

AN EXAMPLE ARTICLE*

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Abstract. This is an example SIAM L^AT_EX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

Key words. example, L^AT_EX

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1. Introduction. The introduction introduces the context and summarizes the manuscript. It is importantly to clearly state the contributions of this piece of work.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$, suppose $f \in C^\beta(\Omega)$, $\beta > 4 - \alpha$, $\|f\|_\beta^{(\alpha/2)} < \infty$

$$(1.1) \quad \begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases}$$

where

$$(1.2) \quad (-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy$$

$$(1.3) \quad \kappa_\alpha = -\frac{1}{2 \cos(\alpha\pi/2)} > 0$$

and the solution $u \in C^{\alpha/2}(\Omega)$.

2. Regularity.

Remark 2.1. 1. $C^k(U)$ is the set of all k -times continuously differentiable functions on open set U .

2. $C^\beta(U)$ is the collection of function f which for any $V \subset\subset U$ $f|_V \in C^\beta(\bar{V})$.

THEOREM 2.2. If $f \in C^\beta(\Omega)$, $\beta > 2$ and $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$

$$(2.1) \quad |f^{(l)}(x)| \leq \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \leq T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \leq x < 2T \end{cases}$$

THEOREM 2.3 (Regularity up to the boundary [1]).

$$(2.2) \quad \|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right)$$

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29 **COROLLARY 2.4.** *Let u be a solution of (1.1) on Ω . Then, for any $x \in \Omega$ and*
 30 *$l = 0, 1, 2, 3, 4$*

$$31 \quad (2.3) \quad |u^{(l)}(x)| \leq \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases}$$

32 The paper is organized as follows. Our main results are in section 4, experimental
 33 results are in section 7, and the conclusions follow in section 8.

3. Numeric Format.

$$34 \quad (3.1) \quad x_i = \begin{cases} T \left(\frac{i}{N} \right)^r, & 0 \leq i \leq N \\ 2T - T \left(\frac{2N-i}{N} \right)^r, & N \leq i \leq 2N \end{cases}$$

35 where $r \geq 1$. And let

$$36 \quad (3.2) \quad h_j = x_j - x_{j-1}, \quad 1 \leq j \leq 2N$$

37 Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear
 38 function space.

$$39 \quad (3.3) \quad \phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \leq x \leq x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \leq x \leq x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

40 And then, we can approximate $u(x)$ with

$$41 \quad (3.4) \quad u_h(x) := \sum_{j=1}^{2N-1} u(x_j) \phi_j(x)$$

42 For convience, we denote

$$43 \quad (3.5) \quad I_h^{2-\alpha}(x_i) := \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u_h(y) dy$$

44 And now, we can approximate the operator (1.2) at x_i with

$$45 \quad (3.6) \quad \begin{aligned} D_h^\alpha u_h(x_i) &:= D_h^2 I_h^{2-\alpha}(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} I_h^{2-\alpha}(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) I_h^{2-\alpha}(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha}(x_{i+1}) \right) \end{aligned}$$

46 Finally, we approximate the equation (1.1) with

$$47 \quad (3.7) \quad -\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i), \quad 1 \leq i \leq 2N-1$$

48 The discrete equation (3.7) can be written in matrix form

$$49 \quad (3.8) \quad AU = F$$

where U is unknown, $F = (f(x_1), \dots, f(x_{2N-1}))$. The matrix A is constructed as follows: Since

(3.9)

$$\begin{aligned} I_h^{2-\alpha}(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u_h(y) dy \\ &= \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u(x_j) \phi_j(y) dy \\ &= \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y) dy \\ &= \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\ &=: \sum_{j=1}^{2N-1} \tilde{a}_{ij} u(x_j), \quad 0 \leq i \leq 2N \end{aligned}$$

Then, substitute in (3.6), we have

$$(3.10) \quad -\kappa_{\alpha} D_h^{\alpha} u_h(x_i) = \sum_{j=1}^{2N-1} a_{ij} u(x_j)$$

where

$$(3.11) \quad a_{ij} = -\kappa_{\alpha} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right)$$

4. Main results. Here we state our main results; the proof is deferred to section 5 and section 6.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 4.1 (Truncation Error). *If $f \in C^2(\Omega)$ and $\alpha \in (1, 2)$, and $u(x)$ is a solution of the equation (1.1), then there exists a constant $C_1, C_2 = C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{C^2(\Omega)}), C_2(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$, such that the truncation error of the discrete format satisfies*

$$(4.1) \quad \begin{aligned} |-\kappa_{\alpha} D_h^{\alpha} u_h(x_i) - f(x_i)| &\leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} (x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \\ &\quad + C_2 h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \leq i \leq N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \leq 2N-1 \end{cases} \end{aligned}$$

where $C_2 = 0$ if $r = 1$.

