

# A SECOND ORDER NUMERICAL METHODS FOR REISZ-FRACTIONAL ELLIPTIC EQUATION ON GRADED MESH\*

JIANXING HAN<sup>†</sup> AND MINGHUA CHEN<sup>‡</sup>

**Abstract.** This is an example SIAM L<sup>A</sup>T<sub>E</sub>X article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

**Key words.** example, L<sup>A</sup>T<sub>E</sub>X

**MSC codes.** 68Q25, 68R10, 68U05

**1. Introduction.** The introduction introduces the context and summarizes the manuscript. It is importantly to clearly state the contributions of this piece of work.

For  $\Omega = (0, 2T)$ ,  $1 < \alpha < 2$ , suppose  $f \in C^\beta(\Omega)$ ,  $\beta > 4 - \alpha$ ,  $\|f\|_\beta^{(\alpha/2)} < \infty$

$$(1.1) \quad \begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases}$$

where

$$(1.2) \quad (-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy$$

$$(1.3) \quad \kappa_\alpha = -\frac{1}{2 \cos(\alpha\pi/2)} > 0$$

and the solution  $u \in C^{\alpha/2}(\Omega)$ .

**2. Regularity.** For any  $\beta > 0$ , we use the standard notation  $C^\beta(\bar{\Omega})$ ,  $C^\beta(\mathbb{R})$ , etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation  $C^\beta(\Omega)$  to refer to  $C^{k,\beta'}(\Omega)$ , where  $k$  is the greatest integer such that  $k < \beta$  and where  $\beta' = \beta - k$ . The Hölder spaces  $C^{k,\beta'}(\Omega)$  are defined as the subspaces of  $C^k(\Omega)$  consisting of functions whose  $k$ -th order partial derivatives are locally Hölder continuous[1] with exponent  $\beta'$  in  $\Omega$ , where  $C^k(\Omega)$  is the set of all  $k$ -times continuously differentiable functions on open set  $\Omega$ .

**DEFINITION 2.1** (delta dependent norm [2]). ...

**THEOREM 2.2.** Let  $f \in C^\beta(\Omega)$ ,  $\beta > 2$  be such that  $\|f\|_\beta^{(\alpha/2)} < \infty$ , then for  $l = 0, 1, 2$

$$(2.1) \quad |f^{(l)}(x)| \leq \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \leq T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \leq x < 2T \end{cases}$$

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<sup>†</sup>School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China (hanjx2023@mail.lzu.edu.cn).

<sup>‡</sup>School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China (chen@mail.lzu.edu.cn).

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33 **THEOREM 2.3** (Regularity up to the boundary [2]). *Let  $\Omega$  be a bounded domain,*  
 34 *and  $\beta > 0$  be such that neither  $\beta$  nor  $\beta + \alpha$  is an integer. Let  $f \in C^\beta(\Omega)$  be such*  
 35 *that  $\|f\|_\beta^{(\alpha/2)} < \infty$ , and  $u \in C^{\alpha/2}(\mathbb{R}^n)$  be a solution of (1.1). Then,  $u \in C^{\beta+\alpha}(\Omega)$  and*

$$36 \quad (2.2) \quad \|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C \left( \|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right)$$

37 **COROLLARY 2.4.** *Let  $u$  be a solution of (1.1) on  $\Omega$ . Then, for any  $x \in \Omega$  and*  
 38  *$l = 0, 1, 2, 3, 4$*

$$39 \quad (2.3) \quad |u^{(l)}(x)| \leq \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases}$$

40 The paper is organized as follows. Our main results are in section 4, experimental  
 41 results are in section 7. Readers would better see section 6 before section 5 to avoid  
 42 details.

### 3. Numeric Format.

$$43 \quad (3.1) \quad x_i = \begin{cases} T \left( \frac{i}{N} \right)^r, & 0 \leq i \leq N \\ 2T - T \left( \frac{2N-i}{N} \right)^r, & N \leq i \leq 2N \end{cases}$$

44 where  $r \geq 1$ . And let

$$45 \quad (3.2) \quad h_j = x_j - x_{j-1}, \quad 1 \leq j \leq 2N$$

46 Let  $\{\phi_j(x)\}_{j=1}^{2N-1}$  be standard hat functions, which are basis of the piecewise linear  
 47 function space.

$$48 \quad (3.3) \quad \phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \leq x \leq x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \leq x \leq x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

49 And then, we can approximate  $u(x)$  with

$$50 \quad (3.4) \quad u_h(x) := \sum_{j=1}^{2N-1} u(x_j) \phi_j(x)$$

51 For convience, we denote

$$52 \quad (3.5) \quad I_h^{2-\alpha}(x_i) := \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u_h(y) dy$$

53 And now, we can approximate the operator (1.2) at  $x_i$  with

$$54 \quad (3.6) \quad \begin{aligned} D_h^\alpha u_h(x_i) &:= D_h^2 I_h^{2-\alpha}(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} I_h^{2-\alpha}(x_{i-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) I_h^{2-\alpha}(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha}(x_{i+1}) \right) \end{aligned}$$

55 Finally, we approximate the equation (1.1) with

$$56 \quad (3.7) \quad -\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i), \quad 1 \leq i \leq 2N-1$$

57 The discrete equation (3.7) can be written in matrix form

$$58 \quad (3.8) \quad AU = F$$

59 where  $U$  is unknown,  $F = (f(x_1), \dots, f(x_{2N-1}))$ . The matrix  $A$  is constructed as  
60 follows: Since

$$\begin{aligned} (3.9) \quad I_h^{2-\alpha}(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u_h(y) dy \\ &= \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u(x_j) \phi_j(y) dy \\ 61 \quad &= \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y) dy \\ &= \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left( \frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\ &=: \sum_{j=1}^{2N-1} \tilde{a}_{ij} u(x_j), \quad 0 \leq i \leq 2N \end{aligned}$$

62 Then, substitute in (3.6), we have

$$63 \quad (3.10) \quad -\kappa_{\alpha} D_h^{\alpha} u_h(x_i) = \sum_{j=1}^{2N-1} a_{ij} u(x_j)$$

64 where  
(3.11)

$$65 \quad a_{ij} = -\kappa_{\alpha} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \tilde{a}_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right), \quad 1 \leq i \leq 2N-1$$

66 **4. Main results.** Here we state our main results; the proof is deferred to sec-  
67 tion 5 and section 6.

68 Let's denote  $h = \frac{1}{N}$ , we have

69 **THEOREM 4.1 (Truncation Error).** *If  $f$  satisfy that  $f \in C^{\beta}(\Omega)$ ,  $\beta > 4 - \alpha$ ,  
70  $\|f\|_{\beta}^{(\alpha/2)} < \infty$ ,  $\alpha \in (1, 2)$ , and  $u(x)$  is a solution of the equation (1.1), where  $\|u\|_{\beta+\alpha}^{(-\alpha/2)} < \infty$ ,  
71 then there exists constants  $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ ,  $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ ,  
72 such that the truncation error of the discrete format satisfies*

$$\begin{aligned} (4.1) \quad \tau_i &:= |-\kappa_{\alpha} D_h^{\alpha} u_h(x_i) - f(x_i)| \\ &\leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} \begin{cases} x_i^{-\alpha}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha}, & N < i \leq 2N-1 \end{cases} \\ &\quad + C_2(r-1)h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \leq i \leq N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \leq 2N-1 \end{cases} \end{aligned}$$

74

75 **THEOREM 4.2 (Convergence).** *The discrete equation (3.7) has solution  $U$ , and  
76 there exists a positive constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$  such that the error*

between the numerical solution  $U$  with the exact solution  $u(x_i)$  satisfies

$$(4.2) \quad \max_{1 \leq i \leq 2N-1} |U_i - u(x_i)| \leq Ch^{\min\{\frac{r\alpha}{2}, 2\}}$$

That means the numerical method has convergence order  $\min\{\frac{r\alpha}{2}, 2\}$ .

**5. Proof of Theorem 4.1.** For convenience, let's denote

$$(5.1) \quad I^{2-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x-y|^{1-\alpha} u(y) dy$$

Then, the truncation error of the discrete format can be written as

$$(5.2) \quad \begin{aligned} -\kappa_{\alpha} D_h^{\alpha} u_h(x_i) - f(x_i) &= -\kappa_{\alpha} (D_h^2 I_h^{2-\alpha}(x_i) - \frac{d^2}{dx^2} I^{2-\alpha}(x_i)) \\ &= -\kappa_{\alpha} D_h^2 (I_h^{2-\alpha} - I^{2-\alpha})(x_i) - \kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \end{aligned}$$

**5.1. Estimate of  $-\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i)$ .**

**THEOREM 5.1.** *There exists a constant  $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$  such that*

$$(5.3) \quad \left| -\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \right| \leq Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N-1 \end{cases}$$

*Proof.* Since  $f \in C^2(\Omega)$  and

$$(5.4) \quad \frac{d^2}{dx^2} (-\kappa_{\alpha} I^{2-\alpha}(x)) = f(x), \quad x \in \Omega,$$

we have  $I^{2-\alpha} \in C^4(\Omega)$ . Therefore, using equation (A.3) of Lemma A.1, for  $1 \leq i \leq 2N-1$ , we have

$$(5.5) \quad \begin{aligned} -\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) &= \frac{h_{i+1} - h_i}{3} f'(x_i) \\ &\quad + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right) \end{aligned}$$

where  $\eta_1 \in [x_{i-1}, x_i]$ ,  $\eta_2 \in [x_i, x_{i+1}]$ . By Lemma B.2 and Theorem 2.2 we have 1.

$$(5.6) \quad \left| \frac{h_{i+1} - h_i}{3} f'(x_i) \right| \leq \frac{C(r-1) \|f\|_{\beta}^{(\alpha/2)}}{3} h^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{-\alpha/2-2/r}, & N < i \leq 2N-1 \end{cases}$$

2. See Proof 25, there is a constant  $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$  such that

$$(5.7) \quad \begin{aligned} &\frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right) \\ &\leq Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N-1 \end{cases} \end{aligned}$$

Summarizes, we get the result.  $\square$

**5.2. Estimate of  $R_i$ .** Now, we study the first part of (5.2)

$$(5.8) \quad D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i) = D_h^2\left(\int_0^{2T} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy\right)$$

For convience, let's denote

$$(5.9) \quad T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy, \quad i = 0, \dots, 2N, \quad j = 1, \dots, 2N$$

And define

$$(5.10) \quad \begin{aligned} R_i &:= D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \sum_{j=1}^{2N} \left( \frac{1}{h_i} T_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right), \quad 1 \leq i \leq 2N-1 \end{aligned}$$

We have some results about the estimate of  $R_i$

**THEOREM 5.2.** *For  $1 \leq i < N/2$ , there exists  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

$$(5.11) \quad R_i \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

**THEOREM 5.3.** *For  $N/2 \leq i \leq N$ , there exists constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

$$(5.12) \quad R_i \leq C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

And for  $N < i \leq 2N-1$ , it is symmetric to the previous case.

To prove these results, we need some utils. Also for simplicity, we denote

**DEFINITION 5.4.**

$$(5.13) \quad S_{ij} = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} T_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right)$$

then

$$(5.14) \quad R_i = \sum_{j=1}^{2N} S_{ij}$$

**5.3. Proof of Theorem 5.2.**

**LEMMA 5.5.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $1 \leq i < N/2$ ,*

$$(5.15) \quad \sum_{j=\max\{2i+1, i+3\}}^N S_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r}$$

*Proof.* Let

$$K_y(x) = \frac{|y-x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

For  $\max\{2i+1, i+3\} \leq j \leq N$ , by Lemma C.1 and Lemma C.2

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 K_y(x_i) dy \\ &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy \end{aligned} \quad (5.16)$$

Therefore,

$$\begin{aligned} \sum_{j=\max\{2i+1, i+3\}}^N S_{ij} &\leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\ &= \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\ &\leq \frac{C}{\alpha/2+2/r} 2^{r(-\alpha/2-2/r)} h^2 x_i^{-\alpha/2-2/r} \end{aligned} \quad (5.17) \quad \square$$

LEMMA 5.6. *Thert exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $1 \leq i < N/2$ ,*

$$\sum_{j=N+1}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \quad (5.18)$$

*Proof.* For  $1 \leq i < N/2, N+1 \leq j \leq 2N-1$ , by equation (C.2) and Lemma C.2

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 K_y(x_i) dy \\ &\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T-y)^{\alpha/2-2/r} y^{-1-\alpha} dy \\ &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T-y)^{\alpha/2-2/r} dy \end{aligned} \quad (5.19)$$

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$$\begin{aligned}
 \sum_{j=N+1}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r} dy \\
 &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2-2/r+1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 < 0 \end{cases} \\
 &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r} h^2, & \alpha/2-2/r+1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2-2/r+1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r} h^{r\alpha/2+r}, & \alpha/2-2/r+1 < 0 \end{cases}
 \end{aligned}
 \tag{5.19}$$

133 And by Lemma A.3

$$S_{i,2N} \leq CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}$$

 135 And when  $\alpha/2 - 2/r + 1 \geq 0$ ,

$$h^{r\alpha/2+r} \leq h^2$$

 137 Summarizes, we get the result. □

 138 For  $i = 1, 2$ .

 139 LEMMA 5.7. *By Lemma C.5 , Lemma 5.5 and Lemma 5.6 we get*

$$\begin{aligned}
 R_1 &= \sum_{j=1}^3 S_{1j} + \sum_{j=4}^{2N} S_{1j} \\
 &\leq Ch^2 x_1^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1 > 0 \\ Ch^2 \ln(N), & \alpha/2-2/r+1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1 < 0 \end{cases}
 \end{aligned}
 \tag{5.20}$$

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$$\begin{aligned}
 R_2 &= \sum_{j=1}^4 S_{2j} + \sum_{j=5}^{2N} S_{2j} \\
 &\leq Ch^2 x_2^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1 > 0 \\ Ch^2 \ln(N), & \alpha/2-2/r+1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1 < 0 \end{cases}
 \end{aligned}
 \tag{5.21}$$

