# SECOND-ORDER DIFFERENCE-QUADRATURE APPROACH ON GRADED MESHES FOR FRACTIONAL LAPLACIAN

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Abstract. This paper deals with the *integral-differential* version of the fractional Laplacian via the Riesz fractional derivative. The numerical analysis presents significant challenges, partly because the solution to the problem has a weak singularity at the boundary, and the model equation can involve a singular source term. In such cases, many prevalent numerical methods may suffer from a severe order reduction. To fill in this gap, we combine finite difference method and numerical quadrature, called difference-quadrature method, to approximate the *differential* and *integral* operator of the fractional Laplacian on graded meshes, respectively. We design a grid mapping function and a natural-skew ordering to handle local truncation errors, and construct an appropriate right-preconditioner for the resulting matrix algebraic equation. By utilizing the Hölder regularity of the data, we prove that the proposed scheme is second-order convergence on graded meshes even if the source term is hypersingular. Numerical experiments illustrate the theoretical results.

## 1. Introduction

Fractional Laplacian is a powerful tool in modeling phenomena for anomalous diffusion, which appears naturally in the $\alpha$-stable Lévy process instead of the standard Brownian motion [1, 3, 17, 35, 24]. It can be found in many applications, such as porous media flow [11], image processing [16], biophysics [2]. In this work, we study a second-order difference-quadrature scheme on graded meshes for the integral-differential version of the fractional Laplacian

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x) & x \in \Omega, \\ u(x) = 0 & x \in \mathbb{R} \setminus \Omega. \end{cases} \tag{1.1}$$

Here $(-\Delta)^{\frac{\alpha}{2}}$ is the integral-differential fractional Laplacian, in terms of the Riesz (left and right Riemann-Liouville) fractional derivative [1, 20, 23, 33], defined by

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{d^2}{dx^2} I^{2-\alpha} u(x) \quad \text{with} \quad 1 < \alpha < 2. \tag{1.2}$$

Note that the Riesz fractional integration can be realized in the form of the Riesz potential defined as the Fourier convolution of the form [32, eq. (1.30)], namely,

$$I^{2-\alpha} u(x) = \int_\Omega K(x-y) u(y) dy \quad \text{with} \quad K(x) = \frac{|x|^{1-\alpha}}{2\cos((2-\alpha)\pi/2)\Gamma(2-\alpha)}. \tag{1.3}$$

The fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ can be defined in several equivalent ways [21, 23] on the whole space $\mathbb{R}^n$. For example, it can be defined as a pseudo-differential operator via the Fourier transform

$$\mathcal{F}[(-\Delta)^{\frac{\alpha}{2}} u](\xi) = |\xi|^\alpha \mathcal{F}[u](\xi),$$

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or in terms of the hypersingular integral operator

$$(*) \qquad (-\Delta)^{\frac{\alpha}{2}} u(x) = C_{n,\alpha} \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy.$$

The major challenges of fractional Laplacian arise partly because typical solutions $u$ have a weak singularity at the boundary; for example in the special case where $\Omega$ is a bounded interval $(a,b) \subset \mathbb{R}$ and $f \equiv 1$, the exact solution is [17, 20, 25]

$$u(x) = \frac{2^{-\alpha} \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right) \Gamma\left(\frac{1+\alpha}{2}\right)} \left[(x-a)(b-x)\right]^{\frac{\alpha}{2}}.$$

Moreover, the model equation (1.1) can involve a singular/hypersingualr source term, even if the exact solution $u$ is absolutely continuous [29, 30, 39]. This leads to a severe order reduction for many numerical methods.

Among various techniques for approximating *integral* version of the fractional Laplacian $(*)$, numerical quadrature with piecewise linear polynomials (collocation) is the simplest, since it only need a single integration and are much simpler to implement on a computer. In [20], Huang and Oberman first proposed a quadrature-based finite difference method for solving the 1 dimensional (1D) integral fractional Laplacian. The numerical solution obtained from this method is $\mathcal{O}\left(h^{2-\alpha}\right)$ accurate in the discrete $L^\infty(\mathbb{R}^n)$ norm if the solution is sufficiently smooth, while this accuracy reduces to $\mathcal{O}\left(h^{\alpha/2}\right)$ in the case $f \equiv 1$, since $u$ has a boundary singularity. Inspired by [20], $\mathcal{O}\left(|\log h| h^{2-\alpha/2}\right)$ convergence for $0 < \alpha < 2$ and $\mathcal{O}\left(h^{\alpha}\right)$ for $\alpha \leq 4/3$, respectively, is proved [19] in the discrete $L^\infty(\mathbb{R}^n)$ norm on graded meshes for $n = 1, 2$ by means of a discrete barrier function. Recently, $\mathcal{O}\left(h^{2-\alpha}\right)$ convergence for $0 < \alpha < 1$ is given in [9] by collocation method on graded meshes, where it remains to be proved for $1 < \alpha < 2$. It seems that achieving a second-order accurate scheme using piecewise linear polynomials collocation method for fractional Laplacian $(*)$ with $1 < \alpha < 2$ is not an easy task.

Nevertheless, there are already some important progress for numerically solving *integral-differential* version of the fractional Laplacian (1.2) with $1 < \alpha < 2$ via the Riesz (left and right Riemann-Liouville) fractional derivative. Take, for example, the finite difference method [5, 14, 27, 28, 37, 8, 7, 6, 31, 34, 38], finite element method [4, 15, 13], and spectral method [10, 12, 36]. However, these methods may suffer from a severe order reduction when the exact solution has a weak singularity at the boundary and the source term is singular/hypersingualr.

How to design/restore the second-order convergence with a singualr/hypersingular source term for the model (1.1) still has not been addressed in the literature. To fill in this gap, we combine finite difference method and numerical quadrature, called difference-quadrature method, to approximate the *differential* and *integral* operator of the fractional Laplacian on graded meshes. This method was proposed by the authors for solving the fractional partial differential equations on uniform mesh [7, 10] when the solution is smooth with $u \in C^4(\bar{\Omega})$. In this work, we design a grid mapping function and a natural-skew ordering to handle local truncation errors, and construct an appropriate right-preconditioner for the resulting matrix algebraic equation. By utilizing the Hölder regularity of the data, we prove that the proposed scheme is second-order convergence on graded meshes even if the source term is hypersingular. Numerical experiments illustrate the theoretical results.

## 2. The main results

In this section, we describe the difference-quadrature scheme on graded meshes for fractional Laplacian (1.1) via the Riesz fractional derivative and state our main results about the convergence rate of the numerical solutions.

### 2.1. **Difference-quadrature scheme.**

To keep the expressions simple below we assume we are on the interval $\Omega = (0, 2T)$, but everything can be shifted to an arbitrary interval $(a, b)$. Partition $\Omega$ by the graded mesh

$$\pi_h : 0 = x_0 < x_1 < x_2 < \cdots < x_{2N-1} < x_{2N} = 2T,$$

where we set

$$x_j = \begin{cases} T\left(\frac{j}{N}\right)^r & \text{for } j = 0, 1, ..., N, \\ 2T - T\left(\frac{2N-j}{N}\right)^r & \text{for } j = N+1, N+2, ..., 2N, \end{cases} \tag{2.1}$$

with a bounded grading exponent $r \geq 1$ . When $r > 1$, the mesh points are clustered near $x = 0$ and $x = 2T$.

Set $h_j = x_j - x_{j-1}$ for $j = 1, 2, ..., 2N$ and define $h := \frac{1}{N}$. Let $S_h$ be the space of globally continuous piecewise linear functions on the mesh $\pi_h$ that vanish at $x = 0, 2T$. In this space, we choose as a basis the standard hat functions

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}) & \text{for } x_{j-1} \leq x \leq x_j, \\ \frac{1}{h_{j+1}}(x_{j+1} - x) & \text{for } x_j \leq x \leq x_{j+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Then, define the piecewise linear interpolant of the true solution $u$ to be

$$\Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x). \tag{2.3}$$

Now, we discretise (1.1) by replacing $u(x)$ by a continuous piecewise linear function

$$u_h(x) := \sum_{j=1}^{2N-1} u_j \phi_j(x), \tag{2.4}$$

whose nodal values $u_j$ are to be determined by collocation at each mesh point $x_i$ for $i = 1, 2, ..., 2N-1$:

$$-D_h^\alpha u_h(x_i) := -D_h^2 I^{2-\alpha} u_h(x_i) = f(x_i) =: f_i. \tag{2.5}$$

Here the approximation of second order derivatives can be found by interpolating by a quadratic function and differentiating twice [22, eq. (1.14)]

$$D_h^2 u(x_i) := \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} u(x_{i-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) u(x_i) + \frac{1}{h_{i+1}} u(x_{i+1}) \right). \tag{2.6}$$

Moreover, the Riesz fractional derivatives in (1.2) can be approximated by

$$-D_h^\alpha u_h(x_i) = -D_h^2 I^{2-\alpha} \sum_{j=1}^{2N-1} u_j \phi_j(x_i) = \sum_{j=1}^{2N-1} a_{ij}\, u_j. \tag{2.7}$$

We have replaced $-\frac{d^2}{dx^2}I^{2-\alpha}u(x_i) = f(x_i)$ in (1.2) by $-D_h^\alpha u_h(x_i) = f(x_i)$ in (2.5), with truncation error

$$\tau_i := -D_h^\alpha \Pi_h u(x_i) - f(x_i) \quad \text{for} \quad i = 1, 2, ..., 2N-1, \tag{2.8}$$

where

$$-D_h^\alpha \Pi_h u(x_i) = -\sum_{j=1}^{2N-1} D_h^\alpha \phi_j(x_i)\, u(x_j) = \sum_{j=1}^{2N-1} a_{ij} u(x_j). \tag{2.9}$$

The discrete equation (2.5) can be written in matrix form

$$AU = F, \tag{2.10}$$

where the coefficient matrix $A$ and the vectors $U$ and $F$ are defined by $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$, $U = (u_1, \cdots, u_{2N-1})^T$ and $F = (f_1, \cdots, f_{2N-1})^T$. In particular, the coefficient $a_{ij}$ can be explicitly expressed as

$$\begin{aligned} a_{ij} &= -D_h^2 I^{2-\alpha} \phi_j(x_i) \\ &= -\frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \tilde{a}_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \end{aligned} \tag{2.11}$$

with the quadrature coefficients

$$\begin{aligned} \tilde{a}_{ij} &= I^{2-\alpha}\phi_j(x_i) \\ &= \frac{\kappa_\alpha}{\Gamma(4-\alpha)} \left( \frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \left( \frac{1}{h_j} + \frac{1}{h_{j+1}} \right) |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right), \end{aligned}$$

and $\kappa_\alpha = \frac{1}{2\cos((2-\alpha)\pi/2)} = -\frac{1}{2\cos(\alpha\pi/2)} > 0$.

2.2. **Regularity of the true solution.** For any $\beta > 0$, we use the standard notation $C^\beta(\bar{\Omega}), C^\beta(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^\beta(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where $k$ is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose $k$-th order partial derivatives are locally Hölder continuous [18, p. 52] with exponent $\beta'$ in $\Omega$. Here, $C^k(\Omega)$ is the set of all $k$-times continuously differentiable functions on open set $\Omega$.

For convenience, we define

$$\delta(x) = \operatorname{dist}(x, \partial\Omega) = \begin{cases} x & 0 < x \le T, \\ 2T - x & T < x < 2T, \end{cases} \tag{2.12}$$

and $\delta(x, y) = \min\{\delta(x), \delta(y)\}$. To bound the derivatives of $u$, we introduce the following $\delta$-dependent Hölder norms.

**Definition 2.1** ($\delta$-dependent Hölder norms [26])**.** For any $\beta > 0$, write $\beta = k + \beta'$, where $k$ is an integer and $\beta' \in (0, 1]$. Given $\sigma \ge -\beta$, define the seminorm

$$|w|_\beta^{(\sigma)} = \sup_{x,y\in\Omega} \left( \delta(x,y)^{\beta+\sigma} \frac{|w^{(k)}(x) - w^{(k)}(y)|}{|x-y|^{\beta'}} \right).$$

For $\sigma > -1$, we also define the norm $\|\cdot\|_\beta^{(\sigma)}$ as follows: in case that $\sigma \ge 0$,

$$\|w\|_\beta^{(\sigma)} = \sum_{l=0}^{k} \sup_{x\in\Omega} \left( \delta(x)^{l+\sigma} |w^{(l)}(x)| \right) + |w|_\beta^{(\sigma)},$$

while for $-1 < \sigma < 0$,

$$\|w\|_\beta^{(\sigma)} = \|w\|_{C^{-\sigma}(\bar\Omega)} + \sum_{l=1}^{k} \sup_{x\in\Omega} \left(\delta(x)^{l+\sigma}|w^{(l)}(x)|\right) + |w|_\beta^{(\sigma)}.$$

**Lemma 2.2.** [26, pp. 276-277] *Let $f \in L^\infty(\Omega)$ and $u$ be a solution of (1.1). Then, $u \in C^{\alpha/2}(\mathbb{R})$ and $u/\delta^{\alpha/2} \in C^\sigma(\bar\Omega)$ for some $\sigma \in (0, 1-\alpha/2)$, $\alpha \in (1,2)$ with*

$$\|u\|_{C^{\alpha/2}(\mathbb{R})} \le C\|f\|_{L^\infty(\Omega)} \quad \text{and} \quad \|u/\delta^{\alpha/2}\|_{C^\sigma(\bar\Omega)} \le C\|f\|_{L^\infty(\Omega)}$$

*for some positive constant $C = C(\Omega, \alpha)$.*

In particular, this result says that if $f \in L^\infty(\Omega)$, then

$$|u(x)| \le C\delta(x)^{\alpha/2} \quad \text{for all } x \in \bar\Omega. \tag{2.13}$$

**Lemma 2.3.** [26, Proposition 1.4] *Let $\beta > 0$ be such that neither $\beta$ nor $\beta+\alpha$ is an integer. Let $f \in C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R})$ be a solution of (1.1). Then, $u \in C^{\beta+\alpha}(\Omega)$ and*

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)}\right)$$

*for some positive constant $C = C(\Omega, \alpha, \beta)$.*

By definition of $\delta$-dependent Hölder norms, we have following results obviously.

