AN EXAMPLE ARTICLE*

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Key words. example, LaTeX

MSC codes. 68Q25, 68R10, 68U05

1. Introduction.

The introduction introduces the context and summarizes the manuscript. It is importantly to clearly state the contributions of this piece of work.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$, suppose $f \in C^\beta(\Omega) \cap L^\infty(\Omega), \beta > 4 - \alpha, \|f\|_\beta^{\alpha/2} < \infty$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

2. Regularity.

Remark 2.1. 1. $C^k(U)$ is the set of all k-times continuously differentiable functions on open set U.
2. $C^\beta(U)$ is the collection of function f which for any $V \subset\subset U$ $f|_V \in C^\beta(\bar{V})$.

THEOREM 2.2. *If $f \in C^\beta(\Omega), \beta > 2$ and $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

$$|f^{(l)}(x)| \le \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \le T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \le x < 2T \end{cases} \tag{2.1}$$

THEOREM 2.3 (Regularity up to the boundary [1]).

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right) \tag{2.2}$$

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*Submitted to the editors DATE.

Funding: This work was funded by the Fog Research Institute under contract no. FRI-454.

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COROLLARY 2.4. *Let u be a solution of (1.1) on Ω. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \leq C \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases} \tag{2.3}$$

The paper is organized as follows. Our main results are in section 4, experimental results are in section 7, and the conclusions follow in section 8.

3. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \leq i \leq N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \leq i \leq 2N \end{cases} \tag{3.1}$$

where $r \geq 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \leq j \leq 2N \tag{3.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \leq x \leq x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \leq x \leq x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

And then, we can approximate $u(x)$ with

$$u_h(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{3.4}$$

For convience, we denote

$$I_h^{2-\alpha}(x_i) := \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y)dy \tag{3.5}$$

And now, we can approximate the operator (1.2) at x_i with

$$\begin{aligned} D_h^\alpha u_h(x_i) &:= D_h^2 I_h^{2-\alpha}(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} I_h^{2-\alpha}(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) I_h^{2-\alpha}(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha}(x_{i+1}) \right) \end{aligned} \tag{3.6}$$

Finally, we approximate the equation (1.1) with

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i), \quad 1 \leq i \leq 2N-1 \tag{3.7}$$

The discrete equation (3.7) can be written in matrix form

$$AU = F \tag{3.8}$$

where U is unknown, $F = (f(x_1), \cdots, f(x_{2N-1}))$. The matrix A is constructed as follows: Since

(3.9)
$$
\begin{aligned}
I_h^{2-\alpha}(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u(x_j) \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\
&=: \sum_{j=1}^{2N-1} \tilde{a}_{ij}\, u(x_j), \quad 0 \le i \le 2N
\end{aligned}
$$

Then, substitute in (3.6), we have

$$
-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} a_{ij}\, u(x_j) \tag{3.10}
$$

where

$$
a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \tag{3.11}
$$

4. Main results. Here we state our main results; the proof is deferred to section 5 and section 6.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 4.1 (Truncation Error). *If $f \in C^2(\Omega)$ and $\alpha \in (1,2)$, and $u(x)$ is a solution of the equation (1.1), then there exists a constant $C_1, C_2 = C_1(T, \alpha, r, \|f\|_{C^2(\Omega)}), C_2(T, \alpha, r, \|f\|_{C^2(\Omega)})$, such that the truncation error of the discrete format satisfies*

$$
\begin{aligned}
|-\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i)| \le & C_1 (h^{r\alpha/2 + r} (x_i^{-1-\alpha} + (2T - x_i)^{-1-\alpha}) \\
& + h^2 (x_i^{-\alpha/2 - 2/r} + (2T - x_i)^{-\alpha/2 - 2/r})) \\
& + C_2 h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \le i \le N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \le 2N-1 \end{cases}
\end{aligned} \tag{4.1}
$$

where $C_2 = 0$ if $r = 1$.

