SECOND-ORDER ERROR ANALYSIS FOR FRACTIONAL LAPLACIAN VIA RIESZ DERIVATIVES ON GRADED MESHES*

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Abstract. This is an example SIAM LaTeX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

Key words. example, LaTeX

MSC codes. ??????????????????

1. Introduction.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$,

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x) & x \in \Omega, \\ u(x) = 0 & x \in \mathbb{R} \setminus \Omega, \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = \frac{-\kappa_\alpha}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_\Omega |x-y|^{1-\alpha} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

2. Preliminaries: Numeric scheme and main results.

2.1. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r & 0 \le i \le N, \\ 2T - T\left(\frac{2N-i}{N}\right)^r & N \le i \le 2N, \end{cases} \tag{2.1}$$

where $r \ge 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N \tag{2.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

And then, define the piecewise linear interpolant of the true solution u to be

$$\Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{2.4}$$

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For convience, we denote

$$I^{2-\alpha}u(x) := \frac{1}{\Gamma(2-\alpha)}\int_{\Omega}|x-y|^{1-\alpha}u(y)dy \tag{2.5}$$

and

$$D_h^2u(x_i) := \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}u(x_{i-1}) - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)u(x_i) + \frac{1}{h_{i+1}}u(x_{i+1})\right) \tag{2.6}$$

Now, we discretise (1.1) by replacing $u(x)$ by a continuous piecewise linear function

$$u_h(x) := \sum_{j=1}^{2N-1} u_j\phi_j(x) \tag{2.7}$$

whose nodal values u_j are to be determined by collocation at each mesh point x_i for $i = 1, 2, ..., 2N-1$:

$$-\kappa_\alpha D_h^\alpha u_h(x_i) := -\kappa_\alpha D_h^2 I^{2-\alpha}u_h(x_i) = f(x_i) =: f_i \tag{2.8}$$

Here,

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i)\, u_j = \sum_{j=1}^{2N-1} a_{ij}\, u_j \tag{2.9}$$

where

$$a_{ij} = -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i) \quad \text{for} \quad i,j = 1,2,...,2N-1 \tag{2.10}$$

We have replaced $(-\Delta)^{\alpha/2}u(x_i) = f(x_i)$ in (1.1) by $-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i)$ in (2.8), with truncation error

$$\tau_i := -\kappa_\alpha\left(D_h^\alpha \Pi_h u(x_i) - \frac{d^2}{dx^2}I^{2-\alpha}u(x_i)\right) \quad \text{for} \quad i = 1,2,...,2N-1 \tag{2.11}$$

where $-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^\alpha \phi_j(x_i)u(x_j) = \sum_{j=1}^{2N-1} a_{ij}u(x_j)$.

The discrete equation (2.8) can be written in matrix form

$$AU = F \tag{2.12}$$

where $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$, $U = (u_1, \cdots, u_{2N-1})^T$ is unknown and $F = (f_1, \cdots, f_{2N-1})^T$.

We can deduce a_{ij},

$$\begin{aligned} a_{ij} &= -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i) \\ &= -\kappa_\alpha \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}\tilde{a}_{i-1,j} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)\tilde{a}_{i,j} + \frac{1}{h_{i+1}}\tilde{a}_{i+1,j}\right) \end{aligned} \tag{2.13}$$

where

$$I^{2-\alpha}\Pi_h u(x_i) = \sum_{j=1}^{2N-1} I^{2-\alpha}\phi_j(x_i)u(x_j) = \sum_{j=1}^{2N-1}\tilde{a}_{ij}u(x_j) \tag{2.14}$$

and

(2.15)

$$\begin{aligned} \tilde{a}_{ij} &= I^{2-\alpha}\phi_j(x_i) \\ &= \frac{1}{\Gamma(4-\alpha)}\left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)|x_i - x_j|^{3-\alpha} + \frac{|x_i-x_{j+1}|^{3-\alpha}}{h_{j+1}}\right) \end{aligned}$$

2.2. Regularity of the true solution. For any $\beta > 0$, we use the standard notation $C^\beta(\bar\Omega), C^\beta(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^\beta(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous [1, p. 52] with exponent β' in Ω, where $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on open set Ω.

For $x \in \Omega = (0, 2T)$, define

$$\delta(x) = \operatorname{dist}(x, \partial\Omega) = \begin{cases} x & 0 < x \le T, \\ 2T - x & T < x < 2T, \end{cases}, \tag{2.16}$$

and $\delta(x,y) = \min\{\delta(x), \delta(y)\}$. Then we have the following δ-dependent Hölder norms.

Definition 2.1 (δ-dependent Hölder norms [2]). *Let $\beta = k+\beta' > 0$ and $\sigma \ge -\beta$, with k integer and $\beta' \in (0,1]$. For $w \in C^\beta(\Omega) = C^{k,\beta'}(\Omega)$, define the seminorm*

$$|w|_\beta^{(\sigma)} = \sup_{x,y\in\Omega} \left(\delta(x,y)^{\beta+\sigma} \frac{|w^{(k)}(x) - w^{(k)}(y)|}{|x-y|^{\beta'}} \right).$$

For $\sigma > -1$, we also define the norm $\|\cdot\|_\beta^{(\sigma)}$ as follows: in case that $\sigma \ge 0$,

$$\|w\|_\beta^{(\sigma)} = \sum_{l=0}^{k} \sup_{x\in\Omega} \left(\delta(x)^{l+\sigma} |w^{(l)}(x)| \right) + |w|_\beta^{(\sigma)},$$

while for $-1 < \sigma < 0$,

$$\|w\|_\beta^{(\sigma)} = \|w\|_{C^{-\sigma}(\bar\Omega)} + \sum_{l=1}^{k} \sup_{x\in\Omega} \left(\delta(x)^{l+\sigma} |D^l w(x)| \right) + |w|_\beta^{(\sigma)}.$$

Lemma 2.2. *[2, pp. 276-277] Assume $f \in L^\infty(\Omega)$. Let u be a solution of (1.1). Then, $u \in C^{\alpha/2}(\mathbb{R})$ and $u/\delta^{\alpha/2} \in C^\sigma(\bar\Omega)$ for some $\sigma \in (0, 1-\alpha/2)$, with*

$$\|u\|_{C^{\alpha/2}(\mathbb{R})} \le C\|f\|_{L^\infty(\Omega)} \quad \text{and} \quad \|u/\delta^{\alpha/2}\|_{C^\sigma(\bar\Omega)} \le C\|f\|_{L^\infty(\Omega)},$$

for some positive constant $C = C(\Omega, \alpha)$.

In particular, if $f \in L^\infty(\Omega)$, then

$$|u(x)| \le C\delta(x)^{\alpha/2} \quad \text{for all } x \in \Omega. \tag{2.17}$$

Lemma 2.3. *[2, Proposition 1.4] Let Ω be a bounded domain, and $\beta > 0$ be such that neither β nor $\beta + \alpha$ is an integer. Let $f \in C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R})$ be a solution of (1.1). Then, $u \in C^{\beta+\alpha}(\Omega)$ and*

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right), \tag{2.18}$$

for some positive constant $C = C(\Omega, \alpha, \beta)$.

By defination of δ-dependent Hölder norms, we have following result obviusly.

LEMMA 2.4. *Let $\beta = 4 - \alpha + \gamma$ with $0 < \gamma < \alpha - 1$. Assume that $f \in L^\infty(\Omega) \cap C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and u be a solution of (1.1). Then*

$$|u^{(l)}(x)| \leq C\delta(x)^{\alpha/2-l}, \quad l = 0, 1, 2, 3, 4, \tag{2.19}$$

where $C = C(\Omega, \alpha, \beta, f)$.

Proof. Our hypotheses imply that $2 < \beta < 3$, and $4 < \beta + \alpha < 5$. By Lemma 2.3, we have

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)}\right).$$

And by Definition 2.1 and Lemma 2.2

$$\sum_{l=1}^{4} \sup_{x\in\Omega}\left(\delta(x)^{l-\alpha/2}|w^{(l)}(x)|\right) \leq C\left(\|f\|_{L^\infty(\Omega)} + \|f\|_\beta^{(\alpha/2)}\right),$$

which is desired result $l = 1, 2, 3, 4$. The case $l = 0$ is covered by (2.17). ∎

LEMMA 2.5. *Let $\beta = 4 - \alpha + \gamma$ with $0 < \gamma < \alpha - 1$. Assume that $f \in C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, then*

$$|f^{(l)}(x)| \leq C\delta(x)^{-\alpha/2-l}, \quad l = 0, 1, 2, \tag{2.20}$$

where $C = C(\Omega, \alpha, \beta, f)$.

Proof. By Definition 2.1, with $2 < \beta < 3$ ∎

$$\sum_{l=0}^{2} \sup_{x\in\Omega}\left(\delta(x)^{l+\alpha/2}|f^{(l)}(x)|\right) \leq \|f\|_\beta^{(\alpha/2)}.$$

And in this paper bellow, without special instructions, we allways assume that $\beta = 4 - \alpha + \gamma$ with $0 < \gamma < \alpha - 1$, $f \in L^\infty(\Omega) \cap C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$.

