A SECOND ORDER NUMERICAL METHODS FOR REISZ-FRACTIONAL ELLIPTIC EQUATION ON GRADED MESH*

JIANXING HAN† AND MINGHUA CHEN‡

Abstract. This is an example SIAM LaTeX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

Key words. example, LaTeX

MSC codes. 68Q25, 68R10, 68U05

1. Introduction.

The introduction introduces the context and summarizes the manuscript. It is importantly to clearly state the contributions of this piece of work.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$, suppose $f \in C^{\beta}(\Omega), \beta > 4 - \alpha, \|f\|_{\beta}^{(\alpha/2)} < \infty$

$$(1.1)\qquad \begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases}$$

where

$$(1.2)\qquad (-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^{\alpha} u}{\partial |x|^{\alpha}} = -\kappa_{\alpha} \frac{d^2}{dx^2} \int_{\Omega} \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy$$

$$(1.3)\qquad \kappa_{\alpha} = -\frac{1}{2\cos(\alpha\pi/2)} > 0$$

and the solution $u \in C^{\alpha/2}(\Omega)$.

2. Regularity.

For any $\beta > 0$, we use the standard notation $C^{\beta}(\bar{\Omega}), C^{\beta}(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^{\beta}(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous cite with exponent β' in Ω, where $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on open set Ω.

DEFINITION 2.1 (delta dependent norm [1]). ...

THEOREM 2.2. *Let $f \in C^{\beta}(\Omega), \beta > 2$ be such that $\|f\|_{\beta}^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

$$(2.1)\qquad |f^{(l)}(x)| \le \|f\|_{\beta}^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \le T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \le x < 2T \end{cases}$$

*Submitted to the editors DATE.

†School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China (hanjx2023@mail.lzu.edu.cn).

‡School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China (chen@mail.lzu.edu.cn).

THEOREM 2.3 (Regularity up to the boundary [1]). *Let Ω be a bounded domain, and $\beta > 0$ be such that neither β nor $\beta + \alpha$ is an integer. Let $f \in C^{\beta}(\Omega)$ be such that$\|f\|_{\beta}^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R}^n)$ be a solution of (1.1). Then, $u \in C^{\beta+\alpha}(\Omega)$ and*

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_{\beta}^{(\alpha/2)}\right) \tag{2.2}$$

COROLLARY 2.4. *Let u be a solution of (1.1) on Ω. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \le \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \le T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \le x < 2T \end{cases} \tag{2.3}$$

The paper is organized as follows. Our main results are in section 4, experimental results are in section 7. Readers would better see section 6 before section 5 to avoid details.

3. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases} \tag{3.1}$$

where $r \ge 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N \tag{3.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

And then, we can approximate $u(x)$ with

$$u_h(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{3.4}$$

For convience, we denote

$$I_h^{2-\alpha}(x_i) := \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u_h(y)dy \tag{3.5}$$

And now, we can approximate the operator (1.2) at x_i with

$$\begin{aligned} D_h^{\alpha} u_h(x_i) &:= D_h^2 I_h^{2-\alpha}(x_i) \\ &= \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_i} I_h^{2-\alpha}(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right) I_h^{2-\alpha}(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha}(x_{i+1})\right) \end{aligned} \tag{3.6}$$

Finally, we approximate the equation (1.1) with

$$-\kappa_{\alpha} D_h^{\alpha} u_h(x_i) = f(x_i), \quad 1 \le i \le 2N-1 \tag{3.7}$$

The discrete equation (3.7) can be written in matrix form

$$AU = F \tag{3.8}$$

where U is unknown, $F = (f(x_1), \cdots, f(x_{2N-1}))$. The matrix A is constructed as follows: Since

$$\begin{aligned}
I_h^{2-\alpha}(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u(x_j) \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\
&=: \sum_{j=1}^{2N-1} \tilde{a}_{ij}\, u(x_j), \quad 0 \le i \le 2N
\end{aligned} \tag{3.9}$$

Then, substitute in (3.6), we have

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} a_{ij}\, u(x_j) \tag{3.10}$$

where

$$a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \tag{3.11}$$

4. Main results.

Here we state our main results; the proof is deferred to section 5 and section 6.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 4.1 (Truncation Error). *If f satisfy that $f \in C^\beta(\Omega), \beta > 4 - \alpha$, $\|f\|_\beta^{(\alpha/2)} < \infty, \alpha \in (1,2)$, and $u(x)$ is a solution of the equation (1.1), where $\|u\|_{\beta+\alpha}^{(-\alpha/2)} < \infty$, then there exists constants $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$, $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$, such that the truncation error of the discrete format satisfies*

$$\begin{aligned}
\tau_i :=& | -\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i)| \\
\le& C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} \begin{cases} x_i^{-\alpha}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha}, & N < i \le 2N-1 \end{cases} \\
&+ C_2 (r-1) h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \le i \le N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \le 2N-1 \end{cases}
\end{aligned} \tag{4.1}$$

THEOREM 4.2 (Convergence). *The discrete equation (3.7) has sulotion U, and there exists a positive constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$ such that the error* *between the numerial solution U with the exact solution $u(x_i)$ satisfies*

$$\max_{1\le i\le 2N-1} |U_i - u(x_i)| \le Ch^{\min\{\frac{r\alpha}{2},2\}} \tag{4.2}$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2},2\}$.

5. Proof of Theorem 4.1. For convience, let's denote

$$I^{2-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)}\int_\Omega |x-y|^{1-\alpha}u(y)dy \tag{5.1}$$

Then, the truncation error of the discrete format can be written as

$$\begin{aligned} -\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i) &= -\kappa_\alpha(D_h^2 I_h^{2-\alpha}(x_i) - \frac{d^2}{dx^2}I^{2-\alpha}(x_i)) \\ &= -\kappa_\alpha D_h^2(I_h^{2-\alpha} - I^{2-\alpha})(x_i) - \kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}(x_i) \end{aligned} \tag{5.2}$$

5.1. Estimate of $-\kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}(x_i)$.

THEOREM 5.1. *There exits a constant $C = C(T,\alpha,r,\|f\|_\beta^{(\alpha/2)})$ such that*

$$\left|-\kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}(x_i)\right| \le Ch^2(x_i^{-\alpha/2-2/r} + (2T-x_i)^{-\alpha/2-2/r}) \tag{5.3}$$

Proof. Since $f \in C^2(\Omega)$ and

$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha}(x)) = f(x), \quad x \in \Omega, \tag{5.4}$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.3) of Lemma A.1, for $1 \le i \le 2N-1$, we have

$$-\kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}(x_i) = \frac{h_{i+1}-h_i}{3}f'(x_i) + \frac{1}{4!}\frac{2}{h_i+h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \tag{5.5}$$

where $\eta_1 \in [x_{i-1},x_i], \eta_2 \in [x_i,x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

$$\left|\frac{h_{i+1}-h_i}{3}f'(x_i)\right| \le \frac{\|f\|_\beta^{(\alpha/2)}}{3}Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T-x_i)^{-\alpha/2-2/r}, & N < i \le 2N-1 \end{cases} \tag{5.6}$$

2. See Proof 25, there is a constant $C = C(T,\alpha,r,\|f\|_\beta^{\alpha/2})$ such that

$$\begin{aligned} &\left|\frac{1}{4!}\frac{2}{h_i+h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2))\right| \\ &\quad \le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T-x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases} \end{aligned} \tag{5.7}$$

Summarizes, we get the result. ∎

5.2. Estimate of R_i. Now, we study the first part of (5.2)

$$D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i) = D_h^2\Big(\int_0^{2T} (u(y) - u_h(y))\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy\Big) \tag{5.8}$$

For convience, let's denote

$$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - u_h(y))\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy \tag{5.9}$$

And define

$$\begin{aligned} R_i &:= D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i) \\ &= \frac{2}{h_i + h_{i+1}}\sum_{j=1}^{2N}\left(\frac{1}{h_i}T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right)T_{i,j} + \frac{1}{h_{i+1}}T_{i+1,j}\right) \end{aligned} \tag{5.10}$$

We have some results about the estimate of R_i

Theorem 5.2. *For $1 \le i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$R_i \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2}x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.11}$$

Theorem 5.3. *For $N/2 \le i \le N$, there exists constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$R_i \le C(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.12}$$

And for $N < i \le 2N - 1$, it is symmetric to the previous case.

To prove these results, we need some utils. Also for simplicity, we denote

Definition 5.4.

$$S_{ij} = \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_i}T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right)T_{i,j} + \frac{1}{h_{i+1}}T_{i+1,j}\right) \tag{5.13}$$

then

$$R_i = \sum_{j=1}^{2N} S_{ij} \tag{5.14}$$

5.3. Proof of Theorem 5.2.

Lemma 5.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \le i < N/2$,*

$$\sum_{j=\max\{2i+1, i+3\}}^{N} S_{ij} \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.15}$$

Proof. For $\max\{2i+1, i+3\} \le j \le N$, by Lemma C.1 and Lemma C.2

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) dy \\ &\le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2 - 2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2 - 2/r - 1} dy \end{aligned} \tag{5.16}$$

Therefore,

$$\begin{aligned} \sum_{j=\max\{2i+1,i+3\}}^{N} S_{ij} &\le Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\ &= \frac{C}{\alpha/2 + 2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\ &\le \frac{C}{\alpha/2+2/r} 2^{r(-\alpha/2-2/r)} h^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{5.17}$$

∎

LEMMA 5.6. *Thert exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \le i < N/2$,*

$$\sum_{j=N+1}^{2N} S_{ij} \le \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2 + r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.18}$$

Proof. For $1 \le i < N/2, N+1 \le j \le 2N-1$, by equation (C.2) and Lemma C.2

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) dy \\ &\le \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2-2/r} y^{-1-\alpha} dy \\ &\le Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T-y)^{\alpha/2-2/r} dy \end{aligned}$$

$$\begin{aligned} \sum_{j=N+1}^{2N-1} S_{ij} &\le CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r} dy \\ &\le CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 < 0 \end{cases} \\ &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r}\ h^2, & \alpha/2-2/r+1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2-2/r+1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2-2/r+1 < 0 \end{cases} \end{aligned} \tag{5.19}$$

And by Lemma A.3

$$S_{i,2N} \leq CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}$$

And when $\alpha/2 - 2/r + 1 \geq 0$,

$$h^{r\alpha/2+r} \leq h^2$$

Summarizes, we get the result. ∎

For $i = 1, 2$.

LEMMA 5.7. *By Lemma* C.5 *, Lemma* 5.5 *and Lemma* 5.6 *we get*

$$R_1 = \sum_{j=1}^{3} S_{1j} + \sum_{j=4}^{2N} S_{1j} \leq Ch^2 x_1^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.20}$$

$$R_2 = \sum_{j=1}^{4} S_{2j} + \sum_{j=5}^{2N} S_{2j} \leq Ch^2 x_2^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.21}$$

For $3 \leq i < N/2$, we have a new separation of R_i, Let's denote $k = \lceil \frac{i}{2} \rceil$.

$$\begin{aligned} R_i =& \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\ &+ \sum_{j=k+1}^{2i-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \\ &+ \sum_{j=2i+1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned} \tag{5.22}$$

LEMMA 5.8. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that for $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil$*

$$|I_1| = |\sum_{j=1}^{k-1} S_{ij}| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.23}$$

Proof. For $2 \leq j \leq k-1$, by Lemma C.1 and Lemma C.3

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\ &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 x_i^{-1-\alpha} \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} dy \end{aligned} \tag{5.24}$$

And by Lemma A.3 , Lemma C.3

$$S_{i1} \leq C x_1^{\alpha/2} x_1 x_i^{-1-\alpha} = C x_1^{\alpha/2+1} x_i^{-1-\alpha} = C T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha} \tag{5.25}$$

Therefore,

$$\begin{aligned} I_1 &= \sum_{j=1}^{k-1} S_{ij} = S_{i1} + \sum_{j=2}^{k-1} S_{ij} \\ &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \\ &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \end{aligned} \tag{5.26}$$

But

$$\int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \leq \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.27}$$

So we have

$$I_1 \leq \begin{cases} \frac{C}{\alpha/2-2/r+1} h^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} h^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.28}$$

$\square$

DEFINITION 5.9. *For convience, let's denote*

$$V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \tag{5.29}$$

THEOREM 5.10. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that for* $3 \le i < N/2, k = \lceil \frac{i}{2} \rceil$,

$$I_3 = \sum_{j=k+1}^{2i-1} V_{ij} \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.30}$$

To estimete V_{ij}, we need some preparations.

LEMMA 5.11. *Denote* $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0,1]$, *by Lemma* A.2

$$\begin{aligned} T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= \int_{x_{j-1}}^{x_j} -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!} h_j^3 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) dy_j^\theta \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta \end{aligned} \tag{5.31}$$

where $\eta_{j1}^\theta \in [x_{j-1}, y_j^\theta], \eta_{j2}^\theta \in [y_j^\theta, x_j]$.

Now Let's construct a series of functions to represent T_{ij}.

DEFINITION 5.12.

$$y_{j-i}(x) = (x^{1/r} + Z_{j-i})^r, \quad Z_{j-i} = T^{1/r} \frac{j-i}{N} \tag{5.32}$$

$$y_{j-i}^\theta(x) = \theta y_{j-1-i}(x) + (1-\theta) y_{j-i}(x) \tag{5.33}$$

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{5.34}$$

Now, we define

$$P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.35}$$

$$Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.36}$$

And now we can rewrite T_{ij}

LEMMA 5.13. *For* $2 \le i \le N, 2 \le j \le N$,

$$\begin{aligned} T_{ij} &= \int_0^1 -\frac{\theta(1-\theta)}{2} P_{j-i}^\theta(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)}{3!} (\theta^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j1}^\theta) - (1-\theta)^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j2}^\theta)) d\theta \end{aligned} \tag{5.37}$$

Immediately, we can see from (5.29) that

LEMMA 5.14. *For $3 \le i \le N-1$, $3 \le j \le N-1$,*

$$V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right)$$
$$= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^\theta(x_i) d\theta$$
$$+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta$$
$$- \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta$$
$$- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,2}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta$$
$$+ \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta \tag{5.38}$$

To estimate V_{ij}, we first estimate $D_h^2 P_{j-i}^\theta(x_i)$, but By Lemma A.1,

$$D_h^2 P_{j-i}^\theta(x_i) = {P_{j-i}^\theta}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \tag{5.39}$$

By Leibniz formula, we calculate and estimate the derivations of h_{j-i}^3, $u''(y_{j-i}^\theta(x))$ and $\frac{|y_{j-i}^\theta(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)}$ separately.