THEOREM 4.2 (Convergence). *The discrete equation (3.7) has solution U , and there exists a positive constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ such that the error between the numerical solution U with the exact solution $u(x_i)$ satisfies*

$$(4.2) \quad \max_{1 \leq i \leq 2N-1} |U_i - u(x_i)| \leq C h^{\min\{\frac{r\alpha}{2}, 2\}}$$

That means the numerical method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

5. Proof of Theorem 4.1. For convience, let's denote

$$(5.1) \quad I^{2-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x-y|^{1-\alpha} u(y) dy$$

Then, the truncation error of the discrete format can be written as

$$(5.2) \quad \begin{aligned} -\kappa_{\alpha} D_h^{\alpha} u_h(x_i) - f(x_i) &= -\kappa_{\alpha} (D_h^2 I_h^{2-\alpha}(x_i) - \frac{d^2}{dx^2} I^{2-\alpha}(x_i)) \\ &= -\kappa_{\alpha} D_h^2 (I_h^{2-\alpha} - I^{2-\alpha})(x_i) - \kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \end{aligned}$$

5.1. Estimate of $-\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i)$.

THEOREM 5.1. *There exists a constant $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$ such that*

$$(5.3) \quad \left| -\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \right| \leq Ch^2 (x_i^{-\alpha/2-2/r} + (2T-x_i)^{-\alpha/2-2/r})$$

Proof. Since $f \in C^2(\Omega)$ and

$$(5.4) \quad \frac{d^2}{dx^2} (-\kappa_{\alpha} I^{2-\alpha}(x)) = f(x), \quad x \in \Omega,$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.3) of Lemma A.1, for $1 \leq i \leq 2N-1$, we have

$$(5.5) \quad -\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2))$$

where $\eta_1 \in [x_{i-1}, x_i]$, $\eta_2 \in [x_i, x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

$$(5.6) \quad \left| \frac{h_{i+1} - h_i}{3} f'(x_i) \right| \leq \frac{\|f\|_{\beta}^{(\alpha/2)}}{3} Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T-x_i)^{-\alpha/2-2/r}, & N < i \leq 2N-1 \end{cases}$$

2. See Proof 9, there is a constant $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$ such that

$$(5.7) \quad \begin{aligned} &\left| \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \right| \\ &\leq Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T-x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N-1 \end{cases} \end{aligned}$$

Summarizes, we get the result. \square

5.2. Estimate of R_i . Now, we study the first part of (5.2)

$$(5.8) \quad D_h^2 (I^{2-\alpha} - I_h^{2-\alpha})(x_i) = D_h^2 \left(\int_0^{2T} (u(y) - u_h(y)) \frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \right)$$

For convience, let's denote

$$(5.9) \quad T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy$$

92 And define

$$93 \quad (5.10) \quad R_i := D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i) \\ = \frac{2}{h_i + h_{i+1}} \sum_{j=1}^{2N} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right)$$

94 We have some results about the estimate of R_i

95 **THEOREM 5.2.** *For $1 \leq i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$96 \quad (5.11) \quad R_i \leq C(h^{r\alpha/2+r} x_i^{-1-\alpha} + h^2 x_i^{-\alpha/2-2/r})$$

97

98 **THEOREM 5.3.** *For $N/2 \leq i \leq N$, there exists constant C, C_2 such that*

$$99 \quad (5.12) \quad R_i \leq Ch^2 x_i^{-\alpha/2-2/r} + C_2 h^2 |T - x_{i-1}|^{1-\alpha}$$

100 where $C_2 = 0$ if $r = 1$.

