A SECOND ORDER NUMERICAL METHODS FOR REISZ-FRACTIONAL ELLIPTIC EQUATION ON GRADED MESH*

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Key words. example, LaTeX

MSC codes. ??????????????????

1. Introduction.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$,

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

2. Preliminaries: Numeric scheme and main results.

2.1. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases} \tag{2.1}$$

where $r \ge 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N \tag{2.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

And then, define the piecewise linear interpolant of the true solution u to be

$$\Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{2.4}$$

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For convience, we denote

$$I^{2-\alpha}u(x) := \frac{1}{\Gamma(2-\alpha)}\int_\Omega |x-y|^{1-\alpha}u(y)dy \tag{2.5}$$

and

$$D_h^2 u(x_i) := \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}u(x_{i-1}) - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)u(x_i) + \frac{1}{h_{i+1}}u(x_{i+1})\right) \tag{2.6}$$

Now, we discretise (1.1) by replacing $u(x)$ by a continuous piecewise linear function

$$u_h(x) := \sum_{j=1}^{2N-1} u_j\phi_j(x) \tag{2.7}$$

whose nodal values u_j are to be determined by collocation at each mesh point x_i for $i = 1, 2, ..., 2N-1$:

$$-\kappa_\alpha D_h^\alpha u_h(x_i) := -\kappa_\alpha D_h^2 I^{2-\alpha}u_h(x_i) = f(x_i) =: f_i \tag{2.8}$$

Here,

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i)\ u_j = \sum_{j=1}^{2N-1} a_{ij}\ u_j \tag{2.9}$$

where

$$a_{ij} = -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i) \quad \text{for} \quad i, j = 1, 2, ..., 2N-1 \tag{2.10}$$

We have replaced $(-\Delta)^{\alpha/2}u(x_i) = f(x_i)$ in (1.1) by $-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i)$ in (2.8), with truncation error

$$\tau_i := -\kappa_\alpha\left(D_h^\alpha \Pi_h u(x_i) - \frac{d^2}{dx^2}I^{2-\alpha}u(x_i)\right) \quad \text{for} \quad i = 1, 2, ..., 2N-1 \tag{2.11}$$

where $-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^\alpha \phi_j(x_i)u(x_j) = \sum_{j=1}^{2N-1} a_{ij}u(x_j)$.

The discrete equation (2.8) can be written in matrix form

$$AU = F \tag{2.12}$$

where $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$, $U = (u_1, \cdots, u_{2N-1})^T$ is unknown and $F = (f_1, \cdots, f_{2N-1})^T$.

We can deduce a_{ij},

$$\begin{aligned} a_{ij} &= -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i) \\ &= -\kappa_\alpha \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}\tilde{a}_{i-1,j} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)\tilde{a}_{i,j} + \frac{1}{h_{i+1}}\tilde{a}_{i+1,j}\right) \end{aligned} \tag{2.13}$$

where

$$I^{2-\alpha}\Pi_h u(x_i) = \sum_{j=1}^{2N-1} I^{2-\alpha}\phi_j(x_i)u(x_j) = \sum_{j=1}^{2N-1}\tilde{a}_{ij}u(x_j) \tag{2.14}$$

and

$$\begin{aligned} \tilde{a}_{ij} &= I^{2-\alpha}\phi_j(x_i) \\ &= \frac{1}{\Gamma(4-\alpha)}\left(\frac{|x_i-x_{j-1}|^{3-\alpha}}{h_j} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)|x_i-x_j|^{3-\alpha} + \frac{|x_i-x_{j+1}|^{3-\alpha}}{h_{j+1}}\right) \end{aligned} \tag{2.15}$$

2.2. Regularity of the true solution. For any $\beta > 0$, we use the standard notation $C^\beta(\bar{\Omega}), C^\beta(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^\beta(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous[1] with exponent β' in Ω, where $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on open set Ω.

DEFINITION 2.1 (delta dependent norm [2]).

$$\delta(x) = dist(x, \partial\Omega) = \begin{cases} x, & 0 < x \le T \\ 2T - x, & T < x < 2T \end{cases}, \quad x \in \Omega \tag{2.16}$$

$$\delta(x, y) = \min\{\delta(x), \delta(y)\}, \quad x, y \in \Omega \tag{2.17}$$

LEMMA 2.2. *Let $f \in C^\beta(\Omega), \beta > 2$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

$$|f^{(l)}(x)| \le \|f\|_\beta^{(\alpha/2)} \delta(x)^{-l-\alpha/2} \tag{2.18}$$

THEOREM 2.3 (Regularity up to the boundary [2]). *Let Ω be a bounded domain, and $\beta > 0$ be such that neither β nor $\beta + \alpha$ is an integer. Let $f \in C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R}^n)$ be a solution of (1.1). Then, $u \in C^{\beta+\alpha}(\Omega)$ and*

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)}\right) \tag{2.19}$$

where C is a constant depending only on Ω, α, and β.

COROLLARY 2.4. *Let u be a solution of (1.1) where $f \in L^\infty(\Omega)$ and $\|f\|_\beta^{(\alpha/2)} < \infty$. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \le \|u\|_{\beta+\alpha}^{(-\alpha/2)} \delta(x)^{\alpha/2-l} \tag{2.20}$$

And in this paper bellow, without special instructions, we allways assume that

$$f \in L^\infty(\Omega) \cap C^\beta(\Omega) \quad \text{and} \quad \|f\|_\beta^{(\alpha/2)} < \infty, \text{ with } \alpha + \beta > 4 \tag{2.21}$$

2.3. Main results. Here we state our main results; the proof is deferred to section 3 and section 4.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 2.5 (Local Truncation Error). *If $u(x)$ is a solution of the equation (1.1) where f satisfy the regular condition (2.21), then there exists $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$ and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$, such that the truncation error (2.11) satisfies*

$$\begin{aligned} |\tau_i| :=& |-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i)| \\ \le& C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} \delta(x_i)^{-\alpha} + C_2(r-1)h^2(T - \delta(x_i) + h_N)^{1-\alpha} \end{aligned} \tag{2.22}$$

THEOREM 2.6 (Global Error). *The discrete equation* (2.8) *has sulotion and there exists a positive constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ *such that the error between the numerial solution* U *with the exact solution* $u(x_i)$ *satisfies*

$$\max_{1\le i\le 2N-1} |u_i - u(x_i)| \le Ch^{\min\{\frac{r\alpha}{2},2\}} \tag{2.23}$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

Remark 2.7. ...

3. Local Truncation Error.

We shall first introduce some notations.

For convenience, we use the notation $\simeq$. That $x_1 \simeq y_1$, means that $c_1 x_1 \le y_1 \le C_1 x_1$ for some positive constants c_1 and C_1 that are independent of N.

And for $1 \le j \le 2N$, we define

$$y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \quad \theta \in (0,1) \tag{3.1}$$

Then we have

LEMMA 3.1. *For* $1 \le i \le 2N-1$

$$h_i \simeq h_{i+1} \simeq h\delta(x_i)^{1-1/r}, \quad \delta(x_i) \simeq \delta(x_{i+1}) \simeq \delta(y_{i+1}^\theta) \tag{3.2}$$

Since $i^r - (i-1)^r \simeq i^{r-1}$, *for* $i \ge 1$, *where* $\theta \in (0,1)$.