Firstly, we have

LEMMA 5.15. *There exists a constant $C = C(T, r)$ such that For $3 \le i \le N - 1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,*

$$h_{j-i}^3(\xi) \le C h^2 x_i^{2-2/r} h_j \tag{5.40}$$
$$(h_{j-i}^3(\xi))' \le C(r-1) h^2 x_i^{1-2/r} h_j \tag{5.41}$$
$$(h_{j-i}^3(\xi))'' \le C(r-1) h^2 x_i^{-2/r} h_j \tag{5.42}$$

The proof of this theorem see Lemma C.6 and Lemma C.7

Second,

LEMMA 5.16. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,*

$$u''(y_{j-i}^\theta(\xi)) \le C x_i^{\alpha/2-2} \tag{5.43}$$
$$(u''(y_{j-i}^\theta(\xi)))' \le C x_i^{\alpha/2-3} \tag{5.44}$$
$$(u''(y_{j-i}^\theta(\xi)))'' \le C x_i^{\alpha/2-4} \tag{5.45}$$

The proof of this theorem see Proof 32

And Finally, we have

LEMMA 5.17. *There exists a constant* $C = C(T, \alpha, r)$ *such that For* $3 \le i \le N-1, 1 \le j \le \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha} \le C|y_j^{\theta} - x_i|^{1-\alpha} \tag{5.46}$$

$$(|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha})' \le C|y_j^{\theta} - x_i|^{1-\alpha} x_i^{-1} \tag{5.47}$$

$$(|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha})'' \le C|y_j^{\theta} - x_i|^{1-\alpha} x_i^{-2} \tag{5.48}$$

where $y_j^{\theta} = \theta x_{j-1} + (1-\theta)x_j$

The proof of this theorem see Proof 33

LEMMA 5.18. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$,

$$D_h^2 P_{j-i}^{\theta}(x_i) \le Ch^2 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \tag{5.49}$$

where $y_j^{\theta} = \theta x_{j-1} + (1-\theta)x_j$

Proof. Since

$$D_h^2 P_{j-i}^{\theta}(x_i) = {P_{j-i}^{\theta}}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \tag{5.50}$$

From (5.35), using Leibniz formula and Lemma 5.15, Lemma 5.16 and Lemma 5.17∎

LEMMA 5.19. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i < N, k = \lceil \frac{i}{2} \rceil$.

For $k \le j \le \min\{2i-1, N-1\}$,

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta}) - Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}} \right) \\ &\qquad \le Ch^2 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned} \tag{5.51}$$

And for $k+1 \le j \le \min\{2i, N\}$,

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta}) - Q_{j-i}^{\theta}(x_{i-1})u'''(\eta_{j-1}^{\theta})}{h_i} \right) \\ &\qquad \le Ch^2 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned} \tag{5.52}$$

where $\eta_j^{\theta} \in [x_{j-1}, x_j]$.

proof see Proof 34

LEMMA 5.20. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i < N, k = \lceil \frac{i}{2} \rceil, k+1 \le j \le \min\{2i-1, N-1\}$,

$$\begin{aligned} V_{ij} &\le Ch^2 \int_0^1 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j d\theta \\ &= Ch^2 \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \end{aligned} \tag{5.53}$$

Proof. Since Lemma 5.14, by Lemma 5.18 and Lemma 5.19, we get the result immediately. ◻

Now we can prove Theorem 5.10 using Lemma 5.20, $k = \lceil \frac{i}{2} \rceil$

$$\begin{aligned} I_3 = \sum_{k+1}^{2i-1} V_{ij} &\le Ch^2 \int_{x_k}^{x_{2i-1}} \frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{2i-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\ &\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{5.54}$$

Lemma 5.21.

$$D_h P_{j-i}^{\theta}(x_i) := \frac{P_{k-i}^{\theta}(x_{i+1}) - P_{k-i}^{\theta}(x_i)}{h_{i+1}} = {P_{j-i}^{\theta}}'(\xi), \quad \xi \in [x_i, x_{i+1}] \tag{5.55}$$

Then, for $3 \le i \le N-1$, $k = \lceil \frac{i}{2} \rceil$,

$$D_h P_{k-i}^{\theta}(x_i) \le Ch^2 x_i^{-\alpha/2-2/r} h_j \tag{5.56}$$

Proof. Using Leibniz formula, by Lemma 5.15, Lemma 5.16 and Lemma 5.17, we take $j = k+1, i = i+1$, we get

$$\begin{aligned} D_h P_{k-i}^{\theta}(x_i) &\le Ch^2 x_{i+1}^{\alpha/2-2/r-1} |y_{k+1}^{\theta} - x_{i+1}|^{1-\alpha} h_{j+1} \\ &\le Ch^2 x_i^{\alpha/2-2/r-1} |y_k^{\theta} - x_i|^{1-\alpha} h_j \\ &\le Ch^2 x_i^{-\alpha/2-2/r} h_j \end{aligned} \tag{5.57}$$ ◻

Lemma 5.22. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \le i < N, k = \lceil \frac{i}{2} \rceil$,*

$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.58}$$

And for $3 \le i < N/2$,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.59}$$

Proof. In fact,

$$\begin{aligned} &\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \\ &= \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i}) T_{i,k} \end{aligned} \tag{5.60}$$

While, by Lemma A.2

$$
\begin{aligned}
\frac{1}{h_{i+1}}(T_{i+1,k}-T_{i,k}) &= \int_{x_{k-1}}^{x_k} (u(y)-u_h(y))\frac{|x_{i+1}-y|^{1-\alpha}-|x_i-y|^{1-\alpha}}{h_{i+1}\Gamma(2-\alpha)}dy \\
&\le \int_{x_{k-1}}^{x_k} h_k^2 u''(\eta)\,\frac{|\xi-y|^{-\alpha}}{\Gamma(1-\alpha)}dy \\
&\le Ch_k h^2 x_k^{2-2/r} x_{k-1}^{\alpha/2-2}|x_i-x_k|^{-\alpha} \\
&\le Ch_k h^2 x_i^{-\alpha/2-2/r}
\end{aligned}
\tag{5.61}
$$

Thus,

$$
\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}(T_{i+1,k}-T_{i,k}) \le Ch^2 x_i^{-\alpha/2-2/r}
\tag{5.62}
$$

For

$$
\begin{aligned}
\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k}) &= \int_0^1 -\frac{\theta(1-\theta)}{2}\frac{P^\theta_{k-i}(x_{i+1})-P^\theta_{k-i}(x_i)}{h_{i+1}}d\theta \\
&+\int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{Q^\theta_{k-i}(x_{i+1})u'''(\eta^\theta_{k+1,1})-Q^\theta_{k-i}(x_i)u'''(\eta^\theta_{k,1})}{h_{i+1}}d\theta \\
&-\int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{Q^\theta_{k-i}(x_{i+1})u'''(\eta^\theta_{k+1,2})-Q^\theta_{k-i}(x_i)u'''(\eta^\theta_{k,2})}{h_{i+1}}d\theta
\end{aligned}
\tag{5.63}
$$

And by Lemma 5.21

$$
\frac{P^\theta_{k-i}(x_{i+1})-P^\theta_{k-i}(x_i)}{h_{i+1}} \le Ch^2 x_i^{-\alpha/2-2/r}h_k
\tag{5.64}
$$

And with Lemma 5.19, we can get

$$
\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k}) \le Ch^2 x_i^{-\alpha/2-2/r}
\tag{5.65}
$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$
\begin{aligned}
\frac{2}{h_i+h_{i+1}}\frac{h_{i+1}-h_i}{h_i h_{i+1}}T_{i,k} &\le h_i^{-3}h^2 x_i^{1-2/r}h_k C h_k^2 x_{k-1}^{\alpha/2-2}|x_k-x_i|^{1-\alpha} \\
&\le Ch^2 x_i^{-\alpha/2-2/r}
\end{aligned}
\tag{5.66}
$$

Summarizes, we have

$$
I_2 \le Ch^2 x_i^{-\alpha/2-2/r}
\tag{5.67}
$$

The case for I_4 is similar. □

Now combine Lemma 5.8, Lemma 5.22, Theorem 5.10, Lemma 5.5 and Lemma 5.6 to get the final result.

For $3 \le i < N/2$

$$
\begin{aligned}
R_i &= I_1+I_2+I_3+I_4+I_5 \\
&\le Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2+r-2>0 \\ Ch^2(x_i^{-1-\alpha}\ln(i)+\ln(N)), & r\alpha/2+r-2=0 \\ Ch^{r\alpha/2+r}x_i^{-1-\alpha}, & r\alpha/2+r-2<0 \end{cases}
\end{aligned}
\tag{5.68}
$$

Combine with $i = 1, 2$, we get for $1 \le i \le N/2$

$$R_i \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases} \tag{5.69}$$

5.4. Proof of Theorem 5.3. For $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$, we have

$$\begin{aligned} R_i =& \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\ &+ \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right) \\ &+ \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 \end{aligned} \tag{5.70}$$

We have estimate I_1 in Lemma 5.8 and I_2 in Lemma 5.22. We can control I_3 in similar with Theorem 5.10 by Lemma 5.20 where $2i - 1 \ge N - 1$

LEMMA 5.23. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$,*

$$\begin{aligned} I_3 = \sum_{j=k+1}^{N-1} V_{ij} &\le Ch^2 \int_{x_k}^{x_{N-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{N-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\ &\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{5.71}$$

Let's study I_5 before I_4.

$$I_5 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} V_{ij} \tag{5.72}$$

Similarly, Let's define a new series of functions

DEFINITION 5.24. *For $i < N, j \ge N$,*

$$y_{j-i}(x) = 2T - (Z_{2N-j+i} - x^{1/r})^r, \quad Z_{2N-j+i} = T^{1/r} \frac{2N - j + i}{N} \tag{5.73}$$

$$y_{j-i}'(x) = (2T - y_{j-i}(x))^{1-1/r} x^{1/r-1} \tag{5.74}$$

$$y_{j-i}''(x) = \frac{1-r}{r}(2T - y_{j-i}(x))^{1-2/r} x^{1/r-2} Z_{2N-j+i} \tag{5.75}$$

(5.76)

$$y^{\theta}_{j-i}(x) = \theta y_{j-i-1}(x) + (1-\theta) y_{j-i}(x) \tag{5.77}$$

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{5.78}$$

$$P^{\theta}_{j-i}(x) = (h_{j-i}(x))^3 u''(y^{\theta}_{j-i}(x)) \frac{|y^{\theta}_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.79}$$

$$Q^{\theta}_{j-i}(x) = (h_{j-i}(x))^4 \frac{|y^{\theta}_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.80}$$

Now we have, for $i < N, j \geq N+2$,

(5.81)

$$\begin{aligned}
V_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P^{\theta}_{j-i}(x_i) d\theta \\
&\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_{i+1}) u'''(\eta^{\theta}_{j+1,1}) - Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,1})}{h_{i+1}} \right) d\theta \\
&\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,1}) - Q^{\theta}_{j-i}(x_{i-1}) u'''(\eta^{\theta}_{j-1,1})}{h_i} \right) d\theta \\
&\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_{i+1}) u'''(\eta^{\theta}_{j+1,2}) - Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,2})}{h_{i+1}} \right) d\theta \\
&\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,2}) - Q^{\theta}_{j-i}(x_{i-1}) u'''(\eta^{\theta}_{j-1,2})}{h_i} \right) d\theta
\end{aligned}$$

Similarly, we first estimate

$$D_h^2 P^{\theta}_{j-i}(\xi) = {P^{\theta}_{j-i}}'(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \tag{5.82}$$

Combine lemmas Lemma C.8, Lemma C.9 and Lemma C.10 , we have

LEMMA 5.25. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that For* $N/2 \leq i < N$, $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in [x_{i-1}, x_{i+1}]$, *we have*

$$\begin{aligned}
|{P^{\theta}_{j-i}}''(\xi)| \leq C h_j h^2 (&|y^{\theta}_j - x_i|^{1-\alpha} \\
&+ |y^{\theta}_j - x_i|^{-\alpha} (|2T - x_i - y^{\theta}_j| + h_N) \\
&+ |y^{\theta}_j - x_i|^{-1-\alpha} (|2T - x_i - y^{\theta}_j| + h_N)^2 \\
&+ (r-1)|y^{\theta}_j - x_i|^{-\alpha})
\end{aligned} \tag{5.83}$$

And

LEMMA 5.26. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that For* $N/2 \le i < N$, , $\xi \in [x_{i-1}, x_{i+1}]$, *we have for* $N+1 \le j \le 2N - \lceil \frac{N}{2} \rceil$

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{Q^\theta_{j-i}(x_{i+1}) u'''(\eta^\theta_{j+1}) - Q^\theta_{j-i}(x_i) u'''(\eta^\theta_j)}{h_{i+1}} \right) \\ &\qquad \le C h^2 h_j (|y^\theta_j - x_i|^{1-\alpha} + |y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_N)) \end{aligned} \tag{5.84}$$

for $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{Q^\theta_{j-i}(x_i) u'''(\eta^\theta_j) - Q^\theta_{j-i}(x_{i-1}) u'''(\eta^\theta_{j-1})}{h_{i+1}} \right) \\ &\qquad \le C h^2 h_j (|y^\theta_j - x_i|^{1-\alpha} + |y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_N)) \end{aligned} \tag{5.85}$$

The proof see Proof 38.

Combine (5.81), Lemma 5.25 and Lemma 5.26, we have

THEOREM 5.27. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that For* $N/2 \le i < N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$

$$\begin{aligned} V_{ij} \le & C h^2 \int_{x_{j-1}}^{x_j} (|y - x_i|^{1-\alpha} \\ & + |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2 \\ & + (r-1)|y - x_i|^{-\alpha}) dy \end{aligned} \tag{5.86}$$

We can esitmate I_5 Now.

