1 问题

对于 $\Omega = (0,1)$, $1 < \alpha < 2$, 假设 $f \in C^2(\Omega)$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1}$$

其中

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = C_R \frac{d^2}{dx^2} \int_\Omega \frac{u(y)}{|x-y|^{\alpha-1}} dy \tag{2}$$

2 数值格式

用线性插值代替原函数，中心差分代替二阶导数，记 $u_h(x)$ 为 $u(x)$ 在网格点上的线性插值。

我们解这样的数值解

$$\begin{aligned} & C_R\Big(\frac{2}{h_{i+1}(h_i + h_{i+1})} \int_\Omega \frac{u_h(x)}{|x_{i+1} - y|^{\alpha-1}} dy \\ & \quad - \frac{2}{h_i h_{i+1}} \int_\Omega \frac{u_h(x)}{|x_i - y|^{\alpha-1}} dy \\ & \quad + \frac{2}{h_i(h_i + h_{i+1})} \int_\Omega \frac{u_h(x)}{|x_{i-1} - y|^{\alpha-1}} dy\Big) \\ & = F_i \end{aligned} \tag{3}$$

矩阵 A 是 M 矩阵，主队角正，其他负，严格对角占优。

3 一致网格

当 $r = 1$, 网格成为一致网格, $x_i = ih, h = \frac{1}{2N}, i = 0, ..., 2N$.

A 等于

$$\begin{aligned} a_{ij} = & \frac{C_R}{(2-\alpha)(3-\alpha)} h^{-\alpha} \\ & \left(|i-j-2|^{3-\alpha} - 4|i-j-1|^{3-\alpha} + 6|i-j|^{3-\alpha} - 4|i-j+1|^{3-\alpha} + |i-j+2|^{3-\alpha}\right) \end{aligned} \tag{4}$$

矩阵行和

$$S_i = \sum_{j=1}^{2N-1} a_{ij} = \frac{C_R}{(2-\alpha)(3-\alpha)} h^{-\alpha}(|i+1|^{3-\alpha} - 3|i|^{3-\alpha} + 3|i-1|^{3-\alpha} - |i-2|^{3-\alpha} + ...2N) \tag{5}$$

我们得到

$$S_i \geq C(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \tag{6}$$

下面估计截断误差 R_i.

$$R_i = \int_0^1 D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \tag{7}$$

目标是

$$R_i \leq Ch^{\alpha/2} S_i \tag{8}$$

这样我们就有

$$\epsilon \leq \max_i \frac{R_i}{S_i} \leq Ch^{\alpha/2} \tag{9}$$

考虑 R_1

$$R_1 = \int_\Omega (u(y) - u_h(y)) \frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} dy \tag{10}$$

我们有

$$R_1 = \int_0^{3h} + \int_{3h}^{1/2} \tag{11}$$

当 $y > 3h$,

$$\frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} \leq C|y|^{-1-\alpha} \tag{12}$$

那么

$$
\begin{aligned}
I_2 &\le C\int_{3h}^{1/2}|y|^{-1-\alpha}u''(y)h^2dy\\
&\le C\int_{3h}^{1}|y|^{-1-\alpha}y^{\alpha/2-2}h^2dy\\
&\le Ch^2\int_{3h}^{1}y^{-3-\alpha/2}dy\\
&\le Ch^2h^{-2-\alpha/2}=Ch^{-\alpha/2}\\
&\le Ch^{\alpha/2}x_1^{-\alpha}\le Ch^{\alpha/2}S_1
\end{aligned}
\tag{13}
$$

在考虑

$$
\begin{aligned}
I_1 &= \int_0^{3h}\frac{u(y)-u_h(y)}{h^2}(|y|^{1-\alpha}-2|y-h|^{1-\alpha}+|y-2h|^{1-\alpha})dy\\
&= \int_0^h+\int_h^{3h}=J_1+J_2
\end{aligned}
\tag{14}
$$

$$
J_2\le Cu''(\eta)h^{2-\alpha}\le Ch^{\alpha/2-2}h^{2-\alpha}\le Ch^{-\alpha/2}
\tag{15}
$$

因为

$$
\begin{aligned}
|u(x)-u_h(x)| &\le \int_0^{x_1}|u'(y)|dy\\
&\le C\int_0^{x_1}y^{\alpha/2-1}dy\\
&\le Cx_1^{\alpha/2}\quad ,x\in(0,h)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
J_1 &= \int_0^{h}\frac{u(y)-u_h(y)}{h^2}(|y|^{1-\alpha}-2|y-h|^{1-\alpha}+|y-2h|^{1-\alpha})dy\\
&\le Ch^{\alpha/2-2}h^{2-\alpha}=Ch^{-\alpha/2}
\end{aligned}
\tag{17}
$$

所以有

$$
R_1\le Ch^{-\alpha/2}\le Ch^{\alpha/2}h^{-\alpha}\le Ch^{\alpha/2}S_1,\quad (S_1\ge Cx_1^{-\alpha})
\tag{18}
$$

R_1,R_2,R_3 全部类似。

3.1 猜想

$$R_i \le Ch^{\alpha/2+1}(x_i^{-\alpha-1} + (1-x_i)^{-\alpha-1}) \quad (then \le Ch^{\alpha/2} S_i) \tag{19}$$

为了简便，我们记 $D(y) := u(y) - u_h(y)$.

