x - x_N) + Z_{2N-j} \right)^r & i = N, j > N \\ (Z_{2N+j-i} - (2T - x)^{1/r})^r & i > N, j < N \\ \dfrac{Z_{2N-j} - (2T - x)^{1/r}}{Z_1} h_N + x_N & i > N, j = N \\ 2T - ((2T - x)^{1/r} - Z_{j-i})^r & i > N, j > N \end{cases}$$

180 *where $Z_j := T^{1/r} \frac{j}{N}$. And*

181 (3.21) $$h_{i,j}(x) = y_{i,j}(x) - y_{i,j-1}(x)$$

182

183 (3.22) $$y_{i,j}^\theta(x) = (1 - \theta) y_{i,j-1}(x) + \theta y_{i,j-1}(x), \quad \theta \in (0, 1)$$

184

185 (3.23) $$P_{i,j}^\theta(x) = (h_{i,j}(x))^3 \frac{|y_{i,j}^\theta(x) - x|^{1-\alpha}}{\Gamma(2 - \alpha)} u''(y_{i,j}^\theta(x))$$

186

187 (3.24) $$Q_{i,j;l}^\theta(x) = (h_{i,j}(x))^l \frac{|y_{i,j}^\theta(x) - x|^{1-\alpha}}{\Gamma(2 - \alpha)}$$

188   Obviously,

189 (3.25) $$y_{i,j}(x_{i-1}) = x_{j-1}, \quad y_{i,j}(x_i) = x_j, \quad y_{i,j}(x_{i+1}) = x_{j+1}$$
190 (3.26) $$h_{i,j}(x_{i-1}) = h_{j-1}, \quad h_{i,j}(x_i) = h_j, \quad h_{i,j}(x_{i+1}) = h_{j+1}$$
191 (3.27) $$y_{i,j}^\theta(x_{i-1}) = y_{j-1}^\theta, \quad y_{i,j}^\theta(x_i) = y_j^\theta, \quad y_{i,j}^\theta(x_{i+1}) = y_{j+1}^\theta$$

192   And now we can rewrite T_{ij}

     LEMMA 3.8.

193 (3.28) $$\begin{aligned} T_{ij} &= \int_0^1 -\frac{\theta(1 - \theta)}{2} P_{i,j}^\theta(x_i) d\theta \\ &+ \int_0^1 \frac{\theta(1 - \theta)}{3!} Q_{i,j;l}^\theta(x_i) \left[\theta^2 u'''(\eta_{j,1}^\theta) - (1 - \theta)^2 u'''(\eta_{j,2}^\theta) \right] d\theta \end{aligned}$$

194 *Immediately, we can see from (3.16) and Lemma 3.6 that For $1 \leq i \leq 2N - 1$,*
195 $2 \leq j \leq 2N - 1$,

(3.29)

$$
\begin{aligned}
S_{ij} = \int_0^1 & -\frac{\theta(1-\theta)}{2} D_h^2 P_{i,j}^\theta(x_i) d\theta \\
& + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j;4}^\theta(x_{i+1})u'''(\eta_{j+1,1}^\theta) - Q_{i,j;4}^\theta(x_i)u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\
& - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j;4}^\theta(x_i)u'''(\eta_{j,1}^\theta) - Q_{i,j;4}^\theta(x_{i-1})u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\
& - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j;4}^\theta(x_{i+1})u'''(\eta_{j+1,2}^\theta) - Q_{i,j;4}^\theta(x_i)u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\
& + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j;4}^\theta(x_i)u'''(\eta_{j,2}^\theta) - Q_{i,j;4}^\theta(x_{i-1})u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta
\end{aligned}
$$

197   We give some properties of mesh transport functions.

198   LEMMA 3.9. *For $2 \leq i, j \leq 2N - 2$ and $\xi \in (x_{i-1}, x_{i+1})$*

199 (3.30) $$\xi \simeq x_i, \quad \delta(y_{i,j}(\xi)) \simeq \delta(x_j), \quad h_{i,j}(\xi) \simeq h_j$$

200

201 (3.31) $$|y_{i,j}(\xi) - \xi| \simeq |x_j - x_i|, \quad |y_{i,j-1}(\xi) - \xi| \simeq |x_{j-1} - x_i|$$

202 *then*

203 (3.32) $$|y_{i,j}^\theta(\xi) - \xi| = (1-\theta)|y_{i,j-1}(\xi) - \xi| + \theta|y_{i,j}(\xi) - \xi| \simeq |y_j^\theta - x_i|$$

204 *since $y_{i,j-1}(\xi) - \xi$, $y_{i,j}(\xi) - \xi$ have the same sign (≥ 0 or ≤ 0)*

        LEMMA 3.10.

205 (3.33) $$y_{i,j}'(x) = \begin{cases} y_{i,j}^{1-1/r}(x)x^{1/r-1} & i < N, j < N \\ \dfrac{h_N}{rZ_1}x^{1/r-1} & i < N, j = N \\ (2T - y_{i,j}(x))^{1-1/r}x^{1/r-1} & i < N, j > N \\ y_{i,j}^{1-1/r}(x)\dfrac{rZ_1}{h_N} & i = N, j < N \\ 1 & i = N, j = N \end{cases}$$

206

207 (3.34) $$y_{i,j}''(x) = \frac{1-r}{r} \begin{cases} y_{i,j}^{1-2/r}(x)x^{1/r-2}Z_{j-i} & i < N, j < N \\ \dfrac{h_N}{rZ_1}x^{1/r-2} & i < N, j = N \\ (2T - y_{i,j}(x))^{1-2/r}x^{1/r-2}Z_{2N-j+i} & i < N, j > N \\ -y_{i,j}^{1-2/r}(x)\left(\dfrac{rZ_1}{h_N}\right)^2 & i = N, j < N \\ 0 & i = N, j = N \end{cases}$$

208   LEMMA 3.11. *For $2 \leq i \leq N, 2 \leq j \leq 2N - 2$, $\xi \in (x_{i-1}, x_{i+1})$*

209 (3.35) $$|h'_{i,j}(\xi)| \leq C(r-1)Z_1 x_i^{1/r-1} \delta(x_j)^{1-2/r} \leq C(r-1)h_j x_i^{1/r-1} \delta(x_j)^{-1/r}$$

