

A SECOND ORDER NUMERICAL METHODS FOR REISZ-FRACTIONAL ELLIPTIC EQUATION ON GRADED MESH*

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Abstract. This is an example SIAM L^AT_EX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

Key words. example, L^AT_EX

MSC codes. 68Q25, 68R10, 68U05

1. Introduction. The introduction introduces the context and summarizes the manuscript. It is importantly to clearly state the contributions of this piece of work.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$, suppose $f \in C^\beta(\Omega)$, $\beta > 4 - \alpha$, $\|f\|_\beta^{(\alpha/2)} < \infty$

$$(1.1) \quad \begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases}$$

where

$$(1.2) \quad (-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy$$

$$(1.3) \quad \kappa_\alpha = -\frac{1}{2 \cos(\alpha\pi/2)} > 0$$

and the solution $u \in C^{\alpha/2}(\Omega)$.

2. Regularity.

Remark 2.1. 1. $C^k(U)$ is the set of all k -times continuously differentiable functions on open set U .

2. $C^\beta(U)$ is the collection of function f which for any $V \subset\subset U$ $f|_V \in C^\beta(\bar{V})$.

THEOREM 2.2. If $f \in C^\beta(\Omega)$, $\beta > 2$ and $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$

$$(2.1) \quad |f^{(l)}(x)| \leq \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \leq T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \leq x < 2T \end{cases}$$

THEOREM 2.3 (Regularity up to the boundary [1]).

$$(2.2) \quad \|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right)$$

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COROLLARY 2.4. Let u be a solution of (1.1) on Ω . Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$

$$(2.3) \quad |u^{(l)}(x)| \leq \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases}$$

The paper is organized as follows. Our main results are in section 4, experimental results are in section 7, and the conclusions follow in section 8.

3. Numeric Format.

$$(3.1) \quad x_i = \begin{cases} T \left(\frac{i}{N} \right)^r, & 0 \leq i \leq N \\ 2T - T \left(\frac{2N-i}{N} \right)^r, & N \leq i \leq 2N \end{cases}$$

where $r \geq 1$. And let

$$(3.2) \quad h_j = x_j - x_{j-1}, \quad 1 \leq j \leq 2N$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$(3.3) \quad \phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \leq x \leq x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \leq x \leq x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

And then, we can approximate $u(x)$ with

$$(3.4) \quad u_h(x) := \sum_{j=1}^{2N-1} u(x_j) \phi_j(x)$$

For convience, we denote

$$(3.5) \quad I_h^{2-\alpha}(x_i) := \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u_h(y) dy$$

And now, we can approximate the operator (1.2) at x_i with

$$(3.6) \quad \begin{aligned} D_h^\alpha u_h(x_i) &:= D_h^2 I_h^{2-\alpha}(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} I_h^{2-\alpha}(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) I_h^{2-\alpha}(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha}(x_{i+1}) \right) \end{aligned}$$

Finally, we approximate the equation (1.1) with

$$(3.7) \quad -\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i), \quad 1 \leq i \leq 2N-1$$

The discrete equation (3.7) can be written in matrix form

$$(3.8) \quad AU = F$$

where U is unknown, $F = (f(x_1), \dots, f(x_{2N-1}))$. The matrix A is constructed as follows: Since

$$\begin{aligned}
 (3.9) \quad I_h^{2-\alpha}(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u_h(y) dy \\
 &= \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u(x_j) \phi_j(y) dy \\
 &= \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y) dy \\
 &= \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\
 &=: \sum_{j=1}^{2N-1} \tilde{a}_{ij} u(x_j), \quad 0 \leq i \leq 2N
 \end{aligned}$$

Then, substitute in (3.6), we have

$$(3.10) \quad -\kappa_{\alpha} D_h^{\alpha} u_h(x_i) = \sum_{j=1}^{2N-1} a_{ij} u(x_j)$$

where

$$(3.11) \quad a_{ij} = -\kappa_{\alpha} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right)$$

4. Main results. Here we state our main results; the proof is deferred to section 5 and section 6.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 4.1 (Truncation Error). *If $f \in C^2(\Omega)$ and $\alpha \in (1, 2)$, and $u(x)$ is a solution of the equation (1.1), then there exists a constant $C_1, C_2 = C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{C^2(\Omega)}), C_2(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$, such that the truncation error of the discrete format satisfies*

$$\begin{aligned}
 (4.1) \quad |-\kappa_{\alpha} D_h^{\alpha} u_h(x_i) - f(x_i)| &\leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} \begin{cases} x_i^{-\alpha}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha}, & N < i \leq 2N-1 \end{cases} \\
 &\quad + C_2(r-1)h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \leq i \leq N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \leq 2N-1 \end{cases}
 \end{aligned}$$

THEOREM 4.2 (Convergence). *The discrete equation (3.7) has solution U , and there exists a positive constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ such that the error between the numerical solution U with the exact solution $u(x_i)$ satisfies*

$$(4.2) \quad \max_{1 \leq i \leq 2N-1} |U_i - u(x_i)| \leq C h^{\min\{\frac{r\alpha}{2}, 2\}}$$

That means the numerical method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

5. Proof of Theorem 4.1. For convience, let's denote

$$(5.1) \quad I^{2-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x-y|^{1-\alpha} u(y) dy$$

Then, the truncation error of the discrete format can be written as

$$(5.2) \quad \begin{aligned} -\kappa_{\alpha} D_h^{\alpha} u_h(x_i) - f(x_i) &= -\kappa_{\alpha} (D_h^2 I_h^{2-\alpha}(x_i) - \frac{d^2}{dx^2} I^{2-\alpha}(x_i)) \\ &= -\kappa_{\alpha} D_h^2 (I_h^{2-\alpha} - I^{2-\alpha})(x_i) - \kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \end{aligned}$$

5.1. Estimate of $-\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i)$.

THEOREM 5.1. *There exists a constant $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$ such that*

$$(5.3) \quad \left| -\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \right| \leq Ch^2 (x_i^{-\alpha/2-2/r} + (2T-x_i)^{-\alpha/2-2/r})$$

Proof. Since $f \in C^2(\Omega)$ and

$$(5.4) \quad \frac{d^2}{dx^2} (-\kappa_{\alpha} I^{2-\alpha}(x)) = f(x), \quad x \in \Omega,$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.3) of Lemma A.1, for $1 \leq i \leq 2N-1$, we have

$$(5.5) \quad -\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2))$$

where $\eta_1 \in [x_{i-1}, x_i]$, $\eta_2 \in [x_i, x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

$$(5.6) \quad \left| \frac{h_{i+1} - h_i}{3} f'(x_i) \right| \leq \frac{\|f\|_{\beta}^{(\alpha/2)}}{3} Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T-x_i)^{-\alpha/2-2/r}, & N < i \leq 2N-1 \end{cases}$$

2. See Proof 20, there is a constant $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$ such that

$$(5.7) \quad \begin{aligned} &\left| \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \right| \\ &\leq Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T-x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N-1 \end{cases} \end{aligned}$$

Summarizes, we get the result. \square

5.2. Estimate of R_i . Now, we study the first part of (5.2)

$$(5.8) \quad D_h^2 (I^{2-\alpha} - I_h^{2-\alpha})(x_i) = D_h^2 \left(\int_0^{2T} (u(y) - u_h(y)) \frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \right)$$

For convience, let's denote

$$(5.9) \quad T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy$$

92 And define

$$93 \quad (5.10) \quad \begin{aligned} R_i &:= D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \sum_{j=1}^{2N} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \end{aligned}$$

94 We have some results about the estimate of R_i

95 **THEOREM 5.2.** *For $1 \leq i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$96 \quad (5.11) \quad R_i \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

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98 **THEOREM 5.3.** *For $N/2 \leq i \leq N$, there exists constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$*
 99 *such that*

$$100 \quad (5.12) \quad R_i \leq C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

101 And for $N < i \leq 2N - 1$, it is symmetric to the previous case.

102 To prove these results, we need some utils. Also for simplicity, we denote

DEFINITION 5.4.

$$103 \quad (5.13) \quad S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right)$$

104 then

$$105 \quad (5.14) \quad R_i = \sum_{j=1}^{2N} S_{ij}$$

106 5.3. Proof of Theorem 5.2.

107 **LEMMA 5.5.** *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \leq$*
 108 *$i < N/2$,*

$$109 \quad (5.15) \quad \sum_{j=\max\{2i+1, i+3\}}^N S_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r}$$

110 *Proof.* For $\max\{2i+1, i+3\} \leq j \leq N$, by Lemma C.1 and Lemma C.2

$$111 \quad (5.16) \quad \begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\ &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy \end{aligned}$$

Therefore,

$$\begin{aligned}
 \sum_{j=\max\{2i+1, i+3\}}^N S_{ij} &\leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\
 &= \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\
 &\leq \frac{C}{\alpha/2+2/r} 2^{r(-\alpha/2-2/r)} h^2 x_i^{-\alpha/2-2/r}
 \end{aligned}
 \tag{5.17}$$

LEMMA 5.6. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \leq i < N/2$,*

$$\sum_{j=N+1}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
 \tag{5.18}$$

Proof. For $1 \leq i < N/2, N+1 \leq j \leq 2N-1$, by equation (C.2) and Lemma C.2

$$\begin{aligned}
 S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\
 &\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2-2/r} y^{-1-\alpha} dy \\
 &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy \\
 \sum_{j=N+1}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T - y)^{\alpha/2-2/r} dy \\
 &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\
 &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r} h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r} h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
 \end{aligned}
 \tag{5.19}$$

And by Lemma A.3

$$S_{i,2N} \leq CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}$$

And when $\alpha/2 - 2/r + 1 \geq 0$,

$$h^{r\alpha/2+r} \leq h^2$$

Summarizes, we get the result. \square

For $i = 1, 2$.

128 LEMMA 5.7. *By Lemma C.5 , Lemma 5.5 and Lemma 5.6 we get*

$$\begin{aligned}
 R_1 &= \sum_{j=1}^3 S_{1j} + \sum_{j=4}^{2N} S_{1j} \\
 (5.20) \quad &\leq Ch^2 x_1^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
 \end{aligned}$$

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$$\begin{aligned}
 R_2 &= \sum_{j=1}^4 S_{2j} + \sum_{j=5}^{2N} S_{2j} \\
 (5.21) \quad &\leq Ch^2 x_2^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
 \end{aligned}$$

132 For $3 \leq i < N/2$, we have a new separation of R_i , Let's denote $k = \lceil \frac{i}{2} \rceil$.

$$\begin{aligned}
 R_i &= \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &= \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \\
 (5.22) \quad &\quad + \sum_{j=k+1}^{2i-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
 &\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \\
 &\quad + \sum_{j=2i+1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &= I_1 + I_2 + I_3 + I_4 + I_5
 \end{aligned}$$

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135 LEMMA 5.8. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \leq$*
 136 *$i \leq N, k = \lceil \frac{i}{2} \rceil$*

$$(5.23) \quad |I_1| = \left| \sum_{j=1}^{k-1} S_{ij} \right| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

Proof. For $2 \leq j \leq k-1$, by Lemma C.1 and Lemma C.3

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\ &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 x_i^{-1-\alpha} \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} dy \end{aligned}$$

And by Lemma A.3, Lemma C.3

$$S_{i1} \leq C x_1^{\alpha/2} x_1 x_i^{-1-\alpha} = C x_1^{\alpha/2+1} x_i^{-1-\alpha} = C T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha}$$

Therefore,

$$\begin{aligned} I_1 &= \sum_{j=1}^{k-1} S_{ij} = S_{i1} + \sum_{j=2}^{k-1} S_{ij} \\ &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{k}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \\ &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \end{aligned}$$

But

$$\int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \leq \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2-2/r+1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 < 0 \end{cases}$$

So we have

$$I_1 \leq \begin{cases} \frac{C}{\alpha/2-2/r+1} h^2 x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2-2/r+1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} h^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2-2/r+1 < 0 \end{cases} \quad \square$$

DEFINITION 5.9. For convience, let's denote

$$V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right)$$

THEOREM 5.10. There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \leq i < N/2, k = \lceil \frac{i}{2} \rceil$,

$$I_3 = \sum_{j=k+1}^{2i-1} V_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r}$$

To estimate V_{ij} , we need some preparations.

