1   问题

对于 $\Omega = (0,1)$, $1 < \alpha < 2$, 假设 $f \in C^2(\Omega)$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1}$$

其中

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = C_R \frac{d^2}{dx^2} \int_\Omega \frac{u(y)}{|x-y|^{\alpha-1}} dy \tag{2}$$

2   数值格式

用线性插值代替原函数，中心差分代替二阶导数，记 $u_h(x)$ 为 $u(x)$ 在网格点上的线性插值。

我们解这样的数值解

$$\begin{aligned} C_R \Big(&\frac{2}{h_{i+1}(h_i + h_{i+1})} \int_\Omega \frac{u_h(x)}{|x_{i+1} - y|^{\alpha-1}} dy \\ &- \frac{2}{h_i h_{i+1}} \int_\Omega \frac{u_h(x)}{|x_i - y|^{\alpha-1}} dy \\ &+ \frac{2}{h_i(h_i + h_{i+1})} \int_\Omega \frac{u_h(x)}{|x_{i-1} - y|^{\alpha-1}} dy \Big) \\ &= F_i \end{aligned} \tag{3}$$

矩阵 A 是 M 矩阵，主队角正，其他负，严格对角占优。

3   一致网格

当 $r = 1$, 网格成为一致网格, $x_i = ih, h = \frac{1}{2N}, i = 0, ..., 2N.$
A 等于

$$a_{ij} = \frac{C_R}{(2-\alpha)(3-\alpha)} h^{-\alpha}$$
$$\left(|i-j-2|^{3-\alpha} - 4|i-j-1|^{3-\alpha} + 6|i-j|^{3-\alpha} - 4|i-j+1|^{3-\alpha} + |i-j+2|^{3-\alpha} \right) \tag{4}$$

矩阵行和

$$S_i = \sum_{j=1}^{2N-1} a_{ij} = \frac{C_R}{(2-\alpha)(3-\alpha)} h^{-\alpha} (|i+1|^{3-\alpha} - 3|i|^{3-\alpha} + 3|i-1|^{3-\alpha} - |i-2|^{3-\alpha} + ...2N) \tag{5}$$

我们得到

$$S_i \geq C(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \tag{6}$$

下面估计截断误差 R_i.

$$R_i = \int_0^1 D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \tag{7}$$

目标是

$$R_i \leq Ch^{\alpha/2} S_i \tag{8}$$

这样我们就有

$$\epsilon \leq \max_i \frac{R_i}{S_i} \leq Ch^{\alpha/2} \tag{9}$$

考虑 R_1

$$R_1 = \int_\Omega (u(y) - u_h(y)) \frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} dy \tag{10}$$

我们有

$$R_1 = \int_0^{3h} + \int_{3h}^{1/2} \tag{11}$$

当 $y > 3h$,

$$\frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} \leq C|y|^{-1-\alpha} \tag{12}$$

那么

$$I_2 \le C \int_{3h}^{1/2} |y|^{-1-\alpha} u''(y) h^2 dy$$
$$\le C \int_{3h}^{1} |y|^{-1-\alpha} y^{\alpha/2-2} h^2 dy$$
$$\le Ch^2 \int_{3h}^{1} y^{-3-\alpha/2} dy \tag{13}$$
$$\le Ch^2 h^{-2-\alpha/2} = Ch^{-\alpha/2}$$
$$\le Ch^{\alpha/2} x_1^{-\alpha} \le Ch^{\alpha/2} S_1$$

在考虑

$$I_1 = \int_0^{3h} \frac{u(y) - u_h(y)}{h^2} (|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}) dy$$
$$= \int_0^h + \int_h^{3h} = J_1 + J_2 \tag{14}$$

$$J_2 \le Cu''(\eta) h^{2-\alpha} \le Ch^{\alpha/2-2} h^{2-\alpha} \le Ch^{-\alpha/2} \tag{15}$$

因为

$$|u(x) - u_h(x)| \le \int_0^{x_1} |u'(y)| dy$$
$$\le C \int_0^{x_1} y^{\alpha/2-1} dy \tag{16}$$
$$\le C x_1^{\alpha/2} \quad , x \in (0, h)$$

$$J_1 = \int_0^h \frac{u(y) - u_h(y)}{h^2} (|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}) dy$$
$$\le Ch^{\alpha/2-2} h^{2-\alpha} = Ch^{-\alpha/2} \tag{17}$$

所以有

$$R_1 \le Ch^{-\alpha/2} \le Ch^{\alpha/2} h^{-\alpha} \le Ch^{\alpha/2} S_1, \quad (S_1 \ge Cx_1^{-\alpha}) \tag{18}$$

R_1, R_2, R_3 全部类似。

3.1 猜想

$$R_i \leq Ch^{\alpha/2+1}(x_i^{-\alpha-1} + (1-x_i)^{-\alpha-1}) \quad (then \leq Ch^{\alpha/2}S_i) \qquad (19)$$

为了简便，我们记 $D(y) := u(y) - u_h(y)$.

当 $3 < i \leq N$ 时，

$$
\begin{aligned}
R_i &= \int_0^1 D(y)\frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2}dy \\
&= \int_0^{x_1} D(y)\frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2}dy \\
&\quad + \int_{x_1}^{x_{\lceil\frac{i}{2}\rceil}} D(y)\frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2}dy \\
&\quad + \int_{x_{\lceil\frac{i}{2}\rceil}}^{x_{\lceil\frac{i}{2}\rceil+1}} \frac{D(y+h) - D(y)}{h^2}|y-x_i|^{1-\alpha} + D(y)\frac{|y-x_{i+1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2}dy \\
&\quad + \int_{x_{\lceil\frac{i}{2}\rceil+1}}^{x_i} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2}|y-x_i|^{1-\alpha}dy \\
&\quad + \int_{x_i}^{x_{N+\lfloor\frac{i}{2}\rfloor-1}} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2}|y-x_i|^{1-\alpha}dy \\
&\quad + \int_{x_{N+\lfloor\frac{i}{2}\rfloor-1}}^{x_{N+\lfloor\frac{i}{2}\rfloor}} \frac{D(y-h) - D(y)}{h^2}|y-x_i|^{1-\alpha} + D(y)\frac{|y-x_{i-1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2}dy \\
&\quad + \int_{x_{N+\lfloor\frac{i}{2}\rfloor}}^{x_{2N-1}} + \int_{x_{2N-1}}^1 D(y)\frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2}dy \\
&= I_1 + I_2 + I_3 + I_4 + \cdots
\end{aligned}
$$

$$(20)$$

1.

