AN EXAMPLE ARTICLE*

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Abstract. This is an example SIAM LaTeX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

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1. Introduction.

The introduction introduces the context and summarizes the manuscript. It is importantly to clearly state the contributions of this piece of work.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$, suppose $f \in C^\beta(\Omega) \cap L^\infty(\Omega), \beta > 4, \|f\|_\beta^{\alpha/2} < \infty$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

2. Regularity.

Remark 2.1. 1. $C^k(U)$ is the set of all k-times continuously differentiable functions on open set U.
2. $C^\beta(U)$ is the collection of function f which for any $V \subset\subset U$ $f|_V \in C^\beta(\bar{V})$.

THEOREM 2.2. *If $f \in C^\beta(\Omega), \beta > 2$ and $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

$$|f^{(l)}(x)| \le \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \le T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \le x < 2T \end{cases} \tag{2.1}$$

THEOREM 2.3 (Regularity up to the boundary [1]).

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)}\right) \tag{2.2}$$

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Corollary 2.4. *Let u be a solution of (1.1) on Ω. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \le C[x(2T-x)]^{\alpha/2-l} \tag{2.3}$$

The paper is organized as follows. Our main results are in section 4, experimental results are in section 7, and the conclusions follow in section 8.

3. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases} \tag{3.1}$$

where $r \ge 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N \tag{3.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

And then, we can approximate $u(x)$ with

$$u_h(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{3.4}$$

For convience, we denote

$$I_h^{2-\alpha}(x) := \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x-y|^{1-\alpha} u_h(y)dy \tag{3.5}$$

And now, we can approximate the operator (1.2) at x_i with

$$\begin{aligned} D_h^\alpha u_h(x_i) &:= D_h^2 I_h^{2-\alpha}(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} I_h^{2-\alpha}(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) I_h^{2-\alpha}(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha}(x_{i+1}) \right) \end{aligned} \tag{3.6}$$

Finally, we approximate the equation (1.1) with

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i), \quad 1 \le i \le 2N-1 \tag{3.7}$$

The discrete equation (3.7) can be written in matrix form

$$AU = F \tag{3.8}$$

where U is unknown, $F = (f(x_1), \cdots, f(x_{2N-1}))$. The matrix A is constructed as follows: Since

$$
\begin{aligned}
I_h^{2-\alpha}(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u(x_j) \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\
&=: \sum_{j=1}^{2N-1} \tilde{a}_{ij}\, u(x_j), \quad 0 \le i \le 2N
\end{aligned}
\tag{3.9}
$$

Then, substitute in (3.6), we have

$$
-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} a_{ij}\, u(x_j) \tag{3.10}
$$

where

$$
a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \tag{3.11}
$$

4. Main results. Here we state our main results; the proof is deferred to section 5 and section 6.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 4.1 (Truncation Error). *If $f \in C^2(\Omega)$ and $\alpha \in (1,2)$, and $u(x)$ is a solution of the equation (1.1), then there exists a constant $C_1, C_2 = C_1(T, \alpha, r, \|f\|_{C^2(\Omega)}), C_2(T, \alpha, r, \|f\|_{C^2(\Omega)})$, such that the truncation error of the discrete format satisfies*

$$
\begin{aligned}
|-\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i)| \le & C_1 (h^{r\alpha/2 + r} (x_i^{-1-\alpha} + (2T - x_i)^{-1-\alpha}) \\
& + h^2 (x_i^{-\alpha/2 - 2/r} + (2T - x_i)^{-\alpha/2 - 2/r})) \\
& + C_2 h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \le i \le N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \le 2N - 1 \end{cases}
\end{aligned}
\tag{4.1}
$$

where $C_2 = 0$ if $r = 1$.

