A SECOND ORDER NUMERICAL METHODS FOR REISZ-FRACTIONAL ELLIPTIC EQUATION ON GRADED MESH*

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Key words. example, LaTeX

MSC codes. ??????????????????

1. Introduction.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$,

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

2. Preliminaries: Numeric scheme and main results.

2.1. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases} \tag{2.1}$$

where $r \ge 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N \tag{2.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

And then, define the piecewise linear interpolant of the true solution u to be

$$\Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{2.4}$$

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For convience, we denote

$$I^{2-\alpha}u(x) := \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x-y|^{1-\alpha} u(y)dy \tag{2.5}$$

and

$$D_h^2 u(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} u(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) u(x_i) + \frac{1}{h_{i+1}} u(x_{i+1}) \right) \tag{2.6}$$

Now, we discretise (1.1) by replacing $u(x)$ by a continuous piecewise linear function

$$u_h(x) := \sum_{j=1}^{2N-1} u_j \phi_j(x) \tag{2.7}$$

whose nodal values u_j are to be determined by collocation at each mesh point x_i for $i = 1, 2, ..., 2N-1$:

$$-\kappa_\alpha D_h^\alpha u_h(x_i) := -\kappa_\alpha D_h^2 I^{2-\alpha} u_h(x_i) = f(x_i) =: f_i \tag{2.8}$$

Here,

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^2 I^{2-\alpha} \phi_j(x_i)\, u_j = \sum_{j=1}^{2N-1} a_{ij}\, u_j \tag{2.9}$$

where

$$a_{ij} = -\kappa_\alpha D_h^2 I^{2-\alpha} \phi_j(x_i) \quad \text{for} \quad i, j = 1, 2, ..., 2N-1 \tag{2.10}$$

We have replaced $(-\Delta)^{\alpha/2} u(x_i) = f(x_i)$ in (1.1) by $-\kappa_\alpha u_h(x_i) = f(x_i)$ in (2.8), with truncation error

$$\tau_i := -\kappa_\alpha \left(D_h^\alpha \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i) \right) \quad \text{for} \quad i = 1, 2, ..., 2N-1 \tag{2.11}$$

where $-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^\alpha \phi_j(x_i) u(x_j) = \sum_{j=1}^{2N-1} a_{ij} u(x_j)$.

The discrete equation (2.8) can be written in matrix form

$$AU = F \tag{2.12}$$

where $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$, $U = (u_1, \cdots, u_{2N-1})^T$ is unknown and $F = (f_1, \cdots, f_{2N-1})^T$.

2.2. Regularity of the true solution. For any $\beta > 0$, we use the standard notation $C^\beta(\bar{\Omega}), C^\beta(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^\beta(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous[1] with exponent β' in Ω, where $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on open set Ω.

DEFINITION 2.1 (delta dependent norm [2]). ...

THEOREM 2.2. *Let $f \in C^\beta(\Omega), \beta > 2$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, then for* $l = 0, 1, 2$

$$|f^{(l)}(x)| \le \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \le T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \le x < 2T \end{cases} \tag{2.13}$$

THEOREM 2.3 (Regularity up to the boundary [2]). *Let Ω be a bounded domain, and $\beta > 0$ be such that neither β nor $\beta + \alpha$ is an integer. Let $f \in C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R}^n)$ be a solution of* (1.1)*. Then, $u \in C^{\beta+\alpha}(\Omega)$ and*

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)}\right) \tag{2.14}$$

COROLLARY 2.4. *Let u be a solution of* (1.1) *where $f \in L^\infty(\Omega)$ and $\|f\|_\beta^{(\alpha/2)} < \infty$. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \le \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \le T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \le x < 2T \end{cases} \tag{2.15}$$

And in this paper bellow, without special instructions, we allways assume that

$$f \in L^\infty(\Omega) \cap C^\beta(\Omega) \quad \text{and} \quad \|f\|_\beta^{(\alpha/2)} < \infty, \text{ with } \alpha + \beta > 4 \tag{2.16}$$

2.3. Main results. Here we state our main results; the proof is deferred to section 3 and section 4.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 2.5 (Local Truncation Error). *If $u(x)$ is a solution of the equation* (1.1) *where f satisfy the regular condition* (2.16)*, then there exists $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$ and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$, such that the truncation error* (2.11) *satisfies*

$$\begin{aligned} |\tau_i| :=& |-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i)| \\ \le& C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} \begin{cases} x_i^{-\alpha}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha}, & N < i \le 2N-1 \end{cases} \\ &+ C_2 (r-1) h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \le i \le N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \le 2N-1 \end{cases} \end{aligned} \tag{2.17}$$

THEOREM 2.6 (Global Error). *The discrete equation* (2.8) *has sulotion and there exists a positive constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$ such that the error between the numerial solution U with the exact solution $u(x_i)$ satisfies*

$$\max_{1 \le i \le 2N-1} |u_i - u(x_i)| \le C h^{\min\{\frac{r\alpha}{2}, 2\}} \tag{2.18}$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

3. Truncation Error.

The truncation error of the discrete format can be written as

$$
-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i) = -\kappa_\alpha (D_h^2 I^{2-\alpha}\Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i)) \\
= -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) - \kappa_\alpha D_h^2 I^{2-\alpha}(\Pi_h u - u)(x_i) \tag{3.1}
$$

3.1. Estimate of $-\kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}(x_i)$.

THEOREM 3.1. *There exits a constant $C = C(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$ such that*

$$
\left| -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \right| \le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases} \tag{3.2}
$$

Proof. Since $f \in C^2(\Omega)$ and

$$
\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha}(x)) = f(x), \quad x \in \Omega, \tag{3.3}
$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.3) of Lemma A.1, for $1 \le i \le 2N-1$, we have

$$
-\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i) \\
+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right) \tag{3.4}
$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

$$
\left| \frac{h_{i+1} - h_i}{3} f'(x_i) \right| \le \frac{C(r-1)\|f\|_\beta^{(\alpha/2)}}{3} h^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{-\alpha/2-2/r}, & N < i \le 2N-1 \end{cases} \tag{3.5}
$$

2. See Proof 25, there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

$$
\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right) \\
\le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases} \tag{3.6}
$$

Summarizes, we get the result. ∎

3.2. Estimate of R_i.

Now, we study the first part of (3.1)

$$
D_h^2 (I^{2-\alpha} - I_h^{2-\alpha} u)(x_i) = D_h^2 (\int_0^{2T} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy) \tag{3.7}
$$

For convience, let's denote

$$
T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy, \quad i = 0, \cdots, 2N, \ j = 1, \cdots, 2N \tag{3.8}
$$

And define

$$R_i := D_h^2(I^{2-\alpha} - I_h^{2-\alpha}u)(x_i) \tag{3.9}$$

$$= \frac{2}{h_i + h_{i+1}} \sum_{j=1}^{2N} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right), \quad 1 \le i \le 2N-1$$

We have some results about the estimate of R_i

THEOREM 3.2. *For $1 \le i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$R_i \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.10}$$

THEOREM 3.3. *For $N/2 \le i \le N$, there exists constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$R_i \le C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.11}$$

And for $N < i \le 2N-1$, it is symmetric to the previous case.

To prove these results, we need some utils. Also for simplicity, we denote

DEFINITION 3.4.

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \tag{3.12}$$

then

$$R_i = \sum_{j=1}^{2N} S_{ij} \tag{3.13}$$

3.3. Proof of Theorem 3.2.

LEMMA 3.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \le i < N/2$,*

$$\sum_{j=\max\{2i+1, i+3\}}^{N} S_{ij} \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.14}$$

Proof. Let

$$K_y(x) = \frac{|y-x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

For $\max\{2i+1, i+3\} \le j \le N$, by Lemma C.1 and Lemma C.2

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \\ &\le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy \end{aligned} \tag{3.15}$$

Therefore,

$$
\begin{aligned}
\sum_{j=\max\{2i+1,i+3\}}^{N} S_{ij} &\leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\
&= \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\
&\leq \frac{C}{\alpha/2+2/r} 2^{r(-\alpha/2-2/r)} h^2 x_i^{-\alpha/2-2/r}
\end{aligned}
\tag{3.16}
$$

∎

Lemma 3.6. *Thert exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \leq i < N/2$,*

$$
\sum_{j=N+1}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\tag{3.17}
$$

Proof. For $1 \leq i < N/2, N+1 \leq j \leq 2N-1$, by equation (C.2) and Lemma C.2

$$
\begin{aligned}
S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \\
&\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T-y)^{\alpha/2-2/r} y^{-1-\alpha} dy \\
&\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T-y)^{\alpha/2-2/r} dy
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j=N+1}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r} dy \\
&\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2-2/r+1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 < 0 \end{cases} \\
&= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r}\ h^2, & \alpha/2-2/r+1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2-2/r+1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2-2/r+1 < 0 \end{cases}
\end{aligned}
\tag{3.18}
$$

And by Lemma A.3

$$
S_{i,2N} \leq CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}
$$

And when $\alpha/2 - 2/r + 1 \geq 0$,

$$
h^{r\alpha/2+r} \leq h^2
$$

Summarizes, we get the result. ∎

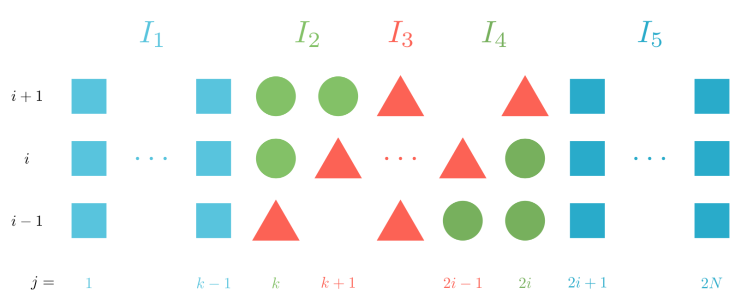
For $i = 1, 2$.

LEMMA 3.7. *By Lemma* C.5 , *Lemma* 3.5 *and Lemma* 3.6 *we get*

$$
\begin{aligned}
R_1 &= \sum_{j=1}^{3} S_{1j} + \sum_{j=4}^{2N} S_{1j} \\
&\leq Ch^2 x_1^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1>0 \\ Ch^2\ln(N), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0 \end{cases}
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
R_2 &= \sum_{j=1}^{4} S_{2j} + \sum_{j=5}^{2N} S_{2j} \\
&\leq Ch^2 x_2^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1>0 \\ Ch^2\ln(N), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0 \end{cases}
\end{aligned} \tag{3.20}
$$

For $3 \leq i < N/2$, we have a new separation of R_i, Let's denote $k = \lceil \frac{i}{2} \rceil$.



$$
\begin{aligned}
R_i =&\sum_{j=1}^{2N}\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,j}+\frac{1}{h_i}T_{i-1,j}\right)\\
=&\sum_{j=1}^{k-1}\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,j}+\frac{1}{h_i}T_{i-1,j}\right)\\
&+\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}(T_{i+1,k}+T_{i+1,k+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,k}\right)\\
&+\sum_{j=k+1}^{2i-1}\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j+1}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,j}+\frac{1}{h_i}T_{i-1,j-1}\right)\\
&+\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}(T_{i-1,2i}+T_{i-1,2i-1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,2i}\right)\\
&+\sum_{j=2i+1}^{2N}\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,j}+\frac{1}{h_i}T_{i-1,j}\right)\\
=&I_1+I_2+I_3+I_4+I_5
\end{aligned}
\tag{3.21}
$$

LEMMA 3.8. *There exists a constant $C=C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that for $3\le i\le N, k=\lceil\frac{i}{2}\rceil$*

$$
|I_1|=|\sum_{j=1}^{k-1}S_{ij}|\le\begin{cases}Ch^2x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1>0\\ Ch^2x_i^{-1-\alpha}\ln(i), & \alpha/2-2/r+1=0\\ Ch^{r\alpha/2+r}x_i^{-1-\alpha}, & \alpha/2-2/r+1<0\end{cases}
\tag{3.22}
$$

Proof. by Lemma A.3 , Lemma C.3

$$
S_{i1}\le Cx_1^{\alpha/2}x_1x_i^{-1-\alpha}=Cx_1^{\alpha/2+1}x_i^{-1-\alpha}=CT^{\alpha/2+1}h^{r\alpha/2+r}x_i^{-1-\alpha}
\tag{3.23}
$$

For $2\le j\le k-1$, by Lemma C.1 and Lemma C.3

$$
\begin{aligned}
S_{ij}&=\int_{x_{j-1}}^{x_j}(u(y)-\Pi_hu(y))D_h^2K_y(x_i)dy\\
&\le Ch^2\int_{x_{j-1}}^{x_j}y^{\alpha/2-2/r}\frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)}dy\\
&=Ch^2x_i^{-1-\alpha}\int_{x_{j-1}}^{x_j}y^{\alpha/2-2/r}dy
\end{aligned}
\tag{3.24}
$$

Therefore,

$$
\begin{aligned}
I_1&=\sum_{j=1}^{k-1}S_{ij}=S_{i1}+\sum_{j=2}^{k-1}S_{ij}\\
&\le Ch^{r\alpha/2+r}x_i^{-1-\alpha}+Ch^2x_i^{-1-\alpha}\int_{x_1}^{x_{\lceil\frac{i}{2}\rceil-1}}y^{\alpha/2-2/r}dy\\
&\le Ch^{r\alpha/2+r}x_i^{-1-\alpha}+Ch^2x_i^{-1-\alpha}\int_{x_1}^{2^{-r}x_i}y^{\alpha/2-2/r}dy
\end{aligned}
\tag{3.25}
$$

But

$$\int_{x_1}^{2^{-r}x_i} y^{\alpha/2-2/r}dy \le \begin{cases} \frac{1}{\alpha/2-2/r+1}(2^{-r}x_i)^{\alpha/2-2/r+1}, & \alpha/2-2/r+1>0 \\ \ln(2^{-r}x_i)-\ln(x_1), & \alpha/2-2/r+1=0 \\ \frac{1}{|\alpha/2-2/r+1|}x_1^{\alpha/2-2/r+1}, & \alpha/2-2/r+1<0 \end{cases} \tag{3.26}$$

So we have

$$I_1 \le \begin{cases} \frac{C}{\alpha/2-2/r+1}h^2 x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1>0 \\ Ch^2 x_i^{-1-\alpha}\ln(i), & \alpha/2-2/r+1=0 \\ \frac{C}{|\alpha/2-2/r+1|}h^{r\alpha/2+r}x_i^{-1-\alpha}, & \alpha/2-2/r+1<0 \end{cases} \tag{3.27}$$

□

DEFINITION 3.9. *For convience, let's denote*

$$V_{ij} = \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j+1} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)T_{i,j} + \frac{1}{h_i}T_{i-1,j-1}\right) \tag{3.28}$$

THEOREM 3.10. *There exists a constant $C = C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that for $3 \le i < N/2, k = \lceil \frac{i}{2} \rceil$,*

$$I_3 = \sum_{j=k+1}^{2i-1} V_{ij} \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.29}$$

To estimete V_{ij}, we need some preparations.