101 And for $N < i \leq 2N - 1$, it is symmetric to the previous case.

102 To prove these results, we need some utils. Also for simplicity, we denote

$$103 \quad (5.13) \quad S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right)$$

104 then

$$105 \quad (5.14) \quad R_i = \sum_{j=1}^{2N} S_{ij}$$

106 **5.3. Proof of Theorem 5.2.**

107 **LEMMA 5.4.** *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \leq$
108 $i < N/2$,*

$$109 \quad (5.15) \quad \sum_{j=\max\{2i+1, i+3\}}^N S_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r}$$

110 *Proof.* For $\max\{2i+1, i+3\} \leq j \leq N$, by Lemma C.1 and Lemma C.2

$$111 \quad (5.16) \quad S_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\ \leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ = Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy$$

112 Therefore,

$$113 \quad (5.17) \quad \sum_{j=\max\{2i+1, i+3\}}^N S_{ij} \leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\ = \frac{C}{\alpha/2 + 2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\ \leq \frac{C}{\alpha/2 + 2/r} 2^{r(-\alpha/2-2/r)} h^2 x_i^{-\alpha/2-2/r} \quad \square$$

LEMMA 5.5. *Thert exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \leq i < N/2$,*

$$(5.18) \quad \sum_{j=N+1}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

Proof. For $1 \leq i < N/2, N+1 \leq j \leq 2N-1$, by equation (C.2) and Lemma C.2

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\ &\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2-2/r} y^{-1-\alpha} dy \\ &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy \end{aligned}$$

$$\begin{aligned} \sum_{j=N+1}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T - y)^{\alpha/2-2/r} dy \\ &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\ &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r} h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r} h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned}$$

And by Lemma A.3

$$S_{i,2N} \leq CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}$$

And when $\alpha/2 - 2/r + 1 \geq 0$,

$$h^{r\alpha/2+r} \leq h^2$$

Summarizes, we get the result. \square

For $i = 1, 2$.

LEMMA 5.6. *By Lemma C.5 , Lemma 5.4 and Lemma 5.5 we get*

$$(5.20) \quad \begin{aligned} R_1 &= \sum_{j=1}^3 S_{1j} + \sum_{j=4}^{2N} S_{1j} \\ &\leq Ch^2 x_1^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned}$$

130

$$\begin{aligned}
R_2 &= \sum_{j=1}^4 S_{2j} + \sum_{j=5}^{2N} S_{2j} \\
(5.21) \quad &\leq Ch^2 x_2^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\end{aligned}$$

132 For $3 \leq i < N/2$, we have a new separation of R_i , Let's denote $k = \lceil \frac{i}{2} \rceil$.

$$\begin{aligned}
R_i &= \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&= \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \\
(5.22) \quad &\quad + \sum_{j=k+1}^{2i-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i-1}} (T_{i-1,2i} + T_{i-1,2i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \\
&\quad + \sum_{j=2i+1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&= I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}$$

134

135 **LEMMA 5.7.** *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \leq$*
 136 *$i < N/2, k = \lceil \frac{i}{2} \rceil$*

$$(5.23) \quad |I_1| = \left| \sum_{j=1}^{k-1} S_{ij} \right| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

138 *Proof.* For $2 \leq j \leq k-1$, by Lemma C.1 and Lemma C.3

$$\begin{aligned}
S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\
(5.24) \quad &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} dy \\
&= Ch^2 x_i^{-1-\alpha} \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} dy
\end{aligned}$$

140 And by Lemma A.3 , Lemma C.3

$$(5.25) \quad S_{i1} \leq C x_1^{\alpha/2} x_1 x_i^{-1-\alpha} = C x_1^{\alpha/2+1} x_i^{-1-\alpha} = C T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha}$$

142 Therefore,

$$\begin{aligned}
 I_1 &= \sum_{j=1}^{k-1} S_{ij} = S_{i1} + \sum_{j=2}^{k-1} S_{ij} \\
 (5.26) \quad &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \\
 &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy
 \end{aligned}$$

144 But

$$(5.27) \quad \int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \leq \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2-2/r+1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 < 0 \end{cases}$$

146 So we have

$$(5.28) \quad I_1 \leq \begin{cases} \frac{C}{\alpha/2-2/r+1} h^2 x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2-2/r+1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} h^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2-2/r+1 < 0 \end{cases} \quad \square$$

148 For convience, let's denote

$$(5.29) \quad V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right)$$

150

151 **THEOREM 5.8.** *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for*
 152 *$3 \leq i < N/2, k = \lceil \frac{i}{2} \rceil$,*

$$(5.30) \quad I_3 = \sum_{j=k+1}^{2i-1} V_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r} := \text{Sorry}$$