$$I_1 = \int_0^{x_1} (u(y) - u_h(y)) \frac{|y - x_{i-1}|^{1-\alpha} - 2|y - x_i|^{1-\alpha} + |y - x_{i+1}|^{1-\alpha}}{h^2} dy$$
$$\leq Ch^{\alpha/2} \int_0^h |y - x_i|^{-1-\alpha} dy$$
$$\leq Ch^{\alpha/2+1} x_i^{-1-\alpha}$$

$$(21)$$

2.

$$I_2 = \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} D(y) \frac{|y - x_{i-1}|^{1-\alpha} - 2|y - x_i|^{1-\alpha} + |y - x_{i+1}|^{1-\alpha}}{h^2} dy$$
$$\leq C \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2} h^2 |x_i - y|^{-1-\alpha} dy \qquad (22)$$
$$\leq Ch^{\alpha/2-1} h^2 x_i^{-1-\alpha} \leq Ch^{\alpha/2+1} x_i^{-1-\alpha}$$

3.

$$I_3 = \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} \frac{D(y+h) - D(y)}{h^2} |y - x_i|^{1-\alpha} + D(y) \frac{|y - x_{i+1}|^{1-\alpha} - |y - x_i|^{1-\alpha}}{h^2} dy$$
$$\leq \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} u'''(\eta_1) h |x_i - y|^{1-\alpha} + u''(\eta_2) h |x_i - y|^{-\alpha} dy$$
$$\leq Ch^2 x_i^{-2-\alpha/2} \leq Ch^{1+\alpha/2} x_i^{-1-\alpha}$$

$$(23)$$

4.

$$I_4 = \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y - x_i|^{1-\alpha} dy$$
$$\leq \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} u''''(\eta) h^2 |x_i - y|^{1-\alpha} dy \qquad (24)$$
$$\leq Cx_i^{\alpha/2-4} h^2 x_i^{2-\alpha}$$
$$\leq Ch^2 x_i^{-2-\alpha/2} \leq Ch^{1+\alpha/2} x_i^{-1-\alpha}$$

猜想证毕，一致网格证完。

4 非一致

$r > 1,$

$$\begin{cases} x_i = \frac{1}{2} \left(\frac{i}{N} \right)^r, & 0 \le i \le N \\ x_i = 1 - \frac{1}{2} \left(\frac{2N-i}{N} \right)^r, & N \le i \le 2N \end{cases} \tag{25}$$

令 $h = \frac{1}{2N}$ ，那么

当 $i < N, x_i < \frac{1}{2}$ 时

$$h_i = \frac{1}{2} \left(\left(\frac{i}{N} \right)^r - \left(\frac{i-1}{N} \right)^r \right) \le C(r) \left(\frac{i}{N} \right)^{r-1} \frac{1}{N} = Chx_i^{(r-1)/r} \tag{26}$$

当 $i \ge N, x_i \ge \frac{1}{2}$ 时

$$h_i = \frac{1}{2} \left(\left(\frac{2N-i+1}{N} \right)^r - \left(\frac{2N-i}{N} \right)^r \right) \le C(r) \left(\frac{2N-i+1}{N} \right)^{r-1} \frac{1}{N} = Ch(1-x_{i-1})^{(r-1)/r} \tag{27}$$

我们声明

$$\begin{aligned} S_i = \sum_{j=1}^{2N-1} a_{ij} =& \frac{C_R}{(2-\alpha)(3-\alpha)} \frac{2}{h_i + h_{i+1}} \\ & (\frac{1}{h_{i+1}} \frac{|x_{i+1}-x_0|^{3-\alpha} - |x_{i+1}-x_1|^{3-\alpha}}{x_1 - x_0} \\ & - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) \frac{|x_i-x_0|^{3-\alpha} - |x_i-x_1|^{3-\alpha}}{x_1 - x_0} \\ & + \frac{1}{h_i} \frac{|x_{i-1}-x_0|^{3-\alpha} - |x_{i-1}-x_1|^{3-\alpha}}{x_1 - x_0}) \\ & + ... \\ \ge & C(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \end{aligned} \tag{28}$$

$$R_i = \int_0^1 D(y) \frac{2}{h_i + h_{i+1}} (\frac{1}{h_{i+1}} |x_{i+1}-y|^{1-\alpha} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) |x_i-y|^{1-\alpha} + \frac{1}{h_i} |x_{i-1}-y|^{1-\alpha}) dy \tag{29}$$

我们声明下面的命题并在这一节中证明

$$R_i \leq C(h^{r\alpha/2+r}x_i^{-1-\alpha} + h^2 x_i^{-\alpha/2-2/r} + h^2(|\frac{1}{2} - x_i| + h)^{1-\alpha}), \quad i \leq N \quad (30)$$

4.1 i=1

下面讨论 R_1

$$R_1 = \int_0^{x_1} + \int_{x_1}^{x_3} + \int_{x_3}^{1/2} + \int_{1/2}^{x_{2N-1}} + \int_{x_{2N-1}}^1$$
$$D(y)\frac{2}{h_1 + h_2}(\frac{1}{h_2}|x_2 - y|^{1-\alpha} - (\frac{1}{h_1} + \frac{1}{h_2})|x_1 - y|^{1-\alpha} + \frac{1}{h_1}|y|^{1-\alpha})dy$$
$$:= I_1 + I_2 + I_3 + I_4 + I_5$$
$$(31)$$

与一致网格时相似，

1.

$$|u(x) - u_h(x)| \leq \int_0^{x_1} |u'(y)|dy$$
$$\leq C \int_0^{x_1} y^{\alpha/2-1}dy \qquad (32)$$
$$\leq Cx_1^{\alpha/2} \quad , x \in (0, x_1)$$

因为 $1 - \alpha > -1$

$$I_1 \leq C \int_0^{x_1} \frac{D(y)}{h_1^2}(|x_2 - y|^{1-\alpha} + 2|x_1 - y|^{1-\alpha} + |y|^{1-\alpha})dy$$
$$\leq Cx_1^{\alpha/2-2}x_1^{2-\alpha} = Cx_1^{-\alpha/2} = Ch^{-r\alpha/2} \qquad (33)$$

2.

$$I_2 \leq Cu''(\eta)x_3^{2-\alpha} \leq Cx_1^{\alpha/2-2}x_3^{2-\alpha} \leq Ch^{-r\alpha/2} \qquad (34)$$

3.

