AN EXAMPLE ARTICLE*

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Key words. example, LaTeX

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1. Introduction.

The introduction introduces the context and summarizes the manuscript. It is importantly to clearly state the contributions of this piece of work.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$, suppose $f \in C^{\beta}(\Omega), \beta > 4 - \alpha, \|f\|_{\beta}^{(\alpha/2)} < \infty$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^{\alpha} u}{\partial |x|^{\alpha}} = -\kappa_{\alpha} \frac{d^2}{dx^2} \int_{\Omega} \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_{\alpha} = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

and the solution $u \in C^{\alpha/2}(\Omega)$.

2. Regularity.

Remark 2.1. 1. $C^k(U)$ is the set of all k-times continuously differentiable functions on open set U.
2. $C^{\beta}(U)$ is the collection of function f which for any $V \subset\subset U$ $f|_V \in C^{\beta}(\bar{V})$.

THEOREM 2.2. *If $f \in C^{\beta}(\Omega), \beta > 2$ and $\|f\|_{\beta}^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

$$|f^{(l)}(x)| \le \|f\|_{\beta}^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \le T \\ (2T - x)^{-l-\alpha/2}, & \text{if } T \le x < 2T \end{cases} \tag{2.1}$$

THEOREM 2.3 (Regularity up to the boundary [1]).

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_{\beta}^{(\alpha/2)} \right) \tag{2.2}$$

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Corollary 2.4. *Let u be a solution of (1.1) on Ω. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \le C \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \le T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \le x < 2T \end{cases} \tag{2.3}$$

The paper is organized as follows. Our main results are in section 4, experimental results are in section 7, and the conclusions follow in section 8.

3. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases} \tag{3.1}$$

where $r \ge 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N \tag{3.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

And then, we can approximate $u(x)$ with

$$u_h(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{3.4}$$

For convience, we denote

$$I_h^{2-\alpha}(x_i) := \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y)dy \tag{3.5}$$

And now, we can approximate the operator (1.2) at x_i with

$$\begin{aligned} D_h^\alpha u_h(x_i) &:= D_h^2 I_h^{2-\alpha}(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} I_h^{2-\alpha}(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) I_h^{2-\alpha}(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha}(x_{i+1}) \right) \end{aligned} \tag{3.6}$$

Finally, we approximate the equation (1.1) with

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i), \quad 1 \le i \le 2N-1 \tag{3.7}$$

The discrete equation (3.7) can be written in matrix form

$$AU = F \tag{3.8}$$

where U is unknown, $F = (f(x_1), \cdots, f(x_{2N-1}))$. The matrix A is constructed as follows: Since

$$
\begin{aligned}
I_h^{2-\alpha}(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u(x_j) \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\
&=: \sum_{j=1}^{2N-1} \tilde{a}_{ij}\, u(x_j), \quad 0 \le i \le 2N
\end{aligned}
\tag{3.9}
$$

Then, substitute in (3.6), we have

$$
-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} a_{ij}\, u(x_j) \tag{3.10}
$$

where

$$
a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \tag{3.11}
$$

4. Main results. Here we state our main results; the proof is deferred to section 5 and section 6.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 4.1 (Truncation Error). *If $f \in C^2(\Omega)$ and $\alpha \in (1,2)$, and $u(x)$ is a solution of the equation* (1.1)*, then there exists a constant $C_1, C_2 = C_1(T, \alpha, r, \|f\|_{C^2(\Omega)}), C_2(T, \alpha, r, \|f\|_{C^2(\Omega)})$, such that the truncation error of the discrete format satisfies*

$$
\begin{aligned}
|-\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i)| \le & C_1 (h^{r\alpha/2 + r} (x_i^{-1-\alpha} + (2T - x_i)^{-1-\alpha}) \\
& + h^2 (x_i^{-\alpha/2 - 2/r} + (2T - x_i)^{-\alpha/2 - 2/r})) \\
& + C_2 h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \le i \le N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \le 2N-1 \end{cases}
\end{aligned}
\tag{4.1}
$$

where $C_2 = 0$ if $r = 1$.

