1 问题

对于 $\Omega=(0,1)$, $1<\alpha<2$,

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}u(x)=f(x), & x\in\Omega\\ u(x)=0, & x\in\mathbb{R}\setminus\Omega\end{cases} \tag{1}$$

其中

$$(-\Delta)^{\frac{\alpha}{2}}u(x)=-\frac{\partial^\alpha u}{\partial|x|^\alpha}=C_R\frac{d^2}{dx^2}\int_\Omega\frac{u(y)}{|x-y|^{\alpha-1}}dy \tag{2}$$

2 数值格式

用线性插值代替原函数，中心差分代替二阶导数，记 $u_h(x)$ 为 $u(x)$ 在网格点上的线性插值。

我们解这样的数值解

$$\begin{aligned}&C_R(\frac{2}{h_{i+1}(h_i+h_{i+1})}\int_\Omega\frac{u_h(x)}{|x_{i+1}-y|^{\alpha-1}}dy\\&\quad-\frac{2}{h_ih_{i+1}}\int_\Omega\frac{u_h(x)}{|x_i-y|^{\alpha-1}}dy\\&\quad+\frac{2}{h_i(h_i+h_{i+1})}\int_\Omega\frac{u_h(x)}{|x_{i-1}-y|^{\alpha-1}}dy)\\&=F_i\end{aligned} \tag{3}$$

矩阵 A 是 M 矩阵，主队角正，其他负，严格对角占优。

3 一致网格

当 $r=1$, 网格成为一致网格, $x_i=ih, h=\frac{1}{2N}, i=0,...,2N$.

A 等于

$$\begin{aligned}a_{ij}=&\frac{C_R}{(2-\alpha)(3-\alpha)}h^{-\alpha}\\&\left(|i-j-2|^{3-\alpha}-4|i-j-1|^{3-\alpha}+6|i-j|^{3-\alpha}-4|i-j+1|^{3-\alpha}+|i-j+2|^{3-\alpha}\right)\end{aligned} \tag{4}$$

矩阵行和

$$S_i = \sum_{j=1}^{2N-1} a_{ij} = \frac{C_R}{(2-\alpha)(3-\alpha)} h^{-\alpha}(|i+1|^{3-\alpha} - 3|i|^{3-\alpha} + 3|i-1|^{3-\alpha} - |i-2|^{3-\alpha} + ...2N) \tag{5}$$

我们得到

$$S_i \geq C(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \tag{6}$$

下面估计截断误差 R_i. 目标是

$$R_i \leq Ch^{\alpha/2} S_i \tag{7}$$

这样我们就有

$$\epsilon \leq \max_i \frac{R_i}{S_i} \leq Ch^{\alpha/2} \tag{8}$$

考虑 R_1

$$R_1 = \int_\Omega (u(y) - u_h(y)) \frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} dy \tag{9}$$

我们有

$$R_1 = \int_0^{3h} + \int_{3h}^1 \tag{10}$$

当 $y > 3h$,

$$\frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} \leq C|y|^{-1-\alpha} \tag{11}$$

那么

$$
\begin{aligned}
I_2 &\le C\int_{3h}^{1}|y|^{-1-\alpha}u''(\eta)h^2dy \\
&\le C\int_{3h}^{1}|y|^{-1-\alpha}h^{\alpha/2-2}h^2dy \\
&\le Ch^{\alpha/2}\int_{3h}^{1}y^{-1-\alpha}dy \\
&\le Ch^{\alpha/2}h^{-\alpha}=Ch^{-\alpha/2} \\
&\le Ch^{\alpha/2}x_1^{-\alpha}\le Ch^{\alpha/2}S_1
\end{aligned}
\tag{12}
$$

在考虑

$$
\begin{aligned}
I_1 &= \int_0^{3h}\frac{u(y)-u_h(y)}{h^2}(|y|^{1-\alpha}-2|y-h|^{1-\alpha}+|y-2h|^{1-\alpha})dy \\
&= \int_0^h+\int_h^{3h}=J_1+J_2
\end{aligned}
\tag{13}
$$

$$
J_2\le Cu''(\eta)h^{2-\alpha}\le Ch^{\alpha/2-2}h^{2-\alpha}\le Ch^{-\alpha/2} \tag{14}
$$

