# AN EXAMPLE ARTICLE*

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**Abstract.** This is an example SIAM LaTeX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

**Key words.** example, LaTeX

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## 1. Introduction.

The introduction introduces the context and summarizes the manuscript. It is importantly to clearly state the contributions of this piece of work.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$, suppose $f \in C^\beta(\Omega), \beta > 4 - \alpha, \|f\|_\beta^{(\alpha/2)} < \infty$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

and the solution $u \in C^{\alpha/2}(\Omega)$.

## 2. Regularity.

*Remark* 2.1. 1. $C^k(U)$ is the set of all $k$-times continuously differentiable functions on open set $U$.
2. $C^\beta(U)$ is the collection of function $f$ which for any $V \subset\subset U$ $f|_V \in C^\beta(\bar{V})$.

THEOREM 2.2. *If $f \in C^\beta(\Omega), \beta > 2$ and $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

$$|f^{(l)}(x)| \leq \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \leq T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \leq x < 2T \end{cases} \tag{2.1}$$

THEOREM 2.3 (Regularity up to the boundary [1]).

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C \left( \|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right) \tag{2.2}$$

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COROLLARY 2.4. *Let $u$ be a solution of (1.1) on $\Omega$. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \leq \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases} \tag{2.3}$$

The paper is organized as follows. Our main results are in section 4, experimental results are in section 7, and the conclusions follow in section 8.

## 3. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \leq i \leq N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \leq i \leq 2N \end{cases} \tag{3.1}$$

where $r \geq 1$ . And let

$$h_j = x_j - x_{j-1}, \quad 1 \leq j \leq 2N \tag{3.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \leq x \leq x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \leq x \leq x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

And then, we can approximate $u(x)$ with

$$u_h(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{3.4}$$

For convience, we denote

$$I_h^{2-\alpha}(x_i) := \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y)dy \tag{3.5}$$

And now, we can approximate the operator (1.2) at $x_i$ with

$$\begin{aligned} D_h^\alpha u_h(x_i) &:= D_h^2 I_h^{2-\alpha}(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} I_h^{2-\alpha}(x_{i-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) I_h^{2-\alpha}(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha}(x_{i+1}) \right) \end{aligned} \tag{3.6}$$

Finally, we approximate the equation (1.1) with

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i), \quad 1 \leq i \leq 2N-1 \tag{3.7}$$

The discrete equation (3.7) can be written in matrix form

$$AU = F \tag{3.8}$$

where $U$ is unknown, $F = (f(x_1), \cdots, f(x_{2N-1}))$. The matrix $A$ is constructed as follows: Since

(3.9)

$$\begin{aligned}
I_h^{2-\alpha}(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u(x_j) \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left( \frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\
&=: \sum_{j=1}^{2N-1} \tilde{a}_{ij}\, u(x_j), \quad 0 \le i \le 2N
\end{aligned}$$

Then, substitute in (3.6), we have

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} a_{ij}\, u(x_j) \tag{3.10}$$

where

$$a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \tilde{a}_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \tag{3.11}$$

**4. Main results.** Here we state our main results; the proof is deferred to section 5 and section 6.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 4.1 (Truncation Error). *If $f \in C^2(\Omega)$ and $\alpha \in (1,2)$, and $u(x)$ is a solution of the equation (1.1), then there exists a constant $C_1, C_2 = C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{C^2(\Omega)}), C_2(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$, such that the truncation error of the discrete format satisfies*

$$\begin{aligned}
|-\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i)| \le & C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} (x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \\
& + C_2 h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \le i \le N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \le 2N-1 \end{cases}
\end{aligned} \tag{4.1}$$

*where $C_2 = 0$ if $r = 1$.*

THEOREM 4.2 (Convergence). *The discrete equation (3.7) has sulotion $U$, and there exists a positive constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$ such that the error between the numerial solution $U$ with the exact solution $u(x_i)$ satisfies*

$$\max_{1 \le i \le 2N-1} |U_i - u(x_i)| \le C h^{\min\{\frac{r\alpha}{2}, 2\}} \tag{4.2}$$

*That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.*

## 5. Proof of Theorem 4.1.

For convience, let's denote

$$I^{2-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x-y|^{1-\alpha} u(y) dy \tag{5.1}$$

Then, the truncation error of the discrete format can be written as

$$\begin{aligned} -\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I_h^{2-\alpha}(x_i) - \frac{d^2}{dx^2} I^{2-\alpha}(x_i)) \\ &= -\kappa_\alpha D_h^2 (I_h^{2-\alpha} - I^{2-\alpha})(x_i) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \end{aligned} \tag{5.2}$$

### 5.1. Estimate of $-\kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}(x_i)$.

THEOREM 5.1. *There exits a constant $C = C(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$ such that*

$$\left| -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \right| \le Ch^2 (x_i^{-\alpha/2-2/r} + (2T - x_i)^{-\alpha/2-2/r}) \tag{5.3}$$

*Proof.* Since $f \in C^2(\Omega)$ and

$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha}(x)) = f(x), \quad x \in \Omega, \tag{5.4}$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.3) of Lemma A.1, for $1 \le i \le 2N-1$, we have

$$-\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \tag{5.5}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

$$\left| \frac{h_{i+1} - h_i}{3} f'(x_i) \right| \le \frac{\|f\|_\beta^{(\alpha/2)}}{3} Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{-\alpha/2-2/r}, & N < i \le 2N-1 \end{cases} \tag{5.6}$$

2. See Proof 17, there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

$$\begin{aligned} &\left| \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \right| \\ &\le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases} \end{aligned} \tag{5.7}$$

Summarizes, we get the result. ∎

### 5.2. Estimate of $R_i$.

Now, we study the first part of (5.2)

$$D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i) = D_h^2 (\int_0^{2T} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy) \tag{5.8}$$

For convience, let's denote

$$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \tag{5.9}$$

And define

$$R_i := D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i)$$
$$= \frac{2}{h_i + h_{i+1}} \sum_{j=1}^{2N} \left( \frac{1}{h_i} T_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \tag{5.10}$$

We have some results about the estimate of $R_i$

THEOREM 5.2. *For $1 \le i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$R_i \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2} x_i^{-1-\alpha}, & \alpha/2 - 2/r < 0 \end{cases} \tag{5.11}$$

THEOREM 5.3. *For $N/2 \le i \le N$, there exists constant $C, C_2$ such that*

$$R_i \le Ch^2 x_i^{-\alpha/2-2/r} + C_2 h^2 |T - x_{i-1}|^{1-\alpha} \tag{5.12}$$

*where $C_2 = 0$ if $r = 1$.*

And for $N < i \le 2N - 1$, it is symmetric to the previous case.

To prove these results, we need some utils. Also for simplicity, we denote

DEFINITION 5.4.

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} T_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \tag{5.13}$$

*then*

$$R_i = \sum_{j=1}^{2N} S_{ij} \tag{5.14}$$

### 5.3. Proof of Theorem 5.2.

LEMMA 5.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \le i < N/2$,*

$$\sum_{j=\max\{2i+1, i+3\}}^{N} S_{ij} \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.15}$$

*Proof.* For $\max\{2i+1, i+3\} \le j \le N$, by Lemma C.1 and Lemma C.2

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left( \frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) dy \\ &\le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy \end{aligned} \tag{5.16}$$

Therefore,

$$
\begin{aligned}
\sum_{j=\max\{2i+1,i+3\}}^{N} S_{ij} &\leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\
&= \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\
&\leq \frac{C}{\alpha/2+2/r} 2^{r(-\alpha/2-2/r)} h^2 x_i^{-\alpha/2-2/r}
\end{aligned}
\tag{5.17}
$$

∎

LEMMA 5.6. *Thert exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \leq i < N/2$,*

$$
\sum_{j=N+1}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\tag{5.18}
$$

*Proof.* For $1 \leq i < N/2, N+1 \leq j \leq 2N-1$, by equation (C.2) and Lemma C.2

$$
\begin{aligned}
S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left( \frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\
&\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2-2/r} y^{-1-\alpha} dy \\
&\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j=N+1}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T - y)^{\alpha/2-2/r} dy \\
&\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\
&= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r}\ h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\end{aligned}
\tag{5.19}
$$

And by Lemma A.3

$$S_{i,2N} \leq CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}$$

And when $\alpha/2 - 2/r + 1 \geq 0$,

$$h^{r\alpha/2+r} \leq h^2$$

Summarizes, we get the result. ∎

For $i = 1, 2$.