当 $3 < i \le N$ 时，

$$\begin{aligned}
R_i =& \int_0^1 D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
=& \int_0^{x_1} D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
&+ \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
&+ \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} \frac{D(y+h) - D(y)}{h^2} |y-x_i|^{1-\alpha} + D(y) \frac{|y-x_{i+1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2} dy \\
&+ \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y-x_i|^{1-\alpha} dy \\
&+ \int_{x_i}^{x_{N+\lfloor \frac{i}{2} \rfloor-1}} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y-x_i|^{1-\alpha} dy \\
&+ \int_{x_{N+\lfloor \frac{i}{2} \rfloor-1}}^{x_{N+\lfloor \frac{i}{2} \rfloor}} \frac{D(y-h) - D(y)}{h^2} |y-x_i|^{1-\alpha} + D(y) \frac{|y-x_{i-1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2} dy \\
&+ \int_{x_{N+\lfloor \frac{i}{2} \rfloor}}^{x_{2N-1}} + \int_{x_{2N-1}}^1 D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
=& I_1 + I_2 + I_3 + I_4 + \cdots
\end{aligned} \tag{20}$$

1.

$$\begin{aligned}
I_1 &= \int_0^{x_1} (u(y)-u_h(y)) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
&\le Ch^{\alpha/2} \int_0^h |y-x_i|^{-1-\alpha} dy \\
&\le Ch^{\alpha/2+1} x_i^{-1-\alpha}
\end{aligned} \tag{21}$$

2.

$$\begin{aligned}
I_2 &= \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
&\le C \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2} h^2 |x_i - y|^{-1-\alpha} dy \\
&\le Ch^{\alpha/2-1} h^2 x_i^{-1-\alpha} \le Ch^{\alpha/2+1} x_i^{-1-\alpha}
\end{aligned} \tag{22}$$

3.

$$\begin{aligned}
I_3 &= \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} \frac{D(y+h)-D(y)}{h^2} |y-x_i|^{1-\alpha} + D(y) \frac{|y-x_{i+1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2} dy \\
&\le \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} u'''(\eta_1) h |x_i - y|^{1-\alpha} + u''(\eta_2) h |x_i - y|^{-\alpha} dy \\
&\le Ch^2 x_i^{-2-\alpha/2} \le Ch^{1+\alpha/2} x_i^{-1-\alpha}
\end{aligned} \tag{23}$$

4.

$$\begin{aligned}
I_4 &= \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y-x_i|^{1-\alpha} dy \\
&\le \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} u''''(\eta) h^2 |x_i - y|^{1-\alpha} dy \\
&\le C x_i^{\alpha/2-4} h^2 x_i^{2-\alpha} \\
&\le Ch^2 x_i^{-2-\alpha/2} \le Ch^{1+\alpha/2} x_i^{-1-\alpha}
\end{aligned} \tag{24}$$

猜想证毕，一致网格证完。

4 非一致

$r > 1$,

$$\begin{cases} x_i = \frac{1}{2}\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ x_i = 1 - \frac{1}{2}\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases} \tag{25}$$

令 $h = \frac{1}{2N}$, 那么

当 $i < N, x_i < \frac{1}{2}$ 时

$$h_i = \frac{1}{2}\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right) \le C(r)\left(\frac{i}{N}\right)^{r-1}\frac{1}{N} = Chx_i^{(r-1)/r} \tag{26}$$

当 $i \ge N, x_i \ge \frac{1}{2}$ 时

$$h_i = \frac{1}{2}\left(\left(\frac{2N-i+1}{N}\right)^r - \left(\frac{2N-i}{N}\right)^r\right) \le C(r)\left(\frac{2N-i+1}{N}\right)^{r-1}\frac{1}{N} = Ch(1-x_{i-1})^{(r-1)/r} \tag{27}$$

