AN EXAMPLE ARTICLE*

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Abstract. This is an example SIAM LaTeX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

Key words. example, LaTeX

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1. Introduction.

The introduction introduces the context and summarizes the manuscript. It is importantly to clearly state the contributions of this piece of work.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$, suppose $f \in C^\beta(\Omega) \cap L^\infty(\Omega), \beta > 4, \|f\|_\beta^{\alpha/2} < \infty$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

2. Regularity.

Remark 2.1. 1. $C^k(U)$ is the set of all k-times continuously differentiable functions on open set U.
2. $C^\beta(U)$ is the collection of function f which for any $V \subset\subset U$ $f|_V \in C^\beta(\bar{V})$.

THEOREM 2.2. *If $f \in C^\beta(\Omega), \beta > 2$ and $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

$$|f^{(l)}(x)| \le \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \le T \\ (2T - x)^{-l-\alpha/2}, & \text{if } T \le x < 2T \end{cases} \tag{2.1}$$

THEOREM 2.3 (Regularity up to the boundary [1]).

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right) \tag{2.2}$$

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Corollary 2.4. *Let u be a solution of (1.1) on Ω. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \leq C[x(2T - x)]^{\alpha/2-l} \tag{2.3}$$

The paper is organized as follows. Our main results are in section 4, experimental results are in section 7, and the conclusions follow in section 9.

3. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \leq i \leq N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \leq i \leq 2N \end{cases} \tag{3.1}$$

where $r \geq 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \leq j \leq 2N \tag{3.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \leq x \leq x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \leq x \leq x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

And then, we can approximate $u(x)$ with

$$u_h(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{3.4}$$

For convience, we denote

$$I_h^{2-\alpha}(x) := \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x - y|^{1-\alpha} u_h(y) dy \tag{3.5}$$

And now, we can approximate the operator (1.2) at x_i with

$$\begin{aligned} D_h^\alpha u_h(x_i) &:= D_h^2 I_h^{2-\alpha}(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} I_h^{2-\alpha}(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) I_h^{2-\alpha}(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha}(x_{i+1}) \right) \end{aligned} \tag{3.6}$$

Finally, we approximate the equation (1.1) with

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i), \quad 1 \leq i \leq 2N - 1 \tag{3.7}$$

The discrete equation (3.7) can be written in matrix form

$$AU = F \tag{3.8}$$

where U is unknown, $F = (f(x_1), \cdots, f(x_{2N-1}))$. The matrix A is constructed as follows: Since

$$
\begin{aligned}
I_h^{2-\alpha}(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u(x_j) \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\
&=: \sum_{j=1}^{2N-1} \tilde{a}_{ij}\, u(x_j), \quad 0 \le i \le 2N
\end{aligned}
\tag{3.9}
$$

Then, substitute in (3.6), we have

$$
-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} a_{ij}\, u(x_j) \tag{3.10}
$$

where

$$
a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \tag{3.11}
$$

4. Main results. Here we state our main results; the proof is deferred to section 5 and section 6.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 4.1 (Truncation Error). *If $f \in C^2(\Omega)$ and $\alpha \in (1, 2)$, and $u(x)$ is a solution of the equation (1.1), then there exists a constant $C_1, C_2 = C_1(T, \alpha, r, \|f\|_{C^2(\Omega)}), C_2(T, \alpha, r, \|f\|_{C^2(\Omega)})$, such that the truncation error of the discrete format satisfies*

$$
\begin{aligned}
|-\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i)| \le & C_1 (h^{r\alpha/2 + r} (x_i^{-1-\alpha} + (2T - x_i)^{-1-\alpha}) \\
& + h^2 (x_i^{-\alpha/2 - 2/r} + (2T - x_i)^{-\alpha/2-2/r})) \\
& + C_2 h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \le i \le N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \le 2N-1 \end{cases}
\end{aligned}
\tag{4.1}
$$

where $C_2 = 0$ if $r = 1$.