THEOREM 4.2 (Convergence). *The discrete equation (3.7) has sulotion U, and there exists a positive constant $C = C(T, \alpha, r, \|f\|_{C^2(\Omega)})$ such that the error between the numerial solution U with the exact solution $u(x_i)$ satisfies*

$$
\max_{1 \le i \le 2N-1} |U_i - u(x_i)| \le C h^{\min\{\frac{r\alpha}{2}, 2\}} \tag{4.2}
$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

5. Proof of Theorem 4.1.

For convience, let's denote

$$I^{2-\alpha}(x)=\frac{1}{\Gamma(2-\alpha)}\int_\Omega |x-y|^{1-\alpha}u(y)dy \tag{5.1}$$

Then, the truncation error of the discrete format can be written as

$$\begin{aligned}-\kappa_\alpha D_h^\alpha u_h(x_i)-f(x_i)&=-\kappa_\alpha(D_h^2 I_h^{2-\alpha}(x_i)-\frac{d^2}{dx^2}I^{2-\alpha}(x_i))\\&=-\kappa_\alpha D_h^2(I_h^{2-\alpha}-I^{2-\alpha})(x_i)-\kappa_\alpha(D_h^2-\frac{d^2}{dx^2})I^{2-\alpha}(x_i)\end{aligned} \tag{5.2}$$

5.1. Estimate of $-\kappa_\alpha(D_h^2-\frac{d^2}{dx^2})I^{2-\alpha}(x_i)$.

THEOREM 5.1. *There exits a constant $C=C(T,\alpha,r,\|f\|_\beta^{(\alpha/2)})$ such that*

$$\left|-\kappa_\alpha(D_h^2-\frac{d^2}{dx^2})I^{2-\alpha}(x_i)\right|\le Ch^2(x_i^{-\alpha/2-2/r}+(2T-x_i)^{-\alpha/2-2/r}) \tag{5.3}$$

Proof. Since $f\in C^2(\Omega)$ and

$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha}(x))=f(x),\quad x\in\Omega, \tag{5.4}$$

we have $I^{2-\alpha}\in C^4(\Omega)$. Therefore, using equation (A.3) of Lemma A.1, for $1\le i\le 2N-1$, we have

$$-\kappa_\alpha(D_h^2-\frac{d^2}{dx^2})I^{2-\alpha}(x_i)=\frac{h_{i+1}-h_i}{3}f'(x_i)+\frac{1}{4!}\frac{2}{h_i+h_{i+1}}(h_i^3f''(\eta_1)+h_{i+1}^3f''(\eta_2)) \tag{5.5}$$

where $\eta_1\in[x_{i-1},x_i],\eta_2\in[x_i,x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

$$\left|\frac{h_{i+1}-h_i}{3}f'(x_i)\right|\le\frac{\|f\|_\beta^{(\alpha/2)}}{3}Ch^2\begin{cases}x_i^{-\alpha/2-2/r}, & 1\le i\le N-1\\ 0, & i=N\\ (2T-x_i)^{-\alpha/2-2/r}, & N<i\le 2N-1\end{cases} \tag{5.6}$$

2. See Proof 7, there is a constant $C=C(T,\alpha,r,\|f\|_\beta^{\alpha/2})$ such that

$$\begin{aligned}&\left|\frac{1}{4!}\frac{2}{h_i+h_{i+1}}(h_i^3f''(\eta_1)+h_{i+1}^3f''(\eta_2))\right|\\&\le Ch^2\begin{cases}x_i^{-\alpha/2-2/r}, & 1\le i\le N\\ (2T-x_i)^{-\alpha/2-2/r}, & N\le i\le 2N-1\end{cases}\end{aligned} \tag{5.7}$$

Summarizes, we get the result. ∎

5.2. Estimate of R_i.

Now, we study the first part of (5.2)

$$D_h^2(I^{2-\alpha}-I_h^{2-\alpha})(x_i)=D_h^2(\int_0^{2T}(u(y)-u_h(y))\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy) \tag{5.8}$$

For convience, let's denote

$$T_{ij}=\int_{x_{j-1}}^{x_j}(u(y)-u_h(y))\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy \tag{5.9}$$

And define

$$R_i := D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i)$$
$$= \frac{2}{h_i + h_{i+1}} \sum_{j=1}^{2N} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \tag{5.10}$$

We have some results about the estimate of R_i

THEOREM 5.2. *For $1 \le i < N/2$, there exists a constant C such that*

$$R_i \le C(h^{r\alpha/2+r} x_i^{-1-\alpha} + h^2 x_i^{-\alpha/2-2/r}) \tag{5.11}$$

THEOREM 5.3. *For $N/2 \le i \le N$, there exists constant C, C_2 such that*

$$R_i \le Ch^2 x_i^{-\alpha/2-2/r} + C_2 h^2 |T - x_{i-1}|^{1-\alpha} \tag{5.12}$$

where $C_2 = 0$ if $r = 1$.

