

A SECOND ORDER NUMERICAL METHODS FOR REISZ-FRACTIONAL ELLIPTIC EQUATION ON GRADED MESH*

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Abstract. This is an example SIAM L^AT_EX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

Key words. example, L^AT_EX

MSC codes. 68Q25, 68R10, 68U05

1. Introduction. The introduction introduces the context and summarizes the manuscript. It is importantly to clearly state the contributions of this piece of work.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$, suppose $f \in C^\beta(\Omega)$, $\beta > 4 - \alpha$, $\|f\|_\beta^{(\alpha/2)} < \infty$

$$(1.1) \quad \begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases}$$

where

$$(1.2) \quad (-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy$$

$$(1.3) \quad \kappa_\alpha = -\frac{1}{2 \cos(\alpha\pi/2)} > 0$$

and the solution $u \in C^{\alpha/2}(\Omega)$.

2. Regularity.

Remark 2.1. 1. $C^k(U)$ is the set of all k -times continuously differentiable functions on open set U .

2. $C^\beta(U)$ is the collection of function f which for any $V \subset\subset U$ $f|_V \in C^\beta(\bar{V})$.

THEOREM 2.2. If $f \in C^\beta(\Omega)$, $\beta > 2$ and $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$

$$(2.1) \quad |f^{(l)}(x)| \leq \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \leq T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \leq x < 2T \end{cases}$$

THEOREM 2.3 (Regularity up to the boundary [1]).

$$(2.2) \quad \|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right)$$

*Submitted to the editors DATE.

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30 **COROLLARY 2.4.** *Let u be a solution of (1.1) on Ω . Then, for any $x \in \Omega$ and*
 31 *$l = 0, 1, 2, 3, 4$*

$$32 \quad (2.3) \quad |u^{(l)}(x)| \leq \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases}$$

33 The paper is organized as follows. Our main results are in section 4, experimental
 34 results are in section 7. Readers would better see section 6 before section 5 to avoid
 35 details.

3. Numeric Format.

$$36 \quad (3.1) \quad x_i = \begin{cases} T \left(\frac{i}{N} \right)^r, & 0 \leq i \leq N \\ 2T - T \left(\frac{2N-i}{N} \right)^r, & N \leq i \leq 2N \end{cases}$$

37 where $r \geq 1$. And let

$$38 \quad (3.2) \quad h_j = x_j - x_{j-1}, \quad 1 \leq j \leq 2N$$

39 Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear
 40 function space.

$$41 \quad (3.3) \quad \phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \leq x \leq x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \leq x \leq x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

42 And then, we can approximate $u(x)$ with

$$43 \quad (3.4) \quad u_h(x) := \sum_{j=1}^{2N-1} u(x_j) \phi_j(x)$$

44 For convience, we denote

$$45 \quad (3.5) \quad I_h^{2-\alpha}(x_i) := \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u_h(y) dy$$

46 And now, we can approximate the operator (1.2) at x_i with

$$47 \quad (3.6) \quad \begin{aligned} D_h^\alpha u_h(x_i) &:= D_h^2 I_h^{2-\alpha}(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} I_h^{2-\alpha}(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) I_h^{2-\alpha}(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha}(x_{i+1}) \right) \end{aligned}$$

48 Finally, we approximate the equation (1.1) with

$$49 \quad (3.7) \quad -\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i), \quad 1 \leq i \leq 2N-1$$

50 The discrete equation (3.7) can be written in matrix form

$$51 \quad (3.8) \quad AU = F$$

where U is unknown, $F = (f(x_1), \dots, f(x_{2N-1}))$. The matrix A is constructed as follows: Since

(3.9)

$$\begin{aligned}
 I_h^{2-\alpha}(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u_h(y) dy \\
 &= \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u(x_j) \phi_j(y) dy \\
 &= \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y) dy \\
 &= \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\
 &=: \sum_{j=1}^{2N-1} \tilde{a}_{ij} u(x_j), \quad 0 \leq i \leq 2N
 \end{aligned}$$

Then, substitute in (3.6), we have

$$(3.10) \quad -\kappa_{\alpha} D_h^{\alpha} u_h(x_i) = \sum_{j=1}^{2N-1} a_{ij} u(x_j)$$

where

$$(3.11) \quad a_{ij} = -\kappa_{\alpha} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right)$$

4. Main results. Here we state our main results; the proof is deferred to section 5 and section 6.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 4.1 (Truncation Error). *If $f \in C^2(\Omega)$ and $\alpha \in (1, 2)$, and $u(x)$ is a solution of the equation (1.1), then there exists constants $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$, $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$, such that the truncation error of the discrete format satisfies*

$$\begin{aligned}
 \tau_i &:= |-\kappa_{\alpha} D_h^{\alpha} u_h(x_i) - f(x_i)| \\
 &\leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} \begin{cases} x_i^{-\alpha}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha}, & N < i \leq 2N-1 \end{cases} \\
 &\quad + C_2(r-1)h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \leq i \leq N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \leq 2N-1 \end{cases}
 \end{aligned}$$

THEOREM 4.2 (Convergence). *The discrete equation (3.7) has solution U , and there exists a positive constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ such that the error between the numerical solution U with the exact solution $u(x_i)$ satisfies*

$$(4.2) \quad \max_{1 \leq i \leq 2N-1} |U_i - u(x_i)| \leq C h^{\min\{\frac{r\alpha}{2}, 2\}}$$

That means the numerical method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

5. Proof of Theorem 4.1. For convience, let's denote

$$(5.1) \quad I^{2-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x-y|^{1-\alpha} u(y) dy$$

Then, the truncation error of the discrete format can be written as

$$(5.2) \quad \begin{aligned} -\kappa_{\alpha} D_h^{\alpha} u_h(x_i) - f(x_i) &= -\kappa_{\alpha} (D_h^2 I_h^{2-\alpha}(x_i) - \frac{d^2}{dx^2} I^{2-\alpha}(x_i)) \\ &= -\kappa_{\alpha} D_h^2 (I_h^{2-\alpha} - I^{2-\alpha})(x_i) - \kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \end{aligned}$$

5.1. Estimate of $-\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i)$.

THEOREM 5.1. *There exists a constant $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$ such that*

$$(5.3) \quad \left| -\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \right| \leq Ch^2 (x_i^{-\alpha/2-2/r} + (2T-x_i)^{-\alpha/2-2/r})$$

Proof. Since $f \in C^2(\Omega)$ and

$$(5.4) \quad \frac{d^2}{dx^2} (-\kappa_{\alpha} I^{2-\alpha}(x)) = f(x), \quad x \in \Omega,$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.3) of Lemma A.1, for $1 \leq i \leq 2N-1$, we have

$$(5.5) \quad -\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2))$$

where $\eta_1 \in [x_{i-1}, x_i]$, $\eta_2 \in [x_i, x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

$$(5.6) \quad \left| \frac{h_{i+1} - h_i}{3} f'(x_i) \right| \leq \frac{\|f\|_{\beta}^{(\alpha/2)}}{3} Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T-x_i)^{-\alpha/2-2/r}, & N < i \leq 2N-1 \end{cases}$$

2. See Proof 25, there is a constant $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$ such that

$$(5.7) \quad \begin{aligned} &\left| \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \right| \\ &\leq Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T-x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N-1 \end{cases} \end{aligned}$$

Summarizes, we get the result. \square

5.2. Estimate of R_i . Now, we study the first part of (5.2)

$$(5.8) \quad D_h^2 (I^{2-\alpha} - I_h^{2-\alpha})(x_i) = D_h^2 \left(\int_0^{2T} (u(y) - u_h(y)) \frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \right)$$

For convience, let's denote

$$(5.9) \quad T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy$$

93 And define

$$94 \quad (5.10) \quad \begin{aligned} R_i &:= D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \sum_{j=1}^{2N} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \end{aligned}$$

95 We have some results about the estimate of R_i

96 **THEOREM 5.2.** *For $1 \leq i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$97 \quad (5.11) \quad R_i \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

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99 **THEOREM 5.3.** *For $N/2 \leq i \leq N$, there exists constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$*
 100 *such that*

$$101 \quad (5.12) \quad R_i \leq C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

102 And for $N < i \leq 2N - 1$, it is symmetric to the previous case.

103 To prove these results, we need some utils. Also for simplicity, we denote

DEFINITION 5.4.

$$104 \quad (5.13) \quad S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right)$$

105 then

$$106 \quad (5.14) \quad R_i = \sum_{j=1}^{2N} S_{ij}$$

107 5.3. Proof of Theorem 5.2.

108 **LEMMA 5.5.** *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \leq$*
 109 *$i < N/2$,*

$$110 \quad (5.15) \quad \sum_{j=\max\{2i+1, i+3\}}^N S_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r}$$

111 *Proof.* For $\max\{2i+1, i+3\} \leq j \leq N$, by Lemma C.1 and Lemma C.2

$$112 \quad (5.16) \quad \begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\ &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy \end{aligned}$$

Therefore,

$$\begin{aligned}
 \sum_{j=\max\{2i+1, i+3\}}^N S_{ij} &\leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\
 &= \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\
 &\leq \frac{C}{\alpha/2+2/r} 2^{r(-\alpha/2-2/r)} h^2 x_i^{-\alpha/2-2/r}
 \end{aligned}
 \tag{5.17}$$

LEMMA 5.6. *Thert exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \leq i < N/2$,*

$$\sum_{j=N+1}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
 \tag{5.18}$$

Proof. For $1 \leq i < N/2, N+1 \leq j \leq 2N-1$, by equation (C.2) and Lemma C.2

$$\begin{aligned}
 S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\
 &\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2-2/r} y^{-1-\alpha} dy \\
 &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy \\
 \sum_{j=N+1}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T - y)^{\alpha/2-2/r} dy \\
 &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\
 &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r} h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r} h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
 \end{aligned}
 \tag{5.19}$$

And by Lemma A.3

$$S_{i,2N} \leq CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}$$

And when $\alpha/2 - 2/r + 1 \geq 0$,

$$h^{r\alpha/2+r} \leq h^2$$

Summarizes, we get the result. \square

For $i = 1, 2$.

129 LEMMA 5.7. *By Lemma C.5 , Lemma 5.5 and Lemma 5.6 we get*

$$\begin{aligned}
 R_1 &= \sum_{j=1}^3 S_{1j} + \sum_{j=4}^{2N} S_{1j} \\
 (5.20) \quad &\leq Ch^2 x_1^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
 \end{aligned}$$

131

$$\begin{aligned}
 R_2 &= \sum_{j=1}^4 S_{2j} + \sum_{j=5}^{2N} S_{2j} \\
 (5.21) \quad &\leq Ch^2 x_2^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
 \end{aligned}$$

133 For $3 \leq i < N/2$, we have a new separation of R_i , Let's denote $k = \lceil \frac{i}{2} \rceil$.

$$\begin{aligned}
 R_i &= \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &= \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \\
 (5.22) \quad &\quad + \sum_{j=k+1}^{2i-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
 &\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \\
 &\quad + \sum_{j=2i+1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &= I_1 + I_2 + I_3 + I_4 + I_5
 \end{aligned}$$

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136 LEMMA 5.8. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \leq$*
 137 *$i \leq N, k = \lceil \frac{i}{2} \rceil$*

$$(5.23) \quad |I_1| = \left| \sum_{j=1}^{k-1} S_{ij} \right| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

Proof. For $2 \leq j \leq k-1$, by Lemma C.1 and Lemma C.3

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\ &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 x_i^{-1-\alpha} \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} dy \end{aligned}$$

And by Lemma A.3, Lemma C.3

$$S_{i1} \leq C x_1^{\alpha/2} x_1 x_i^{-1-\alpha} = C x_1^{\alpha/2+1} x_i^{-1-\alpha} = C T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha}$$

Therefore,

$$\begin{aligned} I_1 &= \sum_{j=1}^{k-1} S_{ij} = S_{i1} + \sum_{j=2}^{k-1} S_{ij} \\ &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{k}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \\ &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \end{aligned}$$

But

$$\int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \leq \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2-2/r+1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 < 0 \end{cases}$$

So we have

$$I_1 \leq \begin{cases} \frac{C}{\alpha/2-2/r+1} h^2 x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2-2/r+1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} h^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2-2/r+1 < 0 \end{cases} \quad \square$$

DEFINITION 5.9. For convience, let's denote

$$V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right)$$

THEOREM 5.10. There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for

$$3 \leq i < N/2, k = \lceil \frac{i}{2} \rceil,$$

$$I_3 = \sum_{j=k+1}^{2i-1} V_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r}$$

To estimate V_{ij} , we need some preparations.

156 LEMMA 5.11. Denote $y_j^\theta = \theta x_{j-1} + (1 - \theta)x_j, \theta \in [0, 1]$, by Lemma A.2

$$\begin{aligned}
 T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
 &= \int_{x_{j-1}}^{x_j} -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
 &\quad + \frac{\theta(1-\theta)}{3!} h_j^3 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) dy_j^\theta \\
 &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
 &\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta
 \end{aligned}
 \tag{5.31}$$

158 where $\eta_{j1}^\theta \in [x_{j-1}, y_j^\theta], \eta_{j2}^\theta \in [y_j^\theta, x_j]$.