THEOREM 4.2 (Convergence). *The discrete equation (3.7) has sulotion U, and there exists a positive constant $C = C(T, \alpha, r, \|f\|_{C^2(\Omega)})$ such that the error between the numerial solution U with the exact solution $u(x_i)$ satisfies*

$$
\max_{1 \le i \le 2N-1} |U_i - u(x_i)| \le C h^{\min\{\frac{r\alpha}{2}, 2\}} \tag{4.2}
$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

5. Proof of Theorem 4.1. For convience, let's denote

$$I^{2-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x-y|^{1-\alpha} u(y)dy \tag{5.1}$$

Then, the truncation error of the discrete format can be written as

$$\begin{aligned} -\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I_h^{2-\alpha}(x_i) - \frac{d^2}{dx^2} I^{2-\alpha}(x_i)) \\ &= -\kappa_\alpha D_h^2 (I_h^{2-\alpha}(x_i) - I^{2-\alpha}(x_i)) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \end{aligned} \tag{5.2}$$

THEOREM 5.1. *There exits a constant $C = C(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$ such that*

$$-\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \le C h^2 (x_i^{-\alpha/2-2/r} + (2T - x_i)^{-\alpha/2-2/r}) \tag{5.3}$$

Proof. Since $f \in C^2(\Omega)$ and

$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha}(x)) = f(x), \quad x \in \Omega, \tag{5.4}$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.2) of Lemma A.1, for $1 \le i \le 2N-1$, we have

$$-\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \tag{5.5}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

$$\left| \frac{h_{i+1} - h_i}{3} f'(x_i) \right| \le \frac{\|f\|_\beta^{(\alpha/2)}}{3} 2^{|r-2|} r(r-1) T^{2/r} h^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{-\alpha/2-2/r}, & N < i \le 2N-1 \end{cases} \tag{5.6}$$

2. By

$$\frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \tag{5.7}$$ ☐

6. Proof of Theorem 4.2. aaaaaaaaaa

7. Experimental results. Figure 1 shows some example results. Additional results are available in the supplement in Table 1. Table 1 shows additional supporting evidence.

8. Discussion of $Z = X \cup Y$.

9. Conclusions. Some conclusions here.

Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x)$ is twice differentable continous function on open set Ω, there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &= g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \end{aligned} \tag{A.1}$$

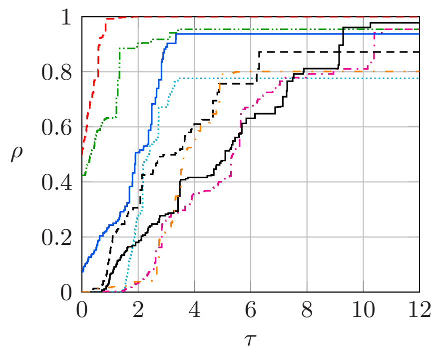

FIG. 1. *Example figure using external image files.*

TABLE 1
Example table.

Species	**Mean**	**Std. Dev.**
1	3.4	1.2
2	5.4	0.6
3	7.4	2.4
4	9.4	1.8

And if $g(x) \in C^4(\Omega)$, then

$$\frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})g(x_i) + \frac{1}{h_i}g(x_{i-1})\right) = g''(x_i) + \frac{h_{i+1} - h_i}{3}g'''(x_i) + \frac{1}{4!}\frac{2}{h_i + h_{i+1}}(h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2)) \tag{A.2}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$.

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \frac{(x_i - x_{i-1})^2}{2}g''(\xi_1), \quad \xi_1 \in [x_{i-1}, x_i]$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2}g''(\xi_2), \quad \xi_2 \in [x_i, x_{i+1}]$$

Subsitute them in the left side of (A.1), we have

$$\frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})g(x_i) + \frac{1}{h_i}g(x_{i-1})\right) = \frac{h_i}{h_i + h_{i+1}}g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}}g''(\xi_2)$$

Now, using intermediate value theorem , there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}}g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}}g''(\xi_2) = g''(\xi)$$

And for the second equation, similarly

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2}g''(x_i) - \frac{h_i^3}{3!}g'''(x_i) + \frac{h_i^4}{4!}g''''(\eta_1)$$

$$g(x_{i+1}) = g(x_i) + h_{i+1}g'(x_i) + \frac{h_{i+1}^2}{2}g''(x_i) + \frac{h_{i+1}^3}{3!}g'''(x_i) + \frac{h_{i+1}^4}{4!}g''''(\eta_2)$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. Subsitute them to the left side of (A.2), we can get the result. ∎

Lemma A.2. *If $y \in [x_{j-1}, x_j]$, denote $y = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0,1]$,*

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2}h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \tag{A.3}$$

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2}h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!}h_j^3(\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \tag{A.4}$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!}u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$

$$u(x_j) = u(y_j^\theta) + (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!}u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2}h_j^2(\theta u''(\xi_1) + (1-\theta)u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2}h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!}u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!}u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!}u''(\xi_2) + \frac{(1-\theta)^3 h_j^3}{3!}u'''(\eta_2)$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$. Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2}h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!}h_j^3(\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

∎

Appendix B. Inequality.