因为

$$
\begin{aligned}
|u(x)-u_h(x)| &\le \int_0^{x_1}|u'(y)|dy \\
&\le C\int_0^{x_1}y^{\alpha/2-1}dy \\
&\le Cx_1^{\alpha/2} \quad ,x\in(0,h)
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
J_1 &= \int_0^{h}\frac{u(y)-u_h(y)}{h^2}(|y|^{1-\alpha}-2|y-h|^{1-\alpha}+|y-2h|^{1-\alpha})dy \\
&\le Ch^{\alpha/2-2}h^{2-\alpha}=Ch^{-\alpha/2}
\end{aligned}
\tag{16}
$$

所以有

$$
R_1\le Ch^{-\alpha/2}\le Ch^{\alpha/2}h^{-\alpha}\le Ch^{\alpha/2}S_1,\quad (S_1\ge Cx_1^{-\alpha}) \tag{17}
$$

R_1,R_2,R_3 全部类似。

3.1 猜想

$$R_i \leq Ch^{\alpha/2+1}(x_i^{-\alpha-1} + (1-x_i)^{-\alpha-1}) \quad (then \leq Ch^{\alpha/2}S_i) \tag{18}$$

为了简便，我们记 $D(y) := u(y) - u_h(y)$.

当 $3 < i \leq N$ 时，

$$\begin{aligned}
R_i = &\int_0^1 D(y)\frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2}dy \\
= &\int_0^{x_1} D(y)\frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2}dy \\
&+ \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} D(y)\frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2}dy \\
&+ \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} \frac{D(y+h)-D(y)}{h^2}|y-x_i|^{1-\alpha} + D(y)\frac{|y-x_{i+1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2}dy \\
&+ \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} \frac{D(y-h)-2D(y)+D(y+h)}{h^2}|y-x_i|^{1-\alpha}dy \\
&+ \cdots (2N-i) \\
= &I_1 + I_2 + I_3 + I_4 + \cdots
\end{aligned} \tag{19}$$

1.

$$\begin{aligned}
I_1 = &\int_0^{x_1} (u(y)-u_h(y))\frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2}dy \\
&\leq Ch^{\alpha/2}\int_0^h |y-x_i|^{-1-\alpha}dy \\
&\leq Ch^{\alpha/2+1}x_i^{-1-\alpha}
\end{aligned} \tag{20}$$

2.

$$
\begin{aligned}
I_2 &= \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
&\leq C \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2} h^2 |x_i - y|^{-1-\alpha} dy \\
&\leq C h^{\alpha/2-1} h^2 x_i^{-1-\alpha} \leq C h^{\alpha/2+1} x_i^{-1-\alpha}
\end{aligned} \tag{21}
$$

3.

$$
\begin{aligned}
I_3 &= \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} \frac{D(y+h)-D(y)}{h^2} |y-x_i|^{1-\alpha} + D(y) \frac{|y-x_{i+1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2} dy \\
&\leq \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} u'''(\eta_1) h |x_i - y|^{1-\alpha} + u''(\eta_2) h |x_i - y|^{-\alpha} dy \\
&\leq C h^2 x_i^{-2-\alpha/2} \leq C h^{1+\alpha/2} x_i^{-1-\alpha}
\end{aligned} \tag{22}
$$

4.

$$
\begin{aligned}
I_4 &= \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y-x_i|^{1-\alpha} dy \\
&\leq \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} u''''(\eta) h^2 |x_i - y|^{1-\alpha} dy \\
&\leq C x_i^{\alpha/2-4} h^2 x_i^{2-\alpha} \\
&\leq C h^2 x_i^{-2-\alpha/2} \leq C h^{1+\alpha/2} x_i^{-1-\alpha}
\end{aligned} \tag{23}
$$

猜想证毕，一致网格证完。

4 非一致

$r > 1$,

$$
\begin{cases}
x_i = \frac{1}{2} \left(\frac{i}{N} \right)^r, & 0 \leq i \leq N \\
x_i = 1 - \frac{1}{2} \left(\frac{2N-i}{N} \right)^r, & N \leq i \leq 2N
\end{cases} \tag{24}
$$

令 $h = \frac{1}{2N}$ ，那么

当 $i > N, x_i > \frac{1}{2}$ 时

$$h_i = \frac{1}{2}\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right) \leq C(r)\left(\frac{i}{N}\right)^{r-1}\frac{1}{N} = Chx_i^{(r-1)/r} \tag{25}$$