$$I_3 = \int_{x_3}^{1/2} D(y) \frac{2}{h_1 + h_2} (\frac{1}{h_2}|x_2 - y|^{1-\alpha} - (\frac{1}{h_1} + \frac{1}{h_2})|x_1 - y|^{1-\alpha} + \frac{1}{h_1}|y|^{1-\alpha}) dy$$

$$\leq C \int_{x_3}^{1/2} y^{\alpha/2-2} (hy^{(r-1)/r})^2 y^{-1-\alpha} dy$$

$$\leq Ch^2 \int_{x_3}^{1/2} y^{\alpha/2-2/r-1-\alpha} dy$$

$$\leq Ch^2 (h^r)^{-2/r-\alpha/2} = Ch^{-r\alpha/2}$$

$$(35)$$

4.

$$I_4 = \int_{1/2}^{x_{2N-1}} D(y) \frac{2}{h_1 + h_2} (\frac{1}{h_2}|x_2 - y|^{1-\alpha} - (\frac{1}{h_1} + \frac{1}{h_2})|x_1 - y|^{1-\alpha} + \frac{1}{h_1}|y|^{1-\alpha}) dy$$

$$\leq C \int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2} (h(1-y)^{(r-1)/r})^2 y^{-1-\alpha} dy$$

$$\leq Ch^2 \int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2+2-2/r}$$

$$\leq Ch^2 (C + h_{2N}^{\alpha/2-2/r+1})$$

$$= Ch^2 (C + h^{r\alpha/2-2+r}) \leq Ch^{\min\{2, r\alpha/2+r\}}$$

$$(36)$$

5.

$$I_5 \leq Ch_{2N}^{\alpha/2+1} \leq Ch^{r\alpha/2+r} \tag{37}$$

综合有

$$R_1 \leq Ch^{-r\alpha/2} \tag{38}$$

R_1, R_2, R_3 一样。

$R_i, 3 < i < N$ 比较困难。

我们记 $D(y) = u(y) - u_h(y)$

$$T_{ij} = \int_{x_{j-1}}^{x_j} D(y)|x_i - y|^{1-\alpha} dy \tag{39}$$

4.2 i<N/2

当 $3 < i < N/2$，即 $x_i < (\frac{1}{4})^r$ 时。

$$
\begin{aligned}
R_i =& \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& \sum_{j=1}^{i/2} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}}(T_{i+1,i/2+1} + T_{i+1,i/2+2}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,i/2+1} \right) \\
&+ \sum_{j=i/2+2}^{2i-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i-1}}(T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,2i} \right) \\
&+ \sum_{j=2i+1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
I_1 =& \int_0^{x_1} + \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} \\
& D(y)\frac{2}{h_i + h_{i+1}}(\frac{1}{h_{i+1}}|x_{i+1} - y|^{1-\alpha} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})|x_i - y|^{1-\alpha} + \frac{1}{h_i}|x_{i-1} - y|^{1-\alpha})dy
\end{aligned}
\tag{41}
$$

1.

$$J_1 \le C x_1^{\alpha/2+1} x_i^{-1-\alpha} \le C h^{r\alpha/2+r} x_i^{-1-\alpha} \tag{42}$$

2.

$$J_2 \le C \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2}(hy^{(r-1)/r})^2 |x_i - y|^{-1-\alpha} dy$$

$$\le Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2/r} dy \tag{43}$$

$$\le Ch^2 x_i^{-1-\alpha}(h^{r\alpha/2-2+r} + x_i^{\alpha/2-2/r+1})$$

$$= C(h^{r\alpha/2+r} x_i^{-1-\alpha} + h^2 x_i^{-\alpha/2-2/r})$$

我们先研究 I_3，考虑

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \tag{44}$$

在此之前我们做一些准备工作。

对于 $y \in [x_{j-1}, x_j]$，我们记 $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

$$D(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \tag{45}$$

那么

$$T_{ij} = \int_{x_{j-1}}^{x_j} D(y)|x_i - y|^{1-\alpha} dy$$

$$= \int_0^1 \frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) |x_i - y_j^\theta|^{1-\alpha} d\theta \tag{46}$$

$$+ \int_0^1 \frac{\theta(1-\theta)}{3!} h_j^4 |x_i - y_j^\theta|^{1-\alpha} (\theta^2 u'''(\eta_{1,j}^\theta) - (1-\theta)^2 u'''(\eta_{2,j}^\theta)) d\theta$$

现在回到原来的问题，我们要研究

$$\frac{2}{h_i + h_{i+1}}(\frac{1}{h_{i+1}}h_{j+1}^3 u''(y_{j+1}^\theta)|x_{i+1} - y_{j+1}^\theta|^{1-\alpha}$$

$$-(\frac{1}{h_i}+\frac{1}{h_{i+1}})h_j^3\, u''(y_j^\theta)|x_i - y_j^\theta|^{1-\alpha} \tag{47}$$

$$+ \frac{1}{h_i}h_{j-1}^3 u''(y_{j-1}^\theta)|x_{i-1} - y_{j-1}^\theta|^{1-\alpha})$$

我们希望把他看成一个函数的二阶导，注意到当 $i/2 \le j \le 2i$ 时

$$x_i^{1/r} - x_j^{1/r} = x_{i+1}^{1/r} - x_{j+1}^{1/r} = 2^{-1/r}\frac{i-j}{N} \tag{48}$$

那么我们将其他的相都表示成 x_i 的函数。

$$y_L(x) = (x^{1/r} + z_L)^r, \quad y_R(x) = (x^{1/r} + z_R)^r \tag{49}$$

其中 $z_L = 2^{-1/r}\frac{j-i-1}{N}, z_R = 2^{-1/r}\frac{j-i}{N}$.

$$y_R(x_i) = x_j, \quad y_R(x_{i+1}) = x_{j+1}, \quad y_R(x_{i-1}) = x_{j-1} \tag{50}$$

$$y_L(x_i) = x_{j-1}, \quad y_L(x_{i+1}) = x_j, \quad y_L(x_{i-1}) = x_{j-2} \tag{51}$$

$$y_\theta(x) = \theta y_L(x) + (1-\theta)y_R(x) \tag{52}$$

$$h_J(x) = y_R(x) - y_L(x) \tag{53}$$

那么我么要研究的就是函数

$$K_1(x) = h_J^3(x)|x - y_\theta(x)|^{1-\alpha}u''(y_\theta(x)) \tag{54}$$

在网格 x_{i-1}, x_i, x_{i+1} 的数值二阶差商。

$$\frac{2}{h_i + h_{i+1}}(\frac{1}{h_{i+1}}K_1(x_{i+1}) - (\frac{1}{h_i}+\frac{1}{h_{i+1}})K_1(x_i) + \frac{1}{h_i}K_1(x_{i-1})) = K_1''(\xi), \; \xi \in [x_{i-1}, x_{i+1}] \tag{55}$$

