A SECOND ORDER NUMERICAL METHODS FOR REISZ-FRACTIONAL ELLIPTIC EQUATION ON GRADED MESH*

JIANXING HAN† AND MINGHUA CHEN‡

Abstract. This is an example SIAM LaTeX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

Key words. example, LaTeX

MSC codes. ??????????????????

1. Introduction.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$,

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

2. Preliminaries: Numeric scheme and main results.

2.1. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases} \tag{2.1}$$

where $r \ge 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N \tag{2.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

And then, define the piecewise linear interpolant of the true solution u to be

$$\Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{2.4}$$

*Submitted to the editors DATE.

†School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China (hanjx2023@mail.lzu.edu.cn).

‡School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China (chen@mail.lzu.edu.cn).

For convience, we denote

$$I^{2-\alpha}u(x) := \frac{1}{\Gamma(2-\alpha)}\int_\Omega |x-y|^{1-\alpha}u(y)dy \tag{2.5}$$

and

$$D_h^2 u(x_i) := \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}u(x_{i-1}) - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)u(x_i) + \frac{1}{h_{i+1}}u(x_{i+1})\right) \tag{2.6}$$

Now, we discretise (1.1) by replacing $u(x)$ by a continuous piecewise linear function

$$u_h(x) := \sum_{j=1}^{2N-1} u_j\phi_j(x) \tag{2.7}$$

whose nodal values u_j are to be determined by collocation at each mesh point x_i for $i = 1, 2, ..., 2N-1$:

$$-\kappa_\alpha D_h^\alpha u_h(x_i) := -\kappa_\alpha D_h^2 I^{2-\alpha}u_h(x_i) = f(x_i) =: f_i \tag{2.8}$$

Here,

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i)\; u_j = \sum_{j=1}^{2N-1} a_{ij}\; u_j \tag{2.9}$$

where

$$a_{ij} = -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i) \quad \text{for} \quad i,j = 1,2,...,2N-1 \tag{2.10}$$

We have replaced $(-\Delta)^{\alpha/2}u(x_i) = f(x_i)$ in (1.1) by $-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i)$ in (2.8), with truncation error

$$\tau_i := -\kappa_\alpha\left(D_h^\alpha \Pi_h u(x_i) - \frac{d^2}{dx^2}I^{2-\alpha}u(x_i)\right) \quad \text{for} \quad i = 1,2,...,2N-1 \tag{2.11}$$

where $-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^\alpha \phi_j(x_i)u(x_j) = \sum_{j=1}^{2N-1} a_{ij}u(x_j)$.

The discrete equation (2.8) can be written in matrix form

$$AU = F \tag{2.12}$$

where $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$, $U = (u_1, \cdots, u_{2N-1})^T$ is unknown and $F = (f_1, \cdots, f_{2N-1})^T$.

We can deduce a_{ij},

$$\begin{aligned} a_{ij} &= -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i) \\ &= -\kappa_\alpha \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}\tilde{a}_{i-1,j} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)\tilde{a}_{i,j} + \frac{1}{h_{i+1}}\tilde{a}_{i+1,j}\right) \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \tilde{a}_{ij} &= I^{2-\alpha}\phi_j(x_i) \\ &= \frac{1}{\Gamma(4-\alpha)}\left(\frac{|x_i-x_{j-1}|^{3-\alpha}}{h_j} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)|x_i-x_j|^{3-\alpha} + \frac{|x_i-x_{j+1}|^{3-\alpha}}{h_{j+1}}\right) \end{aligned} \tag{2.14}$$

We shall finally introduce some notations.

For convenience, we use the notation $\simeq$. That $x_1 \simeq y_1$, means that $c_1 x_1 \leq y_1 \leq C_1 x_1$ for some constants c_1 and C_1 that are independent of N.

Meanwhile, let's define kernel functions

$$K_y(x) := \frac{|y-x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{2.15}$$

We define the difference quotients

$$D_h g(x_i) := \frac{g(x_{i+1}) - g(x_i)}{h_{i+1}}, \quad D_{\bar{h}} g(x_i) := \frac{g(x_i) - g(x_{i-1})}{h_i} \tag{2.16}$$

Thus

$$D_h g(x_i) = D_{\bar{h}} g(x_{i+1})$$

$$D_h^2 g(x_i) = \frac{2}{h_i + h_{i+1}} \left(D_h g(x_i) - D_{\bar{h}} g(x_i) \right) = \frac{2}{h_i + h_{i+1}} \left(D_h g(x_i) - D_h g(x_{i-1}) \right)$$

And for $j = 1, 2, ..., 2N$, we define

$$y_j^\theta = (1-\theta) x_{j-1} + \theta x_j, \quad \theta \in (0,1) \tag{2.17}$$

2.2. Regularity of the true solution. For any $\beta > 0$, we use the standard notation $C^\beta(\bar{\Omega}), C^\beta(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^\beta(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous[1] with exponent β' in Ω, where $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on open set Ω.

Definition 2.1 (delta dependent norm [2]). ...

Lemma 2.2. *Let $f \in C^\beta(\Omega), \beta > 2$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

$$|f^{(l)}(x)| \leq \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \leq T \\ (2T - x)^{-l-\alpha/2}, & \text{if } T \leq x < 2T \end{cases} \tag{2.18}$$

Theorem 2.3 (Regularity up to the boundary [2]). *Let Ω be a bounded domain, and $\beta > 0$ be such that neither β nor $\beta + \alpha$ is an integer. Let $f \in C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R}^n)$ be a solution of (1.1). Then, $u \in C^{\beta+\alpha}(\Omega)$ and*

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right) \tag{2.19}$$

where C is a constant depending only on Ω, α, and β.

Corollary 2.4. *Let u be a solution of (1.1) where $f \in L^\infty(\Omega)$ and $\|f\|_\beta^{(\alpha/2)} < \infty$. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \leq \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T - x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases} \tag{2.20}$$

And in this paper bellow, without special instructions, we allways assume that

$$f \in L^\infty(\Omega) \cap C^\beta(\Omega) \quad \text{and} \quad \|f\|_\beta^{(\alpha/2)} < \infty, \text{ with } \alpha + \beta > 4 \tag{2.21}$$

2.3. Main results. Here we state our main results; the proof is deferred to section 3 and section 4.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 2.5 (Local Truncation Error). *If $u(x)$ is a solution of the equation* (1.1) *where f satisfy the regular condition* (2.21)*, then there exists $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$, such that the truncation error* (2.11) *satisfies*

$$
\begin{aligned}
|\tau_i| :=& |-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i)| \\
\leq& C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} \begin{cases} x_i^{-\alpha}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha}, & N < i \leq 2N-1 \end{cases} \\
&+ C_2(r-1)h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \leq i \leq N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \leq 2N-1 \end{cases}
\end{aligned}
\tag{2.22}
$$

THEOREM 2.6 (Global Error). *The discrete equation* (2.8) *has sulotion and there exists a positive constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ such that the error between the numerial solution U with the exact solution $u(x_i)$ satisfies*

$$
\max_{1 \leq i \leq 2N-1} |u_i - u(x_i)| \leq C h^{\min\{\frac{r\alpha}{2}, 2\}}
\tag{2.23}
$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

Remark 2.7. ...

3. Local Truncation Error.

3.1. Proof of Theorem 2.5. The truncation error of the discrete format can be written as

$$
\begin{aligned}
-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I^{2-\alpha} \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i)) \\
&= -\kappa_\alpha D_h^2 I^{2-\alpha} (\Pi_h u - u)(x_i) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i)
\end{aligned}
\tag{3.1}
$$

THEOREM 3.1. *There exits a constant $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$ such that*

$$
\left| -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) \right| \leq C h^2 \begin{cases} x_i^{-\alpha/2 - 2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2 - 2/r}, & N \leq i \leq 2N-1 \end{cases}
\tag{3.2}
$$

Proof. Since $f \in C^2(\Omega)$ and

$$
\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha} u(x)) = f(x), \quad x \in \Omega,
\tag{3.3}
$$

we have $I^{2-\alpha} u \in C^4(\Omega)$. Therefore, using equation (A.2) of Lemma A.1, for $1 \leq i \leq$

$2N-1$, we have

(3.4)
$$
\begin{aligned}
&-\kappa_\alpha\left(D_h^2-\frac{d^2}{dx^2}\right)I^{2-\alpha}u(x_i)=\frac{h_{i+1}-h_i}{3}f'(x_i)\\
&\quad+\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i}f''(y)\frac{(y-x_{i-1})^3}{3!}dy+\frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}}f''(y)\frac{(y-x_{i+1})^3}{3!}dy\right)
\end{aligned}
$$

By Lemma B.2, Lemma 2.2 and Lemma B.3, we get the result. □

And now define

$$R_i := D_h^2 I^{2-\alpha}(u-\Pi_h u)(x_i),\quad 1\le i\le 2N-1 \tag{3.5}$$

We have some results about the estimate of R_i

Theorem 3.2. *For $1\le i<N/2$, there exists $C=C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that*

$$|R_i|\le\begin{cases}Ch^2x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1>0\\ Ch^2(x_i^{-1-\alpha}\ln(i)+\ln(N)), & \alpha/2-2/r+1=0\\ Ch^{r\alpha/2+r}x_i^{-1-\alpha}, & \alpha/2-2/r+1<0\end{cases} \tag{3.6}$$

Theorem 3.3. *For $N/2\le i\le N$, there exists constant $C=C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that*

$$|R_i|\le C(r-1)h^2|T-x_{i-1}|^{1-\alpha}+\begin{cases}Ch^2, & \alpha/2-2/r+1>0\\ Ch^2\ln(N), & \alpha/2-2/r+1=0\\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0\end{cases} \tag{3.7}$$

And for $N<i\le 2N-1$, it is symmetric to the previous case.

Combine Theorem 3.1, Theorem 3.2 and Theorem 3.3, and for $1\le i\le N$, we have

$$h^2x_i^{-\alpha/2-2/r}\le T^{\alpha/2-2/r}h^{\min\{\frac{r\alpha}{2},2\}}x_i^{-\alpha} \tag{3.8}$$
$$h^{r\alpha/2+r}x_i^{-1-\alpha}\le T^{-1}h^{r\alpha/2}x_i^{-\alpha} \tag{3.9}$$
$$h^r x_i^{-1}\ln(i)=T^{-1}\frac{\ln(i)}{i^r}\le T^{-1},\quad h^r\ln(N)=\frac{\ln(N)}{N^r}\le 1 \tag{3.10}$$

the proof of Theorem 2.5 completed.

We prove Theorem 3.2 and Theorem 3.3 in next subsections below.

3.2. Mesh Transport Functions.

Definition 3.4 (Mesh Transport Functions). *For $1 \le i, j \le 2N-1$.*

$$y_{i,j}(x) = \begin{cases} (x^{1/r} + Z_{j-i})^r & i<N, j<N \\ \dfrac{x^{1/r} - Z_i}{Z_1} h_N + x_N & i<N, j=N \\ 2T - (Z_{2N-(j-i)} - x^{1/r})^r & i<N, j>N \\ \left(\dfrac{Z_1}{h_N}(x - x_N) + Z_j\right)^r & i=N, j<N \\ x, & i=N, j=N \\ 2T - \left(\dfrac{Z_1}{h_N}(2T - x - x_N) + Z_{2N-j}\right)^r & i=N, j>N \\ (Z_{2N+j-i} - (2T-x)^{1/r})^r & i>N, j<N \\ \dfrac{Z_{2N-j} - (2T-x)^{1/r}}{Z_1} h_N + x_N & i>N, j=N \\ 2T - ((2T-x)^{1/r} - Z_{j-i})^r & i>N, j>N \end{cases} \tag{3.11}$$

where

$$Z_j := T^{1/r}\frac{j}{N} \tag{3.12}$$

And

$$h_{i,j}(x) = y_{i,j}(x) - y_{i,j-1}(x) \tag{3.13}$$

$$y^{\theta}_{i,j}(x) = (1-\theta) y_{i,j-1}(x) + \theta y_{i,j-1}(x), \quad \theta \in (0,1) \tag{3.14}$$

We give some properties of mesh transport functions.

Obviously,

$$y_{i,j}(x_{i-1}) = x_{j-1}, \quad y_{i,j}(x_i) = x_j, \quad y_{i,j}(x_{i+1}) = x_{j+1} \tag{3.15}$$

$$h_{i,j}(x_{i-1}) = h_{j-1}, \quad h_{i,j}(x_i) = h_j, \quad h_{i,j}(x_{i+1}) = h_{j+1} \tag{3.16}$$

$$y^{\theta}_{i,j}(x_{i-1}) = y^{\theta}_{j-1}, \quad y^{\theta}_{i,j}(x_i) = y^{\theta}_j, \quad y^{\theta}_{i,j}(x_{i+1}) = y^{\theta}_{j+1} \tag{3.17}$$

Lemma 3.5. *For $1 \le i \le 2N-1, 2 \le j \le 2N-1$,*

$$h_{i,j}(\xi) \simeq h_j, \quad \textit{for } \xi \in (x_{i-1}, x_{i+1}) \tag{3.18}$$

For $1 \le i, j \le 2N-1$,

$$|y_{i,j}(\xi) - \xi| \simeq |x_j - x_i| \quad \textit{for } \xi \in (x_{i-1}, x_{i+1}) \tag{3.19}$$

Lemma 3.6. *derivatives...*

3.3. Proof of Theorem 3.2. For convience, let's denote

Definition 3.7.

$$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy, \quad i = 0, \cdots, 2N, \ j = 1, \cdots, 2N \tag{3.20}$$

Also , we denote vertical difference quotients of T_{ij}

$$V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \tag{3.21}$$

And skew difference quotients of T_{ij}

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j-1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j+1} \right) \tag{3.22}$$

then $R_i = \sum_{j=1}^{2N} V_{ij}$.