LEMMA 3.11. *For $y \in [x_{j-1}, x_j]$, we can rewrite $y = x_{j-1} + \theta h_j = (1-\theta)x_{j-1} + \theta x_j =: y_j^\theta$, $\theta \in [0,1]$, by Lemma* A.2

$$\begin{aligned} T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y))\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy \\ &= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta))\frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}h_j d\theta \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2}h_j^3 u''(y_j^\theta)\frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!}h_j^4\frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}(\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta))d\theta \end{aligned} \tag{3.30}$$

where $\eta_{j1}^\theta \in [x_{j-1}, y_j^\theta], \eta_{j2}^\theta \in [y_j^\theta, x_j]$.

Now Let's construct a series of functions to represent T_{ij}.

DEFINITION 3.12. *For $2 \le i,j \le N-1$,*

$$y_{j-i}(x) = (x^{1/r} + Z_{j-i})^r, \quad Z_{j-i} = T^{1/r}\frac{j-i}{N} \tag{3.31}$$

$$y_{j-i}^\theta(x) = (1-\theta)y_{j-1-i}(x) + \theta y_{j-i}(x) \tag{3.32}$$

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{3.33}$$

Now, we define

$$P^{\theta}_{j-i}(x) = (h_{j-i}(x))^3 u''(y^{\theta}_{j-i}(x)) \frac{|y^{\theta}_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.34}$$

$$Q^{\theta}_{j-i}(x) = (h_{j-i}(x))^4 \frac{|y^{\theta}_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.35}$$

And now we can rewrite T_{ij}

LEMMA 3.13. *For $2 \le i \le N, 2 \le j \le N$,*

$$\begin{aligned} T_{ij} = &\int_0^1 -\frac{\theta(1-\theta)}{2} P^{\theta}_{j-i}(x_i) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} Q^{\theta}_{j-i}(x_i)(\theta^2 u'''(\eta^{\theta}_{j,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{j,2})) d\theta \end{aligned} \tag{3.36}$$

Immediately, we can see from (3.28) that

LEMMA 3.14. *For $3 \le i, j \le N-1$,*

$$\begin{aligned} V_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P^{\theta}_{j-i}(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_{i+1}) u'''(\eta^{\theta}_{j+1,1}) - Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,1})}{h_{i+1}} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,1}) - Q^{\theta}_{j-i}(x_{i-1}) u'''(\eta^{\theta}_{j-1,1})}{h_i} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_{i+1}) u'''(\eta^{\theta}_{j+1,2}) - Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,2})}{h_{i+1}} \right) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,2}) - Q^{\theta}_{j-i}(x_{i-1}) u'''(\eta^{\theta}_{j-1,2})}{h_i} \right) d\theta \end{aligned} \tag{3.37}$$

To estimate V_{ij}, we first estimate $D_h^2 P^{\theta}_{j-i}(x_i)$, but By Lemma A.1,

$$D_h^2 P^{\theta}_{j-i}(x_i) = {P^{\theta}_{j-i}}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \tag{3.38}$$

By Leibniz formula, we calculate and estimate the derivations of $h^3_{j-i}(x)$, $u''(y^{\theta}_{j-i}(x))$ and $\frac{|y^{\theta}_{j-i}(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)}$ separately.

Firstly, we have

LEMMA 3.15. *There exists a constant $C = C(T, r)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,*

$$h^3_{j-i}(\xi) \le C h^2 x_i^{2-2/r} h_j \tag{3.39}$$

$$(h^3_{j-i}(\xi))' \le C(r-1) h^2 x_i^{1-2/r} h_j \tag{3.40}$$

$$(h^3_{j-i}(\xi))'' \le C(r-1) h^2 x_i^{-2/r} h_j \tag{3.41}$$

The proof of this theorem see Lemma C.6 and Lemma C.7

Second,

LEMMA 3.16. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$u''(y_{j-i}^{\theta}(\xi)) \leq C x_i^{\alpha/2-2} \tag{3.42}$$

$$(u''(y_{j-i}^{\theta}(\xi)))' \leq C x_i^{\alpha/2-3} \tag{3.43}$$

$$(u''(y_{j-i}^{\theta}(\xi)))'' \leq C x_i^{\alpha/2-4} \tag{3.44}$$

The proof of this theorem see Proof 31

And Finally, we have

LEMMA 3.17. *There exists a constant* $C = C(T, \alpha, r)$ *such that For* $3 \leq i \leq N-1, 1 \leq j \leq \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha} \leq C|y_j^{\theta} - x_i|^{1-\alpha} \tag{3.45}$$

$$(|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha})' \leq C|y_j^{\theta} - x_i|^{1-\alpha} x_i^{-1} \tag{3.46}$$

$$(|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha})'' \leq C|y_j^{\theta} - x_i|^{1-\alpha} x_i^{-2} \tag{3.47}$$

where $y_j^{\theta} = \theta x_{j-1} + (1-\theta) x_j$

The proof of this theorem see Proof 32

LEMMA 3.18. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$,

$$D_h^2 P_{j-i}^{\theta}(x_i) \leq C h^2 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \tag{3.48}$$

where $y_j^{\theta} = \theta x_{j-1} + (1-\theta) x_j$

Proof. Since

$$D_h^2 P_{j-i}^{\theta}(x_i) = {P_{j-i}^{\theta}}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \tag{3.49}$$

From (3.34), using Leibniz formula and Lemma 3.15, Lemma 3.16 and Lemma 3.17 ∎

LEMMA 3.19. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \leq i < N, k = \lceil \frac{i}{2} \rceil$.
For $k \leq j \leq \min\{2i-1, N-1\}$,

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_{i+1}) u'''(\eta_{j+1}^{\theta}) - Q_{j-i}^{\theta}(x_i) u'''(\eta_j^{\theta})}{h_{i+1}} \right) \\ &\qquad \leq C h^2 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned} \tag{3.50}$$

And for $k+1 \leq j \leq \min\{2i, N\}$,

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_i) u'''(\eta_j^{\theta}) - Q_{j-i}^{\theta}(x_{i-1}) u'''(\eta_{j-1}^{\theta})}{h_i} \right) \\ &\qquad \leq C h^2 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned} \tag{3.51}$$

where $\eta_j^\theta \in [x_{j-1}, x_j]$.

proof see Proof 33

LEMMA 3.20. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i < N, k = \lceil \frac{i}{2} \rceil, k+1 \le j \le \min\{2i-1, N-1\}$,

$$V_{ij} \le Ch^2 \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j d\theta \\ = Ch^2 \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \tag{3.52}$$

Proof. Since Lemma 3.14, by Lemma 3.18 and Lemma 3.19, we get the result immediately. ∎

Now we can prove Theorem 3.10 using Lemma 3.20, $k = \lceil \frac{i}{2} \rceil$

$$\begin{aligned} I_3 = \sum_{k+1}^{2i-1} V_{ij} &\le Ch^2 \int_{x_k}^{x_{2i-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{2i-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\ &\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{3.53}$$

LEMMA 3.21.

$$D_h P_{j-i}^\theta(x_i) := \frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} = {P_{j-i}^\theta}'(\xi), \quad \xi \in [x_i, x_{i+1}] \tag{3.54}$$

Then, for $3 \le i \le N-1$, $k = \lceil \frac{i}{2} \rceil$,

$$D_h P_{k-i}^\theta(x_i) \le Ch^2 x_i^{-\alpha/2-2/r} h_j \tag{3.55}$$

Proof. Using Leibniz formula, by Lemma 3.15, Lemma 3.16 and Lemma 3.17, we take $j = k+1, i = i+1$, we get

$$\begin{aligned} D_h P_{k-i}^\theta(x_i) &\le Ch^2 x_{i+1}^{\alpha/2-2/r-1} |y_{k+1}^\theta - x_{i+1}|^{1-\alpha} h_{j+1} \\ &\le Ch^2 x_i^{\alpha/2-2/r-1} |y_k^\theta - x_i|^{1-\alpha} h_j \\ &\le Ch^2 x_i^{-\alpha/2-2/r} h_j \end{aligned} \tag{3.56}$$ ∎

LEMMA 3.22. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i < N, k = \lceil \frac{i}{2} \rceil$,

$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.57}$$

And for $3 \le i < N/2$,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.58}$$

Proof. In fact,

$$\begin{aligned}(3.59)\qquad &\frac{1}{h_{i+1}}(T_{i+1,k}+T_{i+1,k+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,k}\\ &=\frac{1}{h_{i+1}}(T_{i+1,k}-T_{i,k})+\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k})+(\frac{1}{h_{i+1}}-\frac{1}{h_i})T_{i,k}\end{aligned}$$

While, by Lemma A.2

$$\begin{aligned}\frac{1}{h_{i+1}}(T_{i+1,k}-T_{i,k})&=\int_{x_{k-1}}^{x_k}(u(y)-\Pi_h u(y))\frac{|x_{i+1}-y|^{1-\alpha}-|x_i-y|^{1-\alpha}}{h_{i+1}\Gamma(2-\alpha)}dy\\ (3.60)\qquad &\le\int_{x_{k-1}}^{x_k}h_k^2u''(\eta)\,\frac{|\xi-y|^{-\alpha}}{\Gamma(1-\alpha)}dy\\ &\le Ch_kh^2x_k^{2-2/r}x_{k-1}^{\alpha/2-2}|x_i-x_k|^{-\alpha}\\ &\le Ch_kh^2x_i^{-\alpha/2-2/r}\end{aligned}$$

Thus,

$$(3.61)\qquad \frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}(T_{i+1,k}-T_{i,k})\le Ch^2x_i^{-\alpha/2-2/r}$$

For

$$\begin{aligned}(3.62)\qquad \frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k})&=\int_0^1-\frac{\theta(1-\theta)}{2}\frac{P_{k-i}^{\theta}(x_{i+1})-P_{k-i}^{\theta}(x_i)}{h_{i+1}}d\theta\\ &+\int_0^1\frac{\theta^3(1-\theta)}{3!}\frac{Q_{k-i}^{\theta}(x_{i+1})u'''(\eta_{k+1,1}^{\theta})-Q_{k-i}^{\theta}(x_i)u'''(\eta_{k,1}^{\theta})}{h_{i+1}}d\theta\\ &-\int_0^1\frac{\theta(1-\theta)^3}{3!}\frac{Q_{k-i}^{\theta}(x_{i+1})u'''(\eta_{k+1,2}^{\theta})-Q_{k-i}^{\theta}(x_i)u'''(\eta_{k,2}^{\theta})}{h_{i+1}}d\theta\end{aligned}$$

And by Lemma 3.21

$$(3.63)\qquad \frac{P_{k-i}^{\theta}(x_{i+1})-P_{k-i}^{\theta}(x_i)}{h_{i+1}}\le Ch^2x_i^{-\alpha/2-2/r}h_k$$

And with Lemma 3.19, we can get

$$(3.64)\qquad \frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k})\le Ch^2x_i^{-\alpha/2-2/r}$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned}(3.65)\qquad \frac{2}{h_i+h_{i+1}}\frac{h_{i+1}-h_i}{h_ih_{i+1}}T_{i,k}&\le h_i^{-3}h^2x_i^{1-2/r}h_kCh_k^2x_{k-1}^{\alpha/2-2}|x_k-x_i|^{1-\alpha}\\ &\le Ch^2x_i^{-\alpha/2-2/r}\end{aligned}$$

Summarizes, we have

$$(3.66)\qquad I_2\le Ch^2x_i^{-\alpha/2-2/r}$$

The case for I_4 is similar. $\square$

Now combine Lemma 3.8, Lemma 3.22, Theorem 3.10, Lemma 3.5 and Lemma 3.6 to get the final result.