LEMMA 5.7. *By Lemma* C.5 , *Lemma* 5.5 *and Lemma* 5.6 *we get*

$$\begin{aligned} R_1 &= \sum_{j=1}^{3} S_{1j} + \sum_{j=4}^{2N} S_{1j} \\ &\le Ch^2 x_1^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1>0 \\ Ch^2\ln(N), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0 \end{cases} \end{aligned} \tag{5.20}$$

$$\begin{aligned} R_2 &= \sum_{j=1}^{4} S_{2j} + \sum_{j=5}^{2N} S_{2j} \\ &\le Ch^2 x_2^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1>0 \\ Ch^2\ln(N), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0 \end{cases} \end{aligned} \tag{5.21}$$

For $3 \le i < N/2$, we have a new separation of $R_i$, Let's denote $k = \lceil \frac{i}{2} \rceil$.

$$\begin{aligned} R_i =& \sum_{j=1}^{2N} \frac{2}{h_i+h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& \sum_{j=1}^{k-1} \frac{2}{h_i+h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &+ \frac{2}{h_i+h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\ &+ \sum_{j=k+1}^{2i-1} \frac{2}{h_i+h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &+ \frac{2}{h_i+h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \\ &+ \sum_{j=2i+1}^{2N} \frac{2}{h_i+h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned} \tag{5.22}$$

LEMMA 5.8. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i \le N, k = \lceil \frac{i}{2} \rceil$

$$|I_1| = |\sum_{j=1}^{k-1} S_{ij}| \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1>0 \\ Ch^2 x_i^{-1-\alpha}\ln(i), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2-2/r+1<0 \end{cases} \tag{5.23}$$

*Proof.* For $2 \leq j \leq k-1$, by Lemma C.1 and Lemma C.3

$$
\begin{aligned}
S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left( \frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\
&\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} dy \qquad (5.24) \\
&= Ch^2 x_i^{-1-\alpha} \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} dy
\end{aligned}
$$

And by Lemma A.3 , Lemma C.3

$$
S_{i1} \leq C x_1^{\alpha/2} x_1 x_i^{-1-\alpha} = C x_1^{\alpha/2+1} x_i^{-1-\alpha} = C T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha} \tag{5.25}
$$

Therefore,

$$
\begin{aligned}
I_1 &= \sum_{j=1}^{k-1} S_{ij} = S_{i1} + \sum_{j=2}^{k-1} S_{ij} \\
&\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \qquad (5.26) \\
&\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy
\end{aligned}
$$

But

$$
\int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \leq \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.27}
$$

So we have

$$
I_1 \leq \begin{cases} \frac{C}{\alpha/2-2/r+1} h^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} h^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.28}
$$

□

DEFINITION 5.9. *For convience, let's denote*

$$
V_{ij} = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \tag{5.29}
$$

THEOREM 5.10. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \leq i < N/2, k = \lceil \frac{i}{2} \rceil$,*

$$
I_3 = \sum_{j=k+1}^{2i-1} V_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r} \tag{5.30}
$$

To estimete $V_{ij}$, we need some preparations.

LEMMA 5.11. *Denote* $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0,1]$, *by Lemma* A.2

$$\begin{aligned} T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= \int_{x_{j-1}}^{x_j} -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!} h_j^3 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) dy_j^\theta \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta \end{aligned} \tag{5.31}$$

*where* $\eta_{j1}^\theta \in [x_{j-1}, y_j^\theta], \eta_{j2}^\theta \in [y_j^\theta, x_j]$.

Now Let's construct a series of functions to represent $T_{ij}$.

DEFINITION 5.12.

$$y_{j-i}(x) = (x^{1/r} + Z_{j-i})^r, \quad Z_{j-i} = T^{1/r}\frac{j-i}{N} \tag{5.32}$$

$$y_{j-i}^\theta(x) = \theta y_{j-1-i}(x) + (1-\theta) y_{j-i}(x) \tag{5.33}$$

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{5.34}$$

*Now, we define*

$$P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.35}$$

$$Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.36}$$

And now we can rewrite $T_{ij}$

LEMMA 5.13. *For* $2 \le i \le N, 2 \le j \le N$,

$$\begin{aligned} T_{ij} &= \int_0^1 -\frac{\theta(1-\theta)}{2} P_{j-i}^\theta(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)}{3!} (\theta^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j1}^\theta) - (1-\theta)^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j2}^\theta)) d\theta \end{aligned} \tag{5.37}$$

Immediately, we can see from (5.29) that

LEMMA 5.14. *For* $3 \leq i \leq N-1$, $3 \leq j \leq N-1$,

$$
\begin{aligned}
V_{ij} &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^\theta(x_i) d\theta \\
&\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\
&\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\
&\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,2}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\
&\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta
\end{aligned} \tag{5.38}
$$

To estimate $V_{ij}$, we first estimate $D_h^2 P_{j-i}^\theta(x_i)$ , but By Lemma A.1,

$$D_h^2 P_{j-i}^\theta(x_i) = {P_{j-i}^\theta}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \tag{5.39}$$

By Leibniz formula, we calculate and estimate the derivations of $h_{j-i}^3$, $u''(y_{j-i}^\theta(x))$ and $\frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$ separately.

Firstly, we have

LEMMA 5.15. *There exists a constant* $C = C(T, r)$ *such that For* $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$h_{j-i}^3(\xi) \leq C h^2 x_i^{2-2/r} h_j \tag{5.40}$$

$$(h_{j-i}^3(\xi))' \leq C(r-1) h^2 x_i^{1-2/r} h_j \tag{5.41}$$

$$(h_{j-i}^3(\xi))'' \leq C(r-1) h^2 x_i^{-2/r} h_j \tag{5.42}$$

The proof of this theorem see Lemma C.6 and Lemma C.7

Second,

LEMMA 5.16. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$u''(y_{j-i}^\theta(\xi)) \leq C x_i^{\alpha/2-2} \tag{5.43}$$

$$(u''(y_{j-i}^\theta(\xi)))' \leq C x_i^{\alpha/2-3} \tag{5.44}$$

$$(u''(y_{j-i}^\theta(\xi)))'' \leq C x_i^{\alpha/2-4} \tag{5.45}$$

The proof of this theorem see Proof 24

And Finally, we have

LEMMA 5.17. *There exists a constant* $C = C(T, \alpha, r)$ *such that For* $3 \leq i \leq N-1, 1 \leq j \leq \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha} \tag{5.46}$$

$$(|y^\theta_{j-i}(\xi)-\xi|^{1-\alpha})' \le C|y^\theta_j - x_i|^{1-\alpha}x_i^{-1} \tag{5.47}$$

$$(|y^\theta_{j-i}(\xi)-\xi|^{1-\alpha})'' \le C|y^\theta_j - x_i|^{1-\alpha}x_i^{-2} \tag{5.48}$$

*where* $y^\theta_j = \theta x_{j-1} + (1-\theta)x_j$

The proof of this theorem see Proof 25

Lemma 5.18. *There exists a constant* $C = C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that For* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$,

$$D_h^2 P^\theta_{j-i}(x_i) \le Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \tag{5.49}$$

*where* $y^\theta_j = \theta x_{j-1} + (1-\theta)x_j$

*Proof.* Since

$$D_h^2 P^\theta_{j-i}(x_i) = {P^\theta_{j-i}}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \tag{5.50}$$

From (5.35), using Leibniz formula and Lemma 5.15, Lemma 5.16 and Lemma 5.17∎

Lemma 5.19. *There exists a constant* $C = C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that for* $3 \le i < N, k = \lceil \frac{i}{2} \rceil$.
*For* $k \le j \le \min\{2i-1, N-1\}$,

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}&\left(\frac{Q^\theta_{j-i}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{j-i}(x_i)u'''(\eta^\theta_j)}{h_{i+1}}\right)\\ &\le Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j\end{aligned} \tag{5.51}$$

*And for* $k+1 \le j \le \min\{2i, N\}$,

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}&\left(\frac{Q^\theta_{j-i}(x_i)u'''(\eta^\theta_j) - Q^\theta_{j-i}(x_{i-1})u'''(\eta^\theta_{j-1})}{h_i}\right)\\ &\le Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j\end{aligned} \tag{5.52}$$

*where* $\eta^\theta_j \in [x_{j-1}, x_j]$.

proof see Proof 26

Lemma 5.20. *There exists a constant* $C = C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that for* $3 \le i < N, k = \lceil \frac{i}{2} \rceil, k+1 \le j \le \min\{2i-1, N-1\}$,

$$\begin{aligned}V_{ij} &\le Ch^2 \int_0^1 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j d\theta\\ &= Ch^2 \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy\end{aligned} \tag{5.53}$$

*Proof.* Since Lemma 5.14, by Lemma 5.18 and Lemma 5.19, we get the result immediately. ∎

Now we can prove Theorem 5.10 using Lemma 5.20, $k = \lceil \frac{i}{2} \rceil$

$$
\begin{aligned}
I_3 = \sum_{k+1}^{2i-1} V_{ij} &\le Ch^2 \int_{x_k}^{x_{2i-1}} \frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\
&= Ch^2 \left( \frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{2i-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\
&\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{5.54}
$$