Our main idea to esitmate R_i is to depart it by V_{ij} and S_{ij}.

For $3 \le i < N/2$, let's denote $k = \lceil \frac{i}{2} \rceil$, then

$$\begin{aligned} R_i =& \sum_{j=1}^{2N} V_{ij} \\ =& \sum_{j=1}^{k-1} V_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\ &+ \sum_{j=k+1}^{2i-1} S_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \\ &+ \sum_{j=2i+1}^{2N} V_{ij} \\ =& I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned} \tag{3.23}$$

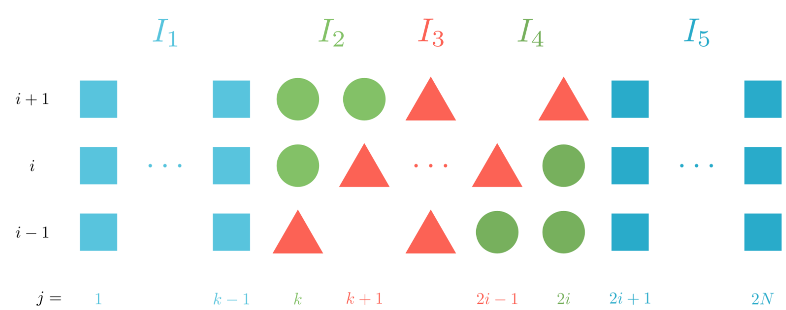

FIG. 1. *The departure of R_i for $3 \le i < N/2$*

and discuss $i = 1, 2$ separately, where

$$R_1 = \sum_{i=1}^{3} V_{1,j} + \sum_{i=4}^{N} V_{i,j}, \quad R_2 = \sum_{i=1}^{4} V_{1,j} + \sum_{i=5}^{N} V_{i,j} \tag{3.24}$$

Then we esrimate each part from easy to hard.

For I_5

LEMMA 3.8. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \le i < N/2$,*

$$\sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| \le C h^2 x_i^{-\alpha/2 - 2/r} \tag{3.25}$$

Proof. For $\max\{2i+1,4\} \le j \le N$, by Lemma A.4 and Lemma B.4 with $y - x_i \simeq y$, we have

$$|V_{ij}| = \left| \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \right| \le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} y^{-1-\alpha} dy = Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy \tag{3.26}$$

With $x_i \simeq x_{2i}$,

$$\sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| \le Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy = \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.27}$$

□

LEMMA 3.9. *Thert exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \le i < N/2$,*

$$\sum_{j=N+1}^{2N} |V_{ij}| \le \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.28}$$

and for $N/2 \le i \le N$,

$$\sum_{j=N-\lceil \frac{N}{2} \rceil+2}^{2N} |V_{ij}| \le \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.29}$$

Proof. For $1 \le i < N/2, N+1 \le j \le 2N-1$, by Lemma A.4, Lemma B.4 and $y - x_i \simeq T$

$$|V_{ij}| = \left| \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \right| \le C \int_{x_{j-1}}^{x_j} h^2 (2T-y)^{\alpha/2-2/r} |y - x_i|^{-1-\alpha} dy \le Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T-y)^{\alpha/2-2/r} dy$$

$$
\begin{aligned}
\sum_{j=N+1}^{2N-1}|V_{ij}| &\le CT^{-1-\alpha}h^2\int_{x_N}^{x_{2N-1}}(2T-y)^{\alpha/2-2/r}dy\\
&\le CT^{-1-\alpha}h^2\begin{cases}\frac{1}{\alpha/2-2/r+1}T^{\alpha/2-2/r+1}, & \alpha/2-2/r+1>0\\ \ln(T)-\ln(h_{2N}), & \alpha/2-2/r+1=0\\ \frac{1}{|\alpha/2-2/r+1|}h_{2N}^{\alpha/2-2/r+1}, & \alpha/2-2/r+1<0\end{cases}\\
&=\begin{cases}\frac{C}{\alpha/2-2/r+1}T^{-\alpha/2-2/r}\ h^2, & \alpha/2-2/r+1>0\\ CrT^{-1-\alpha}h^2\ln(N), & \alpha/2-2/r+1=0\\ \frac{C}{|\alpha/2-2/r+1|}T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2-2/r+1<0\end{cases}
\end{aligned}
\tag{3.30}
$$

And by Lemma A.3

$$|V_{i,2N}|\le CT^{-1-\alpha}h_{2N}^{\alpha/2+1}=CT^{-\alpha/2}h^{r\alpha/2+r}$$

Summarizes, we get the result. Similar for the second inequality. □

For $i=1,2$.

LEMMA 3.10. *From* (3.24)*, by Lemma* B.6*, Lemma* 3.8 *and Lemma* 3.9 *we get for* $i=1,2$

$$|R_i|\le Ch^2x_i^{-\alpha/2-2/r}+\begin{cases}Ch^2, & \alpha/2-2/r+1>0\\ Ch^2\ln(N), & \alpha/2-2/r+1=0\\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0\end{cases}\tag{3.31}$$

LEMMA 3.11. *There exists a constant* $C=C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3\le i\le N, k=\lceil\frac{i}{2}\rceil$

$$|I_1|=|\sum_{j=1}^{k-1}V_{ij}|\le\begin{cases}Ch^2x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1>0\\ Ch^2x_i^{-1-\alpha}\ln(i), & \alpha/2-2/r+1=0\\ Ch^{r\alpha/2+r}x_i^{-1-\alpha}, & \alpha/2-2/r+1<0\end{cases}\tag{3.32}$$

Proof. by Lemma A.3 , Lemma B.4

$$|V_{i1}|\le C\int_0^{x_1}x_1^{\alpha/2}|x_i-y|^{-1-\alpha}dy\simeq x_1^{\alpha/2+1}x_i^{-1-\alpha}=T^{\alpha/2+1}h^{r\alpha/2+r}x_i^{-1-\alpha}\tag{3.33}$$

For $2\le j\le k-1$, by Lemma A.4 and Lemma B.4 with $x_i-y\simeq x_i$, we have

$$
\begin{aligned}
|V_{ij}|&=\left|\int_{x_{j-1}}^{x_j}(u(y)-\Pi_hu(y))D_h^2K_y(x_i)dy\right|\\
&\le Ch^2\int_{x_{j-1}}^{x_j}y^{\alpha/2-2/r}x_i^{-1-\alpha}dy
\end{aligned}
\tag{3.34}
$$

Therefore,

$$\sum_{j=2}^{k-1}|V_{ij}|\le Ch^{r\alpha/2+r}x_i^{-1-\alpha}+Ch^2x_i^{-1-\alpha}\int_{x_1}^{x_{\lceil\frac{i}{2}\rceil-1}}y^{\alpha/2-2/r}dy\tag{3.35}$$

But $x_{\lceil \frac{i}{2} \rceil -1} \le 2^{-r} x_i$, so we have

$$\int_{x_1}^{x_{\lceil \frac{i}{2}\rceil -1}} y^{\alpha/2-2/r} dy \le \begin{cases} \frac{1}{\alpha/2-2/r+1}(2^{-r}x_i)^{\alpha/2-2/r+1}, & \alpha/2-2/r+1>0 \\ \ln(2^{-r}x_i)-\ln(x_1), & \alpha/2-2/r+1=0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2-2/r+1<0 \end{cases} \tag{3.36}$$

Combine the results above, we get the lemma. ∎

THEOREM 3.12. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \le i < N/2, k = \lceil \frac{i}{2} \rceil$,*

$$|I_3| = \left| \sum_{j=k+1}^{2i-1} S_{ij} \right| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.37}$$

To estimete S_{ij}, we need some preparations.

LEMMA 3.13. *For $y \in (x_{j-1}, x_j)$, we can rewrite*

$$y = x_{j-1} + \theta h_j = (1-\theta)x_{j-1} + \theta x_j =: y_j^\theta,\ \theta \in (0,1) \tag{3.38}$$

by Lemma A.2,

$$\begin{aligned} T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta \end{aligned} \tag{3.39}$$

where $\eta_{j1}^\theta \in (x_{j-1}, y_j^\theta), \eta_{j2}^\theta \in (y_j^\theta, x_j)$.

Now Let's construct a series of functions to represent T_{ij}.

DEFINITION 3.14.

$$y_{j-i}(x) = (x^{1/r} + Z_{j-i})^r, \quad Z_{j-i} = T^{1/r} \frac{j-i}{N} \tag{3.40}$$

Particularly, for $i, j \le N-1$,

$$y_{j-i}(x_{i-1}) = x_{j-1}, \quad y_{j-i}(x_i) = x_j, \quad y_{j-i}(x_{i+1}) = x_{j+1}$$

$$y_{j-i}{}'(x) = y_{j-i}(x)^{1-1/r} x^{1/r-1} \tag{3.41}$$

$$y_{j-i}{}''(x) = \frac{1-r}{r} y_{j-i}(x)^{1-2/r} x^{1/r-2} Z_{j-i} \tag{3.42}$$

(3.43)

$$y_{j-i}^{\theta}(x) = (1-\theta)y_{j-1-i}(x) + \theta y_{j-i}(x) \tag{3.44}$$

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{3.45}$$

Now, we define

$$P_{j-i}^{\theta}(x) = (h_{j-i}(x))^3 u''(y_{j-i}^{\theta}(x)) \frac{|y_{j-i}^{\theta}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.46}$$

$$Q_{j-i}^{\theta}(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^{\theta}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.47}$$

And now we can rewrite T_{ij}

LEMMA 3.15. *For $2 \le i \le N, 2 \le j \le N$,*

$$\begin{aligned} T_{ij} &= \int_0^1 -\frac{\theta(1-\theta)}{2} P_{j-i}^{\theta}(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)}{3!} Q_{j-i}^{\theta}(x_i) \left[\theta^2 u'''(\eta_{j,1}^{\theta}) - (1-\theta)^2 u'''(\eta_{j,2}^{\theta})\right] d\theta \end{aligned} \tag{3.48}$$

Immediately, we can see from (3.22) that

LEMMA 3.16. *For $3 \le i, j \le N-1$,*

(3.49)

$$\begin{aligned} S_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^{\theta}(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_{i+1}) u'''(\eta_{j+1,1}^{\theta}) - Q_{j-i}^{\theta}(x_i) u'''(\eta_{j,1}^{\theta})}{h_{i+1}} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_i) u'''(\eta_{j,1}^{\theta}) - Q_{j-i}^{\theta}(x_{i-1}) u'''(\eta_{j-1,1}^{\theta})}{h_i} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_{i+1}) u'''(\eta_{j+1,2}^{\theta}) - Q_{j-i}^{\theta}(x_i) u'''(\eta_{j,2}^{\theta})}{h_{i+1}} \right) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_i) u'''(\eta_{j,2}^{\theta}) - Q_{j-i}^{\theta}(x_{i-1}) u'''(\eta_{j-1,2}^{\theta})}{h_i} \right) d\theta \end{aligned}$$

To estimate S_{ij}, we first estimate $D_h^2 P_{j-i}^{\theta}(x_i)$, but By Lemma A.1,

$$D_h^2 P_{j-i}^{\theta}(x_i) = {P_{j-i}^{\theta}}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{3.50}$$

By Leibniz formula, we calculate and estimate the derivations of $h_{j-i}^3(x)$, $u''(y_{j-i}^{\theta}(x))$ and $\frac{|y_{j-i}^{\theta}(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)}$ separately.

Firstly, we have

Lemma 3.17. *There exists a constant $C = C(T, r)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil \le j \le \min\{2i, N\}$, $\xi \in (x_{i-1}, x_{i+1})$,*

$$h_{j-i}^3(\xi) \le Ch^2 x_i^{2-2/r} h_j \tag{3.51}$$

$$(h_{j-i}^3(\xi))' \le C(r-1)h^2 x_i^{1-2/r} h_j \tag{3.52}$$

$$(h_{j-i}^3(\xi))'' \le C(r-1)h^2 x_i^{-2/r} h_j \tag{3.53}$$

The proof of this theorem see Lemma B.7 and Lemma B.8

Second,

Lemma 3.18. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil \le j \le \min\{2i, N\}$, $\xi \in (x_{i-1}, x_{i+1})$,*

$$u''(y_{j-i}^\theta(\xi)) \le Cx_i^{\alpha/2-2} \tag{3.54}$$

$$(u''(y_{j-i}^\theta(\xi)))' \le Cx_i^{\alpha/2-3} \tag{3.55}$$

$$(u''(y_{j-i}^\theta(\xi)))'' \le Cx_i^{\alpha/2-4} \tag{3.56}$$

The proof of this theorem see Proof 28

And Finally, we have

Lemma 3.19. *There exists a constant $C = C(T, \alpha, r)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil \le j \le \min\{2i, N\}$, $\xi \in (x_{i-1}, x_{i+1})$,*

$$|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \le C|y_j^\theta - x_i|^{1-\alpha} \tag{3.57}$$

$$\left|(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'\right| \le C|y_j^\theta - x_i|^{1-\alpha} x_i^{-1} \tag{3.58}$$

$$\left|(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})''\right| \le C|y_j^\theta - x_i|^{1-\alpha} x_i^{-2} \tag{3.59}$$

where $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

The proof of this theorem see Proof 29

Lemma 3.20. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$,*

$$D_h^2 P_{j-i}^\theta(x_i) \le Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \tag{3.60}$$

where $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

Proof. Since Lemma A.1

$$D_h^2 P_{j-i}^\theta(x_i) = {P_{j-i}^\theta}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{3.61}$$

From (3.46), using Leibniz formula and Lemma 3.17, Lemma 3.18 and Lemma 3.19□

Lemma 3.21. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \le i \le N-1$.*

For $\lceil \frac{i}{2} \rceil \le j \le \min\{2i-1, N-1\}$,

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}&\left(\frac{Q^\theta_{j-i}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{j-i}(x_i)u'''(\eta^\theta_j)}{h_{i+1}}\right)\\&\le Ch^2\frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j\end{aligned}\tag{3.62}$$

And for $\lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i, N\}$,

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}&\left(\frac{Q^\theta_{j-i}(x_i)u'''(\eta^\theta_j) - Q^\theta_{j-i}(x_{i-1})u'''(\eta^\theta_{j-1})}{h_i}\right)\\&\le Ch^2\frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j\end{aligned}\tag{3.63}$$

where $\eta^\theta_j \in (x_{j-1}, x_j)$.

proof see Proof 30

Lemma 3.22. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that for* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$,

$$\begin{aligned}S_{ij} &\le Ch^2\int_0^1 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j d\theta\\&= Ch^2\int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}dy\end{aligned}\tag{3.64}$$

Proof. Since Lemma 3.16, by Lemma 3.20 and Lemma 3.21, we get the result immediately. ∎

Now we can prove Theorem 3.12 using Lemma 3.22, $k = \lceil \frac{i}{2} \rceil$

$$\begin{aligned}I_3 &= \sum_{k+1}^{2i-1} S_{ij} \le Ch^2\int_{x_k}^{x_{2i-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}dy\\&= Ch^2\left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{2i-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)}\right)x_i^{\alpha/2-2-2/r}\\&\le Ch^2x_i^{2-\alpha}x_i^{\alpha/2-2-2/r} = Ch^2x_i^{-\alpha/2-2/r}\end{aligned}\tag{3.65}$$

Now we study I_2, I_4.