Meanwhile, let's define kernel functions

$$K_y(x) := \frac{|y-x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.3}$$

3.1. Proof of Theorem 2.5.

The truncation error of the discrete format can be written as

$$\begin{aligned} -\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I^{2-\alpha} \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i)) \\ &= -\kappa_\alpha D_h^2 I^{2-\alpha} (\Pi_h u - u)(x_i) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) \end{aligned} \tag{3.4}$$

THEOREM 3.2. *There exits a constant* $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$ *such that*

$$\left| -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) \right| \le Ch^2 \delta(x_i)^{-\alpha/2-2/r} \tag{3.5}$$

Proof. Since $f \in C^2(\Omega)$ and

$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha} u(x)) = f(x), \quad x \in \Omega, \tag{3.6}$$

we have $I^{2-\alpha} u \in C^4(\Omega)$. Therefore, using equation (A.2) of Lemma A.1, for $1 \le i \le 2N-1$, we have

$$\begin{aligned} &-\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i) \\ &\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right) \end{aligned} \tag{3.7}$$

By Lemma B.1, Lemma 2.2 and Lemma B.2, we get the result. ∎

And now define

$$R_i := D_h^2 I^{2-\alpha}(u - \Pi_h u)(x_i), \quad 1 \le i \le 2N-1 \tag{3.8}$$

We have some results about the estimate of R_i

THEOREM 3.3. *For $1 \le i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$|R_i| \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.9}$$

THEOREM 3.4. *For $N/2 \le i \le N$, there exists constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$|R_i| \le C(r-1)h^2(T - x_i + h_N)^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.10}$$

And for $N < i \le 2N-1$, it is symmetric to the previous case.

Combine Theorem 3.2, Theorem 3.3 and Theorem 3.4, and for $1 \le i \le N$, we have

$$h^2 x_i^{-\alpha/2-2/r} \le T^{\alpha/2-2/r} h^{\min\{\frac{r\alpha}{2}, 2\}} x_i^{-\alpha} \tag{3.11}$$

$$h^{r\alpha/2+r} x_i^{-1-\alpha} \le T^{-1} h^{r\alpha/2} x_i^{-\alpha} \tag{3.12}$$

$$h^r x_i^{-1} \ln(i) = T^{-1}\frac{\ln(i)}{i^r} \le T^{-1}, \quad h^r \ln(N) = \frac{\ln(N)}{N^r} \le 1 \tag{3.13}$$

the proof of Theorem 2.5 completed.

We prove Theorem 3.3 and Theorem 3.4 in next subsections.

3.2. Outlines and Mesh Transport Functions. For convience, let's denote

DEFINITION 3.5.

$$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy, \quad i = 0, \cdots, 2N,\ j = 1, \cdots, 2N \tag{3.14}$$

Also, we denote vertical difference quotients of T_{ij}

$$\begin{aligned} V_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \\ &= \int_{x_{i-1}}^{x_i} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \end{aligned} \tag{3.15}$$

And skew difference quotients of T_{ij}

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j-1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j+1} \right) \tag{3.16}$$

then $R_i = \sum_{j=1}^{2N} V_{ij}$.

Our main idea is to depart R_i by V_{ij} and S_{ij}. For $3 \leq i < N/2$, let's denote $k = \lceil \frac{i}{2} \rceil$, and take some suitable integer m, then

$$
\begin{aligned}
R_i =& \sum_{j=1}^{2N} V_{ij} \\
=& \sum_{j=1}^{k-1} V_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\
& + \sum_{j=k+1}^{m-1} S_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,m} + T_{i-1,m-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,m} \right) \\
& + \sum_{j=m+1}^{2N} V_{ij} \\
=& I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}
\tag{3.17}
$$

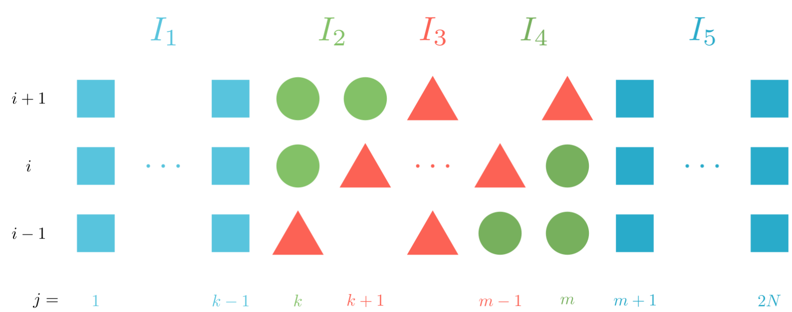

FIG. 1. *The departure of R_i for $i \geq 3$*

and discuss $i = 1, 2$ separately, where

$$
R_1 = \sum_{i=1}^{3} V_{1,j} + \sum_{i=4}^{N} V_{i,j}, \quad R_2 = \sum_{i=1}^{4} V_{1,j} + \sum_{i=5}^{N} V_{i,j} \tag{3.18}
$$

The difficulty for esitmating S_{ij} is that $T_{i-1,j-1}, T_{i,j}$ and $T_{i+1,j+1}$ have different integral region. We first make them normalized.

LEMMA 3.6. *For $y \in (x_{j-1}, x_j)$, we can rewrite $y = y_j^\theta$, from* (3.14)*, and Lemma* A.2*,*

$$
\begin{aligned}
T_{ij} =& \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
=& \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\
=& \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
& + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta
\end{aligned}
\tag{3.19}
$$

where $\eta_{j1}^\theta \in (x_{j-1}, y_j^\theta), \eta_{j2}^\theta \in (y_j^\theta, x_j)$.

Since j changes with i at indices of elements in S_{ij} by (3.16), we create some functions satisfy the property.

DEFINITION 3.7 (Mesh Transport Functions). *For* $1 \le i, j \le 2N-1$.

$$y_{i,j}(x) = \begin{cases} (x^{1/r} + Z_{j-i})^r & i<N, j<N \\ \dfrac{x^{1/r} - Z_i}{Z_1} h_N + x_N & i<N, j=N \\ 2T - (Z_{2N-(j-i)} - x^{1/r})^r & i<N, j>N \\ \left(\dfrac{Z_1}{h_N}(x - x_N) + Z_j\right)^r & i=N, j<N \\ x, & i=N, j=N \\ 2T - \left(\dfrac{Z_1}{h_N}(2T - x - x_N) + Z_{2N-j}\right)^r & i=N, j>N \\ (Z_{2N+j-i} - (2T-x)^{1/r})^r & i>N, j<N \\ \dfrac{Z_{2N-j} - (2T-x)^{1/r}}{Z_1} h_N + x_N & i>N, j=N \\ 2T - ((2T-x)^{1/r} - Z_{j-i})^r & i>N, j>N \end{cases} \tag{3.20}$$

where $Z_j := T^{1/r}\frac{j}{N}$. *And*

$$h_{i,j}(x) = y_{i,j}(x) - y_{i,j-1}(x) \tag{3.21}$$

$$y^{\theta}_{i,j}(x) = (1-\theta) y_{i,j-1}(x) + \theta y_{i,j-1}(x), \quad \theta \in (0,1) \tag{3.22}$$

$$P^{\theta}_{i,j}(x) = (h_{i,j}(x))^3 \frac{|y^{\theta}_{i,j}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y^{\theta}_{i,j}(x)) \tag{3.23}$$

$$Q^{\theta}_{i,j;l}(x) = (h_{i,j}(x))^l \frac{|y^{\theta}_{i,j}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.24}$$

Obviously,

$$y_{i,j}(x_{i-1}) = x_{j-1}, \quad y_{i,j}(x_i) = x_j, \quad y_{i,j}(x_{i+1}) = x_{j+1} \tag{3.25}$$

$$h_{i,j}(x_{i-1}) = h_{j-1}, \quad h_{i,j}(x_i) = h_j, \quad h_{i,j}(x_{i+1}) = h_{j+1} \tag{3.26}$$

$$y^{\theta}_{i,j}(x_{i-1}) = y^{\theta}_{j-1}, \quad y^{\theta}_{i,j}(x_i) = y^{\theta}_j, \quad y^{\theta}_{i,j}(x_{i+1}) = y^{\theta}_{j+1} \tag{3.27}$$

And now we can rewrite T_{ij}

LEMMA 3.8.

