

# A SECOND ORDER NUMERICAL METHODS FOR REISZ-FRACTIONAL ELLIPTIC EQUATION ON GRADED MESH\*

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**Abstract.** This is an example SIAM L<sup>A</sup>T<sub>E</sub>X article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

**Key words.** example, L<sup>A</sup>T<sub>E</sub>X

**MSC codes.** ?????????????????

**1. Introduction.** For  $\Omega = (0, 2T)$ ,  $1 < \alpha < 2$ ,

$$(1.1) \quad \begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases}$$

where

$$(1.2) \quad (-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy$$

$$(1.3) \quad \kappa_\alpha = -\frac{1}{2 \cos(\alpha\pi/2)} > 0$$

**2. Preliminaries: Numeric scheme and main results.**

**2.1. Numeric Format.**

$$(2.1) \quad x_i = \begin{cases} T \left( \frac{i}{N} \right)^r, & 0 \leq i \leq N \\ 2T - T \left( \frac{2N-i}{N} \right)^r, & N \leq i \leq 2N \end{cases}$$

where  $r \geq 1$ . And let

$$(2.2) \quad h_j = x_j - x_{j-1}, \quad 1 \leq j \leq 2N$$

Let  $\{\phi_j(x)\}_{j=1}^{2N-1}$  be standard hat functions, which are basis of the piecewise linear function space.

$$(2.3) \quad \phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \leq x \leq x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \leq x \leq x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

And then, define the piecewise linear interpolant of the true solution  $u$  to be

$$(2.4) \quad \Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j) \phi_j(x)$$

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For convience, we denote

$$(2.5) \quad I^{2-\alpha}u(x) := \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x-y|^{1-\alpha} u(y) dy$$

and

$$(2.6) \quad D_h^2 u(x_i) := \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} u(x_{i-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) u(x_i) + \frac{1}{h_{i+1}} u(x_{i+1}) \right)$$

Now, we discretise (1.1) by replacing  $u(x)$  by a continuous piecewise linear function

$$(2.7) \quad u_h(x) := \sum_{j=1}^{2N-1} u_j \phi_j(x)$$

whose nodal values  $u_j$  are to be determined by collocation at each mesh point  $x_i$  for  $i = 1, 2, \dots, 2N-1$ :

$$(2.8) \quad -\kappa_{\alpha} D_h^{\alpha} u_h(x_i) := -\kappa_{\alpha} D_h^2 I^{2-\alpha} u_h(x_i) = f(x_i) =: f_i$$

Here,

$$(2.9) \quad -\kappa_{\alpha} D_h^{\alpha} u_h(x_i) = \sum_{j=1}^{2N-1} -\kappa_{\alpha} D_h^2 I^{2-\alpha} \phi_j(x_i) u_j = \sum_{j=1}^{2N-1} a_{ij} u_j$$

where

$$(2.10) \quad a_{ij} = -\kappa_{\alpha} D_h^2 I^{2-\alpha} \phi_j(x_i) \quad \text{for } i, j = 1, 2, \dots, 2N-1$$

We have replaced  $(-\Delta)^{\alpha/2} u(x_i) = f(x_i)$  in (1.1) by  $-\kappa_{\alpha} D_h^{\alpha} u_h(x_i) = f(x_i)$  in (2.8), with truncation error

$$(2.11) \quad \tau_i := -\kappa_{\alpha} \left( D_h^{\alpha} \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i) \right) \quad \text{for } i = 1, 2, \dots, 2N-1$$

where  $-\kappa_{\alpha} D_h^{\alpha} \Pi_h u(x_i) = \sum_{j=1}^{2N-1} -\kappa_{\alpha} D_h^{\alpha} \phi_j(x_i) u(x_j) = \sum_{j=1}^{2N-1} a_{ij} u(x_j)$ .

The discrete equation (2.8) can be written in matrix form

$$(2.12) \quad AU = F$$

where  $A = (a_{ij}) \in \mathbb{R}^{(2N-1) \times (2N-1)}$ ,  $U = (u_1, \dots, u_{2N-1})^T$  is unknown and  $F = (f_1, \dots, f_{2N-1})^T$ .

We can deduce  $a_{ij}$ ,

$$(2.13) \quad \begin{aligned} a_{ij} &= -\kappa_{\alpha} D_h^2 I^{2-\alpha} \phi_j(x_i) \\ &= -\kappa_{\alpha} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \tilde{a}_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \end{aligned}$$

where

$$(2.14) \quad \begin{aligned} \tilde{a}_{ij} &= I^{2-\alpha} \phi_j(x_i) \\ &= \frac{1}{\Gamma(4-\alpha)} \left( \frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \end{aligned}$$

We shall finally introduce some notations.

For convenience, we use the notation  $\simeq$ . That  $x_1 \simeq y_1$ , means that  $c_1 x_1 \leq y_1 \leq C_1 x_1$  for some constants  $c_1$  and  $C_1$  that are independent of  $N$ .

Meanwhile, let's define kernel functions

$$(2.15) \quad K_y(x) := \frac{|y-x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

We define the difference quotients

$$(2.16) \quad D_h g(x_i) := \frac{g(x_{i+1}) - g(x_i)}{h_{i+1}}, \quad D_{\bar{h}} g(x_i) := \frac{g(x_i) - g(x_{i-1})}{h_i}$$

Thus

$$D_h g(x_i) = D_{\bar{h}} g(x_{i+1})$$

$$D_h^2 g(x_i) = \frac{2}{h_i + h_{i+1}} (D_h g(x_i) - D_{\bar{h}} g(x_i)) = \frac{2}{h_i + h_{i+1}} (D_h g(x_i) - D_h g(x_{i-1}))$$

And for  $j = 1, 2, \dots, 2N$ , we define

$$(2.17) \quad y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \quad \theta \in (0, 1)$$

**2.2. Regularity of the true solution.** For any  $\beta > 0$ , we use the standard notation  $C^\beta(\bar{\Omega})$ ,  $C^\beta(\mathbb{R})$ , etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation  $C^\beta(\Omega)$  to refer to  $C^{k,\beta'}(\Omega)$ , where  $k$  is the greatest integer such that  $k < \beta$  and where  $\beta' = \beta - k$ . The Hölder spaces  $C^{k,\beta'}(\Omega)$  are defined as the subspaces of  $C^k(\Omega)$  consisting of functions whose  $k$ -th order partial derivatives are locally Hölder continuous[1] with exponent  $\beta'$  in  $\Omega$ , where  $C^k(\Omega)$  is the set of all  $k$ -times continuously differentiable functions on open set  $\Omega$ .

**DEFINITION 2.1** (delta dependent norm [2]). ...

**LEMMA 2.2.** Let  $f \in C^\beta(\Omega)$ ,  $\beta > 2$  be such that  $\|f\|_\beta^{(\alpha/2)} < \infty$ , then for  $l = 0, 1, 2$

$$(2.18) \quad |f^{(l)}(x)| \leq \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \leq T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \leq x < 2T \end{cases}$$

**THEOREM 2.3** (Regularity up to the boundary [2]). Let  $\Omega$  be a bounded domain, and  $\beta > 0$  be such that neither  $\beta$  nor  $\beta + \alpha$  is an integer. Let  $f \in C^\beta(\Omega)$  be such that  $\|f\|_\beta^{(\alpha/2)} < \infty$ , and  $u \in C^{\alpha/2}(\mathbb{R}^n)$  be a solution of (1.1). Then,  $u \in C^{\beta+\alpha}(\Omega)$  and

$$(2.19) \quad \|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C \left( \|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right)$$

where  $C$  is a constant depending only on  $\Omega$ ,  $\alpha$ , and  $\beta$ .

**COROLLARY 2.4.** Let  $u$  be a solution of (1.1) where  $f \in L^\infty(\Omega)$  and  $\|f\|_\beta^{(\alpha/2)} < \infty$ . Then, for any  $x \in \Omega$  and  $l = 0, 1, 2, 3, 4$

$$(2.20) \quad |u^{(l)}(x)| \leq \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases}$$

And in this paper bellow, without special instructions, we allways assume that

$$(2.21) \quad f \in L^\infty(\Omega) \cap C^\beta(\Omega) \quad \text{and} \quad \|f\|_\beta^{(\alpha/2)} < \infty, \quad \text{with } \alpha + \beta > 4$$

**2.3. Main results.** Here we state our main results; the proof is deferred to section 3 and section 4.

Let's denote  $h = \frac{1}{N}$ , we have

**THEOREM 2.5 (Local Truncation Error).** *If  $u(x)$  is a solution of the equation (1.1) where  $f$  satisfy the regular condition (2.21), then there exists  $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$  and  $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ , such that the truncation error (2.11) satisfies*

$$\begin{aligned} |\tau_i| &:= |-\kappa_{\alpha} D_h^{\alpha} \Pi_h u(x_i) - f(x_i)| \\ &\leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} \begin{cases} x_i^{-\alpha}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha}, & N < i \leq 2N - 1 \end{cases} \\ &\quad + C_2(r-1)h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \leq i \leq N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \leq 2N - 1 \end{cases} \end{aligned} \quad (2.22)$$

**THEOREM 2.6 (Global Error).** *The discrete equation (2.8) has solution and there exists a positive constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$  such that the error between the numerical solution  $U$  with the exact solution  $u(x_i)$  satisfies*

$$(2.23) \quad \max_{1 \leq i \leq 2N-1} |u_i - u(x_i)| \leq Ch^{\min\{\frac{r\alpha}{2}, 2\}}$$

That means the numerical method has convergence order  $\min\{\frac{r\alpha}{2}, 2\}$ .

**Remark 2.7.** ...

### 3. Local Truncation Error.

**3.1. Proof of Theorem 2.5.** The truncation error of the discrete format can be written as

$$\begin{aligned} -\kappa_{\alpha} D_h^{\alpha} \Pi_h u(x_i) - f(x_i) &= -\kappa_{\alpha} (D_h^2 I^{2-\alpha} \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i)) \\ &= -\kappa_{\alpha} D_h^2 I^{2-\alpha} (\Pi_h u - u)(x_i) - \kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) \end{aligned}$$

**THEOREM 3.1.** *There exists a constant  $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$  such that*

$$(3.2) \quad \left| -\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) \right| \leq Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N - 1 \end{cases}$$

*Proof.* Since  $f \in C^2(\Omega)$  and

$$(3.3) \quad \frac{d^2}{dx^2} (-\kappa_{\alpha} I^{2-\alpha} u(x)) = f(x), \quad x \in \Omega,$$

we have  $I^{2-\alpha} u \in C^4(\Omega)$ . Therefore, using equation (A.2) of Lemma A.1, for  $1 \leq i \leq$

111  $2N - 1$ , we have  
 (3.4)

$$112 \quad -\kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}u(x_i) = \frac{h_{i+1} - h_i}{3}f'(x_i) \\ + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right)$$

113 By Lemma B.2, Lemma 2.2 and Lemma B.3, we get the result.  $\square$

114 And now define

$$115 \quad (3.5) \quad R_i := D_h^2 I^{2-\alpha}(u - \Pi_h u)(x_i), \quad 1 \leq i \leq 2N - 1$$

116 We have some results about the estimate of  $R_i$

117 **THEOREM 3.2.** *For  $1 \leq i < N/2$ , there exists  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

$$118 \quad (3.6) \quad |R_i| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2(x_i^{-1-\alpha} \ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

119  
 120 **THEOREM 3.3.** *For  $N/2 \leq i \leq N$ , there exists constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$*   
 121 *such that*

$$122 \quad (3.7) \quad |R_i| \leq C(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

123 And for  $N < i \leq 2N - 1$ , it is symmetric to the previous case.

124 Combine Theorem 3.1, Theorem 3.2 and Theorem 3.3, and for  $1 \leq i \leq N$ , we  
 125 have

$$126 \quad (3.8) \quad h^2 x_i^{-\alpha/2-2/r} \leq T^{\alpha/2-2/r} h^{\min\{\frac{r\alpha}{2}, 2\}} x_i^{-\alpha}$$

$$127 \quad (3.9) \quad h^{r\alpha/2+r} x_i^{-1-\alpha} \leq T^{-1} h^{r\alpha/2} x_i^{-\alpha}$$

$$128 \quad (3.10) \quad h^r x_i^{-1} \ln(i) = T^{-1} \frac{\ln(i)}{i^r} \leq T^{-1}, \quad h^r \ln(N) = \frac{\ln(N)}{N^r} \leq 1$$

129 the proof of Theorem 2.5 completed.

130 We prove Theorem 3.2 and Theorem 3.3 in next subsections.

131 **3.2. Outlines and Mesh Transport Functions.** For convience, let's denote

DEFINITION 3.4.

$$132 \quad (3.11) \quad T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy, \quad i = 0, \dots, 2N, \quad j = 1, \dots, 2N$$

133 Also, we denote vertical difference quotients of  $T_{ij}$

$$134 \quad (3.12) \quad V_{ij} = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} T_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right)$$

135 And skew difference quotients of  $T_{ij}$

$$136 \quad (3.13) \quad S_{ij} = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} T_{i-1,j-1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j+1} \right)$$

137 then  $R_i = \sum_{j=1}^{2N} V_{ij}$ .