For $3 \le i < N/2$

$$
\begin{aligned}
R_i &= I_1 + I_2 + I_3 + I_4 + I_5 \\
&\le Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2 (x_i^{-1-\alpha}\ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases}
\end{aligned} \tag{3.67}
$$

Combine with $i = 1, 2$, we get for $1 \le i < N/2$

$$
R_i \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2 (x_i^{-1-\alpha}\ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases} \tag{3.68}
$$

3.4. Proof of Theorem 3.3. For $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$, we have

$$
\begin{aligned}
R_i =& \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\
&+ \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right) \\
&+ \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& I_1 + I_2 + I_3^1 + I_3^2 + I_3^3 + I_4 + I_5
\end{aligned} \tag{3.69}
$$

We have estimate I_1 in Lemma 3.8 and I_2 in Lemma 3.22. We can control I_3 in similar with Theorem 3.10 by Lemma 3.20 where $2i - 1 \ge N - 1$

Lemma 3.23. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$,*

$$
\begin{aligned}
I_3 = \sum_{j=k+1}^{N-1} V_{ij} &\le Ch^2 \int_{x_k}^{x_{N-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\
&= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{N-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\
&\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{3.70}
$$

Let's study I_5 before I_4.

$$I_5 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} V_{ij} \tag{3.71}$$

Similarly, Let's define a new series of functions

Definition 3.24. *For $i < N, j \geq N$, with no confusion, we also denote in this section*

$$y_{j-i}(x) = 2T - (Z_{2N-j+i} - x^{1/r})^r, \quad Z_{2N-j+i} = T^{1/r}\frac{2N-j+i}{N} \tag{3.72}$$

$$y_{j-i}{}'(x) = (2T - y_{j-i}(x))^{1-1/r}x^{1/r-1} \tag{3.73}$$

$$y_{j-i}{}''(x) = \frac{1-r}{r}(2T - y_{j-i}(x))^{1-2/r}x^{1/r-2}Z_{2N-j+i} \tag{3.74}$$

(3.75)

$$y^\theta_{j-i}(x) = (1-\theta)y_{j-i-1}(x) + \theta y_{j-i}(x) \tag{3.76}$$

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{3.77}$$

$$P^\theta_{j-i}(x) = (h_{j-i}(x))^3 u''(y^\theta_{j-i}(x))\frac{|y^\theta_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.78}$$

$$Q^\theta_{j-i}(x) = (h_{j-i}(x))^4 \frac{|y^\theta_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.79}$$

Now we have, for $i < N, j \geq N+2$,

$$\begin{aligned}
V_{ij} &= \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right)T_{i,j} + \frac{1}{h_i}T_{i-1,j-1}\right)\\
&= \int_0^1 -\frac{\theta(1-\theta)}{2}D_h^2 P^\theta_{j-i}(x_i)d\theta\\
&\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q^\theta_{j-i}(x_{i+1})u'''(\eta^\theta_{j+1,1}) - Q^\theta_{j-i}(x_i)u'''(\eta^\theta_{j,1})}{h_{i+1}}\right)d\theta\\
&\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q^\theta_{j-i}(x_i)u'''(\eta^\theta_{j,1}) - Q^\theta_{j-i}(x_{i-1})u'''(\eta^\theta_{j-1,1})}{h_i}\right)d\theta\\
&\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q^\theta_{j-i}(x_{i+1})u'''(\eta^\theta_{j+1,2}) - Q^\theta_{j-i}(x_i)u'''(\eta^\theta_{j,2})}{h_{i+1}}\right)d\theta\\
&\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q^\theta_{j-i}(x_i)u'''(\eta^\theta_{j,2}) - Q^\theta_{j-i}(x_{i-1})u'''(\eta^\theta_{j-1,2})}{h_i}\right)d\theta
\end{aligned} \tag{3.80}$$

Similarly, we first estimate

$$D_h^2 P_{j-i}^{\theta}(\xi) = {P_{j-i}^{\theta}}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \tag{3.81}$$

Combine lemmas Lemma C.8, Lemma C.9 and Lemma C.10 , we have

LEMMA 3.25. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le i < N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in [x_{i-1}, x_{i+1}]$, we have*

$$\begin{aligned}|{P_{j-i}^{\theta}}''(\xi)| \le C h_j h^2 (&|y_j^{\theta} - x_i|^{1-\alpha} \\ &+ |y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N) \\ &+ |y_j^{\theta} - x_i|^{-1-\alpha}(|2T - x_i - y_j^{\theta}| + h_N)^2 \\ &+ (r-1)|y_j^{\theta} - x_i|^{-\alpha})\end{aligned} \tag{3.82}$$

And

LEMMA 3.26. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le i < N$, , $\xi \in [x_{i-1}, x_{i+1}]$, we have for $N+1 \le j \le 2N - \lceil \frac{N}{2} \rceil$*

$$\begin{aligned}&\frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_{i+1}) u'''(\eta_{j+1}^{\theta}) - Q_{j-i}^{\theta}(x_i) u'''(\eta_j^{\theta})}{h_{i+1}} \right) \\ &\qquad \le C h^2 h_j (|y_j^{\theta} - x_i|^{1-\alpha} + |y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N))\end{aligned} \tag{3.83}$$

for $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$

$$\begin{aligned}&\frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_i) u'''(\eta_j^{\theta}) - Q_{j-i}^{\theta}(x_{i-1}) u'''(\eta_{j-1}^{\theta})}{h_{i+1}} \right) \\ &\qquad \le C h^2 h_j (|y_j^{\theta} - x_i|^{1-\alpha} + |y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N))\end{aligned} \tag{3.84}$$

The proof see Proof 37.

Combine (3.80), Lemma 3.25 and Lemma 3.26, we have

THEOREM 3.27. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le i < N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$*

$$\begin{aligned}V_{ij} \le & C h^2 \int_{x_{j-1}}^{x_j} (|y - x_i|^{1-\alpha} \\ & + |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2 \\ & + (r-1)|y - x_i|^{-\alpha}) dy\end{aligned} \tag{3.85}$$

We can esitmate I_5 Now.

THEOREM 3.28. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le i < N$, we have*

$$I_5 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \le C h^2 + C(r-1) h^2 |T - x_{i-1}|^{1-\alpha} \tag{3.86}$$

Proof.

$$
\begin{aligned}
I_5 =& \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} V_{ij} \\
\le& Ch^2 \int_{x_{N+1}}^{x_{2N-i}} + \int_{x_{2N-i}}^{x_{2N-\lceil \frac{N}{2} \rceil}} (|y-x_i|^{1-\alpha} \\
&+ |y-x_i|^{-\alpha}(|2T-x_i-y|+h_N) + |y-x_i|^{-1-\alpha}(|2T-x_i-y|+h_N)^2 \\
&+ (r-1)|y-x_i|^{-\alpha})dy \\
=& J_1 + J_2
\end{aligned} \tag{3.87}
$$

While $x_{N+1} \le y \le x_{2N-i} = 2T - x_i$,

$$
T - x_{i-1} \le x_{N+1} - x_i \le y - x_i \le x_{2N-i} - x_i \le 2(T - x_{i-1}) \tag{3.88}
$$

and

$$
2T - x_i - y + h_N \le 2T - x_i - x_{N+1} + h_N = T - x_i \le T - x_{i-1} \tag{3.89}
$$

So

$$
\begin{aligned}
J_1 &\le Ch^2(x_{2N-i} - x_{N+1})(|T-x_{i-1}|^{1-\alpha} + (r-1)|T-x_{i-1}|^{-\alpha}) \\
&\le Ch^2(|T-x_{i-1}|^{2-\alpha} + (r-1)|T-x_{i-1}|^{1-\alpha}) \\
&\le Ch^2T^{2-\alpha} + C(r-1)h^2|T-x_{i-1}|^{1-\alpha}
\end{aligned} \tag{3.90}
$$

Otherwise, when $x_{2N-i} \le y \le x_{2N-\lceil \frac{N}{2} \rceil}$

$$
x_i + y - 2T + h_N \le y - x_i \tag{3.91}
$$

$$
\begin{aligned}
J_2 &\le Ch^2 \int_{x_{2N-i}}^{(2-2^{-r})T} |y-x_i|^{1-\alpha} + (r-1)|y-x_i|^{-\alpha} \\
&\le Ch^2(T^{2-\alpha} + (r-1)|x_{2N-i} - x_i|^{1-\alpha}) \\
&= Ch^2 + C(r-1)h^2|T-x_i|^{1-\alpha} \le Ch^2 + C(r-1)h^2|T-x_{i-1}|^{1-\alpha}
\end{aligned} \tag{3.92}
$$

Summarizes two cases, we get the result. ∎

For I_4, we have

THEOREM 3.29. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that, for $N/2 \le i \le N-1$*

$$V_{iN} = \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,N+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,N} + \frac{1}{h_i}T_{i-1,N-1}\right) \le Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \tag{3.93}$$

Proof. We use the similar skill in the last section, but more complicated. for $j = N$, Let

$$_Ly_{N-1-i}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r}\frac{N-1-i}{N} \tag{3.94}$$

$$_0y_{N-i}(x) = \frac{x^{1/r} - Z_i}{Z_1}h_N + T, \quad Z_i = T^{1/r}\frac{i}{N}, x_N = T \tag{3.95}$$

and

$$_Ry_{N+1-i}(x) = 2T - (Z_{N-1+i} - x^{1/r})^r, \quad Z_{N-1+i} = T^{1/r}\frac{N-1+i}{N} \tag{3.96}$$

Thus,

$$\begin{aligned}
&_Ly_{N-1-i}(x_{i-1}) = x_{N-2}, \quad {_Ly_{N-1-i}}(x_i) = x_{N-1}, \quad {_Ly_{N-1-i}}(x_{i+1}) = x_N \\
&_0y_{N-i}(x_{i-1}) = x_{N-1}, \quad {_0y_{N-i}}(x_i) = x_N, \quad {_0y_{N-i}}(x_{i+1}) = x_{N+1} \\
&_Ry_{N+1-i}(x_{i-1}) = x_N, \quad {_Ry_{N+1-i}}(x_i) = x_{N+1}, \quad {_Ry_{N+1-i}}(x_{i+1}) = x_{N+2}
\end{aligned}$$

Then, define

$$_Ly^{\theta}_{N-i}(x) = \theta\, {_Ly_{N-1-i}}(x) + (1-\theta)\, {_0y_{N-i}}(x) \tag{3.97}$$

$$_Ry^{\theta}_{N+1-i}(x) = \theta\, {_0y_{N-i}}(x) + (1-\theta)\, {_Ry_{N+1-i}}(x) \tag{3.98}$$

$$_Lh_{N-i}(x) = {_0y_{N-i}}(x) - {_Ly_{N-1-i}}(x) \tag{3.99}$$

$$_Rh_{N+1-i}(x) = {_Ry_{N+1-i}}(x) - {_0y_{N-i}}(x) \tag{3.100}$$

We have

$$_Ly_{N-1-i}{}'(x) = {_Ly^{1-1/r}_{N-1-i}}(x)x^{1/r-1} \tag{3.101}$$

$$_Ly_{N-1-i}{}''(x) = \frac{1-r}{r}\, {_Ly^{1-2/r}_{N-1-i}}(x)x^{1/r-2}Z_{N-1-i} \tag{3.102}$$

$$_0y_{N-i}{}'(x) = \frac{1}{r}\frac{h_N}{Z_1}x^{1/r-1} \tag{3.103}$$

$$_0y_{N-i}{}''(x) = \frac{1-r}{r^2}\frac{h_N}{Z_1}x^{1/r-2} \tag{3.104}$$

$$_Ry_{N+1-i}{}'(x) = (2T - {_Ry_{N+1-i}}(x))^{1-1/r}x^{1/r-1} \tag{3.105}$$

$$_Ry_{N+1-i}{}''(x) = \frac{1-r}{r}(2T - {_Ry_{N+1-i}}(x))^{1-2/r}x^{1/r-2}Z_{N-1+i} \tag{3.106}$$

$$ {}_L P^{\theta}_{N-i}(x) = ({}_L h_{N-i}(x))^3 \frac{|{}_L y^{\theta}_{N-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_L y^{\theta}_{N-i}(x)) \tag{3.107} $$

$$ {}_R P^{\theta}_{N+1-i}(x) = ({}_R h_{N+1-i}(x))^3 \frac{|{}_R y^{\theta}_{N+1-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_R y^{\theta}_{N+1-i}(x)) \tag{3.108} $$

$$ {}_L Q^{\theta}_{N-i}(x) = ({}_L h_{N-i}(x))^4 \frac{|{}_L y^{\theta}_{N-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.109} $$

$$ {}_R Q^{\theta}_{N+1-i}(x) = ({}_R h_{N+1-i}(x))^4 \frac{|{}_R y^{\theta}_{N+1-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.110} $$

Similar with Lemma 3.13, we can get for $l = -1, 0, 1$,

$$ \begin{aligned} T_{i+l,N+l} = &\int_0^1 -\frac{\theta(1-\theta)}{2} {}_L P^{\theta}_{N-i}(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} {}_L Q^{\theta}_{N-i}(x_{i+l})(\theta^2 u'''(\eta^{\theta}_{N+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+l,2})) d\theta \end{aligned} \tag{3.111} $$

(3.112)

$$ \begin{aligned} T_{i+l,N+1+l} = &\int_0^1 -\frac{\theta(1-\theta)}{2} {}_R P^{\theta}_{N+1-i}(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} {}_R Q^{\theta}_{N+1-i}(x_{i+l})(\theta^2 u'''(\eta^{\theta}_{N+1+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+1+l,2})) d\theta \end{aligned} $$

So we have

(3.113)

$$ \begin{aligned} V_{i,N} = &\int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 {}_L P^{\theta}_{N-i}(x_i) d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_L Q^{\theta}_{N-i}(x_{i+1}) u'''(\eta^{\theta}_{N+1,1}) - {}_L Q^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,1})}{h_{i+1}} \right) d\theta \\ &- \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_L Q^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,1}) - {}_L Q^{\theta}_{N-i}(x_{i-1}) u'''(\eta^{\theta}_{N-1,1})}{h_i} \right) d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_L Q^{\theta}_{N-i}(x_{i+1}) u'''(\eta^{\theta}_{N+1,2}) - {}_L Q^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,2})}{h_{i+1}} \right) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_L Q^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,2}) - {}_L Q^{\theta}_{N-i}(x_{i-1}) u'''(\eta^{\theta}_{N-1,2})}{h_i} \right) d\theta \end{aligned} $$

$N+1$ is similar.