$$\begin{aligned} T_{ij} = &\int_0^1 -\frac{\theta(1-\theta)}{2} P^{\theta}_{i,j}(x_i) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} Q^{\theta}_{i,j;l}(x_i) \left[\theta^2 u'''(\eta^{\theta}_{j,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{j,2})\right] d\theta \end{aligned} \tag{3.28}$$

Immediately, we can see from (3.16) and Lemma 3.6 that For $1 \le i \le 2N-1$, $2 \le j \le 2N-1$,

(3.29)
$$
\begin{aligned}
S_{ij} = &\int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{i,j}^{\theta}(x_i)d\theta \\
&+ \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{i,j;4}^{\theta}(x_{i+1})u'''(\eta_{j+1,1}^{\theta}) - Q_{i,j;4}^{\theta}(x_i)u'''(\eta_{j,1}^{\theta})}{h_{i+1}}\right)d\theta \\
&- \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{i,j;4}^{\theta}(x_i)u'''(\eta_{j,1}^{\theta}) - Q_{i,j;4}^{\theta}(x_{i-1})u'''(\eta_{j-1,1}^{\theta})}{h_i}\right)d\theta \\
&- \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{i,j;4}^{\theta}(x_{i+1})u'''(\eta_{j+1,2}^{\theta}) - Q_{i,j;4}^{\theta}(x_i)u'''(\eta_{j,2}^{\theta})}{h_{i+1}}\right)d\theta \\
&+ \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{i,j;4}^{\theta}(x_i)u'''(\eta_{j,2}^{\theta}) - Q_{i,j;4}^{\theta}(x_{i-1})u'''(\eta_{j-1,2}^{\theta})}{h_i}\right)d\theta
\end{aligned}
$$

We give some properties of mesh transport functions.

LEMMA 3.9. *For* $2 \le i, j \le 2N-2$ *and* $\xi \in (x_{i-1}, x_{i+1})$

$$\xi \simeq x_i, \quad \delta(y_{i,j}(\xi)) \simeq \delta(x_j), \quad h_{i,j}(\xi) \simeq h_j \tag{3.30}$$

$$|y_{i,j}(\xi) - \xi| \simeq |x_j - x_i|, \quad |y_{i,j-1}(\xi) - \xi| \simeq |x_{j-1} - x_i| \tag{3.31}$$

then

$$|y_{i,j}^{\theta}(\xi) - \xi| = (1-\theta)|y_{i,j-1}(\xi) - \xi| + \theta|y_{i,j}(\xi) - \xi| \simeq |y_j^{\theta} - x_i| \tag{3.32}$$

since $y_{i,j-1}(\xi) - \xi$, $y_{i,j}(\xi) - \xi$ *have the same sign* (≥ 0 *or* ≤ 0)

LEMMA 3.10.

$$y_{i,j}'(x) = \begin{cases} y_{i,j}^{1-1/r}(x)x^{1/r-1} & i<N, j<N \\ \dfrac{h_N}{rZ_1}x^{1/r-1} & i<N, j=N \\ (2T - y_{i,j}(x))^{1-1/r}x^{1/r-1} & i<N, j>N \\ y_{i,j}^{1-1/r}(x)\dfrac{rZ_1}{h_N} & i=N, j<N \\ 1 & i=N, j=N \end{cases} \tag{3.33}$$

$$y_{i,j}''(x) = \frac{1-r}{r}\begin{cases} y_{i,j}^{1-2/r}(x)x^{1/r-2}Z_{j-i} & i<N, j<N \\ \dfrac{h_N}{rZ_1}x^{1/r-2} & i<N, j=N \\ (2T - y_{i,j}(x))^{1-2/r}x^{1/r-2}Z_{2N-j+i} & i<N, j>N \\ -y_{i,j}^{1-2/r}(x)\left(\dfrac{rZ_1}{h_N}\right)^2 & i=N, j<N \\ 0 & i=N, j=N \end{cases} \tag{3.34}$$

LEMMA 3.11. *For $2 \le i \le N, 2 \le j \le 2N-2$, $\xi \in (x_{i-1}, x_{i+1})$*

$$|h'_{i,j}(\xi)| \le C(r-1)Z_1 x_i^{1/r-1}\delta(x_j)^{1-2/r} \le C(r-1)h_j x_i^{1/r-1}\delta(x_j)^{-1/r} \tag{3.35}$$

$$|(y_{i,j}(\xi)-\xi)'| \le Cx_i^{-1}|x_j - x_i| \tag{3.36}$$

Proof. From (3.21) and Lemma 3.10, we can see that

$$h'_{i,j}(x) = y'_{i,j}(x) - y'_{i,j-1}(x)$$
$$= \begin{cases} x^{1/r-1}(y_{i,j}^{1-1/r}(x) - y_{i,j-1}^{1-1/r}(x)) & i<N, j<N \\ x^{1/r-1}(\dfrac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(x)) & i<N, j=N \\ x^{1/r-1}\left((2T - y_{i,N+1}(x))^{1-1/r} - \dfrac{h_N}{rZ_1}\right) & i<N, j=N+1 \\ x^{1/r-1}\left((2T-y_{i,j}(x))^{1-1/r} - (2T - y_{i,j-1}(x))^{1-1/r}\right) & i<N, j>N+1 \\ \dfrac{rZ_1}{h_N}\left(y_{N,j}^{1-1/r}(x) - y_{N,j-1}^{1-1/r}(x)\right) & i=N, j<N \\ \dfrac{rZ_1}{h_N}\left(\dfrac{h_N}{rZ_1} - y_{N,N-1}^{1-1/r}(x)\right) & i=N, j=N \end{cases} \tag{3.37}$$

While for $2 \le i \le N$, if $2 \le j < N$, $\xi \in (x_{i-1}, x_{i+1})$,

$$\begin{aligned} y_{i,j}^{1-1/r}(\xi) - y_{i,j-1}^{1-1/r}(\xi) &\le x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r} \\ &= T^{1-1/r}N^{1-r}\left((j+1)^{r-1} - (j-2)^{r-1}\right) \\ &\le CT^{1-1/r}(r-1)N^{1-r}j^{r-2} = C(r-1)Z_1 x_j^{1-2/r} \end{aligned} \tag{3.38}$$

if $j = N$, $\xi \in (x_{i-1}, x_{i+1})$, we have $y_{i,N-1}(\xi) \in (x_{N-2}, x_N)$. And

$$\frac{h_N}{rZ_1} = T^{1-1/r}\frac{1-(1-h)^r}{rh} = \eta^{1-1/r} \simeq x_N^{1-1/r}, \quad \eta \in (x_{N-1}, x_N) \tag{3.39}$$

Then

$$|\frac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(\xi)| \le x_N^{1-1/r} - x_{N-2}^{1-1/r} \simeq (r-1)Z_1 x_N^{1-2/r} \tag{3.40}$$

and similar for $j \ge N+1$. Combine with Lemma 3.1, Lemma 3.9, $\eta \simeq x_N$, we get the first result.