210

211 (3.36) $$|(y_{i,j}(\xi) - \xi)'| \leq C x_i^{-1}|x_j - x_i|$$

212   *Proof.* From (3.21) and Lemma 3.10, we can see that

(3.37)
$$h'_{i,j}(x) = y'_{i,j}(x) - y'_{i,j-1}(x)$$

213
$$= \begin{cases} x^{1/r-1}(y_{i,j}^{1-1/r}(x) - y_{i,j-1}^{1-1/r}(x)) & i < N, j < N \\ x^{1/r-1}\left(\dfrac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(x)\right) & i < N, j = N \\ x^{1/r-1}\left((2T - y_{i,N+1}(x))^{1-1/r} - \dfrac{h_N}{rZ_1}\right) & i < N, j = N+1 \\ x^{1/r-1}\left((2T - y_{i,j}(x))^{1-1/r} - (2T - y_{i,j-1}(x))^{1-1/r}\right) & i < N, j > N+1 \\ \dfrac{rZ_1}{h_N}\left(y_{N,j}^{1-1/r}(x) - y_{N,j-1}^{1-1/r}(x)\right) & i = N, j < N \\ \dfrac{rZ_1}{h_N}\left(\dfrac{h_N}{rZ_1} - y_{N,N-1}^{1-1/r}(x)\right) & i = N, j = N \end{cases}$$

214 While for $2 \leq i \leq N$, if $2 \leq j < N$, $\xi \in (x_{i-1}, x_{i+1})$,

215 (3.38)
$$\begin{aligned} y_{i,j}^{1-1/r}(\xi) - y_{i,j-1}^{1-1/r}(\xi) &\leq x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r} \\ &= T^{1-1/r}N^{1-r}\left((j+1)^{r-1} - (j-2)^{r-1}\right) \\ &\leq CT^{1-1/r}(r-1)N^{1-r}j^{r-2} = C(r-1)Z_1 x_j^{1-2/r} \end{aligned}$$

216 if $j = N$, $\xi \in (x_{i-1}, x_{i+1})$, we have $y_{i,N-1}(\xi) \in (x_{N-2}, x_N)$. And

217 (3.39) $$\frac{h_N}{rZ_1} = T^{1-1/r}\frac{1 - (1-h)^r}{rh} = \eta^{1-1/r} \simeq x_N^{1-1/r}, \quad \eta \in (x_{N-1}, x_N)$$

218 Then

219 (3.40) $$|\frac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(\xi)| \leq x_N^{1-1/r} - x_{N-2}^{1-1/r} \simeq (r-1)Z_1 x_N^{1-2/r}$$

220 and similar for $j \geq N + 1$. Combine with Lemma 3.1, Lemma 3.9, $\eta \simeq x_N$, we get
221 the first result.
222   For the second estimate, we have

223 (3.41) $$(y_{i,j}(x) - x)' = y'_{i,j}(x) - 1$$

224 Then, for $2 \leq i < N$, if $2 \leq j < N$, $\xi \in (x_{i-1}, x_{i+1})$, by Lemma A.5

225 (3.42) $$\xi^{1/r}|y_{i,j}^{1-1/r}(\xi) - \xi^{1-1/r}| \leq |y_{i,j}(\xi) - \xi|$$

226 $j > N$ is symmetric to it, that is

227 (3.43)
$$\begin{aligned} \xi^{1/r}|(2T - y_{i,j}(\xi))^{1-1/r} - \xi^{1-1/r}| &\leq |2T - y_{i,j}(\xi) - \xi| \\ &\leq |2T - x_j - x_i| + |y_{i,j}(\xi) - x_j| + |\xi - x_i| \leq |2T - x_j - x_i| + 2h_N \\ &\leq |x_j - T| + |T - x_i| + 2h_N \leq 2|x_j - x_i| \end{aligned}$$

228   But if $j = N$, with (3.39) and Lemma A.5,

229   (3.44)
$$\eta^{1/r}|\frac{h_N}{rZ_1} - \xi^{1-1/r}| \le |\eta - \xi|, \quad \eta \in (x_{N-1}, x_N)$$
$$\le |x_N - x_i| + |h_N| + |h_{i+1}| \le 3|x_N - x_i|$$

230   For $i = N$, if $j < N$, similarly with (3.44),

231   (3.45)
$$\eta^{1/r}|y_{N,j}^{1-1/r}(\xi) - \frac{h_N}{rZ_1}| \le C|x_j - x_N|$$

232   And if $j = N$, it is obviously $\equiv 0$.

233   Similarly, by Lemma 3.10 and Lemma 3.9, we get the second result.   □

234   LEMMA 3.12. *For* $2 \le i \le N, 2 \le j \le 2N - 2$, $\xi \in (x_{i-1}, x_{i+1})$

235   (3.46)
$$|y_{i,j}''(\xi)| \le C(r-1) \begin{cases} x_j^{-1/r} x_i^{1/r-2}|x_j - x_i| & i < N, j < N \\ x_N^{1-1/r} x_i^{1/r-2} & i < N, j = N \\ \delta(x_j)^{1-2/r} x_i^{1/r-2} x_N^{1/r} & i < N, j > N \\ \delta(x_j)^{1-2/r} x_N^{2/r-2} & i = N, j \ne N \\ 0 & i = N, j = N \end{cases}$$

236   *And* $2 \le i \le N, 3 \le j \le 2N - 2$, $\xi \in (x_{i-1}, x_{i+1})$

    (3.47)

237    $$|h_{i,j}''(\xi)| \le C(r-1) \begin{cases} Z_1 x_i^{1/r-2} x_j^{-2/r}(|x_j - x_i| + x_j) & i < N, j < N \\ x_i^{1/r-2} x_N^{1-1/r} & i < N, j = N, N+1 \\ Z_1 x_i^{1/r-2} \delta(x_j)^{1-3/r} x_N^{1/r} & i < N, j > N+1 \\ Z_1 x_N^{2/r-2} \delta(x_j)^{1-3/r} & i = N, j < N \text{ or } j > N+1 \\ x_N^{-1} & i = N, j = N \end{cases}$$