我们声明

$$\begin{aligned} S_i = \sum_{j=1}^{2N-1} a_{ij} =& \frac{C_R}{(2-\alpha)(3-\alpha)}\frac{2}{h_i+h_{i+1}} \\ & (\frac{1}{h_{i+1}}\frac{|x_{i+1}-x_0|^{3-\alpha} - |x_{i+1}-x_1|^{3-\alpha}}{x_1 - x_0} \\ & -(\frac{1}{h_i}+\frac{1}{h_{i+1}})\frac{|x_i-x_0|^{3-\alpha} - |x_i-x_1|^{3-\alpha}}{x_1-x_0} \\ & +\frac{1}{h_i}\frac{|x_{i-1}-x_0|^{3-\alpha} - |x_{i-1}-x_1|^{3-\alpha}}{x_1-x_0}) \\ & + ... \\ \ge & C(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \end{aligned} \tag{28}$$

$$R_i = \int_0^1 D(y)\frac{2}{h_i+h_{i+1}}(\frac{1}{h_{i+1}}|x_{i+1}-y|^{1-\alpha} - (\frac{1}{h_i}+\frac{1}{h_{i+1}})|x_i-y|^{1-\alpha} + \frac{1}{h_i}|x_{i-1}-y|^{1-\alpha})dy \tag{29}$$

下面讨论 R_1

$$
\begin{aligned}
R_1 =& \int_0^{x_1} + \int_{x_1}^{x_3} + \int_{x_3}^{1/2} + \int_{1/2}^{x_{2N-1}} + \int_{x_{2N-1}}^{1} \\
& D(y)\frac{2}{h_1+h_2}\left(\frac{1}{h_2}|x_2-y|^{1-\alpha} - \left(\frac{1}{h_1}+\frac{1}{h_2}\right)|x_1-y|^{1-\alpha} + \frac{1}{h_1}|y|^{1-\alpha}\right)dy \\
:=& I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}
\tag{30}
$$

与一致网格时相似，

1.

$$
\begin{aligned}
|u(x)-u_h(x)| &\le \int_0^{x_1} |u'(y)|dy \\
&\le C\int_0^{x_1} y^{\alpha/2-1}dy \\
&\le Cx_1^{\alpha/2} \quad ,x\in(0,x_1)
\end{aligned}
\tag{31}
$$

因为 $1-\alpha > -1$

$$
\begin{aligned}
I_1 &\le C\int_0^{x_1} \frac{D(y)}{h_1^2}(|x_2-y|^{1-\alpha} + 2|x_1-y|^{1-\alpha} + |y|^{1-\alpha})dy \\
&\le Cx_1^{\alpha/2-2}x_1^{2-\alpha} = Cx_1^{-\alpha/2} = Ch^{-r\alpha/2}
\end{aligned}
\tag{32}
$$

2.

$$
I_2 \le Cu''(\eta)x_3^{2-\alpha} \le Cx_1^{\alpha/2-2}x_3^{2-\alpha} \le Ch^{-r\alpha/2} \tag{33}
$$

3.

$$
\begin{aligned}
I_3 &= \int_{x_3}^{1/2} D(y)\frac{2}{h_1+h_2}(\frac{1}{h_2}|x_2-y|^{1-\alpha} - (\frac{1}{h_1}+\frac{1}{h_2})|x_1-y|^{1-\alpha} + \frac{1}{h_1}|y|^{1-\alpha})dy \\
&\leq C\int_{x_3}^{1/2} y^{\alpha/2-2}(hy^{(r-1)/r})^2 y^{-1-\alpha}dy \\
&\leq Ch^2\int_{x_3}^{1/2} y^{\alpha/2-2/r-1-\alpha}dy \\
&\leq Ch^2(h^r)^{-2/r-\alpha/2} = Ch^{-r\alpha/2}
\end{aligned}
\tag{34}
$$

4.

$$
\begin{aligned}
I_4 &= \int_{1/2}^{x_{2N-1}} D(y)\frac{2}{h_1+h_2}(\frac{1}{h_2}|x_2-y|^{1-\alpha} - (\frac{1}{h_1}+\frac{1}{h_2})|x_1-y|^{1-\alpha} + \frac{1}{h_1}|y|^{1-\alpha})dy \\
&\leq C\int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2}(h(1-y)^{(r-1)/r})^2 y^{-1-\alpha}dy \\
&\leq Ch^2\int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2+2-2/r} \\
&\leq Ch^2(C+h_{2N}^{\alpha/2-2/r+1}) \\
&= Ch^2(C+h^{r\alpha/2-2+r}) \leq Ch^{\min\{2,r\alpha/2+r\}}
\end{aligned}
\tag{35}
$$

5.