由 Leibniz 公式

$$(uvw)'' = u''vw + uv''w + uvw'' + 2u'v'w + 2uv'w' + 2u'vw' \tag{56}$$

由 $y_R^{1/r} = x^{1/r} + z_R$, 我们得到

$$\frac{dy_R}{dx} = x^{1/r-1}y_R^{1-1/r} \tag{57}$$

$$\frac{d^2y_R}{dx^2} = \frac{1-r}{r}x^{1/r-2}y_R^{1-2/r}z_R \tag{58}$$

因此

1.

$$h_J^3 \sim h^3 y_R^{3-3/r} \sim h^3 x^{3-3/r} \tag{59}$$

$$\begin{aligned}(h_J^3)' &= 3h_J^2(y_R' - y_L') \\ &= 3h_J^2 x^{1/r-1}(y_R^{1-1/r} - y_L^{1-1/r}) \\ &\sim h^3 y_R^{2-2/r} x^{1/r-1} y_R^{1-2/r} \\ &\sim h^3 x^{2-3/r}\end{aligned} \tag{60}$$

$$\begin{aligned}(h_J^3)'' &= 6h_J x^{2/r-2}(y_R^{1-1/r} - y_L^{1-1/r})^2 + 3h_J^2\frac{1-r}{r}x^{1/r-2}(y_R^{1-2/r}z_R - y_L^{1-2/r}z_L) \\ &\sim hy_R^{1-1/r}x^{2/r-2}(hy_R^{1-2/r})^2 + \frac{1-r}{r}h^2 y_R^{2-2/r}x^{1/r-2}(z_R h y_R^{1-3/r} + h y_L^{1-2/r}) \\ &\sim h^3 y_R^{3-5/r}x^{2/r-2} + h^3(y_R^{3-5/r}x^{1/r-2}z_R + y_R^{3-4/r}x^{1/r-2}) \\ &\sim h^3(y_R^{3-5/r}x^{2/r-2} + y_R^{3-4/r}x^{1/r-2} + y_R^{3-5/r}x^{1/r-2}z_R) \\ &\sim h^3 x^{1-3/r}\end{aligned} \tag{61}$$

2.

由于

$$x - y_L = (x^{1/r})^r - (x^{1/r} + z_L)^r = -z_L\xi^{r-1} \sim z_L x^{(r-1)/r} \tag{62}$$

$$\begin{aligned}|x - y_\theta|^{1-\alpha} &= |x - \theta y_L - (1-\theta)y_R|^{1-\alpha} \sim |(\theta z_L + (1-\theta)z_R)\xi^{1-1/r}|^{1-\alpha} \\ &\sim z_\theta^{1-\alpha}\xi^{1-\alpha+(\alpha-1)/r}, \quad \xi \in [y_L, x]\end{aligned} \tag{63}$$

$$
\begin{aligned}
(|x - y_\theta|^{1-\alpha})' &= \mathrm{sign}(x - y_\theta)(1 - \alpha)|x - y_\theta|^{-\alpha}(1 - x^{1/r-1}(\theta y_L^{1-1/r} + (1 - \theta)y_R^{1-1/r})) \\
&= (1 - \alpha)|x - y_\theta|^{-\alpha}x^{1/r-1}(x^{1-1/r} - (\theta y_L^{1-1/r} + (1 - \theta)y_R^{1-1/r})) \\
&\sim |x - y_\theta|^{-\alpha}x^{1/r-1}(\theta z_L + (1 - \theta)z_R)\xi_2^{1-2/r} \\
&\sim |(\theta z_L + (1 - \theta)z_R)\xi_1^{1-1/r}|^{-\alpha}x^{1/r-1}(\theta z_L + (1 - \theta)z_R)\xi_2^{1-2/r} \\
&\sim z_\theta^{1-\alpha}x^{-\alpha+(\alpha-1)/r}
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
(|x - y_\theta|^{1-\alpha})'' =& \alpha(\alpha - 1)|x - y_\theta|^{-1-\alpha}(1 - x^{1/r-1}(\theta y_L^{1-1/r} + (1 - \theta)y_R^{1-1/r}))^2 \\
&- \mathrm{sign}(x - y_\theta)(1 - \alpha)|x - y_\theta|^{-\alpha}(\frac{1-r}{r}x^{1/r-2}(\theta y_L^{1-2/r}z_L + (1 - \theta)y_R^{1-2/r}z_R)) \\
\sim& |(\theta z_L + (1 - \theta)z_R)\xi_1^{1-1/r}|^{-1-\alpha}x^{2/r-2}\xi_2^{2-4/r}(\theta z_L + (1 - \theta)z_R)^2 \\
&+ |(\theta z_L + (1 - \theta)z_R)\xi_1^{1-1/r}|^{-\alpha}x^{1/r-2}(\theta z_L + (1 - \theta)z_R)y_R^{1-2/r} \\
\sim& z_\theta^{1-\alpha}x^{-1-\alpha+(\alpha-1)/r}
\end{aligned}
\tag{65}
$$

3.

$$
u''(y_\theta) \le Cy_\theta^{\alpha/2-2} \sim x^{\alpha/2-2}
\tag{66}
$$

$$
\begin{aligned}
(u''(y_\theta))' &= u'''(y_\theta)x^{1/r-1}(\theta y_L^{1-1/r} + (1 - \theta)y_R^{1-1/r}) \\
&\le Cy_\theta^{\alpha/2-3}x^{1/r-1}y_R^{1-1/r} \sim x^{\alpha/2-3}
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
(u''(y_\theta))'' =& u''''(y_\theta)(x^{1/r-1}(\theta y_L^{1-1/r} + (1 - \theta)y_R^{1-1/r}))^2 \\
&+ u'''(y_\theta)\frac{1-r}{r}x^{1/r-2}(\theta y_L^{1-2/r}z_L + (1 - \theta)y_R^{1-2/r}z_R) \\
\sim& y_\theta^{\alpha/2-4}(x^{1/r-1}y_R^{1-1/r})^2 + z_\theta y_R^{\alpha/2-3+1-2/r}x^{1/r-2} \\
<& x^{\alpha/2-4}
\end{aligned}
\tag{68}
$$

$$u''vw \sim h^3 x^{1-3/r} z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} x^{\alpha/2-2} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r}$$
$$(69)$$

$$uv''w \sim h^3 x^{3-3/r} z_\theta^{1-\alpha} x^{-1-\alpha+(\alpha-1)/r} x^{\alpha/2-2} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r}$$
$$(70)$$

$$uvw'' \sim h^3 x^{3-3/r} z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} x^{\alpha/2-4} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r}$$
$$(71)$$

$$u'v'w \sim h^3 x^{2-3/r} z_\theta^{1-\alpha} x^{-\alpha+(\alpha-1)/r} x^{\alpha/2-2} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \quad (72)$$

$$uv'w' \sim h^3 x^{3-3/r} z_\theta^{1-\alpha} x^{-\alpha+(\alpha-1)/r} x^{\alpha/2-3} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \quad (73)$$

$$u'vw' \sim h^3 x^{2-3/r} z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} x^{\alpha/2-3} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r}$$
$$(74)$$