Lemma B.1.

$$h_i \le rT^{1/r}h \begin{cases} x_i^{1-1/r}, & 1 \le i \le N \\ (2T - x_{i-1})^{1-1/r}, & N < i \le 2N-1 \end{cases} \tag{B.1}$$

Proof. For $1 \le i \le N$,

$$\begin{aligned} h_i &= T\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right) \\ &\le rT\frac{1}{N}\left(\frac{i}{N}\right)^{r-1} = rT^{1/r}hx_i^{1-1/r} \end{aligned}$$

For $N < i \le 2N-1$,

$$\begin{aligned} h_i &= T\left(\left(\frac{2N-i+1}{N}\right)^r - \left(\frac{2N-i}{N}\right)^r\right) \\ &\le rT\frac{1}{N}\left(\frac{2N-i+1}{N}\right)^{r-1} = rT^{1/r}h(2T-x_{i-1})^{1-1/r} \end{aligned}$$

∎

LEMMA B.2. *There is a constant $C = 2^{|r-2|}r(r-1)T^{2/r}$ such that for all $i \in \{1, 2, \cdots, 2N-1\}$*

$$|h_{i+1} - h_i| \le Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N-1 \end{cases} \tag{B.2}$$

Proof.

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \le i \le N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \le i \le 2N-1 \end{cases}$$

For $i = 1$,

$$h_2 - h_1 = T(2^r - 2)\left(\frac{1}{N}\right)^r = (2^r - 2)T^{2/r}h^2x_1^{1-2/r}$$

For $2 \le i \le N-1$,

$$h_{i+1} - h_i = r(r-1)T\ N^{-2}\eta^{r-2}, \quad \eta \in [\frac{i-1}{N}, \frac{i+1}{N}]$$

If $r \in [1, 2]$,

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T\ N^{-2}\eta^{r-2} \le r(r-1)T\ h^2\left(\frac{i-1}{N}\right)^{r-2} \\ &\le r(r-1)T\ h^2 2^{2-r}\left(\frac{i}{N}\right)^{r-2} \\ &= 2^{2-r}r(r-1)T^{2/r}h^2x_i^{1-2/r} \end{aligned}$$

else if $r > 2$,

$$\begin{aligned}
h_{i+1} - h_i = r(r-1)T\ N^{-2}\eta^{r-2} &\leq r(r-1)T\ h^2 \left(\frac{i+1}{N}\right)^{r-2} \\
&\leq r(r-1)T\ h^2 2^{r-2} \left(\frac{i}{N}\right)^{r-2} \\
&= 2^{r-2} r(r-1) T^{2/r} h^2 x_i^{1-2/r}
\end{aligned}$$

Since

$$2^r - 2 \leq 2^{|r-2|} r(r-1), \quad r \geq 1$$

we have

$$h_{i+1} - h_i \leq 2^{|r-2|} r(r-1) T^{2/r} h^2 x_i^{1-2/r}, \quad 1 \leq i \leq N-1$$

For $i = N$, $h_{N+1} - h_N = 0$. For $N < i \leq 2N-1$, it's central symmetric to the first half of the proof, which is

$$h_i - h_{i+1} \leq 2^{|r-2|} r(r-1) T^{2/r} h^2 (2T - x_i)^{1-2/r}$$

Summarizes the inequalities, we can get

$$\text{(B.3)} \qquad |h_{i+1} - h_i| \leq 2^{|r-2|} r(r-1) T^{2/r} h^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases} \qquad \square$$

LEMMA B.3.

(B.4)

$$\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq C \begin{cases} x_i^{\alpha/2 - 2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2 - 2/r}, & N < i \leq 2N-1 \end{cases}$$

Proof.

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