

# SECOND-ORDER ERROR ANALYSIS FOR FRACTIONAL LAPLACIAN VIA RIESZ DERIVATIVES ON GRADED MESHES\*

JIANXING HAN<sup>†</sup> AND MINGHUA CHEN<sup>‡</sup>

**Abstract.** This is an example SIAM L<sup>A</sup>T<sub>E</sub>X article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

**Key words.** example, L<sup>A</sup>T<sub>E</sub>X

**MSC codes.** ?????????????????

**1. Introduction.** For  $\Omega = (0, 2T)$ ,  $1 < \alpha < 2$ ,

$$(1.1) \quad \begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x) & x \in \Omega, \\ u(x) = 0 & x \in \mathbb{R} \setminus \Omega, \end{cases}$$

where

$$(1.2) \quad (-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = \frac{-\kappa_\alpha}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_\Omega |x-y|^{1-\alpha} u(y) dy$$

$$(1.3) \quad \kappa_\alpha = -\frac{1}{2 \cos(\alpha\pi/2)} > 0$$

**2. Preliminaries: Numeric scheme and main results.**

**2.1. Numeric Format.**

$$(2.1) \quad x_i = \begin{cases} T \left(\frac{i}{N}\right)^r & 0 \leq i \leq N, \\ 2T - T \left(\frac{2N-i}{N}\right)^r & N \leq i \leq 2N, \end{cases}$$

where  $r \geq 1$ . And let

$$(2.2) \quad h_j = x_j - x_{j-1}, \quad 1 \leq j \leq 2N$$

Let  $\{\phi_j(x)\}_{j=1}^{2N-1}$  be standard hat functions, which are basis of the piecewise linear function space.

$$(2.3) \quad \phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \leq x \leq x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \leq x \leq x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

And then, define the piecewise linear interpolant of the true solution  $u$  to be

$$(2.4) \quad \Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j) \phi_j(x)$$

---

\*Submitted to the editors DATE.

<sup>†</sup>School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China (hanjx2023@lzu.edu.cn).

<sup>‡</sup>School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China (chenmh@lzu.edu.cn).

For convience, we denote

$$(2.5) \quad I^{2-\alpha}u(x) := \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x-y|^{1-\alpha}u(y)dy$$

and

$$(2.6) \quad D_h^2u(x_i) := \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i}u(x_{i-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) u(x_i) + \frac{1}{h_{i+1}}u(x_{i+1}) \right)$$

Now, we discretise (1.1) by replacing  $u(x)$  by a continuous piecewise linear function

$$(2.7) \quad u_h(x) := \sum_{j=1}^{2N-1} u_j \phi_j(x)$$

whose nodal values  $u_j$  are to be determined by collocation at each mesh point  $x_i$  for  $i = 1, 2, \dots, 2N-1$ :

$$(2.8) \quad -\kappa_{\alpha} D_h^{\alpha} u_h(x_i) := -\kappa_{\alpha} D_h^2 I^{2-\alpha} u_h(x_i) = f(x_i) =: f_i$$

Here,

$$(2.9) \quad -\kappa_{\alpha} D_h^{\alpha} u_h(x_i) = \sum_{j=1}^{2N-1} -\kappa_{\alpha} D_h^2 I^{2-\alpha} \phi_j(x_i) u_j = \sum_{j=1}^{2N-1} a_{ij} u_j$$

where

$$(2.10) \quad a_{ij} = -\kappa_{\alpha} D_h^2 I^{2-\alpha} \phi_j(x_i) \quad \text{for } i, j = 1, 2, \dots, 2N-1$$

We have replaced  $(-\Delta)^{\alpha/2}u(x_i) = f(x_i)$  in (1.1) by  $-\kappa_{\alpha} D_h^{\alpha} u_h(x_i) = f(x_i)$  in (2.8), with truncation error

$$(2.11) \quad \tau_i := -\kappa_{\alpha} \left( D_h^{\alpha} \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i) \right) \quad \text{for } i = 1, 2, \dots, 2N-1$$

where  $-\kappa_{\alpha} D_h^{\alpha} \Pi_h u(x_i) = \sum_{j=1}^{2N-1} -\kappa_{\alpha} D_h^{\alpha} \phi_j(x_i) u(x_j) = \sum_{j=1}^{2N-1} a_{ij} u(x_j)$ .

The discrete equation (2.8) can be written in matrix form

$$(2.12) \quad AU = F$$

where  $A = (a_{ij}) \in \mathbb{R}^{(2N-1) \times (2N-1)}$ ,  $U = (u_1, \dots, u_{2N-1})^T$  is unknown and  $F = (f_1, \dots, f_{2N-1})^T$ .

We can deduce  $a_{ij}$ ,

$$(2.13) \quad \begin{aligned} a_{ij} &= -\kappa_{\alpha} D_h^2 I^{2-\alpha} \phi_j(x_i) \\ &= -\kappa_{\alpha} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \tilde{a}_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \end{aligned}$$

where

$$(2.14) \quad I^{2-\alpha} \Pi_h u(x_i) = \sum_{j=1}^{2N-1} I^{2-\alpha} \phi_j(x_i) u(x_j) = \sum_{j=1}^{2N-1} \tilde{a}_{ij} u(x_j)$$

and

$$(2.15) \quad \begin{aligned} \tilde{a}_{ij} &= I^{2-\alpha} \phi_j(x_i) \\ &= \frac{1}{\Gamma(4-\alpha)} \left( \frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \end{aligned}$$

**2.2. Regularity of the true solution.** For any  $\beta > 0$ , we use the standard notation  $C^\beta(\bar{\Omega})$ ,  $C^\beta(\mathbb{R})$ , etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation  $C^\beta(\Omega)$  to refer to  $C^{k,\beta'}(\Omega)$ , where  $k$  is the greatest integer such that  $k < \beta$  and where  $\beta' = \beta - k$ . The Hölder spaces  $C^{k,\beta'}(\Omega)$  are defined as the subspaces of  $C^k(\Omega)$  consisting of functions whose  $k$ -th order partial derivatives are locally Hölder continuous [1, p. 52] with exponent  $\beta'$  in  $\Omega$ , where  $C^k(\Omega)$  is the set of all  $k$ -times continuously differentiable functions on open set  $\Omega$ .

For  $x \in \Omega = (0, 2T)$ , define

$$(2.16) \quad \delta(x) = \text{dist}(x, \partial\Omega) = \begin{cases} x & 0 < x \leq T, \\ 2T - x & T < x < 2T, \end{cases}$$

and  $\delta(x, y) = \min\{\delta(x), \delta(y)\}$ . Then we have the following  $\delta$ -dependent Hölder norms.

**DEFINITION 2.1** ( $\delta$ -dependent Hölder norms [2]). *Let  $\beta = k + \beta' > 0$  and  $\sigma \geq -\beta$ , with  $k$  integer and  $\beta' \in (0, 1]$ . For  $w \in C^\beta(\Omega) = C^{k,\beta'}(\Omega)$ , define the seminorm*

$$|w|_\beta^{(\sigma)} = \sup_{x, y \in \Omega} \left( \delta(x, y)^{\beta+\sigma} \frac{|w^{(k)}(x) - w^{(k)}(y)|}{|x - y|^{\beta'}} \right).$$

For  $\sigma > -1$ , we also define the norm  $\|\cdot\|_\beta^{(\sigma)}$  as follows: in case that  $\sigma \geq 0$ ,

$$\|w\|_\beta^{(\sigma)} = \sum_{l=0}^k \sup_{x \in \Omega} \left( \delta(x)^{l+\sigma} |w^{(l)}(x)| \right) + |w|_\beta^{(\sigma)},$$

while for  $-1 < \sigma < 0$ ,

$$\|w\|_\beta^{(\sigma)} = \|w\|_{C^{-\sigma}(\bar{\Omega})} + \sum_{l=1}^k \sup_{x \in \Omega} \left( \delta(x)^{l+\sigma} |D^l w(x)| \right) + |w|_\beta^{(\sigma)}.$$

**LEMMA 2.2.** [2, pp. 276-277] *Assume  $f \in L^\infty(\Omega)$ . Let  $u$  be a solution of (1.1). Then,  $u \in C^{\alpha/2}(\mathbb{R})$  and  $u/\delta^{\alpha/2} \in C^\sigma(\Omega)$  for some  $\sigma \in (0, 1 - \alpha/2)$ , with*

$$\|u\|_{C^{\alpha/2}(\mathbb{R})} \leq C \|f\|_{L^\infty(\Omega)} \quad \text{and} \quad \|u/\delta^{\alpha/2}\|_{C^\sigma(\bar{\Omega})} \leq C \|f\|_{L^\infty(\Omega)},$$

for some positive constant  $C = C(\Omega, \alpha)$ .