**Lemma 2.4.** *Let $\beta = 4-\alpha+\gamma$ with $0 < \gamma < \alpha-1$. Assume that $f \in L^\infty(\Omega)\cap C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and $u$ be a solution of (1.1). Then*

$$\begin{aligned} &|u^{(l)}(x)| \le C\delta(x)^{\alpha/2-l} \quad \text{for } x \in \Omega \text{ and } l = 0,1,2,3,4,\\ &|f^{(l)}(x)| \le C\delta(x)^{-\alpha/2-l} \text{ for } x \in \Omega \text{ and } l = 0,1,2, \end{aligned}$$

*for some positive constant $C = C(\Omega, \alpha, \beta, f)$.*

*Proof.* Our hypotheses imply that $2 < \beta < 3$, and $4 < \beta+\alpha < 5$. By Lemma 2.3, we have

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)}\right).$$

By Definition 2.1 and Lemma 2.2, it yields

$$\sum_{l=1}^{4} \sup_{x\in\Omega} \left(\delta(x)^{l-\alpha/2}\left|u^{(l)}(x)\right|\right) \le C\left(\|f\|_{L^\infty(\Omega)} + \|f\|_\beta^{(\alpha/2)}\right),$$

which is desired result $l = 1,2,3,4$. The case $l = 0$ is covered by (2.13).

The second inequality can be obtained by Definition 2.1, namely,

$$\sum_{l=0}^{2} \sup_{x\in\Omega} \left(\delta(x)^{l+\alpha/2}|f^{(l)}(x)|\right) \le \|f\|_\beta^{(\alpha/2)}.$$

The proof is completed. □

2.3. **Main results.** The main results of this paper consist of the following theorems, which will be proved in Section 3 and Section 4, respectively.

**Theorem 2.5** (Local Truncation Error)**.** *Let $\alpha \in (1,2)$ and $f \in L^\infty(\Omega) \cap C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, where $\beta = 4-\alpha+\gamma$ with $0<\gamma<\alpha-1$. Then,*

$$\begin{aligned}|\tau_i| &= |-D_h^\alpha \Pi_h u(x_i) - f(x_i)| \\ &\le C h^{\min\{\frac{r\alpha}{2},2\}} \delta(x_i)^{-\alpha} + C(r-1)h^2(T-\delta(x_i)+h_N)^{1-\alpha}\end{aligned}$$

*for some positive constant $C = C(\Omega,\alpha,\beta,r,f)$.*

**Theorem 2.6** (Global Error)**.** *Let $\alpha \in (1,2)$ and $f \in L^\infty(\Omega) \cap C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, where $\beta = 4-\alpha+\gamma$ with $0<\gamma<\alpha-1$. Let $u_i$ be the approximate solution of $u(x_i)$ computed by the discretization scheme* (2.5)*. Then,*

$$\max_{1\le i\le 2N-1} |u_i - u(x_i)| \le C h^{\min\{\frac{r\alpha}{2},2\}}$$

*for some positive constant $C = C(\Omega,\alpha,\beta,r,f)$.*

## 3. Local Truncation Error

For convenience, we use the notation $\simeq$ , where $x \simeq y$ means that $C_1 x \le y \le C_2 x$ for some positive constants $C_1$ and $C_2$ independent of $h$.

For $1 \le j \le 2N$, we define the combination of adjacent grid points as

$$y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \quad \theta \in (0,1). \tag{3.1}$$

Then, using the definition of grid points $\{x_j\}$ in (2.1), it follows that

**Lemma 3.1.** *Let $h = \frac{1}{N}$ and $\delta(x_j)$ be defined by* (2.12)*. Then we have*

$$\begin{aligned}&h_j \simeq h_{j+1} \simeq h\delta(x_j)^{1-1/r}, \quad 1 \le j \le 2N-1,\\ &\delta(x_j) \simeq \delta(x_{j+1}) \simeq \delta(y_{j+1}^\theta), \;\; 1 \le j \le 2N-2.\end{aligned}$$

We next give a detailed analysis of the local truncation error.

3.1. **Proof of Theorem 2.5.** The local truncation error (2.8) can be expressed by

$$\begin{aligned}\tau_i &= -D_h^2 I^{2-\alpha}\Pi_h u(x_i) + \frac{d^2}{dx^2} I^{2-\alpha}u(x_i)\\ &= D_h^2 I^{2-\alpha}\left(u - \Pi_h u\right)(x_i) - \left(D_h^2 - \frac{d^2}{dx^2}\right) I^{2-\alpha} u(x_i).\end{aligned} \tag{3.2}$$

We estimate each component of this partition.

**Theorem 3.2.** *There exists a constant $C$ such that*

$$\left|\left(D_h^2 - \frac{d^2}{dx^2}\right) I^{2-\alpha}u(x_i)\right| \le Ch^2\delta(x_i)^{-\alpha/2-2/r}. \tag{3.3}$$

*Proof.* Since $f \in C^2(\Omega)$ and $-\frac{d^2}{dx^2} I^{2-\alpha}u(x) = f(x)$ for $x \in \Omega$, it implies $I^{2-\alpha}u \in C^4(\Omega)$. From Lemma A.1 in Appendix A, we have for $1 \le i \le 2N-1$,

$$\begin{aligned}&-\left(D_h^2 - \frac{d^2}{dx^2}\right) I^{2-\alpha}u(x_i) = \frac{h_{i+1}-h_i}{3} f'(x_i)\\ &\quad + \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy + \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}} f''(y)\frac{(y-x_{i+1})^3}{3!}dy\right).\end{aligned}$$

According to Lemmas 2.4, B.1 and B.2, the desired result is obtained. □

Now we consider the first term of the local truncation error in (3.2), which we denote for simplicity

$$R_i := D_h^2 I^{2-\alpha}(u - \Pi_h u)(x_i), \quad 1 \le i \le 2N-1. \tag{3.4}$$

We have derived the following results concerning the estimation of $R_i$ including Theorems 3.3 and 3.4, which will be demonstrated in Subsection 3.3.

**Theorem 3.3.** *For $1 \le i < N/2$, there exists a constant $C$ such that*

$$|R_i| \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

**Theorem 3.4.** *For $N/2 \le i \le N$, there exists a constant $C$ such that*

$$|R_i| \le C(r-1)h^2(T - x_i + h_N)^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

*Remark* 3.5. And for $N < i \le 2N-1$, observe first that the mesh (2.1) is symmetric about $x = T$ (i.e., $x = x_i$ is a mesh point if and only if $x = 2T - x_i = x_{2N-i}$ is a mesh point), and the a priori derivative bounds of Lemma 2.4 are also symmetric about $x = T$. But the locations of the mesh points and these bounds on derivatives are the only ingredients used in the analysis of the case $1 \le i \le N$. Thus, one can define $\tilde{u}(x) = u(2T - x)$, and now, the truncation error of $u(x)$ at $x = x_i$ for $i = N+1, N+2, ..., 2N-1$ is exactly the same as the truncation error of $\tilde{u}(x)$ at $x = x_i$ for $i = N-1, N-2, ...1$, which can be handled in exactly the same way as the truncation error analysis of $u(x)$ for $i = 1, 2, ..., N-1$. Transforming back via $x \mapsto 2T - x$, we get the desired result for $i = N+1, N+2, ..., 2N-1$. This technique will be used several times.

Combine Theorems 3.2 to 3.4 and remark 3.5, and for $1 \le i \le N$, we have

$$h^2 x_i^{-\alpha/2-2/r} \le T^{\alpha/2-2/r} h^{\min\{\frac{r\alpha}{2},2\}} x_i^{-\alpha},$$
$$h^{r\alpha/2+r} x_i^{-1-\alpha} \le T^{-1} h^{r\alpha/2} x_i^{-\alpha},$$
$$h^r x_i^{-1}\ln(i) = T^{-1}\frac{\ln(i)}{i^r} \le T^{-1}, \quad h^r \ln(N) = \frac{\ln(N)}{N^r} \le 1,$$

the proof of Theorem 2.5 completed.

3.2. **Grid mapping functions.** In this subsection, we offer an overview of the framework for estimating $R_i$, where we introduce the *natural-skew ordering* and *grid mapping functions*.

From (1.3) and (3.4), we know that

$$I^{2-\alpha}(u - \Pi_h u)(x_i) = \sum_{j=1}^{2N} \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) K(x_i - y) dy = \sum_{j=1}^{2N} T_{ij} \tag{3.5}$$

with

$$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) K(x_i - y) dy, \quad i = 0, \cdots, 2N, \; j = 1, \cdots, 2N. \tag{3.6}$$

To estimate $R_i$ more precisely, we define the *vertical difference quotients* of $T_{ij}$

$$V_{ij} = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} T_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right), \tag{3.7}$$

and the *skew difference quotients* of $T_{ij}$

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} T_{i-1,j-1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j+1} \right). \tag{3.8}$$

From (3.4), (3.5) and (3.6), we have

$$R_1 = \sum_{j=1}^{3} V_{1,j} + \sum_{j=4}^{2N} V_{1,j} \quad \text{and} \quad R_2 = \sum_{j=1}^{4} V_{2,j} + \sum_{j=5}^{2N} V_{2,j}. \tag{3.9}$$

Moreover, using (3.4)-(3.8), we can express $R_i$ based on the natural-skew ordering, as shown in Figure 1:

$$R_i = I_1 + I_2 + I_3 + I_4 + I_5 \quad \text{for} \quad 3 \le i \le N. \tag{3.10}$$

Here,

$$I_1 = \sum_{j=1}^{k-1} V_{ij}, \quad I_3 = \sum_{j=k+1}^{m-1} S_{ij}, \quad I_5 = \sum_{j=m+1}^{2N} V_{ij} \quad \text{for} \quad k = \lceil i/2 \rceil,$$

and

$$\begin{aligned} I_2 &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right), \\ I_4 &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,m} + T_{i-1,m-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,m} \right) \end{aligned}$$

with

$$m = \begin{cases} 2i, & 3 \le i < N/2, \\ 2N - \lceil N/2 \rceil + 1, & N/2 \le i \le N. \end{cases} \tag{3.11}$$

Noted that $I_1$ and $I_5$ along with $V_{ij}$ as defined in (3.7), represent natural (vertical) ordering, while $I_3$, along with $S_{ij}$ as defined in (3.8), represents skew ordering, which is referred to as natural-skew ordering here.

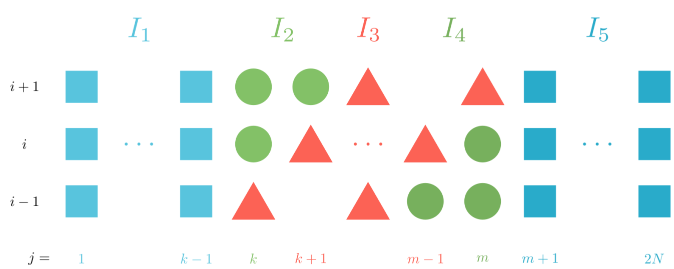


FIGURE 1. Natural-Skew ordering of $R_i$.

The complexity in estimating $S_{ij}$ in (3.8) lies in the fact that the integral domains for $T_{i-1,j-1}, T_{i,j}$ and $T_{i+1,j+1}$ are distinct. We first normalize $T_{ij}$ to the unit interval.

**Lemma 3.6.** *For any $y \in (x_{j-1}, x_j)$, there exits*

$$\begin{aligned}
T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) K(x_i - y) dy \\
&= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) K(x_i - y_j^\theta) h_j d\theta \\
&= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) K(x_i - y_j^\theta) d\theta \\
&\quad + \int_0^1 \frac{\theta(1-\theta)}{3!} h_j^4 K(x_i - y_j^\theta) \left(\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)\right) d\theta
\end{aligned}$$

*with $\eta_{j1}^\theta \in (x_{j-1}, y_j^\theta), \eta_{j2}^\theta \in (y_j^\theta, x_j)$.*

*Proof.* By (3.6) and Lemma A.2, the desired result is obtained. □

To estimate the local truncation error more concisely, we construct the following grid mapping functions.