155 LEMMA 5.11. Denote $y_j^\theta = \theta x_{j-1} + (1 - \theta)x_j, \theta \in [0, 1]$, by Lemma A.2

$$\begin{aligned}
 T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
 &= \int_{x_{j-1}}^{x_j} -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
 &\quad + \frac{\theta(1-\theta)}{3!} h_j^3 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) dy_j^\theta \\
 &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
 &\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta
 \end{aligned}
 \tag{5.31}$$

157 where $\eta_{j1}^\theta \in [x_{j-1}, y_j^\theta], \eta_{j2}^\theta \in [y_j^\theta, x_j]$.

158 Now Let's construct a series of functions to represent T_{ij} .

DEFINITION 5.12.

$$y_{j-i}(x) = (x^{1/r} + Z_{j-i})^r, \quad Z_{j-i} = T^{1/r} \frac{j-i}{N}
 \tag{5.32}$$

160

$$y_{j-i}^\theta(x) = \theta y_{j-1-i}(x) + (1-\theta) y_{j-i}(x)
 \tag{5.33}$$

162

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x)
 \tag{5.34}$$

164 Now, we define

$$P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}
 \tag{5.35}$$

166

$$Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}
 \tag{5.36}$$

168 And now we can rewrite T_{ij}

169 LEMMA 5.13. For $2 \leq i \leq N, 2 \leq j \leq N$,

$$\begin{aligned}
 T_{ij} &= \int_0^1 -\frac{\theta(1-\theta)}{2} P_{j-i}^\theta(x_i) d\theta \\
 &\quad + \int_0^1 \frac{\theta(1-\theta)}{3!} (\theta^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j1}^\theta) - (1-\theta)^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j2}^\theta)) d\theta
 \end{aligned}
 \tag{5.37}$$

171 Immediately, we can see from (5.29) that

LEMMA 5.14. For $3 \leq i \leq N-1$, $3 \leq j \leq N-1$,

$$\begin{aligned}
 (5.38) \quad V_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
 &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^\theta(x_i) d\theta \\
 &\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\
 &\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\
 &\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,2}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\
 &\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta
 \end{aligned}$$

To estimate V_{ij} , we first estimate $D_h^2 P_{j-i}^\theta(x_i)$, but By Lemma A.1,

$$(5.39) \quad D_h^2 P_{j-i}^\theta(x_i) = P_{j-i}^{\theta''}(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

By Leibniz formula, we calculate and estimate the derivations of h_{j-i}^3 , $u''(y_{j-i}^\theta(x))$

and $\frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$ separately.

Firstly, we have

LEMMA 5.15. There exists a constant $C = C(T, r)$ such that For $3 \leq i \leq N-1$, $\lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$(5.40) \quad h_{j-i}^3(\xi) \leq C h^2 x_i^{2-2/r} h_j$$

$$(5.41) \quad (h_{j-i}^3(\xi))' \leq C(r-1) h^2 x_i^{1-2/r} h_j$$

$$(5.42) \quad (h_{j-i}^3(\xi))'' \leq C(r-1) h^2 x_i^{-2/r} h_j$$

The proof of this theorem see Lemma C.6 and Lemma C.7

Second,

LEMMA 5.16. There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $3 \leq i \leq N-1$, $\lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$(5.43) \quad u''(y_{j-i}^\theta(\xi)) \leq C x_i^{\alpha/2-2}$$

$$(5.44) \quad (u''(y_{j-i}^\theta(\xi)))' \leq C x_i^{\alpha/2-3}$$

$$(5.45) \quad (u''(y_{j-i}^\theta(\xi)))'' \leq C x_i^{\alpha/2-4}$$

The proof of this theorem see Proof 27

And Finally, we have

LEMMA 5.17. There exists a constant $C = C(T, \alpha, r)$ such that For $3 \leq i \leq N-1$, $1 \leq j \leq \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$(5.46) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C |y_j^\theta - x_i|^{1-\alpha}$$

$$(196) \quad (5.47) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C|y_j^\theta - x_i|^{1-\alpha}x_i^{-1}$$

$$(197) \quad (5.48) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C|y_j^\theta - x_i|^{1-\alpha}x_i^{-2}$$

(198) where $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

(199) The proof of this theorem see Proof 28

(200)

(201) **LEMMA 5.18.** *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For*
(202) $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\},$

$$(203) \quad (5.49) \quad D_h^2 P_{j-i}^\theta(x_i) \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j$$

(204) where $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

(205) *Proof.* Since

$$(206) \quad (5.50) \quad D_h^2 P_{j-i}^\theta(x_i) = P_{j-i}^{\theta''}(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

(207) From (5.35), using Leibniz formula and Lemma 5.15, Lemma 5.16 and Lemma 5.17□

(208)

(209) **LEMMA 5.19.** *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for*
(210) $3 \leq i < N, k = \lceil \frac{i}{2} \rceil.$
(211) *For $k \leq j \leq \min\{2i-1, N-1\},$*

$$(212) \quad (5.51) \quad \begin{aligned} & \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ & \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned}$$

(213) *And for $k+1 \leq j \leq \min\{2i, N\},$*

$$(214) \quad (5.52) \quad \begin{aligned} & \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta) - Q_{j-i}^\theta(x_{i-1})u'''(\eta_{j-1}^\theta)}{h_i} \right) \\ & \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned}$$

(215) where $\eta_j^\theta \in [x_{j-1}, x_j].$

(216) proof see Proof 29

(217)

(218) **LEMMA 5.20.** *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for*
(219) $3 \leq i < N, k = \lceil \frac{i}{2} \rceil, k+1 \leq j \leq \min\{2i-1, N-1\},$

$$(220) \quad (5.53) \quad \begin{aligned} V_{ij} & \leq Ch^2 \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j d\theta \\ & = Ch^2 \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \end{aligned}$$

Proof. Since Lemma 5.14, by Lemma 5.18 and Lemma 5.19, we get the result immediately. \square

Now we can prove Theorem 5.10 using Lemma 5.20, $k = \lceil \frac{i}{2} \rceil$

$$\begin{aligned}
 I_3 &= \sum_{k+1}^{2i-1} V_{ij} \leq Ch^2 \int_{x_k}^{x_{2i-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\
 &= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{2i-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\
 &\leq Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r}
 \end{aligned}$$

LEMMA 5.21.

$$D_h P_{j-i}^\theta(x_i) := \frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} = P_{j-i}^{\theta'}(\xi), \quad \xi \in [x_i, x_{i+1}]$$

Then, for $3 \leq i \leq N-1$, $k = \lceil \frac{i}{2} \rceil$,

$$D_h P_{k-i}^\theta(x_i) \leq Ch^2 x_i^{-\alpha/2-2/r} h_j$$

Proof. Using Leibniz formula, by Lemma 5.15, Lemma 5.16 and Lemma 5.17, we take $j = k+1$, $i = i+1$, we get

$$\begin{aligned}
 D_h P_{k-i}^\theta(x_i) &\leq Ch^2 x_{i+1}^{\alpha/2-2/r-1} |y_{k+1}^\theta - x_{i+1}|^{1-\alpha} h_{j+1} \\
 &\leq Ch^2 x_i^{\alpha/2-2/r-1} |y_k^\theta - x_i|^{1-\alpha} h_j \\
 &\leq Ch^2 x_i^{-\alpha/2-2/r} h_j
 \end{aligned}$$

LEMMA 5.22. There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \leq i < N$, $k = \lceil \frac{i}{2} \rceil$,

$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

And for $3 \leq i < N/2$,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

Proof. In fact,

$$\begin{aligned}
 &\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \\
 &= \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + \left(\frac{1}{h_{i+1}} - \frac{1}{h_i} \right) T_{i,k}
 \end{aligned}$$

241 While, by Lemma A.2

$$\begin{aligned}
 \frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) &= \int_{x_{k-1}}^{x_k} (u(y) - u_h(y)) \frac{|x_{i+1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_{i+1}\Gamma(2-\alpha)} dy \\
 &\leq \int_{x_{k-1}}^{x_k} h_k^2 u''(\eta) \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy \\
 &\leq Ch_k h^2 x_k^{2-2/r} x_{k-1}^{\alpha/2-2} |x_i - x_k|^{-\alpha} \\
 &\leq Ch_k h^2 x_i^{-\alpha/2-2/r}
 \end{aligned}
 \tag{5.61}$$

243 Thus,

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) \leq Ch^2 x_i^{-\alpha/2-2/r}
 \tag{5.62}$$

245 For

$$\begin{aligned}
 \frac{1}{h_{i+1}}(T_{i+1,k+1} - T_{i,k}) &= \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} d\theta \\
 &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{Q_{k-i}^\theta(x_{i+1})u'''(\eta_{k+1,1}^\theta) - Q_{k-i}^\theta(x_i)u'''(\eta_{k,1}^\theta)}{h_{i+1}} d\theta \\
 &- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{Q_{k-i}^\theta(x_{i+1})u'''(\eta_{k+1,2}^\theta) - Q_{k-i}^\theta(x_i)u'''(\eta_{k,2}^\theta)}{h_{i+1}} d\theta
 \end{aligned}
 \tag{5.63}$$

247 And by Lemma 5.21

$$\frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} \leq Ch^2 x_i^{-\alpha/2-2/r} h_k
 \tag{5.64}$$

249 And with Lemma 5.19, we can get

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) \leq Ch^2 x_i^{-\alpha/2-2/r}
 \tag{5.65}$$

251 For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned}
 \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,k} &\leq h_i^{-3} h^2 x_i^{1-2/r} h_k Ch_k^2 x_{k-1}^{\alpha/2-2} |x_k - x_i|^{1-\alpha} \\
 &\leq Ch^2 x_i^{-\alpha/2-2/r}
 \end{aligned}
 \tag{5.66}$$

253 Summarizes, we have

$$I_2 \leq Ch^2 x_i^{-\alpha/2-2/r}
 \tag{5.67}$$

255 The case for I_4 is similar. □

256 Now combine Lemma 5.8, Lemma 5.22, Theorem 5.10, Lemma 5.5 and Lemma 5.6
 257 to get the final result.

258 For $3 \leq i < N/2$

$$\begin{aligned}
 R_i &= I_1 + I_2 + I_3 + I_4 + I_5 \\
 &\leq Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2 (x_i^{1-\alpha} \ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases}
 \end{aligned}
 \tag{5.68}$$

Combine with $i = 1, 2$, we get for $1 \leq i \leq N/2$

$$(5.69) \quad R_i \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases}$$

5.4. Proof of Theorem 5.3. For $N/2 \leq i < N, k = \lceil \frac{i}{2} \rceil$, we have

$$(5.70) \quad \begin{aligned} R_i &= \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &= \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \\ &\quad + \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right) \\ &\quad + \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 \end{aligned}$$

We have estimate I_1 in Lemma 5.8 and I_2 in Lemma 5.22. We can control I_3 in similar with Theorem 5.10 by Lemma 5.20 where $2i - 1 \geq N - 1$

LEMMA 5.23. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $N/2 \leq i < N, k = \lceil \frac{i}{2} \rceil$,*

$$(5.71) \quad \begin{aligned} I_3 &= \sum_{j=k+1}^{N-1} V_{ij} \leq Ch^2 \int_{x_k}^{x_{N-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{N-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\ &\leq Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned}$$

Let's study I_5 before I_4 .