In particular, if  $f \in L^\infty(\Omega)$ , then

$$(2.17) \quad |u(x)| \leq C \delta(x)^{\alpha/2} \quad \text{for all } x \in \Omega.$$

**LEMMA 2.3.** [2, Proposition 1.4] *Let  $\Omega$  be a bounded domain, and  $\beta > 0$  be such that neither  $\beta$  nor  $\beta + \alpha$  is an integer. Let  $f \in C^\beta(\Omega)$  be such that  $\|f\|_\beta^{(\alpha/2)} < \infty$ , and  $u \in C^{\alpha/2}(\mathbb{R})$  be a solution of (1.1). Then,  $u \in C^{\beta+\alpha}(\Omega)$  and*

$$(2.18) \quad \|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C \left( \|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right),$$

for some positive constant  $C = C(\Omega, \alpha, \beta)$ .

By definition of  $\delta$ -dependent Hölder norms, we have following result obviously.

LEMMA 2.4. Let  $\beta = 4 - \alpha + \gamma$  with  $0 < \gamma < \alpha - 1$ . Assume that  $f \in L^\infty(\Omega) \cap C^\beta(\Omega)$  be such that  $\|f\|_\beta^{(\alpha/2)} < \infty$ , and  $u$  be a solution of (1.1). Then

$$(2.19) \quad |u^{(l)}(x)| \leq C\delta(x)^{\alpha/2-l}, \quad l = 0, 1, 2, 3, 4,$$

where  $C = C(\Omega, \alpha, \beta, f)$ .

*Proof.* Our hypotheses imply that  $2 < \beta < 3$ , and  $4 < \beta + \alpha < 5$ . By Lemma 2.3, we have

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C \left( \|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right).$$

And by Definition 2.1 and Lemma 2.2

$$\sum_{l=1}^4 \sup_{x \in \Omega} \left( \delta(x)^{l-\alpha/2} |w^{(l)}(x)| \right) \leq C \left( \|f\|_{L^\infty(\Omega)} + \|f\|_\beta^{(\alpha/2)} \right),$$

which is desired result  $l = 1, 2, 3, 4$ . The case  $l = 0$  is covered by (2.17).  $\square$

LEMMA 2.5. Let  $\beta = 4 - \alpha + \gamma$  with  $0 < \gamma < \alpha - 1$ . Assume that  $f \in C^\beta(\Omega)$  be such that  $\|f\|_\beta^{(\alpha/2)} < \infty$ , then

$$(2.20) \quad |f^{(l)}(x)| \leq C\delta(x)^{-\alpha/2-l}, \quad l = 0, 1, 2,$$

where  $C = C(\Omega, \alpha, \beta, f)$ .

*Proof.* By Definition 2.1, with  $2 < \beta < 3$   $\square$

$$\sum_{l=0}^2 \sup_{x \in \Omega} \left( \delta(x)^{l+\alpha/2} |f^{(l)}(x)| \right) \leq \|f\|_\beta^{(\alpha/2)}.$$

And in this paper bellow, without special instructions, we allways assume that  $\beta = 4 - \alpha + \gamma$  with  $0 < \gamma < \alpha - 1$ ,  $f \in L^\infty(\Omega) \cap C^\beta(\Omega)$  be such that  $\|f\|_\beta^{(\alpha/2)} < \infty$ .

**2.3. Main results.** Here we state our main results; the proof is deferred to section 3 and section 4.

Let's denote  $h = \frac{1}{N}$ , we have

THEOREM 2.6 (Local Truncation Error). Let  $\alpha \in (1, 2)$  and  $f \in L^\infty(\Omega) \cap C^\beta(\Omega)$  be such that  $\|f\|_\beta^{(\alpha/2)} < \infty$ , where  $\beta = 4 - \alpha + \gamma$  with  $0 < \gamma < \alpha - 1$ . Then,

$$(2.21) \quad \begin{aligned} |\tau_i| &= |-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i)| \\ &\leq Ch^{\min\{\frac{r_\alpha}{2}, 2\}} \delta(x_i)^{-\alpha} + C(r-1)h^2(T - \delta(x_i) + h_N)^{1-\alpha}. \end{aligned}$$

THEOREM 2.7 (Global Error). Let  $\alpha \in (1, 2)$  and  $f \in L^\infty(\Omega) \cap C^\beta(\Omega)$  be such that  $\|f\|_\beta^{(\alpha/2)} < \infty$ , where  $\beta = 4 - \alpha + \gamma$  with  $0 < \gamma < \alpha - 1$ . Let  $u_i$  be the approximate solution of  $u(x_i)$  computed by the discretization scheme (2.12). Then,

$$(2.22) \quad \max_{1 \leq i \leq 2N-1} |u_i - u(x_i)| \leq Ch^{\min\{\frac{r_\alpha}{2}, 2\}}.$$

**3. Local Truncation Error.** We shall first introduce some notations.

For convenience, we use the notation  $\simeq$ . That  $x_1 \simeq y_1$ , means that  $c_1 x_1 \leq y_1 \leq C_1 x_1$  for some positive constants  $c_1$  and  $C_1$  that are independent of  $N$ .

And for  $1 \leq j \leq 2N$ , we define

$$(3.1) \quad y_j^\theta = (1 - \theta)x_{j-1} + \theta x_j, \quad \theta \in (0, 1)$$

Then we have

LEMMA 3.1. For  $1 \leq i \leq 2N - 1$

$$(3.2) \quad h_i \simeq h_{i+1} \simeq h\delta(x_i)^{1-1/r}, \quad \delta(x_i) \simeq \delta(x_{i+1}) \simeq \delta(y_{i+1}^\theta)$$

Since  $i^r - (i-1)^r \simeq i^{r-1}$ , for  $i \geq 1$ , where  $\theta \in (0, 1)$ .

Meanwhile, let's define kernel functions

$$(3.3) \quad K_y(x) := \frac{|y-x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

**3.1. Proof of Theorem 2.6.** The truncation error of the discrete format can be written as

$$(3.4) \quad \begin{aligned} -\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I^{2-\alpha} \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i)) \\ &= -\kappa_\alpha D_h^2 I^{2-\alpha} (\Pi_h u - u)(x_i) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) \end{aligned}$$

THEOREM 3.2. There exists a constant  $C = C(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$  such that

$$(3.5) \quad \left| -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) \right| \leq Ch^2 \delta(x_i)^{-\alpha/2-2/r}$$

*Proof.* Since  $f \in C^2(\Omega)$  and

$$(3.6) \quad \frac{d^2}{dx^2} (-\kappa_\alpha I^{2-\alpha} u(x)) = f(x), \quad x \in \Omega,$$

we have  $I^{2-\alpha} u \in C^4(\Omega)$ . Therefore, using equation (A.2) of Lemma A.1, for  $1 \leq i \leq 2N - 1$ , we have

$$(3.7) \quad \begin{aligned} -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) &= \frac{h_{i+1} - h_i}{3} f'(x_i) \\ &+ \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right) \end{aligned}$$

By Lemma B.1, Lemma 2.5 and Lemma B.2, we get the result. □

And now define

$$(3.8) \quad R_i := D_h^2 I^{2-\alpha} (u - \Pi_h u)(x_i), \quad 1 \leq i \leq 2N - 1$$

We have some results about the estimate of  $R_i$

THEOREM 3.3. For  $1 \leq i < N/2$ , there exists  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that

$$(3.9) \quad |R_i| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

THEOREM 3.4. For  $N/2 \leq i \leq N$ , there exists constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that

$$(3.10) \quad |R_i| \leq C(r-1)h^2(T - x_i + h_N)^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

And for  $N < i \leq 2N - 1$ , it is symmetric to the previous case.

Combine Theorem 3.2, Theorem 3.3 and Theorem 3.4, and for  $1 \leq i \leq N$ , we have

$$(3.11) \quad h^2 x_i^{-\alpha/2-2/r} \leq T^{\alpha/2-2/r} h^{\min\{\frac{r\alpha}{2}, 2\}} x_i^{-\alpha}$$

$$(3.12) \quad h^{r\alpha/2+r} x_i^{-1-\alpha} \leq T^{-1} h^{r\alpha/2} x_i^{-\alpha}$$

$$(3.13) \quad h^r x_i^{-1} \ln(i) = T^{-1} \frac{\ln(i)}{i^r} \leq T^{-1}, \quad h^r \ln(N) = \frac{\ln(N)}{N^r} \leq 1$$

the proof of Theorem 2.6 completed.

We prove Theorem 3.3 and Theorem 3.4 in next subsections.