For the second estimate, we have

$$(y_{i,j}(x) - x)' = y'_{i,j}(x) - 1 \tag{3.41}$$

Then, for $2 \le i < N$, if $2 \le j < N$, $\xi \in (x_{i-1}, x_{i+1})$, by Lemma A.5

$$\xi^{1/r}|y_{i,j}^{1-1/r}(\xi) - \xi^{1-1/r}| \le |y_{i,j}(\xi) - \xi| \tag{3.42}$$

$j > N$ is symmetric to it, that is

$$\begin{aligned} &\xi^{1/r}|(2T - y_{i,j}(\xi))^{1-1/r} - \xi^{1-1/r}| \le |2T - y_{i,j}(\xi) - \xi| \\ &\quad\le |2T - x_j - x_i| + |y_{i,j}(\xi) - x_j| + |\xi - x_i| \le |2T - x_j - x_i| + 2h_N \\ &\quad\le |x_j - T| + |T - x_i| + 2h_N \le 2|x_j - x_i| \end{aligned} \tag{3.43}$$

But if $j = N$, with (3.39) and Lemma A.5,

$$\begin{aligned}\eta^{1/r}\left|\frac{h_N}{rZ_1} - \xi^{1-1/r}\right| &\le |\eta - \xi|, \quad \eta \in (x_{N-1}, x_N) \\ &\le |x_N - x_i| + |h_N| + |h_{i+1}| \le 3|x_N - x_i|\end{aligned} \tag{3.44}$$

For $i = N$, if $j < N$, similarly with (3.44),

$$\eta^{1/r}\left|y_{N,j}^{1-1/r}(\xi) - \frac{h_N}{rZ_1}\right| \le C|x_j - x_N| \tag{3.45}$$

And if $j = N$, it is obviously $\equiv 0$.

Similarly, by Lemma 3.10 and Lemma 3.9, we get the second result. ∎

Lemma 3.12. *For* $2 \le i \le N, 2 \le j \le 2N-2$, $\xi \in (x_{i-1}, x_{i+1})$

$$|y''_{i,j}(\xi)| \le C(r-1)\begin{cases} x_j^{-1/r} x_i^{1/r-2}|x_j - x_i| & i < N, j < N \\ x_N^{1-1/r} x_i^{1/r-2} & i < N, j = N \\ \delta(x_j)^{1-2/r} x_i^{1/r-2} x_N^{1/r} & i < N, j > N \\ \delta(x_j)^{1-2/r} x_N^{2/r-2} & i = N, j \ne N \\ 0 & i = N, j = N \end{cases} \tag{3.46}$$

And $2 \le i \le N, 3 \le j \le 2N-2$, $\xi \in (x_{i-1}, x_{i+1})$

$$|h''_{i,j}(\xi)| \le C(r-1)\begin{cases} Z_1 x_i^{1/r-2} x_j^{-2/r}(|x_j - x_i| + x_j) & i < N, j < N \\ x_i^{1/r-2} x_N^{1-1/r} & i < N, j = N, N+1 \\ Z_1 x_i^{1/r-2}\delta(x_j)^{1-3/r} x_N^{1/r} & i < N, j > N+1 \\ Z_1 x_N^{2/r-2}\delta(x_j)^{1-3/r} & i = N, j < N \text{ or } j > N+1 \\ x_N^{-1} & i = N, j = N \end{cases} \tag{3.47}$$

Proof. Since by Lemma A.5, for $2 \le i, j < N$

$$x_j^{1-1/r}|Z_{j-i}| = x_j^{1-1/r}|x_j^{1/r} - x_i^{1/r}| \le |x_j - x_i| \tag{3.48}$$

and by (3.39), $\dfrac{h_N}{rZ_1} \simeq x_N^{1-1/r}$. And

$$Z_{2N-j+i} \le Z_{2N} = 2T^{1/r} \tag{3.49}$$

Then by Lemma 3.10 and Lemma 3.9, we get the first result.

For the second part, by Lemma 3.10

$$h''_{i,j}(x) = y''_{i,j}(x) - y''_{i,j-1}(x) \tag{3.50}$$

while for $2 \le i < N$, if $3 \le j < N$, $\xi \in (x_{i-1}, x_{i+1})$,

$$y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1} = \left(y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi)\right)Z_{j-i} + y_{i,j-1}^{1-2/r}(\xi)Z_1 \tag{3.51}$$

where $y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi) \simeq (r-2)Z_1 x_j^{1-3/r}$ similar with (3.38). Combine with (3.48), we get

$$|y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1}| \le CZ_1\left(|r-2|x_j^{-2/r}|x_j - x_i| + x_j^{1-2/r}\right) \tag{3.52}$$

if $j = N$,

$$|h''_{i,N}(x)| \le |y''_{i,N}(x)| + |y''_{i,N-1}(x)| \le C(r-1)x_i^{1/r-2}x_N^{1-1/r} \tag{3.53}$$

similarly if $j = N+1$.

However, if $j > N+1$, similar with (3.51), we get

$$\begin{aligned}&(2T - y_{i,j}(\xi))^{1-2/r}Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)}\\ &= \left((2T - y_{i,j}(\xi))^{1-2/r} - (2T - y_{i,j-1}(\xi))^{1-2/r}\right)Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r}Z_1\end{aligned} \tag{3.54}$$

thus,

$$\begin{aligned}&\left|(2T - y_{i,j}(\xi))^{1-2/r}Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)}\right|\\ &\le CZ_1\left(|r-2|(2T - x_j)^{1-3/r}x_N^{1/r} + (2T - x_j)^{1-2/r}\right) \le CZ_1(2T - x_j)^{1-3/r}x_N^{1/r}\end{aligned} \tag{3.55}$$

For $i = N$, it's obvious. Combine with Lemma 3.10 and Lemma 3.9, we get the second result. ∎

3.3. Proof of Theorem 3.3. Then we esrimate each part of (3.17). And We take $m = 2i$ for $3 \le i < N/2$, and $m = N - \lceil N/2 \rceil + 1$ for $N/2 \le i \le N$.

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LEMMA 3.13. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

Case 1. For $1 \le i < N/2$,

$$\sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.56}$$

Case 2. For $1 \le i < N/2$,

$$\sum_{j=N+1}^{2N} |V_{ij}| \le \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0\\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0\\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0\end{cases} \tag{3.57}$$

Case 3. For $N/2 \le i \le N$,

$$\sum_{j=N-\lceil \frac{N}{2} \rceil+2}^{2N} |V_{ij}| \le \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0\\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0\\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0\end{cases} \tag{3.58}$$

Proof. For i, j in each case, by (3.15), Lemma A.3 and Lemma B.3, we have

$$|V_{ij}| \le Ch^2 \int_{x_{j-1}}^{x_j} \delta(y)^{\alpha/2-2/r}|y - x_i|^{-1-\alpha}dy \tag{3.59}$$

For Case 1, with $x_i \simeq x_{2i}$,

$$\begin{aligned}\sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| &\le Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1}dy\\ &= \frac{C}{\alpha/2 + 2/r}h^2(x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r})\\ &\le Ch^2 x_i^{-\alpha/2-2/r}\end{aligned} \tag{3.60}$$

For Case 2 , by (3.15), Lemma A.3, Lemma B.3 and $y - x_i \simeq T$

$$|V_{ij}| \leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T-y)^{\alpha/2-2/r} dy$$

$$\tag{3.61} \begin{aligned} \sum_{j=N+1}^{2N-1} |V_{ij}| &\leq CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r} dy \\ &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2-2/r+1>0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2-2/r+1=0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2-2/r+1<0 \end{cases} \\ &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r}\ h^2, & \alpha/2-2/r+1>0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2-2/r+1=0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2-2/r+1<0 \end{cases} \end{aligned}$$

And by Lemma A.4

$$|V_{i,2N}| \leq CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}$$

Summarizes, we get the result. Similar for Case 3. ∎

For $i = 1, 2$.