159 Now Let's construct a series of functions to represent T_{ij} .

DEFINITION 5.12.

$$y_{j-i}(x) = (x^{1/r} + Z_{j-i})^r, \quad Z_{j-i} = T^{1/r} \frac{j-i}{N}
 \tag{5.32}$$

161

$$y_{j-i}^\theta(x) = \theta y_{j-1-i}(x) + (1-\theta) y_{j-i}(x)
 \tag{5.33}$$

163

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x)
 \tag{5.34}$$

165 Now, we define

$$P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}
 \tag{5.35}$$

167

$$Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}
 \tag{5.36}$$

169 And now we can rewrite T_{ij}

170 LEMMA 5.13. For $2 \leq i \leq N, 2 \leq j \leq N$,

$$\begin{aligned}
 T_{ij} &= \int_0^1 -\frac{\theta(1-\theta)}{2} P_{j-i}^\theta(x_i) d\theta \\
 &\quad + \int_0^1 \frac{\theta(1-\theta)}{3!} (\theta^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j1}^\theta) - (1-\theta)^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j2}^\theta)) d\theta
 \end{aligned}
 \tag{5.37}$$

172 Immediately, we can see from (5.29) that

LEMMA 5.14. For $3 \leq i \leq N-1$, $3 \leq j \leq N-1$,

$$\begin{aligned}
 (5.38) \quad V_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
 &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^\theta(x_i) d\theta \\
 &\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\
 &\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\
 &\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,2}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\
 &\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta
 \end{aligned}$$

To estimate V_{ij} , we first estimate $D_h^2 P_{j-i}^\theta(x_i)$, but By Lemma A.1,

$$(5.39) \quad D_h^2 P_{j-i}^\theta(x_i) = P_{j-i}^{\theta''}(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

By Leibniz formula, we calculate and estimate the derivations of h_{j-i}^3 , $u''(y_{j-i}^\theta(x))$

and $\frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$ separately.

Firstly, we have

LEMMA 5.15. There exists a constant $C = C(T, r)$ such that For $3 \leq i \leq N-1$, $\lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$(5.40) \quad h_{j-i}^3(\xi) \leq C h^2 x_i^{2-2/r} h_j$$

$$(5.41) \quad (h_{j-i}^3(\xi))' \leq C(r-1) h^2 x_i^{1-2/r} h_j$$

$$(5.42) \quad (h_{j-i}^3(\xi))'' \leq C(r-1) h^2 x_i^{-2/r} h_j$$

The proof of this theorem see Lemma C.6 and Lemma C.7

Second,

LEMMA 5.16. There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $3 \leq i \leq N-1$, $\lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$(5.43) \quad u''(y_{j-i}^\theta(\xi)) \leq C x_i^{\alpha/2-2}$$

$$(5.44) \quad (u''(y_{j-i}^\theta(\xi)))' \leq C x_i^{\alpha/2-3}$$

$$(5.45) \quad (u''(y_{j-i}^\theta(\xi)))'' \leq C x_i^{\alpha/2-4}$$

The proof of this theorem see Proof 32

And Finally, we have

LEMMA 5.17. There exists a constant $C = C(T, \alpha, r)$ such that For $3 \leq i \leq N-1$, $1 \leq j \leq \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$(5.46) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C |y_j^\theta - x_i|^{1-\alpha}$$

$$(197) \quad (5.47) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C|y_j^\theta - x_i|^{1-\alpha}x_i^{-1}$$

$$(198) \quad (5.48) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C|y_j^\theta - x_i|^{1-\alpha}x_i^{-2}$$

(199) where $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

(200) The proof of this theorem see Proof 33

(201)

(202) **LEMMA 5.18.** *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For*
(203) $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\},$

$$(204) \quad (5.49) \quad D_h^2 P_{j-i}^\theta(x_i) \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j$$

(205) where $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

(206) *Proof.* Since

$$(207) \quad (5.50) \quad D_h^2 P_{j-i}^\theta(x_i) = P_{j-i}^{\theta''}(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

(208) From (5.35), using Leibniz formula and Lemma 5.15, Lemma 5.16 and Lemma 5.17□

(209)

(210) **LEMMA 5.19.** *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for*
(211) $3 \leq i < N, k = \lceil \frac{i}{2} \rceil.$
(212) *For $k \leq j \leq \min\{2i-1, N-1\},$*

$$(213) \quad (5.51) \quad \begin{aligned} & \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ & \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned}$$

(214) *And for $k+1 \leq j \leq \min\{2i, N\},$*

$$(215) \quad (5.52) \quad \begin{aligned} & \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta) - Q_{j-i}^\theta(x_{i-1})u'''(\eta_{j-1}^\theta)}{h_i} \right) \\ & \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned}$$

(216) where $\eta_j^\theta \in [x_{j-1}, x_j].$

(217) proof see Proof 34

(218)

(219) **LEMMA 5.20.** *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for*
(220) $3 \leq i < N, k = \lceil \frac{i}{2} \rceil, k+1 \leq j \leq \min\{2i-1, N-1\},$

$$(221) \quad (5.53) \quad \begin{aligned} V_{ij} & \leq Ch^2 \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j d\theta \\ & = Ch^2 \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \end{aligned}$$

Proof. Since Lemma 5.14, by Lemma 5.18 and Lemma 5.19, we get the result immediately. \square

Now we can prove Theorem 5.10 using Lemma 5.20, $k = \lceil \frac{i}{2} \rceil$

$$\begin{aligned}
 I_3 &= \sum_{k+1}^{2i-1} V_{ij} \leq Ch^2 \int_{x_k}^{x_{2i-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\
 &= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{2i-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\
 &\leq Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r}
 \end{aligned}$$

LEMMA 5.21.

$$D_h P_{j-i}^\theta(x_i) := \frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} = P_{j-i}^{\theta'}(\xi), \quad \xi \in [x_i, x_{i+1}]$$

Then, for $3 \leq i \leq N-1$, $k = \lceil \frac{i}{2} \rceil$,

$$D_h P_{k-i}^\theta(x_i) \leq Ch^2 x_i^{-\alpha/2-2/r} h_j$$

Proof. Using Leibniz formula, by Lemma 5.15, Lemma 5.16 and Lemma 5.17, we take $j = k+1, i = i+1$, we get

$$\begin{aligned}
 D_h P_{k-i}^\theta(x_i) &\leq Ch^2 x_{i+1}^{\alpha/2-2/r-1} |y_{k+1}^\theta - x_{i+1}|^{1-\alpha} h_{j+1} \\
 &\leq Ch^2 x_i^{\alpha/2-2/r-1} |y_k^\theta - x_i|^{1-\alpha} h_j \\
 &\leq Ch^2 x_i^{-\alpha/2-2/r} h_j
 \end{aligned}$$

LEMMA 5.22. There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \leq i < N, k = \lceil \frac{i}{2} \rceil$,

$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

And for $3 \leq i < N/2$,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

Proof. In fact,

$$\begin{aligned}
 &\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \\
 &= \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + \left(\frac{1}{h_{i+1}} - \frac{1}{h_i} \right) T_{i,k}
 \end{aligned}$$

242 While, by Lemma A.2

$$\begin{aligned}
 \frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) &= \int_{x_{k-1}}^{x_k} (u(y) - u_h(y)) \frac{|x_{i+1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_{i+1}\Gamma(2-\alpha)} dy \\
 &\leq \int_{x_{k-1}}^{x_k} h_k^2 u''(\eta) \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy \\
 &\leq Ch_k h^2 x_k^{2-2/r} x_{k-1}^{\alpha/2-2} |x_i - x_k|^{-\alpha} \\
 &\leq Ch_k h^2 x_i^{-\alpha/2-2/r}
 \end{aligned}
 \tag{5.61}$$

244 Thus,

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) \leq Ch^2 x_i^{-\alpha/2-2/r}
 \tag{5.62}$$

246 For

$$\begin{aligned}
 \frac{1}{h_{i+1}}(T_{i+1,k+1} - T_{i,k}) &= \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} d\theta \\
 &\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{Q_{k-i}^\theta(x_{i+1})u'''(\eta_{k+1,1}^\theta) - Q_{k-i}^\theta(x_i)u'''(\eta_{k,1}^\theta)}{h_{i+1}} d\theta \\
 &\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{Q_{k-i}^\theta(x_{i+1})u'''(\eta_{k+1,2}^\theta) - Q_{k-i}^\theta(x_i)u'''(\eta_{k,2}^\theta)}{h_{i+1}} d\theta
 \end{aligned}
 \tag{5.63}$$

248 And by Lemma 5.21

$$\frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} \leq Ch^2 x_i^{-\alpha/2-2/r} h_k
 \tag{5.64}$$

250 And with Lemma 5.19, we can get

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) \leq Ch^2 x_i^{-\alpha/2-2/r}
 \tag{5.65}$$

252 For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned}
 \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,k} &\leq h_i^{-3} h^2 x_i^{1-2/r} h_k Ch_k^2 x_{k-1}^{\alpha/2-2} |x_k - x_i|^{1-\alpha} \\
 &\leq Ch^2 x_i^{-\alpha/2-2/r}
 \end{aligned}
 \tag{5.66}$$

254 Summarizes, we have

$$I_2 \leq Ch^2 x_i^{-\alpha/2-2/r}
 \tag{5.67}$$

256 The case for I_4 is similar. □

257 Now combine Lemma 5.8, Lemma 5.22, Theorem 5.10, Lemma 5.5 and Lemma 5.6
 258 to get the final result.

259 For $3 \leq i < N/2$

$$\begin{aligned}
 R_i &= I_1 + I_2 + I_3 + I_4 + I_5 \\
 &\leq Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2 (x_i^{1-\alpha} \ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases}
 \end{aligned}
 \tag{5.68}$$

Combine with $i = 1, 2$, we get for $1 \leq i \leq N/2$

$$(5.69) \quad R_i \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases}$$

5.4. Proof of Theorem 5.3. For $N/2 \leq i < N, k = \lceil \frac{i}{2} \rceil$, we have

$$(5.70) \quad \begin{aligned} R_i &= \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &= \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \\ &\quad + \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right) \\ &\quad + \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 \end{aligned}$$

We have estimate I_1 in Lemma 5.8 and I_2 in Lemma 5.22. We can control I_3 in similar with Theorem 5.10 by Lemma 5.20 where $2i - 1 \geq N - 1$

LEMMA 5.23. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $N/2 \leq i < N, k = \lceil \frac{i}{2} \rceil$,*

$$(5.71) \quad \begin{aligned} I_3 &= \sum_{j=k+1}^{N-1} V_{ij} \leq Ch^2 \int_{x_k}^{x_{N-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{N-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\ &\leq Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned}$$

Let's study I_5 before I_4 .

$$(5.72) \quad I_5 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} V_{ij}$$

Similarly, Let's define a new series of functions

DEFINITION 5.24. *For $i < N, j \geq N$,*

$$(5.73) \quad y_{j-i}(x) = 2T - (Z_{2N-j+i} - x^{1/r})^r, \quad Z_{2N-j+i} = T^{1/r} \frac{2N-j+i}{N}$$

275

276 (5.74)
$$y_{j-i}'(x) = (2T - y_{j-i}(x))^{1-1/r} x^{1/r-1}$$

277 (5.75)
$$y_{j-i}''(x) = \frac{1-r}{r} (2T - y_{j-i}(x))^{1-2/r} x^{1/r-2} Z_{2N-j+i}$$

278 (5.76)

279

280 (5.77)
$$y_{j-i}^\theta(x) = \theta y_{j-i-1}(x) + (1-\theta) y_{j-i}(x)$$

281

282 (5.78)
$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x)$$

283

284 (5.79)
$$P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

285

286 (5.80)
$$Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

 287 Now we have, for $i < N, j \geq N+2$,

(5.81)

$$\begin{aligned} V_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^\theta(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,2}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta \end{aligned}$$

288

289 Similarly, we first estimate

290 (5.82)
$$D_h^2 P_{j-i}^\theta(\xi) = P_{j-i}^{\theta'}(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

 291 Combine lemmas [Lemma C.8](#), [Lemma C.9](#) and [Lemma C.10](#), we have

 292 **LEMMA 5.25.** *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For*

 293 *$N/2 \leq i < N, N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1, \xi \in [x_{i-1}, x_{i+1}]$, we have*

294 (5.83)
$$\begin{aligned} |P_{j-i}^{\theta''}(\xi)| &\leq Ch_j h^2 (|y_j^\theta - x_i|^{1-\alpha} \\ &\quad + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N) \\ &\quad + |y_j^\theta - x_i|^{-1-\alpha} (|2T - x_i - y_j^\theta| + h_N)^2 \\ &\quad + (r-1) |y_j^\theta - x_i|^{-\alpha}) \end{aligned}$$

And

LEMMA 5.26. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i < N$, $\xi \in [x_{i-1}, x_{i+1}]$, we have for $N+1 \leq j \leq 2N - \lceil \frac{N}{2} \rceil$*

$$(5.84) \quad \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ \leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N))$$

for $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$

$$(5.85) \quad \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta) - Q_{j-i}^\theta(x_{i-1})u'''(\eta_{j-1}^\theta)}{h_{i+1}} \right) \\ \leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N))$$

The proof see Proof 38.

Combine (5.81), Lemma 5.25 and Lemma 5.26, we have

THEOREM 5.27. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i < N$, $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$*

$$(5.86) \quad V_{ij} \leq Ch^2 \int_{x_{j-1}}^{x_j} (|y - x_i|^{1-\alpha} \\ + |y - x_i|^{-\alpha} (|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha} (|2T - x_i - y| + h_N)^2 \\ + (r-1)|y - x_i|^{-\alpha}) dy$$

We can estimate I_5 Now.