143 For  $3 \leq i < N/2$ , we have a new separation of  $R_i$ , Let's denote  $k = \lceil \frac{i}{2} \rceil$ .

$$\begin{aligned}
 R_i &= \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &= \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &\quad + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \\
 144 \quad (5.22) \quad &+ \sum_{j=k+1}^{2i-1} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
 &+ \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \\
 &+ \sum_{j=2i+1}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &= I_1 + I_2 + I_3 + I_4 + I_5
 \end{aligned}$$

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146 **LEMMA 5.8.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $3 \leq$*   
 147  *$i \leq N, k = \lceil \frac{i}{2} \rceil$*

$$148 \quad (5.23) \quad |I_1| = \left| \sum_{j=1}^{k-1} S_{ij} \right| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

149 *Proof.* by Lemma A.3 , Lemma C.3

$$150 \quad (5.24) \quad S_{i1} \leq C x_1^{\alpha/2} x_1 x_i^{-1-\alpha} = C x_1^{\alpha/2+1} x_i^{-1-\alpha} = C T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha}$$

151 For  $2 \leq j \leq k-1$ , by Lemma C.1 and Lemma C.3

$$\begin{aligned}
 S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 K_y(x_i) dy \\
 152 \quad (5.25) \quad &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} dy \\
 &= Ch^2 x_i^{-1-\alpha} \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} dy
 \end{aligned}$$

153 Therefore,

$$\begin{aligned}
 I_1 &= \sum_{j=1}^{k-1} S_{ij} = S_{i1} + \sum_{j=2}^{k-1} S_{ij} \\
 154 \quad (5.26) \quad &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \\
 &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy
 \end{aligned}$$



155 But

$$156 \quad (5.27) \quad \int_{x_1}^{2^{-r}x_i} y^{\alpha/2-2/r} dy \leq \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r}x_i)^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 > 0 \\ \ln(2^{-r}x_i) - \ln(x_1), & \alpha/2-2/r+1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 < 0 \end{cases}$$

157 So we have

$$158 \quad (5.28) \quad I_1 \leq \begin{cases} \frac{C}{\alpha/2-2/r+1} h^2 x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2-2/r+1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} h^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2-2/r+1 < 0 \end{cases} \quad \square$$

159 DEFINITION 5.9. For convience, let's denote

$$160 \quad (5.29) \quad V_{ij} = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right)$$

161

162 THEOREM 5.10. There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  
 163  $3 \leq i < N/2, k = \lceil \frac{i}{2} \rceil$ ,

$$164 \quad (5.30) \quad I_3 = \sum_{j=k+1}^{2i-1} V_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r}$$

165 To estimate  $V_{ij}$ , we need some preparations.

166 LEMMA 5.11. For  $y \in [x_{j-1}, x_j]$ , we can rewrite  $y = x_{j-1} + \theta h_j = (1-\theta)x_{j-1} +$   
 167  $\theta x_j =: y_j^\theta$ ,  $\theta \in [0, 1]$ , by Lemma A.2

$$168 \quad (5.31) \quad \begin{aligned} T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= \int_0^1 (u(y_j^\theta) - u_h(y_j^\theta)) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta \end{aligned}$$

169 where  $\eta_{j1}^\theta \in [x_{j-1}, y_j^\theta]$ ,  $\eta_{j2}^\theta \in [y_j^\theta, x_j]$ .

170 Now Let's construct a series of functions to represent  $T_{ij}$ .

171 DEFINITION 5.12. For  $2 \leq i, j \leq N-1$ ,

$$172 \quad (5.32) \quad y_{j-i}(x) = (x^{1/r} + Z_{j-i})^r, \quad Z_{j-i} = T^{1/r} \frac{j-i}{N}$$

173

$$174 \quad (5.33) \quad y_{j-i}^\theta(x) = (1-\theta)y_{j-1-i}(x) + \theta y_{j-i}(x)$$

175

$$176 \quad (5.34) \quad h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x)$$

Now, we define

$$(5.35) \quad P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$(5.36) \quad Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

And now we can rewrite  $T_{ij}$

LEMMA 5.13. For  $2 \leq i \leq N, 2 \leq j \leq N$ ,

$$(5.37) \quad T_{ij} = \int_0^1 -\frac{\theta(1-\theta)}{2} P_{j-i}^\theta(x_i) d\theta \\ + \int_0^1 \frac{\theta(1-\theta)}{3!} Q_{j-i}^\theta(x_i) (\theta^2 u'''(\eta_{j,1}^\theta) - (1-\theta)^2 u'''(\eta_{j,2}^\theta)) d\theta$$

Immediately, we can see from (5.29) that

LEMMA 5.14. For  $3 \leq i, j \leq N-1$ ,

$$(5.38) \quad V_{ij} = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ = \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^\theta(x_i) d\theta \\ + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\ - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\ - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,2}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\ + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta$$

To estimate  $V_{ij}$ , we first estimate  $D_h^2 P_{j-i}^\theta(x_i)$ , but By Lemma A.1,

$$(5.39) \quad D_h^2 P_{j-i}^\theta(x_i) = P_{j-i}^{\theta''}(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

By Leibniz formula, we calculate and estimate the derivations of  $h_{j-i}^3(x)$ ,  $u''(y_{j-i}^\theta(x))$

and  $\frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$  separately.

Firstly, we have

LEMMA 5.15. There exists a constant  $C = C(T, r)$  such that For  $3 \leq i \leq N -$

$1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$ ,  $\xi \in [x_{i-1}, x_{i+1}]$ ,

$$(5.40) \quad h_{j-i}^3(\xi) \leq C h^2 x_i^{2-2/r} h_j$$

$$(5.41) \quad (h_{j-i}^3(\xi))' \leq C(r-1) h^2 x_i^{1-2/r} h_j$$

$$(5.42) \quad (h_{j-i}^3(\xi))'' \leq C(r-1) h^2 x_i^{-2/r} h_j$$

The proof of this theorem see Lemma C.6 and Lemma C.7

Second,

LEMMA 5.16. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For*  
 $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}, \xi \in [x_{i-1}, x_{i+1}],$

$$(5.43) \quad u''(y_{j-i}^\theta(\xi)) \leq Cx_i^{\alpha/2-2}$$

$$(5.44) \quad (u''(y_{j-i}^\theta(\xi)))' \leq Cx_i^{\alpha/2-3}$$

$$(5.45) \quad (u''(y_{j-i}^\theta(\xi)))'' \leq Cx_i^{\alpha/2-4}$$

The proof of this theorem see Proof 31

And Finally, we have

LEMMA 5.17. *There exists a constant  $C = C(T, \alpha, r)$  such that For  $3 \leq i \leq$   
 $N-1, 1 \leq j \leq \min\{2i-1, N-1\}, \xi \in [x_{i-1}, x_{i+1}],$*

$$(5.46) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}$$

$$(5.47) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C|y_j^\theta - x_i|^{1-\alpha}x_i^{-1}$$

$$(5.48) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C|y_j^\theta - x_i|^{1-\alpha}x_i^{-2}$$

where  $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

The proof of this theorem see Proof 32

LEMMA 5.18. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For*  
 $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\},$

$$(5.49) \quad D_h^2 P_{j-i}^\theta(x_i) \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j$$

where  $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

*Proof.* Since

$$(5.50) \quad D_h^2 P_{j-i}^\theta(x_i) = P_{j-i}^{\theta}{}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

From (5.35), using Leibniz formula and Lemma 5.15, Lemma 5.16 and Lemma 5.17

LEMMA 5.19. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for*  
 $3 \leq i < N, k = \lceil \frac{i}{2} \rceil.$   
*For  $k \leq j \leq \min\{2i-1, N-1\},$*

$$(5.51) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j$$

And for  $k+1 \leq j \leq \min\{2i, N\},$

$$(5.52) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta) - Q_{j-i}^\theta(x_{i-1})u'''(\eta_{j-1}^\theta)}{h_i} \right) \\ \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j$$

where  $\eta_j^\theta \in [x_{j-1}, x_j]$ .

proof see Proof 33

LEMMA 5.20. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $3 \leq i < N, k = \lceil \frac{i}{2} \rceil, k+1 \leq j \leq \min\{2i-1, N-1\}$ ,*

$$\begin{aligned} V_{ij} &\leq Ch^2 \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j d\theta \\ &= Ch^2 \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \end{aligned}$$

*Proof.* Since Lemma 5.14, by Lemma 5.18 and Lemma 5.19, we get the result immediately.  $\square$

Now we can prove Theorem 5.10 using Lemma 5.20,  $k = \lceil \frac{i}{2} \rceil$

$$\begin{aligned} I_3 &= \sum_{k+1}^{2i-1} V_{ij} \leq Ch^2 \int_{x_k}^{x_{2i-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &= Ch^2 \left( \frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{2i-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\ &\leq Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned}$$

LEMMA 5.21.

$$D_h P_{j-i}^\theta(x_i) := \frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} = P_{j-i}^{\theta'}(\xi), \quad \xi \in [x_i, x_{i+1}]$$

Then, for  $3 \leq i \leq N-1, k = \lceil \frac{i}{2} \rceil$ ,

$$D_h P_{k-i}^\theta(x_i) \leq Ch^2 x_i^{-\alpha/2-2/r} h_j$$

*Proof.* Using Leibniz formula, by Lemma 5.15, Lemma 5.16 and Lemma 5.17, we take  $j = k+1, i = i+1$ , we get

$$\begin{aligned} D_h P_{k-i}^\theta(x_i) &\leq Ch^2 x_{i+1}^{\alpha/2-2/r-1} |y_{k+1}^\theta - x_{i+1}|^{1-\alpha} h_{j+1} \\ &\leq Ch^2 x_i^{\alpha/2-2/r-1} |y_k^\theta - x_i|^{1-\alpha} h_j \\ &\leq Ch^2 x_i^{-\alpha/2-2/r} h_j \end{aligned}$$

LEMMA 5.22. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $3 \leq i < N, k = \lceil \frac{i}{2} \rceil$ ,*

$$I_2 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

And for  $3 \leq i < N/2$ ,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

*Proof.* In fact,

$$\begin{aligned} & \frac{1}{h_{i+1}}(T_{i+1,k} + T_{i+1,k+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right)T_{i,k} \\ &= \frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}}(T_{i+1,k+1} - T_{i,k}) + \left(\frac{1}{h_{i+1}} - \frac{1}{h_i}\right)T_{i,k} \end{aligned} \quad (5.60)$$

While, by Lemma A.2

$$\begin{aligned} \frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) &= \int_{x_{k-1}}^{x_k} (u(y) - u_h(y)) \frac{|x_{i+1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_{i+1}\Gamma(2-\alpha)} dy \\ &\leq \int_{x_{k-1}}^{x_k} h_k^2 u''(\eta) \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy \\ &\leq Ch_k h^2 x_k^{2-2/r} x_{k-1}^{\alpha/2-2} |x_i - x_k|^{-\alpha} \\ &\leq Ch_k h^2 x_i^{-\alpha/2-2/r} \end{aligned} \quad (5.61)$$

Thus,

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) \leq Ch^2 x_i^{-\alpha/2-2/r} \quad (5.62)$$

For

$$\begin{aligned} \frac{1}{h_{i+1}}(T_{i+1,k+1} - T_{i,k}) &= \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{Q_{k-i}^\theta(x_{i+1})u'''(\eta_{k+1,1}^\theta) - Q_{k-i}^\theta(x_i)u'''(\eta_{k,1}^\theta)}{h_{i+1}} d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{Q_{k-i}^\theta(x_{i+1})u'''(\eta_{k+1,2}^\theta) - Q_{k-i}^\theta(x_i)u'''(\eta_{k,2}^\theta)}{h_{i+1}} d\theta \end{aligned} \quad (5.63)$$

And by Lemma 5.21

$$\frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} \leq Ch^2 x_i^{-\alpha/2-2/r} h_k \quad (5.64)$$

And with Lemma 5.19, we can get

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) \leq Ch^2 x_i^{-\alpha/2-2/r} \quad (5.65)$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned} \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,k} &\leq h_i^{-3} h^2 x_i^{1-2/r} h_k Ch_k^2 x_{k-1}^{\alpha/2-2} |x_k - x_i|^{1-\alpha} \\ &\leq Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \quad (5.66)$$

Summarizes, we have

$$I_2 \leq Ch^2 x_i^{-\alpha/2-2/r} \quad (5.67)$$

The case for  $I_4$  is similar.  $\square$

269 Now combine Lemma 5.8, Lemma 5.22, Theorem 5.10, Lemma 5.5 and Lemma 5.6  
 270 to get the final result.  
 271 For  $3 \leq i < N/2$

$$\begin{aligned}
 R_i &= I_1 + I_2 + I_3 + I_4 + I_5 \\
 (5.68) \quad &\leq Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases}
 \end{aligned}$$

273 Combine with  $i = 1, 2$ , we get for  $1 \leq i < N/2$

$$(5.69) \quad R_i \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases}$$

275 **5.4. Proof of Theorem 5.3.** For  $N/2 \leq i < N, k = \lceil \frac{i}{2} \rceil$ , we have

$$\begin{aligned}
 (5.70) \quad R_i &= \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &= \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &\quad + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \\
 276 \quad &+ \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
 &+ \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right) \\
 &+ \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
 &= I_1 + I_2 + I_3^1 + I_3^2 + I_3^3 + I_4 + I_5
 \end{aligned}$$

277 We have estimate  $I_1$  in Lemma 5.8 and  $I_2$  in Lemma 5.22. We can control  $I_3$  in  
 278 similar with Theorem 5.10 by Lemma 5.20 where  $2i - 1 \geq N - 1$

279 **LEMMA 5.23.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for*  
 280  $N/2 \leq i < N, k = \lceil \frac{i}{2} \rceil$ ,

$$\begin{aligned}
 (5.71) \quad I_3 &= \sum_{j=k+1}^{N-1} V_{ij} \leq Ch^2 \int_{x_k}^{x_{N-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\
 &= Ch^2 \left( \frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{N-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\
 &\leq Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r}
 \end{aligned}$$

Let's study  $I_5$  before  $I_4$ .

$$(5.72) \quad I_5 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} V_{ij}$$