LEMMA 3.14. *From* (3.18)*, by Lemma* B.4*, Lemma* 3.13 *and* **??** *we get for* $i = 1, 2$

$$\tag{3.62} |R_i| \leq Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1>0 \\ Ch^2 \ln(N), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0 \end{cases}$$

LEMMA 3.15. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil$

$$\tag{3.63} |I_1| = |\sum_{j=1}^{k-1} V_{ij}| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1>0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2-2/r+1<0 \end{cases}$$

Proof. by (3.15), Lemma A.4 , Lemma B.3

$$\tag{3.64} |V_{i1}| \leq C \int_0^{x_1} x_1^{\alpha/2} |x_i - y|^{-1-\alpha} dy \simeq x_1^{\alpha/2+1} x_i^{-1-\alpha} = T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha}$$

For $2 \leq j \leq k-1$, by Lemma A.3 and Lemma B.3 with $x_i - y \simeq x_i$, we have

$$\tag{3.65} |V_{ij}| \leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} x_i^{-1-\alpha} dy$$

Therefore,

$$\tag{3.66} \sum_{j=2}^{k-1} |V_{ij}| \leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil -1}} y^{\alpha/2-2/r} dy$$

But $x_{\lceil \frac{i}{2}\rceil - 1} \le 2^{-r}x_i$, so we have

$$\int_{x_1}^{x_{\lceil \frac{i}{2}\rceil-1}} y^{\alpha/2-2/r}dy \le \begin{cases} \frac{1}{\alpha/2-2/r+1}(2^{-r}x_i)^{\alpha/2-2/r+1}, & \alpha/2-2/r+1>0 \\ \ln(2^{-r}x_i)-\ln(x_1), & \alpha/2-2/r+1=0 \\ \frac{1}{|\alpha/2-2/r+1|}x_1^{\alpha/2-2/r+1}, & \alpha/2-2/r+1<0 \end{cases} \tag{3.67}$$

Combine the results above, we get the lemma. ◻

LEMMA 3.16. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that*
Case 1. For $3 \le i < N, \lceil \frac{i}{2}\rceil + 1 \le j \le \min\{2i-1, N-1\}$,

$$|D_h^2 P_{i,j}^{\theta}(x_i)| \le Ch_j^3 \frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-4} \tag{3.68}$$

Case 2. For $N/2 \le i \le N$, $j = N, N+1$

$$|D_h^2 P_{i,j}^{\theta}(\xi)| \le Ch_j^3|y_j^{\theta}-x_i|^{1-\alpha} + C(r-1)h_j^2\Big(|y_j^{\theta}-x_i|^{1-\alpha} + h_j|y_j^{\theta}-x_i|^{-\alpha}\Big) \tag{3.69}$$

Case 3. For $N/2 \le i \le N$, $N+2 \le j \le 2N - \lceil \frac{N}{2}\rceil$,

$$|D_h^2 P_{i,j}^{\theta}(\xi)| \le Ch_j^3\Big(|y_j^{\theta}-x_i|^{1-\alpha} + (r-1)|y_j^{\theta}-x_i|^{-\alpha}\Big) \tag{3.70}$$

Proof. Since $\mathrm{sign}(y_{i,j}^{\theta}(\xi)-\xi)$ is independent of ξ, we can derivate it. Then by Lemma A.1

$$D_h^2 P_{i,j}^{\theta}(x_i) = {P_{i,j}^{\theta}}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{3.71}$$

From (3.23), using Leibniz formula and chain rules, and Lemma 3.9, Lemma 3.10, Lemma 3.11, Lemma 3.12, Corollary 2.4, Lemma 3.1

For every case, we have $x_i \simeq \delta(x_j)$, so we have

$$h_{i,j}(\xi) \le Ch_j, \quad |h'_{i,j}(\xi)| \le C(r-1)h_j x_i^{-1} \tag{3.72}$$

$$|y_{i,j}^{\theta}(\xi)-\xi| \le C|y_j^{\theta}-x_i|, \quad \big|(y_{i.j}^{\theta}(\xi)-x_i)'\big| \le C|y_j^{\theta}-x_i|x_i^{-1} \tag{3.73}$$

(3.74)
$$|u''(y_{i,j}^{\theta}(\xi))| \le Cx_i^{\alpha/2-2}, \quad \Big|\big(u''(y_{i,j}^{\theta}(\xi))\big)'\Big| \le Cx_i^{\alpha/2-3}, \quad \Big|\big(u''(y_{i,j}^{\theta}(\xi))\big)''\Big| \le Cx_i^{\alpha/2-4}$$

By Lemma 3.12, we have

For Case 1,

$$|h''_{i,j}(\xi)| \le C(r-1)h_j x_i^{-2}, \quad \big|(y_{i,j}^{\theta}(\xi)-x_i)''\big| \le C(r-1)|y_j^{\theta}-x_i|x_i^{-2} \tag{3.75}$$

For Case 2, since $x_i \simeq x_j \simeq T$

$$|h''_{i,j}(\xi)| \le C(r-1), \quad \big|(y_{i,j}^{\theta}(\xi)-x_i)''\big| \le C(r-1) \tag{3.76}$$

For Case 3, since $x_i \simeq \delta(x_j) \simeq T$, we have

$$|h''_{i,j}(\xi)| \le C(r-1)h_j, \quad \big|(y_{i,j}^{\theta}(\xi)-x_i)''\big| \le C(r-1) \tag{3.77}$$ ◻

Combine them, we get the result.

Lemma 3.17. There exists a constant $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $2 \le i \le N$, $2 \le j \le 2N-2$,

$$\left|\frac{Q^{\theta}_{i,j;l}(x_{i+1})u^{(l-1)}(\eta^{\theta}_{j+1}) - Q^{\theta}_{i,j;l}(x_i)u^{(l-1)}(\eta^{\theta}_{j})}{h_{i+1}}\right| \le Ch_j^l\frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{-1}\delta(x_j)^{\alpha/2-l+1-1/r}(x_i^{1/r}+\delta(x_j)^{1/r}) \tag{3.78}$$

And

$$\left|\frac{Q^{\theta}_{i,j;l}(x_{i})u^{(l-1)}(\eta^{\theta}_{j}) - Q^{\theta}_{i,j;l}(x_{i-1})u^{(l-1)}(\eta^{\theta}_{j-1})}{h_{i}}\right| \le Ch_j^l\frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{-1}\delta(x_j)^{\alpha/2-l+1-1/r}(x_i^{1/r}+\delta(x_j)^{1/r}) \tag{3.79}$$

where $\eta^{\theta}_j \in (x_{j-1}, x_j)$.

Proof.

$$\begin{aligned}&\frac{Q^{\theta}_{i,j;l}(x_{i+1})u'''(\eta^{\theta}_{j+1}) - Q^{\theta}_{i,j;l}(x_i)u'''(\eta^{\theta}_{j})}{h_{i+1}}\\ &\quad= \frac{Q^{\theta}_{i,j;l}(x_{i+1}) - Q^{\theta}_{i,j;l}(x_i)}{h_{i+1}}u'''(\eta^{\theta}_{j+1}) + Q^{\theta}_{i,j;l}(x_i)\frac{u'''(\eta^{\theta}_{j+1}) - u'''(\eta^{\theta}_{j})}{h_{i+1}}\end{aligned} \tag{3.80}$$

Using mean value theorem

$$D_hQ^{\theta}_{i,j;l}(x_i) := \frac{Q^{\theta}_{i,j;l}(x_{i+1}) - Q^{\theta}_{i,j;l}(x_i)}{h_{i+1}} = {Q^{\theta}_{i,j;l}}'(\xi), \quad \xi \in (x_i, x_{i+1}) \tag{3.81}$$

From (3.24) and Leibniz rule, by Lemma 3.9, Lemma 3.11 and Lemma 3.1, we have

$$|{Q^{\theta}_{i,j;l}}'(\xi)| \le Ch_j^l\frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}(x_i^{-1} + x_i^{1/r-1}\delta(x_j)^{-1/r}) \tag{3.82}$$

$$Q^{\theta}_{i,j;l}(x_i) = h_j^l\frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.83}$$

