1 问题

对于 $\Omega = (0,1)$, $1 < \alpha < 2$,

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1}$$

其中

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = C_R \frac{d^2}{dx^2} \int_\Omega \frac{u(y)}{|x-y|^{\alpha-1}} dy \tag{2}$$

2 数值格式

用线性插值代替原函数，中心差分代替二阶导数，记 $u_h(x)$ 为 $u(x)$ 在网格点上的线性插值。

我们解这样的数值解

$$\begin{aligned} C_R(& \frac{2}{h_{i+1}(h_i + h_{i+1})} \int_\Omega \frac{u_h(x)}{|x_{i+1} - y|^{\alpha-1}} dy \\ & - \frac{2}{h_i h_{i+1}} \int_\Omega \frac{u_h(x)}{|x_i - y|^{\alpha-1}} dy \\ & + \frac{2}{h_i(h_i + h_{i+1})} \int_\Omega \frac{u_h(x)}{|x_{i-1} - y|^{\alpha-1}} dy) \\ = F_i & \end{aligned} \tag{3}$$

矩阵 A 是 M 矩阵，主队角正，其他负，严格对角占优。

3 一致网格

当 $r = 1$, 网格成为一致网格, $x_i = ih, h = \frac{1}{2N}, i = 0, ..., 2N$.

A 等于

$$\begin{aligned} a_{ij} = & \frac{C_R}{(2-\alpha)(3-\alpha)} h^{-\alpha} \\ & \left(|i-j-2|^{3-\alpha} - 4|i-j-1|^{3-\alpha} + 6|i-j|^{3-\alpha} - 4|i-j+1|^{3-\alpha} + |i-j+2|^{3-\alpha}\right) \end{aligned} \tag{4}$$

矩阵行和

$$S_i = \sum_{j=1}^{2N-1} a_{ij} = \frac{C_R}{(2-\alpha)(3-\alpha)} h^{-\alpha}(|i+1|^{3-\alpha} - 3|i|^{3-\alpha} + 3|i-1|^{3-\alpha} - |i-2|^{3-\alpha} + ...2N) \tag{5}$$

我们得到

$$S_i \geq C(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \tag{6}$$

下面估计截断误差 R_i. 目标是

$$R_i \leq Ch^{\alpha/2} S_i \tag{7}$$

这样我们就有

$$\epsilon \leq \max_i \frac{R_i}{S_i} \leq Ch^{\alpha/2} \tag{8}$$

考虑 R_1

$$R_1 = \int_\Omega (u(y) - u_h(y)) \frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} dy \tag{9}$$

我们有

$$R_1 = \int_0^{3h} + \int_{3h}^1 \tag{10}$$

当 $y > 3h$,

$$\frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} \leq C|y|^{-1-\alpha} \tag{11}$$

那么

$$
\begin{aligned}
I_2 &\leq C\int_{3h}^{1}|y|^{-1-\alpha}u''(\eta)h^2dy \\
&\leq C\int_{3h}^{1}|y|^{-1-\alpha}h^{\alpha/2-2}h^2dy \\
&\leq Ch^{\alpha/2}\int_{3h}^{1}y^{-1-\alpha}dy \\
&\leq Ch^{\alpha/2}h^{-\alpha} = Ch^{-\alpha/2} \\
&\leq Ch^{\alpha/2}x_1^{-\alpha} \leq Ch^{\alpha/2}S_1
\end{aligned} \tag{12}
$$

在考虑

$$
\begin{aligned}
I_1 &= \int_0^{3h}\frac{u(y)-u_h(y)}{h^2}(|y|^{1-\alpha}-2|y-h|^{1-\alpha}+|y-2h|^{1-\alpha})dy \\
&= \int_0^h + \int_h^{3h} = J_1 + J_2
\end{aligned} \tag{13}
$$

$$
J_2 \leq Cu''(\eta)h^{2-\alpha} \leq Ch^{\alpha/2-2}h^{2-\alpha} \leq Ch^{-\alpha/2} \tag{14}
$$

