

## 1 问题

对于  $\Omega = (0, 1)$ ,  $1 < \alpha < 2$ , 假设  $f \in C^2(\Omega)$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \quad (1)$$

其中

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = C_R \frac{d^2}{dx^2} \int_{\Omega} \frac{u(y)}{|x-y|^{\alpha-1}} dy \quad (2)$$

## 2 数值格式

用线性插值代替原函数, 中心差分代替二阶导数, 记  $u_h(x)$  为  $u(x)$  在网格点上的线性插值。

我们解这样的数值解

$$\begin{aligned} & C_R \left( \frac{2}{h_{i+1}(h_i + h_{i+1})} \int_{\Omega} \frac{u_h(x)}{|x_{i+1} - y|^{\alpha-1}} dy \right. \\ & \quad - \frac{2}{h_i h_{i+1}} \int_{\Omega} \frac{u_h(x)}{|x_i - y|^{\alpha-1}} dy \\ & \quad \left. + \frac{2}{h_i(h_i + h_{i+1})} \int_{\Omega} \frac{u_h(x)}{|x_{i-1} - y|^{\alpha-1}} dy \right) \\ & = F_i \end{aligned} \quad (3)$$

矩阵  $A$  是  $M$  矩阵, 主对角正, 其他负, 严格对角占优。

## 3 一致网格

当  $r = 1$ , 网格成为一致网格,  $x_i = ih, h = \frac{1}{2N}, i = 0, \dots, 2N$ .

$A$  等于

$$\begin{aligned} a_{ij} &= \frac{C_R}{(2-\alpha)(3-\alpha)} h^{-\alpha} \\ & \quad (|i-j-2|^{3-\alpha} - 4|i-j-1|^{3-\alpha} + 6|i-j|^{3-\alpha} - 4|i-j+1|^{3-\alpha} + |i-j+2|^{3-\alpha}) \end{aligned} \quad (4)$$

矩阵行和

$$S_i = \sum_{j=1}^{2N-1} a_{ij} = \frac{C_R}{(2-\alpha)(3-\alpha)} h^{-\alpha} (|i+1|^{3-\alpha} - 3|i|^{3-\alpha} + 3|i-1|^{3-\alpha} - |i-2|^{3-\alpha} + \dots + 2N) \quad (5)$$

我们得到

$$S_i \geq C(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \quad (6)$$

下面估计截断误差  $R_i$ .

$$R_i = \int_0^1 D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \quad (7)$$

目标是

$$R_i \leq Ch^{\alpha/2} S_i \quad (8)$$

这样我们就有

$$\epsilon \leq \max_i \frac{R_i}{S_i} \leq Ch^{\alpha/2} \quad (9)$$

考虑  $R_1$

$$R_1 = \int_{\Omega} (u(y) - u_h(y)) \frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} dy \quad (10)$$

我们有

$$R_1 = \int_0^{3h} + \int_{3h}^{1/2} \quad (11)$$

当  $y > 3h$ ,

$$\frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} \leq C|y|^{-1-\alpha} \quad (12)$$

那么

$$\begin{aligned}
I_2 &\leq C \int_{3h}^{1/2} |y|^{-1-\alpha} u''(y) h^2 dy \\
&\leq C \int_{3h}^1 |y|^{-1-\alpha} y^{\alpha/2-2} h^2 dy \\
&\leq Ch^2 \int_{3h}^1 y^{-3-\alpha/2} dy \\
&\leq Ch^2 h^{-2-\alpha/2} = Ch^{-\alpha/2} \\
&\leq Ch^{\alpha/2} x_1^{-\alpha} \leq Ch^{\alpha/2} S_1
\end{aligned} \tag{13}$$

在考虑

$$\begin{aligned}
I_1 &= \int_0^{3h} \frac{u(y) - u_h(y)}{h^2} (|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}) dy \\
&= \int_0^h + \int_h^{3h} = J_1 + J_2
\end{aligned} \tag{14}$$

$$J_2 \leq Cu''(\eta) h^{2-\alpha} \leq Ch^{\alpha/2-2} h^{2-\alpha} \leq Ch^{-\alpha/2} \tag{15}$$

因为

$$\begin{aligned}
|u(x) - u_h(x)| &\leq \int_0^{x_1} |u'(y)| dy \\
&\leq C \int_0^{x_1} y^{\alpha/2-1} dy \\
&\leq C x_1^{\alpha/2}, \quad x \in (0, h)
\end{aligned} \tag{16}$$

$$\begin{aligned}
J_1 &= \int_0^h \frac{u(y) - u_h(y)}{h^2} (|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}) dy \\
&\leq Ch^{\alpha/2-2} h^{2-\alpha} = Ch^{-\alpha/2}
\end{aligned} \tag{17}$$

所以有

$$R_1 \leq Ch^{-\alpha/2} \leq Ch^{\alpha/2} h^{-\alpha} \leq Ch^{\alpha/2} S_1, \quad (S_1 \geq C x_1^{-\alpha}) \tag{18}$$

$R_1, R_2, R_3$  全部类似。

## 3.1 猜想

$$R_i \leq Ch^{\alpha/2+1}(x_i^{-\alpha-1} + (1-x_i)^{-\alpha-1}) \quad (\text{then } \leq Ch^{\alpha/2}S_i) \quad (19)$$

为了简便, 我们记  $D(y) := u(y) - u_h(y)$ .