THEOREM 5.28. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that For* $N/2 \le i < N$, *we have*

$$I_5 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \le C h^2 + C(r-1) h^2 |T - x_{i-1}|^{1-\alpha} \tag{5.87}$$

Proof.

$$\begin{aligned} I_5 = & \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \\ \le & C h^2 \int_{x_{N+1}}^{x_{2N-i}} + \int_{x_{2N-i}}^{x_{2N - \lceil \frac{N}{2} \rceil}} (|y - x_i|^{1-\alpha} \\ & + |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2 \\ & + (r-1)|y - x_i|^{-\alpha}) dy \\ = & J_1 + J_2 \end{aligned} \tag{5.88}$$

While $x_{N+1} \le y \le x_{2N-i} = 2T - x_i$,

$$T - x_{i-1} \le x_{N+1} - x_i \le y - x_i \le x_{2N-i} - x_i \le 2(T - x_{i-1}) \tag{5.89}$$

and

$$2T - x_i - y + h_N \leq 2T - x_i - x_{N+1} + h_N = T - x_i \leq T - x_{i-1} \tag{5.90}$$

So

$$\begin{aligned} J_1 &\leq Ch^2(x_{2N-i} - x_{N+1})(|T - x_{i-1}|^{1-\alpha} + (r-1)|T - x_{i-1}|^{-\alpha}) \\ &\leq Ch^2(|T - x_{i-1}|^{2-\alpha} + (r-1)|T - x_{i-1}|^{1-\alpha}) \\ &\leq Ch^2T^{2-\alpha} + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned} \tag{5.91}$$

Otherwise, when $x_{2N-i} \leq y \leq x_{2N-\lceil \frac{N}{2} \rceil}$

$$x_i + y - 2T + h_N \leq y - x_i \tag{5.92}$$

$$\begin{aligned} J_2 &\leq Ch^2 \int_{x_{2N-i}}^{(2-2^{-r})T} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} \\ &\leq Ch^2(T^{2-\alpha} + (r-1)|x_{2N-i} - x_i|^{1-\alpha}) \\ &= Ch^2 + C(r-1)h^2|T - x_i|^{1-\alpha} \leq Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned} \tag{5.93}$$

Summarizes two cases, we get the result. ∎

For I_4, we have

THEOREM 5.29. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that, for $N/2 \le i \le N-1$*

$$V_{iN} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,N+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,N} + \frac{1}{h_i} T_{i-1,N-1} \right) \le Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \tag{5.94}$$

Proof. We use the similar skill in the last section, but more complicated. for $j = N$, Let

$$y_{i\to N-1}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r} \frac{N-1-i}{N} \tag{5.95}$$

$$y_{i\to N}(x) = \frac{x^{1/r} - Z_i}{Z_1} h_N + T, \quad Z_i = T^{1/r} \frac{i}{N}, x_N = T \tag{5.96}$$

and

$$y_{i\to N+1}(x) = 2T - (Z_{N-1+i} - x^{1/r})^r, \quad Z_{N-1+i} = T^{1/r} \frac{N-1+i}{N} \tag{5.97}$$

Thus,

$$\begin{gathered} y_{i\to N-1}(x_{i-1}) = x_{N-2}, \quad y_{i\to N-1}(x_i) = x_{N-1}, \quad y_{i\to N-1}(x_{i+1}) = x_N \\ y_{i\to N}(x_{i-1}) = x_{N-1}, \quad y_{i\to N}(x_i) = x_N, \quad y_{i\to N}(x_{i+1}) = x_{N+1} \\ y_{i\to N+1}(x_{i-1}) = x_N, \quad y_{i\to N+1}(x_i) = x_{N+1}, \quad y_{i\to N+1}(x_{i+1}) = x_{N+2} \end{gathered}$$

Then, define

$$y^\theta_{i\to N}(x) = \theta y_{i\to N-1}(x) + (1-\theta) y_{i\to N}(x) \tag{5.98}$$

$$y^\theta_{i\to N+1}(x) = \theta y_{i\to N}(x) + (1-\theta) y_{i\to N+1}(x) \tag{5.99}$$

$$h_{i\to N}(x) = y_{i\to N}(x) - y_{i\to N-1}(x) \tag{5.100}$$

$$h_{i\to N+1}(x) = y_{i\to N+1}(x) - y_{i\to N}(x) \tag{5.101}$$

We have

$$y_{i\to N-1}{}'(x) = y^{1-1/r}_{i\to N-1}(x) x^{1/r-1} \tag{5.102}$$

$$y_{i\to N-1}{}''(x) = \frac{1-r}{r} y^{1-2/r}_{i\to N-1}(x) x^{1/r-2} Z_{N-1-i} \tag{5.103}$$

$$y_{i\to N}{}'(x) = \frac{1}{r} \frac{h_N}{Z_1} x^{1/r-1} \tag{5.104}$$

$$y_{i\to N}{}''(x) = \frac{1-r}{r^2} \frac{h_N}{Z_1} x^{1/r-2} \tag{5.105}$$

$$y_{i\to N+1}{}'(x) = (2T - y_{i\to N+1}(x))^{1-1/r} x^{1/r-1} \tag{5.106}$$

$$y_{i\to N+1}{}''(x) = \frac{1-r}{r} (2T - y_{i\to N+1}(x))^{1-2/r} x^{1/r-2} Z_{N-1+i} \tag{5.107}$$

$$P^{\theta}_{i\to N}(x) = (h_{i\to N}(x))^3 \frac{|y^{\theta}_{i\to N}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y^{\theta}_{i\to N}(x)) \tag{5.108}$$

$$P^{\theta}_{i\to N+1}(x) = (h_{i\to N+1}(x))^3 \frac{|y^{\theta}_{i\to N+1}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y^{\theta}_{i\to N+1}(x)) \tag{5.109}$$

$$Q^{\theta}_{i\to N}(x) = (h_{i\to N}(x))^4 \frac{|y^{\theta}_{i\to N}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.110}$$

$$Q^{\theta}_{i\to N+1}(x) = (h_{i\to N+1}(x))^4 \frac{|y^{\theta}_{i\to N+1}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.111}$$

Similar with Lemma 5.13, we can get for $l = -1, 0, 1$,

$$\begin{aligned} T_{i+l,N+l} = &\int_0^1 -\frac{\theta(1-\theta)}{2} P^{\theta}_{i\to N}(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} Q^{\theta}_{i\to N}(x_{i+l})(\theta^2 u'''(\eta^{\theta}_{N+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+l,2})) d\theta \end{aligned} \tag{5.112}$$

(5.113)

$$\begin{aligned} T_{i+l,N+1+l} = &\int_0^1 -\frac{\theta(1-\theta)}{2} P^{\theta}_{i\to N+1}(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} Q^{\theta}_{i\to N+1}(x_{i+l})(\theta^2 u'''(\eta^{\theta}_{N+1+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+1+l,2})) d\theta \end{aligned}$$

So we have

(5.114)

$$\begin{aligned} V_{i,N} = &\int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P^{\theta}_{i\to N}(x_i) d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{i\to N}(x_{i+1}) u'''(\eta^{\theta}_{N+1,1}) - Q^{\theta}_{i\to N}(x_i) u'''(\eta^{\theta}_{N,1})}{h_{i+1}} \right) d\theta \\ &- \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{i\to N}(x_i) u'''(\eta^{\theta}_{N,1}) - Q^{\theta}_{i\to N}(x_{i-1}) u'''(\eta^{\theta}_{N-1,1})}{h_i} \right) d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{i\to N}(x_{i+1}) u'''(\eta^{\theta}_{N+1,2}) - Q^{\theta}_{i\to N}(x_i) u'''(\eta^{\theta}_{N,2})}{h_{i+1}} \right) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{i\to N}(x_i) u'''(\eta^{\theta}_{N,2}) - Q^{\theta}_{i\to N}(x_{i-1}) u'''(\eta^{\theta}_{N-1,2})}{h_i} \right) d\theta \end{aligned}$$

$N+1$ is similar.

We estimate $D_h^2 P^{\theta}_{i\to N}(x_i) = {P^{\theta}_{i\to N}}''(\xi), \xi \in [x_{i-1}, x_{i+1}]$,

LEMMA 5.30.

$$h^3_{i\to N}(\xi) \le C h_N^3 \le C h^3 \tag{5.115}$$

$$h^3_{i\to N+1}(\xi) \le C h_N^3 \le C h^3 \tag{5.116}$$

$$(h^3_{i\to N}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{5.117}$$

$$(h^3_{i\to N+1}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{5.118}$$

$$(h^3_{i\to N}(\xi))'' \le C(r-1)h^2 \tag{5.119}$$

$$(h^3_{i\to N+1}(\xi))'' \le C(r-1)h^2 \tag{5.120}$$

Proof.

$$h_{i\to N}(\xi) \le 2h_N, \quad h_{i\to N+1}(\xi) \le 2h_N \tag{5.121}$$

$$\begin{aligned}(h^l_{i\to N}(\xi))' &= l h^{l-1}_{i\to N}(\xi)({y_{i\to N}}'(\xi) - {y_{i\to N-1}}'(\xi))\\ &= l h^{l-1}_{i\to N}(\xi) x_i^{1/r-1}(\frac{1}{r}\frac{h_N}{Z_1} - y^{1-1/r}_{i\to N-1}(\xi))\end{aligned} \tag{5.122}$$

while

$$\begin{aligned}|\frac{1}{r}\frac{h_N}{Z_1} - y^{1-1/r}_{i\to N-1}(\xi)| &= |\frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}| \quad \eta\in[x_{N-2}, x_N]\\ &= T^{1-1/r}|(\frac{N-t}{N})^{r-1} - (\frac{N-s}{N})^{r-1}| \quad t\in[0,1], s\in[0,2]\\ &\le T^{1-1/r}|1-(\frac{N-2}{N})^{r-1}| \le CT^{1-1/r}(r-1)\frac{2}{N}\end{aligned} \tag{5.123}$$

Thus,

$$(h^l_{i\to N}(\xi))' \le C(r-1)h_N^{l-1}x_i^{1/r-1}h \tag{5.124}$$

$$\begin{aligned}(h^l_{i\to N+1}(\xi))' &= l h^{l-1}_{i\to N+1}(\xi)({y_{i\to N+1}}'(\xi) - {y_{i\to N}}'(\xi))\\ &= l h^{l-1}_{i\to N+1}(\xi) x_i^{1/r-1}((2T - y_{i\to N+1}(\xi))^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1})\end{aligned} \tag{5.125}$$

Similarly,

$$\begin{aligned}|(2T - y_{i\to N+1})^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1}| &= |\eta^{1-1/r} - \frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1}| \quad \eta\in[x_{N-2}, x_N]\\ &= T^{1-1/r}|(\frac{N-s}{N})^{r-1} - (\frac{N-t}{N})^{r-1}| \quad t\in[0,1], s\in[0,2]\\ &\le T^{1-1/r}|(\frac{N-2}{N})^{r-1} - 1| \le CT^{1-1/r}(r-1)\frac{2}{N}\end{aligned} \tag{5.126}$$

And

$$\begin{aligned}(h^3_{i\to N}(\xi))'' &= 3h^2_{i\to N}(\xi){h_{i\to N}}''(\xi) + 6h_{i\to N}(\xi)({h_{i\to N}}'(\xi))^2\\ &\le Ch_N^2\frac{1-r}{r}x_i^{1/r-2}(\frac{1}{r}\frac{h_N}{Z_1} - y^{1-2/r}_{i\to N-1}(\xi)Z_{N-1-i}) + Ch_N(r-1)^2h^2x_i^{2/r-2}\end{aligned} \tag{5.127}$$

$$|\frac{h_N}{rZ_1} - y^{1-2/r}_{i\to N-1}(\xi)Z_{N-1-i}| \le T^{1-1/r} + Cx_N^{1-2/r}x_N^{1/r} = CT^{1-1/r}$$

So

$$(h^3_{i\to N}(\xi))'' \leq Ch_N^2 \frac{1-r}{r} x_i^{1/r-2} + C(r-1)^2 h_N x_i^{2/r-2} h^2 \\ \leq C(r-1)h_N^2 \tag{5.128}$$

$h^3_{i\to N+1}(\xi)$ is similar. □

LEMMA 5.31.

$$u''(y^\theta_{i\to N}(\xi)) \leq C x_{N-2}^{-\alpha/2-2} \leq C \tag{5.129}$$

$$(u''(y^\theta_{i\to N}(\xi)))' \leq C \tag{5.130}$$

$$(u''(y^\theta_{i\to N}(\xi)))'' \leq C \tag{5.131}$$

Proof.

$$\begin{aligned}(u''(y^\theta_{i\to N}(\xi)))' &= u'''(y^\theta_{i\to N}(\xi)) {y^\theta_{i\to N}}'(\xi) \\ &\leq C(\theta {y_{i\to N-1}}'(\xi) + (1-\theta){y_{i\to N}}'(\xi)) \\ &\leq C x_i^{1/r-1}(\theta y^{1-1/r}_{i\to N-1}(\xi) + (1-\theta)\frac{h_N}{rZ_1}) \\ &\leq C x_i^{1/r-1} x_N^{1-1/r}\end{aligned} \tag{5.132}$$

And

(5.133)

$$\begin{aligned}(u''(y^\theta_{i\to N}(\xi)))'' &= u''''(y^\theta_{i\to N}(\xi))({y^\theta_{i\to N}}'(\xi))^2 + u'''(y^\theta_{i\to N}(\xi)){y^\theta_{i\to N}}''(\xi) \\ &\leq C x_i^{2/r-2} x_N^{2-2/r} + C\frac{r-1}{r} x_i^{1/r-2}(\theta x_N^{1-2/r} Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \\ &\leq C x_i^{2/r-2} + C(r-1)x_i^{1/r-2} T^{1-1/r}\end{aligned}$$

□

LEMMA 5.32.

$$|y^\theta_{i\to N}(\xi) - \xi|^{1-\alpha} \leq C|y^\theta_N - x_i|^{1-\alpha} \tag{5.134}$$

$$(|y^\theta_{i\to N}(\xi) - \xi|^{1-\alpha})' \leq C|y^\theta_N - x_i|^{1-\alpha} \tag{5.135}$$

$$(|y^\theta_{i\to N}(\xi) - \xi|^{1-\alpha})'' \leq C(r-1)|y^\theta_N - x_i|^{-\alpha} + |y^\theta_N - x_i|^{1-\alpha} \tag{5.136}$$

Proof.