THEOREM 4.2 (Convergence). *The discrete equation* (3.7) *has sulotion U, and there exists a positive constant $C = C(T, \alpha, r, \|f\|_{C^2(\Omega)})$ such that the error between the numerial solution U with the exact solution $u(x_i)$ satisfies*

$$
\max_{1 \le i \le 2N-1} |U_i - u(x_i)| \le C h^{\min\{\frac{r\alpha}{2}, 2\}} \tag{4.2}
$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

5. Proof of Theorem 4.1.

For convience, let's denote

$$I^{2-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x-y|^{1-\alpha} u(y) dy \tag{5.1}$$

Then, the truncation error of the discrete format can be written as

$$\begin{aligned} -\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I_h^{2-\alpha}(x_i) - \frac{d^2}{dx^2} I^{2-\alpha}(x_i)) \\ &= -\kappa_\alpha D_h^2 (I_h^{2-\alpha} - I^{2-\alpha})(x_i) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \end{aligned} \tag{5.2}$$

5.1. Estimate of $-\kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}(x_i)$.

Theorem 5.1. *There exits a constant $C = C(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$ such that*

$$\left| -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \right| \le Ch^2 (x_i^{-\alpha/2-2/r} + (2T - x_i)^{-\alpha/2-2/r}) \tag{5.3}$$

Proof. Since $f \in C^2(\Omega)$ and

$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha}(x)) = f(x), \quad x \in \Omega, \tag{5.4}$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.3) of Lemma A.1, for $1 \le i \le 2N-1$, we have

$$-\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \tag{5.5}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

$$\left| \frac{h_{i+1} - h_i}{3} f'(x_i) \right| \le \frac{\|f\|_\beta^{(\alpha/2)}}{3} Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{-\alpha/2-2/r}, & N < i \le 2N-1 \end{cases} \tag{5.6}$$

2. See Proof 9, there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

$$\begin{aligned} &\left| \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \right| \\ &\le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases} \end{aligned} \tag{5.7}$$

Summarizes, we get the result. ∎

5.2. Estimate of R_i.

Now, we study the first part of (5.2)

$$D_h^2 (I^{2-\alpha} - I_h^{2-\alpha})(x_i) = D_h^2 (\int_0^{2T} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy) \tag{5.8}$$

For convience, let's denote

$$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \tag{5.9}$$

And define

$$R_i := D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i)$$
$$= \frac{2}{h_i + h_{i+1}} \sum_{j=1}^{2N} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \tag{5.10}$$

We have some results about the estimate of R_i

THEOREM 5.2. *For $1 \le i < N/2$, there exists a constant $C = C(T, \alpha, r, f)$ such that*

$$R_i \le C(h^{r\alpha/2+r} x_i^{-1-\alpha} + h^2 x_i^{-\alpha/2-2/r}) \tag{5.11}$$

THEOREM 5.3. *For $N/2 \le i \le N$, there exists constant C, C_2 such that*

$$R_i \le Ch^2 x_i^{-\alpha/2-2/r} + C_2 h^2 |T - x_{i-1}|^{1-\alpha} \tag{5.12}$$

where $C_2 = 0$ if $r = 1$.

And for $N < i \le 2N-1$, it is symmetric to the previous case.

