AN EXAMPLE ARTICLE*

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1. Introduction.

The introduction introduces the context and summarizes the manuscript. It is importantly to clearly state the contributions of this piece of work.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$, suppose $f \in C^\beta(\Omega), \beta > 4 - \alpha, \|f\|_\beta^{(\alpha/2)} < \infty$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

and the solution $u \in C^{\alpha/2}(\Omega)$.

2. Regularity.

Remark 2.1. 1. $C^k(U)$ is the set of all k-times continuously differentiable functions on open set U.
2. $C^\beta(U)$ is the collection of function f which for any $V \subset\subset U$ $f|_V \in C^\beta(\bar{V})$.

THEOREM 2.2. *If $f \in C^\beta(\Omega), \beta > 2$ and $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

$$|f^{(l)}(x)| \le \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \le T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \le x < 2T \end{cases} \tag{2.1}$$

THEOREM 2.3 (Regularity up to the boundary [1]).

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right) \tag{2.2}$$

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Corollary 2.4. *Let u be a solution of (1.1) on Ω. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \leq \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T - x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases} \tag{2.3}$$

The paper is organized as follows. Our main results are in section 4, experimental results are in section 7, and the conclusions follow in section 8.

3. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \leq i \leq N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \leq i \leq 2N \end{cases} \tag{3.1}$$

where $r \geq 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \leq j \leq 2N \tag{3.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \leq x \leq x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \leq x \leq x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

And then, we can approximate $u(x)$ with

$$u_h(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{3.4}$$

For convience, we denote

$$I_h^{2-\alpha}(x_i) := \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y) dy \tag{3.5}$$

And now, we can approximate the operator (1.2) at x_i with

$$\begin{aligned} D_h^\alpha u_h(x_i) &:= D_h^2 I_h^{2-\alpha}(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} I_h^{2-\alpha}(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) I_h^{2-\alpha}(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha}(x_{i+1}) \right) \end{aligned} \tag{3.6}$$

Finally, we approximate the equation (1.1) with

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i), \quad 1 \leq i \leq 2N - 1 \tag{3.7}$$

The discrete equation (3.7) can be written in matrix form

$$AU = F \tag{3.8}$$

where U is unknown, $F = (f(x_1), \cdots, f(x_{2N-1}))$. The matrix A is constructed as follows: Since

(3.9)
$$\begin{aligned}
I_h^{2-\alpha}(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u(x_j)\phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\
&=: \sum_{j=1}^{2N-1} \tilde{a}_{ij}\, u(x_j), \quad 0 \le i \le 2N
\end{aligned}$$

Then, substitute in (3.6), we have

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} a_{ij}\, u(x_j) \tag{3.10}$$

where

$$a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \tag{3.11}$$

4. Main results.

Here we state our main results; the proof is deferred to section 5 and section 6.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 4.1 (Truncation Error). *If $f \in C^2(\Omega)$ and $\alpha \in (1,2)$, and $u(x)$ is a solution of the equation* (1.1)*, then there exists a constant $C_1, C_2 = C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{C^2(\Omega)}), C_2(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$, such that the truncation error of the discrete format satisfies*

$$\begin{aligned}
|-\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i)| \le & C_1 h^{\min\{\frac{r\alpha}{2},2\}} (x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \\
& + C_2 h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \le i \le N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \le 2N-1 \end{cases}
\end{aligned} \tag{4.1}$$

where $C_2 = 0$ if $r = 1$.

THEOREM 4.2 (Convergence). *The discrete equation* (3.7) *has sulotion U, and there exists a positive constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$ such that the error between the numerial solution U with the exact solution $u(x_i)$ satisfies*

$$\max_{1 \le i \le 2N-1} |U_i - u(x_i)| \le C h^{\min\{\frac{r\alpha}{2},2\}} \tag{4.2}$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

5. Proof of Theorem 4.1.

For convience, let's denote

$$I^{2-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)}\int_\Omega |x-y|^{1-\alpha}u(y)dy \tag{5.1}$$

Then, the truncation error of the discrete format can be written as

$$\begin{aligned} -\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I_h^{2-\alpha}(x_i) - \frac{d^2}{dx^2}I^{2-\alpha}(x_i)) \\ &= -\kappa_\alpha D_h^2(I_h^{2-\alpha} - I^{2-\alpha})(x_i) - \kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}(x_i) \end{aligned} \tag{5.2}$$

5.1. Estimate of $-\kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}(x_i)$.

THEOREM 5.1. *There exits a constant $C = C(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$ such that*

$$\left| -\kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}(x_i)\right| \le Ch^2(x_i^{-\alpha/2-2/r} + (2T-x_i)^{-\alpha/2-2/r}) \tag{5.3}$$

Proof. Since $f \in C^2(\Omega)$ and

$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha}(x)) = f(x), \quad x \in \Omega, \tag{5.4}$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.3) of Lemma A.1, for $1 \le i \le 2N-1$, we have

$$-\kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}(x_i) = \frac{h_{i+1}-h_i}{3}f'(x_i) + \frac{1}{4!}\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \tag{5.5}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

$$\left|\frac{h_{i+1}-h_i}{3}f'(x_i)\right| \le \frac{\|f\|_\beta^{(\alpha/2)}}{3}Ch^2\begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T-x_i)^{-\alpha/2-2/r}, & N < i \le 2N-1 \end{cases} \tag{5.6}$$

2. See Proof 18, there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

$$\begin{aligned} &\left|\frac{1}{4!}\frac{2}{h_i+h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2))\right| \\ &\le Ch^2\begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T-x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases} \end{aligned} \tag{5.7}$$

Summarizes, we get the result. ∎

5.2. Estimate of R_i.

Now, we study the first part of (5.2)

$$D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i) = D_h^2(\int_0^{2T}(u(y) - u_h(y))\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy) \tag{5.8}$$

For convience, let's denote

$$T_{ij} = \int_{x_{j-1}}^{x_j}(u(y) - u_h(y))\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy \tag{5.9}$$

And define

$$R_i := D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i)$$
$$= \frac{2}{h_i + h_{i+1}} \sum_{j=1}^{2N} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \tag{5.10}$$

We have some results about the estimate of R_i

THEOREM 5.2. *For $1 \le i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$R_i \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2(x_i^{-1-\alpha} \ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2} x_i^{-1-\alpha}, & \alpha/2 - 2/r < 0 \end{cases} \tag{5.11}$$

THEOREM 5.3. *For $N/2 \le i \le N$, there exists constant C, C_2 such that*

$$R_i \le Ch^2 x_i^{-\alpha/2-2/r} + C_2 h^2 |T - x_{i-1}|^{1-\alpha} \tag{5.12}$$

where $C_2 = 0$ if $r = 1$.

And for $N < i \le 2N - 1$, it is symmetric to the previous case.

To prove these results, we need some utils. Also for simplicity, we denote

DEFINITION 5.4.

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \tag{5.13}$$

then

$$R_i = \sum_{j=1}^{2N} S_{ij} \tag{5.14}$$

5.3. Proof of Theorem 5.2.

LEMMA 5.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \le i < N/2$,*

$$\sum_{j=\max\{2i+1, i+3\}}^{N} S_{ij} \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.15}$$

Proof. For $\max\{2i+1, i+3\} \le j \le N$, by Lemma C.1 and Lemma C.2

$$S_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) dy$$
$$\le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy \tag{5.16}$$
$$= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy$$

Therefore,

$$
\begin{aligned}
\sum_{j=\max\{2i+1,i+3\}}^{N} S_{ij} &\leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\
&= \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\
&\leq \frac{C}{\alpha/2+2/r} 2^{r(-\alpha/2-2/r)} h^2 x_i^{-\alpha/2-2/r}
\end{aligned}
\tag{5.17}
$$

∎

Lemma 5.6. *Thert exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \leq i < N/2$,*

$$
\sum_{j=N+1}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\tag{5.18}
$$

Proof. For $1 \leq i < N/2, N+1 \leq j \leq 2N-1$, by equation (C.2) and Lemma C.2

$$
\begin{aligned}
S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\
&\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2-2/r} y^{-1-\alpha} dy \\
&\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j=N+1}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T - y)^{\alpha/2-2/r} dy \\
&\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\
&= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r}\ h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\end{aligned}
\tag{5.19}
$$

And by Lemma A.3

$$
S_{i,2N} \leq CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}
$$

And when $\alpha/2 - 2/r + 1 \geq 0$,

$$
h^{r\alpha/2+r} \leq h^2
$$

Summarizes, we get the result. ∎

For $i = 1, 2$.

LEMMA 5.7. *By Lemma* C.5 , *Lemma* 5.5 *and Lemma* 5.6 *we get*

$$
\begin{aligned}
R_1 &= \sum_{j=1}^{3} S_{1j} + \sum_{j=4}^{2N} S_{1j} \\
&\leq Ch^2 x_1^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1>0 \\ Ch^2\ln(N), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0 \end{cases}
\end{aligned} \tag{5.20}
$$

$$
\begin{aligned}
R_2 &= \sum_{j=1}^{4} S_{2j} + \sum_{j=5}^{2N} S_{2j} \\
&\leq Ch^2 x_2^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1>0 \\ Ch^2\ln(N), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0 \end{cases}
\end{aligned} \tag{5.21}
$$

For $3 \leq i < N/2$, we have a new separation of R_i, Let's denote $k = \lceil \frac{i}{2} \rceil$.

$$
\begin{aligned}
R_i =& \sum_{j=1}^{2N} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i}+\frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& \sum_{j=1}^{k-1} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i}+\frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&+ \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k}+T_{i+1,k+1}) - (\frac{1}{h_i}+\frac{1}{h_{i+1}}) T_{i,k} \right) \\
&+ \sum_{j=k+1}^{2i-1} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i}+\frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&+ \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i}+T_{i-1,2i-1}) - (\frac{1}{h_i}+\frac{1}{h_{i+1}}) T_{i,2i} \right) \\
&+ \sum_{j=2i+1}^{2N} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i}+\frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& I_1+I_2+I_3+I_4+I_5
\end{aligned} \tag{5.22}
$$

LEMMA 5.8. *There exists a constant $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil$*

$$
|I_1| = |\sum_{j=1}^{k-1} S_{ij}| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1>0 \\ Ch^2 x_i^{-1-\alpha}\ln(i), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2-2/r+1<0 \end{cases} \tag{5.23}
$$

Proof. For $2 \leq j \leq k-1$, by Lemma C.1 and Lemma C.3

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) dy \\ &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 x_i^{-1-\alpha} \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} dy \end{aligned} \tag{5.24}$$

And by Lemma A.3 , Lemma C.3

$$S_{i1} \leq C x_1^{\alpha/2} x_1 x_i^{-1-\alpha} = C x_1^{\alpha/2+1} x_i^{-1-\alpha} = C T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha} \tag{5.25}$$

Therefore,

$$\begin{aligned} I_1 &= \sum_{j=1}^{k-1} S_{ij} = S_{i1} + \sum_{j=2}^{k-1} S_{ij} \\ &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \\ &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \end{aligned} \tag{5.26}$$

But

$$\int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \leq \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.27}$$

So we have

$$I_1 \leq \begin{cases} \frac{C}{\alpha/2-2/r+1} h^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} h^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.28}$$

□

DEFINITION 5.9. *For convience, let's denote*

$$V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \tag{5.29}$$

THEOREM 5.10. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \leq i < N/2, k = \lceil \frac{i}{2} \rceil$,*

$$I_3 = \sum_{j=k+1}^{2i-1} V_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r} \tag{5.30}$$

To estimete V_{ij}, we need some preparations.

LEMMA 5.11. *Denote* $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0,1]$, *by Lemma* A.2

$$\begin{aligned} T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= \int_{x_{j-1}}^{x_j} -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!} h_j^3 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) dy_j^\theta \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta \end{aligned} \tag{5.31}$$

where $\eta_{j1}^\theta \in [x_{j-1}, y_j^\theta], \eta_{j2}^\theta \in [y_j^\theta, x_j]$.