We estimate $D_h^2 {}_L P^{\theta}_{N-i}(x_i) = {}_L P^{\theta}_{N-i}{}''(\xi), \xi \in [x_{i-1}, x_{i+1}]$,

Lemma 3.30.

$$ {}_L h^3_{N-i}(\xi) \le C h_N^3 \le C h^3 \tag{3.114} $$

$$ {}_R h^3_{N+1-i}(\xi) \le C h_N^3 \le C h^3 \tag{3.115} $$

$$({}_Lh^3_{N-i}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{3.116}$$

$$({}_Rh^3_{N+1-i}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{3.117}$$

$$({}_Lh^3_{N-i}(\xi))'' \le C(r-1)h^2 \tag{3.118}$$

$$({}_Rh^3_{N+1-i}(\xi))'' \le C(r-1)h^2 \tag{3.119}$$

Proof.

$$ {}_Lh_{N-i}(\xi) \le 2h_N, \quad {}_Rh_{N+1-i}(\xi) \le 2h_N \tag{3.120}$$

$$\begin{aligned} ({}_Lh^l_{N-i}(\xi))' &= l\,{}_Lh^{l-1}_{N-i}(\xi)({}_0y_{N-i}{}'(\xi) - {}_Ly_{N-1-i}{}'(\xi)) \\ &= l\,{}_Lh^{l-1}_{N-i}(\xi)x_i^{1/r-1}(\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-1/r}_{N-1-i}(\xi)) \end{aligned} \tag{3.121}$$

while

$$\begin{aligned} |\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-1/r}_{N-1-i}(\xi)| &= |\frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r}|(\frac{N-t}{N})^{r-1} - (\frac{N-s}{N})^{r-1}| \quad t \in [0,1], s \in [0,2] \\ &\le T^{1-1/r}|1 - (\frac{N-2}{N})^{r-1}| \le CT^{1-1/r}(r-1)\frac{2}{N} \end{aligned} \tag{3.122}$$

Thus,

$$({}_Lh^l_{N-i}(\xi))' \le C(r-1)h_N^{l-1}x_i^{1/r-1}h \tag{3.123}$$

$$\begin{aligned} ({}_Rh^l_{N+1-i}(\xi))' &= l\,{}_Rh^{l-1}_{N+1-i}(\xi)({}_Ry_{N+1-i}{}'(\xi) - {}_0y_{N-i}{}'(\xi)) \\ &= l\,{}_Rh^{l-1}_{N+1-i}(\xi)x_i^{1/r-1}((2T - {}_Ry_{N+1-i}(\xi))^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1}) \end{aligned} \tag{3.124}$$

Similarly,

$$\begin{aligned} |(2T - {}_Ry_{N+1-i})^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1}| &= |\eta^{1-1/r} - \frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r}|(\frac{N-s}{N})^{r-1} - (\frac{N-t}{N})^{r-1}| \quad t \in [0,1], s \in [0,2] \\ &\le T^{1-1/r}|(\frac{N-2}{N})^{r-1} - 1| \le CT^{1-1/r}(r-1)\frac{2}{N} \end{aligned} \tag{3.125}$$

And

$$\begin{aligned} ({}_Lh^3_{N-i}(\xi))'' &= 3\,{}_Lh^2_{N-i}(\xi)\,{}_Lh_{N-i}{}''(\xi) + 6\,{}_Lh_{N-i}(\xi)({}_Lh_{N-i}{}'(\xi))^2 \\ &\le Ch_N^2\frac{1-r}{r}x_i^{1/r-2}(\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i}) + Ch_N(r-1)^2h^2x_i^{2/r-2} \end{aligned} \tag{3.126}$$

$$|\frac{h_N}{rZ_1} - {}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i}| \le T^{1-1/r} + Cx_N^{1-2/r}x_N^{1/r} = CT^{1-1/r}$$

So

$$
\begin{aligned}
({}_Lh^3_{N-i}(\xi))'' &\le Ch_N^2\frac{1-r}{r}x_i^{1/r-2} + C(r-1)^2h_Nx_i^{2/r-2}h^2 \\
&\le C(r-1)h_N^2
\end{aligned}
\tag{3.127}
$$

${}_Rh^3_{N+1-i}(\xi)$ is similar. ∎

LEMMA 3.31.

$$u''({}_Ly^\theta_{N-i}(\xi)) \le Cx_{N-2}^{-\alpha/2-2} \le C \tag{3.128}$$

$$(u''({}_Ly^\theta_{N-i}(\xi)))' \le C \tag{3.129}$$

$$(u''({}_Ly^\theta_{N-i}(\xi)))'' \le C \tag{3.130}$$

Proof.

$$
\begin{aligned}
(u''({}_Ly^\theta_{N-i}(\xi)))' &= u'''({}_Ly^\theta_{N-i}(\xi)){}_L{y^\theta_{N-i}}'(\xi) \\
&\le C(\theta{}_L{y_{N-1-i}}'(\xi) + (1-\theta){}_0{y_{N-i}}'(\xi)) \\
&\le Cx_i^{1/r-1}(\theta{}_Ly^{1-1/r}_{N-1-i}(\xi) + (1-\theta)\frac{h_N}{rZ_1}) \\
&\le Cx_i^{1/r-1}x_N^{1-1/r}
\end{aligned}
\tag{3.131}
$$

And ∎

$$
\begin{aligned}
(u''({}_Ly^\theta_{N-i}(\xi)))'' &= u''''({}_Ly^\theta_{N-i}(\xi))({}_L{y^\theta_{N-i}}'(\xi))^2 + u'''({}_Ly^\theta_{N-i}(\xi)){}_L{y^\theta_{N-i}}''(\xi) \\
&\le Cx_i^{2/r-2}x_N^{2-2/r} + C\frac{r-1}{r}x_i^{1/r-2}(\theta x_N^{1-2/r}Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \\
&\le Cx_i^{2/r-2} + C(r-1)x_i^{1/r-2}T^{1-1/r}
\end{aligned}
\tag{3.132}
$$

LEMMA 3.32.

$$|{}_Ly^\theta_{N-i}(\xi) - \xi|^{1-\alpha} \le C|y^\theta_N - x_i|^{1-\alpha} \tag{3.133}$$

$$(|{}_Ly^\theta_{N-i}(\xi) - \xi|^{1-\alpha})' \le C|y^\theta_N - x_i|^{1-\alpha} \tag{3.134}$$

$$(|{}_Ly^\theta_{N-i}(\xi) - \xi|^{1-\alpha})'' \le C(r-1)|y^\theta_N - x_i|^{-\alpha} + |y^\theta_N - x_i|^{1-\alpha} \tag{3.135}$$

Proof.

$$
\begin{aligned}
({}_Ly^\theta_{N-i}(\xi) - \xi)' &= (\theta({}_Ly_{N-1-i}(\xi) - \xi) + (1-\theta)({}_0y_{N-i}(\xi) - \xi))' \\
&= \theta({}_L{y_{N-1-i}}'(\xi) - 1) + (1-\theta)({}_0{y_{N-i}}'(\xi) - 1) \\
&= \theta\xi^{1/r-1}({}_Ly^{1-1/r}_{N-1-i}(\xi) - \xi^{1-1/r}) + (1-\theta)\xi^{1/r-1}(\frac{h_N}{rZ_1} - \xi^{1-1/r})
\end{aligned}
\tag{3.136}
$$

$$
\begin{aligned}
({}_Ly^\theta_{N-i}(\xi) - \xi)'' &= \theta({}_L{y_{N-1-i}}''(\xi)) + (1-\theta)({}_0{y_{N-i}}''(\xi)) \\
&= \frac{1-r}{r}\xi^{1/r-2}(\theta{}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \le 0
\end{aligned}
\tag{3.137}
$$

And

$$|({}_Ly^\theta_{N-i}(\xi) - \xi)''| \le C(r-1)\xi^{1/r-2}T^{1-1/r} \tag{3.138}$$

We have known

$$C|x_{N-1}-x_i| \le |{}_Ly_{N-1-i}(\xi)-\xi| \le C|x_{N-1}-x_i| \tag{3.139}$$

If $\xi \le x_{N-1}$, then $({}_0y_{N-i}(\xi)-\xi)' \ge 0$, so

$$C|x_N-x_i| \le |x_{N-1}-x_{i-1}| \le |{}_Ly^{\theta}_{N-i}(\xi)-\xi| \le |x_{N+1}-x_{i+1}| \le C|x_N-x_i| \tag{3.140}$$

If $i = N-1$ and $\xi \in [x_{N-1}, x_N]$, then ${}_0y_{N-i}(\xi)-\xi$ is concave, bigger than its two neighboring points, which are equal to h_N, so

$$h_N = |x_N - x_{N-1}| \le |{}_0y_{N-i}(\xi)-\xi| \le |x_{N+1}-x_{N-1}| = 2h_N \tag{3.141}$$

So we have

$$|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha} \le C|y^{\theta}_N - x_i|^{1-\alpha} \tag{3.142}$$

While

$${}_Ly^{1-1/r}_{N-1-i}(\xi)-\xi^{1-1/r} \le ({}_Ly_{N-1-i}(\xi)-\xi)\xi^{-1/r} \tag{3.143}$$

and

$$\begin{aligned}
&|\frac{h_N}{rZ_1}-\xi^{1-1/r}| \le \max\{|\frac{h_N}{rZ_1}-x^{1-1/r}_{i-1}|, |\frac{h_N}{rZ_1}-x^{1-1/r}_{i+1}|\} \\
&\qquad \le \max\begin{cases} T^{1-1/r}-x^{1-1/r}_{i-1} \le |x_N - x_{i-1}|T^{-1/r} \le C|x_N-x_i| \\ |x^{1-1/r}_{i+1}-x^{1-1/r}_{N-1}| \le |x_{i+1}-x_{N-1}|x^{-1/r}_{N-1} \le C|x_N-x_i| \end{cases}
\end{aligned} \tag{3.144}$$

So we have

$$({}_Ly^{\theta}_{N-i}(\xi)-\xi)' \le C|y^{\theta}_N - x_i| \tag{3.145}$$

$$\begin{aligned}
(|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha})' &= |{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{-\alpha}({}_Ly^{\theta}_{N-i}(\xi)-\xi)' \\
&\le |y^{\theta}_N - x_i|^{1-\alpha}
\end{aligned} \tag{3.146}$$

Finally,

$$\begin{aligned}
(|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha})'' = (1-\alpha)&|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{-\alpha}({}_Ly^{\theta}_{N-i}(\xi)-\xi)'' \\
&+ \alpha(\alpha-1)|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{-1-\alpha}(({}_Ly^{\theta}_{N-i}(\xi)-\xi)')^2 \\
&\le C(r-1)|y^{\theta}_N - x_i|^{-\alpha} + C|y^{\theta}_N - x_i|^{1-\alpha}
\end{aligned} \tag{3.147}$$ $\square$

By the three lemmas above, for $N/2 \le i \le N-1$, we have

Lemma 3.33.

$$\begin{aligned}
&D^2_h {}_LP^{\theta}_{N-i}(x_i) = {}_LP^{\theta\,\prime\prime}_{N-i}(\xi) \quad \xi \in [x_{i-1}, x_{i+1}] \\
&\qquad \le Ch^3|y^{\theta}_N - x_i|^{1-\alpha} + C(r-1)(h^3|y^{\theta}_N - x_i|^{-\alpha} + h^2|y^{\theta}_N - x_i|^{1-\alpha})
\end{aligned} \tag{3.148}$$

And

LEMMA 3.34.

$$\frac{2}{h_i+h_{i+1}}\left(\frac{{}_LQ^{\theta}_{N-i}(x_{i+1})u'''(\eta^{\theta}_{N+1})-{}_LQ^{\theta}_{N-i}(x_i)u'''(\eta^{\theta}_{N})}{h_{i+1}}\right) \le Ch^3|y^{\theta}_N-x_i|^{1-\alpha} \tag{3.149}$$

And immediately, For $N/2 \le i \le N-2$

$$\begin{aligned} V_{iN} &\le C\int_{x_{N-1}}^{x_N} h^2|y-x_i|^{1-\alpha}+C(r-1)h^2|y-x_i|^{-\alpha}+h|y-x_i|^{1-\alpha}dy \\ &\le Ch^2h_N|T-x_i|^{1-\alpha}+C(r-1)h^2|x_{N-1}-x_i|^{1-\alpha}+Chh_N|T-x_i|^{1-\alpha} \\ &\le Ch^2+C(r-1)h^2|T-x_{i-1}|^{1-\alpha} \end{aligned} \tag{3.150}$$

But expecially, when $i=N-1$,

$$\begin{aligned} V_{N-1,N} = &\int_0^1 -\frac{\theta^{2-\alpha}(1-\theta)}{2}\frac{2}{h_{N-1}+h_N}\left(\frac{1}{h_{N-1}}h^{4-\alpha}_{N-1}u''(y^{\theta}_{N-1})-(\frac{1}{h_{N-1}}+\frac{1}{h_N})h^{4-\alpha}_N u''(y^{\theta}_N)+\frac{1}{h_N}h^{4-\alpha}_{N+1}u''(y^{\theta}_{N+1})\right)d\theta \\ &+\int_0^1\frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{{}_LQ^{\theta}_{N-i}(x_{i+1})u'''(\eta^{\theta}_{N+1,1})-{}_LQ^{\theta}_{N-i}(x_i)u'''(\eta^{\theta}_{N,1})}{h_{i+1}}\right)d\theta \\ &-\int_0^1\frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{{}_LQ^{\theta}_{N-i}(x_i)u'''(\eta^{\theta}_{N,1})-{}_LQ^{\theta}_{N-i}(x_{i-1})u'''(\eta^{\theta}_{N-1,1})}{h_i}\right)d\theta \\ &-\int_0^1\frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{{}_LQ^{\theta}_{N-i}(x_{i+1})u'''(\eta^{\theta}_{N+1,2})-{}_LQ^{\theta}_{N-i}(x_i)u'''(\eta^{\theta}_{N,2})}{h_{i+1}}\right)d\theta \\ &+\int_0^1\frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{{}_LQ^{\theta}_{N-i}(x_i)u'''(\eta^{\theta}_{N,2})-{}_LQ^{\theta}_{N-i}(x_{i-1})u'''(\eta^{\theta}_{N-1,2})}{h_i}\right)d\theta \end{aligned} \tag{3.151}$$

while combine Lemma 3.30

$$\begin{aligned} &\frac{2}{h_{N-1}+h_N}\left(\frac{1}{h_{N-1}}h^{4-\alpha}_{N-1}u''(y^{\theta}_{N-1})-(\frac{1}{h_{N-1}}+\frac{1}{h_N})h^{4-\alpha}_N u''(y^{\theta}_N)+\frac{1}{h_N}h^{4-\alpha}_{N+1}u''(y^{\theta}_{N+1})\right) \\ &=D^2_h(h^{4-\alpha}_{N-1\to N}(x_i)u''(y^{\theta}_{N-1\to N}(x_i))) \\ &\le Ch^{4-\alpha}_N+C(r-1)h^{3-\alpha}_N \le Ch^{4-\alpha}+C(r-1)h^2|T-x_{N-1-1}|^{1-\alpha} \end{aligned} \tag{3.152}$$

Similarly with $j=N+1$. □

I_6, I_7 is easy. Similar with Lemma 3.22 and Lemma 3.6, we have

THEOREM 3.35. *There is a constant* $C = C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that For* $N/2 \le i \le N$,

$$I_6 = \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}(T_{i-1,2N-\lceil\frac{N}{2}\rceil+1}+T_{i-1,2N-\lceil\frac{N}{2}\rceil})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,2N-\lceil\frac{N}{2}\rceil+1}\right) \le Ch^2 \tag{3.153}$$

Proof. In fact, let $l = 2N - \lceil\frac{N}{2}\rceil + 1$

$$\begin{aligned}&\frac{1}{h_i}(T_{i-1,l}+T_{i-1,l-1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,l}\\&=\frac{1}{h_i}(T_{i-1,l}-T_{i,l})+\frac{1}{h_i}(T_{i-1,l-1}-T_{i,l})+(\frac{1}{h_i}-\frac{1}{h_{i+1}})T_{i,l}\end{aligned} \tag{3.154}$$