THEOREM 5.28. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i < N$, we have*

$$(5.87) \quad I_5 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \leq Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha}$$

Proof.

$$(5.88) \quad I_5 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \\ \leq Ch^2 \int_{x_{N+1}}^{x_{2N-i}} + \int_{x_{2N-i}}^{x_{2N - \lceil \frac{N}{2} \rceil}} (|y - x_i|^{1-\alpha} \\ + |y - x_i|^{-\alpha} (|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha} (|2T - x_i - y| + h_N)^2 \\ + (r-1)|y - x_i|^{-\alpha}) dy \\ = J_1 + J_2$$

While $x_{N+1} \leq y \leq x_{2N-i} = 2T - x_i$,

$$(5.89) \quad T - x_{i-1} \leq x_{N+1} - x_i \leq y - x_i \leq x_{2N-i} - x_i \leq 2(T - x_{i-1})$$

313 and

$$314 \quad (5.90) \quad 2T - x_i - y + h_N \leq 2T - x_i - x_{N+1} + h_N = T - x_i \leq T - x_{i-1}$$

315 So

$$\begin{aligned} 316 \quad (5.91) \quad J_1 &\leq Ch^2(x_{2N-i} - x_{N+1})(|T - x_{i-1}|^{1-\alpha} + (r-1)|T - x_{i-1}|^{-\alpha}) \\ &\leq Ch^2(|T - x_{i-1}|^{2-\alpha} + (r-1)|T - x_{i-1}|^{1-\alpha}) \\ &\leq Ch^2T^{2-\alpha} + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned}$$

317 Otherwise, when $x_{2N-i} \leq y \leq x_{2N-\lceil \frac{N}{2} \rceil}$

$$318 \quad (5.92) \quad x_i + y - 2T + h_N \leq y - x_i$$

319

$$\begin{aligned} 320 \quad (5.93) \quad J_2 &\leq Ch^2 \int_{x_{2N-i}}^{(2-2^{-r})T} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} \\ &\leq Ch^2(T^{2-\alpha} + (r-1)|x_{2N-i} - x_i|^{1-\alpha}) \\ &= Ch^2 + C(r-1)h^2|T - x_i|^{1-\alpha} \leq Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned}$$

321 Summarizes two cases, we get the result. \square

For I_4 , we have

THEOREM 5.29. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that, for $N/2 \leq i \leq N-1$*

$$(5.94) \quad \begin{aligned} V_{iN} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1, N+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, N} + \frac{1}{h_i} T_{i-1, N-1} \right) \\ &\leq Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \end{aligned}$$

Proof. We use the similar skill in the last section, but more complicated. for $j = N$, Let

$$(5.95) \quad y_{i \rightarrow N-1}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r} \frac{N-1-i}{N}$$

$$(5.96) \quad y_{i \rightarrow N}(x) = \frac{x^{1/r} - Z_i}{Z_1} h_N + T, \quad Z_i = T^{1/r} \frac{i}{N}, x_N = T$$

and

$$(5.97) \quad y_{i \rightarrow N+1}(x) = 2T - (Z_{N-1+i} - x^{1/r})^r, \quad Z_{N-1+i} = T^{1/r} \frac{N-1+i}{N}$$

Thus,

$$y_{i \rightarrow N-1}(x_{i-1}) = x_{N-2}, \quad y_{i \rightarrow N}(x_i) = x_{N-1}, \quad y_{i \rightarrow N}(x_{i+1}) = x_N$$

$$y_{i \rightarrow N}(x_{i-1}) = x_{N-1}, \quad y_{i \rightarrow N}(x_i) = x_N, \quad y_{i \rightarrow N}(x_{i+1}) = x_{N+1}$$

$$y_{i \rightarrow N-1}(x_{i-1}) = x_N, \quad y_{i \rightarrow N}(x_i) = x_{N+1}, \quad y_{i \rightarrow N}(x_{i+1}) = x_{N+2}$$

Then, define

$$(5.98) \quad y_{i \rightarrow N}^\theta(x) = \theta y_{i \rightarrow N-1}(x) + (1-\theta) y_{i \rightarrow N}(x)$$

$$(5.99) \quad y_{i \rightarrow N+1}^\theta(x) = \theta y_{i \rightarrow N}(x) + (1-\theta) y_{i \rightarrow N+1}(x)$$

$$(5.100) \quad h_{i \rightarrow N}(x) = y_{i \rightarrow N}(x) - y_{i \rightarrow N-1}(x)$$

$$(5.101) \quad h_{i \rightarrow N+1}(x) = y_{i \rightarrow N+1}(x) - y_{i \rightarrow N}(x)$$

We have

$$(5.102) \quad y_{i \rightarrow N-1}'(x) = y_{i \rightarrow N-1}^{1-1/r}(x) x^{1/r-1}$$

$$(5.103) \quad y_{i \rightarrow N-1}''(x) = \frac{1-r}{r} y_{i \rightarrow N-1}^{1-2/r}(x) x^{1/r-2} Z_{N-1-i}$$

$$(5.104) \quad y_{i \rightarrow N}'(x) = \frac{1}{r} \frac{h_N}{Z_1} x^{1/r-1}$$

$$(5.105) \quad y_{i \rightarrow N}''(x) = \frac{1-r}{r^2} \frac{h_N}{Z_1} x^{1/r-2}$$

$$(5.106) \quad y_{i \rightarrow N+1}'(x) = (2T - y_{i \rightarrow N+1}(x))^{1-1/r} x^{1/r-1}$$

$$(5.107) \quad y_{i \rightarrow N+1}''(x) = \frac{1-r}{r} (2T - y_{i \rightarrow N+1}(x))^{1-2/r} x^{1/r-2} Z_{N-1+i}$$

350

$$351 \quad (5.108) \quad P_{i \rightarrow N}^\theta(x) = (h_{i \rightarrow N}(x))^3 \frac{|y_{i \rightarrow N}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y_{i \rightarrow N}^\theta(x))$$

$$352 \quad (5.109) \quad P_{i \rightarrow N+1}^\theta(x) = (h_{i \rightarrow N+1}(x))^3 \frac{|y_{i \rightarrow N+1}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y_{i \rightarrow N+1}^\theta(x))$$

$$353 \quad (5.110) \quad Q_{i \rightarrow N}^\theta(x) = (h_{i \rightarrow N}(x))^4 \frac{|y_{i \rightarrow N}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$354 \quad (5.111) \quad Q_{i \rightarrow N+1}^\theta(x) = (h_{i \rightarrow N+1}(x))^4 \frac{|y_{i \rightarrow N+1}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

 355 Similar with Lemma 5.13, we can get for $l = -1, 0, 1$,

$$356 \quad (5.112) \quad T_{i+l, N+l} = \int_0^1 -\frac{\theta(1-\theta)}{2} P_{i \rightarrow N}^\theta(x_{i+l}) d\theta \\ + \int_0^1 \frac{\theta(1-\theta)}{3!} Q_{i \rightarrow N}^\theta(x_{i+l}) (\theta^2 u'''(\eta_{N+l,1}^\theta) - (1-\theta)^2 u'''(\eta_{N+l,2}^\theta)) d\theta$$

357

$$(5.113) \quad T_{i+l, N+1+l} = \int_0^1 -\frac{\theta(1-\theta)}{2} P_{i \rightarrow N+1}^\theta(x_{i+l}) d\theta \\ + \int_0^1 \frac{\theta(1-\theta)}{3!} Q_{i \rightarrow N+1}^\theta(x_{i+l}) (\theta^2 u'''(\eta_{N+1+l,1}^\theta) - (1-\theta)^2 u'''(\eta_{N+1+l,2}^\theta)) d\theta$$

358

359 So we have

$$(5.114) \quad V_{i,N} = \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{i \rightarrow N}^\theta(x_i) d\theta \\ + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_{i+1}) u'''(\eta_{N+1,1}^\theta) - Q_{i \rightarrow N}^\theta(x_i) u'''(\eta_{N,1}^\theta)}{h_{i+1}} \right) d\theta \\ - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_i) u'''(\eta_{N,1}^\theta) - Q_{i \rightarrow N}^\theta(x_{i-1}) u'''(\eta_{N-1,1}^\theta)}{h_i} \right) d\theta \\ - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_{i+1}) u'''(\eta_{N+1,2}^\theta) - Q_{i \rightarrow N}^\theta(x_i) u'''(\eta_{N,2}^\theta)}{h_{i+1}} \right) d\theta \\ + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_i) u'''(\eta_{N,2}^\theta) - Q_{i \rightarrow N}^\theta(x_{i-1}) u'''(\eta_{N-1,2}^\theta)}{h_i} \right) d\theta$$

360

 361 $N+1$ is similar.

 362 We estimate $D_h^2 P_{i \rightarrow N}^\theta(x_i) = P_{i \rightarrow N}^{\theta''}(\xi)$, $\xi \in [x_{i-1}, x_{i+1}]$,

363

LEMMA 5.30.

$$364 \quad (5.115) \quad h_{i \rightarrow N}^3(\xi) \leq Ch_N^3 \leq Ch^3$$

$$365 \quad (5.116) \quad h_{i \rightarrow N+1}^3(\xi) \leq Ch_N^3 \leq Ch^3$$

$$(h_{i \rightarrow N}^3(\xi))' \leq C(r-1)h_N^2 h \leq C(r-1)h^3 \quad (5.117)$$

$$(h_{i \rightarrow N+1}^3(\xi))' \leq C(r-1)h_N^2 h \leq C(r-1)h^3 \quad (5.118)$$

$$(h_{i \rightarrow N}^3(\xi))'' \leq C(r-1)h^2 \quad (5.119)$$

$$(h_{i \rightarrow N+1}^3(\xi))'' \leq C(r-1)h^2 \quad (5.120)$$

Proof.

$$h_{i \rightarrow N}(\xi) \leq 2h_N, \quad h_{i \rightarrow N+1}(\xi) \leq 2h_N \quad (5.121)$$

371

$$\begin{aligned} (h_{i \rightarrow N}^l(\xi))' &= lh_{i \rightarrow N}^{l-1}(\xi)(y_{i \rightarrow N}'(\xi) - y_{i \rightarrow N-1}'(\xi)) \\ &= lh_{i \rightarrow N}^{l-1}(\xi)x_i^{1/r-1}(\frac{1}{r}\frac{h_N}{Z_1} - y_{i \rightarrow N-1}^{1-1/r}(\xi)) \end{aligned} \quad (5.122)$$

373 while

(5.123)

$$\begin{aligned} |\frac{1}{r}\frac{h_N}{Z_1} - y_{i \rightarrow N-1}^{1-1/r}(\xi)| &= |\frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r}|(\frac{N-t}{N})^{r-1} - (\frac{N-s}{N})^{r-1}| \quad t \in [0, 1], s \in [0, 2] \\ &\leq T^{1-1/r}|1 - (\frac{N-2}{N})^{r-1}| \leq CT^{1-1/r}(r-1)\frac{2}{N} \end{aligned}$$

374

375 Thus,

$$(h_{i \rightarrow N}^l(\xi))' \leq C(r-1)h_N^{l-1}x_i^{1/r-1}h \quad (5.124)$$

$$\begin{aligned} (h_{i \rightarrow N+1}^l(\xi))' &= lh_{i \rightarrow N+1}^{l-1}(\xi)(y_{i \rightarrow N+1}'(\xi) - y_{i \rightarrow N}'(\xi)) \\ &= lh_{i \rightarrow N+1}^{l-1}(\xi)x_i^{1/r-1}((2T - y_{i \rightarrow N+1}(\xi))^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1}) \end{aligned} \quad (5.125)$$

378 Similarly,

(5.126)

$$\begin{aligned} |(2T - y_{i \rightarrow N+1})^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1}| &= |\eta^{1-1/r} - \frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r}|(\frac{N-s}{N})^{r-1} - (\frac{N-t}{N})^{r-1}| \quad t \in [0, 1], s \in [0, 2] \\ &\leq T^{1-1/r}|(\frac{N-2}{N})^{r-1} - 1| \leq CT^{1-1/r}(r-1)\frac{2}{N} \end{aligned}$$

379

380 And

(5.127)

$$\begin{aligned} (h_{i \rightarrow N}^3(\xi))'' &= 3h_{i \rightarrow N}^2(\xi)h_{i \rightarrow N}''(\xi) + 6h_{i \rightarrow N}(\xi)(h_{i \rightarrow N}'(\xi))^2 \\ &\leq Ch_N^2\frac{1-r}{r}x_i^{1/r-2}(\frac{1}{r}\frac{h_N}{Z_1} - y_{i \rightarrow N-1}^{1-2/r}(\xi)Z_{N-1-i}) + Ch_N(r-1)^2h^2x_i^{2/r-2} \end{aligned}$$

381

$$|\frac{h_N}{rZ_1} - y_{i \rightarrow N-1}^{1-2/r}(\xi)Z_{N-1-i}| \leq T^{1-1/r} + Cx_N^{1-2/r}x_N^{1/r} = CT^{1-1/r} \quad (5.128)$$

382

383 So

$$384 \quad (5.128) \quad \begin{aligned} (h_{i \rightarrow N}^3(\xi))'' &\leq C h_N^2 \frac{1-r}{r} x_i^{1/r-2} + C(r-1)^2 h_N x_i^{2/r-2} h^2 \\ &\leq C(r-1) h_N^2 x_i^{1/r-1} \end{aligned}$$

385 $h_{i \rightarrow N+1}^3(\xi)$ is similar. □

LEMMA 5.31.

$$386 \quad (5.129) \quad u''(y_{i \rightarrow N}^\theta(\xi)) \leq C x_{N-2}^{-\alpha/2-2} \leq C$$

$$387 \quad (5.130) \quad (u''(y_{i \rightarrow N}^\theta(\xi)))' \leq C$$

$$388 \quad (5.131) \quad (u''(y_{i \rightarrow N}^\theta(\xi)))'' \leq C$$

Proof.

$$389 \quad (5.132) \quad \begin{aligned} (u''(y_{i \rightarrow N}^\theta(\xi)))' &= u'''(y_{i \rightarrow N}^\theta(\xi)) y_{i \rightarrow N}^{\theta'}(\xi) \\ &\leq C(\theta y_{i \rightarrow N-1}'(\xi) + (1-\theta) y_{i \rightarrow N}'(\xi)) \\ &\leq C x_i^{1/r-1} (\theta y_{i \rightarrow N-1}^{1-1/r}(\xi) + (1-\theta) \frac{h_N}{r Z_1}) \\ &\leq C x_i^{1/r-1} x_N^{1-1/r} \end{aligned}$$

390 And

(5.133)

$$391 \quad \begin{aligned} (u''(y_{i \rightarrow N}^\theta(\xi)))'' &= u''''(y_{i \rightarrow N}^\theta(\xi)) (y_{i \rightarrow N}^{\theta'}(\xi))^2 + u'''(y_{i \rightarrow N}^\theta(\xi)) y_{i \rightarrow N}^{\theta''}(\xi) \\ &\leq C x_i^{2/r-2} x_N^{2-2/r} + C \frac{r-1}{r} x_i^{1/r-2} (\theta x_N^{1-2/r} Z_{N-1-i} + (1-\theta) \frac{h_N}{r Z_1}) \\ &\leq C x_i^{2/r-2} + C(r-1) x_i^{1/r-2} T^{1-1/r} \end{aligned} \quad \square$$

LEMMA 5.32.

$$392 \quad (5.134) \quad |y_{i \rightarrow N}^\theta(\xi) - \xi|^{1-\alpha} \leq C |y_N^\theta - x_i|^{1-\alpha}$$

$$393 \quad (5.135) \quad (|y_{i \rightarrow N}^\theta(\xi) - \xi|^{1-\alpha})' \leq C |y_N^\theta - x_i|^{1-\alpha}$$

$$394 \quad (5.136) \quad (|y_{i \rightarrow N}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C(r-1) |y_N^\theta - x_i|^{-\alpha} + |y_N^\theta - x_i|^{1-\alpha}$$

Proof.