With Lemma 3.1 and Corollary 2.4

$$|u^{(l-1)}(\eta^{\theta}_{j+1})| \le C(\eta^{\theta}_{j+1})^{\alpha/2-l+1} \simeq \delta(x_j)^{\alpha/2-l+1}$$

and by Lemma 3.1

$$\begin{aligned}\frac{|u^{(l-1)}(\eta^{\theta}_{j+1}) - u^{(l-1)}(\eta^{\theta}_{j})|}{h_{i+1}} &= |u^{(l)}(\eta)|\frac{\eta^{\theta}_{j+1} - \eta^{\theta}_j}{h_{i+1}}, \quad \eta \in (x_{j-1}, x_{j+1})\\ &\le C\delta(\eta)^{\alpha/2-l}\frac{x_{j+1} - x_{j-1}}{h_{i+1}} = C\delta(\eta)^{\alpha/2-l}\frac{h_{j+1} + h_j}{h_{i+1}}\\ &\simeq x_i^{1/r-1}\delta(x_j)^{\alpha/2-l+1-1/r}\end{aligned}$$

Combine the results above, we get the first term. While, the later is similar. ∎

LEMMA 3.18. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that for* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$,

$$\begin{aligned} |S_{ij}| &\le Ch^2 \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j d\theta \\ &= Ch^2 \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \end{aligned} \tag{3.84}$$

Proof. Since (3.29), by $x_i \simeq x_j$, Lemma 3.1, Lemma 3.16, Lemma 3.17, we get the result immediately. ∎

THEOREM 3.19. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that for* $3 \le i \le N-1, k = \lceil \frac{i}{2} \rceil$,

$$\sum_{j=k+1}^{\min\{2i-1,N-1\}} |S_{ij}| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.85}$$

Proof. By Lemma 3.18, while $x_k \simeq x_i \simeq x_{\min\{2i-1,N-1\}}$, we have

$$\begin{aligned} \sum_{k+1}^{\min\{2i-1,N-1\}} |S_{ij}| &\le Ch^2 \int_{x_k}^{x_{\min\{2i-1,N-1\}}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{3.86}$$ ∎

Now we study I_2, I_4.

LEMMA 3.20. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that for* $3 \le i \le N, k = \lceil \frac{i}{2} \rceil$,

$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.87}$$

And for $3 \le i < N/2$,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.88}$$

Proof. In fact,

$$\begin{aligned} &\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \\ &= \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i}) T_{i,k} \end{aligned} \tag{3.89}$$

While, by Lemma A.3, Lemma B.3, Lemma 3.1 and $x_k \simeq x_i$, we have

$$\begin{aligned} \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) &= \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y)) D_h K_y(x_i) dy \\ &\le Ch_k^2 x_k^{\alpha/2-2} h_k |x_i - x_k|^{-\alpha} \le Ch^2 x_i^{-\alpha/2-2/r} h_k \end{aligned} \tag{3.90}$$

Thus,

$$\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}|T_{i+1,k}-T_{i,k}| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.91}$$

From (3.14), Lemma A.2 and normalizzation, we have

$$\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k}) = \int_0^1 -\frac{\theta(1-\theta)}{2}\frac{Q^{\theta}_{i,k;3}(x_{i+1})u''(\eta^{\theta}_{k+1}) - Q^{\theta}_{i,k;3}(x_i)u''(\eta^{\theta}_k)}{h_{i+1}}d\theta \tag{3.92}$$

where $\eta^{\theta}_k \in (x_{k-1}, x_k)$ and $\eta^{\theta}_{k+1} \in (x_k, x_{k+1})$. And with Lemma 3.17, we can get

$$\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}|T_{i+1,k+1}-T_{i,k}| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.93}$$

For the third term, by Lemma 3.1, Lemma B.1, Lemma A.3 and $x_k \simeq x_i$, we have

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}\frac{h_{i+1}-h_i}{h_i h_{i+1}}T_{i,k} &\le h_i^{-3}h^2 x_i^{1-2/r} C h_k^3 x_k^{\alpha/2-2}|x_k-x_i|^{1-\alpha}\\ &\le Ch^2 x_i^{-\alpha/2-2/r}\end{aligned} \tag{3.94}$$

Summarizes, we have

$$I_2 \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.95}$$

The case for I_4 is similar. ∎

Now combine Lemma 3.14, Lemma 3.15, Lemma 3.20, Theorem 3.19, Lemma 3.13 and **??**, we get Theorem 3.3.

For $N/2 \le i < N$, we take $m = 2N - \lceil \frac{N}{2} \rceil + 1$. And depart I_3 to three parts:

$$\begin{aligned} I_3 = \sum_{j=k+1}^{m} S_{ij} &= \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{m-1} S_{ij}\\ &= I_3^1 + I_3^2 + I_3^3\end{aligned} \tag{3.96}$$

We have estimated I_3^1 in Theorem 3.19.

Combine **??**, **??** and formula (3.29) for $i \le N-1, j \ge N+2$, we have

Lemma 3.21. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \le i \le N-1$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$*

$$S_{ij} \le Ch^2 \int_{x_{j-1}}^{x_j} \Big(|y-x_i|^{1-\alpha} + (r-1)|y-x_i|^{-\alpha}\Big)dy \tag{3.97}$$

We can esitmate I_3^3 Now.

Lemma 3.22. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \le i \le N-1$, we have*

$$I_3^3 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2}\rceil} S_{ij} \le Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \tag{3.98}$$

Proof.

$$
\begin{aligned}
I_3^3 = &\sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} S_{ij} \\
\leq &Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil \frac{N}{2} \rceil}} \Big(|y-x_i|^{1-\alpha} + (r-1)|y-x_i|^{-\alpha} \Big) dy
\end{aligned} \tag{3.99}
$$

$$
\begin{aligned}
I_3^3 &\leq Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil \frac{N}{2} \rceil}} |y-x_i|^{1-\alpha} + (r-1)|y-x_i|^{-\alpha} \\
&\leq Ch^2 (T^{2-\alpha} + (r-1)|x_{N+1} - x_i|^{1-\alpha}) \\
&= Ch^2 + C(r-1)h^2 |T - \delta(x_i) + h_N|^{1-\alpha}
\end{aligned} \tag{3.100}
$$

∎

For I_3^2, we have

THEOREM 3.23. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that, for $N/2 \le i \le N-1$*

$$\begin{aligned} V_{iN} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,N+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,N} + \frac{1}{h_i} T_{i-1,N-1} \right) \\ &\le Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \end{aligned} \tag{3.101}$$

I_4, I_5 is easy. Similar with Lemma 3.20 and **??**, we have

Now we can conclude a part of the theorem Theorem 3.4 at the beginning of this section.

By Lemma 3.15, Lemma 3.20, **??**, Theorem 3.23, Lemma 3.22, **??**, **??**, we have

THEOREM 3.24. *there exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $N/2 \leq i \leq N-1$,

$$R_i = I_1 + I_2 + I_3^1 + I_3^2 + I_3^3 + I_4 + I_5$$
$$\leq C(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.102}$$

and with Theorem 3.24 we prove the Theorem 3.4

4. Convergence analysis.

4.1. Properties of some Matrices. Review subsection 2.1, we have got (2.10).

DEFINITION 4.1. *We call one matrix an M matrix , which means its entries are positive on major diagonal and nonpositive on others, and strictly diagonally dominant in rows.*

Now we have

LEMMA 4.2. *Matrix A defined by* (2.12) *where* (2.13) *is an M matrix. And there exists a constant $C_A = C(T, \alpha, r)$ such that*

$$S_i := \sum_{j=1}^{2N-1} a_{ij} \geq C_A(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \tag{4.1}$$

Proof. From (2.15), we have

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right) \tag{4.2}$$

Let

$$g(x) = g_0(x) + g_{2N}(x) \tag{4.3}$$

where

$$g_0(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x - x_0|^{3-\alpha} - |x - x_1|^{3-\alpha}}{h_1}$$

$$g_{2N}(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N} - x|^{3-\alpha} - |x_{2N-1} - x|^{3-\alpha}}{h_{2N}}$$