Similarly, Let's define a new series of functions

DEFINITION 5.24. For  $i < N, j \geq N$ , with no confusion, we also denote in this section

$$(5.73) \quad y_{j-i}(x) = 2T - (Z_{2N-j+i} - x^{1/r})^r, \quad Z_{2N-j+i} = T^{1/r} \frac{2N-j+i}{N}$$

$$(5.74) \quad y_{j-i}'(x) = (2T - y_{j-i}(x))^{1-1/r} x^{1/r-1}$$

$$(5.75) \quad y_{j-i}''(x) = \frac{1-r}{r} (2T - y_{j-i}(x))^{1-2/r} x^{1/r-2} Z_{2N-j+i}$$

$$(5.76)$$

$$(5.77) \quad y_{j-i}^\theta(x) = (1-\theta)y_{j-i-1}(x) + \theta y_{j-i}(x)$$

$$(5.78) \quad h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x)$$

$$(5.79) \quad P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$(5.80) \quad Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

Now we have, for  $i < N, j \geq N+2$ ,

$$(5.81) \quad \begin{aligned} V_{ij} &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^\theta(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,2}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta \end{aligned}$$

Similarly, we first estimate

$$(5.82) \quad D_h^2 P_{j-i}^\theta(\xi) = P_{j-i}^{\theta''}(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

Combine lemmas Lemma C.8, Lemma C.9 and Lemma C.10 , we have

LEMMA 5.25. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i < N$ ,  $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$ ,  $\xi \in [x_{i-1}, x_{i+1}]$ , we have*

$$(5.83) \quad \begin{aligned} |P_{j-i}^{\theta''}(\xi)| &\leq Ch_j h^2 (|y_j^\theta - x_i|^{1-\alpha} \\ &\quad + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N) \\ &\quad + |y_j^\theta - x_i|^{-1-\alpha} (|2T - x_i - y_j^\theta| + h_N)^2 \\ &\quad + (r-1)|y_j^\theta - x_i|^{-\alpha}) \end{aligned}$$

And

LEMMA 5.26. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i < N$ ,  $\xi \in [x_{i-1}, x_{i+1}]$ , we have for  $N+1 \leq j \leq 2N - \lceil \frac{N}{2} \rceil$*

$$(5.84) \quad \begin{aligned} &\frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ &\leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N)) \end{aligned}$$

for  $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$

$$(5.85) \quad \begin{aligned} &\frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta) - Q_{j-i}^\theta(x_{i-1})u'''(\eta_{j-1}^\theta)}{h_{i+1}} \right) \\ &\leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N)) \end{aligned}$$

The proof see Proof 37.

Combine (5.81), Lemma 5.25 and Lemma 5.26, we have

THEOREM 5.27. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i < N$ ,  $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$*

$$(5.86) \quad \begin{aligned} V_{ij} &\leq Ch^2 \int_{x_{j-1}}^{x_j} (|y - x_i|^{1-\alpha} \\ &\quad + |y - x_i|^{-\alpha} (|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha} (|2T - x_i - y| + h_N)^2 \\ &\quad + (r-1)|y - x_i|^{-\alpha}) dy \end{aligned}$$

We can estimate  $I_5$  Now.

THEOREM 5.28. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i < N$ , we have*

$$(5.87) \quad I_5 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \leq Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha}$$



*Proof.*

$$\begin{aligned}
 I_5 &= \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} V_{ij} \\
 &\leq Ch^2 \int_{x_{N+1}}^{x_{2N-i}} + \int_{x_{2N-i}}^{x_{2N-\lceil \frac{N}{2} \rceil}} (|y - x_i|^{1-\alpha} \\
 &\quad + |y - x_i|^{-\alpha} (|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha} (|2T - x_i - y| + h_N)^2 \\
 &\quad + (r-1)|y - x_i|^{-\alpha}) dy \\
 &= J_1 + J_2
 \end{aligned}
 \tag{5.88}$$

While  $x_{N+1} \leq y \leq x_{2N-i} = 2T - x_i$ ,

$$T - x_{i-1} \leq x_{N+1} - x_i \leq y - x_i \leq x_{2N-i} - x_i \leq 2(T - x_{i-1})$$

and

$$2T - x_i - y + h_N \leq 2T - x_i - x_{N+1} + h_N = T - x_i \leq T - x_{i-1}$$

So

$$\begin{aligned}
 J_1 &\leq Ch^2 (x_{2N-i} - x_{N+1}) (|T - x_{i-1}|^{1-\alpha} + (r-1)|T - x_{i-1}|^{-\alpha}) \\
 &\leq Ch^2 (|T - x_{i-1}|^{2-\alpha} + (r-1)|T - x_{i-1}|^{1-\alpha}) \\
 &\leq Ch^2 T^{2-\alpha} + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha}
 \end{aligned}
 \tag{5.91}$$

Otherwise, when  $x_{2N-i} \leq y \leq x_{2N-\lceil \frac{N}{2} \rceil}$

$$x_i + y - 2T + h_N \leq y - x_i$$

332

$$\begin{aligned}
 J_2 &\leq Ch^2 \int_{x_{2N-i}}^{(2-2^{-r})T} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} \\
 &\leq Ch^2 (T^{2-\alpha} + (r-1)|x_{2N-i} - x_i|^{1-\alpha}) \\
 &= Ch^2 + C(r-1)h^2 |T - x_i|^{1-\alpha} \leq Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha}
 \end{aligned}
 \tag{5.93}$$

Summarizes two cases, we get the result.  $\square$

For  $I_4$ , we have

**THEOREM 5.29.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that, for  $N/2 \leq i \leq N-1$*

$$(5.94) \quad \begin{aligned} V_{iN} &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1, N+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, N} + \frac{1}{h_i} T_{i-1, N-1} \right) \\ &\leq Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \end{aligned}$$

*Proof.* We use the similar skill in the last section, but more complicated. for  $j = N$ , Let

$$(5.95) \quad {}_L y_{N-1-i}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r} \frac{N-1-i}{N}$$

$$(5.96) \quad {}_0 y_{N-i}(x) = \frac{x^{1/r} - Z_i}{Z_1} h_N + T, \quad Z_i = T^{1/r} \frac{i}{N}, x_N = T$$

and

$$(5.97) \quad {}_R y_{N+1-i}(x) = 2T - (Z_{N-1+i} - x^{1/r})^r, \quad Z_{N-1+i} = T^{1/r} \frac{N-1+i}{N}$$

Thus,

$$\begin{aligned} {}_L y_{N-1-i}(x_{i-1}) &= x_{N-2}, \quad {}_L y_{N-1-i}(x_i) = x_{N-1}, \quad {}_L y_{N-1-i}(x_{i+1}) = x_N \\ {}_0 y_{N-i}(x_{i-1}) &= x_{N-1}, \quad {}_0 y_{N-i}(x_i) = x_N, \quad {}_0 y_{N-i}(x_{i+1}) = x_{N+1} \\ {}_R y_{N+1-i}(x_{i-1}) &= x_N, \quad {}_R y_{N+1-i}(x_i) = x_{N+1}, \quad {}_R y_{N+1-i}(x_{i+1}) = x_{N+2} \end{aligned}$$

Then, define

$$(5.98) \quad {}_L y_{N-i}^\theta(x) = \theta {}_L y_{N-1-i}(x) + (1-\theta) {}_0 y_{N-i}(x)$$

$$(5.99) \quad {}_R y_{N+1-i}^\theta(x) = \theta {}_0 y_{N-i}(x) + (1-\theta) {}_R y_{N+1-i}(x)$$

$$(5.100) \quad {}_L h_{N-i}(x) = {}_0 y_{N-i}(x) - {}_L y_{N-1-i}(x)$$

$$(5.101) \quad {}_R h_{N+1-i}(x) = {}_R y_{N+1-i}(x) - {}_0 y_{N-i}(x)$$

We have

$$(5.102) \quad {}_L y_{N-1-i}'(x) = {}_L y_{N-1-i}^{1-1/r}(x) x^{1/r-1}$$

$$(5.103) \quad {}_L y_{N-1-i}''(x) = \frac{1-r}{r} {}_L y_{N-1-i}^{1-2/r}(x) x^{1/r-2} Z_{N-1-i}$$

$$(5.104) \quad {}_0 y_{N-i}'(x) = \frac{1}{r} \frac{h_N}{Z_1} x^{1/r-1}$$

$$(5.105) \quad {}_0 y_{N-i}''(x) = \frac{1-r}{r^2} \frac{h_N}{Z_1} x^{1/r-2}$$

$$(5.106) \quad {}_R y_{N+1-i}'(x) = (2T - {}_R y_{N+1-i}(x))^{1-1/r} x^{1/r-1}$$

$$(5.107) \quad {}_R y_{N+1-i}''(x) = \frac{1-r}{r} (2T - {}_R y_{N+1-i}(x))^{1-2/r} x^{1/r-2} Z_{N-1+i}$$

363

$$364 \quad (5.108) \quad {}_L P_{N-i}^\theta(x) = ({}_L h_{N-i}(x))^3 \frac{|{}_L y_{N-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_L y_{N-i}^\theta(x))$$

$$365 \quad (5.109) \quad {}_R P_{N+1-i}^\theta(x) = ({}_R h_{N+1-i}(x))^3 \frac{|{}_R y_{N+1-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_R y_{N+1-i}^\theta(x))$$

$$366 \quad (5.110) \quad {}_L Q_{N-i}^\theta(x) = ({}_L h_{N-i}(x))^4 \frac{|{}_L y_{N-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$367 \quad (5.111) \quad {}_R Q_{N+1-i}^\theta(x) = ({}_R h_{N+1-i}(x))^4 \frac{|{}_R y_{N+1-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

 368 Similar with Lemma 5.13, we can get for  $l = -1, 0, 1$ ,

$$369 \quad (5.112) \quad \begin{aligned} T_{i+l, N+l} &= \int_0^1 -\frac{\theta(1-\theta)}{2} {}_L P_{N-i}^\theta(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} {}_L Q_{N-i}^\theta(x_{i+l}) (\theta^2 u'''(\eta_{N+l,1}^\theta) - (1-\theta)^2 u'''(\eta_{N+l,2}^\theta)) d\theta \end{aligned}$$

370

$$(5.113) \quad \begin{aligned} T_{i+l, N+1+l} &= \int_0^1 -\frac{\theta(1-\theta)}{2} {}_R P_{N+1-i}^\theta(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} {}_R Q_{N+1-i}^\theta(x_{i+l}) (\theta^2 u'''(\eta_{N+1+l,1}^\theta) - (1-\theta)^2 u'''(\eta_{N+1+l,2}^\theta)) d\theta \end{aligned}$$

371

372 So we have

$$(5.114) \quad \begin{aligned} V_{i,N} &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_{hL}^2 {}_L P_{N-i}^\theta(x_i) d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_{i+1}) u'''(\eta_{N+1,1}^\theta) - {}_L Q_{N-i}^\theta(x_i) u'''(\eta_{N,1}^\theta)}{h_{i+1}} \right) d\theta \\ &- \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_i) u'''(\eta_{N,1}^\theta) - {}_L Q_{N-i}^\theta(x_{i-1}) u'''(\eta_{N-1,1}^\theta)}{h_i} \right) d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_{i+1}) u'''(\eta_{N+1,2}^\theta) - {}_L Q_{N-i}^\theta(x_i) u'''(\eta_{N,2}^\theta)}{h_{i+1}} \right) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_i) u'''(\eta_{N,2}^\theta) - {}_L Q_{N-i}^\theta(x_{i-1}) u'''(\eta_{N-1,2}^\theta)}{h_i} \right) d\theta \end{aligned}$$

373

 374  $N+1$  is similar.

 375 We estimate  $D_{hL}^2 {}_L P_{N-i}^\theta(x_i) = {}_L P_{N-i}^{\theta''}(\xi)$ ,  $\xi \in [x_{i-1}, x_{i+1}]$ ,

376

LEMMA 5.30.

$$377 \quad (5.115) \quad {}_L h_{N-i}^3(\xi) \leq Ch_N^3 \leq Ch^3$$

$$378 \quad (5.116) \quad {}_R h_{N+1-i}^3(\xi) \leq Ch_N^3 \leq Ch^3$$

$$(5.117) \quad ({}_L h_{N-i}^3(\xi))' \leq C(r-1)h_N^2 h \leq C(r-1)h^3$$

$$(5.118) \quad ({}_R h_{N+1-i}^3(\xi))' \leq C(r-1)h_N^2 h \leq C(r-1)h^3$$

$$(5.119) \quad ({}_L h_{N-i}^3(\xi))'' \leq C(r-1)h^2$$

$$(5.120) \quad ({}_R h_{N+1-i}^3(\xi))'' \leq C(r-1)h^2$$

*Proof.*

$$(5.121) \quad {}_L h_{N-i}(\xi) \leq 2h_N, \quad {}_R h_{N+1-i}(\xi) \leq 2h_N$$

384

$$(5.122) \quad \begin{aligned} ({}_L h_{N-i}^l(\xi))' &= {}_L h_{N-i}^{l-1}(\xi)({}_0 y_{N-i}'(\xi) - {}_L y_{N-1-i}'(\xi)) \\ &= {}_L h_{N-i}^{l-1}(\xi)x_i^{1/r-1}(\frac{1}{r} \frac{h_N}{Z_1} - {}_L y_{N-1-i}^{1-1/r}(\xi)) \end{aligned}$$

386 while

(5.123)

$$\begin{aligned} |\frac{1}{r} \frac{h_N}{Z_1} - {}_L y_{N-1-i}^{1-1/r}(\xi)| &= |\frac{1}{r} \frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r} |(\frac{N-t}{N})^{r-1} - (\frac{N-s}{N})^{r-1}| \quad t \in [0, 1], s \in [0, 2] \\ &\leq T^{1-1/r} |1 - (\frac{N-2}{N})^{r-1}| \leq CT^{1-1/r}(r-1)\frac{2}{N} \end{aligned}$$

387

388 Thus,

$$(5.124) \quad ({}_L h_{N-i}^l(\xi))' \leq C(r-1)h_N^{l-1}x_i^{1/r-1}h$$

$$(5.125) \quad \begin{aligned} ({}_R h_{N+1-i}^l(\xi))' &= {}_R h_{N+1-i}^{l-1}(\xi)({}_R y_{N+1-i}'(\xi) - {}_0 y_{N-i}'(\xi)) \\ &= {}_R h_{N+1-i}^{l-1}(\xi)x_i^{1/r-1}((2T - {}_R y_{N+1-i}(\xi))^{1-1/r} - \frac{1}{r} \frac{h_N}{Z_1}) \end{aligned}$$