And for $N < i \le 2N - 1$, it is symmetric to the previous case.

To prove these results, we need some utils. Also for simplicity, we denote

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \tag{5.13}$$

then

$$R_i = \sum_{j=1}^{2N} S_{ij} \tag{5.14}$$

LEMMA 5.4. *While $0 \le i < N/2$, By Lemma* A.3

$$|T_{i1}| \le C \int_0^{x_1} x_1^{\alpha/2} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} dy$$
$$= C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2} \left| x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha} \right| \tag{5.15}$$
$$\le C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2+2-\alpha} = C \frac{1}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \quad 0 < 2 - \alpha < 1$$

For $2 \le j \le N$, by Lemma A.2

$$|T_{ij}| \le \frac{C}{4} \int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy$$
$$\le \frac{C}{4\Gamma(3-\alpha)} h_j^2 x_{j-1}^{\alpha/2-2} \left| |x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha} \right| \tag{5.16}$$

LEMMA 5.5. *While $1 \le i < N/2$,*

$$\sum_{j=\max\{2i+1, i+3\}}^{N} S_{ij} \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.17}$$

Proof. For $\max\{2i+1, i+3\} \le j \le N$, by Lemma C.1 and Lemma C.2

$$S_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) dy \tag{5.18}$$
$$\le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2 - 2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy$$
$$= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2 - 2/r - 1} dy$$

Therefore,

$$\sum_{j=\max\{2i+1, i+3\}}^{N} S_{ij} \le Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \tag{5.19}$$
$$= \frac{C}{\alpha/2 + 2/r} h^2 (2^{r(-\alpha/2-2/r)} x_i^{-\alpha/2-2/r} - T^{-\alpha/2-2/r})$$

∎

6. Proof of Theorem 4.2.

7. Experimental results.

8. Conclusions.

Some conclusions here.

Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x)$ is twice differentable continous function on open set Ω, there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

$$D_h^2 g(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \tag{A.1}$$
$$= g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \tag{A.2}$$
$$= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right)$$

And if $g(x) \in C^4(\Omega)$, then

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \tag{A.3}$$
$$= g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2))$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$.

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in [x_{i-1}, x_i]$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in [x_i, x_{i+1}]$$

Subsitute them in the left side of (A.1), we have

$$\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}g(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})g(x_i)+\frac{1}{h_i}g(x_{i-1})\right)$$
$$=\frac{h_i}{h_i+h_{i+1}}g''(\xi_1)+\frac{h_{i+1}}{h_i+h_{i+1}}g''(\xi_2)$$

Now, using intermediate value theorem , there exists $\xi\in[\xi_1,\xi_2]$ such that

$$\frac{h_i}{h_i+h_{i+1}}g''(\xi_1)+\frac{h_{i+1}}{h_i+h_{i+1}}g''(\xi_2)=g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1})=g(x_i)-(x_i-x_{i-1})g'(x_i)+\int_{x_{i-1}}^{x_i}g''(y)(y-x_{i-1})dy$$
$$g(x_{i+1})=g(x_i)+(x_{i+1}-x_i)g'(x_i)+\int_{x_i}^{x_{i+1}}g''(y)(x_{i+1}-y)dy$$