To prove these results, we need some utils. Also for simplicity, we denote

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \tag{5.13}$$

then

$$R_i = \sum_{j=1}^{2N} S_{ij} \tag{5.14}$$

5.3. Proof of Theorem 5.2.

LEMMA 5.4. *There exists a constant $C = C(T, \alpha, r, f)$ such that for $1 \le i < N/2$,*

$$\sum_{j=\max\{2i+1, i+3\}}^{2N} S_{ij} \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.15}$$

Proof. For $\max\{2i+1, i+3\} \le j \le N$, by Lemma C.1 and Lemma C.2

$$S_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) dy$$
$$\le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy \tag{5.16}$$
$$= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy$$

Therefore,

$$\sum_{j=\max\{2i+1, i+3\}}^{N} S_{ij} \le Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy$$
$$= \frac{C}{\alpha/2 + 2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \tag{5.17}$$
$$\le \frac{C}{\alpha/2 + 2/r} 2^{r(-\alpha/2-2/r)} h^2 x_i^{-\alpha/2-2/r}$$

Otherwise, for $N+1 \le j \le 2N-1$, by equation (C.2) and Lemma C.2

$$\begin{aligned}\sum_{j=N+1}^{2N-1} S_{ij} &\le Ch^2 \int_{x_N}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r} y^{-1-\alpha} dy \\ &\le CT^{-1-\alpha} \frac{1}{|\alpha/2-2/r+1|} h^2\end{aligned} \tag{5.18}$$

$\square$

For $i = 1, 2$.

Lemma 5.5. *By Lemma* C.5 *and Lemma* 5.4 *we get*

$$\begin{aligned}R_1 &= \sum_{j=1}^{3} S_{1j} + \sum_{j=4}^{2N} S_{1j} \\ &\le Ch^2 x_1^{-\alpha/2-2/r}\end{aligned} \tag{5.19}$$

$$\begin{aligned}R_2 &= \sum_{j=1}^{4} S_{2j} + \sum_{j=5}^{2N} S_{2j} \\ &\le Ch^2 x_2^{-\alpha/2-2/r}\end{aligned} \tag{5.20}$$

For $3 \le i < N/2$, we have a new separation of R_i, Let's denote $k = \lceil \frac{i}{2} \rceil$.

$$\begin{aligned}R_i =& \sum_{j=1}^{2N} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& \sum_{j=1}^{k-1} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &+ \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\ &+ \sum_{j=k+1}^{2i-1} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &+ \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i-1}} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \\ &+ \sum_{j=2i+1}^{2N} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& I_1 + I_2 + I_3 + I_4 + I_5\end{aligned} \tag{5.21}$$

where I_1 makes sence only if $i \ge 3$.

For convience, let's denote

$$V_{ij} = \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \tag{5.22}$$

Lemma 5.6. *There exists a constant* $C = C(T, \alpha, r, f)$ *such that for* $3 \le i < N/2, k = \lceil \frac{i}{2} \rceil$

$$I_1 \le C(h^{r\alpha/2+r} x_i^{-1-\alpha} + h^2 x_i^{-\alpha/2-2/r}) \tag{5.23}$$

Proof. For $2 \le j \le k-1$, by Lemma C.1 and Lemma C.3

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\ &\le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2 - 2/r} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 x_i^{-1-\alpha} \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} dy \end{aligned} \tag{5.24}$$

And by Lemma A.3 , Lemma C.3

$$S_{i1} \le C x_1^{\alpha/2} x_1 x_i^{-1-\alpha} = C x_1^{\alpha/2+1} x_i^{-1-\alpha} = C T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha} \tag{5.25}$$

Therefore,

$$\begin{aligned} I_1 &= \sum_{j=1}^{k-1} S_{ij} = S_{i1} + \sum_{j=2}^{k-1} S_{ij} \\ &\le Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \\ &\le Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \end{aligned} \tag{5.26}$$

But

$$\int_{x_1}^{2^{-r}x_i} y^{\alpha/2-2/r} dy \le \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.27}$$

So we have

$$I_1 \le \begin{cases} Ch^2 x_i^{-1-\alpha} + \frac{C}{\alpha/2-2/r+1} h^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} (1 + r \ln(\frac{i}{2})), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.28}$$ ∎

THEOREM 5.7. *There exists a constant $C = C(T, \alpha, r, f)$ such that for $2 \le i < N/2, k = \lceil \frac{i}{2} \rceil$,*

$$I_3 = \sum_{j=k+1}^{2i-1} V_{ij} \le Ch^2 x_i^{-\alpha/2-2/r} \tag{5.29}$$

To estimete V_{ij}, we need some preparations.