Lemma 5.21.

$$
D_h P_{j-i}^{\theta}(x_i) := \frac{P_{k-i}^{\theta}(x_{i+1}) - P_{k-i}^{\theta}(x_i)}{h_{i+1}} = {P_{j-i}^{\theta}}'(\xi), \quad \xi \in [x_i, x_{i+1}] \tag{5.55}
$$

Then, for $3 \le i \le N-1$, $k = \lceil \frac{i}{2} \rceil$,

$$
D_h P_{k-i}^{\theta}(x_i) \le Ch^2 x_i^{-\alpha/2-2/r} h_j \tag{5.56}
$$

*Proof.* Using Leibniz formula, by Lemma 5.15, Lemma 5.16 and Lemma 5.17, we take $j = k+1, i = i+1$, we get

$$
\begin{aligned}
D_h P_{k-i}^{\theta}(x_i) &\le Ch^2 x_{i+1}^{\alpha/2-2/r-1} |y_{k+1}^{\theta} - x_{i+1}|^{1-\alpha} h_{j+1} \\
&\le Ch^2 x_i^{\alpha/2-2/r-1} |y_k^{\theta} - x_i|^{1-\alpha} h_j \\
&\le Ch^2 x_i^{-\alpha/2-2/r} h_j
\end{aligned} \tag{5.57}
$$
∎

Lemma 5.22. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \le i < N, k = \lceil \frac{i}{2} \rceil$,*

$$
I_2 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.58}
$$

*And for $3 \le i < N/2$,*

$$
I_4 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.59}
$$

*Proof.* In fact,

$$
\begin{aligned}
&\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \\
&= \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i}) T_{i,k}
\end{aligned} \tag{5.60}
$$

While, by Lemma A.2

$$\begin{aligned}\frac{1}{h_{i+1}}(T_{i+1,k}-T_{i,k}) &= \int_{x_{k-1}}^{x_k}(u(y)-u_h(y))\frac{|x_{i+1}-y|^{1-\alpha}-|x_i-y|^{1-\alpha}}{h_{i+1}\Gamma(2-\alpha)}dy\\ &\le \int_{x_{k-1}}^{x_k} h_j^2 u''(\eta)\,\frac{|\xi-y|^{-\alpha}}{\Gamma(1-\alpha)}dy\\ &\le Ch_jh^2x_j^{2-2/r}x_{k-1}^{\alpha/2-2}|x_i-x_k|^{-\alpha}\\ &\le Ch_jh^2x_i^{-\alpha/2-2/r}\end{aligned}\tag{5.61}$$

Thus,

$$\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}(T_{i+1,k}-T_{i,k})\le Ch^2x_i^{-\alpha/2-2/r}\tag{5.62}$$

For

$$\begin{aligned}\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k}) = \int_0^1 -\frac{\theta(1-\theta)}{2}\,\frac{P_{k-i}^{\theta}(x_{i+1})-P_{k-i}^{\theta}(x_i)}{h_{i+1}}d\theta\\ +\int_0^1\frac{\theta^3(1-\theta)}{3!}\,\frac{Q_{k-i}^{\theta}(x_{i+1})u'''(\eta_{j+1,1}^{\theta})-Q_{k-i}^{\theta}(x_i)u'''(\eta_{j,1}^{\theta})}{h_{i+1}}d\theta\\ -\int_0^1\frac{\theta(1-\theta)^3}{3!}\,\frac{Q_{k-i}^{\theta}(x_{i+1})u'''(\eta_{j+1,2}^{\theta})-Q_{k-i}^{\theta}(x_i)u'''(\eta_{j,2}^{\theta})}{h_{i+1}}d\theta\end{aligned}\tag{5.63}$$

And by Lemma 5.21

$$\frac{P_{k-i}^{\theta}(x_{i+1})-P_{k-i}^{\theta}(x_i)}{h_{i+1}}\le Ch^2x_i^{-\alpha/2-2/r}h_j\tag{5.64}$$

And with Lemma 5.19, we can get

$$\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k})\le Ch^2x_i^{-\alpha/2-2/r}\tag{5.65}$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}\frac{h_{i+1}-h_i}{h_ih_{i+1}}T_{i,k} &\le h_i^{-3}h^2x_i^{1-2/r}h_kCh_k^2x_{k-1}^{\alpha/2-2}|x_k-x_i|^{1-\alpha}\\ &\le Ch^2x_i^{-\alpha/2-2/r}\end{aligned}\tag{5.66}$$

Summarizes, we have

$$I_2\le Ch^2x_i^{-\alpha/2-2/r}\tag{5.67}$$

The case for $I_4$ is similar. ∎

Now combine Lemma 5.8, Lemma 5.22, Theorem 5.10, Lemma 5.5 and Lemma 5.6 to get the final result.

For $3\le i<N/2$

$$\begin{aligned}R_i &= I_1+I_2+I_3+I_4+I_5\\ &\le Ch^2x_i^{-\alpha/2-2/r}+\begin{cases}Ch^2x_i^{-\alpha/2-2/r}, & r\alpha/2+r-2>0\\ Ch^2(x_i^{-1-\alpha}\ln(i)+\ln(N)), & r\alpha/2+r-2=0\\ Ch^{r\alpha/2+r}x_i^{-1-\alpha}, & r\alpha/2+r-2<0\end{cases}\end{aligned}\tag{5.68}$$

Combine with $i = 1, 2$, we get for $1 \le i \le N/2$

$$R_i \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases} \tag{5.69}$$

**5.4. Proof of Theorem 5.3.** For $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$, we have

$$\begin{aligned} R_i =& \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &+ \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\ &+ \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &+ \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \\ &+ \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 \end{aligned} \tag{5.70}$$

We have estimate $I_1$ in Lemma 5.8 and $I_2$ in Lemma 5.22. We can control $I_3$ in similar with Theorem 5.10 by Lemma 5.20 where $2i - 1 \ge N - 1$

$$\begin{aligned} I_3 &= \sum_{j=k+1}^{N-1} V_{ij} \le Ch^2 \int_{x_k}^{x_{N-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &= Ch^2 \left( \frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{N-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\ &\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{5.71}$$

For $I_4$, we have

Lemma 5.23. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that, for $N/2 \le i < N - 1$*

$$\begin{aligned} V_{iN} &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,N+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,N} + \frac{1}{h_i} T_{i-1,N-1} \right) \\ &\le Ch^2 |T - x_{i-1}|^{1-\alpha} \end{aligned} \tag{5.72}$$

*Proof.* We use the similar skill in the last section, but more complicated. for $j = N$, Let

$$y_{i \to N}(x) = \frac{x^{1/r} - Z_i}{Z_1} h_N + T, \quad Z_i = T^{1/r} \frac{i}{N}, x_N = T \tag{5.73}$$

and

$$y_{i\to N-1}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r}\frac{N-1-i}{N} \tag{5.74}$$

Thus,

$$\begin{aligned} y_{i\to N}(x_{i-1}) = x_{N-1}, \quad y_{i\to N}(x_i) = x_N, \quad y_{i\to N}(x_{i+1}) = x_{N+1} \\ y_{i\to N-1}(x_{i-1}) = x_{N-2}, \quad y_{i\to N}(x_i) = x_{N-1}, \quad y_{i\to N}(x_{i+1}) = x_N \end{aligned}$$

Then, define

$$y^{\theta}_{i\to N}(x) = \theta y_{i\to N-1}(x) + (1-\theta) y_{i\to N}(x) \tag{5.75}$$

$$h_{i\to N}(x) = y_{i\to N}(x) - y_{i\to N-1}(x) \tag{5.76}$$

We have

$$y_{i\to N-1}{}'(x) = y^{1-1/r}_{i\to N-1}(x) x^{1/r-1} \tag{5.77}$$

$$y_{i\to N-1}{}''(x) = \frac{1-r}{r} y^{1-2/r}_{i\to N-1}(x) x^{1/r-2} Z_{N-1-i} \tag{5.78}$$

$$y_{i\to N}{}'(x) = \frac{1}{r}\frac{h_N}{Z_1} x^{1/r-1} \tag{5.79}$$

$$y_{i\to N}{}''(x) = \frac{1-r}{r^2}\frac{h_N}{Z_1} x^{1/r-2} \tag{5.80}$$

$$P^{\theta}_{i\to N}(x) = (h_{i\to N}(x))^3 \frac{|y^{\theta}_{i\to N}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y^{\theta}_{i\to N}(x)) \tag{5.81}$$

$$Q^{\theta}_{i\to N}(x) = (h_{i\to N}(x))^4 \frac{|y^{\theta}_{i\to N}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.82}$$