And the last equation can be obtained by

$$g(x_{i-1})=g(x_i)-h_ig'(x_i)+\frac{h_i^2}{2}g''(x_i)-\frac{h_i^3}{3!}g'''(x_i)+\frac{h_i^4}{4!}g''''(\eta_1)$$
$$g(x_{i+1})=g(x_i)+h_{i+1}g'(x_i)+\frac{h_{i+1}^2}{2}g''(x_i)+\frac{h_{i+1}^3}{3!}g'''(x_i)+\frac{h_{i+1}^4}{4!}g''''(\eta_2)$$

where $\eta_1\in[x_{i-1},x_i],\eta_2\in[x_i,x_{i+1}]$. Expecially,

$$\text{(A.4)}\qquad \begin{aligned}\frac{h_i^4}{4!}g''''(\eta_1)&=\int_{x_{i-1}}^{x_i}g''''(y)\frac{(y-x_{i-1})^3}{3!}dy\\ \frac{h_{i+1}^4}{4!}g''''(\eta_2)&=\int_{x_i}^{x_{i+1}}g''''(y)\frac{(x_{i+1}-y)^3}{3!}dy\end{aligned}$$

Subsitute them to the left side of (A.3), we can get the result. ∎

Lemma A.2. *If $y\in[x_{j-1},x_j]$, denote $y=\theta x_{j-1}+(1-\theta)x_j,\theta\in[0,1]$,*

$$\text{(A.5)}\qquad u(y_j^\theta)-u_h(y_j^\theta)=-\frac{\theta(1-\theta)}{2}h_j^2u''(\xi),\quad \xi\in[x_{j-1},x_j]$$

(A.6)
$$u(y_j^\theta)-u_h(y_j^\theta)=-\frac{\theta(1-\theta)}{2}h_j^2u''(y_j^\theta)+\frac{\theta(1-\theta)}{3!}h_j^3(\theta^2u'''(\eta_1)-(1-\theta)^2u'''(\eta_2))$$

where $\eta_1\in[x_{j-1},y_j^\theta],\eta_2\in[y_j^\theta,x_j]$.

Proof. By Taylor expansion, we have

$$u(x_{j-1})=u(y_j^\theta)-\theta h_ju'(y_j^\theta)+\frac{\theta^2h_j^2}{2!}u''(\xi_1),\quad \xi_1\in[x_{j-1},y_j^\theta]$$
$$u(x_j)=u(y_j^\theta)+(1-\theta)h_ju'(y_j^\theta)+\frac{(1-\theta)^2h_j^2}{2!}u''(\xi_2),\quad \xi_2\in[y_j^\theta,x_j]$$

Thus

$$
\begin{aligned}
u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\
&= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta)u''(\xi_2)) \\
&= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2]
\end{aligned}
$$

The second equation is similar,

$$
u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)
$$

$$
u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)
$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$. Thus

$\square$

$$
\begin{aligned}
u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\
&= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))
\end{aligned}
$$

LEMMA A.3. *For $x \in [x_{j-1}, x_j]$*

$$
\begin{aligned}
|u(x) - u_h(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_{x}^{x_j} u'(y)dy \right| \\
&\le \int_{x_{j-1}}^{x_j} |u'(y)|dy
\end{aligned} \tag{A.7}
$$

If $x \in [0, x_1]$, with Corollary 2.4, we have

$$
|u(x) - u_h(x)| \le \int_0^{x_1} |u'(y)|dy \le \int_0^{x_1} C y^{\alpha/2-1} dy \le C\frac{2}{\alpha} x_1^{\alpha/2} \tag{A.8}
$$

Similarly, if $x \in [x_{2N-1}, 1]$, we have

$$
|u(x) - u_h(x)| \le C\frac{2}{\alpha}(2T - x_{2N-1})^{\alpha/2} = C\frac{2}{\alpha} x_1^{\alpha/2} \tag{A.9}
$$

Appendix B. Inequality.

LEMMA B.1.

$$
h_i \le r T^{1/r} h \begin{cases} x_i^{1-1/r}, & 1 \le i \le N \\ (2T - x_{i-1})^{1-1/r}, & N < i \le 2N-1 \end{cases} \tag{B.1}
$$

Proof. For $1 \le i \le N$,

$$
\begin{aligned}
h_i &= T\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right) \\
&\le rT\frac{1}{N}\left(\frac{i}{N}\right)^{r-1} = rT^{1/r} h x_i^{1-1/r}
\end{aligned}
$$

For $N < i \leq 2N-1$,

$$h_i = T\left(\left(\frac{2N-i+1}{N}\right)^r - \left(\frac{2N-i}{N}\right)^r\right)$$
$$\leq rT\frac{1}{N}\left(\frac{2N-i+1}{N}\right)^{r-1} = rT^{1/r}h(2T-x_{i-1})^{1-1/r}$$ ∎

Lemma B.2. *There is a constant $C = 2^{|r-2|}r(r-1)T^{2/r}$ such that for all $i \in \{1, 2, \cdots, 2N-1\}$*

$$|h_{i+1} - h_i| \leq Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases} \tag{B.2}$$

Proof.