2.3. Main results. Here we state our main results; the proof is deferred to section 3 and section 4.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 2.6 (Local Truncation Error). *Let $\alpha \in (1, 2)$ and $f \in L^\infty(\Omega) \cap C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, where $\beta = 4 - \alpha + \gamma$ with $0 < \gamma < \alpha - 1$. Then,*

$$\begin{aligned} |\tau_i| &= |-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i)| \\ &\leq Ch^{\min\{\frac{r\alpha}{2},2\}}\delta(x_i)^{-\alpha} + C(r-1)h^2(T - \delta(x_i) + h_N)^{1-\alpha}. \end{aligned} \tag{2.21}$$

THEOREM 2.7 (Global Error). *Let $\alpha \in (1, 2)$ and $f \in L^\infty(\Omega) \cap C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, where $\beta = 4-\alpha+\gamma$ with $0 < \gamma < \alpha-1$. Let u_i be the approximate solution of $u(x_i)$ computed by the discretization scheme (2.12). Then,*

$$\max_{1\leq i\leq 2N-1} |u_i - u(x_i)| \leq Ch^{\min\{\frac{r\alpha}{2},2\}}. \tag{2.22}$$

3. Local Truncation Error. We shall first introduce some notations.

For convenience, we use the notation $\simeq$. That $x_1 \simeq y_1$, means that $c_1 x_1 \le y_1 \le C_1 x_1$ for some positive constants c_1 and C_1 that are independent of N.

And for $1 \le j \le 2N$, we define

$$y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \quad \theta \in (0,1) \tag{3.1}$$

Then we have

LEMMA 3.1. *For $1 \le i \le 2N-1$*

$$h_i \simeq h_{i+1} \simeq h\delta(x_i)^{1-1/r}, \quad \delta(x_i) \simeq \delta(x_{i+1}) \simeq \delta(y_{i+1}^\theta) \tag{3.2}$$

Since $i^r - (i-1)^r \simeq i^{r-1}$, for $i \ge 1$, where $\theta \in (0,1)$.

Meanwhile, let's define kernel functions

$$K_y(x) := \frac{|y-x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.3}$$

3.1. Proof of Theorem 2.6. The truncation error of the discrete format can be written as

$$\begin{aligned} -\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I^{2-\alpha}\Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i)) \\ &= -\kappa_\alpha D_h^2 I^{2-\alpha}(\Pi_h u - u)(x_i) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) \end{aligned} \tag{3.4}$$

THEOREM 3.2. *There exits a constant $C = C(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$ such that*

$$\left| -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) \right| \le C h^2 \delta(x_i)^{-\alpha/2 - 2/r} \tag{3.5}$$

Proof. Since $f \in C^2(\Omega)$ and

$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha} u(x)) = f(x), \quad x \in \Omega, \tag{3.6}$$

we have $I^{2-\alpha} u \in C^4(\Omega)$. Therefore, using equation (A.2) of Lemma A.1, for $1 \le i \le 2N-1$, we have

$$\begin{aligned} &-\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i) \\ &+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right) \end{aligned} \tag{3.7}$$

By Lemma B.1, Lemma 2.5 and Lemma B.2, we get the result. ∎

And now define

$$R_i := D_h^2 I^{2-\alpha}(u - \Pi_h u)(x_i), \quad 1 \le i \le 2N-1 \tag{3.8}$$

We have some results about the estimate of R_i

THEOREM 3.3. *For $1 \le i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$|R_i| \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.9}$$

THEOREM 3.4. *For $N/2 \le i \le N$, there exists constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$|R_i| \le C(r-1)h^2(T - x_i + h_N)^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.10}$$

And for $N < i \le 2N - 1$, it is symmetric to the previous case.

Combine Theorem 3.2, Theorem 3.3 and Theorem 3.4, and for $1 \le i \le N$, we have

$$h^2 x_i^{-\alpha/2-2/r} \le T^{\alpha/2-2/r} h^{\min\{\frac{r\alpha}{2},2\}} x_i^{-\alpha} \tag{3.11}$$

$$h^{r\alpha/2+r} x_i^{-1-\alpha} \le T^{-1} h^{r\alpha/2} x_i^{-\alpha} \tag{3.12}$$

$$h^r x_i^{-1}\ln(i) = T^{-1}\frac{\ln(i)}{i^r} \le T^{-1}, \quad h^r \ln(N) = \frac{\ln(N)}{N^r} \le 1 \tag{3.13}$$

the proof of Theorem 2.6 completed.

We prove Theorem 3.3 and Theorem 3.4 in next subsections.

3.2. Grid Mapping Functions. For convience, let's denote

DEFINITION 3.5.

$$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy, \quad i = 0, \cdots, 2N, \ j = 1, \cdots, 2N. \tag{3.14}$$

Also, we denote vertical difference quotients of T_{ij}

$$\begin{aligned} V_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \\ &= \int_{x_{i-1}}^{x_i} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy. \end{aligned} \tag{3.15}$$

Then by (3.8) $R_i = \sum_{j=1}^{2N} V_{ij}$. *And define skew difference quotients of T_{ij}*

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j-1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j+1} \right). \tag{3.16}$$

Our main idea is to depart R_i by V_{ij} and S_{ij}. For $3 \le i < N/2$, let's denote

$k = \lceil \frac{i}{2} \rceil$, and take some suitable integer m, then

$$
\begin{aligned}
R_i =& \sum_{j=1}^{2N} V_{ij} \\
=& \sum_{j=1}^{k-1} V_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\
&+ \sum_{j=k+1}^{m-1} S_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,m} + T_{i-1,m-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,m} \right) \\
&+ \sum_{j=m+1}^{2N} V_{ij} \\
=& I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned}
\tag{3.17}
$$

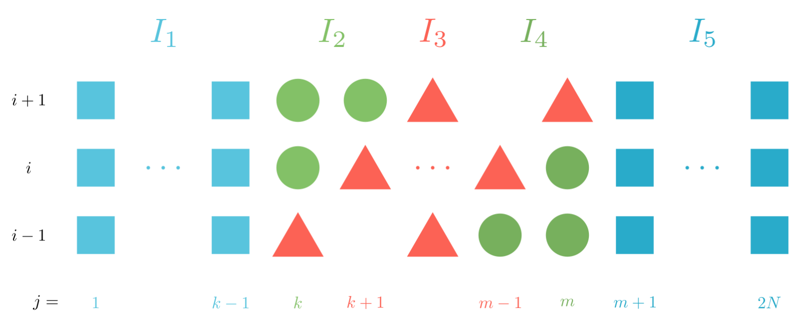


FIG. 1. *The departure of R_i for $i \geq 3$*

and discuss $i = 1, 2$ separately, where

$$
R_1 = \sum_{i=1}^{3} V_{1,j} + \sum_{i=4}^{N} V_{i,j}, \quad R_2 = \sum_{i=1}^{4} V_{1,j} + \sum_{i=5}^{N} V_{i,j}. \tag{3.18}
$$

The difficulty for esitmating S_{ij} is that $T_{i-1,j-1}, T_{i,j}$ and $T_{i+1,j+1}$ have different integral region. We first make them normalized.

LEMMA 3.6. *For $y \in (x_{j-1}, x_j)$, we can rewrite $y = y_j^\theta$, by* (3.14) *and Lemma* A.2*,*

$$
\begin{aligned}
T_{ij} =& \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
=& \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\
=& \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
&+ \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta,
\end{aligned}
\tag{3.19}
$$

where $\eta_{j1}^\theta \in (x_{j-1}, y_j^\theta), \eta_{j2}^\theta \in (y_j^\theta, x_j)$.

Since j changes with i at indices of elements in S_{ij} by (3.16), we create some functions satisfy the property.

DEFINITION 3.7 (Grid Mapping Functions). *For* $1 \leq i, j \leq 2N-1$.

$$
y_{i,j}(x) = \begin{cases}
(x^{1/r} + Z_{j-i})^r & i < N, j < N, \\
\dfrac{x^{1/r} - Z_i}{Z_1} h_N + x_N & i < N, j = N, \\
2T - (Z_{2N-(j-i)} - x^{1/r})^r & i < N, j > N, \\
\left(\dfrac{Z_1}{h_N}(x - x_N) + Z_j\right)^r & i = N, j < N, \\
x, & i = N, j = N, \\
2T - \left(\dfrac{Z_1}{h_N}(2T - x - x_N) + Z_{2N-j}\right)^r & i = N, j > N, \\
(Z_{2N+j-i} - (2T - x)^{1/r})^r & i > N, j < N, \\
\dfrac{Z_{2N-j} - (2T - x)^{1/r}}{Z_1} h_N + x_N & i > N, j = N, \\
2T - ((2T - x)^{1/r} - Z_{j-i})^r & i > N, j > N,
\end{cases} \tag{3.20}
$$

where $Z_j := T^{1/r} \frac{j}{N}$. *And*

$$
h_{i,j}(x) = y_{i,j}(x) - y_{i,j-1}(x), \tag{3.21}
$$

$$
y_{i,j}^{\theta}(x) = (1 - \theta) y_{i,j-1}(x) + \theta y_{i,j-1}(x), \quad \theta \in (0, 1), \tag{3.22}
$$

$$
P_{i,j}^{\theta}(x) = (h_{i,j}(x))^3 \frac{|y_{i,j}^{\theta}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y_{i,j}^{\theta}(x)), \tag{3.23}
$$

$$
Q_{i,j;l}^{\theta}(x) = (h_{i,j}(x))^l \frac{|y_{i,j}^{\theta}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}. \tag{3.24}
$$

Obviously,

$$
y_{i,j}(x_{i-1}) = x_{j-1}, \quad y_{i,j}(x_i) = x_j, \quad y_{i,j}(x_{i+1}) = x_{j+1}, \tag{3.25}
$$

$$
h_{i,j}(x_{i-1}) = h_{j-1}, \quad h_{i,j}(x_i) = h_j, \quad h_{i,j}(x_{i+1}) = h_{j+1}, \tag{3.26}
$$

$$
y_{i,j}^{\theta}(x_{i-1}) = y_{j-1}^{\theta}, \quad y_{i,j}^{\theta}(x_i) = y_j^{\theta}, \quad y_{i,j}^{\theta}(x_{i+1}) = y_{j+1}^{\theta}. \tag{3.27}
$$

And now we can rewrite T_{ij}.