(5.137)

$$\begin{aligned}(y^\theta_{i\to N}(\xi) - \xi)' &= (\theta(y_{i\to N-1}(\xi) - \xi) + (1-\theta)(y_{i\to N}(\xi) - \xi))' \\ &= \theta({y_{i\to N-1}}'(\xi) - 1) + (1-\theta)({y_{i\to N}}'(\xi) - 1) \\ &= \theta\xi^{1/r-1}(y^{1-1/r}_{i\to N-1}(\xi) - \xi^{1-1/r}) + (1-\theta)\xi^{1/r-1}(\frac{h_N}{rZ_1} - \xi^{1-1/r})\end{aligned}$$

$$\begin{aligned}(y^\theta_{i\to N}(\xi) - \xi)'' &= \theta({y_{i\to N-1}}''(\xi)) + (1-\theta)({y_{i\to N}}''(\xi)) \\ &= \frac{1-r}{r}\xi^{1/r-2}(\theta y^{1-2/r}_{i\to N-1}(\xi) Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \leq 0\end{aligned} \tag{5.138}$$

And

$$|(y^\theta_{i\to N}(\xi) - \xi)''| \leq C(r-1)\xi^{1/r-2} T^{1-1/r} \tag{5.139}$$

We have known

$$C|x_{N-1}-x_i| \le |y_{i\to N-1}(\xi)-\xi| \le C|x_{N-1}-x_i| \tag{5.140}$$

If $\xi \le x_{N-1}$, then $(y_{i\to N}(\xi)-\xi)' \ge 0$, so

$$C|x_N-x_i| \le |x_{N-1}-x_{i-1}| \le |y^\theta_{i\to N}(\xi)-\xi| \le |x_{N+1}-x_{i+1}| \le C|x_N-x_i| \tag{5.141}$$

If $i=N-1$ and $\xi\in[x_{N-1},x_N]$, then $y_{i\to N}(\xi)-\xi$ is concave, bigger than its two neighboring points, which are equal to h_N, so

$$h_N=|x_N-x_{N-1}| \le |y_{i\to N}(\xi)-\xi| \le |x_{N+1}-x_{N-1}|=2h_N \tag{5.142}$$

So we have

$$|y^\theta_{i\to N}(\xi)-\xi|^{1-\alpha} \le C|y^\theta_N-x_i|^{1-\alpha} \tag{5.143}$$

While

$$y^{1-1/r}_{i\to N-1}(\xi)-\xi^{1-1/r} \le (y_{i\to N-1}(\xi)-\xi)\xi^{-1/r} \tag{5.144}$$

and

$$\begin{aligned}|\frac{h_N}{rZ_1}-\xi^{1-1/r}| &\le \max\{|\frac{h_N}{rZ_1}-x^{1-1/r}_{i-1}|,|\frac{h_N}{rZ_1}-x^{1-1/r}_{i+1}|\}\\ &\le \max\begin{cases}T^{1-1/r}-x^{1-1/r}_{i-1} \le |x_N-x_{i-1}|T^{-1/r} \le C|x_N-x_i|\\ |x^{1-1/r}_{i+1}-x^{1-1/r}_{N-1}| \le |x_{i+1}-x_{N-1}|x^{-1/r}_{N-1} \le C|x_N-x_i|\end{cases}\end{aligned} \tag{5.145}$$

So we have

$$(y^\theta_{i\to N}(\xi)-\xi)' \le C|y^\theta_N-x_i| \tag{5.146}$$

$$\begin{aligned}(|y^\theta_{i\to N}(\xi)-\xi|^{1-\alpha})' &= |y^\theta_{i\to N}(\xi)-\xi|^{-\alpha}(y^\theta_{i\to N}(\xi)-\xi)'\\ &\le |y^\theta_N-x_i|^{1-\alpha}\end{aligned} \tag{5.147}$$

Finally,

$$\begin{aligned}(|y^\theta_{i\to N}(\xi)-\xi|^{1-\alpha})'' &= (1-\alpha)|y^\theta_{i\to N}(\xi)-\xi|^{-\alpha}(y^\theta_{i\to N}(\xi)-\xi)''\\ &\quad+\alpha(\alpha-1)|y^\theta_{i\to N}(\xi)-\xi|^{-1-\alpha}((y^\theta_{i\to N}(\xi)-\xi)')^2\\ &\le C(r-1)|y^\theta_N-x_i|^{-\alpha}+C|y^\theta_N-x_i|^{1-\alpha}\end{aligned} \tag{5.148}$$ □

By the three lemmas above, for $N/2\le i\le N-1$, we have

LEMMA 5.33.

$$\begin{aligned}D^2_hP^\theta_{i\to N}(x_i) &= {P^\theta_{i\to N}}''(\xi) \quad \xi\in[x_{i-1},x_{i+1}]\\ &\le Ch^3|y^\theta_N-x_i|^{1-\alpha}+C(r-1)(h^3|y^\theta_N-x_i|^{-\alpha}+h^2|y^\theta_N-x_i|^{1-\alpha})\end{aligned} \tag{5.149}$$

And

LEMMA 5.34.

$$
\frac{2}{h_i+h_{i+1}}\left(\frac{Q^{\theta}_{i\to N}(x_{i+1})u'''(\eta^{\theta}_{N+1})-Q^{\theta}_{i\to N}(x_i)u'''(\eta^{\theta}_{N})}{h_{i+1}}\right) \le Ch^3|y^{\theta}_N-x_i|^{1-\alpha} \tag{5.150}
$$

And immediately, For $N/2\le i\le N-2$

$$
\begin{aligned}
V_{iN} &\le C\int_{x_{N-1}}^{x_N} h^2|y-x_i|^{1-\alpha}+C(r-1)h^2|y-x_i|^{-\alpha}+h|y-x_i|^{1-\alpha}dy\\
&\le Ch^2h_N|T-x_i|^{1-\alpha}+C(r-1)h^2|x_{N-1}-x_i|^{1-\alpha}+Chh_N|T-x_i|^{1-\alpha}\\
&\le Ch^2+C(r-1)h^2|T-x_{i-1}|^{1-\alpha}
\end{aligned} \tag{5.151}
$$

But expecially, when $i=N-1$,

$$
\begin{aligned}
V_{N-1,N} = &\int_0^1 -\frac{\theta^{2-\alpha}(1-\theta)}{2}\frac{2}{h_{N-1}+h_N}\left(\frac{1}{h_{N-1}}h^{4-\alpha}_{N-1}u''(y^{\theta}_{N-1})-(\frac{1}{h_{N-1}}+\frac{1}{h_N})h^{4-\alpha}_N u''(y^{\theta}_N)+\frac{1}{h_N}h^{4-\alpha}_{N+1}u''(y^{\theta}_{N+1})\right)d\theta\\
&+\int_0^1\frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q^{\theta}_{i\to N}(x_{i+1})u'''(\eta^{\theta}_{N+1,1})-Q^{\theta}_{i\to N}(x_i)u'''(\eta^{\theta}_{N,1})}{h_{i+1}}\right)d\theta\\
&-\int_0^1\frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q^{\theta}_{i\to N}(x_i)u'''(\eta^{\theta}_{N,1})-Q^{\theta}_{i\to N}(x_{i-1})u'''(\eta^{\theta}_{N-1,1})}{h_i}\right)d\theta\\
&-\int_0^1\frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q^{\theta}_{i\to N}(x_{i+1})u'''(\eta^{\theta}_{N+1,2})-Q^{\theta}_{i\to N}(x_i)u'''(\eta^{\theta}_{N,2})}{h_{i+1}}\right)d\theta\\
&+\int_0^1\frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q^{\theta}_{i\to N}(x_i)u'''(\eta^{\theta}_{N,2})-Q^{\theta}_{i\to N}(x_{i-1})u'''(\eta^{\theta}_{N-1,2})}{h_i}\right)d\theta
\end{aligned} \tag{5.152}
$$

while combine Lemma 5.30

$$
\begin{aligned}
&\frac{2}{h_{N-1}+h_N}\left(\frac{1}{h_{N-1}}h^{4-\alpha}_{N-1}u''(y^{\theta}_{N-1})-(\frac{1}{h_{N-1}}+\frac{1}{h_N})h^{4-\alpha}_N u''(y^{\theta}_N)+\frac{1}{h_N}h^{4-\alpha}_{N+1}u''(y^{\theta}_{N+1})\right)\\
&=D^2_h(h^{4-\alpha}_{N-1\to N}(x_i)u''(y^{\theta}_{N-1\to N}(x_i)))\\
&\le Ch^{4-\alpha}_N+C(r-1)h^{3-\alpha}_N\le Ch^{4-\alpha}+C(r-1)h^2|T-x_{N-1-1}|^{1-\alpha}
\end{aligned} \tag{5.153}
$$

Similarly with $j=N+1$. ∎

I_6, I_7 is easy. Similar with Lemma 5.22 and Lemma 5.6, we have

THEOREM 5.35. *There is a constant $C = C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \le i \le N$,*

$$
\begin{aligned}
I_6 &= \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}(T_{i-1,2N-\lceil\frac{N}{2}\rceil+1}+T_{i-1,2N-\lceil\frac{N}{2}\rceil})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,2N-\lceil\frac{N}{2}\rceil+1}\right)\\
&\le Ch^2
\end{aligned}
\tag{5.154}
$$

Proof. In fact, let $l = 2N-\lceil\frac{N}{2}\rceil+1$

$$
\begin{aligned}
&\frac{1}{h_i}(T_{i-1,l}+T_{i-1,l-1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,l}\\
&=\frac{1}{h_i}(T_{i-1,l}-T_{i,l})+\frac{1}{h_i}(T_{i-1,l-1}-T_{i,l})+(\frac{1}{h_i}-\frac{1}{h_{i+1}})T_{i,l}
\end{aligned}
\tag{5.155}
$$

While, by Lemma A.2

$$
\begin{aligned}
\frac{1}{h_i}(T_{i-1,l}-T_{i,l}) &= \int_{x_{l-1}}^{x_l}(u(y)-u_h(y))\frac{|x_{i-1}-y|^{1-\alpha}-|x_i-y|^{1-\alpha}}{h_i\Gamma(2-\alpha)}dy\\
&\le C\int_{x_{l-1}}^{x_l} h_l^2u''(\eta)\ \frac{|\xi-y|^{-\alpha}}{\Gamma(1-\alpha)}dy\\
&\le Ch_l^3x_{l-1}^{\alpha/2-2}T^{-\alpha}\\
&\le Ch_l^3
\end{aligned}
\tag{5.156}
$$

Thus,

$$
\frac{2}{h_i+h_{i+1}}\frac{1}{h_i}(T_{i-1,l}-T_{i,l}) \le Ch_l^2
\tag{5.157}
$$

For

$$
\frac{1}{h_i}(T_{i-1,l-1}-T_{i,l}) = \int_0^1 -\frac{\theta(1-\theta)}{2}\frac{h_{l-1}^3|y_{l-1}^\theta-x_{i-1}|^{1-\alpha}u''(\eta_{l-1}^\theta)-h_l^3|y_l^\theta-x_i|^{1-\alpha}u''(\eta_l^\theta)}{h_i}d\theta
\tag{5.158}
$$

And Similar with Lemma 5.19, we can get

$$
\frac{h_{l-1}^3|y_{l-1}^\theta-x_{i-1}|^{1-\alpha}u''(\eta_{l-1}^\theta)-h_l^3|y_l^\theta-x_i|^{1-\alpha}u''(\eta_l^\theta)}{(h_i+h_{i+1})h_i} \le Ch_l^2|y_l^\theta-x_i|^{1-\alpha}
\tag{5.159}
$$

So

$$
\frac{2}{h_i+h_{i+1}}\frac{1}{h_i}(T_{i-1,l-1}-T_{i,l}) \le Ch^2
\tag{5.160}
$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$
\begin{aligned}
\frac{2}{h_i+h_{i+1}}\frac{h_{i+1}-h_i}{h_ih_{i+1}}T_{i,l} &\le h_i^{-3}h^2x_i^{1-2/r}h_lCh_l^2x_{l-1}^{\alpha/2-2}|x_l-x_i|^{1-\alpha}\\
&\le Ch^2
\end{aligned}
\tag{5.161}
$$

Summarizes, we have

$$
I_6 \le Ch^2
\tag{5.162}
$$

□

And

LEMMA 5.36. *There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le i \le N$,*

$$
\begin{aligned}
I_7 &= \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} S_{ij} \\
&\le \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\end{aligned} \tag{5.163}
$$

Proof. For $i \le N, j \ge 2N - \lceil \frac{N}{2} \rceil + 2$, we have

$$
\begin{aligned}
S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\
&\le \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2 - 2/r} |y - x_{i+1}^{-1-\alpha} dy \\
&\le Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2 - 2/r} dy
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N-1} S_{ij} &\le CT^{-1-\alpha} h^2 \int_{(2-2^{-r})T}^{x_{2N-1}} (2T - y)^{\alpha/2 - 2/r} dy \\
&\le CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2 - 2/r + 1} T^{\alpha/2 - 2/r + 1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r}T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2 - 2/r + 1|} h_{2N}^{\alpha/2 - 2/r + 1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\
&= \begin{cases} \frac{C}{\alpha/2 - 2/r + 1} T^{-\alpha/2 - 2/r}\ h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2 - 2/r + 1|} T^{-\alpha/2 - 2/r}\ h^{r\alpha/2 + r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\end{aligned} \tag{5.164}
$$

∎

Now we can conclude a part of the theorem Theorem 5.3 at the beginning of this section.

By Lemma 5.8 Lemma 5.22 Lemma 5.23 Theorem 5.29 Theorem 5.28 Theorem 5.35 Lemma 5.36 , we have

THEOREM 5.37. *there exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $N/2 \le i < N$,*

$$
\begin{aligned}
R_i &= \sum_{j=1}^{7} I_j \\
&\le C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\end{aligned} \tag{5.165}
$$

And what we left is the case $i = N$. Fortunately, we can use the same department of R_i above, and it is symmetric. Most of the item has been esitmated by Lemma 5.8 and Theorem 5.35, we just need to consider I_3, I_4.