当  $3 < i \leq N$  时,

$$\begin{aligned} R_i &= \int_0^1 D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\ &= \int_0^{x_1} D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\ &\quad + \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\ &\quad + \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} \frac{D(y+h) - D(y)}{h^2} |y-x_i|^{1-\alpha} + D(y) \frac{|y-x_{i+1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2} dy \\ &\quad + \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y-x_i|^{1-\alpha} dy \\ &\quad + \cdots (2N-i) \\ &= I_1 + I_2 + I_3 + I_4 + \cdots \end{aligned} \quad (20)$$

1.

$$\begin{aligned} I_1 &= \int_0^{x_1} (u(y) - u_h(y)) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\ &\leq Ch^{\alpha/2} \int_0^h |y-x_i|^{-1-\alpha} dy \\ &\leq Ch^{\alpha/2+1} x_i^{-1-\alpha} \end{aligned} \quad (21)$$

2.

$$\begin{aligned}
I_2 &= \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} D(y) \frac{|y - x_{i-1}|^{1-\alpha} - 2|y - x_i|^{1-\alpha} + |y - x_{i+1}|^{1-\alpha}}{h^2} dy \\
&\leq C \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2} h^2 |x_i - y|^{-1-\alpha} dy \\
&\leq Ch^{\alpha/2-1} h^2 x_i^{-1-\alpha} \leq Ch^{\alpha/2+1} x_i^{-1-\alpha}
\end{aligned} \tag{22}$$

3.

$$\begin{aligned}
I_3 &= \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil}+1} \frac{D(y+h) - D(y)}{h^2} |y - x_i|^{1-\alpha} + D(y) \frac{|y - x_{i+1}|^{1-\alpha} - |y - x_i|^{1-\alpha}}{h^2} dy \\
&\leq \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil}+1} u'''(\eta_1) h |x_i - y|^{1-\alpha} + u''(\eta_2) h |x_i - y|^{-\alpha} dy \\
&\leq Ch^2 x_i^{-2-\alpha/2} \leq Ch^{1+\alpha/2} x_i^{-1-\alpha}
\end{aligned} \tag{23}$$

4.

$$\begin{aligned}
I_4 &= \int_{x_{\lceil \frac{i}{2} \rceil}+1}^{x_i} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y - x_i|^{1-\alpha} dy \\
&\leq \int_{x_{\lceil \frac{i}{2} \rceil}+1}^{x_i} u''''(\eta) h^2 |x_i - y|^{1-\alpha} dy \\
&\leq C x_i^{\alpha/2-4} h^2 x_i^{2-\alpha} \\
&\leq Ch^2 x_i^{-2-\alpha/2} \leq Ch^{1+\alpha/2} x_i^{-1-\alpha}
\end{aligned} \tag{24}$$

猜想证毕，一致网格证完。

## 4 非一致

 $r > 1$ ,

$$\begin{cases} x_i = \frac{1}{2} \left( \frac{i}{N} \right)^r, & 0 \leq i \leq N \\ x_i = 1 - \frac{1}{2} \left( \frac{2N-i}{N} \right)^r, & N \leq i \leq 2N \end{cases} \tag{25}$$

令  $h = \frac{1}{2N}$ ，那么

当  $i < N, x_i < \frac{1}{2}$  时

$$h_i = \frac{1}{2} \left( \left( \frac{i}{N} \right)^r - \left( \frac{i-1}{N} \right)^r \right) \leq C(r) \left( \frac{i}{N} \right)^{r-1} \frac{1}{N} = Chx_i^{(r-1)/r} \quad (26)$$

当  $i \geq N, x_i \geq \frac{1}{2}$  时

$$h_i = \frac{1}{2} \left( \left( \frac{2N-i+1}{N} \right)^r - \left( \frac{2N-i}{N} \right)^r \right) \leq C(r) \left( \frac{2N-i+1}{N} \right)^{r-1} \frac{1}{N} = Ch(1-x_{i-1})^{(r-1)/r} \quad (27)$$

我们声明

$$\begin{aligned} S_i &= \sum_{j=1}^{2N-1} a_{ij} = \frac{C_R}{(2-\alpha)(3-\alpha)} \frac{2}{h_i + h_{i+1}} \\ &\quad \left( \frac{1}{h_{i+1}} \frac{|x_{i+1} - x_0|^{3-\alpha} - |x_{i+1} - x_1|^{3-\alpha}}{x_1 - x_0} \right. \\ &\quad \left. - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{x_1 - x_0} \right. \\ &\quad \left. + \frac{1}{h_i} \frac{|x_{i-1} - x_0|^{3-\alpha} - |x_{i-1} - x_1|^{3-\alpha}}{x_1 - x_0} \right) \\ &\geq C(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \end{aligned} \quad (28)$$

$$R_i = \int_0^1 D(y) \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} |x_{i+1} - y|^{1-\alpha} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) |x_i - y|^{1-\alpha} + \frac{1}{h_i} |x_{i-1} - y|^{1-\alpha} \right) dy \quad (29)$$