Similar with Lemma 5.13, we can get for $l = -1, 0, 1$,

$$\begin{aligned} T_{i+l,N+l} = &\int_0^1 -\frac{\theta(1-\theta)}{2} P^{\theta}_{i\to N}(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} Q^{\theta}_{i\to N}(x_{i+l})(\theta^2 u'''(\eta^{\theta}_{N+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+l,2})) d\theta \end{aligned} \tag{5.83}$$

So we have

$$V_{iN} = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,N+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,N} + \frac{1}{h_i} T_{i-1,N-1} \right)$$
$$= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{i\to N}^\theta(x_i) d\theta$$
$$+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i\to N}^\theta(x_{i+1}) u'''(\eta_{N+1,1}^\theta) - Q_{i\to N}^\theta(x_i) u'''(\eta_{N,1}^\theta)}{h_{i+1}} \right) d\theta$$
$$- \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i\to N}^\theta(x_i) u'''(\eta_{N,1}^\theta) - Q_{i\to N}^\theta(x_{i-1}) u'''(\eta_{N-1,1}^\theta)}{h_i} \right) d\theta$$
$$- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i\to N}^\theta(x_{i+1}) u'''(\eta_{N+1,2}^\theta) - Q_{i\to N}^\theta(x_i) u'''(\eta_{N,2}^\theta)}{h_{i+1}} \right) d\theta$$
$$+ \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i\to N}^\theta(x_i) u'''(\eta_{N,2}^\theta) - Q_{i\to N}^\theta(x_{i-1}) u'''(\eta_{N-1,2}^\theta)}{h_i} \right) d\theta \tag{5.84}$$

To estimate $D_h^2 P_{i\to N}^\theta(x_i) = {P_{i\to N}^\theta}''(\xi), \xi \in [x_{i-1}, x_{i+1}]$,

$$h_{i\to N}(\xi) \le 2h_N$$

Lemma 5.24.

$$h_{i\to N}^3(\xi) \le C h_N^3 \le C h^3 \tag{5.85}$$
$$(h_{i\to N}^3(\xi))' \le C(r-1) h_N^2 h \le C(r-1) h^3 \tag{5.86}$$
$$(h_{i\to N}^3(\xi))'' \le C(r-1) h_N^2 \le C(r-1) h^2 \tag{5.87}$$

*Proof.*

$$\begin{aligned} (h_{i\to N}^l(\xi))' &= l h_{i\to N}^{l-1}(\xi)({y_{i\to N}}'(\xi) - {y_{i\to N-1}}'(\xi)) \\ &= l h_{i\to N}^{l-1}(\xi) x_i^{1/r-1} (\frac{1}{r}\frac{h_N}{Z_1} - y_{i\to N-1}^{1-1/r}(\xi)) \end{aligned} \tag{5.88}$$

while

$$\begin{aligned} |\frac{1}{r}\frac{h_N}{Z_1} - y_{i\to N-1}^{1-1/r}(\xi)| &= |\frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r} |(\frac{N-t}{N})^{r-1} - (\frac{N-s}{N})^{r-1}| \quad t \in [0,1], s \in [0,2] \\ &\le T^{1-1/r} |1 - (\frac{N-2}{N})^{r-1}| \le C T^{1-1/r}(r-1)\frac{2}{N} \end{aligned} \tag{5.89}$$

Thus,

$$(h_{i\to N}^l(\xi))' \le C(r-1) h_N^{l-1} x_i^{1/r-1} h \tag{5.90}$$

And

$$(h^3_{i\to N}(\xi))'' = 6h_{i\to N}(\xi)({y_{i\to N}}'(\xi) - {y_{i\to N-1}}'(\xi))^2 + 3h^2_{i\to N}(\xi)({y_{i\to N}}''(\xi) - {y_{i\to N-1}}''(\xi)) \le C(r-1)^2 h_N x_i^{2/r-2} h^2 + Ch_N^2 \frac{r-1}{r} x_i^{1/r-2}(\frac{h_N}{rZ_1} - y^{1-2/r}_{i\to N-1}(\xi) Z_{N-1-i}) \tag{5.91}$$

$$|\frac{h_N}{rZ_1} - y^{1-2/r}_{i\to N-1}(\xi)Z_{N-1-i}| \le T^{1-1/r} + Cx_N^{1-2/r} x_N^{1/r} = CT^{1-1/r}$$

So

$$(h^3_{i\to N}(\xi))'' \le C(r-1)^2 h_N x_i^{2/r-2} h^2 + Ch_N^2 \frac{1-r}{r} x_i^{1/r-2} \le C(r-1)h_N^2 x_i^{1/r-1} \tag{5.92}$$

∎

Lemma 5.25.

$$u''(y^\theta_{i\to N}(\xi)) \le Cx_{N-2}^{-\alpha/2-2} \le C \tag{5.93}$$

$$(u''(y^\theta_{i\to N}(\xi)))' \le C \tag{5.94}$$

$$(u''(y^\theta_{i\to N}(\xi)))'' \le C \tag{5.95}$$

*Proof.*

$$(u''(y^\theta_{i\to N}(\xi)))' = u'''(y^\theta_{i\to N}(\xi)){y^\theta_{i\to N}}'(\xi) \le C(\theta {y_{i\to N-1}}'(\xi) + (1-\theta){y_{i\to N}}'(\xi)) \le Cx_i^{1/r-1}(\theta y^{1-1/r}_{i\to N-1}(\xi) + (1-\theta)\frac{h_N}{rZ_1}) \le Cx_i^{1/r-1}x_N^{1-1/r} \tag{5.96}$$

And

$$(u''(y^\theta_{i\to N}(\xi)))'' = u''''(y^\theta_{i\to N}(\xi))({y^\theta_{i\to N}}'(\xi))^2 + u'''(y^\theta_{i\to N}(\xi)){y^\theta_{i\to N}}''(\xi) \le Cx_i^{2/r-2}x_N^{2-2/r} + C\frac{r-1}{r}x_i^{1/r-2}(\theta x_N^{1-2/r}Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \le Cx_i^{2/r-2} + C(r-1)x_i^{1/r-2}T^{1-1/r} \tag{5.97}$$

∎

Lemma 5.26.

$$|y^\theta_{i\to N}(\xi) - \xi|^{1-\alpha} \le C|y^\theta_N - x_i|^{1-\alpha} \tag{5.98}$$

$$(|y^\theta_{i\to N}(\xi) - \xi|^{1-\alpha})' \le C|y^\theta_N - x_i|^{1-\alpha} \tag{5.99}$$

$$(|y^\theta_{i\to N}(\xi) - \xi|^{1-\alpha})'' \le C(r-1)|y^\theta_N - x_i|^{-\alpha} + |y^\theta_N - x_i|^{1-\alpha} \tag{5.100}$$

*Proof.*

$$(y^\theta_{i\to N}(\xi) - \xi)' = (\theta(y_{i\to N-1}(\xi) - \xi) + (1-\theta)(y_{i\to N}(\xi) - \xi))' = \theta({y_{i\to N-1}}'(\xi) - 1) + (1-\theta)({y_{i\to N}}'(\xi) - 1) = \theta\xi^{1/r-1}(y^{1-1/r}_{i\to N-1}(\xi) - \xi^{1-1/r}) + (1-\theta)\xi^{1/r-1}(\frac{h_N}{rZ_1} - \xi^{1-1/r}) \tag{5.101}$$

$$(5.102)\quad \begin{aligned}(y^\theta_{i\to N}(\xi)-\xi)'' &= \theta(y_{i\to N-1}''(\xi)) + (1-\theta)(y_{i\to N}''(\xi))\\ &= \frac{1-r}{r}\xi^{1/r-2}(\theta y^{1-2/r}_{i\to N-1}(\xi)Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \le 0\end{aligned}$$

And

$$(5.103)\quad |(y^\theta_{i\to N}(\xi)-\xi)''| \le C(r-1)\xi^{1/r-2}T^{1-1/r}$$

We have known

$$(5.104)\quad C|x_{N-1}-x_i| \le |y_{i\to N-1}(\xi)-\xi| \le C|x_{N-1}-x_i|$$

If $\xi \le x_{N-1}$, then $(y_{i\to N}(\xi)-\xi)' \ge 0$, so

$$(5.105)\quad C|x_N-x_i| \le |x_{N-1}-x_{i-1}| \le |y^\theta_{i\to N}(\xi)-\xi| \le |x_{N+1}-x_{i+1}| \le C|x_N-x_i|$$

If $i = N-1$ and $\xi \in [x_{N-1}, x_N]$, then $y_{i\to N}(\xi)-\xi$ is concave, bigger than its two neighboring points, which are equal to $h_N$, so

$$(5.106)\quad h_N = |x_N - x_{N-1}| \le |y_{i\to N}(\xi)-\xi| \le |x_{N+1}-x_{N-1}| = 2h_N$$