Now Let's construct a series of functions to represent T_{ij}.

DEFINITION 5.12.

$$y_{j-i}(x) = (x^{1/r} + Z_{j-i})^r, \quad Z_{j-i} = T^{1/r} \frac{j-i}{N} \tag{5.32}$$

$$y_{j-i}^\theta(x) = \theta y_{j-1-i}(x) + (1-\theta) y_{j-i}(x) \tag{5.33}$$

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{5.34}$$

Now, we define

$$P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.35}$$

$$Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.36}$$

And now we can rewrite T_{ij}

LEMMA 5.13. *For* $2 \le i \le N, 2 \le j \le N$,

$$\begin{aligned} T_{ij} &= \int_0^1 -\frac{\theta(1-\theta)}{2} P_{j-i}^\theta(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)}{3!} (\theta^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j1}^\theta) - (1-\theta)^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j2}^\theta)) d\theta \end{aligned} \tag{5.37}$$

Immediately, we can see from (5.29) that

LEMMA 5.14. *For $3 \le i \le N-1$, $3 \le j \le N-1$,*

$$\begin{aligned}
V_{ij} &= \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j+1} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)T_{i,j} + \frac{1}{h_i}T_{i-1,j-1}\right)\\
&= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^{\theta}(x_i)d\theta\\
&\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1,1}^{\theta}) - Q_{j-i}^{\theta}(x_i)u'''(\eta_{j,1}^{\theta})}{h_{i+1}}\right)d\theta\\
&\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{j-i}^{\theta}(x_i)u'''(\eta_{j,1}^{\theta}) - Q_{j-i}^{\theta}(x_{i-1})u'''(\eta_{j-1,1}^{\theta})}{h_i}\right)d\theta\\
&\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1,2}^{\theta}) - Q_{j-i}^{\theta}(x_i)u'''(\eta_{j,2}^{\theta})}{h_{i+1}}\right)d\theta\\
&\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{j-i}^{\theta}(x_i)u'''(\eta_{j,2}^{\theta}) - Q_{j-i}^{\theta}(x_{i-1})u'''(\eta_{j-1,2}^{\theta})}{h_i}\right)d\theta
\end{aligned} \tag{5.38}$$

To estimate V_{ij}, we first estimate $D_h^2 P_{j-i}^{\theta}(x_i)$, but By Lemma A.1,

$$D_h^2 P_{j-i}^{\theta}(x_i) = {P_{j-i}^{\theta}}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \tag{5.39}$$

By Leibniz formula, we calculate and estimate the derivations of h_{j-i}^3, $u''(y_{j-i}^{\theta}(x))$ and $\frac{|y_{j-i}^{\theta}(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)}$ separately.

Firstly, we have

LEMMA 5.15. *There exists a constant $C = C(T, r)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,*

$$h_{j-i}^3(\xi) \le Ch^2 x_i^{2-2/r} h_j \tag{5.40}$$

$$(h_{j-i}^3(\xi))' \le C(r-1)h^2 x_i^{1-2/r} h_j \tag{5.41}$$

$$(h_{j-i}^3(\xi))'' \le C(r-1)h^2 x_i^{-2/r} h_j \tag{5.42}$$

The proof of this theorem see Lemma C.6 and Lemma C.7

Second,

LEMMA 5.16. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,*

$$u''(y_{j-i}^{\theta}(\xi)) \le Cx_i^{\alpha/2-2} \tag{5.43}$$

$$(u''(y_{j-i}^{\theta}(\xi)))' \le Cx_i^{\alpha/2-3} \tag{5.44}$$

$$(u''(y_{j-i}^{\theta}(\xi)))'' \le Cx_i^{\alpha/2-4} \tag{5.45}$$

The proof of this theorem see Proof 25

And Finally, we have

LEMMA 5.17. *There exists a constant $C = C(T, \alpha, r)$ such that For $3 \le i \le N-1, 1 \le j \le \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,*

$$|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha} \le C|y_j^{\theta} - x_i|^{1-\alpha} \tag{5.46}$$

$$(|y_{j-i}^{\theta}(\xi)-\xi|^{1-\alpha})' \le C|y_j^{\theta}-x_i|^{1-\alpha}x_i^{-1} \tag{5.47}$$

$$(|y_{j-i}^{\theta}(\xi)-\xi|^{1-\alpha})'' \le C|y_j^{\theta}-x_i|^{1-\alpha}x_i^{-2} \tag{5.48}$$

where $y_j^{\theta}=\theta x_{j-1}+(1-\theta)x_j$

The proof of this theorem see Proof 26

Lemma 5.18. *There exists a constant* $C=C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $3\le i\le N-1, \lceil\frac{i}{2}\rceil+1\le j\le \min\{2i-1,N-1\}$,

$$D_h^2P_{j-i}^{\theta}(x_i)\le Ch^2\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j \tag{5.49}$$

where $y_j^{\theta}=\theta x_{j-1}+(1-\theta)x_j$

Proof. Since

$$D_h^2P_{j-i}^{\theta}(x_i)=P_{j-i}^{\theta}{}''(\xi),\quad \xi\in[x_{i-1},x_{i+1}] \tag{5.50}$$

From (5.35), using Leibniz formula and Lemma 5.15, Lemma 5.16 and Lemma 5.17 ∎

Lemma 5.19. *There exists a constant* $C=C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3\le i<N, k=\lceil\frac{i}{2}\rceil$.
For $k\le j\le\min\{2i-1,N-1\}$,

$$\begin{aligned}&\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta})-Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}}\right)\\&\qquad\le Ch^2\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j\end{aligned} \tag{5.51}$$

And for $k+1\le j\le\min\{2i,N\}$,

$$\begin{aligned}&\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})-Q_{j-i}^{\theta}(x_{i-1})u'''(\eta_{j-1}^{\theta})}{h_i}\right)\\&\qquad\le Ch^2\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j\end{aligned} \tag{5.52}$$

where $\eta_j^{\theta}\in[x_{j-1},x_j]$.

proof see Proof 27

Lemma 5.20. *There exists a constant* $C=C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3\le i<N, k=\lceil\frac{i}{2}\rceil, k+1\le j\le\min\{2i-1,N-1\}$,

$$\begin{aligned}V_{ij}&\le Ch^2\int_0^1\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j\,d\theta\\&=Ch^2\int_{x_{j-1}}^{x_j}\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}dy\end{aligned} \tag{5.53}$$

Proof. Since Lemma 5.14, by Lemma 5.18 and Lemma 5.19, we get the result immediately. ∎

Now we can prove Theorem 5.10 using Lemma 5.20, $k = \lceil \frac{i}{2} \rceil$

$$\begin{aligned} I_3 &= \sum_{k+1}^{2i-1} V_{ij} \le Ch^2 \int_{x_k}^{x_{2i-1}} \frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{2i-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\ &\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{5.54}$$

Lemma 5.21.

$$D_h P_{j-i}^{\theta}(x_i) := \frac{P_{k-i}^{\theta}(x_{i+1}) - P_{k-i}^{\theta}(x_i)}{h_{i+1}} = {P_{j-i}^{\theta}}'(\xi), \quad \xi \in [x_i, x_{i+1}] \tag{5.55}$$

Then, for $3 \le i \le N-1$, $k = \lceil \frac{i}{2} \rceil$,

$$D_h P_{k-i}^{\theta}(x_i) \le Ch^2 x_i^{-\alpha/2-2/r} h_j \tag{5.56}$$

Proof. Using Leibniz formula, by Lemma 5.15, Lemma 5.16 and Lemma 5.17, we take $j = k+1, i = i+1$, we get

$$\begin{aligned} D_h P_{k-i}^{\theta}(x_i) &\le Ch^2 x_{i+1}^{\alpha/2-2/r-1} |y_{k+1}^{\theta} - x_{i+1}|^{1-\alpha} h_{j+1} \\ &\le Ch^2 x_i^{\alpha/2-2/r-1} |y_k^{\theta} - x_i|^{1-\alpha} h_j \\ &\le Ch^2 x_i^{-\alpha/2-2/r} h_j \end{aligned} \tag{5.57}$$ ∎

Lemma 5.22. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \le i < N, k = \lceil \frac{i}{2} \rceil$,*

$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.58}$$

And for $3 \le i < N/2$,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.59}$$

Proof. In fact,

$$\begin{aligned} &\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \\ &= \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i}) T_{i,k} \end{aligned} \tag{5.60}$$

While, by Lemma A.2

$$
\begin{aligned}
\frac{1}{h_{i+1}}(T_{i+1,k}-T_{i,k}) &= \int_{x_{k-1}}^{x_k}(u(y)-u_h(y))\frac{|x_{i+1}-y|^{1-\alpha}-|x_i-y|^{1-\alpha}}{h_{i+1}\Gamma(2-\alpha)}dy \\
&\le \int_{x_{k-1}}^{x_k} h_j^2 u''(\eta)\ \frac{|\xi-y|^{-\alpha}}{\Gamma(1-\alpha)}dy \\
&\le Ch_j h^2 x_j^{2-2/r} x_{k-1}^{\alpha/2-2}|x_i-x_k|^{-\alpha} \\
&\le Ch_j h^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{5.61}
$$

Thus,

$$
\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}(T_{i+1,k}-T_{i,k}) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.62}
$$

For

$$
\begin{aligned}
\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k}) = \int_0^1 -\frac{\theta(1-\theta)}{2}\frac{P_{k-i}^{\theta}(x_{i+1})-P_{k-i}^{\theta}(x_i)}{h_{i+1}}d\theta \\
+\int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{Q_{k-i}^{\theta}(x_{i+1})u'''(\eta_{j+1,1}^{\theta})-Q_{k-i}^{\theta}(x_i)u'''(\eta_{j,1}^{\theta})}{h_{i+1}}d\theta \\
-\int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{Q_{k-i}^{\theta}(x_{i+1})u'''(\eta_{j+1,2}^{\theta})-Q_{k-i}^{\theta}(x_i)u'''(\eta_{j,2}^{\theta})}{h_{i+1}}d\theta
\end{aligned} \tag{5.63}
$$

And by Lemma 5.21

$$
\frac{P_{k-i}^{\theta}(x_{i+1})-P_{k-i}^{\theta}(x_i)}{h_{i+1}} \le Ch^2 x_i^{-\alpha/2-2/r}h_j \tag{5.64}
$$

And with Lemma 5.19, we can get

$$
\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k}) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.65}
$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$
\begin{aligned}
\frac{2}{h_i+h_{i+1}}\frac{h_{i+1}-h_i}{h_i h_{i+1}}T_{i,k} &\le h_i^{-3}h^2 x_i^{1-2/r}h_k C h_k^2 x_{k-1}^{\alpha/2-2}|x_k-x_i|^{1-\alpha} \\
&\le Ch^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{5.66}
$$

Summarizes, we have

$$
I_2 \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.67}
$$

The case for I_4 is similar. ∎

Now combine Lemma 5.8, Lemma 5.22, Theorem 5.10, Lemma 5.5 and Lemma 5.6 to get the final result.