**3.2. Grid Mapping Functions.** For convience, let's denote

DEFINITION 3.5.

$$(3.14) \quad T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy, \quad i = 0, \dots, 2N, \quad j = 1, \dots, 2N.$$

Also, we denote vertical difference quotients of  $T_{ij}$

$$(3.15) \quad \begin{aligned} V_{ij} &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} T_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \\ &= \int_{x_{i-1}}^{x_i} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy. \end{aligned}$$

Then by (3.8)  $R_i = \sum_{j=1}^{2N} V_{ij}$ . And define skew difference quotients of  $T_{ij}$

$$(3.16) \quad S_{ij} = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} T_{i-1,j-1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j+1} \right).$$

Our main idea is to depart  $R_i$  by  $V_{ij}$  and  $S_{ij}$ . For  $3 \leq i < N/2$ , let's denote

160  $k = \lceil \frac{i}{2} \rceil$ , and take some suitable integer  $m$ , then

$$\begin{aligned}
 R_i &= \sum_{j=1}^{2N} V_{ij} \\
 &= \sum_{j=1}^{k-1} V_{ij} + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \\
 161 \quad (3.17) \quad &+ \sum_{j=k+1}^{m-1} S_{ij} + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,m} + T_{i-1,m-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,m} \right) \\
 &+ \sum_{j=m+1}^{2N} V_{ij} \\
 &= I_1 + I_2 + I_3 + I_4 + I_5,
 \end{aligned}$$

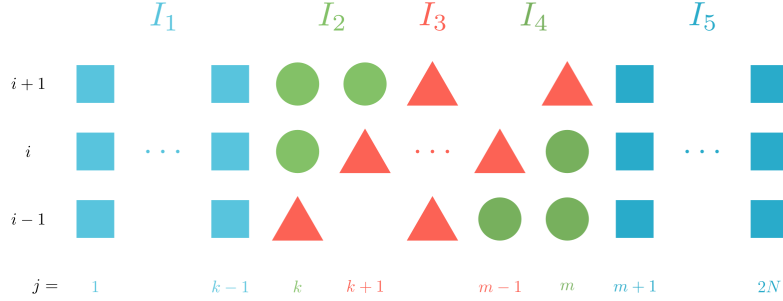


FIG. 1. The departure of  $R_i$  for  $i \geq 3$

162 and discuss  $i = 1, 2$  separately, where

$$163 \quad (3.18) \quad R_1 = \sum_{i=1}^3 V_{1,i} + \sum_{i=4}^N V_{i,j}, \quad R_2 = \sum_{i=1}^4 V_{1,i} + \sum_{i=5}^N V_{i,j}.$$

164 The difficulty for estimating  $S_{ij}$  is that  $T_{i-1,j-1}$ ,  $T_{i,j}$  and  $T_{i+1,j+1}$  have different  
 165 integral region. We first make them normalized.

166 LEMMA 3.6. For  $y \in (x_{j-1}, x_j)$ , we can rewrite  $y = y_j^\theta$ , by (3.14) and Lemma A.2,

$$\begin{aligned}
 T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
 &= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\
 167 \quad (3.19) \quad &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
 &\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta,
 \end{aligned}$$

168 where  $\eta_{j1}^\theta \in (x_{j-1}, y_j^\theta)$ ,  $\eta_{j2}^\theta \in (y_j^\theta, x_j)$ .

Since  $j$  changes with  $i$  at indices of elements in  $S_{ij}$  by (3.16), we create some functions satisfy the property.

**DEFINITION 3.7** (Grid Mapping Functions). *For  $1 \leq i, j \leq 2N - 1$ .*

$$(3.20) \quad y_{i,j}(x) = \begin{cases} (x^{1/r} + Z_{j-i})^r & i < N, j < N, \\ \frac{x^{1/r} - Z_i}{Z_1} h_N + x_N & i < N, j = N, \\ 2T - (Z_{2N-(j-i)} - x^{1/r})^r & i < N, j > N, \\ \left( \frac{Z_1}{h_N} (x - x_N) + Z_j \right)^r & i = N, j < N, \\ x, & i = N, j = N, \\ 2T - \left( \frac{Z_1}{h_N} (2T - x - x_N) + Z_{2N-j} \right)^r & i = N, j > N, \\ (Z_{2N+j-i} - (2T - x)^{1/r})^r & i > N, j < N, \\ \frac{Z_{2N-j} - (2T - x)^{1/r}}{Z_1} h_N + x_N & i > N, j = N, \\ 2T - ((2T - x)^{1/r} - Z_{j-i})^r & i > N, j > N, \end{cases}$$

where  $Z_j := T^{1/r} \frac{j}{N}$ . And

$$(3.21) \quad h_{i,j}(x) = y_{i,j}(x) - y_{i,j-1}(x),$$

$$(3.22) \quad y_{i,j}^\theta(x) = (1 - \theta)y_{i,j-1}(x) + \theta y_{i,j}(x), \quad \theta \in (0, 1),$$

$$(3.23) \quad P_{i,j}^\theta(x) = (h_{i,j}(x))^3 \frac{|y_{i,j}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y_{i,j}^\theta(x)),$$

$$(3.24) \quad Q_{i,j;l}^\theta(x) = (h_{i,j}(x))^l \frac{|y_{i,j}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}.$$

Obviously,

$$(3.25) \quad y_{i,j}(x_{i-1}) = x_{j-1}, \quad y_{i,j}(x_i) = x_j, \quad y_{i,j}(x_{i+1}) = x_{j+1},$$

$$(3.26) \quad h_{i,j}(x_{i-1}) = h_{j-1}, \quad h_{i,j}(x_i) = h_j, \quad h_{i,j}(x_{i+1}) = h_{j+1},$$

$$(3.27) \quad y_{i,j}^\theta(x_{i-1}) = y_{j-1}^\theta, \quad y_{i,j}^\theta(x_i) = y_j^\theta, \quad y_{i,j}^\theta(x_{i+1}) = y_{j+1}^\theta.$$

And now we can rewrite  $T_{ij}$ .

**LEMMA 3.8.**

$$(3.28) \quad T_{ij} = \int_0^1 -\frac{\theta(1-\theta)}{2} P_{i,j}^\theta(x_i) d\theta + \int_0^1 \frac{\theta(1-\theta)}{3!} Q_{i,j;l}^\theta(x_i) [\theta^2 u'''(\eta_{j,1}^\theta) - (1-\theta)^2 u'''(\eta_{j,2}^\theta)] d\theta.$$



188 *Immediately, we can see from (3.16) and Lemma 3.6 that For  $1 \leq i \leq 2N - 1$ ,*  
 189  $2 \leq j \leq 2N - 1$ ,  
 (3.29)

$$\begin{aligned}
 S_{ij} = & \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{i,j}^\theta(x_i) d\theta \\
 & + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i,j;4}^\theta(x_{i+1})u'''(\eta_{j+1,1}^\theta) - Q_{i,j;4}^\theta(x_i)u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\
 & - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i,j;4}^\theta(x_i)u'''(\eta_{j,1}^\theta) - Q_{i,j;4}^\theta(x_{i-1})u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\
 & - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i,j;4}^\theta(x_{i+1})u'''(\eta_{j+1,2}^\theta) - Q_{i,j;4}^\theta(x_i)u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\
 & + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i,j;4}^\theta(x_i)u'''(\eta_{j,2}^\theta) - Q_{i,j;4}^\theta(x_{i-1})u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta.
 \end{aligned}$$

191 We give some properties of the grid mapping functions.

192 LEMMA 3.9. *For  $2 \leq i, j \leq 2N - 2$  and  $\xi \in (x_{i-1}, x_{i+1})$*

$$193 \quad (3.30) \quad \xi \simeq x_i, \quad \delta(y_{i,j}(\xi)) \simeq \delta(x_j), \quad h_{i,j}(\xi) \simeq h_j,$$

194

$$195 \quad (3.31) \quad |y_{i,j}(\xi) - \xi| \simeq |x_j - x_i|, \quad |y_{i,j-1}(\xi) - \xi| \simeq |x_{j-1} - x_i|,$$

196 *then*

$$197 \quad (3.32) \quad |y_{i,j}^\theta(\xi) - \xi| = (1-\theta)|y_{i,j-1}(\xi) - \xi| + \theta|y_{i,j}(\xi) - \xi| \simeq |y_j^\theta - x_i|,$$

198 *since  $y_{i,j-1}(\xi) - \xi$ ,  $y_{i,j}(\xi) - \xi$  have the same sign ( $\geq 0$  or  $\leq 0$ ).*

LEMMA 3.10.

$$199 \quad (3.33) \quad y'_{i,j}(x) = \begin{cases} y_{i,j}^{1-1/r}(x)x^{1/r-1} & i < N, j < N, \\ \frac{h_N}{rZ_1}x^{1/r-1} & i < N, j = N, \\ (2T - y_{i,j}(x))^{1-1/r}x^{1/r-1} & i < N, j > N, \\ y_{i,j}^{1-1/r}(x)\frac{rZ_1}{h_N} & i = N, j < N, \\ 1 & i = N, j = N, \end{cases}$$

200

$$201 \quad (3.34) \quad y''_{i,j}(x) = \frac{1-r}{r} \begin{cases} y_{i,j}^{1-2/r}(x)x^{1/r-2}Z_{j-i} & i < N, j < N, \\ \frac{h_N}{rZ_1}x^{1/r-2} & i < N, j = N, \\ (2T - y_{i,j}(x))^{1-2/r}x^{1/r-2}Z_{2N-j+i} & i < N, j > N, \\ -y_{i,j}^{1-2/r}(x)\left(\frac{rZ_1}{h_N}\right)^2 & i = N, j < N, \\ 0 & i = N, j = N. \end{cases}$$