Lemma 3.23. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that for* $3 \le i \le N-1, k = \lceil \frac{i}{2} \rceil$,

$$I_2 = \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}(T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,k}\right) \le Ch^2x_i^{-\alpha/2-2/r}\tag{3.66}$$

And for $3 \le i < N/2$,

$$I_4 = \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}(T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,2i}\right) \le Ch^2x_i^{-\alpha/2-2/r}\tag{3.67}$$

Proof. In fact,

$$
\begin{aligned}
&\frac{1}{h_{i+1}}(T_{i+1,k}+T_{i+1,k+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,k} \\
&=\frac{1}{h_{i+1}}(T_{i+1,k}-T_{i,k})+\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k})+(\frac{1}{h_{i+1}}-\frac{1}{h_i})T_{i,k}
\end{aligned} \tag{3.68}
$$

While, by Lemma A.2 and Lemma B.1

$$
\begin{aligned}
\frac{1}{h_{i+1}}(T_{i+1,k}-T_{i,k}) &= \int_{x_{k-1}}^{x_k}(u(y)-\Pi_h u(y))\frac{|x_{i+1}-y|^{1-\alpha}-|x_i-y|^{1-\alpha}}{h_{i+1}\Gamma(2-\alpha)}dy \\
&\le h_k^2 \max_{\eta\in(x_{k-1},x_k)}|u''(\eta)| \int_{x_{k-1}}^{x_k}\frac{|\xi-y|^{-\alpha}}{\Gamma(1-\alpha)}dy, \quad \xi\in(x_i,x_{i+1}) \\
&\le Ch^2 x_k^{2-2/r}x_{k-1}^{\alpha/2-2}\ h_k|x_i-x_k|^{-\alpha} \\
&\le Ch^2 x_i^{-\alpha/2-2/r}h_k
\end{aligned} \tag{3.69}
$$

Thus,

$$
\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}|T_{i+1,k}-T_{i,k}|\le Ch^2x_i^{-\alpha/2-2/r} \tag{3.70}
$$

From Lemma 3.15

$$
\begin{aligned}
\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k}) &= \int_0^1 -\frac{\theta(1-\theta)}{2}\frac{P_{k-i}^{\theta}(x_{i+1})-P_{k-i}^{\theta}(x_i)}{h_{i+1}}d\theta \\
&+\int_0^1\frac{\theta^3(1-\theta)}{3!}\frac{Q_{k-i}^{\theta}(x_{i+1})u'''(\eta_{k+1,1}^{\theta})-Q_{k-i}^{\theta}(x_i)u'''(\eta_{k,1}^{\theta})}{h_{i+1}}d\theta \\
&-\int_0^1\frac{\theta(1-\theta)^3}{3!}\frac{Q_{k-i}^{\theta}(x_{i+1})u'''(\eta_{k+1,2}^{\theta})-Q_{k-i}^{\theta}(x_i)u'''(\eta_{k,2}^{\theta})}{h_{i+1}}d\theta
\end{aligned} \tag{3.71}
$$

and

$$
D_hP_{k-i}^{\theta}(x_i):=\frac{P_{k-i}^{\theta}(x_{i+1})-P_{k-i}^{\theta}(x_i)}{h_{i+1}}={P_{k-i}^{\theta}}'(\xi), \quad \xi\in(x_i,x_{i+1}) \tag{3.72}
$$

Similar with Lemma 3.20, from Lemma 3.15, using Leibniz formula, by Lemma B.7, Lemma 3.18 and Lemma 3.19 we get

$$
|D_hP_{k-i}^{\theta}(x_i)|\le Ch^2x_i^{-\alpha/2-2/r}h_k \tag{3.73}
$$

And with Lemma 3.21, we can get

$$
\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}|T_{i+1,k+1}-T_{i,k}|\le Ch^2x_i^{-\alpha/2-2/r} \tag{3.74}
$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$
\begin{aligned}
\frac{2}{h_i+h_{i+1}}\frac{h_{i+1}-h_i}{h_ih_{i+1}}T_{i,k} &\le h_i^{-3}h^2x_i^{1-2/r}h_kCh_k^2x_{k-1}^{\alpha/2-2}|x_k-x_i|^{1-\alpha} \\
&\le Ch^2x_i^{-\alpha/2-2/r}
\end{aligned} \tag{3.75}
$$

Summarizes, we have

$$I_2 \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.76}$$

The case for I_4 is similar. ∎

Now combine Lemma 3.10, Lemma 3.11, Lemma 3.23, Theorem 3.12, Lemma 3.8 and Lemma 3.9, we get Theorem 3.2.

3.4. Proof of Theorem 3.3. For $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$, we have

$$\begin{aligned}
R_i =& \sum_{j=1}^{2N} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& \sum_{j=1}^{k-1} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&+ \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\
&+ \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&+ \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right) \\
&+ \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& I_1 + I_2 + I_3^1 + I_3^2 + I_3^3 + I_4 + I_5
\end{aligned} \tag{3.77}$$

We have estimate I_1 in Lemma 3.11 and I_2 in Lemma 3.23. We can control I_3^1 similar with Theorem 3.12 by Lemma 3.22 where $2i-1 \ge N-1$

LEMMA 3.24. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$,*

$$\begin{aligned}
I_3^1 =& \sum_{j=k+1}^{N-1} S_{ij} \le Ch^2 \int_{x_k}^{x_{N-1}} \frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\
&= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{N-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\
&\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{3.78}$$

Let's study I_3^3 before I_3^2.

$$I_3^3 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} S_{ij} \tag{3.79}$$

Similarly, Let's define a new series of functions

DEFINITION 3.25. *For $i \leq N-1, j \geq N+1$, with no confusion, we also denote in this section*

$$y_{j-i}(x) = 2T - (Z_{2N-j+i} - x^{1/r})^r, \quad Z_{2N-j+i} = T^{1/r}\frac{2N-j+i}{N} \tag{3.80}$$

Particularly

$$y_{j-i}(x_{i-1}) = x_{j-1}, \quad y_{j-i}(x_i) = x_j, \quad y_{j-i}(x_{i+1}) = x_{j+1}$$

$y \to z$?

$$y_{j-i}{}'(x) = (2T - y_{j-i}(x))^{1-1/r}x^{1/r-1} \tag{3.81}$$

$$y_{j-i}{}''(x) = \frac{1-r}{r}(2T - y_{j-i}(x))^{1-2/r}x^{1/r-2}Z_{2N-j+i} \tag{3.82}$$

(3.83)

$$y^\theta_{j-i}(x) = (1-\theta)y_{j-i-1}(x) + \theta y_{j-i}(x) \tag{3.84}$$

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{3.85}$$

$$P^\theta_{j-i}(x) = (h_{j-i}(x))^3 u''(y^\theta_{j-i}(x))\frac{|y^\theta_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.86}$$

$$Q^\theta_{j-i}(x) = (h_{j-i}(x))^4\frac{|y^\theta_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.87}$$

Now we have the same formula Lemma 3.16 for $i \leq N-1, j \geq N+2$,

Similarly, we first estimate

$$D^2_h P^\theta_{j-i}(\xi) = {P^\theta_{j-i}}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{3.88}$$

Combine Definition 3.25, Lemma B.9, Lemma B.10 and Lemma B.11, using Leibniz formula, we have

LEMMA 3.26. *There exists a constant $C = C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \leq i \leq N-1$, $N+2 \leq j \leq 2N - \lceil\frac{N}{2}\rceil + 1$,, we have*

$$\begin{aligned} |D^2_h P^\theta_{j-i}(\xi)| \leq Ch_j h^2\Big(&|y^\theta_j - x_i|^{1-\alpha} \\ &+ |y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_N) \\ &+ |y^\theta_j - x_i|^{-1-\alpha}(|2T - x_i - y^\theta_j| + h_N)^2 \\ &+ (r-1)|y^\theta_j - x_i|^{-\alpha}\Big) \end{aligned} \tag{3.89}$$

And

LEMMA 3.27. *There exists a constant* $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le i \le N-1,\ N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil,\ \xi \in (x_{i-1}, x_{i+1})$ *, we have*

$$\frac{2}{h_i + h_{i+1}} \left| \frac{Q^{\theta}_{j-i}(x_{i+1})u'''(\eta^{\theta}_{j+1}) - Q^{\theta}_{j-i}(x_i)u'''(\eta^{\theta}_{j})}{h_{i+1}} \right| \le Ch^2 h_j \left(|y^{\theta}_j - x_i|^{1-\alpha} + |y^{\theta}_j - x_i|^{-\alpha}(|2T - x_i - y^{\theta}_j| + h_N)\right) \tag{3.90}$$

and

$$\frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_i)u'''(\eta^{\theta}_{j}) - Q^{\theta}_{j-i}(x_{i-1})u'''(\eta^{\theta}_{j-1})}{h_{i+1}} \right) \le Ch^2 h_j (|y^{\theta}_j - x_i|^{1-\alpha} + |y^{\theta}_j - x_i|^{-\alpha}(|2T - x_i - y^{\theta}_j| + h_N)) \tag{3.91}$$

Proof. From Definition 3.25, by Lemma B.9 and Lemma B.11, for $\xi \in (x_i, x_{i+1})$, by Leibniz formula, we have

$$\left| {Q^{\theta}_{j-i}}'(\xi) \right| \le Ch^2 h_j^2 ((r-1)|y^{\theta}_j - x_i|^{1-\alpha} + |y^{\theta}_j - x_i|^{-\alpha}(|2T - x_i - y^{\theta}_j| + h_N)) \tag{3.92}$$

$$\left| Q^{\theta}_{j-i}(\xi) \right| \le Ch^2 h_j^2 |y^{\theta}_j - x_i|^{1-\alpha} \tag{3.93}$$

So use the skill in Proof 30 with Lemma B.10

$$\frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_{i+1})u'''(\eta^{\theta}_{j+1}) - Q^{\theta}_{j-i}(x_i)u'''(\eta^{\theta}_{j})}{h_{i+1}} \right) \le Ch^2 h_j (|y^{\theta}_j - x_i|^{1-\alpha} + |y^{\theta}_j - x_i|^{-\alpha}(|2T - x_i - y^{\theta}_j| + h_N)) \tag{3.94}$$

□

Combine Lemma 3.26, Lemma 3.27 and formula Lemma 3.16 for $i \le N-1, j \ge N+2$, we have

LEMMA 3.28. *There exists a constant* $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le i \le N-1,\ N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$

$$\begin{aligned} S_{ij} \le & Ch^2 \int_{x_{j-1}}^{x_j} \Big(|y - x_i|^{1-\alpha} \\ & + |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2 \\ & + (r-1)|y - x_i|^{-\alpha} \Big) dy \end{aligned} \tag{3.95}$$

We can esitmate I_3^3 Now.