For $3 \le i < N/2$

$$
\begin{aligned}
R_i &= I_1+I_2+I_3+I_4+I_5 \\
&\le Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2+r-2>0 \\ Ch^2(x_i^{-1-\alpha}\ln(i)+\ln(N)), & r\alpha/2+r-2=0 \\ Ch^{r\alpha/2+r}x_i^{-1-\alpha}, & r\alpha/2+r-2<0 \end{cases}
\end{aligned} \tag{5.68}
$$

Combine with $i = 1, 2$, we get for $1 \le i \le N/2$

$$R_i \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases} \tag{5.69}$$

5.4. Proof of Theorem 5.3. For $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$, we have

$$\begin{aligned} R_i =& \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\ &+ \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \\ &+ \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 \end{aligned} \tag{5.70}$$

We have estimate I_1 in Lemma 5.8 and I_2 in Lemma 5.22. We can control I_3 in similar with Theorem 5.10 by Lemma 5.20 where $2i - 1 \ge N - 1$

$$\begin{aligned} I_3 = \sum_{j=k+1}^{N-1} V_{ij} &\le Ch^2 \int_{x_k}^{x_{N-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{N-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\ &\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{5.71}$$

Let's study I_5 before I_4.

$$I_5 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} V_{ij} \tag{5.72}$$

Similarly, Let's define a new series of functions

Definition 5.23. *For $i < N, j \ge N$,*

$$y_{j-i}(x) = 2T - (Z_{2N-j+i} - x^{1/r})^r, \quad Z_{2N-j+i} = T^{1/r} \frac{2N - j + i}{N} \tag{5.73}$$

$$y_{j-i}{}'(x) = (2T - y_{j-i}(x))^{1-1/r} x^{1/r-1} \tag{5.74}$$

$$y_{j-i}''(x) = \frac{1-r}{r}(2T - y_{j-i}(x))^{1-2/r} x^{1/r-2} Z_{2N-j+i} \tag{5.75}$$

(5.76)

$$y^{\theta}_{j-i}(x) = \theta y_{j-i-1}(x) + (1-\theta) y_{j-i}(x) \tag{5.77}$$

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{5.78}$$

$$P^{\theta}_{j-i}(x) = (h_{j-i}(x))^3 u''(y^{\theta}_{j-i}(x)) \frac{|y^{\theta}_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.79}$$

$$Q^{\theta}_{j-i}(x) = (h_{j-i}(x))^4 \frac{|y^{\theta}_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.80}$$

Now we have, for $i < N, j \geq N+2$,

(5.81)

$$\begin{aligned}
V_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P^{\theta}_{j-i}(x_i) d\theta \\
&\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_{i+1}) u'''(\eta^{\theta}_{j+1,1}) - Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,1})}{h_{i+1}} \right) d\theta \\
&\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,1}) - Q^{\theta}_{j-i}(x_{i-1}) u'''(\eta^{\theta}_{j-1,1})}{h_i} \right) d\theta \\
&\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_{i+1}) u'''(\eta^{\theta}_{j+1,2}) - Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,2})}{h_{i+1}} \right) d\theta \\
&\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,2}) - Q^{\theta}_{j-i}(x_{i-1}) u'''(\eta^{\theta}_{j-1,2})}{h_i} \right) d\theta
\end{aligned}$$

Similarly, we first estimate

$$D_h^2 P^{\theta}_{j-i}(\xi) = {P^{\theta}_{j-i}}'(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \tag{5.82}$$

Combine lemmas Lemma C.8, Lemma C.9 and Lemma C.10 , we have

Lemma 5.24. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that For* $N/2 \leq i < N$, $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in [x_{i-1}, x_{i+1}]$*, we have*

$$\begin{aligned}
|{P^{\theta}_{j-i}}''(\xi)| \leq C h_j h^2 (&|y^{\theta}_j - x_i|^{1-\alpha} \\
&+ |y^{\theta}_j - x_i|^{-\alpha} (|2T - x_i - y^{\theta}_j| + h_N) \\
&+ |y^{\theta}_j - x_i|^{-1-\alpha} (|2T - x_i - y^{\theta}_j| + h_N)^2 \\
&+ (r-1) |y^{\theta}_j - x_i|^{-\alpha})
\end{aligned} \tag{5.83}$$

And

Lemma 5.25. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le i < N$, , $\xi \in [x_{i-1}, x_{i+1}]$, we have for $N+1 \le j \le 2N - \lceil \frac{N}{2} \rceil$*

$$\begin{aligned}\frac{2}{h_i + h_{i+1}} &\left(\frac{Q_{j-i}^{\theta}(x_{i+1}) u'''(\eta_{j+1}^{\theta}) - Q_{j-i}^{\theta}(x_i) u'''(\eta_j^{\theta})}{h_{i+1}} \right) \\ &\le C h^2 h_j (|y_j^{\theta} - x_i|^{1-\alpha} + |y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N))\end{aligned} \tag{5.84}$$

for $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$

$$\begin{aligned}\frac{2}{h_i + h_{i+1}} &\left(\frac{Q_{j-i}^{\theta}(x_i) u'''(\eta_j^{\theta}) - Q_{j-i}^{\theta}(x_{i-1}) u'''(\eta_{j-1}^{\theta})}{h_{i+1}} \right) \\ &\le C h^2 h_j (|y_j^{\theta} - x_i|^{1-\alpha} + |y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N))\end{aligned} \tag{5.85}$$

The proof see Proof 31.

Combine (5.81), Lemma 5.24 and Lemma 5.25, we have

Theorem 5.26. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le i < N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$*

$$\begin{aligned}V_{ij} \le & C h^2 \int_{x_{j-1}}^{x_j} (|y - x_i|^{1-\alpha} \\ & + |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2 \\ & + (r-1)|y - x_i|^{-\alpha}) dy\end{aligned} \tag{5.86}$$

We can esitmate I_5 Now.

Theorem 5.27. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le i < N$, we have*

$$I_5 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \le C h^2 + C(r-1) h^2 |T - x_{i-1}|^{1-\alpha} \tag{5.87}$$

Proof.

$$\begin{aligned}I_5 = & \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \\ \le & C h^2 \int_{x_{N+1}}^{x_{2N-i}} + \int_{x_{2N-i}}^{x_{2N-\lceil \frac{N}{2} \rceil}} (|y - x_i|^{1-\alpha} \\ & + |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2 \\ & + (r-1)|y - x_i|^{-\alpha}) dy \\ = & J_1 + J_2\end{aligned} \tag{5.88}$$

While $x_{N+1} \le y \le x_{2N-i} = 2T - x_i$,

$$T - x_{i-1} \le x_{N+1} - x_i \le y - x_i \le x_{2N-i} - x_i \le 2(T - x_{i-1}) \tag{5.89}$$

and

$$2T - x_i - y + h_N \le 2T - x_i - x_{N+1} + h_N = T - x_i \le T - x_{i-1} \tag{5.90}$$

So

$$
\begin{aligned}
J_1 &\le Ch^2(x_{2N-i} - x_{N+1})(|T - x_{i-1}|^{1-\alpha} + (r-1)|T - x_{i-1}|^{-\alpha}) \\
&\le Ch^2(|T - x_{i-1}|^{2-\alpha} + (r-1)|T - x_{i-1}|^{1-\alpha}) \\
&\le Ch^2T^{2-\alpha} + C(r-1)h^2|T - x_{i-1}|^{1-\alpha}
\end{aligned} \tag{5.91}
$$

Otherwise, when $x_{2N-i} \le y \le x_{2N-\lceil \frac{N}{2} \rceil}$

$$
x_i + y - 2T + h_N \le y - x_i \tag{5.92}
$$

$$
\begin{aligned}
J_2 &\le Ch^2 \int_{x_{2N-i}}^{(2-2^{-r})T} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} \\
&\le Ch^2(T^{2-\alpha} + (r-1)|x_{2N-i} - x_i|^{1-\alpha}) \\
&= Ch^2 + C(r-1)h^2|T - x_i|^{1-\alpha} \le Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha}
\end{aligned} \tag{5.93}
$$

Summarizes two cases, we get the result. ∎

For I_4, we have

THEOREM 5.28. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that, for $N/2 \le i < N-1$*

$$V_{iN} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,N+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,N} + \frac{1}{h_i} T_{i-1,N-1} \right) \le Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \tag{5.94}$$

Proof. We use the similar skill in the last section, but more complicated. for $j = N$, Let

$$y_{i\to N-1}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r} \frac{N-1-i}{N} \tag{5.95}$$

$$y_{i\to N}(x) = \frac{x^{1/r} - Z_i}{Z_1} h_N + T, \quad Z_i = T^{1/r} \frac{i}{N}, x_N = T \tag{5.96}$$

and

$$y_{i\to N+1}(x) = 2T - (Z_{N-1+i} - x^{1/r})^r, \quad Z_{N-1+i} = T^{1/r} \frac{N-1+i}{N} \tag{5.97}$$

Thus,

$$\begin{aligned} y_{i\to N-1}(x_{i-1}) = x_{N-2}, \quad y_{i\to N}(x_i) = x_{N-1}, \quad y_{i\to N}(x_{i+1}) = x_N \\ y_{i\to N}(x_{i-1}) = x_{N-1}, \quad y_{i\to N}(x_i) = x_N, \quad y_{i\to N}(x_{i+1}) = x_{N+1} \\ y_{i\to N-1}(x_{i-1}) = x_N, \quad y_{i\to N}(x_i) = x_{N+1}, \quad y_{i\to N}(x_{i+1}) = x_{N+2} \end{aligned}$$

Then, define

$$y^\theta_{i\to N}(x) = \theta y_{i\to N-1}(x) + (1-\theta) y_{i\to N}(x) \tag{5.98}$$

$$y^\theta_{i\to N+1}(x) = \theta y_{i\to N}(x) + (1-\theta) y_{i\to N+1}(x) \tag{5.99}$$

$$h_{i\to N}(x) = y_{i\to N}(x) - y_{i\to N-1}(x) \tag{5.100}$$

$$h_{i\to N+1}(x) = y_{i\to N+1}(x) - y_{i\to N}(x) \tag{5.101}$$

We have

$$y_{i\to N-1}'(x) = y^{1-1/r}_{i\to N-1}(x) x^{1/r-1} \tag{5.102}$$

$$y_{i\to N-1}''(x) = \frac{1-r}{r} y^{1-2/r}_{i\to N-1}(x) x^{1/r-2} Z_{N-1-i} \tag{5.103}$$

$$y_{i\to N}'(x) = \frac{1}{r} \frac{h_N}{Z_1} x^{1/r-1} \tag{5.104}$$

$$y_{i\to N}''(x) = \frac{1-r}{r^2} \frac{h_N}{Z_1} x^{1/r-2} \tag{5.105}$$

$$y_{i\to N+1}'(x) = (2T - y_{i\to N+1}(x))^{1-1/r} x^{1/r-1} \tag{5.106}$$

$$y_{i\to N+1}''(x) = \frac{1-r}{r} (2T - y_{i\to N+1}(x))^{1-2/r} x^{1/r-2} Z_{N-1+i} \tag{5.107}$$

$$P^{\theta}_{i\to N}(x) = (h_{i\to N}(x))^3 \frac{|y^{\theta}_{i\to N}(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y^{\theta}_{i\to N}(x)) \tag{5.108}$$

$$P^{\theta}_{i\to N+1}(x) = (h_{i\to N+1}(x))^3 \frac{|y^{\theta}_{i\to N+1}(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y^{\theta}_{i\to N+1}(x)) \tag{5.109}$$

$$Q^{\theta}_{i\to N}(x) = (h_{i\to N}(x))^4 \frac{|y^{\theta}_{i\to N}(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.110}$$

$$Q^{\theta}_{i\to N+1}(x) = (h_{i\to N+1}(x))^4 \frac{|y^{\theta}_{i\to N+1}(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.111}$$

Similar with Lemma 5.13, we can get for $l=-1,0,1$,

$$\begin{aligned} T_{i+l,N+l} &= \int_0^1 -\frac{\theta(1-\theta)}{2} P^{\theta}_{i\to N}(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} Q^{\theta}_{i\to N}(x_{i+l})(\theta^2 u'''(\eta^{\theta}_{N+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+l,2})) d\theta \end{aligned} \tag{5.112}$$

$$\begin{aligned} T_{i+l,N+1+l} &= \int_0^1 -\frac{\theta(1-\theta)}{2} P^{\theta}_{i\to N+1}(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} Q^{\theta}_{i\to N+1}(x_{i+l})(\theta^2 u'''(\eta^{\theta}_{N+1+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+1+l,2})) d\theta \end{aligned} \tag{5.113}$$

So we have

$$\begin{aligned} V_{i,N} &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P^{\theta}_{i\to N}(x_i) d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i+h_{i+1}} \left(\frac{Q^{\theta}_{i\to N}(x_{i+1})u'''(\eta^{\theta}_{N+1,1}) - Q^{\theta}_{i\to N}(x_i)u'''(\eta^{\theta}_{N,1})}{h_{i+1}} \right) d\theta \\ &- \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i+h_{i+1}} \left(\frac{Q^{\theta}_{i\to N}(x_i)u'''(\eta^{\theta}_{N,1}) - Q^{\theta}_{i\to N}(x_{i-1})u'''(\eta^{\theta}_{N-1,1})}{h_i} \right) d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i+h_{i+1}} \left(\frac{Q^{\theta}_{i\to N}(x_{i+1})u'''(\eta^{\theta}_{N+1,2}) - Q^{\theta}_{i\to N}(x_i)u'''(\eta^{\theta}_{N,2})}{h_{i+1}} \right) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i+h_{i+1}} \left(\frac{Q^{\theta}_{i\to N}(x_i)u'''(\eta^{\theta}_{N,2}) - Q^{\theta}_{i\to N}(x_{i-1})u'''(\eta^{\theta}_{N-1,2})}{h_i} \right) d\theta \end{aligned} \tag{5.114}$$

$N+1$ is similar.