LEMMA 5.8. *Denote* $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0,1]$, *by Lemma* A.2

$$\begin{aligned} T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= \int_{x_{j-1}}^{x_j} -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!} h_j^3 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) dy_j^\theta \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta \end{aligned} \tag{5.30}$$

where $\eta_{j1}^\theta \in [x_{j-1}, y_j^\theta], \eta_{j2}^\theta \in [y_j^\theta, x_j]$.

Now Let's construct a series of functions to represent T_{ij}.

$$y_{j-i}(x) = (x^{1/r} + z_{j-i})^r, \quad z_{j-i} = T^{1/r} \frac{j-i}{N} \tag{5.31}$$

$$y_{j-i}^\theta(x) = \theta y_{j-1-i}(x) + (1-\theta) y_{j-i}(x) \tag{5.32}$$

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{5.33}$$

Now, we define

$$P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.34}$$

$$Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.35}$$

And now we can rewrite T_{ij}

LEMMA 5.9. *For* $2 \le i \le N, 2 \le j \le \min\{2i-1, N\}$,

$$\begin{aligned} T_{ij} &= \int_0^1 -\frac{\theta(1-\theta)}{2} P_{j-i}^\theta(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)}{3!} (\theta^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j1}^\theta) - (1-\theta)^2 Q_{j-i}^\theta(x_i) u'''(\eta_{j2}^\theta)) d\theta \end{aligned} \tag{5.36}$$

Immediately, we can see that

LEMMA 5.10. *For* $2 \le i < N/2, k = \lceil \frac{i}{2} \rceil, k+1 \le j \le \min\{2i-1, N\}$,

LEMMA 5.11. *There exists a constant* $C = C(T, \alpha, r, f)$ *such that for* $2 \le i < N, k = \lceil \frac{i}{2} \rceil, k+1 \le j \le \min\{2i-1, N\}$,

$$V_{ij} \le \mathit{Sorry} \tag{5.37}$$

Proof.

$$P_{ij}^{\theta} = h_j^3 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y_j^{\theta}) \tag{5.38}$$

$$Q_{ij}^{\theta} = h_j^4 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.39}$$

$$\begin{aligned} V_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} \left(\frac{1}{h_{i+1}} P_{i+1,j+1}^{\theta} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) P_{i,j}^{\theta} + \frac{1}{h_i} P_{i-1,j-1}^{\theta} \right) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)}{3!} \left(\frac{1}{h_{i+1}} Q_{i+1,j+1}^{\theta} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) Q_{i,j}^{\theta} + \frac{1}{h_i} Q_{i-1,j-1}^{\theta} \right) d\theta \end{aligned} \tag{5.40}$$ ∎

6. Proof of Theorem 4.2.

7. Experimental results.

8. Conclusions.

Some conclusions here.

Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x)$ is twice differentable continous function on open set Ω, there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

$$\begin{aligned} D_h^2 g(x_i) &:= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &= g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \end{aligned} \tag{A.1}$$

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\quad = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right) \end{aligned} \tag{A.2}$$

And if $g(x) \in C^4(\Omega)$, then

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\quad = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2)) \end{aligned} \tag{A.3}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$.

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in [x_{i-1}, x_i]$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in [x_i, x_{i+1}]$$

Subsitute them in the left side of (A.1), we have

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\quad = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) \end{aligned}$$

Now, using intermediate value theorem , there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y)dy$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \frac{h_i^4}{4!} g''''(\eta_1)$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \frac{h_{i+1}^4}{4!} g''''(\eta_2)$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. Expecially,

$$\begin{aligned} \frac{h_i^4}{4!} g''''(\eta_1) &= \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy \\ \frac{h_{i+1}^4}{4!} g''''(\eta_2) &= \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \end{aligned} \tag{A.4}$$