138 Our main idea is to depart  $R_i$  by  $V_{ij}$  and  $S_{ij}$ . For  $3 \leq i < N/2$ , let's denote

139  $k = \lceil \frac{i}{2} \rceil$ , and take some suitable integer  $m$ , then

$$\begin{aligned}
 R_i &= \sum_{j=1}^{2N} V_{ij} \\
 &= \sum_{j=1}^{k-1} V_{ij} + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \\
 140 \quad (3.14) \quad &+ \sum_{j=k+1}^{m-1} S_{ij} + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,m} + T_{i-1,m-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,m} \right) \\
 &+ \sum_{j=m+1}^{2N} V_{ij} \\
 &= I_1 + I_2 + I_3 + I_4 + I_5
 \end{aligned}$$

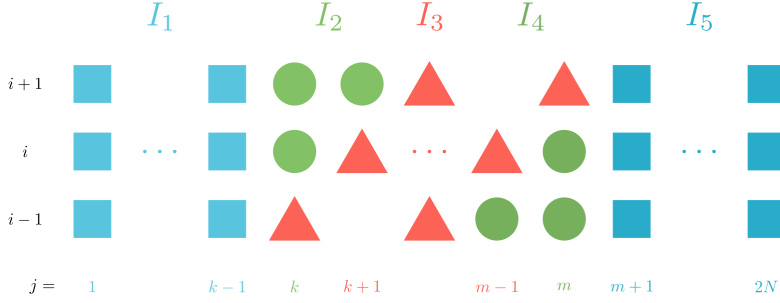


FIG. 1. The departure of  $R_i$  for  $i \geq 3$

141 and discuss  $i = 1, 2$  separately, where

$$142 \quad (3.15) \quad R_1 = \sum_{i=1}^3 V_{1,j} + \sum_{i=4}^N V_{i,j}, \quad R_2 = \sum_{i=1}^4 V_{1,j} + \sum_{i=5}^N V_{i,j}$$

143 The difficulty for estimating  $S_{ij}$  is that  $T_{i-1,j-1}, T_{i,j}$  and  $T_{i+1,j+1}$  have different  
 144 integral region. We first make them normalized.

145 LEMMA 3.5. For  $y \in (x_{j-1}, x_j)$ , we can rewrite  $y = y_j^\theta$ , from (3.11), and Lemma A.2, ■

$$\begin{aligned}
 T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
 &= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\
 146 \quad (3.16) \quad &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
 &\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta
 \end{aligned}$$

147 where  $\eta_{j1}^\theta \in (x_{j-1}, y_j^\theta), \eta_{j2}^\theta \in (y_j^\theta, x_j)$ .

Since  $j$  changes with  $i$  at indices of elements in  $S_{ij}$  by (3.13), we create some functions satisfy the property.

**DEFINITION 3.6** (Mesh Transport Functions). *For  $1 \leq i, j \leq 2N - 1$ .*

$$(3.17) \quad y_{i,j}(x) = \begin{cases} (x^{1/r} + Z_{j-i})^r & i < N, j < N \\ \frac{x^{1/r} - Z_i}{Z_1} h_N + x_N & i < N, j = N \\ 2T - (Z_{2N-(j-i)} - x^{1/r})^r & i < N, j > N \\ \left( \frac{Z_1}{h_N} (x - x_N) + Z_j \right)^r & i = N, j < N \\ x, & i = N, j = N \\ 2T - \left( \frac{Z_1}{h_N} (2T - x - x_N) + Z_{2N-j} \right)^r & i = N, j > N \\ (Z_{2N+j-i} - (2T - x)^{1/r})^r & i > N, j < N \\ \frac{Z_{2N-j} - (2T - x)^{1/r}}{Z_1} h_N + x_N & i > N, j = N \\ 2T - ((2T - x)^{1/r} - Z_{j-i})^r & i > N, j > N \end{cases}$$

where  $Z_j := T^{1/r} \frac{j}{N}$ ,  $x \in [x_{i-1}, x_{i+1}]$ . And

$$(3.18) \quad h_{i,j}(x) = y_{i,j}(x) - y_{i,j-1}(x)$$

$$(3.19) \quad y_{i,j}^\theta(x) = (1 - \theta)y_{i,j-1}(x) + \theta y_{i,j}(x), \quad \theta \in (0, 1)$$

$$(3.20) \quad P_{i,j}^\theta(x) = (h_{i,j}(x))^3 \frac{|y_{i,j}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y_{i,j}^\theta(x))$$

$$(3.21) \quad Q_{i,j}^\theta(x) = (h_{i,j}(x))^4 \frac{|y_{i,j}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

Obviously,

$$(3.22) \quad y_{i,j}(x_{i-1}) = x_{j-1}, \quad y_{i,j}(x_i) = x_j, \quad y_{i,j}(x_{i+1}) = x_{j+1}$$

$$(3.23) \quad h_{i,j}(x_{i-1}) = h_{j-1}, \quad h_{i,j}(x_i) = h_j, \quad h_{i,j}(x_{i+1}) = h_{j+1}$$

$$(3.24) \quad y_{i,j}^\theta(x_{i-1}) = y_{j-1}^\theta, \quad y_{i,j}^\theta(x_i) = y_j^\theta, \quad y_{i,j}^\theta(x_{i+1}) = y_{j+1}^\theta$$

And now we can rewrite  $T_{ij}$

**LEMMA 3.7.** *For  $0 \leq i \leq 2N, 1 \leq j \leq 2N$ ,*

$$(3.25) \quad T_{ij} = \int_0^1 -\frac{\theta(1-\theta)}{2} P_{i,j}^\theta(x_i) d\theta + \int_0^1 \frac{\theta(1-\theta)}{3!} Q_{i,j}^\theta(x_i) [\theta^2 u'''(\eta_{j,1}^\theta) - (1-\theta)^2 u'''(\eta_{j,2}^\theta)] d\theta$$

168 *Immediately, we can see from (3.13) that For  $1 \leq i \leq 2N-1, 2 \leq j \leq 2N-1$ ,*  
 (3.26)

$$\begin{aligned}
 S_{ij} = & \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{i,j}^\theta(x_i) d\theta \\
 & + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i,j}^\theta(x_{i+1}) u'''(\eta_{j+1,1}^\theta) - Q_{i,j}^\theta(x_i) u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\
 169 & - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i,j}^\theta(x_i) u'''(\eta_{j,1}^\theta) - Q_{i,j}^\theta(x_{i-1}) u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\
 & - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i,j}^\theta(x_{i+1}) u'''(\eta_{j+1,2}^\theta) - Q_{i,j}^\theta(x_i) u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\
 & + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i,j}^\theta(x_i) u'''(\eta_{j,2}^\theta) - Q_{i,j}^\theta(x_{i-1}) u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta
 \end{aligned}$$

170 We give some properties of mesh transport functions.

171 LEMMA 3.8. *For  $1 \leq i \leq 2N-1, 2 \leq j \leq 2N-1$  and  $\xi \in (x_{i-1}, x_{i+1})$*

$$172 \quad (3.27) \quad y_{i,j}(\xi) \simeq x_j, \quad h_{i,j}(\xi) \simeq h_j$$

173 *For  $1 \leq i, j \leq 2N-1$ ,*

$$174 \quad (3.28) \quad |y_{i,j}(\xi) - \xi| \simeq |x_j - x_i|$$

175 *And for  $1 \leq i \leq 2N-1, 2 \leq j \leq 2N-1$ ,*

$$176 \quad (3.29) \quad |y_{i,j}^\theta(\xi) - \xi| = (1-\theta)|y_{i,j-1}(\xi) - \xi| + \theta|y_{i,j}(\xi) - \xi| \simeq |y_j^\theta - x_i|$$

177 *since  $y_{i,j-1}(\xi) - \xi, y_{i,j}(\xi) - \xi$  have the same sign ( $\geq 0$  or  $\leq 0$ )*

LEMMA 3.9.

$$178 \quad (3.30) \quad y'_{i,j}(x) = \begin{cases} y_{i,j}^{1-1/r}(x) x^{1/r-1} & i < N, j < N \\ \frac{h_N}{rZ_1} x^{1/r-1} & i < N, j = N \\ (2T - y_{i,j}^{1-1/r}(x)) x^{1/r-1} & i < N, j > N \\ y_{i,j}^{1-1/r}(x) \frac{rZ_1}{h_N} & i = N, j < N \\ 1 & i = N, j = N \end{cases}$$

179

$$180 \quad (3.31) \quad y''_{i,j}(x) = \frac{1-r}{r} \begin{cases} y_{i,j}^{1-2/r}(x) x^{1/r-2} Z_{i-j} & i < N, j < N \\ \frac{h_N}{rZ_1} x^{1/r-2} & i < N, j = N \\ (2T - y_{i,j}^{1-2/r}(x)) x^{1/r-2} Z_{2N-j+i} & i < N, j > N \\ -y_{i,j}^{1-2/r}(x) \left( \frac{rZ_1}{h_N} \right)^2 & i = N, j < N \\ 0 & i = N, j = N \end{cases}$$



LEMMA 3.10. For  $2 \leq i \leq N, 2 \leq j \leq 2N - 1$

$$(3.32) \quad |h'_{i,j}(\xi)| \leq C(r-1)hx_i^{1/r-1} \begin{cases} x_j^{1-2/r} & j \leq N \\ (2T-x_j)^{1-2/r} & j > N \end{cases}$$

And

$$(3.33) \quad |(y_{i,j}(\xi) - \xi)'| \leq C(r-1)\xi^{-1} \begin{cases} |y_{i,j}(\xi) - \xi| & j \leq N \\ |2T - y_{i,j}(\xi) - \xi| & j > N \end{cases}$$

**3.3. Proof of Theorem 3.2.** Then we estimate each part of (3.14) from easy to hard.

For  $I_5$

LEMMA 3.11. There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $1 \leq i < N/2$ ,

$$(3.34) \quad \sum_{j=\max\{2i+1, 4\}}^N |V_{ij}| \leq Ch^2 x_i^{-\alpha/2-2/r}$$

*Proof.* For  $\max\{2i+1, 4\} \leq j \leq N$ , by Lemma A.4 and Lemma B.4 with  $y - x_i \simeq y$ , we have

$$(3.35) \quad \begin{aligned} |V_{ij}| &= \left| \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \right| \\ &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} y^{-1-\alpha} dy \\ &= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy \end{aligned}$$

With  $x_i \simeq x_{2i}$ ,

$$(3.36) \quad \begin{aligned} \sum_{j=\max\{2i+1, 4\}}^N |V_{ij}| &\leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\ &= \frac{C}{\alpha/2 + 2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\ &\leq Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \quad \square$$

LEMMA 3.12. There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $1 \leq i < N/2$ ,

$$(3.37) \quad \sum_{j=N+1}^{2N} |V_{ij}| \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

and for  $N/2 \leq i \leq N$ ,

$$(3.38) \quad \sum_{j=N-\lceil \frac{N}{2} \rceil + 2}^{2N} |V_{ij}| \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

202 *Proof.* For  $1 \leq i < N/2, N+1 \leq j \leq 2N-1$ , by Lemma A.4, Lemma B.4 and  
 203  $y - x_i \simeq T$

$$\begin{aligned}
 |V_{ij}| &= \left| \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \right| \\
 &\leq C \int_{x_{j-1}}^{x_j} h^2 (2T - y)^{\alpha/2-2/r} |y - x_i|^{-1-\alpha} dy \\
 &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy \\
 \sum_{j=N+1}^{2N-1} |V_{ij}| &\leq CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T - y)^{\alpha/2-2/r} dy \\
 &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\
 &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r} h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r} h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
 \end{aligned}
 \tag{3.39}$$

207 And by Lemma A.3

$$208 \quad |V_{i,2N}| \leq CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}$$

209 Summarizes, we get the result. Similar for the second inequality.  $\square$

210 For  $i = 1, 2$ .

211 LEMMA 3.13. From (3.15), by Lemma B.6, Lemma 3.11 and Lemma 3.12 we get  
 212 for  $i = 1, 2$

$$213 \quad |R_i| \leq Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

214

215 LEMMA 3.14. There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  
 216  $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil$

$$217 \quad |I_1| = \left| \sum_{j=1}^{k-1} V_{ij} \right| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

218 *Proof.* by Lemma A.3 , Lemma B.4

$$219 \quad |V_{i1}| \leq C \int_0^{x_1} x_1^{\alpha/2} |x_i - y|^{-1-\alpha} dy \simeq x_1^{\alpha/2+1} x_i^{-1-\alpha} = T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha}$$

For  $2 \leq j \leq k-1$ , by Lemma A.4 and Lemma B.4 with  $x_i - y \simeq x_i$ , we have

$$(3.43) \quad |V_{ij}| = \left| \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \right| \\ \leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} x_i^{-1-\alpha} dy$$

Therefore,

$$(3.44) \quad \sum_{j=2}^{k-1} |V_{ij}| \leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil-1}} y^{\alpha/2-2/r} dy$$

But  $x_{\lceil \frac{i}{2} \rceil-1} \leq 2^{-r} x_i$ , so we have

$$(3.45) \quad \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil-1}} y^{\alpha/2-2/r} dy \leq \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2-2/r+1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 < 0 \end{cases}$$

Combine the results above, we get the lemma. □

**THEOREM 3.15.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $3 \leq i < N/2, k = \lceil \frac{i}{2} \rceil$ ,*

$$(3.46) \quad |I_3| = \left| \sum_{j=k+1}^{2i-1} S_{ij} \right| \leq Ch^2 x_i^{-\alpha/2-2/r}$$

To estimate  $S_{ij}$ , we first estimate  $D_h^2 P_{i,j}^\theta(x_i)$ , but By Lemma A.1,

$$(3.47) \quad D_h^2 P_{i,j}^\theta(x_i) = P_{i,j}^{\theta''}(\xi), \quad \xi \in (x_{i-1}, x_{i+1})$$

By Leibniz formula, we calculate and estimate the derivations of  $h_{j-i}^3(x)$ ,  $u''(y_{j-i}^\theta(x))$

and  $\frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$  separately.