LEMMA 3.29. *There exists a constant* $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le i \le N-1$, *we have*

$$I_3^3 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} S_{ij} \le Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \tag{3.96}$$

Proof.

$$
\begin{aligned}
I_3^3 = &\sum_{j=N+2}^{2N-\lceil\frac{N}{2}\rceil} S_{ij} \\
\le &Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil\frac{N}{2}\rceil}} \Big(|y-x_i|^{1-\alpha} \\
&+ |y-x_i|^{-\alpha}(|2T-x_i-y|+h_N) + |y-x_i|^{-1-\alpha}(|2T-x_i-y|+h_N)^2 \\
&+ (r-1)|y-x_i|^{-\alpha}\Big)dy
\end{aligned}
\tag{3.97}
$$

Since

$$
|2T-x_i-y|+h_N \le y-x_i \tag{3.98}
$$

$$
\begin{aligned}
I_3^3 &\le Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil\frac{N}{2}\rceil}} |y-x_i|^{1-\alpha} + (r-1)|y-x_i|^{-\alpha} \\
&\le Ch^2(T^{2-\alpha} + (r-1)|x_{N+1}-x_i|^{1-\alpha}) \\
&\le Ch^2 + C(r-1)h^2|T-x_{i-1}|^{1-\alpha}
\end{aligned}
\tag{3.99}
$$

□

For I_3^2, we have

THEOREM 3.30. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that, for $N/2 \le i \le N-1$*

$$V_{iN} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,N+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,N} + \frac{1}{h_i} T_{i-1,N-1} \right) \le Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \tag{3.100}$$

Proof. We use the similar skill in the last section, but more complicated. for $j = N$, Let

$$_L y_{N-1-i}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r} \frac{N-1-i}{N} \tag{3.101}$$

$$_0 y_{N-i}(x) = \frac{x^{1/r} - Z_i}{Z_1} h_N + T, \quad Z_i = T^{1/r} \frac{i}{N}, x_N = T \tag{3.102}$$

and

$$_R y_{N+1-i}(x) = 2T - (Z_{N-1+i} - x^{1/r})^r, \quad Z_{N-1+i} = T^{1/r} \frac{N-1+i}{N} \tag{3.103}$$

Thus,

$$\begin{gathered}
_L y_{N-1-i}(x_{i-1}) = x_{N-2}, \quad _L y_{N-1-i}(x_i) = x_{N-1}, \quad _L y_{N-1-i}(x_{i+1}) = x_N \\
_0 y_{N-i}(x_{i-1}) = x_{N-1}, \quad _0 y_{N-i}(x_i) = x_N, \quad _0 y_{N-i}(x_{i+1}) = x_{N+1} \\
_R y_{N+1-i}(x_{i-1}) = x_N, \quad _R y_{N+1-i}(x_i) = x_{N+1}, \quad _R y_{N+1-i}(x_{i+1}) = x_{N+2}
\end{gathered}$$

Then, define

$$_L y^\theta_{N-i}(x) = \theta {_L y_{N-1-i}}(x) + (1-\theta) {_0 y_{N-i}}(x) \tag{3.104}$$

$$_R y^\theta_{N+1-i}(x) = \theta {_0 y_{N-i}}(x) + (1-\theta) {_R y_{N+1-i}}(x) \tag{3.105}$$

$$_L h_{N-i}(x) = {_0 y_{N-i}}(x) - {_L y_{N-1-i}}(x) \tag{3.106}$$

$$_R h_{N+1-i}(x) = {_R y_{N+1-i}}(x) - {_0 y_{N-i}}(x) \tag{3.107}$$

We have

$$_L y_{N-1-i}{}'(x) = {_L y^{1-1/r}_{N-1-i}}(x) x^{1/r-1} \tag{3.108}$$

$$_L y_{N-1-i}{}''(x) = \frac{1-r}{r} {_L y^{1-2/r}_{N-1-i}}(x) x^{1/r-2} Z_{N-1-i} \tag{3.109}$$

$$_0 y_{N-i}{}'(x) = \frac{1}{r} \frac{h_N}{Z_1} x^{1/r-1} \tag{3.110}$$

$$_0 y_{N-i}{}''(x) = \frac{1-r}{r^2} \frac{h_N}{Z_1} x^{1/r-2} \tag{3.111}$$

$$_R y_{N+1-i}{}'(x) = (2T - {_R y_{N+1-i}}(x))^{1-1/r} x^{1/r-1} \tag{3.112}$$

$$_R y_{N+1-i}{}''(x) = \frac{1-r}{r} (2T - {_R y_{N+1-i}}(x))^{1-2/r} x^{1/r-2} Z_{N-1+i} \tag{3.113}$$

$$
{}_{L}P^{\theta}_{N-i}(x) = ({}_{L}h_{N-i}(x))^3 \frac{|{}_{L}y^{\theta}_{N-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_{L}y^{\theta}_{N-i}(x)) \tag{3.114}
$$

$$
{}_{R}P^{\theta}_{N+1-i}(x) = ({}_{R}h_{N+1-i}(x))^3 \frac{|{}_{R}y^{\theta}_{N+1-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_{R}y^{\theta}_{N+1-i}(x)) \tag{3.115}
$$

$$
{}_{L}Q^{\theta}_{N-i}(x) = ({}_{L}h_{N-i}(x))^4 \frac{|{}_{L}y^{\theta}_{N-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.116}
$$

$$
{}_{R}Q^{\theta}_{N+1-i}(x) = ({}_{R}h_{N+1-i}(x))^4 \frac{|{}_{R}y^{\theta}_{N+1-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.117}
$$

Similar with Lemma 3.15, we can get for $l = -1, 0, 1$,

$$
\begin{aligned}
T_{i+l,N+l} = & \int_0^1 -\frac{\theta(1-\theta)}{2} {}_{L}P^{\theta}_{N-i}(x_{i+l}) d\theta \\
& + \int_0^1 \frac{\theta(1-\theta)}{3!} {}_{L}Q^{\theta}_{N-i}(x_{i+l}) (\theta^2 u'''(\eta^{\theta}_{N+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+l,2})) d\theta
\end{aligned} \tag{3.118}
$$

$$
\begin{aligned}
T_{i+l,N+1+l} = & \int_0^1 -\frac{\theta(1-\theta)}{2} {}_{R}P^{\theta}_{N+1-i}(x_{i+l}) d\theta \\
& + \int_0^1 \frac{\theta(1-\theta)}{3!} {}_{R}Q^{\theta}_{N+1-i}(x_{i+l}) (\theta^2 u'''(\eta^{\theta}_{N+1+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+1+l,2})) d\theta
\end{aligned} \tag{3.119}
$$

So we have

$$
\begin{aligned}
V_{i,N} = & \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 {}_{L}P^{\theta}_{N-i}(x_i) d\theta \\
& + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_{L}Q^{\theta}_{N-i}(x_{i+1}) u'''(\eta^{\theta}_{N+1,1}) - {}_{L}Q^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,1})}{h_{i+1}} \right) d\theta \\
& - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_{L}Q^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,1}) - {}_{L}Q^{\theta}_{N-i}(x_{i-1}) u'''(\eta^{\theta}_{N-1,1})}{h_i} \right) d\theta \\
& - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_{L}Q^{\theta}_{N-i}(x_{i+1}) u'''(\eta^{\theta}_{N+1,2}) - {}_{L}Q^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,2})}{h_{i+1}} \right) d\theta \\
& + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_{L}Q^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,2}) - {}_{L}Q^{\theta}_{N-i}(x_{i-1}) u'''(\eta^{\theta}_{N-1,2})}{h_i} \right) d\theta
\end{aligned} \tag{3.120}
$$

$N+1$ is similar.

We estimate $D_h^2 {}_{L}P^{\theta}_{N-i}(x_i) = {}_{L}{P^{\theta}_{N-i}}''(\xi), \xi \in (x_{i-1}, x_{i+1})$,

Lemma 3.31.

$$
{}_{L}h^3_{N-i}(\xi) \leq C h^3_N \leq C h^3 \tag{3.121}
$$

$$
{}_{R}h^3_{N+1-i}(\xi) \leq C h^3_N \leq C h^3 \tag{3.122}
$$

$$({}_Lh^3_{N-i}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{3.123}$$

$$({}_Rh^3_{N+1-i}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{3.124}$$

$$({}_Lh^3_{N-i}(\xi))'' \le C(r-1)h^2 \tag{3.125}$$

$$({}_Rh^3_{N+1-i}(\xi))'' \le C(r-1)h^2 \tag{3.126}$$

Proof.

$$ {}_Lh_{N-i}(\xi) \le 2(C?)h_N, \quad {}_Rh_{N+1-i}(\xi) \le 2h_N \tag{3.127}$$

$$\begin{aligned} ({}_Lh^l_{N-i}(\xi))' &= l\,{}_Lh^{l-1}_{N-i}(\xi)({}_0y_{N-i}{}'(\xi) - {}_Ly_{N-1-i}{}'(\xi)) \\ &= l\,{}_Lh^{l-1}_{N-i}(\xi)\xi^{1/r-1}(\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-1/r}_{N-1-i}(\xi)) \end{aligned} \tag{3.128}$$

while

$$\begin{aligned} |\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-1/r}_{N-1-i}(\xi)| &= |\frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r}|(\frac{N-t}{N})^{r-1} - (\frac{N-s}{N})^{r-1}| \quad t \in [0,1], s \in [0,2] \\ &\le T^{1-1/r}|1 - (\frac{N-2}{N})^{r-1}| \le CT^{1-1/r}(r-1)\frac{2}{N} \end{aligned} \tag{3.129}$$

Thus,

$$({}_Lh^l_{N-i}(\xi))' \le C(r-1)h_N^{l-1}x_i^{1/r-1}h \tag{3.130}$$

And

$$\begin{aligned} ({}_Lh^3_{N-i}(\xi))'' &= 3{}_Lh^2_{N-i}(\xi){}_Lh_{N-i}{}''(\xi) + 6{}_Lh_{N-i}(\xi)({}_Lh_{N-i}{}'(\xi))^2 \\ &\le Ch_N^2\frac{1-r}{r}x_i^{1/r-2}(\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i}) + Ch_N(r-1)^2h^2x_i^{2/r-2} \end{aligned} \tag{3.131}$$

$$|\frac{h_N}{rZ_1} - {}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i}| \le T^{1-1/r} + Cx_N^{1-2/r}x_N^{1/r} = CT^{1-1/r}$$

So

$$\begin{aligned} ({}_Lh^3_{N-i}(\xi))'' &\le Ch_N^2\frac{1-r}{r}x_i^{1/r-2} + C(r-1)^2h_Nx_i^{2/r-2}h^2 \\ &\le C(r-1)h_N^2 \end{aligned} \tag{3.132}$$

${}_Rh^3_{N+1-i}(\xi)$ is similar. ∎

LEMMA 3.32.

$$u''({}_Ly^\theta_{N-i}(\xi)) \le Cx_{N-2}^{-\alpha/2-2} \le C \tag{3.133}$$

$$(u''({}_Ly^\theta_{N-i}(\xi)))' \le C \tag{3.134}$$

$$(u''({}_Ly^\theta_{N-i}(\xi)))'' \le C \tag{3.135}$$

Proof.

$$
\begin{aligned}
(u''({}_Ly^{\theta}_{N-i}(\xi)))' &= u'''({}_Ly^{\theta}_{N-i}(\xi))\,{}_L{y^{\theta}_{N-i}}'(\xi)\\
&\le C(\theta\,{}_L{y_{N-1-i}}'(\xi)+(1-\theta)\,{}_0{y_{N-i}}'(\xi))\\
&\le Cx_i^{1/r-1}(\theta\,{}_Ly^{1-1/r}_{N-1-i}(\xi)+(1-\theta)\frac{h_N}{rZ_1})\\
&\le Cx_i^{1/r-1}x_N^{1-1/r}
\end{aligned}
\tag{3.136}
$$

And

$$
\begin{aligned}
(u''({}_Ly^{\theta}_{N-i}(\xi)))'' &= u''''({}_Ly^{\theta}_{N-i}(\xi))({}_L{y^{\theta}_{N-i}}'(\xi))^2+u'''({}_Ly^{\theta}_{N-i}(\xi))\,{}_L{y^{\theta}_{N-i}}''(\xi)\\
&\le Cx_i^{2/r-2}x_N^{2-2/r}+C\frac{r-1}{r}x_i^{1/r-2}(\theta x_N^{1-2/r}Z_{N-1-i}+(1-\theta)\frac{h_N}{rZ_1})\\
&\le Cx_i^{2/r-2}+C(r-1)x_i^{1/r-2}T^{1-1/r}
\end{aligned}
\tag{3.137}
$$

$\square$

Lemma 3.33.

$$|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha}\le C|y^{\theta}_N-x_i|^{1-\alpha} \tag{3.138}$$

$$(|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha})'\le C|y^{\theta}_N-x_i|^{1-\alpha} \tag{3.139}$$

$$(|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha})''\le C(r-1)|y^{\theta}_N-x_i|^{-\alpha}+|y^{\theta}_N-x_i|^{1-\alpha} \tag{3.140}$$

Proof.

$$
\begin{aligned}
({}_Ly^{\theta}_{N-i}(\xi)-\xi)' &= (\theta({}_Ly_{N-1-i}(\xi)-\xi)+(1-\theta)({}_0y_{N-i}(\xi)-\xi))'\\
&= \theta({}_L{y_{N-1-i}}'(\xi)-1)+(1-\theta)({}_0{y_{N-i}}'(\xi)-1)\\
&= \theta\xi^{1/r-1}({}_Ly^{1-1/r}_{N-1-i}(\xi)-\xi^{1-1/r})+(1-\theta)\xi^{1/r-1}(\frac{h_N}{rZ_1}-\xi^{1-1/r})
\end{aligned}
\tag{3.141}
$$

$$
\begin{aligned}
({}_Ly^{\theta}_{N-i}(\xi)-\xi)'' &= \theta({}_L{y_{N-1-i}}''(\xi))+(1-\theta)({}_0{y_{N-i}}''(\xi))\\
&= \frac{1-r}{r}\xi^{1/r-2}(\theta\,{}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i}+(1-\theta)\frac{h_N}{rZ_1})\le 0
\end{aligned}
\tag{3.142}
$$

And

$$|({}_Ly^{\theta}_{N-i}(\xi)-\xi)''|\le C(r-1)\xi^{1/r-2}T^{1-1/r} \tag{3.143}$$

We have known

$$C|x_{N-1}-x_i|\le|{}_Ly_{N-1-i}(\xi)-\xi|\le C|x_{N-1}-x_i| \tag{3.144}$$

If $\xi\le x_{N-1}$, then $({}_0y_{N-i}(\xi)-\xi)'\ge 0$, so

$$C|x_N-x_i|\le|x_{N-1}-x_{i-1}|\le|{}_Ly^{\theta}_{N-i}(\xi)-\xi|\le|x_{N+1}-x_{i+1}|\le C|x_N-x_i| \tag{3.145}$$

If $i=N-1$ and $\xi\in[x_{N-1},x_N]$, then ${}_0y_{N-i}(\xi)-\xi$ is concave, bigger than its two neighboring points, which are equal to h_N, so

$$h_N=|x_N-x_{N-1}|\le|{}_0y_{N-i}(\xi)-\xi|\le|x_{N+1}-x_{N-1}|=2h_N \tag{3.146}$$