While, by Lemma A.2

$$\begin{aligned}\frac{1}{h_i}(T_{i-1,l}-T_{i,l}) &= \int_{x_{l-1}}^{x_l}(u(y)-\Pi_h u(y))\frac{|x_{i-1}-y|^{1-\alpha}-|x_i-y|^{1-\alpha}}{h_i\Gamma(2-\alpha)}dy\\&\le C\int_{x_{l-1}}^{x_l}h_l^2u''(\eta)\,\frac{|\xi-y|^{-\alpha}}{\Gamma(1-\alpha)}dy\\&\le Ch_l^3x_{l-1}^{\alpha/2-2}T^{-\alpha}\\&\le Ch_l^3\end{aligned} \tag{3.155}$$

Thus,

$$\frac{2}{h_i+h_{i+1}}\frac{1}{h_i}(T_{i-1,l}-T_{i,l}) \le Ch_l^2 \tag{3.156}$$

For

$$\frac{1}{h_i}(T_{i-1,l-1}-T_{i,l}) = \int_0^1 -\frac{\theta(1-\theta)}{2}\frac{h_{l-1}^3|y_{l-1}^\theta-x_{i-1}|^{1-\alpha}u''(\eta_{l-1}^\theta)-h_l^3|y_l^\theta-x_i|^{1-\alpha}u''(\eta_l^\theta)}{h_i}d\theta \tag{3.157}$$

And Similar with Lemma 3.19, we can get

$$\frac{h_{l-1}^3|y_{l-1}^\theta-x_{i-1}|^{1-\alpha}u''(\eta_{l-1}^\theta)-h_l^3|y_l^\theta-x_i|^{1-\alpha}u''(\eta_l^\theta)}{(h_i+h_{i+1})h_i} \le Ch_l^2|y_l^\theta-x_i|^{1-\alpha} \tag{3.158}$$

So

$$\frac{2}{h_i+h_{i+1}}\frac{1}{h_i}(T_{i-1,l-1}-T_{i,l}) \le Ch^2 \tag{3.159}$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}\frac{h_{i+1}-h_i}{h_ih_{i+1}}T_{i,l} &\le h_i^{-3}h^2x_i^{1-2/r}h_lCh_l^2x_{l-1}^{\alpha/2-2}|x_l-x_i|^{1-\alpha}\\&\le Ch^2\end{aligned} \tag{3.160}$$

Summarizes, we have

$$I_6 \le Ch^2 \tag{3.161}$$ ∎

And

LEMMA 3.36. *There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i \leq N$,*

$$I_7 = \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.162}$$

Proof. For $i \leq N, j \geq 2N - \lceil \frac{N}{2} \rceil + 2$, we have

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) dy \\ &\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2 - 2/r} |y - x_{i+1}^{-1-\alpha} dy \\ &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy \end{aligned}$$

$$\begin{aligned} \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{(2-2^{-r})T}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r} dy \\ &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r}T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\ &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r}\ h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned} \tag{3.163}$$

∎

Now we can conclude a part of the theorem Theorem 3.3 at the beginning of this section.

By Lemma 3.8 Lemma 3.22 Lemma 3.23 Theorem 3.29 Theorem 3.28 Theorem 3.35 Lemma 3.36 , we have

THEOREM 3.37. *there exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $N/2 \leq i < N$,*

$$R_i = \sum_{j=1}^{7} I_j \leq C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.164}$$

And what we left is the case $i = N$. Fortunately, we can use the same department of R_i above, and it is symmetric. Most of the item has been esitmated by Lemma 3.8 and Theorem 3.35, we just need to consider I_3, I_4.

THEOREM 3.38. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$I_3 = \sum_{j=\lceil \frac{N}{2} \rceil+1}^{N-1} V_{Nj} \le Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha} \tag{3.165}$$

Proof. DEFINITION 3.39. *For $N/2 \le j < N$, Let's define*

$$y_j(x) = (\frac{Z_1}{h_N}(x - x_N) + Z_j)^r, \quad Z_j = T^{1/r}\frac{j}{N} \tag{3.166}$$

We can see that is the inverse of the function ${}_0y_{N-i}(x)$ defined in Theorem 3.29.

$$y_j'(x) = y_j^{1-1/r}(x)\frac{rZ_1}{h_N} \tag{3.167}$$

$$y_j''(x) = y_j^{1-2/r}(x)\frac{r(r-1)Z_1}{h_N} \tag{3.168}$$

With the scheme we used several times, we can get

LEMMA 3.40. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le j < N$, $\xi \in [x_{N-1}, x_{N+1}]$,*

$$h_j(\xi)^3 \le Ch^3 \tag{3.169}$$

$$(h_j^3(\xi))' \le C(r-1)h^3 \tag{3.170}$$

$$(h_j^3(\xi))'' \le C(r-1)h^3 \tag{3.171}$$

$$u''(y_j^\theta(\xi)) \le C \tag{3.172}$$

$$(u''(y_j^\theta(\xi)))' \le C \tag{3.173}$$

$$(u''(y_j^\theta(\xi)))'' \le C \tag{3.174}$$

$$|\xi - y_j^\theta(\xi)|^{1-\alpha} \le C|x_N - y_j^\theta|^{1-\alpha} \tag{3.175}$$

$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})' \le C|x_N - y_j^\theta|^{1-\alpha} \tag{3.176}$$

$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})'' \le C|x_N - y_j^\theta|^{1-\alpha} + C(r-1)|x_N - y_j^\theta|^{-\alpha} \tag{3.177}$$

LEMMA 3.41. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le j < N$,*

$$V_{Nj} \le Ch^2 \int_{x_{j-1}}^{x_j} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \tag{3.178}$$

Therefore,

$$\begin{aligned} I_3 &\le Ch^2 \int_{x_{\lceil \frac{N}{2} \rceil}}^{x_{N-1}} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \\ &\le Ch^2(|T - x_{N-1}|^{2-\alpha} + (r-1)|T - x_{N-1}|^{1-\alpha}) \end{aligned} \tag{3.179}$$

$\square$

For $j = N$,

LEMMA 3.42.
(3.180)
$$V_{N,N} = \frac{1}{h_N^2}\left(T_{N-1,N-1} - 2T_{N,N} + T_{N+1,N+1}\right) \le Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha}$$

Proof.
(3.181)
$$\begin{aligned}
V_{N,N} = &\int_0^1 -\frac{\theta(1-\theta)^{2-\alpha}}{2}\frac{1}{h_N^2}\left(h_{N-1}^{4-\alpha}u''(y_{N-1}^\theta) - 2h_N^{4-\alpha}u''(y_N^\theta) + h_{N+1}^{4-\alpha}u''(y_{N+1}^\theta)\right)d\theta\\
&+\int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,1}^\theta) - Q_{N\to N}^\theta(x_i)u'''(\eta_{N,1}^\theta)}{h_N}\right)d\theta\\
&-\int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,1}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,1}^\theta)}{h_N}\right)d\theta\\
&-\int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,2}^\theta) - Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta)}{h_N}\right)d\theta\\
&+\int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,2}^\theta)}{h_N}\right)d\theta
\end{aligned}$$

∎

So combine Lemma 3.8, Theorem 3.35, Theorem 3.38, Lemma 3.42 We have

LEMMA 3.43.
$$R_N \le C(r-1)h^2|T - x_{N-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0\\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0\\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0\end{cases} \tag{3.182}$$

and with Theorem 3.37 we prove the Theorem 3.3

3.5. Truncation error. combine Theorem 3.1, Theorem 3.2 and Theorem 3.3 we get For $1 \le i \le N$
(3.183)
$$R_i \le C_2(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} C_1h^2x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0\\ C_1h^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0\\ C_1h^{r\alpha/2+r}x_i^{-1-\alpha/2}, & r\alpha/2 + r - 2 < 0\end{cases}$$

But,
$$h^2x_i^{-\alpha/2-2/r} \le T^{\alpha/2-2/r}\begin{cases} h^2x_i^{-\alpha}, & \text{if} \quad r\alpha/2 - 2 \ge 0\\ h^{r\alpha/2}x_i^{-\alpha}, & \text{if} \quad r\alpha/2 - 2 \le 0\end{cases} \tag{3.184}$$
$$h^{r\alpha/2+r}x_i^{-1-\alpha} \le T^{-1}h^{r\alpha/2}x_i^{-\alpha}, \quad \text{if} \quad r\alpha/2 - 2 \le 0 \tag{3.185}$$
(3.186)

And when $r\alpha/2 - 2 = -r < 0$,
$$\begin{aligned} h^2x_i^{-1-\alpha}\ln(i)h^{-r\alpha/2}x_i^\alpha &= h^r x_i^{-1}\ln(i)\\ &= T^{-1}\frac{\ln(i)}{i^r} \le C(T,r)\end{aligned} \tag{3.187}$$

and

$$h^2 \ln(N) h^{-r\alpha/2} x_i^\alpha = h^r \ln(N) x_i^\alpha \leq T^\alpha \frac{\ln(N)}{N^r} \leq C(T,\alpha,r) \tag{3.188}$$

So for $1 \leq i \leq N$,

$$R_i \leq C_2(r-1)h^2|T - x_{i-1}|^{1-\alpha} + C_1 h^{\min\{\frac{r\alpha}{2},2\}} x_i^{-\alpha} \tag{3.189}$$

And for $i \geq N$, it is symmetric for i and $2N - i$.

The proof of Theorem 2.5 completed.

4. Proof of Theorem 2.6. Review subsection 2.1,we have (**??**) and (**??**),

$$\tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \tag{4.1}$$

$$a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \tag{4.2}$$

Thus

LEMMA 4.1.

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right) \tag{4.3}$$

DEFINITION 4.2. *We call one matrix a M matrix , which means its entries are positive on major diagonal and nonpositive on others, and Strictly diagonally dominant in rows.*

Now we have

LEMMA 4.3. *The matrix A defined by* (**??**) *is a M matrix. and*

$$\begin{aligned} S_i :=& \sum_{j=1}^{2N-1} a_{ij} \\ =& -\kappa_\alpha \sum_{j=1}^{2N-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \tilde{a}_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) \tilde{a}_{i,j} + \frac{1}{h_i} \tilde{a}_{i-1,j} \right) \\ \geq& C_A (x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \end{aligned} \tag{4.4}$$

Proof. Let

$$g(x) = g_0(x) + g_{2N}(x) \tag{4.5}$$

where

$$g_0(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x - x_0|^{3-\alpha} - |x - x_1|^{3-\alpha}}{h_1}$$

$$g_{2N}(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N} - x|^{3-\alpha} - |x_{2N-1} - x|^{3-\alpha}}{h_{2N}}$$

Thus

$$-\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

Then

$$\begin{aligned} S_i :=& \sum_{j=1}^{2N-1} a_{ij} \\ =& \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ =& D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i) \end{aligned} \tag{4.6}$$

When $i = 1$

$$\begin{aligned} D_h^2 g_0(x_1) &= \frac{2}{h_1+h_2}\left(\frac{1}{h_2}g_0(x_2) - (\frac{1}{h_1}+\frac{1}{h_2})g_0(x_1) + \frac{1}{h_1}g_0(x_0)\right) \\ &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)}\frac{h_1^{3-\alpha}+h_2^{3-\alpha}+2h_1^{2-\alpha}h_2-(h_1+h_2)^{3-\alpha}}{(h_1+h_2)h_1h_2} \\ &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)}\frac{h_1^{3-\alpha}+h_2^{3-\alpha}+2h_1^{2-\alpha}h_2-(h_1+h_2)^{3-\alpha}}{(h_1+h_2)h_1^{1-\alpha}h_2}h_1^{-\alpha} \\ &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)}\frac{1+(2^r-1)^{3-\alpha}+2(2^r-1)-(2^r)^{3-\alpha}}{2^r(2^r-1)}h_1^{-\alpha} \end{aligned} \tag{4.7}$$

but

$$1+(2^r-1)^{3-\alpha}+2(2^r-1)-(2^r)^{3-\alpha} > 0 \tag{4.8}$$

While for $i \geq 2$

$$\begin{aligned} D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \\ &= -\kappa_\alpha \frac{|\xi - x_0|^{1-\alpha} - |\xi - x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1} \\ &= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)}|\xi-\eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\ &\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)}x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)}2^{-r\alpha}x_i^{-\alpha} \end{aligned} \tag{4.9}$$

So

$$\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}g_0(x_{i+1}) - (\frac{1}{h_i}+\frac{1}{h_{i+1}})g_0(x_i) + \frac{1}{h_i}g_0(x_{i-1})\right) \geq Cx_i^{-\alpha} \tag{4.10}$$

symmetricly,

$$\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}g_{2N}(x_{i+1}) - (\frac{1}{h_i}+\frac{1}{h_{i+1}})g_{2N}(x_i) + \frac{1}{h_i}g_{2N}(x_{i-1})\right) \geq C(\alpha,r)(2T-x_i)^{-\alpha} \tag{4.11}$$

Let

$$g(x) = \begin{cases} x, & 0 < x \leq T \\ 2T - x, & T < x < 2T \end{cases} \tag{4.12}$$

And define

$$G = \mathrm{diag}(g(x_1), ..., g(x_{2N-1})) \tag{4.13}$$

Then

LEMMA 4.4. *The matrix $B := AG$, the major diagnal is positive, and nonpositive on others. And there is a constant $C_{AG}, C = C(\alpha, r)$ such that*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C_{AG}(x_i^{1-\alpha} + (2T-x_i)^{1-\alpha}) + C\begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \tag{4.14}$$

Proof.

$$b_{ij} = a_{ij}g(x_j) = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}}\tilde{a}_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})\tilde{a}_{i,j} + \frac{1}{h_i}\tilde{a}_{i-1,j} \right) g(x_j)$$

Since

$$g(x) \equiv \Pi_h g(x) \tag{4.15}$$

by **??**, we have

$$\begin{aligned}
\tilde{M}_i &:= \sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} \tilde{a}_{ij} g(x_j) \\
&= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} \Pi_h g(y) dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} g(y) dy \\
&= \frac{-2}{\Gamma(4-\alpha)} |T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)} (x_i^{3-\alpha} + (2T - x_i)^{3-\alpha}) \\
&:= w(x_i) = p(x_i) + q(x_i)
\end{aligned} \tag{4.16}$$

Thus,

$$\begin{aligned}
M_i &:= \sum_{j=1}^{2N-1} b_{ij} = \sum_{j=1}^{2N-1} a_{ij} g(x_j) \\
&= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \tilde{M}_{i+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) \tilde{M}_i + \frac{1}{h_i} \tilde{M}_{i-1} \right) \\
&= D_h^2(-\kappa_\alpha p)(x_i) - \kappa_\alpha D_h^2 q(x_i)
\end{aligned} \tag{4.17}$$

for $1 \leq i \leq N-1$,by Lemma A.1

$$\begin{aligned}
D_h^2(-\kappa_\alpha p)(x_i) &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} p(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) p(x_i) + \frac{1}{h_i} p(x_{i-1}) \right) \\
&= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1}) \\
&\geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - x_{i-1}|^{1-\alpha}
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
D_h^2(-\kappa_\alpha p)(x_N) &:= -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left(\frac{1}{h_{N+1}} p(x_{N+1}) - (\frac{1}{h_N} + \frac{1}{h_{N+1}}) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right) \\
&= \frac{4\kappa_\alpha}{\Gamma(4-\alpha) h_N^2} h_N^{3-\alpha} \\
&= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - x_{N-1})^{1-\alpha}
\end{aligned} \tag{4.19}$$

Symmetricly for $i \geq N$, we get

$$D_h^2(-\kappa_\alpha p)(x_i) \geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \tag{4.20}$$

Similarly, we can get

$$
\begin{aligned}
D_h^2 q(x_i) &:= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} q(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right) \\
&\leq \frac{2^{r(\alpha-1)+1}}{\Gamma(2-\alpha)} (x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}), \quad i = 1, \cdots, 2N-1
\end{aligned} \tag{4.21}
$$

□

So, we get the result.