THEOREM 4.2 (Convergence). *The discrete equation (3.7) has sulotion U, and there exists a positive constant $C = C(T, \alpha, r, \|f\|_{C^2(\Omega)})$ such that the error between the numerial solution U with the exact solution $u(x_i)$ satisfies*

$$
\max_{1 \le i \le 2N-1} |U_i - u(x_i)| \le C h^{\min\{\frac{r\alpha}{2}, 2\}} \tag{4.2}
$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

5. Proof of Theorem 4.1. For convience, let's denote

$$I^{2-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x-y|^{1-\alpha} u(y)dy \tag{5.1}$$

Then, the truncation error of the discrete format can be written as

$$\begin{aligned} -\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I_h^{2-\alpha}(x_i) - \frac{d^2}{dx^2} I^{2-\alpha}(x_i)) \\ &= -\kappa_\alpha D_h^2 (I_h^{2-\alpha}(x_i) - I^{2-\alpha}(x_i)) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \end{aligned} \tag{5.2}$$

THEOREM 5.1. *There exits a constant $C = C(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$ such that*

$$-\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \le Ch^2 (x_i^{-\alpha/2-2/r} + (2T - x_i)^{-\alpha/2-2/r}) \tag{5.3}$$

Proof. Since $f \in C^2(\Omega)$ and

$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha}(x)) = f(x), \quad x \in \Omega, \tag{5.4}$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.3) of Lemma A.1, for $1 \le i \le 2N-1$, we have

$$-\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \tag{5.5}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

$$\left| \frac{h_{i+1} - h_i}{3} f'(x_i) \right| \le \frac{\|f\|_\beta^{(\alpha/2)}}{3} 2^{|r-2|} r(r-1) T^{2/r} h^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{-\alpha/2-2/r}, & N < i \le 2N-1 \end{cases} \tag{5.6}$$

2. See Proof 6, there is a constant C such that

$$\begin{aligned} &\frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\ &\qquad \le Ch^2 (x_i^{-\alpha/2-2/r} + (2T - x_i)^{-\alpha/2-2/r}) \end{aligned} \tag{5.7}$$

□

6. Proof of Theorem 4.2. aaaaaaaaaa

7. Experimental results.

8. Conclusions. Some conclusions here.

Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x)$ is twice differentable continous function on open set Ω, there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\qquad = g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \end{aligned} \tag{A.1}$$

(A.2)
$$\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}g(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})g(x_i)+\frac{1}{h_i}g(x_{i-1})\right)$$
$$=\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i}g''(y)(y-x_{i-1})dy+\frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}}g''(y)(x_{i+1}-y)dy\right)$$

And if $g(x)\in C^4(\Omega)$, then
(A.3)
$$\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}g(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})g(x_i)+\frac{1}{h_i}g(x_{i-1})\right)$$
$$=g''(x_i)+\frac{h_{i+1}-h_i}{3}g'''(x_i)+\frac{1}{4!}\frac{2}{h_i+h_{i+1}}(h_i^3g''''(\eta_1)+h_{i+1}^3g''''(\eta_2))$$

where $\eta_1\in[x_{i-1},x_i],\eta_2\in[x_i,x_{i+1}]$.

Proof.

$$g(x_{i-1})=g(x_i)-(x_i-x_{i-1})g'(x_i)+\frac{(x_i-x_{i-1})^2}{2}g''(\xi_1),\quad \xi_1\in[x_{i-1},x_i]$$
$$g(x_{i+1})=g(x_i)+(x_{i+1}-x_i)g'(x_i)+\frac{(x_{i+1}-x_i)^2}{2}g''(\xi_2),\quad \xi_2\in[x_i,x_{i+1}]$$

Subsitute them in the left side of (A.1), we have

$$\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}g(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})g(x_i)+\frac{1}{h_i}g(x_{i-1})\right)$$
$$=\frac{h_i}{h_i+h_{i+1}}g''(\xi_1)+\frac{h_{i+1}}{h_i+h_{i+1}}g''(\xi_2)$$

Now, using intermediate value theorem , there exists $\xi\in[\xi_1,\xi_2]$ such that

$$\frac{h_i}{h_i+h_{i+1}}g''(\xi_1)+\frac{h_{i+1}}{h_i+h_{i+1}}g''(\xi_2)=g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1})=g(x_i)-(x_i-x_{i-1})g'(x_i)+\int_{x_{i-1}}^{x_i}g''(y)(y-x_{i-1})dy$$
$$g(x_{i+1})=g(x_i)+(x_{i+1}-x_i)g'(x_i)+\int_{x_i}^{x_{i+1}}g''(y)(x_{i+1}-y)dy$$