So we have

$$|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha}\le C|y^{\theta}_N-x_i|^{1-\alpha} \tag{3.147}$$

While

$$ {}_Ly^{1-1/r}_{N-1-i}(\xi)-\xi^{1-1/r}\le({}_Ly_{N-1-i}(\xi)-\xi)\xi^{-1/r} \tag{3.148}$$

and

$$\begin{aligned}|\frac{h_N}{rZ_1}-\xi^{1-1/r}|&\le\max\{|\frac{h_N}{rZ_1}-x^{1-1/r}_{i-1}|,|\frac{h_N}{rZ_1}-x^{1-1/r}_{i+1}|\}\\&\le\max\begin{cases}T^{1-1/r}-x^{1-1/r}_{i-1}\le|x_N-x_{i-1}|T^{-1/r}\le C|x_N-x_i|\\|x^{1-1/r}_{i+1}-x^{1-1/r}_{N-1}|\le|x_{i+1}-x_{N-1}|x^{-1/r}_{N-1}\le C|x_N-x_i|\end{cases}\end{aligned} \tag{3.149}$$

So we have

$$({}_Ly^{\theta}_{N-i}(\xi)-\xi)'\le C|y^{\theta}_N-x_i| \tag{3.150}$$

$$\begin{aligned}(|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha})'&=|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{-\alpha}({}_Ly^{\theta}_{N-i}(\xi)-\xi)'\\&\le|y^{\theta}_N-x_i|^{1-\alpha}\end{aligned} \tag{3.151}$$

Finally,

$$\begin{aligned}(|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha})''&=(1-\alpha)|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{-\alpha}({}_Ly^{\theta}_{N-i}(\xi)-\xi)''\\&\quad+\alpha(\alpha-1)|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{-1-\alpha}(({}_Ly^{\theta}_{N-i}(\xi)-\xi)')^2\\&\le C(r-1)|y^{\theta}_N-x_i|^{-\alpha}+C|y^{\theta}_N-x_i|^{1-\alpha}\end{aligned} \tag{3.152}$$

□

By the three lemmas above, for $N/2\le i\le N-1$, we have

LEMMA 3.34.

$$\begin{aligned}D^2_h{}_LP^{\theta}_{N-i}(x_i)&={}_L{P^{\theta}_{N-i}}''(\xi)\quad\xi\in(x_{i-1},x_{i+1})\\&\le Ch^3|y^{\theta}_N-x_i|^{1-\alpha}+C(r-1)(h^3|y^{\theta}_N-x_i|^{-\alpha}+h^2|y^{\theta}_N-x_i|^{1-\alpha})\end{aligned} \tag{3.153}$$

while $\theta h_N=y^{\theta}_N-x_{N-1}\le y^{\theta}_N-x_i$, we have

$$\theta D^2_h{}_LP^{\theta}_{N-i}(x_i)\le Ch^3|y^{\theta}_N-x_i|^{1-\alpha}+C(r-1)(h^2|y^{\theta}_N-x_i|^{1-\alpha}) \tag{3.154}$$

And

LEMMA 3.35.

$$\begin{aligned}&\frac{2}{h_i+h_{i+1}}\left(\frac{{}_LQ^{\theta}_{N-i}(x_{i+1})u'''(\eta^{\theta}_{N+1})-{}_LQ^{\theta}_{N-i}(x_i)u'''(\eta^{\theta}_N)}{h_{i+1}}\right)\\&\le Ch^3|y^{\theta}_N-x_i|^{1-\alpha}\end{aligned} \tag{3.155}$$

And immediately with Lemma 3.16, For $N/2\le i\le N-1$

$$\begin{aligned}V_{iN}&\le C\int_{x_{N-1}}^{x_N}h^2|y-x_i|^{1-\alpha}+C(r-1)h|y-x_i|^{1-\alpha}dy\\&\le Ch^2h_N|T-x_i|^{1-\alpha}+C(r-1)h^2|x_N-x_i|^{1-\alpha}\\&\le Ch^2+C(r-1)h^2|T-x_{i-1}|^{1-\alpha}\end{aligned} \tag{3.156}$$

Similarly with $j=N+1$. □

I_4, I_5 is easy. Similar with Lemma 3.23 and Lemma 3.9, we have

THEOREM 3.36. *There is a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le i \le N$,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right) \le Ch^2 \tag{3.157}$$

Proof. Similar with Lemma 3.23. In fact, let $m = 2N - \lceil \frac{N}{2} \rceil + 1$

$$\begin{aligned} &\frac{1}{h_i}(T_{i-1,l} + T_{i-1,l-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,l} \\ &= \frac{1}{h_i}(T_{i-1,l} - T_{i,l}) + \frac{1}{h_i}(T_{i-1,l-1} - T_{i,l}) + (\frac{1}{h_i} - \frac{1}{h_{i+1}})T_{i,l} \end{aligned} \tag{3.158}$$

While, by Lemma A.2

$$\begin{aligned} \frac{1}{h_i}(T_{i-1,l} - T_{i,l}) &= \int_{x_{l-1}}^{x_l} (u(y) - \Pi_h u(y)) \frac{|x_{i-1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_i \Gamma(2-\alpha)} dy \\ &\le C \int_{x_{l-1}}^{x_l} h_l^2 u''(\eta) \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy, \quad \xi \in (x_{i-1}, x_i) \\ &\le C h_l^3 (2T - x_{l-1})^{\alpha/2-2} T^{-\alpha} \\ &\le C h_l^3 \end{aligned} \tag{3.159}$$

Thus,

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_i}(T_{i-1,l} - T_{i,l}) \le C h_l^2 \tag{3.160}$$

For

$$\frac{1}{h_i}(T_{i-1,l-1} - T_{i,l}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{h_i} d\theta \tag{3.161}$$

And Similar with Lemma 3.21, we can get

$$\frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{(h_i + h_{i+1})h_i} \le C h_l^2 |y_l^\theta - x_i|^{1-\alpha} \tag{3.162}$$

So

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_i}(T_{i-1,l-1} - T_{i,l}) \le Ch^2 \tag{3.163}$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned} \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,l} &\le h_i^{-3} h^2 x_i^{1-2/r} h_l C h_l^2 x_{l-1}^{\alpha/2-2} |x_l - x_i|^{1-\alpha} \\ &\le Ch^2 \end{aligned} \tag{3.164}$$

Summarizes, we have

$$I_4 \le Ch^2 \tag{3.165}$$

∎

Now we can conclude a part of the theorem Theorem 3.3 at the beginning of this section.

By Lemma 3.11, Lemma 3.23, Lemma 3.24, Theorem 3.30, Lemma 3.29, Theorem 3.36, Lemma 3.9, we have

THEOREM 3.37. *there exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $N/2 \le i \le N-1$,

$$
\begin{aligned}
R_i &= I_1 + I_2 + I_3^1 + I_3^2 + I_3^3 + I_4 + I_5 \\
&\le C(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\end{aligned}
\tag{3.166}
$$

And what we left is the case $i = N$. Fortunately, we can use the same department of R_i above, and it is symmetric. Most of the item has been esitmated by Lemma 3.11 and Theorem 3.36, we just need to consider I_3, I_4.

THEOREM 3.38. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that*

$$I_3 = \sum_{j=\lceil \frac{N}{2} \rceil + 1}^{N-1} V_{Nj} \le Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha} \tag{3.167}$$

Proof. DEFINITION 3.39. *For $N/2 \le j < N$, Let's define*

$$y_j(x) = (\frac{Z_1}{h_N}(x - x_N) + Z_j)^r, \quad Z_j = T^{1/r}\frac{j}{N} \tag{3.168}$$

We can see that is the inverse of the function ${}_0y_{N-i}(x)$ defined in Theorem 3.30.

$$y_j'(x) = y_j^{1-1/r}(x)\frac{rZ_1}{h_N} \tag{3.169}$$

$$y_j''(x) = y_j^{1-2/r}(x)\frac{r(r-1)Z_1}{h_N} \tag{3.170}$$

With the scheme we used several times, we can get

LEMMA 3.40. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \le j < N$, $\xi \in [x_{N-1}, x_{N+1}]$,*

$$h_j(\xi)^3 \le Ch^3 \tag{3.171}$$

$$(h_j^3(\xi))' \le C(r-1)h^3 \tag{3.172}$$

$$(h_j^3(\xi))'' \le C(r-1)h^3 \tag{3.173}$$

$$u''(y_j^\theta(\xi)) \le C \tag{3.174}$$

$$(u''(y_j^\theta(\xi)))' \le C \tag{3.175}$$

$$(u''(y_j^\theta(\xi)))'' \le C \tag{3.176}$$

$$|\xi - y_j^\theta(\xi)|^{1-\alpha} \le C|x_N - y_j^\theta|^{1-\alpha} \tag{3.177}$$

$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})' \le C|x_N - y_j^\theta|^{1-\alpha} \tag{3.178}$$

$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})'' \le C|x_N - y_j^\theta|^{1-\alpha} + C(r-1)|x_N - y_j^\theta|^{-\alpha} \tag{3.179}$$

LEMMA 3.41. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \le j < N$,*

$$V_{Nj} \le Ch^2 \int_{x_{j-1}}^{x_j} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \tag{3.180}$$

Therefore,

$$\begin{aligned} I_3 &\le Ch^2 \int_{x_{\lceil \frac{N}{2} \rceil}}^{x_{N-1}} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \\ &\le Ch^2(|T - x_{N-1}|^{2-\alpha} + (r-1)|T - x_{N-1}|^{1-\alpha}) \end{aligned} \tag{3.181}$$

$\square$

For $j = N$,

LEMMA 3.42.

$$V_{N,N} = \frac{1}{h_N^2}\left(T_{N-1,N-1} - 2T_{N,N} + T_{N+1,N+1}\right) \le Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha} \tag{3.182}$$

Proof. ∎

$$\begin{aligned} V_{N,N} = &\int_0^1 -\frac{\theta(1-\theta)^{2-\alpha}}{2}\frac{1}{h_N^2}\left(h_{N-1}^{4-\alpha}u''(y_{N-1}^\theta) - 2h_N^{4-\alpha}u''(y_N^\theta) + h_{N+1}^{4-\alpha}u''(y_{N+1}^\theta)\right)d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,1}^\theta) - Q_{N\to N}^\theta(x_i)u'''(\eta_{N,1}^\theta)}{h_N}\right)d\theta \\ &- \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,1}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,1}^\theta)}{h_N}\right)d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,2}^\theta) - Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta)}{h_N}\right)d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,2}^\theta)}{h_N}\right)d\theta \end{aligned} \tag{3.183}$$

So combine Lemma 3.11, Theorem 3.36, Theorem 3.38, Lemma 3.42 We have

LEMMA 3.43.

$$R_N \le C(r-1)h^2|T - x_{N-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.184}$$

and with Theorem 3.37 we prove the Theorem 3.3

4. Convergence analysis.

4.1. Properties of some Matrices. Review subsection 2.1, we have got (2.10).

DEFINITION 4.1. *We call one matrix an M matrix , which means its entries are positive on major diagonal and nonpositive on others, and strictly diagonally dominant in rows.*

Now we have

LEMMA 4.2. *Matrix A defined by (2.12) where (2.13) is an M matrix. And there exists a constant $C_A = C(T, \alpha, r)$ such that*

$$S_i := \sum_{j=1}^{2N-1} a_{ij} \geq C_A(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \tag{4.1}$$

Proof. From (2.14), we have

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right) \tag{4.2}$$

Let

$$g(x) = g_0(x) + g_{2N}(x) \tag{4.3}$$

where

$$g_0(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x - x_0|^{3-\alpha} - |x - x_1|^{3-\alpha}}{h_1}$$

$$g_{2N}(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N} - x|^{3-\alpha} - |x_{2N-1} - x|^{3-\alpha}}{h_{2N}}$$

Thus

$$-\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

Then

$$\begin{aligned} S_i :=& \sum_{j=1}^{2N-1} a_{ij} \\ =& \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ =& D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i) \end{aligned} \tag{4.4}$$

When $i = 1$

$$\begin{aligned} D_h^2 g_0(x_1) = & \frac{2}{h_1 + h_2} \left(\frac{1}{h_2} g_0(x_2) - (\frac{1}{h_1} + \frac{1}{h_2}) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right) \\ =& \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2) h_1 h_2} \\ =& \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2) h_1^{1-\alpha} h_2} h_1^{-\alpha} \\ =& \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha}}{2^r (2^r - 1)} h_1^{-\alpha} \end{aligned} \tag{4.5}$$

but

$$1+(2^r-1)^{3-\alpha}+2(2^r-1)-(2^r)^{3-\alpha}>0 \tag{4.6}$$

While for $i \geq 2$

$$\begin{aligned} D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \\ &= -\kappa_\alpha \frac{|\xi-x_0|^{1-\alpha}-|\xi-x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1} \\ &= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)}|\xi-\eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\ &\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha} \end{aligned} \tag{4.7}$$

So

$$\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}g_0(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})g_0(x_i)+\frac{1}{h_i}g_0(x_{i-1})\right) \geq C x_i^{-\alpha} \tag{4.8}$$

symmetricly,

(4.9)
$$\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}g_{2N}(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})g_{2N}(x_i)+\frac{1}{h_i}g_{2N}(x_{i-1})\right) \geq C(\alpha,r)(2T-x_i)^{-\alpha}$$