$$(5.72) \quad I_5 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} V_{ij}$$

Similarly, Let's define a new series of functions

DEFINITION 5.24. *For $i < N, j \geq N$,*

$$(5.73) \quad y_{j-i}(x) = 2T - (Z_{2N-j+i} - x^{1/r})^r, \quad Z_{2N-j+i} = T^{1/r} \frac{2N-j+i}{N}$$

274

275 (5.74)
$$y_{j-i}'(x) = (2T - y_{j-i}(x))^{1-1/r} x^{1/r-1}$$

276 (5.75)
$$y_{j-i}''(x) = \frac{1-r}{r} (2T - y_{j-i}(x))^{1-2/r} x^{1/r-2} Z_{2N-j+i}$$

277 (5.76)

278

279 (5.77)
$$y_{j-i}^\theta(x) = \theta y_{j-i-1}(x) + (1-\theta) y_{j-i}(x)$$

280

281 (5.78)
$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x)$$

282

283 (5.79)
$$P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

284

285 (5.80)
$$Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

286

 Now we have, for $i < N, j \geq N+2$,

(5.81)

$$\begin{aligned} V_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^\theta(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,2}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta \end{aligned}$$

287

288

Similarly, we first estimate

289 (5.82)
$$D_h^2 P_{j-i}^\theta(\xi) = P_{j-i}^{\theta'}(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

290

 Combine lemmas [Lemma C.8](#), [Lemma C.9](#) and [Lemma C.10](#), we have

291

 LEMMA 5.25. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For*

292

 $N/2 \leq i < N, N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1, \xi \in [x_{i-1}, x_{i+1}]$, we have

293 (5.83)

$$\begin{aligned} |P_{j-i}^{\theta''}(\xi)| &\leq Ch_j h^2 (|y_j^\theta - x_i|^{1-\alpha} \\ &\quad + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N) \\ &\quad + |y_j^\theta - x_i|^{-1-\alpha} (|2T - x_i - y_j^\theta| + h_N)^2 \\ &\quad + (r-1) |y_j^\theta - x_i|^{-\alpha}) \end{aligned}$$

And

LEMMA 5.26. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i < N$, $\xi \in [x_{i-1}, x_{i+1}]$, we have for $N+1 \leq j \leq 2N - \lceil \frac{N}{2} \rceil$*

$$(5.84) \quad \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ \leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N))$$

for $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$

$$(5.85) \quad \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta) - Q_{j-i}^\theta(x_{i-1})u'''(\eta_{j-1}^\theta)}{h_{i+1}} \right) \\ \leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N))$$

The proof see Proof 33.

Combine (5.81), Lemma 5.25 and Lemma 5.26, we have

THEOREM 5.27. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i < N$, $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$*

$$(5.86) \quad V_{ij} \leq Ch^2 \int_{x_{j-1}}^{x_j} (|y - x_i|^{1-\alpha} \\ + |y - x_i|^{-\alpha} (|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha} (|2T - x_i - y| + h_N)^2 \\ + (r-1)|y - x_i|^{-\alpha}) dy$$

We can estimate I_5 Now.

THEOREM 5.28. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i < N$, we have*

$$(5.87) \quad I_5 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \leq Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha}$$

Proof.

$$(5.88) \quad I_5 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \\ \leq Ch^2 \int_{x_{N+1}}^{x_{2N-i}} + \int_{x_{2N-i}}^{x_{2N - \lceil \frac{N}{2} \rceil}} (|y - x_i|^{1-\alpha} \\ + |y - x_i|^{-\alpha} (|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha} (|2T - x_i - y| + h_N)^2 \\ + (r-1)|y - x_i|^{-\alpha}) dy \\ = J_1 + J_2$$

While $x_{N+1} \leq y \leq x_{2N-i} = 2T - x_i$,

$$(5.89) \quad T - x_{i-1} \leq x_{N+1} - x_i \leq y - x_i \leq x_{2N-i} - x_i \leq 2(T - x_{i-1})$$

312 and

$$313 \quad (5.90) \quad 2T - x_i - y + h_N \leq 2T - x_i - x_{N+1} + h_N = T - x_i \leq T - x_{i-1}$$

314 So

$$\begin{aligned} 315 \quad (5.91) \quad J_1 &\leq Ch^2(x_{2N-i} - x_{N+1})(|T - x_{i-1}|^{1-\alpha} + (r-1)|T - x_{i-1}|^{-\alpha}) \\ &\leq Ch^2(|T - x_{i-1}|^{2-\alpha} + (r-1)|T - x_{i-1}|^{1-\alpha}) \\ &\leq Ch^2T^{2-\alpha} + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned}$$

316 Otherwise, when $x_{2N-i} \leq y \leq x_{2N-\lceil \frac{N}{2} \rceil}$

$$317 \quad (5.92) \quad x_i + y - 2T + h_N \leq y - x_i$$

318

$$\begin{aligned} 319 \quad (5.93) \quad J_2 &\leq Ch^2 \int_{x_{2N-i}}^{(2-2^{-r})T} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} \\ &\leq Ch^2(T^{2-\alpha} + (r-1)|x_{2N-i} - x_i|^{1-\alpha}) \\ &= Ch^2 + C(r-1)h^2|T - x_i|^{1-\alpha} \leq Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned}$$

320 Summarizes two cases, we get the result. \square

For I_4 , we have

THEOREM 5.29. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that, for $N/2 \leq i < N-1$*

$$(5.94) \quad \begin{aligned} V_{iN} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1, N+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, N} + \frac{1}{h_i} T_{i-1, N-1} \right) \\ &\leq Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \end{aligned}$$

Proof. We use the similar skill in the last section, but more complicated. for $j = N$, Let

$$(5.95) \quad y_{i \rightarrow N-1}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r} \frac{N-1-i}{N}$$

$$(5.96) \quad y_{i \rightarrow N}(x) = \frac{x^{1/r} - Z_i}{Z_1} h_N + T, \quad Z_i = T^{1/r} \frac{i}{N}, x_N = T$$

and

$$(5.97) \quad y_{i \rightarrow N+1}(x) = 2T - (Z_{N-1+i} - x^{1/r})^r, \quad Z_{N-1+i} = T^{1/r} \frac{N-1+i}{N}$$

Thus,

$$\begin{aligned} y_{i \rightarrow N-1}(x_{i-1}) &= x_{N-2}, \quad y_{i \rightarrow N}(x_i) = x_{N-1}, \quad y_{i \rightarrow N}(x_{i+1}) = x_N \\ y_{i \rightarrow N}(x_{i-1}) &= x_{N-1}, \quad y_{i \rightarrow N}(x_i) = x_N, \quad y_{i \rightarrow N}(x_{i+1}) = x_{N+1} \\ y_{i \rightarrow N-1}(x_{i-1}) &= x_N, \quad y_{i \rightarrow N}(x_i) = x_{N+1}, \quad y_{i \rightarrow N}(x_{i+1}) = x_{N+2} \end{aligned}$$

Then, define

$$(5.98) \quad y_{i \rightarrow N}^\theta(x) = \theta y_{i \rightarrow N-1}(x) + (1-\theta) y_{i \rightarrow N}(x)$$

$$(5.99) \quad y_{i \rightarrow N+1}^\theta(x) = \theta y_{i \rightarrow N}(x) + (1-\theta) y_{i \rightarrow N+1}(x)$$

$$(5.100) \quad h_{i \rightarrow N}(x) = y_{i \rightarrow N}(x) - y_{i \rightarrow N-1}(x)$$

$$(5.101) \quad h_{i \rightarrow N+1}(x) = y_{i \rightarrow N+1}(x) - y_{i \rightarrow N}(x)$$

We have

$$(5.102) \quad y_{i \rightarrow N-1}'(x) = y_{i \rightarrow N-1}^{1-1/r}(x) x^{1/r-1}$$

$$(5.103) \quad y_{i \rightarrow N-1}''(x) = \frac{1-r}{r} y_{i \rightarrow N-1}^{1-2/r}(x) x^{1/r-2} Z_{N-1-i}$$

$$(5.104) \quad y_{i \rightarrow N}'(x) = \frac{1}{r} \frac{h_N}{Z_1} x^{1/r-1}$$

$$(5.105) \quad y_{i \rightarrow N}''(x) = \frac{1-r}{r^2} \frac{h_N}{Z_1} x^{1/r-2}$$

$$(5.106) \quad y_{i \rightarrow N+1}'(x) = (2T - y_{i \rightarrow N+1}(x))^{1-1/r} x^{1/r-1}$$

$$(5.107) \quad y_{i \rightarrow N+1}''(x) = \frac{1-r}{r} (2T - y_{i \rightarrow N+1}(x))^{1-2/r} x^{1/r-2} Z_{N-1+i}$$

349

$$350 \quad (5.108) \quad P_{i \rightarrow N}^\theta(x) = (h_{i \rightarrow N}(x))^3 \frac{|y_{i \rightarrow N}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y_{i \rightarrow N}^\theta(x))$$

$$351 \quad (5.109) \quad P_{i \rightarrow N+1}^\theta(x) = (h_{i \rightarrow N+1}(x))^3 \frac{|y_{i \rightarrow N+1}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y_{i \rightarrow N+1}^\theta(x))$$

$$352 \quad (5.110) \quad Q_{i \rightarrow N}^\theta(x) = (h_{i \rightarrow N}(x))^4 \frac{|y_{i \rightarrow N}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$353 \quad (5.111) \quad Q_{i \rightarrow N+1}^\theta(x) = (h_{i \rightarrow N+1}(x))^4 \frac{|y_{i \rightarrow N+1}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

 354 Similar with Lemma 5.13, we can get for $l = -1, 0, 1$,

$$355 \quad (5.112) \quad T_{i+l, N+l} = \int_0^1 -\frac{\theta(1-\theta)}{2} P_{i \rightarrow N}^\theta(x_{i+l}) d\theta \\ + \int_0^1 \frac{\theta(1-\theta)}{3!} Q_{i \rightarrow N}^\theta(x_{i+l}) (\theta^2 u'''(\eta_{N+l,1}^\theta) - (1-\theta)^2 u'''(\eta_{N+l,2}^\theta)) d\theta$$

356

$$(5.113) \quad T_{i+l, N+1+l} = \int_0^1 -\frac{\theta(1-\theta)}{2} P_{i \rightarrow N+1}^\theta(x_{i+l}) d\theta \\ 357 \quad + \int_0^1 \frac{\theta(1-\theta)}{3!} Q_{i \rightarrow N+1}^\theta(x_{i+l}) (\theta^2 u'''(\eta_{N+1+l,1}^\theta) - (1-\theta)^2 u'''(\eta_{N+1+l,2}^\theta)) d\theta$$

358 So we have

$$(5.114) \quad V_{i,N} = \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{i \rightarrow N}^\theta(x_i) d\theta \\ + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_{i+1}) u'''(\eta_{N+1,1}^\theta) - Q_{i \rightarrow N}^\theta(x_i) u'''(\eta_{N,1}^\theta)}{h_{i+1}} \right) d\theta \\ 359 \quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_i) u'''(\eta_{N,1}^\theta) - Q_{i \rightarrow N}^\theta(x_{i-1}) u'''(\eta_{N-1,1}^\theta)}{h_i} \right) d\theta \\ - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_{i+1}) u'''(\eta_{N+1,2}^\theta) - Q_{i \rightarrow N}^\theta(x_i) u'''(\eta_{N,2}^\theta)}{h_{i+1}} \right) d\theta \\ + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_i) u'''(\eta_{N,2}^\theta) - Q_{i \rightarrow N}^\theta(x_{i-1}) u'''(\eta_{N-1,2}^\theta)}{h_i} \right) d\theta$$

 360 $N+1$ is similar.