238   *Proof.* Since by Lemma A.5, for $2 \le i, j < N$

239   (3.48)
$$x_j^{1-1/r}|Z_{j-i}| = x_j^{1-1/r}|x_j^{1/r} - x_i^{1/r}| \le |x_j - x_i|$$

240   and by (3.39), $\frac{h_N}{rZ_1} \simeq x_N^{1-1/r}$. And

241   (3.49)
$$Z_{2N-j+i} \le Z_{2N} = 2T^{1/r}$$

242   Then by Lemma 3.10 and Lemma 3.9, we get the first result.

243   For the second part, by Lemma 3.10

244   (3.50)
$$h_{i,j}''(x) = y_{i,j}''(x) - y_{i,j-1}''(x)$$

245   while for $2 \le i < N$, if $3 \le j < N$, $\xi \in (x_{i-1}, x_{i+1})$,

    (3.51)

246    $$y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1} = \left(y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi)\right)Z_{j-i} + y_{i,j-1}^{1-2/r}(\xi)Z_1$$

247   where $y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi) \simeq (r-2)Z_1 x_j^{1-3/r}$ similar with (3.38). Combine with
248   (3.48), we get

249   (3.52) $$|y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1}| \le CZ_1\left(|r-2|x_j^{-2/r}|x_j - x_i| + x_j^{1-2/r}\right)$$

if $j = N$,

(3.53) $|h''_{i,N}(x)| \leq |y''_{i,N}(x)| + |y''_{i,N-1}(x)| \leq C(r-1)x_i^{1/r-2}x_N^{1-1/r}$

similarly if $j = N + 1$.

However, if $j > N + 1$, similar with (3.51), we get

(3.54)
$(2T - y_{i,j}(\xi))^{1-2/r}Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)}$

$= \left((2T - y_{i,j}(\xi))^{1-2/r} - (2T - y_{i,j-1}(\xi))^{1-2/r}\right)Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r}Z_1$

thus,

(3.55)
$\left|(2T - y_{i,j}(\xi))^{1-2/r}Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)}\right|$

$\leq CZ_1\left(|r-2|(2T-x_j)^{1-3/r}x_N^{1/r} + (2T-x_j)^{1-2/r}\right) \leq CZ_1(2T-x_j)^{1-3/r}x_N^{1/r}$

For $i = N$, it's obvious. Combine with Lemma 3.10 and Lemma 3.9, we get the second result.   □

3.3. Proof of Theorems. Then we esrimate each part of (3.17). And We take $m = 2i$ for $3 \leq i < N/2$, and $m = N - \lceil N/2 \rceil + 1$ for $N/2 \leq i \leq N$.

For I_5

LEMMA 3.13. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

Case 1. For $1 \leq i < N/2$,

(3.56) $\sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| \leq Ch^2 x_i^{-\alpha/2-2/r}$

Case 2. For $1 \leq i < N/2$,

(3.57) $\sum_{j=N+1}^{2N} |V_{ij}| \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$

Case 3. For $N/2 \leq i \leq N$,

(3.58) $\sum_{j=N-\lceil\frac{N}{2}\rceil+2}^{2N} |V_{ij}| \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$

Proof. For i, j in each case, by (3.15), Lemma A.3 and Lemma B.3, we have

(3.59) $|V_{ij}| \leq Ch^2 \int_{x_{j-1}}^{x_j} \delta(y)^{\alpha/2-2/r}|y - x_i|^{-1-\alpha}dy$

For Case 1, with $x_i \simeq x_{2i}$,

$$\sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| \leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1}dy$$

(3.60)
$$= \frac{C}{\alpha/2 + 2/r}h^2(x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r})$$

$$\leq Ch^2 x_i^{-\alpha/2-2/r}$$

For Case 2 , by (3.15), Lemma A.3, Lemma B.3 and $y - x_i \simeq T$

$$|V_{ij}| \le Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2 - 2/r} dy$$

$$\sum_{j=N+1}^{2N-1} |V_{ij}| \le CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T - y)^{\alpha/2 - 2/r} dy$$

(3.61)
$$\le CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2 - 2/r + 1} T^{\alpha/2 - 2/r + 1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2 - 2/r + 1|} h_{2N}^{\alpha/2 - 2/r + 1}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

$$= \begin{cases} \frac{C}{\alpha/2 - 2/r + 1} T^{-\alpha/2 - 2/r} h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2 - 2/r + 1|} T^{-\alpha/2 - 2/r} h^{r\alpha/2 + r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

And by Lemma A.4

$$|V_{i,2N}| \le CT^{-1-\alpha} h_{2N}^{\alpha/2 + 1} = CT^{-\alpha/2} h^{r\alpha/2 + r}$$

Summarizes, we get the result. Similar for Case 3. □

For $i = 1, 2$.

LEMMA 3.14. *From* (3.18)*, by Lemma* B.4*, Lemma* 3.13 *Case 1 2, we get for* $i = 1, 2$

(3.62)
$$|R_i| \le Ch^2 x_i^{-\alpha/2 - 2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2 + r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

LEMMA 3.15. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i \le N, k = \lceil \frac{i}{2} \rceil$

(3.63)
$$|I_1| = |\sum_{j=1}^{k-1} V_{ij}| \le \begin{cases} Ch^2 x_i^{-\alpha/2 - 2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2 + r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

Proof. by (3.15), Lemma A.4 , Lemma B.3

(3.64) $\quad |V_{i1}| \le C \int_0^{x_1} x_1^{\alpha/2} |x_i - y|^{-1-\alpha} dy \simeq x_1^{\alpha/2 + 1} x_i^{-1-\alpha} = T^{\alpha/2 + 1} h^{r\alpha/2 + r} x_i^{-1-\alpha}$

For $2 \le j \le k - 1$, by Lemma A.3 and Lemma B.3 with $x_i - y \simeq x_i$, we have

(3.65)
$$|V_{ij}| \le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2 - 2/r} x_i^{-1-\alpha} dy$$

Therefore,

(3.66)
$$\sum_{j=2}^{k-1} |V_{ij}| \le Ch^{r\alpha/2 + r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2 - 2/r} dy$$