Let

$$g(x)=\begin{cases} x, & 0<x\leq T \\ 2T-x, & T<x<2T \end{cases} \tag{4.10}$$

And define

$$G=\mathrm{diag}(g(x_1),...,g(x_{2N-1})) \tag{4.11}$$

Then

LEMMA 4.3. *The matrix $B:=AG$, the major diagnal is positive, and nonpositive on others. And there is a constant $C_{AG}, C=C(\alpha,r)$ such that*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C_{AG}(x_i^{1-\alpha}+(2T-x_i)^{1-\alpha})+C\begin{cases} |T-x_{i-1}|^{1-\alpha}, & i\leq N \\ |x_{i+1}-T|^{1-\alpha}, & i\geq N \end{cases} \tag{4.12}$$

Proof.

$$b_{ij}=a_{ij}g(x_j)=-\kappa_\alpha\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}\tilde{a}_{i+1,j}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})\tilde{a}_{i,j}+\frac{1}{h_i}\tilde{a}_{i-1,j}\right)g(x_j)$$

Since

$$g(x)\equiv \Pi_h g(x) \tag{4.13}$$

by **??**, we have

$$
\begin{aligned}
\tilde{M}_i &:= \sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} \tilde{a}_{ij} g(x_j) \\
&= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} \Pi_h g(y) dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} g(y) dy \\
&= \frac{-2}{\Gamma(4-\alpha)} |T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)} (x_i^{3-\alpha} + (2T - x_i)^{3-\alpha}) \\
&:= w(x_i) = p(x_i) + q(x_i)
\end{aligned} \tag{4.14}
$$

Thus,

$$
\begin{aligned}
M_i &:= \sum_{j=1}^{2N-1} b_{ij} = \sum_{j=1}^{2N-1} a_{ij} g(x_j) \\
&= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \tilde{M}_{i+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) \tilde{M}_i + \frac{1}{h_i} \tilde{M}_{i-1} \right) \\
&= D_h^2(-\kappa_\alpha p)(x_i) - \kappa_\alpha D_h^2 q(x_i)
\end{aligned} \tag{4.15}
$$

for $1 \leq i \leq N-1$,by Lemma A.1

$$
\begin{aligned}
D_h^2(-\kappa_\alpha p)(x_i) &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} p(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) p(x_i) + \frac{1}{h_i} p(x_{i-1}) \right) \\
&= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1}) \\
&\geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - x_{i-1}|^{1-\alpha}
\end{aligned} \tag{4.16}
$$

$$
\begin{aligned}
D_h^2(-\kappa_\alpha p)(x_N) &:= -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left(\frac{1}{h_{N+1}} p(x_{N+1}) - (\frac{1}{h_N} + \frac{1}{h_{N+1}}) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right) \\
&= \frac{4\kappa_\alpha}{\Gamma(4-\alpha) h_N^2} h_N^{3-\alpha} \\
&= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - x_{N-1})^{1-\alpha}
\end{aligned} \tag{4.17}
$$

Symmetricly for $i \geq N$, we get

$$
D_h^2(-\kappa_\alpha p)(x_i) \geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \tag{4.18}
$$

Similarly, we can get

$$
\begin{aligned}
D_h^2 q(x_i) &:= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} q(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right) \\
&\leq \frac{2^{r(\alpha-1)+1}}{\Gamma(2-\alpha)} (x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}), \quad i = 1, \cdots, 2N-1
\end{aligned} \tag{4.19}
$$

□

So, we get the result.

Notice that

$$x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha} \tag{4.20}$$

We can get

THEOREM 4.4. *There exists a real* $\lambda = \lambda(T, \alpha, r) > 0$ *and* $C = C(T, \alpha, r) > 0$ *such that* $B := A(\lambda I + G)$ *is an* M *matrix. And*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \tag{4.21}$$

Proof. By Lemma 4.2 with C_A and Lemma 4.3 with C_{AG} , it's sufficient to take $\lambda = (C + 2TC_{AG})/C_A$, then

$$M_i \geq C \left((x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \right) \tag{4.22}$$ □

4.2. Proof of Theorem 2.6. For equation

$$AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1} U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1} U = F \tag{4.23}$$

which means

$$\sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + g(x_j)} = -\tau_i \tag{4.24}$$

where $\epsilon_i = u(x_i) - u_i$.

And if

$$|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| = \max_{1 \leq i \leq 2N-1} |\frac{\epsilon_i}{\lambda + g(x_i)}| \tag{4.25}$$

Then, since $B = A(\lambda I + G)$ is an M matrix, it is Strictly diagonally dominant. Thus,

$$\begin{aligned} |\tau_{i_0}| &= |\sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + g(x_j)}| \\ &\geq b_{i_0,i_0} |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| - \sum_{j \neq i_0} |b_{i_0,j}| |\frac{\epsilon_j}{\lambda + g(x_j)}| \\ &\geq b_{i_0,i_0} |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| - \sum_{j \neq i_0} |b_{i_0,j}| |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| \\ &= \sum_{j=1}^{2N-1} b_{i_0,j} |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| \\ &= M_{i_0} |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| \end{aligned} \tag{4.26}$$

By Theorem 2.5 and Theorem 4.4,

We kown that there exists constants $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$, and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that

$$|\frac{\epsilon_i}{\lambda + g(x_i)}| \leq |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| \leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} + C_2(r-1)h^2 \tag{4.27}$$

as $\lambda + g(x_i) \leq \lambda + T$

So, we can get

$$|\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{r\alpha}{2},2\}} \tag{4.28}$$

The convergency has been proved.

Remarks:

5. Experimental results.

5.1. $f \equiv 1$.

5.2. $f = x^\gamma, \gamma < 0$. Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x) \in C^2(\Omega)$, there exists $\xi \in (x_{i-1}, x_{i+1})$ such that*

$$D_h^2 g(x_i) = g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{A.1}$$

And if $g(x) \in C^4(\Omega)$, then

$$\begin{aligned} D_h^2 g(x_i) &= g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) \\ &\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right) \end{aligned} \tag{A.2}$$

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})$$

Subsitute them in the left side of (A.1), we have

$$\begin{aligned} D_h^2 g(x_i) &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (g(x_{i+1}) - g(x_i) + \frac{1}{h_i} (g(x_{i-1}) - g(x_i)) \right) \\ &= \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) \end{aligned}$$

Now, using intermediate value theorem, there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy$$

Expecially,

$$\begin{aligned} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy &= \frac{h_i^4}{4!} g''''(\eta_1) \\ \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy &= \frac{h_{i+1}^4}{4!} g''''(\eta_2) \end{aligned} \tag{A.3}$$

where $\eta_1 \in (x_{i-1}, x_i), \eta_2 \in (x_i, x_{i+1})$. ∎

Lemma A.2. *Denote* $y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \theta \in (0,1)$,

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in (x_{j-1}, x_j) \tag{A.4}$$

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \tag{A.5}$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in (x_{j-1}, y_j^\theta)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in (y_j^\theta, x_j)$$

Thus

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta) u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$. Thus ∎

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

Lemma A.3. *For* $x \in [x_{j-1}, x_j]$

$$\begin{aligned} |u(x) - \Pi_h u(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_{x}^{x_j} u'(y)dy \right| \\ &\le \int_{x_{j-1}}^{x_j} |u'(y)|dy \end{aligned} \tag{A.6}$$

If $x \in [0, x_1]$, *with Corollary* 2.4, *we have*

$$|u(x) - \Pi_h u(x)| \le \int_0^{x_1} |u'(y)|dy \le \int_0^{x_1} C y^{\alpha/2-1} dy \le C\frac{2}{\alpha} x_1^{\alpha/2} = C\frac{2}{\alpha} h_1^{\alpha/2} \tag{A.7}$$

Similarly, if $x \in [x_{2N-1}, 1]$, *we have*

$$|u(x) - \Pi_h u(x)| \le C\frac{2}{\alpha}(2T - x_{2N-1})^{\alpha/2} = C\frac{2}{\alpha} h_{2N}^{\alpha/2} \tag{A.8}$$

LEMMA A.4. *By Lemma* A.2*, Corollary* 2.4 *and Lemma* B.1 *, There is a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *for* $2 \le j \le N$*,*

$$|u(y) - \Pi_h u(y)| \le h_j^2 \max_{\xi \in [x_{j-1}, x_j]} |u''(\xi)| \le Ch^2 y^{\alpha/2-2/r}, \quad \textit{for } y \in (x_{j-1}, x_j) \tag{A.9}$$

symmetricly, for $N < j \le 2N-1$*, we have*

$$|u(y) - \Pi_h u(y)| \le Ch^2 (2T-y)^{\alpha/2-2/r}, \quad \textit{for } y \in (x_{j-1}, x_j) \tag{A.10}$$

LEMMA A.5.

$$b^{1-\theta}|a^\theta - b^\theta| \le |a-b| \ (\ \textit{also } a^{1-\theta}|a^\theta - b^\theta| \le |a-b|), \quad a, b \ge 0, \ \theta \in [0,1] \tag{A.11}$$

Appendix B. Proofs of some technical details. Review that $h = \frac{1}{N}$ and the defination of $\simeq$ in subsection 2.1

LEMMA B.1.

$$h_i \simeq \begin{cases} hx_i^{1-1/r}, & 1 \le i \le N \\ h(2T - x_{i-1})^{1-1/r}, & N < i \le 2N \end{cases} \tag{B.1}$$

Since $i^r - (i-1)^r \simeq i^{r-1}$*, for* $i \ge 1$*.*

And

$$h_i \simeq h_{i+1}, \quad x_i \simeq x_{i+1} \simeq y_i^\theta, \quad \textit{for } 1 \le i \le 2N-1, \ \theta \in (0,1) \tag{B.2}$$

LEMMA B.2. *There is a constant* C *such that for* $i = 1, 2, \cdots, 2N-1$

$$|h_{i+1} - h_i| \le Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N-1 \end{cases} \tag{B.3}$$

Proof. By (2.2),

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \le i \le N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \le i \le 2N-1 \end{cases} \tag{B.4}$$

Since

$$(i+1)^r - 2i^r + (i-1)^r \simeq r(r-1)i^{r-2}, \quad \text{for } i \ge 1 \tag{B.5}$$

We get the result. ∎

LEMMA B.3. *there is a constant* $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ *such that*

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right| \\ &\le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases} \end{aligned} \tag{B.6}$$

Proof. By Lemma 2.2, we have for $1 \le i \le N$

$$\left|\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy\right| \le \frac{\|f\|_\beta^{(\alpha/2)}}{3!}\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y-x_{i-1})^3 dy \tag{B.7}$$

For $i=1$,

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y-x_{i-1})^3 dy = \int_0^{x_1} y^{1-\alpha/2}dy = \frac{1}{2-\alpha/2}x_1^{2-\alpha/2} = \frac{1}{2-\alpha/2}x_1^{-\alpha/2-2}h_1^4$$

And for $2 \le i \le N$, since $x_i \simeq x_{i-1} \le y \le x_i$, we have

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y-x_{i-1})^3 dy \simeq \int_{x_{i-1}}^{x_i} x_i^{-\alpha/2-2}(y-x_{i-1})^3 dy = \frac{1}{4!}x_i^{-\alpha/2-2}h_i^4$$

So for $1 \le i \le N$, we have

$$\left|\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy\right| \le Cx_i^{-\alpha/2-2}h_i^4 \tag{B.8}$$

and similarly,

$$\left|\int_{x_i}^{x_{i+1}} f''(y)\frac{(x_{i+1}-y)^3}{3!}dy\right| \le Cx_i^{-\alpha/2-2}h_{i+1}^4 \tag{B.9}$$

Thus for $1 \le i \le N$, with Lemma B.1 we have

$$\begin{aligned} &\frac{2}{h_i+h_{i+1}}\left|\frac{1}{h_i}\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy + \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}} f''(y)\frac{(y-x_{i+1})^3}{3!}dy\right| \\ &\le Cx_i^{-\alpha/2-2}\frac{2}{h_i+h_{i+1}}(h_i^3+h_{i+1}^3) \simeq x_i^{-\alpha/2-2}h_i^2 \simeq x_i^{-\alpha/2-2}h^2x_i^{2-2/r} \\ &= Ch^2x_i^{-\alpha/2-2/r} \end{aligned} \tag{B.10}$$

It's symmetric for $N < i \le 2N-1$. ∎

LEMMA B.4. *There is a constant $C = C(\alpha, r)$ such that for all $1 \le i \le 2N-1$, $1 \le j \le 2N$ s.t. $\min\{|j-i|, |j-1-i|\} \ge 2$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_h^2K_y(x_i) \simeq |y-x_i|^{-1-\alpha} \tag{B.11}$$

Proof. Sinec $y-x_{i-1}, y-x_i, y-x_{i+1}$ have the same sign, by Lemma A.1,

$$D_h^2K_y(x_i) = \frac{|y-\xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in (x_{i-1}, x_{i+1})$$

however, $|y-\xi| \simeq |y-x_i|$, we get the result. ∎

LEMMA B.5. *While $0 \le i < N/2$, By Lemma* A.3

$$\begin{aligned} |T_{i1}| &\le C\int_0^{x_1} x_1^{\alpha/2}\frac{|x_i-y|^{1-\alpha}}{\Gamma(2-\alpha)}dy \\ &= C\frac{1}{\Gamma(3-\alpha)}x_1^{\alpha/2}\left|x_i^{2-\alpha} - |x_i-x_1|^{2-\alpha}\right| \\ &\le C\frac{1}{\Gamma(3-\alpha)}x_1^{\alpha/2+2-\alpha} = C\frac{1}{\Gamma(3-\alpha)}x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1 \end{aligned} \tag{B.12}$$

For $2 \le j \le N$, by Lemma A.2 and Corollary 2.4

$$
\begin{aligned}
|T_{ij}| &\le \frac{C}{4}\int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2}\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy \\
&\le \frac{C}{4\Gamma(3-\alpha)}h_j^2 x_{j-1}^{\alpha/2-2}\left||x_j-x_i|^{2-\alpha}-|x_{j-1}-x_i|^{2-\alpha}\right|
\end{aligned}
\tag{B.13}
$$