So we have

$$(5.107)\quad |y^\theta_{i\to N}(\xi)-\xi|^{1-\alpha} \le C|y^\theta_N - x_i|^{1-\alpha}$$

While

$$(5.108)\quad y^{1-1/r}_{i\to N-1}(\xi) - \xi^{1-1/r} \le (y_{i\to N-1}(\xi)-\xi)\xi^{-1/r}$$

and

$$(5.109)\quad \begin{aligned}&|\frac{h_N}{rZ_1} - \xi^{1-1/r}| \le \max\{|\frac{h_N}{rZ_1} - x^{1-1/r}_{i-1}|, |\frac{h_N}{rZ_1} - x^{1-1/r}_{i+1}|\}\\ &\qquad \le \max\begin{cases} T^{1-1/r} - x^{1-1/r}_{i-1} \le |x_N - x_{i-1}|T^{-1/r} \le C|x_N - x_i|\\ |x^{1-1/r}_{i+1} - x^{1-1/r}_{N-1}| \le |x_{i+1}-x_{N-1}|x^{-1/r}_{N-1} \le C|x_N-x_i|\end{cases}\end{aligned}$$

So we have

$$(5.110)\quad (y^\theta_{i\to N}(\xi)-\xi)' \le C|y^\theta_N - x_i|$$

$$(5.111)\quad \begin{aligned}(|y^\theta_{i\to N}(\xi)-\xi|^{1-\alpha})' &= |y^\theta_{i\to N}(\xi)-\xi|^{-\alpha}(y^\theta_{i\to N}(\xi)-\xi)'\\ &\le |y^\theta_N - x_i|^{1-\alpha}\end{aligned}$$

Finally,

$$(5.112)\quad \begin{aligned}(|y^\theta_{i\to N}(\xi)-\xi|^{1-\alpha})'' &= (1-\alpha)|y^\theta_{i\to N}(\xi)-\xi|^{-\alpha}(y^\theta_{i\to N}(\xi)-\xi)''\\ &\quad + \alpha(\alpha-1)|y^\theta_{i\to N}(\xi)-\xi|^{-1-\alpha}((y^\theta_{i\to N}(\xi)-\xi)')^2\\ &\le C(r-1)|y^\theta_N - x_i|^{-\alpha} + C|y^\theta_N - x_i|^{1-\alpha}\end{aligned}$$ ∎

By the three lemmas above, for $N/2 \le i \le N-1$, we have

$$(5.113)\quad \begin{aligned}&D^2_h P^\theta_{i\to N}(x_i) = P^\theta_{i\to N}(\xi) \quad \xi \in [x_{i-1}, x_{i+1}]\\ &\qquad \le Ch^3|y^\theta_N - x_i|^{1-\alpha} + C(r-1)(h^3|y^\theta_N - x_i|^{-\alpha} + h^2|y^\theta_N - x_i|^{1-\alpha})\end{aligned}$$ ∎

## 6. Proof of Theorem 4.2.

## 7. Experimental results.

**8. Conclusions.** Some conclusions here.

## Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x)$ is twice differentable continous function on open set $\Omega$, there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

$$D_h^2 g(x_i) := \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) = g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \tag{A.1}$$

$$\frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right) \tag{A.2}$$

*And if $g(x) \in C^4(\Omega)$, then*

$$\frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2)) \tag{A.3}$$

*where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$.*

*Proof.*

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in [x_{i-1}, x_i]$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in [x_i, x_{i+1}]$$

Subsitute them in the left side of (A.1), we have

$$\frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2)$$

Now, using intermediate value theorem , there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y)dy$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \frac{h_i^4}{4!} g''''(\eta_1)$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \frac{h_{i+1}^4}{4!} g''''(\eta_2)$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. Expecially,

$$\begin{aligned} \frac{h_i^4}{4!} g''''(\eta_1) &= \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy \\ \frac{h_{i+1}^4}{4!} g''''(\eta_2) &= \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \end{aligned} \tag{A.4}$$

Subsitute them to the left side of (A.3), we can get the result. ∎

LEMMA A.2. *If $y \in [x_{j-1}, x_j]$, denote $y = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0,1]$,*

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \tag{A.5}$$

(A.6)

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

*where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$.*

*Proof.* By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta)u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$. Thus ∎

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

Lemma A.3. *For $x \in [x_{j-1}, x_j]$*

$$\begin{aligned} |u(x) - u_h(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_{x}^{x_j} u'(y)dy \right| \\ &\leq \int_{x_{j-1}}^{x_j} |u'(y)|dy \end{aligned} \tag{A.7}$$

*If $x \in [0, x_1]$, with Corollary 2.4, we have*

$$|u(x) - u_h(x)| \leq \int_0^{x_1} |u'(y)|dy \leq \int_0^{x_1} Cy^{\alpha/2-1}dy \leq C\frac{2}{\alpha}x_1^{\alpha/2} \tag{A.8}$$

*Similarly, if $x \in [x_{2N-1}, 1]$, we have*

$$|u(x) - u_h(x)| \leq C\frac{2}{\alpha}(2T - x_{2N-1})^{\alpha/2} = C\frac{2}{\alpha}x_1^{\alpha/2} \tag{A.9}$$

## Appendix B. Inequality.

Lemma B.1.

$$h_i \leq rT^{1/r}h \begin{cases} x_i^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_{i-1})^{1-1/r}, & N < i \leq 2N - 1 \end{cases} \tag{B.1}$$

$$h_i \geq rT^{1/r}h \begin{cases} x_{i-1}^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_i)^{1-1/r}, & N < i \leq 2N - 1 \end{cases} \tag{B.2}$$

*Proof.* For $1 \leq i \leq N$,

$$\begin{aligned} h_i &= T\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right) \\ &\leq rT\frac{1}{N}\left(\frac{i}{N}\right)^{r-1} = rT^{1/r}hx_i^{1-1/r} \end{aligned}$$

$$h_i \geq rT\frac{1}{N}\left(\frac{i-1}{N}\right)^{r-1} = rT^{1/r}hx_{i-1}^{1-1/r}$$

For $N < i \leq 2N$,

$$\begin{aligned} h_i &= T\left(\left(\frac{2N-i+1}{N}\right)^r - \left(\frac{2N-i}{N}\right)^r\right) \\ &\leq rT\frac{1}{N}\left(\frac{2N-i+1}{N}\right)^{r-1} = rT^{1/r}h(2T - x_{i-1})^{1-1/r} \end{aligned}$$

$$h_i \geq rT\frac{1}{N}\left(\frac{2N-i}{N}\right)^{r-1} = rT^{1/r}h(2T - x_i)^{1-1/r}$$

∎

LEMMA B.2. *There is a constant* $C = 2^{|r-2|}r(r-1)T^{2/r}$ *such that for all* $i \in \{1, 2, \cdots, 2N-1\}$

$$|h_{i+1} - h_i| \le Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N-1 \end{cases} \tag{B.3}$$

*Proof.*

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \le i \le N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \le i \le 2N-1 \end{cases}$$

For $i = 1$,

$$h_2 - h_1 = T(2^r - 2)\left(\frac{1}{N}\right)^r = (2^r - 2)T^{2/r}h^2 x_1^{1-2/r}$$

For $2 \le i \le N-1$,

$$h_{i+1} - h_i = r(r-1)T\ N^{-2}\eta^{r-2}, \quad \eta \in [\frac{i-1}{N}, \frac{i+1}{N}]$$

If $r \in [1, 2]$,

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T\ N^{-2}\eta^{r-2} \le r(r-1)T\ h^2\left(\frac{i-1}{N}\right)^{r-2} \\ &\le r(r-1)T\ h^2 2^{2-r}\left(\frac{i}{N}\right)^{r-2} \\ &= 2^{2-r}r(r-1)T^{2/r}h^2 x_i^{1-2/r} \end{aligned}$$

else if $r > 2$,

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T\ N^{-2}\eta^{r-2} \le r(r-1)T\ h^2\left(\frac{i+1}{N}\right)^{r-2} \\ &\le r(r-1)T\ h^2 2^{r-2}\left(\frac{i}{N}\right)^{r-2} \\ &= 2^{r-2}r(r-1)T^{2/r}h^2 x_i^{1-2/r} \end{aligned}$$

Since

$$2^r - 2 \le 2^{|r-2|}r(r-1), \quad r \ge 1$$

we have

$$h_{i+1} - h_i \le 2^{|r-2|}r(r-1)T^{2/r}h^2 x_i^{1-2/r}, \quad 1 \le i \le N-1$$