因此

$$K_1''(\xi) \sim h^3 z_\theta^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r}, \xi \in [x_{i-1}, x_{i+1}] \tag{75}$$

现在我们处理第二部分

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} h_{j+1}^4 u'''(\eta_{1,j+1}^\theta) |x_{i+1} - y_{j+1}^\theta|^{1-\alpha} \right.$$
$$- \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) h_j^4 \, u'''(\eta_{1,j}^\theta) |x_i - y_j^\theta|^{1-\alpha} \tag{76}$$
$$\left. + \frac{1}{h_i} h_{j-1}^4 u'''(\eta_{1,j-1}^\theta) |x_{i-1} - y_{j-1}^\theta|^{1-\alpha} \right)$$

这次我们只用一阶差分

$$\frac{1}{h_i} \left(h_j^4 u'''(\eta_{1,j}^\theta) |x_i - y_j^\theta|^{1-\alpha} - h_{j-1}^4 \, u'''(\eta_{1,j-1}^\theta) |x_{i-1} - y_{j-1}^\theta|^{1-\alpha} \right) \tag{77}$$

为了方便计算，我们还是用辅助函数来对上面一项进行估计。

$$K_2(x) = h_J^4(x) |x - y_\theta(x)|^{1-\alpha} \tag{78}$$

$$
\begin{aligned}
K_2'(x) &= (h_J^4)'|x - y_\theta(x)|^{1-\alpha} + h_J^4(|x - y_\theta(x)|^{1-\alpha})' \\
&\sim h^3 x^{3-3/r} x^{1/r-1} h y_R^{1-2/r} z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} \\
&\quad + h^4 y_R^{4-4/r} z_\theta^{1-\alpha} x^{-\alpha+(\alpha-1)/r} \\
&\sim h^4 z_\theta^{1-\alpha} x^{4-5/r-\alpha+\alpha/r}
\end{aligned}
\tag{79}
$$

那么，上面就等于

$$
\begin{aligned}
&\frac{1}{h_i}(K_2(x_i)u'''(\eta_{1,j}^\theta) - K_2(x_{i-1})u'''(\eta_{1,j-1}^\theta)) \\
&= \frac{1}{h_i}K_2(x_i)(u'''(\eta_{1,j}^\theta) - u'''(\eta_{1,j-1}^\theta)) + \frac{1}{h_i}(K_2(x_i) - K_2(x_{i-1})u'''(\eta_{1,j-1}^\theta)) \\
&\leq h_i^{-1}K_2(x_i)u''''(\eta_j^\theta)(x_j - x_{j-2}) + K_2'(\xi)u'''(\eta_{1,j-1}^\theta) \quad (\eta_j^\theta \in [x_{j-2}, x_j], \xi \in [x_{j-1}, x_j]) \\
&\sim h_i^{-1}h_j^4|x_i - y_j^\theta|^{1-\alpha} C(\eta_j^\theta)^{\alpha/2-4} 2h_j \\
&\quad + h^4 z_\theta^{1-\alpha} \xi^{4-5/r-\alpha+\alpha/r}(\eta_j^\theta)^{\alpha/2-3} \\
&\sim h^4 x_i^{4-4/r} z_\theta^{1-\alpha} x_i^{1-\alpha+(\alpha-1)/r} x_i^{\alpha/2-4} + h^4 z_\theta^{1-\alpha} x_i^{4-5/r-\alpha+\alpha/r} x_i^{\alpha/2-3} \\
&\sim h x_i^{1-1/r} h^3 z_\theta^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r}
\end{aligned}
\tag{80}
$$

因此，

$$
\begin{aligned}
&\frac{2}{h_i + h_{i+1}} \frac{1}{h_i}(K_2(x_i)u'''(\eta_{1,j}^\theta) - K_2(x_{i-1})u'''(\eta_{1,j-1}^\theta)) \\
&\sim h^3 z_\theta^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r}
\end{aligned}
\tag{81}
$$

最终我们得到，当 $i/2 + 2 \leq j \leq 2i - 1 < N$ 时，有

$$
\begin{aligned}
&\frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,j} + \frac{1}{h_i}T_{i-1,j-1}\right) \\
&\leq Ch^3\left(\frac{|i-j|+1}{N}\right)^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r}
\end{aligned}
\tag{82}
$$

那么我们得到

$$I_3 \leq C \sum_{j=i/2+2}^{2i-1} \left(\frac{1}{N}\right)^3 \left(\frac{|i-j|+1}{N}\right)^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r}$$

$$\leq C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r+(\alpha-2)/r} \left(\frac{2i}{N}\right)^{2-\alpha}$$

$$\leq C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r+(\alpha-2)/r} x_i^{(2-\alpha)/r} \tag{83}$$

$$= C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r}$$

现在我们处理 I_2, 记 $k = i/2+1$

$$I_2 = \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}}(T_{i+1,i/2+1} + T_{i+1,i/2+2}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})(T_{i,i/2+1})\right)$$

$$= \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}}(T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i})T_{i,k}\right)$$

$$= J_1 + J_2 + J_3 \tag{84}$$

$$J_1 = \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k})\right)$$

$$= \frac{2}{h_i+h_{i+1}} \int_{x_{k-1}}^{x_k} D(y) \frac{|x_{i+1}-y|^{1-\alpha} - |x_i-y|^{1-\alpha}}{h_{i+1}} dy \tag{85}$$

$$\leq C x_i^{\alpha/2-2} h_k^2 x_i^{-\alpha}$$

$$\leq C h^2 x_i^{-\alpha/2-2/r}$$

$$J_2 = \frac{2}{h_i+h_{i+1}} \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k})$$

$$= \frac{2}{h_i+h_{i+1}} \int_0^1 \frac{h_{k+1}D(y_{k+1}^\theta)|x_{i+1}-y_{k+1}^\theta|^{1-\alpha} - h_k D(y_k^\theta)|x_i-y_k^\theta|^{1-\alpha}}{h_{i+1}} d\theta \tag{86}$$