LEMMA B.6. *There exists a constant $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$\sum_{j=1}^{3} V_{1j} \le Ch^2 x_1^{-\alpha/2-2/r} \tag{B.14}$$

$$\sum_{j=1}^{4} V_{2j} \le Ch^2 x_2^{-\alpha/2-2/r} \tag{B.15}$$

Proof. For $0 \le i \le 3, 1 \le j \le 4$, by Lemma A.3, Lemma A.4 and (3.20)

$$T_{ij} \le Cx_1^{2-\alpha/2} \simeq h_1^2\, h^2 x_1^{-\alpha/2-2/r} \simeq h_1^2\, h^2 x_2^{-\alpha/2-2/r} \tag{B.16}$$

Therefore, by (3.21), we get the result. ∎

LEMMA B.7. *There exists a constant $C = C(T,r,l)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2}\rceil \le j \le \min\{2i,N\}$,*
when $\xi \in (x_{i-1},x_{i+1})$,

$$(h_{j-i}^3(\xi))' \le (r-1)Ch^2 x_i^{1-2/r} h_j \tag{B.17}$$

$$(h_{j-i}^4(\xi))' \le (r-1)Ch^2 x_i^{1-2/r} h_j^2 \tag{B.18}$$

Proof. From (3.40)

$$y'_{j-i}(x) = y_{j-i}^{1-1/r}(x)x^{1/r-1} \tag{B.19}$$

$$y''_{j-i}(x) = \frac{1-r}{r}y_{j-i}^{1-2/r}(x)x^{1/r-2}Z_{j-i} \tag{B.20}$$

For $\xi \in (x_{i-1},x_{i+1})$ and $2 \le k \le j \le \min\{2i-1,N-1\}$, using Lemma B.1

$$\xi \simeq x_i \simeq x_j$$

$$h_{j-i}(\xi) \simeq h_j \simeq hx_j^{1-1/r} \simeq hx_i^{1-1/r}$$

$$
\begin{aligned}
h'_{j-i}(\xi) &= y'_{j-i}(\xi) - y'_{j-i-1}(\xi) \\
&= \xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi))
\end{aligned}
\tag{B.21}
$$

Since

$$
\begin{aligned}
y_{j-i}^{1-1/r}(\xi)-y_{j-i-1}^{1-1/r}(\xi) &\le x_{j+1}^{1-1/r}-x_{j-2}^{1-1/r}\\
&= T^{1-1/r}N^{1-r}((j+1)^{r-1}-(j-2)^{r-1})\\
&\le C(r-1)j^{r-2}N^{1-r}\\
&= C(r-1)hx_j^{1-2/r}
\end{aligned}
\tag{B.22}
$$

Therefore,

$$
h'_{j-i}(\xi)\le Cx_i^{1/r-1}(r-1)hx_j^{1-2/r}\simeq (r-1)hx_i^{-1/r} \tag{B.23}
$$

for $l=3,4$

$$
\begin{aligned}
(h_{j-i}^l(\xi))' &= lh_{j-i}^{l-1}(\xi)h'_{j-i}(\xi)\\
&\le Ch_{j-i}^{l-1}(\xi)(r-1)hx_i^{-1/r}\\
&\simeq Ch_j^{l-2}hx_j^{1-1/r}(r-1)hx_i^{-1/r}\\
&\simeq C(r-1)h^2x_i^{1-2/r}h_j^{l-2}
\end{aligned}
\tag{B.24}
$$

Meanwhile, we can get

$$
h_{j-i}^3(\xi)\simeq h_j^3\le Ch^2x_i^{2-2/r}h_j \tag{B.25}
$$

$$
h_{j-i}^4(\xi)\simeq h_j^4\le Ch^2x_i^{2-2/r}h_j^2 \tag{B.26}
$$

∎

LEMMA B.8. *There exists a constant $C=C(T,r,l)$ such that For $3\le i\le N-1, \lceil \frac{i}{2}\rceil \le j\le \min\{2i,N\}$, when $\xi\in(x_{i-1},x_{i+1})$,*

$$
(h_{j-i}^3(\xi))''\le C(r-1)h^2x_i^{-2/r}h_j \tag{B.27}
$$

Proof.

$$
(h_{j-i}^3(\xi))''=6h_{j-i}(\xi)(h'_{j-i}(\xi))^2+3h_{j-i}^2(\xi)h''_{j-i}(\xi) \tag{B.28}
$$

By (B.23)

$$
h_{j-i}(\xi)(h'_{j-i}(\xi))^2\le Ch_j(r-1)^2h^2x_i^{-2/r} \tag{B.29}
$$

For the second partial

$$
\begin{aligned}
h''_{j-i}(\xi) &= y''_{j-i}(\xi)-y''_{j-i-1}(\xi)\\
&= \frac{1-r}{r}\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i}-y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1})\\
&= \frac{1-r}{r}\xi^{1/r-2}((y_{j-i}^{1-2/r}(\xi)-y_{j-i-1}^{1-2/r}(\xi))Z_{j-i}+y_{j-i-1}^{1-2/r}(\xi)Z_1)
\end{aligned}
\tag{B.30}
$$

but

$$
\begin{aligned}
|y_{j-i}^{1-2/r}(\xi)-y_{j-i-1}^{1-2/r}(\xi)| &\le |x_{j+1}^{1-2/r}-x_{j-2}^{1-2/r}|\\
&= T^{1-2/r}N^{2-r}|(j+1)^{r-2}-(j-2)^{r-2}|\\
&\le C|r-2|N^{2-r}j^{r-3}\\
&= C|r-2|hx_j^{1-3/r}
\end{aligned}
\tag{B.31}
$$

So we can get

$$\begin{aligned} |h''_{j-i}(\xi)| &\le C(r-1)x_i^{1/r-2}(|r-2|hx_i^{1-3/r}x_i^{1/r}+x_i^{1-2/r}h) \\ &\le C(r-1)hx_i^{-1-1/r} \end{aligned} \tag{B.32}$$

Summarizes, we have

$$(h^3_{j-i}(\xi))'' \le C(r-1)h^2x_i^{-2/r}h_j \tag{B.33}$$ ∎

proof of Lemma 3.18. From (3.40)

$$y'_{j-i}(x) = y_{j-i}^{1-1/r}(x)x^{1/r-1} \tag{B.34}$$

$$y''_{j-i}(x) = \frac{1-r}{r}y_{j-i}^{1-2/r}(x)x^{1/r-2}Z_{j-i} \tag{B.35}$$

Since

$$y^\theta_{j-i}(\xi) \simeq x_j \simeq x_i$$

We have known

$$u''(y^\theta_{j-i}(\xi)) \le C(y^\theta_{j-i}(\xi))^{\alpha/2-2} \simeq x_j^{\alpha/2-2} \simeq x_i^{\alpha/2-2} \tag{B.36}$$

$$\begin{aligned} (u''(y^\theta_{j-i}(\xi)))' &= u'''(y^\theta_{j-i}(\xi))(y^\theta_{j-i}(\xi))' \\ &\le Cx_i^{\alpha/2-3}\xi^{1/r-1}y_{j-i}^{1-1/r}(\xi) \\ &\simeq x_i^{\alpha/2-3}x_i^{1/r-1}x_i^{1-1/r} = Cx_i^{\alpha/2-3} \end{aligned} \tag{B.37}$$

$$\begin{aligned} (u''(y^\theta_{j-i}(\xi)))'' &= u''''(y^\theta_{j-i}(\xi))({y^\theta_{j-i}}'(\xi))^2 + u'''(y^\theta_{j-i}(\xi)){y^\theta_{j-i}}''(\xi) \\ &\le Cx_i^{\alpha/2-4} + Cx_i^{\alpha/2-3}\frac{r-1}{r}x_i^{1-2/r}x_i^{1/r-2}Z_{|j-i|+1} \\ &\le Cx_i^{\alpha/2-4} + C\frac{r-1}{r}x_i^{\alpha/2-3}x_i^{-1-1/r}x_i^{1/r} \\ &= Cx_i^{\alpha/2-4} \end{aligned} \tag{B.38}$$ ∎

Proof of Lemma 3.19.

$$\begin{aligned} |y^\theta_{j-i}(\xi)-\xi| &= |\theta(y_{j-i-1}(\xi)-\xi)+(1-\theta)(y_{j-i}(\xi)-\xi)| \\ &= \theta|y_{j-i-1}(\xi)-\xi| + (1-\theta)|y_{j-i}(\xi)-\xi| \end{aligned} \tag{B.39}$$

where $y_{j-i-1}(\xi)-\xi$ and $y_{j-i}(\xi)-\xi$ have the same sign (≥ 0 or ≤ 0), independent with ξ.

Since $|y_{j-i}(\xi)-\xi| = \mathrm{sign}(j-i)(y_{j-i}(\xi)-\xi)$ is increasing with ξ,

$$(\frac{i-1}{i})^r|x_j-x_i| \le |x_{j-1}-x_{i-1}| \le |y_{j-i}(\xi)-\xi| \le |x_{j+1}-x_{i+1}| \le (\frac{i+1}{i})^r|x_j-x_i| \tag{B.40}$$

we have

$$|y_{j-i}(\xi)-\xi| \simeq |x_j-x_i| \tag{B.41}$$

Similarly, $|y_{j-1-i}(\xi)-\xi| \simeq |x_{j-1}-x_i|$. Thus, with (B.39) , (B.41) and (2.17) we get

$$|y^{\theta}_{j-i}(\xi)-\xi| \simeq |y^{\theta}_j - x_i| \tag{B.42}$$

Next, since $|y^{\theta}_{j-i}(\xi)-\xi| = \mathrm{sign}(j-i-1+\theta)(y^{\theta}_{j-i}(\xi)-\xi)$, so we can derivate it.

$$|(|y^{\theta}_{j-i}(\xi)-\xi|^{1-\alpha})'| = (\alpha-1)|y^{\theta}_{j-i}(\xi)-\xi|^{-\alpha}|(y^{\theta}_{j-i}(\xi))'-1| \tag{B.43}$$

While, similar with (B.39), we have

$$|(y^{\theta}_{j-i}(\xi))'-1| = (1-\theta)|y'_{j-i-1}(\xi)-1| + \theta|y'_{j-i}(\xi)-1| \tag{B.44}$$

By Lemma A.5 and (B.41), we have

$$\begin{aligned} |y'_{j-i}(\xi)-1| &= \xi^{1/r-1}|y^{1-1/r}_{j-i}(\xi)-\xi^{1-1/r}| \\ &\leq \xi^{-1}|y_{j-i}(\xi)-\xi| \\ &\simeq x_i^{-1}|x_j-x_i| \end{aligned} \tag{B.45}$$

So similar with (B.42), we can get

$$|(y^{\theta}_{j-i}(\xi))'-1| \leq C x_i^{-1}|y^{\theta}_j - x_i| \tag{B.46}$$

Combine with (B.42), we get

$$|(|y^{\theta}_{j-i}(\xi)-\xi|^{1-\alpha})'| \leq C|y^{\theta}_j - x_i|^{-\alpha} x_i^{-1}|y^{\theta}_j - x_i| = C|y^{\theta}_j - x_i|^{1-\alpha}x_i^{-1} \tag{B.47}$$

Finally, we have

$$\begin{aligned} (|y^{\theta}_{j-i}(\xi)-\xi|^{1-\alpha})'' = \alpha(\alpha-1)&|y^{\theta}_{j-i}(\xi)-\xi|^{-\alpha-1}((y^{\theta}_{j-i}(\xi))'-1)^2 \\ &+ \mathrm{sign}(j-i-1+\theta)(1-\alpha)|y^{\theta}_{j-i}(\xi)-\xi|^{-\alpha}(y^{\theta}_{j-i}(\xi))'' \end{aligned} \tag{B.48}$$

For

$$(y^{\theta}_{j-i}(\xi))'' = (1-\theta)y''_{j-i-1}(\xi) + \theta y''_{j-i}(\xi) \tag{B.49}$$

and

$$\begin{aligned} y''_{j-i}(\xi) &= \frac{1-r}{r} y^{1-2/r}_{j-i}(x)x^{1/r-2}Z_{j-i} \\ &\simeq \frac{1-r}{r} x_j^{1-2/r}x_i^{1/r-2}Z_{j-i} \end{aligned} \tag{B.50}$$

while by Lemma A.5

$$|Z_{j-i}| = |x_j^{1/r} - x_i^{1/r}| \leq |x_j - x_i|x_i^{1/r-1} \tag{B.51}$$

we have

$$|y''_{j-i}(\xi)| \leq C(r-1)x_i^{-2}|x_j-x_i| \tag{B.52}$$

Therefore

$$|(y^{\theta}_{j-i}(\xi))''| \leq C(r-1)x_i^{-2}|y^{\theta}_j - x_i| \tag{B.53}$$

Then, combine with (B.46),

$$|(|y^{\theta}_{j-i}(\xi)-\xi|^{1-\alpha})''| \leq C|y^{\theta}_j - x_i|^{1-\alpha}x_i^{-2} \tag{B.54}$$

□

proof of Lemma 3.21. For $\lceil \frac{i}{2} \rceil \le j \le \min\{2i-1, N-1\}$

$$\begin{aligned} &\frac{Q^{\theta}_{j-i}(x_{i+1})u'''(\eta^{\theta}_{j+1}) - Q^{\theta}_{j-i}(x_i)u'''(\eta^{\theta}_{j})}{h_{i+1}} \\ &\quad = \frac{Q^{\theta}_{j-i}(x_{i+1}) - Q^{\theta}_{j-i}(x_i)}{h_{i+1}} u'''(\eta^{\theta}_{j+1}) + Q^{\theta}_{j-i}(x_i) \frac{u'''(\eta^{\theta}_{j+1}) - u'''(\eta^{\theta}_{j})}{h_{i+1}} \end{aligned} \tag{B.55}$$