**Definition 3.7.** For $1 \le i, j \le 2N - 1$, we define the grid mapping functions

$$y_{i,j}(x) = \begin{cases}
(x^{1/r} + Z_{j-i})^r & i < N, j < N, \\
\dfrac{x^{1/r} - Z_i}{Z_1} h_N + x_N & i < N, j = N, \\
2T - (Z_{2N-(j-i)} - x^{1/r})^r & i < N, j > N, \\
\left(\dfrac{Z_1}{h_N}(x - x_N) + Z_j\right)^r & i = N, j < N, \\
x & i = N, j = N, \\
2T - \left(\dfrac{Z_1}{h_N}(2T - x - x_N) + Z_{2N-j}\right)^r & i = N, j > N, \\
(Z_{2N+j-i} - (2T - x)^{1/r})^r & i > N, j < N, \\
\dfrac{Z_{2N-j} - (2T-x)^{1/r}}{Z_1} h_N + x_N & i > N, j = N, \\
2T - ((2T - x)^{1/r} - Z_{j-i})^r & i > N, j > N
\end{cases} \tag{3.12}$$

with $Z_j := T^{1/r} \frac{j}{N}$.

Let us further define

$$h_{i,j}(x) = y_{i,j}(x) - y_{i,j-1}(x), \tag{3.13}$$

$$y_{i,j}^\theta(x) = (1-\theta) y_{i,j-1}(x) + \theta y_{i,j-1}(x), \quad \theta \in (0,1), \tag{3.14}$$

$$P_{i,j}^\theta(x) = (h_{i,j}(x))^3 K(x - y_{i,j}^\theta(x)) u''(y_{i,j}^\theta(x)), \tag{3.15}$$

$$Q_{i,j,l}^\theta(x) = (h_{i,j}(x))^l K(x - y_{i,j}^\theta(x)), \quad l = 3, 4. \tag{3.16}$$

Then, we can check that

$$\begin{aligned}
&y_{i,j}(x_{i-1}) = x_{j-1}, \quad y_{i,j}(x_i) = x_j, \quad y_{i,j}(x_{i+1}) = x_{j+1}, \\
&h_{i,j}(x_{i-1}) = h_{j-1}, \quad h_{i,j}(x_i) = h_j, \quad h_{i,j}(x_{i+1}) = h_{j+1}, \\
&y_{i,j}^\theta(x_{i-1}) = y_{j-1}^\theta, \quad y_{i,j}^\theta(x_i) = y_j^\theta, \quad y_{i,j}^\theta(x_{i+1}) = y_{j+1}^\theta.
\end{aligned} \tag{3.17}$$

Now, we can rewrite $T_{ij}$ by (3.15) in (3.6) as

$$
\begin{aligned}
T_{ij} = & \int_0^1 -\frac{\theta(1-\theta)}{2} P^{\theta}_{i,j}(x_i) d\theta \\
& + \int_0^1 \frac{\theta(1-\theta)}{3!} Q^{\theta}_{i,j,4}(x_i) \left[\theta^2 u'''(\eta^{\theta}_{j,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{j,2})\right] d\theta.
\end{aligned} \tag{3.18}
$$

From (2.6), (3.8) and (3.18), for $1 \le i \le 2N-1$, $2 \le j \le 2N-1$, we have

$$
\begin{aligned}
S_{ij} = & \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P^{\theta}_{i,j}(x_i) d\theta \\
& + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q^{\theta}_{i,j,4}(x_{i+1}) u'''(\eta^{\theta}_{j+1,1}) - Q^{\theta}_{i,j,4}(x_i) u'''(\eta^{\theta}_{j,1})}{h_{i+1}} \right) d\theta \\
& - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q^{\theta}_{i,j,4}(x_i) u'''(\eta^{\theta}_{j,1}) - Q^{\theta}_{i,j,4}(x_{i-1}) u'''(\eta^{\theta}_{j-1,1})}{h_i} \right) d\theta \\
& - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q^{\theta}_{i,j,4}(x_{i+1}) u'''(\eta^{\theta}_{j+1,2}) - Q^{\theta}_{i,j,4}(x_i) u'''(\eta^{\theta}_{j,2})}{h_{i+1}} \right) d\theta \\
& + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q^{\theta}_{i,j,4}(x_i) u'''(\eta^{\theta}_{j,2}) - Q^{\theta}_{i,j,4}(x_{i-1}) u'''(\eta^{\theta}_{j-1,2})}{h_i} \right) d\theta.
\end{aligned} \tag{3.19}
$$

The derivatives of the grid mapping functions are calculated as follows.

**Lemma 3.8.** *For $1 \le i, j \le 2N-1$, there exist,*

$$
y'_{i,j}(x) = \begin{cases}
y_{i,j}^{1-1/r}(x) x^{1/r-1}, & i < N, j < N, \\
\frac{h_N}{rZ_1} x^{1/r-1}, & i < N, j = N, \\
(2T - y_{i,j}(x))^{1-1/r} x^{1/r-1}, & i < N, j > N, \\
y_{i,j}^{1-1/r}(x) \frac{rZ_1}{h_N}, & i = N, j < N, \\
1 & i = N, j = N,
\end{cases}
$$

*and*

$$
y''_{i,j}(x) = \frac{1-r}{r} \begin{cases}
y_{i,j}^{1-2/r}(x) x^{1/r-2} Z_{j-i}, & i < N, j < N, \\
\frac{h_N}{rZ_1} x^{1/r-2}, & i < N, j = N, \\
(2T - y_{i,j}(x))^{1-2/r} x^{1/r-2} Z_{2N-j+i}, & i < N, j > N, \\
-y_{i,j}^{1-2/r}(x) \left(\frac{rZ_1}{h_N}\right)^2, & i = N, j < N, \\
0, & i = N, j = N.
\end{cases}
$$

*Proof.* The desired results can be obtained by Definition 3.7 directly. □

The following lemmas about the grid mapping functions will be used in next subsection. They are proved in Appendix C.

**Lemma 3.9.** *For any $\xi \in (x_{i-1}, x_{i+1})$, $2 \le i, j \le 2N-2$, there exist*

$$\xi \simeq x_i, \quad \delta(y_{i,j}(\xi)) \simeq \delta(x_j), \quad h_{i,j}(\xi) \simeq h_j,$$
$$|y_{i,j}(\xi) - \xi| \simeq |x_j - x_i|, \quad |y_{i,j-1}(\xi) - \xi| \simeq |x_{j-1} - x_i|,$$
$$|y_{i,j}^{\theta}(\xi) - \xi| = (1-\theta)|y_{i,j-1}(\xi) - \xi| + \theta |y_{i,j}(\xi) - \xi| \simeq |y_j^{\theta} - x_i|.$$

**Lemma 3.10.** *For any $\xi \in (x_{i-1}, x_{i+1})$, $2 \le i \le N, 2 \le j \le 2N-2$, there exist*

$$|h_{i,j}'(\xi)| \le C(r-1) Z_1 x_i^{1/r-1} \delta(x_j)^{1-2/r} \le C(r-1) h_j x_i^{1/r-1} \delta(x_j)^{-1/r},$$
$$|(y_{i,j}(\xi) - \xi)'| \le C x_i^{-1} |x_j - x_i|.$$

**Lemma 3.11.** *For any $\xi \in (x_{i-1}, x_{i+1})$, $2 \le i \le N, 2 \le j \le 2N-2$, there exist*

$$|y_{i,j}''(\xi)| \le C(r-1) \begin{cases} x_j^{-1/r} x_i^{1/r-2} |x_j - x_i|, & i < N, j < N, \\ x_N^{1-1/r} x_i^{1/r-2}, & i < N, j = N, \\ \delta(x_j)^{1-2/r} x_i^{1/r-2} x_N^{1/r}, & i < N, j > N, \\ \delta(x_j)^{1-2/r} x_N^{2/r-2}, & i = N, j \ne N, \\ 0 & i = N, j = N. \end{cases}$$

*For $2 \le i \le N, 3 \le j \le 2N-2$, there exist*

$$|h_{i,j}''(\xi)| \le C(r-1) \begin{cases} Z_1 x_i^{1/r-2} x_j^{-2/r} (|x_j - x_i| + x_j), & i < N, j < N, \\ x_i^{1/r-2} x_N^{1-1/r}, & i < N, j = N, N+1, \\ Z_1 x_i^{1/r-2} \delta(x_j)^{1-3/r} x_N^{1/r}, & i < N, j > N+1, \\ Z_1 x_N^{2/r-2} \delta(x_j)^{1-3/r}, & i = N, j \ne N, N+1, \\ x_N^{-1}, & i = N, j = N, N+1. \end{cases}$$

**Lemma 3.12.** *Let $P_{i,j}^{\theta}(x_i)$ be defined by* (3.15) *and the difference quotient operator $D_h^2$ be defined by* (2.6). *Then we have*
Case 1. *For $3 \le i < N, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, there exists*

$$|D_h^2 P_{i,j}^{\theta}(x_i)| \le C h_j^3 |y_j^{\theta} - x_i|^{1-\alpha} x_i^{\alpha/2-4}.$$

Case 2. *For $N/2 \le i \le N$, $j = N, N+1$, there exists*

$$|D_h^2 P_{i,j}^{\theta}(\xi)| \le C h_j^3 |y_j^{\theta} - x_i|^{1-\alpha} + C(r-1) h_j^2 \Big( |y_j^{\theta} - x_i|^{1-\alpha} + h_j |y_j^{\theta} - x_i|^{-\alpha} \Big).$$

Case 3. *For $N/2 \le i \le N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil$, there exists*

$$|D_h^2 P_{i,j}^{\theta}(\xi)| \le C h_j^3 \Big( |y_j^{\theta} - x_i|^{1-\alpha} + (r-1) |y_j^{\theta} - x_i|^{-\alpha} \Big).$$

**Lemma 3.13.** *Let $Q_{i,j,l}^{\theta}(x_i)$ be defined by* (3.16). *Then we have for $2 \le i \le N$, $2 \le j \le 2N-2$, $l = 3, 4$, there exist*

$$\left| \frac{Q_{i,j,l}^{\theta}(x_{i+1}) u^{(l-1)}(\eta_{j+1}^{\theta}) - Q_{i,j,l}^{\theta}(x_i) u^{(l-1)}(\eta_j^{\theta})}{h_{i+1}} \right|$$
$$\le C h_j^l |y_j^{\theta} - x_i|^{1-\alpha} x_i^{-1} \delta(x_j)^{\alpha/2-l+1-1/r} (x_i^{1/r} + \delta(x_j)^{1/r}),$$

*and*

$$\left| \frac{Q_{i,j,l}^{\theta}(x_i) u^{(l-1)}(\eta_j^{\theta}) - Q_{i,j,l}^{\theta}(x_{i-1}) u^{(l-1)}(\eta_{j-1}^{\theta})}{h_i} \right|$$
$$\le C h_j^l |y_j^{\theta} - x_i|^{1-\alpha} x_i^{-1} \delta(x_j)^{\alpha/2-l+1-1/r} (x_i^{1/r} + \delta(x_j)^{1/r})$$

*with* $\eta_j^\theta \in (x_{j-1}, x_j)$.

3.3. **Error analysis of** $R_i$**.** In this subsection, we estimate the first term of the local truncation error $R_i$ in (3.4) through (3.9) and (3.10). We denote

$$K_y(x) := K(x-y) = \frac{\kappa_\alpha}{\Gamma(2-\alpha)}|x-y|^{1-\alpha}, \quad 1<\alpha<2, \tag{3.20}$$

where the kernel function $K(x)$ is given in (1.3) and $\kappa_\alpha$ is given in (2.11).

**Lemma 3.14.** *Let* $I_5 = \sum_{j=m+1}^{2N} V_{ij}$ *be defined by* (3.10)*. Then we have*
Case 1. *For* $1 \le i < N/2$ *and* $m = \max\{2i, 3\}$*, there exists*

$$\sum_{j=m+1}^{2N} |V_{ij}| \le Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1>0, \\ Ch^2\ln(N), & \alpha/2-2/r+1=0, \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0. \end{cases}$$

Case 2. *For* $N/2 \le i \le N$ *and* $m = 2N - \lceil \frac{N}{2} \rceil + 1$*, there exists*

$$\sum_{j=m+1}^{2N} |V_{ij}| \le \begin{cases} Ch^2, & \alpha/2-2/r+1>0, \\ Ch^2\ln(N), & \alpha/2-2/r+1=0, \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0. \end{cases}$$

*Proof.* For $1 \le i < N/2$, $m+1 \le j \le 2N$ with $m = \max\{2i, 3\}$, using (3.6), (3.7), (3.20), Lemmas A.3 and B.3, we have

$$\begin{aligned} |V_{ij}| &= \left| \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \right| \\ &\le Ch^2 \int_{x_{j-1}}^{x_j} \delta(y)^{\alpha/2-2/r} |x_i - y|^{-1-\alpha} dy. \end{aligned}$$

Since $y \ge x_{j-1} \ge x_{2i}$, $y - x_i \simeq y$, and $x_i \simeq x_{2i}$, it yields

$$\begin{aligned} \sum_{j=m+1}^{N} |V_{ij}| &\le Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\ &= \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\ &\le Ch^2 x_i^{-\alpha/2-2/r}. \end{aligned}$$

On the other hand, since $y - x_i \simeq T$ if $y \ge x_N = T$, there exist

$$\begin{aligned} \sum_{j=N+1}^{2N-1} |V_{ij}| &\le CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r} dy \\ &\le \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r}\, h^2, & \alpha/2-2/r+1>0, \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2-2/r+1=0, \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r}\, h^{r\alpha/2+r}, & \alpha/2-2/r+1<0. \end{cases} \end{aligned}$$

Finally, by Lemma A.4, one has

$$|V_{i,2N}| \le CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}.$$

Then, the desired result in Case 1 is obtained. We can similarly prove for Case 2, the details are omitted here. ∎

Immediately, we can calculate $R_1, R_2$ from (3.9).