For $i = N$, $h_{N+1} - h_N = 0$. For $N < i \le 2N-1$, it's central symmetric to the first half of the proof, which is

$$h_i - h_{i+1} \le 2^{|r-2|}r(r-1)T^{2/r}h^2(2T - x_i)^{1-2/r}$$

Summarizes the inequalities, we can get

$$\text{(B.4)} \qquad |h_{i+1} - h_i| \leq 2^{|r-2|} r(r-1) T^{2/r} h^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases} \qquad \square$$

## Appendix C. Proofs of some technical details.

*Additional proof of Theorem* 5.1. For $2 \leq i \leq N-1$,

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\ &\qquad \leq C \frac{2}{h_i + h_{i+1}} (h_i^3 x_{i-1}^{-2-\alpha/2} + h_{i+1}^3 x_i^{-2-\alpha/2}) \\ &\qquad \leq 2C (h_i^2 x_{i-1}^{-2-\alpha/2} + h_{i+1}^2 x_i^{-2-\alpha/2}) \end{aligned}$$

Since Lemma B.1, we have

$$h_i \leq r T^{1/r} h x_i^{1-1/r}, \quad 1 \leq i \leq N$$

$$h_{i+1} \leq r T^{1/r} h x_{i+1}^{1-1/r}, \quad 1 \leq i \leq N-1$$

and

$$x_{i-1}^{-2-\alpha/2} \leq 2^{-r(-2-\alpha/2)} x_i^{-2-\alpha/2} \quad 2 \leq i \leq N-1$$

$$x_{i+1}^{1-1/r} \leq 2^{r-1} x_i^{1-1/r} \quad 1 \leq i \leq N-1$$

So there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

$$\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq C h^2 x_i^{-\alpha/2 - 2/r}, \quad 2 \leq i \leq N-1$$

For $i = 1$, by (A.4)

$$\begin{aligned} &\frac{1}{4!} \frac{2}{h_1 + h_2} (h_1^3 f''(\eta_1) + h_2^3 f''(\eta_2)) \\ &\qquad = \frac{2}{h_1 + h_2} \left( \frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy + \frac{1}{4!} h_2^3 f''(\eta_2) \right) \end{aligned}$$

We have proved above that

$$\frac{2}{h_1 + h_2} h_2^3 f''(\eta_2) \leq C h^2 x_1^{-\alpha/2 - 2/r}$$

and we can get

$$\begin{aligned} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy &\leq C \frac{1}{3!} \int_0^{x_1} y^{1-\alpha/2} dy \\ &= C \frac{1}{3!(2-\alpha/2)} x_1^{2-\alpha/2} \end{aligned}$$

so

$$\frac{2}{h_1 + h_2} \frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy = \frac{C 2^{1-r}}{3!(2-\alpha/2)} x_1^{-\alpha/2} = \frac{C 2^{1-r}}{3!(2-\alpha/2)} T^{2/r} h^2 x_1^{-\alpha/2 - 2/r}$$

And for $i = N$, we have

$$\begin{aligned}
&\frac{2}{h_N + h_{N+1}}(h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2)) \\
&\qquad = h_N^2(f''(\eta_1) + f''(\eta_2)) \\
&\qquad \le r^2 T^{2/r} h^2 x_N^{2-2/r} \; 2C x_{N-1}^{-2-\alpha/2} \\
&\qquad \le 2r^2 T^{2/r} C 2^{-r(-2-\alpha/2)} \; h^2 x_N^{-\alpha/2-2/r}
\end{aligned}$$

Finally, $N+1 \le i \le 2N-1$ is symmetric to the first half of the proof, so we can conclude that

$$\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases}$$

□

LEMMA C.1. *There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ for $2 \le j \le N$, if $y \in [x_{j-1}, x_j]$,*

$$|u(y) - u_h(y)| \le Ch^2 y^{\alpha/2-2/r} \tag{C.1}$$

*Proof.* For $2 \le j \le N$, we have

$$x_j \le 2^r y, \quad x_{j-1} \ge 2^{-r} y$$

And by Lemma A.2, Lemma B.1 and Corollary 2.4, we have

$$\begin{aligned}
u(y) - u_h(y) &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \\
&\le \frac{\|u\|_{\beta+\alpha}^{(-\alpha/2)}}{4} r^2 T^{2/r} \; h^2 x_j^{2-2/r} x_{j-1}^{\alpha/2-2} \\
&\le Ch^2 \; 2^{2r-2} y^{2-2/r} \; 2^{-r(\alpha/2-2)} y^{\alpha/2-2} \\
&= C 2^{-r\alpha/2+4r-2} \; h^2 y^{\alpha/2-2/r}
\end{aligned}$$

symmetricly, for $N < j \le 2N-1$, we have

$$|u(y) - u_h(y)| \le Ch^2 (2T - y)^{\alpha/2-2/r} \tag{C.2}$$

□

LEMMA C.2. *There is a constant $C = C(\alpha, r)$ such that for all $1 \le i < N/2$, $\max\{2i+1, i+3\} \le j \le 2N$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_h^2\left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) \le C \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} \tag{C.3}$$

*Proof.* Sinec $y \ge x_{j-1} > x_{i+1}$, by Lemma A.1, if $j - 1 > i + 1$

□

$$\begin{aligned}
D_h^2\left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) &= \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\
&\le \frac{(y - x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)} \\
&\le (1 - (\frac{2}{3})^r)^{-1-\alpha} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}
\end{aligned}$$

LEMMA C.3. *There is a constant $C = C(\alpha, r)$ such that for all $3 \le i < N/2, k = \lceil \frac{i}{2} \rceil$, $1 \le j \le k-1$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_h^2\left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) \le C\frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \tag{C.4}$$

*Proof.* Sinec $y \le x_j < x_{i-1}$, by Lemma A.1,

$$\begin{aligned} D_h^2\left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) &= \frac{|\xi - y|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\le \frac{(x_{i-1} - x_j)^{-1-\alpha}}{\Gamma(-\alpha)} \le \frac{(x_{i-1} - x_{k-1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\le ((\frac{2}{3})^r - (\frac{1}{2})^r)^{-1-\alpha} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

$\square$

LEMMA C.4. *While $0 \le i < N/2$, By Lemma A.3*

$$\begin{aligned} |T_{i1}| &\le C \int_0^{x_1} x_1^{\alpha/2} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= C\frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2} \left| x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha} \right| \\ &\le C\frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2+2-\alpha} = C\frac{1}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1 \end{aligned} \tag{C.5}$$

*For $2 \le j \le N$, by Lemma A.2 and Corollary 2.4*

$$\begin{aligned} |T_{ij}| &\le \frac{C}{4} \int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &\le \frac{C}{4\Gamma(3-\alpha)} h_j^2 x_{j-1}^{\alpha/2-2} \left| |x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha} \right| \end{aligned} \tag{C.6}$$

LEMMA C.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$\sum_{j=1}^{3} S_{1j} \le Ch^2 x_1^{-\alpha/2-2/r} \tag{C.7}$$

$$\sum_{j=1}^{4} S_{2j} \le Ch^2 x_2^{-\alpha/2-2/r} \tag{C.8}$$

*Proof.*

$$S_{1j} = \frac{2}{x_2}\left(\frac{1}{x_1}T_{0j} - \left(\frac{1}{x_1} + \frac{1}{h_2}\right)T_{1j} + \frac{1}{h_2}T_{2j}\right)$$

So, by Lemma C.4

$$S_{11} \le \frac{2}{x_2 x_1} 4 \frac{C}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \le C x_1^{-\alpha/2}$$

$$S_{12} \le \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_2^2 x_1^{\alpha/2-2} \left(x_2^{2-\alpha} + 2h_2^{2-\alpha} + h_2^{2-\alpha}\right) \le C x_1^{-\alpha/2}$$

$$S_{13} \le \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_3^2 x_2^{\alpha/2-2} \left(x_3^{2-\alpha} + 2x_3^{2-\alpha} + h_3^{2-\alpha}\right) \le C x_1^{-\alpha/2}$$

But

$$x_1^{-\alpha/2} = T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

For $i = 2$, Sorry ∎

Lemma C.6. *There exists a constant $C = C(T, r, l)$ such that For $3 \le i \le N - 1, k+1 = \lceil \frac{i}{2} \rceil, k \le j \le \min\{2i-1, N-1\}$, $l = 3, 4$, when $\xi \in [x_{i-1}, x_{i+1}]$,*

$$(h_{j-i}^3(\xi))' \le (r-1) C h^2 x_i^{1-2/r} h_j \tag{C.9}$$

$$(h_{j-i}^4(\xi))' \le (r-1) C h^2 x_i^{1-2/r} h_j^2 \tag{C.10}$$

*Proof.* From (5.32)

$$y'_{j-i}(x) = y_{j-i}^{1-1/r}(x) x^{1/r-1} \tag{C.11}$$

$$y''_{j-i}(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i} \tag{C.12}$$

for $l = 3, 4$, by (5.34)

$$\begin{aligned} (h_{j-i}^l(\xi))' &= l\ h_{j-i}^{l-1}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi)) \\ &= l\ h_{j-i}^{l-1}(\xi) \xi^{1/r-1} (y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \ge 0 \end{aligned} \tag{C.13}$$