下面讨论  $R_1$

$$\begin{aligned} R_1 &= \int_0^{x_1} + \int_{x_1}^{x_3} + \int_{x_3}^{1/2} + \int_{1/2}^{x_{2N-1}} + \int_{x_{2N-1}}^1 \\ &\quad D(y) \frac{2}{h_1 + h_2} \left( \frac{1}{h_2} |x_2 - y|^{1-\alpha} - \left( \frac{1}{h_1} + \frac{1}{h_2} \right) |x_1 - y|^{1-\alpha} + \frac{1}{h_1} |y|^{1-\alpha} \right) dy \\ &:= I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned} \quad (30)$$

与一致网格时相似,

1.

$$\begin{aligned}
 |u(x) - u_h(x)| &\leq \int_0^{x_1} |u'(y)| dy \\
 &\leq C \int_0^{x_1} y^{\alpha/2-1} dy \\
 &\leq C x_1^{\alpha/2}, \quad x \in (0, x_1)
 \end{aligned} \tag{31}$$

因为  $1 - \alpha > -1$

$$\begin{aligned}
 I_1 &\leq C \int_0^{x_1} \frac{D(y)}{h_1^2} (|x_2 - y|^{1-\alpha} + 2|x_1 - y|^{1-\alpha} + |y|^{1-\alpha}) dy \\
 &\leq C x_1^{\alpha/2-2} x_1^{2-\alpha} = C x_1^{-\alpha/2} = C h^{-r\alpha/2}
 \end{aligned} \tag{32}$$

2.

$$I_2 \leq C u''(\eta) x_3^{2-\alpha} \leq C x_1^{\alpha/2-2} x_3^{2-\alpha} \leq C h^{-r\alpha/2} \tag{33}$$

3.

$$\begin{aligned}
 I_3 &= \int_{x_3}^{1/2} D(y) \frac{2}{h_1 + h_2} \left( \frac{1}{h_2} |x_2 - y|^{1-\alpha} - \left( \frac{1}{h_1} + \frac{1}{h_2} \right) |x_1 - y|^{1-\alpha} + \frac{1}{h_1} |y|^{1-\alpha} \right) dy \\
 &\leq C \int_{x_3}^{1/2} y^{\alpha/2-2} (h y^{(r-1)/r})^2 y^{-1-\alpha} dy \\
 &\leq C h^2 \int_{x_3}^{1/2} y^{\alpha/2-2/r-1-\alpha} dy \\
 &\leq C h^2 (h^r)^{-2/r-\alpha/2} = C h^{-r\alpha/2}
 \end{aligned} \tag{34}$$

4.

$$\begin{aligned}
I_4 &= \int_{1/2}^{x_{2N-1}} D(y) \frac{2}{h_1 + h_2} \left( \frac{1}{h_2} |x_2 - y|^{1-\alpha} - \left( \frac{1}{h_1} + \frac{1}{h_2} \right) |x_1 - y|^{1-\alpha} + \frac{1}{h_1} |y|^{1-\alpha} \right) dy \\
&\leq C \int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2} (h(1-y)^{(r-1)/r})^2 y^{-1-\alpha} dy \\
&\leq Ch^2 \int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2+2-2/r} \\
&\leq Ch^2 (C + h_{2N}^{\alpha/2-2/r+1}) \\
&= Ch^2 (C + h^{r\alpha/2-2+r}) \leq Ch^{\min\{2, r\alpha/2+r\}}
\end{aligned} \tag{35}$$

5.

$$I_5 \leq Ch_{2N}^{\alpha/2+1} \leq Ch^{r\alpha/2+r} \tag{36}$$

综合有

$$R_1 \leq Ch^{-r\alpha/2} \tag{37}$$

 $R_1, R_2, R_3$  一样。 $R_i, 3 < i \leq N$  比较困难。

$$\begin{aligned}
R_i &= \int_0^{x_1} + \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} + \dots \\
&\quad D(y) \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} |x_{i+1} - y|^{1-\alpha} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) |x_i - y|^{1-\alpha} + \frac{1}{h_i} |x_{i-1} - y|^{1-\alpha} \right) dy
\end{aligned} \tag{38}$$

1.

$$I_1 \leq Cx_1^{\alpha/2+1} x_i^{-1-\alpha} \leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} \tag{39}$$

2.



$$\begin{aligned}
I_2 &\leq C \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2} (hy^{(r-1)/r})^2 |x_i - y|^{-1-\alpha} dy \\
&\leq Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2/r} dy \\
&\leq Ch^2 x_i^{-1-\alpha} (h^{r\alpha/2-2+r} + x_i^{\alpha/2-2/r+1})
\end{aligned} \tag{40}$$