154 To estimate V_{ij} , we need some preparations.

155 **LEMMA 5.9.** *Denote $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0, 1]$, by Lemma A.2*

$$\begin{aligned}
 T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
 &= \int_{x_{j-1}}^{x_j} -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
 (5.31) \quad &+ \frac{\theta(1-\theta)}{3!} h_j^3 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) dy_j^\theta \\
 &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
 &+ \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta
 \end{aligned}$$

157 where $\eta_{j1}^\theta \in [x_{j-1}, y_j^\theta], \eta_{j2}^\theta \in [y_j^\theta, x_j]$.

158 Now Let's construct a series of functions to represent T_{ij} .

$$159 \quad (5.32) \quad y_{j-i}(x) = (x^{1/r} + Z_{j-i})^r, \quad Z_{j-i} = T^{1/r} \frac{j-i}{N}$$

$$161 \quad (5.33) \quad y_{j-i}^\theta(x) = \theta y_{j-1-i}(x) + (1-\theta) y_{j-i}(x)$$

$$163 \quad (5.34) \quad h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x)$$

164 Now, we define

$$165 \quad (5.35) \quad P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$167 \quad (5.36) \quad Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

168 And now we can rewrite T_{ij}

169 LEMMA 5.10. For $2 \leq i \leq N, 2 \leq j \leq N$,

$$170 \quad (5.37) \quad T_{ij} = \int_0^1 -\frac{\theta(1-\theta)}{2} P_{j-i}^\theta(x_i) d\theta \\ + \int_0^1 \frac{\theta(1-\theta)}{3!} (\theta^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j1}^\theta) - (1-\theta)^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j2}^\theta)) d\theta$$

171 Immediately, we can see from (5.29) that

172 LEMMA 5.11. For $3 \leq i \leq N-1, 3 \leq j \leq N-1$,

$$173 \quad (5.38) \quad V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ = \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^\theta(x_i) d\theta \\ + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\ - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\ - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,2}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\ + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta$$

174

175 LEMMA 5.12. There exists a constant $C = C(T, \alpha, r, f)$ such that for $3 \leq i <$
 176 $N, k = \lceil \frac{i}{2} \rceil, k+1 \leq j \leq \min\{2i-1, N\}$,

$$177 \quad (5.39) \quad V_{ij} \leq \text{Sorry}$$

6. Proof of Theorem 4.2.

7. Experimental results.

8. Conclusions. Some conclusions here.

Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x)$ is twice differentiable continuous function on open set Ω , there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

$$(A.1) \quad D_h^2 g(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

$$(A.2) \quad \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right)$$

And if $g(x) \in C^4(\Omega)$, then

$$(A.3) \quad \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2))$$

where $\eta_1 \in [x_{i-1}, x_i]$, $\eta_2 \in [x_i, x_{i+1}]$.

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in [x_{i-1}, x_i]$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in [x_i, x_{i+1}]$$

Substitute them in the left side of (A.1), we have

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2)$$

Now, using intermediate value theorem, there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y)dy$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \frac{h_i^4}{4!} g''''(\eta_1)$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \frac{h_{i+1}^4}{4!} g''''(\eta_2)$$

where $\eta_1 \in [x_{i-1}, x_i]$, $\eta_2 \in [x_i, x_{i+1}]$. Expecially,

$$\begin{aligned} \frac{h_i^4}{4!} g''''(\eta_1) &= \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy \\ \frac{h_{i+1}^4}{4!} g''''(\eta_2) &= \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \end{aligned}$$

Substitute them to the left side of (A.3), we can get the result. \square

LEMMA A.2. If $y \in [x_{j-1}, x_j]$, denote $y = \theta x_{j-1} + (1 - \theta)x_j$, $\theta \in [0, 1]$,

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j]$$

(A.6)