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \leq i \leq 2N-1 \end{cases}$$

For $i = 1$,

$$h_2 - h_1 = T(2^r - 2)\left(\frac{1}{N}\right)^r = (2^r - 2)T^{2/r}h^2x_1^{1-2/r}$$

For $2 \leq i \leq N-1$,

$$h_{i+1} - h_i = r(r-1)T\ N^{-2}\eta^{r-2}, \quad \eta \in [\frac{i-1}{N}, \frac{i+1}{N}]$$

If $r \in [1, 2]$,

$$h_{i+1} - h_i = r(r-1)T\ N^{-2}\eta^{r-2} \leq r(r-1)T\ h^2\left(\frac{i-1}{N}\right)^{r-2}$$
$$\leq r(r-1)T\ h^2 2^{2-r}\left(\frac{i}{N}\right)^{r-2}$$
$$= 2^{2-r}r(r-1)T^{2/r}h^2x_i^{1-2/r}$$

else if $r > 2$,

$$h_{i+1} - h_i = r(r-1)T\ N^{-2}\eta^{r-2} \leq r(r-1)T\ h^2\left(\frac{i+1}{N}\right)^{r-2}$$
$$\leq r(r-1)T\ h^2 2^{r-2}\left(\frac{i}{N}\right)^{r-2}$$
$$= 2^{r-2}r(r-1)T^{2/r}h^2x_i^{1-2/r}$$

Since

$$2^r - 2 \leq 2^{|r-2|}r(r-1), \quad r \geq 1$$

we have

$$h_{i+1} - h_i \le 2^{|r-2|} r(r-1) T^{2/r} h^2 x_i^{1-2/r}, \quad 1 \le i \le N-1$$

For $i = N$, $h_{N+1} - h_N = 0$. For $N < i \le 2N-1$, it's central symmetric to the first half of the proof, which is

$$h_i - h_{i+1} \le 2^{|r-2|} r(r-1) T^{2/r} h^2 (2T - x_i)^{1-2/r}$$

Summarizes the inequalities, we can get

$$\text{(B.3)} \qquad |h_{i+1} - h_i| \le 2^{|r-2|} r(r-1) T^{2/r} h^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N-1 \end{cases} \qquad \square$$

Appendix C. Proofs of some technical details.

Additional proof of Theorem 5.1. For $2 \le i \le N-1$,

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\ &\qquad \le C \frac{2}{h_i + h_{i+1}} (h_i^3 x_{i-1}^{-2-\alpha/2} + h_{i+1}^3 x_i^{-2-\alpha/2}) \\ &\qquad \le 2C (h_i^2 x_{i-1}^{-2-\alpha/2} + h_{i+1}^2 x_i^{-2-\alpha/2}) \end{aligned}$$

Since Lemma B.1, we have

$$\begin{aligned} h_i &\le r T^{1/r} h x_i^{1-1/r}, \quad 1 \le i \le N \\ h_{i+1} &\le r T^{1/r} h x_{i+1}^{1-1/r}, \quad 1 \le i \le N-1 \end{aligned}$$

and

$$\begin{aligned} x_{i-1}^{-2-\alpha/2} &\le 2^{-r(-2-\alpha/2)} x_i^{-2-\alpha/2} \quad 2 \le i \le N-1 \\ x_{i+1}^{1-1/r} &\le 2^{r-1} x_i^{1-1/r} \quad 1 \le i \le N-1 \end{aligned}$$

So there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

$$\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \le C h^2 x_i^{-\alpha/2 - 2/r}, \quad 2 \le i \le N-1$$