LEMMA 3.8.

$$
\begin{aligned}
T_{ij} = & \int_0^1 -\frac{\theta(1-\theta)}{2} P_{i,j}^{\theta}(x_i) d\theta \\
& + \int_0^1 \frac{\theta(1-\theta)}{3!} Q_{i,j;l}^{\theta}(x_i) \left[\theta^2 u'''(\eta_{j,1}^{\theta}) - (1-\theta)^2 u'''(\eta_{j,2}^{\theta})\right] d\theta.
\end{aligned} \tag{3.28}
$$

Immediately, we can see from (3.16) and Lemma 3.6 that For $1 \le i \le 2N-1$, $2 \le j \le 2N-1$,

$$
\begin{aligned}
S_{ij} = &\int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{i,j}^{\theta}(x_i)d\theta \\
&+ \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{i,j;4}^{\theta}(x_{i+1})u'''(\eta_{j+1,1}^{\theta}) - Q_{i,j;4}^{\theta}(x_i)u'''(\eta_{j,1}^{\theta})}{h_{i+1}}\right)d\theta \\
&- \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{i,j;4}^{\theta}(x_i)u'''(\eta_{j,1}^{\theta}) - Q_{i,j;4}^{\theta}(x_{i-1})u'''(\eta_{j-1,1}^{\theta})}{h_i}\right)d\theta \\
&- \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{i,j;4}^{\theta}(x_{i+1})u'''(\eta_{j+1,2}^{\theta}) - Q_{i,j;4}^{\theta}(x_i)u'''(\eta_{j,2}^{\theta})}{h_{i+1}}\right)d\theta \\
&+ \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{i,j;4}^{\theta}(x_i)u'''(\eta_{j,2}^{\theta}) - Q_{i,j;4}^{\theta}(x_{i-1})u'''(\eta_{j-1,2}^{\theta})}{h_i}\right)d\theta.
\end{aligned}
\tag{3.29}
$$

We give some properties of the grid mapping functions.

LEMMA 3.9. *For* $2 \le i,j \le 2N-2$ *and* $\xi \in (x_{i-1}, x_{i+1})$

$$\xi \simeq x_i, \quad \delta(y_{i,j}(\xi)) \simeq \delta(x_j), \quad h_{i,j}(\xi) \simeq h_j, \tag{3.30}$$

$$|y_{i,j}(\xi) - \xi| \simeq |x_j - x_i|, \quad |y_{i,j-1}(\xi) - \xi| \simeq |x_{j-1} - x_i|, \tag{3.31}$$

then

$$|y_{i,j}^{\theta}(\xi) - \xi| = (1-\theta)|y_{i,j-1}(\xi) - \xi| + \theta|y_{i,j}(\xi) - \xi| \simeq |y_j^{\theta} - x_i|, \tag{3.32}$$

since $y_{i,j-1}(\xi) - \xi$, $y_{i,j}(\xi) - \xi$ *have the same sign* (≥ 0 *or* ≤ 0).

LEMMA 3.10.

$$
y_{i,j}'(x) = \begin{cases}
y_{i,j}^{1-1/r}(x)x^{1/r-1} & i<N, j<N,\\
\dfrac{h_N}{rZ_1}x^{1/r-1} & i<N, j=N,\\
(2T - y_{i,j}(x))^{1-1/r}x^{1/r-1} & i<N, j>N,\\
y_{i,j}^{1-1/r}(x)\dfrac{rZ_1}{h_N} & i=N, j<N,\\
1 & i=N, j=N,
\end{cases}
\tag{3.33}
$$

$$
y_{i,j}''(x) = \frac{1-r}{r}\begin{cases}
y_{i,j}^{1-2/r}(x)x^{1/r-2}Z_{j-i} & i<N, j<N,\\
\dfrac{h_N}{rZ_1}x^{1/r-2} & i<N, j=N,\\
(2T - y_{i,j}(x))^{1-2/r}x^{1/r-2}Z_{2N-j+i} & i<N, j>N,\\
-y_{i,j}^{1-2/r}(x)\left(\dfrac{rZ_1}{h_N}\right)^2 & i=N, j<N,\\
0 & i=N, j=N.
\end{cases}
\tag{3.34}
$$

LEMMA 3.11. *For $2 \le i \le N, 2 \le j \le 2N-2$, $\xi \in (x_{i-1}, x_{i+1})$*

$$|h'_{i,j}(\xi)| \le C(r-1)Z_1 x_i^{1/r-1}\delta(x_j)^{1-2/r} \le C(r-1)h_j x_i^{1/r-1}\delta(x_j)^{-1/r}, \tag{3.35}$$

$$|(y_{i,j}(\xi)-\xi)'| \le Cx_i^{-1}|x_j - x_i|. \tag{3.36}$$

Proof. From (3.21) and Lemma 3.10, we can see that

$$\begin{aligned} h'_{i,j}(x) &= y'_{i,j}(x) - y'_{i,j-1}(x) \\ &= \begin{cases} x^{1/r-1}(y_{i,j}^{1-1/r}(x) - y_{i,j-1}^{1-1/r}(x)) & i<N, j<N, \\ x^{1/r-1}(\dfrac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(x)) & i<N, j=N, \\ x^{1/r-1}\left((2T-y_{i,N+1}(x))^{1-1/r} - \dfrac{h_N}{rZ_1}\right) & i<N, j=N+1, \\ x^{1/r-1}\left((2T-y_{i,j}(x))^{1-1/r} - (2T-y_{i,j-1}(x))^{1-1/r}\right) & i<N, j>N+1, \\ \dfrac{rZ_1}{h_N}\left(y_{N,j}^{1-1/r}(x) - y_{N,j-1}^{1-1/r}(x)\right) & i=N, j<N, \\ \dfrac{rZ_1}{h_N}\left(\dfrac{h_N}{rZ_1} - y_{N,N-1}^{1-1/r}(x)\right) & i=N, j=N. \end{cases} \end{aligned} \tag{3.37}$$

While for $2 \le i \le N$, if $2 \le j < N$, $\xi \in (x_{i-1}, x_{i+1})$,

$$\begin{aligned} y_{i,j}^{1-1/r}(\xi) - y_{i,j-1}^{1-1/r}(\xi) &\le x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r} \\ &= T^{1-1/r}N^{1-r}\left((j+1)^{r-1} - (j-2)^{r-1}\right) \\ &\le CT^{1-1/r}(r-1)N^{1-r}j^{r-2} = C(r-1)Z_1 x_j^{1-2/r}. \end{aligned} \tag{3.38}$$

If $j = N$, $\xi \in (x_{i-1}, x_{i+1})$, we have $y_{i,N-1}(\xi) \in (x_{N-2}, x_N)$. And

$$\frac{h_N}{rZ_1} = T^{1-1/r}\frac{1-(1-h)^r}{rh} = \eta^{1-1/r} \simeq x_N^{1-1/r}, \quad \eta \in (x_{N-1}, x_N). \tag{3.39}$$

Then

$$|\frac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(\xi)| \le x_N^{1-1/r} - x_{N-2}^{1-1/r} \simeq (r-1)Z_1 x_N^{1-2/r}. \tag{3.40}$$

And similar for $j \ge N+1$. Combine with Lemma 3.1, Lemma 3.9, $\eta \simeq x_N$, we get the first result.

For the second estimate, we have

$$(y_{i,j}(x) - x)' = y'_{i,j}(x) - 1. \tag{3.41}$$

Then, for $2 \le i < N$, if $2 \le j < N$, $\xi \in (x_{i-1}, x_{i+1})$, by Lemma A.5

$$\xi^{1/r}|y_{i,j}^{1-1/r}(\xi) - \xi^{1-1/r}| \le |y_{i,j}(\xi) - \xi|. \tag{3.42}$$

$j > N$ is symmetric to it, that is

$$\begin{aligned} &\xi^{1/r}|(2T - y_{i,j}(\xi))^{1-1/r} - \xi^{1-1/r}| \le |2T - y_{i,j}(\xi) - \xi| \\ &\quad \le |2T - x_j - x_i| + |y_{i,j}(\xi) - x_j| + |\xi - x_i| \le |2T - x_j - x_i| + 2h_N \\ &\quad \le |x_j - T| + |T - x_i| + 2h_N \le 2|x_j - x_i|. \end{aligned} \tag{3.43}$$

But if $j = N$, with (3.39) and Lemma A.5,

$$\begin{aligned}\eta^{1/r}\left|\frac{h_N}{rZ_1} - \xi^{1-1/r}\right| &\le |\eta - \xi|, \quad \eta \in (x_{N-1}, x_N) \\ &\le |x_N - x_i| + |h_N| + |h_{i+1}| \le 3|x_N - x_i|.\end{aligned} \tag{3.44}$$

For $i = N$, if $j < N$, similarly with (3.44),

$$\eta^{1/r}\left|y_{N,j}^{1-1/r}(\xi) - \frac{h_N}{rZ_1}\right| \le C|x_j - x_N|. \tag{3.45}$$

And if $j = N$, it is obviously $\equiv 0$.

