AN EXAMPLE ARTICLE*

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1. Introduction.

The introduction introduces the context and summarizes the manuscript. It is importantly to clearly state the contributions of this piece of work.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$, suppose $f \in C^2(\Omega)$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

THEOREM 1.1. *Let u be a solution of (1.1) on Ω. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \le C[x(2T-x)]^{\alpha/2-l} \tag{1.4}$$

The paper is organized as follows. Our main results are in section 3, experimental results are in section 6, and the conclusions follow in section 8.

2. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases} \tag{2.1}$$

where $r \ge 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N \tag{2.2}$$

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Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

And then, we can approximate $u(x)$ with

$$u_h(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{2.4}$$

For convience, we denote

$$I_h^{2-\alpha}(x) := \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x-y|^{1-\alpha} u_h(y)dy \tag{2.5}$$

And now, we can approximate the operator (1.2) at x_i with

$$\begin{aligned} D_h^\alpha u_h(x_i) &:= D_h^2 I_h^{2-\alpha}(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} I_h^{2-\alpha}(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) I_h^{2-\alpha}(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha}(x_{i+1}) \right) \end{aligned} \tag{2.6}$$

Finally, we approximate the equation (1.1) with

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i), \quad 1 \le i \le 2N-1 \tag{2.7}$$

The discrete equation (2.7) can be written in matrix form

$$AU = F \tag{2.8}$$

where U is unknown, $F = (f(x_1), \cdots, f(x_{2N-1}))$. The matrix A is constructed as follows: Since

$$\begin{aligned} I_h^{2-\alpha}(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y)dy \\ &= \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u(x_j)\phi_j(y)dy \\ &= \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y)dy \\ &= \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\ &=: \sum_{j=1}^{2N-1} \tilde{a}_{ij}\, u(x_j), \quad 0 \le i \le 2N \end{aligned} \tag{2.9}$$

Then, substitute in (2.6), we have

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} a_{ij}\, u(x_j) \tag{2.10}$$

where

$$a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}\tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}}\tilde{a}_{i+1,j} \right) \tag{2.11}$$

3. Main results. Here we state our main results; the proof is deferred to section 4 and section 5.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 3.1 (Truncation Error). *If $f \in C^2(\Omega)$ and $\alpha \in (1,2)$, and $u(x)$ is a solution of the equation* (1.1)*, then there exists a constant $C = C(T, \alpha, r, \|f\|_{C^2(\Omega)})$, such that the truncation error of the discrete format satisfies*

$$\begin{aligned} |-\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i)| \le & C(h^{r\alpha/2+r}(x_i^{-1-\alpha} + (2T - x_i)^{-1-\alpha}) \\ & + h^2(x_i^{-\alpha/2-2/r} + (2T - x_i)^{-\alpha/2-2/r}) \\ & + h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \le i \le N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \le 2N-1 \end{cases} \quad) \end{aligned} \tag{3.1}$$

THEOREM 3.2 (Convergence). *The discrete equation* (2.7) *has sulotion U, and there exists a positive constant $C = C(T, \alpha, r, \|f\|_{C^2(\Omega)})$ such that the error between the numerial solution U with the exact solution $u(x_i)$ satisfies*

$$\max_{1 \le i \le 2N-1} |U_i - x(x_i)| \le C h^{\min\{\frac{r\alpha}{2}, 2\}} \tag{3.2}$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

4. Proof of Theorem 3.1. For convience, let's denote

$$I^{2-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x - y|^{1-\alpha} u(y) dy \tag{4.1}$$

Then, the truncation error of the discrete format can be written as

$$\begin{aligned} -\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I_h^{2-\alpha}(x_i) - \frac{d^2}{dx^2} I^{2-\alpha}(x_i)) \\ &= -\kappa_\alpha D_h^2 (I_h^{2-\alpha}(x_i) - I^{2-\alpha}(x_i)) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \end{aligned} \tag{4.2}$$

THEOREM 4.1. *There exits a constant $C = C(T, r, \|f\|_{C^2(\Omega)})$ such that*

$$-\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \le C h^2 (x_i^{1-2/r} + (2T - x_i)^{1-2/r}) \tag{4.3}$$