(5.137)

$$395 \quad \begin{aligned} (y_{i \rightarrow N}^\theta(\xi) - \xi)' &= (\theta(y_{i \rightarrow N-1}(\xi) - \xi) + (1-\theta)(y_{i \rightarrow N}(\xi) - \xi))' \\ &= \theta(y_{i \rightarrow N-1}'(\xi) - 1) + (1-\theta)(y_{i \rightarrow N}'(\xi) - 1) \\ &= \theta \xi^{1/r-1} (y_{i \rightarrow N-1}^{1-1/r}(\xi) - \xi^{1-1/r}) + (1-\theta) \xi^{1/r-1} \left(\frac{h_N}{r Z_1} - \xi^{1-1/r} \right) \end{aligned}$$

396

$$397 \quad (5.138) \quad \begin{aligned} (y_{i \rightarrow N}^\theta(\xi) - \xi)'' &= \theta(y_{i \rightarrow N-1}''(\xi)) + (1-\theta)(y_{i \rightarrow N}''(\xi)) \\ &= \frac{1-r}{r} \xi^{1/r-2} (\theta y_{i \rightarrow N-1}^{1-2/r}(\xi) Z_{N-1-i} + (1-\theta) \frac{h_N}{r Z_1}) \leq 0 \end{aligned}$$

398 And

$$399 \quad (5.139) \quad |(y_{i \rightarrow N}^\theta(\xi) - \xi)''| \leq C(r-1) \xi^{1/r-2} T^{1-1/r}$$

We have known

$$(5.140) \quad C|x_{N-1} - x_i| \leq |y_{i \rightarrow N-1}(\xi) - \xi| \leq C|x_{N-1} - x_i|$$

If $\xi \leq x_{N-1}$, then $(y_{i \rightarrow N}(\xi) - \xi)' \geq 0$, so

$$(5.141) \quad C|x_N - x_i| \leq |x_{N-1} - x_{i-1}| \leq |y_{i \rightarrow N}^\theta(\xi) - \xi| \leq |x_{N+1} - x_{i+1}| \leq C|x_N - x_i|$$

If $i = N - 1$ and $\xi \in [x_{N-1}, x_N]$, then $y_{i \rightarrow N}(\xi) - \xi$ is concave, bigger than its two neighboring points, which are equal to h_N , so

$$(5.142) \quad h_N = |x_N - x_{N-1}| \leq |y_{i \rightarrow N}(\xi) - \xi| \leq |x_{N+1} - x_{N-1}| = 2h_N$$

So we have

$$(5.143) \quad |y_{i \rightarrow N}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_N^\theta - x_i|^{1-\alpha}$$

While

$$(5.144) \quad y_{i \rightarrow N-1}^{1-1/r}(\xi) - \xi^{1-1/r} \leq (y_{i \rightarrow N-1}(\xi) - \xi)\xi^{-1/r}$$

and

$$(5.145) \quad \begin{aligned} \left| \frac{h_N}{rZ_1} - \xi^{1-1/r} \right| &\leq \max\left\{ \left| \frac{h_N}{rZ_1} - x_{i-1}^{1-1/r} \right|, \left| \frac{h_N}{rZ_1} - x_{i+1}^{1-1/r} \right| \right\} \\ &\leq \max \begin{cases} T^{1-1/r} - x_{i-1}^{1-1/r} \leq |x_N - x_{i-1}|T^{-1/r} \leq C|x_N - x_i| \\ |x_{i+1}^{1-1/r} - x_{N-1}^{1-1/r}| \leq |x_{i+1} - x_{N-1}|x_{N-1}^{-1/r} \leq C|x_N - x_i| \end{cases} \end{aligned}$$

So we have

$$(5.146) \quad (y_{i \rightarrow N}^\theta(\xi) - \xi)' \leq C|y_N^\theta - x_i|$$

$$(5.147) \quad \begin{aligned} (|y_{i \rightarrow N}^\theta(\xi) - \xi|^{1-\alpha})' &= |y_{i \rightarrow N}^\theta(\xi) - \xi|^{-\alpha} (y_{i \rightarrow N}^\theta(\xi) - \xi)' \\ &\leq |y_N^\theta - x_i|^{1-\alpha} \end{aligned}$$

Finally,

$$(5.148) \quad \begin{aligned} (|y_{i \rightarrow N}^\theta(\xi) - \xi|^{1-\alpha})'' &= (1-\alpha)|y_{i \rightarrow N}^\theta(\xi) - \xi|^{-\alpha} (y_{i \rightarrow N}^\theta(\xi) - \xi)'' \\ &\quad + \alpha(\alpha-1)|y_{i \rightarrow N}^\theta(\xi) - \xi|^{-1-\alpha} ((y_{i \rightarrow N}^\theta(\xi) - \xi)')^2 \\ &\leq C(r-1)|y_N^\theta - x_i|^{-\alpha} + C|y_N^\theta - x_i|^{1-\alpha} \end{aligned} \quad \square$$

By the three lemmas above, for $N/2 \leq i \leq N-1$, we have

LEMMA 5.33.

$$(5.149) \quad \begin{aligned} D_h^2 P_{i \rightarrow N}^\theta(x_i) &= P_{i \rightarrow N}^{\theta''}(\xi) \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq Ch^3|y_N^\theta - x_i|^{1-\alpha} + C(r-1)(h^3|y_N^\theta - x_i|^{-\alpha} + h^2|y_N^\theta - x_i|^{1-\alpha}) \end{aligned}$$

And

LEMMA 5.34.

$$\begin{aligned} (5.150) \quad & \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_{i+1})u'''(\eta_{N+1}^\theta) - Q_{i \rightarrow N}^\theta(x_i)u'''(\eta_N^\theta)}{h_{i+1}} \right) \\ & \leq Ch^3|y_N^\theta - x_i|^{1-\alpha} \end{aligned}$$

And immediately, For $N/2 \leq i \leq N-2$

$$\begin{aligned} (5.151) \quad & V_{iN} \leq C \int_{x_{N-1}}^{x_N} h^2|y - x_i|^{1-\alpha} + C(r-1)h^2|y - x_i|^{-\alpha} + h|y - x_i|^{1-\alpha} dy \\ & \leq Ch^2h_N|T - x_i|^{1-\alpha} + C(r-1)h^2|x_{N-1} - x_i|^{1-\alpha} + Chh_N|T - x_i|^{1-\alpha} \\ & \leq Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned}$$

But especially, when $i = N-1$,

$$\begin{aligned} (5.152) \quad & V_{N-1,N} = \int_0^1 -\frac{\theta^{2-\alpha}(1-\theta)}{2} \frac{2}{h_{N-1} + h_N} \left(\frac{1}{h_{N-1}} h_{N-1}^{4-\alpha} u''(y_{N-1}^\theta) - \left(\frac{1}{h_{N-1}} + \frac{1}{h_N} \right) h_N^{4-\alpha} u''(y_N^\theta) + \frac{1}{h_N} h_{N+1}^{4-\alpha} u''(y_{N+1}^\theta) \right) d\theta \\ & + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_{i+1})u'''(\eta_{N+1,1}^\theta) - Q_{i \rightarrow N}^\theta(x_i)u'''(\eta_{N,1}^\theta)}{h_{i+1}} \right) d\theta \\ & - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_i)u'''(\eta_{N,1}^\theta) - Q_{i \rightarrow N}^\theta(x_{i-1})u'''(\eta_{N-1,1}^\theta)}{h_i} \right) d\theta \\ & - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_{i+1})u'''(\eta_{N+1,2}^\theta) - Q_{i \rightarrow N}^\theta(x_i)u'''(\eta_{N,2}^\theta)}{h_{i+1}} \right) d\theta \\ & + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_i)u'''(\eta_{N,2}^\theta) - Q_{i \rightarrow N}^\theta(x_{i-1})u'''(\eta_{N-1,2}^\theta)}{h_i} \right) d\theta \end{aligned}$$

while combine Lemma 5.30

$$\begin{aligned} (5.153) \quad & \frac{2}{h_{N-1} + h_N} \left(\frac{1}{h_{N-1}} h_{N-1}^{4-\alpha} u''(y_{N-1}^\theta) - \left(\frac{1}{h_{N-1}} + \frac{1}{h_N} \right) h_N^{4-\alpha} u''(y_N^\theta) + \frac{1}{h_N} h_{N+1}^{4-\alpha} u''(y_{N+1}^\theta) \right) \\ & = D_h^2(h_{N-1 \rightarrow N}^{4-\alpha}(x_i)u''(y_{N-1 \rightarrow N}^\theta(x_i))) \\ & \leq Ch_N^{4-\alpha} + C(r-1)h_N^{3-\alpha} \leq Ch^{4-\alpha} + C(r-1)h^2|T - x_{N-1-1}|^{1-\alpha} \end{aligned}$$

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430 Similarly with $j = N+1$. □

I_6, I_7 is easy. Similar with Lemma 5.22 and Lemma 5.6, we have

THEOREM 5.35. *There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i \leq N$,*

$$(5.154) \quad I_6 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1, 2N - \lceil \frac{N}{2} \rceil + 1} + T_{i-1, 2N - \lceil \frac{N}{2} \rceil}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, 2N - \lceil \frac{N}{2} \rceil + 1} \right) \leq Ch^2$$

Proof. In fact, let $l = 2N - \lceil \frac{N}{2} \rceil + 1$

$$(5.155) \quad \begin{aligned} & \frac{1}{h_i} (T_{i-1, l} + T_{i-1, l-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, l} \\ &= \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) + \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) + \left(\frac{1}{h_i} - \frac{1}{h_{i+1}} \right) T_{i, l} \end{aligned}$$

While, by Lemma A.2

$$(5.156) \quad \begin{aligned} \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) &= \int_{x_{l-1}}^{x_l} (u(y) - u_h(y)) \frac{|x_{i-1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_i \Gamma(2-\alpha)} dy \\ &\leq C \int_{x_{l-1}}^{x_l} h_l^2 u''(\eta) \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy \\ &\leq Ch_l^3 x_{l-1}^{\alpha/2-2} T^{-\alpha} \\ &\leq Ch_l^3 \end{aligned}$$

Thus,

$$(5.157) \quad \frac{2}{h_i + h_{i+1}} \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) \leq Ch_l^3$$

For

$$(5.158) \quad \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{h_i} d\theta$$

And Similar with Lemma 5.19, we can get

$$(5.159) \quad \frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{(h_i + h_{i+1}) h_i} \leq Ch_l^2 |y_l^\theta - x_i|^{1-\alpha}$$

So

$$(5.160) \quad \frac{2}{h_i + h_{i+1}} \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) \leq Ch^2$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$(5.161) \quad \begin{aligned} \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i, l} &\leq h_i^{-3} h^2 x_i^{1-2/r} h_l C h_l^2 x_{l-1}^{\alpha/2-2} |x_l - x_i|^{1-\alpha} \\ &\leq Ch^2 \end{aligned}$$

Summarizes, we have

$$(5.162) \quad I_6 \leq Ch^2$$

□

And

LEMMA 5.36. *There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i \leq N$,*

$$I_7 = \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

Proof. For $i \leq N, j \geq 2N - \lceil \frac{N}{2} \rceil + 2$, we have

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\ &\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2-2/r} |y - x_{i+1}^{-1-\alpha}| dy \\ &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy \end{aligned}$$

$$\begin{aligned} \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{(2-2^{-r})T}^{x_{2N-1}} (2T - y)^{\alpha/2-2/r} dy \\ &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r}T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\ &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r} h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r} h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned}$$

Now we can conclude a part of the theorem Theorem 5.3 at the beginning of this section.

By Lemma 5.8 Lemma 5.22 Lemma 5.23 Theorem 5.29 Theorem 5.28 Theorem 5.35 Lemma 5.36, we have

THEOREM 5.37. *there exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $N/2 \leq i < N$,*

$$R_i = \sum_{j=1}^7 I_j \leq C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

And what we left is the case $i = N$. Fortunately, we can use the same department of R_i above, and it is symmetric. Most of the item has been esitimated by Lemma 5.8 and Theorem 5.35, we just need to consider I_3, I_4 .