 361 We estimate $D_h^2 P_{i \rightarrow N}^\theta(x_i) = P_{i \rightarrow N}^{\theta''}(\xi)$, $\xi \in [x_{i-1}, x_{i+1}]$,

362

LEMMA 5.30.

$$363 \quad (5.115) \quad h_{i \rightarrow N}^3(\xi) \leq Ch_N^3 \leq Ch^3$$

$$364 \quad (5.116) \quad h_{i \rightarrow N+1}^3(\xi) \leq Ch_N^3 \leq Ch^3$$

$$(h_{i \rightarrow N}^3(\xi))' \leq C(r-1)h_N^2 h \leq C(r-1)h^3 \quad (5.117)$$

$$(h_{i \rightarrow N+1}^3(\xi))' \leq C(r-1)h_N^2 h \leq C(r-1)h^3 \quad (5.118)$$

$$(h_{i \rightarrow N}^3(\xi))'' \leq C(r-1)h^2 \quad (5.119)$$

$$(h_{i \rightarrow N+1}^3(\xi))'' \leq C(r-1)h^2 \quad (5.120)$$

Proof.

$$h_{i \rightarrow N}(\xi) \leq 2h_N, \quad h_{i \rightarrow N+1}(\xi) \leq 2h_N \quad (5.121)$$

370

$$\begin{aligned} (h_{i \rightarrow N}^l(\xi))' &= lh_{i \rightarrow N}^{l-1}(\xi)(y_{i \rightarrow N}'(\xi) - y_{i \rightarrow N-1}'(\xi)) \\ &= lh_{i \rightarrow N}^{l-1}(\xi)x_i^{1/r-1}(\frac{1}{r}\frac{h_N}{Z_1} - y_{i \rightarrow N-1}^{1-1/r}(\xi)) \end{aligned} \quad (5.122)$$

372 while

(5.123)

$$\begin{aligned} |\frac{1}{r}\frac{h_N}{Z_1} - y_{i \rightarrow N-1}^{1-1/r}(\xi)| &= |\frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r}|(\frac{N-t}{N})^{r-1} - (\frac{N-s}{N})^{r-1}| \quad t \in [0, 1], s \in [0, 2] \\ &\leq T^{1-1/r}|1 - (\frac{N-2}{N})^{r-1}| \leq CT^{1-1/r}(r-1)\frac{2}{N} \end{aligned} \quad (5.123)$$

374 Thus,

$$(h_{i \rightarrow N}^l(\xi))' \leq C(r-1)h_N^{l-1}x_i^{1/r-1}h \quad (5.124)$$

$$\begin{aligned} (h_{i \rightarrow N+1}^l(\xi))' &= lh_{i \rightarrow N+1}^{l-1}(\xi)(y_{i \rightarrow N+1}'(\xi) - y_{i \rightarrow N}'(\xi)) \\ &= lh_{i \rightarrow N+1}^{l-1}(\xi)x_i^{1/r-1}((2T - y_{i \rightarrow N+1}(\xi))^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1}) \end{aligned} \quad (5.125)$$

377 Similarly,

(5.126)

$$\begin{aligned} |(2T - y_{i \rightarrow N+1})^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1}| &= |\eta^{1-1/r} - \frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r}|(\frac{N-s}{N})^{r-1} - (\frac{N-t}{N})^{r-1}| \quad t \in [0, 1], s \in [0, 2] \\ &\leq T^{1-1/r}|(\frac{N-2}{N})^{r-1} - 1| \leq CT^{1-1/r}(r-1)\frac{2}{N} \end{aligned} \quad (5.126)$$

379 And

(5.127)

$$\begin{aligned} (h_{i \rightarrow N}^3(\xi))'' &= 3h_{i \rightarrow N}^2(\xi)h_{i \rightarrow N}''(\xi) + 6h_{i \rightarrow N}(\xi)(h_{i \rightarrow N}'(\xi))^2 \\ &\leq Ch_N^2\frac{1-r}{r}x_i^{1/r-2}(\frac{1}{r}\frac{h_N}{Z_1} - y_{i \rightarrow N-1}^{1-2/r}(\xi)Z_{N-1-i}) + Ch_N(r-1)^2h^2x_i^{2/r-2} \end{aligned} \quad (5.127)$$

$$|\frac{h_N}{rZ_1} - y_{i \rightarrow N-1}^{1-2/r}(\xi)Z_{N-1-i}| \leq T^{1-1/r} + Cx_N^{1-2/r}x_N^{1/r} = CT^{1-1/r} \quad (5.128)$$

382 So

$$383 \quad (5.128) \quad \begin{aligned} (h_{i \rightarrow N}^3(\xi))'' &\leq C h_N^2 \frac{1-r}{r} x_i^{1/r-2} + C(r-1)^2 h_N x_i^{2/r-2} h^2 \\ &\leq C(r-1) h_N^2 x_i^{1/r-1} \end{aligned}$$

384 $h_{i \rightarrow N+1}^3(\xi)$ is similar. □

LEMMA 5.31.

$$385 \quad (5.129) \quad u''(y_{i \rightarrow N}^\theta(\xi)) \leq C x_{N-2}^{-\alpha/2-2} \leq C$$

$$386 \quad (5.130) \quad (u''(y_{i \rightarrow N}^\theta(\xi)))' \leq C$$

$$387 \quad (5.131) \quad (u''(y_{i \rightarrow N}^\theta(\xi)))'' \leq C$$

Proof.

$$388 \quad (5.132) \quad \begin{aligned} (u''(y_{i \rightarrow N}^\theta(\xi)))' &= u'''(y_{i \rightarrow N}^\theta(\xi)) y_{i \rightarrow N}^{\theta'}(\xi) \\ &\leq C(\theta y_{i \rightarrow N-1}'(\xi) + (1-\theta) y_{i \rightarrow N}'(\xi)) \\ &\leq C x_i^{1/r-1} (\theta y_{i \rightarrow N-1}^{1-1/r}(\xi) + (1-\theta) \frac{h_N}{r Z_1}) \\ &\leq C x_i^{1/r-1} x_N^{1-1/r} \end{aligned}$$

389 And

(5.133)

$$390 \quad \begin{aligned} (u''(y_{i \rightarrow N}^\theta(\xi)))'' &= u''''(y_{i \rightarrow N}^\theta(\xi)) (y_{i \rightarrow N}^{\theta'}(\xi))^2 + u'''(y_{i \rightarrow N}^\theta(\xi)) y_{i \rightarrow N}^{\theta''}(\xi) \\ &\leq C x_i^{2/r-2} x_N^{2-2/r} + C \frac{r-1}{r} x_i^{1/r-2} (\theta x_N^{1-2/r} Z_{N-1-i} + (1-\theta) \frac{h_N}{r Z_1}) \\ &\leq C x_i^{2/r-2} + C(r-1) x_i^{1/r-2} T^{1-1/r} \end{aligned}$$

LEMMA 5.32.

$$391 \quad (5.134) \quad |y_{i \rightarrow N}^\theta(\xi) - \xi|^{1-\alpha} \leq C |y_N^\theta - x_i|^{1-\alpha}$$

$$392 \quad (5.135) \quad (|y_{i \rightarrow N}^\theta(\xi) - \xi|^{1-\alpha})' \leq C |y_N^\theta - x_i|^{1-\alpha}$$

$$393 \quad (5.136) \quad (|y_{i \rightarrow N}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C(r-1) |y_N^\theta - x_i|^{-\alpha} + |y_N^\theta - x_i|^{1-\alpha}$$

Proof.

(5.137)

$$394 \quad \begin{aligned} (y_{i \rightarrow N}^\theta(\xi) - \xi)' &= (\theta(y_{i \rightarrow N-1}(\xi) - \xi) + (1-\theta)(y_{i \rightarrow N}(\xi) - \xi))' \\ &= \theta(y_{i \rightarrow N-1}'(\xi) - 1) + (1-\theta)(y_{i \rightarrow N}'(\xi) - 1) \\ &= \theta \xi^{1/r-1} (y_{i \rightarrow N-1}^{1-1/r}(\xi) - \xi^{1-1/r}) + (1-\theta) \xi^{1/r-1} \left(\frac{h_N}{r Z_1} - \xi^{1-1/r} \right) \end{aligned}$$

395

$$396 \quad (5.138) \quad \begin{aligned} (y_{i \rightarrow N}^\theta(\xi) - \xi)'' &= \theta(y_{i \rightarrow N-1}''(\xi)) + (1-\theta)(y_{i \rightarrow N}''(\xi)) \\ &= \frac{1-r}{r} \xi^{1/r-2} (\theta y_{i \rightarrow N-1}^{1-2/r}(\xi) Z_{N-1-i} + (1-\theta) \frac{h_N}{r Z_1}) \leq 0 \end{aligned}$$

397 And

$$398 \quad (5.139) \quad |(y_{i \rightarrow N}^\theta(\xi) - \xi)''| \leq C(r-1) \xi^{1/r-2} T^{1-1/r}$$

We have known

$$(5.140) \quad C|x_{N-1} - x_i| \leq |y_{i \rightarrow N-1}(\xi) - \xi| \leq C|x_{N-1} - x_i|$$

If $\xi \leq x_{N-1}$, then $(y_{i \rightarrow N}(\xi) - \xi)' \geq 0$, so

$$(5.141) \quad C|x_N - x_i| \leq |x_{N-1} - x_{i-1}| \leq |y_{i \rightarrow N}^\theta(\xi) - \xi| \leq |x_{N+1} - x_{i+1}| \leq C|x_N - x_i|$$

If $i = N - 1$ and $\xi \in [x_{N-1}, x_N]$, then $y_{i \rightarrow N}(\xi) - \xi$ is concave, bigger than its two neighboring points, which are equal to h_N , so

$$(5.142) \quad h_N = |x_N - x_{N-1}| \leq |y_{i \rightarrow N}(\xi) - \xi| \leq |x_{N+1} - x_{N-1}| = 2h_N$$

So we have

$$(5.143) \quad |y_{i \rightarrow N}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_N^\theta - x_i|^{1-\alpha}$$

While

$$(5.144) \quad y_{i \rightarrow N-1}^{1-1/r}(\xi) - \xi^{1-1/r} \leq (y_{i \rightarrow N-1}(\xi) - \xi)\xi^{-1/r}$$

and

$$(5.145) \quad \begin{aligned} \left| \frac{h_N}{rZ_1} - \xi^{1-1/r} \right| &\leq \max\left\{ \left| \frac{h_N}{rZ_1} - x_{i-1}^{1-1/r} \right|, \left| \frac{h_N}{rZ_1} - x_{i+1}^{1-1/r} \right| \right\} \\ &\leq \max \begin{cases} T^{1-1/r} - x_{i-1}^{1-1/r} \leq |x_N - x_{i-1}|T^{-1/r} \leq C|x_N - x_i| \\ |x_{i+1}^{1-1/r} - x_{N-1}^{1-1/r}| \leq |x_{i+1} - x_{N-1}|x_{N-1}^{-1/r} \leq C|x_N - x_i| \end{cases} \end{aligned}$$

So we have

$$(5.146) \quad (y_{i \rightarrow N}^\theta(\xi) - \xi)' \leq C|y_N^\theta - x_i|$$

$$(5.147) \quad \begin{aligned} (|y_{i \rightarrow N}^\theta(\xi) - \xi|^{1-\alpha})' &= |y_{i \rightarrow N}^\theta(\xi) - \xi|^{-\alpha} (y_{i \rightarrow N}^\theta(\xi) - \xi)' \\ &\leq |y_N^\theta - x_i|^{1-\alpha} \end{aligned}$$