Similarly, by Lemma 3.10 and Lemma 3.9, we get the second result. ∎

LEMMA 3.12. *For $2 \le i \le N, 2 \le j \le 2N-2$, $\xi \in (x_{i-1}, x_{i+1})$*

$$|y''_{i,j}(\xi)| \le C(r-1)\begin{cases} x_j^{-1/r}x_i^{1/r-2}|x_j - x_i| & i < N, j < N, \\ x_N^{1-1/r}x_i^{1/r-2} & i < N, j = N, \\ \delta(x_j)^{1-2/r}x_i^{1/r-2}x_N^{1/r} & i < N, j > N, \\ \delta(x_j)^{1-2/r}x_N^{2/r-2} & i = N, j \ne N, \\ 0 & i = N, j = N. \end{cases} \tag{3.46}$$

And $2 \le i \le N, 3 \le j \le 2N-2$, $\xi \in (x_{i-1}, x_{i+1})$

$$|h''_{i,j}(\xi)| \le C(r-1)\begin{cases} Z_1 x_i^{1/r-2}x_j^{-2/r}(|x_j - x_i| + x_j) & i < N, j < N, \\ x_i^{1/r-2}x_N^{1-1/r} & i < N, j = N, N+1, \\ Z_1 x_i^{1/r-2}\delta(x_j)^{1-3/r}x_N^{1/r} & i < N, j > N+1, \\ Z_1 x_N^{2/r-2}\delta(x_j)^{1-3/r} & i = N, j < N \text{ or } j > N+1, \\ x_N^{-1} & i = N, j = N. \end{cases} \tag{3.47}$$

Proof. Since by Lemma A.5, for $2 \le i, j < N$

$$x_j^{1-1/r}|Z_{j-i}| = x_j^{1-1/r}|x_j^{1/r} - x_i^{1/r}| \le |x_j - x_i|, \tag{3.48}$$

and by (3.39), $\dfrac{h_N}{rZ_1} \simeq x_N^{1-1/r}$. And

$$Z_{2N-j+i} \le Z_{2N} = 2T^{1/r}. \tag{3.49}$$

Then by Lemma 3.10 and Lemma 3.9, we get the first result.

For the second part, by Lemma 3.10

$$h''_{i,j}(x) = y''_{i,j}(x) - y''_{i,j-1}(x), \tag{3.50}$$

while for $2 \le i < N$, if $3 \le j < N$, $\xi \in (x_{i-1}, x_{i+1})$,

$$y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1} = \left(y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi)\right)Z_{j-i} + y_{i,j-1}^{1-2/r}(\xi)Z_1, \tag{3.51}$$

where $y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi) \simeq (r-2)Z_1 x_j^{1-3/r}$ similar with (3.38). Combine with (3.48), we get

$$|y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1}| \le CZ_1\left(|r-2|x_j^{-2/r}|x_j - x_i| + x_j^{1-2/r}\right). \tag{3.52}$$

If $j = N$,

$$|h''_{i,N}(x)| \le |y''_{i,N}(x)| + |y''_{i,N-1}(x)| \le C(r-1)x_i^{1/r-2}x_N^{1-1/r}. \tag{3.53}$$

Similarly if $j = N+1$.

However, if $j > N+1$, similar with (3.51), we get

$$\begin{aligned} &(2T - y_{i,j}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)} \\ &= \left((2T - y_{i,j}(\xi))^{1-2/r} - (2T - y_{i,j-1}(\xi))^{1-2/r}\right) Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r} Z_1, \end{aligned} \tag{3.54}$$

thus,

$$\begin{aligned} &\left|(2T - y_{i,j}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)}\right| \\ &\le CZ_1 \left(|r-2|(2T - x_j)^{1-3/r}x_N^{1/r} + (2T - x_j)^{1-2/r}\right) \le CZ_1(2T - x_j)^{1-3/r}x_N^{1/r}. \end{aligned} \tag{3.55}$$

For $i = N$, it's obvious. Combine with Lemma 3.10 and Lemma 3.9, we get the second result. ∎

3.3. Proof of Theorems. Then we esrimate each part of (3.17). And We take $m = 2i$ for $3 \le i < N/2$, and $m = N - \lceil N/2 \rceil + 1$ for $N/2 \le i \le N$.

For I_5

Lemma 3.13. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that*

Case 1. For $1 \le i < N/2$,

$$\sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| \le Ch^2 x_i^{-\alpha/2-2/r}. \tag{3.56}$$

Case 2. For $1 \le i < N/2$,

$$\sum_{j=N+1}^{2N} |V_{ij}| \le \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0. \end{cases} \tag{3.57}$$

Case 3. For $N/2 \le i \le N$,

$$\sum_{j=N-\lceil \frac{N}{2} \rceil+2}^{2N} |V_{ij}| \le \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0. \end{cases} \tag{3.58}$$

Proof. For i, j in each case, by (3.15), Lemma A.3 and Lemma B.3, we have

$$|V_{ij}| \le Ch^2 \int_{x_{j-1}}^{x_j} \delta(y)^{\alpha/2-2/r}|y - x_i|^{-1-\alpha}dy. \tag{3.59}$$

For Case 1, with $x_i \simeq x_{2i}$,

$$\begin{aligned} \sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| &\le Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1}dy \\ &= \frac{C}{\alpha/2 + 2/r} h^2(x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\ &\le Ch^2 x_i^{-\alpha/2-2/r}. \end{aligned} \tag{3.60}$$

For Case 2 , by (3.15), Lemma A.3, Lemma B.3 and $y - x_i \simeq T$,

$$|V_{ij}| \le Ch^2T^{-1-\alpha}\int_{x_{j-1}}^{x_j}(2T-y)^{\alpha/2-2/r}dy,$$

$$\sum_{j=N+1}^{2N-1}|V_{ij}| \le CT^{-1-\alpha}h^2\int_{x_N}^{x_{2N-1}}(2T-y)^{\alpha/2-2/r}dy$$

$$\le CT^{-1-\alpha}h^2\begin{cases}\frac{1}{\alpha/2-2/r+1}T^{\alpha/2-2/r+1}, & \alpha/2-2/r+1>0,\\ \ln(T)-\ln(h_{2N}), & \alpha/2-2/r+1=0,\\ \frac{1}{|\alpha/2-2/r+1|}h_{2N}^{\alpha/2-2/r+1}, & \alpha/2-2/r+1<0,\end{cases} \tag{3.61}$$

$$=\begin{cases}\frac{C}{\alpha/2-2/r+1}T^{-\alpha/2-2/r}\ h^2, & \alpha/2-2/r+1>0,\\ CrT^{-1-\alpha}h^2\ln(N), & \alpha/2-2/r+1=0,\\ \frac{C}{|\alpha/2-2/r+1|}T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2-2/r+1<0.\end{cases}$$

And by Lemma A.4

$$|V_{i,2N}| \le CT^{-1-\alpha}h_{2N}^{\alpha/2+1} = CT^{-\alpha/2}h^{r\alpha/2+r}.$$

Summarizes, we get the result. Similar for Case 3. ∎

For $i = 1, 2$.

Lemma 3.14. *From* (3.18)*, by Lemma* B.4*, Lemma* 3.13 *Case 1 2, we get for* $i = 1, 2$

$$|R_i| \le Ch^2x_i^{-\alpha/2-2/r} + \begin{cases}Ch^2, & \alpha/2-2/r+1>0,\\ Ch^2\ln(N), & \alpha/2-2/r+1=0,\\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0.\end{cases} \tag{3.62}$$

Lemma 3.15. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i \le N, k = \lceil \frac{i}{2} \rceil$

$$|I_1| = |\sum_{j=1}^{k-1}V_{ij}| \le \begin{cases}Ch^2x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1>0,\\ Ch^2x_i^{-1-\alpha}\ln(i), & \alpha/2-2/r+1=0,\\ Ch^{r\alpha/2+r}x_i^{-1-\alpha}, & \alpha/2-2/r+1<0.\end{cases} \tag{3.63}$$

Proof. by (3.15), Lemma A.4 , Lemma B.3

$$|V_{i1}| \le C\int_0^{x_1}x_1^{\alpha/2}|x_i-y|^{-1-\alpha}dy \simeq x_1^{\alpha/2+1}x_i^{-1-\alpha} = T^{\alpha/2+1}h^{r\alpha/2+r}x_i^{-1-\alpha} \tag{3.64}$$

For $2 \le j \le k-1$, by Lemma A.3 and Lemma B.3 with $x_i - y \simeq x_i$, we have

$$|V_{ij}| \le Ch^2\int_{x_{j-1}}^{x_j}y^{\alpha/2-2/r}x_i^{-1-\alpha}dy \tag{3.65}$$

Therefore,

$$\sum_{j=2}^{k-1}|V_{ij}| \le Ch^{r\alpha/2+r}x_i^{-1-\alpha} + Ch^2x_i^{-1-\alpha}\int_{x_1}^{x_{\lceil\frac{i}{2}\rceil-1}}y^{\alpha/2-2/r}dy \tag{3.66}$$

But $x_{\lceil \frac{i}{2} \rceil - 1} \le 2^{-r} x_i$, so we have

$$\int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \le \begin{cases} \frac{1}{\alpha/2-2/r+1}(2^{-r}x_i)^{\alpha/2-2/r+1}, & \alpha/2-2/r+1>0 \\ \ln(2^{-r}x_i) - \ln(x_1), & \alpha/2-2/r+1=0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2-2/r+1<0 \end{cases} \tag{3.67}$$