Firstly, we have

**LEMMA 3.16.** *There exists a constant  $C = C(T, r)$  such that For  $3 \leq i \leq N - 1, \lceil \frac{i}{2} \rceil \leq j \leq \min\{2i, N\}, \xi \in (x_{i-1}, x_{i+1})$ ,*

$$(3.48) \quad h_{j-i}^3(\xi) \leq Ch^2 x_i^{2-2/r} h_j$$

$$(3.49) \quad (h_{j-i}^3(\xi))' \leq C(r-1) h^2 x_i^{1-2/r} h_j$$

$$(3.50) \quad (h_{j-i}^3(\xi))'' \leq C(r-1) h^2 x_i^{-2/r} h_j$$

The proof of this theorem see Lemma B.7 and Lemma B.8

Second,

**LEMMA 3.17.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $3 \leq i \leq N - 1, \lceil \frac{i}{2} \rceil \leq j \leq \min\{2i, N\}, \xi \in (x_{i-1}, x_{i+1})$ ,*

$$(3.51) \quad u''(y_{j-i}^\theta(\xi)) \leq C x_i^{\alpha/2-2}$$

$$(3.52) \quad (u''(y_{j-i}^\theta(\xi)))' \leq Cx_i^{\alpha/2-3}$$

$$(3.53) \quad (u''(y_{j-i}^\theta(\xi)))'' \leq Cx_i^{\alpha/2-4}$$

The proof of this theorem see Proof 28

And Finally, we have

LEMMA 3.18. *There exists a constant  $C = C(T, \alpha, r)$  such that For  $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil \leq j \leq \min\{2i, N\}, \xi \in (x_{i-1}, x_{i+1})$ ,*

$$(3.54) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}$$

$$(3.55) \quad |(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'| \leq C|y_j^\theta - x_i|^{1-\alpha}x_i^{-1}$$

$$(3.56) \quad |(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})''| \leq C|y_j^\theta - x_i|^{1-\alpha}x_i^{-2}$$

where  $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

The proof of this theorem see Proof 29

LEMMA 3.19. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$ ,*

$$(3.57) \quad D_h^2 P_{i,j}^\theta(x_i) \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j$$

where  $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

*Proof.* Since Lemma A.1

$$(3.58) \quad D_h^2 P_{i,j}^\theta(x_i) = P_{i,j}^{\theta''}(\xi), \quad \xi \in (x_{i-1}, x_{i+1})$$

From (3.20), using Leibniz formula and Lemma 3.16, Lemma 3.17 and Lemma 3.18□

LEMMA 3.20. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $3 \leq i \leq N-1$ . For  $\lceil \frac{i}{2} \rceil \leq j \leq \min\{2i-1, N-1\}$ ,*

$$(3.59) \quad \begin{aligned} & \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i,j}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{i,j}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ & \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned}$$

And for  $\lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i, N\}$ ,

$$(3.60) \quad \begin{aligned} & \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i,j}^\theta(x_i)u'''(\eta_j^\theta) - Q_{i,j}^\theta(x_{i-1})u'''(\eta_{j-1}^\theta)}{h_i} \right) \\ & \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned}$$

where  $\eta_j^\theta \in (x_{j-1}, x_j)$ .

proof see Proof 30

LEMMA 3.21. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for*  
 $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\},$

$$\begin{aligned} S_{ij} &\leq Ch^2 \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j d\theta \\ &= Ch^2 \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \end{aligned}$$

*Proof.* Since (3.26), by Lemma 3.19 and Lemma 3.20, we get the result immediately.  $\square$

Now we can prove Theorem 3.15 using Lemma 3.21,  $k = \lceil \frac{i}{2} \rceil$

$$\begin{aligned} I_3 &= \sum_{k+1}^{2i-1} S_{ij} \leq Ch^2 \int_{x_k}^{x_{2i-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &= Ch^2 \left( \frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{2i-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\ &\leq Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned}$$

Now we study  $I_2, I_4$ .

LEMMA 3.22. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for*  
 $3 \leq i \leq N-1, k = \lceil \frac{i}{2} \rceil,$   
 (3.63)

$$I_2 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

And for  $3 \leq i < N/2$ ,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

*Proof.* In fact,

$$\begin{aligned} &\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \\ &= \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + \left( \frac{1}{h_{i+1}} - \frac{1}{h_i} \right) T_{i,k} \end{aligned}$$

While, by Lemma A.2 and Lemma B.1

$$\begin{aligned} &\frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) = \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y)) \frac{|x_{i+1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_{i+1} \Gamma(2-\alpha)} dy \\ &\leq h_k^2 \max_{\eta \in (x_{k-1}, x_k)} |u''(\eta)| \int_{x_{k-1}}^{x_k} \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy, \quad \xi \in (x_i, x_{i+1}) \\ &\leq Ch^2 x_k^{2-2/r} x_{k-1}^{\alpha/2-2} h_k |x_i - x_k|^{-\alpha} \\ &\leq Ch^2 x_i^{-\alpha/2-2/r} h_k \end{aligned}$$

292 Thus,

$$293 \quad (3.67) \quad \frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} |T_{i+1,k} - T_{i,k}| \leq Ch^2 x_i^{-\alpha/2-2/r}$$

294 From (3.25)  
 (3.68)

$$\begin{aligned} \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) &= \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{Q_{k-i}^\theta(x_{i+1})u'''(\eta_{k+1,1}^\theta) - Q_{k-i}^\theta(x_i)u'''(\eta_{k,1}^\theta)}{h_{i+1}} d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{Q_{k-i}^\theta(x_{i+1})u'''(\eta_{k+1,2}^\theta) - Q_{k-i}^\theta(x_i)u'''(\eta_{k,2}^\theta)}{h_{i+1}} d\theta \end{aligned}$$

296 and

$$297 \quad (3.69) \quad D_h P_{k-i}^\theta(x_i) := \frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} = P_{k-i}^{\theta'}(\xi), \quad \xi \in (x_i, x_{i+1})$$

298 Similar with Lemma 3.19, from (3.25), using Leibniz formula, by Lemma B.7, Lemma 3.17  
 299 and Lemma 3.18 we get

$$300 \quad (3.70) \quad |D_h P_{k-i}^\theta(x_i)| \leq Ch^2 x_i^{-\alpha/2-2/r} h_k$$

301 And with Lemma 3.20, we can get

$$302 \quad (3.71) \quad \frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} |T_{i+1,k+1} - T_{i,k}| \leq Ch^2 x_i^{-\alpha/2-2/r}$$

303 For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned} 304 \quad (3.72) \quad \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,k} &\leq h_i^{-3} h^2 x_i^{1-2/r} h_k Ch_k^2 x_{k-1}^{\alpha/2-2} |x_k - x_i|^{1-\alpha} \\ &\leq Ch^2 x_i^{-\alpha/2-2/r} \end{aligned}$$

305 Summarizes, we have

$$306 \quad (3.73) \quad I_2 \leq Ch^2 x_i^{-\alpha/2-2/r}$$

307 The case for  $I_4$  is similar. □

308 Now combine Lemma 3.13, Lemma 3.14, Lemma 3.22, Theorem 3.15, Lemma 3.11  
 309 and Lemma 3.12, we get Theorem 3.2.

**3.4. Proof of Theorem 3.3.** For  $N/2 \leq i < N, k = \lceil \frac{i}{2} \rceil$ , we have

$$\begin{aligned}
 (3.74) \quad R_i &= \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &= \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &\quad + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \\
 &\quad + \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
 &\quad + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right) \\
 &\quad + \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &= I_1 + I_2 + I_3^1 + I_3^2 + I_3^3 + I_4 + I_5
 \end{aligned}$$

We have estimate  $I_1$  in Lemma 3.14 and  $I_2$  in Lemma 3.22. We can control  $I_3^1$  similar with Theorem 3.15 by Lemma 3.21 where  $2i - 1 \geq N - 1$

LEMMA 3.23. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $N/2 \leq i < N, k = \lceil \frac{i}{2} \rceil$ ,*

$$\begin{aligned}
 (3.75) \quad I_3^1 &= \sum_{j=k+1}^{N-1} S_{ij} \leq Ch^2 \int_{x_k}^{x_{N-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\
 &= Ch^2 \left( \frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{N-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\
 &\leq Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r}
 \end{aligned}$$

Let's study  $I_3^3$  before  $I_3^2$ .

$$(3.76) \quad I_3^3 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} S_{ij}$$

Similarly, Let's define a new series of functions

DEFINITION 3.24. *For  $i \leq N - 1, j \geq N + 1$ , with no confusion, we also denote in this section*

$$(3.77) \quad y_{j-i}(x) = 2T - (Z_{2N-j+i} - x^{1/r})^r, \quad Z_{2N-j+i} = T^{1/r} \frac{2N-j+i}{N}$$

*Particularly*

$$y_{j-i}(x_{i-1}) = x_{j-1}, \quad y_{j-i}(x_i) = x_j, \quad y_{j-i}(x_{i+1}) = x_{j+1}$$

325  $y \rightarrow z?$

326 (3.78)  $y_{j-i}'(x) = (2T - y_{j-i}(x))^{1-1/r} x^{1/r-1}$

327 (3.79)  $y_{j-i}''(x) = \frac{1-r}{r} (2T - y_{j-i}(x))^{1-2/r} x^{1/r-2} Z_{2N-j+i}$

328 (3.80)

329

330 (3.81)  $y_{j-i}^\theta(x) = (1-\theta)y_{j-i-1}(x) + \theta y_{j-i}(x)$

331

332 (3.82)  $h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x)$

333

334 (3.83)  $P_{i,j}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$

335

336 (3.84)  $Q_{i,j}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$

337 Now we have the same formula (3.26) for  $i \leq N-1, j \geq N+2$ ,

338 Similarly, we first estimate

339 (3.85)  $D_h^2 P_{i,j}^\theta(\xi) = P_{i,j}^{\theta''}(\xi), \quad \xi \in (x_{i-1}, x_{i+1})$

340 Combine Definition 3.24, Lemma B.9, Lemma B.10 and Lemma B.11, using Leib-  
341 niz formula, we have

342 LEMMA 3.25. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For*  
343  *$N/2 \leq i \leq N-1, N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$ , we have*

344 (3.86) 
$$\begin{aligned} |D_h^2 P_{i,j}^\theta(\xi)| &\leq Ch_j h^2 \left( |y_j^\theta - x_i|^{1-\alpha} \right. \\ &\quad + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N) \\ &\quad + |y_j^\theta - x_i|^{-1-\alpha} (|2T - x_i - y_j^\theta| + h_N)^2 \\ &\quad \left. + (r-1) |y_j^\theta - x_i|^{-\alpha} \right) \end{aligned}$$

345 And

346 LEMMA 3.26. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For*  
347  *$N/2 \leq i \leq N-1, N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil, \xi \in (x_{i-1}, x_{i+1})$ , we have*

348 (3.87) 
$$\begin{aligned} \frac{2}{h_i + h_{i+1}} \left| \frac{Q_{i,j}^\theta(x_{i+1}) u'''(\eta_{j+1}^\theta) - Q_{i,j}^\theta(x_i) u'''(\eta_j^\theta)}{h_{i+1}} \right| \\ \leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N)) \end{aligned}$$

349 and

350 (3.88) 
$$\begin{aligned} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i,j}^\theta(x_i) u'''(\eta_j^\theta) - Q_{i,j}^\theta(x_{i-1}) u'''(\eta_{j-1}^\theta)}{h_{i+1}} \right) \\ \leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N)) \end{aligned}$$



*Proof.* From Definition 3.24, by Lemma B.9 and Lemma B.11, for  $\xi \in (x_i, x_{i+1})$ , by Leibniz formula, we have

$$(3.89) \quad \left| Q_{i,j}^{\theta}{}'(\xi) \right| \leq Ch^2 h_j^2 ((r-1)|y_j^{\theta} - x_i|^{1-\alpha} + |y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N))$$

$$(3.90) \quad |Q_{i,j}^{\theta}(\xi)| \leq Ch^2 h_j^2 |y_j^{\theta} - x_i|^{1-\alpha}$$

So use the skill in Proof 30 with Lemma B.10

$$(3.91) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i,j}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta}) - Q_{i,j}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}} \right) \leq Ch^2 h_j (|y_j^{\theta} - x_i|^{1-\alpha} + |y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N)) \quad \square$$

Combine Lemma 3.25, Lemma 3.26 and formula (3.26) for  $i \leq N-1, j \geq N+2$ , we have

LEMMA 3.27. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i \leq N-1, N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$*

$$(3.92) \quad S_{ij} \leq Ch^2 \int_{x_{j-1}}^{x_j} \left( |y - x_i|^{1-\alpha} + |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2 + (r-1)|y - x_i|^{-\alpha} \right) dy$$

We can estimate  $I_3^3$  Now.