THEOREM 5.38. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$I_3 = \sum_{j=\lceil \frac{N}{2} \rceil+1}^{N-1} V_{Nj} \le Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha} \tag{5.166}$$

Proof. DEFINITION 5.39. *For $N/2 \le j < N$, Let's define*

$$y_j(x) = (\frac{Z_1}{h_N}(x - x_N) + Z_j)^r \tag{5.167}$$

We can see that is the inverse of the function $y_{i\to N}(x)$ defined in Theorem 5.29.

$$y_j'(x) = y_j^{1-1/r}(x)\frac{rZ_1}{h_N} \tag{5.168}$$

$$y_j''(x) = y_j^{1-2/r}(x)\frac{r(r-1)Z_1}{h_N} \tag{5.169}$$

With the scheme we used several times, we can get

LEMMA 5.40. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le j < N$, $\xi \in [x_{N-1}, x_{N+1}]$,*

$$h_j(\xi)^3 \le Ch^3 \tag{5.170}$$

$$(h_j^3(\xi))' \le C(r-1)h^3 \tag{5.171}$$

$$(h_j^3(\xi))'' \le C(r-1)h^3 \tag{5.172}$$

$$u''(y_j^\theta(\xi)) \le C \tag{5.173}$$

$$(u''(y_j^\theta(\xi)))' \le C \tag{5.174}$$

$$(u''(y_j^\theta(\xi)))'' \le C \tag{5.175}$$

$$|\xi - y_j^\theta(\xi)|^{1-\alpha} \le C|x_N - y_j^\theta|^{1-\alpha} \tag{5.176}$$

$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})' \le C|x_N - y_j^\theta|^{1-\alpha} \tag{5.177}$$

$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})'' \le C|x_N - y_j^\theta|^{1-\alpha} + C(r-1)|x_N - y_j^\theta|^{-\alpha} \tag{5.178}$$

LEMMA 5.41. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le j < N$,*

$$V_{Nj} \le Ch^2 \int_{x_{j-1}}^{x_j} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \tag{5.179}$$

Therefore,

$$\begin{aligned} I_3 &\le Ch^2 \int_{x_{\lceil \frac{N}{2} \rceil}}^{x_{N-1}} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \\ &\le Ch^2(|T - x_{N-1}|^{2-\alpha} + (r-1)|T - x_{N-1}|^{1-\alpha}) \end{aligned} \tag{5.180}$$

$\square$

For $j = N$,

LEMMA 5.42.

$$V_{N,N} = \frac{1}{h_N^2}\left(T_{N-1,N-1} - 2T_{N,N} + T_{N+1,N+1}\right) \le Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha} \tag{5.181}$$

Proof. ∎

$$\begin{aligned} V_{N,N} = &\int_0^1 -\frac{\theta(1-\theta)^{2-\alpha}}{2}\frac{1}{h_N^2}\left(h_{N-1}^{4-\alpha}u''(y_{N-1}^\theta) - 2h_N^{4-\alpha}u''(y_N^\theta) + h_{N+1}^{4-\alpha}u''(y_{N+1}^\theta)\right)d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,1}^\theta) - Q_{N\to N}^\theta(x_i)u'''(\eta_{N,1}^\theta)}{h_N}\right)d\theta \\ &- \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,1}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,1}^\theta)}{h_N}\right)d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,2}^\theta) - Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta)}{h_N}\right)d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,2}^\theta)}{h_N}\right)d\theta \end{aligned} \tag{5.182}$$

So combine Lemma 5.8, Theorem 5.35, Theorem 5.38, Lemma 5.42 We have

LEMMA 5.43.

$$R_N \le C(r-1)h^2|T - x_{N-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.183}$$

and with Theorem 5.37 we prove the Theorem 5.3

5.5. Truncation error. combine Theorem 5.1, Theorem 5.2 and Theorem 5.3 we get For $1 \le i \le N$

$$R_i \le C_2(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} C_1h^2x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ C_1h^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ C_1h^{r\alpha/2+r}x_i^{-1-\alpha/2}, & r\alpha/2 + r - 2 < 0 \end{cases} \tag{5.184}$$

But,

$$h^2x_i^{-\alpha/2-2/r} \le T^{\alpha/2-2/r}\begin{cases} h^2x_i^{-\alpha}, & \text{if} \quad r\alpha/2 - 2 \ge 0 \\ h^{r\alpha/2}x_i^{-\alpha}, & \text{if} \quad r\alpha/2 - 2 \le 0 \end{cases} \tag{5.185}$$

$$h^{r\alpha/2+r}x_i^{-1-\alpha} \le T^{-1}h^{r\alpha/2}x_i^{-\alpha}, \quad \text{if} \quad r\alpha/2 - 2 \le 0 \tag{5.186}$$

$$\tag{5.187}$$

And when $r\alpha/2 - 2 = -r < 0$,

$$\begin{aligned} h^2x_i^{-1-\alpha}\ln(i)h^{-r\alpha/2}x_i^{\alpha} &= h^rx_i^{-1}\ln(i) \\ &= T^{-1}\frac{\ln(i)}{i^r} \le C(T,r) \end{aligned} \tag{5.188}$$

and

$$h^2 \ln(N) h^{-r\alpha/2} x_i^\alpha = h^r \ln(N) x_i^\alpha \leq T^\alpha \frac{\ln(N)}{N^r} \leq C(T,\alpha,r) \tag{5.189}$$

So for $1 \leq i \leq N$,

$$R_i \leq C_2(r-1)h^2|T - x_{i-1}|^{1-\alpha} + C_1 h^{\min\{\frac{r\alpha}{2},2\}} x_i^{-\alpha} \tag{5.190}$$

And for $i \geq N$, it is symmetric for i and $2N - i$.

The proof of Theorem 4.1 completed.

6. Proof of Theorem 4.2. Review section 3,we have (3.9) and (3.11),

$$\tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)}\left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}}|x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}}\right) \tag{6.1}$$

$$a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_i}\tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right)\tilde{a}_{i,j} + \frac{1}{h_{i+1}}\tilde{a}_{i+1,j}\right) \tag{6.2}$$

Thus

LEMMA 6.1.

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)}\left(\frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}}\right) \tag{6.3}$$

DEFINITION 6.2. *We call one matrix a M matrix , which means its entries are positive on major diagonal and nonpositive on others, and Strictly diagonally dominant in rows.*

Now we have

LEMMA 6.3. *The matrix A defined by* (3.11) *is a M matrix. and*

$$\begin{aligned} S_i :=& \sum_{j=1}^{2N-1} a_{ij} \\ =& \sum_{j=1}^{2N-1} \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}\tilde{a}_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})\tilde{a}_{i,j} + \frac{1}{h_i}\tilde{a}_{i-1,j}\right) \\ \geq& C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \end{aligned} \tag{6.4}$$

Proof. Let

$$g(x) = g_0(x) + g_{2N}(x) \tag{6.5}$$

where

$$g_0(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)}\frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1}$$

$$g_{2N}(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)}\frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}}$$

Then, for $2 \leq i \leq 2N-2$,

$$\begin{aligned} S_i :=& \sum_{j=1}^{2N-1} a_{ij} \\ =& \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})g(x_i) + \frac{1}{h_i}g(x_{i-1})\right) \\ =& g_0''(\xi) + g_{2N}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \end{aligned} \tag{6.6}$$

While for $i \geq 2$

$$g_0''(\xi) = -\kappa_\alpha \frac{|\xi - x_0|^{1-\alpha} - |\xi - x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1}$$
$$(6.7) \qquad = \frac{\kappa_\alpha}{-\Gamma(1-\alpha)}|\xi - \eta|^{-\alpha}, \quad \eta \in [x_0, x_1]$$
$$\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha}$$

when $i = 1$

$$\frac{2}{h_1 + h_2}\left(\frac{1}{h_2}g_0(x_2) - (\frac{1}{h_1} + \frac{1}{h_2})g_0(x_1) + \frac{1}{h_1}g_0(x_0)\right)$$
$$= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1+h_2)^{3-\alpha}}{(h_1+h_2)h_1h_2}$$
$$(6.8) \qquad = \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1+h_2)^{3-\alpha}}{(h_1+h_2)h_1^{1-\alpha}h_2} h_1^{-\alpha}$$
$$= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r-1)^{3-\alpha} + 2(2^r-1) - (2^r)^{3-\alpha}}{2^r(2^r-1)} h_1^{-\alpha}$$

but

$$(6.9) \qquad 1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha} > 0$$

So

$$(6.10) \qquad \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}g_0(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})g_0(x_i) + \frac{1}{h_i}g_0(x_{i-1})\right) \geq C x_i^{-\alpha}$$

symmetricly,

$$(6.11) \qquad \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}g_{2N}(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})g_{2N}(x_i) + \frac{1}{h_i}g_{2N}(x_{i-1})\right) \geq C(\alpha, r)(2T - x_i)^{-\alpha}$$

$\square$

Let

$$(6.12) \qquad g(x) = \begin{cases} x, & 0 < x \leq T \\ 2T - x, & T < x < 2T \end{cases}$$

And define

$$(6.13) \qquad G = \mathrm{diag}(g(x_1), ..., g(x_{2N-1}))$$

Then

LEMMA 6.4. *The matrix $B := AG$, the major diagnal is positive, and nonpositive on others. And there is a constant $C = C(\alpha, r)$ such that*

$$(6.14) \qquad M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases}$$

Proof. Since

$$g(x) \equiv g_h(x) \tag{6.15}$$

by (3.9), we have

$$\begin{aligned}\tilde{M}_i &:= \sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} \tilde{a}_{ij} g(x_j) \\ &= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} g_h(y)dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} g(y)dy \\ &= \frac{-2}{\Gamma(4-\alpha)}|T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)}(x_i^{3-\alpha} + (2T - x_i)^{3-\alpha}) \\ &:= w(x_i) = p(x_i) + q(x_i)\end{aligned} \tag{6.16}$$

Thus,

$$\begin{aligned}M_i &:= \sum_{j=1}^{2N-1} a_{ij} g(x_j) \\ &= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} w(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) w(x_i) + \frac{1}{h_i} w(x_{i-1}) \right)\end{aligned} \tag{6.17}$$

for $1 \leq i < N-1$,by Lemma A.1

$$\begin{aligned}P_i &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} p(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) p(x_i) + \frac{1}{h_i} p(x_{i-1}) \right) \\ &= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}|T - \xi|^{1-\alpha} \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}|T - x_{i-1}|^{1-\alpha}\end{aligned} \tag{6.18}$$

and

$$\begin{aligned}P_{N-1} &:= \frac{-2\kappa_\alpha}{h_{N-1} + h_N} \left(\frac{1}{h_N} p(x_N) - (\frac{1}{h_{N-1}} + \frac{1}{h_N}) p(x_{N-1}) + \frac{1}{h_{N-1}} p(x_{N-2}) \right) \\ &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{2}{h_{N-1} + h_N} \left(-(\frac{1}{h_{N-1}} + \frac{1}{h_N}) h_N^{3-\alpha} + \frac{1}{h_{N-1}} (h_{N-1} + h_N)^{3-\alpha} \right) \\ &= \frac{4\kappa_\alpha}{\Gamma(4-\alpha) h_{N-1}} \left(-h_N^{2-\alpha} + (h_{N-1} + h_N)^{2-\alpha} \right) \\ &= \frac{4\kappa_\alpha}{(3-\alpha)\Gamma(2-\alpha)} \xi^{1-\alpha} \quad \xi \in [h_N, h_{N-1} + h_N] \\ &\geq \frac{4\kappa_\alpha}{(3-\alpha)\Gamma(2-\alpha)} (h_{N-1} + h_N)^{1-\alpha} = \frac{4\kappa_\alpha}{(3-\alpha)\Gamma(2-\alpha)} (T - x_{N-2})^{1-\alpha}\end{aligned} \tag{6.19}$$

$$\begin{aligned}P_N &:= -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left(\frac{1}{h_{N+1}} p(x_{N+1}) - (\frac{1}{h_N} + \frac{1}{h_{N+1}}) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right) \\ &= \frac{4\kappa_\alpha}{\Gamma(4-\alpha) h_N^2} h_N^{3-\alpha} \\ &= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - x_{N-1})^{1-\alpha}\end{aligned} \tag{6.20}$$

Symmetricly for $i \geq N$, we get

$$P_i \geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \tag{6.21}$$

Similarly, we can get

$$\begin{aligned} Q_i &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} q(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right) \\ &\geq \frac{-2^{r(\alpha-1)+1}\kappa_\alpha}{\Gamma(2-\alpha)} (x_{i-1}^{1-\alpha} + (1 - x_{i+1})^{1-\alpha}) \end{aligned} \tag{6.22}$$

□

Notice that

$$x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha} \tag{6.23}$$

We can get

THEOREM 6.5. *There exists a real $\lambda = \lambda(T, \alpha, r) > 0$ and $C = C(T, \alpha, r) > 0$ such that $B := A(\lambda I + G)$ is an M matrix. And*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + C \begin{cases} |\frac{1}{2} - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - \frac{1}{2}|^{1-\alpha}, & i \geq N \end{cases} \tag{6.24}$$

Proof. By 6.3 with C_1 and 6.4 with C_2 , it's sufficient to take $\lambda = 4TC_2/C_1$, then

$$M_i \geq C_2 \left((x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \right) \tag{6.25}$$

□

Now, we can prove the convergency Theorem 4.2.

For equation

$$AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1} U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1} U = F \tag{6.26}$$

which means

$$\sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + g(x_j)} = \tau_i \tag{6.27}$$

where $\epsilon_i = u(x_i) - U_i$.

And if

$$\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| = \max_{1 \leq i \leq 2N-1} \left|\frac{\epsilon_i}{\lambda + g(x_i)}\right| \tag{6.28}$$

Then, since $B = A(\lambda I + G)$ is an M matrix, it is Strictly diagonally dominant. Thus,

$$
\begin{aligned}
|\tau_{i_0}| &= |\sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + g(x_i)}| \\
&\geq b_{i_0,i_0} |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| - \sum_{j \neq i_0} |b_{i_0,j}| |\frac{\epsilon_j}{\lambda + g(x_j)}| \\
&\geq b_{i_0,i_0} |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| - \sum_{j \neq i_0} |b_{i_0,j}| |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| \\
&= \sum_{j=1}^{2N-1} b_{i_0,j} |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| \\
&= M_{i_0} |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}|
\end{aligned}
\tag{6.29}
$$

By Theorem 4.1 and Theorem 6.5,

We kown that there exists constants $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$, and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that

$$
|\frac{\epsilon_i}{\lambda + g(x_i)}| \leq |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| \leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} + C_2(r-1)h^2 \tag{6.30}
$$

as $\lambda + g(x_i) \leq \lambda + T$

So, we can get

$$
|\epsilon_i| \leq C(\lambda + T) h^{\min\{\frac{r\alpha}{2}, 2\}} \tag{6.31}
$$

The convergency has been proved.

7. Experimental results.

8. Remarks.
some remarks.

In Theorem 2.3 If $f \in L^\infty(\Omega)$ then $u \in C_{\alpha/2}(\Omega)$, which is Proposition 1.1 in [1].