Subsitute them to the left side of (A.3), we can get the result. ∎

LEMMA A.2. *If $y \in [x_{j-1}, x_j]$, denote $y = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0,1]$,*

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \tag{A.5}$$

(A.6)

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta) u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$. Thus ∎

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

LEMMA A.3. *For $x \in [x_{j-1}, x_j]$*

$$\begin{aligned} |u(x) - u_h(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_{x}^{x_j} u'(y)dy \right| \\ &\le \int_{x_{j-1}}^{x_j} |u'(y)|dy \end{aligned} \tag{A.7}$$

If $x \in [0, x_1]$, with Corollary 2.4, we have

$$|u(x) - u_h(x)| \le \int_0^{x_1} |u'(y)|dy \le \int_0^{x_1} Cy^{\alpha/2-1}dy \le C\frac{2}{\alpha}x_1^{\alpha/2} \tag{A.8}$$

Similarly, if $x \in [x_{2N-1}, 1]$, we have

$$|u(x) - u_h(x)| \le C\frac{2}{\alpha}(2T - x_{2N-1})^{\alpha/2} = C\frac{2}{\alpha}x_1^{\alpha/2} \tag{A.9}$$

Appendix B. Inequality.

LEMMA B.1.

$$h_i \le rT^{1/r}h \begin{cases} x_i^{1-1/r}, & 1 \le i \le N \\ (2T - x_{i-1})^{1-1/r}, & N < i \le 2N - 1 \end{cases} \tag{B.1}$$

Proof. For $1 \le i \le N$,

$$\begin{aligned} h_i &= T\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right) \\ &\le rT\frac{1}{N}\left(\frac{i}{N}\right)^{r-1} = rT^{1/r}hx_i^{1-1/r} \end{aligned}$$

For $N < i \le 2N - 1$,

$$\begin{aligned} h_i &= T\left(\left(\frac{2N-i+1}{N}\right)^r - \left(\frac{2N-i}{N}\right)^r\right) \\ &\le rT\frac{1}{N}\left(\frac{2N-i+1}{N}\right)^{r-1} = rT^{1/r}h(2T - x_{i-1})^{1-1/r} \end{aligned}$$

□

LEMMA B.2. *There is a constant $C = 2^{|r-2|}r(r-1)T^{2/r}$ such that for all $i \in \{1, 2, \cdots, 2N-1\}$*

$$|h_{i+1} - h_i| \le Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N-1 \end{cases} \tag{B.2}$$

Proof.

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \le i \le N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \le i \le 2N-1 \end{cases}$$

For $i = 1$,

$$h_2 - h_1 = T(2^r - 2)\left(\frac{1}{N}\right)^r = (2^r - 2)T^{2/r}h^2 x_1^{1-2/r}$$

For $2 \le i \le N - 1$,

$$h_{i+1} - h_i = r(r-1)T\ N^{-2}\eta^{r-2}, \quad \eta \in [\frac{i-1}{N}, \frac{i+1}{N}]$$

If $r \in [1, 2]$,

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T\ N^{-2}\eta^{r-2} \le r(r-1)T\ h^2\left(\frac{i-1}{N}\right)^{r-2} \\ &\le r(r-1)T\ h^2 2^{2-r}\left(\frac{i}{N}\right)^{r-2} \\ &= 2^{2-r}r(r-1)T^{2/r}h^2 x_i^{1-2/r} \end{aligned}$$

else if $r > 2$,

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T\ N^{-2}\eta^{r-2} \le r(r-1)T\ h^2\left(\frac{i+1}{N}\right)^{r-2} \\ &\le r(r-1)T\ h^2 2^{r-2}\left(\frac{i}{N}\right)^{r-2} \\ &= 2^{r-2}r(r-1)T^{2/r}h^2 x_i^{1-2/r} \end{aligned}$$

Since

$$2^r - 2 \le 2^{|r-2|}r(r-1), \quad r \ge 1$$

we have

$$h_{i+1} - h_i \le 2^{|r-2|}r(r-1)T^{2/r}h^2 x_i^{1-2/r}, \quad 1 \le i \le N-1$$