$$
I_5 \leq Ch_{2N}^{\alpha/2+1} \leq Ch^{r\alpha/2+r} \tag{36}
$$

综合有

$$
R_1 \leq Ch^{-r\alpha/2} \tag{37}
$$

R_1, R_2, R_3 一样。

4.1 一般的 i

$R_i, 3 < i < N$ 比较困难。$i = N$ 再处理。

我们记 $D(y) = u(y) - u_h(y)$

$$T_{ij} = \int_{x_{j-1}}^{x_j} D(y)|x_i - y|^{1-\alpha} dy \tag{38}$$

那么

$$\begin{aligned}
R_i =&\sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=&\sum_{j=1}^{i/2} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,i/2+1} + T_{i+1,i/2+2}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})(T_{i,i/2+1}) \right) \\
&+ \sum_{j=i/2+2}^{i} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&+ \sum_{j=i+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{N+2}^{N+i/2-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i-1}} (T_{i-1,N+i/2} + T_{i-1,N+i/2-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})(T_{i,N+i/2}) \right) \\
&+ \sum_{j=N+i/2+1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=&I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8
\end{aligned} \tag{39}$$

$$\begin{aligned}
I_1 =& \int_0^{x_1} + \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} \\
& D(y) \frac{2}{h_i + h_{i+1}} (\frac{1}{h_{i+1}} |x_{i+1} - y|^{1-\alpha} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) |x_i - y|^{1-\alpha} + \frac{1}{h_i} |x_{i-1} - y|^{1-\alpha}) dy
\end{aligned} \tag{40}$$

1.

$$J_1 \le C x_1^{\alpha/2+1} x_i^{-1-\alpha} \le C h^{r\alpha/2+r} x_i^{-1-\alpha} \tag{41}$$

2.

$$
\begin{aligned}
J_2 &\le C\int_{x_1}^{x_{\lceil \frac{i}{2}\rceil}} y^{\alpha/2-2}(hy^{(r-1)/r})^2|x_i-y|^{-1-\alpha}dy \\
&\le Ch^2x_i^{-1-\alpha}\int_{x_1}^{x_{\lceil \frac{i}{2}\rceil}} y^{\alpha/2-2/r}dy \\
&\le Ch^2x_i^{-1-\alpha}(h^{r\alpha/2-2+r}+x_i^{\alpha/2-2/r+1})
\end{aligned}
\tag{42}
$$

我们先研究 I_3，考虑

$$
\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j+1}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,j}+\frac{1}{h_i}T_{i-1,j-1}\right) \tag{43}
$$

在此之前我们做一些准备工作。

对于 $y\in[x_{j-1},x_j]$，我们记 $y_j^\theta=\theta x_{j-1}+(1-\theta)x_j$

$$
D(y_j^\theta)=-\frac{\theta(1-\theta)}{2}h_j^2u''(y_j^\theta)+\frac{\theta(1-\theta)}{3!}h_j^3(\theta^2u'''(\eta_1)-(1-\theta)^2u'''(\eta_2)) \tag{44}
$$

那么

$$
\begin{aligned}
T_{ij} &= \int_{x_{j-1}}^{x_j} D(y)|x_i-y|^{1-\alpha}dy \\
&= \int_0^1 \frac{\theta(1-\theta)}{2}h_j^3u''(y_j^\theta)|x_i-y_j^\theta|^{1-\alpha}d\theta \\
&\quad + \int_0^1 \frac{\theta(1-\theta)}{3!}h_j^4|x_i-y_j^\theta|^{1-\alpha}(\theta^2u'''(\eta_{1,j}^\theta)-(1-\theta)^2u'''(\eta_{2,j}^\theta))d\theta
\end{aligned}
\tag{45}
$$

现在回到原来的问题，我们要研究

$$
\begin{aligned}
&\frac{2}{h_i+h_{i+1}}(\frac{1}{h_{i+1}}h_{j+1}^3u''(y_{j+1}^\theta)|x_{i+1}-y_{j+1}^\theta|^{1-\alpha} \\
&\quad -(\frac{1}{h_i}+\frac{1}{h_{i+1}})h_j^3\,u''(y_j^\theta)|x_i-y_j^\theta|^{1-\alpha} \\
&\qquad +\frac{1}{h_i}h_{j-1}^3u''(y_{j-1}^\theta)|x_{i-1}-y_{j-1}^\theta|^{1-\alpha})
\end{aligned}
\tag{46}
$$

我们希望把他看成一个函数的二阶导，注意到当 $j \le i \le N$ 时

$$x_i^{1/r} - x_j^{1/r} = x_{i+1}^{1/r} - x_{j+1}^{1/r} = 2^{-1/r}\frac{i-j}{N} \tag{47}$$

那么我们将其他的相都表示成 x_i 的函数。

$$y_L(x) = (x^{1/r} - z_L)^r, \quad y_R(x) = (x^{1/r} - z_R)^r \tag{48}$$

其中 $z_R = 2^{-1/r}\frac{i-j}{N}, z_L = 2^{-1/r}\frac{i-j+1}{N}$.