我们看他的两个积分项

$$
\begin{aligned}
\frac{K_1(x_{i+1}) - K_1(x_i)}{h_{i+1}} &= K_1'(\xi) \\
&\sim h^3 x^{2-3/r} \, z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} \, x^{\alpha/2-2} \\
&\quad + h^3 x^{3-3/r} \, z_\theta^{1-\alpha} x^{-\alpha+(\alpha-1)/r} \, x^{\alpha/2-2} \\
&\quad + h^3 x^{3-3/r} \, z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} \, x^{\alpha/2-3} \\
&\sim hx^{1-1/r} h^2 z_\theta^{1-\alpha} x^{-\alpha/2+\alpha/r-3/r} \\
&\sim hx^{1-1/r} h^2 x^{(1-\alpha)/r} x^{-\alpha/2+\alpha/r-3/r} \\
&\sim hx^{1-1/r} h^2 x^{-\alpha/2-2/r}
\end{aligned}
\tag{87}
$$

第二部分研究过了

$$
\begin{aligned}
\frac{1}{h_i}(K_2(x_{i+1})u'''(\eta_{1,k+1}^\theta) &- K_2(x_i)u'''(\eta_{1,k}^\theta)) \\
&\sim hx_i^{1-1/r} \, h^3 z_\theta^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r} \\
&\sim hx_i^{1-1/r} \, h^3 x_i^{(1-\alpha)/r} x_i^{-\alpha/2-2/r+(\alpha-2)/r} \\
&\sim hx_i^{1-1/r} \, h^3 x_i^{-\alpha/2-2/r-1/r}
\end{aligned}
\tag{88}
$$

因此

$$
J_2 \le Ch^2 x^{-\alpha/2-2/r}
\tag{89}
$$

现在考虑 J_3

$$
\begin{aligned}
J_3 &= \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}} - \frac{1}{h_i}\right)T_{i,k} \\
&= -\frac{2}{h_i + h_{i+1}}\frac{h_{i+1} - h_i}{h_i h_{i+1}}\int_{x_{k-1}}^{x_k} D(y_k^\theta)|x_i - y_k^\theta|^{1-\alpha}dy \\
&\sim h_i^{-1} x_i^{-1} h_k^3 x_i^{\alpha/2-2} x_i^{1-\alpha} \\
&\sim h^2 x_i^{-\alpha/2-2/r}
\end{aligned}
\tag{90}
$$

因此我们有

$$
I_2 \le Ch^2 x_i^{-\alpha/2-2/r}
\tag{91}
$$

I_4 类似。

现在考虑 I_5

$$
\begin{aligned}
I_5 &= \sum_{j=2i+1}^{N} + \sum_{j=N+1}^{2N-1} + \sum_{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&= \int_{x_{2i}}^{x_N} + \int_{x_N}^{x_{2N-1}} + \int_{x_{2N-1}}^{2N} \\
&\quad D(y) \frac{2}{h_i + h_{i+1}} (\frac{1}{h_{i+1}} |x_{i+1} - y|^{1-\alpha} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) |x_i - y|^{1-\alpha} + \frac{1}{h_i} |x_{i-1} - y|^{1-\alpha}) dy \\
&= J_1 + J_2 + J_3
\end{aligned}
\tag{92}
$$

$$
\begin{aligned}
J_1 &= \int_{x_{2i}}^{1/2} D(y) \frac{2}{h_i + h_{i+1}} (\frac{1}{h_{i+1}} |x_{i+1} - y|^{1-\alpha} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) |x_i - y|^{1-\alpha} + \frac{1}{h_i} |x_{i-1} - y|^{1-\alpha}) dy \\
&\leq C \int_{x_{2i}}^{1/2} y^{\alpha/2-2} (h y^{1-1/r})^2 |y - x_i|^{-1-\alpha} dy \\
&\leq C \int_{x_{2i}}^{1/2} h^2 y^{\alpha/2-2+2-2/r-1-\alpha} dy \\
&\leq C h^2 x_i^{-\alpha/2-2/r}
\end{aligned}
\tag{93}
$$

$$
\begin{aligned}
J_2 &= \int_{1/2}^{x_{2N-1}} D(y) \frac{2}{h_i + h_{i+1}} (\frac{1}{h_{i+1}} |x_{i+1} - y|^{1-\alpha} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) |x_i - y|^{1-\alpha} + \frac{1}{h_i} |x_{i-1} - y|^{1-\alpha}) dy \\
&\leq C \int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2} (h(1-y)^{1-1/r})^2 |y - x_i|^{-1-\alpha} dy \\
&\leq C \int_{1/2}^{x_{2N-1}} h^2 (1-y)^{\alpha/2-2+2-2/r} dy \\
&\leq C h^2 (C + h^{r(\alpha/2-2/r+1)}) \\
&\leq C h^2 + C h^{r\alpha/2+r}
\end{aligned}
\tag{94}
$$

$$
J_3 \leq C h_{2N}^{\alpha/2+1} \leq C h^{r\alpha/2+r}
\tag{95}
$$

全部加起来，我们得到

$$
R_i \leq C h^{r\alpha/2+r} x_i^{-1-\alpha} + C h^2 x_i^{-\alpha/2-2/r}
\tag{96}
$$

4.3 N/2<i<N

当 $N/2 < i < N$ ，即 $(\frac{1}{4})^r < x_i < \frac{1}{2}$ 时。

$$
\begin{aligned}
R_i &= \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&= \sum_{j=1}^{i/2} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,i/2+1} + T_{i+1,i/2+2}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,i/2+1} \right) \\
&\quad + \sum_{j=i/2+2}^{3N/2-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i-1}} (T_{i-1,3N/2} + T_{i-1,3N/2-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,3N/2} \right) \\
&\quad + \sum_{j=3N/2+1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&= I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}
\tag{97}
$$

I_1, I_2, I_4, I_5 略。

$$
\begin{aligned}
I_3 &= \sum_{j=i/2+2}^{N-1} + \sum_{j=N}^{N+1} + \sum_{N+2}^{2N-i} + \sum_{2N-i+1}^{3N/2-1} \\
&\quad \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&= J_1 + J_2 + J_3 + J_4
\end{aligned}
\tag{98}
$$

$$
\begin{aligned}
J_1 &\leq C \sum_{j=i/2+2}^{N-1} \left(\frac{1}{N}\right)^3 \left(\frac{|i-j|+1}{N}\right)^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r} \\
&\leq C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r+(\alpha-2)/r} \left(\left(\frac{i}{2N}\right)^{2-\alpha} + \left(\frac{N-i}{N}\right)^{2-\alpha}\right) \\
&\leq C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r+(\alpha-2)/r} \left(x_i^{(2-\alpha)/r} + |\tfrac{1}{2}-x_i|^{2-\alpha} x_i^{(2-\alpha)(1/r-1)}\right) \\
&= Ch^2 |\tfrac{1}{2}-x_i|^{2-\alpha} + Ch^2 \leq Ch^2
\end{aligned}
\tag{99}
$$