LEMMA 3.11. For  $2 \leq i \leq N, 2 \leq j \leq 2N - 2, \xi \in (x_{i-1}, x_{i+1})$

$$(3.35) \quad |h'_{i,j}(\xi)| \leq C(r-1)Z_1x_i^{1/r-1}\delta(x_j)^{1-2/r} \leq C(r-1)h_jx_i^{1/r-1}\delta(x_j)^{-1/r},$$

$$(3.36) \quad |(y_{i,j}(\xi) - \xi)'| \leq Cx_i^{-1}|x_j - x_i|.$$

*Proof.* From (3.21) and Lemma 3.10, we can see that

$$(3.37) \quad h'_{i,j}(x) = y'_{i,j}(x) - y'_{i,j-1}(x) = \begin{cases} x^{1/r-1}(y_{i,j}^{1-1/r}(x) - y_{i,j-1}^{1-1/r}(x)) & i < N, j < N, \\ x^{1/r-1}(\frac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(x)) & i < N, j = N, \\ x^{1/r-1}\left((2T - y_{i,N+1}(x))^{1-1/r} - \frac{h_N}{rZ_1}\right) & i < N, j = N+1, \\ x^{1/r-1}\left((2T - y_{i,j}(x))^{1-1/r} - (2T - y_{i,j-1}(x))^{1-1/r}\right) & i < N, j > N+1, \\ \frac{rZ_1}{h_N}\left(y_{N,j}^{1-1/r}(x) - y_{N,j-1}^{1-1/r}(x)\right) & i = N, j < N, \\ \frac{rZ_1}{h_N}\left(\frac{h_N}{rZ_1} - y_{N,N-1}^{1-1/r}(x)\right) & i = N, j = N. \end{cases}$$

While for  $2 \leq i \leq N$ , if  $2 \leq j < N, \xi \in (x_{i-1}, x_{i+1})$ ,

$$(3.38) \quad \begin{aligned} y_{i,j}^{1-1/r}(\xi) - y_{i,j-1}^{1-1/r}(\xi) &\leq x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r} \\ &= T^{1-1/r}N^{1-r}((j+1)^{r-1} - (j-2)^{r-1}) \\ &\leq CT^{1-1/r}(r-1)N^{1-r}j^{r-2} = C(r-1)Z_1x_j^{1-2/r}. \end{aligned}$$

If  $j = N, \xi \in (x_{i-1}, x_{i+1})$ , we have  $y_{i,N-1}(\xi) \in (x_{N-2}, x_N)$ . And

$$(3.39) \quad \frac{h_N}{rZ_1} = T^{1-1/r} \frac{1 - (1-h)^r}{rh} = \eta^{1-1/r} \simeq x_N^{1-1/r}, \quad \eta \in (x_{N-1}, x_N).$$

Then

$$(3.40) \quad \left| \frac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(\xi) \right| \leq x_N^{1-1/r} - x_{N-2}^{1-1/r} \simeq (r-1)Z_1x_N^{1-2/r}.$$

And similar for  $j \geq N+1$ . Combine with Lemma 3.1, Lemma 3.9,  $\eta \simeq x_N$ , we get the first result.

For the second estimate, we have

$$(3.41) \quad (y_{i,j}(x) - x)' = y'_{i,j}(x) - 1.$$

Then, for  $2 \leq i < N$ , if  $2 \leq j < N, \xi \in (x_{i-1}, x_{i+1})$ , by Lemma A.5

$$(3.42) \quad \xi^{1/r}|y_{i,j}^{1-1/r}(\xi) - \xi^{1-1/r}| \leq |y_{i,j}(\xi) - \xi|.$$

$j > N$  is symmetric to it, that is

$$(3.43) \quad \begin{aligned} \xi^{1/r}|(2T - y_{i,j}(\xi))^{1-1/r} - \xi^{1-1/r}| &\leq |2T - y_{i,j}(\xi) - \xi| \\ &\leq |2T - x_j - x_i| + |y_{i,j}(\xi) - x_j| + |\xi - x_i| \leq |2T - x_j - x_i| + 2h_N \\ &\leq |x_j - T| + |T - x_i| + 2h_N \leq 2|x_j - x_i|. \end{aligned}$$

But if  $j = N$ , with (3.39) and Lemma A.5,

$$(3.44) \quad \begin{aligned} \eta^{1/r} \left| \frac{h_N}{rZ_1} - \xi^{1-1/r} \right| &\leq |\eta - \xi|, \quad \eta \in (x_{N-1}, x_N) \\ &\leq |x_N - x_i| + |h_N| + |h_{i+1}| \leq 3|x_N - x_i|. \end{aligned}$$

For  $i = N$ , if  $j < N$ , similarly with (3.44),

$$(3.45) \quad \eta^{1/r} |y_{N,j}^{1-1/r}(\xi) - \frac{h_N}{rZ_1}| \leq C|x_j - x_N|.$$

And if  $j = N$ , it is obviously  $\equiv 0$ .

Similarly, by Lemma 3.10 and Lemma 3.9, we get the second result.  $\square$

LEMMA 3.12. For  $2 \leq i \leq N, 2 \leq j \leq 2N - 2, \xi \in (x_{i-1}, x_{i+1})$

$$(3.46) \quad |y''_{i,j}(\xi)| \leq C(r-1) \begin{cases} x_j^{-1/r} x_i^{1/r-2} |x_j - x_i| & i < N, j < N, \\ x_N^{1-1/r} x_i^{1/r-2} & i < N, j = N, \\ \delta(x_j)^{1-2/r} x_i^{1/r-2} x_N^{1/r} & i < N, j > N, \\ \delta(x_j)^{1-2/r} x_N^{2/r-2} & i = N, j \neq N, \\ 0 & i = N, j = N. \end{cases}$$

And  $2 \leq i \leq N, 3 \leq j \leq 2N - 2, \xi \in (x_{i-1}, x_{i+1})$

$$(3.47) \quad |h''_{i,j}(\xi)| \leq C(r-1) \begin{cases} Z_1 x_i^{1/r-2} x_j^{-2/r} (|x_j - x_i| + x_j) & i < N, j < N, \\ x_i^{1/r-2} x_N^{1-1/r} & i < N, j = N, N+1, \\ Z_1 x_i^{1/r-2} \delta(x_j)^{1-3/r} x_N^{1/r} & i < N, j > N+1, \\ Z_1 x_N^{2/r-2} \delta(x_j)^{1-3/r} & i = N, j < N \text{ or } j > N+1, \\ x_N^{-1} & i = N, j = N. \end{cases}$$

*Proof.* Since by Lemma A.5, for  $2 \leq i, j < N$

$$(3.48) \quad x_j^{1-1/r} |Z_{j-i}| = x_j^{1-1/r} |x_j^{1/r} - x_i^{1/r}| \leq |x_j - x_i|,$$

and by (3.39),  $\frac{h_N}{rZ_1} \simeq x_N^{1-1/r}$ . And

$$(3.49) \quad Z_{2N-j+i} \leq Z_{2N} = 2T^{1/r}.$$

Then by Lemma 3.10 and Lemma 3.9, we get the first result.

For the second part, by Lemma 3.10

$$(3.50) \quad h''_{i,j}(x) = y''_{i,j}(x) - y''_{i,j-1}(x),$$

while for  $2 \leq i < N$ , if  $3 \leq j < N, \xi \in (x_{i-1}, x_{i+1})$ ,

$$(3.51) \quad y_{i,j}^{1-2/r}(\xi) Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi) Z_{j-i-1} = \left( y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi) \right) Z_{j-i} + y_{i,j-1}^{1-2/r}(\xi) Z_1,$$

where  $y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi) \simeq (r-2)Z_1 x_j^{1-3/r}$  similar with (3.38). Combine with

(3.48), we get

$$(3.52) \quad |y_{i,j}^{1-2/r}(\xi) Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi) Z_{j-i-1}| \leq CZ_1 \left( |r-2|x_j^{-2/r}|x_j - x_i| + x_j^{1-2/r} \right).$$

244 If  $j = N$ ,

$$245 \quad (3.53) \quad |h''_{i,N}(x)| \leq |y''_{i,N}(x)| + |y''_{i,N-1}(x)| \leq C(r-1)x_i^{1/r-2}x_N^{1-1/r}.$$

246 Similarly if  $j = N+1$ .

247 However, if  $j > N+1$ , similar with (3.51), we get

$$248 \quad (3.54) \quad \begin{aligned} & (2T - y_{i,j}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)} \\ & = \left( (2T - y_{i,j}(\xi))^{1-2/r} - (2T - y_{i,j-1}(\xi))^{1-2/r} \right) Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r} Z_1, \end{aligned}$$

249 thus,

$$250 \quad (3.55) \quad \begin{aligned} & \left| (2T - y_{i,j}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)} \right| \\ & \leq CZ_1 \left( |r-2|(2T - x_j)^{1-3/r} x_N^{1/r} + (2T - x_j)^{1-2/r} \right) \leq CZ_1 (2T - x_j)^{1-3/r} x_N^{1/r}. \end{aligned}$$

251 For  $i = N$ , it's obvious. Combine with Lemma 3.10 and Lemma 3.9, we get the second  
252 result.  $\square$

253 **3.3. Proof of Theorems.** Then we estimate each part of (3.17). And We take

254  $m = 2i$  for  $3 \leq i < N/2$ , and  $m = N - \lceil N/2 \rceil + 1$  for  $N/2 \leq i \leq N$ .