And the last equation can be obtained by

$$g(x_{i-1})=g(x_i)-h_ig'(x_i)+\frac{h_i^2}{2}g''(x_i)-\frac{h_i^3}{3!}g'''(x_i)+\frac{h_i^4}{4!}g''''(\eta_1)$$
$$g(x_{i+1})=g(x_i)+h_{i+1}g'(x_i)+\frac{h_{i+1}^2}{2}g''(x_i)+\frac{h_{i+1}^3}{3!}g'''(x_i)+\frac{h_{i+1}^4}{4!}g''''(\eta_2)$$

where $\eta_1\in[x_{i-1},x_i],\eta_2\in[x_i,x_{i+1}]$. Expecially,

(A.4)
$$\frac{h_i^4}{4!}g''''(\eta_1)=\int_{x_{i-1}}^{x_i}g''''(y)\frac{(y-x_{i-1})^3}{3!}dy$$
$$\frac{h_{i+1}^4}{4!}g''''(\eta_2)=\int_{x_i}^{x_{i+1}}g''''(y)\frac{(x_{i+1}-y)^3}{3!}dy$$

Subsitute them to the left side of (A.3), we can get the result. ∎

Lemma A.2. *If $y \in [x_{j-1}, x_j]$, denote $y = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0,1]$,*

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \tag{A.5}$$

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \tag{A.6}$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$
$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta) u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$
$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$. Thus ∎

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

Appendix B. Inequality.

Lemma B.1.

$$h_i \le r T^{1/r} h \begin{cases} x_i^{1-1/r}, & 1 \le i \le N \\ (2T - x_{i-1})^{1-1/r}, & N < i \le 2N-1 \end{cases} \tag{B.1}$$

Proof. For $1 \le i \le N$,

$$\begin{aligned} h_i &= T\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right) \\ &\le rT\frac{1}{N}\left(\frac{i}{N}\right)^{r-1} = rT^{1/r} h x_i^{1-1/r} \end{aligned}$$

For $N < i \leq 2N-1$,

$$h_i = T\left(\left(\frac{2N-i+1}{N}\right)^r - \left(\frac{2N-i}{N}\right)^r\right)$$
$$\leq rT\frac{1}{N}\left(\frac{2N-i+1}{N}\right)^{r-1} = rT^{1/r}h(2T-x_{i-1})^{1-1/r}$$ ∎

LEMMA B.2. *There is a constant $C = 2^{|r-2|}r(r-1)T^{2/r}$ such that for all $i \in \{1,2,\cdots,2N-1\}$*

$$|h_{i+1}-h_i| \leq Ch^2\begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T-x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases} \tag{B.2}$$

Proof.

$$h_{i+1}-h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \leq i \leq 2N-1 \end{cases}$$

For $i = 1$,

$$h_2 - h_1 = T(2^r-2)\left(\frac{1}{N}\right)^r = (2^r-2)T^{2/r}h^2x_1^{1-2/r}$$

For $2 \leq i \leq N-1$,

$$h_{i+1}-h_i = r(r-1)T\ N^{-2}\eta^{r-2}, \quad \eta \in [\frac{i-1}{N}, \frac{i+1}{N}]$$

If $r \in [1,2]$,

$$h_{i+1}-h_i = r(r-1)T\ N^{-2}\eta^{r-2} \leq r(r-1)T\ h^2\left(\frac{i-1}{N}\right)^{r-2}$$
$$\leq r(r-1)T\ h^2 2^{2-r}\left(\frac{i}{N}\right)^{r-2}$$
$$= 2^{2-r}r(r-1)T^{2/r}h^2x_i^{1-2/r}$$

else if $r > 2$,

$$h_{i+1}-h_i = r(r-1)T\ N^{-2}\eta^{r-2} \leq r(r-1)T\ h^2\left(\frac{i+1}{N}\right)^{r-2}$$
$$\leq r(r-1)T\ h^2 2^{r-2}\left(\frac{i}{N}\right)^{r-2}$$
$$= 2^{r-2}r(r-1)T^{2/r}h^2x_i^{1-2/r}$$