For $\xi \in [x_{i-1}, x_{i+1}]$ and $2 \le k \le j \le \min\{2i-1, N-1\}$, using Lemma B.1

$$\begin{aligned} h_{j-i}(\xi) &\le h_{j-i}(x_{i+1}) = h_{j+1} \\ &\le r T^{1/r}\ h x_{j+1}^{1-1/r} \le r T^{1/r} 2^{r-1}\ h x_i^{1-1/r} \end{aligned}$$

And

$$2^{-r} x_i \le x_{i-1} \le \xi \le x_{i+1} \le 2^r x_i \tag{C.14}$$

We have

$$\xi^{1/r-m} \le 2^{|mr-1|} x_i^{1/r-m}, \quad m = 1, 2 \tag{C.15}$$

but

$$\begin{aligned} y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) &= (\xi^{1/r} + Z_{j-i})^{r-1} - (\xi^{1/r} + Z_{j-i-1})^{r-1} \\ &= (r-1) Z_1 (\xi^{1/r} + Z_{j-i-\gamma})^{r-2}, \quad \gamma \in [0, 1] \\ &= (r-1) T^{1/r} h y_{j-i-\gamma}^{1-2/r}(\xi) \end{aligned} \tag{C.16}$$

And

$$4^{-r}x_i \le x_{\lceil \frac{i}{2}\rceil -1} \le x_{j-2} = y_{j-i-1}(x_{i-1}) \le y_{j-i-\gamma}(\xi) \le y_{j-i}(x_{i+1}) = x_{j+1} \le x_{2i} \le 2^r x_i \tag{C.17}$$

Therefore,

$$y_{j-i-\gamma}^{1-2/r}(\xi) \le 2^{2|r-2|} x_i^{1-2/r} \tag{C.18}$$

So we can get

$$y'_{j-i}(\xi) - y'_{j-i-1}(\xi) \le (r-1)C(T,r)hx_i^{-1/r} \tag{C.19}$$

We get

$$(h_{j-i}^l(\xi))' \le l(r-1)C\, h_{j+1}^{l-1} h x_i^{-1/r} \tag{C.20}$$

And by Lemma B.1,

$$h_{j+1} \le rTh\left(\frac{j+1}{N}\right)^{r-1} \le rTh2^{r-1}\left(\frac{j-1}{N}\right) = 2^{r-1}h_j \tag{C.21}$$

$$h_{j+1} \le rT^{1/r}hx_{j+1}^{1-1/r} \le rT^{1/r}hx_{2i}^{1-1/r} \le rT^{1/r}2^{r-1}hx_i^{1-1/r} \tag{C.22}$$

We can get

$$\begin{aligned} (h_{j-i}^l(\xi))' &\le l(r-1)C\, h_j^{l-2} h_{j+1} h x_i^{-1/r} \\ &\le l(r-1)Chh_j^{l-2}(hx_i^{1-1/r})x_i^{-1/r} \\ &= (r-1)C\, h^2 x_i^{1-2/r} h_j^{l-2} \end{aligned} \tag{C.23}$$

Meanwhile, we can get

$$h_{j-i}^3(\xi) \le h_{j+1}^3 \le Ch^2 x_i^{2-2/r} h_j \tag{C.24}$$

$$h_{j-i}^4(\xi) \le h_{j+1}^4 \le Ch^2 x_i^{2-2/r} h_j^2 \tag{C.25}$$

$\square$

LEMMA C.7. *There exists a constant $C = C(T,r,l)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2}\rceil + 1 \le j \le \min\{2i-1, N-1\}$, when $\xi \in [x_{i-1}, x_{i+1}]$,*

$$(h_{j-i}^3(\xi))'' \le C(r-1)h^2 x_i^{-2/r} h_j \tag{C.26}$$

*Proof.* From (C.11)

$$\begin{aligned} (h_{j-i}^3(\xi))'' &= 6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 + 3h_{j-i}^2(\xi)(y''_{j-i}(\xi) - y''_{j-i-1}(\xi)) \\ &= 6h_{j-i}(\xi)\xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \\ &\quad + 3\frac{1-r}{r}h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \end{aligned} \tag{C.27}$$

Using the inequalities of the proof of Lemma C.6

$$\begin{aligned} &6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 \\ &\quad \le 6h_{j+1}((r-1)Chx_i^{-1/r})^2 \\ &\quad \le C(r-1)^2\, h^2 x_i^{-2/r} h_j \end{aligned} \tag{C.28}$$

For the second partial

$$\begin{aligned} &h^2_{j-i}(\xi)\xi^{1/r-2}(y^{1-2/r}_{j-i}(\xi)Z_{j-i} - y^{1-2/r}_{j-i-1}(\xi)Z_{j-i-1}) \\ &\quad \le Ch^2_{j+1}x_i^{1/r-2}((y^{1-2/r}_{j-i}(\xi) - y^{1-2/r}_{j-i-1}(\xi))Z_{j-i} + y^{1-2/r}_{j-i-1}(\xi)Z_1) \end{aligned} \tag{C.29}$$

but

$$\begin{aligned} y^{1-2/r}_{j-i}(\xi) - y^{1-2/r}_{j-i-1}(\xi) &= (\xi^{1/r} + Z_{j-i})^{r-2} - (\xi^{1/r} + Z_{j-i-1})^{r-2} \\ &= (r-2)Z_1(\xi^{1/r} + Z_{j-i-\gamma})^{r-3} \\ &= (r-2)T^{-r}hy^{1-3/r}_{j-i-\gamma}(\xi) \\ &\le C(r-2)hx_i^{1-3/r} \end{aligned} \tag{C.30}$$

So we can get

$$\begin{aligned} &h^2_{j-i}(\xi)\xi^{1/r-2}(y^{1-2/r}_{j-i}(\xi)Z_{j-i} - y^{1-2/r}_{j-i-1}(\xi)Z_{j-i-1}) \\ &\quad \le Ch_jhx_i^{1-1/r}x_i^{1/r-2}(C(r-2)hx_i^{1-3/r}Z_{j-i} + Cx_i^{1-2/r}T^{1/r}h) \\ &\quad \le Ch^2((r-2)x_i^{-3/r}x^{1/r}_{|j-i|} + x_i^{-2/r})h_j \\ &\quad \le Ch^2x_i^{-2/r}h_j \end{aligned} \tag{C.31}$$

Summarizes, we have

$$(h^3_{j-i}(\xi))'' \le C(r-1)h^2x_i^{-2/r}h_j \tag{C.32}$$

∎

*proof of Lemma* 5.16. From (5.32)

$$y'_{j-i}(x) = y^{1-1/r}_{j-i}(x)x^{1/r-1} \tag{C.33}$$

$$y''_{j-i}(x) = \frac{1-r}{r}y^{1-2/r}_{j-i}(x)x^{1/r-2}Z_{j-i} \tag{C.34}$$

Since

$$x_{j-2} \le y_{j-i-1}(x_{i-1}) \le y^{\theta}_{j-i}(\xi) \le y^{\theta}_{j-i-1}(x_{i+1}) \le x_{j+1}$$

We have known (C.17)

$$u''(y^{\theta}_{j-i}(\xi)) \le C(y^{\theta}_{j-i}(\xi))^{\alpha/2-2} \le Cx^{\alpha/2-2}_{j-2} \le Cx^{\alpha/2-2}_{\lceil\frac{i}{2}\rceil-1} \le C4^{r(2-\alpha/2)}x_i^{\alpha/2-2} \tag{C.35}$$

$$\begin{aligned} (u''(y^{\theta}_{j-i}(\xi)))' &= u'''(y^{\theta}_{j-i}(\xi)){y^{\theta}_{j-i}}'(\xi) \\ &\le Cx_i^{\alpha/2-3}\xi^{1/r-1}y^{1-1/r}_{j-i}(\xi) \\ &\le Cx_i^{\alpha/2-3}x_i^{1/r-1}x_i^{1-1/r} = Cx_i^{\alpha/2-3} \end{aligned} \tag{C.36}$$

$$
\begin{aligned}
(u''(y^\theta_{j-i}(\xi)))'' &= u''''(y^\theta_{j-i}(\xi))({y^\theta_{j-i}}''(\xi))^2 + u'''(y^\theta_{j-i}(\xi)){y^\theta_{j-i}}''(\xi) \\
&\le Cx_i^{\alpha/2-4} + Cx_i^{\alpha/2-3}\frac{r-1}{r}x_i^{1-2/r}x_i^{1/r-2}Z_{|j-i|+1} \\
&\le Cx_i^{\alpha/2-4} + C\frac{r-1}{r}x_i^{\alpha/2-3}x_i^{-1/r}x_i^{1/r} \\
&= Cx_i^{\alpha/2-4}
\end{aligned} \tag{C.37}
$$