对于 J_3, J_4，由于 $i < N < j-1$

$$
x_i^{1/r}+(1-x_j)^{1/r} = 2^{-1/r}\left(\frac{i}{N}+\frac{2N-j}{N}\right) = 2^{-1/r}\left(2-\frac{j-i}{N}\right) = x_{i-1}^{1/r}+(1-x_{j-1})^{1/r}
\tag{100}
$$

那么令

$$
y_L = 1 - (z_L - x^{1/r})^r, \quad y_R = 1 - (z_R - x^{1/r})^r
\tag{101}
$$

其中

$$
z_L = 2^{-1/r}\left(2 - \frac{j-i-1}{N}\right), \quad z_R = 2^{-1/r}\left(2 - \frac{j-i}{N}\right)
\tag{102}
$$

那么类似的

$$
\frac{dy_R}{dx} = x^{1/r-1}(1-y_R)^{1-1/r}
\tag{103}
$$

$$
\frac{d^2y_R}{dx^2} = \frac{1-r}{r} x^{1/r-2}(1-y_R)^{1-2/r} z_R
\tag{104}
$$

那么我们考虑

$$
K_1(x) = h_J^3 |y_\theta - x|^{1-\alpha} u''(y_\theta)
\tag{105}
$$

的二阶导数

其中 $\frac{1}{4} < x < \frac{1}{2}, \frac{1}{2} < y_\theta < \frac{3}{4}$.

1.

$$h_J^3 \sim h^3(1 - y_R)^{3-3/r} \sim h^3 \tag{106}$$

$$
\begin{aligned}
(h_J^3)' &= 3h_J^2(y_R' - y_L') \\
&= 3h_J^2 x^{1/r-1}((1 - y_R)^{1-1/r} - (1 - y_L)^{1-1/r}) \\
&\sim h^3(1 - y_R)^{2-2/r} x^{1/r-1}(1 - y_R)^{1-2/r} \\
&\sim h^3
\end{aligned}
\tag{107}
$$

$$
\begin{aligned}
(h_J^3)'' &= 6h_J x^{2/r-2}((1 - y_R)^{1-1/r} - (1 - y_L)^{1-1/r})^2 \\
&\quad + 3h_J^2 \frac{1-r}{r} x^{1/r-2}((1 - y_R)^{1-2/r} z_R - (1 - y_L)^{1-2/r} z_L) \\
&\sim h(1 - y_R)^{1-1/r} x^{2/r-2}(h(1 - y_R)^{1-2/r})^2 \\
&\quad - h^2(1 - y_R)^{2-2/r} x^{1/r-2}(z_R h(1 - y_R)^{1-3/r} + h(1 - y_L)^{1-2/r}) \\
&\sim h^3
\end{aligned}
\tag{108}
$$

2.
由于

$$
y - x \sim \begin{cases} \frac{1}{2} - x, & 1 < y < 1 - x \\ \frac{1}{2} - y, & y > 1 - x \end{cases}
\tag{109}
$$

$$
\begin{aligned}
(|y_\theta - x|^{1-\alpha})' &= (1 - \alpha)|y_\theta - x|^{-\alpha}(x^{1/r-1}(\theta(1 - y_L)^{1-1/r} + (1 - \theta)(1 - y_R)^{1-1/r}) - 1) \\
&= (1 - \alpha)|y_\theta - x|^{-\alpha} x^{1/r-1}(\theta(1 - y_L)^{1-1/r} + (1 - \theta)(1 - y_R)^{1-1/r} - (1 - (1 - x))^{1-1/r}) \\
&\sim |y_\theta - x|^{-\alpha}(y_\theta - (1 - x))
\end{aligned}
\tag{110}
$$

$$
\begin{aligned}
(|y_\theta - x|^{1-\alpha})'' &= \alpha(\alpha - 1)|y_\theta - x|^{-1-\alpha}(x^{1/r-1}(\theta(1 - y_L)^{1-1/r} + (1 - \theta)(1 - y_R)^{1-1/r}) - 1)^2 \\
&\quad - (1 - \alpha)|y_\theta - x|^{-\alpha}(\frac{1-r}{r} x^{1/r-2}(\theta(1 - y_L)^{1-2/r} z_L + (1 - \theta)(1 - y_R)^{1-2/r} z_R)) \\
&\sim |y_\theta - x|^{-1-\alpha}(y_\theta - (1 - x))^2 + |y_\theta - x|^{-\alpha}
\end{aligned}
\tag{111}
$$

3.

$$u''(y_\theta) \le C(1 - y_\theta)^{\alpha/2-2} \sim 1 \tag{112}$$

$$
\begin{aligned}
(u''(y_\theta))' &= u'''(y_\theta)x^{1/r-1}(\theta(1-y_L)^{1-1/r} + (1-\theta)(1-y_R)^{1-1/r}) \\
&\sim 1
\end{aligned}
\tag{113}
$$

$$
\begin{aligned}
(u''(y_\theta))'' =& u''''(y_\theta)(x^{1/r-1}(\theta(1-y_L)^{1-1/r} + (1-\theta)(1-y_R)^{1-1/r}))^2 \\
&+ u'''(y_\theta)\frac{1-r}{r}x^{1/r-2}(\theta(1-y_L)^{1-2/r}z_L + (1-\theta)(1-y_R)^{1-2/r}z_R) \\
\sim& 1
\end{aligned}
\tag{114}
$$

那么 大概齐，要补上细节

$$
\begin{aligned}
J_3 &\le C\sum_{j=N+2}^{2N-i} h^3(|\frac{1}{2} - x_i|^{1-\alpha} + |\frac{1}{2} - x_i|^{-\alpha}) \\
&\le Ch^2|\frac{1}{2} - x_i|^{-\alpha}\int_{1/2}^{1-x_i} 1\ dy \\
&\le Ch^2|\frac{1}{2} - x_i|^{1-\alpha}
\end{aligned}
\tag{115}
$$

$$
\begin{aligned}
J_4 &\le C\int_{1-x_i}^{3/4} h^2|y - x_i|^{-\alpha} \\
&\le Ch^2 + Ch^2|\frac{1}{2} - x_i|^{1-\alpha}
\end{aligned}
\tag{116}
$$