We estimate $D_h^2 P^{\theta}_{i\to N}(x_i) = {P^{\theta}_{i\to N}}''(\xi), \xi \in [x_{i-1}, x_{i+1}]$,

Lemma 5.29.

$$h^3_{i\to N}(\xi) \le C h_N^3 \le C h^3 \tag{5.115}$$

$$h^3_{i\to N+1}(\xi) \le C h_N^3 \le C h^3 \tag{5.116}$$

$$(h^3_{i\to N}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{5.117}$$

$$(h^3_{i\to N+1}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{5.118}$$

$$(h^3_{i\to N}(\xi))'' \le C(r-1)h^2 \tag{5.119}$$

$$(h^3_{i\to N+1}(\xi))'' \le C(r-1)h^2 \tag{5.120}$$

Proof.

$$h_{i\to N}(\xi) \le 2h_N, \quad h_{i\to N+1}(\xi) \le 2h_N \tag{5.121}$$

$$\begin{aligned}(h^l_{i\to N}(\xi))' &= l h^{l-1}_{i\to N}(\xi)({y_{i\to N}}'(\xi) - {y_{i\to N-1}}'(\xi))\\ &= l h^{l-1}_{i\to N}(\xi) x_i^{1/r-1}(\frac{1}{r}\frac{h_N}{Z_1} - y^{1-1/r}_{i\to N-1}(\xi))\end{aligned} \tag{5.122}$$

while

$$\begin{aligned}|\frac{1}{r}\frac{h_N}{Z_1} - y^{1-1/r}_{i\to N-1}(\xi)| &= |\frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}| \quad \eta \in [x_{N-2}, x_N]\\ &= T^{1-1/r}|(\frac{N-t}{N})^{r-1} - (\frac{N-s}{N})^{r-1}| \quad t\in[0,1], s\in[0,2]\\ &\le T^{1-1/r}|1-(\frac{N-2}{N})^{r-1}| \le CT^{1-1/r}(r-1)\frac{2}{N}\end{aligned} \tag{5.123}$$

Thus,

$$(h^l_{i\to N}(\xi))' \le C(r-1)h_N^{l-1}x_i^{1/r-1}h \tag{5.124}$$

$$\begin{aligned}(h^l_{i\to N+1}(\xi))' &= l h^{l-1}_{i\to N+1}(\xi)({y_{i\to N+1}}'(\xi) - {y_{i\to N}}'(\xi))\\ &= l h^{l-1}_{i\to N+1}(\xi) x_i^{1/r-1}((2T - y_{i\to N+1}(\xi))^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1})\end{aligned} \tag{5.125}$$

Similarly,

$$\begin{aligned}|(2T - y_{i\to N+1})^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1}| &= |\eta^{1-1/r} - \frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1}| \quad \eta\in[x_{N-2}, x_N]\\ &= T^{1-1/r}|(\frac{N-s}{N})^{r-1} - (\frac{N-t}{N})^{r-1}| \quad t\in[0,1], s\in[0,2]\\ &\le T^{1-1/r}|(\frac{N-2}{N})^{r-1} - 1| \le CT^{1-1/r}(r-1)\frac{2}{N}\end{aligned} \tag{5.126}$$

And

$$\begin{aligned}(h^3_{i\to N}(\xi))'' &= 3h^2_{i\to N}(\xi){h_{i\to N}}''(\xi) + 6h_{i\to N}(\xi)({h_{i\to N}}'(\xi))^2\\ &\le Ch_N^2\frac{1-r}{r}x_i^{1/r-2}(\frac{1}{r}\frac{h_N}{Z_1} - y^{1-2/r}_{i\to N-1}(\xi)Z_{N-1-i}) + Ch_N(r-1)^2h^2x_i^{2/r-2}\end{aligned} \tag{5.127}$$

$$|\frac{h_N}{rZ_1} - y^{1-2/r}_{i\to N-1}(\xi)Z_{N-1-i}| \le T^{1-1/r} + Cx_N^{1-2/r}x_N^{1/r} = CT^{1-1/r}$$

So

$$(h^3_{i\to N}(\xi))'' \le Ch_N^2\frac{1-r}{r}x_i^{1/r-2} + C(r-1)^2h_Nx_i^{2/r-2}h^2 \\ \le C(r-1)h_N^2x_i^{1/r-1} \tag{5.128}$$

$h^3_{i\to N+1}(\xi)$ is similar. ∎

Lemma 5.30.

$$u''(y^\theta_{i\to N}(\xi)) \le Cx_{N-2}^{-\alpha/2-2} \le C \tag{5.129}$$

$$(u''(y^\theta_{i\to N}(\xi)))' \le C \tag{5.130}$$

$$(u''(y^\theta_{i\to N}(\xi)))'' \le C \tag{5.131}$$

Proof.

$$\begin{aligned}(u''(y^\theta_{i\to N}(\xi)))' &= u'''(y^\theta_{i\to N}(\xi)){y^\theta_{i\to N}}'(\xi)\\ &\le C(\theta {y_{i\to N-1}}'(\xi) + (1-\theta){y_{i\to N}}'(\xi))\\ &\le Cx_i^{1/r-1}(\theta y^{1-1/r}_{i\to N-1}(\xi) + (1-\theta)\frac{h_N}{rZ_1})\\ &\le Cx_i^{1/r-1}x_N^{1-1/r}\end{aligned} \tag{5.132}$$

And

$$\begin{aligned}(u''(y^\theta_{i\to N}(\xi)))'' &= u''''(y^\theta_{i\to N}(\xi))({y^\theta_{i\to N}}'(\xi))^2 + u'''(y^\theta_{i\to N}(\xi)){y^\theta_{i\to N}}''(\xi)\\ &\le Cx_i^{2/r-2}x_N^{2-2/r} + C\frac{r-1}{r}x_i^{1/r-2}(\theta x_N^{1-2/r}Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1})\\ &\le Cx_i^{2/r-2} + C(r-1)x_i^{1/r-2}T^{1-1/r}\end{aligned} \tag{5.133}$$

∎

Lemma 5.31.

$$|y^\theta_{i\to N}(\xi) - \xi|^{1-\alpha} \le C|y^\theta_N - x_i|^{1-\alpha} \tag{5.134}$$

$$(|y^\theta_{i\to N}(\xi) - \xi|^{1-\alpha})' \le C|y^\theta_N - x_i|^{1-\alpha} \tag{5.135}$$

$$(|y^\theta_{i\to N}(\xi) - \xi|^{1-\alpha})'' \le C(r-1)|y^\theta_N - x_i|^{-\alpha} + |y^\theta_N - x_i|^{1-\alpha} \tag{5.136}$$

Proof.

$$\begin{aligned}(y^\theta_{i\to N}(\xi) - \xi)' &= (\theta(y_{i\to N-1}(\xi) - \xi) + (1-\theta)(y_{i\to N}(\xi) - \xi))'\\ &= \theta({y_{i\to N-1}}'(\xi) - 1) + (1-\theta)({y_{i\to N}}'(\xi) - 1)\\ &= \theta\xi^{1/r-1}(y^{1-1/r}_{i\to N-1}(\xi) - \xi^{1-1/r}) + (1-\theta)\xi^{1/r-1}(\frac{h_N}{rZ_1} - \xi^{1-1/r})\end{aligned} \tag{5.137}$$

$$\begin{aligned}(y^\theta_{i\to N}(\xi) - \xi)'' &= \theta({y_{i\to N-1}}''(\xi)) + (1-\theta)({y_{i\to N}}''(\xi))\\ &= \frac{1-r}{r}\xi^{1/r-2}(\theta y^{1-2/r}_{i\to N-1}(\xi)Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \le 0\end{aligned} \tag{5.138}$$

And

$$|(y^\theta_{i\to N}(\xi) - \xi)''| \le C(r-1)\xi^{1/r-2}T^{1-1/r} \tag{5.139}$$

We have known

$$C|x_{N-1}-x_i| \leq |y_{i\to N-1}(\xi)-\xi| \leq C|x_{N-1}-x_i| \tag{5.140}$$

If $\xi \leq x_{N-1}$, then $(y_{i\to N}(\xi)-\xi)' \geq 0$, so

$$C|x_N-x_i| \leq |x_{N-1}-x_{i-1}| \leq |y^\theta_{i\to N}(\xi)-\xi| \leq |x_{N+1}-x_{i+1}| \leq C|x_N-x_i| \tag{5.141}$$

If $i=N-1$ and $\xi \in [x_{N-1},x_N]$, then $y_{i\to N}(\xi)-\xi$ is concave, bigger than its two neighboring points, which are equal to h_N, so

$$h_N = |x_N-x_{N-1}| \leq |y_{i\to N}(\xi)-\xi| \leq |x_{N+1}-x_{N-1}| = 2h_N \tag{5.142}$$

So we have

$$|y^\theta_{i\to N}(\xi)-\xi|^{1-\alpha} \leq C|y^\theta_N-x_i|^{1-\alpha} \tag{5.143}$$

While

$$y^{1-1/r}_{i\to N-1}(\xi)-\xi^{1-1/r} \leq (y_{i\to N-1}(\xi)-\xi)\xi^{-1/r} \tag{5.144}$$

and

$$\begin{aligned} |\frac{h_N}{rZ_1}-\xi^{1-1/r}| &\leq \max\{|\frac{h_N}{rZ_1}-x^{1-1/r}_{i-1}|,|\frac{h_N}{rZ_1}-x^{1-1/r}_{i+1}|\} \\ &\leq \max\begin{cases} T^{1-1/r}-x^{1-1/r}_{i-1} \leq |x_N-x_{i-1}|T^{-1/r} \leq C|x_N-x_i| \\ |x^{1-1/r}_{i+1}-x^{1-1/r}_{N-1}| \leq |x_{i+1}-x_{N-1}|x^{-1/r}_{N-1} \leq C|x_N-x_i| \end{cases} \end{aligned} \tag{5.145}$$

So we have

$$(y^\theta_{i\to N}(\xi)-\xi)' \leq C|y^\theta_N-x_i| \tag{5.146}$$

$$\begin{aligned} (|y^\theta_{i\to N}(\xi)-\xi|^{1-\alpha})' &= |y^\theta_{i\to N}(\xi)-\xi|^{-\alpha}(y^\theta_{i\to N}(\xi)-\xi)' \\ &\leq |y^\theta_N-x_i|^{1-\alpha} \end{aligned} \tag{5.147}$$