THEOREM 5.38. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$(5.166) \quad I_3 = \sum_{j=\lceil \frac{N}{2} \rceil + 1}^{N-1} V_{Nj} \leq Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha}$$

Proof. **DEFINITION 5.39.** *For $N/2 \leq j < N$, Let's define*

$$(5.167) \quad y_j(x) = \left(\frac{Z_1}{h_N}(x - x_N) + Z_j \right)^r$$

We can see that is the inverse of the function $y_{i \rightarrow N}(x)$ defined in Theorem 5.29.

$$(5.168) \quad y'_j(x) = y_j^{1-1/r}(x) \frac{rZ_1}{h_N}$$

$$(5.169) \quad y''_j(x) = y_j^{1-2/r}(x) \frac{r(r-1)Z_1}{h_N}$$

With the scheme we used several times, we can get

LEMMA 5.40. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq j < N$, $\xi \in [x_{N-1}, x_{N+1}]$,*

$$(5.170) \quad h_j(\xi)^3 \leq Ch^3$$

$$(5.171) \quad (h_j^3(\xi))' \leq C(r-1)h^3$$

$$(5.172) \quad (h_j^3(\xi))'' \leq C(r-1)h^3$$

$$(5.173) \quad u''(y_j^\theta(\xi)) \leq C$$

$$(5.174) \quad (u''(y_j^\theta(\xi)))' \leq C$$

$$(5.175) \quad (u''(y_j^\theta(\xi)))'' \leq C$$

$$(5.176) \quad |\xi - y_j^\theta(\xi)|^{1-\alpha} \leq C|x_N - y_j^\theta|^{1-\alpha}$$

$$(5.177) \quad (|\xi - y_j^\theta(\xi)|^{1-\alpha})' \leq C|x_N - y_j^\theta|^{1-\alpha}$$

$$(5.178) \quad (|\xi - y_j^\theta(\xi)|^{1-\alpha})'' \leq C|x_N - y_j^\theta|^{1-\alpha} + C(r-1)|x_N - y_j^\theta|^{-\alpha}$$

LEMMA 5.41. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq j < N$,*

$$(5.179) \quad V_{Nj} \leq Ch^2 \int_{x_{j-1}}^{x_j} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy$$

Therefore,

$$(5.180) \quad \begin{aligned} I_3 &\leq Ch^2 \int_{\lceil \frac{N}{2} \rceil}^{N-1} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \\ &\leq Ch^2(|T - x_{N-1}|^{2-\alpha} + (r-1)|T - x_{N-1}|^{1-\alpha}) \end{aligned}$$

□

For $j = N$,

LEMMA 5.42.

(5.181)

$$V_{N,N} = \frac{1}{h_N^2} (T_{N-1,N-1} - 2T_{N,N} + T_{N+1,N+1}) \leq Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha}$$

Proof.

(5.182)

$$\begin{aligned} V_{N,N} = & \int_0^1 -\frac{\theta(1-\theta)^{2-\alpha}}{2} \frac{1}{h_N^2} (h_{N-1}^{4-\alpha} u''(y_{N-1}^\theta) - 2h_N^{4-\alpha} u''(y_N^\theta) + h_{N+1}^{4-\alpha} u''(y_{N+1}^\theta)) d\theta \\ & + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{1}{h_N} \left(\frac{Q_{N \rightarrow N}^\theta(x_{N+1}) u'''(\eta_{N+1,1}^\theta) - Q_{N \rightarrow N}^\theta(x_i) u'''(\eta_{N,1}^\theta)}{h_N} \right) d\theta \\ & - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{1}{h_N} \left(\frac{Q_{N \rightarrow N}^\theta(x_N) u'''(\eta_{N,1}^\theta) - Q_{N \rightarrow N}^\theta(x_{N-1}) u'''(\eta_{N-1,1}^\theta)}{h_N} \right) d\theta \\ & - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{1}{h_N} \left(\frac{Q_{N \rightarrow N}^\theta(x_{N+1}) u'''(\eta_{N+1,2}^\theta) - Q_{N \rightarrow N}^\theta(x_N) u'''(\eta_{N,2}^\theta)}{h_N} \right) d\theta \\ & + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{1}{h_N} \left(\frac{Q_{N \rightarrow N}^\theta(x_N) u'''(\eta_{N,2}^\theta) - Q_{N \rightarrow N}^\theta(x_{N-1}) u'''(\eta_{N-1,2}^\theta)}{h_N} \right) d\theta \end{aligned}$$

So combine Lemma 5.8, Theorem 5.35, Theorem 5.38, Lemma 5.42 We have

LEMMA 5.43.

$$R_N \leq C(r-1)h^2|T - x_{N-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

and with Theorem 5.37 we prove the Theorem 5.3

5.5. Truncation error. combine Theorem 5.1, Theorem 5.2 and Theorem 5.3

we get For $1 \leq i \leq N$

(5.184)

$$R_i \leq C_2(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} C_1 h^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ C_1 h^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ C_1 h^{r\alpha/2+r} x_i^{-1-\alpha/2}, & r\alpha/2 + r - 2 < 0 \end{cases}$$

But,

$$h^2 x_i^{-\alpha/2-2/r} \leq T^{\alpha/2-2/r} \begin{cases} h^2 x_i^{-\alpha}, & \text{if } r\alpha/2 - 2 \geq 0 \\ h^{r\alpha/2} x_i^{-\alpha}, & \text{if } r\alpha/2 - 2 \leq 0 \end{cases}$$

$$h^{r\alpha/2+r} x_i^{-1-\alpha} \leq T^{-1} h^{r\alpha/2} x_i^{-\alpha}, \quad \text{if } r\alpha/2 - 2 \leq 0$$

(5.187)

And when $r\alpha/2 - 2 = -r < 0$,

$$\begin{aligned} h^2 x_i^{-1-\alpha} \ln(i) h^{-r\alpha/2} x_i^\alpha &= h^r x_i^{-1} \ln(i) \\ &= T^{-1} \frac{\ln(i)}{i^r} \leq C(T, r) \end{aligned}$$

(5.188)

512 and

513 (5.189)
$$h^2 \ln(N) h^{-r\alpha/2} x_i^\alpha = h^r \ln(N) x_i^\alpha \leq T^\alpha \frac{\ln(N)}{N^r} \leq C(T, \alpha, r)$$

514 So for $1 \leq i \leq N$,

515 (5.190)
$$R_i \leq C_2(r-1)h^2|T-x_{i-1}|^{1-\alpha} + C_1h^{\min\{\frac{r\alpha}{2}, 2\}}x_i^{-\alpha}$$

516 And for $i \geq N$, it is symmetric for i and $2N-i$.

517 The proof of Theorem 4.1 completed.

6. Proof of Theorem 4.2. Review section 3, we have (3.9) and (3.11),

$$(6.1) \quad \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right)$$

$$(6.2) \quad a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right)$$

Thus

LEMMA 6.1.

$$(6.3) \quad \sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right)$$

DEFINITION 6.2. We call one matrix a M matrix, which means its entries are positive on major diagonal and nonpositive on others, and Strictly diagonally dominant in rows.

Now we have

LEMMA 6.3. The matrix A defined by (3.11) is a M matrix. and

$$(6.4) \quad \begin{aligned} S_i &:= \sum_{j=1}^{2N-1} a_{ij} \\ &= \sum_{j=1}^{2N-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \tilde{a}_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_i} \tilde{a}_{i-1,j} \right) \\ &\geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \end{aligned}$$

Proof. Let

$$(6.5) \quad g(x) = g_0(x) + g_{2N}(x)$$

where

$$\begin{aligned} g_0(x) &:= \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} \\ g_{2N}(x) &:= \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \end{aligned}$$

Then, for $2 \leq i \leq 2N - 2$,

$$(6.6) \quad \begin{aligned} S_i &:= \sum_{j=1}^{2N-1} a_{ij} \\ &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &= g_0''(\xi) + g_{2N}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \end{aligned}$$

While for $i \geq 2$

$$\begin{aligned} g_0''(\xi) &= -\kappa_\alpha \frac{|\xi - x_0|^{1-\alpha} - |\xi - x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1} \\ &= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} |\xi - \eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\ &\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha} \end{aligned}$$

when $i = 1$

$$\begin{aligned} &\frac{2}{h_1 + h_2} \left(\frac{1}{h_2} g_0(x_2) - \left(\frac{1}{h_1} + \frac{1}{h_2} \right) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right) \\ &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1h_2} \\ &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1^{1-\alpha}h_2} h_1^{-\alpha} \\ &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha}}{2^r(2^r - 1)} h_1^{-\alpha} \end{aligned}$$

but

$$1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha} > 0$$

So

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g_0(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g_0(x_i) + \frac{1}{h_i} g_0(x_{i-1}) \right) \geq C x_i^{-\alpha}$$

symmetricly,

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g_{2N}(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g_{2N}(x_i) + \frac{1}{h_i} g_{2N}(x_{i-1}) \right) \geq C(\alpha, r)(2T - x_i)^{-\alpha} \quad \square$$

Let

$$g(x) = \begin{cases} x, & 0 < x \leq T \\ 2T - x, & T < x < 2T \end{cases}$$

And define

$$G = \text{diag}(g(x_1), \dots, g(x_{2N-1}))$$

Then

LEMMA 6.4. *The matrix $B := AG$, the major diagonal is positive, and nonpositive on others. And there is a constant $C = C(\alpha, r)$ such that*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases}$$

Proof. Since

$$(6.15) \quad g(x) \equiv g_h(x)$$

by (3.9), we have

$$\begin{aligned} \tilde{M}_i &:= \sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} \tilde{a}_{ij} g(x_j) \\ (6.16) \quad &= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} g_h(y) dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} g(y) dy \\ &= \frac{-2}{\Gamma(4-\alpha)} |T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)} (x_i^{3-\alpha} + (2T - x_i)^{3-\alpha}) \\ &:= w(x_i) = p(x_i) + q(x_i) \end{aligned}$$

Thus,

$$\begin{aligned} (6.17) \quad M_i &:= \sum_{j=1}^{2N-1} a_{ij} g(x_j) \\ &= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} w(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) w(x_i) + \frac{1}{h_i} w(x_{i-1}) \right) \end{aligned}$$

for $1 \leq i < N-1$, by Lemma A.1

$$\begin{aligned} (6.18) \quad P_i &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} p(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) p(x_i) + \frac{1}{h_i} p(x_{i-1}) \right) \\ &= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - \xi|^{1-\alpha} \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - x_{i-1}|^{1-\alpha} \end{aligned}$$

and

$$\begin{aligned} (6.19) \quad P_{N-1} &:= \frac{-2\kappa_\alpha}{h_{N-1} + h_N} \left(\frac{1}{h_N} p(x_N) - \left(\frac{1}{h_{N-1}} + \frac{1}{h_N} \right) p(x_{N-1}) + \frac{1}{h_{N-1}} p(x_{N-2}) \right) \\ &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{2}{h_{N-1} + h_N} \left(-\left(\frac{1}{h_{N-1}} + \frac{1}{h_N} \right) h_N^{3-\alpha} + \frac{1}{h_{N-1}} (h_{N-1} + h_N)^{3-\alpha} \right) \\ (6.20) \quad &= \frac{4\kappa_\alpha}{\Gamma(4-\alpha) h_{N-1}} (-h_N^{2-\alpha} + (h_{N-1} + h_N)^{2-\alpha}) \\ &= \frac{4\kappa_\alpha}{(3-\alpha)\Gamma(2-\alpha)} \xi^{1-\alpha} \quad \xi \in [h_N, h_{N-1} + h_N] \\ &\geq \frac{4\kappa_\alpha}{(3-\alpha)\Gamma(2-\alpha)} (h_{N-1} + h_N)^{1-\alpha} = \frac{4\kappa_\alpha}{(3-\alpha)\Gamma(2-\alpha)} (T - x_{N-2})^{1-\alpha} \end{aligned}$$

$$\begin{aligned} (6.20) \quad P_N &:= -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left(\frac{1}{h_{N+1}} p(x_{N+1}) - \left(\frac{1}{h_N} + \frac{1}{h_{N+1}} \right) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right) \\ (6.20) \quad &= \frac{4\kappa_\alpha}{\Gamma(4-\alpha) h_N^2} h_N^{3-\alpha} \\ &= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - x_{N-1})^{1-\alpha} \end{aligned}$$

Symmetricly for $i \geq N$, we get

$$(6.21) \quad P_i \geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases}$$

Similarly, we can get

$$(6.22) \quad \begin{aligned} Q_i &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} q(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right) \\ &\geq \frac{-2^{r(\alpha-1)+1}\kappa_\alpha}{\Gamma(2-\alpha)} (x_{i-1}^{1-\alpha} + (1 - x_{i+1})^{1-\alpha}) \end{aligned} \quad \square$$

Notice that

$$(6.23) \quad x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha}$$

We can get

THEOREM 6.5. *There exists a real $\lambda = \lambda(T, \alpha, r) > 0$ and $C = C(T, \alpha, r) > 0$ such that $B := A(\lambda I + G)$ is an M matrix. And*

$$(6.24) \quad M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + C \begin{cases} |\frac{1}{2} - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - \frac{1}{2}|^{1-\alpha}, & i \geq N \end{cases}$$

Proof. By 6.3 with C_1 and 6.4 with C_2 , it's sufficient to take $\lambda = 4TC_2/C_1$, then

$$(6.25) \quad M_i \geq C_2 \left((x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \right) \quad \square$$

Now, we can prove the convergency Theorem 4.2.

For equation

$$(6.26) \quad AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1}U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1}U = F$$

which means

$$(6.27) \quad \sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + g(x_j)} = \tau_i$$

where $\epsilon_i = u(x_i) - U_i$.