But $x_{\lceil \frac{i}{2} \rceil - 1} \le 2^{-r} x_i$, so we have

(3.67)
$$\int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2 - 2/r} dy \le \begin{cases} \frac{1}{\alpha/2 - 2/r + 1}(2^{-r} x_i)^{\alpha/2 - 2/r + 1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2 - 2/r + 1|} x_1^{\alpha/2 - 2/r + 1}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

Combine the results above, we get the lemma. □

LEMMA 3.16. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta + \alpha}^{(-\alpha/2)})$ *such that*
Case 1. For $3 \le i < N, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i - 1, N - 1\}$,

(3.68)
$$|D_h^2 P_{i,j}^\theta(x_i)| \le Ch_j^3 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2 - 4}$$

Case 2. For $N/2 \le i \le N, j = N, N+1$

(3.69) $|D_h^2 P_{i,j}^\theta(\xi)| \le Ch_j^3 |y_j^\theta - x_i|^{1-\alpha} + C(r-1)h_j^2 \left(|y_j^\theta - x_i|^{1-\alpha} + h_j |y_j^\theta - x_i|^{-\alpha} \right)$

Case 3. For $N/2 \le i \le N, N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil$,

(3.70)
$$|D_h^2 P_{i,j}^\theta(\xi)| \le Ch_j^3 \left(|y_j^\theta - x_i|^{1-\alpha} + (r-1)|y_j^\theta - x_i|^{-\alpha} \right)$$

Proof. Since $\text{sign}(y_{i,j}^\theta(\xi) - \xi)$ is independent of ξ, we can derivate it. Then by Lemma A.1

(3.71)
$$D_h^2 P_{i,j}^\theta(x_i) = P_{i,j}^{\theta}{}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1})$$

From (3.23), using Leibniz formula and chain rules, and Lemma 3.9, Lemma 3.10, Lemma 3.11, Lemma 3.12, **??**, Lemma 3.1
For every case, we have $x_i \simeq \delta(x_j)$, so we have

(3.72)
$$h_{i,j}(\xi) \le Ch_j, \quad |h_{i,j}'(\xi)| \le C(r-1)h_j x_i^{-1}$$

(3.73)
$$|y_{i,j}^\theta(\xi) - \xi| \le C|y_j^\theta - x_i|, \quad \left| (y_{i,j}^\theta(\xi) - x_i)' \right| \le C|y_j^\theta - x_i| x_i^{-1}$$

(3.74)
$$|u''(y_{i,j}^\theta(\xi))| \le Cx_i^{\alpha/2 - 2}, \quad \left| \left(u''(y_{i,j}^\theta(\xi)) \right)' \right| \le Cx_i^{\alpha/2 - 3}, \quad \left| \left(u''(y_{i,j}^\theta(\xi)) \right)'' \right| \le Cx_i^{\alpha/2 - 4}$$

By Lemma 3.12, we have
For Case 1,

(3.75)
$$|h_{i,j}''(\xi)| \le C(r-1)h_j x_i^{-2}, \quad \left| (y_{i,j}^\theta(\xi) - x_i)'' \right| \le C(r-1)|y_j^\theta - x_i| x_i^{-2}$$

For Case 2, since $x_i \simeq x_j \simeq T$

(3.76)
$$|h_{i,j}''(\xi)| \le C(r-1), \quad \left| (y_{i,j}^\theta(\xi) - x_i)'' \right| \le C(r-1)$$

For Case 3, since $x_i \simeq \delta(x_j) \simeq T$, we have

(3.77)
$$|h_{i,j}''(\xi)| \le C(r-1)h_j, \quad \left| (y_{i,j}^\theta(\xi) - x_i)'' \right| \le C(r-1)$$ □

Combine them, we get the result.

324

LEMMA 3.17. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $2 \leq i \leq N$, $2 \leq j \leq 2N - 2$,

(3.78)
$$\left| \frac{Q_{i,j;l}^{\theta}(x_{i+1}) u^{(l-1)}(\eta_{j+1}^{\theta}) - Q_{i,j;l}^{\theta}(x_i) u^{(l-1)}(\eta_j^{\theta})}{h_{i+1}} \right| \leq C h_j^l \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{-1} \delta(x_j)^{\alpha/2-l+1-1/r} (x_i^{1/r} + \delta(x_j)^{1/r})$$

And

(3.79)
$$\left| \frac{Q_{i,j;l}^{\theta}(x_i) u^{(l-1)}(\eta_j^{\theta}) - Q_{i,j;l}^{\theta}(x_{i-1}) u^{(l-1)}(\eta_{j-1}^{\theta})}{h_i} \right| \leq C h_j^l \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{-1} \delta(x_j)^{\alpha/2-l+1-1/r} (x_i^{1/r} + \delta(x_j)^{1/r})$$

where $\eta_j^{\theta} \in (x_{j-1}, x_j)$.

Proof.

(3.80)
$$\frac{Q_{i,j;l}^{\theta}(x_{i+1}) u'''(\eta_{j+1}^{\theta}) - Q_{i,j;l}^{\theta}(x_i) u'''(\eta_j^{\theta})}{h_{i+1}}$$
$$= \frac{Q_{i,j;l}^{\theta}(x_{i+1}) - Q_{i,j;l}^{\theta}(x_i)}{h_{i+1}} u'''(\eta_{j+1}^{\theta}) + Q_{i,j;l}^{\theta}(x_i) \frac{u'''(\eta_{j+1}^{\theta}) - u'''(\eta_j^{\theta})}{h_{i+1}}$$

Using mean value theorem

(3.81)
$$D_h Q_{i,j;l}^{\theta}(x_i) := \frac{Q_{i,j;l}^{\theta}(x_{i+1}) - Q_{i,j;l}^{\theta}(x_i)}{h_{i+1}} = Q_{i,j;l}^{\theta}{}'(\xi), \quad \xi \in (x_i, x_{i+1})$$

From (3.24) and Leibniz rule, by Lemma 3.9, Lemma 3.11 and Lemma 3.1, we have

(3.82)
$$|Q_{i,j;l}^{\theta}{}'(\xi)| \leq C h_j^l \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (x_i^{-1} + x_i^{1/r-1} \delta(x_j)^{-1/r})$$