255 For  $I_5$

256 **LEMMA 3.13.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

257 *Case 1. For  $1 \leq i < N/2$ ,*

$$258 \quad (3.56) \quad \sum_{j=\max\{2i+1,4\}}^N |V_{ij}| \leq Ch^2 x_i^{-\alpha/2-2/r}.$$

259 *Case 2. For  $1 \leq i < N/2$ ,*

$$260 \quad (3.57) \quad \sum_{j=N+1}^{2N} |V_{ij}| \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

261 *Case 3. For  $N/2 \leq i \leq N$ ,*

$$262 \quad (3.58) \quad \sum_{j=N-\lceil \frac{N}{2} \rceil+2}^{2N} |V_{ij}| \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

263 *Proof.* For  $i, j$  in each case, by (3.15), Lemma A.3 and Lemma B.3, we have

$$264 \quad (3.59) \quad |V_{ij}| \leq Ch^2 \int_{x_{j-1}}^{x_j} \delta(y)^{\alpha/2-2/r} |y - x_i|^{-1-\alpha} dy.$$

265 For Case 1, with  $x_i \simeq x_{2i}$ ,

$$266 \quad (3.60) \quad \begin{aligned} \sum_{j=\max\{2i+1,4\}}^N |V_{ij}| & \leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\ & = \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\ & \leq Ch^2 x_i^{-\alpha/2-2/r}. \end{aligned}$$

For Case 2 , by (3.15), Lemma A.3, Lemma B.3 and  $y - x_i \simeq T$ ,

$$\begin{aligned}
 |V_{ij}| &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy, \\
 \sum_{j=N+1}^{2N-1} |V_{ij}| &\leq CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T - y)^{\alpha/2-2/r} dy \\
 &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0, \\ \ln(T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0, \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0, \end{cases} \\
 &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r} h^2, & \alpha/2 - 2/r + 1 > 0, \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0, \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r} h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0. \end{cases}
 \end{aligned}
 \tag{3.61}$$

And by Lemma A.4

$$|V_{i,2N}| \leq CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}.$$

Summarizes, we get the result. Similar for Case 3. □

For  $i = 1, 2$ .

LEMMA 3.14. From (3.18), by Lemma B.4, Lemma 3.13 Case 1 2, we get for  $i = 1, 2$

$$|R_i| \leq Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

LEMMA 3.15. There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil$

$$|I_1| = \left| \sum_{j=1}^{k-1} V_{ij} \right| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

*Proof.* by (3.15), Lemma A.4 , Lemma B.3

$$|V_{i1}| \leq C \int_0^{x_1} x_1^{\alpha/2} |x_i - y|^{-1-\alpha} dy \simeq x_1^{\alpha/2+1} x_i^{-1-\alpha} = T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha}$$

For  $2 \leq j \leq k-1$ , by Lemma A.3 and Lemma B.3 with  $x_i - y \simeq x_i$ , we have

$$|V_{ij}| \leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} x_i^{-1-\alpha} dy$$

Therefore,

$$\sum_{j=2}^{k-1} |V_{ij}| \leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil-1}} y^{\alpha/2-2/r} dy$$

288 But  $x_{\lceil \frac{i}{2} \rceil - 1} \leq 2^{-r} x_i$ , so we have

$$289 \quad (3.67) \quad \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2 - 2/r} dy \leq \begin{cases} \frac{1}{\alpha/2 - 2/r + 1} (2^{-r} x_i)^{\alpha/2 - 2/r + 1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2 - 2/r + 1|} x_1^{\alpha/2 - 2/r + 1}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

290 Combine the results above, we get the lemma.  $\square$

291

292 **LEMMA 3.16.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

293 *Case 1. For  $3 \leq i < N$ ,  $\lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i - 1, N - 1\}$ ,*

$$294 \quad (3.68) \quad |D_h^2 P_{i,j}^\theta(x_i)| \leq C h_j^3 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-4}$$

295 *Case 2. For  $N/2 \leq i \leq N$ ,  $j = N, N + 1$*

$$296 \quad (3.69) \quad |D_h^2 P_{i,j}^\theta(\xi)| \leq C h_j^3 |y_j^\theta - x_i|^{1-\alpha} + C(r-1) h_j^2 \left( |y_j^\theta - x_i|^{1-\alpha} + h_j |y_j^\theta - x_i|^{-\alpha} \right)$$

297 *Case 3. For  $N/2 \leq i \leq N$ ,  $N + 2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil$ ,*

$$298 \quad (3.70) \quad |D_h^2 P_{i,j}^\theta(\xi)| \leq C h_j^3 \left( |y_j^\theta - x_i|^{1-\alpha} + (r-1) |y_j^\theta - x_i|^{-\alpha} \right)$$

299 *Proof.* Since  $\text{sign}(y_{i,j}^\theta(\xi) - \xi)$  is independent of  $\xi$ , we can derivate it. Then by  
300 Lemma A.1

$$301 \quad (3.71) \quad D_h^2 P_{i,j}^\theta(x_i) = P_{i,j}^{\theta''}(\xi), \quad \xi \in (x_{i-1}, x_{i+1})$$

302 From (3.23), using Leibniz formula and chain rules, and Lemma 3.9, Lemma 3.10,  
303 Lemma 3.11, Lemma 3.12, Lemma 2.4, Lemma 3.1

304 For every case, we have  $x_i \simeq \delta(x_j)$ , so we have

$$305 \quad (3.72) \quad h_{i,j}(\xi) \leq C h_j, \quad |h'_{i,j}(\xi)| \leq C(r-1) h_j x_i^{-1}$$

306

$$307 \quad (3.73) \quad |y_{i,j}^\theta(\xi) - \xi| \leq C |y_j^\theta - x_i|, \quad |(y_{i,j}^\theta(\xi) - x_i)'| \leq C |y_j^\theta - x_i| x_i^{-1}$$

308

$$309 \quad (3.74) \quad |u''(y_{i,j}^\theta(\xi))| \leq C x_i^{\alpha/2-2}, \quad |(u''(y_{i,j}^\theta(\xi)))'| \leq C x_i^{\alpha/2-3}, \quad |(u''(y_{i,j}^\theta(\xi)))''| \leq C x_i^{\alpha/2-4}$$

310 By Lemma 3.12, we have

311 For Case 1,

$$312 \quad (3.75) \quad |h''_{i,j}(\xi)| \leq C(r-1) h_j x_i^{-2}, \quad |(y_{i,j}^\theta(\xi) - x_i)''| \leq C(r-1) |y_j^\theta - x_i| x_i^{-2}$$

313 For Case 2, since  $x_i \simeq x_j \simeq T$

$$314 \quad (3.76) \quad |h''_{i,j}(\xi)| \leq C(r-1), \quad |(y_{i,j}^\theta(\xi) - x_i)''| \leq C(r-1)$$

315 For Case 3, since  $x_i \simeq \delta(x_j) \simeq T$ , we have

$$316 \quad (3.77) \quad |h''_{i,j}(\xi)| \leq C(r-1) h_j, \quad |(y_{i,j}^\theta(\xi) - x_i)''| \leq C(r-1) \quad \square$$

317 Combine them, we get the result.

LEMMA 3.17. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $2 \leq i \leq N, 2 \leq j \leq 2N - 2$ ,*

$$(3.78) \quad \left| \frac{Q_{i,j;l}^\theta(x_{i+1})u^{(l-1)}(\eta_{j+1}^\theta) - Q_{i,j;l}^\theta(x_i)u^{(l-1)}(\eta_j^\theta)}{h_{i+1}} \right| \leq Ch_j^l \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{-1} \delta(x_j)^{\alpha/2-l+1-1/r} (x_i^{1/r} + \delta(x_j)^{1/r})$$

And

$$(3.79) \quad \left| \frac{Q_{i,j;l}^\theta(x_i)u^{(l-1)}(\eta_j^\theta) - Q_{i,j;l}^\theta(x_{i-1})u^{(l-1)}(\eta_{j-1}^\theta)}{h_i} \right| \leq Ch_j^l \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{-1} \delta(x_j)^{\alpha/2-l+1-1/r} (x_i^{1/r} + \delta(x_j)^{1/r})$$

where  $\eta_j^\theta \in (x_{j-1}, x_j)$ .