Notice that

$$
x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha} \tag{4.22}
$$

We can get

THEOREM 4.5. *There exists a real $\lambda = \lambda(T, \alpha, r) > 0$ and $C = C(T, \alpha, r) > 0$ such that $B := A(\lambda I + G)$ is an M matrix. And*

$$
M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \tag{4.23}
$$

Proof. By Lemma 4.3 with C_A and Lemma 4.4 with C_{AG} , it's sufficient to take $\lambda = (C + 2TC_{AG})/C_A$, then

$$
M_i \geq C \left((x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \right) \tag{4.24}
$$

□

Now, we can prove the convergency Theorem 2.6.

For equation

$$
AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1} U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1} U = F \tag{4.25}
$$

which means

$$
\sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + g(x_j)} = -\tau_i \tag{4.26}
$$

where $\epsilon_i = u(x_i) - u_i$.

And if

$$
\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| = \max_{1 \leq i \leq 2N-1} \left|\frac{\epsilon_i}{\lambda + g(x_i)}\right| \tag{4.27}
$$

Then, since $B = A(\lambda I + G)$ is an M matrix, it is Strictly diagonally dominant. Thus,

$$
\begin{aligned}
|\tau_{i_0}| &= \left| \sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + g(x_j)} \right| \\
&\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_j}{\lambda + g(x_j)} \right| \\
&\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \\
&= \sum_{j=1}^{2N-1} b_{i_0,j} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \\
&= M_{i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right|
\end{aligned} \tag{4.28}
$$

By Theorem 2.5 and Theorem 4.5,

We kown that there exists constants $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$, and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that

$$|\frac{\epsilon_i}{\lambda + g(x_i)}| \leq |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| \leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} + C_2(r-1)h^2 \tag{4.29}$$

as $\lambda + g(x_i) \leq \lambda + T$

So, we can get

$$|\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{r\alpha}{2}, 2\}} \tag{4.30}$$

The convergency has been proved.

5. Experimental results.

6. Remarks.
some remarks.

In Theorem 2.3 If $f \in L^\infty(\Omega)$ then $u \in C_{\alpha/2}(\Omega)$, which is Proposition 1.1 in [2].

When $\|f\|_\beta^{(\gamma)} < \infty$, where $\beta > 2-\alpha$ and $\gamma \in [-\alpha, -\alpha/2]$, we observed convergent order $\min\{r(\alpha+\gamma), 2\}$ in numerical experiments. And we can prove that kind theorems with the techneque we used in this paper.

Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x) \in C^2\Omega$, there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

$$
\begin{aligned}
D_h^2 g(x_i) &:= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\
&= g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1})
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
&\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\
&\quad = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right)
\end{aligned} \tag{A.2}
$$

And if $g(x) \in C^4(\Omega)$, then

$$
\begin{aligned}
&\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\
&\quad = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2))
\end{aligned} \tag{A.3}
$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$.

Proof.

$$
g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)
$$

$$
g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})
$$

Subsitute them in the left side of (A.1), we have

$$
\begin{aligned}
&\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\
&\quad = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2)
\end{aligned}
$$

Now, using intermediate value theorem , there exists $\xi \in [\xi_1, \xi_2]$ such that

$$
\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)
$$

For the second equation, similarly

$$
g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy
$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y)dy$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy$$

Expecially,

$$\begin{aligned} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy &= \frac{h_i^4}{4!} g''''(\eta_1) \\ \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy &= \frac{h_{i+1}^4}{4!} g''''(\eta_2) \end{aligned} \tag{A.4}$$

where $\eta_1 \in (x_{i-1}, x_i), \eta_2 \in (x_i, x_{i+1})$. Subsitute them to the left side of (A.3), we can get the result. □

LEMMA A.2. *Denote* $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0, 1]$,

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \tag{A.5}$$

(A.6)
$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3((1-\theta)^2 u'''(\eta_1) - \theta^2 u'''(\eta_2))$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$

$$u(x_j) = u(y_j^\theta) + \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

Thus

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - \theta u(x_{j-1}) - (1-\theta)u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2((1-\theta)u''(\xi_1) + \theta u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) - \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) + \frac{\theta^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$. Thus

$$
\begin{aligned}
u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - \theta u(x_{j-1}) - (1-\theta)u(x_j) \\
&= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3((1-\theta)^2 u'''(\eta_1) - \theta^2 u'''(\eta_2))
\end{aligned}
$$

□

LEMMA A.3. *For $x \in [x_{j-1}, x_j]$*

$$
\begin{aligned}
|u(x) - \Pi_h u(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_x^{x_j} u'(y)dy \right| \\
&\le \int_{x_{j-1}}^{x_j} |u'(y)|dy
\end{aligned}
\tag{A.7}
$$

If $x \in [0, x_1]$, with Corollary 2.4, we have

$$
|u(x) - \Pi_h u(x)| \le \int_0^{x_1} |u'(y)|dy \le \int_0^{x_1} C y^{\alpha/2-1} dy \le C \frac{2}{\alpha} x_1^{\alpha/2}
\tag{A.8}
$$

Similarly, if $x \in [x_{2N-1}, 1]$, we have

$$
|u(x) - \Pi_h u(x)| \le C\frac{2}{\alpha}(2T - x_{2N-1})^{\alpha/2} = C\frac{2}{\alpha} x_1^{\alpha/2}
\tag{A.9}
$$

Appendix B. Inequality. For convenience, we use the notation and $\simeq$. That $x_1 \simeq y_1$, means that $c_1 x_1 \le y_1 \le C_1 x_1$ for some constants c_1 and C_1 that are independent of mesh parameters.

LEMMA B.1.

$$
h_i \le rT^{1/r} h \begin{cases} x_i^{1-1/r}, & 1 \le i \le N \\ (2T - x_{i-1})^{1-1/r}, & N < i \le 2N-1 \end{cases}
\tag{B.1}
$$

$$
h_i \ge rT^{1/r} h \begin{cases} x_{i-1}^{1-1/r}, & 1 \le i \le N \\ (2T - x_i)^{1-1/r}, & N < i \le 2N-1 \end{cases}
\tag{B.2}
$$

Proof. For $1 \le i \le N$,

$$
\begin{aligned}
h_i &= T\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right) \\
&\le rT\frac{1}{N}\left(\frac{i}{N}\right)^{r-1} = rT^{1/r} h x_i^{1-1/r}
\end{aligned}
$$

$$
h_i \ge rT\frac{1}{N}\left(\frac{i-1}{N}\right)^{r-1} = rT^{1/r} h x_{i-1}^{1-1/r}
$$

For $N < i \le 2N$,

$$
\begin{aligned}
h_i &= T\left(\left(\frac{2N-i+1}{N}\right)^r - \left(\frac{2N-i}{N}\right)^r\right) \\
&\le rT\frac{1}{N}\left(\frac{2N-i+1}{N}\right)^{r-1} = rT^{1/r} h (2T - x_{i-1})^{1-1/r}
\end{aligned}
$$

$$h_i \geq rT\frac{1}{N}\left(\frac{2N-i}{N}\right)^{r-1} = rT^{1/r}h(2T-x_i)^{1-1/r} \qquad \square$$

LEMMA B.2. *There is a constant* $C = 2^{|r-2|}r(r-1)T^{2/r}$ *such that for all* $i \in \{1, 2, \cdots, 2N-1\}$

$$|h_{i+1} - h_i| \leq Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases} \tag{B.3}$$

Proof.

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \leq i \leq 2N-1 \end{cases}$$

For $i = 1$,

$$h_2 - h_1 = T(2^r - 2)\left(\frac{1}{N}\right)^r = (2^r - 2)T^{2/r}h^2 x_1^{1-2/r}$$

For $2 \leq i \leq N-1$, by Lemma A.1, we have

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T\ N^{-2}\eta^{r-2}, \quad \eta \in [\frac{i-1}{N}, \frac{i+1}{N}] \\ &= C(r-1)h^2 x_i^{1-2/r} \end{aligned}$$

Summarizes the inequalities, we can get

$$|h_{i+1} - h_i| \leq 2^{|r-2|}r(r-1)T^{2/r}h^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases} \tag{B.4} \qquad \square$$

Appendix C. Proofs of some technical details.

Additional proof of Theorem 3.1. For $2 \leq i \leq N-1$,

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\ &\qquad \leq C\frac{2}{h_i + h_{i+1}}(h_i^3 x_{i-1}^{-2-\alpha/2} + h_{i+1}^3 x_i^{-2-\alpha/2}) \\ &\qquad \leq 2C(h_i^2 x_{i-1}^{-2-\alpha/2} + h_{i+1}^2 x_i^{-2-\alpha/2}) \end{aligned}$$

There is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

$$\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq Ch^2 x_i^{-\alpha/2-2/r}, \quad 2 \leq i \leq N-1$$

For $i = 1$, by (A.4)

$$\begin{aligned}&\frac{1}{4!}\frac{2}{h_1+h_2}(h_1^3 f''(\eta_1) + h_2^3 f''(\eta_2))\\ &\quad = \frac{2}{h_1+h_2}\left(\frac{1}{h_1}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy + \frac{1}{4!}h_2^3 f''(\eta_2)\right)\end{aligned}$$

We have proved above that

$$\frac{2}{h_1+h_2}h_2^3 f''(\eta_2) \leq Ch^2 x_1^{-\alpha/2-2/r}$$

and we can get

$$\begin{aligned}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy &\leq C\frac{1}{3!}\int_0^{x_1} y^{1-\alpha/2}dy\\ &= C\frac{1}{3!(2-\alpha/2)}x_1^{2-\alpha/2}\end{aligned}$$

so

$$\frac{2}{h_1+h_2}\frac{1}{h_1}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy = \frac{C2^{1-r}}{3!(2-\alpha/2)}x_1^{-\alpha/2} = \frac{C2^{1-r}}{3!(2-\alpha/2)}T^{2/r}h^2 x_1^{-\alpha/2-2/r}$$

And for $i = N$, we have

$$\begin{aligned}&\frac{2}{h_N+h_{N+1}}(h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2))\\ &\qquad = h_N^2(f''(\eta_1) + f''(\eta_2))\\ &\qquad \leq r^2 T^{2/r} h^2 x_N^{2-2/r}\ 2Cx_{N-1}^{-2-\alpha/2}\\ &\qquad \leq 2r^2 T^{2/r} C2^{-r(-2-\alpha/2)}\ h^2 x_N^{-\alpha/2-2/r}\end{aligned}$$

Finally, $N+1 \leq i \leq 2N-1$ is symmetric to the first half of the proof, so we can conclude that

$$\frac{2}{h_i+h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq Ch^2\begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N\\ (2T-x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N-1\end{cases}$$

∎

LEMMA C.1. *By a standard error estimate for linear interpolation, and Corollary 2.4, There is a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *for* $2 \leq j \leq N$,

(C.1) $$|u(y) - \Pi_h u(y)| \leq Ch^2 y^{\alpha/2-2/r}, \quad \text{for } y \in [x_{j-1}, x_j]$$

symmetricly, for $N < j \leq 2N-1$, *we have*

(C.2) $$|u(y) - \Pi_h u(y)| \leq Ch^2(2T-y)^{\alpha/2-2/r}$$

LEMMA C.2. *There is a constant* $C = C(\alpha, r)$ *such that for all* $1 \leq i < N/2$, $\max\{2i+1, i+3\} \leq j \leq 2N$ *, we have*

(C.3) $$D_h^2 K_y(x_i) \leq C\frac{y^{-1-\alpha}}{\Gamma(-\alpha)}, \quad y \in [x_{j-1}, x_j]$$

Proof. Sinec $y \geq x_{j-1} > x_{i+1}$, by Lemma A.1, if $j-1 > i+1$ ∎

$$\begin{aligned}
D_h^2 K_y(x_i) = K_y''(\xi) &= \frac{|y-\xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\
&\leq \frac{(y-x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)} \\
&\leq (1-(\frac{2}{3})^r)^{-1-\alpha} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}
\end{aligned}$$

LEMMA C.3. *There is a constant $C = C(\alpha, r)$ such that for all $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil$, $1 \leq j \leq k-1$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_h^2 K_y(x_i) \leq C \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \tag{C.4}$$

Proof. Sinec $y \leq x_j < x_{i-1}$, by Lemma A.1, ∎

$$\begin{aligned}
D_h^2 K_y(x_i) &= \frac{|\xi-y|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\
&\leq \frac{(x_{i-1}-x_j)^{-1-\alpha}}{\Gamma(-\alpha)} \leq \frac{(x_{i-1}-x_{k-1})^{-1-\alpha}}{\Gamma(-\alpha)} \\
&\leq ((\frac{2}{3})^r - (\frac{1}{2})^r)^{-1-\alpha} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)}
\end{aligned}$$