Since mean value theorem

$$\frac{Q^{\theta}_{j-i}(x_{i+1}) - Q^{\theta}_{j-i}(x_i)}{h_{i+1}} = {Q^{\theta}_{j-i}}'(\xi), \quad \xi \in (x_i, x_{i+1}) \tag{B.56}$$

From (3.47) and Leibniz rule, by Lemma B.7 and Lemma 3.19, we have

$$|{Q^{\theta}_{j-i}}'(\xi)| \le Ch^2 \frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 \tag{B.57}$$

And by Definition 3.14 and Lemma B.1

$$Q^{\theta}_{j-i}(x_i) = h_j^4 \frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \simeq Ch^2 x_i^{2-2/r} \frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 \tag{B.58}$$

With $\eta^{\theta}_j \in (x_{j-1}, x_j)$

$$u'''(\eta^{\theta}_{j+1}) \le C(\eta^{\theta}_{j+1})^{\alpha/2-3} \simeq x_j^{\alpha/2-3} \simeq x_i^{\alpha/2-3}$$

and

$$\begin{aligned} \frac{u'''(\eta^{\theta}_{j+1}) - u'''(\eta^{\theta}_{j})}{h_{i+1}} &= u''''(\eta) \frac{\eta^{\theta}_{j+1} - \eta^{\theta}_{j}}{h_{i+1}} \\ &\le C\eta^{\alpha/2-4} \frac{x_{j+1} - x_{j-1}}{h_{i+1}} = C\eta^{\alpha/2-4} \frac{h_{j+1} + h_j}{h_{i+1}} \\ &\simeq x_j^{\alpha/2-4} \simeq x_i^{\alpha/2-4} \end{aligned}$$

So we have

$$\begin{aligned} &\frac{Q^{\theta}_{j-i}(x_{i+1})u'''(\eta^{\theta}_{j+1}) - Q^{\theta}_{j-i}(x_i)u'''(\eta^{\theta}_{j})}{h_{i+1}} \\ &\le Ch^2 \frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 \, x_i^{\alpha/2-3} + Ch^2 x_i^{2-2/r} \frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 x_{j-1}^{\alpha/2-4} \\ &= Ch^2 \frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j^2 \end{aligned} \tag{B.59}$$

while $h_j \simeq h_i$, subsitute into the inequality above, we get the goal

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_{i+1})u'''(\eta^{\theta}_{j+1}) - Q^{\theta}_{j-i}(x_i)u'''(\eta^{\theta}_{j})}{h_{i+1}} \right) \\ &\le \frac{1}{h_i} Ch^2 \frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j h_i \\ &= Ch^2 \frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned} \tag{B.60}$$

While, the later is similar. ∎

Lemma B.9. *There exists a constant $C = C(T, r)$ such that For $N/2 \le i \le N-1$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $l = 3, 4$, $\xi \in (x_{i-1}, x_{i+1})$, we have*

$$h^l_{j-i}(\xi) \le Ch^l_j \le Ch^2 h^{l-2}_j \tag{B.61}$$

$$(h^l_{j-i-1}(\xi))' \le C(r-1)h^2 h^{l-2}_j \tag{B.62}$$

$$(h^3_{j-i}(\xi))'' \le C(r-1)h^2 h_j \tag{B.63}$$

Proof.

$$\begin{aligned}(h_{j-i}(\xi))' &= {y_{j-i}}'(\xi) - {y_{j-i-1}}'(\xi)\\ &= \xi^{1/r-1}((2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}) \le 0\end{aligned} \tag{B.64}$$

Thus,

$$Ch_j \le h_{j+1} \le h_{j-i}(\xi) \le h_{j-i}(x_{i-1}) = h_{j-1} \le Ch_j \tag{B.65}$$

So as $4^{-r}T \le 2T - x_j \le T, 2^{-r}T \le x_i \le T$, we have

$$h^l_{j-i}(\xi) \le Ch^l_j \le Ch^2(2T - x_j)^{2-2/r} h^{l-2}_j \le Ch^2 h^{l-2}_j \tag{B.66}$$

Since

$$\begin{aligned}&|(2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}|\\ &= |(Z_{2N-(j-i)} - \xi^{1/r})^{r-1} - (Z_{2N-(j-1-i)} - \xi^{1/r})^{r-1}|\\ &= (r-1)Z_1(Z_{2N-(j-i-\gamma)} - \xi^{1/r})^{r-2} \quad \gamma \in [0, 1]\\ &\le C(r-1)h(2T - x_j)^{1-2/r}\end{aligned} \tag{B.67}$$

we have

$$|(h_{j-i}(\xi))'| \le C(r-1)h(2T - x_j)^{1-2/r} x_i^{1/r-1} \tag{B.68}$$

And

$$\begin{aligned}(h^l_{j-i}(\xi))' &= lh^{l-1}_{j-i}(\xi){h_{j-i}}'(\xi)\\ &\le C(r-1)h^{l-1}_j \; h(2T - x_j)^{1-2/r} x_i^{1/r-1}\\ &\le C(r-1)h^2 h^{l-2}_j (2T - x_j)^{2-3/r} x_i^{1-1/r}\\ &\le C(r-1)h^2 h^{l-2}_j\end{aligned} \tag{B.69}$$

(B.70)

$$\begin{aligned}(h^3_{j-i}(\xi))'' &= 6h_{j-i}(\xi)({y_{j-i}}'(\xi) - {y_{j-i-1}}'(\xi))^2 + 3h^2_{j-i}(\xi)({y_{j-i}}''(\xi) - {y_{j-i-1}}''(\xi))\\ &\le C(r-1)h_j h^2 + Ch^2_j \frac{1-r}{r} \xi^{1/r-2}((2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-1-i)})\\ &\le C(r-1)h_j h^2 + C(r-1)h^2_j(C(r-2)h(2T - x_j)^{1-3/r} Z_{2N-(j-i)} + Z_1(2T - x_{j-1})^{1-2/r})\\ &\le C(r-1)h_j h^2 + C(r-1)h^2_j h = Ch^2 h_j\end{aligned}$$

∎

Lemma B.10. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le i \le N-1$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in (x_{i-1}, x_{i+1})$, we have*

$$u''(y_{j-i}^{\theta}(\xi)) \le C \tag{B.71}$$

$$(u''(y_{j-i}^{\theta}(\xi)))' \le C \tag{B.72}$$

$$(u''(y_{j-i}^{\theta}(\xi)))'' \le C \tag{B.73}$$

Proof.

$$x_{j-2} \le y_{j-i}^{\theta}(\xi) \le x_{j+1} \Rightarrow 4^{-r}T \le 2T - y_{j-i}^{\theta}(\xi) \le T \tag{B.74}$$

Thus, for $l = 2, 3, 4$,

$$u^{(l)}(y_{j-i}^{\theta}(\xi)) \le C(2T - y_{j-i}^{\theta}(\xi))^{\alpha/2-l} \le C \tag{B.75}$$

and

$$\begin{aligned}(y_{j-i}^{\theta}(\xi))' &= \theta {y_{j-1-i}}'(\xi) + (1-\theta){y_{j-i-1}}'(\xi)\\ &= \xi^{1/r-1}(\theta(2T - y_{j-1-i}(\xi))^{1-1/r} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-1/r})\\ &\le C(2T - x_{j-2})^{1-1/r} \le C\end{aligned} \tag{B.76}$$

With

$$Z_{2N-j-i} \le 2T^{1/r} \tag{B.77}$$

$$\begin{aligned}(y_{j-i}^{\theta}(\xi))'' &= \theta {y_{j-1-i}}''(\xi) + (1-\theta){y_{j-i-1}}''(\xi)\\ &= \frac{1-r}{r}\xi^{1/r-2}(\theta(2T - y_{j-i-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)} + (1-\theta)(2T - y_{j-i}(\xi))^{1-2/r}Z_{2N-(j-i)})\\ &\le C(r-1)\end{aligned} \tag{B.78}$$

Therefore,

$$\begin{aligned}(u''(y_{j-i}^{\theta}(\xi)))' &= u'''(y_{j-i}^{\theta}(\xi))(y_{j-i}^{\theta}(\xi))'\\ &\le C\end{aligned} \tag{B.79}$$

$$\begin{aligned}(u''(y_{j-i}^{\theta}(\xi)))'' &= u'''(y_{j-i}^{\theta}(\xi))({y_{j-i}^{\theta}}'(\xi))^2 + u''''(y_{j-i}^{\theta}(\xi)){y_{j-i}^{\theta}}''(\xi)\\ &\le C + C(r-1) = C\end{aligned} \tag{B.80}$$

□

Lemma B.11. *There exists a constant $C = C(T, \alpha, r)$ such that For $N/2 \le i \le N-1$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in (x_{i-1}, x_{i+1})$*

$$|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha} \le C|y_j^{\theta} - x_i|^{1-\alpha} \tag{B.81}$$

$$\left|(|y_{j-i}^{\theta}(\xi) - \xi)^{1-\alpha}|'\right| \le C|y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N) \tag{B.82}$$

$$\left|(|y_{j-i}^{\theta}(\xi) - \xi)^{1-\alpha}|''\right| \le C(r-1)|y_j^{\theta} - x_i|^{-\alpha} + C|y_j^{\theta} - x_i|^{-1-\alpha}(|2T - x_i - y_j^{\theta}| + h_N)^2 \tag{B.83}$$

Proof. Since $y_{j-i-1}(\xi) > x_{j-2} \geq x_N > \xi$

$$y^{\theta}_{j-i}(\xi) - \xi = (1-\theta)(y_{j-1-i}(\xi) - \xi) + \theta(y_{j-i}(\xi) - \xi) > 0 \tag{B.84}$$

$$\begin{aligned} (y_{j-i}(\xi) - \xi)'' &= {y_{j-i}}''(\xi) \\ &= \frac{1-r}{r}\xi^{1/r-2}(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} \leq 0 \end{aligned} \tag{B.85}$$

It's concave, so

$$y_{j-i}(\xi) - \xi \geq \min_{\xi \in \{x_{i-1}, x_{i+1}\}} y_{j-i}(\xi) - \xi = \min\{x_{j+1} - x_{i+1}, x_{j-1} - x_{i-1}\} \geq C(x_j - x_i) \tag{B.86}$$

With (B.84), we have

$$|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha} \leq C|y^{\theta}_j - x_i|^{1-\alpha} \tag{B.87}$$

By Lemma A.5

$$\begin{aligned} |{y_{j-i}}'(\xi) - 1| &= \xi^{1/r-1}|(2T - y_{j-i}(\xi))^{1-1/r} - \xi^{1-1/r}| \\ &\leq \xi^{-1}|2T - y_{j-i}(\xi) - \xi| \end{aligned} \tag{B.88}$$

$$\begin{aligned} |2T - \xi - y_{j-i}(\xi)| &\leq |2T - x_i - x_j| + |x_i - \xi| + |x_j - y_{j-i}(\xi)| \\ &\leq |2T - x_i - x_j| + h_{i+1} + h_j \\ &\leq C\left(|2T - x_i - x_j| + h_N\right) \end{aligned} \tag{B.89}$$

With $\xi \simeq x_i \simeq 1$,

$$|{y_{j-i}}'(\xi) - 1| \leq C\left(|2T - x_i - x_j| + h_N\right) \tag{B.90}$$

Thus,

$$\begin{aligned} |(y^{\theta}_{j-i}(\xi))' - 1| &\leq (1-\theta)|{y_{j-i-1}}'(\xi) - 1| + \theta|{y_{j-i}}'(\xi) - 1| \\ &\leq C\left((1-\theta)|2T - x_i - x_{j-1}| + \theta|2T - x_i - x_j| + h_N\right) \\ &= C\left(|2T - x_i - y^{\theta}_j| + h_N\right) \end{aligned} \tag{B.91}$$

So

$$\begin{aligned} \left|(|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})'\right| &= |1-\alpha||y^{\theta}_{j-i}(\xi) - \xi|^{-\alpha}|(y^{\theta}_{j-i}(\xi))' - 1| \\ &\leq C|y^{\theta}_j - x_i|^{-\alpha}(|2T - x_i - y^{\theta}_j| + h_N) \end{aligned} \tag{B.92}$$

$$\begin{aligned} \left|(|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})''\right| &\leq |1-\alpha||y^{\theta}_{j-i}(\xi) - \xi|^{-\alpha}|(y^{\theta}_{j-i}(\xi) - \xi)''| + \alpha(\alpha-1)|y^{\theta}_{j-i}(\xi) - \xi|^{-1-\alpha}({y^{\theta}_{j-i}}'(\xi) - 1)^2 \\ &\leq C(r-1)|y^{\theta}_j - x_i|^{-\alpha} + C|y^{\theta}_j - x_i|^{-1-\alpha}(|2T - x_i - y^{\theta}_j| + h_N)^2 \end{aligned} \tag{B.93}$$

$\square$

Acknowledgments. We would like to acknowledge the assistance of volunteers in putting together this example manuscript and supplement.

REFERENCES

[1] D. Gilbarg and N. S. Trudinger, *Poisson's Equation and the Newtonian Potential*, Springer Berlin Heidelberg, Berlin, Heidelberg, 1977, pp. 50–67, https://doi.org/10.1007/978-3-642-96379-7_4, https://doi.org/10.1007/978-3-642-96379-7_4.

[2] X. Ros-Oton and J. Serra, *The dirichlet problem for the fractional laplacian: Regularity up to the boundary*, Journal de Mathématiques Pures et Appliquées, 101 (2014), pp. 275–302, https://doi.org/https://doi.org/10.1016/j.matpur.2013.06.003, https://www.sciencedirect.com/science/article/pii/S0021782413000895.