Thus

$$-\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

Then

$$\begin{aligned} S_i :=& \sum_{j=1}^{2N-1} a_{ij} \\ =&\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ =&D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i) \end{aligned} \tag{4.4}$$

When $i = 1$

$$\begin{aligned} D_h^2 g_0(x_1) = & \frac{2}{h_1 + h_2} \left(\frac{1}{h_2} g_0(x_2) - (\frac{1}{h_1} + \frac{1}{h_2}) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right) \\ =&\frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2) h_1 h_2} \\ =&\frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2) h_1^{1-\alpha} h_2} h_1^{-\alpha} \\ =&\frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha}}{2^r (2^r - 1)} h_1^{-\alpha} \end{aligned} \tag{4.5}$$

but

$$1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha} > 0 \tag{4.6}$$

While for $i \geq 2$

$$\begin{aligned} D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \\ &= -\kappa_\alpha \frac{|\xi - x_0|^{1-\alpha} - |\xi - x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1} \\ &= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} |\xi - \eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\ &\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha} \end{aligned} \tag{4.7}$$

So

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g_0(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g_0(x_i) + \frac{1}{h_i} g_0(x_{i-1}) \right) \geq C x_i^{-\alpha} \tag{4.8}$$

symmetricly,

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g_{2N}(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g_{2N}(x_i) + \frac{1}{h_i} g_{2N}(x_{i-1}) \right) \geq C(\alpha, r)(2T - x_i)^{-\alpha} \tag{4.9}$$

Let

$$\delta(x) = \begin{cases} x, & 0 < x \leq T \\ 2T - x, & T < x < 2T \end{cases} \tag{4.10}$$

And define

$$G = \mathrm{diag}(\delta(x_1), ..., \delta(x_{2N-1})) \tag{4.11}$$

Then

LEMMA 4.3. *The matrix $B := AG$, the major diagnal is positive, and nonpositive on others. And there is a constant $C_{AG}, C = C(\alpha, r)$ such that*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C_{AG}(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}) + C(T - \delta(x_i) + h_N)^{1-\alpha} \tag{4.12}$$

Proof.

$$b_{ij} = a_{ij}\delta(x_j) = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \tilde{a}_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) \tilde{a}_{i,j} + \frac{1}{h_i} \tilde{a}_{i-1,j} \right) \delta(x_j)$$

Since

$$\delta(x) \equiv \Pi_h \delta(x) \tag{4.13}$$

by (2.14) and (2.5), we have

$$
\begin{aligned}
\tilde{M}_i &:= \sum_{j=1}^{2N-1} \tilde{b}_{ij} := \sum_{j=1}^{2N-1} \tilde{a}_{ij}\delta(x_j) \\
&= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} \Pi_h \delta(y) dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} \delta(y) dy \\
&= \frac{-2}{\Gamma(4-\alpha)} |T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)} (x_i^{3-\alpha} + (2T - x_i)^{3-\alpha}) \\
&:= w(x_i) = p(x_i) + q(x_i)
\end{aligned} \tag{4.14}
$$

Thus,

$$
\begin{aligned}
M_i &:= \sum_{j=1}^{2N-1} b_{ij} = \sum_{j=1}^{2N-1} a_{ij}\delta(x_j) \\
&= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \tilde{M}_{i+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) \tilde{M}_i + \frac{1}{h_i} \tilde{M}_{i-1} \right) \\
&= D_h^2(-\kappa_\alpha p)(x_i) - \kappa_\alpha D_h^2 q(x_i)
\end{aligned} \tag{4.15}
$$

for $1 \leq i \leq N-1$,by Lemma A.1

$$
\begin{aligned}
D_h^2(-\kappa_\alpha p)(x_i) &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} p(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) p(x_i) + \frac{1}{h_i} p(x_{i-1}) \right) \\
&= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1}) \\
&\geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} (T - \delta(x_i) + h_N)^{1-\alpha}
\end{aligned} \tag{4.16}
$$

$$
\begin{aligned}
D_h^2(-\kappa_\alpha p)(x_N) &:= -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left(\frac{1}{h_{N+1}} p(x_{N+1}) - (\frac{1}{h_N} + \frac{1}{h_{N+1}}) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right) \\
&= \frac{4\kappa_\alpha}{\Gamma(4-\alpha) h_N^2} h_N^{3-\alpha} = \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - \delta(x_N) + h_N)^{1-\alpha}
\end{aligned} \tag{4.17}
$$

Symmetricly for $i \geq N$, we get

$$
D_h^2(-\kappa_\alpha p)(x_i) \geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} (T - \delta(x_i) + h_N)^{1-\alpha} \tag{4.18}
$$

Similarly, we can get

$$
\begin{aligned}
D_h^2 q(x_i) &:= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} q(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right) \\
&\leq \frac{2^{r(\alpha-1)+1}}{\Gamma(2-\alpha)} (x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}), \quad i = 1, \cdots, 2N-1
\end{aligned} \tag{4.19}
$$

□

So, we get the result.

Notice that

$$
x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha} \tag{4.20}
$$

We can get

THEOREM 4.4. *There exists a real* $\lambda = \lambda(T, \alpha, r) > 0$ *and* $C = C(T, \alpha, r) > 0$ *such that* $B := A(\lambda I + G)$ *is an* M *matrix. And*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) + C(T - \delta(x_i) + h_N)^{1-\alpha} \tag{4.21}$$

Proof. By Lemma 4.2 with C_A and Lemma 4.3 with C_{AG} , it's sufficient to take $\lambda = (C + 2TC_{AG})/C_A$, then

$$M_i \geq C\left((x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + (T - \delta(x_i) + h_N)^{1-\alpha}\right) \tag{4.22}$$ ☐

4.2. Proof of Theorem 2.6. For equation

$$AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1}U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1}U = F \tag{4.23}$$

which means

$$\sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + \delta(x_j)} = -\tau_i \tag{4.24}$$

where $\epsilon_i = u(x_i) - u_i$.

And if

$$\left|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}\right| = \max_{1 \leq i \leq 2N-1} \left|\frac{\epsilon_i}{\lambda + \delta(x_i)}\right| \tag{4.25}$$

Then, since $B = A(\lambda I + G)$ is an M matrix, it is Strictly diagonally dominant. Thus,

$$\begin{aligned}
|\tau_{i_0}| &= \left|\sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + \delta(x_j)}\right| \\
&\geq b_{i_0,i_0}\left|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}\right| - \sum_{j \neq i_0} |b_{i_0,j}|\left|\frac{\epsilon_j}{\lambda + \delta(x_j)}\right| \\
&\geq b_{i_0,i_0}\left|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}\right| - \sum_{j \neq i_0} |b_{i_0,j}|\left|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}\right| \\
&= \sum_{j=1}^{2N-1} b_{i_0,j}\left|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}\right| \\
&= M_{i_0}\left|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}\right|
\end{aligned} \tag{4.26}$$

By Theorem 2.5 and Theorem 4.4,

We kown that there exists constants $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$, and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that

$$\left|\frac{\epsilon_i}{\lambda + \delta(x_i)}\right| \leq \left|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}\right| \leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} + C_2(r-1)h^2 \tag{4.27}$$

as $\lambda + \delta(x_i) \leq \lambda + T$

So, we can get

$$|\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{r\alpha}{2}, 2\}} \tag{4.28}$$

The convergency has been proved.

Remarks:

5. Experimental results.

5.1. $f \equiv 1$.