(3.83)
$$Q_{i,j;l}^{\theta}(x_i) = h_j^l \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}$$

With Lemma 3.1 and **??**

$$|u^{(l-1)}(\eta_{j+1}^{\theta})| \leq C (\eta_{j+1}^{\theta})^{\alpha/2-l+1} \simeq \delta(x_j)^{\alpha/2-l+1}$$

and by Lemma 3.1

$$\frac{|u^{(l-1)}(\eta_{j+1}^{\theta}) - u^{(l-1)}(\eta_j^{\theta})|}{h_{i+1}} = |u^{(l)}(\eta)| \frac{\eta_{j+1}^{\theta} - \eta_j^{\theta}}{h_{i+1}}, \quad \eta \in (x_{j-1}, x_{j+1})$$
$$\leq C \delta(\eta)^{\alpha/2-l} \frac{x_{j+1} - x_{j-1}}{h_{i+1}} = C \delta(\eta)^{\alpha/2-l} \frac{h_{j+1} + h_j}{h_{i+1}}$$
$$\simeq x_i^{1/r-1} \delta(x_j)^{\alpha/2-l+1-1/r}$$

Combine the results above, we get the first term. While, the later is similar. □

343

LEMMA 3.18. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that*

Case 1. For $3 \leq i \leq N - 1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i - 1, N - 1\}$,

(3.84)
$$\begin{aligned}
|S_{ij}| &\leq Ch_j^2 x_i^{\alpha/2-4} \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\
&= Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy
\end{aligned}$$

Thus,

(3.85)
$$\sum_{j=k+1}^{\min\{2i-1,N-1\}} |S_{ij}| \leq Ch^2 x_i^{-\alpha/2-2/r}$$

Case 2. For $N/2 \leq i \leq N, j = N, N+1$, since $\theta(1-\theta)h_j \leq |y_j^\theta - x_i|$, we have

(3.86)
$$|S_{ij}| \leq C(h^3 + (r-1)h^2)(T - x_i + h_N)^{1-\alpha}$$

Case 3. For $N/2 \leq i \leq N, N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil$,

(3.87)
$$|S_{ij}| \leq Ch^2 \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy$$

Thus,

(3.88)
$$\sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} |S_{ij}| \leq Ch^2 + C(r-1)h^2(T - x_i + h_N)^{1-\alpha}$$

Expecially, for $i = N$, the estimate of $\lceil \frac{N}{2} \rceil + 1 \leq j \leq N - 1$ is symmetric with $N + 2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil$.

Proof. Since (3.29), by $x_i \simeq x_j$, Lemma 3.1, Lemma 3.16, Lemma 3.17

For Case 1, we get the first result immediately. While $x_k \simeq x_i \simeq x_{\min\{2i-1,N-1\}}$, we have

(3.89)
$$\begin{aligned}
\sum_{k+1}^{\min\{2i-1,N-1\}} |S_{ij}| &\leq Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_k}^{x_{\min\{2i-1,N-1\}}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
&\leq Ch^2 x_i^{\alpha/2-2-2/r} x_i^{2-\alpha} = Ch^2 x_i^{-\alpha/2-2/r}
\end{aligned}$$

For Case 2,

(3.90)
$$\begin{aligned}
|S_{ij}| &\leq C(h_j^3 + (r-1)h_j^2) \int_0^1 |y_j^\theta - x_i|^{1-\alpha} d\theta \\
&= C(h_j^2 + (r-1)h_j) \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} dy
\end{aligned}$$

however,

(3.91)
$$\begin{aligned}
\int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} dy &= \frac{1}{2-\alpha} \left((x_j - x_i)^{2-\alpha} - (x_{j-1} - x_i)^{2-\alpha} \right) \\
&\simeq h_N(|x_j - x_i + h_N|)^{1-\alpha}
\end{aligned}$$

For Case 3,

$$(3.92) \qquad |S_{ij}| \leq Ch_j^2 \int_0^1 \left(|y_j^\theta - x_i|^{1-\alpha} + (r-1)|y_j^\theta - x_i|^{-\alpha} \right) h_j d\theta$$

$$\leq Ch^2 \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy$$

Thus,

$$(3.93) \qquad \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} |S_{ij}| = Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil \frac{N}{2} \rceil}} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy \qquad \qquad \square$$

$$\leq Ch^2 (T^{2-\alpha} + (r-1)(T - x_i + h_N)^{1-\alpha})$$

Now we study I_2, I_4.

LEMMA 3.19. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that*

Case 1. For $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil$,

$$(3.94)$$
$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}}(T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,k} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

Case 2. For $3 \leq i < N/2$,

$$(3.95)$$
$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,2i} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

Case 3. For $N/2 \leq i \leq N, m = N - \lceil \frac{N}{2} \rceil + 1$,

$$(3.96) \qquad I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,m} + T_{i-1,m-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,m} \right) \leq Ch^2$$

Proof. In fact,

$$(3.97) \qquad \frac{1}{h_{i+1}}(T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,k}$$

$$= \frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}}(T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i})T_{i,k}$$

While, by Lemma A.3, Lemma B.3, Lemma 3.1 and $x_k \simeq x_i$, we have

$$(3.98) \qquad \frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) = \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y))D_h K_y(x_i)dy$$

$$\leq Ch_k^2 x_k^{\alpha/2-2} h_k |x_i - x_k|^{-\alpha} \leq Ch^2 x_i^{-\alpha/2-2/r} h_k$$

Thus,

$$(3.99) \qquad \frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} |T_{i+1,k} - T_{i,k}| \leq Ch^2 x_i^{-\alpha/2-2/r}$$

From (3.14), Lemma A.2 and normalization, we have

$$(3.100)$$
$$\frac{1}{h_{i+1}}(T_{i+1,k+1} - T_{i,k}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{Q_{i,k;3}^\theta(x_{i+1})u''(\eta_{k+1}^\theta) - Q_{i,k;3}^\theta(x_i)u''(\eta_k^\theta)}{h_{i+1}} d\theta$$