Combine the results above, we get the lemma. ∎

LEMMA 3.16. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

Case 1. For $3 \le i < N, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$,

$$|D_h^2 P_{i,j}^\theta(x_i)| \le C h_j^3 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-4} \tag{3.68}$$

Case 2. For $N/2 \le i \le N$, $j = N, N+1$

$$|D_h^2 P_{i,j}^\theta(\xi)| \le C h_j^3 |y_j^\theta - x_i|^{1-\alpha} + C(r-1) h_j^2 \Big(|y_j^\theta - x_i|^{1-\alpha} + h_j |y_j^\theta - x_i|^{-\alpha} \Big) \tag{3.69}$$

Case 3. For $N/2 \le i \le N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil$,

$$|D_h^2 P_{i,j}^\theta(\xi)| \le C h_j^3 \Big(|y_j^\theta - x_i|^{1-\alpha} + (r-1)|y_j^\theta - x_i|^{-\alpha} \Big) \tag{3.70}$$

Proof. Since $\mathrm{sign}(y_{i,j}^\theta(\xi) - \xi)$ is independent of ξ, we can derivate it. Then by Lemma A.1

$$D_h^2 P_{i,j}^\theta(x_i) = {P_{i,j}^\theta}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{3.71}$$

From (3.23), using Leibniz formula and chain rules, and Lemma 3.9, Lemma 3.10, Lemma 3.11, Lemma 3.12, Lemma 2.4, Lemma 3.1

For every case, we have $x_i \simeq \delta(x_j)$, so we have

$$h_{i,j}(\xi) \le C h_j, \quad |h'_{i,j}(\xi)| \le C(r-1) h_j x_i^{-1} \tag{3.72}$$

$$|y_{i,j}^\theta(\xi) - \xi| \le C|y_j^\theta - x_i|, \quad \big|(y_{i.j}^\theta(\xi) - x_i)'\big| \le C|y_j^\theta - x_i| x_i^{-1} \tag{3.73}$$

$$|u''(y_{i,j}^\theta(\xi))| \le C x_i^{\alpha/2-2}, \quad \Big|\big(u''(y_{i,j}^\theta(\xi))\big)'\Big| \le C x_i^{\alpha/2-3}, \quad \Big|\big(u''(y_{i,j}^\theta(\xi))\big)''\Big| \le C x_i^{\alpha/2-4} \tag{3.74}$$

By Lemma 3.12, we have

For Case 1,

$$|h''_{i,j}(\xi)| \le C(r-1) h_j x_i^{-2}, \quad \big|(y_{i,j}^\theta(\xi) - x_i)''\big| \le C(r-1)|y_j^\theta - x_i| x_i^{-2} \tag{3.75}$$

For Case 2, since $x_i \simeq x_j \simeq T$

$$|h''_{i,j}(\xi)| \le C(r-1), \quad \big|(y_{i,j}^\theta(\xi) - x_i)''\big| \le C(r-1) \tag{3.76}$$

For Case 3, since $x_i \simeq \delta(x_j) \simeq T$, we have

$$|h''_{i,j}(\xi)| \le C(r-1) h_j, \quad \big|(y_{i,j}^\theta(\xi) - x_i)''\big| \le C(r-1) \tag{3.77}$$

∎

Combine them, we get the result.

LEMMA 3.17. *There exists a constant $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $2 \le i \le N$, $2 \le j \le 2N-2$,*

$$\left|\frac{Q^{\theta}_{i,j;l}(x_{i+1})u^{(l-1)}(\eta^{\theta}_{j+1}) - Q^{\theta}_{i,j;l}(x_i)u^{(l-1)}(\eta^{\theta}_j)}{h_{i+1}}\right| \le Ch_j^l \frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{-1}\delta(x_j)^{\alpha/2-l+1-1/r}(x_i^{1/r}+\delta(x_j)^{1/r}) \tag{3.78}$$

And

$$\left|\frac{Q^{\theta}_{i,j;l}(x_i)u^{(l-1)}(\eta^{\theta}_j) - Q^{\theta}_{i,j;l}(x_{i-1})u^{(l-1)}(\eta^{\theta}_{j-1})}{h_i}\right| \le Ch_j^l \frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{-1}\delta(x_j)^{\alpha/2-l+1-1/r}(x_i^{1/r}+\delta(x_j)^{1/r}) \tag{3.79}$$

where $\eta^{\theta}_j \in (x_{j-1},x_j)$.

Proof.

$$\begin{aligned}&\frac{Q^{\theta}_{i,j;l}(x_{i+1})u'''(\eta^{\theta}_{j+1}) - Q^{\theta}_{i,j;l}(x_i)u'''(\eta^{\theta}_j)}{h_{i+1}}\\ &= \frac{Q^{\theta}_{i,j;l}(x_{i+1}) - Q^{\theta}_{i,j;l}(x_i)}{h_{i+1}}u'''(\eta^{\theta}_{j+1}) + Q^{\theta}_{i,j;l}(x_i)\frac{u'''(\eta^{\theta}_{j+1}) - u'''(\eta^{\theta}_j)}{h_{i+1}}\end{aligned} \tag{3.80}$$

Using mean value theorem

$$D_h Q^{\theta}_{i,j;l}(x_i) := \frac{Q^{\theta}_{i,j;l}(x_{i+1}) - Q^{\theta}_{i,j;l}(x_i)}{h_{i+1}} = {Q^{\theta}_{i,j;l}}'(\xi), \quad \xi \in (x_i,x_{i+1}) \tag{3.81}$$

From (3.24) and Leibniz rule, by Lemma 3.9, Lemma 3.11 and Lemma 3.1, we have

$$|{Q^{\theta}_{i,j;l}}'(\xi)| \le Ch_j^l \frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}(x_i^{-1} + x_i^{1/r-1}\delta(x_j)^{-1/r}) \tag{3.82}$$

$$Q^{\theta}_{i,j;l}(x_i) = h_j^l \frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.83}$$

With Lemma 3.1 and Lemma 2.4

$$|u^{(l-1)}(\eta^{\theta}_{j+1})| \le C(\eta^{\theta}_{j+1})^{\alpha/2-l+1} \simeq \delta(x_j)^{\alpha/2-l+1}$$

and by Lemma 3.1

$$\begin{aligned}\frac{|u^{(l-1)}(\eta^{\theta}_{j+1}) - u^{(l-1)}(\eta^{\theta}_j)|}{h_{i+1}} &= |u^{(l)}(\eta)|\frac{\eta^{\theta}_{j+1}-\eta^{\theta}_j}{h_{i+1}}, \quad \eta \in (x_{j-1},x_{j+1})\\ &\le C\delta(\eta)^{\alpha/2-l}\frac{x_{j+1}-x_{j-1}}{h_{i+1}} = C\delta(\eta)^{\alpha/2-l}\frac{h_{j+1}+h_j}{h_{i+1}}\\ &\simeq x_i^{1/r-1}\delta(x_j)^{\alpha/2-l+1-1/r}\end{aligned}$$

Combine the results above, we get the first term. While, the later is similar. ∎

LEMMA 3.18. *There exists a constant $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

Case 1. For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$,

$$\begin{aligned} |S_{ij}| &\le Ch_j^2 x_i^{\alpha/2-4} \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\ &= Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \end{aligned} \tag{3.84}$$

Thus,

$$\sum_{j=k+1}^{\min\{2i-1,N-1\}} |S_{ij}| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.85}$$

Case 2. For $N/2 \le i \le N$, $j = N, N+1$, *since* $\theta(1-\theta)h_j \le |y_j^\theta - x_i|$, *we have*

$$|S_{ij}| \le C(h^3 + (r-1)h^2)(T - x_i + h_N)^{1-\alpha} \tag{3.86}$$

Case 3. For $N/2 \le i \le N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil$,

$$|S_{ij}| \le Ch^2 \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy \tag{3.87}$$

Thus,

$$\sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} |S_{ij}| \le Ch^2 + C(r-1)h^2(T - x_i + h_N)^{1-\alpha} \tag{3.88}$$

Expecially, for $i = N$, *the estimate of* $\lceil \frac{N}{2} \rceil + 1 \le j \le N-1$ *is symmetric with* $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil$.

Proof. Since (3.29), by $x_i \simeq x_j$, Lemma 3.1, Lemma 3.16, Lemma 3.17

For Case 1, we get the first result immediately. While $x_k \simeq x_i \simeq x_{\min\{2i-1,N-1\}}$, we have

$$\begin{aligned} \sum_{k+1}^{\min\{2i-1,N-1\}} |S_{ij}| &\le Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_k}^{x_{\min\{2i-1,N-1\}}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &\le Ch^2 x_i^{\alpha/2-2-2/r} x_i^{2-\alpha} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{3.89}$$

For Case 2,

$$\begin{aligned} |S_{ij}| &\le C(h_j^3 + (r-1)h_j^2) \int_0^1 |y_j^\theta - x_i|^{1-\alpha} d\theta \\ &= C(h_j^2 + (r-1)h_j) \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} dy \end{aligned} \tag{3.90}$$

however,

$$\begin{aligned} \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} dy &= \frac{1}{2-\alpha} \left((x_j - x_i)^{2-\alpha} - (x_{j-1} - x_i)^{2-\alpha} \right) \\ &\simeq h_N(|x_j - x_i + h_N)^{1-\alpha} \end{aligned} \tag{3.91}$$

For Case 3,

$$|S_{ij}| \le Ch_j^2 \int_0^1 \left(|y_j^\theta - x_i|^{1-\alpha} + (r-1)|y_j^\theta - x_i|^{-\alpha}\right) h_j d\theta \\ \le Ch^2 \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} + (r-1)|y-x_i|^{-\alpha} dy \tag{3.92}$$

Thus,

$$\sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} |S_{ij}| = Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil \frac{N}{2} \rceil}} |y-x_i|^{1-\alpha} + (r-1)|y-x_i|^{-\alpha} dy \\ \le Ch^2 (T^{2-\alpha} + (r-1)(T - x_i + h_N)^{1-\alpha}) \tag{3.93}$$

□

Now we study I_2, I_4.