LEMMA 3.28. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i \leq N-1$ , we have*

$$(3.93) \quad I_3^3 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} S_{ij} \leq Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha}$$

*Proof.*

$$(3.94) \quad I_3^3 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} S_{ij} \leq Ch^2 \int_{x_{N+1}}^{x_{2N - \lceil \frac{N}{2} \rceil}} \left( |y - x_i|^{1-\alpha} + |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2 + (r-1)|y - x_i|^{-\alpha} \right) dy$$

Since

$$(3.95) \quad |2T - x_i - y| + h_N \leq y - x_i$$

$$(3.96) \quad \begin{aligned} I_3^3 &\leq Ch^2 \int_{x_{N+1}}^{x_{2N - \lceil \frac{N}{2} \rceil}} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} \\ &\leq Ch^2 (T^{2-\alpha} + (r-1)|x_{N+1} - x_i|^{1-\alpha}) \\ &\leq Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \end{aligned} \quad \square$$

For  $I_3^2$ , we have

**THEOREM 3.29.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that, for  $N/2 \leq i \leq N-1$*

$$(3.97) \quad \begin{aligned} V_{iN} &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1, N+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, N} + \frac{1}{h_i} T_{i-1, N-1} \right) \\ &\leq Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \end{aligned}$$

*Proof.* We use the similar skill in the last section, but more complicated. for  $j = N$ , Let

$$(3.98) \quad {}_L y_{N-1-i}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r} \frac{N-1-i}{N}$$

$$(3.99) \quad {}_0 y_{N-i}(x) = \frac{x^{1/r} - Z_i}{Z_1} h_N + T, \quad Z_i = T^{1/r} \frac{i}{N}, x_N = T$$

and

$$(3.100) \quad {}_R y_{N+1-i}(x) = 2T - (Z_{N-1+i} - x^{1/r})^r, \quad Z_{N-1+i} = T^{1/r} \frac{N-1+i}{N}$$

Thus,

$${}_L y_{N-1-i}(x_{i-1}) = x_{N-2}, \quad {}_L y_{N-1-i}(x_i) = x_{N-1}, \quad {}_L y_{N-1-i}(x_{i+1}) = x_N$$

$${}_0 y_{N-i}(x_{i-1}) = x_{N-1}, \quad {}_0 y_{N-i}(x_i) = x_N, \quad {}_0 y_{N-i}(x_{i+1}) = x_{N+1}$$

$${}_R y_{N+1-i}(x_{i-1}) = x_N, \quad {}_R y_{N+1-i}(x_i) = x_{N+1}, \quad {}_R y_{N+1-i}(x_{i+1}) = x_{N+2}$$

Then, define

$$(3.101) \quad {}_L y_{N-i}^\theta(x) = \theta {}_L y_{N-1-i}(x) + (1-\theta) {}_0 y_{N-i}(x)$$

$$(3.102) \quad {}_R y_{N+1-i}^\theta(x) = \theta {}_0 y_{N-i}(x) + (1-\theta) {}_R y_{N+1-i}(x)$$

$$(3.103) \quad {}_L h_{N-i}(x) = {}_0 y_{N-i}(x) - {}_L y_{N-1-i}(x)$$

$$(3.104) \quad {}_R h_{N+1-i}(x) = {}_R y_{N+1-i}(x) - {}_0 y_{N-i}(x)$$

We have

$$(3.105) \quad {}_L y_{N-1-i}'(x) = {}_L y_{N-1-i}^{1-1/r}(x) x^{1/r-1}$$

$$(3.106) \quad {}_L y_{N-1-i}''(x) = \frac{1-r}{r} {}_L y_{N-1-i}^{1-2/r}(x) x^{1/r-2} Z_{N-1-i}$$

$$(3.107) \quad {}_0 y_{N-i}'(x) = \frac{1}{r} \frac{h_N}{Z_1} x^{1/r-1}$$

$$(3.108) \quad {}_0 y_{N-i}''(x) = \frac{1-r}{r^2} \frac{h_N}{Z_1} x^{1/r-2}$$

$$(3.109) \quad {}_R y_{N+1-i}'(x) = (2T - {}_R y_{N+1-i}(x))^{1-1/r} x^{1/r-1}$$

$$(3.110) \quad {}_R y_{N+1-i}''(x) = \frac{1-r}{r} (2T - {}_R y_{N+1-i}(x))^{1-2/r} x^{1/r-2} Z_{N-1+i}$$

400

$$(3.111) \quad {}_L P_{N-i}^\theta(x) = ({}_L h_{N-i}(x))^3 \frac{|{}_L y_{N-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_L y_{N-i}^\theta(x))$$

$$(3.112) \quad {}_R P_{N+1-i}^\theta(x) = ({}_R h_{N+1-i}(x))^3 \frac{|{}_R y_{N+1-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_R y_{N+1-i}^\theta(x))$$

$$(3.113) \quad {}_L Q_{N-i}^\theta(x) = ({}_L h_{N-i}(x))^4 \frac{|{}_L y_{N-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$(3.114) \quad {}_R Q_{N+1-i}^\theta(x) = ({}_R h_{N+1-i}(x))^4 \frac{|{}_R y_{N+1-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

 405 Similar with (3.25), we can get for  $l = -1, 0, 1$ ,

$$(3.115) \quad \begin{aligned} T_{i+l, N+l} &= \int_0^1 -\frac{\theta(1-\theta)}{2} {}_L P_{N-i}^\theta(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} {}_L Q_{N-i}^\theta(x_{i+l}) (\theta^2 u'''(\eta_{N+l,1}^\theta) - (1-\theta)^2 u'''(\eta_{N+l,2}^\theta)) d\theta \end{aligned}$$

407

$$(3.116) \quad \begin{aligned} T_{i+l, N+1+l} &= \int_0^1 -\frac{\theta(1-\theta)}{2} {}_R P_{N+1-i}^\theta(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} {}_R Q_{N+1-i}^\theta(x_{i+l}) (\theta^2 u'''(\eta_{N+1+l,1}^\theta) - (1-\theta)^2 u'''(\eta_{N+1+l,2}^\theta)) d\theta \end{aligned}$$

409 So we have

$$(3.117) \quad \begin{aligned} V_{i,N} &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_{hL}^2 {}_L P_{N-i}^\theta(x_i) d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_{i+1}) u'''(\eta_{N+1,1}^\theta) - {}_L Q_{N-i}^\theta(x_i) u'''(\eta_{N,1}^\theta)}{h_{i+1}} \right) d\theta \\ &- \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_i) u'''(\eta_{N,1}^\theta) - {}_L Q_{N-i}^\theta(x_{i-1}) u'''(\eta_{N-1,1}^\theta)}{h_i} \right) d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_{i+1}) u'''(\eta_{N+1,2}^\theta) - {}_L Q_{N-i}^\theta(x_i) u'''(\eta_{N,2}^\theta)}{h_{i+1}} \right) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_i) u'''(\eta_{N,2}^\theta) - {}_L Q_{N-i}^\theta(x_{i-1}) u'''(\eta_{N-1,2}^\theta)}{h_i} \right) d\theta \end{aligned}$$

 411  $N+1$  is similar.

 412 We estimate  $D_{hL}^2 {}_L P_{N-i}^\theta(x_i) = {}_L P_{N-i}^{\theta''}(\xi)$ ,  $\xi \in (x_{i-1}, x_{i+1})$ ,

413

LEMMA 3.30.

$$(3.118) \quad {}_L h_{N-i}^3(\xi) \leq C h_N^3 \leq C h^3$$

$$(3.119) \quad {}_R h_{N+1-i}^3(\xi) \leq C h_N^3 \leq C h^3$$

$$\begin{aligned}
(3.120) \quad & ({}_L h_{N-i}^3(\xi))' \leq C(r-1)h_N^2 h \leq C(r-1)h^3 \\
(3.121) \quad & ({}_R h_{N+1-i}^3(\xi))' \leq C(r-1)h_N^2 h \leq C(r-1)h^3 \\
(3.122) \quad & ({}_L h_{N-i}^3(\xi))'' \leq C(r-1)h^2 \\
(3.123) \quad & ({}_R h_{N+1-i}^3(\xi))'' \leq C(r-1)h^2
\end{aligned}$$

*Proof.*

$$(3.124) \quad {}_L h_{N-i}(\xi) \leq 2(C?)h_N, \quad {}_R h_{N+1-i}(\xi) \leq 2h_N$$

$$\begin{aligned}
(3.125) \quad & ({}_L h_{N-i}^l(\xi))' = {}_L h_{N-i}^{l-1}(\xi)({}_0 y_{N-i}'(\xi) - {}_L y_{N-1-i}'(\xi)) \\
& = {}_L h_{N-i}^{l-1}(\xi)\xi^{1/r-1}\left(\frac{1}{r}\frac{h_N}{Z_1} - {}_L y_{N-1-i}^{1-1/r}(\xi)\right)
\end{aligned}$$

while

$$\begin{aligned}
(3.126) \quad & \left|\frac{1}{r}\frac{h_N}{Z_1} - {}_L y_{N-1-i}^{1-1/r}(\xi)\right| = \left|\frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}\right| \quad \eta \in [x_{N-2}, x_N] \\
& = T^{1-1/r} \left| \left(\frac{N-t}{N}\right)^{r-1} - \left(\frac{N-s}{N}\right)^{r-1} \right| \quad t \in [0, 1], s \in [0, 2] \\
& \leq T^{1-1/r} \left| 1 - \left(\frac{N-2}{N}\right)^{r-1} \right| \leq CT^{1-1/r}(r-1)\frac{2}{N}
\end{aligned}$$

Thus,

$$(3.127) \quad ({}_L h_{N-i}^l(\xi))' \leq C(r-1)h_N^{l-1}x_i^{1/r-1}h$$

And

$$\begin{aligned}
(3.128) \quad & ({}_L h_{N-i}^3(\xi))'' = 3{}_L h_{N-i}^2(\xi){}_L h_{N-i}''(\xi) + 6{}_L h_{N-i}(\xi)({}_L h_{N-i}'(\xi))^2 \\
& \leq Ch_N^2 \frac{1-r}{r} x_i^{1/r-2} \left( \frac{1}{r}\frac{h_N}{Z_1} - {}_L y_{N-1-i}^{1-2/r}(\xi)Z_{N-1-i} \right) + Ch_N(r-1)^2 h^2 x_i^{2/r-2}
\end{aligned}$$

$$\left| \frac{h_N}{rZ_1} - {}_L y_{N-1-i}^{1-2/r}(\xi)Z_{N-1-i} \right| \leq T^{1-1/r} + Cx_N^{1-2/r}x_N^{1/r} = CT^{1-1/r}$$

So

$$\begin{aligned}
(3.129) \quad & ({}_L h_{N-i}^3(\xi))'' \leq Ch_N^2 \frac{1-r}{r} x_i^{1/r-2} + C(r-1)^2 h_N x_i^{2/r-2} h^2 \\
& \leq C(r-1)h_N^2
\end{aligned}$$

${}_R h_{N+1-i}^3(\xi)$  is similar. □

LEMMA 3.31.

$$(3.130) \quad u''({}_L y_{N-i}^\theta(\xi)) \leq Cx_{N-2}^{-\alpha/2-2} \leq C$$

$$(3.131) \quad (u''({}_L y_{N-i}^\theta(\xi)))' \leq C$$

$$(3.132) \quad (u''({}_L y_{N-i}^\theta(\xi)))'' \leq C$$

*Proof.*

$$\begin{aligned}
 (u''(Ly_{N-i}^\theta(\xi)))' &= u'''(Ly_{N-i}^\theta(\xi))Ly_{N-i}^{\theta'}(\xi) \\
 &\leq C(\theta Ly_{N-1-i}'(\xi) + (1-\theta)_0y_{N-i}'(\xi)) \\
 &\leq Cx_i^{1/r-1}(\theta Ly_{N-1-i}^{1-1/r}(\xi) + (1-\theta)\frac{h_N}{rZ_1}) \\
 &\leq Cx_i^{1/r-1}x_N^{1-1/r}
 \end{aligned}
 \tag{3.133}$$

And

$$\begin{aligned}
 (u''(Ly_{N-i}^\theta(\xi)))'' &= u''''(Ly_{N-i}^\theta(\xi))(Ly_{N-i}^{\theta'}(\xi))^2 + u'''(Ly_{N-i}^\theta(\xi))Ly_{N-i}^{\theta''}(\xi) \\
 &\leq Cx_i^{2/r-2}x_N^{2-2/r} + C\frac{r-1}{r}x_i^{1/r-2}(\theta x_N^{1-2/r}Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \\
 &\leq Cx_i^{2/r-2} + C(r-1)x_i^{1/r-2}T^{1-1/r}
 \end{aligned}
 \tag{3.134}$$

LEMMA 3.32.

$$|Ly_{N-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_N^\theta - x_i|^{1-\alpha} \tag{3.135}$$

$$(|Ly_{N-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C|y_N^\theta - x_i|^{1-\alpha} \tag{3.136}$$

$$(|Ly_{N-i}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C(r-1)|y_N^\theta - x_i|^{-\alpha} + |y_N^\theta - x_i|^{1-\alpha} \tag{3.137}$$

*Proof.*

$$\begin{aligned}
 (Ly_{N-i}^\theta(\xi) - \xi)' &= (\theta(Ly_{N-1-i}(\xi) - \xi) + (1-\theta)(_0y_{N-i}(\xi) - \xi))' \\
 &= \theta(Ly_{N-1-i}'(\xi) - 1) + (1-\theta)(_0y_{N-i}'(\xi) - 1) \\
 &= \theta\xi^{1/r-1}(Ly_{N-1-i}^{1-1/r}(\xi) - \xi^{1-1/r}) + (1-\theta)\xi^{1/r-1}(\frac{h_N}{rZ_1} - \xi^{1-1/r})
 \end{aligned}
 \tag{3.138}$$

$$\begin{aligned}
 (Ly_{N-i}^\theta(\xi) - \xi)'' &= \theta(Ly_{N-1-i}''(\xi)) + (1-\theta)(_0y_{N-i}''(\xi)) \\
 &= \frac{1-r}{r}\xi^{1/r-2}(\theta Ly_{N-1-i}^{1-2/r}(\xi)Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \leq 0
 \end{aligned}
 \tag{3.139}$$

And

$$|(Ly_{N-i}^\theta(\xi) - \xi)''| \leq C(r-1)\xi^{1/r-2}T^{1-1/r} \tag{3.140}$$

We have known

$$C|x_{N-1} - x_i| \leq |Ly_{N-1-i}(\xi) - \xi| \leq C|x_{N-1} - x_i| \tag{3.141}$$