When $\|f\|_\beta^{(\gamma)} < \infty$, where $\beta > 2-\alpha$ and $\gamma \in [-\alpha, -\alpha/2]$, we observed convergent order $\min\{r(\alpha+\gamma), 2\}$ in numerical experiments. And we can prove that kind theorems with the techneque we used in this paper.

Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x) \in C^2\Omega$, there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

$$\begin{aligned} D_h^2 g(x_i) &:= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &= g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \end{aligned} \tag{A.1}$$

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\quad = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right) \end{aligned} \tag{A.2}$$

And if $g(x) \in C^4(\Omega)$, then

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\quad = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2)) \end{aligned} \tag{A.3}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$.

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in [x_{i-1}, x_i]$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in [x_i, x_{i+1}]$$

Subsitute them in the left side of (A.1), we have

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\quad = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) \end{aligned}$$

Now, using intermediate value theorem , there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y)dy$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy$$

Expecially,

$$\begin{aligned} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy &= \frac{h_i^4}{4!} g''''(\eta_1) \\ \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy &= \frac{h_{i+1}^4}{4!} g''''(\eta_2) \end{aligned} \tag{A.4}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. Subsitute them to the left side of (A.3), we can get the result. ∎

LEMMA A.2. *If $y \in [x_{j-1}, x_j]$, denote $y = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0,1]$,*

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \tag{A.5}$$

(A.6)

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta) u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$. Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

∎

LEMMA A.3. *For $x \in [x_{j-1}, x_j]$*

$$\begin{aligned} |u(x) - u_h(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_{x}^{x_j} u'(y)dy \right| \\ &\le \int_{x_{j-1}}^{x_j} |u'(y)|dy \end{aligned} \tag{A.7}$$

If $x \in [0, x_1]$, with Corollary 2.4, we have

$$|u(x) - u_h(x)| \le \int_0^{x_1} |u'(y)|dy \le \int_0^{x_1} C y^{\alpha/2-1} dy \le C\frac{2}{\alpha} x_1^{\alpha/2} \tag{A.8}$$

Similarly, if $x \in [x_{2N-1}, 1]$, we have

$$|u(x) - u_h(x)| \le C\frac{2}{\alpha}(2T - x_{2N-1})^{\alpha/2} = C\frac{2}{\alpha} x_1^{\alpha/2} \tag{A.9}$$

Appendix B. Inequality.

LEMMA B.1.

$$h_i \le rT^{1/r} h \begin{cases} x_i^{1-1/r}, & 1 \le i \le N \\ (2T - x_{i-1})^{1-1/r}, & N < i \le 2N-1 \end{cases} \tag{B.1}$$

$$h_i \ge rT^{1/r} h \begin{cases} x_{i-1}^{1-1/r}, & 1 \le i \le N \\ (2T - x_i)^{1-1/r}, & N < i \le 2N-1 \end{cases} \tag{B.2}$$

Proof. For $1 \le i \le N$,

$$\begin{aligned} h_i &= T\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right) \\ &\le rT\frac{1}{N}\left(\frac{i}{N}\right)^{r-1} = rT^{1/r} h x_i^{1-1/r} \end{aligned}$$

$$h_i \ge rT\frac{1}{N}\left(\frac{i-1}{N}\right)^{r-1} = rT^{1/r} h x_{i-1}^{1-1/r}$$

For $N < i \le 2N$,

$$\begin{aligned} h_i &= T\left(\left(\frac{2N-i+1}{N}\right)^r - \left(\frac{2N-i}{N}\right)^r\right) \\ &\le rT\frac{1}{N}\left(\frac{2N-i+1}{N}\right)^{r-1} = rT^{1/r} h (2T - x_{i-1})^{1-1/r} \end{aligned}$$

$$h_i \ge rT\frac{1}{N}\left(\frac{2N-i}{N}\right)^{r-1} = rT^{1/r} h (2T - x_i)^{1-1/r}$$

∎

LEMMA B.2. *There is a constant* $C = 2^{|r-2|}r(r-1)T^{2/r}$ *such that for all* $i \in \{1, 2, \cdots, 2N-1\}$

$$|h_{i+1} - h_i| \le Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N-1 \end{cases} \tag{B.3}$$

Proof.

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \le i \le N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \le i \le 2N-1 \end{cases}$$

For $i = 1$,

$$h_2 - h_1 = T(2^r - 2)\left(\frac{1}{N}\right)^r = (2^r - 2)T^{2/r}h^2 x_1^{1-2/r}$$

For $2 \le i \le N-1$,

$$h_{i+1} - h_i = r(r-1)T\ N^{-2}\eta^{r-2}, \quad \eta \in [\frac{i-1}{N}, \frac{i+1}{N}]$$

If $r \in [1, 2]$,

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T\ N^{-2}\eta^{r-2} \le r(r-1)T\ h^2\left(\frac{i-1}{N}\right)^{r-2} \\ &\le r(r-1)T\ h^2 2^{2-r}\left(\frac{i}{N}\right)^{r-2} \\ &= 2^{2-r}r(r-1)T^{2/r}h^2 x_i^{1-2/r} \end{aligned}$$

else if $r > 2$,

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T\ N^{-2}\eta^{r-2} \le r(r-1)T\ h^2\left(\frac{i+1}{N}\right)^{r-2} \\ &\le r(r-1)T\ h^2 2^{r-2}\left(\frac{i}{N}\right)^{r-2} \\ &= 2^{r-2}r(r-1)T^{2/r}h^2 x_i^{1-2/r} \end{aligned}$$

Since

$$2^r - 2 \le 2^{|r-2|}r(r-1), \quad r \ge 1$$

we have

$$h_{i+1} - h_i \le 2^{|r-2|}r(r-1)T^{2/r}h^2 x_i^{1-2/r}, \quad 1 \le i \le N-1$$

For $i = N$, $h_{N+1} - h_N = 0$. For $N < i \le 2N-1$, it's central symmetric to the first half of the proof, which is

$$h_i - h_{i+1} \le 2^{|r-2|}r(r-1)T^{2/r}h^2(2T - x_i)^{1-2/r}$$

Summarizes the inequalities, we can get

$$\text{(B.4)}\qquad |h_{i+1}-h_i| \le 2^{|r-2|} r(r-1)T^{2/r}h^2 \begin{cases} x_i^{1-2/r}, & 1\le i\le N-1\\ 0, & i=N \\ (2T-x_i)^{1-2/r}, & N<i\le 2N-1 \end{cases} \qquad \square$$

Appendix C. Proofs of some technical details.

Additional proof of Theorem 5.1. For $2\le i\le N-1$,

$$\begin{aligned} &\frac{2}{h_i+h_{i+1}}(h_i^3 f''(\eta_1)+h_{i+1}^3 f''(\eta_2))\\ &\qquad \le C\frac{2}{h_i+h_{i+1}}(h_i^3 x_{i-1}^{-2-\alpha/2}+h_{i+1}^3 x_i^{-2-\alpha/2})\\ &\qquad \le 2C(h_i^2 x_{i-1}^{-2-\alpha/2}+h_{i+1}^2 x_i^{-2-\alpha/2}) \end{aligned}$$

Since Lemma B.1, we have

$$h_i \le rT^{1/r}hx_i^{1-1/r},\quad 1\le i\le N$$

$$h_{i+1} \le rT^{1/r}hx_{i+1}^{1-1/r},\quad 1\le i\le N-1$$

and

$$x_{i-1}^{-2-\alpha/2} \le 2^{-r(-2-\alpha/2)}x_i^{-2-\alpha/2}\quad 2\le i\le N-1$$

$$x_{i+1}^{1-1/r} \le 2^{r-1}x_i^{1-1/r}\quad 1\le i\le N-1$$

So there is a constant $C=C(T,\alpha,r,\|f\|_\beta^{\alpha/2})$ such that

$$\frac{2}{h_i+h_{i+1}}(h_i^3 f''(\eta_1)+h_{i+1}^3 f''(\eta_2)) \le Ch^2 x_i^{-\alpha/2-2/r},\quad 2\le i\le N-1$$

For $i=1$, by (A.4)

$$\begin{aligned} &\frac{1}{4!}\frac{2}{h_1+h_2}(h_1^3 f''(\eta_1)+h_2^3 f''(\eta_2))\\ &\qquad =\frac{2}{h_1+h_2}\left(\frac{1}{h_1}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy+\frac{1}{4!}h_2^3 f''(\eta_2)\right) \end{aligned}$$

We have proved above that

$$\frac{2}{h_1+h_2}h_2^3 f''(\eta_2)\le Ch^2 x_1^{-\alpha/2-2/r}$$

and we can get

$$\begin{aligned} \int_0^{x_1} f''(y)\frac{y^3}{3!}dy &\le C\frac{1}{3!}\int_0^{x_1} y^{1-\alpha/2}dy\\ &= C\frac{1}{3!(2-\alpha/2)}x_1^{2-\alpha/2} \end{aligned}$$

so

$$\frac{2}{h_1+h_2}\frac{1}{h_1}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy=\frac{C2^{1-r}}{3!(2-\alpha/2)}x_1^{-\alpha/2}=\frac{C2^{1-r}}{3!(2-\alpha/2)}T^{2/r}h^2 x_1^{-\alpha/2-2/r}$$

And for $i = N$, we have

$$\begin{aligned}
&\frac{2}{h_N + h_{N+1}}(h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2)) \\
&\quad = h_N^2(f''(\eta_1) + f''(\eta_2)) \\
&\quad \le r^2 T^{2/r} h^2 x_N^{2-2/r}\ 2C x_{N-1}^{-2-\alpha/2} \\
&\quad \le 2r^2 T^{2/r} C 2^{-r(-2-\alpha/2)}\ h^2 x_N^{-\alpha/2-2/r}
\end{aligned}$$

Finally, $N+1 \le i \le 2N-1$ is symmetric to the first half of the proof, so we can conclude that

$$\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases}$$

∎

LEMMA C.1. *There is a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *for* $2 \le j \le N$, *if* $y \in [x_{j-1}, x_j]$,

$$|u(y) - u_h(y)| \le Ch^2 y^{\alpha/2-2/r} \tag{C.1}$$

Proof. For $2 \le j \le N$, we have

$$x_j \le 2^r y, \quad x_{j-1} \ge 2^{-r} y$$

And by Lemma A.2, Lemma B.1 and Corollary 2.4, we have

$$\begin{aligned}
u(y) - u_h(y) &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \\
&\le \frac{\|u\|_{\beta+\alpha}^{(-\alpha/2)}}{4} r^2 T^{2/r}\ h^2 x_j^{2-2/r} x_{j-1}^{\alpha/2-2} \\
&\le Ch^2\ 2^{2r-2} y^{2-2/r}\ 2^{-r(\alpha/2-2)} y^{\alpha/2-2} \\
&= C 2^{-r\alpha/2+4r-2}\ h^2 y^{\alpha/2-2/r}
\end{aligned}$$

symmetricly, for $N < j \le 2N-1$, we have

$$|u(y) - u_h(y)| \le Ch^2 (2T - y)^{\alpha/2-2/r} \tag{C.2}$$

∎

LEMMA C.2. *There is a constant* $C = C(\alpha, r)$ *such that for all* $1 \le i < N/2$, $\max\{2i+1, i+3\} \le j \le 2N$ *and* $y \in [x_{j-1}, x_j]$, *we have*

$$D_h^2\left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) \le C \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} \tag{C.3}$$

Proof. Sinec $y \ge x_{j-1} > x_{i+1}$, by Lemma A.1, if $j - 1 > i + 1$

∎

$$\begin{aligned}
D_h^2\left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) &= \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\
&\le \frac{(y - x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)} \\
&\le (1 - (\frac{2}{3})^r)^{-1-\alpha} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}
\end{aligned}$$

LEMMA C.3. *There is a constant $C = C(\alpha, r)$ such that for all $3 \le i < N/2, k = \lceil \frac{i}{2} \rceil$, $1 \le j \le k-1$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_h^2\left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) \le C\frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \tag{C.4}$$

Proof. Sinec $y \le x_j < x_{i-1}$, by Lemma A.1, $\square$

$$\begin{aligned} D_h^2\left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) &= \frac{|\xi - y|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\le \frac{(x_{i-1} - x_j)^{-1-\alpha}}{\Gamma(-\alpha)} \le \frac{(x_{i-1} - x_{k-1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\le ((\frac{2}{3})^r - (\frac{1}{2})^r)^{-1-\alpha}\frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

LEMMA C.4. *While $0 \le i < N/2$, By Lemma* A.3

$$\begin{aligned} |T_{i1}| &\le C\int_0^{x_1} x_1^{\alpha/2}\frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)}dy \\ &= C\frac{1}{\Gamma(3-\alpha)}x_1^{\alpha/2}\left|x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha}\right| \\ &\le C\frac{1}{\Gamma(3-\alpha)}x_1^{\alpha/2+2-\alpha} = C\frac{1}{\Gamma(3-\alpha)}x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1 \end{aligned} \tag{C.5}$$

For $2 \le j \le N$, by Lemma A.2 *and Corollary* 2.4

$$\begin{aligned} |T_{ij}| &\le \frac{C}{4}\int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2}\frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy \\ &\le \frac{C}{4\Gamma(3-\alpha)}h_j^2 x_{j-1}^{\alpha/2-2}\left||x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha}\right| \end{aligned} \tag{C.6}$$

LEMMA C.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$\sum_{j=1}^{3} S_{1j} \le Ch^2 x_1^{-\alpha/2-2/r} \tag{C.7}$$

$$\sum_{j=1}^{4} S_{2j} \le Ch^2 x_2^{-\alpha/2-2/r} \tag{C.8}$$

Proof.

$$S_{1j} = \frac{2}{x_2}\left(\frac{1}{x_1}T_{0j} - \left(\frac{1}{x_1} + \frac{1}{h_2}\right)T_{1j} + \frac{1}{h_2}T_{2j}\right)$$

So, by Lemma C.4

$$S_{11} \le \frac{2}{x_2 x_1}4\frac{C}{\Gamma(3-\alpha)}x_1^{2-\alpha/2} \le Cx_1^{-\alpha/2}$$

$$S_{12} \le \frac{2}{x_2x_1}\frac{C}{4\Gamma(3-\alpha)}h_2^2x_1^{\alpha/2-2}\left(x_2^{2-\alpha}+2h_2^{2-\alpha}+h_2^{2-\alpha}\right) \le Cx_1^{-\alpha/2}$$

$$S_{13} \le \frac{2}{x_2x_1}\frac{C}{4\Gamma(3-\alpha)}h_3^2x_2^{\alpha/2-2}\left(x_3^{2-\alpha}+2x_3^{2-\alpha}+h_3^{2-\alpha}\right) \le Cx_1^{-\alpha/2}$$