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

where $\eta_1 \in [x_{j-1}, y_j^\theta]$, $\eta_2 \in [y_j^\theta, x_j]$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta)u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in [x_{j-1}, y_j^\theta]$, $\eta_2 \in [y_j^\theta, x_j]$. Thus \square

$$u(y_j^\theta) - u_h(y_j^\theta) = u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j)$$

$$= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

LEMMA A.3. For $x \in [x_{j-1}, x_j]$

$$|u(x) - u_h(x)| = \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^x u'(y) dy - \frac{x - x_{j-1}}{h_j} \int_x^{x_j} u'(y) dy \right|$$

$$\leq \int_{x_{j-1}}^{x_j} |u'(y)| dy$$

If $x \in [0, x_1]$, with Corollary 2.4, we have

$$|u(x) - u_h(x)| \leq \int_0^{x_1} |u'(y)| dy \leq \int_0^{x_1} C y^{\alpha/2-1} dy \leq C \frac{2}{\alpha} x_1^{\alpha/2}$$

Similarly, if $x \in [x_{2N-1}, 1]$, we have

$$|u(x) - u_h(x)| \leq C \frac{2}{\alpha} (2T - x_{2N-1})^{\alpha/2} = C \frac{2}{\alpha} x_1^{\alpha/2}$$

Appendix B. Inequality.

LEMMA B.1.

$$h_i \leq r T^{1/r} h \begin{cases} x_i^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_{i-1})^{1-1/r}, & N < i \leq 2N-1 \end{cases}$$

Proof. For $1 \leq i \leq N$,

$$h_i = T \left(\left(\frac{i}{N} \right)^r - \left(\frac{i-1}{N} \right)^r \right)$$

$$\leq r T \frac{1}{N} \left(\frac{i}{N} \right)^{r-1} = r T^{1/r} h x_i^{1-1/r}$$

For $N < i \leq 2N-1$,

$$h_i = T \left(\left(\frac{2N-i+1}{N} \right)^r - \left(\frac{2N-i}{N} \right)^r \right)$$

$$\leq r T \frac{1}{N} \left(\frac{2N-i+1}{N} \right)^{r-1} = r T^{1/r} h (2T - x_{i-1})^{1-1/r}$$

□

LEMMA B.2. There is a constant $C = 2^{|r-2|} r(r-1) T^{2/r}$ such that for all $i \in \{1, 2, \dots, 2N-1\}$

$$|h_{i+1} - h_i| \leq C h^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases}$$

Proof.

$$h_{i+1} - h_i = \begin{cases} T \left(\left(\frac{i+1}{N} \right)^r - 2 \left(\frac{i}{N} \right)^r + \left(\frac{i-1}{N} \right)^r \right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T \left(\left(\frac{2N-i-1}{N} \right)^r - 2 \left(\frac{2N-i}{N} \right)^r + \left(\frac{2N-i+1}{N} \right)^r \right), & N+1 \leq i \leq 2N-1 \end{cases}$$

237 For $i = 1$,

$$238 \quad h_2 - h_1 = T(2^r - 2) \left(\frac{1}{N} \right)^r = (2^r - 2)T^{2/r}h^2x_1^{1-2/r}$$

239 For $2 \leq i \leq N - 1$,

$$240 \quad h_{i+1} - h_i = r(r-1)T N^{-2}\eta^{r-2}, \quad \eta \in \left[\frac{i-1}{N}, \frac{i+1}{N} \right]$$

241 If $r \in [1, 2]$,

$$\begin{aligned} 242 \quad h_{i+1} - h_i &= r(r-1)T N^{-2}\eta^{r-2} \leq r(r-1)T h^2 \left(\frac{i-1}{N} \right)^{r-2} \\ &\leq r(r-1)T h^2 2^{2-r} \left(\frac{i}{N} \right)^{r-2} \\ &= 2^{2-r}r(r-1)T^{2/r}h^2x_i^{1-2/r} \end{aligned}$$

243 else if $r > 2$,

$$\begin{aligned} 244 \quad h_{i+1} - h_i &= r(r-1)T N^{-2}\eta^{r-2} \leq r(r-1)T h^2 \left(\frac{i+1}{N} \right)^{r-2} \\ &\leq r(r-1)T h^2 2^{r-2} \left(\frac{i}{N} \right)^{r-2} \\ &= 2^{r-2}r(r-1)T^{2/r}h^2x_i^{1-2/r} \end{aligned}$$

245 Since

$$246 \quad 2^r - 2 \leq 2^{|r-2|}r(r-1), \quad r \geq 1$$

247 we have

$$248 \quad h_{i+1} - h_i \leq 2^{|r-2|}r(r-1)T^{2/r}h^2x_i^{1-2/r}, \quad 1 \leq i \leq N-1$$