当 $i \geq N, x_i \geq \frac{1}{2}$ 时

$$h_i = \frac{1}{2}\left(\left(\frac{2N-i+1}{N}\right)^r - \left(\frac{2N-i}{N}\right)^r\right) \leq C(r)\left(\frac{2N-i+1}{N}\right)^{r-1}\frac{1}{N} = Ch(1-x_{i-1})^{(r-1)/r} \tag{26}$$

我们声明

$$\begin{aligned}
S_i = \sum_{j=1}^{2N-1} a_{ij} =& \frac{C_R}{(2-\alpha)(3-\alpha)}\frac{2}{h_i+h_{i+1}} \\
&(\frac{1}{h_{i+1}}\frac{|x_{i+1}-x_0|^{3-\alpha}-|x_{i+1}-x_1|^{3-\alpha}}{x_1-x_0} \\
&-(\frac{1}{h_i}+\frac{1}{h_{i+1}})\frac{|x_i-x_0|^{3-\alpha}-|x_i-x_1|^{3-\alpha}}{x_1-x_0} \\
&+\frac{1}{h_i}\frac{|x_{i-1}-x_0|^{3-\alpha}-|x_{i-1}-x_1|^{3-\alpha}}{x_1-x_0}) \\
\geq& C(x_i^{-\alpha}+(1-x_i)^{-\alpha})
\end{aligned} \tag{27}$$

$$R_i = \int_0^1 D(y)\frac{2}{h_i+h_{i+1}}(\frac{1}{h_{i+1}}|x_{i+1}-y|^{1-\alpha}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})|x_i-y|^{1-\alpha}+\frac{1}{h_i}|x_{i-1}-y|^{1-\alpha})dy \tag{28}$$

下面讨论 R_1

$$\begin{aligned}
R_1 =& \int_0^{x_1} + \int_{x_1}^{x_3} + \int_{x_3}^{1/2} + \int_{1/2}^{1} \\
&D(y)\frac{2}{h_1+h_2}(\frac{1}{h_2}|x_2-y|^{1-\alpha}-(\frac{1}{h_1}+\frac{1}{h_2})|x_1-y|^{1-\alpha}+\frac{1}{h_1}|y|^{1-\alpha})dy \\
:=& I_1+I_2+I_3+I_4
\end{aligned} \tag{29}$$

与一致网格时相似，

1.

$$\begin{aligned}|u(x)-u_h(x)| &\le \int_0^{x_1}|u'(y)|dy \\ &\le C\int_0^{x_1} y^{\alpha/2-1}dy \\ &\le Cx_1^{\alpha/2} \quad ,x\in(0,h)\end{aligned} \tag{30}$$

因为 $1-\alpha>-1$

$$\begin{aligned}I_1 &\le C\int_0^{x_1}\frac{D(y)}{h_1^2}(|x_2-y|^{1-\alpha}+2|x_1-y|^{1-\alpha}+|y|^{1-\alpha})dy \\ &\le Cx_1^{\alpha/2-2}x_1^{2-\alpha}=Cx_1^{-\alpha/2}=Ch^{-r\alpha/2}\end{aligned} \tag{31}$$

2.

$$I_2\le Cu''(\eta)x_3^{2-\alpha}\le Cx_1^{\alpha/2-2}x_3^{2-\alpha}\le Ch^{-r\alpha/2} \tag{32}$$

3.

$$\begin{aligned}I_3 &= \int_{x_3}^{1/2}D(y)\frac{2}{h_1+h_2}(\frac{1}{h_2}|x_2-y|^{1-\alpha}-(\frac{1}{h_1}+\frac{1}{h_2})|x_1-y|^{1-\alpha}+\frac{1}{h_1}|y|^{1-\alpha})dy \\ &\le C\int_{x_3}^{1/2}y^{\alpha/2-2}(hy^{(r-1)/r})^2y^{-1-\alpha}dy \\ &\le Ch^2\int_{x_3}^{1/2}y^{\alpha/2-2/r-1-\alpha}dy \\ &\le Ch^2(h^r)^{-2/r-\alpha/2}=Ch^{-r\alpha/2}\end{aligned} \tag{33}$$