$$y_R(x_i) = x_j, \quad y_R(x_{i+1}) = x_{j+1}, \quad y_R(x_{i-1}) = x_{j-1} \tag{49}$$

$$y_L(x_i) = x_{j-1}, \quad y_L(x_{i+1}) = x_j, \quad y_L(x_{i-1}) = x_{j-2} \tag{50}$$

$$y_\theta(x) = \theta y_L(x) + (1-\theta) y_R(x) \tag{51}$$

$$h_J(x) = y_R(x) - y_L(x) \tag{52}$$

那么我么要研究的就是函数

$$K_1(x) = h_J^3(x)|x - y_\theta(x)|^{1-\alpha} u''(y_\theta(x)) \tag{53}$$

在网格 x_{i-1}, x_i, x_{i+1} 的数值二阶差商。

$$\frac{2}{h_i + h_{i+1}}(\frac{1}{h_{i+1}}K_1(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})K_1(x_i) + \frac{1}{h_i}K_1(x_{i-1})) = K_1''(\xi),\ \xi \in [x_{i-1}, x_{i+1}] \tag{54}$$

由 Leibniz 公式

$$(uvw)'' = u''vw + uv''w + uvw'' + 2u'v'w + 2uv'w' + 2u'vw' \tag{55}$$

由 $y_R^{1/r} = x^{1/r} - z_R$, 我们得到

$$\frac{dy_R}{dx} = x^{1/r-1} y_R^{1-1/r} \tag{56}$$

$$\frac{d^2y_R}{dx^2} = \frac{r-1}{r} x^{1/r-2} y_R^{1-2/r} z_R \tag{57}$$

因此

1.

$$h_J^3 \sim h^3 y_R^{3-3/r} \sim h^3 x^{3-3/r} \tag{58}$$

$$\begin{aligned}
(h_J^3)' &= 3h_J^2(y_R' - y_L') \\
&= 3h_J^2 x^{1/r-1}(y_R^{1-1/r} - y_L^{1-1/r}) \\
&\sim h^3 y_R^{2-2/r} x^{1/r-1} y_R^{1-2/r} \\
&\sim h^3 x^{2-3/r}
\end{aligned} \tag{59}$$

$$\begin{aligned}
(h_J^3)'' &= 6h_J x^{2/r-2}(y_R^{1-1/r} - y_L^{1-1/r})^2 + 3h_J^2 \frac{r-1}{r} x^{1/r-2}(y_R^{1-2/r} z_R - y_L^{1-2/r} z_L) \\
&\sim h y_R^{1-1/r} x^{2/r-2}(h y_R^{1-2/r})^2 + \frac{r-1}{r} h^2 y_R^{2-2/r} x^{1/r-2}(z_R h y_R^{1-3/r} - h y_L^{1-2/r}) \\
&\sim h^3 y_R^{3-5/r} x^{2/r-2} + \frac{r-1}{r} h^3 (y_R^{3-5/r} x^{1/r-2} z_R - y_R^{3-4/r} x^{1/r-2}) \\
&\sim h^3 (y_R^{3-5/r} x^{2/r-2} + y_R^{3-4/r} x^{1/r-2} + y_R^{3-5/r} x^{1/r-2} z_R) \\
&< h^3 x^{1-3/r}
\end{aligned} \tag{60}$$

2.

由于

$$x - y_L = (x^{1/r})^r - (x^{1/r} - z_L)^r = z_L \xi^{r-1} \sim z_L x^{(r-1)/r} \tag{61}$$

$$\begin{aligned}
|x - y_\theta|^{1-\alpha} &= |x - \theta y_L - (1-\theta) y_R|^{1-\alpha} \sim |(\theta z_L + (1-\theta) z_R) \xi^{1-1/r}|^{1-\alpha} \\
&\sim z_\theta^{1-\alpha} \xi^{1-\alpha+(\alpha-1)/r}, \quad \xi \in [y_L, x]
\end{aligned} \tag{62}$$