**Lemma 3.15.** *For $i = 1, 2$, we have*

$$|R_i| \le Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

*Proof.* According to (3.9), Lemmas 3.14 and B.4, the desired result is obtained. □

For $R_i$ with $3 \le i \le N$, the terms $\{I_1, I_2, I_3, I_4\}$ in (3.10) remain to be estimated.

**Lemma 3.16.** *Let $I_1 = \sum_{j=1}^{k-1} V_{ij}$ be defined by* (3.10)*. Then we have, for $3 \le i \le N, k = \lceil \frac{i}{2} \rceil$,*

$$\sum_{j=1}^{k-1} |V_{ij}| \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

*Proof.* According to (3.7), Lemmas A.4 and B.3, it yields

$$|V_{i1}| \le C \int_0^{x_1} x_1^{\alpha/2} |x_i - y|^{-1-\alpha} dy \simeq x_1^{\alpha/2+1} x_i^{-1-\alpha} = T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha}.$$

Using Lemma A.3, Lemma B.3 and $y \le x_{k-1} < 2^{-r} x_i$, $x_i - y \simeq x_i$, we have

$$|V_{ij}| \le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} x_i^{-1-\alpha} dy, \quad 2 \le j \le k-1,$$

and

$$\sum_{j=2}^{k-1} |V_{ij}| \le Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy.$$

Moreover we can check that

$$\int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \le \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0, \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2 - 2/r + 1 = 0, \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

The proof is completed. □

Subsequently, we turn our attention to $I_3 = \sum_{j=k+1}^{m-1} S_{ij}$ with $m = 2i$ for $3 \le i < N/2$ and $m = 2N - \lceil N/2 \rceil + 1$ for $N/2 \le i \le N$ in (3.11).

**Lemma 3.17.** *Let $I_3 = \sum_{j=k+1}^{m-1} S_{ij}$ be defined by* (3.10)*. Then we have*
Case 1. *For $N/2 \le i \le N$, $m = 2N - \lceil N/2 \rceil + 1$, there exist*

$$|S_{ij}| \le C(h^3 + (r-1)h^2)(T - x_i + h_N)^{1-\alpha}, \quad j = N, N+1,$$

*and*

$$\sum_{j=N+2}^{m-1} |S_{ij}| \le Ch^2 + C(r-1)h^2 (T - x_i + h_N)^{1-\alpha}.$$

Case 2. *For* $3 \le i \le N-1$, $k = \lceil \frac{i}{2} \rceil$, *there exist*

$$\sum_{j=k+1}^{\min\{m-1,N-1\}} |S_{ij}| \le Ch^2 x_i^{-\alpha/2-2/r},$$

*and*

$$\sum_{j=\lceil \frac{N}{2} \rceil+1}^{N-1} |S_{Nj}| \le Ch^2 + C(r-1)h^2 h_N^{1-\alpha}.$$

*Proof.* Case 1: From (3.19), using $\theta(1-\theta)h_j \le |y_j^\theta - x_i|$, Lemmas 3.1, 3.12 and 3.13, it yields

$$|S_{ij}| \le C(h_j^3 + (r-1)h_j^2) \int_0^1 |y_j^\theta - x_i|^{1-\alpha} d\theta, \quad j = N, N+1$$

with

$$\int_0^1 |y_j^\theta - x_i|^{1-\alpha} dy \simeq (|x_j - x_i| + h_N)^{1-\alpha}.$$

On the other hand, for $j \ge N+2$, $x_i \simeq x_j \simeq T$, we have

$$\begin{aligned} |S_{ij}| &\le Ch_j^2 \int_0^1 \left( |y_j^\theta - x_i|^{1-\alpha} + (r-1)|y_j^\theta - x_i|^{-\alpha} \right) h_j d\theta \\ &\le Ch^2 \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy. \end{aligned}$$

It implies that

$$\begin{aligned} \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} |S_{ij}| &= Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil \frac{N}{2} \rceil}} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy \\ &\le Ch^2 \left( T^{2-\alpha} + (r-1)(T - x_i + h_N)^{1-\alpha} \right). \end{aligned}$$

Case 2: for $3 \le i \le N-1$, $k+1 \le j \le \min\{m-1, N-1\}$, using Lemmas 3.1, 3.12 and 3.13, $x_i \simeq x_j$ and $h_i \simeq h_j$, we have

$$\begin{aligned} |S_{ij}| &\le Ch_j^2 x_i^{\alpha/2-4} \int_0^1 |y_j^\theta - x_i|^{1-\alpha} h_j d\theta \\ &= Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} dy, \end{aligned}$$

and

$$\begin{aligned} \sum_{k+1}^{\min\{2i-1,N-1\}} |S_{ij}| &\le Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_k}^{x_{\min\{2i-1,N-1\}}} |y - x_i|^{1-\alpha} dy \\ &\le Ch^2 x_i^{\alpha/2-2-2/r} x_i^{2-\alpha} = Ch^2 x_i^{-\alpha/2-2/r}. \end{aligned}$$

We can similarly prove the last inequality by Case 1. The proof is completed. $\square$

Finally, we focus our error analysis on the terms $I_2$ and $I_4$.

**Lemma 3.18.** *Let* $I_2, I_4$ *be defined by* (3.10). *Then we have*
Case 1. *For* $3 \le i \le N$, $k = \lceil \frac{i}{2} \rceil$, *there exists*

$$I_2 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \le Ch^2 x_i^{-\alpha/2-2/r}.$$

Case 2. *For $3 \le i < N/2$, $m = 2i$, there exists*

$$I_4 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i}(T_{i-1,2i} + T_{i-1,2i-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \le Ch^2 x_i^{-\alpha/2-2/r}.$$

Case 3. *For $N/2 \le i \le N$, $m = N - \lceil \frac{N}{2} \rceil + 1$, there exists*

$$I_4 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i}(T_{i-1,m} + T_{i-1,m-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,m} \right) \le Ch^2.$$

*Proof.* Since

$$\begin{aligned} &\frac{1}{h_{i+1}} \left(T_{i+1,k} + T_{i+1,k+1}\right) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \\ &= \frac{1}{h_{i+1}} \left(T_{i+1,k} - T_{i,k}\right) + \frac{1}{h_{i+1}} \left(T_{i+1,k+1} - T_{i,k}\right) + \left( \frac{1}{h_{i+1}} - \frac{1}{h_i} \right) T_{i,k}. \end{aligned} \tag{3.21}$$

According to $x_i - x_k \simeq x_i \simeq x_k$, Lemmas 3.1, A.3 and B.3, we have

$$\begin{aligned} \frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) &= \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y)) D_h K_y(x_i) dy \\ &\le Ch_k^2 x_k^{\alpha/2-2}\ h_k |x_i - x_k|^{-\alpha} \le Ch^2 x_i^{-\alpha/2-2/r} h_k. \end{aligned}$$

From Lemmas 3.6 and A.2 and (3.16), we can obtain

$$\frac{1}{h_{i+1}} \left(T_{i+1,k+1} - T_{i,k}\right) = \int_0^1 \frac{\theta(\theta-1)}{2} \frac{Q^{\theta}_{i,k;3}(x_{i+1}) u''(\eta^{\theta}_{k+1}) - Q^{\theta}_{i,k;3}(x_i) u''(\eta^{\theta}_{k})}{h_{i+1}} d\theta$$

with $\eta_k^\theta \in (x_{k-1}, x_k)$ and $\eta_{k+1}^\theta \in (x_k, x_{k+1})$. Using Lemmas 3.1 and 3.13, we have

$$\frac{1}{h_{i+1}} |T_{i+1,k+1} - T_{i,k}| \le Ch^2 x_i^{-\alpha/2-2/r} h_k.$$

For the third term in (3.21), using $h_i \simeq h_k$, Lemmas 3.1, A.3 and B.1, it yields

$$\begin{aligned} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,k} &\le C(r-1) h_i^{-2} h^2 x_i^{1-2/r}\ h_k^3 x_k^{\alpha/2-2} |x_k - x_i|^{1-\alpha} \\ &\le C(r-1) h^2 x_i^{-\alpha/2-2/r} h_k. \end{aligned}$$

Then, the desired result of Case 1 is obtained. The Case 2 and Case 3 for $I_4$ can be similarly proven as the way in Case 1; the details are omitted here. □

**Proof of Theorem 3.3.** For $1 \le i < N/2$ with $m = 2i$ in (3.10), combining Lemma 3.15, Lemma 3.16, Cases 1 and 2 of Lemma 3.18, Case 2 of Lemma 3.17 and Case 1 of Lemma 3.14, the proof is completed. □

**Proof of Theorem 3.4.** For $N/2 \le i \le N$ with $m = 2N - \lceil N/2 \rceil + 1$ in (3.10), we split $I_3$ as

$$I_3 = \sum_{j=k+1}^{m-1} S_{ij} = \sum_{j=k+1}^{N-1} S_{ij} + (S_{iN} + S_{i,N+1}) + \sum_{j=N+2}^{m-1} S_{ij}. \tag{3.22}$$

According to Lemma 3.16, Cases 1 and 3 of Lemma 3.18, Lemma 3.17 and Case 2 of Lemma 3.14, the desired result is obtained. □

## 4. Convergence analysis

We can now prove our main convergence result for Theorem 2.6.

4.1. **Some properties of the stiffness matrix.** In this subsection, we show some properties of the stiffness matrix $A$ defined by (2.10) and construct an appropriate right-preconditioner for the resulting matrix algebraic equation.

**Lemma 4.1.** *The stiffness matrix $A$ defined by* (2.10) *is strictly diagonally dominant, with positive entries on the main diagonal and negative off-diagonal entries. In particular, there exists a constant $C_A$ such that*

$$\sum_{j=1}^{2N-1} a_{ij} \geq C_A(x_i^{-\alpha} + (2T - x_i)^{-\alpha})$$

*with $C_A = \frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)} 2^{-r\alpha}$ .*

*Proof.* Let $C_j := \left(\frac{1}{h_j}, -\frac{1}{h_j} - \frac{1}{h_{j+1}}, \frac{1}{h_{j+1}}\right)$ and

$$D_{ij} := \begin{pmatrix} |x_{i-1} - x_{j-1}|^{3-\alpha} & |x_{i-1} - x_j|^{3-\alpha} & |x_{i-1} - x_{j+1}|^{3-\alpha} \\ |x_i - x_{j-1}|^{3-\alpha} & |x_i - x_j|^{3-\alpha} & |x_i - x_{j+1}|^{3-\alpha} \\ |x_{i+1} - x_{j-1}|^{3-\alpha} & |x_{i+1} - x_j|^{3-\alpha} & |x_{i+1} - x_{j+1}|^{3-\alpha} \end{pmatrix}.$$

From (2.11), we have

$$a_{ij} = \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{2}{h_i + h_{i+1}} C_i D_{ij} C_j^T$$

with $\mathrm{sign}(a_{ij}) = \mathrm{sign}(a_{ji})$. For $i = j$, there exists

$$a_{ii} = \frac{\kappa_\alpha}{\Gamma(4-\alpha)} \frac{4}{h_i h_{i+1}} \left(h_i^{2-\alpha} + h_{i+1}^{2-\alpha} - (h_i + h_{i+1})^{2-\alpha}\right) > 0,$$

where we use $1 + t^\theta > (1+t)^\theta$ with $t = \frac{h_{i+1}}{h_i}$ for $\theta \in (0, 1)$.