□

*Proof of Lemma* 5.17.

$$
\begin{aligned}
|y^\theta_{j-i}(\xi)-\xi| &= |\theta(y_{j-i-1}(\xi)-\xi) + (1-\theta)(y_{j-i}(\xi)-\xi)| \\
&= \theta|y_{j-i-1}(\xi)-\xi| + (1-\theta)|y_{j-i}(\xi)-\xi|
\end{aligned} \tag{C.38}
$$

Since $|y_{j-i}(\xi)-\xi|$ is increasing about $\xi$, we have

$$
(\frac{i-1}{i})^r|x_j-x_i| \le |x_{j-1}-x_{i-1}| \le |y_{j-i}(\xi)-\xi| \le |x_{j+1}-x_{i+1}| \le (\frac{i+1}{i})^r|x_j-x_i| \tag{C.39}
$$

Thus,

$$
(\frac{2}{3})^r|y^\theta_j - x_i| \le |y^\theta_{j-i}(\xi)-\xi| \le (\frac{3}{4})^r(\theta|x_j-x_i| + (1-\theta)|x_{j-1}-x_i|) = (\frac{3}{4})^r|y^\theta_j-x_i| \tag{C.40}
$$

$$
|y^\theta_{j-i}(\xi)-\xi|^{1-\alpha} \le C|y^\theta_j - x_i|^{1-\alpha} \tag{C.41}
$$

Next,

$$
\begin{aligned}
(|y^\theta_{j-i}(\xi)-\xi|^{1-\alpha})' &= (1-\alpha)|y^\theta_{j-i}(\xi)-\xi|^{-\alpha}|\xi^{1/r-1}(\theta y^{1-1/r}_{j-i-1}(\xi) + (1-\theta)y^{1-1/r}_{j-i}(\xi)) - 1| \\
&\le C|y^\theta_j - x_i|^{-\alpha}\xi^{1/r-1}|\theta y^{1-1/r}_{j-i-1}(\xi) + (1-\theta)y^{1-1/r}_{j-i}(\xi) - \xi^{1-1/r}|
\end{aligned} \tag{C.42}
$$

Similar with (C.40), we have

$$
\begin{aligned}
|y^{1-1/r}_{j-i}(\xi) - \xi^{1-1/r}| &\le C|x_j^{1-1/r} - x_i^{1-1/r}| \\
&\le C|x_j - x_i|x_i^{-1/r}
\end{aligned} \tag{C.43}
$$

So we can get

$$
\begin{aligned}
&|\theta y^{1-1/r}_{j-i-1}(\xi) + (1-\theta)y^{1-1/r}_{j-i}(\xi) - \xi^{1-1/r}| \\
&\quad\le Cx_i^{-1/r}(\theta|x_{j-1}-x_i| + (1-\theta)|x_j-x_i|) \\
&\quad= Cx_i^{-1/r}|y^\theta_j - x_i|
\end{aligned} \tag{C.44}
$$

Combine them, we get

$$
\begin{aligned}
(|y^\theta_{j-i}(\xi)-\xi|^{1-\alpha})' &\le C|y^\theta_j - x_i|^{-\alpha}x_i^{1/r-1}x_i^{-1/r}|y^\theta_j - x_i| \\
&= C|y^\theta_j - x_i|^{1-\alpha}x_i^{-1}
\end{aligned} \tag{C.45}
$$

Finally, we have

$$
\begin{aligned}
(|y^\theta_{j-i}(\xi)-\xi|^{1-\alpha})'' &= \alpha(\alpha-1)|y^\theta_{j-i}(\xi)-\xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y^{1-1/r}_{j-i-1}(\xi) + (1-\theta)y^{1-1/r}_{j-i}(\xi)) - 1)^2 \\
&\quad + (1-\alpha)|y^\theta_{j-i}(\xi)-\xi|^{-\alpha}\frac{1-r}{r}\xi^{1/r-2}|\theta y^{1-2/r}_{j-i-1}(\xi)Z_{j-i-1} + (1-\theta)y^{1-2/r}_{j-i}(\xi)Z_{j-i}|
\end{aligned} \tag{C.46}
$$

Using the inequalities above ,we have

$$\begin{aligned}
&|y_{j-i}^{\theta}(\xi)-\xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi)+(1-\theta)y_{j-i}^{1-1/r}(\xi))-1)^2\\
&\quad\le C|y_j^{\theta}-x_i|^{-\alpha-1}(x_i^{-1}|y_j^{\theta}-x_i|)^2\\
&\quad= C|y_j^{\theta}-x_i|^{1-\alpha}x_i^{-2}
\end{aligned}\tag{C.47}$$

And by

$$|Z_{j-i}|=|x_j^{1/r}-x_i^{1/r}|\le|x_j-x_i|x_i^{1/r-1}\tag{C.48}$$

we have

$$\begin{aligned}
&|y_{j-i}^{\theta}(\xi)-\xi|^{-\alpha}\xi^{1/r-2}|\theta y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}+(1-\theta)y_{j-i}^{1-2/r}(\xi)Z_{j-i}|\\
&\le C|y_j^{\theta}-x_i|^{-\alpha}x_i^{1/r-2}x_i^{1-2/r}|\theta Z_{j-i-1}+(1-\theta)Z_{j-i}|\\
&\le C|y_j^{\theta}-x_i|^{-\alpha}x_i^{-2}|y_j^{\theta}-x_i|\\
&= C|y_j^{\theta}-x_i|^{1-\alpha}x_i^{-2}
\end{aligned}\tag{C.49}$$

□

*proof of Lemma* 5.19. For $k\le j<\min\{2i-1,N-1\}$

$$\begin{aligned}
&\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta})-Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}}\\
&\quad\frac{Q_{j-i}^{\theta}(x_{i+1})-Q_{j-i}^{\theta}(x_i)}{h_{i+1}}u'''(\eta_{j+1}^{\theta})+Q_{j-i}^{\theta}(x_i)\frac{u'''(\eta_{j+1}^{\theta})-u'''(\eta_j^{\theta})}{h_{i+1}}\\
&\quad\le {Q_{j-i}^{\theta}}'(\xi)Cx_j^{\alpha/2-3}+Q_{j-i}^{\theta}(x_i)Cu''''(\eta)\frac{h_i+h_{i+1}}{h_{i+1}}
\end{aligned}\tag{C.50}$$

where $\xi\in[x_i,x_{i+1}],\eta\in[x_{j-1},x_{j+1}]$.

From (5.36), by Lemma C.6 and Lemma 5.17, we have

$$\begin{aligned}
{Q_{j-i}^{\theta}}'(\xi)&\le Ch^2\frac{|y_{j+1}^{\theta}-x_{i+1}|^{1-\alpha}}{\Gamma(2-\alpha)}x_{i+1}^{1-2/r}h_{j+1}^2\\
&\le Ch^2\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{1-2/r}h_j^2
\end{aligned}\tag{C.51}$$

And by defination

$$Q_{j-i}^{\theta}(x_i)=h_j^4\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}\le Ch^2x_i^{2-2/r}\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}h_j^2\tag{C.52}$$

With , we have

$$4^{-r}x_i\le x_{k-1}\le x_{j-1}<x_j\le x_{2i-1}\le 2^rx_i\tag{C.53}$$

So we have

$$\begin{aligned}
&\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta})-Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}}\\
&\le Ch^2\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{1-2/r}h_j^2\,x_i^{\alpha/2-3}+Ch^2x_i^{2-2/r}\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}h_j^2x_{j-1}^{\alpha/2-4}\\
&= Ch^2\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j^2
\end{aligned}\tag{C.54}$$

while

$$h_j \leq h_{2i-1} \leq 2^r h_i$$

Subsitute into the inequality above, we get the goal

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} \left( \frac{Q^\theta_{j-i}(x_{i+1}) u'''(\eta^\theta_{j+1}) - Q^\theta_{j-i}(x_i) u'''(\eta^\theta_j)}{h_{i+1}} \right) \\ & \leq \frac{1}{h_i} C h^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j 2^r h_i \\ & = C h^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned} \tag{C.55}$$

While, the later is similar. ∎

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## REFERENCES

[1] X. Ros-Oton and J. Serra, *The dirichlet problem for the fractional laplacian: Regularity up to the boundary*, Journal de Mathématiques Pures et Appliquées, 101 (2014), pp. 275–302, https://doi.org/https://doi.org/10.1016/j.matpur.2013.06.003, https://www.sciencedirect.com/science/article/pii/S0021782413000895.