Proof. Since $f \in C^2(\Omega)$ and

$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha}(x)) = f(x), \quad x \in \Omega, \tag{4.4}$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.2) of Lemma A.1, for $1 \leq i \leq 2N-1$, we have

$$-\kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}(x_i) = \frac{h_{i+1}-h_i}{3}f'(x_i) + \frac{1}{4!}\frac{2}{h_i+h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \tag{4.5}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. By Lemma B.2, we have 1.

$$\left|\frac{h_{i+1}-h_i}{3}f'(x_i)\right| \leq \frac{\|f\|_{C^1(\Omega)}}{3}2^{|r-2|}r(r-1)T^{2/r}h^2(x_i^{1-2/r} + (2T-x_i)^{1-2/r}) \tag{4.6}$$

2.

$$\begin{aligned} &\frac{1}{4!}\frac{2}{h_i+h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\ &\qquad \leq \frac{\|f\|_{C^2(\Omega)}}{12}(h_i^2 - h_i h_{i+1} + h_{i+1}^2) \end{aligned} \tag{4.7}$$

$\square$

5. Proof of Theorem 3.2. aaaaaaaaaa

6. Experimental results. Figure 1 shows some example results. Additional results are available in the supplement in Table 1. Table 1 shows additional supporting evidence.

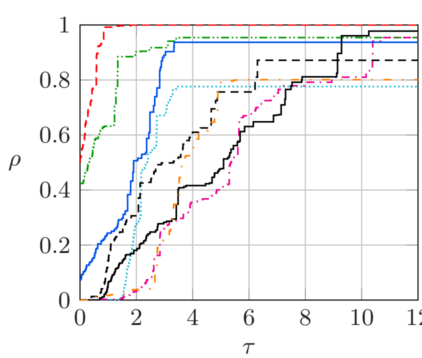

FIG. 1. *Example figure using external image files.*

TABLE 1
Example table.

Species	**Mean**	**Std. Dev.**
1	3.4	1.2
2	5.4	0.6
3	7.4	2.4
4	9.4	1.8

7. Discussion of $Z = X \cup Y$.

8. Conclusions. Some conclusions here.

Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x)$ is twice differentable continous function on open set Ω, there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

$$\begin{aligned}\text{(A.1)}\qquad &\frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})g(x_i) + \frac{1}{h_i}g(x_{i-1})\right)\\ &= g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]\end{aligned}$$

And if $g(x) \in C^4(\Omega)$, then

(A.2)

$$\begin{aligned}&\frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})g(x_i) + \frac{1}{h_i}g(x_{i-1})\right)\\ &= g''(x_i) + \frac{h_{i+1} - h_i}{3}g'''(x_i) + \frac{1}{4!}\frac{2}{h_i + h_{i+1}}(h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2))\end{aligned}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$.

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \frac{(x_i - x_{i-1})^2}{2}g''(\xi_1), \quad \xi_1 \in [x_{i-1}, x_i]$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2}g''(\xi_2), \quad \xi_2 \in [x_i, x_{i+1}]$$

Subsitute them in the left side of (A.1), we have

$$\begin{aligned}&\frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})g(x_i) + \frac{1}{h_i}g(x_{i-1})\right)\\ &= \frac{h_i}{h_i + h_{i+1}}g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}}g''(\xi_2)\end{aligned}$$

Now, using intermediate value theorem , there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}}g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}}g''(\xi_2) = g''(\xi)$$

And for the second equation, similarly

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2}g''(x_i) - \frac{h_i^3}{3!}g'''(x_i) + \frac{h_i^4}{4!}g''''(\eta_1)$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2}g''(x_i) + \frac{h_{i+1}^3}{3!}g'''(x_i) + \frac{h_{i+1}^4}{4!}g''''(\eta_2)$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. Subsitute them to the left side of (A.2), we can get the result. □

LEMMA A.2. *If $y \in [x_{j-1}, x_j]$, denote $y = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0, 1]$,*

$$\text{(A.3)}\qquad u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2}h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j]$$

(A.4)
$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2}h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!}h_j^3(\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$
where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$.