LEMMA 3.19. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

Case 1. For $3 \le i \le N, k = \lceil \frac{i}{2} \rceil$,

$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \le Ch^2 x_i^{-\alpha/2 - 2/r} \tag{3.94}$$

Case 2. For $3 \le i < N/2$,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \le Ch^2 x_i^{-\alpha/2 - 2/r} \tag{3.95}$$

Case 3. For $N/2 \le i \le N$, $m = N - \lceil \frac{N}{2} \rceil + 1$,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,m} + T_{i-1,m-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,m} \right) \le Ch^2 \tag{3.96}$$

Proof. In fact,

$$\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \\ = \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i}) T_{i,k} \tag{3.97}$$

While, by Lemma A.3, Lemma B.3, Lemma 3.1 and $x_k \simeq x_i$, we have

$$\frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) = \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y)) D_h K_y(x_i) dy \\ \le Ch_k^2 x_k^{\alpha/2-2}\ h_k |x_i - x_k|^{-\alpha} \le Ch^2 x_i^{-\alpha/2-2/r} h_k \tag{3.98}$$

Thus,

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} |T_{i+1,k} - T_{i,k}| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.99}$$

From (3.14), Lemma A.2 and normalization, we have

$$\frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{Q_{i,k;3}^\theta (x_{i+1}) u''(\eta_{k+1}^\theta) - Q_{i,k;3}^\theta (x_i) u''(\eta_k^\theta)}{h_{i+1}} d\theta \tag{3.100}$$

where $\eta_k^\theta \in (x_{k-1}, x_k)$ and $\eta_{k+1}^\theta \in (x_k, x_{k+1})$. And with Lemma 3.17, we can get

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} |T_{i+1,k+1} - T_{i,k}| \leq Ch^2 x_i^{-\alpha/2-2/r} \tag{3.101}$$

For the third term, by Lemma 3.1, Lemma B.1, Lemma A.3 and $x_k \simeq x_i$, we have

$$\begin{aligned} \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,k} &\leq h_i^{-3} h^2 x_i^{1-2/r} C h_k^3 x_k^{\alpha/2-2} |x_k - x_i|^{1-\alpha} \\ &\leq Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{3.102}$$

Summarizes, we have

$$I_2 \leq Ch^2 x_i^{-\alpha/2-2/r} \tag{3.103}$$

The case for I_4 is similar. ∎

Now we have study everr part to prove Theorem 3.3 and Theorem 3.4.

For $1 \leq i < N/2$, combine Lemma 3.14, Lemma 3.15, Lemma 3.19 Cases 1 2, Lemma 3.18 Case 1, Lemma 3.13 Case 1 2, we get Theorem 3.3.

For $N/2 \leq i \leq N$, we take $m = 2N - \lceil \frac{N}{2} \rceil + 1$. And depart I_3 to three parts:

$$I_3 = \sum_{j=k+1}^{m} S_{ij} = \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{m-1} S_{ij} \tag{3.104}$$

combine Lemma 3.15, Lemma 3.19 Cases 1 3, Lemma 3.18, Lemma 3.13 Case 1 2, we get Theorem 3.4.

4. Convergence analysis.

4.1. Properties of some Matrices.

Review subsection 2.1, we have got (2.10).

Definition 4.1. *We call one matrix an M matrix , which means its entries are positive on major diagonal and nonpositive on others, and strictly diagonally dominant in rows.*

Now we have

Lemma 4.2. *Matrix A defined by* (2.12) *where* (2.13) *is an M matrix. And there exists a constant $C_A = C(T, \alpha, r)$ such that*

$$S_i := \sum_{j=1}^{2N-1} a_{ij} \geq C_A (x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \tag{4.1}$$

Proof. From (2.15), we have

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right) \tag{4.2}$$

Let

$$g(x) = g_0(x) + g_{2N}(x) \tag{4.3}$$

where

$$g_0(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x-x_0|^{3-\alpha} - |x-x_1|^{3-\alpha}}{h_1}$$

$$g_{2N}(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N}-x|^{3-\alpha} - |x_{2N-1}-x|^{3-\alpha}}{h_{2N}}$$

Thus

$$-\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

Then

$$\begin{aligned} S_i := & \sum_{j=1}^{2N-1} a_{ij} \\ = & \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = & D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i) \end{aligned} \tag{4.4}$$

When $i = 1$

$$\begin{aligned} D_h^2 g_0(x_1) = & \frac{2}{h_1 + h_2} \left(\frac{1}{h_2} g_0(x_2) - (\frac{1}{h_1} + \frac{1}{h_2}) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right) \\ = & \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1+h_2)^{3-\alpha}}{(h_1+h_2) h_1 h_2} \\ = & \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1+h_2)^{3-\alpha}}{(h_1+h_2) h_1^{1-\alpha} h_2} h_1^{-\alpha} \\ = & \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r-1)^{3-\alpha} + 2(2^r-1) - (2^r)^{3-\alpha}}{2^r(2^r-1)} h_1^{-\alpha} \end{aligned} \tag{4.5}$$

but

$$1 + (2^r-1)^{3-\alpha} + 2(2^r-1) - (2^r)^{3-\alpha} > 0 \tag{4.6}$$

While for $i \geq 2$

$$\begin{aligned} D_h^2 g_0(x_i) = & g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \\ = & -\kappa_\alpha \frac{|\xi-x_0|^{1-\alpha} - |\xi-x_1|^{1-\alpha}}{\Gamma(2-\alpha) h_1} \\ = & \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} |\xi-\eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\ \geq & \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha} \end{aligned} \tag{4.7}$$

So

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g_0(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g_0(x_i) + \frac{1}{h_i} g_0(x_{i-1}) \right) \geq C x_i^{-\alpha} \tag{4.8}$$

symmetricly,

$$\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}g_{2N}(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})g_{2N}(x_i)+\frac{1}{h_i}g_{2N}(x_{i-1})\right)\geq C(\alpha,r)(2T-x_i)^{-\alpha} \tag{4.9}$$

$\square$

Let

$$G=\operatorname{diag}(\delta(x_1),...,\delta(x_{2N-1})) \tag{4.10}$$

Then

Lemma 4.3. *The matrix $B:=AG$, the major diagnal is positive, and nonpositive on others. And there is a constant $C_{AG}, C=C(\alpha,r)$ such that*

$$M_i:=\sum_{j=1}^{2N-1}b_{ij}\geq -C_{AG}(x_i^{1-\alpha}+(2T-x_i)^{1-\alpha})+C(T-\delta(x_i)+h_N)^{1-\alpha} \tag{4.11}$$

Proof.

$$b_{ij}=a_{ij}\delta(x_j)=-\kappa_\alpha\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}\tilde{a}_{i+1,j}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})\tilde{a}_{i,j}+\frac{1}{h_i}\tilde{a}_{i-1,j}\right)\delta(x_j)$$

Since

$$\delta(x)\equiv\Pi_h\delta(x) \tag{4.12}$$

by (2.14) and (2.5), we have

$$\begin{aligned}\tilde{M}_i&:=\sum_{j=1}^{2N-1}\tilde{b}_{ij}:=\sum_{j=1}^{2N-1}\tilde{a}_{ij}\delta(x_j)\\&=\int_0^{2T}\frac{|x_i-y|^{1-\alpha}}{\Gamma(2-\alpha)}\Pi_h\delta(y)dy=\int_0^{2T}\frac{|x_i-y|^{1-\alpha}}{\Gamma(2-\alpha)}\delta(y)dy\\&=\frac{-2}{\Gamma(4-\alpha)}|T-x_i|^{3-\alpha}+\frac{1}{\Gamma(4-\alpha)}(x_i^{3-\alpha}+(2T-x_i)^{3-\alpha})\\&:=w(x_i)=p(x_i)+q(x_i)\end{aligned} \tag{4.13}$$

Thus,

$$\begin{aligned}M_i&:=\sum_{j=1}^{2N-1}b_{ij}=\sum_{j=1}^{2N-1}a_{ij}\delta(x_j)\\&=-\kappa_\alpha\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}\tilde{M}_{i+1}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})\tilde{M}_i+\frac{1}{h_i}\tilde{M}_{i-1}\right)\\&=D_h^2(-\kappa_\alpha p)(x_i)-\kappa_\alpha D_h^2q(x_i)\end{aligned} \tag{4.14}$$

for $1\leq i\leq N-1$,by Lemma A.1

$$\begin{aligned}D_h^2(-\kappa_\alpha p)(x_i)&:=-\kappa_\alpha\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}p(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})p(x_i)+\frac{1}{h_i}p(x_{i-1})\right)\\&=\frac{2\kappa_\alpha}{\Gamma(2-\alpha)}|T-\xi|^{1-\alpha}\quad \xi\in(x_{i-1},x_{i+1})\\&\geq\frac{2\kappa_\alpha}{\Gamma(2-\alpha)}(T-\delta(x_i)+h_N)^{1-\alpha}\end{aligned} \tag{4.15}$$