Finally,

$$(5.148) \quad \begin{aligned} (|y_{i \rightarrow N}^\theta(\xi) - \xi|^{1-\alpha})'' &= (1-\alpha)|y_{i \rightarrow N}^\theta(\xi) - \xi|^{-\alpha} (y_{i \rightarrow N}^\theta(\xi) - \xi)'' \\ &\quad + \alpha(\alpha-1)|y_{i \rightarrow N}^\theta(\xi) - \xi|^{-1-\alpha} ((y_{i \rightarrow N}^\theta(\xi) - \xi)')^2 \\ &\leq C(r-1)|y_N^\theta - x_i|^{-\alpha} + C|y_N^\theta - x_i|^{1-\alpha} \end{aligned} \quad \square$$

By the three lemmas above, for $N/2 \leq i \leq N-1$, we have

LEMMA 5.33.

$$(5.149) \quad \begin{aligned} D_h^2 P_{i \rightarrow N}^\theta(x_i) &= P_{i \rightarrow N}^{\theta''}(\xi) \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq Ch^3|y_N^\theta - x_i|^{1-\alpha} + C(r-1)(h^3|y_N^\theta - x_i|^{-\alpha} + h^2|y_N^\theta - x_i|^{1-\alpha}) \end{aligned}$$

And

LEMMA 5.34.

$$\begin{aligned} 421 \quad (5.150) \quad & \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_{i+1})u'''(\eta_{N+1}^\theta) - Q_{i \rightarrow N}^\theta(x_i)u'''(\eta_N^\theta)}{h_{i+1}} \right) \\ & \leq Ch^3|y_N^\theta - x_i|^{1-\alpha} \end{aligned}$$

422 And immediately, For $N/2 \leq i \leq N-2$

$$\begin{aligned} 423 \quad (5.151) \quad & V_{iN} \leq C \int_{x_{N-1}}^{x_N} h^2|y - x_i|^{1-\alpha} + C(r-1)h^2|y - x_i|^{-\alpha} + h|y - x_i|^{1-\alpha} dy \\ & \leq Ch^2h_N|T - x_i|^{1-\alpha} + C(r-1)h^2|x_{N-1} - x_i|^{1-\alpha} + Chh_N|T - x_i|^{1-\alpha} \\ & \leq Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned}$$

424 But expecially, when $i = N-1$,

$$\begin{aligned} 425 \quad (5.152) \quad & V_{N-1,N} = \int_0^1 -\frac{\theta^{2-\alpha}(1-\theta)}{2} \frac{2}{h_{N-1} + h_N} \left(\frac{1}{h_{N-1}} h_{N-1}^{4-\alpha} u''(y_{N-1}^\theta) - \left(\frac{1}{h_{N-1}} + \frac{1}{h_N} \right) h_N^{4-\alpha} u''(y_N^\theta) + \frac{1}{h_N} h_{N+1}^{4-\alpha} u''(y_{N+1}^\theta) \right) d\theta \\ & + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_{i+1})u'''(\eta_{N+1,1}^\theta) - Q_{i \rightarrow N}^\theta(x_i)u'''(\eta_{N,1}^\theta)}{h_{i+1}} \right) d\theta \\ & - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_i)u'''(\eta_{N,1}^\theta) - Q_{i \rightarrow N}^\theta(x_{i-1})u'''(\eta_{N-1,1}^\theta)}{h_i} \right) d\theta \\ & - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_{i+1})u'''(\eta_{N+1,2}^\theta) - Q_{i \rightarrow N}^\theta(x_i)u'''(\eta_{N,2}^\theta)}{h_{i+1}} \right) d\theta \\ & + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i \rightarrow N}^\theta(x_i)u'''(\eta_{N,2}^\theta) - Q_{i \rightarrow N}^\theta(x_{i-1})u'''(\eta_{N-1,2}^\theta)}{h_i} \right) d\theta \end{aligned}$$

426 while combine Lemma 5.30

$$\begin{aligned} 427 \quad (5.153) \quad & \frac{2}{h_{N-1} + h_N} \left(\frac{1}{h_{N-1}} h_{N-1}^{4-\alpha} u''(y_{N-1}^\theta) - \left(\frac{1}{h_{N-1}} + \frac{1}{h_N} \right) h_N^{4-\alpha} u''(y_N^\theta) + \frac{1}{h_N} h_{N+1}^{4-\alpha} u''(y_{N+1}^\theta) \right) \\ & = D_h^2(h_{N-1 \rightarrow N}^{4-\alpha}(x_i)u''(y_{N-1 \rightarrow N}^\theta(x_i))) \\ & \leq Ch_N^{4-\alpha} + C(r-1)h_N^{3-\alpha} \leq Ch^{4-\alpha} + C(r-1)h^2|T - x_{N-1-1}|^{1-\alpha} \end{aligned}$$

428

429 Similarly with $j = N+1$. □

I_6, I_7 is easy. Similar with Lemma 5.22 and Lemma 5.6, we have

THEOREM 5.35. *There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For*
 $N/2 \leq i \leq N,$
(5.154)

$$I_6 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1, 2N - \lceil \frac{N}{2} \rceil + 1} + T_{i-1, 2N - \lceil \frac{N}{2} \rceil}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, 2N - \lceil \frac{N}{2} \rceil + 1} \right) \\ \leq Ch^2$$

Proof. In fact, let $l = 2N - \lceil \frac{N}{2} \rceil + 1$

$$(5.155) \quad \begin{aligned} & \frac{1}{h_i} (T_{i-1, l} + T_{i-1, l-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, l} \\ &= \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) + \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) + \left(\frac{1}{h_i} - \frac{1}{h_{i+1}} \right) T_{i, l} \end{aligned}$$

While, by Lemma A.2

$$(5.156) \quad \begin{aligned} \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) &= \int_{x_{l-1}}^{x_l} (u(y) - u_h(y)) \frac{|x_{i-1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_i \Gamma(2-\alpha)} dy \\ &\leq C \int_{x_{l-1}}^{x_l} h_l^2 u''(\eta) \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy \\ &\leq Ch_l^3 x_{l-1}^{\alpha/2-2} T^{-\alpha} \\ &\leq Ch_l^3 \end{aligned}$$

Thus,

$$(5.157) \quad \frac{2}{h_i + h_{i+1}} \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) \leq Ch_l^3$$

For

$$(5.158) \quad \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{h_i} d\theta$$

And Similar with Lemma 5.19, we can get

$$(5.159) \quad \frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{(h_i + h_{i+1}) h_i} \leq Ch_l^2 |y_l^\theta - x_i|^{1-\alpha}$$

So

$$(5.160) \quad \frac{2}{h_i + h_{i+1}} \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) \leq Ch^2$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$(5.161) \quad \begin{aligned} \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i, l} &\leq h_i^{-3} h^2 x_i^{1-2/r} h_l C h_l^2 x_{l-1}^{\alpha/2-2} |x_l - x_i|^{1-\alpha} \\ &\leq Ch^2 \end{aligned}$$

Summarizes, we have

$$(5.162) \quad I_6 \leq Ch^2$$

□

And

LEMMA 5.36. *There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i \leq N$,*

$$I_7 = \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

Proof. For $i \leq N, j \geq 2N - \lceil \frac{N}{2} \rceil + 2$, we have

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\ &\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2-2/r} |y - x_{i+1}^{-1-\alpha}| dy \\ &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy \end{aligned}$$

$$\begin{aligned} \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{(2-2^{-r})T}^{x_{2N-1}} (2T - y)^{\alpha/2-2/r} dy \\ &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r}T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\ &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r} h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r} h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned}$$

Now we can conclude a part of the theorem Theorem 5.3 at the beginning of this section.

By Lemma 5.8 Lemma 5.22 Lemma 5.23 Theorem 5.29 Theorem 5.28 Theorem 5.35 Lemma 5.36, we have there exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $N/2 \leq i < N$,

$$\begin{aligned} R_i &= \sum_{j=1}^7 I_j \\ &\leq C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned}$$

6. Proof of Theorem 4.2.

7. Experimental results.

8. Conclusions. Some conclusions here.

Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x)$ is twice differentiable continuous function on open set Ω , there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

$$(A.1) \quad D_h^2 g(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

$$(A.2) \quad \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right)$$

And if $g(x) \in C^4(\Omega)$, then

$$(A.3) \quad \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2))$$

where $\eta_1 \in [x_{i-1}, x_i]$, $\eta_2 \in [x_i, x_{i+1}]$.

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in [x_{i-1}, x_i]$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in [x_i, x_{i+1}]$$

Substitute them in the left side of (A.1), we have

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2)$$

Now, using intermediate value theorem, there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y)dy$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \frac{h_i^4}{4!} g''''(\eta_1)$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \frac{h_{i+1}^4}{4!} g''''(\eta_2)$$

where $\eta_1 \in [x_{i-1}, x_i]$, $\eta_2 \in [x_i, x_{i+1}]$. Expecially,

$$\begin{aligned} \frac{h_i^4}{4!} g''''(\eta_1) &= \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy \\ \frac{h_{i+1}^4}{4!} g''''(\eta_2) &= \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \end{aligned}$$

Substitute them to the left side of (A.3), we can get the result. \square

LEMMA A.2. If $y \in [x_{j-1}, x_j]$, denote $y = \theta x_{j-1} + (1 - \theta)x_j$, $\theta \in [0, 1]$,

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j]$$

(A.6)

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

where $\eta_1 \in [x_{j-1}, y_j^\theta]$, $\eta_2 \in [y_j^\theta, x_j]$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta)u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in [x_{j-1}, y_j^\theta]$, $\eta_2 \in [y_j^\theta, x_j]$. Thus \square

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

LEMMA A.3. For $x \in [x_{j-1}, x_j]$

$$\begin{aligned} |u(x) - u_h(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^x u'(y) dy - \frac{x - x_{j-1}}{h_j} \int_x^{x_j} u'(y) dy \right| \\ &\leq \int_{x_{j-1}}^{x_j} |u'(y)| dy \end{aligned}$$

If $x \in [0, x_1]$, with Corollary 2.4, we have

$$|u(x) - u_h(x)| \leq \int_0^{x_1} |u'(y)| dy \leq \int_0^{x_1} C y^{\alpha/2-1} dy \leq C \frac{2}{\alpha} x_1^{\alpha/2}$$

Similarly, if $x \in [x_{2N-1}, 1]$, we have

$$|u(x) - u_h(x)| \leq C \frac{2}{\alpha} (2T - x_{2N-1})^{\alpha/2} = C \frac{2}{\alpha} x_1^{\alpha/2}$$

Appendix B. Inequality.