If  $\xi \leq x_{N-1}$ , then  $(_0y_{N-i}(\xi) - \xi)' \geq 0$ , so

$$C|x_N - x_i| \leq |x_{N-1} - x_{i-1}| \leq |Ly_{N-i}^\theta(\xi) - \xi| \leq |x_{N+1} - x_{i+1}| \leq C|x_N - x_i| \tag{3.142}$$

If  $i = N-1$  and  $\xi \in [x_{N-1}, x_N]$ , then  $_0y_{N-i}(\xi) - \xi$  is concave, bigger than its two neighboring points, which are equal to  $h_N$ , so

$$h_N = |x_N - x_{N-1}| \leq |_0y_{N-i}(\xi) - \xi| \leq |x_{N+1} - x_{N-1}| = 2h_N \tag{3.143}$$

So we have

$$(3.144) \quad |Ly_{N-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_N^\theta - x_i|^{1-\alpha}$$

While

$$(3.145) \quad Ly_{N-1-i}^{1-1/r}(\xi) - \xi^{1-1/r} \leq (Ly_{N-1-i}(\xi) - \xi)\xi^{-1/r}$$

and

$$(3.146) \quad \left| \frac{h_N}{rZ_1} - \xi^{1-1/r} \right| \leq \max\left\{ \left| \frac{h_N}{rZ_1} - x_{i-1}^{1-1/r} \right|, \left| \frac{h_N}{rZ_1} - x_{i+1}^{1-1/r} \right| \right\}$$

$$\leq \max \begin{cases} T^{1-1/r} - x_{i-1}^{1-1/r} \leq |x_N - x_{i-1}|T^{-1/r} \leq C|x_N - x_i| \\ |x_{i+1}^{1-1/r} - x_{N-1}^{1-1/r}| \leq |x_{i+1} - x_{N-1}|x_{N-1}^{-1/r} \leq C|x_N - x_i| \end{cases}$$

So we have

$$(3.147) \quad (Ly_{N-i}^\theta(\xi) - \xi)' \leq C|y_N^\theta - x_i|$$

$$(3.148) \quad (|Ly_{N-i}^\theta(\xi) - \xi|^{1-\alpha})' = |Ly_{N-i}^\theta(\xi) - \xi|^{-\alpha} (Ly_{N-i}^\theta(\xi) - \xi)' \leq |y_N^\theta - x_i|^{1-\alpha}$$

Finally,

$$(3.149) \quad \begin{aligned} (|Ly_{N-i}^\theta(\xi) - \xi|^{1-\alpha})'' &= (1-\alpha)|Ly_{N-i}^\theta(\xi) - \xi|^{-\alpha} (Ly_{N-i}^\theta(\xi) - \xi)'' \\ &\quad + \alpha(\alpha-1)|Ly_{N-i}^\theta(\xi) - \xi|^{-1-\alpha} ((Ly_{N-i}^\theta(\xi) - \xi)')^2 \\ &\leq C(r-1)|y_N^\theta - x_i|^{-\alpha} + C|y_N^\theta - x_i|^{1-\alpha} \end{aligned} \quad \square$$

By the three lemmas above, for  $N/2 \leq i \leq N-1$ , we have

LEMMA 3.33.

$$(3.150) \quad \begin{aligned} D_{hL}^2 P_{N-i}^\theta(x_i) &= {}_L P_{N-i}^\theta''(\xi) \quad \xi \in (x_{i-1}, x_{i+1}) \\ &\leq Ch^3|y_N^\theta - x_i|^{1-\alpha} + C(r-1)(h^3|y_N^\theta - x_i|^{-\alpha} + h^2|y_N^\theta - x_i|^{1-\alpha}) \end{aligned}$$

while  $\theta h_N = y_N^\theta - x_{N-1} \leq y_N^\theta - x_i$ , we have

$$(3.151) \quad \theta D_{hL}^2 P_{N-i}^\theta(x_i) \leq Ch^3|y_N^\theta - x_i|^{1-\alpha} + C(r-1)(h^2|y_N^\theta - x_i|^{1-\alpha})$$

And

LEMMA 3.34.

$$(3.152) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_{i+1})u'''(\eta_{N+1}^\theta) - {}_L Q_{N-i}^\theta(x_i)u'''(\eta_N^\theta)}{h_{i+1}} \right) \leq Ch^3|y_N^\theta - x_i|^{1-\alpha}$$

And immediately with (3.26), For  $N/2 \leq i \leq N-1$

$$(3.153) \quad \begin{aligned} V_{iN} &\leq C \int_{x_{N-1}}^{x_N} h^2|y - x_i|^{1-\alpha} + C(r-1)h|y - x_i|^{1-\alpha} dy \\ &\leq Ch^2h_N|T - x_i|^{1-\alpha} + C(r-1)h^2|x_N - x_i|^{1-\alpha} \\ &\leq Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned}$$

Similarly with  $j = N+1$ .  $\square$

$I_4, I_5$  is easy. Similar with Lemma 3.22 and Lemma 3.12, we have

**THEOREM 3.35.** *There is a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i \leq N$ ,*

$$(3.154) \quad \begin{aligned} I_4 &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1, 2N - \lceil \frac{N}{2} \rceil + 1} + T_{i-1, 2N - \lceil \frac{N}{2} \rceil}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, 2N - \lceil \frac{N}{2} \rceil + 1} \right) \\ &\leq Ch^2 \end{aligned}$$

*Proof.* Similar with Lemma 3.22. In fact, let  $m = 2N - \lceil \frac{N}{2} \rceil + 1$

$$(3.155) \quad \begin{aligned} &\frac{1}{h_i} (T_{i-1, l} + T_{i-1, l-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, l} \\ &= \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) + \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) + \left( \frac{1}{h_i} - \frac{1}{h_{i+1}} \right) T_{i, l} \end{aligned}$$

While, by Lemma A.2

$$(3.156) \quad \begin{aligned} \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) &= \int_{x_{l-1}}^{x_l} (u(y) - \Pi_h u(y)) \frac{|x_{i-1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_i \Gamma(2-\alpha)} dy \\ &\leq C \int_{x_{l-1}}^{x_i} h_l^2 u''(\eta) \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy, \quad \xi \in (x_{i-1}, x_i) \\ &\leq Ch_l^3 (2T - x_{l-1})^{\alpha/2-2} T^{-\alpha} \\ &\leq Ch_l^3 \end{aligned}$$

Thus,

$$(3.157) \quad \frac{2}{h_i + h_{i+1}} \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) \leq Ch_l^2$$

For

$$(3.158) \quad \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{h_i} d\theta$$

And Similar with Lemma 3.20, we can get

$$(3.159) \quad \frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{(h_i + h_{i+1}) h_i} \leq Ch_l^2 |y_l^\theta - x_i|^{1-\alpha}$$

So

$$(3.160) \quad \frac{2}{h_i + h_{i+1}} \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) \leq Ch^2$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$(3.161) \quad \begin{aligned} \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i, l} &\leq h_i^{-3} h^2 x_i^{1-2/r} h_l C h_l^2 x_{l-1}^{\alpha/2-2} |x_l - x_i|^{1-\alpha} \\ &\leq Ch^2 \end{aligned}$$

Summarizes, we have

$$(3.162) \quad I_4 \leq Ch^2 \quad \square$$

496 Now we can conclude a part of the theorem Theorem 3.3 at the beginning of this  
 497 section.

498 By Lemma 3.14, Lemma 3.22, Lemma 3.23, Theorem 3.29, Lemma 3.28, Theo-  
 499 rem 3.35, Lemma 3.12, we have

500 THEOREM 3.36. *there exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for*  
 501  $N/2 \leq i \leq N-1$ ,

$$\begin{aligned}
 R_i &= I_1 + I_2 + I_3^1 + I_3^2 + I_3^3 + I_4 + I_5 \\
 (3.163) \quad &\leq C(r-1)h^2|T-x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
 \end{aligned}$$



And what we left is the case  $i = N$ . Fortunately, we can use the same department of  $R_i$  above, and it is symmetric. Most of the item has been esitimated by Lemma 3.14 and Theorem 3.35, we just need to consider  $I_3, I_4$ .

**THEOREM 3.37.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

$$(3.164) \quad I_3 = \sum_{j=\lceil \frac{N}{2} \rceil + 1}^{N-1} V_{Nj} \leq Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha}$$

*Proof.* **DEFINITION 3.38.** *For  $N/2 \leq j < N$ , Let's define*

$$(3.165) \quad y_j(x) = \left( \frac{Z_1}{h_N}(x - x_N) + Z_j \right)^r, \quad Z_j = T^{1/r} \frac{j}{N}$$

We can see that is the inverse of the function  ${}_0y_{N-i}(x)$  defined in Theorem 3.29.

$$(3.166) \quad y'_j(x) = y_j^{1-1/r}(x) \frac{rZ_1}{h_N}$$

$$(3.167) \quad y''_j(x) = y_j^{1-2/r}(x) \frac{r(r-1)Z_1}{h_N}$$

With the scheme we used several times, we can get

**LEMMA 3.39.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq j < N$ ,  $\xi \in [x_{N-1}, x_{N+1}]$ ,*

$$(3.168) \quad h_j(\xi)^3 \leq Ch^3$$

$$(3.169) \quad (h_j^3(\xi))' \leq C(r-1)h^3$$

$$(3.170) \quad (h_j^3(\xi))'' \leq C(r-1)h^3$$

$$(3.171) \quad u''(y_j^\theta(\xi)) \leq C$$

$$(3.172) \quad (u''(y_j^\theta(\xi)))' \leq C$$

$$(3.173) \quad (u''(y_j^\theta(\xi)))'' \leq C$$

$$(3.174) \quad |\xi - y_j^\theta(\xi)|^{1-\alpha} \leq C|x_N - y_j^\theta|^{1-\alpha}$$

$$(3.175) \quad (|\xi - y_j^\theta(\xi)|^{1-\alpha})' \leq C|x_N - y_j^\theta|^{1-\alpha}$$

$$(3.176) \quad (|\xi - y_j^\theta(\xi)|^{1-\alpha})'' \leq C|x_N - y_j^\theta|^{1-\alpha} + C(r-1)|x_N - y_j^\theta|^{-\alpha}$$

**LEMMA 3.40.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq j < N$ ,*

$$(3.177) \quad V_{Nj} \leq Ch^2 \int_{x_{j-1}}^{x_j} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy$$

Therefore,

$$(3.178) \quad \begin{aligned} I_3 &\leq Ch^2 \int_{\lceil \frac{N}{2} \rceil}^{N-1} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \\ &\leq Ch^2(|T - x_{N-1}|^{2-\alpha} + (r-1)|T - x_{N-1}|^{1-\alpha}) \end{aligned}$$

□

For  $j = N$ ,

LEMMA 3.41.

(3.179)

$$V_{N,N} = \frac{1}{h_N^2} (T_{N-1,N-1} - 2T_{N,N} + T_{N+1,N+1}) \leq Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha}$$

*Proof.*

(3.180)

□

$$\begin{aligned} V_{N,N} = & \int_0^1 -\frac{\theta(1-\theta)^{2-\alpha}}{2} \frac{1}{h_N^2} (h_{N-1}^{4-\alpha} u''(y_{N-1}^\theta) - 2h_N^{4-\alpha} u''(y_N^\theta) + h_{N+1}^{4-\alpha} u''(y_{N+1}^\theta)) d\theta \\ & + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{1}{h_N} \left( \frac{Q_{N \rightarrow N}^\theta(x_{N+1}) u'''(\eta_{N+1,1}^\theta) - Q_{N \rightarrow N}^\theta(x_i) u'''(\eta_{N,1}^\theta)}{h_N} \right) d\theta \\ & - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{1}{h_N} \left( \frac{Q_{N \rightarrow N}^\theta(x_N) u'''(\eta_{N,1}^\theta) - Q_{N \rightarrow N}^\theta(x_{N-1}) u'''(\eta_{N-1,1}^\theta)}{h_N} \right) d\theta \\ & - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{1}{h_N} \left( \frac{Q_{N \rightarrow N}^\theta(x_{N+1}) u'''(\eta_{N+1,2}^\theta) - Q_{N \rightarrow N}^\theta(x_N) u'''(\eta_{N,2}^\theta)}{h_N} \right) d\theta \\ & + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{1}{h_N} \left( \frac{Q_{N \rightarrow N}^\theta(x_N) u'''(\eta_{N,2}^\theta) - Q_{N \rightarrow N}^\theta(x_{N-1}) u'''(\eta_{N-1,2}^\theta)}{h_N} \right) d\theta \end{aligned}$$

So combine Lemma 3.14, Theorem 3.35, Theorem 3.37, Lemma 3.41 We have

LEMMA 3.42.

$$R_N \leq C(r-1)h^2|T - x_{N-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

and with Theorem 3.36 we prove the Theorem 3.3

#### 4. Convergence analysis.

**4.1. Properties of some Matrices.** Review subsection 2.1, we have got (2.10).

DEFINITION 4.1. We call one matrix an  $M$  matrix, which means its entries are positive on major diagonal and nonpositive on others, and strictly diagonally dominant in rows.