And if

$$(6.28) \quad \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| = \max_{1 \leq i \leq 2N-1} \left| \frac{\epsilon_i}{\lambda + g(x_i)} \right|$$

Then, since $B = A(\lambda I + G)$ is an M matrix, it is Strictly diagonally dominant. Thus,

$$\begin{aligned}
 |\tau_{i_0}| &= \left| \sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + g(x_i)} \right| \\
 &\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_j}{\lambda + g(x_j)} \right| \\
 &\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \\
 &= \sum_{j=1}^{2N-1} b_{i_0,j} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \\
 &= M_{i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right|
 \end{aligned}
 \tag{6.29}$$

By Theorem 4.1 and Theorem 6.5,

We know that there exists constants $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$, and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that

$$\left| \frac{\epsilon_i}{\lambda + g(x_i)} \right| \leq \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} + C_2(r-1)h^2
 \tag{6.30}$$

as $\lambda + g(x_i) \leq \lambda + T$

So, we can get

$$|\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{r\alpha}{2}, 2\}}
 \tag{6.31}$$

The convergency has been proved.

7. Experimental results.

8. Remarks. some remarks.

In Theorem 2.3 If $f \in L^\infty(\Omega)$ then $u \in C_{\alpha/2}(\Omega)$, which is Proposition 1.1 in [1].

When $\|f\|_\beta^{(\gamma)} < \infty$, where $\beta > 2 - \alpha$ and $\gamma \in [-\alpha, -\alpha/2]$, we observed convergent order $\min\{r(\alpha+\gamma), 2\}$ in numerical experiments. And we can prove that kind theorems with the techneque we used in this paper.

Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x)$ is twice differentiable continous function on open set Ω , there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

$$(A.1) \quad D_h^2 g(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

$$(A.2) \quad \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right)$$

And if $g(x) \in C^4(\Omega)$, then

$$(A.3) \quad \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2))$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$.

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in [x_{i-1}, x_i]$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in [x_i, x_{i+1}]$$

Substitute them in the left side of (A.1), we have

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2)$$

Now, using intermediate value theorem, there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y)dy$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \frac{h_i^4}{4!} g''''(\eta_1)$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \frac{h_{i+1}^4}{4!} g''''(\eta_2)$$

where $\eta_1 \in [x_{i-1}, x_i]$, $\eta_2 \in [x_i, x_{i+1}]$. Expecially,

$$\begin{aligned} \frac{h_i^4}{4!} g''''(\eta_1) &= \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy \\ \frac{h_{i+1}^4}{4!} g''''(\eta_2) &= \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \end{aligned}$$

Substitute them to the left side of (A.3), we can get the result. \square

LEMMA A.2. If $y \in [x_{j-1}, x_j]$, denote $y = \theta x_{j-1} + (1 - \theta)x_j$, $\theta \in [0, 1]$,

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j]$$

(A.6)

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

where $\eta_1 \in [x_{j-1}, y_j^\theta]$, $\eta_2 \in [y_j^\theta, x_j]$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta)u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in [x_{j-1}, y_j^\theta]$, $\eta_2 \in [y_j^\theta, x_j]$. Thus \square

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

LEMMA A.3. For $x \in [x_{j-1}, x_j]$

$$\begin{aligned} |u(x) - u_h(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^x u'(y) dy - \frac{x - x_{j-1}}{h_j} \int_x^{x_j} u'(y) dy \right| \\ &\leq \int_{x_{j-1}}^{x_j} |u'(y)| dy \end{aligned}$$

If $x \in [0, x_1]$, with Corollary 2.4, we have

$$|u(x) - u_h(x)| \leq \int_0^{x_1} |u'(y)| dy \leq \int_0^{x_1} C y^{\alpha/2-1} dy \leq C \frac{2}{\alpha} x_1^{\alpha/2}$$

Similarly, if $x \in [x_{2N-1}, 1]$, we have

$$|u(x) - u_h(x)| \leq C \frac{2}{\alpha} (2T - x_{2N-1})^{\alpha/2} = C \frac{2}{\alpha} x_1^{\alpha/2}$$

Appendix B. Inequality.

LEMMA B.1.

$$h_i \leq rT^{1/r} h \begin{cases} x_i^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_{i-1})^{1-1/r}, & N < i \leq 2N-1 \end{cases}$$

$$h_i \geq rT^{1/r} h \begin{cases} x_{i-1}^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_i)^{1-1/r}, & N < i \leq 2N-1 \end{cases}$$

Proof. For $1 \leq i \leq N$,

$$\begin{aligned} h_i &= T \left(\left(\frac{i}{N} \right)^r - \left(\frac{i-1}{N} \right)^r \right) \\ &\leq rT \frac{1}{N} \left(\frac{i}{N} \right)^{r-1} = rT^{1/r} h x_i^{1-1/r} \end{aligned}$$

$$h_i \geq rT \frac{1}{N} \left(\frac{i-1}{N} \right)^{r-1} = rT^{1/r} h x_{i-1}^{1-1/r}$$

For $N < i \leq 2N$,

$$\begin{aligned} h_i &= T \left(\left(\frac{2N-i+1}{N} \right)^r - \left(\frac{2N-i}{N} \right)^r \right) \\ &\leq rT \frac{1}{N} \left(\frac{2N-i+1}{N} \right)^{r-1} = rT^{1/r} h (2T - x_{i-1})^{1-1/r} \end{aligned}$$

$$h_i \geq rT \frac{1}{N} \left(\frac{2N-i}{N} \right)^{r-1} = rT^{1/r} h (2T - x_i)^{1-1/r}$$

□

661 LEMMA B.2. *There is a constant $C = 2^{|r-2|}r(r-1)T^{2/r}$ such that for all $i \in$*
 662 *$\{1, 2, \dots, 2N-1\}$*

$$663 \quad (B.3) \quad |h_{i+1} - h_i| \leq Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases}$$

Proof.

$$664 \quad h_{i+1} - h_i = \begin{cases} T \left(\left(\frac{i+1}{N} \right)^r - 2 \left(\frac{i}{N} \right)^r + \left(\frac{i-1}{N} \right)^r \right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T \left(\left(\frac{2N-i-1}{N} \right)^r - 2 \left(\frac{2N-i}{N} \right)^r + \left(\frac{2N-i+1}{N} \right)^r \right), & N+1 \leq i \leq 2N-1 \end{cases}$$

665 For $i = 1$,

$$666 \quad h_2 - h_1 = T(2^r - 2) \left(\frac{1}{N} \right)^r = (2^r - 2)T^{2/r}h^2x_1^{1-2/r}$$

667 For $2 \leq i \leq N-1$,

$$668 \quad h_{i+1} - h_i = r(r-1)T N^{-2}\eta^{r-2}, \quad \eta \in \left[\frac{i-1}{N}, \frac{i+1}{N} \right]$$

669 If $r \in [1, 2]$,

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T N^{-2}\eta^{r-2} \leq r(r-1)T h^2 \left(\frac{i-1}{N} \right)^{r-2} \\ 670 \quad &\leq r(r-1)T h^2 2^{2-r} \left(\frac{i}{N} \right)^{r-2} \\ &= 2^{2-r}r(r-1)T^{2/r}h^2x_i^{1-2/r} \end{aligned}$$

671 else if $r > 2$,

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T N^{-2}\eta^{r-2} \leq r(r-1)T h^2 \left(\frac{i+1}{N} \right)^{r-2} \\ 672 \quad &\leq r(r-1)T h^2 2^{r-2} \left(\frac{i}{N} \right)^{r-2} \\ &= 2^{r-2}r(r-1)T^{2/r}h^2x_i^{1-2/r} \end{aligned}$$

673 Since

$$674 \quad 2^r - 2 \leq 2^{|r-2|}r(r-1), \quad r \geq 1$$

675 we have

$$676 \quad h_{i+1} - h_i \leq 2^{|r-2|}r(r-1)T^{2/r}h^2x_i^{1-2/r}, \quad 1 \leq i \leq N-1$$

677 For $i = N$, $h_{N+1} - h_N = 0$. For $N < i \leq 2N-1$, it's central symmetric to the first
 678 half of the proof, which is

$$679 \quad h_i - h_{i+1} \leq 2^{|r-2|}r(r-1)T^{2/r}h^2(2T - x_i)^{1-2/r}$$

680 Summarizes the inequalities, we can get

$$681 \quad (B.4) \quad |h_{i+1} - h_i| \leq 2^{|r-2|} r(r-1) T^{2/r} h^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases} \quad \square$$

682 Appendix C. Proofs of some technical details.

683 *Additional proof of Theorem 5.1.* For $2 \leq i \leq N-1$,

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\ 684 \quad & \leq C \frac{2}{h_i + h_{i+1}} (h_i^3 x_{i-1}^{-2-\alpha/2} + h_{i+1}^3 x_i^{-2-\alpha/2}) \\ & \leq 2C (h_i^2 x_{i-1}^{-2-\alpha/2} + h_{i+1}^2 x_i^{-2-\alpha/2}) \end{aligned}$$

685 Since Lemma B.1, we have

$$\begin{aligned} 686 \quad & h_i \leq r T^{1/r} h x_i^{1-1/r}, \quad 1 \leq i \leq N \\ 687 \quad & h_{i+1} \leq r T^{1/r} h x_{i+1}^{1-1/r}, \quad 1 \leq i \leq N-1 \end{aligned}$$

688 and

$$\begin{aligned} 689 \quad & x_{i-1}^{-2-\alpha/2} \leq 2^{-r(-2-\alpha/2)} x_i^{-2-\alpha/2} \quad 2 \leq i \leq N-1 \\ 690 \quad & x_{i+1}^{1-1/r} \leq 2^{r-1} x_i^{1-1/r} \quad 1 \leq i \leq N-1 \end{aligned}$$

691 So there is a constant $C = C(T, \alpha, r, \|f\|_{\beta}^{\alpha/2})$ such that

$$692 \quad \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq C h^2 x_i^{-\alpha/2-2/r}, \quad 2 \leq i \leq N-1$$

693 For $i = 1$, by (A.4)

$$\begin{aligned} & \frac{1}{4!} \frac{2}{h_1 + h_2} (h_1^3 f''(\eta_1) + h_2^3 f''(\eta_2)) \\ 694 \quad & = \frac{2}{h_1 + h_2} \left(\frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy + \frac{1}{4!} h_2^3 f''(\eta_2) \right) \end{aligned}$$

695 We have proved above that

$$696 \quad \frac{2}{h_1 + h_2} h_2^3 f''(\eta_2) \leq C h^2 x_1^{-\alpha/2-2/r}$$

697 and we can get

$$\begin{aligned} & \int_0^{x_1} f''(y) \frac{y^3}{3!} dy \leq C \frac{1}{3!} \int_0^{x_1} y^{1-\alpha/2} dy \\ 698 \quad & = C \frac{1}{3!(2-\alpha/2)} x_1^{2-\alpha/2} \end{aligned}$$

699 so

$$700 \quad \frac{2}{h_1 + h_2} \frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy = \frac{C 2^{1-r}}{3!(2-\alpha/2)} x_1^{-\alpha/2} = \frac{C 2^{1-r}}{3!(2-\alpha/2)} T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

And for $i = N$, we have

$$\begin{aligned} & \frac{2}{h_N + h_{N+1}} (h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2)) \\ &= h_N^2 (f''(\eta_1) + f''(\eta_2)) \\ &\leq r^2 T^{2/r} h^2 x_N^{2-2/r} 2C x_{N-1}^{-2-\alpha/2} \\ &\leq 2r^2 T^{2/r} C 2^{-r(-2-\alpha/2)} h^2 x_N^{-\alpha/2-2/r} \end{aligned}$$

Finally, $N + 1 \leq i \leq 2N - 1$ is symmetric to the first half of the proof, so we can conclude that \square

$$\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq C h^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N - 1 \end{cases}$$

LEMMA C.1. *There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ for $2 \leq j \leq N$, if $y \in [x_{j-1}, x_j]$,*

$$(C.1) \quad |u(y) - u_h(y)| \leq C h^2 y^{\alpha/2-2/r}$$

Proof. For $2 \leq j \leq N$, we have

$$x_j \leq 2^r y, \quad x_{j-1} \geq 2^{-r} y$$

And by Lemma A.2, Lemma B.1 and Corollary 2.4, we have

$$\begin{aligned} u(y) - u_h(y) &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \\ &\leq \frac{\|u\|_{\beta+\alpha}^{(-\alpha/2)}}{4} r^2 T^{2/r} h^2 x_j^{2-2/r} x_{j-1}^{\alpha/2-2} \\ &\leq C h^2 2^{2r-2} y^{2-2/r} 2^{-r(\alpha/2-2)} y^{\alpha/2-2} \\ &= C 2^{-r\alpha/2+4r-2} h^2 y^{\alpha/2-2/r} \end{aligned}$$

symmetricly, for $N < j \leq 2N - 1$, we have

$$(C.2) \quad |u(y) - u_h(y)| \leq C h^2 (2T - y)^{\alpha/2-2/r} \quad \square$$

LEMMA C.2. *There is a constant $C = C(\alpha, r)$ such that for all $1 \leq i < N/2$, $\max\{2i + 1, i + 3\} \leq j \leq 2N$ and $y \in [x_{j-1}, x_j]$, we have*

$$(C.3) \quad D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) \leq C \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}$$

Proof. Sinec $y \geq x_{j-1} > x_{i+1}$, by Lemma A.1, if $j - 1 > i + 1$ \square

$$\begin{aligned} D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) &= \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq \frac{(y - x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\leq \left(1 - \left(\frac{2}{3}\right)^r\right)^{-1-\alpha} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

LEMMA C.3. *There is a constant $C = C(\alpha, r)$ such that for all $3 \leq i < N/2, k = \lceil \frac{i}{2} \rceil, 1 \leq j \leq k-1$ and $y \in [x_{j-1}, x_j]$, we have*

$$(C.4) \quad D_h^2\left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) \leq C \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)}$$