综上，

$$I_3 \le Ch^2|\frac{1}{2} - x_i|^{1-\alpha} \tag{117}$$

4.4　i=N

$$h_N = h_{N+1}, \quad x_N = \tfrac{1}{2}.$$

$$R_N = \frac{1}{h_N^2} \int_0^1 D(y)(|x_{N+1}-y|^{1-\alpha} - 2|x_N-y|^{1-\alpha} + |x_{N-1}-y|^{1-\alpha})dy \quad (118)$$

5　收敛性分析

我们声明一个命题：

1. A 是 M 矩阵。

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{(2-\alpha)(3-\alpha)} \left(\frac{|x_i-x_0|^{3-\alpha} - |x_i-x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N}-x_i|^{3-\alpha} - |x_{2N-1}-x_i|^{3-\alpha}}{h_{2N}} \right)$$
$$(119)$$

2. 令

$$G = \mathrm{diag}(x_1,...,x_N(=1-x_N), 1-x_{N+1},...,1-x_{2N-1}) \quad (120)$$

则存在 $\lambda > 0$ 使得 $B := A(\lambda I + G)$ 也是 M 矩阵，且

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (1-x_i)^{-\alpha} + (|\tfrac{1}{2}-x_i|+h)^{1-\alpha}) \quad (121)$$

那么我们可以得到

$$\max_i |\frac{\epsilon_i}{\lambda+g(x_i)}| = |\frac{\epsilon_{i_0}}{\lambda+g(x_{i_0})}| \leq \frac{|R_{i_0}|}{M_{i_0}} \leq Ch^{\min\{2,r\alpha/2\}} \quad (122)$$

其中

$$g(x) = \begin{cases} x, & 0 < x \leq 1/2 \\ 1-x, & 1/2 < x < 1 \end{cases} \quad (123)$$

从而

$$|\epsilon_i| \leq C(\lambda+\frac{1}{2})h^{\min\{2,r\alpha/2\}} \quad (124)$$

这是因为 $g(x)$ 的线性插值就是他自身 $g(x) \equiv g_h(x)$，所以有

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij} g(x_j) = \int_0^1 |x_i - y|^{1-\alpha} g(y) dy$$

$$= -\frac{2}{(2-\alpha)(3-\alpha)}|\frac{1}{2} - x_i|^{3-\alpha} + \frac{1}{(2-\alpha)(3-\alpha)}(x_i^{3-\alpha} + (1-x_i)^{3-\alpha})$$

$$:= w(x_i) = p(x_i) + q(x_i) \tag{125}$$

所以

$$M_i := \sum_{j=1}^{2N-1} a_{ij} g(x_j)$$

$$= C_R \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} w(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) w(x_i) + \frac{1}{h_i} w(x_{i-1}) \right) \tag{126}$$

其中 $C_R < 0$.

特别的,

$$P_N := C_R \frac{2}{h_N + h_{N+1}} \left(\frac{1}{h_{N+1}} p(x_{N+1}) - (\frac{1}{h_N} + \frac{1}{h_{N+1}}) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right)$$

$$= \frac{-2C_R}{(2-\alpha)(3-\alpha)h_N^2} h_N^{3-\alpha}$$

$$= \frac{-2C_R}{(2-\alpha)(3-\alpha)} (\frac{1}{2} - x_{N-1})^{1-\alpha} \tag{127}$$

$$P_{N-1} := \frac{2C_R}{h_{N-1} + h_N} \left(\frac{1}{h_N} p(x_N) - (\frac{1}{h_{N-1}} + \frac{1}{h_N}) p(x_{N-1}) + \frac{1}{h_{N-1}} p(x_{N-2}) \right)$$

$$= \frac{-2C_R}{(2-\alpha)(3-\alpha)} \frac{2}{h_{N-1} + h_N} \left(-(\frac{1}{h_{N-1}} + \frac{1}{h_N}) h_N^{3-\alpha} + \frac{1}{h_{N-1}} (h_{N-1} + h_N)^{3-\alpha} \right)$$

$$= \frac{-4C_R}{(2-\alpha)(3-\alpha)h_{N-1}} \left(-h_N^{2-\alpha} + (h_{N-1} + h_N)^{2-\alpha} \right)$$

$$= \frac{-4C_R}{(3-\alpha)} \xi^{1-\alpha} \quad \xi \in [h_N, h_{N-1} + h_N]$$

$$\geq \frac{-4C_R}{(3-\alpha)} (h_{N-1} + h_N)^{1-\alpha} = \frac{-4C_R}{(3-\alpha)} (\frac{1}{2} - x_{N-2})^{1-\alpha} \tag{128}$$

对于 $1 \leq i < N - 1$

$$
\begin{aligned}
P_i &= -2C_R|\frac{1}{2} - \xi|^{1-\alpha} \quad \xi \in [x_{i-1}, x_i] \\
&\geq -2C_R|\frac{1}{2} - x_{i-1}|^{1-\alpha}
\end{aligned} \tag{129}
$$

综合起来

$$
P_i \geq C \begin{cases} |\frac{1}{2} - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - \frac{1}{2}|^{1-\alpha}, & i \geq N \end{cases} \tag{130}
$$

并且对于 $1 < i \leq N$, 我们有不等式

$$
\begin{aligned}
Q_i &:= C_R \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}}q(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})q(x_i) + \frac{1}{h_i}q(x_{i-1}) \right) \\
&= C_R q''(\xi) \quad \xi \in [x_{i-1}, x_{i+1}] \\
&= C_R(\xi^{1-\alpha} + (1-\xi)^{1-\alpha}) \\
&\geq C_R(x_{i-1}^{1-\alpha} + (1-x_{i+1})^{1-\alpha})
\end{aligned} \tag{131}
$$

$$
\begin{aligned}
{}_L Q_1 &= \frac{C_R}{(2-\alpha)(3-\alpha)} \frac{2}{h_1 + h_2} \left(-(\frac{1}{h_1} + \frac{1}{h_2})h_1^{3-\alpha} + \frac{1}{h_2}(h_1 + h_2)^{3-\alpha} \right) \\
&= \frac{2C_R}{(2-\alpha)(3-\alpha)h_2} \left(-h_1^{2-\alpha} + (h_1 + h_2)^{2-\alpha} \right) \\
&= \frac{2C_R}{(3-\alpha)}\xi^{1-\alpha} \quad \xi \in [h_N, h_{N-1} + h_N] \\
&\geq \frac{2C_R}{(3-\alpha)}h_1^{1-\alpha}
\end{aligned} \tag{132}
$$

能得到

$$
Q_i \geq -C(x_i^{1-\alpha} + (1-x_i)^{1-\alpha}) \tag{133}
$$