Now we have

LEMMA 4.2. Matrix  $A$  defined by (2.12) where (2.13) is an  $M$  matrix. And there exists a constant  $C_A = C(T, \alpha, r)$  such that

$$(4.1) \quad S_i := \sum_{j=1}^{2N-1} a_{ij} \geq C_A(x_i^{-\alpha} + (2T - x_i)^{-\alpha})$$

*Proof.* From (2.14), we have

$$(4.2) \quad \sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left( \frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right)$$

Let

$$(4.3) \quad g(x) = g_0(x) + g_{2N}(x)$$

where

$$g_0(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x - x_0|^{3-\alpha} - |x - x_1|^{3-\alpha}}{h_1}$$

$$g_{2N}(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N} - x|^{3-\alpha} - |x_{2N-1} - x|^{3-\alpha}}{h_{2N}}$$

Thus

$$-\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

Then

$$(4.4) \quad S_i := \sum_{j=1}^{2N-1} a_{ij}$$

$$= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right)$$

$$= D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i)$$

When  $i = 1$

$$(4.5) \quad D_h^2 g_0(x_1) = \frac{2}{h_1 + h_2} \left( \frac{1}{h_2} g_0(x_2) - \left( \frac{1}{h_1} + \frac{1}{h_2} \right) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right)$$

$$= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1h_2}$$

$$= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1^{1-\alpha}h_2} h_1^{-\alpha}$$

$$= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha}}{2^r(2^r - 1)} h_1^{-\alpha}$$

but

$$(4.6) \quad 1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha} > 0$$

While for  $i \geq 2$

$$(4.7) \quad \begin{aligned} D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \\ &= -\kappa_\alpha \frac{|\xi - x_0|^{1-\alpha} - |\xi - x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1} \\ &= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} |\xi - \eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\ &\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha} \end{aligned}$$

So

$$(4.8) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g_0(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g_0(x_i) + \frac{1}{h_i} g_0(x_{i-1}) \right) \geq C x_i^{-\alpha}$$

symmetricly,

$$(4.9) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g_{2N}(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g_{2N}(x_i) + \frac{1}{h_i} g_{2N}(x_{i-1}) \right) \geq C(\alpha, r)(2T - x_i)^{-\alpha} \quad \square$$

Let

$$(4.10) \quad g(x) = \begin{cases} x, & 0 < x \leq T \\ 2T - x, & T < x < 2T \end{cases}$$

And define

$$(4.11) \quad G = \text{diag}(g(x_1), \dots, g(x_{2N-1}))$$

Then

LEMMA 4.3. *The matrix  $B := AG$ , the major diagonal is positive, and nonpositive on others. And there is a constant  $C_{AG}, C = C(\alpha, r)$  such that*

$$(4.12) \quad M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C_{AG}(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases}$$

*Proof.*

$$b_{ij} = a_{ij}g(x_j) = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_i} \tilde{a}_{i-1,j} \right) g(x_j)$$

Since

$$(4.13) \quad g(x) \equiv \Pi_h g(x)$$

by ??, we have

$$\begin{aligned}
 \tilde{M}_i &:= \sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} \tilde{a}_{ij} g(x_j) \\
 &= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} \Pi_h g(y) dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} g(y) dy \\
 &= \frac{-2}{\Gamma(4-\alpha)} |T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)} (x_i^{3-\alpha} + (2T - x_i)^{3-\alpha}) \\
 &:= w(x_i) = p(x_i) + q(x_i)
 \end{aligned}
 \tag{4.14}$$

Thus,

$$\begin{aligned}
 M_i &:= \sum_{j=1}^{2N-1} b_{ij} = \sum_{j=1}^{2N-1} a_{ij} g(x_j) \\
 &= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} \tilde{M}_{i+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{M}_i + \frac{1}{h_i} \tilde{M}_{i-1} \right) \\
 &= D_h^2(-\kappa_\alpha p)(x_i) - \kappa_\alpha D_h^2 q(x_i)
 \end{aligned}
 \tag{4.15}$$

for  $1 \leq i \leq N-1$ , by Lemma A.1

$$\begin{aligned}
 D_h^2(-\kappa_\alpha p)(x_i) &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} p(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) p(x_i) + \frac{1}{h_i} p(x_{i-1}) \right) \\
 &= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1}) \\
 &\geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - x_{i-1}|^{1-\alpha}
 \end{aligned}
 \tag{4.16}$$

$$\begin{aligned}
 D_h^2(-\kappa_\alpha p)(x_N) &:= -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left( \frac{1}{h_{N+1}} p(x_{N+1}) - \left( \frac{1}{h_N} + \frac{1}{h_{N+1}} \right) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right) \\
 &= \frac{4\kappa_\alpha}{\Gamma(4-\alpha) h_N^2} h_N^{3-\alpha} \\
 &= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - x_{N-1})^{1-\alpha}
 \end{aligned}
 \tag{4.17}$$

Symmetricly for  $i \geq N$ , we get

$$D_h^2(-\kappa_\alpha p)(x_i) \geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases}
 \tag{4.18}$$

Similarly, we can get

$$\begin{aligned}
 D_h^2 q(x_i) &:= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} q(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right) \\
 &\leq \frac{2^{r(\alpha-1)+1}}{\Gamma(2-\alpha)} (x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}), \quad i = 1, \dots, 2N-1
 \end{aligned}
 \tag{4.19}$$

So, we get the result.  $\square$

Notice that

$$(4.20) \quad x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha}$$

We can get

THEOREM 4.4. *There exists a real  $\lambda = \lambda(T, \alpha, r) > 0$  and  $C = C(T, \alpha, r) > 0$  such that  $B := A(\lambda I + G)$  is an  $M$  matrix. And*

$$(4.21) \quad M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases}$$

*Proof.* By Lemma 4.2 with  $C_A$  and Lemma 4.3 with  $C_{AG}$ , it's sufficient to take  $\lambda = (C + 2TC_{AG})/C_A$ , then

$$(4.22) \quad M_i \geq C \left( (x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \right) \quad \square$$

**4.2. Proof of Theorem 2.6.** For equation

$$(4.23) \quad AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1}U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1}U = F$$

which means

$$(4.24) \quad \sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + g(x_j)} = -\tau_i$$

where  $\epsilon_i = u(x_i) - u_i$ .

And if

$$(4.25) \quad \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| = \max_{1 \leq i \leq 2N-1} \left| \frac{\epsilon_i}{\lambda + g(x_i)} \right|$$

Then, since  $B = A(\lambda I + G)$  is an  $M$  matrix, it is Strictly diagonally dominant. Thus,

$$\begin{aligned} |\tau_{i_0}| &= \left| \sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + g(x_j)} \right| \\ &\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_j}{\lambda + g(x_j)} \right| \\ &\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \\ &= \sum_{j=1}^{2N-1} b_{i_0,j} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \\ &= M_{i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \end{aligned} \quad (4.26)$$

By Theorem 2.5 and Theorem 4.4,

We know that there exists constants  $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ , and  $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that

$$(4.27) \quad \left| \frac{\epsilon_i}{\lambda + g(x_i)} \right| \leq \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} + C_2(r-1)h^2$$

615 as  $\lambda + g(x_i) \leq \lambda + T$

616 So, we can get

$$617 \quad (4.28) \quad |\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{\alpha}{2}, 2\}}$$

618 The convergency has been proved.

619 Remarks:

## 5. Experimental results.

### 5.1. $f \equiv 1$ .

### 5.2. $f = x^\gamma, \gamma < 0$ . Appendix A. Approximate of difference quotients.

LEMMA A.1. If  $g(x) \in C^2(\Omega)$ , there exists  $\xi \in (x_{i-1}, x_{i+1})$  such that

$$(A.1) \quad D_h^2 g(x_i) = g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1})$$

And if  $g(x) \in C^4(\Omega)$ , then  
(A.2)

$$D_h^2 g(x_i) = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \int_{x_{i-1}}^{x_i} g'''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g'''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right)$$

*Proof.*

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})$$

Substitute them in the left side of (A.1), we have

$$\begin{aligned} D_h^2 g(x_i) &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (g(x_{i+1}) - g(x_i) + \frac{1}{h_i} (g(x_{i-1}) - g(x_i))) \right) \\ &= \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) \end{aligned}$$

Now, using [intermediate value theorem](#), there exists  $\xi \in [\xi_1, \xi_2]$  such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g'''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g'''(y) \frac{(x_{i+1} - y)^3}{3!} dy$$

Especially,

$$(A.3) \quad \begin{aligned} \int_{x_{i-1}}^{x_i} g'''(y) \frac{(y - x_{i-1})^3}{3!} dy &= \frac{h_i^4}{4!} g'''(\eta_1) \\ \int_{x_i}^{x_{i+1}} g'''(y) \frac{(x_{i+1} - y)^3}{3!} dy &= \frac{h_{i+1}^4}{4!} g'''(\eta_2) \end{aligned}$$

where  $\eta_1 \in (x_{i-1}, x_i), \eta_2 \in (x_i, x_{i+1})$ .  $\square$



LEMMA A.2. Denote  $y_j^\theta = (1 - \theta)x_{j-1} + \theta x_j, \theta \in (0, 1)$ ,

$$(A.4) \quad u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in (x_{j-1}, x_j)$$

$$(A.5) \quad u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

where  $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$ .

*Proof.* By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in (x_{j-1}, y_j^\theta)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in (y_j^\theta, x_j)$$

Thus

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta)u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where  $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$ . Thus

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

LEMMA A.3. For  $x \in [x_{j-1}, x_j]$

$$(A.6) \quad \begin{aligned} |u(x) - \Pi_h u(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^x u'(y) dy - \frac{x - x_{j-1}}{h_j} \int_x^{x_j} u'(y) dy \right| \\ &\leq \int_{x_{j-1}}^{x_j} |u'(y)| dy \end{aligned}$$

If  $x \in [0, x_1]$ , with Corollary 2.4, we have

$$(A.7) \quad |u(x) - \Pi_h u(x)| \leq \int_0^{x_1} |u'(y)| dy \leq \int_0^{x_1} C y^{\alpha/2-1} dy \leq C \frac{2}{\alpha} x_1^{\alpha/2} = C \frac{2}{\alpha} h_1^{\alpha/2}$$

Similarly, if  $x \in [x_{2N-1}, 1]$ , we have

$$(A.8) \quad |u(x) - \Pi_h u(x)| \leq C \frac{2}{\alpha} (2T - x_{2N-1})^{\alpha/2} = C \frac{2}{\alpha} h_{2N}^{\alpha/2}$$

LEMMA A.4. *By Lemma A.2, Corollary 2.4 and Lemma B.1, There is a constant*  
 $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  for  $2 \leq j \leq N$ ,

$$(A.9) \quad |u(y) - \Pi_h u(y)| \leq h_j^2 \max_{\xi \in [x_{j-1}, x_j]} |u''(\xi)| \leq Ch^2 y^{\alpha/2-2/r}, \quad \text{for } y \in (x_{j-1}, x_j)$$

symmetricly, for  $N < j \leq 2N - 1$ , we have

$$(A.10) \quad |u(y) - \Pi_h u(y)| \leq Ch^2 (2T - y)^{\alpha/2-2/r}, \quad \text{for } y \in (x_{j-1}, x_j)$$

LEMMA A.5.

$$(A.11) \quad b^{1-\theta} |a^\theta - b^\theta| \leq |a - b| \quad (\text{also } a^{1-\theta} |a^\theta - b^\theta| \leq |a - b|), \quad a, b \geq 0, \theta \in [0, 1]$$

**Appendix B. Proofs of some technical details.** Review that  $h = \frac{1}{N}$  and the definition of  $\simeq$  in subsection 2.1

LEMMA B.1.

$$(B.1) \quad h_i \simeq \begin{cases} hx_i^{1-1/r}, & 1 \leq i \leq N \\ h(2T - x_{i-1})^{1-1/r}, & N < i \leq 2N \end{cases}$$

Since  $i^r - (i-1)^r \simeq i^{r-1}$ , for  $i \geq 1$ .

And

$$(B.2) \quad h_i \simeq h_{i+1}, \quad x_i \simeq x_{i+1} \simeq y_i^\theta, \quad \text{for } 1 \leq i \leq 2N - 1, \theta \in (0, 1)$$

LEMMA B.2. *There is a constant  $C$  such that for  $i = 1, 2, \dots, 2N - 1$*

$$(B.3) \quad |h_{i+1} - h_i| \leq Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N - 1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N - 1 \end{cases}$$

*Proof.* By (2.2),

$$(B.4) \quad h_{i+1} - h_i = \begin{cases} T \left( \left( \frac{i+1}{N} \right)^r - 2 \left( \frac{i}{N} \right)^r + \left( \frac{i-1}{N} \right)^r \right), & 1 \leq i \leq N - 1 \\ 0, & i = N \\ -T \left( \left( \frac{2N-i-1}{N} \right)^r - 2 \left( \frac{2N-i}{N} \right)^r + \left( \frac{2N-i+1}{N} \right)^r \right), & N + 1 \leq i \leq 2N - 1 \end{cases}$$

Since

$$(B.5) \quad (i+1)^r - 2i^r + (i-1)^r \simeq r(r-1)i^{r-2}, \quad \text{for } i \geq 1$$

We get the result.  $\square$

LEMMA B.3. *there is a constant  $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$  such that*

$$(B.6) \quad \begin{aligned} & \frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right| \\ & \leq Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N - 1 \end{cases} \end{aligned}$$

*Proof.* By Lemma 2.2, we have for  $1 \leq i \leq N$

$$(B.7) \quad \left| \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy \right| \leq \frac{\|f\|_{\beta}^{(\alpha/2)}}{3!} \int_{x_{i-1}}^{x_i} y^{-\alpha/2-2} (y - x_{i-1})^3 dy$$

For  $i = 1$ ,

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2} (y - x_{i-1})^3 dy = \int_0^{x_1} y^{1-\alpha/2} dy = \frac{1}{2 - \alpha/2} x_1^{2-\alpha/2} = \frac{1}{2 - \alpha/2} x_1^{-\alpha/2-2} h_1^4$$

And for  $2 \leq i \leq N$ , since  $x_i \simeq x_{i-1} \leq y \leq x_i$ , we have

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2} (y - x_{i-1})^3 dy \simeq \int_{x_{i-1}}^{x_i} x_i^{-\alpha/2-2} (y - x_{i-1})^3 dy = \frac{1}{4!} x_i^{-\alpha/2-2} h_i^4$$