249 For $i = N$, $h_{N+1} - h_N = 0$. For $N < i \leq 2N - 1$, it's central symmetric to the first
250 half of the proof, which is

$$251 \quad h_i - h_{i+1} \leq 2^{|r-2|}r(r-1)T^{2/r}h^2(2T - x_i)^{1-2/r}$$

252 Summarizes the inequalities, we can get

$$253 \quad (\text{B.3}) \quad |h_{i+1} - h_i| \leq 2^{|r-2|}r(r-1)T^{2/r}h^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases} \quad \square$$

254 **Appendix C. Proofs of some technical details.**

Additional proof of Theorem 5.1. For $2 \leq i \leq N-1$,

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\ & \leq C \frac{2}{h_i + h_{i+1}} (h_i^3 x_{i-1}^{-2-\alpha/2} + h_{i+1}^3 x_i^{-2-\alpha/2}) \\ & \leq 2C (h_i^2 x_{i-1}^{-2-\alpha/2} + h_{i+1}^2 x_i^{-2-\alpha/2}) \end{aligned}$$

Since Lemma B.1, we have

$$\begin{aligned} h_i & \leq r T^{1/r} h x_i^{1-1/r}, \quad 1 \leq i \leq N \\ h_{i+1} & \leq r T^{1/r} h x_{i+1}^{1-1/r}, \quad 1 \leq i \leq N-1 \end{aligned}$$

and

$$\begin{aligned} x_{i-1}^{-2-\alpha/2} & \leq 2^{-r(-2-\alpha/2)} x_i^{-2-\alpha/2} \quad 2 \leq i \leq N-1 \\ x_{i+1}^{1-1/r} & \leq 2^{r-1} x_i^{1-1/r} \quad 1 \leq i \leq N-1 \end{aligned}$$

So there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

$$\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq C h^2 x_i^{-\alpha/2-2/r}, \quad 2 \leq i \leq N-1$$

For $i = 1$, by (A.4)

$$\begin{aligned} & \frac{1}{4!} \frac{2}{h_1 + h_2} (h_1^3 f''(\eta_1) + h_2^3 f''(\eta_2)) \\ & = \frac{2}{h_1 + h_2} \left(\frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy + \frac{1}{4!} h_2^3 f''(\eta_2) \right) \end{aligned}$$

We have proved above that

$$\frac{2}{h_1 + h_2} h_2^3 f''(\eta_2) \leq C h^2 x_1^{-\alpha/2-2/r}$$

and we can get

$$\begin{aligned} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy & \leq C \frac{1}{3!} \int_0^{x_1} y^{1-\alpha/2} dy \\ & = C \frac{1}{3!(2-\alpha/2)} x_1^{2-\alpha/2} \end{aligned}$$

so

$$\frac{2}{h_1 + h_2} \frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy = \frac{C 2^{1-r}}{3!(2-\alpha/2)} x_1^{-\alpha/2} = \frac{C 2^{1-r}}{3!(2-\alpha/2)} T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

And for $i = N$, we have

$$\begin{aligned} & \frac{2}{h_N + h_{N+1}} (h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2)) \\ & = h_N^2 (f''(\eta_1) + f''(\eta_2)) \\ & \leq r^2 T^{2/r} h^2 x_N^{2-2/r} 2C x_{N-1}^{-2-\alpha/2} \\ & \leq 2r^2 T^{2/r} C 2^{-r(-2-\alpha/2)} h^2 x_N^{-\alpha/2-2/r} \end{aligned}$$

275 Finally, $N + 1 \leq i \leq 2N - 1$ is symmetric to the first half of the proof, so we can
 276 conclude that □

$$277 \quad \frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N - 1 \end{cases}$$