$$
\begin{aligned}
(|x-y_\theta|^{1-\alpha})' &= (1-\alpha)|x-y_\theta|^{-\alpha}(1-x^{1/r-1}(\theta y_L^{1-1/r}+(1-\theta)y_R^{1-1/r}))\\
&= (1-\alpha)|x-y_\theta|^{-\alpha}x^{1/r-1}(x^{1-1/r}-(\theta y_L^{1-1/r}+(1-\theta)y_R^{1-1/r}))\\
&\sim |x-y_\theta|^{-\alpha}x^{1/r-1}(\theta z_L+(1-\theta)z_R)\xi_2^{1-2/r}\\
&\sim |(\theta z_L+(1-\theta)z_R)\xi_1^{1-1/r}|^{-\alpha}x^{1/r-1}(\theta z_L+(1-\theta)z_R)\xi_2^{1-2/r}\\
&\sim z_\theta^{1-\alpha}x^{-\alpha+(\alpha-1)/r}
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
(|x-y_\theta|^{1-\alpha})'' =&\alpha(\alpha-1)|x-y_\theta|^{-1-\alpha}(1-x^{1/r-1}(\theta y_L^{1-1/r}+(1-\theta)y_R^{1-1/r}))^2\\
&+(1-\alpha)|x-y_\theta|^{-\alpha}(\frac{1-r}{r}x^{1/r-2}(\theta y_L^{1-2/r}z_L+(1-\theta)y_R^{1-2/r}z_R))\\
\sim&|(\theta z_L+(1-\theta)z_R)\xi_1^{1-1/r}|^{-1-\alpha}x^{2/r-2}\xi_2^{2-4/r}(\theta z_L+(1-\theta)z_R)^2\\
&+|(\theta z_L+(1-\theta)z_R)\xi_1^{1-1/r}|^{-\alpha}x^{1/r-2}(\theta z_L+(1-\theta)z_R)y_R^{1-2/r}\\
\sim&z_\theta^{1-\alpha}x^{-1-\alpha+(\alpha-1)/r}
\end{aligned}
\tag{64}
$$

3.

$$
u''(y_\theta)\le Cy_\theta^{\alpha/2-2}\sim x^{\alpha/2-2}
\tag{65}
$$

$$
\begin{aligned}
(u''(y_\theta))' &= u'''(y_\theta)x^{1/r-1}(\theta y_L^{1-1/r}+(1-\theta)y_R^{1-1/r})\\
&\le Cy_\theta^{\alpha/2-3}x^{1/r-1}y_R^{1-1/r}\sim x^{\alpha/2-3}
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
(u''(y_\theta))'' =&u''''(y_\theta)(x^{1/r-1}(\theta y_L^{1-1/r}+(1-\theta)y_R^{1-1/r}))^2\\
&+u'''(y_\theta)\frac{r-1}{r}x^{1/r-2}(\theta y_L^{1-2/r}z_L+(1-\theta)y_R^{1-2/r}z_R)\\
\sim&y_\theta^{\alpha/2-4}(x^{1/r-1}y_R^{1-1/r})^2+z_\theta y_R^{\alpha/2-3+1-2/r}x^{1/r-2}\\
<&x^{\alpha/2-4}
\end{aligned}
\tag{67}
$$

$$u''vw \sim h^3x^{1-3/r}\ z_\theta^{1-\alpha}x^{1-\alpha+(\alpha-1)/r}\ x^{\alpha/2-2} \sim h^3z_\theta^{1-\alpha}x^{-\alpha/2-2/r+(\alpha-2)/r} \tag{68}$$

$$uv''w \sim h^3x^{3-3/r}\ z_\theta^{1-\alpha}x^{-1-\alpha+(\alpha-1)/r}\ x^{\alpha/2-2} \sim h^3z_\theta^{1-\alpha}x^{-\alpha/2-2/r+(\alpha-2)/r} \tag{69}$$

$$uvw'' \sim h^3x^{3-3/r}\ z_\theta^{1-\alpha}x^{1-\alpha+(\alpha-1)/r}\ x^{\alpha/2-4} \sim h^3z_\theta^{1-\alpha}x^{-\alpha/2-2/r+(\alpha-2)/r} \tag{70}$$

$$u'v'w \sim h^3x^{2-3/r}\ z_\theta^{1-\alpha}x^{-\alpha+(\alpha-1)/r}\ x^{\alpha/2-2} \sim h^3z_\theta^{1-\alpha}x^{-\alpha/2-2/r+(\alpha-2)/r} \tag{71}$$

$$uv'w' \sim h^3x^{3-3/r}\ z_\theta^{1-\alpha}x^{-\alpha+(\alpha-1)/r}\ x^{\alpha/2-3} \sim h^3z_\theta^{1-\alpha}x^{-\alpha/2-2/r+(\alpha-2)/r} \tag{72}$$

$$u'vw' \sim h^3x^{2-3/r}\ z_\theta^{1-\alpha}x^{1-\alpha+(\alpha-1)/r}\ x^{\alpha/2-3} \sim h^3z_\theta^{1-\alpha}x^{-\alpha/2-2/r+(\alpha-2)/r} \tag{73}$$