But

$$x_1^{-\alpha/2} = T^{2/r}h^2x_1^{-\alpha/2-2/r}$$

$i=2$ is similar. ∎

LEMMA C.6. *There exists a constant $C = C(T,r,l)$ such that For $3 \le i \le N-1, k+1 = \lceil \frac{i}{2} \rceil, k \le j \le \min\{2i-1, N-1\}$, $l = 3,4$, when $\xi \in [x_{i-1}, x_{i+1}]$,*

$$(h_{j-i}^3(\xi))' \le (r-1)Ch^2x_i^{1-2/r}h_j \tag{C.9}$$

$$(h_{j-i}^4(\xi))' \le (r-1)Ch^2x_i^{1-2/r}h_j^2 \tag{C.10}$$

Proof. From (5.32)

$$y'_{j-i}(x) = y_{j-i}^{1-1/r}(x)x^{1/r-1} \tag{C.11}$$

$$y''_{j-i}(x) = \frac{1-r}{r}y_{j-i}^{1-2/r}(x)x^{1/r-2}Z_{j-i} \tag{C.12}$$

for $l=3,4$, by (5.34)

$$\begin{aligned}(h_{j-i}^l(\xi))' &= l\ h_{j-i}^{l-1}(\xi)(y'_{j-i}(\xi)-y'_{j-i-1}(\xi))\\ &= l\ h_{j-i}^{l-1}(\xi)\xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi)-y_{j-i-1}^{1-1/r}(\xi)) \ge 0\end{aligned} \tag{C.13}$$

For $\xi \in [x_{i-1}, x_{i+1}]$ and $2 \le k \le j \le \min\{2i-1, N-1\}$, using Lemma B.1

$$\begin{aligned}h_{j-i}(\xi) &\le h_{j-i}(x_{i+1}) = h_{j+1}\\ &\le rT^{1/r}\ hx_{j+1}^{1-1/r} \le rT^{1/r}2^{r-1}\ hx_i^{1-1/r}\end{aligned}$$

And

$$2^{-r}x_i \le x_{i-1} \le \xi \le x_{i+1} \le 2^rx_i \tag{C.14}$$

We have

$$\xi^{1/r-m} \le 2^{|mr-1|}x_i^{1/r-m}, \quad m=1,2 \tag{C.15}$$

but

$$\begin{aligned}y_{j-i}^{1-1/r}(\xi)-y_{j-i-1}^{1-1/r}(\xi) &= (\xi^{1/r}+Z_{j-i})^{r-1}-(\xi^{1/r}+Z_{j-i-1})^{r-1}\\ &= (r-1)Z_1(\xi^{1/r}+Z_{j-i-\gamma})^{r-2}, \quad \gamma \in [0,1]\\ &= (r-1)T^{1/r}hy_{j-i-\gamma}^{1-2/r}(\xi)\end{aligned} \tag{C.16}$$

And

(C.17)
$$4^{-r}x_i \le x_{\lceil \frac{i}{2}\rceil -1} \le x_{j-2} = y_{j-i-1}(x_{i-1}) \le y_{j-i-\gamma}(\xi) \le y_{j-i}(x_{i+1}) = x_{j+1} \le x_{2i} \le 2^r x_i$$

Therefore,

$$y_{j-i-\gamma}^{1-2/r}(\xi) \le 2^{2|r-2|} x_i^{1-2/r} \tag{C.18}$$

So we can get

$$y'_{j-i}(\xi) - y'_{j-i-1}(\xi) \le (r-1)C(T,r)hx_i^{-1/r} \tag{C.19}$$

We get

$$(h_{j-i}^l(\xi))' \le l(r-1)C\ h_{j+1}^{l-1} h x_i^{-1/r} \tag{C.20}$$

And by Lemma B.1,

$$h_{j+1} \le rTh\left(\frac{j+1}{N}\right)^{r-1} \le rTh2^{r-1}\left(\frac{j-1}{N}\right) = 2^{r-1}h_j \tag{C.21}$$

$$h_{j+1} \le rT^{1/r}hx_{j+1}^{1-1/r} \le rT^{1/r}hx_{2i}^{1-1/r} \le rT^{1/r}2^{r-1}hx_i^{1-1/r} \tag{C.22}$$

We can get

$$\begin{aligned}(h_{j-i}^l(\xi))' &\le l(r-1)C\ h_j^{l-2}h_{j+1}hx_i^{-1/r}\\ &\le l(r-1)Chh_j^{l-2}(hx_i^{1-1/r})x_i^{-1/r}\\ &= (r-1)C\ h^2 x_i^{1-2/r}h_j^{l-2}\end{aligned} \tag{C.23}$$

Meanwhile, we can get

$$h_{j-i}^3(\xi) \le h_{j+1}^3 \le Ch^2 x_i^{2-2/r}h_j \tag{C.24}$$

$$h_{j-i}^4(\xi) \le h_{j+1}^4 \le Ch^2 x_i^{2-2/r}h_j^2 \tag{C.25}$$

∎

LEMMA C.7. *There exists a constant $C = C(T,r,l)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2}\rceil + 1 \le j \le \min\{2i-1, N-1\}$, when $\xi \in [x_{i-1}, x_{i+1}]$,*

$$(h_{j-i}^3(\xi))'' \le C(r-1)h^2 x_i^{-2/r}h_j \tag{C.26}$$

Proof. From (C.11)

$$\begin{aligned}(h_{j-i}^3(\xi))'' &= 6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 + 3h_{j-i}^2(\xi)(y''_{j-i}(\xi) - y''_{j-i-1}(\xi))\\ &= 6h_{j-i}(\xi)(\xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)))^2\\ &\quad + 3\frac{1-r}{r}h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1})\end{aligned} \tag{C.27}$$

Using the inequalities of the proof of Lemma C.6

$$\begin{aligned}
&6h_{j-i}(\xi)(y'_{j-i}(\xi)-y'_{j-i-1}(\xi))^2\\
&\quad\le 6h_{j+1}((r-1)Chx_i^{-1/r})^2\\
&\quad\le C(r-1)^2\ h^2x_i^{-2/r}h_j
\end{aligned}\tag{C.28}$$

For the second partial

$$\begin{aligned}
&h^2_{j-i}(\xi)\xi^{1/r-2}(y^{1-2/r}_{j-i}(\xi)Z_{j-i}-y^{1-2/r}_{j-i-1}(\xi)Z_{j-i-1})\\
&\quad\le Ch^2_{j+1}x_i^{1/r-2}((y^{1-2/r}_{j-i}(\xi)-y^{1-2/r}_{j-i-1}(\xi))Z_{j-i}+y^{1-2/r}_{j-i-1}(\xi)Z_1)
\end{aligned}\tag{C.29}$$

but

$$\begin{aligned}
y^{1-2/r}_{j-i}(\xi)-y^{1-2/r}_{j-i-1}(\xi)&=(\xi^{1/r}+Z_{j-i})^{r-2}-(\xi^{1/r}+Z_{j-i-1})^{r-2}\\
&=(r-2)Z_1(\xi^{1/r}+Z_{j-i-\gamma})^{r-3}\\
&=(r-2)T^{1/r}hy^{1-3/r}_{j-i-\gamma}(\xi)\\
&\le C(r-2)hx_i^{1-3/r}
\end{aligned}\tag{C.30}$$

So we can get

$$\begin{aligned}
&h^2_{j-i}(\xi)\xi^{1/r-2}(y^{1-2/r}_{j-i}(\xi)Z_{j-i}-y^{1-2/r}_{j-i-1}(\xi)Z_{j-i-1})\\
&\quad\le Ch_jhx_i^{1-1/r}x_i^{1/r-2}(C(r-2)hx_i^{1-3/r}Z_{j-i}+Cx_i^{1-2/r}T^{1/r}h)\\
&\quad\le Ch^2((r-2)x_i^{-3/r}x^{1/r}_{|j-i|}+x_i^{-2/r})h_j\\
&\quad\le Ch^2x_i^{-2/r}h_j
\end{aligned}\tag{C.31}$$

Summarizes, we have

$$(h^3_{j-i}(\xi))''\le C(r-1)h^2x_i^{-2/r}h_j \tag{C.32}$$ □

proof of Lemma 5.16. From (5.32)

$$y'_{j-i}(x)=y^{1-1/r}_{j-i}(x)x^{1/r-1}\tag{C.33}$$

$$y''_{j-i}(x)=\frac{1-r}{r}y^{1-2/r}_{j-i}(x)x^{1/r-2}Z_{j-i}\tag{C.34}$$

Since

$$x_{j-2}\le y_{j-i-1}(x_{i-1})\le y^{\theta}_{j-i}(\xi)\le y^{\theta}_{j-i}(x_{i+1})\le x_{j+1}$$

We have known (C.17)

$$u''(y^{\theta}_{j-i}(\xi))\le C(y^{\theta}_{j-i}(\xi))^{\alpha/2-2}\le Cx^{\alpha/2-2}_{j-2}\le Cx^{\alpha/2-2}_{\lceil\frac{i}{2}\rceil-1}\le C4^{r(2-\alpha/2)}x_i^{\alpha/2-2}\tag{C.35}$$

$$\begin{aligned}
(u''(y^{\theta}_{j-i}(\xi)))'&=u'''(y^{\theta}_{j-i}(\xi)){y^{\theta}_{j-i}}'(\xi)\\
&\le Cx_i^{\alpha/2-3}\xi^{1/r-1}y^{1-1/r}_{j-i}(\xi)\\
&\le Cx_i^{\alpha/2-3}x_i^{1/r-1}x_i^{1-1/r}=Cx_i^{\alpha/2-3}
\end{aligned}\tag{C.36}$$

$$\begin{aligned}(u''(y^{\theta}_{j-i}(\xi)))'' &= u''''(y^{\theta}_{j-i}(\xi))({y^{\theta}_{j-i}}'(\xi))^2 + u'''(y^{\theta}_{j-i}(\xi)){y^{\theta}_{j-i}}''(\xi)\\ &\leq Cx_i^{\alpha/2-4} + Cx_i^{\alpha/2-3}\frac{r-1}{r}x_i^{1-2/r}x_i^{1/r-2}Z_{|j-i|+1}\\ &\leq Cx_i^{\alpha/2-4} + C\frac{r-1}{r}x_i^{\alpha/2-3}x_i^{-1/r}x_i^{1/r}\\ &= Cx_i^{\alpha/2-4}\end{aligned} \tag{C.37}$$

□

Proof of Lemma 5.17.

$$\begin{aligned}|y^{\theta}_{j-i}(\xi)-\xi| &= |\theta(y_{j-i-1}(\xi)-\xi)+(1-\theta)(y_{j-i}(\xi)-\xi)|\\ &= \theta|y_{j-i-1}(\xi)-\xi| + (1-\theta)|y_{j-i}(\xi)-\xi|\end{aligned} \tag{C.38}$$

Since $|y_{j-i}(\xi)-\xi|$ is increasing about ξ, we have

$$(\frac{i-1}{i})^r|x_j-x_i| \leq |x_{j-1}-x_{i-1}| \leq |y_{j-i}(\xi)-\xi| \leq |x_{j+1}-x_{i+1}| \leq (\frac{i+1}{i})^r|x_j-x_i| \tag{C.39}$$

Thus,

$$(\frac{2}{3})^r|y^{\theta}_j-x_i| \leq |y^{\theta}_{j-i}(\xi)-\xi| \leq (\frac{3}{4})^r(\theta|x_j-x_i|+(1-\theta)|x_{j-1}-x_i|) = (\frac{3}{4})^r|y^{\theta}_j-x_i| \tag{C.40}$$

$$|y^{\theta}_{j-i}(\xi)-\xi|^{1-\alpha} \leq C|y^{\theta}_j-x_i|^{1-\alpha} \tag{C.41}$$

Next,

$$\begin{aligned}(|y^{\theta}_{j-i}(\xi)-\xi|^{1-\alpha})' &= (1-\alpha)|y^{\theta}_{j-i}(\xi)-\xi|^{-\alpha}|\xi^{1/r-1}(\theta y^{1-1/r}_{j-i-1}(\xi)+(1-\theta)y^{1-1/r}_{j-i}(\xi))-1|\\ &\leq C|y^{\theta}_j-x_i|^{-\alpha}\xi^{1/r-1}|\theta y^{1-1/r}_{j-i-1}(\xi)+(1-\theta)y^{1-1/r}_{j-i}(\xi)-\xi^{1-1/r}|\end{aligned} \tag{C.42}$$

Similar with (C.39), we have

$$\begin{aligned}|y^{1-1/r}_{j-i}(\xi)-\xi^{1-1/r}| &\leq C|x_j^{1-1/r}-x_i^{1-1/r}|\\ &\leq C|x_j-x_i|x_i^{-1/r}\end{aligned} \tag{C.43}$$

So we can get

$$\begin{aligned}&|\theta y^{1-1/r}_{j-i-1}(\xi)+(1-\theta)y^{1-1/r}_{j-i}(\xi)-\xi^{1-1/r}|\\ &\quad\leq Cx_i^{-1/r}(\theta|x_{j-1}-x_i|+(1-\theta)|x_j-x_i|)\\ &\quad= Cx_i^{-1/r}|y^{\theta}_j-x_i|\end{aligned} \tag{C.44}$$

Combine them, we get

$$\begin{aligned}(|y^{\theta}_{j-i}(\xi)-\xi|^{1-\alpha})' &\leq C|y^{\theta}_j-x_i|^{-\alpha}x_i^{1/r-1}x_i^{-1/r}|y^{\theta}_j-x_i|\\ &= C|y^{\theta}_j-x_i|^{1-\alpha}x_i^{-1}\end{aligned} \tag{C.45}$$

Finally, we have

$$\begin{aligned}(|y^{\theta}_{j-i}(\xi)-\xi|^{1-\alpha})'' &= \alpha(\alpha-1)|y^{\theta}_{j-i}(\xi)-\xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y^{1-1/r}_{j-i-1}(\xi)+(1-\theta)y^{1-1/r}_{j-i}(\xi))-1)^2\\ &\quad+(1-\alpha)|y^{\theta}_{j-i}(\xi)-\xi|^{-\alpha}\frac{1-r}{r}\xi^{1/r-2}|\theta y^{1-2/r}_{j-i-1}(\xi)Z_{j-i-1}+(1-\theta)y^{1-2/r}_{j-i}(\xi)Z_{j-i}|\end{aligned} \tag{C.46}$$