Finally,

$$\begin{aligned} (|y^\theta_{i\to N}(\xi)-\xi|^{1-\alpha})'' = (1-\alpha)&|y^\theta_{i\to N}(\xi)-\xi|^{-\alpha}(y^\theta_{i\to N}(\xi)-\xi)'' \\ &+\alpha(\alpha-1)|y^\theta_{i\to N}(\xi)-\xi|^{-1-\alpha}((y^\theta_{i\to N}(\xi)-\xi)')^2 \\ \leq C(r-1)&|y^\theta_N-x_i|^{-\alpha}+C|y^\theta_N-x_i|^{1-\alpha} \end{aligned} \tag{5.148}$$ ∎

By the three lemmas above, for $N/2 \leq i \leq N-1$, we have

LEMMA 5.32.

$$\begin{aligned} D^2_h P^\theta_{i\to N}(x_i) &= {P^\theta_{i\to N}}''(\xi) \quad \xi \in [x_{i-1},x_{i+1}] \\ &\leq Ch^3|y^\theta_N-x_i|^{1-\alpha}+C(r-1)(h^3|y^\theta_N-x_i|^{-\alpha}+h^2|y^\theta_N-x_i|^{1-\alpha}) \end{aligned} \tag{5.149}$$

And

LEMMA 5.33.

$$\frac{2}{h_i+h_{i+1}}\left(\frac{Q^{\theta}_{i\to N}(x_{i+1})u'''(\eta^{\theta}_{N+1})-Q^{\theta}_{i\to N}(x_i)u'''(\eta^{\theta}_{N})}{h_{i+1}}\right) \le Ch^3|y^{\theta}_N-x_i|^{1-\alpha} \tag{5.150}$$

And immediately, For $N/2\le i\le N-2$

$$\begin{aligned} V_{iN} &\le C\int_{x_{N-1}}^{x_N} h^2|y-x_i|^{1-\alpha}+C(r-1)h^2|y-x_i|^{-\alpha}+h|y-x_i|^{1-\alpha}dy \\ &\le Ch^2h_N|T-x_i|^{1-\alpha}+C(r-1)h^2|x_{N-1}-x_i|^{1-\alpha}+Chh_N|T-x_i|^{1-\alpha} \\ &\le Ch^2+C(r-1)h^2|T-x_{i-1}|^{1-\alpha} \end{aligned} \tag{5.151}$$

But expecially, when $i=N-1$,

$$\begin{aligned} V_{N-1,N} &= \int_0^1 -\frac{\theta^{2-\alpha}(1-\theta)}{2}\frac{2}{h_{N-1}+h_N}\left(\frac{1}{h_{N-1}}h^{4-\alpha}_{N-1}u''(y^{\theta}_{N-1})-(\frac{1}{h_{N-1}}+\frac{1}{h_N})h^{4-\alpha}_N u''(y^{\theta}_N)+\frac{1}{h_N}h^{4-\alpha}_{N+1}u''(y^{\theta}_{N+1})\right)d\theta \\ &+\int_0^1\frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q^{\theta}_{i\to N}(x_{i+1})u'''(\eta^{\theta}_{N+1,1})-Q^{\theta}_{i\to N}(x_i)u'''(\eta^{\theta}_{N,1})}{h_{i+1}}\right)d\theta \\ &-\int_0^1\frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q^{\theta}_{i\to N}(x_{i})u'''(\eta^{\theta}_{N,1})-Q^{\theta}_{i\to N}(x_{i-1})u'''(\eta^{\theta}_{N-1,1})}{h_{i}}\right)d\theta \\ &-\int_0^1\frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q^{\theta}_{i\to N}(x_{i+1})u'''(\eta^{\theta}_{N+1,2})-Q^{\theta}_{i\to N}(x_i)u'''(\eta^{\theta}_{N,2})}{h_{i+1}}\right)d\theta \\ &+\int_0^1\frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q^{\theta}_{i\to N}(x_{i})u'''(\eta^{\theta}_{N,2})-Q^{\theta}_{i\to N}(x_{i-1})u'''(\eta^{\theta}_{N-1,2})}{h_{i}}\right)d\theta \end{aligned} \tag{5.152}$$

while

$$\begin{aligned} &\frac{2}{h_{N-1}+h_N}\left(\frac{1}{h_{N-1}}h^{4-\alpha}_{N-1}u''(y^{\theta}_{N-1})-(\frac{1}{h_{N-1}}+\frac{1}{h_N})h^{4-\alpha}_N u''(y^{\theta}_N)+\frac{1}{h_N}h^{4-\alpha}_{N+1}u''(y^{\theta}_{N+1})\right) \\ &=\frac{2}{h_{N-1}+h_N}\frac{1}{h_{N-1}}\left(h^{4-\alpha}_{N-1}u''(y^{\theta}_{N-1})-h^{4-\alpha}_N u''(y^{\theta}_N)\right) \\ &\quad-\frac{2}{h_{N-1}+h_N}\frac{1}{h_N}\left(h^{4-\alpha}_N u''(y^{\theta}_N)-h^{4-\alpha}_{N+1}u''(y^{\theta}_{N+1})\right) \\ &\le Ch^{4-\alpha}_N+C(r-1)h^{3-\alpha}_N\le Ch^{4-\alpha}+C(r-1)h^2|T-x_{N-1-1}|^{1-\alpha} \end{aligned} \tag{5.153}$$

$\square$

6. Proof of Theorem 4.2.

7. Experimental results.

8. Conclusions.
Some conclusions here.

Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x)$ is twice differentable continous function on open set Ω, there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

$$\begin{aligned} D_h^2 g(x_i) &:= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &= g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \end{aligned} \tag{A.1}$$

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\quad = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right) \end{aligned} \tag{A.2}$$

And if $g(x) \in C^4(\Omega)$, then

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\qquad = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2)) \end{aligned} \tag{A.3}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$.

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in [x_{i-1}, x_i]$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in [x_i, x_{i+1}]$$

Subsitute them in the left side of (A.1), we have

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\qquad = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) \end{aligned}$$

Now, using intermediate value theorem , there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y)dy$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \frac{h_i^4}{4!} g''''(\eta_1)$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \frac{h_{i+1}^4}{4!} g''''(\eta_2)$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. Expecially,

$$\begin{aligned} \frac{h_i^4}{4!} g''''(\eta_1) &= \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy \\ \frac{h_{i+1}^4}{4!} g''''(\eta_2) &= \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \end{aligned} \tag{A.4}$$

Subsitute them to the left side of (A.3), we can get the result. ∎

LEMMA A.2. *If $y \in [x_{j-1}, x_j]$, denote $y = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0,1]$,*

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \tag{A.5}$$

(A.6)

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta)u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$. Thus ∎

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

LEMMA A.3. *For $x \in [x_{j-1}, x_j]$*

$$|u(x) - u_h(x)| = \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_{x}^{x_j} u'(y)dy \right| \tag{A.7}$$
$$\leq \int_{x_{j-1}}^{x_j} |u'(y)|dy$$

If $x \in [0, x_1]$, with Corollary 2.4, we have

$$|u(x) - u_h(x)| \leq \int_0^{x_1} |u'(y)|dy \leq \int_0^{x_1} Cy^{\alpha/2-1}dy \leq C\frac{2}{\alpha}x_1^{\alpha/2} \tag{A.8}$$

Similarly, if $x \in [x_{2N-1}, 1]$, we have

$$|u(x) - u_h(x)| \leq C\frac{2}{\alpha}(2T - x_{2N-1})^{\alpha/2} = C\frac{2}{\alpha}x_1^{\alpha/2} \tag{A.9}$$

Appendix B. Inequality.

LEMMA B.1.

$$h_i \leq rT^{1/r}h \begin{cases} x_i^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_{i-1})^{1-1/r}, & N < i \leq 2N-1 \end{cases} \tag{B.1}$$

$$h_i \geq rT^{1/r}h \begin{cases} x_{i-1}^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_i)^{1-1/r}, & N < i \leq 2N-1 \end{cases} \tag{B.2}$$

Proof. For $1 \leq i \leq N$,

$$h_i = T\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right)$$
$$\leq rT\frac{1}{N}\left(\frac{i}{N}\right)^{r-1} = rT^{1/r}hx_i^{1-1/r}$$

$$h_i \geq rT\frac{1}{N}\left(\frac{i-1}{N}\right)^{r-1} = rT^{1/r}hx_{i-1}^{1-1/r}$$

For $N < i \leq 2N$,

$$h_i = T\left(\left(\frac{2N-i+1}{N}\right)^r - \left(\frac{2N-i}{N}\right)^r\right)$$
$$\leq rT\frac{1}{N}\left(\frac{2N-i+1}{N}\right)^{r-1} = rT^{1/r}h(2T - x_{i-1})^{1-1/r}$$

$$h_i \geq rT\frac{1}{N}\left(\frac{2N-i}{N}\right)^{r-1} = rT^{1/r}h(2T - x_i)^{1-1/r}$$ ∎

Lemma B.2. *There is a constant* $C = 2^{|r-2|}r(r-1)T^{2/r}$ *such that for all* $i \in \{1, 2, \cdots, 2N-1\}$

$$|h_{i+1} - h_i| \le Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N-1 \end{cases} \tag{B.3}$$

Proof.

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \le i \le N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \le i \le 2N-1 \end{cases}$$

For $i = 1$,

$$h_2 - h_1 = T(2^r - 2)\left(\frac{1}{N}\right)^r = (2^r - 2)T^{2/r}h^2x_1^{1-2/r}$$

For $2 \le i \le N-1$,

$$h_{i+1} - h_i = r(r-1)T\ N^{-2}\eta^{r-2}, \quad \eta \in [\frac{i-1}{N}, \frac{i+1}{N}]$$

If $r \in [1, 2]$,

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T\ N^{-2}\eta^{r-2} \le r(r-1)T\ h^2\left(\frac{i-1}{N}\right)^{r-2} \\ &\le r(r-1)T\ h^2 2^{2-r}\left(\frac{i}{N}\right)^{r-2} \\ &= 2^{2-r}r(r-1)T^{2/r}h^2x_i^{1-2/r} \end{aligned}$$

else if $r > 2$,

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T\ N^{-2}\eta^{r-2} \le r(r-1)T\ h^2\left(\frac{i+1}{N}\right)^{r-2} \\ &\le r(r-1)T\ h^2 2^{r-2}\left(\frac{i}{N}\right)^{r-2} \\ &= 2^{r-2}r(r-1)T^{2/r}h^2x_i^{1-2/r} \end{aligned}$$

Since

$$2^r - 2 \le 2^{|r-2|}r(r-1), \quad r \ge 1$$

we have

$$h_{i+1} - h_i \le 2^{|r-2|}r(r-1)T^{2/r}h^2x_i^{1-2/r}, \quad 1 \le i \le N-1$$

For $i = N$, $h_{N+1} - h_N = 0$. For $N < i \le 2N-1$, it's central symmetric to the first half of the proof, which is

$$h_i - h_{i+1} \le 2^{|r-2|}r(r-1)T^{2/r}h^2(2T - x_i)^{1-2/r}$$

Summarizes the inequalities, we can get

$$|h_{i+1} - h_i| \le 2^{|r-2|} r(r-1) T^{2/r} h^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N-1 \end{cases} \tag{B.4}$$ ∎

Appendix C. Proofs of some technical details.