因此

$$K_1''(\xi) \sim h^3z_\theta^{1-\alpha}x_i^{-\alpha/2-2/r+(\alpha-2)/r}, \xi \in [x_{i-1}, x_{i+1}] \tag{74}$$

现在我们处理第二部分

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}(&\frac{1}{h_{i+1}}h_{j+1}^4u'''(\eta_{1,j+1}^\theta)|x_{i+1}-y_{j+1}^\theta|^{1-\alpha}\\ &-(\frac{1}{h_i}+\frac{1}{h_{i+1}})h_j^4\ u'''(\eta_{1,j}^\theta)|x_i-y_j^\theta|^{1-\alpha}\\ &\qquad+\frac{1}{h_i}h_{j-1}^4u'''(\eta_{1,j-1}^\theta)|x_{i-1}-y_{j-1}^\theta|^{1-\alpha})\end{aligned} \tag{75}$$

这次我们只用一阶差分

$$\frac{1}{h_i}(h_j^4u'''(\eta_{1,j}^\theta)|x_i-y_j^\theta|^{1-\alpha} - h_{j-1}^4\ u'''(\eta_{1,j-1}^\theta)|x_{i-1}-y_{j-1}^\theta|^{1-\alpha}) \tag{76}$$

为了方便计算，我们还是用辅助函数来对上面一项进行估计。

$$K_2(x) = h_J^4(x)|x-y_\theta(x)|^{1-\alpha} \tag{77}$$

$$
\begin{aligned}
K_2'(x) &= (h_J^4)'|x-y_\theta(x)|^{1-\alpha} + h_J^4(|x-y_\theta(x)|^{1-\alpha})' \\
&\sim h^3 x^{3-3/r} x^{1/r-1} h y_R^{1-2/r}\ z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} \\
&\quad + h^4 y_R^{4-4/r} z_\theta^{1-\alpha} x^{-\alpha+(\alpha-1)/r} \\
&\sim h^4 z_\theta^{1-\alpha} x^{4-5/r-\alpha+\alpha/r}
\end{aligned} \tag{78}
$$

那么，上面就等于

$$
\begin{aligned}
&\frac{1}{h_i}(K_2(x_i)u'''(\eta^\theta_{1,j}) - K_2(x_{i-1})u'''(\eta^\theta_{1,j-1})) \\
=&\frac{1}{h_i}K_2(x_i)(u'''(\eta^\theta_{1,j}) - u'''(\eta^\theta_{1,j-1})) + \frac{1}{h_i}(K_2(x_i) - K_2(x_{i-1})u'''(\eta^\theta_{1,j-1})) \\
\le& h_i^{-1}K_2(x_i)u''''(\eta_j^\theta)(x_j - x_{j-2}) + K_2'(\xi)u'''(\eta^\theta_{1,j-1}) \quad (\eta_j^\theta \in [x_{j-2}, x_j], \xi \in [x_{j-1}, x_j]) \\
\sim& h_i^{-1}h_j^4|x_i - y_j^\theta|^{1-\alpha}\ C(\eta_j^\theta)^{\alpha/2-4} 2h_j \\
&+ h^4 z_\theta^{1-\alpha}\xi^{4-5/r-\alpha+\alpha/r}(\eta_j^\theta)^{\alpha/2-3} \\
\sim& h^4 x_i^{4-4/r} z_\theta^{1-\alpha} x_i^{1-\alpha+(\alpha-1)/r} x_i^{\alpha/2-4} + h^4 z_\theta^{1-\alpha} x_i^{4-5/r-\alpha+\alpha/r} x_i^{\alpha/2-3} \\
\sim& h x_i^{1-1/r}\ h^3 z_\theta^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r}
\end{aligned} \tag{79}
$$

因此，

$$
\begin{aligned}
&\frac{2}{h_i+h_{i+1}}\frac{1}{h_i}(K_2(x_i)u'''(\eta^\theta_{1,j}) - K_2(x_{i-1})u'''(\eta^\theta_{1,j-1})) \\
&\qquad \sim h^3 z_\theta^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r}
\end{aligned} \tag{80}
$$

最终我们得到，当 $i/2+2 \le j \le i < N$ 时，有

$$
\begin{aligned}
&\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,j} + \frac{1}{h_i}T_{i-1,j-1}\right) \\
&\qquad \le Ch^3\left(\frac{i-j+1}{N}\right)^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r}
\end{aligned} \tag{81}
$$

那么我们得到

$$
\begin{aligned}
I_3 &\le C \sum_{j=i/2+2}^{i} \left(\frac{1}{N}\right)^3 \left(\frac{i-j+1}{N}\right)^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r} \\
&\le C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r+(\alpha-2)/r} \left(\frac{i}{2N}\right)^{2-\alpha} \\
&\le C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r+(\alpha-2)/r} x_i^{(2-\alpha)/r} \\
&= C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{82}
$$