278 LEMMA C.1. *There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ for $2 \leq j \leq N$, if*
 279 $y \in [x_{j-1}, x_j]$,

$$280 \quad (C.1) \quad |u(y) - u_h(y)| \leq Ch^2 y^{\alpha/2-2/r}$$

281 *Proof.* For $2 \leq j \leq N$, we have

$$282 \quad x_j \leq 2^r y, \quad x_{j-1} \geq 2^{-r} y$$

283 And by Lemma A.2, Lemma B.1 and Corollary 2.4, we have

$$\begin{aligned} u(y) - u_h(y) &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \\ 284 \quad &\leq \frac{\|u\|_{\beta+\alpha}^{(-\alpha/2)}}{4} r^2 T^{2/r} h^2 x_j^{2-2/r} x_{j-1}^{\alpha/2-2} \\ &\leq Ch^2 2^{2r-2} y^{2-2/r} 2^{-r(\alpha/2-2)} y^{\alpha/2-2} \\ &= C 2^{-r\alpha/2+4r-2} h^2 y^{\alpha/2-2/r} \end{aligned}$$

285 symmetricly, for $N < j \leq 2N - 1$, we have

$$286 \quad (C.2) \quad |u(y) - u_h(y)| \leq Ch^2 (2T - y)^{\alpha/2-2/r} \quad \square$$

287 LEMMA C.2. *There is a constant $C = C(\alpha, r)$ such that for all $1 \leq i < N/2$,*
 288 $\max\{2i + 1, i + 3\} \leq j \leq 2N$ and $y \in [x_{j-1}, x_j]$, we have

$$289 \quad (C.3) \quad D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) \leq C \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}$$

290 *Proof.* Since $y \geq x_{j-1} > x_{i+1}$, by Lemma A.1, if $j - 1 > i + 1$ □

$$\begin{aligned} D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) &= \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ 291 \quad &\leq \frac{(y - x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\leq (1 - (\frac{2}{3})^r)^{-1-\alpha} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

292 LEMMA C.3. *There is a constant $C = C(\alpha, r)$ such that for all $3 \leq i < N/2, k =$*
 293 $\lceil \frac{i}{2} \rceil, 1 \leq j \leq k - 1$ and $y \in [x_{j-1}, x_j]$, we have

$$294 \quad (C.4) \quad D_h^2 \left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) \leq C \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)}$$

Proof. Since $y \leq x_j < x_{i-1}$, by Lemma A.1,

$$\begin{aligned} D_h^2\left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) &= \frac{|\xi - y|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq \frac{(x_{i-1} - x_j)^{-1-\alpha}}{\Gamma(-\alpha)} \leq \frac{(x_{i-1} - x_{k-1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\leq \left(\left(\frac{2}{3}\right)^r - \left(\frac{1}{2}\right)^r\right)^{-1-\alpha} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

□

LEMMA C.4. While $0 \leq i < N/2$, By Lemma A.3

$$\begin{aligned} |T_{i1}| &\leq C \int_0^{x_1} x_1^{\alpha/2} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2} |x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha}| \\ &\leq C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2+2-\alpha} = C \frac{1}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1 \end{aligned}$$

For $2 \leq j \leq N$, by Lemma A.2 and Corollary 2.4

$$\begin{aligned} |T_{ij}| &\leq \frac{C}{4} \int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &\leq \frac{C}{4\Gamma(3-\alpha)} h_j^2 x_{j-1}^{\alpha/2-2} ||x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha}| \end{aligned}$$

LEMMA C.5. There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that

$$\sum_{j=1}^3 S_{1j} \leq C h^2 x_1^{-\alpha/2-2/r}$$

$$\sum_{j=1}^4 S_{2j} \leq C h^2 x_2^{-\alpha/2-2/r}$$

Proof.

$$S_{1j} = \frac{2}{x_2} \left(\frac{1}{x_1} T_{0j} - \left(\frac{1}{x_1} + \frac{1}{h_2} \right) T_{1j} + \frac{1}{h_2} T_{2j} \right)$$

So, by Lemma C.4

$$\begin{aligned} S_{11} &\leq \frac{2}{x_2 x_1} 4 \frac{C}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \leq C x_1^{-\alpha/2} \\ S_{12} &\leq \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_2^2 x_1^{\alpha/2-2} (x_2^{2-\alpha} + 2h_2^{2-\alpha} + h_2^{2-\alpha}) \leq C x_1^{-\alpha/2} \\ S_{13} &\leq \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_3^2 x_2^{\alpha/2-2} (x_3^{2-\alpha} + 2x_3^{2-\alpha} + h_3^{2-\alpha}) \leq C x_1^{-\alpha/2} \end{aligned}$$

314 But

315
$$x_1^{-\alpha/2} = T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

316 For $i = 2$, Sorry

□

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319

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