因为

$$
\begin{aligned}
|u(x)-u_h(x)| &\leq \int_0^h|u'(y)|dy \\
&\leq C\int_0^h y^{\alpha/2-1}dy \\
&\leq Ch^{\alpha/2} \quad ,x\in(0,h)
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
J_1 &= \int_0^{h}\frac{u(y)-u_h(y)}{h^2}(|y|^{1-\alpha}-2|y-h|^{1-\alpha}+|y-2h|^{1-\alpha})dy \\
&\leq Ch^{\alpha/2-2}h^{2-\alpha} = Ch^{-\alpha/2}
\end{aligned} \tag{16}
$$

所以有

$$
R_1 \leq Ch^{-\alpha/2} \leq Ch^{\alpha/2}h^{-\alpha} \leq Ch^{\alpha/2}S_1, \quad (S_1 \geq Cx_1^{-\alpha}) \tag{17}
$$

R_1, R_2, R_3 全部类似。

3.1 猜想

$$R_i \le Ch^{\alpha/2+1}(x_i^{-\alpha-1} + (1-x_i)^{-\alpha-1}) \quad (then \le Ch^{\alpha/2}S_i) \tag{18}$$

为了简便，我们记 $D(y) := u(y) - u_h(y)$.

当 $3 < i \le N$ 时，

$$\begin{aligned}
R_i =& \int_0^1 D(y)\frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2}dy \\
=& \int_0^{x_1} D(y)\frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2}dy \\
&+ \int_{x_1}^{x_{\lceil \frac{i}{2}\rceil}} D(y)\frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2}dy \\
&+ \int_{x_{\lceil \frac{i}{2}\rceil}}^{x_{\lceil \frac{i}{2}\rceil+1}} \frac{D(y+h)-D(y)}{h^2}|y-x_i|^{1-\alpha} + D(y)\frac{|y-x_{i+1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2}dy \\
&+ \int_{x_{\lceil \frac{i}{2}\rceil+1}}^{x_i} \frac{D(y-h)-2D(y)+D(y+h)}{h^2}|y-x_i|^{1-\alpha}dy \\
&+ \cdots (2N-i) \\
=& I_1 + I_2 + I_3 + I_4 + \cdots
\end{aligned} \tag{19}$$

1.

$$\begin{aligned}
I_1 =& \int_0^{x_1} (u(y)-u_h(y))\frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2}dy \\
&\le Ch^{\alpha/2}\int_0^h |y-x_i|^{-1-\alpha}dy \\
&\le Ch^{\alpha/2+1}x_i^{-1-\alpha}
\end{aligned} \tag{20}$$

2.

$$
\begin{aligned}
I_2 &= \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} D(y)\frac{|y-x_{i-1}|^{1-\alpha}-2|y-x_i|^{1-\alpha}+|y-x_{i+1}|^{1-\alpha}}{h^2}dy \\
&\leq C\int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2}h^2|x_i-y|^{-1-\alpha}dy \\
&\leq Ch^{\alpha/2-1}h^2x_i^{-1-\alpha} \leq Ch^{\alpha/2+1}x_i^{-1-\alpha}
\end{aligned}
\tag{21}
$$

3.

$$
\begin{aligned}
I_3 &= \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} \frac{D(y+h)-D(y)}{h^2}|y-x_i|^{1-\alpha}+D(y)\frac{|y-x_{i+1}|^{1-\alpha}-|y-x_i|^{1-\alpha}}{h^2}dy \\
&\leq \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} u'''(\eta_1)h|x_i-y|^{1-\alpha}+u''(\eta_2)h|x_i-y|^{-\alpha}dy \\
&\leq Ch^2x_i^{-2-\alpha/2} \leq Ch^{1+\alpha/2}x_i^{-1-\alpha}
\end{aligned}
\tag{22}
$$

4.

$$
\begin{aligned}
I_4 &= \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} \frac{D(y-h)-2D(y)+D(y+h)}{h^2}|y-x_i|^{1-\alpha}dy \\
&\leq \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} u''''(\eta)h^2|x_i-y|^{1-\alpha}dy \\
&\leq Cx_i^{\alpha/2-4}h^2x_i^{2-\alpha} \\
&\leq Ch^2x_i^{-2-\alpha/2} \leq Ch^{1+\alpha/2}x_i^{-1-\alpha}
\end{aligned}
\tag{23}
$$

猜想证毕，一致网格证完。

4 非一致

$r > 1$