391 Similarly,

(5.126)

$$\begin{aligned} |(2T - {}_R y_{N+1-i})^{1-1/r} - \frac{1}{r} \frac{h_N}{Z_1}| &= |\eta^{1-1/r} - \frac{1}{r} \frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r} |(\frac{N-s}{N})^{r-1} - (\frac{N-t}{N})^{r-1}| \quad t \in [0, 1], s \in [0, 2] \\ &\leq T^{1-1/r} |(\frac{N-2}{N})^{r-1} - 1| \leq CT^{1-1/r}(r-1)\frac{2}{N} \end{aligned}$$

392

393 And

(5.127)

$$\begin{aligned} ({}_L h_{N-i}^3(\xi))'' &= 3{}_L h_{N-i}^2(\xi){}_L h_{N-i}''(\xi) + 6{}_L h_{N-i}(\xi)({}_L h_{N-i}'(\xi))^2 \\ &\leq Ch_N^2 \frac{1-r}{r} x_i^{1/r-2} (\frac{1}{r} \frac{h_N}{Z_1} - {}_L y_{N-1-i}^{1-2/r}(\xi)Z_{N-1-i}) + Ch_N(r-1)^2 h^2 x_i^{2/r-2} \end{aligned}$$

394

$$(5.128) \quad |\frac{h_N}{rZ_1} - {}_L y_{N-1-i}^{1-2/r}(\xi)Z_{N-1-i}| \leq T^{1-1/r} + Cx_N^{1-2/r}x_N^{1/r} = CT^{1-1/r}$$

395

396 So

$$397 \quad (5.128) \quad \begin{aligned} ({}_L h_{N-i}^3(\xi))'' &\leq C h_N^2 \frac{1-r}{r} x_i^{1/r-2} + C(r-1)^2 h_N x_i^{2/r-2} h^2 \\ &\leq C(r-1) h_N^2 \end{aligned}$$

398  ${}_R h_{N+1-i}^3(\xi)$  is similar. □

LEMMA 5.31.

$$399 \quad (5.129) \quad u''({}_L y_{N-i}^\theta(\xi)) \leq C x_{N-2}^{-\alpha/2-2} \leq C$$

$$400 \quad (5.130) \quad (u''({}_L y_{N-i}^\theta(\xi)))' \leq C$$

$$401 \quad (5.131) \quad (u''({}_L y_{N-i}^\theta(\xi)))'' \leq C$$

*Proof.*

$$402 \quad (5.132) \quad \begin{aligned} (u''({}_L y_{N-i}^\theta(\xi)))' &= u'''({}_L y_{N-i}^\theta(\xi)) {}_L y_{N-i}^{\theta'}(\xi) \\ &\leq C(\theta {}_L y_{N-1-i}'(\xi) + (1-\theta) {}_0 y_{N-i}'(\xi)) \\ &\leq C x_i^{1/r-1} (\theta {}_L y_{N-1-i}^{1-1/r}(\xi) + (1-\theta) \frac{h_N}{r Z_1}) \\ &\leq C x_i^{1/r-1} x_N^{1-1/r} \end{aligned}$$

403 And

(5.133)

$$404 \quad \begin{aligned} (u''({}_L y_{N-i}^\theta(\xi)))'' &= u''''({}_L y_{N-i}^\theta(\xi)) ({}_L y_{N-i}^{\theta'})^2 + u'''({}_L y_{N-i}^\theta(\xi)) {}_L y_{N-i}^{\theta''}(\xi) \\ &\leq C x_i^{2/r-2} x_N^{2-2/r} + C \frac{r-1}{r} x_i^{1/r-2} (\theta x_N^{1-2/r} Z_{N-1-i} + (1-\theta) \frac{h_N}{r Z_1}) \\ &\leq C x_i^{2/r-2} + C(r-1) x_i^{1/r-2} T^{1-1/r} \end{aligned}$$

LEMMA 5.32.

$$405 \quad (5.134) \quad |{}_L y_{N-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C |y_N^\theta - x_i|^{1-\alpha}$$

$$406 \quad (5.135) \quad (|{}_L y_{N-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C |y_N^\theta - x_i|^{1-\alpha}$$

$$407 \quad (5.136) \quad (|{}_L y_{N-i}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C(r-1) |y_N^\theta - x_i|^{-\alpha} + |y_N^\theta - x_i|^{1-\alpha}$$

*Proof.*

(5.137)

$$408 \quad \begin{aligned} ({}_L y_{N-i}^\theta(\xi) - \xi)' &= (\theta ({}_L y_{N-1-i}(\xi) - \xi) + (1-\theta) ({}_0 y_{N-i}(\xi) - \xi))' \\ &= \theta ({}_L y_{N-1-i}'(\xi) - 1) + (1-\theta) ({}_0 y_{N-i}'(\xi) - 1) \\ &= \theta \xi^{1/r-1} ({}_L y_{N-1-i}^{1-1/r}(\xi) - \xi^{1-1/r}) + (1-\theta) \xi^{1/r-1} \left( \frac{h_N}{r Z_1} - \xi^{1-1/r} \right) \end{aligned}$$

409

$$410 \quad (5.138) \quad \begin{aligned} ({}_L y_{N-i}^\theta(\xi) - \xi)'' &= \theta ({}_L y_{N-1-i}''(\xi)) + (1-\theta) ({}_0 y_{N-i}''(\xi)) \\ &= \frac{1-r}{r} \xi^{1/r-2} (\theta {}_L y_{N-1-i}^{1-2/r}(\xi) Z_{N-1-i} + (1-\theta) \frac{h_N}{r Z_1}) \leq 0 \end{aligned}$$

411 And

$$412 \quad (5.139) \quad |({}_L y_{N-i}^\theta(\xi) - \xi)''| \leq C(r-1) \xi^{1/r-2} T^{1-1/r}$$

We have known

$$(5.140) \quad C|x_{N-1} - x_i| \leq |Ly_{N-1-i}(\xi) - \xi| \leq C|x_{N-1} - x_i|$$

If  $\xi \leq x_{N-1}$ , then  $(_0y_{N-i}(\xi) - \xi)' \geq 0$ , so

$$(5.141) \quad C|x_N - x_i| \leq |x_{N-1} - x_{i-1}| \leq |Ly_{N-i}^\theta(\xi) - \xi| \leq |x_{N+1} - x_{i+1}| \leq C|x_N - x_i|$$

If  $i = N - 1$  and  $\xi \in [x_{N-1}, x_N]$ , then  $_0y_{N-i}(\xi) - \xi$  is concave, bigger than its two neighboring points, which are equal to  $h_N$ , so

$$(5.142) \quad h_N = |x_N - x_{N-1}| \leq |_0y_{N-i}(\xi) - \xi| \leq |x_{N+1} - x_{N-1}| = 2h_N$$

So we have

$$(5.143) \quad |Ly_{N-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_N^\theta - x_i|^{1-\alpha}$$

While

$$(5.144) \quad Ly_{N-1-i}^{1-1/r}(\xi) - \xi^{1-1/r} \leq (Ly_{N-1-i}(\xi) - \xi)\xi^{-1/r}$$

and

$$(5.145) \quad \left| \frac{h_N}{rZ_1} - \xi^{1-1/r} \right| \leq \max\left\{ \left| \frac{h_N}{rZ_1} - x_{i-1}^{1-1/r} \right|, \left| \frac{h_N}{rZ_1} - x_{i+1}^{1-1/r} \right| \right\}$$

$$\leq \max \begin{cases} T^{1-1/r} - x_{i-1}^{1-1/r} \leq |x_N - x_{i-1}|T^{-1/r} \leq C|x_N - x_i| \\ |x_{i+1}^{1-1/r} - x_{N-1}^{1-1/r}| \leq |x_{i+1} - x_{N-1}|x_{N-1}^{-1/r} \leq C|x_N - x_i| \end{cases}$$

So we have

$$(5.146) \quad (Ly_{N-i}^\theta(\xi) - \xi)' \leq C|y_N^\theta - x_i|$$

$$(5.147) \quad (|Ly_{N-i}^\theta(\xi) - \xi|^{1-\alpha})' = |Ly_{N-i}^\theta(\xi) - \xi|^{-\alpha} (Ly_{N-i}^\theta(\xi) - \xi)' \leq |y_N^\theta - x_i|^{1-\alpha}$$

Finally,

$$(5.148) \quad \begin{aligned} (|Ly_{N-i}^\theta(\xi) - \xi|^{1-\alpha})'' &= (1-\alpha)|Ly_{N-i}^\theta(\xi) - \xi|^{-\alpha} (Ly_{N-i}^\theta(\xi) - \xi)'' \\ &\quad + \alpha(\alpha-1)|Ly_{N-i}^\theta(\xi) - \xi|^{-1-\alpha} ((Ly_{N-i}^\theta(\xi) - \xi)')^2 \quad \square \\ &\leq C(r-1)|y_N^\theta - x_i|^{-\alpha} + C|y_N^\theta - x_i|^{1-\alpha} \end{aligned}$$

By the three lemmas above, for  $N/2 \leq i \leq N-1$ , we have

LEMMA 5.33.

$$(5.149) \quad \begin{aligned} D_h^2 {}_L P_{N-i}^\theta(x_i) &= {}_L P_{N-i}^{\theta''}(\xi) \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq Ch^3|y_N^\theta - x_i|^{1-\alpha} + C(r-1)(h^3|y_N^\theta - x_i|^{-\alpha} + h^2|y_N^\theta - x_i|^{1-\alpha}) \end{aligned}$$

And

LEMMA 5.34.

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_{i+1})u'''(\eta_{N+1}^\theta) - {}_L Q_{N-i}^\theta(x_i)u'''(\eta_N^\theta)}{h_{i+1}} \right) \\ & \leq Ch^3|y_N^\theta - x_i|^{1-\alpha} \end{aligned} \quad (5.150)$$

And immediately, For  $N/2 \leq i \leq N-2$

$$\begin{aligned} V_{iN} & \leq C \int_{x_{N-1}}^{x_N} h^2|y - x_i|^{1-\alpha} + C(r-1)h^2|y - x_i|^{-\alpha} + h|y - x_i|^{1-\alpha} dy \\ & \leq Ch^2h_N|T - x_i|^{1-\alpha} + C(r-1)h^2|x_{N-1} - x_i|^{1-\alpha} + Chh_N|T - x_i|^{1-\alpha} \\ & \leq Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned} \quad (5.151)$$

But expecially, when  $i = N-1$ ,

$$\begin{aligned} V_{N-1,N} & = \int_0^1 -\frac{\theta^{2-\alpha}(1-\theta)}{2} \frac{2}{h_{N-1} + h_N} \left( \frac{1}{h_{N-1}} h_{N-1}^{4-\alpha} u''(y_{N-1}^\theta) - \left( \frac{1}{h_{N-1}} + \frac{1}{h_N} \right) h_N^{4-\alpha} u''(y_N^\theta) + \frac{1}{h_N} h_{N+1}^{4-\alpha} u''(y_{N+1}^\theta) \right) d\theta \\ & \quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_{i+1})u'''(\eta_{N+1,1}^\theta) - {}_L Q_{N-i}^\theta(x_i)u'''(\eta_{N,1}^\theta)}{h_{i+1}} \right) d\theta \\ & \quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_i)u'''(\eta_{N,1}^\theta) - {}_L Q_{N-i}^\theta(x_{i-1})u'''(\eta_{N-1,1}^\theta)}{h_i} \right) d\theta \\ & \quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_{i+1})u'''(\eta_{N+1,2}^\theta) - {}_L Q_{N-i}^\theta(x_i)u'''(\eta_{N,2}^\theta)}{h_{i+1}} \right) d\theta \\ & \quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_i)u'''(\eta_{N,2}^\theta) - {}_L Q_{N-i}^\theta(x_{i-1})u'''(\eta_{N-1,2}^\theta)}{h_i} \right) d\theta \end{aligned} \quad (5.152)$$

while combine Lemma 5.30

$$\begin{aligned} & \frac{2}{h_{N-1} + h_N} \left( \frac{1}{h_{N-1}} h_{N-1}^{4-\alpha} u''(y_{N-1}^\theta) - \left( \frac{1}{h_{N-1}} + \frac{1}{h_N} \right) h_N^{4-\alpha} u''(y_N^\theta) + \frac{1}{h_N} h_{N+1}^{4-\alpha} u''(y_{N+1}^\theta) \right) \\ & = D_h^2(h_{N-1 \rightarrow N}^{4-\alpha}(x_i)u''(y_{N-1 \rightarrow N}^\theta(x_i))) \\ & \leq Ch_N^{4-\alpha} + C(r-1)h_N^{3-\alpha} \leq Ch^{4-\alpha} + C(r-1)h^2|T - x_{N-1-1}|^{1-\alpha} \end{aligned} \quad (5.153)$$

442

443 Similarly with  $j = N+1$ . □

$I_6, I_7$  is easy. Similar with Lemma 5.22 and Lemma 5.6, we have

**THEOREM 5.35.** *There is a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For*  
 $N/2 \leq i \leq N,$   
(5.154)

$$I_6 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1, 2N - \lceil \frac{N}{2} \rceil + 1} + T_{i-1, 2N - \lceil \frac{N}{2} \rceil}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, 2N - \lceil \frac{N}{2} \rceil + 1} \right) \\ \leq Ch^2$$

*Proof.* In fact, let  $l = 2N - \lceil \frac{N}{2} \rceil + 1$

$$\begin{aligned} & \frac{1}{h_i} (T_{i-1, l} + T_{i-1, l-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, l} \\ &= \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) + \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) + \left( \frac{1}{h_i} - \frac{1}{h_{i+1}} \right) T_{i, l} \end{aligned}$$

While, by Lemma A.2

$$\begin{aligned} \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) &= \int_{x_{l-1}}^{x_l} (u(y) - u_h(y)) \frac{|x_{i-1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_i \Gamma(2-\alpha)} dy \\ &\leq C \int_{x_{l-1}}^{x_l} h_l^2 u''(\eta) \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy \\ &\leq Ch_l^3 x_{l-1}^{\alpha/2-2} T^{-\alpha} \\ &\leq Ch_l^3 \end{aligned}$$

Thus,

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) \leq Ch_l^2$$

For

$$\frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{h_i} d\theta$$

And Similar with Lemma 5.19, we can get

$$\frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{(h_i + h_{i+1}) h_i} \leq Ch_l^2 |y_l^\theta - x_i|^{1-\alpha}$$

So

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) \leq Ch^2$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned} \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i, l} &\leq h_i^{-3} h^2 x_i^{1-2/r} h_l C h_l^2 x_{l-1}^{\alpha/2-2} |x_l - x_i|^{1-\alpha} \\ &\leq Ch^2 \end{aligned}$$

Summarizes, we have

$$I_6 \leq Ch^2$$

□



And

LEMMA 5.36. *There is a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i \leq N$ ,*

$$I_7 = \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

*Proof.* For  $i \leq N, j \geq 2N - \lceil \frac{N}{2} \rceil + 2$ , we have

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left( \frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\ &\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2-2/r} |y - x_{i+1}^{-1-\alpha}| dy \\ &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy \end{aligned}$$

$$\begin{aligned} \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{(2-2^{-r})T}^{x_{2N-1}} (2T - y)^{\alpha/2-2/r} dy \\ &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r}T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\ &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r} h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r} h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned}$$

Now we can conclude a part of the theorem Theorem 5.3 at the beginning of this section.