Additional proof of Theorem 5.1. For $2 \le i \le N-1$,

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\ &\qquad \le C \frac{2}{h_i + h_{i+1}} (h_i^3 x_{i-1}^{-2-\alpha/2} + h_{i+1}^3 x_i^{-2-\alpha/2}) \\ &\qquad \le 2C (h_i^2 x_{i-1}^{-2-\alpha/2} + h_{i+1}^2 x_i^{-2-\alpha/2}) \end{aligned}$$

Since Lemma B.1, we have

$$h_i \le r T^{1/r} h x_i^{1-1/r}, \quad 1 \le i \le N$$
$$h_{i+1} \le r T^{1/r} h x_{i+1}^{1-1/r}, \quad 1 \le i \le N-1$$

and

$$x_{i-1}^{-2-\alpha/2} \le 2^{-r(-2-\alpha/2)} x_i^{-2-\alpha/2} \quad 2 \le i \le N-1$$
$$x_{i+1}^{1-1/r} \le 2^{r-1} x_i^{1-1/r} \quad 1 \le i \le N-1$$

So there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

$$\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \le C h^2 x_i^{-\alpha/2 - 2/r}, \quad 2 \le i \le N-1$$

For $i = 1$, by (A.4)

$$\begin{aligned} &\frac{1}{4!} \frac{2}{h_1 + h_2} (h_1^3 f''(\eta_1) + h_2^3 f''(\eta_2)) \\ &\qquad = \frac{2}{h_1 + h_2} \left(\frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy + \frac{1}{4!} h_2^3 f''(\eta_2) \right) \end{aligned}$$

We have proved above that

$$\frac{2}{h_1 + h_2} h_2^3 f''(\eta_2) \le C h^2 x_1^{-\alpha/2 - 2/r}$$

and we can get

$$\begin{aligned} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy &\le C \frac{1}{3!} \int_0^{x_1} y^{1-\alpha/2} dy \\ &= C \frac{1}{3!(2 - \alpha/2)} x_1^{2-\alpha/2} \end{aligned}$$

so

$$\frac{2}{h_1 + h_2} \frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy = \frac{C 2^{1-r}}{3!(2-\alpha/2)} x_1^{-\alpha/2} = \frac{C 2^{1-r}}{3!(2-\alpha/2)} T^{2/r} h^2 x_1^{-\alpha/2 - 2/r}$$

And for $i = N$, we have

$$\begin{aligned}
&\frac{2}{h_N + h_{N+1}}(h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2)) \\
&\qquad = h_N^2(f''(\eta_1) + f''(\eta_2)) \\
&\qquad \le r^2 T^{2/r} h^2 x_N^{2-2/r}\ 2C x_{N-1}^{-2-\alpha/2} \\
&\qquad \le 2r^2 T^{2/r} C 2^{-r(-2-\alpha/2)}\ h^2 x_N^{-\alpha/2-2/r}
\end{aligned}$$

Finally, $N+1 \le i \le 2N-1$ is symmetric to the first half of the proof, so we can conclude that

$$\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases}$$

□

LEMMA C.1. *There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ for $2 \le j \le N$, if $y \in [x_{j-1}, x_j]$,*

$$|u(y) - u_h(y)| \le Ch^2 y^{\alpha/2-2/r} \tag{C.1}$$

Proof. For $2 \le j \le N$, we have

$$x_j \le 2^r y, \quad x_{j-1} \ge 2^{-r} y$$

And by Lemma A.2, Lemma B.1 and Corollary 2.4, we have

$$\begin{aligned}
u(y) - u_h(y) &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \\
&\le \frac{\|u\|_{\beta+\alpha}^{(-\alpha/2)}}{4} r^2 T^{2/r}\ h^2 x_j^{2-2/r} x_{j-1}^{\alpha/2-2} \\
&\le Ch^2\ 2^{2r-2} y^{2-2/r}\ 2^{-r(\alpha/2-2)} y^{\alpha/2-2} \\
&= C 2^{-r\alpha/2+4r-2}\ h^2 y^{\alpha/2-2/r}
\end{aligned}$$

symmetricly, for $N < j \le 2N-1$, we have

$$|u(y) - u_h(y)| \le Ch^2(2T - y)^{\alpha/2-2/r} \tag{C.2}$$

□

LEMMA C.2. *There is a constant $C = C(\alpha, r)$ such that for all $1 \le i < N/2$, $\max\{2i+1, i+3\} \le j \le 2N$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_h^2\left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) \le C \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} \tag{C.3}$$

Proof. Sinec $y \ge x_{j-1} > x_{i+1}$, by Lemma A.1, if $j - 1 > i + 1$

□

$$\begin{aligned}
D_h^2\left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) &= \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\
&\le \frac{(y - x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)} \\
&\le (1 - (\frac{2}{3})^r)^{-1-\alpha} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}
\end{aligned}$$

LEMMA C.3. *There is a constant $C = C(\alpha, r)$ such that for all $3 \le i < N/2, k = \lceil \frac{i}{2} \rceil$, $1 \le j \le k-1$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_h^2\Big(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\Big)(x_i) \le C\frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \tag{C.4}$$

Proof. Sinec $y \le x_j < x_{i-1}$, by Lemma A.1,

$$\begin{aligned} D_h^2\Big(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\Big)(x_i) &= \frac{|\xi - y|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\le \frac{(x_{i-1} - x_j)^{-1-\alpha}}{\Gamma(-\alpha)} \le \frac{(x_{i-1} - x_{k-1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\le ((\frac{2}{3})^r - (\frac{1}{2})^r)^{-1-\alpha}\frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

∎

LEMMA C.4. *While $0 \le i < N/2$, By Lemma A.3*

$$\begin{aligned} |T_{i1}| &\le C\int_0^{x_1} x_1^{\alpha/2}\frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)}dy \\ &= C\frac{1}{\Gamma(3-\alpha)}x_1^{\alpha/2}\left|x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha}\right| \\ &\le C\frac{1}{\Gamma(3-\alpha)}x_1^{\alpha/2+2-\alpha} = C\frac{1}{\Gamma(3-\alpha)}x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1 \end{aligned} \tag{C.5}$$

For $2 \le j \le N$, by Lemma A.2 and Corollary 2.4

$$\begin{aligned} |T_{ij}| &\le \frac{C}{4}\int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2}\frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy \\ &\le \frac{C}{4\Gamma(3-\alpha)}h_j^2 x_{j-1}^{\alpha/2-2}\left||x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha}\right| \end{aligned} \tag{C.6}$$

LEMMA C.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$\sum_{j=1}^{3} S_{1j} \le Ch^2 x_1^{-\alpha/2-2/r} \tag{C.7}$$

$$\sum_{j=1}^{4} S_{2j} \le Ch^2 x_2^{-\alpha/2-2/r} \tag{C.8}$$

Proof.

$$S_{1j} = \frac{2}{x_2}\left(\frac{1}{x_1}T_{0j} - \left(\frac{1}{x_1} + \frac{1}{h_2}\right)T_{1j} + \frac{1}{h_2}T_{2j}\right)$$

So, by Lemma C.4

$$S_{11} \le \frac{2}{x_2 x_1}4\frac{C}{\Gamma(3-\alpha)}x_1^{2-\alpha/2} \le Cx_1^{-\alpha/2}$$

$$S_{12} \le \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_2^2 x_1^{\alpha/2-2} \left(x_2^{2-\alpha} + 2h_2^{2-\alpha} + h_2^{2-\alpha}\right) \le C x_1^{-\alpha/2}$$

$$S_{13} \le \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_3^2 x_2^{\alpha/2-2} \left(x_3^{2-\alpha} + 2x_3^{2-\alpha} + h_3^{2-\alpha}\right) \le C x_1^{-\alpha/2}$$

But

$$x_1^{-\alpha/2} = T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

For $i = 2$, Sorry ◻

Lemma C.6. *There exists a constant $C = C(T, r, l)$ such that For $3 \le i \le N - 1, k+1 = \lceil \frac{i}{2} \rceil, k \le j \le \min\{2i-1, N-1\}$, $l = 3, 4$, when $\xi \in [x_{i-1}, x_{i+1}]$,*

$$(h_{j-i}^3(\xi))' \le (r-1) C h^2 x_i^{1-2/r} h_j \tag{C.9}$$

$$(h_{j-i}^4(\xi))' \le (r-1) C h^2 x_i^{1-2/r} h_j^2 \tag{C.10}$$

Proof. From (5.32)

$$y'_{j-i}(x) = y_{j-i}^{1-1/r}(x) x^{1/r-1} \tag{C.11}$$

$$y''_{j-i}(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i} \tag{C.12}$$

for $l = 3, 4$, by (5.34)

$$\begin{aligned} (h_{j-i}^l(\xi))' &= l\, h_{j-i}^{l-1}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi)) \\ &= l\, h_{j-i}^{l-1}(\xi) \xi^{1/r-1} (y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \ge 0 \end{aligned} \tag{C.13}$$

For $\xi \in [x_{i-1}, x_{i+1}]$ and $2 \le k \le j \le \min\{2i-1, N-1\}$, using Lemma B.1

$$\begin{aligned} h_{j-i}(\xi) &\le h_{j-i}(x_{i+1}) = h_{j+1} \\ &\le r T^{1/r}\, h x_{j+1}^{1-1/r} \le r T^{1/r} 2^{r-1}\, h x_i^{1-1/r} \end{aligned}$$

And

$$2^{-r} x_i \le x_{i-1} \le \xi \le x_{i+1} \le 2^r x_i \tag{C.14}$$

We have

$$\xi^{1/r-m} \le 2^{|mr-1|} x_i^{1/r-m}, \quad m = 1, 2 \tag{C.15}$$

but

$$\begin{aligned} y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) &= (\xi^{1/r} + Z_{j-i})^{r-1} - (\xi^{1/r} + Z_{j-i-1})^{r-1} \\ &= (r-1) Z_1 (\xi^{1/r} + Z_{j-i-\gamma})^{r-2}, \quad \gamma \in [0, 1] \\ &= (r-1) T^{1/r} h y_{j-i-\gamma}^{1-2/r}(\xi) \end{aligned} \tag{C.16}$$

And

$$4^{-r}x_i \le x_{\lceil \frac{i}{2}\rceil -1} \le x_{j-2} = y_{j-i-1}(x_{i-1}) \le y_{j-i-\gamma}(\xi) \le y_{j-i}(x_{i+1}) = x_{j+1} \le x_{2i} \le 2^r x_i \tag{C.17}$$

Therefore,

$$y_{j-i-\gamma}^{1-2/r}(\xi) \le 2^{2|r-2|}x_i^{1-2/r} \tag{C.18}$$

So we can get

$$y'_{j-i}(\xi) - y'_{j-i-1}(\xi) \le (r-1)C(T,r)hx_i^{-1/r} \tag{C.19}$$

We get

$$(h_{j-i}^l(\xi))' \le l(r-1)C\ h_{j+1}^{l-1}hx_i^{-1/r} \tag{C.20}$$

And by Lemma B.1,

$$h_{j+1} \le rTh\left(\frac{j+1}{N}\right)^{r-1} \le rTh2^{r-1}\left(\frac{j-1}{N}\right) = 2^{r-1}h_j \tag{C.21}$$

$$h_{j+1} \le rT^{1/r}hx_{j+1}^{1-1/r} \le rT^{1/r}hx_{2i}^{1-1/r} \le rT^{1/r}2^{r-1}hx_i^{1-1/r} \tag{C.22}$$

We can get

$$\begin{aligned}(h_{j-i}^l(\xi))' &\le l(r-1)C\ h_j^{l-2}h_{j+1}hx_i^{-1/r}\\ &\le l(r-1)Chh_j^{l-2}(hx_i^{1-1/r})x_i^{-1/r}\\ &= (r-1)C\ h^2x_i^{1-2/r}h_j^{l-2}\end{aligned} \tag{C.23}$$