*Proof.*

$$(3.80) \quad \begin{aligned} & \frac{Q_{i,j;l}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{i,j;l}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \\ &= \frac{Q_{i,j;l}^\theta(x_{i+1}) - Q_{i,j;l}^\theta(x_i)}{h_{i+1}} u'''(\eta_{j+1}^\theta) + Q_{i,j;l}^\theta(x_i) \frac{u'''(\eta_{j+1}^\theta) - u'''(\eta_j^\theta)}{h_{i+1}} \end{aligned}$$

Using mean value theorem

$$(3.81) \quad D_h Q_{i,j;l}^\theta(x_i) := \frac{Q_{i,j;l}^\theta(x_{i+1}) - Q_{i,j;l}^\theta(x_i)}{h_{i+1}} = Q_{i,j;l}^{\theta'}(\xi), \quad \xi \in (x_i, x_{i+1})$$

From (3.24) and Leibniz rule, by Lemma 3.9, Lemma 3.11 and Lemma 3.1, we have

$$(3.82) \quad |Q_{i,j;l}^{\theta'}(\xi)| \leq Ch_j^l \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (x_i^{-1} + x_i^{1/r-1} \delta(x_j)^{-1/r})$$

$$(3.83) \quad Q_{i,j;l}^\theta(x_i) = h_j^l \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}$$

With Lemma 3.1 and Lemma 2.4

$$|u^{(l-1)}(\eta_{j+1}^\theta)| \leq C(\eta_{j+1}^\theta)^{\alpha/2-l+1} \simeq \delta(x_j)^{\alpha/2-l+1}$$

and by Lemma 3.1

$$\begin{aligned} \frac{|u^{(l-1)}(\eta_{j+1}^\theta) - u^{(l-1)}(\eta_j^\theta)|}{h_{i+1}} &= |u^{(l)}(\eta)| \frac{\eta_{j+1}^\theta - \eta_j^\theta}{h_{i+1}}, \quad \eta \in (x_{j-1}, x_{j+1}) \\ &\leq C\delta(\eta)^{\alpha/2-l} \frac{x_{j+1} - x_{j-1}}{h_{i+1}} = C\delta(\eta)^{\alpha/2-l} \frac{h_{j+1} + h_j}{h_{i+1}} \\ &\simeq x_i^{1/r-1} \delta(x_j)^{\alpha/2-l+1-1/r} \end{aligned}$$

Combine the results above, we get the first term. While, the later is similar.  $\square$

LEMMA 3.18. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

Case 1. For  $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$ ,

$$\begin{aligned} |S_{ij}| &\leq Ch_j^2 x_i^{\alpha/2-4} \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\ &= Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \end{aligned}$$

Thus,

$$\sum_{j=k+1}^{\min\{2i-1, N-1\}} |S_{ij}| \leq Ch^2 x_i^{-\alpha/2-2/r}$$

Case 2. For  $N/2 \leq i \leq N, j = N, N+1$ , since  $\theta(1-\theta)h_j \leq |y_j^\theta - x_i|$ , we have

$$|S_{ij}| \leq C(h^3 + (r-1)h^2)(T - x_i + h_N)^{1-\alpha}$$

Case 3. For  $N/2 \leq i \leq N, N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil$ ,

$$|S_{ij}| \leq Ch^2 \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy$$

Thus,

$$\sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} |S_{ij}| \leq Ch^2 + C(r-1)h^2(T - x_i + h_N)^{1-\alpha}$$

Especially, for  $i = N$ , the estimate of  $\lceil \frac{N}{2} \rceil + 1 \leq j \leq N-1$  is symmetric with  $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil$ .

*Proof.* Since (3.29), by  $x_i \simeq x_j$ , Lemma 3.1, Lemma 3.16, Lemma 3.17

For Case 1, we get the first result immediately. While  $x_k \simeq x_i \simeq x_{\min\{2i-1, N-1\}}$ , we have

$$\begin{aligned} \sum_{k+1}^{\min\{2i-1, N-1\}} |S_{ij}| &\leq Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_k}^{x_{\min\{2i-1, N-1\}}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &\leq Ch^2 x_i^{\alpha/2-2-2/r} x_i^{2-\alpha} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned}$$

For Case 2,

$$\begin{aligned} |S_{ij}| &\leq C(h_j^3 + (r-1)h_j^2) \int_0^1 |y_j^\theta - x_i|^{1-\alpha} d\theta \\ &= C(h_j^2 + (r-1)h_j) \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} dy \end{aligned}$$

however,

$$\begin{aligned} \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} dy &= \frac{1}{2-\alpha} ((x_j - x_i)^{2-\alpha} - (x_{j-1} - x_i)^{2-\alpha}) \\ &\simeq h_N (|x_j - x_i + h_N|)^{1-\alpha} \end{aligned}$$



For Case 3,

$$\begin{aligned} |S_{ij}| &\leq Ch_j^2 \int_0^1 (|y_j^\theta - x_i|^{1-\alpha} + (r-1)|y_j^\theta - x_i|^{-\alpha}) h_j d\theta \\ &\leq Ch^2 \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} |S_{ij}| &= Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil \frac{N}{2} \rceil}} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy \\ &\leq Ch^2 (T^{2-\alpha} + (r-1)(T - x_i + h_N)^{1-\alpha}) \end{aligned} \quad \square$$

Now we study  $I_2, I_4$ .

LEMMA 3.19. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

Case 1. For  $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil$ ,

$$I_2 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

Case 2. For  $3 \leq i < N/2$ ,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

Case 3. For  $N/2 \leq i \leq N, m = N - \lceil \frac{N}{2} \rceil + 1$ ,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,m} + T_{i-1,m-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,m} \right) \leq Ch^2$$

*Proof.* In fact,

$$\begin{aligned} &\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \\ &= \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + \left( \frac{1}{h_{i+1}} - \frac{1}{h_i} \right) T_{i,k} \end{aligned}$$

While, by Lemma A.3, Lemma B.3, Lemma 3.1 and  $x_k \simeq x_i$ , we have

$$\begin{aligned} \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) &= \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y)) D_h K_y(x_i) dy \\ &\leq Ch_k^2 x_k^{\alpha/2-2} h_k |x_i - x_k|^{-\alpha} \leq Ch^2 x_i^{-\alpha/2-2/r} h_k \end{aligned}$$

Thus,

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} |T_{i+1,k} - T_{i,k}| \leq Ch^2 x_i^{-\alpha/2-2/r}$$

From (3.14), Lemma A.2 and normalization, we have

(3.100)

$$\frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{Q_{i,k;3}^\theta(x_{i+1}) u''(\eta_{k+1}^\theta) - Q_{i,k;3}^\theta(x_i) u''(\eta_k^\theta)}{h_{i+1}} d\theta$$

where  $\eta_k^\theta \in (x_{k-1}, x_k)$  and  $\eta_{k+1}^\theta \in (x_k, x_{k+1})$ . And with Lemma 3.17, we can get

$$(3.101) \quad \frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} |T_{i+1,k+1} - T_{i,k}| \leq Ch^2 x_i^{-\alpha/2-2/r}$$

For the third term, by Lemma 3.1, Lemma B.1, Lemma A.3 and  $x_k \simeq x_i$ , we have

$$(3.102) \quad \begin{aligned} \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,k} &\leq h_i^{-3} h^2 x_i^{1-2/r} Ch_k^3 x_k^{\alpha/2-2} |x_k - x_i|^{1-\alpha} \\ &\leq Ch^2 x_i^{-\alpha/2-2/r} \end{aligned}$$

Summarizes, we have

$$(3.103) \quad I_2 \leq Ch^2 x_i^{-\alpha/2-2/r}$$

The case for  $I_4$  is similar. □

Now we have study evrr part to prove Theorem 3.3 and Theorem 3.4.

For  $1 \leq i < N/2$ , combine Lemma 3.14, Lemma 3.15, Lemma 3.19 Cases 1 2, Lemma 3.18 Case 1, Lemma 3.13 Case 1 2, we get Theorem 3.3.

For  $N/2 \leq i \leq N$ , we take  $m = 2N - \lceil \frac{N}{2} \rceil + 1$ . And depart  $I_3$  to three parts:

$$(3.104) \quad I_3 = \sum_{j=k+1}^m S_{ij} = \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{m-1} S_{ij}$$

combine Lemma 3.15, Lemma 3.19 Cases 1 3, Lemma 3.18, Lemma 3.13 Case 1 2, we get Theorem 3.4.

#### 4. Convergence analysis.

**4.1. Properties of some Matrices.** Review subsection 2.1, we have got (2.10).

**DEFINITION 4.1.** We call one matrix an  $M$  matrix, which means its entries are positive on major diagonal and nonpositive on others, and strictly diagonally dominant in rows.