By Lemma 5.8 Lemma 5.22 Lemma 5.23 Theorem 5.29 Theorem 5.28 Theorem 5.35 Lemma 5.36, we have

THEOREM 5.37. *there exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $N/2 \leq i < N$ ,*

$$R_i = \sum_{j=1}^7 I_j \leq C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

And what we left is the case  $i = N$ . Fortunately, we can use the same department of  $R_i$  above, and it is symmetric. Most of the item has been esitimated by Lemma 5.8 and Theorem 5.35, we just need to consider  $I_3, I_4$ .

**THEOREM 5.38.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

$$(5.166) \quad I_3 = \sum_{j=\lceil \frac{N}{2} \rceil + 1}^{N-1} V_{Nj} \leq Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha}$$

*Proof.* **DEFINITION 5.39.** *For  $N/2 \leq j < N$ , Let's define*

$$(5.167) \quad y_j(x) = \left( \frac{Z_1}{h_N}(x - x_N) + Z_j \right)^r, \quad Z_j = T^{1/r} \frac{j}{N}$$

*We can see that is the inverse of the function  ${}_0y_{N-i}(x)$  defined in Theorem 5.29.*

$$(5.168) \quad y'_j(x) = y_j^{1-1/r}(x) \frac{rZ_1}{h_N}$$

$$(5.169) \quad y''_j(x) = y_j^{1-2/r}(x) \frac{r(r-1)Z_1}{h_N}$$

With the scheme we used several times, we can get

**LEMMA 5.40.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq j < N$ ,  $\xi \in [x_{N-1}, x_{N+1}]$ ,*

$$(5.170) \quad h_j(\xi)^3 \leq Ch^3$$

$$(5.171) \quad (h_j^3(\xi))' \leq C(r-1)h^3$$

$$(5.172) \quad (h_j^3(\xi))'' \leq C(r-1)h^3$$

$$(5.173) \quad u''(y_j^\theta(\xi)) \leq C$$

$$(5.174) \quad (u''(y_j^\theta(\xi)))' \leq C$$

$$(5.175) \quad (u''(y_j^\theta(\xi)))'' \leq C$$

$$(5.176) \quad |\xi - y_j^\theta(\xi)|^{1-\alpha} \leq C|x_N - y_j^\theta|^{1-\alpha}$$

$$(5.177) \quad (|\xi - y_j^\theta(\xi)|^{1-\alpha})' \leq C|x_N - y_j^\theta|^{1-\alpha}$$

$$(5.178) \quad (|\xi - y_j^\theta(\xi)|^{1-\alpha})'' \leq C|x_N - y_j^\theta|^{1-\alpha} + C(r-1)|x_N - y_j^\theta|^{-\alpha}$$

**LEMMA 5.41.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq j < N$ ,*

$$(5.179) \quad V_{Nj} \leq Ch^2 \int_{x_{j-1}}^{x_j} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy$$

Therefore,

$$(5.180) \quad \begin{aligned} I_3 &\leq Ch^2 \int_{\lceil \frac{N}{2} \rceil}^{N-1} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \\ &\leq Ch^2(|T - x_{N-1}|^{2-\alpha} + (r-1)|T - x_{N-1}|^{1-\alpha}) \end{aligned}$$

□

For  $j = N$ ,

LEMMA 5.42.

(5.181)

$$V_{N,N} = \frac{1}{h_N^2} (T_{N-1,N-1} - 2T_{N,N} + T_{N+1,N+1}) \leq Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha}$$

*Proof.*

(5.182)

$$\begin{aligned} V_{N,N} = & \int_0^1 -\frac{\theta(1-\theta)^{2-\alpha}}{2} \frac{1}{h_N^2} (h_{N-1}^{4-\alpha} u''(y_{N-1}^\theta) - 2h_N^{4-\alpha} u''(y_N^\theta) + h_{N+1}^{4-\alpha} u''(y_{N+1}^\theta)) d\theta \\ & + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{1}{h_N} \left( \frac{Q_{N \rightarrow N}^\theta(x_{N+1}) u'''(\eta_{N+1,1}^\theta) - Q_{N \rightarrow N}^\theta(x_i) u'''(\eta_{N,1}^\theta)}{h_N} \right) d\theta \\ & - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{1}{h_N} \left( \frac{Q_{N \rightarrow N}^\theta(x_N) u'''(\eta_{N,1}^\theta) - Q_{N \rightarrow N}^\theta(x_{N-1}) u'''(\eta_{N-1,1}^\theta)}{h_N} \right) d\theta \\ & - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{1}{h_N} \left( \frac{Q_{N \rightarrow N}^\theta(x_{N+1}) u'''(\eta_{N+1,2}^\theta) - Q_{N \rightarrow N}^\theta(x_N) u'''(\eta_{N,2}^\theta)}{h_N} \right) d\theta \\ & + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{1}{h_N} \left( \frac{Q_{N \rightarrow N}^\theta(x_N) u'''(\eta_{N,2}^\theta) - Q_{N \rightarrow N}^\theta(x_{N-1}) u'''(\eta_{N-1,2}^\theta)}{h_N} \right) d\theta \end{aligned}$$

So combine Lemma 5.8, Theorem 5.35, Theorem 5.38, Lemma 5.42 We have

LEMMA 5.43.

$$R_N \leq C(r-1)h^2|T - x_{N-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

and with Theorem 5.37 we prove the Theorem 5.3

**5.5. Truncation error.** combine Theorem 5.1, Theorem 5.2 and Theorem 5.3

we get For  $1 \leq i \leq N$

(5.184)

$$R_i \leq C_2(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} C_1 h^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ C_1 h^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ C_1 h^{r\alpha/2+r} x_i^{-1-\alpha/2}, & r\alpha/2 + r - 2 < 0 \end{cases}$$

But,

$$h^2 x_i^{-\alpha/2-2/r} \leq T^{\alpha/2-2/r} \begin{cases} h^2 x_i^{-\alpha}, & \text{if } r\alpha/2 - 2 \geq 0 \\ h^{r\alpha/2} x_i^{-\alpha}, & \text{if } r\alpha/2 - 2 \leq 0 \end{cases}$$

$$h^{r\alpha/2+r} x_i^{-1-\alpha} \leq T^{-1} h^{r\alpha/2} x_i^{-\alpha}, \quad \text{if } r\alpha/2 - 2 \leq 0$$

(5.187)

And when  $r\alpha/2 - 2 = -r < 0$ ,

$$\begin{aligned} h^2 x_i^{-1-\alpha} \ln(i) h^{-r\alpha/2} x_i^\alpha &= h^r x_i^{-1} \ln(i) \\ &= T^{-1} \frac{\ln(i)}{i^r} \leq C(T, r) \end{aligned}$$

525 and

526 (5.189) 
$$h^2 \ln(N) h^{-r\alpha/2} x_i^\alpha = h^r \ln(N) x_i^\alpha \leq T^\alpha \frac{\ln(N)}{N^r} \leq C(T, \alpha, r)$$

527 So for  $1 \leq i \leq N$ ,

528 (5.190) 
$$R_i \leq C_2(r-1)h^2|T - x_{i-1}|^{1-\alpha} + C_1h^{\min\{\frac{r\alpha}{2}, 2\}}x_i^{-\alpha}$$

529 And for  $i \geq N$ , it is symmetric for  $i$  and  $2N - i$ .

530 The proof of Theorem 4.1 completed.

**6. Proof of Theorem 4.2.** Review section 3, we have (3.9) and (3.11),

$$(6.1) \quad \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left( \frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right)$$

$$(6.2) \quad a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \tilde{a}_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right)$$

Thus

LEMMA 6.1.

$$(6.3) \quad \sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left( \frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right)$$

DEFINITION 6.2. We call one matrix a *M matrix*, which means its entries are positive on major diagonal and nonpositive on others, and Strictly diagonally dominant in rows.

Now we have

LEMMA 6.3. The matrix *A* defined by (3.11) is a *M matrix*. and

$$(6.4) \quad \begin{aligned} S_i &:= \sum_{j=1}^{2N-1} a_{ij} \\ &= -\kappa_\alpha \sum_{j=1}^{2N-1} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_i} \tilde{a}_{i-1,j} \right) \\ &\geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \end{aligned}$$

*Proof.* Let

$$(6.5) \quad g(x) = g_0(x) + g_{2N}(x)$$

where

$$\begin{aligned} g_0(x) &:= \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x - x_0|^{3-\alpha} - |x - x_1|^{3-\alpha}}{h_1} \\ g_{2N}(x) &:= \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N} - x|^{3-\alpha} - |x_{2N-1} - x|^{3-\alpha}}{h_{2N}} \end{aligned}$$

Thus

$$-\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

Then

$$(6.6) \quad \begin{aligned} S_i &:= \sum_{j=1}^{2N-1} a_{ij} \\ &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &= D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i) \end{aligned}$$

When  $i = 1$

$$\begin{aligned}
 D_h^2 g_0(x_1) &= \frac{2}{h_1 + h_2} \left( \frac{1}{h_2} g_0(x_2) - \left( \frac{1}{h_1} + \frac{1}{h_2} \right) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right) \\
 &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1h_2} \\
 &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1^{1-\alpha}h_2} h_1^{-\alpha} \\
 &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha}}{2^r(2^r - 1)} h_1^{-\alpha}
 \end{aligned}$$

but

$$1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha} > 0$$

While for  $i \geq 2$

$$\begin{aligned}
 D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \\
 &= -\kappa_\alpha \frac{|\xi - x_0|^{1-\alpha} - |\xi - x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1} \\
 &= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} |\xi - \eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\
 &\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha}
 \end{aligned}$$

So

$$\frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g_0(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g_0(x_i) + \frac{1}{h_i} g_0(x_{i-1}) \right) \geq C x_i^{-\alpha}$$

symmetricly,

(6.11)

$$\frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g_{2N}(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g_{2N}(x_i) + \frac{1}{h_i} g_{2N}(x_{i-1}) \right) \geq C(\alpha, r)(2T - x_i)^{-\alpha}$$

Let

$$g(x) = \begin{cases} x, & 0 < x \leq T \\ 2T - x, & T < x < 2T \end{cases}$$

And define

$$G = \text{diag}(g(x_1), \dots, g(x_{2N-1}))$$

Then

LEMMA 6.4. *The matrix  $B := AG$ , the major diagonal is positive, and nonpositive on others. And there is a constant  $C = C(\alpha, r)$  such that*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases}$$

*Proof.*

$$b_{ij} = a_{ij}g(x_j) = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_i} \tilde{a}_{i-1,j} \right) g(x_j)$$

Since

$$(6.15) \quad g(x) \equiv \Pi_h g(x)$$

by (3.9), we have

$$\begin{aligned} \tilde{M}_i &:= \sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} \tilde{a}_{ij} g(x_j) \\ (6.16) \quad &= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} \Pi_h g(y) dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} g(y) dy \\ &= \frac{-2}{\Gamma(4-\alpha)} |T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)} (x_i^{3-\alpha} + (2T - x_i)^{3-\alpha}) \\ &:= w(x_i) = p(x_i) + q(x_i) \end{aligned}$$

Thus,

$$\begin{aligned} M_i &:= \sum_{j=1}^{2N-1} b_{ij} = \sum_{j=1}^{2N-1} a_{ij} g(x_j) \\ (6.17) \quad &= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} \tilde{M}_{i+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{M}_i + \frac{1}{h_i} \tilde{M}_{i-1} \right) \\ &= -\kappa_\alpha D_h^2(p(x_i) + q(x_i)) \end{aligned}$$

for  $1 \leq i < N-1$ , by Lemma A.1

$$\begin{aligned} D_h^2 p(x_i) &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} p(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) p(x_i) + \frac{1}{h_i} p(x_{i-1}) \right) \\ (6.18) \quad &= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1}) \\ &\geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - x_{i-1}|^{1-\alpha} \end{aligned}$$

and

$$\begin{aligned} (6.19) \quad D_h^2 p(x_{N-1}) &:= \frac{-2\kappa_\alpha}{h_{N-1} + h_N} \left( \frac{1}{h_N} p(x_N) - \left( \frac{1}{h_{N-1}} + \frac{1}{h_N} \right) p(x_{N-1}) + \frac{1}{h_{N-1}} p(x_{N-2}) \right) \\ &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{2}{h_{N-1} + h_N} \left( -\left( \frac{1}{h_{N-1}} + \frac{1}{h_N} \right) h_N^{3-\alpha} + \frac{1}{h_{N-1}} (h_{N-1} + h_N)^{3-\alpha} \right) \\ (580) \quad &= \frac{4\kappa_\alpha}{\Gamma(4-\alpha) h_{N-1}} (-h_N^{2-\alpha} + (h_{N-1} + h_N)^{2-\alpha}) \\ &= \frac{4\kappa_\alpha}{(3-\alpha)\Gamma(2-\alpha)} \xi^{1-\alpha} \quad \xi \in [h_N, h_{N-1} + h_N] \\ &\geq \frac{4\kappa_\alpha}{(3-\alpha)\Gamma(2-\alpha)} (h_{N-1} + h_N)^{1-\alpha} = \frac{4\kappa_\alpha}{(3-\alpha)\Gamma(2-\alpha)} (T - x_{N-2})^{1-\alpha} \end{aligned}$$