Meanwhile, we can get

$$h_{j-i}^3(\xi) \le h_{j+1}^3 \le Ch^2x_i^{2-2/r}h_j \tag{C.24}$$

$$h_{j-i}^4(\xi) \le h_{j+1}^4 \le Ch^2x_i^{2-2/r}h_j^2 \tag{C.25}$$

□

Lemma C.7. *There exists a constant $C = C(T,r,l)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2}\rceil + 1 \le j \le \min\{2i-1, N-1\}$, when $\xi \in [x_{i-1}, x_{i+1}]$,*

$$(h_{j-i}^3(\xi))'' \le C(r-1)h^2x_i^{-2/r}h_j \tag{C.26}$$

Proof. From (C.11)

$$\begin{aligned}(h_{j-i}^3(\xi))'' &= 6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 + 3h_{j-i}^2(\xi)(y''_{j-i}(\xi) - y''_{j-i-1}(\xi))\\ &= 6h_{j-i}(\xi)\xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi))\\ &\quad + 3\frac{1-r}{r}h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1})\end{aligned} \tag{C.27}$$

Using the inequalities of the proof of Lemma C.6

$$\begin{aligned}
&6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 \\
&\qquad \le 6h_{j+1}((r-1)Chx_i^{-1/r})^2 \\
&\qquad \le C(r-1)^2\, h^2 x_i^{-2/r} h_j
\end{aligned} \tag{C.28}$$

For the second partial

$$\begin{aligned}
&h^2_{j-i}(\xi)\xi^{1/r-2}(y^{1-2/r}_{j-i}(\xi)Z_{j-i} - y^{1-2/r}_{j-i-1}(\xi)Z_{j-i-1}) \\
&\qquad \le Ch^2_{j+1}x_i^{1/r-2}((y^{1-2/r}_{j-i}(\xi) - y^{1-2/r}_{j-i-1}(\xi))Z_{j-i} + y^{1-2/r}_{j-i-1}(\xi)Z_1)
\end{aligned} \tag{C.29}$$

but

$$\begin{aligned}
y^{1-2/r}_{j-i}(\xi) - y^{1-2/r}_{j-i-1}(\xi) &= (\xi^{1/r} + Z_{j-i})^{r-2} - (\xi^{1/r} + Z_{j-i-1})^{r-2} \\
&= (r-2)Z_1(\xi^{1/r} + Z_{j-i-\gamma})^{r-3} \\
&= (r-2)T^{-r}hy^{1-3/r}_{j-i-\gamma}(\xi) \\
&\le C(r-2)hx_i^{1-3/r}
\end{aligned} \tag{C.30}$$

So we can get

$$\begin{aligned}
&h^2_{j-i}(\xi)\xi^{1/r-2}(y^{1-2/r}_{j-i}(\xi)Z_{j-i} - y^{1-2/r}_{j-i-1}(\xi)Z_{j-i-1}) \\
&\qquad \le Ch_jhx_i^{1-1/r}x_i^{1/r-2}(C(r-2)hx_i^{1-3/r}Z_{j-i} + Cx_i^{1-2/r}T^{1/r}h) \\
&\qquad \le Ch^2((r-2)x_i^{-3/r}x^{1/r}_{|j-i|} + x_i^{-2/r})h_j \\
&\qquad \le Ch^2x_i^{-2/r}h_j
\end{aligned} \tag{C.31}$$

Summarizes, we have

$$(h^3_{j-i}(\xi))'' \le C(r-1)h^2x_i^{-2/r}h_j \tag{C.32}$$ ∎

proof of Lemma 5.16. From (5.32)

$$y'_{j-i}(x) = y^{1-1/r}_{j-i}(x)x^{1/r-1} \tag{C.33}$$

$$y''_{j-i}(x) = \frac{1-r}{r}y^{1-2/r}_{j-i}(x)x^{1/r-2}Z_{j-i} \tag{C.34}$$

Since

$$x_{j-2} \le y_{j-i-1}(x_{i-1}) \le y^\theta_{j-i}(\xi) \le y^\theta_{j-i-1}(x_{i+1}) \le x_{j+1}$$

We have known (C.17)

$$u''(y^\theta_{j-i}(\xi)) \le C(y^\theta_{j-i}(\xi))^{\alpha/2-2} \le Cx^{\alpha/2-2}_{j-2} \le Cx^{\alpha/2-2}_{\lceil \frac{i}{2}\rceil-1} \le C4^{r(2-\alpha/2)}x_i^{\alpha/2-2} \tag{C.35}$$

$$\begin{aligned}
(u''(y^\theta_{j-i}(\xi)))' &= u'''(y^\theta_{j-i}(\xi)){y^\theta_{j-i}}'(\xi) \\
&\le Cx_i^{\alpha/2-3}\xi^{1/r-1}y^{1-1/r}_{j-i}(\xi) \\
&\le Cx_i^{\alpha/2-3}x_i^{1/r-1}x_i^{1-1/r} = Cx_i^{\alpha/2-3}
\end{aligned} \tag{C.36}$$

$$(u''(y^\theta_{j-i}(\xi)))'' = u''''(y^\theta_{j-i}(\xi))({y^\theta_{j-i}}''(\xi))^2 + u'''(y^\theta_{j-i}(\xi)){y^\theta_{j-i}}''(\xi) \tag{C.37}$$
$$\le Cx_i^{\alpha/2-4} + Cx_i^{\alpha/2-3}\frac{r-1}{r}x_i^{1-2/r}x_i^{1/r-2}Z_{|j-i|+1}$$
$$\le Cx_i^{\alpha/2-4} + C\frac{r-1}{r}x_i^{\alpha/2-3}x_i^{-1/r}x_i^{1/r}$$
$$= Cx_i^{\alpha/2-4}$$
□

Proof of Lemma 5.17.

$$|y^\theta_{j-i}(\xi) - \xi| = |\theta(y_{j-i-1}(\xi) - \xi) + (1-\theta)(y_{j-i}(\xi) - \xi)| \tag{C.38}$$
$$= \theta|y_{j-i-1}(\xi) - \xi| + (1-\theta)|y_{j-i}(\xi) - \xi|$$

Since $|y_{j-i}(\xi) - \xi|$ is increasing about ξ, we have

$$(\frac{i-1}{i})^r|x_j - x_i| \le |x_{j-1} - x_{i-1}| \le |y_{j-i}(\xi) - \xi| \le |x_{j+1} - x_{i+1}| \le (\frac{i+1}{i})^r|x_j - x_i| \tag{C.39}$$

Thus,

$$(\frac{2}{3})^r|y^\theta_j - x_i| \le |y^\theta_{j-i}(\xi) - \xi| \le (\frac{3}{4})^r(\theta|x_j - x_i| + (1-\theta)|x_{j-1} - x_i|) = (\frac{3}{4})^r|y^\theta_j - x_i| \tag{C.40}$$

$$|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha} \le C|y^\theta_j - x_i|^{1-\alpha} \tag{C.41}$$

Next,

$$(|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha})' = (1-\alpha)|y^\theta_{j-i}(\xi) - \xi|^{-\alpha}|\xi^{1/r-1}(\theta y^{1-1/r}_{j-i-1}(\xi) + (1-\theta)y^{1-1/r}_{j-i}(\xi)) - 1| \tag{C.42}$$
$$\le C|y^\theta_j - x_i|^{-\alpha}\xi^{1/r-1}|\theta y^{1-1/r}_{j-i-1}(\xi) + (1-\theta)y^{1-1/r}_{j-i}(\xi) - \xi^{1-1/r}|$$

Similar with (C.40), we have

$$|y^{1-1/r}_{j-i}(\xi) - \xi^{1-1/r}| \le C|x_j^{1-1/r} - x_i^{1-1/r}| \tag{C.43}$$
$$\le C|x_j - x_i|x_i^{-1/r}$$

So we can get

$$|\theta y^{1-1/r}_{j-i-1}(\xi) + (1-\theta)y^{1-1/r}_{j-i}(\xi) - \xi^{1-1/r}| \tag{C.44}$$
$$\le Cx_i^{-1/r}(\theta|x_{j-1} - x_i| + (1-\theta)|x_j - x_i|)$$
$$= Cx_i^{-1/r}|y^\theta_j - x_i|$$

Combine them, we get

$$(|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha})' \le C|y^\theta_j - x_i|^{-\alpha}x_i^{1/r-1}x_i^{-1/r}|y^\theta_j - x_i| \tag{C.45}$$
$$= C|y^\theta_j - x_i|^{1-\alpha}x_i^{-1}$$

Finally, we have

$$(|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha})'' = \alpha(\alpha-1)|y^\theta_{j-i}(\xi) - \xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y^{1-1/r}_{j-i-1}(\xi) + (1-\theta)y^{1-1/r}_{j-i}(\xi)) - 1)^2 \tag{C.46}$$
$$+ (1-\alpha)|y^\theta_{j-i}(\xi) - \xi|^{-\alpha}\frac{1-r}{r}\xi^{1/r-2}|\theta y^{1-2/r}_{j-i-1}(\xi)Z_{j-i-1} + (1-\theta)y^{1-2/r}_{j-i}(\xi)Z_{j-i}|$$

Using the inequalities above ,we have

$$\begin{aligned}
&|y^\theta_{j-i}(\xi)-\xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y^{1-1/r}_{j-i-1}(\xi)+(1-\theta)y^{1-1/r}_{j-i}(\xi))-1)^2\\
&\quad\le C|y^\theta_j-x_i|^{-\alpha-1}(x_i^{-1}|y^\theta_j-x_i|)^2\\
&\quad= C|y^\theta_j-x_i|^{1-\alpha}x_i^{-2}
\end{aligned}\tag{C.47}$$

And by

$$|Z_{j-i}|=|x_j^{1/r}-x_i^{1/r}|\le|x_j-x_i|x_i^{1/r-1}\tag{C.48}$$

we have

$$\begin{aligned}
&|y^\theta_{j-i}(\xi)-\xi|^{-\alpha}\xi^{1/r-2}|\theta y^{1-2/r}_{j-i-1}(\xi)Z_{j-i-1}+(1-\theta)y^{1-2/r}_{j-i}(\xi)Z_{j-i}|\\
&\le C|y^\theta_j-x_i|^{-\alpha}x_i^{1/r-2}x_i^{1-2/r}|\theta Z_{j-i-1}+(1-\theta)Z_{j-i}|\\
&\le C|y^\theta_j-x_i|^{-\alpha}x_i^{-2}|y^\theta_j-x_i|\\
&= C|y^\theta_j-x_i|^{1-\alpha}x_i^{-2}
\end{aligned}\tag{C.49}$$

$\square$

proof of Lemma 5.19. For $k\le j<\min\{2i-1,N-1\}$

$$\begin{aligned}
&\frac{Q^\theta_{j-i}(x_{i+1})u'''(\eta^\theta_{j+1})-Q^\theta_{j-i}(x_i)u'''(\eta^\theta_j)}{h_{i+1}}\\
&\frac{Q^\theta_{j-i}(x_{i+1})-Q^\theta_{j-i}(x_i)}{h_{i+1}}u'''(\eta^\theta_{j+1})+Q^\theta_{j-i}(x_i)\frac{u'''(\eta^\theta_{j+1})-u'''(\eta^\theta_j)}{h_{i+1}}\\
&\le {Q^\theta_{j-i}}'(\xi)Cx_j^{\alpha/2-3}+Q^\theta_{j-i}(x_i)Cu''''(\eta)\frac{h_i+h_{i+1}}{h_{i+1}}
\end{aligned}\tag{C.50}$$

where $\xi\in[x_i,x_{i+1}],\eta\in[x_{j-1},x_{j+1}]$.