So for  $1 \leq i \leq N$ , we have

$$(B.8) \quad \left| \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy \right| \leq C x_i^{-\alpha/2-2} h_i^4$$

and similarly,

$$(B.9) \quad \left| \int_{x_i}^{x_{i+1}} f''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right| \leq C x_i^{-\alpha/2-2} h_{i+1}^4$$

Thus for  $1 \leq i \leq N$ , with Lemma B.1 we have

$$(B.10) \quad \begin{aligned} & \frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right| \\ & \leq C x_i^{-\alpha/2-2} \frac{2}{h_i + h_{i+1}} (h_i^3 + h_{i+1}^3) \simeq x_i^{-\alpha/2-2} h_i^2 \simeq x_i^{-\alpha/2-2} h_i^2 x_i^{2-2/r} \\ & = C h^2 x_i^{-\alpha/2-2/r} \end{aligned}$$

It's symmetric for  $N < i \leq 2N - 1$ .  $\square$

LEMMA B.4. *There is a constant  $C = C(\alpha, r)$  such that for all  $1 \leq i \leq 2N - 1$ ,  $1 \leq j \leq 2N$  s.t.  $\min\{|j - i|, |j - 1 - i|\} \geq 2$  and  $y \in [x_{j-1}, x_j]$ , we have*

$$(B.11) \quad D_h^2 K_y(x_i) \simeq |y - x_i|^{-1-\alpha}$$

*Proof.* Sinec  $y - x_{i-1}, y - x_i, y - x_{i+1}$  have the same sign, by Lemma A.1,

$$D_h^2 K_y(x_i) = \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in (x_{i-1}, x_{i+1})$$

however,  $|y - \xi| \simeq |y - x_i|$ , we get the result.  $\square$

LEMMA B.5. *While  $0 \leq i < N/2$ , By Lemma A.3*

$$(B.12) \quad \begin{aligned} |T_{i1}| & \leq C \int_0^{x_1} x_1^{\alpha/2} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ & = C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2} |x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha}| \\ & \leq C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2+2-\alpha} = C \frac{1}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \quad 0 < 2 - \alpha < 1 \end{aligned}$$

704 For  $2 \leq j \leq N$ , by Lemma A.2 and Corollary 2.4

$$705 \quad (B.13) \quad \begin{aligned} |T_{ij}| &\leq \frac{C}{4} \int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2} \frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &\leq \frac{C}{4\Gamma(3-\alpha)} h_j^2 x_{j-1}^{\alpha/2-2} ||x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha}| \end{aligned}$$

706 LEMMA B.6. There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that

$$707 \quad (B.14) \quad \sum_{j=1}^3 V_{1j} \leq Ch^2 x_1^{-\alpha/2-2/r}$$

$$708 \quad (B.15) \quad \sum_{j=1}^4 V_{2j} \leq Ch^2 x_2^{-\alpha/2-2/r}$$

709 *Proof.* For  $0 \leq i \leq 3, 1 \leq j \leq 4$ , by Lemma A.3, Lemma A.4 and (3.11)

$$710 \quad (B.16) \quad T_{ij} \leq Cx_1^{2-\alpha/2} \simeq h_1^2 h^2 x_1^{-\alpha/2-2/r} \simeq h_1^2 h^2 x_2^{-\alpha/2-2/r}$$

711 Therefore, by (3.12), we get the result.  $\square$

712

713 LEMMA B.7. There exists a constant  $C = C(T, r, l)$  such that For  $3 \leq i \leq N -$   
 714  $1, \lceil \frac{i}{2} \rceil \leq j \leq \min\{2i, N\}$ ,  
 715 when  $\xi \in (x_{i-1}, x_{i+1})$ ,

$$716 \quad (B.17) \quad (h_{j-i}^3(\xi))' \leq (r-1)Ch^2 x_i^{1-2/r} h_j$$

717

$$718 \quad (B.18) \quad (h_{j-i}^4(\xi))' \leq (r-1)Ch^2 x_i^{1-2/r} h_j^2$$

719 *Proof.* From (??)

$$720 \quad (B.19) \quad y'_{j-i}(x) = y_{j-i}^{1-1/r}(x) x^{1/r-1}$$

$$721 \quad (B.20) \quad y''_{j-i}(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i}$$

722 For  $\xi \in (x_{i-1}, x_{i+1})$  and  $2 \leq k \leq j \leq \min\{2i-1, N-1\}$ , using Lemma B.1

$$723 \quad \xi \simeq x_i \simeq x_j$$

724

$$725 \quad h_{j-i}(\xi) \simeq h_j \simeq h x_j^{1-1/r} \simeq h x_i^{1-1/r}$$

$$726 \quad (B.21) \quad \begin{aligned} h'_{j-i}(\xi) &= y'_{j-i}(\xi) - y'_{j-i-1}(\xi) \\ &= \xi^{1/r-1} (y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \end{aligned}$$

727 Since

$$\begin{aligned}
 y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) &\leq x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r} \\
 &= T^{1-1/r} N^{1-r} ((j+1)^{r-1} - (j-2)^{r-1}) \\
 (B.22) \quad &\leq C(r-1)j^{r-2}N^{1-r} \\
 &= C(r-1)hx_j^{1-2/r}
 \end{aligned}$$

729 Therefore,

$$\begin{aligned}
 (B.23) \quad h'_{j-i}(\xi) &\leq Cx_i^{1/r-1}(r-1)hx_j^{1-2/r} \simeq (r-1)hx_i^{-1/r} \\
 731 \quad &\text{for } l = 3, 4
 \end{aligned}$$

$$\begin{aligned}
 (h^l_{j-i}(\xi))' &= lh_{j-i}^{l-1}(\xi)h'_{j-i}(\xi) \\
 &\leq Ch_{j-i}^{l-1}(\xi)(r-1)hx_i^{-1/r} \\
 732 \quad (B.24) \quad &\simeq Ch_j^{l-2}hx_j^{1-1/r}(r-1)hx_i^{-1/r} \\
 &\simeq C(r-1)h^2x_i^{1-2/r}h_j^{l-2}
 \end{aligned}$$

733 Meanwhile, we can get

$$(B.25) \quad h^3_{j-i}(\xi) \simeq h_j^3 \leq Ch^2x_i^{2-2/r}h_j$$

$$(B.26) \quad h^4_{j-i}(\xi) \simeq h_j^4 \leq Ch^2x_i^{2-2/r}h_j^2 \quad \square$$

736

737 **LEMMA B.8.** *There exists a constant  $C = C(T, r, l)$  such that For  $3 \leq i \leq N -$*   
 738  *$1, \lceil \frac{i}{2} \rceil \leq j \leq \min\{2i, N\},$*   
 739 *when  $\xi \in (x_{i-1}, x_{i+1}),$*

$$(B.27) \quad (h^3_{j-i}(\xi))'' \leq C(r-1)h^2x_i^{-2/r}h_j$$

*Proof.*

$$(B.28) \quad (h^3_{j-i}(\xi))'' = 6h_{j-i}(\xi)(h'_{j-i}(\xi))^2 + 3h^2_{j-i}(\xi)h''_{j-i}(\xi)$$

742 By (B.23)

$$(B.29) \quad h_{j-i}(\xi)(h'_{j-i}(\xi))^2 \leq Ch_j(r-1)^2h^2x_i^{-2/r}$$

744 For the second partial

$$\begin{aligned}
 h''_{j-i}(\xi) &= y''_{j-i}(\xi) - y''_{j-i-1}(\xi) \\
 745 \quad (B.30) \quad &= \frac{1-r}{r}\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\
 &= \frac{1-r}{r}\xi^{1/r-2}((y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi))Z_{j-i} + y_{j-i-1}^{1-2/r}(\xi)Z_1)
 \end{aligned}$$

746 but

$$\begin{aligned}
 |y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi)| &\leq |x_{j+1}^{1-2/r} - x_{j-2}^{1-2/r}| \\
 747 \quad (B.31) \quad &= T^{1-2/r}N^{2-r}|(j+1)^{r-2} - (j-2)^{r-2}| \\
 &\leq C|r-2|N^{2-r}j^{r-3} \\
 &= C|r-2|hx_j^{1-3/r}
 \end{aligned}$$

748 So we can get

$$749 \quad (B.32) \quad |h''_{j-i}(\xi)| \leq C(r-1)x_i^{1/r-2}(|r-2|h x_i^{1-3/r} x_i^{1/r} + x_i^{1-2/r} h) \\ \leq C(r-1)h x_i^{-1-1/r}$$

750 Summarizes, we have

$$751 \quad (B.33) \quad (h^3_{j-i}(\xi))'' \leq C(r-1)h^2 x_i^{-2/r} h_j \quad \square$$

752 *proof of Lemma 3.17.* From (??)

$$753 \quad (B.34) \quad y'_{j-i}(x) = y_{j-i}^{1-1/r}(x) x^{1/r-1}$$

$$754 \quad (B.35) \quad y''_{j-i}(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i}$$

755 Since

$$756 \quad y_{j-i}^\theta(\xi) \simeq x_j \simeq x_i$$

757 We have known

$$758 \quad (B.36) \quad u''(y_{j-i}^\theta(\xi)) \leq C(y_{j-i}^\theta(\xi))^{\alpha/2-2} \simeq x_j^{\alpha/2-2} \simeq x_i^{\alpha/2-2}$$

759

$$760 \quad (B.37) \quad (u''(y_{j-i}^\theta(\xi)))' = u'''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta(\xi))' \\ \leq C x_i^{\alpha/2-3} \xi^{1/r-1} y_{j-i}^{1-1/r}(\xi) \\ \simeq x_i^{\alpha/2-3} x_i^{1/r-1} x_i^{1-1/r} = C x_i^{\alpha/2-3}$$

761

$$762 \quad (B.38) \quad (u''(y_{j-i}^\theta(\xi)))'' = u''''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta(\xi))'^2 + u'''(y_{j-i}^\theta(\xi))y_{j-i}^{\theta''}(\xi) \\ \leq C x_i^{\alpha/2-4} + C x_i^{\alpha/2-3} \frac{r-1}{r} x_i^{1-2/r} x_i^{1/r-2} Z_{|j-i|+1} \\ \leq C x_i^{\alpha/2-4} + C \frac{r-1}{r} x_i^{\alpha/2-3} x_i^{-1-1/r} x_i^{1/r} \\ = C x_i^{\alpha/2-4} \quad \square$$

*Proof of Lemma 3.18.*

$$763 \quad (B.39) \quad |y_{j-i}^\theta(\xi) - \xi| = |\theta(y_{j-i-1}(\xi) - \xi) + (1-\theta)(y_{j-i}(\xi) - \xi)| \\ = \theta|y_{j-i-1}(\xi) - \xi| + (1-\theta)|y_{j-i}(\xi) - \xi|$$

764 where  $y_{j-i-1}(\xi) - \xi$  and  $y_{j-i}(\xi) - \xi$  have the same sign ( $\geq 0$  or  $\leq 0$ ), independent  
765 with  $\xi$ .

766 Since  $|y_{j-i}(\xi) - \xi| = \text{sign}(j-i)(y_{j-i}(\xi) - \xi)$  is increasing with  $\xi$ ,

$$(B.40) \quad \left(\frac{i-1}{i}\right)^r |x_j - x_i| \leq |x_{j-1} - x_{i-1}| \leq |y_{j-i}(\xi) - \xi| \leq |x_{j+1} - x_{i+1}| \leq \left(\frac{i+1}{i}\right)^r |x_j - x_i|$$

768 we have

$$769 \quad (B.41) \quad |y_{j-i}(\xi) - \xi| \simeq |x_j - x_i|$$

Similarly,  $|y_{j-1-i}(\xi) - \xi| \simeq |x_{j-1} - x_i|$ . Thus, with (B.39) , (B.41) and (2.17) we get

$$(B.42) \quad |y_{j-i}^\theta(\xi) - \xi| \simeq |y_j^\theta - x_i|$$

Next, since  $|y_{j-i}^\theta(\xi) - \xi| = \text{sign}(j - i - 1 + \theta)(y_{j-i}^\theta(\xi) - \xi)$ , so we can derivate it.

$$(B.43) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' = (\alpha - 1)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha} |(y_{j-i}^\theta(\xi))' - 1|$$

While, similar with (B.39), we have

$$(B.44) \quad |(y_{j-i}^\theta(\xi))' - 1| = (1 - \theta)|y_{j-i-1}'(\xi) - 1| + \theta|y_{j-i}'(\xi) - 1|$$

By Lemma A.5 and (B.41), we have

$$(B.45) \quad \begin{aligned} |y_{j-i}'(\xi) - 1| &= \xi^{1/r-1} |y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| \\ &\leq \xi^{-1} |y_{j-i}(\xi) - \xi| \\ &\simeq x_i^{-1} |x_j - x_i| \end{aligned}$$

So similar with (B.42), we can get

$$(B.46) \quad |(y_{j-i}^\theta(\xi))' - 1| \leq C x_i^{-1} |y_j^\theta - x_i|$$

Combine with (B.42), we get

$$(B.47) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C |y_j^\theta - x_i|^{-\alpha} x_i^{-1} |y_j^\theta - x_i| = C |y_j^\theta - x_i|^{1-\alpha} x_i^{-1}$$

Finally, we have

$$(B.48) \quad \begin{aligned} (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' &= \alpha(\alpha - 1)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha-1} ((y_{j-i}^\theta(\xi))' - 1)^2 \\ &\quad + \text{sign}(j - i - 1 + \theta)(1 - \alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha} (y_{j-i}^\theta(\xi))'' \end{aligned}$$