LEMMA C.4. *While $0 \leq i < N/2$, By Lemma* A.3

$$\begin{aligned}
|T_{i1}| &\leq C \int_0^{x_1} x_1^{\alpha/2} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
&= C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2} \left| x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha} \right| \\
&\leq C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2+2-\alpha} = C \frac{1}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1
\end{aligned} \tag{C.5}$$

For $2 \leq j \leq N$, by Lemma A.2 *and Corollary* 2.4

$$\begin{aligned}
|T_{ij}| &\leq \frac{C}{4} \int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2} \frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
&\leq \frac{C}{4\Gamma(3-\alpha)} h_j^2 x_{j-1}^{\alpha/2-2} \left| |x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha} \right|
\end{aligned} \tag{C.6}$$

LEMMA C.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$\sum_{j=1}^{3} S_{1j} \leq Ch^2 x_1^{-\alpha/2-2/r} \tag{C.7}$$

$$\sum_{j=1}^{4} S_{2j} \leq Ch^2 x_2^{-\alpha/2-2/r} \tag{C.8}$$

Proof.

$$S_{1j} = \frac{2}{x_2}\left(\frac{1}{x_1}T_{0j} - \left(\frac{1}{x_1}+\frac{1}{h_2}\right)T_{1j} + \frac{1}{h_2}T_{2j}\right)$$

So, by Lemma C.4

$$S_{11} \le \frac{2}{x_2 x_1} 4\frac{C}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \le C x_1^{-\alpha/2}$$

$$S_{12} \le \frac{2}{x_2 x_1}\frac{C}{4\Gamma(3-\alpha)} h_2^2 x_1^{\alpha/2-2}\left(x_2^{2-\alpha} + 2h_2^{2-\alpha} + h_2^{2-\alpha}\right) \le C x_1^{-\alpha/2}$$

$$S_{13} \le \frac{2}{x_2 x_1}\frac{C}{4\Gamma(3-\alpha)} h_3^2 x_2^{\alpha/2-2}\left(x_3^{2-\alpha} + 2x_3^{2-\alpha} + h_3^{2-\alpha}\right) \le C x_1^{-\alpha/2}$$

But

$$x_1^{-\alpha/2} = T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

$i = 2$ is similar. ∎

Lemma C.6. *There exists a constant $C = C(T, r, l)$ such that For $3 \le i \le N-1, k = \lceil \frac{i}{2} \rceil, k \le j \le \min\{2i-1, N-1\}$, when $\xi \in [x_{i-1}, x_{i+1}]$,*

$$(h_{j-i}^3(\xi))' \le (r-1) C h^2 x_i^{1-2/r} h_j \tag{C.9}$$

$$(h_{j-i}^4(\xi))' \le (r-1) C h^2 x_i^{1-2/r} h_j^2 \tag{C.10}$$

Proof. From (3.31)

$$y'_{j-i}(x) = y_{j-i}^{1-1/r}(x) x^{1/r-1} \tag{C.11}$$

$$y''_{j-i}(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i} \tag{C.12}$$

for $l = 3, 4$, by (3.33)

$$\begin{aligned}(h_{j-i}^l(\xi))' &= l\, h_{j-i}^{l-1}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi)) \\ &= l\, h_{j-i}^{l-1}(\xi)\xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi))\end{aligned} \tag{C.13}$$

For $\xi \in (x_{i-1}, x_{i+1})$ and $2 \le k \le j \le \min\{2i-1, N-1\}$, using Lemma B.1

$$\xi \simeq x_i \simeq x_j$$

$$\begin{aligned}h_{j-i}(\xi) &\le h_{j-i}(x_{i+1}) = h_{j+1} \simeq h_j \\ &\simeq h x_j^{1-1/r} \simeq h x_i^{1-1/r}\end{aligned}$$

$$h_{j-i}(\xi) \simeq h x_i^{1-1/r}$$

And

$$2^{-r} x_i \le x_{i-1} \le \xi \le x_{i+1} \le 2^r x_i \tag{C.14}$$

We have

$$\xi^{1/r-m} \le 2^{|mr-1|} x_i^{1/r-m}, \quad m = 1, 2 \tag{C.15}$$

but

$$x_j \simeq x_i, \quad \text{for } k-1 \le j \le \min\{2i-1, N-1\} \tag{C.16}$$

Then

$$\begin{aligned} y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) &\le x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r} \\ &= T^{1-1/r} N^{1-r}((j+1)^{r-1} - (j-2)^{r-1}) \\ &\le C(r-1) j^{r-2} N^{1-r} \\ &= C(r-1) h x_j^{1-2/r} \end{aligned} \tag{C.17}$$

Therefore,

$$y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) \le C(r-1) h x_j^{1-2/r} \le C(r-1) h x_i^{1-2/r} \tag{C.18}$$

But expecially for $i = 3, j = 2$,

$$y_{-1}^{1-1/r}(\xi) - y_{-2}^{1-1/r}(\xi) \le \max \begin{cases} x_3^{1-1/r} - x_2^{1-1/r} \\ x_1^{1-1/r} - 0 \end{cases} \le C(r-1) x_1^{1-1/r} \le C(r-1) h x_3^{1-2/r}$$

So we can get

$$y'_{j-i}(\xi) - y'_{j-i-1}(\xi) \le C(r-1) h x_i^{-1/r} \tag{C.19}$$

We get

$$(h_{j-i}^l(\xi))' \le l(r-1) C\, h_j^{l-1} h x_i^{-1/r} \tag{C.20}$$

And by Lemma B.1,

$$h_{j+1} \le rTh \left(\frac{j+1}{N}\right)^{r-1} \le rTh 2^{r-1} \left(\frac{j-1}{N}\right) = 2^{r-1} h_j \tag{C.21}$$

$$h_{j+1} \le rT^{1/r} h x_{j+1}^{1-1/r} \le rT^{1/r} h x_{2i}^{1-1/r} \le rT^{1/r} 2^{r-1} h x_i^{1-1/r} \tag{C.22}$$

We can get

$$\begin{aligned} (h_{j-i}^l(\xi))' &\le l(r-1) C\, h_j^{l-2} h_j h x_i^{-1/r} \\ &\le l(r-1) C h h_j^{l-2} (h x_i^{1-1/r}) x_i^{-1/r} \\ &= (r-1) C\, h^2 x_i^{1-2/r} h_j^{l-2} \end{aligned} \tag{C.23}$$

Meanwhile, we can get

$$h_{j-i}^3(\xi) \simeq h_j^3 \le Ch^2 x_i^{2-2/r} h_j \tag{C.24}$$

$$h_{j-i}^4(\xi) \simeq h_j^4 \le Ch^2 x_i^{2-2/r} h_j^2 \tag{C.25}$$

□

LEMMA C.7. *There exists a constant $C = C(T, r, l)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, when $\xi \in (x_{i-1}, x_{i+1})$,*

$$(h_{j-i}^3(\xi))'' \le C(r-1)h^2 x_i^{-2/r} h_j \tag{C.26}$$

Proof. From (C.11)

$$\begin{aligned}(h_{j-i}^3(\xi))'' &= 6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 + 3h_{j-i}^2(\xi)(y''_{j-i}(\xi) - y''_{j-i-1}(\xi)) \\ &= 6h_{j-i}(\xi)(\xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)))^2 \\ &\quad + 3\frac{1-r}{r} h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1})\end{aligned} \tag{C.27}$$

Using the inequalities of the proof of Lemma C.6

$$\begin{aligned}&6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 \\ &\qquad \le Ch_j((r-1)Chx_i^{-1/r})^2 \\ &\qquad \le C(r-1)^2\, h^2 x_i^{-2/r} h_j\end{aligned} \tag{C.28}$$

For the second partial

$$\begin{aligned}&h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\ &\qquad \le Ch_{j+1}^2 x_i^{1/r-2}((y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi))Z_{j-i} + y_{j-i-1}^{1-2/r}(\xi)Z_1)\end{aligned} \tag{C.29}$$

but

$$\begin{aligned}y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi) &= (\xi^{1/r} + Z_{j-i})^{r-2} - (\xi^{1/r} + Z_{j-i-1})^{r-2} \\ &= (r-2)Z_1(\xi^{1/r} + Z_{j-i-\gamma})^{r-3} \\ &= (r-2)T^{1/r} h y_{j-i-\gamma}^{1-3/r}(\xi) \\ &\le C(r-2)hx_i^{1-3/r}\end{aligned} \tag{C.30}$$

So we can get

$$\begin{aligned}&h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\ &\qquad \le Ch_j h x_i^{1-1/r} x_i^{1/r-2}(C(r-2)hx_i^{1-3/r} Z_{j-i} + Cx_i^{1-2/r} T^{1/r} h) \\ &\qquad \le Ch^2((r-2)x_i^{-3/r} x_{|j-i|}^{1/r} + x_i^{-2/r})h_j \\ &\qquad \le Ch^2 x_i^{-2/r} h_j\end{aligned} \tag{C.31}$$

Summarizes, we have

$$(h_{j-i}^3(\xi))'' \le C(r-1)h^2 x_i^{-2/r} h_j \tag{C.32}$$

□

proof of Lemma 3.16. From (3.31)

$$y'_{j-i}(x) = y_{j-i}^{1-1/r}(x)x^{1/r-1} \tag{C.33}$$

$$y''_{j-i}(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x)x^{1/r-2} Z_{j-i} \tag{C.34}$$

Since

$$y^{\theta}_{j-i}(\xi) \simeq x_j$$

We have known

$$u''(y^{\theta}_{j-i}(\xi)) \le C(y^{\theta}_{j-i}(\xi))^{\alpha/2-2} \le Cx_{j-2}^{\alpha/2-2} \le Cx_{\lceil \frac{i}{2} \rceil -1}^{\alpha/2-2} \le C4^{r(2-\alpha/2)} x_i^{\alpha/2-2} \tag{C.35}$$

$$\begin{aligned}(u''(y^{\theta}_{j-i}(\xi)))' &= u'''(y^{\theta}_{j-i}(\xi)) {y^{\theta}_{j-i}}'(\xi)\\ &\le Cx_i^{\alpha/2-3} \xi^{1/r-1} y_{j-i}^{1-1/r}(\xi)\\ &\le Cx_i^{\alpha/2-3} x_i^{1/r-1} x_i^{1-1/r} = Cx_i^{\alpha/2-3}\end{aligned} \tag{C.36}$$

$$\begin{aligned}(u''(y^{\theta}_{j-i}(\xi)))'' &= u''''(y^{\theta}_{j-i}(\xi))({y^{\theta}_{j-i}}'(\xi))^2 + u'''(y^{\theta}_{j-i}(\xi)){y^{\theta}_{j-i}}''(\xi)\\ &\le Cx_i^{\alpha/2-4} + Cx_i^{\alpha/2-3}\frac{r-1}{r} x_i^{1-2/r} x_i^{1/r-2} Z_{|j-i|+1}\\ &\le Cx_i^{\alpha/2-4} + C\frac{r-1}{r} x_i^{\alpha/2-3} x_i^{-1/r} x_i^{1/r}\\ &= Cx_i^{\alpha/2-4}\end{aligned} \tag{C.37}$$

□

Proof of Lemma 3.17.

$$\begin{aligned}|y^{\theta}_{j-i}(\xi) - \xi| &= |\theta(y_{j-i-1}(\xi) - \xi) + (1-\theta)(y_{j-i}(\xi) - \xi)|\\ &= \theta|y_{j-i-1}(\xi) - \xi| + (1-\theta)|y_{j-i}(\xi) - \xi|\end{aligned} \tag{C.38}$$

Since $|y_{j-i}(\xi) - \xi|$ is increasing about ξ, we have

$$(\frac{i-1}{i})^r |x_j - x_i| \le |x_{j-1} - x_{i-1}| \le |y_{j-i}(\xi) - \xi| \le |x_{j+1} - x_{i+1}| \le (\frac{i+1}{i})^r |x_j - x_i| \tag{C.39}$$

Thus,

$$(\frac{2}{3})^r |y^{\theta}_j - x_i| \le |y^{\theta}_{j-i}(\xi) - \xi| \le (\frac{3}{4})^r(\theta|x_j - x_i| + (1-\theta)|x_{j-1} - x_i|) = (\frac{3}{4})^r |y^{\theta}_j - x_i| \tag{C.40}$$

$$|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha} \le C|y^{\theta}_j - x_i|^{1-\alpha} \tag{C.41}$$

Next,

$$\begin{aligned}(|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})' &= (1-\alpha)|y^{\theta}_{j-i}(\xi) - \xi|^{-\alpha} |\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi)) - 1|\\ &\le C|y^{\theta}_j - x_i|^{-\alpha} \xi^{1/r-1} |\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta) y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}|\end{aligned} \tag{C.42}$$

Similar with (C.39), we have

$$\begin{aligned} |y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| &\leq C|x_j^{1-1/r} - x_i^{1-1/r}| \\ &\leq C|x_j - x_i|x_i^{-1/r} \end{aligned} \tag{C.43}$$

So we can get

$$\begin{aligned} &|\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta) y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| \\ &\quad \leq C x_i^{-1/r}(\theta|x_{j-1} - x_i| + (1-\theta)|x_j - x_i|) \\ &\quad = C x_i^{-1/r}|y_j^\theta - x_i| \end{aligned} \tag{C.44}$$

Combine them, we get

$$\begin{aligned} (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' &\leq C|y_j^\theta - x_i|^{-\alpha} x_i^{1/r-1} x_i^{-1/r}|y_j^\theta - x_i| \\ &= C|y_j^\theta - x_i|^{1-\alpha} x_i^{-1} \end{aligned} \tag{C.45}$$

Finally, we have

$$\begin{aligned} (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' = \alpha(\alpha-1)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta) y_{j-i}^{1-1/r}(\xi)) - 1)^2 \\ + (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}\frac{1-r}{r}\xi^{1/r-2}|\theta y_{j-i-1}^{1-2/r}(\xi) Z_{j-i-1} + (1-\theta) y_{j-i}^{1-2/r}(\xi) Z_{j-i}| \end{aligned} \tag{C.46}$$