385   where $\eta_k^\theta \in (x_{k-1}, x_k)$ and $\eta_{k+1}^\theta \in (x_k, x_{k+1})$. And with Lemma 3.17, we can get

386   (3.101) $$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} |T_{i+1,k+1} - T_{i,k}| \leq Ch^2 x_i^{-\alpha/2 - 2/r}$$

387   For the third term, by Lemma 3.1, Lemma B.1, Lemma A.3 and $x_k \simeq x_i$, we have

388   (3.102) $$\frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,k} \leq h_i^{-3} h^2 x_i^{1-2/r} Ch_k^3 x_k^{\alpha/2-2} |x_k - x_i|^{1-\alpha}$$
$$\leq Ch^2 x_i^{-\alpha/2-2/r}$$

389   Summarizes, we have

390   (3.103) $$I_2 \leq Ch^2 x_i^{-\alpha/2-2/r}$$

391   The case for I_4 is similar.   □

392   Now we have study everr part to prove Theorem 3.3 and Theorem 3.4.
393   For $1 \leq i < N/2$, combine Lemma 3.14, Lemma 3.15, Lemma 3.19 Cases 1 2,
394   Lemma 3.18 Case 1, Lemma 3.13 Case 1 2, we get Theorem 3.3.
395   For $N/2 \leq i \leq N$, we take $m = 2N - \lceil \frac{N}{2} \rceil + 1$. And depart I_3 to three parts:

396   (3.104) $$I_3 = \sum_{j=k+1}^{m} S_{ij} = \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{m-1} S_{ij}$$

397   combine Lemma 3.15, Lemma 3.19 Cases 1 3, Lemma 3.18, Lemma 3.13 Case 1 2, we
398   get Theorem 3.4.

4. Convergence analysis.

4.1. Properties of some Matrices. Review subsection 2.1, we have got (2.10).

DEFINITION 4.1. *We call one matrix an M matrix , which means its entries are positive on major diagonal and nonpositive on others, and strictly diagonally dominant in rows.*

Now we have

LEMMA 4.2. *Matrix A defined by (2.12) where (2.13) is an M matrix. And there exists a constant $C_A = C(T, \alpha, r)$ such that*

407   (4.1) $$S_i := \sum_{j=1}^{2N-1} a_{ij} \geq C_A(x_i^{-\alpha} + (2T - x_i)^{-\alpha})$$

Proof. From (2.15), we have

(4.2)
409   $$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right)$$

Let

411   (4.3) $$g(x) = g_0(x) + g_{2N}(x)$$

412 where

$$g_0(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x-x_0|^{3-\alpha} - |x-x_1|^{3-\alpha}}{h_1}$$

$$g_{2N}(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N}-x|^{3-\alpha} - |x_{2N-1}-x|^{3-\alpha}}{h_{2N}}$$

415 Thus

$$-\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

417 Then

$$(4.4) \quad \begin{aligned} S_i &:= \sum_{j=1}^{2N-1} a_{ij} \\ &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &= D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i) \end{aligned}$$

419 When $i = 1$

$$(4.5) \quad \begin{aligned} D_h^2 g_0(x_1) &= \frac{2}{h_1 + h_2} \left(\frac{1}{h_2} g_0(x_2) - (\frac{1}{h_1} + \frac{1}{h_2}) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right) \\ &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1+h_2)^{3-\alpha}}{(h_1+h_2)h_1 h_2} \\ &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1+h_2)^{3-\alpha}}{(h_1+h_2)h_1^{1-\alpha} h_2} h_1^{-\alpha} \\ &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r-1)^{3-\alpha} + 2(2^r-1) - (2^r)^{3-\alpha}}{2^r(2^r-1)} h_1^{-\alpha} \end{aligned}$$

421 but

$$(4.6) \quad 1 + (2^r-1)^{3-\alpha} + 2(2^r-1) - (2^r)^{3-\alpha} > 0$$

423 While for $i \geq 2$

$$(4.7) \quad \begin{aligned} D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \\ &= -\kappa_\alpha \frac{|\xi-x_0|^{1-\alpha} - |\xi-x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1} \\ &= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} |\xi-\eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\ &\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha} \end{aligned}$$

425 So

$$(4.8) \quad \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g_0(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g_0(x_i) + \frac{1}{h_i} g_0(x_{i-1}) \right) \geq C x_i^{-\alpha}$$

symmetricly,

(4.9)
$$\frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}g_{2N}(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})g_{2N}(x_i) + \frac{1}{h_i}g_{2N}(x_{i-1})\right) \geq C(\alpha, r)(2T - x_i)^{-\alpha}$$ ∎

Let

(4.10)
$$G = \text{diag}(\delta(x_1), ..., \delta(x_{2N-1}))$$

Then

LEMMA 4.3. *The matrix $B := AG$, the major diagnal is positive, and nonpositive on others. And there is a constant $C_{AG}, C = C(\alpha, r)$ such that*

(4.11)
$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C_{AG}(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}) + C(T - \delta(x_i) + h_N)^{1-\alpha}$$

Proof.

$$b_{ij} = a_{ij}\delta(x_j) = -\kappa_\alpha \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}\tilde{a}_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})\tilde{a}_{i,j} + \frac{1}{h_i}\tilde{a}_{i-1,j}\right)\delta(x_j)$$

Since

(4.12)
$$\delta(x) \equiv \Pi_h \delta(x)$$

by (2.14) and (2.5), we have

$$\tilde{M}_i := \sum_{j=1}^{2N-1} \tilde{b}_{ij} := \sum_{j=1}^{2N-1} \tilde{a}_{ij}\delta(x_j)$$