Proof. Since $y \leq x_j < x_{i-1}$, by Lemma A.1,

$$\begin{aligned} D_h^2\left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) &= \frac{|\xi - y|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq \frac{(x_{i-1} - x_j)^{-1-\alpha}}{\Gamma(-\alpha)} \leq \frac{(x_{i-1} - x_{k-1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\leq \left(\left(\frac{2}{3}\right)^r - \left(\frac{1}{2}\right)^r\right)^{-1-\alpha} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

□

LEMMA C.4. *While $0 \leq i < N/2$, By Lemma A.3*

$$\begin{aligned} (C.5) \quad |T_{i1}| &\leq C \int_0^{x_1} x_1^{\alpha/2} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2} |x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha}| \\ &\leq C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2+2-\alpha} = C \frac{1}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1 \end{aligned}$$

For $2 \leq j \leq N$, by Lemma A.2 and Corollary 2.4

$$\begin{aligned} (C.6) \quad |T_{ij}| &\leq \frac{C}{4} \int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &\leq \frac{C}{4\Gamma(3-\alpha)} h_j^2 x_{j-1}^{\alpha/2-2} ||x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha}| \end{aligned}$$

LEMMA C.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$(C.7) \quad \sum_{j=1}^3 S_{1j} \leq C h^2 x_1^{-\alpha/2-2/r}$$

$$(C.8) \quad \sum_{j=1}^4 S_{2j} \leq C h^2 x_2^{-\alpha/2-2/r}$$

Proof.

$$S_{1j} = \frac{2}{x_2} \left(\frac{1}{x_1} T_{0j} - \left(\frac{1}{x_1} + \frac{1}{h_2} \right) T_{1j} + \frac{1}{h_2} T_{2j} \right)$$

So, by Lemma C.4

$$S_{11} \leq \frac{2}{x_2 x_1} 4 \frac{C}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \leq C x_1^{-\alpha/2}$$

$$S_{12} \leq \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_2^2 x_1^{\alpha/2-2} (x_2^{2-\alpha} + 2h_2^{2-\alpha} + h_2^{2-\alpha}) \leq C x_1^{-\alpha/2}$$

$$S_{13} \leq \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_3^2 x_2^{\alpha/2-2} (x_3^{2-\alpha} + 2h_3^{2-\alpha} + h_3^{2-\alpha}) \leq C x_1^{-\alpha/2}$$

But

$$x_1^{-\alpha/2} = T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

For $i = 2$, Sorry □

LEMMA C.6. *There exists a constant $C = C(T, r, l)$ such that For $3 \leq i \leq N - 1$, $k + 1 = \lceil \frac{i}{2} \rceil$, $k \leq j \leq \min\{2i - 1, N - 1\}$, $l = 3, 4$, when $\xi \in [x_{i-1}, x_{i+1}]$,*

$$(C.9) \quad (h_{j-i}^3(\xi))' \leq (r-1) C h^2 x_i^{1-2/r} h_j$$

$$(C.10) \quad (h_{j-i}^4(\xi))' \leq (r-1) C h^2 x_i^{1-2/r} h_j^2$$

Proof. From (5.32)

$$(C.11) \quad y'_{j-i}(x) = y_{j-i}^{1-1/r}(x) x^{1/r-1}$$

$$(C.12) \quad y''_{j-i}(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i}$$

for $l = 3, 4$, by (5.34)

$$(C.13) \quad \begin{aligned} (h_{j-i}^l(\xi))' &= l h_{j-i}^{l-1}(\xi) (y'_{j-i}(\xi) - y'_{j-i-1}(\xi)) \\ &= l h_{j-i}^{l-1}(\xi) \xi^{1/r-1} (y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \geq 0 \end{aligned}$$

For $\xi \in [x_{i-1}, x_{i+1}]$ and $2 \leq k \leq j \leq \min\{2i - 1, N - 1\}$, using Lemma B.1

$$\begin{aligned} h_{j-i}(\xi) &\leq h_{j-i}(x_{i+1}) = h_{j+1} \\ &\leq r T^{1/r} h x_{j+1}^{1-1/r} \leq r T^{1/r} 2^{r-1} h x_i^{1-1/r} \end{aligned}$$

And

$$(C.14) \quad 2^{-r} x_i \leq x_{i-1} \leq \xi \leq x_{i+1} \leq 2^r x_i$$

We have

$$(C.15) \quad \xi^{1/r-m} \leq 2^{|mr-1|} x_i^{1/r-m}, \quad m = 1, 2$$

but

$$(C.16) \quad \begin{aligned} y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) &= (\xi^{1/r} + Z_{j-i})^{r-1} - (\xi^{1/r} + Z_{j-i-1})^{r-1} \\ &= (r-1) Z_1 (\xi^{1/r} + Z_{j-i-\gamma})^{r-2}, \quad \gamma \in [0, 1] \\ &= (r-1) T^{1/r} h y_{j-i-\gamma}^{1-2/r}(\xi) \end{aligned}$$

And
(C.17)

$$4^{-r}x_i \leq x_{\lceil \frac{i}{2} \rceil - 1} \leq x_{j-2} = y_{j-i-1}(x_{i-1}) \leq y_{j-i-\gamma}(\xi) \leq y_{j-i}(x_{i+1}) = x_{j+1} \leq x_{2i} \leq 2^r x_i$$

Therefore,

$$(C.18) \quad y_{j-i-\gamma}^{1-2/r}(\xi) \leq 2^{2|r-2|} x_i^{1-2/r}$$

So we can get

$$(C.19) \quad y'_{j-i}(\xi) - y'_{j-i-1}(\xi) \leq (r-1)C(T, r) h x_i^{-1/r}$$

We get

$$(C.20) \quad (h_{j-i}^l(\xi))' \leq l(r-1)C h_{j+1}^{l-1} h x_i^{-1/r}$$

And by Lemma B.1,

$$(C.21) \quad h_{j+1} \leq rTh \left(\frac{j+1}{N} \right)^{r-1} \leq rTh 2^{r-1} \left(\frac{j-1}{N} \right) = 2^{r-1} h_j$$

$$(C.22) \quad h_{j+1} \leq rT^{1/r} h x_{j+1}^{1-1/r} \leq rT^{1/r} h x_{2i}^{1-1/r} \leq rT^{1/r} 2^{r-1} h x_i^{1-1/r}$$

We can get

$$(C.23) \quad \begin{aligned} (h_{j-i}^l(\xi))' &\leq l(r-1)C h_j^{l-2} h_{j+1} h x_i^{-1/r} \\ &\leq l(r-1)C h h_j^{l-2} (h x_i^{1-1/r}) x_i^{-1/r} \\ &= (r-1)C h^2 x_i^{1-2/r} h_j^{l-2} \end{aligned}$$

Meanwhile, we can get

$$(C.24) \quad h_{j-i}^3(\xi) \leq h_{j+1}^3 \leq C h^2 x_i^{2-2/r} h_j$$

$$(C.25) \quad h_{j-i}^4(\xi) \leq h_{j+1}^4 \leq C h^2 x_i^{2-2/r} h_j^2 \quad \square$$

LEMMA C.7. *There exists a constant $C = C(T, r, l)$ such that For $3 \leq i \leq N - 1$, $\lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i - 1, N - 1\}$, when $\xi \in [x_{i-1}, x_{i+1}]$,*

$$(C.26) \quad (h_{j-i}^3(\xi))'' \leq C(r-1) h^2 x_i^{-2/r} h_j$$

Proof. From (C.11)

$$(C.27) \quad \begin{aligned} (h_{j-i}^3(\xi))'' &= 6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 + 3h_{j-i}^2(\xi)(y''_{j-i}(\xi) - y''_{j-i-1}(\xi)) \\ &= 6h_{j-i}(\xi)\xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \\ &\quad + 3\frac{1-r}{r}h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \end{aligned}$$

789 Using the inequalities of the proof of Lemma C.6

$$\begin{aligned}
 & 6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 \\
 790 \quad (C.28) \quad & \leq 6h_{j+1}((r-1)Chx_i^{-1/r})^2 \\
 & \leq C(r-1)^2 h^2 x_i^{-2/r} h_j
 \end{aligned}$$

791 For the second partial

$$\begin{aligned}
 & h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\
 792 \quad (C.29) \quad & \leq Ch_{j+1}^2 x_i^{1/r-2}((y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi))Z_{j-i} + y_{j-i-1}^{1-2/r}(\xi)Z_1)
 \end{aligned}$$

793 but

$$\begin{aligned}
 & y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi) = (\xi^{1/r} + Z_{j-i})^{r-2} - (\xi^{1/r} + Z_{j-i-1})^{r-2} \\
 & = (r-2)Z_1(\xi^{1/r} + Z_{j-i-\gamma})^{r-3} \\
 794 \quad (C.30) \quad & = (r-2)T^{-r}hy_{j-i-\gamma}^{1-3/r}(\xi) \\
 & \leq C(r-2)hx_i^{1-3/r}
 \end{aligned}$$

795 So we can get

$$\begin{aligned}
 & h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\
 & \leq Ch_j hx_i^{1-1/r} x_i^{1/r-2} (C(r-2)hx_i^{1-3/r}Z_{j-i} + Cx_i^{1-2/r}T^{1/r}h) \\
 796 \quad (C.31) \quad & \leq Ch^2((r-2)x_i^{-3/r}x_{|j-i|}^{1/r} + x_i^{-2/r})h_j \\
 & \leq Ch^2 x_i^{-2/r} h_j
 \end{aligned}$$

797 Summarizes, we have

$$798 \quad (C.32) \quad (h_{j-i}^3(\xi))'' \leq C(r-1)h^2 x_i^{-2/r} h_j \quad \square$$

799 *proof of Lemma 5.16.* From (5.32)

$$800 \quad (C.33) \quad y'_{j-i}(x) = y_{j-i}^{1-1/r}(x)x^{1/r-1}$$

$$801 \quad (C.34) \quad y''_{j-i}(x) = \frac{1-r}{r}y_{j-i}^{1-2/r}(x)x^{1/r-2}Z_{j-i}$$

802 Since

$$803 \quad x_{j-2} \leq y_{j-i-1}(x_{i-1}) \leq y_{j-i}^\theta(\xi) \leq y_{j-i-1}^\theta(x_{i+1}) \leq x_{j+1}$$

804 We have known (C.17)

$$805 \quad (C.35) \quad u''(y_{j-i}^\theta(\xi)) \leq C(y_{j-i}^\theta(\xi))^{\alpha/2-2} \leq Cx_{j-2}^{\alpha/2-2} \leq Cx_{\lfloor \frac{i}{2} \rfloor -1}^{\alpha/2-2} \leq C4^{r(2-\alpha/2)}x_i^{\alpha/2-2}$$

806

$$\begin{aligned}
 & (u''(y_{j-i}^\theta(\xi)))' = u'''(y_{j-i}^\theta(\xi))y_{j-i}^{\theta'}(\xi) \\
 807 \quad (C.36) \quad & \leq Cx_i^{\alpha/2-3}\xi^{1/r-1}y_{j-i}^{1-1/r}(\xi) \\
 & \leq Cx_i^{\alpha/2-3}x_i^{1/r-1}x_i^{1-1/r} = Cx_i^{\alpha/2-3}
 \end{aligned}$$

808

$$\begin{aligned}
& (u''(y_{j-i}^\theta(\xi)))'' = u''''(y_{j-i}^\theta(\xi))(y_{j-i}^{\theta''}(\xi))^2 + u'''(y_{j-i}^\theta(\xi))y_{j-i}^{\theta'''}(\xi) \\
& \leq Cx_i^{\alpha/2-4} + Cx_i^{\alpha/2-3}\frac{r-1}{r}x_i^{1-2/r}x_i^{1/r-2}Z_{|j-i|+1} \\
& \leq Cx_i^{\alpha/2-4} + C\frac{r-1}{r}x_i^{\alpha/2-3}x_i^{-1/r}x_i^{1/r} \\
& = Cx_i^{\alpha/2-4}
\end{aligned}
\tag{C.37}$$

809

□

Proof of Lemma 5.17.

$$\begin{aligned}
& |y_{j-i}^\theta(\xi) - \xi| = |\theta(y_{j-i-1}(\xi) - \xi) + (1-\theta)(y_{j-i}(\xi) - \xi)| \\
& = \theta|y_{j-i-1}(\xi) - \xi| + (1-\theta)|y_{j-i}(\xi) - \xi|
\end{aligned}
\tag{C.38}$$

810

811 Since $|y_{j-i}(\xi) - \xi|$ is increasing about ξ , we have

$$\begin{aligned}
& (\frac{i-1}{i})^r|x_j - x_i| \leq |x_{j-1} - x_{i-1}| \leq |y_{j-i}(\xi) - \xi| \leq |x_{j+1} - x_{i+1}| \leq (\frac{i+1}{i})^r|x_j - x_i|
\end{aligned}
\tag{C.39}$$

812

813 Thus,

$$\begin{aligned}
& (\frac{2}{3})^r|y_j^\theta - x_i| \leq |y_{j-i}^\theta(\xi) - \xi| \leq (\frac{3}{4})^r(\theta|x_j - x_i| + (1-\theta)|x_{j-1} - x_i|) = (\frac{3}{4})^r|y_j^\theta - x_i|
\end{aligned}
\tag{C.40}$$

814

$$|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}
\tag{C.41}$$

815

816 Next,

$$\begin{aligned}
& (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' = (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi)) - 1| \\
& \leq C|y_j^\theta - x_i|^{-\alpha}\xi^{1/r-1}|\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}|
\end{aligned}
\tag{C.42}$$