LEMMA B.1.

$$h_i \leq rT^{1/r} h \begin{cases} x_i^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_{i-1})^{1-1/r}, & N < i \leq 2N-1 \end{cases}$$

$$h_i \geq rT^{1/r} h \begin{cases} x_{i-1}^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_i)^{1-1/r}, & N < i \leq 2N-1 \end{cases}$$

Proof. For $1 \leq i \leq N$,

$$\begin{aligned} h_i &= T \left(\left(\frac{i}{N} \right)^r - \left(\frac{i-1}{N} \right)^r \right) \\ &\leq rT \frac{1}{N} \left(\frac{i}{N} \right)^{r-1} = rT^{1/r} h x_i^{1-1/r} \end{aligned}$$

$$h_i \geq rT \frac{1}{N} \left(\frac{i-1}{N} \right)^{r-1} = rT^{1/r} h x_{i-1}^{1-1/r}$$

For $N < i \leq 2N$,

$$\begin{aligned} h_i &= T \left(\left(\frac{2N-i+1}{N} \right)^r - \left(\frac{2N-i}{N} \right)^r \right) \\ &\leq rT \frac{1}{N} \left(\frac{2N-i+1}{N} \right)^{r-1} = rT^{1/r} h (2T - x_{i-1})^{1-1/r} \end{aligned}$$

$$h_i \geq rT \frac{1}{N} \left(\frac{2N-i}{N} \right)^{r-1} = rT^{1/r} h (2T - x_i)^{1-1/r}$$

□

LEMMA B.2. *There is a constant $C = 2^{|r-2|}r(r-1)T^{2/r}$ such that for all $i \in \{1, 2, \dots, 2N-1\}$*

$$(B.3) \quad |h_{i+1} - h_i| \leq Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases}$$

Proof.

$$h_{i+1} - h_i = \begin{cases} T \left(\left(\frac{i+1}{N} \right)^r - 2 \left(\frac{i}{N} \right)^r + \left(\frac{i-1}{N} \right)^r \right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T \left(\left(\frac{2N-i-1}{N} \right)^r - 2 \left(\frac{2N-i}{N} \right)^r + \left(\frac{2N-i+1}{N} \right)^r \right), & N+1 \leq i \leq 2N-1 \end{cases}$$

For $i = 1$,

$$h_2 - h_1 = T(2^r - 2) \left(\frac{1}{N} \right)^r = (2^r - 2)T^{2/r}h^2x_1^{1-2/r}$$

For $2 \leq i \leq N-1$,

$$h_{i+1} - h_i = r(r-1)T N^{-2}\eta^{r-2}, \quad \eta \in \left[\frac{i-1}{N}, \frac{i+1}{N} \right]$$

If $r \in [1, 2]$,

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T N^{-2}\eta^{r-2} \leq r(r-1)T h^2 \left(\frac{i-1}{N} \right)^{r-2} \\ &\leq r(r-1)T h^2 2^{2-r} \left(\frac{i}{N} \right)^{r-2} \\ &= 2^{2-r}r(r-1)T^{2/r}h^2x_i^{1-2/r} \end{aligned}$$

else if $r > 2$,

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T N^{-2}\eta^{r-2} \leq r(r-1)T h^2 \left(\frac{i+1}{N} \right)^{r-2} \\ &\leq r(r-1)T h^2 2^{r-2} \left(\frac{i}{N} \right)^{r-2} \\ &= 2^{r-2}r(r-1)T^{2/r}h^2x_i^{1-2/r} \end{aligned}$$

Since

$$2^r - 2 \leq 2^{|r-2|}r(r-1), \quad r \geq 1$$

we have

$$h_{i+1} - h_i \leq 2^{|r-2|}r(r-1)T^{2/r}h^2x_i^{1-2/r}, \quad 1 \leq i \leq N-1$$

For $i = N$, $h_{N+1} - h_N = 0$. For $N < i \leq 2N-1$, it's central symmetric to the first half of the proof, which is

$$h_i - h_{i+1} \leq 2^{|r-2|}r(r-1)T^{2/r}h^2(2T - x_i)^{1-2/r}$$

Summarizes the inequalities, we can get

$$(B.4) \quad |h_{i+1} - h_i| \leq 2^{|r-2|} r(r-1) T^{2/r} h^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases} \quad \square$$

Appendix C. Proofs of some technical details.

Additional proof of Theorem 5.1. For $2 \leq i \leq N-1$,

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\ & \leq C \frac{2}{h_i + h_{i+1}} (h_i^3 x_{i-1}^{-2-\alpha/2} + h_{i+1}^3 x_i^{-2-\alpha/2}) \\ & \leq 2C (h_i^2 x_{i-1}^{-2-\alpha/2} + h_{i+1}^2 x_i^{-2-\alpha/2}) \end{aligned}$$

Since Lemma B.1, we have

$$\begin{aligned} h_i & \leq r T^{1/r} h x_i^{1-1/r}, \quad 1 \leq i \leq N \\ h_{i+1} & \leq r T^{1/r} h x_{i+1}^{1-1/r}, \quad 1 \leq i \leq N-1 \end{aligned}$$

and

$$\begin{aligned} x_{i-1}^{-2-\alpha/2} & \leq 2^{-r(-2-\alpha/2)} x_i^{-2-\alpha/2} \quad 2 \leq i \leq N-1 \\ x_{i+1}^{1-1/r} & \leq 2^{r-1} x_i^{1-1/r} \quad 1 \leq i \leq N-1 \end{aligned}$$

So there is a constant $C = C(T, \alpha, r, \|f\|_{\beta}^{\alpha/2})$ such that

$$\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq C h^2 x_i^{-\alpha/2-2/r}, \quad 2 \leq i \leq N-1$$

For $i = 1$, by (A.4)

$$\begin{aligned} & \frac{1}{4!} \frac{2}{h_1 + h_2} (h_1^3 f''(\eta_1) + h_2^3 f''(\eta_2)) \\ & = \frac{2}{h_1 + h_2} \left(\frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy + \frac{1}{4!} h_2^3 f''(\eta_2) \right) \end{aligned}$$

We have proved above that

$$\frac{2}{h_1 + h_2} h_2^3 f''(\eta_2) \leq C h^2 x_1^{-\alpha/2-2/r}$$

and we can get

$$\begin{aligned} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy & \leq C \frac{1}{3!} \int_0^{x_1} y^{1-\alpha/2} dy \\ & = C \frac{1}{3!(2-\alpha/2)} x_1^{2-\alpha/2} \end{aligned}$$

so

$$\frac{2}{h_1 + h_2} \frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy = \frac{C 2^{1-r}}{3!(2-\alpha/2)} x_1^{-\alpha/2} = \frac{C 2^{1-r}}{3!(2-\alpha/2)} T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

And for $i = N$, we have

$$\begin{aligned} & \frac{2}{h_N + h_{N+1}} (h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2)) \\ &= h_N^2 (f''(\eta_1) + f''(\eta_2)) \\ &\leq r^2 T^{2/r} h^2 x_N^{2-2/r} 2C x_{N-1}^{-2-\alpha/2} \\ &\leq 2r^2 T^{2/r} C 2^{-r(-2-\alpha/2)} h^2 x_N^{-\alpha/2-2/r} \end{aligned}$$

Finally, $N + 1 \leq i \leq 2N - 1$ is symmetric to the first half of the proof, so we can conclude that \square

$$\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N - 1 \end{cases}$$

LEMMA C.1. *There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ for $2 \leq j \leq N$, if $y \in [x_{j-1}, x_j]$,*

$$(C.1) \quad |u(y) - u_h(y)| \leq Ch^2 y^{\alpha/2-2/r}$$

Proof. For $2 \leq j \leq N$, we have

$$x_j \leq 2^r y, \quad x_{j-1} \geq 2^{-r} y$$

And by Lemma A.2, Lemma B.1 and Corollary 2.4, we have

$$\begin{aligned} u(y) - u_h(y) &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \\ &\leq \frac{\|u\|_{\beta+\alpha}^{(-\alpha/2)}}{4} r^2 T^{2/r} h^2 x_j^{2-2/r} x_{j-1}^{\alpha/2-2} \\ &\leq Ch^2 2^{2r-2} y^{2-2/r} 2^{-r(\alpha/2-2)} y^{\alpha/2-2} \\ &= C 2^{-r\alpha/2+4r-2} h^2 y^{\alpha/2-2/r} \end{aligned}$$

symmetricly, for $N < j \leq 2N - 1$, we have

$$(C.2) \quad |u(y) - u_h(y)| \leq Ch^2 (2T - y)^{\alpha/2-2/r} \quad \square$$

LEMMA C.2. *There is a constant $C = C(\alpha, r)$ such that for all $1 \leq i < N/2$, $\max\{2i + 1, i + 3\} \leq j \leq 2N$ and $y \in [x_{j-1}, x_j]$, we have*

$$(C.3) \quad D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) \leq C \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}$$

Proof. Since $y \geq x_{j-1} > x_{i+1}$, by Lemma A.1, if $j - 1 > i + 1$ \square

$$\begin{aligned} D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) &= \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq \frac{(y - x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\leq \left(1 - \left(\frac{2}{3}\right)^r\right)^{-1-\alpha} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

LEMMA C.3. *There is a constant $C = C(\alpha, r)$ such that for all $3 \leq i < N/2, k = \lceil \frac{i}{2} \rceil, 1 \leq j \leq k-1$ and $y \in [x_{j-1}, x_j]$, we have*

$$(C.4) \quad D_h^2\left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) \leq C \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)}$$

Proof. Since $y \leq x_j < x_{i-1}$, by Lemma A.1, □

$$\begin{aligned} D_h^2\left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) &= \frac{|\xi - y|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq \frac{(x_{i-1} - x_j)^{-1-\alpha}}{\Gamma(-\alpha)} \leq \frac{(x_{i-1} - x_{k-1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\leq \left(\left(\frac{2}{3}\right)^r - \left(\frac{1}{2}\right)^r\right)^{-1-\alpha} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

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LEMMA C.4. *While $0 \leq i < N/2$, By Lemma A.3*

$$\begin{aligned} |T_{i1}| &\leq C \int_0^{x_1} x_1^{\alpha/2} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ (C.5) \quad &= C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2} |x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha}| \\ &\leq C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2+2-\alpha} = C \frac{1}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1 \end{aligned}$$

For $2 \leq j \leq N$, by Lemma A.2 and Corollary 2.4

$$\begin{aligned} |T_{ij}| &\leq \frac{C}{4} \int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ (C.6) \quad &\leq \frac{C}{4\Gamma(3-\alpha)} h_j^2 x_{j-1}^{\alpha/2-2} ||x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha}| \end{aligned}$$

LEMMA C.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$(C.7) \quad \sum_{j=1}^3 S_{1j} \leq Ch^2 x_1^{-\alpha/2-2/r}$$

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$$(C.8) \quad \sum_{j=1}^4 S_{2j} \leq Ch^2 x_2^{-\alpha/2-2/r}$$

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Proof.

$$S_{1j} = \frac{2}{x_2} \left(\frac{1}{x_1} T_{0j} - \left(\frac{1}{x_1} + \frac{1}{h_2} \right) T_{1j} + \frac{1}{h_2} T_{2j} \right)$$

So, by Lemma C.4

$$S_{11} \leq \frac{2}{x_2 x_1} 4 \frac{C}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \leq C x_1^{-\alpha/2}$$

$$S_{12} \leq \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_2^2 x_1^{\alpha/2-2} (x_2^{2-\alpha} + 2h_2^{2-\alpha} + h_2^{2-\alpha}) \leq C x_1^{-\alpha/2}$$

$$S_{13} \leq \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_3^2 x_2^{\alpha/2-2} (x_3^{2-\alpha} + 2h_3^{2-\alpha} + h_3^{2-\alpha}) \leq C x_1^{-\alpha/2}$$

But

$$x_1^{-\alpha/2} = T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

For $i = 2$, Sorry □

LEMMA C.6. *There exists a constant $C = C(T, r, l)$ such that For $3 \leq i \leq N - 1$, $k + 1 = \lceil \frac{i}{2} \rceil$, $k \leq j \leq \min\{2i - 1, N - 1\}$, $l = 3, 4$, when $\xi \in [x_{i-1}, x_{i+1}]$,*

$$(C.9) \quad (h_{j-i}^3(\xi))' \leq (r-1) C h^2 x_i^{1-2/r} h_j$$

$$(C.10) \quad (h_{j-i}^4(\xi))' \leq (r-1) C h^2 x_i^{1-2/r} h_j^2$$

Proof. From (5.32)

$$(C.11) \quad y'_{j-i}(x) = y_{j-i}^{1-1/r}(x) x^{1/r-1}$$

$$(C.12) \quad y''_{j-i}(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i}$$

for $l = 3, 4$, by (5.34)

$$(C.13) \quad \begin{aligned} (h_{j-i}^l(\xi))' &= l h_{j-i}^{l-1}(\xi) (y'_{j-i}(\xi) - y'_{j-i-1}(\xi)) \\ &= l h_{j-i}^{l-1}(\xi) \xi^{1/r-1} (y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \geq 0 \end{aligned}$$