5.2. $f = x^\gamma, \gamma < 0$. Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x) \in C^2(\Omega)$, there exists $\xi \in (x_{i-1}, x_{i+1})$ such that*

$$D_h^2 g(x_i) = g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{A.1}$$

And if $g(x) \in C^4(\Omega)$, then

$$\begin{aligned} D_h^2 g(x_i) &= g''(x_i) + \frac{h_{i+1}-h_i}{3} g'''(x_i) \\ &\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y-x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1}-y)^3}{3!} dy \right) \end{aligned} \tag{A.2}$$

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})$$

Subsitute them in the left side of (A.1), we have

$$\begin{aligned} D_h^2 g(x_i) &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (g(x_{i+1}) - g(x_i) + \frac{1}{h_i} (g(x_{i-1}) - g(x_i)) \right) \\ &= \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) \end{aligned}$$

Now, using intermediate value theorem, there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy$$

Expecially,

$$\begin{aligned} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy &= \frac{h_i^4}{4!} g''''(\eta_1) \\ \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy &= \frac{h_{i+1}^4}{4!} g''''(\eta_2) \end{aligned} \tag{A.3}$$

where $\eta_1 \in (x_{i-1}, x_i), \eta_2 \in (x_i, x_{i+1})$. ∎

LEMMA A.2. *Denote* $y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \theta \in (0,1)$,

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in (x_{j-1}, x_j) \tag{A.4}$$

(A.5)

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in (x_{j-1}, y_j^\theta)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in (y_j^\theta, x_j)$$

Thus

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta) u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta) u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$. Thus ∎

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta) u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

LEMMA A.3. *By Lemma* A.2*, Corollary* 2.4 *and Lemma* 3.1 *, There is a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *for* $2 \le j \le 2N-1$,

$$|u(y) - \Pi_h u(y)| \le h_j^2 \max_{\xi \in [x_{j-1}, x_j]} |u''(\xi)| \le C h^2 \delta(y)^{\alpha/2 - 2/r}, \quad \textit{for } y \in (x_{j-1}, x_j) \tag{A.6}$$

LEMMA A.4. *For* $x \in [x_{j-1}, x_j]$

$$\begin{aligned} |u(x) - \Pi_h u(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y) dy - \frac{x - x_{j-1}}{h_j} \int_{x}^{x_j} u'(y) dy \right| \\ &\le \int_{x_{j-1}}^{x_j} |u'(y)| dy \end{aligned} \tag{A.7}$$

If $x \in [0, x_1]$, with Corollary 2.4, we have

$$|u(x) - \Pi_h u(x)| \leq \int_0^{x_1} |u'(y)| dy \leq \int_0^{x_1} C y^{\alpha/2-1} dy \leq C \frac{2}{\alpha} x_1^{\alpha/2} = C \frac{2}{\alpha} h_1^{\alpha/2} \tag{A.8}$$

Similarly, if $x \in [x_{2N-1}, 1]$, we have

$$|u(x) - \Pi_h u(x)| \leq C \frac{2}{\alpha} (2T - x_{2N-1})^{\alpha/2} = C \frac{2}{\alpha} h_{2N}^{\alpha/2} \tag{A.9}$$

LEMMA A.5.

$$b^{1-\theta} |a^\theta - b^\theta| \leq |a - b| \ (\text{ also } a^{1-\theta} |a^\theta - b^\theta| \leq |a - b|), \quad a, b \geq 0, \ \theta \in [0, 1] \tag{A.10}$$

Appendix B. Proofs of some technical details. Review that $h = \frac{1}{N}$ and the defination of $\simeq$ in subsection 2.1

LEMMA B.1. *There is a constant C such that for $i = 1, 2, \cdots, 2N - 1$*

$$|h_{i+1} - h_i| \leq C h^2 \delta(x_i)^{1-2/r} \tag{B.1}$$

Proof. By (2.2),

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \leq i \leq 2N-1 \end{cases} \tag{B.2}$$

Since

$$(i+1)^r - 2i^r + (i-1)^r \simeq r(r-1) i^{r-2}, \quad \text{for } i \geq 1 \tag{B.3}$$

We get the result. ∎

LEMMA B.2. *there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that*

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right| \\ &\quad \leq C h^2 \delta(x_i)^{-\alpha/2-2/r} \end{aligned} \tag{B.4}$$

Proof. By Lemma 2.2, we have for $1 \leq i \leq N$

$$\left| \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy \right| \leq \frac{\|f\|_\beta^{(\alpha/2)}}{3!} \int_{x_{i-1}}^{x_i} y^{-\alpha/2-2} (y - x_{i-1})^3 dy \tag{B.5}$$

For $i = 1$,

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2} (y - x_{i-1})^3 dy = \int_0^{x_1} y^{1-\alpha/2} dy = \frac{1}{2 - \alpha/2} x_1^{2-\alpha/2} = \frac{1}{2 - \alpha/2} x_1^{-\alpha/2-2} h_1^4$$

And for $2 \leq i \leq N$, since $x_i \simeq x_{i-1} \leq y \leq x_i$, we have

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2} (y - x_{i-1})^3 dy \simeq \int_{x_{i-1}}^{x_i} x_i^{-\alpha/2-2} (y - x_{i-1})^3 dy = \frac{1}{4!} x_i^{-\alpha/2-2} h_i^4$$

So for $1 \le i \le N$, we have

$$\left|\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy\right| \le Cx_i^{-\alpha/2-2}h_i^4 \tag{B.6}$$

and similarly,

$$\left|\int_{x_i}^{x_{i+1}} f''(y)\frac{(x_{i+1}-y)^3}{3!}dy\right| \le Cx_i^{-\alpha/2-2}h_{i+1}^4 \tag{B.7}$$

Thus for $1 \le i \le N$, with Lemma 3.1 we have

$$\begin{aligned}
&\frac{2}{h_i+h_{i+1}}\left|\frac{1}{h_i}\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy + \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}} f''(y)\frac{(y-x_{i+1})^3}{3!}dy\right| \\
&\le Cx_i^{-\alpha/2-2}\frac{2}{h_i+h_{i+1}}(h_i^3+h_{i+1}^3) \simeq x_i^{-\alpha/2-2}h_i^2 \simeq x_i^{-\alpha/2-2}h^2x_i^{2-2/r} \\
&= Ch^2x_i^{-\alpha/2-2/r}
\end{aligned} \tag{B.8}$$

It's symmetric for $N < i \le 2N-1$. ◻

Lemma B.3. *There is a constant $C = C(\alpha, r)$ such that for all $1 \le i \le 2N-1$, $1 \le j \le 2N$ s.t. $\min\{|j-i|, |j-1-i|\} \ge 2$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_hK_y(x_i) \simeq |y-x_i|^{-\alpha}, \quad D_h^2K_y(x_i) \simeq |y-x_i|^{-1-\alpha} \tag{B.9}$$

Proof. Sinec $y-x_{i-1}, y-x_i, y-x_{i+1}$ have the same sign, by mean value theorem and Lemma A.1,

$$\begin{aligned}
D_hK_y(x_i) &= \frac{|y-\xi|^{-\alpha}}{\Gamma(1-\alpha)}, \quad \xi \in (x_i, x_{i+1}) \\
D_h^2K_y(x_i) &= \frac{|y-\xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in (x_{i-1}, x_{i+1})
\end{aligned}$$

however, $|y-\xi| \simeq |y-x_i|$, we get the result. ◻

Lemma B.4. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$\sum_{j=1}^{3} V_{1j} \le Ch^2x_1^{-\alpha/2-2/r} \tag{B.10}$$

$$\sum_{j=1}^{4} V_{2j} \le Ch^2x_2^{-\alpha/2-2/r} \tag{B.11}$$

Proof. For $0 \le i \le 3, 1 \le j \le 4$, by Lemma A.4, Lemma A.3 and (3.14)

$$T_{ij} \le Cx_1^{2-\alpha/2} \simeq h_1^2\, h^2x_1^{-\alpha/2-2/r} \simeq h_1^2\, h^2x_2^{-\alpha/2-2/r} \tag{B.12}$$

Therefore, by (3.15), we get the result. ◻

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