For $j = i - 1$, we can check that

$$\begin{aligned} C_i D_{i,i-1} C_{i-1}^T = \frac{1}{h_{i-1} h_i h_{i+1}} \Big( & h_{i+1} h_{i-1}^{3-\alpha} - (h_i + h_{i+1})(h_{i-1} + h_i)^{3-\alpha} \\ & + h_i (h_{i-1} + h_i + h_{i-1})^{3-\alpha} + (h_{i-1} + h_i)(h_i + h_{i+1}) h_i^{2-\alpha} \\ & - (h_{i-1} + h_i)(h_i + h_{i+1})^{3-\alpha} + h_{i-1} h_{i+1} h_i^{2-\alpha} + h_{i-1} h_{i+1}^{3-\alpha} \Big). \end{aligned}$$

Let $s = \frac{h_{i-1}}{h_i}$ and $t = \frac{h_{i+1}}{h_i}$. Then by Lemma B.6, we have

$$\begin{aligned} C_i D_{i,i-1} C_{i-1}^T = \frac{h_i^{3-\alpha}}{h_{i-1} h_{i+1}} \Big( & st(1 + s^{2-\alpha} + t^{2-\alpha}) + (1 + s + t)^{3-\alpha} \\ & - (1+s)(1+t)\left((1+s)^{2-\alpha} + (1+t)^{2-\alpha} - 1\right) \Big) > 0, \end{aligned}$$

which implies that

$$a_{i,i-1} = \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{2}{h_i + h_{i+1}} C_i D_{i,i-1} C_{i-1}^T < 0.$$

For $|i - j| \geq 2$, $x_{i+1} - y$, $x_i - y$ and $x_{i-1} - y$ have the same sign ($> 0$ or $< 0$) for $y \in (x_{j-1}, x_{j+1})$, it yields

$$\frac{h_i}{h_i + h_{i+1}} |x_{i+1} - y| + \frac{h_{i+1}}{h_i + h_{i+1}} |x_{i-1} - y| = |x_i - y|.$$

Since $x^{1-\alpha}$ is a convex function for $\alpha \in (1,2)$, by Jensen's inequality, we have

$$\frac{h_i}{h_i+h_{i+1}}|x_{i+1}-y|^{1-\alpha}+\frac{h_{i+1}}{h_i+h_{i+1}}|x_{i-1}-y|^{1-\alpha} > |x_i-y|^{1-\alpha},$$

which implies that $D_h^2 K_y(x_i) > 0$ by (2.6) and (3.20). Thus, from (2.11), we get

$$a_{ij} = -D_h^2 I^{2-\alpha}\phi_j(x_i) = -\int_{x_{j-1}}^{x_{j+1}} \phi_j(y) D_h^2 K_y(x_i)dy < 0.$$

We next prove that the stiffness matrix $A$ defined by (2.10) is strictly diagonally dominant. For the quadrature coefficients $\tilde{a}_{ij}$ in (2.11), we calculate that

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = g_0(x_i) + g_{2N}(x_i)$$

with

$$\begin{aligned}
g_0(x) &= \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x-x_0|^{3-\alpha} - |x-x_1|^{3-\alpha}}{h_1},\\
g_{2N}(x) &= \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N}-x|^{3-\alpha} - |x_{2N-1}-x|^{3-\alpha}}{h_{2N}}.
\end{aligned}$$

It implies that

$$\sum_{j=1}^{2N-1} a_{ij} = D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i).$$

For $i=1$, there exists

$$\begin{aligned}
D_h^2 g_0(x_1) &= \frac{2}{h_1+h_2}\left(\frac{1}{h_2}g_0(x_2) - (\frac{1}{h_1}+\frac{1}{h_2})g_0(x_1) + \frac{1}{h_1}g_0(x_0)\right)\\
&= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1+(2^r-1)^{3-\alpha}+2(2^r-1)-(2^r)^{3-\alpha}}{2^r(2^r-1)} x_1^{-\alpha}\\
&\geq \frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)} 2^{-r\alpha} x_1^{-\alpha},
\end{aligned}$$

since $h(t) = 2\left(1+(t-1)^{3-\alpha}+2(t-1)-t^{3-\alpha}\right) - (3-\alpha)(2-\alpha)(\alpha-1)(t^{2-\alpha}-t^{1-\alpha})$ is a increasing function for $t = 2^r \geq 1$ and $h(1) = 0$.

For $i \geq 2$, using Lemma A.1, it leads to

$$\begin{aligned}
D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1})\\
&= -\kappa_\alpha \frac{|\xi-x_0|^{1-\alpha} - |\xi-x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1}\\
&= \frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)}|\xi-\eta|^{-\alpha}, \quad \eta \in [x_0, x_1]\\
&\geq \frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)} 2^{-r\alpha} x_i^{-\alpha}.
\end{aligned}$$

Then we have $D_h^2 g_0(x_i) \geq C_A x_i^{-\alpha}$ with $C_A = \frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)} 2^{-r\alpha}$ for $i \geq 1$. We can similarly prove $D_h^2 g_{2N}(x_i) \geq C_A(2T-x_i)^{-\alpha}$. The proof is completed. □

Let us first introduce the quasi-preconditioner

$$G = \mathrm{diag}(\delta(x_1), ..., \delta(x_{2N-1})), \tag{4.1}$$

where $\delta(x)$ is defined by (2.12). Then we have

**Lemma 4.2.** *Let $\tilde{B} := AG$ and $A$ be defined by (2.10). Then the matrix $\tilde{B} = (\tilde{b}_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$ has positive entries on the main diagonal and negative off-diagonal entries. In particular, there exist constants $C_{\tilde{B}}, C_B$ such that*

$$\sum_{j=1}^{2N-1} \tilde{b}_{ij} \geq C_B(T-\delta(x_i)+h_N)^{1-\alpha} - C_{\tilde{B}}(x_i^{1-\alpha} + (2T-x_i)^{1-\alpha}).$$

*with $C_B = \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}$, $C_{\tilde{B}} = \frac{\kappa_\alpha}{\Gamma(2-\alpha)} 2^{r(\alpha-1)}$.*

*Proof.* From (2.11) and (4.1), it yields

$$\tilde{b}_{ij} = a_{ij}\delta(x_j) = -\frac{2}{h_i+h_{i+1}} \left( \frac{1}{h_{i+1}}\tilde{a}_{i+1,j} - (\frac{1}{h_i}+\frac{1}{h_{i+1}})\tilde{a}_{i,j} + \frac{1}{h_i}\tilde{a}_{i-1,j} \right) \delta(x_j).$$

Since $\delta(x) \equiv \Pi_h\delta(x) = \sum_{j=1}^{2N-1} \phi_j(x)\delta(x_j)$ by (2.3) and (2.12), from the definition of the quadrature coefficients $\tilde{a}_{ij}$ in (2.11), we have

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij}\delta(x_j) = \sum_{j=1}^{2N-1} I^{2-\alpha}\phi_j(x_i)\delta(x_j) = I^{2-\alpha}\delta(x_i) = -p(x_i) + q(x_i)$$

with

$$p(x) = \frac{2\kappa_\alpha}{\Gamma(4-\alpha)}|T-x|^{3-\alpha} \quad \text{and} \quad q(x) = \frac{\kappa_\alpha}{\Gamma(4-\alpha)}\left(x^{3-\alpha} + (2T-x)^{3-\alpha}\right).$$

Thus, we have

$$\sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} a_{ij}\delta(x_j) = D_h^2 p(x_i) - D_h^2 q(x_i).$$

For $i \neq N$, by Lemma A.1, it leads to

$$\begin{aligned} D_h^2 p(x_i) &= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}|T-\xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1}) \\ &\geq C_B(T-\delta(x_i)+h_N)^{1-\alpha} \ \text{ with } \ C_B = \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}, \end{aligned}$$

and for $i = N$, it yields

$$D_h^2 p(x_N) = \frac{4\kappa_\alpha}{\Gamma(4-\alpha)h_N^2} h_N^{3-\alpha} \geq C_B(T-\delta(x_N)+h_N)^{1-\alpha}.$$

We can similarly prove the following inequality.

$$D_h^2 q(x_i) \leq C_{\tilde{B}}(x_i^{1-\alpha} + (2T-x_i)^{1-\alpha}), \quad i = 1, \cdots, 2N-1$$

The proof is completed. $\square$

Noted that $\tilde{B} = AG$ in Lemma 4.2 is not diagonally dominant, e.g., $\sum_{j=1}^{2N-1} \tilde{b}_{ij} < 0$ if $x_i$ is near the boundary. We introduce the preconditioner $\lambda I + \mu G$ as following.

**Lemma 4.3.** *Let $B := A(\lambda I+\mu G)$ with $\lambda = 1+2^{r(\alpha-1)}T$, $\mu = (\alpha-1)2^{-r\alpha-1}$. Then the matrix $B = (b_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$ is strictly diagonally dominant, with positive entries on the main diagonal and negative off-diagonal entries. In particular, there exists*

$$\sum_{j=1}^{2N-1} b_{ij} \geq C_A \left( (x_i^{-\alpha} + (2T-x_i)^{-\alpha}) + (T-\delta(x_i)+h_N)^{1-\alpha} \right).$$

*Proof.* From Lemmas 4.1 and 4.2, we have

$$\begin{aligned}\sum_{j=1}^{2N-1} b_{ij} = & \sum_{j=1}^{2N-1} \left(\lambda a_{ij} + \mu \tilde{b}_{ij}\right) \\ & \geq \lambda C_A \left(x_i^{-\alpha} + (2T-x_i)^{-\alpha}\right) - \mu C_{\tilde{B}} 2T \left(x_i^{-\alpha} + (2T-x_i)^{-\alpha}\right) \\ & \quad + \mu C_B \left(T-\delta(x_i)+h_N\right)^{1-\alpha},\end{aligned}$$

with $\lambda = 1 + 2TC_{\tilde{B}}/C_B = 1 + 2^{r(\alpha-1)}T$ and $\mu = C_A/C_B = (\alpha-1)2^{-r\alpha-1}$. The proof is completed. $\square$

4.2. **Proof of Theorem 2.6.** Let $\epsilon_i = u(x_i) - u_i$ with $\epsilon_0 = \epsilon_{2N} = 0$. Subtracting (2.7) from (2.9), we get

$$A\epsilon = \tau, \tag{4.2}$$

where $\epsilon = [\epsilon_1, \epsilon_2, ..., \epsilon_{2N-1}]^T$ and $\tau = [\tau_1, \tau_2, ..., \tau_{2N-1}]^T$ with $\tau_i$ in (2.8).

Let $\lambda I + \mu G$ be the right-preconditioner and $B = A(\lambda I + \mu G)$ defined in Lemma 4.3. Then we can rewrite (4.2) as

$$B(\lambda I + \mu G)^{-1}\epsilon = \tau, \quad \text{i.e.} \quad \sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + \mu\delta(x_j)} = \tau_i.$$

Assume that

$$\left|\frac{\epsilon_{i_0}}{\lambda + \mu\delta(x_{i_0})}\right| = \max_{1\leq j\leq 2N-1} \left|\frac{\epsilon_j}{\lambda + \mu\delta(x_j)}\right|.$$

From Lemma 4.3 with $b_{ii} > 0$ and $b_{ij} < 0, i \neq j$, it yields

$$\begin{aligned}|\tau_{i_0}| &= \left|\sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + \mu\delta(x_j)}\right| \\ &\geq b_{i_0,i_0} \left|\frac{\epsilon_{i_0}}{\lambda + \mu\delta(x_{i_0})}\right| - \sum_{j\neq i_0} |b_{i_0,j}| \left|\frac{\epsilon_j}{\lambda + \mu\delta(x_j)}\right| \\ &\geq b_{i_0,i_0} \left|\frac{\epsilon_{i_0}}{\lambda + \mu\delta(x_{i_0})}\right| - \sum_{j\neq i_0} |b_{i_0,j}| \left|\frac{\epsilon_{i_0}}{\lambda + \mu\delta(x_{i_0})}\right| \\ &= \sum_{j=1}^{2N-1} b_{i_0,j} \left|\frac{\epsilon_{i_0}}{\lambda + \mu\delta(x_{i_0})}\right|.\end{aligned}$$

According to the above inequality, Theorem 2.5 and Lemma 4.3, we have

$$\left|\frac{\epsilon_i}{\lambda + \mu\delta(x_i)}\right| \leq \left|\frac{\epsilon_{i_0}}{\lambda + \mu\delta(x_{i_0})}\right| \leq \frac{|\tau_{i_0}|}{\sum_{j=1}^{2N-1} b_{i_0,j}} \leq Ch^{\min\{\frac{r\alpha}{2},2\}} + C(r-1)h^2.$$

Since $\lambda + \mu\delta(x_i) \le \lambda + \mu T$, we can derive

$$|\epsilon_i| \le C(\lambda + \mu T)h^{\min\{\frac{r\alpha}{2},2\}} \le C\left(1 + (2^{r(\alpha-1)} + (\alpha-1)2^{-r\alpha-1})T\right)h^{\min\{\frac{r\alpha}{2},2\}}.$$

The proof is completed.

*Remark* 4.4. Let $B = A(\lambda I + \mu G)$ with $\mu = 0$ in the proof of Theorem 2.6, which means that there is no preconditioning. Thus, from Theorem 2.5 and Lemma 4.1, we can only prove

$$|\epsilon_i| \le Ch^{\min\{\frac{r\alpha}{2},2\}} + C(r-1)h^{3-\alpha} \le Ch^{\min\{\frac{r\alpha}{2},3-\alpha\}}, \quad 1 < \alpha < 2,$$

which may suffer from a severe order reduction.