(6.20)

$$\begin{aligned} D_h^2(x_N) &:= -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left( \frac{1}{h_{N+1}} p(x_{N+1}) - \left( \frac{1}{h_N} + \frac{1}{h_{N+1}} \right) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right) \\ &= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)h_N^2} h_N^{3-\alpha} \\ &= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - x_{N-1})^{1-\alpha} \end{aligned}$$

Symmetricly for  $i \geq N$ , we get

$$(6.21) \quad P_i \geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases}$$

Similarly, we can get

$$\begin{aligned} (6.22) \quad Q_i &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} q(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right) \\ &\geq \frac{-2^{r(\alpha-1)+1}\kappa_\alpha}{\Gamma(2-\alpha)} (x_{i-1}^{1-\alpha} + (1 - x_{i+1})^{1-\alpha}) \end{aligned} \quad \square$$

Notice that

$$(6.23) \quad x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha}$$

We can get

THEOREM 6.5. *There exists a real  $\lambda = \lambda(T, \alpha, r) > 0$  and  $C = C(T, \alpha, r) > 0$  such that  $B := A(\lambda I + G)$  is an  $M$  matrix. And*

$$(6.24) \quad M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + C \begin{cases} |\frac{1}{2} - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - \frac{1}{2}|^{1-\alpha}, & i \geq N \end{cases}$$

*Proof.* By 6.3 with  $C_1$  and 6.4 with  $C_2$ , it's sufficient to take  $\lambda = 4TC_2/C_1$ , then

$$(6.25) \quad M_i \geq C_2 \left( (x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \right) \quad \square$$

Now, we can prove the convergency Theorem 4.2.

For equation

$$(6.26) \quad AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1}U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1}U = F$$

which means

$$(6.27) \quad \sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + g(x_j)} = \tau_i$$

where  $\epsilon_i = u(x_i) - U_i$ .

And if

$$(6.28) \quad \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| = \max_{1 \leq i \leq 2N-1} \left| \frac{\epsilon_i}{\lambda + g(x_i)} \right|$$



Then, since  $B = A(\lambda I + G)$  is an  $M$  matrix, it is Strictly diagonally dominant. Thus,

$$\begin{aligned}
 |\tau_{i_0}| &= \left| \sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + g(x_i)} \right| \\
 &\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_j}{\lambda + g(x_j)} \right| \\
 &\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \\
 &= \sum_{j=1}^{2N-1} b_{i_0,j} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \\
 &= M_{i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right|
 \end{aligned}
 \tag{6.29}$$

By Theorem 4.1 and Theorem 6.5,

We know that there exists constants  $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ ,  
 and  $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that

$$\left| \frac{\epsilon_i}{\lambda + g(x_i)} \right| \leq \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} + C_2(r-1)h^2
 \tag{6.30}$$

as  $\lambda + g(x_i) \leq \lambda + T$

So, we can get

$$|\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{r\alpha}{2}, 2\}}
 \tag{6.31}$$

The convergency has been proved.

## 7. Experimental results.

### 8. Remarks. some remarks.

In Theorem 2.3 If  $f \in L^\infty(\Omega)$  then  $u \in C_{\alpha/2}(\Omega)$ , which is Proposition 1.1 in [2].

When  $\|f\|_\beta^{(\gamma)} < \infty$ , where  $\beta > 2 - \alpha$  and  $\gamma \in [-\alpha, -\alpha/2]$ , we observed convergent order  $\min\{r(\alpha+\gamma), 2\}$  in numerical experiments. And we can prove that kind theorems with the techneque we used in this paper.

## Appendix A. Approximate of difference quotients.

LEMMA A.1. If  $g(x) \in C^2\Omega$ , there exists  $\xi \in [x_{i-1}, x_{i+1}]$  such that

$$(A.1) \quad D_h^2 g(x_i) := \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

$$(A.2) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right)$$

And if  $g(x) \in C^4(\Omega)$ , then

$$(A.3) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2))$$

where  $\eta_1 \in [x_{i-1}, x_i]$ ,  $\eta_2 \in [x_i, x_{i+1}]$ .

*Proof.*

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})$$

Substitute them in the left side of (A.1), we have

$$\frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2)$$

Now, using **intermediate value theorem**, there exists  $\xi \in [\xi_1, \xi_2]$  such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y)dy$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy$$

Especially,

$$(A.4) \quad \begin{aligned} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy &= \frac{h_i^4}{4!} g''''(\eta_1) \\ \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy &= \frac{h_{i+1}^4}{4!} g''''(\eta_2) \end{aligned}$$

where  $\eta_1 \in [x_{i-1}, x_i]$ ,  $\eta_2 \in [x_i, x_{i+1}]$ . Subsitute them to the left side of (A.3), we can get the result.  $\square$

LEMMA A.2. Denote  $y_j^\theta = \theta x_{j-1} + (1 - \theta)x_j$ ,  $\theta \in [0, 1]$ ,

$$(A.5) \quad u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1 - \theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j]$$

$$(A.6) \quad u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1 - \theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1 - \theta)}{3!} h_j^3 ((1 - \theta)^2 u'''(\eta_1) - \theta^2 u'''(\eta_2))$$

where  $\eta_1 \in [x_{j-1}, y_j^\theta]$ ,  $\eta_2 \in [y_j^\theta, x_j]$ .

*Proof.* By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - (1 - \theta)h_j u'(y_j^\theta) + \frac{(1 - \theta)^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$

$$u(x_j) = u(y_j^\theta) + \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - \theta u(x_{j-1}) - (1 - \theta)u(x_j) \\ &= -\frac{\theta(1 - \theta)}{2} h_j^2 ((1 - \theta)u''(\xi_1) + \theta u''(\xi_2)) \\ &= -\frac{\theta(1 - \theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - (1 - \theta)h_j u'(y_j^\theta) + \frac{(1 - \theta)^2 h_j^2}{2!} u''(y_j^\theta) - \frac{(1 - \theta)^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) + \frac{\theta^3 h_j^3}{3!} u'''(\eta_2)$$

where  $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$ . Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - \theta u(x_{j-1}) - (1-\theta)u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 ((1-\theta)^2 u'''(\eta_1) - \theta^2 u'''(\eta_2)) \end{aligned}$$

LEMMA A.3. For  $x \in [x_{j-1}, x_j]$

$$\begin{aligned} |u(x) - u_h(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^x u'(y) dy - \frac{x - x_{j-1}}{h_j} \int_x^{x_j} u'(y) dy \right| \\ &\leq \int_{x_{j-1}}^{x_j} |u'(y)| dy \end{aligned}$$

If  $x \in [0, x_1]$ , with Corollary 2.4, we have

$$|u(x) - u_h(x)| \leq \int_0^{x_1} |u'(y)| dy \leq \int_0^{x_1} C y^{\alpha/2-1} dy \leq C \frac{2}{\alpha} x_1^{\alpha/2}$$

Similarly, if  $x \in [x_{2N-1}, 1]$ , we have

$$|u(x) - u_h(x)| \leq C \frac{2}{\alpha} (2T - x_{2N-1})^{\alpha/2} = C \frac{2}{\alpha} x_1^{\alpha/2}$$

## Appendix B. Inequality.

LEMMA B.1.

$$h_i \leq rT^{1/r} h \begin{cases} x_i^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_{i-1})^{1-1/r}, & N < i \leq 2N-1 \end{cases}$$

$$h_i \geq rT^{1/r} h \begin{cases} x_{i-1}^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_i)^{1-1/r}, & N < i \leq 2N-1 \end{cases}$$

*Proof.* For  $1 \leq i \leq N$ ,

$$\begin{aligned} h_i &= T \left( \left( \frac{i}{N} \right)^r - \left( \frac{i-1}{N} \right)^r \right) \\ &\leq rT \frac{1}{N} \left( \frac{i}{N} \right)^{r-1} = rT^{1/r} h x_i^{1-1/r} \end{aligned}$$

$$h_i \geq rT \frac{1}{N} \left( \frac{i-1}{N} \right)^{r-1} = rT^{1/r} h x_{i-1}^{1-1/r}$$

For  $N < i \leq 2N$ ,

$$\begin{aligned} h_i &= T \left( \left( \frac{2N-i+1}{N} \right)^r - \left( \frac{2N-i}{N} \right)^r \right) \\ &\leq rT \frac{1}{N} \left( \frac{2N-i+1}{N} \right)^{r-1} = rT^{1/r} h (2T - x_{i-1})^{1-1/r} \end{aligned}$$

$$h_i \geq rT \frac{1}{N} \left( \frac{2N-i}{N} \right)^{r-1} = rT^{1/r} h (2T - x_i)^{1-1/r}$$

676

677 **LEMMA B.2.** *There is a constant  $C = 2^{|r-2|r}r(r-1)T^{2/r}$  such that for all  $i \in$*   
678  *$\{1, 2, \dots, 2N-1\}$*

$$679 \quad (B.3) \quad |h_{i+1} - h_i| \leq Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases}$$

*Proof.*

$$680 \quad h_{i+1} - h_i = \begin{cases} T \left( \left( \frac{i+1}{N} \right)^r - 2 \left( \frac{i}{N} \right)^r + \left( \frac{i-1}{N} \right)^r \right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T \left( \left( \frac{2N-i-1}{N} \right)^r - 2 \left( \frac{2N-i}{N} \right)^r + \left( \frac{2N-i+1}{N} \right)^r \right), & N+1 \leq i \leq 2N-1 \end{cases}$$

681 For  $i = 1$ ,

$$682 \quad h_2 - h_1 = T(2^r - 2) \left( \frac{1}{N} \right)^r = (2^r - 2)T^{2/r}h^2x_1^{1-2/r}$$

683 For  $2 \leq i \leq N-1$ , by Lemma A.1, we have

$$684 \quad \begin{aligned} h_{i+1} - h_i &= r(r-1)T N^{-2}\eta^{r-2}, \quad \eta \in \left[ \frac{i-1}{N}, \frac{i+1}{N} \right] \\ &= C(r-1)h^2x_i^{1-2/r} \end{aligned}$$

685 Summarizes the inequalities, we can get

$$686 \quad (B.4) \quad |h_{i+1} - h_i| \leq 2^{|r-2|r}r(r-1)T^{2/r}h^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases} \quad \square$$

## 687 **Appendix C. Proofs of some technical details.**

688 *Additional proof of Theorem 5.1.* For  $2 \leq i \leq N-1$ ,

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\ 689 \quad &\leq C \frac{2}{h_i + h_{i+1}}(h_i^3 x_{i-1}^{-2-\alpha/2} + h_{i+1}^3 x_i^{-2-\alpha/2}) \\ &\leq 2C(h_i^2 x_{i-1}^{-2-\alpha/2} + h_{i+1}^2 x_i^{-2-\alpha/2}) \end{aligned}$$

690 There is a constant  $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$  such that

$$691 \quad \frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq Ch^2 x_i^{-\alpha/2-2/r}, \quad 2 \leq i \leq N-1$$

692 For  $i = 1$ , by (A.4)

$$\begin{aligned} &\frac{1}{4!} \frac{2}{h_1 + h_2}(h_1^3 f''(\eta_1) + h_2^3 f''(\eta_2)) \\ 693 \quad &= \frac{2}{h_1 + h_2} \left( \frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy + \frac{1}{4!} h_2^3 f''(\eta_2) \right) \end{aligned}$$

We have proved above that

$$\frac{2}{h_1 + h_2} h_2^3 f''(\eta_2) \leq C h^2 x_1^{-\alpha/2-2/r}$$

and we can get

$$\begin{aligned} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy &\leq C \frac{1}{3!} \int_0^{x_1} y^{1-\alpha/2} dy \\ &= C \frac{1}{3!(2-\alpha/2)} x_1^{2-\alpha/2} \end{aligned}$$

so

$$\frac{2}{h_1 + h_2} \frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy = \frac{C 2^{1-r}}{3!(2-\alpha/2)} x_1^{-\alpha/2} = \frac{C 2^{1-r}}{3!(2-\alpha/2)} T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

And for  $i = N$ , we have

$$\begin{aligned} \frac{2}{h_N + h_{N+1}} (h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2)) \\ &= h_N^2 (f''(\eta_1) + f''(\eta_2)) \\ &\leq r^2 T^{2/r} h^2 x_N^{2-2/r} 2C x_{N-1}^{-2-\alpha/2} \\ &\leq 2r^2 T^{2/r} C 2^{-r(-2-\alpha/2)} h^2 x_N^{-\alpha/2-2/r} \end{aligned}$$

Finally,  $N + 1 \leq i \leq 2N - 1$  is symmetric to the first half of the proof, so we can conclude that  $\square$

$$\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq C h^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N - 1 \end{cases}$$

LEMMA C.1. *By a standard error estimate for linear interpolation, and Corollary 2.4, There is a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  for  $2 \leq j \leq N$ ,*

$$(C.1) \quad |u(y) - u_h(y)| \leq C h^2 y^{\alpha/2-2/r}, \quad \text{for } y \in [x_{j-1}, x_j]$$

symmetricly, for  $N < j \leq 2N - 1$ , we have

$$(C.2) \quad |u(y) - u_h(y)| \leq C h^2 (2T - y)^{\alpha/2-2/r}$$

LEMMA C.2. *There is a constant  $C = C(\alpha, r)$  such that for all  $1 \leq i < N/2$ ,  $\max\{2i + 1, i + 3\} \leq j \leq 2N$ , we have*

$$(C.3) \quad D_h^2 K_y(x_i) \leq C \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}, \quad y \in [x_{j-1}, x_j]$$