Now we have

**LEMMA 4.2.** Matrix  $A$  defined by (2.12) where (2.13) is an  $M$  matrix. And there exists a constant  $C_A = C(T, \alpha, r)$  such that

$$(4.1) \quad S_i := \sum_{j=1}^{2N-1} a_{ij} \geq C_A (x_i^{-\alpha} + (2T - x_i)^{-\alpha})$$

*Proof.* From (2.15), we have

$$(4.2) \quad \sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left( \frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right)$$

Let

$$(4.3) \quad g(x) = g_0(x) + g_{2N}(x)$$

406 where

$$\begin{aligned}
 407 \quad g_0(x) &:= \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x-x_0|^{3-\alpha} - |x-x_1|^{3-\alpha}}{h_1} \\
 408 \quad g_{2N}(x) &:= \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N}-x|^{3-\alpha} - |x_{2N-1}-x|^{3-\alpha}}{h_{2N}}
 \end{aligned}$$

409 Thus

$$410 \quad -\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

411 Then

$$\begin{aligned}
 412 \quad (4.4) \quad S_i &:= \sum_{j=1}^{2N-1} a_{ij} \\
 &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\
 &= D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i)
 \end{aligned}$$

413 When  $i = 1$

$$\begin{aligned}
 414 \quad (4.5) \quad D_h^2 g_0(x_1) &= \frac{2}{h_1 + h_2} \left( \frac{1}{h_2} g_0(x_2) - \left( \frac{1}{h_1} + \frac{1}{h_2} \right) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right) \\
 &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1h_2} \\
 &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1^{1-\alpha}h_2} h_1^{-\alpha} \\
 &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha}}{2^r(2^r - 1)} h_1^{-\alpha}
 \end{aligned}$$

415 but

$$416 \quad (4.6) \quad 1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha} > 0$$

417 While for  $i \geq 2$

$$\begin{aligned}
 418 \quad (4.7) \quad D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \\
 &= -\kappa_\alpha \frac{|\xi - x_0|^{1-\alpha} - |\xi - x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1} \\
 &= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} |\xi - \eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\
 &\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha}
 \end{aligned}$$

419 So

$$420 \quad (4.8) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g_0(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g_0(x_i) + \frac{1}{h_i} g_0(x_{i-1}) \right) \geq C x_i^{-\alpha}$$

symmetricly,

(4.9)

□

$$\frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g_{2N}(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g_{2N}(x_i) + \frac{1}{h_i} g_{2N}(x_{i-1}) \right) \geq C(\alpha, r)(2T - x_i)^{-\alpha}$$

Let

$$(4.10) \quad G = \text{diag}(\delta(x_1), \dots, \delta(x_{2N-1}))$$

Then

LEMMA 4.3. *The matrix  $B := AG$ , the major diagonal is positive, and nonpositive on others. And there is a constant  $C_{AG}, C = C(\alpha, r)$  such that*

$$(4.11) \quad M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C_{AG}(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}) + C(T - \delta(x_i) + h_N)^{1-\alpha}$$

*Proof.*

$$b_{ij} = a_{ij}\delta(x_j) = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_i} \tilde{a}_{i-1,j} \right) \delta(x_j)$$

Since

$$(4.12) \quad \delta(x) \equiv \Pi_h \delta(x)$$

by (2.14) and (2.5), we have

$$\begin{aligned} \tilde{M}_i &:= \sum_{j=1}^{2N-1} \tilde{b}_{ij} := \sum_{j=1}^{2N-1} \tilde{a}_{ij} \delta(x_j) \\ &= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} \Pi_h \delta(y) dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} \delta(y) dy \\ &= \frac{-2}{\Gamma(4-\alpha)} |T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)} (x_i^{3-\alpha} + (2T - x_i)^{3-\alpha}) \\ &:= w(x_i) = p(x_i) + q(x_i) \end{aligned}$$

Thus,

$$\begin{aligned} (4.14) \quad M_i &:= \sum_{j=1}^{2N-1} b_{ij} = \sum_{j=1}^{2N-1} a_{ij} \delta(x_j) \\ &= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} \tilde{M}_{i+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{M}_i + \frac{1}{h_i} \tilde{M}_{i-1} \right) \\ &= D_h^2(-\kappa_\alpha p)(x_i) - \kappa_\alpha D_h^2 q(x_i) \end{aligned}$$

for  $1 \leq i \leq N-1$ , by Lemma A.1

(4.15)

$$\begin{aligned} D_h^2(-\kappa_\alpha p)(x_i) &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} p(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) p(x_i) + \frac{1}{h_i} p(x_{i-1}) \right) \\ &= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1}) \\ &\geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} (T - \delta(x_i) + h_N)^{1-\alpha} \end{aligned}$$

$$\begin{aligned}
(4.16) \quad D_h^2(-\kappa_\alpha p)(x_N) &:= -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left( \frac{1}{h_{N+1}} p(x_{N+1}) - \left( \frac{1}{h_N} + \frac{1}{h_{N+1}} \right) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right) \\
&= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)h_N^2} h_N^{3-\alpha} = \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - \delta(x_N) + h_N)^{1-\alpha}
\end{aligned}$$

Symmetricly for  $i \geq N$ , we get

$$(4.17) \quad D_h^2(-\kappa_\alpha p)(x_i) \geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} (T - \delta(x_i) + h_N)^{1-\alpha}$$

Similarly, we can get

$$\begin{aligned}
(4.18) \quad D_h^2 q(x_i) &:= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} q(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right) \\
&\leq \frac{2^{r(\alpha-1)+1}}{\Gamma(2-\alpha)} (x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}), \quad i = 1, \dots, 2N-1
\end{aligned}$$

So, we get the result.

Notice that

$$(4.19) \quad x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha}$$

We can get

**THEOREM 4.4.** *There exists a real  $\lambda = \lambda(T, \alpha, r) > 0$  and  $C = C(T, \alpha, r) > 0$  such that  $B := A(\lambda I + G)$  is an  $M$  matrix. And*

$$(4.20) \quad M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) + C(T - \delta(x_i) + h_N)^{1-\alpha}$$

*Proof.* By Lemma 4.2 with  $C_A$  and Lemma 4.3 with  $C_{AG}$ , it's sufficient to take  $\lambda = (C + 2TC_{AG})/C_A$ , then

$$(4.21) \quad M_i \geq C((x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + (T - \delta(x_i) + h_N)^{1-\alpha})$$

**4.2. Proof of Theorem 2.7.** For equation

$$(4.22) \quad AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1}U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1}U = F$$

which means

$$(4.23) \quad \sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + \delta(x_j)} = -\tau_i$$

where  $\epsilon_i = u(x_i) - u_i$ .

And if

$$(4.24) \quad \left| \frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})} \right| = \max_{1 \leq i \leq 2N-1} \left| \frac{\epsilon_i}{\lambda + \delta(x_i)} \right|$$

Then, since  $B = A(\lambda I + G)$  is an  $M$  matrix, it is Strictly diagonally dominant. Thus,

$$\begin{aligned}
 |\tau_{i_0}| &= \left| \sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + \delta(x_j)} \right| \\
 &\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_j}{\lambda + \delta(x_j)} \right| \\
 &\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})} \right| \\
 &= \sum_{j=1}^{2N-1} b_{i_0,j} \left| \frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})} \right| \\
 &= M_{i_0} \left| \frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})} \right|
 \end{aligned}
 \tag{4.25}$$

By Theorem 2.6 and Theorem 4.4,

We know that there exists constants  $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ ,  
and  $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that

$$\left| \frac{\epsilon_i}{\lambda + \delta(x_i)} \right| \leq \left| \frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})} \right| \leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} + C_2(r-1)h^2
 \tag{4.26}$$

as  $\lambda + \delta(x_i) \leq \lambda + T$

So, we can get

$$|\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{r\alpha}{2}, 2\}}
 \tag{4.27}$$

The convergency has been proved.

*Remark 4.5* (Weaker regularity on the derivatives of  $u$ ). Suppose that the bound of Lemma 2.4 is replaced by the more general weaker regularity bound

$$|u^{(l)}(x)| \leq C\delta(x)^{\sigma-l}, \quad l = 0, 1, 2, 3, 4$$

where  $\sigma \in (0, \frac{\alpha}{2}]$  is fixed. Then

$$I^{2-\alpha}u(x) = \int_0^{x/2} + \int_{x/2}^{T+x/2} + \int_{T+x/2}^{2T} u(y)K_y(x)dx$$

and for  $l = 1, 2, 3, 4$ , we have

$$\begin{aligned}
 \frac{d^l}{dx^l} I^{2-\alpha}u(x) &= \int_0^{x/2} + \int_{T+x/2}^{2T} u(y)K_y^{(l)}(x)dy \\
 &+ \sum_{k=0}^{l-1} u^{(k)}\left(\frac{x}{2}\right)K_{x/2}^{(l-1-k)}(x) - u^{(k)}\left(T + \frac{x}{2}\right)K_{T+x/2}^{(l-1-k)}(x) \\
 &+ \int_{x/2}^{T+x/2} u^{(l)}(y)K_y(x)dy
 \end{aligned}$$

Thus, we can get

$$|f^l(x)| \leq C\delta(x)^{\sigma-\alpha-l}, \quad l = 0, 1, 2.$$

480 Examine the proof above, by replacing the regularity condition with the weaker one,  
 481 we can get the similar results:

$$\begin{aligned}
 482 \quad (4.28) \quad |\tau_i| &= |-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i)| \\
 &\leq Ch^{\min\{r\sigma, 2\}} \delta(x_i)^{-\alpha} + C(r-1)h^2(T - \delta(x_i) + h_N)^{1-\alpha}.
 \end{aligned}$$

483 And the convergence result of Theorem 2.7 is changed to

$$484 \quad (4.29) \quad \max_{1 \leq i \leq 2N-1} |u_i - u(x_i)| \leq Ch^{\min\{r\sigma, 2\}}.$$

485

## 5. Experimental results.

### 5.1. $f \equiv 1$ .

### 5.2. $f = x^\gamma, \gamma < 0$ . Appendix A. Approximate of difference quotients.

LEMMA A.1. If  $g(x) \in C^2(\Omega)$ , there exists  $\xi \in (x_{i-1}, x_{i+1})$  such that

$$(A.1) \quad D_h^2 g(x_i) = g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1})$$

And if  $g(x) \in C^4(\Omega)$ , then

$$(A.2) \quad D_h^2 g(x_i) = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right)$$

*Proof.*

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})$$

Substitute them in the left side of (A.1), we have

$$D_h^2 g(x_i) = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (g(x_{i+1}) - g(x_i) + \frac{1}{h_i} (g(x_{i-1}) - g(x_i))) \right) \\ = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2)$$

Now, using [intermediate value theorem](#), there exists  $\xi \in [\xi_1, \xi_2]$  such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy$$

Especially,

$$(A.3) \quad \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy = \frac{h_i^4}{4!} g''''(\eta_1) \\ \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy = \frac{h_{i+1}^4}{4!} g''''(\eta_2)$$

where  $\eta_1 \in (x_{i-1}, x_i), \eta_2 \in (x_i, x_{i+1})$ .  $\square$



LEMMA A.2. Denote  $y_j^\theta = (1 - \theta)x_{j-1} + \theta x_j, \theta \in (0, 1)$ ,

$$(A.4) \quad u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in (x_{j-1}, x_j)$$

$$(A.5) \quad u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

where  $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$ .