Since

$$2^r - 2 \leq 2^{|r-2|}r(r-1), \quad r \geq 1$$

we have

$$h_{i+1} - h_i \le 2^{|r-2|} r(r-1) T^{2/r} h^2 x_i^{1-2/r}, \quad 1 \le i \le N-1$$

For $i = N$, $h_{N+1} - h_N = 0$. For $N < i \le 2N-1$, it's central symmetric to the first half of the proof, which is

$$h_i - h_{i+1} \le 2^{|r-2|} r(r-1) T^{2/r} h^2 (2T - x_i)^{1-2/r}$$

Summarizes the inequalities, we can get

$$\text{(B.3)} \qquad |h_{i+1} - h_i| \le 2^{|r-2|} r(r-1) T^{2/r} h^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N-1 \end{cases} \qquad \square$$

Appendix C. Proofs of some technical details.

Additional proof of Theorem 5.1. For $2 \le i \le N-1$,

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\ &\qquad \le C \frac{2}{h_i + h_{i+1}} (h_i^3 x_{i-1}^{-2-\alpha/2} + h_{i+1}^3 x_i^{-2-\alpha/2}) \\ &\qquad \le 2C (h_i^2 x_{i-1}^{-2-\alpha/2} + h_{i+1}^2 x_i^{-2-\alpha/2}) \end{aligned}$$

Since Lemma B.1, we have

$$h_i \le r T^{1/r} h x_i^{1-1/r}, \quad 1 \le i \le N$$
$$h_{i+1} \le r T^{1/r} h x_{i+1}^{1-1/r}, \quad 1 \le i \le N-1$$

and

$$x_{i-1}^{-2-\alpha/2} \le 2^{-r(-2-\alpha/2)} x_i^{-2-\alpha/2} \quad 2 \le i \le N-1$$
$$x_{i+1}^{1-1/r} \le 2^{r-1} x_i^{1-1/r} \quad 1 \le i \le N-1$$

So there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

$$\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \le C h^2 x_i^{-\alpha/2 - 2/r}, \quad 2 \le i \le N-1$$

For $i = 1$, by (A.4)

$$\begin{aligned} &\frac{1}{4!} \frac{2}{h_1 + h_2} (h_1^3 f''(\eta_1) + h_2^3 f''(\eta_2)) \\ &\qquad = \frac{2}{h_1 + h_2} \left(\frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy + \frac{1}{4!} h_2^3 f''(\eta_2) \right) \end{aligned}$$

We have proved above that

$$\frac{2}{h_1 + h_2} h_2^3 f''(\eta_2) \le C h^2 x_1^{-\alpha/2 - 2/r}$$

and we can get

$$\begin{aligned}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy &\le C\frac{1}{3!}\int_0^{x_1} y^{1-\alpha/2}dy \\ &= C\frac{1}{3!(2-\alpha/2)}x_1^{2-\alpha/2}\end{aligned}$$

so

$$\frac{2}{h_1+h_2}\frac{1}{h_1}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy = \frac{C2^{1-r}}{3!(2-\alpha/2)}x_1^{-\alpha/2} = \frac{C2^{1-r}}{3!(2-\alpha/2)}T^{2/r}h^2x_1^{-\alpha/2-2/r}$$

And for $i = N$, we have

$$\begin{aligned}&\frac{2}{h_N+h_{N+1}}(h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2)) \\ &\qquad = h_N^2(f''(\eta_1)+f''(\eta_2)) \\ &\qquad \le r^2T^{2/r}h^2x_N^{2-2/r}\ 2Cx_{N-1}^{-2-\alpha/2} \\ &\qquad \le 2r^2T^{2/r}C2^{-r(-2-\alpha/2)}\ h^2x_N^{-\alpha/2-2/r}\end{aligned}$$

Finally, $N+1 \le i \le 2N-1$ is symmetric to the first half of the proof, so we can conclude that

$$\frac{2}{h_i+h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \le Ch^2x_i^{-\alpha/2-2/r}, \quad 1 \le i \le 2N-1$$

$\square$

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