*Proof.* Since  $y \geq x_{j-1} > x_{i+1}$ , by Lemma A.1, if  $j - 1 > i + 1$   $\square$

$$\begin{aligned} D_h^2 K_y(x_i) &= K_y''(\xi) = \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq \frac{(y - x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\leq (1 - (\frac{2}{3})^r)^{-1-\alpha} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

LEMMA C.3. *There is a constant  $C = C(\alpha, r)$  such that for all  $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil, 1 \leq j \leq k-1$  and  $y \in [x_{j-1}, x_j]$ , we have*

$$(C.4) \quad D_h^2 K_y(x_i) \leq C \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)}$$

*Proof.* Since  $y \leq x_j < x_{i-1}$ , by Lemma A.1, □

$$\begin{aligned} D_h^2 K_y(x_i) &= \frac{|\xi - y|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq \frac{(x_{i-1} - x_j)^{-1-\alpha}}{\Gamma(-\alpha)} \leq \frac{(x_{i-1} - x_{k-1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\leq \left(\left(\frac{2}{3}\right)^r - \left(\frac{1}{2}\right)^r\right)^{-1-\alpha} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

LEMMA C.4. *While  $0 \leq i < N/2$ , By Lemma A.3*

$$\begin{aligned} |T_{i1}| &\leq C \int_0^{x_1} x_1^{\alpha/2} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ (C.5) \quad &= C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2} |x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha}| \\ &\leq C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2+2-\alpha} = C \frac{1}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1 \end{aligned}$$

For  $2 \leq j \leq N$ , by Lemma A.2 and Corollary 2.4

$$\begin{aligned} |T_{ij}| &\leq \frac{C}{4} \int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ (C.6) \quad &\leq \frac{C}{4\Gamma(3-\alpha)} h_j^2 x_{j-1}^{\alpha/2-2} ||x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha}| \end{aligned}$$

LEMMA C.5. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

$$(C.7) \quad \sum_{j=1}^3 S_{1j} \leq Ch^2 x_1^{-\alpha/2-2/r}$$

$$(C.8) \quad \sum_{j=1}^4 S_{2j} \leq Ch^2 x_2^{-\alpha/2-2/r}$$

*Proof.*

$$S_{1j} = \frac{2}{x_2} \left( \frac{1}{x_1} T_{0j} - \left( \frac{1}{x_1} + \frac{1}{h_2} \right) T_{1j} + \frac{1}{h_2} T_{2j} \right)$$

So, by Lemma C.4

$$S_{11} \leq \frac{2}{x_2 x_1} 4 \frac{C}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \leq C x_1^{-\alpha/2}$$

$$S_{12} \leq \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_2^2 x_1^{\alpha/2-2} (x_2^{2-\alpha} + 2h_2^{2-\alpha} + h_2^{2-\alpha}) \leq C x_1^{-\alpha/2}$$

$$S_{13} \leq \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_3^2 x_2^{\alpha/2-2} (x_3^{2-\alpha} + 2h_3^{2-\alpha} + h_3^{2-\alpha}) \leq C x_1^{-\alpha/2}$$

But

$$x_1^{-\alpha/2} = T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

$i = 2$  is similar.  $\square$

**LEMMA C.6.** *There exists a constant  $C = C(T, r, l)$  such that For  $3 \leq i \leq N - 1$ ,  $k + 1 = \lceil \frac{i}{2} \rceil$ ,  $k \leq j \leq \min\{2i - 1, N - 1\}$ ,  $l = 3, 4$ , when  $\xi \in [x_{i-1}, x_{i+1}]$ ,*

$$(C.9) \quad (h_{j-i}^3(\xi))' \leq (r-1)C h^2 x_i^{1-2/r} h_j$$

$$(C.10) \quad (h_{j-i}^4(\xi))' \leq (r-1)C h^2 x_i^{1-2/r} h_j^2$$

*Proof.* From (5.32)

$$(C.11) \quad y'_{j-i}(x) = y_{j-i}^{1-1/r}(x) x^{1/r-1}$$

$$(C.12) \quad y''_{j-i}(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i}$$

for  $l = 3, 4$ , by (5.34)

$$(C.13) \quad \begin{aligned} (h_{j-i}^l(\xi))' &= l h_{j-i}^{l-1}(\xi) (y'_{j-i}(\xi) - y'_{j-i-1}(\xi)) \\ &= l h_{j-i}^{l-1}(\xi) \xi^{1/r-1} (y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \geq 0 \end{aligned}$$

For  $\xi \in [x_{i-1}, x_{i+1}]$  and  $2 \leq k \leq j \leq \min\{2i - 1, N - 1\}$ , using Lemma B.1

$$\begin{aligned} h_{j-i}(\xi) &\leq h_{j-i}(x_{i+1}) = h_{j+1} \\ &\leq r T^{1/r} h x_{j+1}^{1-1/r} \leq r T^{1/r} 2^{r-1} h x_i^{1-1/r} \end{aligned}$$

And

$$(C.14) \quad 2^{-r} x_i \leq x_{i-1} \leq \xi \leq x_{i+1} \leq 2^r x_i$$

We have

$$(C.15) \quad \xi^{1/r-m} \leq 2^{|mr-1|} x_i^{1/r-m}, \quad m = 1, 2$$

but

$$(C.16) \quad \begin{aligned} y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) &= (\xi^{1/r} + Z_{j-i})^{r-1} - (\xi^{1/r} + Z_{j-i-1})^{r-1} \\ &= (r-1) Z_1 (\xi^{1/r} + Z_{j-i-\gamma})^{r-2}, \quad \gamma \in [0, 1] \\ &= (r-1) T^{1/r} h y_{j-i-\gamma}^{1-2/r}(\xi) \end{aligned}$$



And

(C.17)

$$4^{-r}x_i \leq x_{\lceil \frac{i}{2} \rceil - 1} \leq x_{j-2} = y_{j-i-1}(x_{i-1}) \leq y_{j-i-\gamma}(\xi) \leq y_{j-i}(x_{i+1}) = x_{j+1} \leq x_{2i} \leq 2^r x_i$$

Therefore,

$$(C.18) \quad y_{j-i-\gamma}^{1-2/r}(\xi) \leq 2^{2|r-2|} x_i^{1-2/r}$$

So we can get

$$(C.19) \quad y'_{j-i}(\xi) - y'_{j-i-1}(\xi) \leq (r-1)C(T, r) h x_i^{-1/r}$$

We get

$$(C.20) \quad (h_{j-i}^l(\xi))' \leq l(r-1)C h_{j+1}^{l-1} h x_i^{-1/r}$$

And by Lemma B.1,

$$(C.21) \quad h_{j+1} \leq rTh \left( \frac{j+1}{N} \right)^{r-1} \leq rTh 2^{r-1} \left( \frac{j-1}{N} \right) = 2^{r-1} h_j$$

$$(C.22) \quad h_{j+1} \leq rT^{1/r} h x_{j+1}^{1-1/r} \leq rT^{1/r} h x_{2i}^{1-1/r} \leq rT^{1/r} 2^{r-1} h x_i^{1-1/r}$$

We can get

$$(C.23) \quad \begin{aligned} (h_{j-i}^l(\xi))' &\leq l(r-1)C h_j^{l-2} h_{j+1} h x_i^{-1/r} \\ &\leq l(r-1)C h h_j^{l-2} (h x_i^{1-1/r}) x_i^{-1/r} \\ &= (r-1)C h^2 x_i^{1-2/r} h_j^{l-2} \end{aligned}$$

Meanwhile, we can get

$$(C.24) \quad h_{j-i}^3(\xi) \leq h_{j+1}^3 \leq C h^2 x_i^{2-2/r} h_j$$

$$(C.25) \quad h_{j-i}^4(\xi) \leq h_{j+1}^4 \leq C h^2 x_i^{2-2/r} h_j^2 \quad \square$$

**LEMMA C.7.** *There exists a constant  $C = C(T, r, l)$  such that For  $3 \leq i \leq N - 1$ ,  $\lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i - 1, N - 1\}$ , when  $\xi \in [x_{i-1}, x_{i+1}]$ ,*

$$(C.26) \quad (h_{j-i}^3(\xi))'' \leq C(r-1) h^2 x_i^{-2/r} h_j$$

*Proof.* From (C.11)

$$(C.27) \quad \begin{aligned} (h_{j-i}^3(\xi))'' &= 6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 + 3h_{j-i}^2(\xi)(y''_{j-i}(\xi) - y''_{j-i-1}(\xi)) \\ &= 6h_{j-i}(\xi)(\xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)))^2 \\ &\quad + 3\frac{1-r}{r} h_{j-i}^2(\xi) \xi^{1/r-2} (y_{j-i}^{1-2/r}(\xi) Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi) Z_{j-i-1}) \end{aligned}$$

Using the inequalities of the proof of Lemma C.6

$$\begin{aligned}
 & 6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 \\
 & \leq 6h_{j+1}((r-1)Chx_i^{-1/r})^2 \\
 & \leq C(r-1)^2 h^2 x_i^{-2/r} h_j
 \end{aligned}
 \tag{C.28}$$

For the second partial

$$\begin{aligned}
 & h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\
 & \leq Ch_{j+1}^2 x_i^{1/r-2}((y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi))Z_{j-i} + y_{j-i-1}^{1-2/r}(\xi)Z_1)
 \end{aligned}
 \tag{C.29}$$

but

$$\begin{aligned}
 & y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi) = (\xi^{1/r} + Z_{j-i})^{r-2} - (\xi^{1/r} + Z_{j-i-1})^{r-2} \\
 & = (r-2)Z_1(\xi^{1/r} + Z_{j-i-\gamma})^{r-3} \\
 & = (r-2)T^{1/r}hy_{j-i-\gamma}^{1-3/r}(\xi) \\
 & \leq C(r-2)hx_i^{1-3/r}
 \end{aligned}
 \tag{C.30}$$

So we can get

$$\begin{aligned}
 & h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\
 & \leq Ch_jhx_i^{1-1/r}x_i^{1/r-2}(C(r-2)hx_i^{1-3/r}Z_{j-i} + Cx_i^{1-2/r}T^{1/r}h) \\
 & \leq Ch^2((r-2)x_i^{-3/r}x_{|j-i|}^{1/r} + x_i^{-2/r})h_j \\
 & \leq Ch^2x_i^{-2/r}h_j
 \end{aligned}
 \tag{C.31}$$

Summarizes, we have

$$(h_{j-i}^3(\xi))'' \leq C(r-1)h^2x_i^{-2/r}h_j$$

□

*proof of Lemma 5.16.* From (5.32)

$$y'_{j-i}(x) = y_{j-i}^{1-1/r}(x)x^{1/r-1}$$

$$y''_{j-i}(x) = \frac{1-r}{r}y_{j-i}^{1-2/r}(x)x^{1/r-2}Z_{j-i}$$

Since

$$x_{j-2} \leq y_{j-i-1}(x_{i-1}) \leq y_{j-i}^\theta(\xi) \leq y_{j-i}^\theta(x_{i+1}) \leq x_{j+1}$$

We have known (C.17)

$$u''(y_{j-i}^\theta(\xi)) \leq C(y_{j-i}^\theta(\xi))^{\alpha/2-2} \leq Cx_{j-2}^{\alpha/2-2} \leq Cx_{\lfloor \frac{i}{2} \rfloor -1}^{\alpha/2-2} \leq C4^{r(2-\alpha/2)}x_i^{\alpha/2-2}$$

801

$$\begin{aligned}
 & (u''(y_{j-i}^\theta(\xi)))' = u'''(y_{j-i}^\theta(\xi))y_{j-i}^{\theta'}(\xi) \\
 & \leq Cx_i^{\alpha/2-3}\xi^{1/r-1}y_{j-i}^{1-1/r}(\xi) \\
 & \leq Cx_i^{\alpha/2-3}x_i^{1/r-1}x_i^{1-1/r} = Cx_i^{\alpha/2-3}
 \end{aligned}
 \tag{C.36}$$

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$$\begin{aligned}
(u''(y_{j-i}^\theta(\xi)))'' &= u''''(y_{j-i}^\theta(\xi))(y_{j-i}^{\theta'}(\xi))^2 + u'''(y_{j-i}^\theta(\xi))y_{j-i}^{\theta''}(\xi) \\
&\leq Cx_i^{\alpha/2-4} + Cx_i^{\alpha/2-3}\frac{r-1}{r}x_i^{1-2/r}x_i^{1/r-2}Z_{|j-i|+1} \\
&\leq Cx_i^{\alpha/2-4} + C\frac{r-1}{r}x_i^{\alpha/2-3}x_i^{-1/r}x_i^{1/r} \\
&= Cx_i^{\alpha/2-4}
\end{aligned}
\tag{C.37}$$

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□

*Proof of Lemma 5.17.*

$$\begin{aligned}
|y_{j-i}^\theta(\xi) - \xi| &= |\theta(y_{j-i-1}(\xi) - \xi) + (1-\theta)(y_{j-i}(\xi) - \xi)| \\
&= \theta|y_{j-i-1}(\xi) - \xi| + (1-\theta)|y_{j-i}(\xi) - \xi|
\end{aligned}
\tag{C.38}$$

805

Since  $|y_{j-i}(\xi) - \xi|$  is increasing about  $\xi$ , we have

$$\begin{aligned}
&(\frac{i-1}{i})^r|x_j - x_i| \leq |x_{j-1} - x_{i-1}| \leq |y_{j-i}(\xi) - \xi| \leq |x_{j+1} - x_{i+1}| \leq (\frac{i+1}{i})^r|x_j - x_i|
\end{aligned}
\tag{C.39}$$

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Thus,

$$\begin{aligned}
&(\frac{2}{3})^r|y_j^\theta - x_i| \leq |y_{j-i}^\theta(\xi) - \xi| \leq (\frac{3}{4})^r(\theta|x_j - x_i| + (1-\theta)|x_{j-1} - x_i|) = (\frac{3}{4})^r|y_j^\theta - x_i|
\end{aligned}
\tag{C.40}$$

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$$|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}
\tag{C.41}$$

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Next,

$$\begin{aligned}
&(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' = (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi)) - 1| \\
&\leq C|y_j^\theta - x_i|^{-\alpha}\xi^{1/r-1}|\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}|
\end{aligned}
\tag{C.42}$$