*Proof.* By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in (x_{j-1}, y_j^\theta)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in (y_j^\theta, x_j)$$

Thus

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta)u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$\begin{aligned} u(x_{j-1}) &= u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1) \\ u(x_j) &= u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2) \end{aligned}$$

where  $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$ . Thus □

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

LEMMA A.3. By Lemma A.2, Lemma 2.4 and Lemma 3.1, There is a constant

$C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  for  $2 \leq j \leq 2N-1$ ,

$$(A.6) \quad |u(y) - \Pi_h u(y)| \leq h_j^2 \max_{\xi \in [x_{j-1}, x_j]} |u''(\xi)| \leq C h^2 \delta(y)^{\alpha/2-2/r}, \quad \text{for } y \in (x_{j-1}, x_j)$$

LEMMA A.4. For  $x \in [x_{j-1}, x_j]$

$$\begin{aligned} |u(x) - \Pi_h u(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^x u'(y) dy - \frac{x - x_{j-1}}{h_j} \int_x^{x_j} u'(y) dy \right| \\ &\leq \int_{x_{j-1}}^{x_j} |u'(y)| dy \end{aligned}$$

If  $x \in [0, x_1]$ , with Lemma 2.4, we have

$$(A.8) \quad |u(x) - \Pi_h u(x)| \leq \int_0^{x_1} |u'(y)| dy \leq \int_0^{x_1} C y^{\alpha/2-1} dy \leq C \frac{2}{\alpha} x_1^{\alpha/2} = C \frac{2}{\alpha} h_1^{\alpha/2}$$

Similarly, if  $x \in [x_{2N-1}, 1]$ , we have

$$(A.9) \quad |u(x) - \Pi_h u(x)| \leq C \frac{2}{\alpha} (2T - x_{2N-1})^{\alpha/2} = C \frac{2}{\alpha} h_{2N}^{\alpha/2}$$

LEMMA A.5.

$$(A.10) \quad b^{1-\theta}|a^\theta - b^\theta| \leq |a - b| \quad (\text{also } a^{1-\theta}|a^\theta - b^\theta| \leq |a - b|), \quad a, b \geq 0, \theta \in [0, 1]$$

**Appendix B. Proofs of some technical details.** Review that  $h = \frac{1}{N}$  and the definition of  $\simeq$  in subsection 2.1

LEMMA B.1. *There is a constant  $C$  such that for  $i = 1, 2, \dots, 2N - 1$*

$$(B.1) \quad |h_{i+1} - h_i| \leq Ch^2 \delta(x_i)^{1-2/r}$$

*Proof.* By (2.2),  
(B.2)

$$h_{i+1} - h_i = \begin{cases} T \left( \left( \frac{i+1}{N} \right)^r - 2 \left( \frac{i}{N} \right)^r + \left( \frac{i-1}{N} \right)^r \right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T \left( \left( \frac{2N-i-1}{N} \right)^r - 2 \left( \frac{2N-i}{N} \right)^r + \left( \frac{2N-i+1}{N} \right)^r \right), & N+1 \leq i \leq 2N-1 \end{cases}$$

Since

$$(B.3) \quad (i+1)^r - 2i^r + (i-1)^r \simeq r(r-1)i^{r-2}, \quad \text{for } i \geq 1$$

We get the result.  $\square$

LEMMA B.2. *there is a constant  $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$  such that*

$$(B.4) \quad \frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right| \leq Ch^2 \delta(x_i)^{-\alpha/2-2/r}$$

*Proof.* By Lemma 2.5, we have for  $1 \leq i \leq N$

$$(B.5) \quad \left| \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy \right| \leq \frac{\|f\|_\beta^{(\alpha/2)}}{3!} \int_{x_{i-1}}^{x_i} y^{-\alpha/2-2} (y - x_{i-1})^3 dy$$

For  $i = 1$ ,

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2} (y - x_{i-1})^3 dy = \int_0^{x_1} y^{1-\alpha/2} dy = \frac{1}{2-\alpha/2} x_1^{2-\alpha/2} = \frac{1}{2-\alpha/2} x_1^{-\alpha/2-2} h_1^4$$

And for  $2 \leq i \leq N$ , since  $x_i \simeq x_{i-1} \leq y \leq x_i$ , we have

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2} (y - x_{i-1})^3 dy \simeq \int_{x_{i-1}}^{x_i} x_i^{-\alpha/2-2} (y - x_{i-1})^3 dy = \frac{1}{4!} x_i^{-\alpha/2-2} h_i^4$$

So for  $1 \leq i \leq N$ , we have

$$(B.6) \quad \left| \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy \right| \leq C x_i^{-\alpha/2-2} h_i^4$$

and similarly,

$$(B.7) \quad \left| \int_{x_i}^{x_{i+1}} f''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right| \leq C x_i^{-\alpha/2-2} h_{i+1}^4$$

Thus for  $1 \leq i \leq N$ , with Lemma 3.1 we have

$$(B.8) \quad \begin{aligned} & \frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right| \\ & \leq C x_i^{-\alpha/2-2} \frac{2}{h_i + h_{i+1}} (h_i^3 + h_{i+1}^3) \simeq x_i^{-\alpha/2-2} h_i^2 \simeq x_i^{-\alpha/2-2} h^2 x_i^{2-2/r} \\ & = C h^2 x_i^{-\alpha/2-2/r} \end{aligned}$$

It's symmetric for  $N < i \leq 2N - 1$ .  $\square$

LEMMA B.3. *There is a constant  $C = C(\alpha, r)$  such that for all  $1 \leq i \leq 2N - 1$ ,  $1 \leq j \leq 2N$  s.t.  $\min\{|j - i|, |j - 1 - i|\} \geq 2$  and  $y \in [x_{j-1}, x_j]$ , we have*

$$(B.9) \quad D_h K_y(x_i) \simeq |y - x_i|^{-\alpha}, \quad D_h^2 K_y(x_i) \simeq |y - x_i|^{-1-\alpha}$$

*Proof.* Sinec  $y - x_{i-1}, y - x_i, y - x_{i+1}$  have the same sign, by mean value theorem and Lemma A.1,

$$(B.10) \quad \begin{aligned} D_h K_y(x_i) &= \frac{|y - \xi|^{-\alpha}}{\Gamma(1 - \alpha)}, \quad \xi \in (x_i, x_{i+1}) \\ D_h^2 K_y(x_i) &= \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in (x_{i-1}, x_{i+1}) \end{aligned}$$

however,  $|y - \xi| \simeq |y - x_i|$ , we get the result.  $\square$

LEMMA B.4. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

$$(B.10) \quad \sum_{j=1}^3 V_{1j} \leq C h^2 x_1^{-\alpha/2-2/r}$$

$$(B.11) \quad \sum_{j=1}^4 V_{2j} \leq C h^2 x_2^{-\alpha/2-2/r}$$

*Proof.* For  $0 \leq i \leq 3, 1 \leq j \leq 4$ , by Lemma A.4, Lemma A.3 and (3.14)

$$(B.12) \quad T_{ij} \leq C x_1^{2-\alpha/2} \simeq h_1^2 h^2 x_1^{-\alpha/2-2/r} \simeq h_1^2 h^2 x_2^{-\alpha/2-2/r}$$

Therefore, by (3.15), we get the result.  $\square$

**Acknowledgments.** We would like to acknowledge the assistance of volunteers in putting together this example manuscript and supplement.

## REFERENCES

- [1] D. GILBARG AND N. S. TRUDINGER, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften, Vol. 224, Springer-Verlag, Berlin-New York, 1977.
- [2] X. ROS-OTON AND J. SERRA, *The dirichlet problem for the fractional laplacian: Regularity up to the boundary*, Journal de Mathématiques Pures et Appliquées, 101 (2014), pp. 275–302, <https://doi.org/https://doi.org/10.1016/j.matpur.2013.06.003>, <https://www.sciencedirect.com/science/article/pii/S0021782413000895>.