For $i = 1$, by (A.4)

$$\begin{aligned} &\frac{1}{4!} \frac{2}{h_1 + h_2} (h_1^3 f''(\eta_1) + h_2^3 f''(\eta_2)) \\ &\qquad = \frac{2}{h_1 + h_2} \left(\frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy + \frac{1}{4!} h_2^3 f''(\eta_2) \right) \end{aligned}$$

We have proved above that

$$\frac{2}{h_1 + h_2} h_2^3 f''(\eta_2) \le C h^2 x_1^{-\alpha/2 - 2/r}$$

and we can get

$$\begin{aligned}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy &\le C\frac{1}{3!}\int_0^{x_1} y^{1-\alpha/2}dy\\ &= C\frac{1}{3!(2-\alpha/2)}x_1^{2-\alpha/2}\end{aligned}$$

so

$$\frac{2}{h_1+h_2}\frac{1}{h_1}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy = \frac{C2^{1-r}}{3!(2-\alpha/2)}x_1^{-\alpha/2} = \frac{C2^{1-r}}{3!(2-\alpha/2)}T^{2/r}h^2x_1^{-\alpha/2-2/r}$$

And for $i = N$, we have

$$\begin{aligned}&\frac{2}{h_N+h_{N+1}}(h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2))\\ &\quad = h_N^2(f''(\eta_1) + f''(\eta_2))\\ &\quad \le r^2T^{2/r}h^2x_N^{2-2/r}\ 2Cx_{N-1}^{-2-\alpha/2}\\ &\quad \le 2r^2T^{2/r}C2^{-r(-2-\alpha/2)}\ h^2x_N^{-\alpha/2-2/r}\end{aligned}$$

Finally, $N+1 \le i \le 2N-1$ is symmetric to the first half of the proof, so we can conclude that

∎

$$\frac{2}{h_i+h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \le Ch^2\begin{cases}x_i^{-\alpha/2-2/r}, & 1 \le i \le N\\ (2T-x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1\end{cases}$$

Lemma C.1. *There is a constant C for $2 \le j \le N$, if $y \in [x_{j-1}, x_j]$,*

$$|u(y) - u_h(y)| \le Ch^2y^{\alpha/2-2/r} \tag{C.1}$$

Proof. For $2 \le j \le N$, we have

$$x_j \le 2^r y, \quad x_{j-1} \ge 2^{-r}y$$

And by Lemma A.2 and Lemma B.1, we have

$$\begin{aligned}u(y) - u_h(y) &= -\frac{\theta(1-\theta)}{2}h_j^2u''(\xi), \quad \xi \in [x_{j-1}, x_j]\\ &\le \frac{C}{4}h^2x_j^{2-2/r}x_{j-1}^{\alpha/2-2}\\ &\le \frac{C}{4}h^2\ 2^{2r-2}y^{2-2/r}\ 2^{-r(\alpha/2-2)}y^{\alpha/2-2}\\ &= C2^{-r\alpha/2-r}\ h^2y^{\alpha/2-2/r}\end{aligned}$$

symmetricly, for $N < j \le 2N-1$, we have

$$|u(y) - u_h(y)| \le Ch^2(2T-y)^{\alpha/2-2/r} \tag{C.2}$$

∎

Lemma C.2. *There is a constant C such that for all $1 \le i < N/2$, $\max\{2i+1, i+3\} \le j \le N$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_h^2\left(\frac{|y-\cdot|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) \le Cy^{-1-\alpha} \tag{C.3}$$

Proof. Sinec $y \geq x_{j-1} > x_{i+1}$, by Lemma A.1, if $j-1 > i+1$

$$\begin{aligned} D_h^2(\frac{|y-\cdot|^{1-\alpha}}{\Gamma(2-\alpha)})(x_i) &= \frac{|y-\xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq \frac{(y-x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\leq (1-(\frac{2}{3})^r)^{-1-\alpha}\frac{y^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

But if $i=1, j=3, y \in [x_2, x_3]$

$$D_h^2(\frac{|y-\cdot|^{1-\alpha}}{\Gamma(2-\alpha)})(x_1) = \frac{2}{x_2}\,()$$

∎

Acknowledgments. We would like to acknowledge the assistance of volunteers in putting together this example manuscript and supplement.

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