For $\xi \in [x_{i-1}, x_{i+1}]$ and $2 \leq k \leq j \leq \min\{2i - 1, N - 1\}$, using Lemma B.1

$$\begin{aligned} h_{j-i}(\xi) &\leq h_{j-i}(x_{i+1}) = h_{j+1} \\ &\leq r T^{1/r} h x_{j+1}^{1-1/r} \leq r T^{1/r} 2^{r-1} h x_i^{1-1/r} \end{aligned}$$

And

$$(C.14) \quad 2^{-r} x_i \leq x_{i-1} \leq \xi \leq x_{i+1} \leq 2^r x_i$$

We have

$$(C.15) \quad \xi^{1/r-m} \leq 2^{\lfloor mr-1 \rfloor} x_i^{1/r-m}, \quad m = 1, 2$$

but

$$(C.16) \quad \begin{aligned} y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) &= (\xi^{1/r} + Z_{j-i})^{r-1} - (\xi^{1/r} + Z_{j-i-1})^{r-1} \\ &= (r-1) Z_1 (\xi^{1/r} + Z_{j-i-\gamma})^{r-2}, \quad \gamma \in [0, 1] \\ &= (r-1) T^{1/r} h y_{j-i-\gamma}^{1-2/r}(\xi) \end{aligned}$$

And

(C.17)

$$4^{-r}x_i \leq x_{\lceil \frac{i}{2} \rceil - 1} \leq x_{j-2} = y_{j-i-1}(x_{i-1}) \leq y_{j-i-\gamma}(\xi) \leq y_{j-i}(x_{i+1}) = x_{j+1} \leq x_{2i} \leq 2^r x_i$$

Therefore,

$$(C.18) \quad y_{j-i-\gamma}^{1-2/r}(\xi) \leq 2^{2|r-2|} x_i^{1-2/r}$$

So we can get

$$(C.19) \quad y'_{j-i}(\xi) - y'_{j-i-1}(\xi) \leq (r-1)C(T, r) h x_i^{-1/r}$$

We get

$$(C.20) \quad (h_{j-i}^l(\xi))' \leq l(r-1)C h_{j+1}^{l-1} h x_i^{-1/r}$$

And by Lemma B.1,

$$(C.21) \quad h_{j+1} \leq rTh \left(\frac{j+1}{N} \right)^{r-1} \leq rTh 2^{r-1} \left(\frac{j-1}{N} \right) = 2^{r-1} h_j$$

$$(C.22) \quad h_{j+1} \leq rT^{1/r} h x_{j+1}^{1-1/r} \leq rT^{1/r} h x_{2i}^{1-1/r} \leq rT^{1/r} 2^{r-1} h x_i^{1-1/r}$$

We can get

$$(C.23) \quad \begin{aligned} (h_{j-i}^l(\xi))' &\leq l(r-1)C h_j^{l-2} h_{j+1} h x_i^{-1/r} \\ &\leq l(r-1)C h h_j^{l-2} (h x_i^{1-1/r}) x_i^{-1/r} \\ &= (r-1)C h^2 x_i^{1-2/r} h_j^{l-2} \end{aligned}$$

Meanwhile, we can get

$$(C.24) \quad h_{j-i}^3(\xi) \leq h_{j+1}^3 \leq C h^2 x_i^{2-2/r} h_j$$

$$(C.25) \quad h_{j-i}^4(\xi) \leq h_{j+1}^4 \leq C h^2 x_i^{2-2/r} h_j^2 \quad \square$$

LEMMA C.7. *There exists a constant $C = C(T, r, l)$ such that For $3 \leq i \leq N - 1$, $\lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i - 1, N - 1\}$, when $\xi \in [x_{i-1}, x_{i+1}]$,*

$$(C.26) \quad (h_{j-i}^3(\xi))'' \leq C(r-1) h^2 x_i^{-2/r} h_j$$

Proof. From (C.11)

$$(C.27) \quad \begin{aligned} (h_{j-i}^3(\xi))'' &= 6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 + 3h_{j-i}^2(\xi)(y''_{j-i}(\xi) - y''_{j-i-1}(\xi)) \\ &= 6h_{j-i}(\xi)\xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \\ &\quad + 3\frac{1-r}{r}h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \end{aligned}$$

Using the inequalities of the proof of Lemma C.6

$$\begin{aligned} & 6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 \\ (C.28) \quad & \leq 6h_{j+1}((r-1)Chx_i^{-1/r})^2 \\ & \leq C(r-1)^2 h^2 x_i^{-2/r} h_j \end{aligned}$$

For the second partial

$$\begin{aligned} & h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\ (C.29) \quad & \leq Ch_{j+1}^2 x_i^{1/r-2}((y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi))Z_{j-i} + y_{j-i-1}^{1-2/r}(\xi)Z_1) \end{aligned}$$

but

$$\begin{aligned} & y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi) = (\xi^{1/r} + Z_{j-i})^{r-2} - (\xi^{1/r} + Z_{j-i-1})^{r-2} \\ & = (r-2)Z_1(\xi^{1/r} + Z_{j-i-\gamma})^{r-3} \\ (C.30) \quad & = (r-2)T^{-r}hy_{j-i-\gamma}^{1-3/r}(\xi) \\ & \leq C(r-2)hx_i^{1-3/r} \end{aligned}$$

So we can get

$$\begin{aligned} & h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\ & \leq Ch_j hx_i^{1-1/r} x_i^{1/r-2} (C(r-2)hx_i^{1-3/r}Z_{j-i} + Cx_i^{1-2/r}T^{1/r}h) \\ (C.31) \quad & \leq Ch^2((r-2)x_i^{-3/r}x_{|j-i|}^{1/r} + x_i^{-2/r})h_j \\ & \leq Ch^2 x_i^{-2/r} h_j \end{aligned}$$

Summarizes, we have

$$(C.32) \quad (h_{j-i}^3(\xi))'' \leq C(r-1)h^2 x_i^{-2/r} h_j \quad \square$$

proof of Lemma 5.16. From (5.32)

$$(C.33) \quad y'_{j-i}(x) = y_{j-i}^{1-1/r}(x)x^{1/r-1}$$

$$(C.34) \quad y''_{j-i}(x) = \frac{1-r}{r}y_{j-i}^{1-2/r}(x)x^{1/r-2}Z_{j-i}$$

Since

$$x_{j-2} \leq y_{j-i-1}(x_{i-1}) \leq y_{j-i}^\theta(\xi) \leq y_{j-i-1}^\theta(x_{i+1}) \leq x_{j+1}$$

We have known (C.17)

$$(C.35) \quad u''(y_{j-i}^\theta(\xi)) \leq C(y_{j-i}^\theta(\xi))^{\alpha/2-2} \leq Cx_{j-2}^{\alpha/2-2} \leq Cx_{\lfloor \frac{i}{2} \rfloor -1}^{\alpha/2-2} \leq C4^{r(2-\alpha/2)}x_i^{\alpha/2-2}$$

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$$\begin{aligned} & (u''(y_{j-i}^\theta(\xi)))' = u'''(y_{j-i}^\theta(\xi))y_{j-i}^{\theta'}(\xi) \\ (C.36) \quad & \leq Cx_i^{\alpha/2-3}\xi^{1/r-1}y_{j-i}^{1-1/r}(\xi) \\ & \leq Cx_i^{\alpha/2-3}x_i^{1/r-1}x_i^{1-1/r} = Cx_i^{\alpha/2-3} \end{aligned}$$

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$$\begin{aligned}
& (u''(y_{j-i}^\theta(\xi)))'' = u''''(y_{j-i}^\theta(\xi))(y_{j-i}^{\theta''}(\xi))^2 + u'''(y_{j-i}^\theta(\xi))y_{j-i}^{\theta'''}(\xi) \\
& \leq Cx_i^{\alpha/2-4} + Cx_i^{\alpha/2-3}\frac{r-1}{r}x_i^{1-2/r}x_i^{1/r-2}Z_{|j-i|+1} \\
& \leq Cx_i^{\alpha/2-4} + C\frac{r-1}{r}x_i^{\alpha/2-3}x_i^{-1/r}x_i^{1/r} \\
& = Cx_i^{\alpha/2-4}
\end{aligned}
\tag{C.37}$$

□

Proof of Lemma 5.17.

$$\begin{aligned}
& |y_{j-i}^\theta(\xi) - \xi| = |\theta(y_{j-i-1}(\xi) - \xi) + (1-\theta)(y_{j-i}(\xi) - \xi)| \\
& = \theta|y_{j-i-1}(\xi) - \xi| + (1-\theta)|y_{j-i}(\xi) - \xi|
\end{aligned}
\tag{C.38}$$

Since $|y_{j-i}(\xi) - \xi|$ is increasing about ξ , we have

$$\left(\frac{i-1}{i}\right)^r |x_j - x_i| \leq |x_{j-1} - x_{i-1}| \leq |y_{j-i}(\xi) - \xi| \leq |x_{j+1} - x_{i+1}| \leq \left(\frac{i+1}{i}\right)^r |x_j - x_i|
\tag{C.39}$$

Thus,

$$\left(\frac{2}{3}\right)^r |y_j^\theta - x_i| \leq |y_{j-i}^\theta(\xi) - \xi| \leq \left(\frac{3}{4}\right)^r (\theta|x_j - x_i| + (1-\theta)|x_{j-1} - x_i|) = \left(\frac{3}{4}\right)^r |y_j^\theta - x_i|
\tag{C.40}$$

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$$|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}
\tag{C.41}$$

Next,

$$\begin{aligned}
& (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' = (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi)) - 1 \\
& \leq C|y_j^\theta - x_i|^{-\alpha}\xi^{1/r-1}|\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}|
\end{aligned}
\tag{C.42}$$

Similar with (C.40), we have

$$\begin{aligned}
& |y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| \leq C|x_j^{1-1/r} - x_i^{1-1/r}| \\
& \leq C|x_j - x_i|x_i^{-1/r}
\end{aligned}
\tag{C.43}$$

So we can get

$$\begin{aligned}
& |\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| \\
& \leq Cx_i^{-1/r}(\theta|x_{j-1} - x_i| + (1-\theta)|x_j - x_i|) \\
& = Cx_i^{-1/r}|y_j^\theta - x_i|
\end{aligned}
\tag{C.44}$$

Combine them, we get

$$\begin{aligned}
& (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C|y_j^\theta - x_i|^{-\alpha}x_i^{1/r-1}x_i^{-1/r}|y_j^\theta - x_i| \\
& = C|y_j^\theta - x_i|^{1-\alpha}x_i^{-1}
\end{aligned}
\tag{C.45}$$

Finally, we have

$$\begin{aligned}
& (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' = \alpha(\alpha-1)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi)) - 1)^2 \\
& + (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}\frac{1-r}{r}\xi^{1/r-2}|\theta y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1} + (1-\theta)y_{j-i}^{1-2/r}(\xi)Z_{j-i}|
\end{aligned}
\tag{C.46}$$

Using the inequalities above ,we have

$$\begin{aligned}
 & |y_{j-i}^\theta(\xi) - \xi|^{-\alpha-1} (\xi^{1/r-1} (\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta) y_{j-i}^{1-1/r}(\xi)) - 1)^2 \\
 & \leq C |y_j^\theta - x_i|^{-\alpha-1} (x_i^{-1} |y_j^\theta - x_i|)^2 \\
 & = C |y_j^\theta - x_i|^{1-\alpha} x_i^{-2}
 \end{aligned}
 \tag{C.47}$$

And by

$$|Z_{j-i}| = |x_j^{1/r} - x_i^{1/r}| \leq |x_j - x_i| x_i^{1/r-1}$$

we have

$$\begin{aligned}
 & |y_{j-i}^\theta(\xi) - \xi|^{-\alpha} \xi^{1/r-2} |\theta y_{j-i-1}^{1-2/r}(\xi) Z_{j-i-1} + (1-\theta) y_{j-i}^{1-2/r}(\xi) Z_{j-i}| \\
 & \leq C |y_j^\theta - x_i|^{-\alpha} x_i^{1/r-2} x_i^{1-2/r} |\theta Z_{j-i-1} + (1-\theta) Z_{j-i}| \\
 & \leq C |y_j^\theta - x_i|^{-\alpha} x_i^{-2} |y_j^\theta - x_i| \\
 & = C |y_j^\theta - x_i|^{1-\alpha} x_i^{-2}
 \end{aligned}
 \tag{C.49}$$

proof of Lemma 5.19. For $k \leq j < \min\{2i-1, N-1\}$

$$\begin{aligned}
 & \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_j^\theta)}{h_{i+1}} \\
 & \frac{Q_{j-i}^\theta(x_{i+1}) - Q_{j-i}^\theta(x_i)}{h_{i+1}} u'''(\eta_{j+1}^\theta) + Q_{j-i}^\theta(x_i) \frac{u'''(\eta_{j+1}^\theta) - u'''(\eta_j^\theta)}{h_{i+1}} \\
 & \leq Q_{j-i}^{\theta'}(\xi) C x_j^{\alpha/2-3} + Q_{j-i}^\theta(x_i) C u''''(\eta) \frac{h_i + h_{i+1}}{h_{i+1}}
 \end{aligned}
 \tag{C.50}$$

where $\xi \in [x_i, x_{i+1}]$, $\eta \in [x_{j-1}, x_{j+1}]$.