(4.13)
$$= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)}\Pi_h \delta(y)dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)}\delta(y)dy$$

$$= \frac{-2}{\Gamma(4-\alpha)}|T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)}(x_i^{3-\alpha} + (2T - x_i)^{3-\alpha})$$

$$:= w(x_i) = p(x_i) + q(x_i)$$

Thus,

$$M_i := \sum_{j=1}^{2N-1} b_{ij} = \sum_{j=1}^{2N-1} a_{ij}\delta(x_j)$$

(4.14)
$$= -\kappa_\alpha \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}\tilde{M}_{i+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})\tilde{M}_i + \frac{1}{h_i}\tilde{M}_{i-1}\right)$$

$$= D_h^2(-\kappa_\alpha p)(x_i) - \kappa_\alpha D_h^2 q(x_i)$$

for $1 \leq i \leq N-1$,by Lemma A.1

(4.15)
$$D_h^2(-\kappa_\alpha p)(x_i) := -\kappa_\alpha \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}p(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})p(x_i) + \frac{1}{h_i}p(x_{i-1})\right)$$

$$= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}|T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1})$$

$$\geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}(T - \delta(x_i) + h_N)^{1-\alpha}$$

444

(4.16)
$$D_h^2(-\kappa_\alpha p)(x_N) := -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left(\frac{1}{h_{N+1}} p(x_{N+1}) - (\frac{1}{h_N} + \frac{1}{h_{N+1}})p(x_N) + \frac{1}{h_N}p(x_{N-1}) \right)$$

445
$$= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)h_N^2} h_N^{3-\alpha} = \frac{4\kappa_\alpha}{\Gamma(4-\alpha)}(T - \delta(x_N) + h_N)^{1-\alpha}$$

446   Symmetricly for $i \geq N$, we get

447   (4.17)
$$D_h^2(-\kappa_\alpha p)(x_i) \geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}(T - \delta(x_i) + h_N)^{1-\alpha}$$

448   Similarly, we can get

449   (4.18)
$$D_h^2 q(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} q(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})q(x_i) + \frac{1}{h_i}q(x_{i-1}) \right)$$
$$\leq \frac{2^{r(\alpha-1)+1}}{\Gamma(2-\alpha)}(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}), \quad i = 1, \cdots, 2N-1$$
□

450   So, we get the result.

451   Notice that

452   (4.19)
$$x_i^{-\alpha} \geq (2T)^{-1}x_i^{1-\alpha}$$

453   We can get

454   THEOREM 4.4. *There exists a real* $\lambda = \lambda(T, \alpha, r) > 0$ *and* $C = C(T, \alpha, r) > 0$
455   *such that* $B := A(\lambda I + G)$ *is an M matrix. And*

456   (4.20)
$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) + C(T - \delta(x_i) + h_N)^{1-\alpha}$$

457   *Proof.* By Lemma 4.2 with C_A and Lemma 4.3 with C_{AG} , it's sufficient to take
458    $\lambda = (C + 2TC_{AG})/C_A$, then

459   (4.21)
$$M_i \geq C \left((x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + (T - \delta(x_i) + h_N)^{1-\alpha} \right)$$
□

460   **4.2. Proof of Theorem 2.7.** For equation

461   (4.22)
$$AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1}U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1}U = F$$

462   which means

463   (4.23)
$$\sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + \delta(x_j)} = -\tau_i$$

464   where $\epsilon_i = u(x_i) - u_i$.
465   And if

466   (4.24)
$$|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}| = \max_{1 \leq i \leq 2N-1} |\frac{\epsilon_i}{\lambda + \delta(x_i)}|$$

467   Then, since $B = A(\lambda I + G)$ is an M matrix, it is Strictly diagonally dominant. Thus,

$$
\begin{aligned}
|\tau_{i_0}| = |\sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + \delta(x_j)}| \\
\geq b_{i_0,i_0}|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}| - \sum_{j\neq i_0} |b_{i_0,j}||\frac{\epsilon_j}{\lambda + \delta(x_j)}| \\
\geq b_{i_0,i_0}|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}| - \sum_{j\neq i_0} |b_{i_0,j}||\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}| \\
= \sum_{j=1}^{2N-1} b_{i_0,j}|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}| \\
= M_{i_0}|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}|
\end{aligned}
$$

468   (4.25)

469   By Theorem 2.6 and Theorem 4.4,

470   We kown that there exists constants $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$,

471   and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that

472   (4.26) $$|\frac{\epsilon_i}{\lambda + \delta(x_i)}| \leq |\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}| \leq C_1 h^{\min\{\frac{r\alpha}{2},2\}} + C_2(r-1)h^2$$

473   as $\lambda + \delta(x_i) \leq \lambda + T$

474   So, we can get

475   (4.27) $$|\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{r\alpha}{2},2\}}$$

476   The convergency has been proved.

477   Remarks:

5. Experimental results.

5.1. $f \equiv 1$.

5.2. $f = x^\gamma, \gamma < 0$. Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x) \in C^2(\Omega)$, there exists $\xi \in (x_{i-1}, x_{i+1})$ such that*

(A.1) $$D_h^2 g(x_i) = g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1})$$

And if $g(x) \in C^4(\Omega)$, then

(A.2)
$$D_h^2 g(x_i) = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i)$$
$$+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right)$$

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})$$

Subsitute them in the left side of (A.1), we have

$$D_h^2 g(x_i) = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (g(x_{i+1}) - g(x_i) + \frac{1}{h_i} (g(x_{i-1}) - g(x_i))) \right)$$
$$= \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2)$$

Now, using intermediate value theorem, there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy$$

Expecially,

(A.3)
$$\int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy = \frac{h_i^4}{4!} g''''(\eta_1)$$
$$\int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy = \frac{h_{i+1}^4}{4!} g''''(\eta_2)$$

where $\eta_1 \in (x_{i-1}, x_i), \eta_2 \in (x_i, x_{i+1})$. □

497   LEMMA A.2. *Denote* $y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \theta \in (0,1),$

498   (A.4) $$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in (x_{j-1}, x_j)$$

499

(A.5)

500   $$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

501   *where* $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j).$

502   *Proof.* By Taylor expansion, we have

503          $$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in (x_{j-1}, y_j^\theta)$$

504          $$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in (y_j^\theta, x_j)$$

505   Thus

506   $$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta) u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

507   The second equation is similar,

508          $$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

509          $$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

510   where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j).$ Thus

511   $$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

$\square$

512   LEMMA A.3. *By Lemma A.2,* **??** *and Lemma 3.1 , There is a constant* $C =$
513   $C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *for* $2 \le j \le 2N - 1,$