For

$$(B.49) \quad (y_{j-i}^\theta(\xi))'' = (1 - \theta)y_{j-i-1}''(\xi) + \theta y_{j-i}''(\xi)$$

and

$$(B.50) \quad \begin{aligned} y_{j-i}''(\xi) &= \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i} \\ &\simeq \frac{1-r}{r} x_j^{1-2/r} x_i^{1/r-2} Z_{j-i} \end{aligned}$$

while by Lemma A.5

$$(B.51) \quad |Z_{j-i}| = |x_j^{1/r} - x_i^{1/r}| \leq |x_j - x_i| x_i^{1/r-1}$$

we have

$$(B.52) \quad |y_{j-i}''(\xi)| \leq C(r-1)x_i^{-2}|x_j - x_i|$$

Therefore

$$(B.53) \quad |(y_{j-i}^\theta(\xi))''| \leq C(r-1)x_i^{-2}|y_j^\theta - x_i|$$

Then, combine with (B.46),

$$(B.54) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C |y_j^\theta - x_i|^{1-\alpha} x_i^{-2} \quad \square$$

796 *proof of Lemma 3.20.* For  $\lceil \frac{i}{2} \rceil \leq j \leq \min\{2i-1, N-1\}$

$$\begin{aligned} & \frac{Q_{i,j}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{i,j}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \\ 797 \quad (B.55) \quad &= \frac{Q_{i,j}^\theta(x_{i+1}) - Q_{i,j}^\theta(x_i)}{h_{i+1}} u'''(\eta_{j+1}^\theta) + Q_{i,j}^\theta(x_i) \frac{u'''(\eta_{j+1}^\theta) - u'''(\eta_j^\theta)}{h_{i+1}} \end{aligned}$$

798 Since mean value theorem

$$799 \quad (B.56) \quad \frac{Q_{i,j}^\theta(x_{i+1}) - Q_{i,j}^\theta(x_i)}{h_{i+1}} = Q_{i,j}^{\theta'}(\xi), \quad \xi \in (x_i, x_{i+1})$$

800 From (3.21) and Leibniz rule, by Lemma B.7 and Lemma 3.18, we have

$$801 \quad (B.57) \quad |Q_{i,j}^{\theta'}(\xi)| \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2$$

802 And by ?? and Lemma B.1

$$803 \quad (B.58) \quad Q_{i,j}^\theta(x_i) = h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \simeq Ch^2 x_i^{2-2/r} \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2$$

804 With  $\eta_j^\theta \in (x_{j-1}, x_j)$

$$805 \quad u'''(\eta_{j+1}^\theta) \leq C(\eta_{j+1}^\theta)^{\alpha/2-3} \simeq x_j^{\alpha/2-3} \simeq x_i^{\alpha/2-3}$$

806 and

$$\begin{aligned} & \frac{u'''(\eta_{j+1}^\theta) - u'''(\eta_j^\theta)}{h_{i+1}} = u''''(\eta) \frac{\eta_{j+1}^\theta - \eta_j^\theta}{h_{i+1}} \\ 807 \quad & \leq C\eta^{\alpha/2-4} \frac{x_{j+1} - x_{j-1}}{h_{i+1}} = C\eta^{\alpha/2-4} \frac{h_{j+1} + h_j}{h_{i+1}} \\ & \simeq x_j^{\alpha/2-4} \simeq x_i^{\alpha/2-4} \end{aligned}$$

808 So we have

$$\begin{aligned} & \frac{Q_{i,j}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{i,j}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \\ 809 \quad (B.59) \quad & \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 x_i^{\alpha/2-3} + Ch^2 x_i^{2-2/r} \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 x_{j-1}^{\alpha/2-4} \\ & = Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j^2 \end{aligned}$$

810 while  $h_j \simeq h_i$ , substitute into the inequality above, we get the goal

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{i,j}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{i,j}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ 811 \quad (B.60) \quad & \leq \frac{1}{h_i} Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j h_i \\ & = Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned}$$

812 While, the later is similar. □



813

 814 **LEMMA B.9.** *There exists a constant  $C = C(T, r)$  such that For  $N/2 \leq i \leq N-1$ ,*  
 815  *$N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$ ,  $l = 3, 4$ ,  $\xi \in (x_{i-1}, x_{i+1})$ , we have*

816 (B.61) 
$$h_{j-i}^l(\xi) \leq Ch_j^l \leq Ch^2 h_j^{l-2}$$

817 (B.62) 
$$(h_{j-i-1}^l(\xi))' \leq C(r-1)h^2 h_j^{l-2}$$

818 (B.63) 
$$(h_{j-i}^3(\xi))'' \leq C(r-1)h^2 h_j$$

*Proof.*

819 (B.64) 
$$\begin{aligned} (h_{j-i}(\xi))' &= y_{j-i}'(\xi) - y_{j-i-1}'(\xi) \\ &= \xi^{1/r-1}((2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}) \leq 0 \end{aligned}$$

820 Thus,

821 (B.65) 
$$Ch_j \leq h_{j+1} \leq h_{j-i}(\xi) \leq h_{j-i}(x_{i-1}) = h_{j-1} \leq Ch_j$$

 822 So as  $4^{-r}T \leq 2T - x_j \leq T$ ,  $2^{-r}T \leq x_i \leq T$ , we have

823 (B.66) 
$$h_{j-i}^l(\xi) \leq Ch_j^l \leq Ch^2(2T - x_j)^{2-2/r} h_j^{l-2} \leq Ch^2 h_j^{l-2}$$

824 Since

825 (B.67) 
$$\begin{aligned} & |(2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}| \\ &= |(Z_{2N-(j-i)} - \xi^{1/r})^{r-1} - (Z_{2N-(j-1-i)} - \xi^{1/r})^{r-1}| \\ &= (r-1)Z_1(Z_{2N-(j-i-\gamma)} - \xi^{1/r})^{r-2} \quad \gamma \in [0, 1] \\ &\leq C(r-1)h(2T - x_j)^{1-2/r} \end{aligned}$$

826 we have

827 (B.68) 
$$|(h_{j-i}(\xi))'| \leq C(r-1)h(2T - x_j)^{1-2/r} x_i^{1/r-1}$$

828 And

829 (B.69) 
$$\begin{aligned} (h_{j-i}^l(\xi))' &= l h_{j-i}^{l-1}(\xi) h_{j-i}'(\xi) \\ &\leq C(r-1)h_j^{l-1} h(2T - x_j)^{1-2/r} x_i^{1/r-1} \\ &\leq C(r-1)h^2 h_j^{l-2} (2T - x_j)^{2-3/r} x_i^{1-1/r} \\ &\leq C(r-1)h^2 h_j^{l-2} \end{aligned}$$

(B.70)  $\square$

$$\begin{aligned} (h_{j-i}^3(\xi))'' &= 6h_{j-i}(\xi)(y_{j-i}'(\xi) - y_{j-i-1}'(\xi))^2 + 3h_{j-i}^2(\xi)(y_{j-i}''(\xi) - y_{j-i-1}''(\xi)) \\ &\leq C(r-1)h_j h^2 + Ch_j^2 \frac{1-r}{r} \xi^{1/r-2} ((2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-1-i)}) \\ &\leq C(r-1)h_j h^2 + C(r-1)h_j^2 (C(r-2)h(2T - x_j)^{1-3/r} Z_{2N-(j-i)} + Z_1(2T - x_{j-1})^{1-2/r}) \\ &\leq C(r-1)h_j h^2 + C(r-1)h_j^2 h = Ch^2 h_j \end{aligned}$$

831

832 LEMMA B.10. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For*  
 833  *$N/2 \leq i \leq N-1$ ,  $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$ ,  $\xi \in (x_{i-1}, x_{i+1})$ , we have*

$$834 \quad (B.71) \quad u''(y_{j-i}^\theta(\xi)) \leq C$$

$$835 \quad (B.72) \quad (u''(y_{j-i}^\theta(\xi)))' \leq C$$

$$836 \quad (B.73) \quad (u''(y_{j-i}^\theta(\xi)))'' \leq C$$

*Proof.*

$$837 \quad (B.74) \quad x_{j-2} \leq y_{j-i}^\theta(\xi) \leq x_{j+1} \Rightarrow 4^{-r}T \leq 2T - y_{j-i}^\theta(\xi) \leq T$$

838 Thus, for  $l = 2, 3, 4$ ,

$$839 \quad (B.75) \quad u^{(l)}(y_{j-i}^\theta(\xi)) \leq C(2T - y_{j-i}^\theta(\xi))^{\alpha/2-l} \leq C$$

840 and

$$\begin{aligned} (y_{j-i}^\theta(\xi))' &= \theta y_{j-1-i}'(\xi) + (1-\theta)y_{j-i-1}'(\xi) \\ 841 \quad (B.76) \quad &= \xi^{1/r-1}(\theta(2T - y_{j-1-i}(\xi))^{1-1/r} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-1/r}) \\ &\leq C(2T - x_{j-2})^{1-1/r} \leq C \end{aligned}$$

842 With

$$843 \quad (B.77) \quad Z_{2N-j-i} \leq 2T^{1/r}$$

$$\begin{aligned} 844 \quad (B.78) \quad &(y_{j-i}^\theta(\xi))'' = \theta y_{j-1-i}''(\xi) + (1-\theta)y_{j-i-1}''(\xi) \\ &= \frac{1-r}{r} \xi^{1/r-2}(\theta(2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)} + (1-\theta)(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)}) \\ &\leq C(r-1) \end{aligned}$$

846 Therefore,

$$\begin{aligned} 847 \quad (B.79) \quad &(u''(y_{j-i}^\theta(\xi)))' = u'''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta(\xi))' \\ &\leq C \end{aligned}$$

848

$$\begin{aligned} 849 \quad (B.80) \quad &(u''(y_{j-i}^\theta(\xi)))'' = u'''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta(\xi))'^2 + u''''(y_{j-i}^\theta(\xi))y_{j-i}^\theta(\xi)'' \\ &\leq C + C(r-1) = C \end{aligned} \quad \square$$

850

851 LEMMA B.11. *There exists a constant  $C = C(T, \alpha, r)$  such that For  $N/2 \leq i \leq$*   
 852  *$N-1$ ,  $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$ ,  $\xi \in (x_{i-1}, x_{i+1})$*

$$853 \quad (B.81) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}$$

$$854 \quad (B.82) \quad |(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'| \leq C|y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N)$$

$$\begin{aligned} (B.83) \quad &|(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})''| \leq C(r-1)|y_j^\theta - x_i|^{-\alpha} + C|y_j^\theta - x_i|^{-1-\alpha}(|2T - x_i - y_j^\theta| + h_N)^2 \\ 855 \quad & \end{aligned}$$

*Proof.* Since  $y_{j-i-1}(\xi) > x_{j-2} \geq x_N > \xi$

$$(B.84) \quad y_{j-i}^\theta(\xi) - \xi = (1 - \theta)(y_{j-i-1}(\xi) - \xi) + \theta(y_{j-i}(\xi) - \xi) > 0$$

$$(B.85) \quad \begin{aligned} (y_{j-i}(\xi) - \xi)'' &= y_{j-i}''(\xi) \\ &= \frac{1-r}{r} \xi^{1/r-2} (2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} \leq 0 \end{aligned}$$

It's concave, so

$$(B.86) \quad y_{j-i}(\xi) - \xi \geq \min_{\xi \in \{x_{i-1}, x_{i+1}\}} y_{j-i}(\xi) - \xi = \min\{x_{j+1} - x_{i+1}, x_{j-1} - x_{i-1}\} \geq C(x_j - x_i)$$

With (B.84), we have

$$(B.87) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}$$

By Lemma A.5

$$(B.88) \quad \begin{aligned} |y_{j-i}'(\xi) - 1| &= \xi^{1/r-1} |(2T - y_{j-i}(\xi))^{1-1/r} - \xi^{1-1/r}| \\ &\leq \xi^{-1} |2T - y_{j-i}(\xi) - \xi| \end{aligned}$$

$$(B.89) \quad \begin{aligned} |2T - \xi - y_{j-i}(\xi)| &\leq |2T - x_i - x_j| + |x_i - \xi| + |x_j - y_{j-i}(\xi)| \\ &\leq |2T - x_i - x_j| + h_{i+1} + h_j \\ &\leq C(|2T - x_i - x_j| + h_N) \end{aligned}$$

With  $\xi \simeq x_i \simeq 1$ ,

$$(B.90) \quad |y_{j-i}'(\xi) - 1| \leq C(|2T - x_i - x_j| + h_N)$$

Thus,

$$(B.91) \quad \begin{aligned} |(y_{j-i}^\theta(\xi))' - 1| &\leq (1 - \theta)|y_{j-i-1}'(\xi) - 1| + \theta|y_{j-i}'(\xi) - 1| \\ &\leq C((1 - \theta)|2T - x_i - x_{j-1}| + \theta|2T - x_i - x_j| + h_N) \\ &= C(|2T - x_i - y_j^\theta| + h_N) \end{aligned}$$

So

$$(B.92) \quad \begin{aligned} |(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'| &= |1 - \alpha||y_{j-i}^\theta(\xi) - \xi|^{-\alpha} |(y_{j-i}^\theta(\xi))' - 1| \\ &\leq C|y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N) \end{aligned}$$

(B.93)

$$\begin{aligned} |(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})''| &\leq |1 - \alpha||y_{j-i}^\theta(\xi) - \xi|^{-\alpha} |(y_{j-i}^\theta(\xi) - \xi)''| + \alpha(\alpha - 1)|y_{j-i}^\theta(\xi) - \xi|^{-1-\alpha} (y_{j-i}^\theta(\xi))' - 1|^2 \\ &\leq C(r - 1)|y_j^\theta - x_i|^{-\alpha} + C|y_j^\theta - x_i|^{-1-\alpha} (|2T - x_i - y_j^\theta| + h_N)^2 \end{aligned}$$

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