*Remark* 4.5 (singular source term). From Lemma 2.4, it follows that the source term $|f(x)| \le C\delta(x)^{-\alpha/2}$ could potentially be singular. Suppose that the bound of Lemma 2.4 is replaced by the more general weaker regularity bound, i.e.,

$$\begin{aligned}
&|u^{(l)}(x)| \le C\delta(x)^{\sigma-l} \quad \text{for } x \in \Omega \text{ and } l = 0,1,2,3,4,\\
&|f^{(l)}(x)| \le C\delta(x)^{\sigma-\alpha-l} \text{ for } x \in \Omega \text{ and } l = 0,1,2,
\end{aligned}$$

where $\sigma \in (0, \frac{\alpha}{2}]$ is fixed.

Similar to the performer in Theorems 2.5 and 2.6, it is easy to check the local truncation error

$$\begin{aligned}
|\tau_i| &= |-D_h^\alpha \Pi_h u(x_i) - f(x_i)|\\
&\le Ch^{\min\{r\sigma,2\}}\delta(x_i)^{-\alpha} + C(r-1)h^2(T - \delta(x_i) + h_N)^{1-\alpha},
\end{aligned}$$

and the global error

$$\max_{1\le i\le 2N-1} |u_i - u(x_i)| \le Ch^{\min\{r\sigma,2\}}.$$

## 5. Numerical experiments

We use the difference-quadrature scheme (2.10) to solve the fractional Laplacian boundary value problem (1.1) with both regular and singular source terms on the interval $\Omega = (0,1)$.

5.1. **Regular source term.** If $f \equiv 1$, the exact (Getoor) solution [17, 20, 25] of the problem (1.1) is

$$u(x) = \frac{2^{-\alpha}\Gamma(\frac{1}{2})}{\Gamma(1+\frac{\alpha}{2})\Gamma(\frac{1+\alpha}{2})}\left[x(1-x)\right]^{\frac{\alpha}{2}}, \quad x \in \Omega.$$

In the numerical experiments of this example, we measure the numerical errors by using the maximum nodal error (i.e., the discrete $L^\infty$ norm):

$$E^N := \max_{0\le i\le 2N} |u(x_i) - u_i|.$$

The rate of convergence of $E^N$ is computed in the usual way, viz.,

$$Rate^N = \log_2\left(\frac{E^{N/2}}{E^N}\right).$$

Tables 1 and 2 show that the difference-quadrature method (2.10) has convergence order $\mathcal{O}(h^{\min\{\frac{r\alpha}{2},2\}})$, which agrees exactly with Theorem 2.6.

Table 1. $r = 1$: maximum nodal errors showing convergence rate $\mathcal{O}(h^{\frac{\alpha}{2}})$

| $\alpha$ \ $N$ | 100 | 200 | 400 | 800 |
|---|---|---|---|---|
| 1.2 | 1.1269e-3 | 7.4281e-4 | 4.8986e-4 | 3.2311e-4 |
| | | 0.6013 | 0.6006 | 0.6003 |
| 1.5 | 2.4996e-4 | 1.4876e-4 | 8.8489e-5 | 5.2627e-5 |
| | | 0.7487 | 0.7494 | 0.7497 |
| 1.8 | 2.7320e-5 | 1.4829e-5 | 7.9970e-6 | 4.2989e-6 |
| | | 0.8815 | 0.8909 | 0.8955 |

Table 2. $r = \frac{4}{\alpha}$: maximum nodal errors showing convergence rate $\mathcal{O}(h^2)$

| $\alpha$ \ $N$ | 100 | 200 | 400 | 800 |
|---|---|---|---|---|
| 1.2 | 4.1583e-5 | 1.0628e-5 | 2.6919e-6 | 6.7824e-7 |
| | | 1.9682 | 1.9811 | 1.9888 |
| 1.5 | 2.0681e-5 | 5.3790e-5 | 1.3824e-6 | 3.5239e-7 |
| | | 1.9429 | 1.9601 | 1.9720 |
| 1.8 | 7.6424e-6 | 2.0649e-6 | 5.5008e-7 | 1.4495e-7 |
| | | 1.8880 | 1.9083 | 1.9240 |

5.2. **Singular source term.** We take the singular source term $f(x) = x^{\sigma-\alpha}$, $\sigma \in (0, \frac{\alpha}{2}]$ with $\sigma = 0.4$ in (1.1). Since the analytic solution is unknown, the convergence rate of the numerical results is computed by

$$Rate^N = \log_2\left(\frac{E^{N/2}}{E^N}\right) \quad \text{with} \quad E^N = \max_{0\le i\le N} |u_i^{N/2} - u_{2i}^N|.$$

Table 3. $r = 1$:maximum nodal errors showing convergence rate $\mathcal{O}(h^{\sigma})$

| $\alpha$ \ $N$ | 100 | 200 | 400 | 800 |
|---|---|---|---|---|
| 1.2 | 2.9193e-2 | 2.2619e-2 | 1.7435e-2 | 1.3395e-2 |
| | | 0.3681 | 0.3755 | 0.3804 |
| 1.5 | 4.0497e-2 | 3.1068e-2 | 2.3717e-2 | 1.8057e-2 |
| | | 0.3824 | 0.3895 | 0.3934 |
| 1.8 | 5.6776e-2 | 4.3468e-2 | 3.3112e-2 | 2.5161e-2 |
| | | 0.3853 | 0.3926 | 0.3962 |

Table 4. $r = \frac{2}{\sigma}$:maximum nodal errors showing convergence rate $\mathcal{O}(h^2)$

| $\alpha$ \ $2N$ | 100 | 200 | 400 | 800 |
|---|---|---|---|---|
| 1.2 | 2.7820e-3 | 6.9631e-4 | 1.7418e-4 | 4.3557e-5 |
| | | 1.9983 | 1.9992 | 1.9996 |
| 1.5 | 3.1742e-3 | 8.0150e-4 | 2.0223e-4 | 5.0947e-5 |
| | | 1.9856 | 1.9867 | 1.9889 |
| 1.8 | 5.0311e-3 | 1.3190e-3 | 3.4157e-4 | 8.7691e-5 |
| | | 1.9315 | 1.9492 | 1.9617 |

Tables 3 and 4 show that the difference-quadrature method (2.10) has convergence order $\mathcal{O}(h^{\min\{r\sigma,2\}})$, which agrees with Remark 4.5.

## Appendix A. Approximations of difference and interpolation

In this appendix, we provide some approximations for the second-order difference quotients $D_h^2$ and the interpolation error $u(x)-\Pi_h u(x)$.

**Lemma A.1.** *Let $D_h^2$ be the difference quotient operator defined by* (2.6). *If $g(x)\in C(\bar{\Omega})\cap C^2(\Omega)$, there exists $\xi\in(x_{i-1},x_{i+1})$, $i=1,2,...,2N-1$, such that*

$$D_h^2 g(x_i)=g''(\xi),\quad \xi\in(x_{i-1},x_{i+1}).$$

*Moreover, if $g(x)\in C(\bar{\Omega})\cap C^4(\Omega)$, then we have*

$$\begin{aligned}&D_h^2 g(x_i)=g''(x_i)+\frac{h_{i+1}-h_i}{3}g'''(x_i)\\&+\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i}g''''(y)\frac{(y-x_{i-1})^3}{3!}dy+\frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}}g''''(y)\frac{(x_{i+1}-y)^3}{3!}dy\right).\end{aligned}$$

*Proof.* By Taylor series expansion, we obtain

$$\begin{aligned}g(x_{i-1})&=g(x_i)-(x_i-x_{i-1})g'(x_i)+\frac{(x_i-x_{i-1})^2}{2}g''(\xi_1),\quad \xi_1\in(x_{i-1},x_i),\\g(x_{i+1})&=g(x_i)+(x_{i+1}-x_i)g'(x_i)+\frac{(x_{i+1}-x_i)^2}{2}g''(\xi_2),\quad \xi_2\in(x_i,x_{i+1}).\end{aligned}$$

From (2.6) and the above equations, it yields

$$\begin{aligned}D_h^2 g(x_i)&=\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}\left(g(x_{i+1})-g(x_i)\right)+\frac{1}{h_i}\left(g(x_{i-1})-g(x_i)\right)\right)\\&=\frac{h_i}{h_i+h_{i+1}}g''(\xi_1)+\frac{h_{i+1}}{h_i+h_{i+1}}g''(\xi_2).\end{aligned}$$

According to intermediate value theorem, there exists some $\xi\in[\xi_1,\xi_2]$ such that

$$D_h^2 g(x_i)=\frac{h_i}{h_i+h_{i+1}}g''(\xi_1)+\frac{h_{i+1}}{h_i+h_{i+1}}g''(\xi_2)=g''(\xi).$$

The second equation can also be derived in a similar manner through Taylor expansion. The proof is completed. □

**Lemma A.2.** *Let $y_j^\theta=(1-\theta)x_{j-1}+\theta x_j,\theta\in(0,1)$ with $2\le j\le 2N-1$. Then we have*

$$u(y_j^\theta)-\Pi_h u(y_j^\theta)=-\frac{\theta(1-\theta)}{2}h_j^2u''(\xi),\quad \xi\in(x_{j-1},x_j),$$

*and*

$$u(y_j^\theta)-\Pi_h u(y_j^\theta)=-\frac{\theta(1-\theta)}{2}h_j^2u''(y_j^\theta)+\frac{\theta(1-\theta)}{3!}h_j^3\left(\theta^2u'''(\eta_1)-(1-\theta)^2u'''(\eta_2)\right)$$

*with $\eta_1\in(x_{j-1},y_j^\theta),\eta_2\in(y_j^\theta,x_j)$.*

*Proof.* Using Taylor series expansion, we get

$$\begin{aligned}u(x_{j-1})&=u(y_j^\theta)-\theta h_ju'(y_j^\theta)+\frac{\theta^2h_j^2}{2!}u''(\xi_1),\quad \xi_1\in(x_{j-1},y_j^\theta),\\u(x_j)&=u(y_j^\theta)+(1-\theta)h_ju'(y_j^\theta)+\frac{(1-\theta)^2h_j^2}{2!}u''(\xi_2),\quad \xi_2\in(y_j^\theta,x_j),\end{aligned}$$

which implies that

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2(\theta u''(\xi_1) + (1-\theta)u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2]. \end{aligned}$$

The second equation can be similarly obtained by

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1),$$
$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

with $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$. The proof is completed. □

**Lemma A.3.** *For any $y \in (x_{j-1}, x_j)$, $2 \le j \le 2N-1$, there exists*

$$|u(y) - \Pi_h u(y)| \le h_j^2 \max_{\xi \in [x_{j-1}, x_j]} |u''(\xi)| \le C h^2 \delta(y)^{\alpha/2 - 2/r}.$$

*Proof.* According to Lemmas 2.4, 3.1 and A.2, the desired result is obtained. □

**Lemma A.4.** *For any $x \in [x_{j-1}, x_j]$, $1 \le j \le 2N$, there exists*

$$|u(x) - \Pi_h u(x)| = \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_{x}^{x_j} u'(y)dy \right| \le \int_{x_{j-1}}^{x_j} |u'(y)|dy.$$

*In particular, there exist,*

$$|u(x) - \Pi_h u(x)| \le C \frac{2}{\alpha} h_1^{\alpha/2}, \quad x \in (0, x_1) \cup (x_{2N-1}, 2T).$$

*Proof.* From the definition of $\Pi_h u(x)$ in (2.3) and using $u(x) = u(x_i) + \int_{x_i}^{x} u'(y)dy$, the proof is completed. □

## Appendix B. Bound estimates

Set $h_i = x_i - x_{i-1}$ for $j = 1, 2, ..., 2N$ and define $h := \frac{1}{N}$. The following bounds are needed in several places.