817

818 Similar with (C.40), we have

$$\begin{aligned}
& |y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| \leq C|x_j^{1-1/r} - x_i^{1-1/r}| \\
& \leq C|x_j - x_i|x_i^{-1/r}
\end{aligned}
\tag{C.43}$$

819

820 So we can get

$$\begin{aligned}
& |\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| \\
& \leq Cx_i^{-1/r}(\theta|x_{j-1} - x_i| + (1-\theta)|x_j - x_i|) \\
& = Cx_i^{-1/r}|y_j^\theta - x_i|
\end{aligned}
\tag{C.44}$$

821

822 Combine them, we get

$$\begin{aligned}
& (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C|y_j^\theta - x_i|^{-\alpha}x_i^{1/r-1}x_i^{-1/r}|y_j^\theta - x_i| \\
& = C|y_j^\theta - x_i|^{1-\alpha}x_i^{-1}
\end{aligned}
\tag{C.45}$$

823

824 Finally, we have

$$\begin{aligned}
& (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' = \alpha(\alpha-1)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi)) - 1)^2 \\
& + (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}\frac{1-r}{r}\xi^{1/r-2}|\theta y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1} + (1-\theta)y_{j-i}^{1-2/r}(\xi)Z_{j-i}|
\end{aligned}
\tag{C.46}$$

825

Using the inequalities above ,we have

$$\begin{aligned}
 & |y_{j-i}^\theta(\xi) - \xi|^{-\alpha-1} (\xi^{1/r-1} (\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta) y_{j-i}^{1-1/r}(\xi)) - 1)^2 \\
 & \leq C |y_j^\theta - x_i|^{-\alpha-1} (x_i^{-1} |y_j^\theta - x_i|)^2 \\
 & = C |y_j^\theta - x_i|^{1-\alpha} x_i^{-2}
 \end{aligned}
 \tag{C.47}$$

And by

$$|Z_{j-i}| = |x_j^{1/r} - x_i^{1/r}| \leq |x_j - x_i| x_i^{1/r-1}$$

we have

$$\begin{aligned}
 & |y_{j-i}^\theta(\xi) - \xi|^{-\alpha} \xi^{1/r-2} |\theta y_{j-i-1}^{1-2/r}(\xi) Z_{j-i-1} + (1-\theta) y_{j-i}^{1-2/r}(\xi) Z_{j-i}| \\
 & \leq C |y_j^\theta - x_i|^{-\alpha} x_i^{1/r-2} x_i^{1-2/r} |\theta Z_{j-i-1} + (1-\theta) Z_{j-i}| \\
 & \leq C |y_j^\theta - x_i|^{-\alpha} x_i^{-2} |y_j^\theta - x_i| \\
 & = C |y_j^\theta - x_i|^{1-\alpha} x_i^{-2}
 \end{aligned}
 \tag{C.49}$$

proof of Lemma 5.19. For $k \leq j < \min\{2i-1, N-1\}$

$$\begin{aligned}
 & \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_j^\theta)}{h_{i+1}} \\
 & \frac{Q_{j-i}^\theta(x_{i+1}) - Q_{j-i}^\theta(x_i)}{h_{i+1}} u'''(\eta_{j+1}^\theta) + Q_{j-i}^\theta(x_i) \frac{u'''(\eta_{j+1}^\theta) - u'''(\eta_j^\theta)}{h_{i+1}} \\
 & \leq Q_{j-i}^{\theta'}(\xi) C x_j^{\alpha/2-3} + Q_{j-i}^\theta(x_i) C u''''(\eta) \frac{h_i + h_{i+1}}{h_{i+1}}
 \end{aligned}
 \tag{C.50}$$

where $\xi \in [x_i, x_{i+1}]$, $\eta \in [x_{j-1}, x_{j+1}]$.

From (5.36), by Lemma C.6 and Lemma 5.17, we have

$$\begin{aligned}
 Q_{j-i}^{\theta'}(\xi) & \leq C h^2 \frac{|y_{j+1}^\theta - x_{i+1}|^{1-\alpha}}{\Gamma(2-\alpha)} x_{i+1}^{1-2/r} h_{j+1}^2 \\
 & \leq C h^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2
 \end{aligned}
 \tag{C.51}$$

And by defination

$$Q_{j-i}^\theta(x_i) = h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \leq C h^2 x_i^{2-2/r} \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2$$

With , we have

$$4^{-r} x_i \leq x_{k-1} \leq x_{j-1} < x_j \leq x_{2i-1} \leq 2^r x_i$$

So we have

$$\begin{aligned}
 & \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_j^\theta)}{h_{i+1}} \\
 & \leq C h^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 x_i^{\alpha/2-3} + C h^2 x_i^{2-2/r} \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 x_{j-1}^{\alpha/2-4} \\
 & = C h^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j^2
 \end{aligned}
 \tag{C.54}$$

844 while

$$845 \quad h_j \leq h_{2i-1} \leq 2^r h_i$$

846 Subsitute into the inequality above, we get the goal

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ 847 \quad (C.55) \quad & \leq \frac{1}{h_i} Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j 2^r h_i \\ & = Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned}$$

848 While, the later is similar. \square

849

850 **LEMMA C.8.** *There exists a constant $C = C(T, r)$ such that For $N/2 \leq i < N$,*
 851 *$N + 2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$, $l = 3, 4$, $\xi \in [x_{i-1}, x_{i+1}]$, we have*

$$852 \quad (C.56) \quad h_{j-i}^l(\xi) \leq Ch_j^l \leq Ch^2 h_j^{l-2}$$

$$853 \quad (C.57) \quad (h_{j-i-1}^l(\xi))' \leq C(r-1)h^2 h_j^{l-2}$$

$$854 \quad (C.58) \quad (h_{j-i}^3(\xi))'' \leq C(r-1)h^2 h_j$$

Proof.

$$\begin{aligned} 855 \quad (C.59) \quad & (h_{j-i}(\xi))' = y_{j-i}'(\xi) - y_{j-i-1}'(\xi) \\ & = \xi^{1/r-1}((2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}) \leq 0 \end{aligned}$$

856 Thus,

$$857 \quad (C.60) \quad Ch_j \leq h_{j+1} \leq h_{j-i}(\xi) \leq h_{j-i}(x_{i-1}) = h_{j-1} \leq Ch_j$$

858 So as $4^{-r}T \leq 2T - x_j \leq T$, $2^{-r}T \leq x_i \leq T$, we have

$$859 \quad (C.61) \quad h_{j-i}^l(\xi) \leq Ch_j^l \leq Ch^2(2T - x_j)^{2-2/r} h_j^{l-2} \leq Ch^2 h_j^{l-2}$$

860 Since

$$\begin{aligned} & |(2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}| \\ 861 \quad (C.62) \quad & = |(Z_{2N-(j-i)} - \xi^{1/r})^{r-1} - (Z_{2N-(j-i-1)} - \xi^{1/r})^{r-1}| \\ & = (r-1)Z_1(Z_{2N-(j-i-\gamma)} - \xi^{1/r})^{r-2} \quad \gamma \in [0, 1] \\ & \leq C(r-1)h(2T - x_j)^{1-2/r} \end{aligned}$$

862 we have

$$863 \quad (C.63) \quad |(h_{j-i}(\xi))'| \leq C(r-1)h(2T - x_j)^{1-2/r} x_i^{1/r-1}$$

864 And

$$\begin{aligned} & (h_{j-i}^l(\xi))' = lh_{j-i}^{l-1}(\xi)h_{j-i}'(\xi) \\ 865 \quad (C.64) \quad & \leq C(r-1)h_j^{l-1} h(2T - x_j)^{1-2/r} x_i^{1/r-1} \\ & \leq C(r-1)h^2 h_j^{l-2} (2T - x_j)^{2-3/r} x_i^{1-1/r} \\ & \leq C(r-1)h^2 h_j^{l-2} \end{aligned}$$

(C.65) □

$$\begin{aligned}
 (h_{j-i}^3(\xi))'' &= 6h_{j-i}(\xi)(y_{j-i}'(\xi) - y_{j-i-1}'(\xi))^2 + 3h_{j-i}^2(\xi)(y_{j-i}''(\xi) - y_{j-i-1}''(\xi)) \\
 &\leq C(r-1)h_j h^2 + Ch_j^2 \frac{1-r}{r} \xi^{1/r-2} ((2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-1-i)}) \\
 &\leq C(r-1)h_j h^2 + C(r-1)h_j^2 (C(r-2)h(2T - x_j)^{1-3/r} Z_{2N-(j-i)} + Z_1(2T - x_{j-1})^{1-2/r}) \\
 &\leq C(r-1)h_j h^2 + C(r-1)h_j^2 h = Ch^2 h_j
 \end{aligned}$$

LEMMA C.9. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For*

$N/2 \leq i < N$, $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in [x_{i-1}, x_{i+1}]$, we have

$$(C.66) \quad u''(y_{j-i}^\theta(\xi)) \leq C$$

$$(C.67) \quad (u''(y_{j-i}^\theta(\xi)))' \leq C$$

$$(C.68) \quad (u''(y_{j-i}^\theta(\xi)))'' \leq C$$

Proof.

$$(C.69) \quad x_{j-2} \leq y_{j-i}^\theta(\xi) \leq x_{j+1} \Rightarrow 4^{-r}T \leq 2T - y_{j-i}^\theta(\xi) \leq T$$

Thus, for $l = 2, 3, 4$,

$$(C.70) \quad u^{(l)}(y_{j-i}^\theta(\xi)) \leq C(2T - y_{j-i}^\theta(\xi))^{\alpha/2-l} \leq C$$

and

$$\begin{aligned}
 (y_{j-i}^\theta(\xi))' &= \theta y_{j-1-i}'(\xi) + (1-\theta)y_{j-i-1}'(\xi) \\
 (C.71) \quad &= \xi^{1/r-1}(\theta(2T - y_{j-1-i}(\xi))^{1-1/r} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-1/r}) \\
 &\leq C(2T - x_{j-2})^{1-1/r} \leq C
 \end{aligned}$$

With

$$(C.72) \quad Z_{2N-j-i} \leq 2T^{1/r}$$

$$\begin{aligned}
 (C.73) \quad (y_{j-i}^\theta(\xi))'' &= \theta y_{j-1-i}''(\xi) + (1-\theta)y_{j-i-1}''(\xi) \\
 &= \frac{1-r}{r} \xi^{1/r-2} (\theta(2T - y_{j-1-i}(\xi))^{1-2/r} Z_{2N-(j-i-1)} + (1-\theta)(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)}) \\
 &\leq C(r-1)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (C.74) \quad (u''(y_{j-i}^\theta(\xi)))' &= u'''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta(\xi))' \\
 &\leq C
 \end{aligned}$$

$$\begin{aligned}
 (C.75) \quad (u''(y_{j-i}^\theta(\xi)))'' &= u'''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta(\xi))' + u''''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta(\xi))'' \\
 &\leq C + C(r-1) = C
 \end{aligned}$$

LEMMA C.10. *There exists a constant $C = C(T, \alpha, r)$ such that*

$$(C.76) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}$$

$$(C.77) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C|y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N)$$

$$(C.78) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C(r-1)|y_j^\theta - x_i|^{-\alpha} + C|y_j^\theta - x_i|^{-1-\alpha}(|2T - x_i - y_j^\theta| + h_N)^2$$

Proof.

$$(C.79) \quad (y_{j-i}^\theta(\xi) - \xi)' = \theta y_{j-1-i}'(\xi) + (1-\theta)y_{j-i}'(\xi) - 1$$

$$(C.80) \quad |y_{j-i}'(\xi) - 1| = \xi^{1/r-1} |(2T - y_{j-i}(\xi))^{1-1/r} - \xi^{1-1/r}| \\ \leq \xi^{1/r-1} |2T - \xi - y_{j-i}(\xi)| \xi^{-1/r}$$

$$(C.81) \quad |2T - \xi - y_{j-i}(\xi)| \leq \max \begin{cases} |2T - x_{i-1} - x_{j-1}| \\ |2T - x_{i+1} - x_{j+1}| \end{cases} \\ \leq |2T - x_i - x_j| + h_{i+1} + h_j$$

$$(C.82) \quad (y_{j-i}^\theta(\xi) - \xi)'' = \theta y_{j-1-i}''(\xi) + (1-\theta)y_{j-i}''(\xi) \\ = \frac{1-r}{r} \xi^{1/r-2} (\theta(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)}) \leq 0$$

It's concave, so

$$(C.83) \quad y_{j-i}(\xi) - \xi \geq \min\{x_{j+1} - x_{i+1}, x_{j-1} - x_{i-1}\} \geq C(x_j - x_i)$$

We have

$$(C.84) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}$$

$$(C.85) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' = (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}(y_{j-i}^\theta(\xi) - \xi)' \\ \leq C|y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_{i+1} + h_{j-1})$$

$$(C.86) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' = (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}(y_{j-i}^\theta(\xi) - \xi)'' + \alpha(\alpha-1)|y_{j-i}^\theta(\xi) - \xi|^{-1-\alpha}(y_{j-i}^\theta(\xi) - \xi)'(y_{j-i}^\theta(\xi) - \xi)' \\ \leq C(r-1)|y_j^\theta - x_i|^{-\alpha} + C|y_j^\theta - x_i|^{-1-\alpha}(|2T - x_i - y_j^\theta| + h_{i+1} + h_{j-1})^2$$

Proof. From (5.24), by Lemma C.8 and Lemma C.10, we have $\xi \in [x_i, x_{i+1}]$

$$(C.87) \quad Q_{j-i}^\theta(\xi) \leq Ch^2 h_j^2 ((r-1)|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N))$$

$$(C.88) \quad Q_{j-i}^\theta(\xi) \leq Ch^2 h_j^2 |y_j^\theta - x_i|^{1-\alpha}$$

So use the skill in Proof 34 with Lemma C.9

$$(C.89) \quad \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ \leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N))$$

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