Using the inequalities above ,we have

$$\begin{aligned} &|y_{j-i}^\theta(\xi) - \xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta) y_{j-i}^{1-1/r}(\xi)) - 1)^2 \\ &\quad \leq C|y_j^\theta - x_i|^{-\alpha-1}(x_i^{-1}|y_j^\theta - x_i|)^2 \\ &\quad = C|y_j^\theta - x_i|^{1-\alpha} x_i^{-2} \end{aligned} \tag{C.47}$$

And by

$$|Z_{j-i}| = |x_j^{1/r} - x_i^{1/r}| \leq |x_j - x_i| x_i^{1/r-1} \tag{C.48}$$

we have

$$\begin{aligned} &|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}\xi^{1/r-2}|\theta y_{j-i-1}^{1-2/r}(\xi) Z_{j-i-1} + (1-\theta) y_{j-i}^{1-2/r}(\xi) Z_{j-i}| \\ &\leq C|y_j^\theta - x_i|^{-\alpha} x_i^{1/r-2} x_i^{1-2/r}|\theta Z_{j-i-1} + (1-\theta) Z_{j-i}| \\ &\leq C|y_j^\theta - x_i|^{-\alpha} x_i^{-2}|y_j^\theta - x_i| \\ &= C|y_j^\theta - x_i|^{1-\alpha} x_i^{-2} \end{aligned} \tag{C.49}$$

∎

proof of Lemma 3.19. For $k \leq j < \min\{2i-1, N-1\}$

$$\begin{aligned} &\frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_j^\theta)}{h_{i+1}} \\ &\quad \frac{Q_{j-i}^\theta(x_{i+1}) - Q_{j-i}^\theta(x_i)}{h_{i+1}} u'''(\eta_{j+1}^\theta) + Q_{j-i}^\theta(x_i)\frac{u'''(\eta_{j+1}^\theta) - u'''(\eta_j^\theta)}{h_{i+1}} \\ &\quad \leq {Q_{j-i}^\theta}'(\xi) C x_j^{\alpha/2-3} + Q_{j-i}^\theta(x_i) C u''''(\eta)\frac{h_i + h_{i+1}}{h_{i+1}} \end{aligned} \tag{C.50}$$

where $\xi \in [x_i, x_{i+1}], \eta \in [x_{j-1}, x_{j+1}]$.

From (3.35), by Lemma C.6 and Lemma 3.17, we have

$$Q^{\theta\,\prime}_{j-i}(\xi) \le Ch^2 \frac{|y^\theta_{j+1} - x_{i+1}|^{1-\alpha}}{\Gamma(2-\alpha)} x_{i+1}^{1-2/r} h_{j+1}^2 \le Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 \tag{C.51}$$

And by defination

$$Q^\theta_{j-i}(x_i) = h_j^4 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \le Ch^2 x_i^{2-2/r} \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 \tag{C.52}$$

With , we have

$$4^{-r} x_i \le x_{k-1} \le x_{j-1} < x_j \le x_{2i-1} \le 2^r x_i \tag{C.53}$$

So we have

$$\begin{aligned} &\frac{Q^\theta_{j-i}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{j-i}(x_i)u'''(\eta^\theta_j)}{h_{i+1}} \\ &\le Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2\, x_i^{\alpha/2-3} + Ch^2 x_i^{2-2/r} \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 x_{j-1}^{\alpha/2-4} \\ &= Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j^2 \end{aligned} \tag{C.54}$$

while

$$h_j \le h_{2i-1} \le 2^r h_i$$

Subsitute into the inequality above, we get the goal

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{Q^\theta_{j-i}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{j-i}(x_i)u'''(\eta^\theta_j)}{h_{i+1}} \right) \\ &\le \frac{1}{h_i} Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j 2^r h_i \\ &= Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned} \tag{C.55}$$

While, the later is similar. ∎

Lemma C.8. *There exists a constant $C = C(T, r)$ such that For $N/2 \le i < N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $l = 3, 4$, $\xi \in [x_{i-1}, x_{i+1}]$, we have*

$$h^l_{j-i}(\xi) \le Ch^l_j \le Ch^2 h_j^{l-2} \tag{C.56}$$

$$(h^l_{j-i-1}(\xi))' \le C(r-1)h^2 h_j^{l-2} \tag{C.57}$$

$$(h^3_{j-i}(\xi))'' \le C(r-1)h^2 h_j \tag{C.58}$$

Proof.

$$(h_{j-i}(\xi))' = {y_{j-i}}'(\xi) - {y_{j-i-1}}'(\xi) = \xi^{1/r-1}((2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}) \le 0 \tag{C.59}$$

Thus,

$$Ch_j \le h_{j+1} \le h_{j-i}(\xi) \le h_{j-i}(x_{i-1}) = h_{j-1} \le Ch_j \tag{C.60}$$

So as $4^{-r}T \le 2T - x_j \le T, 2^{-r}T \le x_i \le T$, we have

$$h^l_{j-i}(\xi) \le Ch^l_j \le Ch^2(2T - x_j)^{2-2/r} h^{l-2}_j \le Ch^2 h^{l-2}_j \tag{C.61}$$

Since

$$\begin{aligned} &|(2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}| \\ &\quad = |(Z_{2N-(j-i)} - \xi^{1/r})^{r-1} - (Z_{2N-(j-1-i)} - \xi^{1/r})^{r-1}| \\ &\quad = (r-1)Z_1(Z_{2N-(j-i-\gamma)} - \xi^{1/r})^{r-2} \quad \gamma \in [0,1] \\ &\quad \le C(r-1)h(2T - x_j)^{1-2/r} \end{aligned} \tag{C.62}$$

we have

$$|(h_{j-i}(\xi))'| \le C(r-1)h(2T - x_j)^{1-2/r} x_i^{1/r-1} \tag{C.63}$$

And

$$\begin{aligned} (h^l_{j-i}(\xi))' &= l h^{l-1}_{j-i}(\xi) {h_{j-i}}'(\xi) \\ &\le C(r-1)h^{l-1}_j \, h(2T - x_j)^{1-2/r} x_i^{1/r-1} \\ &\le C(r-1)h^2 h^{l-2}_j (2T - x_j)^{2-3/r} x_i^{1-1/r} \\ &\le C(r-1)h^2 h^{l-2}_j \end{aligned} \tag{C.64}$$

$$\begin{aligned} (h^3_{j-i}(\xi))'' &= 6h_{j-i}(\xi)({y_{j-i}}'(\xi) - {y_{j-i-1}}'(\xi))^2 + 3h^2_{j-i}(\xi)({y_{j-i}}''(\xi) - {y_{j-i-1}}''(\xi)) \\ &\le C(r-1)h_j h^2 + Ch^2_j \frac{1-r}{r} \xi^{1/r-2}((2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-1-i)}) \\ &\le C(r-1)h_j h^2 + C(r-1)h^2_j (C(r-2)h(2T - x_j)^{1-3/r} Z_{2N-(j-i)} + Z_1(2T - x_{j-1})^{1-2/r}) \\ &\le C(r-1)h_j h^2 + C(r-1)h^2_j h = Ch^2 h_j \end{aligned} \tag{C.65}$$

∎

LEMMA C.9. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \le i < N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in [x_{i-1}, x_{i+1}]$, we have*

$$u''(y^\theta_{j-i}(\xi)) \le C \tag{C.66}$$

$$(u''(y^\theta_{j-i}(\xi)))' \le C \tag{C.67}$$

$$(u''(y^\theta_{j-i}(\xi)))'' \le C \tag{C.68}$$

Proof.

$$x_{j-2} \le y^{\theta}_{j-i}(\xi) \le x_{j+1} \Rightarrow 4^{-r}T \le 2T - y^{\theta}_{j-i}(\xi) \le T \tag{C.69}$$

Thus, for $l = 2, 3, 4$,

$$u^{(l)}(y^{\theta}_{j-i}(\xi)) \le C(2T - y^{\theta}_{j-i}(\xi))^{\alpha/2-l} \le C \tag{C.70}$$

and

$$\begin{aligned}(y^{\theta}_{j-i}(\xi))' &= \theta {y_{j-1-i}}'(\xi) + (1-\theta){y_{j-i-1}}'(\xi)\\ &= \xi^{1/r-1}(\theta(2T - y_{j-1-i}(\xi))^{1-1/r} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-1/r})\\ &\le C(2T - x_{j-2})^{1-1/r} \le C\end{aligned} \tag{C.71}$$

With

$$Z_{2N-j-i} \le 2T^{1/r} \tag{C.72}$$

$$\begin{aligned}(y^{\theta}_{j-i}(\xi))'' &= \theta {y_{j-1-i}}''(\xi) + (1-\theta){y_{j-i-1}}''(\xi)\\ &= \frac{1-r}{r}\xi^{1/r-2}(\theta(2T - y_{j-i-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)} + (1-\theta)(2T - y_{j-i}(\xi))^{1-2/r}Z_{2N-(j-i)})\\ &\le C(r-1)\end{aligned} \tag{C.73}$$

Therefore,

$$\begin{aligned}(u''(y^{\theta}_{j-i}(\xi)))' &= u'''(y^{\theta}_{j-i}(\xi))(y^{\theta}_{j-i}(\xi))'\\ &\le C\end{aligned} \tag{C.74}$$

$$\begin{aligned}(u''(y^{\theta}_{j-i}(\xi)))'' &= u'''(y^{\theta}_{j-i}(\xi))({y^{\theta}_{j-i}}'(\xi))^2 + u''''(y^{\theta}_{j-i}(\xi)){y^{\theta}_{j-i}}''(\xi)\\ &\le C + C(r-1) = C\end{aligned} \tag{C.75}$$

$\square$

LEMMA C.10. *There exists a constant $C = C(T, \alpha, r)$ such that*

$$|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha} \le C|y^{\theta}_j - x_i|^{1-\alpha} \tag{C.76}$$

$$(|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})' \le C|y^{\theta}_j - x_i|^{-\alpha}(|2T - x_i - y^{\theta}_j| + h_N) \tag{C.77}$$

$$(|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})'' \le C(r-1)|y^{\theta}_j - x_i|^{-\alpha} + C|y^{\theta}_j - x_i|^{-1-\alpha}(|2T - x_i - y^{\theta}_j| + h_N)^2 \tag{C.78}$$

Proof.

$$(y^{\theta}_{j-i}(\xi) - \xi)' = \theta {y_{j-1-i}}'(\xi) + (1-\theta){y_{j-i}}'(\xi) - 1 \tag{C.79}$$

$$\begin{aligned}|{y_{j-i}}'(\xi) - 1| &= \xi^{1/r-1}|(2T - y_{j-i}(\xi))^{1-1/r} - \xi^{1-1/r}|\\ &\le \xi^{1/r-1}|2T - \xi - y_{j-i}(\xi)|\xi^{-1/r}\end{aligned} \tag{C.80}$$

$$\begin{aligned}|2T - \xi - y_{j-i}(\xi)| &\le \max\begin{cases}|2T - x_{i-1} - x_{j-1}|\\ |2T - x_{i+1} - x_{j+1}|\end{cases}\\ &\le |2T - x_i - x_j| + h_{i+1} + h_j\end{aligned} \tag{C.81}$$

(C.82)
$$
\begin{aligned}
(y_{j-i}^{\theta}(\xi)-\xi)'' &= \theta {y_{j-1-i}}''(\xi)+(1-\theta){y_{j-i}}''(\xi)\\
&= \frac{1-r}{r}\xi^{1/r-2}(\theta(2T-y_{j-i}(\xi))^{1-2/r}Z_{2N-(j-i)}+(1-\theta)(2T-y_{j-i-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)})\leq 0
\end{aligned}
$$

It's concave, so

$$y_{j-i}(\xi)-\xi \geq \min\{x_{j+1}-x_{i+1}, x_{j-1}-x_{i-1}\} \geq C(x_j-x_i) \tag{C.83}$$

We have

$$|y_{j-i}^{\theta}(\xi)-\xi|^{1-\alpha} \leq C|y_j^{\theta}-x_i|^{1-\alpha} \tag{C.84}$$

$$
\begin{aligned}
(|y_{j-i}^{\theta}(\xi)-\xi|^{1-\alpha})' &= (1-\alpha)|y_{j-i}^{\theta}(\xi)-\xi|^{-\alpha}(y_{j-i}^{\theta}(\xi)-\xi)'\\
&\leq C|y_j^{\theta}-x_i|^{-\alpha}(|2T-x_i-y_j^{\theta}|+h_{i+1}+h_{j-1})
\end{aligned} \tag{C.85}
$$

(C.86) ∎
$$
\begin{aligned}
(|y_{j-i}^{\theta}(\xi)-\xi|^{1-\alpha})'' &= (1-\alpha)|y_{j-i}^{\theta}(\xi)-\xi|^{-\alpha}(y_{j-i}^{\theta}(\xi)-\xi)''+\alpha(\alpha-1)|y_{j-i}^{\theta}(\xi)-\xi|^{-1-\alpha}({y_{j-i}^{\theta}}'(\xi)-1)^2\\
&\leq C(r-1)|y_j^{\theta}-x_i|^{-\alpha}+C|y_j^{\theta}-x_i|^{-1-\alpha}(|2T-x_i-y_j^{\theta}|+h_{i+1}+h_{j-1})^2
\end{aligned}
$$

Proof. From (3.24), by Lemma C.8 and Lemma C.10, we have $\xi \in [x_i, x_{i+1}]$

$${Q_{j-i}^{\theta}}'(\xi) \leq Ch^2h_j^2((r-1)|y_j^{\theta}-x_i|^{1-\alpha}+|y_j^{\theta}-x_i|^{-\alpha}(|2T-x_i-y_j^{\theta}|+h_N)) \tag{C.87}$$

$$Q_{j-i}^{\theta}(\xi) \leq Ch^2h_j^2|y_j^{\theta}-x_i|^{1-\alpha} \tag{C.88}$$

So use the skill in Proof 33 with Lemma C.9

$$
\begin{aligned}
&\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta})-Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}}\right)\\
&\leq Ch^2h_j(|y_j^{\theta}-x_i|^{1-\alpha}+|y_j^{\theta}-x_i|^{-\alpha}(|2T-x_i-y_j^{\theta}|+h_N))
\end{aligned} \tag{C.89}
$$
∎

$$a^{1-\theta}|a^{\theta}-b^{\theta}| \leq |a-b|, \theta \in [0,1] \tag{C.90}$$

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