810

Similar with (C.39), we have

$$\begin{aligned}
|y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| &\leq C|x_j^{1-1/r} - x_i^{1-1/r}| \\
&\leq C|x_j - x_i|x_i^{-1/r}
\end{aligned}
\tag{C.43}$$

811

So we can get

$$\begin{aligned}
&|\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| \\
&\leq Cx_i^{-1/r}(\theta|x_{j-1} - x_i| + (1-\theta)|x_j - x_i|) \\
&= Cx_i^{-1/r}|y_j^\theta - x_i|
\end{aligned}
\tag{C.44}$$

812

Combine them, we get

$$\begin{aligned}
(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' &\leq C|y_j^\theta - x_i|^{-\alpha}x_i^{1/r-1}x_i^{-1/r}|y_j^\theta - x_i| \\
&= C|y_j^\theta - x_i|^{1-\alpha}x_i^{-1}
\end{aligned}
\tag{C.45}$$

813

Finally, we have

$$\begin{aligned}
&(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' = \alpha(\alpha-1)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi)) - 1)^2 \\
&\quad + (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}\frac{1-r}{r}\xi^{1/r-2}|\theta y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1} + (1-\theta)y_{j-i}^{1-2/r}(\xi)Z_{j-i}|
\end{aligned}
\tag{C.46}$$

814

822 Using the inequalities above ,we have

$$\begin{aligned}
 & |y_{j-i}^\theta(\xi) - \xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi)) - 1)^2 \\
 823 \quad (C.47) \quad & \leq C|y_j^\theta - x_i|^{-\alpha-1}(x_i^{-1}|y_j^\theta - x_i|)^2 \\
 & = C|y_j^\theta - x_i|^{1-\alpha}x_i^{-2}
 \end{aligned}$$

824 And by

$$825 \quad (C.48) \quad |Z_{j-i}| = |x_j^{1/r} - x_i^{1/r}| \leq |x_j - x_i|x_i^{1/r-1}$$

826 we have

$$\begin{aligned}
 & |y_{j-i}^\theta(\xi) - \xi|^{-\alpha}\xi^{1/r-2}|\theta y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1} + (1-\theta)y_{j-i}^{1-2/r}(\xi)Z_{j-i}| \\
 827 \quad (C.49) \quad & \leq C|y_j^\theta - x_i|^{-\alpha}x_i^{1/r-2}x_i^{1-2/r}|\theta Z_{j-i-1} + (1-\theta)Z_{j-i}| \\
 & \leq C|y_j^\theta - x_i|^{-\alpha}x_i^{-2}|y_j^\theta - x_i| \\
 & = C|y_j^\theta - x_i|^{1-\alpha}x_i^{-2}
 \end{aligned}$$

□

828 *proof of Lemma 5.19.* For  $k \leq j < \min\{2i-1, N-1\}$

$$\begin{aligned}
 & \frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \\
 829 \quad (C.50) \quad & \frac{Q_{j-i}^\theta(x_{i+1}) - Q_{j-i}^\theta(x_i)}{h_{i+1}}u'''(\eta_{j+1}^\theta) + Q_{j-i}^\theta(x_i)\frac{u'''(\eta_{j+1}^\theta) - u'''(\eta_j^\theta)}{h_{i+1}} \\
 & \leq Q_{j-i}^{\theta'}(\xi)Cx_j^{\alpha/2-3} + Q_{j-i}^\theta(x_i)Cu''''(\eta)\frac{h_i + h_{i+1}}{h_{i+1}}
 \end{aligned}$$

830 where  $\xi \in [x_i, x_{i+1}]$ ,  $\eta \in [x_{j-1}, x_{j+1}]$ .

831 From (5.36), by Lemma C.6 and Lemma 5.17, we have

$$\begin{aligned}
 & Q_{j-i}^{\theta'}(\xi) \leq Ch^2 \frac{|y_{j+1}^\theta - x_{i+1}|^{1-\alpha}}{\Gamma(2-\alpha)} x_{i+1}^{1-2/r} h_{j+1}^2 \\
 832 \quad (C.51) \quad & \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2
 \end{aligned}$$

833 And by defination

$$834 \quad (C.52) \quad Q_{j-i}^\theta(x_i) = h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \leq Ch^2 x_i^{2-2/r} \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2$$

835 With , we have

$$836 \quad (C.53) \quad 4^{-r}x_i \leq x_{k-1} \leq x_{j-1} < x_j \leq x_{2i-1} \leq 2^r x_i$$

837 So we have

$$\begin{aligned}
 & \frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \\
 838 \quad (C.54) \quad & \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 x_i^{\alpha/2-3} + Ch^2 x_i^{2-2/r} \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 x_{j-1}^{\alpha/2-4} \\
 & = Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j^2
 \end{aligned}$$

839 while

$$840 \quad h_j \leq h_{2i-1} \leq 2^r h_i$$

841 Subsitute into the inequality above, we get the goal

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ 842 \quad (C.55) \quad & \leq \frac{1}{h_i} Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j 2^r h_i \\ & = Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned}$$

843 While, the later is similar.  $\square$

844

845 **LEMMA C.8.** *There exists a constant  $C = C(T, r)$  such that For  $N/2 \leq i < N$ ,*  
 846  *$N + 2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$ ,  $l = 3, 4$ ,  $\xi \in [x_{i-1}, x_{i+1}]$ , we have*

$$847 \quad (C.56) \quad h_{j-i}^l(\xi) \leq Ch_j^l \leq Ch^2 h_j^{l-2}$$

$$848 \quad (C.57) \quad (h_{j-i-1}^l(\xi))' \leq C(r-1)h^2 h_j^{l-2}$$

$$849 \quad (C.58) \quad (h_{j-i}^3(\xi))'' \leq C(r-1)h^2 h_j$$

*Proof.*

$$\begin{aligned} 850 \quad (C.59) \quad & (h_{j-i}(\xi))' = y_{j-i}'(\xi) - y_{j-i-1}'(\xi) \\ & = \xi^{1/r-1}((2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}) \leq 0 \end{aligned}$$

851 Thus,

$$852 \quad (C.60) \quad Ch_j \leq h_{j+1} \leq h_{j-i}(\xi) \leq h_{j-i}(x_{i-1}) = h_{j-1} \leq Ch_j$$

853 So as  $4^{-r}T \leq 2T - x_j \leq T$ ,  $2^{-r}T \leq x_i \leq T$ , we have

$$854 \quad (C.61) \quad h_{j-i}^l(\xi) \leq Ch_j^l \leq Ch^2(2T - x_j)^{2-2/r} h_j^{l-2} \leq Ch^2 h_j^{l-2}$$

855 Since

$$\begin{aligned} & |(2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}| \\ 856 \quad (C.62) \quad & = |(Z_{2N-(j-i)} - \xi^{1/r})^{r-1} - (Z_{2N-(j-i-1)} - \xi^{1/r})^{r-1}| \\ & = (r-1)Z_1(Z_{2N-(j-i-\gamma)} - \xi^{1/r})^{r-2} \quad \gamma \in [0, 1] \\ & \leq C(r-1)h(2T - x_j)^{1-2/r} \end{aligned}$$

857 we have

$$858 \quad (C.63) \quad |(h_{j-i}(\xi))'| \leq C(r-1)h(2T - x_j)^{1-2/r} x_i^{1/r-1}$$

859 And

$$\begin{aligned} & (h_{j-i}^l(\xi))' = lh_{j-i}^{l-1}(\xi)h_{j-i}'(\xi) \\ 860 \quad (C.64) \quad & \leq C(r-1)h_j^{l-1} h(2T - x_j)^{1-2/r} x_i^{1/r-1} \\ & \leq C(r-1)h^2 h_j^{l-2} (2T - x_j)^{2-3/r} x_i^{1-1/r} \\ & \leq C(r-1)h^2 h_j^{l-2} \end{aligned}$$

(C.65) □

$$\begin{aligned}
 (h_{j-i}^3(\xi))'' &= 6h_{j-i}(\xi)(y_{j-i}'(\xi) - y_{j-i-1}'(\xi))^2 + 3h_{j-i}^2(\xi)(y_{j-i}''(\xi) - y_{j-i-1}''(\xi)) \\
 &\leq C(r-1)h_j h^2 + Ch_j^2 \frac{1-r}{r} \xi^{1/r-2} ((2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-1-i)}) \\
 &\leq C(r-1)h_j h^2 + C(r-1)h_j^2 (C(r-2)h(2T - x_j)^{1-3/r} Z_{2N-(j-i)} + Z_1(2T - x_{j-1})^{1-2/r}) \\
 &\leq C(r-1)h_j h^2 + C(r-1)h_j^2 h = Ch^2 h_j
 \end{aligned}$$

LEMMA C.9. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i < N$ ,  $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$ ,  $\xi \in [x_{i-1}, x_{i+1}]$ , we have*

$$(C.66) \quad u''(y_{j-i}^\theta(\xi)) \leq C$$

$$(C.67) \quad (u''(y_{j-i}^\theta(\xi)))' \leq C$$

$$(C.68) \quad (u''(y_{j-i}^\theta(\xi)))'' \leq C$$

*Proof.*

$$(C.69) \quad x_{j-2} \leq y_{j-i}^\theta(\xi) \leq x_{j+1} \Rightarrow 4^{-r}T \leq 2T - y_{j-i}^\theta(\xi) \leq T$$

Thus, for  $l = 2, 3, 4$ ,

$$(C.70) \quad u^{(l)}(y_{j-i}^\theta(\xi)) \leq C(2T - y_{j-i}^\theta(\xi))^{\alpha/2-l} \leq C$$

and

$$\begin{aligned}
 (y_{j-i}^\theta(\xi))' &= \theta y_{j-1-i}'(\xi) + (1-\theta)y_{j-i-1}'(\xi) \\
 &= \xi^{1/r-1}(\theta(2T - y_{j-1-i}(\xi))^{1-1/r} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-1/r}) \\
 &\leq C(2T - x_{j-2})^{1-1/r} \leq C
 \end{aligned}$$

With

$$(C.72) \quad Z_{2N-j-i} \leq 2T^{1/r}$$

$$\begin{aligned}
 (C.73) \quad (y_{j-i}^\theta(\xi))'' &= \theta y_{j-1-i}''(\xi) + (1-\theta)y_{j-i-1}''(\xi) \\
 &= \frac{1-r}{r} \xi^{1/r-2} (\theta(2T - y_{j-1-i}(\xi))^{1-2/r} Z_{2N-(j-i-1)} + (1-\theta)(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)}) \\
 &\leq C(r-1)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (C.74) \quad (u''(y_{j-i}^\theta(\xi)))' &= u'''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta(\xi))' \\
 &\leq C
 \end{aligned}$$

$$\begin{aligned}
 (C.75) \quad (u''(y_{j-i}^\theta(\xi)))'' &= u'''(y_{j-i}^\theta(\xi))(y_{j-i}^{\theta'}(\xi))^2 + u''''(y_{j-i}^\theta(\xi))y_{j-i}^{\theta''}(\xi) \\
 &\leq C + C(r-1) = C
 \end{aligned}$$

LEMMA C.10. *There exists a constant  $C = C(T, \alpha, r)$  such that*

$$(C.76) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}$$

$$(C.77) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C|y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N)$$

$$(C.78) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C(r-1)|y_j^\theta - x_i|^{-\alpha} + C|y_j^\theta - x_i|^{-1-\alpha}(|2T - x_i - y_j^\theta| + h_N)^2$$

*Proof.*

$$(C.79) \quad (y_{j-i}^\theta(\xi) - \xi)' = \theta y_{j-1-i}'(\xi) + (1-\theta)y_{j-i}'(\xi) - 1$$

$$(C.80) \quad |y_{j-i}'(\xi) - 1| = \xi^{1/r-1} |(2T - y_{j-i}(\xi))^{1-1/r} - \xi^{1-1/r}| \\ \leq \xi^{1/r-1} |2T - \xi - y_{j-i}(\xi)| \xi^{-1/r}$$

$$(C.81) \quad |2T - \xi - y_{j-i}(\xi)| \leq \max \begin{cases} |2T - x_{i-1} - x_{j-1}| \\ |2T - x_{i+1} - x_{j+1}| \end{cases} \\ \leq |2T - x_i - x_j| + h_{i+1} + h_j$$

$$(C.82) \quad (y_{j-i}^\theta(\xi) - \xi)'' = \theta y_{j-1-i}''(\xi) + (1-\theta)y_{j-i}''(\xi) \\ = \frac{1-r}{r} \xi^{1/r-2} (\theta(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)}) \leq 0$$

It's concave, so

$$(C.83) \quad y_{j-i}(\xi) - \xi \geq \min\{x_{j+1} - x_{i+1}, x_{j-1} - x_{i-1}\} \geq C(x_j - x_i)$$

We have

$$(C.84) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}$$

$$(C.85) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' = (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha} (y_{j-i}^\theta(\xi) - \xi)' \\ \leq C|y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_{i+1} + h_{j-1})$$

$$(C.86) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' = (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha} (y_{j-i}^\theta(\xi) - \xi)'' + \alpha(\alpha-1)|y_{j-i}^\theta(\xi) - \xi|^{-1-\alpha} (y_{j-i}^\theta(\xi) - \xi)' (y_{j-i}^\theta(\xi) - \xi)' \\ \leq C(r-1)|y_j^\theta - x_i|^{-\alpha} + C|y_j^\theta - x_i|^{-1-\alpha} (|2T - x_i - y_j^\theta| + h_{i+1} + h_{j-1})^2$$

*Proof.* From (5.24), by Lemma C.8 and Lemma C.10, we have  $\xi \in [x_i, x_{i+1}]$

$$(C.87) \quad Q_{j-i}^\theta(\xi) \leq Ch^2 h_j^2 ((r-1)|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N))$$

$$(C.88) \quad Q_{j-i}^\theta(\xi) \leq Ch^2 h_j^2 |y_j^\theta - x_i|^{1-\alpha}$$

So use the skill in Proof 33 with Lemma C.9

$$(C.89) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ \leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N))$$

$$a^{1-\theta}|a^\theta - b^\theta| \leq |a - b|, \theta \in [0, 1]$$

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