From (5.36), by Lemma C.6 and Lemma 5.17, we have

$$\begin{aligned}
 Q_{j-i}^{\theta'}(\xi) & \leq C h^2 \frac{|y_{j+1}^\theta - x_{i+1}|^{1-\alpha}}{\Gamma(2-\alpha)} x_{i+1}^{1-2/r} h_{j+1}^2 \\
 & \leq C h^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2
 \end{aligned}
 \tag{C.51}$$

And by defination

$$Q_{j-i}^\theta(x_i) = h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \leq C h^2 x_i^{2-2/r} \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2$$

With , we have

$$4^{-r} x_i \leq x_{k-1} \leq x_{j-1} < x_j \leq x_{2i-1} \leq 2^r x_i$$

So we have

$$\begin{aligned}
 & \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_j^\theta)}{h_{i+1}} \\
 & \leq C h^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 x_i^{\alpha/2-3} + C h^2 x_i^{2-2/r} \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 x_{j-1}^{\alpha/2-4} \\
 & = C h^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j^2
 \end{aligned}
 \tag{C.54}$$

709 while

$$710 \quad h_j \leq h_{2i-1} \leq 2^r h_i$$

711 Subsitute into the inequality above, we get the goal

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ 712 \quad (C.55) \quad & \leq \frac{1}{h_i} Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j 2^r h_i \\ & = Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned}$$

713 While, the later is similar. □

714

715 **LEMMA C.8.** *There exists a constant $C = C(T, r)$ such that For $N/2 \leq i < N$,*
 716 *$N + 2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$, $l = 3, 4$, $\xi \in [x_{i-1}, x_{i+1}]$, we have*

$$717 \quad (C.56) \quad h_{j-i}^l(\xi) \leq Ch_j^l \leq Ch^2 h_j^{l-2}$$

$$718 \quad (C.57) \quad (h_{j-i-1}^l(\xi))' \leq C(r-1)h^2 h_j^{l-2}$$

$$719 \quad (C.58) \quad (h_{j-i}^3(\xi))'' \leq C(r-1)h^2 h_j$$

Proof.

$$\begin{aligned} 720 \quad (C.59) \quad & (h_{j-i}(\xi))' = y_{j-i}'(\xi) - y_{j-i-1}'(\xi) \\ & = \xi^{1/r-1}((2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}) \leq 0 \end{aligned}$$

721 Thus,

$$722 \quad (C.60) \quad Ch_j \leq h_{j+1} \leq h_{j-i}(\xi) \leq h_{j-i}(x_{i-1}) = h_{j-1} \leq Ch_j$$

723 So as $4^{-r}T \leq 2T - x_j \leq T$, $2^{-r}T \leq x_i \leq T$, we have

$$724 \quad (C.61) \quad h_{j-i}^l(\xi) \leq Ch_j^l \leq Ch^2(2T - x_j)^{2-2/r} h_j^{l-2} \leq Ch^2 h_j^{l-2}$$

725 Since

$$\begin{aligned} & |(2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}| \\ 726 \quad (C.62) \quad & = |(Z_{2N-(j-i)} - \xi^{1/r})^{r-1} - (Z_{2N-(j-i-1)} - \xi^{1/r})^{r-1}| \\ & = (r-1)Z_1(Z_{2N-(j-i-\gamma)} - \xi^{1/r})^{r-2} \quad \gamma \in [0, 1] \\ & \leq C(r-1)h(2T - x_j)^{1-2/r} \end{aligned}$$

727 we have

$$728 \quad (C.63) \quad |(h_{j-i}(\xi))'| \leq C(r-1)h(2T - x_j)^{1-2/r} x_i^{1/r-1}$$

729 And

$$\begin{aligned} & (h_{j-i}^l(\xi))' = lh_{j-i}^{l-1}(\xi)h_{j-i}'(\xi) \\ 730 \quad (C.64) \quad & \leq C(r-1)h_j^{l-1} h(2T - x_j)^{1-2/r} x_i^{1/r-1} \\ & \leq C(r-1)h^2 h_j^{l-2} (2T - x_j)^{2-3/r} x_i^{1-1/r} \\ & \leq C(r-1)h^2 h_j^{l-2} \end{aligned}$$

$$\begin{aligned}
(C.65) \quad & (h_{j-i}^3(\xi))'' = 6h_{j-i}(\xi)(y_{j-i}'(\xi) - y_{j-i-1}'(\xi))^2 + 3h_{j-i}^2(\xi)(y_{j-i}''(\xi) - y_{j-i-1}''(\xi)) \\
& \leq C(r-1)h_j h^2 + Ch_j^2 \frac{1-r}{r} \xi^{1/r-2} ((2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-1-i)}) \\
& \leq C(r-1)h_j h^2 + C(r-1)h_j^2 (C(r-2)h(2T - x_j)^{1-3/r} Z_{2N-(j-i)} + Z_1(2T - x_{j-1})^{1-2/r}) \\
& \leq C(r-1)h_j h^2 + C(r-1)h_j^2 h = Ch^2 h_j
\end{aligned}$$

LEMMA C.9. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i < N$, $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in [x_{i-1}, x_{i+1}]$, we have*

$$\begin{aligned}
(C.66) \quad & u''(y_{j-i}^\theta(\xi)) \leq C \\
(C.67) \quad & (u''(y_{j-i}^\theta(\xi)))' \leq C \\
(C.68) \quad & (u''(y_{j-i}^\theta(\xi)))'' \leq C
\end{aligned}$$

Proof.

$$(C.69) \quad x_{j-2} \leq y_{j-i}^\theta(\xi) \leq x_{j+1} \Rightarrow 4^{-r}T \leq 2T - y_{j-i}^\theta(\xi) \leq T$$

Thus, for $l = 2, 3, 4$,

$$(C.70) \quad u^{(l)}(y_{j-i}^\theta(\xi)) \leq C(2T - y_{j-i}^\theta(\xi))^{\alpha/2-l} \leq C$$

and

$$\begin{aligned}
(C.71) \quad & (y_{j-i}^\theta(\xi))' = \theta y_{j-1-i}'(\xi) + (1-\theta)y_{j-i-1}'(\xi) \\
& = \xi^{1/r-1}(\theta(2T - y_{j-1-i}(\xi))^{1-1/r} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-1/r}) \\
& \leq C(2T - x_{j-2})^{1-1/r} \leq C
\end{aligned}$$

With

$$(C.72) \quad Z_{2N-j-i} \leq 2T^{1/r}$$

$$\begin{aligned}
(C.73) \quad & (y_{j-i}^\theta(\xi))'' = \theta y_{j-1-i}''(\xi) + (1-\theta)y_{j-i-1}''(\xi) \\
& = \frac{1-r}{r} \xi^{1/r-2} (\theta(2T - y_{j-1-i}(\xi))^{1-2/r} Z_{2N-(j-i-1)} + (1-\theta)(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)}) \\
& \leq C(r-1)
\end{aligned}$$

Therefore,

$$\begin{aligned}
(C.74) \quad & (u''(y_{j-i}^\theta(\xi)))' = u'''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta(\xi))' \\
& \leq C
\end{aligned}$$

$$\begin{aligned}
(C.75) \quad & (u''(y_{j-i}^\theta(\xi)))'' = u'''(y_{j-i}^\theta(\xi))(y_{j-i}^{\theta'}(\xi))^2 + u''''(y_{j-i}^\theta(\xi))y_{j-i}^{\theta''}(\xi) \\
& \leq C + C(r-1) = C
\end{aligned}$$

LEMMA C.10. *There exists a constant $C = C(T, \alpha, r)$ such that*

$$(C.76) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}$$

$$(C.77) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C|y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N)$$

$$(C.78)$$

$$(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C(r-1)|y_j^\theta - x_i|^{-\alpha} + C|y_j^\theta - x_i|^{-1-\alpha}(|2T - x_i - y_j^\theta| + h_N)^2$$

Proof.

$$(C.79) \quad (y_{j-i}^\theta(\xi) - \xi)' = \theta y_{j-1-i}'(\xi) + (1-\theta)y_{j-i}'(\xi) - 1$$

$$(C.80) \quad |y_{j-i}'(\xi) - 1| = \xi^{1/r-1} |(2T - y_{j-i}(\xi))^{1-1/r} - \xi^{1-1/r}| \\ \leq \xi^{1/r-1} |2T - \xi - y_{j-i}(\xi)| \xi^{-1/r}$$

$$(C.81) \quad |2T - \xi - y_{j-i}(\xi)| \leq \max \begin{cases} |2T - x_{i-1} - x_{j-1}| \\ |2T - x_{i+1} - x_{j+1}| \end{cases} \\ \leq |2T - x_i - x_j| + h_{i+1} + h_j$$

$$(C.82) \quad (y_{j-i}^\theta(\xi) - \xi)'' = \theta y_{j-1-i}''(\xi) + (1-\theta)y_{j-i}''(\xi)$$

$$= \frac{1-r}{r} \xi^{1/r-2} (\theta(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)}) \leq 0$$

It's concave, so

$$(C.83) \quad y_{j-i}(\xi) - \xi \geq \min\{x_{j+1} - x_{i+1}, x_{j-1} - x_{i-1}\} \geq C(x_j - x_i)$$

We have

$$(C.84) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}$$

$$(C.85) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' = (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}(y_{j-i}^\theta(\xi) - \xi)' \\ \leq C|y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_{i+1} + h_{j-1})$$

$$(C.86)$$

$$(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' = (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}(y_{j-i}^\theta(\xi) - \xi)'' + \alpha(\alpha-1)|y_{j-i}^\theta(\xi) - \xi|^{-1-\alpha}(y_{j-i}^\theta(\xi) - \xi)'(y_{j-i}^\theta(\xi) - \xi)' \\ \leq C(r-1)|y_j^\theta - x_i|^{-\alpha} + C|y_j^\theta - x_i|^{-1-\alpha}(|2T - x_i - y_j^\theta| + h_{i+1} + h_{j-1})^2$$

Proof. From (5.24), by Lemma C.8 and Lemma C.10, we have $\xi \in [x_i, x_{i+1}]$

$$(C.87) \quad Q_{j-i}^\theta(\xi) \leq Ch^2 h_j^2 ((r-1)|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N))$$

$$(C.88) \quad Q_{j-i}^\theta(\xi) \leq Ch^2 h_j^2 |y_j^\theta - x_i|^{1-\alpha}$$

So use the skill in Proof 29 with Lemma C.9

$$(C.89) \quad \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ \leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N))$$

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