**Lemma B.1.** *For $i = 1, 2, \cdots, 2N-1$, there exists a constant $C$ such that*

$$|h_{i+1} - h_i| \le C(r-1) h^2 \delta(x_i)^{1-2/r}, \quad r \ge 1.$$

*Proof.* According to the definition of $h_i = x_i - x_{i-1}$ as defined in (2.1), we obtain

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \le i \le N-1, \\ 0, & i = N, \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \le i \le 2N-1. \end{cases}$$

Since $(i+1)^r - 2i^r + (i-1)^r \simeq r(r-1)i^{r-2}$ for $i \ge 1$, the desired result is obtained. □

**Lemma B.2.** *For $1 \le i \le 2N-1$, there exists a constant $C$ such that*

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right| \\ &\le C h^2 \delta(x_i)^{-\alpha/2 - 2/r}. \end{aligned}$$

*Proof.* By Lemma 2.4, for $1 \le i \le 2N-1$, we have

$$\left|\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy\right| \le \frac{\|f\|_\beta^{(\alpha/2)}}{3!}\int_{x_{i-1}}^{x_i} \delta(y)^{-\alpha/2-2}(y-x_{i-1})^3 dy.$$

For $i=1$, we get

$$\int_{x_{i-1}}^{x_i} \delta(y)^{-\alpha/2-2}(y-x_{i-1})^3 dy = \int_0^{x_1} y^{1-\alpha/2}dy = \frac{1}{2-\alpha/2}x_1^{2-\alpha/2} \simeq x_1^{-\alpha/2-2}h_1^4.$$

For $2 \le i \le 2N-1$, by Lemma 3.1, we have

$$\int_{x_{i-1}}^{x_i} \delta(y)^{-\alpha/2-2}(y-x_{i-1})^3 dy \simeq \int_{x_{i-1}}^{x_i} \delta(x_i)^{-\alpha/2-2}(y-x_{i-1})^3 dy \simeq \delta(x_i)^{-\alpha/2-2}h_i^4$$

The desired result is obtained. □

**Lemma B.3.** *For all $1 \le i \le 2N-1$, $1 \le j \le 2N$, $y \in (x_{j-1}, x_j)$, there exist*

$$|D_h K_y(x_i)| \simeq |x_i - y|^{-\alpha} \quad \textit{if} \quad [x_{j-1}, x_j] \cap [x_i, x_{i+1}] = \varnothing,$$
$$D_h^2 K_y(x_i) \simeq |x_i - y|^{-1-\alpha} \quad \textit{if} \quad [x_{j-1}, x_j] \cap [x_{i-1}, x_{i+1}] = \varnothing.$$

*Proof.* Since $x_{i-1}-y$, $x_i - y$ and $x_{i+1}-y$ have the same sign, using Lemma A.1 and $K_y(x) = \frac{\kappa_\alpha}{\Gamma(2-\alpha)}|x-y|^{1-\alpha}$ in (3.20), it yields

$$|D_h K_y(x_i)| = \frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)}|\xi - y|^{-\alpha}, \quad \xi \in (x_i, x_{i+1}),$$
$$D_h^2 K_y(x_i) = \frac{\kappa_\alpha \alpha(\alpha-1)}{\Gamma(2-\alpha)}|\xi - y|^{-1-\alpha}, \quad \xi \in (x_{i-1}, x_{i+1}).$$

Moreover, from $|\xi - y| \simeq |x_i - y|$, the desired result is obtained. □

**Lemma B.4.** *There exists a constant $C$ such that*

$$\sum_{j=1}^{3} V_{1j} \le Ch^2 x_1^{-\alpha/2-2/r} \quad \textit{and} \quad \sum_{j=1}^{4} V_{2j} \le Ch^2 x_2^{-\alpha/2-2/r}.$$

*Proof.* According Lemma A.4, Lemma A.3, (3.6), (3.7), it implies,

$$T_{ij} \le Cx_1^{2-\alpha/2} \simeq h_1^2\, h^2 x_1^{-\alpha/2-2/r} \simeq h_1^2\, h^2 x_2^{-\alpha/2-2/r} \quad \text{for} \quad 0 \le i \le 3, 1 \le j \le 4.$$

The proof is completed. □

**Lemma B.5.** *Let $a, b > 0$, $\theta \in [0,1]$. Then we have*

$$b^{1-\theta}|a^\theta - b^\theta| \le |a-b|.$$

*Proof.* Since $|t^\theta - 1| \le |t-1|$ with $t = \frac{a}{b} > 0$, the proof is completed. □

**Lemma B.6.** *Let $x > 0, y \ge 1$ with $\alpha \in (1,2)$. Then we have*

$$\begin{aligned} f(x,y) =& xy(1+x^{2-\alpha}+y^{2-\alpha}) + (1+x+y)^{3-\alpha} \\ &- (1+x)(1+y)\left((1+x)^{2-\alpha}+(1+y)^{2-\alpha}-1\right) > 0. \end{aligned}$$

*Proof.* The first and second derivatives of $f(x,y)$ with respect to $x$ are

$$\begin{aligned}\partial_x f(x,y) =& (3-\alpha)\left[x^{2-\alpha}y + (1+x+y)^{2-\alpha} - (1+x)^{2-\alpha}(1+y)\right]\\ &+ 1 + 2y + y^{3-\alpha} - (1+y)^{3-\alpha},\\ \partial_x^2 f(x,y) =& (3-\alpha)(2-\alpha)\left(yx^{1-\alpha} + (1+x+y)^{1-\alpha} - (1+y)(1+x)^{1-\alpha}\right).\end{aligned}$$

Since $\frac{y}{1+y}x + \frac{1}{1+y}(1+x+y) = 1+x$ and $x^{1-\alpha}$ is convex for $x>0$, using Jensen's inequality, it yields

$$\frac{y}{1+y}x^{1-\alpha} + \frac{1}{1+y}(1+x+y)^{1-\alpha} > (1+x)^{1-\alpha},$$

which implies $\partial_x^2 f(x,y) > 0$ and $\partial_x f(x,y) > \partial_x f(0,y)$.

Since $\partial_x f(0,y) > 0$ for $y \geq 1$, we have $f(x,y) > f(0,y) = 0$. The proof is completed.

□

## Appendix C. Proofs for grid mapping functions

In this appendix, we provide the proofs of Lemmas 3.9-3.13 in subsection 3.2.

**Proof of Lemma 3.9.** The first two approximations can be derived from (2.1) and (3.17) with $2 \leq i,j \leq 2N-2$.

We next prove $|y_{i,j}(\xi) - \xi| \simeq |x_j - x_i|$. From (3.12), we have $y_{i,j}(\xi) - \xi = 0$ if $i = j$. Without loss of generality, if $i < j$, then $y_{i,j}(\xi) - \xi \leq x_{j+1} - x_{i-1} \simeq x_j - x_i$. Since the second derivatives of $|y_{i,j}(\xi) - \xi|$ is less than zero by Lemma 3.8, which implies it is concave. Thus, $|y_{i,j}(\xi) - \xi| \geq \min\{x_{j-1} - x_{i-1}, x_{j+1} - x_{i+1}\} \simeq |x_j - x_i|$.

From (3.12), (3.13), (3.17), using the approximation above, there exists

$$h_{i,j}(\xi) = y_{i,j}(\xi) - y_{i,j-1}(\xi) = y_{j-1,j}(y_{i,j-1}(\xi)) - y_{i,j-1}(\xi) \simeq x_j - x_{j-1} = h_j.$$

The final estimate can be obtained since $y_{i,j-1}(\xi) - \xi$, $y_{i,j}(\xi) - \xi$ have the same sign ($\geq 0$ or $\leq 0$). □

**Proof of Lemma 3.10.** From (3.13) and Lemma 3.8, we can see that

$$\begin{aligned} h'_{i,j}(x) &= y'_{i,j}(x) - y'_{i,j-1}(x)\\ &= \begin{cases} x^{1/r-1}\left(y_{i,j}^{1-1/r}(x) - y_{i,j-1}^{1-1/r}(x)\right), & i<N, j<N,\\ x^{1/r-1}\left(\dfrac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(x)\right), & i<N, j=N,\\ x^{1/r-1}\left((2T - y_{i,N+1}(x))^{1-1/r} - \dfrac{h_N}{rZ_1}\right), & i<N, j=N+1,\\ x^{1/r-1}\left((2T - y_{i,j}(x))^{1-1/r} - (2T - y_{i,j-1}(x))^{1-1/r}\right), & i<N, j>N+1,\\ \dfrac{rZ_1}{h_N}\left(y_{N,j}^{1-1/r}(x) - y_{N,j-1}^{1-1/r}(x)\right), & i=N, j<N,\\ \dfrac{rZ_1}{h_N}\left(\dfrac{h_N}{rZ_1} - y_{N,N-1}^{1-1/r}(x)\right), & i=N, j=N. \end{cases}\end{aligned}$$

For $2 \leq i \leq N$, $2 \leq j < N$, it yields

$$\begin{aligned} y_{i,j}^{1-1/r}(\xi) - y_{i,j-1}^{1-1/r}(\xi) &\leq x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r}\\ &= T^{1-1/r}N^{1-r}\left((j+1)^{r-1} - (j-2)^{r-1}\right)\\ &\leq CT^{1-1/r}(r-1)N^{1-r}j^{r-2} = C(r-1)Z_1 x_j^{1-2/r}. \end{aligned} \tag{C.1}$$

For $2 \le i \le N$, $j = N$, since

$$\frac{h_N}{rZ_1} = T^{1-1/r}\frac{1-(1-h)^r}{rh} = \eta^{1-1/r} \simeq x_N^{1-1/r}, \quad \eta \in (x_{N-1}, x_N), \tag{C.2}$$

we have

$$|\frac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(\xi)| \le x_N^{1-1/r} - x_{N-2}^{1-1/r} \simeq (r-1)Z_1 x_N^{1-2/r}.$$

For $2 \le i \le N$, $N+1 \le j \le 2N-2$, it can be checked

$$|h'_{i,j}(\xi)| \le C(r-1)Z_1\xi^{1/r-1}(2T-x_j)^{1-2/r}.$$

Combine with Lemmas 3.1 and 3.9, the first inequality is obtained.

On the other hand, from (3.12), we have $|y_{i,j}(x) - x| = \operatorname{sign}(j-i)(y_{i,j}(x) - x)$ and $(y_{i,j}(x) - x)' = y'_{i,j}(x) - 1$.

For $2 \le i < N$, $2 \le j < N$, by Lemmas 3.9 and B.5, we have

$$\xi^{1/r}|y_{i,j}^{1-1/r}(\xi) - \xi^{1-1/r}| \le |y_{i,j}(\xi) - \xi| \simeq |x_j - x_i|.$$

For $2 \le i < N$, $j = N$, using (C.2) and Lemma B.5, it yields

$$\begin{aligned}\eta^{1/r}|\frac{h_N}{rZ_1} - \xi^{1-1/r}| &\le |\eta - \xi|, \quad \eta \in (x_{N-1}, x_N)\\ &\le |x_N - x_i| + |h_N| + |h_{i+1}| \le 3|x_N - x_i|.\end{aligned} \tag{C.3}$$

For $2 \le i < N$, $N < j \le 2N-2$, from Lemma B.5, one has

$$\begin{aligned}\xi^{1/r}|(2T - y_{i,j}(\xi))^{1-1/r} - \xi^{1-1/r}| &\le |2T - y_{i,j}(\xi) - \xi|\\ &\le |2T - x_j - x_i| + |y_{i,j}(\xi) - x_j| + |\xi - x_i| \le |2T - x_j - x_i| + 2h_N\\ &\le |x_j - T| + |T - x_i| + 2h_N \le 2|x_j - x_i|.\end{aligned}$$

Similar to proof of (C.3), we have

$$\eta^{1/r}|y_{N,j}^{1-1/r}(\xi) - \frac{h_N}{rZ_1}| \le C|x_j - x_N|, \quad i = N, j < N,$$

$$\eta^{1/r}|(2T - y_{N,j}(\xi))^{1-1/r} - \frac{h_N}{rZ_1}| \le C|x_j - x_N|, \quad i = N, j > N,$$

and $y_{N,N}(x) - x \equiv 0$.

Thus, using Lemmas 3.8 and 3.9, the second inequality is obtained. $\square$

**Proof of Lemma 3.11.** By Definition 3.7, Lemma B.5 and (C.2), there exist

$$\begin{aligned}&x_j^{1-1/r}|Z_{j-i}| = x_j^{1-1/r}|x_j^{1/r} - x_i^{1/r}| \le |x_j - x_i|, \quad i < N, j < N,\\ &Z_{2N-j+i} \le Z_{2N} = 2T^{1/r}, \qquad i < N, j > N,\\ &\frac{h_N}{rZ_1} \simeq x_N^{1-1/r}, \qquad i = N, 2 \le j \le 2N-2.\end{aligned} \tag{C.4}$$

Combine with Lemmas 3.8 and 3.9, the first inequality is obtained.

From (3.13), it yields $h''_{i,j}(x) = y''_{i,j}(x) - y''_{i,j-1}(x)$. For $3 \le j \le 2N-2$, we have

$$\begin{aligned}|y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi)| &\simeq (r-2)Z_1 x_j^{1-3/r},\\ |(2T - y_{i,j}(\xi))^{1-2/r} - (2T - y_{i,j-1}(\xi))^{1-2/r}| &\simeq (r-2)Z_1(2T - x_j)^{1-3/r},\end{aligned} \tag{C.5}$$

which can be similarly proven as (C.1).