最后我们处理 I_2, 记 $k = i/2+1$

$$
\begin{aligned}
I_2 &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,i/2+1} + T_{i+1,i/2+2}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})(T_{i,i/2+1}) \right) \\
&= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i}) T_{i,k} \right) \\
&= J_1 + J_2 + J_3
\end{aligned} \tag{83}
$$

$$
\begin{aligned}
J_1 &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) \right) \\
&= \frac{2}{h_i + h_{i+1}} \int_{x_{k-1}}^{x_k} D(y) \frac{|x_{i+1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_{i+1}} dy \\
&\le C x_i^{\alpha/2-2} h_k^2 x_i^{-\alpha} \\
&\le C h^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{84}
$$

$$
\begin{aligned}
J_2 &= \frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} \left(T_{i+1,k+1} - T_{i,k} \right) \\
&= \frac{2}{h_i + h_{i+1}} \int_0^1 \frac{h_{k+1} D(y_{k+1}^\theta) |x_{i+1} - y_{k+1}^\theta|^{1-\alpha} - h_k D(y_k^\theta) |x_i - y_k^\theta|^{1-\alpha}}{h_{i+1}} d\theta
\end{aligned} \tag{85}
$$

我们看他的两个积分项

$$
\begin{aligned}
\frac{K_1(x_{i+1})-K_1(x_i)}{h_{i+1}} &= K_1'(\xi) \\
&\sim h^3 x^{2-3/r}\ z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r}\ x^{\alpha/2-2} \\
&\quad + h^3 x^{3-3/r}\ z_\theta^{1-\alpha} x^{-\alpha+(\alpha-1)/r}\ x^{\alpha/2-2} \\
&\quad + h^3 x^{3-3/r}\ z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r}\ x^{\alpha/2-3} \\
&\sim h x^{1-1/r} h^2 z_\theta^{1-\alpha} x^{-\alpha/2+\alpha/r-3/r} \\
&\sim h x^{1-1/r} h^2 x^{(1-\alpha)/r} x^{-\alpha/2+\alpha/r-3/r} \\
&\sim h x^{1-1/r} h^2 x^{-\alpha/2-2/r}
\end{aligned} \tag{86}
$$

第二部分研究过了

$$
\begin{aligned}
&\frac{1}{h_i}(K_2(x_{i+1})u'''(\eta^\theta_{1,k+1}) - K_2(x_i)u'''(\eta^\theta_{1,k})) \\
&\sim h x_i^{1-1/r}\ h^3 z_\theta^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r} \\
&\sim h x_i^{1-1/r}\ h^3 x_i^{(1-\alpha)/r} x_i^{-\alpha/2-2/r+(\alpha-2)/r} \\
&\sim h x_i^{1-1/r}\ h^3 x_i^{-\alpha/2-2/r-1/r}
\end{aligned} \tag{87}
$$

因此

$$
J_2 \le C h^2 x^{-\alpha/2-2/r} \tag{88}
$$

现在考虑 J_3

$$
\begin{aligned}
J_3 &= \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}} - \frac{1}{h_i}\right) T_{i,k} \\
&= -\frac{2}{h_i+h_{i+1}} \frac{h_{i+1}-h_i}{h_i h_{i+1}} \int_{x_{k-1}}^{x_k} D(y_k^\theta)|x_i - y_k^\theta|^{1-\alpha} dy \\
&\sim h_i^{-1} x_i^{-1} h_k^3 x_i^{\alpha/2-2} x_i^{1-\alpha} \\
&\sim h^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{89}
$$

因此我们有

$$
I_2 \le C h^2 x_i^{-\alpha/2-2/r} \tag{90}
$$

全部加起来，我们得到

错了！不对称，要补上

$$I_4 = \sum_{j=i+1}^{N-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \quad (91)$$

类似的，我们要研究的就变成
$y_L^{1/r} = x^{1/r} + z_L,\ y_R^{1/r} = x^{1/r} + z_R, \quad z_L = 2^{-1/r} \frac{j-i-1}{N},\ z_R = 2^{-1/r} \frac{j-i}{N}$

$$K_1(x) = h_J^3(x) |y_\theta(x) - x|^{1-\alpha} u''(y_\theta(x)) \quad (92)$$

在网格 x_{i-1}, x_i, x_{i+1} 的数值二阶差商。

$$\frac{2}{h_i + h_{i+1}} (\frac{1}{h_{i+1}} K_1(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) K_1(x_i) + \frac{1}{h_i} K_1(x_{i-1})) = K_1''(\xi),\ \xi \in [x_{i-1}, x_{i+1}] \quad (93)$$