Using the inequalities above ,we have

$$\begin{aligned}
&|y_{j-i}^{\theta}(\xi)-\xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi)+(1-\theta)y_{j-i}^{1-1/r}(\xi))-1)^2\\
&\quad\le C|y_j^{\theta}-x_i|^{-\alpha-1}(x_i^{-1}|y_j^{\theta}-x_i|)^2\\
&\quad= C|y_j^{\theta}-x_i|^{1-\alpha}x_i^{-2}
\end{aligned}\tag{C.47}$$

And by

$$|Z_{j-i}|=|x_j^{1/r}-x_i^{1/r}|\le|x_j-x_i|x_i^{1/r-1}\tag{C.48}$$

we have

$$\begin{aligned}
&|y_{j-i}^{\theta}(\xi)-\xi|^{-\alpha}\xi^{1/r-2}|\theta y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}+(1-\theta)y_{j-i}^{1-2/r}(\xi)Z_{j-i}|\\
&\le C|y_j^{\theta}-x_i|^{-\alpha}x_i^{1/r-2}x_i^{1-2/r}|\theta Z_{j-i-1}+(1-\theta)Z_{j-i}|\\
&\le C|y_j^{\theta}-x_i|^{-\alpha}x_i^{-2}|y_j^{\theta}-x_i|\\
&= C|y_j^{\theta}-x_i|^{1-\alpha}x_i^{-2}
\end{aligned}\tag{C.49}$$

$\square$

proof of Lemma 5.19. For $k\le j<\min\{2i-1,N-1\}$

$$\begin{aligned}
&\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta})-Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}}\\
&\quad\frac{Q_{j-i}^{\theta}(x_{i+1})-Q_{j-i}^{\theta}(x_i)}{h_{i+1}}u'''(\eta_{j+1}^{\theta})+Q_{j-i}^{\theta}(x_i)\frac{u'''(\eta_{j+1}^{\theta})-u'''(\eta_j^{\theta})}{h_{i+1}}\\
&\quad\le {Q_{j-i}^{\theta}}'(\xi)Cx_j^{\alpha/2-3}+Q_{j-i}^{\theta}(x_i)Cu''''(\eta)\frac{h_i+h_{i+1}}{h_{i+1}}
\end{aligned}\tag{C.50}$$

where $\xi\in[x_i,x_{i+1}],\eta\in[x_{j-1},x_{j+1}]$.

From (5.36), by Lemma C.6 and Lemma 5.17, we have

$$\begin{aligned}
{Q_{j-i}^{\theta}}'(\xi)&\le Ch^2\frac{|y_{j+1}^{\theta}-x_{i+1}|^{1-\alpha}}{\Gamma(2-\alpha)}x_{i+1}^{1-2/r}h_{j+1}^2\\
&\le Ch^2\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{1-2/r}h_j^2
\end{aligned}\tag{C.51}$$

And by defination

$$Q_{j-i}^{\theta}(x_i)=h_j^4\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}\le Ch^2x_i^{2-2/r}\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}h_j^2\tag{C.52}$$

With , we have

$$4^{-r}x_i\le x_{k-1}\le x_{j-1}<x_j\le x_{2i-1}\le 2^rx_i\tag{C.53}$$

So we have

$$\begin{aligned}
&\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta})-Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}}\\
&\le Ch^2\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{1-2/r}h_j^2\,x_i^{\alpha/2-3}+Ch^2x_i^{2-2/r}\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}h_j^2x_{j-1}^{\alpha/2-4}\\
&= Ch^2\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j^2
\end{aligned}\tag{C.54}$$

while

$$h_j \le h_{2i-1} \le 2^r h_i$$

Subsitute into the inequality above, we get the goal

$$\begin{aligned}
&\frac{2}{h_i+h_{i+1}}\left(\frac{Q^{\theta}_{j-i}(x_{i+1})u'''(\eta^{\theta}_{j+1}) - Q^{\theta}_{j-i}(x_i)u'''(\eta^{\theta}_j)}{h_{i+1}}\right)\\
&\le \frac{1}{h_i}Ch^2\frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j 2^r h_i\\
&= Ch^2\frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j
\end{aligned} \tag{C.55}$$

While, the later is similar. ∎

Lemma C.8. *There exists a constant $C = C(T, r)$ such that For $N/2 \le i < N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $l = 3, 4$, $\xi \in [x_{i-1}, x_{i+1}]$, we have*

$$h^l_{j-i}(\xi) \le Ch^l_j \le Ch^2 h^{l-2}_j \tag{C.56}$$

$$(h^l_{j-i-1}(\xi))' \le C(r-1)h^2 h^{l-2}_j \tag{C.57}$$

$$(h^3_{j-i}(\xi))'' \le C(r-1)h^2 h_j \tag{C.58}$$

Proof.

$$\begin{aligned}
(h_{j-i}(\xi))' &= {y_{j-i}}'(\xi) - {y_{j-i-1}}'(\xi)\\
&= \xi^{1/r-1}((2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}) \le 0
\end{aligned} \tag{C.59}$$

Thus,

$$Ch_j \le h_{j+1} \le h_{j-i}(\xi) \le h_{j-i}(x_{i-1}) = h_{j-1} \le Ch_j \tag{C.60}$$

So as $4^{-r}T \le 2T - x_j \le T, 2^{-r}T \le x_i \le T$, we have

$$h^l_{j-i}(\xi) \le Ch^l_j \le Ch^2(2T - x_j)^{2-2/r}h^{l-2}_j \le Ch^2 h^{l-2}_j \tag{C.61}$$

Since

$$\begin{aligned}
&|(2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}|\\
&= |(Z_{2N-(j-i)} - \xi^{1/r})^{r-1} - (Z_{2N-(j-1-i)} - \xi^{1/r})^{r-1}|\\
&= (r-1)Z_1(Z_{2N-(j-i-\gamma)} - \xi^{1/r})^{r-2} \quad \gamma \in [0, 1]\\
&\le C(r-1)h(2T - x_j)^{1-2/r}
\end{aligned} \tag{C.62}$$

we have

$$|(h_{j-i}(\xi))'| \le C(r-1)h(2T - x_j)^{1-2/r}x_i^{1/r-1} \tag{C.63}$$

And

$$\begin{aligned}
(h^l_{j-i}(\xi))' &= lh^{l-1}_{j-i}(\xi){h_{j-i}}'(\xi)\\
&\le C(r-1)h^{l-1}_j\ h(2T - x_j)^{1-2/r}x_i^{1/r-1}\\
&\le C(r-1)h^2 h^{l-2}_j(2T - x_j)^{2-3/r}x_i^{1-1/r}\\
&\le C(r-1)h^2 h^{l-2}_j
\end{aligned} \tag{C.64}$$

(C.65)

$$
\begin{aligned}
(h_{j-i}^3(\xi))'' &= 6h_{j-i}(\xi)(y_{j-i}{}'(\xi) - y_{j-i-1}{}'(\xi))^2 + 3h_{j-i}^2(\xi)(y_{j-i}{}''(\xi) - y_{j-i-1}{}''(\xi)) \\
&\le C(r-1)h_j h^2 + Ch_j^2 \frac{1-r}{r}\xi^{1/r-2}((2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-1-i)}) \\
&\le C(r-1)h_j h^2 + C(r-1)h_j^2 (C(r-2)h(2T - x_j)^{1-3/r} Z_{2N-(j-i)} + Z_1(2T - x_{j-1})^{1-2/r}) \\
&\le C(r-1)h_j h^2 + C(r-1)h_j^2 h = Ch^2 h_j
\end{aligned}
$$

□

LEMMA C.9. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le i < N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in [x_{i-1}, x_{i+1}]$, we have*

$$u''(y_{j-i}^\theta(\xi)) \le C \tag{C.66}$$

$$(u''(y_{j-i}^\theta(\xi)))' \le C \tag{C.67}$$

$$(u''(y_{j-i}^\theta(\xi)))'' \le C \tag{C.68}$$

Proof.

$$x_{j-2} \le y_{j-i}^\theta(\xi) \le x_{j+1} \Rightarrow 4^{-r}T \le 2T - y_{j-i}^\theta(\xi) \le T \tag{C.69}$$

Thus, for $l = 2, 3, 4$,

$$u^{(l)}(y_{j-i}^\theta(\xi)) \le C(2T - y_{j-i}^\theta(\xi))^{\alpha/2-l} \le C \tag{C.70}$$

and

$$
\begin{aligned}
(y_{j-i}^\theta(\xi))' &= \theta y_{j-1-i}{}'(\xi) + (1-\theta)y_{j-i-1}{}'(\xi) \\
&= \xi^{1/r-1}(\theta(2T - y_{j-1-i}(\xi))^{1-1/r} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-1/r}) \\
&\le C(2T - x_{j-2})^{1-1/r} \le C
\end{aligned}
\tag{C.71}
$$

With

$$Z_{2N-j-i} \le 2T^{1/r} \tag{C.72}$$

(C.73)

$$
\begin{aligned}
(y_{j-i}^\theta(\xi))'' &= \theta y_{j-1-i}{}''(\xi) + (1-\theta)y_{j-i-1}{}''(\xi) \\
&= \frac{1-r}{r}\xi^{1/r-2}(\theta(2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)} + (1-\theta)(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)}) \\
&\le C(r-1)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(u''(y_{j-i}^\theta(\xi)))' &= u'''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta(\xi))' \\
&\le C
\end{aligned}
\tag{C.74}
$$

$$
\begin{aligned}
(u''(y_{j-i}^\theta(\xi)))'' &= u'''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta{}'(\xi))^2 + u''''(y_{j-i}^\theta(\xi))y_{j-i}^\theta{}''(\xi) \\
&\le C + C(r-1) = C
\end{aligned}
\tag{C.75}
$$

□

LEMMA C.10. *There exists a constant $C = C(T, \alpha, r)$ such that*

$$|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha} \leq C|y_j^{\theta} - x_i|^{1-\alpha} \tag{C.76}$$

$$(|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha})' \leq C|y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N) \tag{C.77}$$

$$(|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha})'' \leq C(r-1)|y_j^{\theta} - x_i|^{-\alpha} + C|y_j^{\theta} - x_i|^{-1-\alpha}(|2T - x_i - y_j^{\theta}| + h_N)^2 \tag{C.78}$$

Proof.

$$(y_{j-i}^{\theta}(\xi) - \xi)' = \theta {y_{j-1-i}}'(\xi) + (1-\theta){y_{j-i}}'(\xi) - 1 \tag{C.79}$$

$$\begin{aligned} |{y_{j-i}}'(\xi) - 1| &= \xi^{1/r-1}|(2T - y_{j-i}(\xi))^{1-1/r} - \xi^{1-1/r}| \\ &\leq \xi^{1/r-1}|2T - \xi - y_{j-i}(\xi)|\xi^{-1/r} \end{aligned} \tag{C.80}$$

$$\begin{aligned} |2T - \xi - y_{j-i}(\xi)| &\leq \max \begin{cases} |2T - x_{i-1} - x_{j-1}| \\ |2T - x_{i+1} - x_{j+1}| \end{cases} \\ &\leq |2T - x_i - x_j| + h_{i+1} + h_j \end{aligned} \tag{C.81}$$

$$\begin{aligned} (y_{j-i}^{\theta}(\xi) - \xi)'' &= \theta {y_{j-1-i}}''(\xi) + (1-\theta){y_{j-i}}''(\xi) \\ &= \frac{1-r}{r}\xi^{1/r-2}(\theta(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)}) \leq 0 \end{aligned} \tag{C.82}$$

It's concave, so

$$y_{j-i}(\xi) - \xi \geq \min\{x_{j+1} - x_{i+1}, x_{j-1} - x_{i-1}\} \geq C(x_j - x_i) \tag{C.83}$$

We have

$$|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha} \leq C|y_j^{\theta} - x_i|^{1-\alpha} \tag{C.84}$$

$$\begin{aligned} (|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha})' &= (1-\alpha)|y_{j-i}^{\theta}(\xi) - \xi|^{-\alpha}(y_{j-i}^{\theta}(\xi) - \xi)' \\ &\leq C|y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_{i+1} + h_{j-1}) \end{aligned} \tag{C.85}$$

$$\begin{aligned} (|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha})'' &= (1-\alpha)|y_{j-i}^{\theta}(\xi) - \xi|^{-\alpha}(y_{j-i}^{\theta}(\xi) - \xi)'' + \alpha(\alpha-1)|y_{j-i}^{\theta}(\xi) - \xi|^{-1-\alpha}({y_{j-i}^{\theta}}'(\xi) - 1)^2 \\ &\leq C(r-1)|y_j^{\theta} - x_i|^{-\alpha} + C|y_j^{\theta} - x_i|^{-1-\alpha}(|2T - x_i - y_j^{\theta}| + h_{i+1} + h_{j-1})^2 \end{aligned} \tag{C.86}$$

□

Proof. From (5.24), by Lemma C.8 and Lemma C.10, we have $\xi \in [x_i, x_{i+1}]$

$${Q_{j-i}^{\theta}}'(\xi) \leq Ch^2 h_j^2((r-1)|y_j^{\theta} - x_i|^{1-\alpha} + |y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N)) \tag{C.87}$$

$$Q_{j-i}^{\theta}(\xi) \leq Ch^2 h_j^2 |y_j^{\theta} - x_i|^{1-\alpha} \tag{C.88}$$

So use the skill in Proof 34 with Lemma C.9

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}}\left(\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta}) - Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}}\right) \\ &\leq Ch^2 h_j(|y_j^{\theta} - x_i|^{1-\alpha} + |y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N)) \end{aligned} \tag{C.89}$$

□

$$a^{1-\theta}|a^\theta - b^\theta| \leq |a-b|, \theta \in [0,1] \tag{C.90}$$

Acknowledgments. We would like to acknowledge the assistance of volunteers in putting together this example manuscript and supplement.

REFERENCES

[1] X. Ros-Oton and J. Serra, *The dirichlet problem for the fractional laplacian: Regularity up to the boundary*, Journal de Mathématiques Pures et Appliquées, 101 (2014), pp. 275–302, https://doi.org/https://doi.org/10.1016/j.matpur.2013.06.003, https://www.sciencedirect.com/science/article/pii/S0021782413000895.