For $2 \le i < N$, $3 \le j < N$, it yields

$$y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1} = \left(y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi)\right) Z_{j-i} + y_{i,j-1}^{1-2/r}(\xi)Z_1.$$

Combine with (C.4) and (C.5), we get

$$|y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1}| \le CZ_1 \left(|r-2|x_j^{-2/r}|x_j - x_i| + x_j^{1-2/r}\right).$$

For $2 \le i < N$, $j = N, N+1$, it leads to

$$|h''_{i,j}(x)| \le |y''_{i,j}(x)| + |y''_{i,j-1}(x)| \le C(r-1)x_i^{1/r-2}x_N^{1-1/r}.$$

For $2 \le i < N$, $j > N+1$, from Lemma 3.1 and (C.5), we have

$$\begin{aligned}
&\left|\delta(y_{i,j}(\xi))^{1-2/r}Z_{2N-(j-i)} - \delta(y_{i,j-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)}\right| \\
&= \left|\left(\delta(y_{i,j}(\xi))^{1-2/r} - \delta(y_{i,j-1}(\xi))^{1-2/r}\right) Z_{2N-(j-i)} - \delta(y_{i,j-1}(\xi))^{1-2/r}Z_1\right| \\
&\le CZ_1 \left(|r-2|\delta(x_j)^{1-3/r}x_N^{1/r} + \delta(x_j)^{1-2/r}\right) \le CZ_1\delta(x_j)^{1-3/r}x_N^{1/r}.
\end{aligned}$$

For $i = N$, one has

$$|h''_{N,j}(\xi)| = \begin{cases} |y''_{N,N-1}(\xi)|, & j = N, \\ |y''_{N,N+1}(\xi)|, & j = N+1 \end{cases} \le Cx_N^{-1}.$$

For $i = N$, $j \ne N, N+1$, using (C.5), Lemmas 3.8 and 3.9, the second inequality is obtained. □

**Proof of Lemma 3.12.** According to $|y_{i,j}^\theta(\xi) - \xi| = \operatorname{sign}(j-i-1+\theta)(y_{i,j}^\theta(\xi) - \xi)$ with $\theta \in (0,1)$, Lemma A.1, (1.3) and (3.15), we have

$$D_h^2 P_{i,j}^\theta(x_i) = {P_{i,j}^\theta}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}).$$

From Lemmas 2.4, 3.1 and 3.8 to 3.11, and regarding the selection process of $i, j$ within Case 1-3, it turns out that

$$\begin{gathered}
h_{i,j}(\xi) \le Ch_j, \quad |h'_{i,j}(\xi)| \le C(r-1)h_j x_i^{-1}, \\
|y_{i,j}^\theta(\xi) - \xi| \le C|y_j^\theta - x_i|, \quad \left|(y_{i.j}^\theta(\xi) - x_i)'\right| \le C|y_j^\theta - x_i|x_i^{-1}, \\
\left|u''(y_{i,j}^\theta(\xi))\right| \le Cx_i^{\alpha/2-2}, \ \left|\left(u''(y_{i,j}^\theta(\xi))\right)'\right| \le Cx_i^{\alpha/2-3}, \ \left|\left(u''(y_{i,j}^\theta(\xi))\right)''\right| \le Cx_i^{\alpha/2-4}.
\end{gathered}$$

By Lemma 3.11, we have

$$\begin{aligned}
&|h''_{i,j}(\xi)| \le C(r-1)h_j x_i^{-2}, && \left|(y_{i,j}^\theta(\xi) - x_i)''\right| \le C(r-1)|y_j^\theta - x_i|x_i^{-2}, && \text{for Case 1,} \\
&|h''_{i,j}(\xi)| \le C(r-1), && \left|(y_{i,j}^\theta(\xi) - x_i)''\right| \le C(r-1), && \text{for Case 2,} \\
&|h''_{i,j}(\xi)| \le C(r-1)h_j, && \left|(y_{i,j}^\theta(\xi) - x_i)''\right| \le C(r-1), && \text{for Case 3.}
\end{aligned}$$

Using Leibniz formula and chain rules, the desired results are obtained. □

**Proof of Lemma 3.13.** Since

$$\begin{aligned}
&\frac{Q_{i,j,l}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{i,j,l}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \\
&\quad = \frac{Q_{i,j,l}^\theta(x_{i+1}) - Q_{i,j,l}^\theta(x_i)}{h_{i+1}}u'''(\eta_{j+1}^\theta) + Q_{i,j,l}^\theta(x_i)\frac{u'''(\eta_{j+1}^\theta) - u'''(\eta_j^\theta)}{h_{i+1}}.
\end{aligned}$$

Using mean value theorem, it yields

$$\frac{Q^{\theta}_{i,j,l}(x_{i+1}) - Q^{\theta}_{i,j,l}(x_i)}{h_{i+1}} = {Q^{\theta}_{i,j,l}}'(\xi), \quad \xi \in (x_i, x_{i+1}).$$

From (3.16), Lemmas 3.1, 3.9 and 3.10, Leibniz formula and chain rule, we have

$$\begin{aligned} |{Q^{\theta}_{i,j,l}}'(\xi)| &\le C h_j^l |y_j^\theta - x_i|^{1-\alpha}(x_i^{-1} + x_i^{1/r-1}\delta(x_j)^{-1/r}), \\ Q^{\theta}_{i,j,l}(x_i) &= C h_j^l |y_j^\theta - x_i|^{1-\alpha}. \end{aligned}$$

According to Lemmas 2.4 and 3.1, it implies

$$|u^{(l-1)}(\eta^\theta_{j+1})| \le C(\eta^\theta_{j+1})^{\alpha/2-l+1} \simeq \delta(x_j)^{\alpha/2-l+1},$$

and

$$\begin{aligned} \frac{|u^{(l-1)}(\eta^\theta_{j+1}) - u^{(l-1)}(\eta^\theta_j)|}{h_{i+1}} &= |u^{(l)}(\eta)| \frac{\eta^\theta_{j+1} - \eta^\theta_j}{h_{i+1}}, \quad \eta \in (x_{j-1}, x_{j+1}) \\ &\le C\delta(\eta)^{\alpha/2-l} \frac{x_{j+1} - x_{j-1}}{h_{i+1}} = C\delta(\eta)^{\alpha/2-l} \frac{h_{j+1} + h_j}{h_{i+1}} \\ &\simeq x_i^{1/r-1} \delta(x_j)^{\alpha/2-l+1-1/r}. \end{aligned}$$

Thus, the first inequality is obtained. The second one can be similarly proven as the way provided above. □

## References


[1] Gabriel Acosta, Juan Pablo Borthagaray, Oscar Bruno, and Martín Maas. Regularity theory and high order numerical methods for the (1D)-fractional Laplacian. *Math. Comp.*, 87(312):1821–1857, 2018. 1, 1

[2] Fuensanta Andreu-Vaillo, José M. Mazón, Julio D. Rossi, and J. Julián Toledo-Melero. *Nonlocal diffusion problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010. 1

[3] Jean Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996. 1

[4] Weiping Bu, Yifa Tang, and Jiye Yang. Galerkin finite element method for two-dimensional Riesz space fractional diffusion equations. *J. Comput. Phys.*, 276:26–38, 2014. 1

[5] Cem Çelik and Melda Duman. Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative. *J. Comput. Phys.*, 231(4):1743–1750, 2012. 1

[6] Minghua Chen and Weihua Deng. Fourth order accurate scheme for the space fractional diffusion equations. *SIAM J. Numer. Anal.*, 52(3):1418–1438, 2014. 1

[7] Minghua Chen and Weihua Deng. A second-order numerical method for two-dimensional two-sided space fractional convection diffusion equation. *Appl. Math. Model.*, 38(13):3244–3259, 2014. 1

[8] Minghua Chen and Weihua Deng. High order algorithms for the fractional substantial diffusion equation with truncated Lévy flights. *SIAM J. Sci. Comput.*, 37(2):A890–A917, 2015. 1

[9] Minghua Chen, Weihua Deng, Chao Min, Jiankang Shi, and Martin Stynes. Error analysis of a collocation method on graded meshes for a fractional Laplacian problem. *Adv. Comput. Math.*, 50(3):Paper No. 49, 27, 2024. 1

[10] Minghua Chen, Yantao Wang, Xiao Cheng, and Weihua Deng. Second-order LOD multigrid method for multidimensional Riesz fractional diffusion equation. *BIT*, 54(3):623–647, 2014. 1

[11] Arturo de Pablo, Fernando Quirós, Ana Rodríguez, and Juan Luis Vázquez. A fractional porous medium equation. *Adv. Math.*, 226(2):1378–1409, 2011. 1

[12] Beichuan Deng, Zhimin Zhang, and Xuan Zhao. Superconvergence points for the spectral interpolation of Riesz fractional derivatives. *J. Sci. Comput.*, 81(3):1577–1601, 2019. 1

[13] Weihua Deng. Finite element method for the space and time fractional Fokker-Planck equation. *SIAM J. Numer. Anal.*, 47(1):204–226, 2008/09. 1

[14] Hengfei Ding and Changpin Li. High-order algorithms for Riesz derivative and their applications (IV). *Fract. Calc. Appl. Anal.*, 22(6):1537–1560, 2019. 1

[15] V. J. Ervin, N. Heuer, and J. P. Roop. Regularity of the solution to 1-D fractional order diffusion equations. *Math. Comp.*, 87(313):2273–2294, 2018. 1

[16] Paolo Gatto and Jan S. Hesthaven. Numerical approximation of the fractional Laplacian via *hp*-finite elements, with an application to image denoising. *J. Sci. Comput.*, 65(1):249–270, 2015. 1

[17] R. K. Getoor. First passage times for symmetric stable processes in space. *Trans. Amer. Math. Soc.*, 101:75–90, 1961. 1, 1, 5.1

[18] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order.* Grundlehren der Mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin-New York, 1977. 2.2

[19] Rubing Han and Shuonan Wu. A monotone discretization for integral fractional Laplacian on bounded Lipschitz domains: pointwise error estimates under Hölder regularity. *SIAM J. Numer. Anal.*, 60(6):3052–3077, 2022. 1

[20] Yanghong Huang and Adam Oberman. Numerical methods for the fractional Laplacian: a finite difference–quadrature approach. *SIAM J. Numer. Anal.*, 52(6):3056–3084, 2014. 1, 1, 5.1

[21] Mateusz Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. *Fract. Calc. Appl. Anal.*, 20(1):7–51, 2017. 1

[22] Randall J. LeVeque. *Finite difference methods for ordinary and partial differential equations.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007. Steady-state and time-dependent problems. 2.1

[23] Anna Lischke, Guofei Pang, Mamikon Gulian, and et al. What is the fractional Laplacian? A comparative review with new results. *J. Comput. Phys.*, 404:109009, 62, 2020. 1, 1

[24] Ralf Metzler and Joseph Klafter. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.*, 339(1):77, 2000. 1

[25] Xavier Ros-Oton and Joaquim Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl. (9)*, 101(3):275–302, 2014. 1, 5.1

[26] Xavier Ros-Oton and Joaquim Serra. The dirichlet problem for the fractional laplacian: Regularity up to the boundary. *Journal de Mathématiques Pures et Appliquées*, 101(3):275–302, 2014. 2.1, 2.2, 2.3

[27] Zi-Hang She, Cheng-Xue Lao, Hong Yang, and Fu-Rong Lin. Banded preconditioners for Riesz space fractional diffusion equations. *J. Sci. Comput.*, 86(3):Paper No. 31, 22, 2021. 1

[28] S. Shen, F. Liu, V. Anh, and I. Turner. The fundamental solution and numerical solution of the Riesz fractional advection-dispersion equation. *IMA J. Appl. Math.*, 73(6):850–872, 2008. 1

[29] Jiankang Shi and Minghua Chen. High-order BDF convolution quadrature for subdiffusion models with a singular source term. *SIAM J. Numer. Anal.*, 61(6):2559–2579, 2023. 1

[30] Jiankang Shi, Minghua Chen, and Jianxiong Cao. High-order BDF convolution quadrature for fractional evolution equations with hyper-singular source term. *J. Sci. Comput.*, 101(1):Paper No. 9, 22, 2024. 1

[31] Ercília Sousa and Can Li. A weighted finite difference method for the fractional diffusion equation based on the Riemann-Liouville derivative. *Appl. Numer. Math.*, 90:22–37, 2015. 1

[32] Vasily Tarasov. *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media.* Springer Berlin, Heidelberg, 10 2011. 1

[33] Vasily E. Tarasov. *Fractional dynamics.* Nonlinear Physical Science. Springer, Heidelberg; Higher Education Press, Beijing, 2010. Applications of fractional calculus to dynamics of particles, fields and media. 1

[34] WenYi Tian, Han Zhou, and Weihua Deng. A class of second order difference approximations for solving space fractional diffusion equations. *Math. Comp.*, 84(294):1703–1727, 2015. 1

[35] Enrico Valdinoci. From the long jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl. SeMA*, (49):33–44, 2009. 1

[36] Nan Wang, Zhiping Mao, Chengming Huang, and George Em Karniadakis. A spectral penalty method for two-sided fractional differential equations with general boundary conditions. *SIAM J. Sci. Comput.*, 41(3):A1840–A1866, 2019. 1

[37] Q. Yang, F. Liu, and I. Turner. Numerical methods for fractional partial differential equations with Riesz space fractional derivatives. *Appl. Math. Model.*, 34(1):200–218, 2010. 1

[38] Fanhai Zeng, Fawang Liu, Changpin Li, Kevin Burrage, Ian Turner, and V. Anh. A Crank-Nicolson ADI spectral method for a two-dimensional Riesz space fractional nonlinear reaction-diffusion equation. *SIAM J. Numer. Anal.*, 52(6):2599–2622, 2014. 1

[39] Han Zhou and Wenyi Tian. Two time-stepping schemes for sub-diffusion equations with singular source terms. *J. Sci. Comput.*, 92(2):Paper No. 70, 28, 2022. 1

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