(4.16)
$$
\begin{aligned}
D_h^2(-\kappa_\alpha p)(x_N) &:= -\kappa_\alpha \frac{2}{h_N+h_{N+1}}\left(\frac{1}{h_{N+1}}p(x_{N+1}) - (\frac{1}{h_N}+\frac{1}{h_{N+1}})p(x_N) + \frac{1}{h_N}p(x_{N-1})\right)\\
&= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)h_N^2}h_N^{3-\alpha} = \frac{4\kappa_\alpha}{\Gamma(4-\alpha)}(T-\delta(x_N)+h_N)^{1-\alpha}
\end{aligned}
$$

Symmetricly for $i \geq N$, we get

$$
D_h^2(-\kappa_\alpha p)(x_i) \geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}(T-\delta(x_i)+h_N)^{1-\alpha} \tag{4.17}
$$

Similarly, we can get

$$
\begin{aligned}
D_h^2 q(x_i) &:= \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}q(x_{i+1}) - (\frac{1}{h_i}+\frac{1}{h_{i+1}})q(x_i) + \frac{1}{h_i}q(x_{i-1})\right)\\
&\leq \frac{2^{r(\alpha-1)+1}}{\Gamma(2-\alpha)}(x_i^{1-\alpha}+(2T-x_i)^{1-\alpha}), \quad i=1,\cdots,2N-1
\end{aligned} \tag{4.18}
$$

$\square$

So, we get the result.

Notice that

$$
x_i^{-\alpha} \geq (2T)^{-1}x_i^{1-\alpha} \tag{4.19}
$$

We can get

THEOREM 4.4. *There exists a real $\lambda = \lambda(T,\alpha,r) > 0$ and $C = C(T,\alpha,r) > 0$ such that $B := A(\lambda I + G)$ is an M matrix. And*

$$
M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha}+(2T-x_i)^{-\alpha}) + C(T-\delta(x_i)+h_N)^{1-\alpha} \tag{4.20}
$$

Proof. By Lemma 4.2 with C_A and Lemma 4.3 with C_{AG} , it's sufficient to take $\lambda = (C+2TC_{AG})/C_A$, then

$$
M_i \geq C\left((x_i^{-\alpha}+(1-x_i)^{-\alpha}) + (T-\delta(x_i)+h_N)^{1-\alpha}\right) \tag{4.21}
$$

$\square$

4.2. Proof of Theorem 2.7. For equation

$$
AU = F \Leftrightarrow A(\lambda I+G)(\lambda I+G)^{-1}U = F \quad \text{i.e.} \quad B(\lambda I+G)^{-1}U = F \tag{4.22}
$$

which means

$$
\sum_{j=1}^{2N-1} b_{ij}\frac{\epsilon_j}{\lambda+\delta(x_j)} = -\tau_i \tag{4.23}
$$

where $\epsilon_i = u(x_i) - u_i$.

And if

$$
|\frac{\epsilon_{i_0}}{\lambda+\delta(x_{i_0})}| = \max_{1\leq i\leq 2N-1}|\frac{\epsilon_i}{\lambda+\delta(x_i)}| \tag{4.24}
$$

Then, since $B = A(\lambda I + G)$ is an M matrix, it is Strictly diagonally dominant. Thus,

$$
\begin{aligned}
|\tau_{i_0}| &= |\sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda+\delta(x_j)}| \\
&\geq b_{i_0,i_0} |\frac{\epsilon_{i_0}}{\lambda+\delta(x_{i_0})}| - \sum_{j\neq i_0} |b_{i_0,j}||\frac{\epsilon_j}{\lambda+\delta(x_j)}| \\
&\geq b_{i_0,i_0} |\frac{\epsilon_{i_0}}{\lambda+\delta(x_{i_0})}| - \sum_{j\neq i_0} |b_{i_0,j}||\frac{\epsilon_{i_0}}{\lambda+\delta(x_{i_0})}| \\
&= \sum_{j=1}^{2N-1} b_{i_0,j} |\frac{\epsilon_{i_0}}{\lambda+\delta(x_{i_0})}| \\
&= M_{i_0} |\frac{\epsilon_{i_0}}{\lambda+\delta(x_{i_0})}|
\end{aligned} \tag{4.25}
$$

By Theorem 2.6 and Theorem 4.4,

We kown that there exists constants $C_1(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha}, \|f\|^{(\alpha/2)}_{\beta})$, and $C_2(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that

$$
|\frac{\epsilon_i}{\lambda+\delta(x_i)}| \leq |\frac{\epsilon_{i_0}}{\lambda+\delta(x_{i_0})}| \leq C_1 h^{\min\{\frac{r\alpha}{2},2\}} + C_2(r-1)h^2 \tag{4.26}
$$

as $\lambda + \delta(x_i) \leq \lambda + T$

So, we can get

$$
|\epsilon_i| \leq C(\lambda+T)h^{\min\{\frac{r\alpha}{2},2\}} \tag{4.27}
$$

The convergency has been proved.

Remark 4.5 (Weaker regularity on the derivatives of u). Suppose that the bound of Lemma 2.4 is replaced by the more general weaker regularity bound

$$
|u^{(l)}(x)| \leq C\delta(x)^{\sigma-l}, \quad l = 0, 1, 2, 3, 4
$$

where $\sigma \in (0, \frac{\alpha}{2}]$ is fixed. Then

$$
I^{2-\alpha}u(x) = \int_0^{x/2} + \int_{x/2}^{T+x/2} + \int_{T+x/2}^{2T} u(y)K_y(x)dx
$$

and for $l = 1, 2, 3, 4$, we have

$$
\begin{aligned}
\frac{d^l}{dx^l} I^{2-\alpha}u(x) = &\int_0^{x/2} + \int_{T+x/2}^{2T} u(y)K_y^{(l)}(x)dy \\
&+ \sum_{k=0}^{l-1} u^{(k)}(\frac{x}{2})K_{x/2}^{(l-1-k)}(x) - u^{(k)}(T+\frac{x}{2})K_{T+x/2}^{(l-1-k)}(x) \\
&+ \int_{x/2}^{T+x/2} u^{(l)}(y)K_y(x)dy
\end{aligned}
$$

Thus, we can get

$$
|f^l(x)| \leq C\delta(x)^{\sigma-\alpha-l}, \quad l = 0, 1, 2.
$$

Examine the proof above, by replacing the regularity condition with the weaker one, we can get the similar results:

$$\begin{aligned} |\tau_i| &= |-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i)| \\ &\le C h^{\min\{r\sigma,2\}} \delta(x_i)^{-\alpha} + C(r-1)h^2(T-\delta(x_i)+h_N)^{1-\alpha}. \end{aligned} \tag{4.28}$$

And the convergence result of Theorem 2.7 is changed to

$$\max_{1\le i\le 2N-1} |u_i - u(x_i)| \le C h^{\min\{r\sigma,2\}}. \tag{4.29}$$

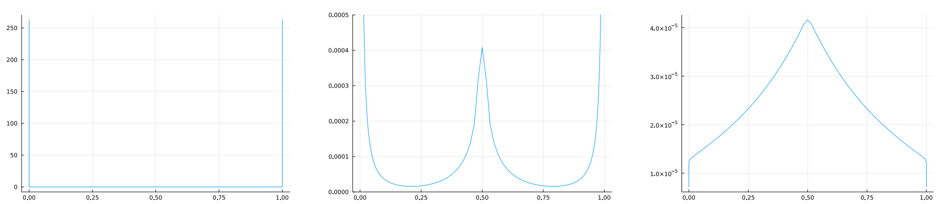
5. Experimental results.

TABLE 1
$r = 1$:

α \ $2N$	200	400	800	1600
1.2	1.127e-3	7.428e-4	4.899e-4	3.231e-4
		0.6013	0.6006	0.6003
1.5	2.500e-4	1.488e-4	8.849e-5	5.263e-5
		0.7487	0.7494	0.7497
1.8	2.732e-5	1.483e-5	7.997e-6	4.299e-6
		0.8815	0.8909	0.8955

TABLE 2
$r = \frac{4}{\alpha}$:

α \ $2N$	200	400	800	1600
1.2	4.158e-5	1.063e-5	2.692e-6	6.782e-7
		1.968	1.981	1.989
1.5	2.068e-5	5.379e-5	1.382e-6	3.524e-7
		1.943	1.960	1.972
1.8	7.642e-6	2.065e-6	5.501e-7	1.450e-7
		1.888	1.908	1.924



(a) LTE.1 (b) LTE.2 (c) GE

FIG. 2. *truncation error and global error for $f \equiv 1$, where $\alpha = 1.2$, $r = 4/\alpha$, $2N = 200$*

5.1. $f \equiv 1$.

And Figure 2(a), Figure 2(b) show the $|\tau_i|$, whose difference is just ylim, and Figure 2(c) shows the global error $|u_i - u(x_i)|$. And that is the Figure 2(c) suggests the technique we used in subsection 4.2

5.2. f is singular.