From (5.36), by Lemma C.6 and Lemma 5.17, we have

$$\begin{aligned}
{Q^\theta_{j-i}}'(\xi)&\le Ch^2\frac{|y^\theta_{j+1}-x_{i+1}|^{1-\alpha}}{\Gamma(2-\alpha)}x_{i+1}^{1-2/r}h^2_{j+1}\\
&\le Ch^2\frac{|y^\theta_j-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{1-2/r}h_j^2
\end{aligned}\tag{C.51}$$

And by defination

$$Q^\theta_{j-i}(x_i)=h_j^4\frac{|y^\theta_j-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}\le Ch^2x_i^{2-2/r}\frac{|y^\theta_j-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}h_j^2\tag{C.52}$$

With , we have

$$4^{-r}x_i\le x_{k-1}\le x_{j-1}<x_j\le x_{2i-1}\le 2^rx_i\tag{C.53}$$

So we have

$$\begin{aligned}
&\frac{Q^\theta_{j-i}(x_{i+1})u'''(\eta^\theta_{j+1})-Q^\theta_{j-i}(x_i)u'''(\eta^\theta_j)}{h_{i+1}}\\
&\le Ch^2\frac{|y^\theta_j-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{1-2/r}h_j^2\,x_i^{\alpha/2-3}+Ch^2x_i^{2-2/r}\frac{|y^\theta_j-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}h_j^2x_{j-1}^{\alpha/2-4}\\
&= Ch^2\frac{|y^\theta_j-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j^2
\end{aligned}\tag{C.54}$$

while

$$h_j \le h_{2i-1} \le 2^r h_i$$

Subsitute into the inequality above, we get the goal

$$\begin{aligned}
&\frac{2}{h_i+h_{i+1}}\left(\frac{Q^\theta_{j-i}(x_{i+1})u'''(\eta^\theta_{j+1})-Q^\theta_{j-i}(x_i)u'''(\eta^\theta_j)}{h_{i+1}}\right)\\
&\qquad\le \frac{1}{h_i}Ch^2\frac{|y^\theta_j-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j 2^r h_i\\
&\qquad= Ch^2\frac{|y^\theta_j-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j
\end{aligned} \tag{C.55}$$

While, the later is similar. ∎

Lemma C.8. *There exists a constant $C = C(T, r)$ such that For $N/2 \le i < N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $l = 3, 4$, $\xi \in [x_{i-1}, x_{i+1}]$, we have*

$$h^l_{j-i}(\xi) \le Ch^l_j \le Ch^2 h^{l-2}_j \tag{C.56}$$

$$(h^l_{j-i-1}(\xi))' \le C(r-1)h^2 h^{l-2}_j \tag{C.57}$$

$$(h^3_{j-i}(\xi))'' \le C(r-1)h^2 h_j \tag{C.58}$$

Proof.

$$\begin{aligned}
(h_{j-i}(\xi))' &= {y_{j-i}}'(\xi) - {y_{j-i-1}}'(\xi)\\
&= \xi^{1/r-1}((2T-y_{j-i}(\xi))^{1-1/r} - (2T-y_{j-i-1}(\xi))^{1-1/r}) \le 0
\end{aligned} \tag{C.59}$$

Thus,

$$Ch_j \le h_{j+1} \le h_{j-i}(\xi) \le h_{j-i}(x_{i-1}) = h_{j-1} \le Ch_j \tag{C.60}$$

So as $4^{-r}T \le 2T - x_j \le T, 2^{-r}T \le x_i \le T$, we have

$$h^l_{j-i}(\xi) \le Ch^l_j \le Ch^2(2T-x_j)^{2-2/r}h^{l-2}_j \le Ch^2 h^{l-2}_j \tag{C.61}$$

Since

$$\begin{aligned}
&|(2T-y_{j-i}(\xi))^{1-1/r} - (2T-y_{j-i-1}(\xi))^{1-1/r}|\\
&\quad= |(Z_{2N-(j-i)}-\xi^{1/r})^{r-1} - (Z_{2N-(j-1-i)}-\xi^{1/r})^{r-1}|\\
&\quad= (r-1)Z_1(Z_{2N-(j-i-\gamma)}-\xi^{1/r})^{r-2} \quad \gamma\in[0,1]\\
&\quad\le C(r-1)h(2T-x_j)^{1-2/r}
\end{aligned} \tag{C.62}$$

we have

$$|(h_{j-i}(\xi))'| \le C(r-1)h(2T-x_j)^{1-2/r}x_i^{1/r-1} \tag{C.63}$$

And

$$\begin{aligned}
(h^l_{j-i}(\xi))' &= lh^{l-1}_{j-i}(\xi){h_{j-i}}'(\xi)\\
&\le C(r-1)h^{l-1}_j\ h(2T-x_j)^{1-2/r}x_i^{1/r-1}\\
&\le C(r-1)h^2 h^{l-2}_j(2T-x_j)^{2-3/r}x_i^{1-1/r}\\
&\le C(r-1)h^2 h^{l-2}_j
\end{aligned} \tag{C.64}$$

$$(h_{j-i}^3(\xi))'' = 6h_{j-i}(\xi)(y_{j-i}{}'(\xi) - y_{j-i-1}{}'(\xi))^2 + 3h_{j-i}^2(\xi)(y_{j-i}{}''(\xi) - y_{j-i-1}{}''(\xi))$$
$$\le C(r-1)h_jh^2 + Ch_j^2\frac{1-r}{r}\xi^{1/r-2}((2T - y_{j-i}(\xi))^{1-2/r}Z_{2N-(j-i)} - (2T - y_{j-i-1}(\xi))^{1-2/r}Z_{2N-(j-1-i)})$$
$$\le C(r-1)h_jh^2 + C(r-1)h_j^2(C(r-2)h(2T-x_j)^{1-3/r}Z_{2N-(j-i)} + Z_1(2T-x_{j-1})^{1-2/r})$$
$$\le C(r-1)h_jh^2 + C(r-1)h_j^2h = Ch^2h_j \tag{C.65}$$
□

Lemma C.9. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le i < N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in [x_{i-1}, x_{i+1}]$, we have*

$$u''(y_{j-i}^\theta(\xi)) \le C \tag{C.66}$$
$$(u''(y_{j-i}^\theta(\xi)))' \le C \tag{C.67}$$
$$(u''(y_{j-i}^\theta(\xi)))'' \le C \tag{C.68}$$

Proof.

$$x_{j-2} \le y_{j-i}^\theta(\xi) \le x_{j+1} \Rightarrow 4^{-r}T \le 2T - y_{j-i}^\theta(\xi) \le T \tag{C.69}$$

Thus, for $l = 2, 3, 4$,

$$u^{(l)}(y_{j-i}^\theta(\xi)) \le C(2T - y_{j-i}^\theta(\xi))^{\alpha/2-l} \le C \tag{C.70}$$

and

$$\begin{aligned}(y_{j-i}^\theta(\xi))' &= \theta y_{j-1-i}{}'(\xi) + (1-\theta)y_{j-i-1}{}'(\xi)\\ &= \xi^{1/r-1}(\theta(2T - y_{j-1-i}(\xi))^{1-1/r} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-1/r})\\ &\le C(2T - x_{j-2})^{1-1/r} \le C\end{aligned} \tag{C.71}$$

With

$$Z_{2N-j-i} \le 2T^{1/r} \tag{C.72}$$

$$(y_{j-i}^\theta(\xi))'' = \theta y_{j-1-i}{}''(\xi) + (1-\theta)y_{j-i-1}{}''(\xi)$$
$$= \frac{1-r}{r}\xi^{1/r-2}(\theta(2T - y_{j-i-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)} + (1-\theta)(2T - y_{j-i}(\xi))^{1-2/r}Z_{2N-(j-i)})$$
$$\le C(r-1) \tag{C.73}$$

Therefore,

$$\begin{aligned}(u''(y_{j-i}^\theta(\xi)))' &= u'''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta(\xi))'\\ &\le C\end{aligned} \tag{C.74}$$

$$\begin{aligned}(u''(y_{j-i}^\theta(\xi)))'' &= u'''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta{}'(\xi))^2 + u''''(y_{j-i}^\theta(\xi))y_{j-i}^\theta{}''(\xi)\\ &\le C + C(r-1) = C\end{aligned} \tag{C.75}$$
□

LEMMA C.10. *There exists a constant $C = C(T, \alpha, r)$ such that*

$$|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha} \le C|y^\theta_j - x_i|^{1-\alpha} \tag{C.76}$$

$$(|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha})' \le C|y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_N) \tag{C.77}$$

$$(|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha})'' \le C(r-1)|y^\theta_j - x_i|^{-\alpha} + C|y^\theta_j - x_i|^{-1-\alpha}(|2T - x_i - y^\theta_j| + h_N)^2 \tag{C.78}$$

Proof.

$$(y^\theta_{j-i}(\xi) - \xi)' = \theta y_{j-1-i}{}'(\xi) + (1-\theta)y_{j-i}{}'(\xi) - 1 \tag{C.79}$$

$$\begin{aligned} |y_{j-i}{}'(\xi) - 1| &= \xi^{1/r-1}|(2T - y_{j-i}(\xi))^{1-1/r} - \xi^{1-1/r}| \\ &\le \xi^{1/r-1}|2T - \xi - y_{j-i}(\xi)|\xi^{-1/r} \end{aligned} \tag{C.80}$$

$$\begin{aligned} |2T - \xi - y_{j-i}(\xi)| &\le \max\begin{cases} |2T - x_{i-1} - x_{j-1}| \\ |2T - x_{i+1} - x_{j+1}| \end{cases} \\ &\le |2T - x_i - x_j| + h_{i+1} + h_j \end{aligned} \tag{C.81}$$

$$\begin{aligned} (y^\theta_{j-i}(\xi) - \xi)'' &= \theta y_{j-1-i}{}''(\xi) + (1-\theta)y_{j-i}{}''(\xi) \\ &= \frac{1-r}{r}\xi^{1/r-2}(\theta(2T - y_{j-i}(\xi))^{1-2/r}Z_{2N-(j-i)} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)}) \le 0 \end{aligned} \tag{C.82}$$

It's concave, so

$$y_{j-i}(\xi) - \xi \ge \min\{x_{j+1} - x_{i+1}, x_{j-1} - x_{i-1}\} \ge C(x_j - x_i) \tag{C.83}$$

We have

$$|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha} \le C|y^\theta_j - x_i|^{1-\alpha} \tag{C.84}$$

$$\begin{aligned} (|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha})' &= (1-\alpha)|y^\theta_{j-i}(\xi) - \xi|^{-\alpha}(y^\theta_{j-i}(\xi) - \xi)' \\ &\le C|y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_{i+1} + h_{j-1}) \end{aligned} \tag{C.85}$$

□

$$\begin{aligned} (|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha})'' &= (1-\alpha)|y^\theta_{j-i}(\xi) - \xi|^{-\alpha}(y^\theta_{j-i}(\xi) - \xi)'' + \alpha(\alpha-1)|y^\theta_{j-i}(\xi) - \xi|^{-1-\alpha}(y^\theta_{j-i}{}'(\xi) - 1)^2 \\ &\le C(r-1)|y^\theta_j - x_i|^{-\alpha} + C|y^\theta_j - x_i|^{-1-\alpha}(|2T - x_i - y^\theta_j| + h_{i+1} + h_{j-1})^2 \end{aligned} \tag{C.86}$$

Proof. From (5.23), by Lemma C.8 and Lemma C.10, we have $\xi \in [x_i, x_{i+1}]$

$$Q^\theta_{j-i}{}'(\xi) \le Ch^2h_j^2((r-1)|y^\theta_j - x_i|^{1-\alpha} + |y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_N)) \tag{C.87}$$

$$Q^\theta_{j-i}(\xi) \le Ch^2h_j^2|y^\theta_j - x_i|^{1-\alpha} \tag{C.88}$$

So use the skill in Proof 27 with Lemma C.9

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}}\left(\frac{Q^\theta_{j-i}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{j-i}(x_i)u'''(\eta^\theta_j)}{h_{i+1}}\right) \\ &\le Ch^2h_j(|y^\theta_j - x_i|^{1-\alpha} + |y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_N)) \end{aligned} \tag{C.89}$$

□

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