514   (A.6) $|u(y) - \Pi_h u(y)| \le h_j^2 \max\limits_{\xi \in [x_{j-1}, x_j]} |u''(\xi)| \le C h^2 \delta(y)^{\alpha/2 - 2/r}, \quad$ *for* $y \in (x_{j-1}, x_j)$

515   LEMMA A.4. *For* $x \in [x_{j-1}, x_j]$

516   (A.7) $$\begin{aligned} |u(x) - \Pi_h u(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^x u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_x^{x_j} u'(y)dy \right| \\ &\le \int_{x_{j-1}}^{x_j} |u'(y)|dy \end{aligned}$$

517   *If $x \in [0, x_1]$, with ??, we have*

518   (A.8) $$|u(x) - \Pi_h u(x)| \leq \int_0^{x_1} |u'(y)| dy \leq \int_0^{x_1} C y^{\alpha/2-1} dy \leq C \frac{2}{\alpha} x_1^{\alpha/2} = C \frac{2}{\alpha} h_1^{\alpha/2}$$

519   *Similarly, if $x \in [x_{2N-1}, 1]$, we have*

520   (A.9) $$|u(x) - \Pi_h u(x)| \leq C \frac{2}{\alpha} (2T - x_{2N-1})^{\alpha/2} = C \frac{2}{\alpha} h_{2N}^{\alpha/2}$$

LEMMA A.5.

521   (A.10) $b^{1-\theta}|a^\theta - b^\theta| \leq |a - b|$ (*also* $a^{1-\theta}|a^\theta - b^\theta| \leq |a - b|$), $a, b \geq 0$, $\theta \in [0, 1]$

522   **Appendix B. Proofs of some technical details.** Review that $h = \frac{1}{N}$ and the
523   defination of $\simeq$ in subsection 2.1

524

525   LEMMA B.1.   *There is a constant C such that for $i = 1, 2, \cdots, 2N - 1$*

526   (B.1) $$|h_{i+1} - h_i| \leq C h^2 \delta(x_i)^{1-2/r}$$

527   *Proof.* By (2.2),

    (B.2)

528    $$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \leq i \leq 2N-1 \end{cases}$$

529   Since

530   (B.3) $$(i+1)^r - 2i^r + (i-1)^r \simeq r(r-1)i^{r-2}, \quad \text{for } i \geq 1$$

531   We get the result.   □

532   LEMMA B.2.   *there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that*

533   (B.4) $$\frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right|$$
    $$\leq C h^2 \delta(x_i)^{-\alpha/2-2/r}$$

534   *Proof.* By Lemma 2.5, we have for $1 \leq i \leq N$

535   (B.5) $$\left| \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy \right| \leq \frac{\|f\|_\beta^{(\alpha/2)}}{3!} \int_{x_{i-1}}^{x_i} y^{-\alpha/2-2} (y - x_{i-1})^3 dy$$

536   For $i = 1$,

537    $$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2} (y - x_{i-1})^3 dy = \int_0^{x_1} y^{1-\alpha/2} dy = \frac{1}{2 - \alpha/2} x_1^{2-\alpha/2} = \frac{1}{2 - \alpha/2} x_1^{-\alpha/2-2} h_1^4$$

538   And for $2 \leq i \leq N$, since $x_i \simeq x_{i-1} \leq y \leq x_i$, we have

539    $$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2} (y - x_{i-1})^3 dy \simeq \int_{x_{i-1}}^{x_i} x_i^{-\alpha/2-2} (y - x_{i-1})^3 dy = \frac{1}{4!} x_i^{-\alpha/2-2} h_i^4$$

So for $1 \leq i \leq N$, we have

(B.6)
$$\left| \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy \right| \leq C x_i^{-\alpha/2-2} h_i^4$$

and similarly,

(B.7)
$$\left| \int_{x_i}^{x_{i+1}} f''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right| \leq C x_i^{-\alpha/2-2} h_{i+1}^4$$

Thus for $1 \leq i \leq N$, with Lemma 3.1 we have

(B.8)
$$\frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right|$$
$$\leq C x_i^{-\alpha/2-2} \frac{2}{h_i + h_{i+1}} (h_i^3 + h_{i+1}^3) \simeq x_i^{-\alpha/2-2} h_i^2 \simeq x_i^{-\alpha/2-2} h^2 x_i^{2-2/r}$$
$$= C h^2 x_i^{-\alpha/2-2/r}$$

It's symmetric for $N < i \leq 2N - 1$. □

LEMMA B.3. *There is a constant $C = C(\alpha, r)$ such that for all $1 \leq i \leq 2N - 1$, $1 \leq j \leq 2N$ s.t. $\min\{|j - i|, |j - 1 - i|\} \geq 2$ and $y \in [x_{j-1}, x_j]$, we have*

(B.9)
$$D_h K_y(x_i) \simeq |y - x_i|^{-\alpha}, \quad D_h^2 K_y(x_i) \simeq |y - x_i|^{-1-\alpha}$$

Proof. Sinec $y - x_{i-1}, y - x_i, y - x_{i+1}$ have the same sign, by mean value theorem and Lemma A.1,

$$D_h K_y(x_i) = \frac{|y - \xi|^{-\alpha}}{\Gamma(1 - \alpha)}, \quad \xi \in (x_i, x_{i+1})$$

$$D_h^2 K_y(x_i) = \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in (x_{i-1}, x_{i+1})$$

however, $|y - \xi| \simeq |y - x_i|$, we get the result. □

LEMMA B.4. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

(B.10)
$$\sum_{j=1}^{3} V_{1j} \leq C h^2 x_1^{-\alpha/2-2/r}$$

(B.11)
$$\sum_{j=1}^{4} V_{2j} \leq C h^2 x_2^{-\alpha/2-2/r}$$

Proof. For $0 \leq i \leq 3, 1 \leq j \leq 4$, by Lemma A.4, Lemma A.3 and (3.14)

(B.12)
$$T_{ij} \leq C x_1^{2-\alpha/2} \simeq h_1^2 h^2 x_1^{-\alpha/2-2/r} \simeq h_1^2 h^2 x_2^{-\alpha/2-2/r}$$

Therefore, by (3.15), we get the result. □

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