While by Remark 4.5, we take $f = x^{\sigma-\alpha}$, where $\sigma \in (0, \frac{\alpha}{2}]$. In these cases, we donnot hnown the exact solution, so we calculate the rate of convergence by

$$Rate^N = \log_2 \left(\frac{RE^N}{RE^{2N}} \right)$$

where

$$RE^N = \max_{1 \le i \le 2N-1} |u_i^N - u_{2i}^{2N}|$$

Let $\sigma = 0.4$, $\alpha = 1.5$, then $f(x) = x^{-1.1}, x \in (0, 1)$, with $2N = [200, 400, 800]$

Appendix A. Approximate of difference quotients.

TABLE 3
$r = 1$:

α \ $2N$	200	400	800	1600
1.2		0.2262	0.01744	0.01339
			0.3755	0.3804
1.5		0.03107	0.02372	0.01806
			0.3895	0.3934
1.8		0.04347	0.03311	0.02516
			0.3926	0.3962

TABLE 4
$r = \frac{2}{\sigma}$:

α \ $2N$	200	400	800	1600
1.2		6.963e-4	1.742e-4	4.356e-5
			1.999	2.000
1.5		8.015e-4	2.022e-4	5.095e-5
			1.987	1.989
1.8		1.319e-3	3.416e-4	8.769e-5
			1.949	1.962

LEMMA A.1. *If $g(x) \in C^2(\Omega)$, there exists $\xi \in (x_{i-1}, x_{i+1})$ such that*

$$D_h^2 g(x_i) = g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{A.1}$$

And if $g(x) \in C^4(\Omega)$, then

(A.2)

$$D_h^2 g(x_i) = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right)$$

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})$$

Subsitute them in the left side of (A.1), we have

$$\begin{aligned} D_h^2 g(x_i) &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (g(x_{i+1}) - g(x_i) + \frac{1}{h_i} (g(x_{i-1}) - g(x_i)) \right) \\ &= \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) \end{aligned}$$

Now, using intermediate value theorem, there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1}g'(x_i) + \frac{h_{i+1}^2}{2}g''(x_i) + \frac{h_{i+1}^3}{3!}g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y)\frac{(x_{i+1}-y)^3}{3!}dy$$

Expecially,

$$\text{(A.3)} \qquad \begin{aligned} &\int_{x_{i-1}}^{x_i} g''''(y)\frac{(y-x_{i-1})^3}{3!}dy = \frac{h_i^4}{4!}g''''(\eta_1) \\ &\int_{x_i}^{x_{i+1}} g''''(y)\frac{(x_{i+1}-y)^3}{3!}dy = \frac{h_{i+1}^4}{4!}g''''(\eta_2) \end{aligned}$$

where $\eta_1 \in (x_{i-1}, x_i), \eta_2 \in (x_i, x_{i+1})$. ∎

Lemma A.2. *Denote* $y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \theta \in (0,1)$,

$$\text{(A.4)} \qquad u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2}h_j^2 u''(\xi), \quad \xi \in (x_{j-1}, x_j)$$

(A.5)

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2}h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!}h_j^3(\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!}u''(\xi_1), \quad \xi_1 \in (x_{j-1}, y_j^\theta)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!}u''(\xi_2), \quad \xi_2 \in (y_j^\theta, x_j)$$

Thus

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2}h_j^2(\theta u''(\xi_1) + (1-\theta)u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2}h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!}u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!}u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!}u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!}u'''(\eta_2)$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$. Thus ∎

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2}h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!}h_j^3(\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

Lemma A.3. *By Lemma* A.2*, Lemma* 2.4 *and Lemma* 3.1 *, There is a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *for* $2 \le j \le 2N-1$,

$$\text{(A.6)} \quad |u(y) - \Pi_h u(y)| \le h_j^2 \max_{\xi \in [x_{j-1}, x_j]} |u''(\xi)| \le Ch^2\delta(y)^{\alpha/2-2/r}, \quad \textit{for } y \in (x_{j-1}, x_j)$$

LEMMA A.4. *For* $x \in [x_{j-1}, x_j]$

$$|u(x) - \Pi_h u(x)| = \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_{x}^{x_j} u'(y)dy \right| \leq \int_{x_{j-1}}^{x_j} |u'(y)|dy \tag{A.7}$$

If $x \in [0, x_1]$, *with Lemma* 2.4, *we have*

$$|u(x) - \Pi_h u(x)| \leq \int_0^{x_1} |u'(y)|dy \leq \int_0^{x_1} Cy^{\alpha/2-1}dy \leq C\frac{2}{\alpha}x_1^{\alpha/2} = C\frac{2}{\alpha}h_1^{\alpha/2} \tag{A.8}$$

Similarly, if $x \in [x_{2N-1}, 1]$, *we have*

$$|u(x) - \Pi_h u(x)| \leq C\frac{2}{\alpha}(2T - x_{2N-1})^{\alpha/2} = C\frac{2}{\alpha}h_{2N}^{\alpha/2} \tag{A.9}$$

LEMMA A.5.

$$b^{1-\theta}|a^\theta - b^\theta| \leq |a - b| \;(\text{ also } a^{1-\theta}|a^\theta - b^\theta| \leq |a - b|), \quad a, b \geq 0,\ \theta \in [0, 1] \tag{A.10}$$

Appendix B. Proofs of some technical details. Review that $h = \frac{1}{N}$ and the defination of $\simeq$ in subsection 2.1

LEMMA B.1. *There is a constant* C *such that for* $i = 1, 2, \cdots, 2N - 1$

$$|h_{i+1} - h_i| \leq Ch^2\delta(x_i)^{1-2/r} \tag{B.1}$$

Proof. By (2.2),

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \leq i \leq 2N-1 \end{cases} \tag{B.2}$$

Since

$$(i+1)^r - 2i^r + (i-1)^r \simeq r(r-1)i^{r-2}, \quad \text{for } i \geq 1 \tag{B.3}$$

We get the result. □

LEMMA B.2. *there is a constant* $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ *such that*

$$\frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y)\frac{(y - x_{i-1})^3}{3!}dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y)\frac{(y - x_{i+1})^3}{3!}dy \right| \leq Ch^2\delta(x_i)^{-\alpha/2-2/r} \tag{B.4}$$

Proof. By Lemma 2.5, we have for $1 \leq i \leq N$

$$\left| \int_{x_{i-1}}^{x_i} f''(y)\frac{(y - x_{i-1})^3}{3!}dy \right| \leq \frac{\|f\|_\beta^{(\alpha/2)}}{3!} \int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y - x_{i-1})^3 dy \tag{B.5}$$

For $i = 1$,

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y - x_{i-1})^3 dy = \int_0^{x_1} y^{1-\alpha/2} dy = \frac{1}{2-\alpha/2} x_1^{2-\alpha/2} = \frac{1}{2-\alpha/2} x_1^{-\alpha/2-2} h_1^4$$

And for $2 \le i \le N$, since $x_i \simeq x_{i-1} \le y \le x_i$, we have

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y - x_{i-1})^3 dy \simeq \int_{x_{i-1}}^{x_i} x_i^{-\alpha/2-2}(y - x_{i-1})^3 dy = \frac{1}{4!} x_i^{-\alpha/2-2} h_i^4$$

So for $1 \le i \le N$, we have

$$\left| \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy \right| \le C x_i^{-\alpha/2-2} h_i^4 \tag{B.6}$$

and similarly,

$$\left| \int_{x_i}^{x_{i+1}} f''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right| \le C x_i^{-\alpha/2-2} h_{i+1}^4 \tag{B.7}$$

Thus for $1 \le i \le N$, with Lemma 3.1 we have

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right| \\ &\le C x_i^{-\alpha/2-2} \frac{2}{h_i + h_{i+1}} (h_i^3 + h_{i+1}^3) \simeq x_i^{-\alpha/2-2} h_i^2 \simeq x_i^{-\alpha/2-2} h^2 x_i^{2-2/r} \\ &= C h^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{B.8}$$

It's symmetric for $N < i \le 2N - 1$. ∎

Lemma B.3. *There is a constant $C = C(\alpha, r)$ such that for all $1 \le i \le 2N - 1$, $1 \le j \le 2N$ s.t. $\min\{|j - i|, |j - 1 - i|\} \ge 2$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_h K_y(x_i) \simeq |y - x_i|^{-\alpha}, \quad D_h^2 K_y(x_i) \simeq |y - x_i|^{-1-\alpha} \tag{B.9}$$

Proof. Sinec $y - x_{i-1}, y - x_i, y - x_{i+1}$ have the same sign, by mean value theorem and Lemma A.1,

$$\begin{aligned} D_h K_y(x_i) &= \frac{|y - \xi|^{-\alpha}}{\Gamma(1-\alpha)}, \quad \xi \in (x_i, x_{i+1}) \\ D_h^2 K_y(x_i) &= \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in (x_{i-1}, x_{i+1}) \end{aligned}$$

however, $|y - \xi| \simeq |y - x_i|$, we get the result. ∎

Lemma B.4. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$\sum_{j=1}^{3} V_{1j} \le C h^2 x_1^{-\alpha/2-2/r} \tag{B.10}$$

$$\sum_{j=1}^{4} V_{2j} \le C h^2 x_2^{-\alpha/2-2/r} \tag{B.11}$$

Proof. For $0 \leq i \leq 3, 1 \leq j \leq 4$, by Lemma A.4, Lemma A.3 and (3.14)

$$T_{ij} \leq C x_1^{2-\alpha/2} \simeq h_1^2\, h^2 x_1^{-\alpha/2-2/r} \simeq h_1^2\, h^2 x_2^{-\alpha/2-2/r} \tag{B.12}$$

Therefore, by (3.15), we get the result. ∎

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