Proof. By Taylor expansion, we have
$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!}u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$
$$u(x_j) = u(y_j^\theta) + (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!}u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$
Thus
$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2}h_j^2(\theta u''(\xi_1) + (1-\theta)u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2}h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$
The second equation is similar,
$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!}u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!}u'''(\eta_1)$$
$$u(x_j) = u(y_j^\theta) + (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!}u''(\xi_2) + \frac{(1-\theta)^3 h_j^3}{3!}u'''(\eta_2)$$
where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$. Thus
$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2}h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!}h_j^3(\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$
□

Appendix B. Inequality.

LEMMA B.1.

(B.1)
$$h_i \le rT^{1/r}h \begin{cases} x_i^{1-1/r}, & 1 \le i \le N \\ (2T - x_{i-1})^{1-1/r}, & N < i \le 2N-1 \end{cases}$$

Proof. For $1 \le i \le N$,
$$\begin{aligned} h_i &= T\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right) \\ &\le rT\frac{1}{N}\left(\frac{i}{N}\right)^{r-1} = rT^{1/r}hx_i^{1-1/r} \end{aligned}$$
For $N < i \le 2N-1$,
$$\begin{aligned} h_i &= T\left(\left(\frac{2N-i}{N}\right)^r - \left(\frac{2N-i+1}{N}\right)^r\right) \\ &\le rT\frac{1}{N}\left(\frac{2N-i+1}{N}\right)^{r-1} = rT^{1/r}h(2T - x_{i-1})^{1-1/r} \end{aligned}$$
□

Lemma B.2. *There is a constant $C = 2^{|r-2|}r(r-1)T^{2/r}$ such that for all $i \in \{1, 2, \cdots, 2N-1\}$*

$$|h_{i+1} - h_i| \le Ch^2(x_i^{1-2/r} + (2T - x_i)^{1-2/r}) \tag{B.2}$$

Proof.

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \le i \le N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \le i \le 2N-1 \end{cases}$$

For $i = 1$,

$$h_2 - h_1 = T(2^r - 2)\left(\frac{1}{N}\right)^r = (2^r - 2)T^{2/r}h^2x_1^{1-2/r}$$

For $2 \le i \le N-1$,

$$h_{i+1} - h_i = r(r-1)T\ N^{-2}\eta^{r-2}, \quad \eta \in [\frac{i-1}{N}, \frac{i+1}{N}]$$

If $r \in [1, 2)$,

$$\begin{aligned} h_{i+1} - h_i &\le r(r-1)T\ N^{-2}\eta^{r-2} \le r(r-1)T\ h^2\left(\frac{i-1}{N}\right)^{r-2} \\ &\le r(r-1)T\ h^2 2^{2-r}\left(\frac{i}{N}\right)^{r-2} \\ &= 2^{2-r}r(r-1)T^{2/r}h^2x_i^{1-2/r} \end{aligned}$$

else if $r > 2$,

$$\begin{aligned} h_{i+1} - h_i &\le r(r-1)T\ N^{-2}\eta^{r-2} \le r(r-1)T\ h^2\left(\frac{i+1}{N}\right)^{r-2} \\ &\le r(r-1)T\ h^2 2^{r-2}\left(\frac{i}{N}\right)^{r-2} \\ &= 2^{r-2}r(r-1)T^{2/r}h^2x_i^{1-2/r} \end{aligned}$$

Since

$$2^r - 2 \le 2^{|r-2|}r(r-1), \quad r \ge 1$$

we have

$$h_{i+1} - h_i \le 2^{|r-2|}r(r-1)T^{2/r}h^2x_i^{1-2/r}, \quad 1 \le i \le N-1$$

For $i = N$, $h_{N+1} - h_N = 0$. For $N < i \le 2N-1$, it's central symmetric to the first half of the proof, which is

$$h_i - h_{i+1} \le 2^{|r-2|}r(r-1)T^{2/r}h^2(2T - x_i)^{1-2/r}$$

Summarizes the inequalities, we can get

$$|h_{i+1} - h_i| \le 2^{|r-2|} r(r-1) T^{2/r} h^2 (x_i^{1-2/r} + (2T - x_i)^{1-2/r}) \tag{B.3}$$

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