For $i = N$, $h_{N+1} - h_N = 0$. For $N < i \le 2N - 1$, it's central symmetric to the first half of the proof, which is

$$h_i - h_{i+1} \le 2^{|r-2|}r(r-1)T^{2/r}h^2(2T - x_i)^{1-2/r}$$

Summarizes the inequalities, we can get

$$|h_{i+1} - h_i| \le 2^{|r-2|}r(r-1)T^{2/r}h^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N-1 \end{cases} \tag{B.3}$$ □

Appendix C. Proofs of some technical details.

Additional proof of Theorem 5.1. For $2 \le i \le N-1$,

$$\begin{aligned}
&\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\
&\qquad \le C\frac{2}{h_i + h_{i+1}}(h_i^3 x_{i-1}^{-2-\alpha/2} + h_{i+1}^3 x_i^{-2-\alpha/2}) \\
&\qquad \le 2C(h_i^2 x_{i-1}^{-2-\alpha/2} + h_{i+1}^2 x_i^{-2-\alpha/2})
\end{aligned}$$

Since Lemma B.1, we have

$$h_i \le rT^{1/r} h x_i^{1-1/r}, \quad 1 \le i \le N$$

$$h_{i+1} \le rT^{1/r} h x_{i+1}^{1-1/r}, \quad 1 \le i \le N-1$$

and

$$x_{i-1}^{-2-\alpha/2} \le 2^{-r(-2-\alpha/2)} x_i^{-2-\alpha/2} \quad 2 \le i \le N-1$$

$$x_{i+1}^{1-1/r} \le 2^{r-1} x_i^{1-1/r} \quad 1 \le i \le N-1$$

So there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

$$\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \le Ch^2 x_i^{-\alpha/2-2/r}, \quad 2 \le i \le N-1$$

For $i = 1$, by (A.4)

$$\begin{aligned}
&\frac{1}{4!}\frac{2}{h_1 + h_2}(h_1^3 f''(\eta_1) + h_2^3 f''(\eta_2)) \\
&\quad = \frac{2}{h_1 + h_2}\left(\frac{1}{h_1}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy + \frac{1}{4!}h_2^3 f''(\eta_2)\right)
\end{aligned}$$

We have proved above that

$$\frac{2}{h_1 + h_2} h_2^3 f''(\eta_2) \le Ch^2 x_1^{-\alpha/2-2/r}$$

and we can get

$$\begin{aligned}
\int_0^{x_1} f''(y)\frac{y^3}{3!}dy &\le C\frac{1}{3!}\int_0^{x_1} y^{1-\alpha/2} dy \\
&= C\frac{1}{3!(2-\alpha/2)} x_1^{2-\alpha/2}
\end{aligned}$$

so

$$\frac{2}{h_1 + h_2}\frac{1}{h_1}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy = \frac{C2^{1-r}}{3!(2-\alpha/2)} x_1^{-\alpha/2} = \frac{C2^{1-r}}{3!(2-\alpha/2)} T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

And for $i = N$, we have

$$\begin{aligned}
&\frac{2}{h_N + h_{N+1}}(h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2)) \\
&\qquad = h_N^2 (f''(\eta_1) + f''(\eta_2)) \\
&\qquad \le r^2 T^{2/r} h^2 x_N^{2-2/r}\ 2C x_{N-1}^{-2-\alpha/2} \\
&\qquad \le 2r^2 T^{2/r} C 2^{-r(-2-\alpha/2)}\ h^2 x_N^{-\alpha/2-2/r}
\end{aligned}$$

Finally, $N+1 \le i \le 2N-1$ is symmetric to the first half of the proof, so we can conclude that

□

$$\frac{2}{h_i+h_{i+1}}(h_i^3 f''(\eta_1)+h_{i+1}^3 f''(\eta_2)) \le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1\le i\le N \\ (2T-x_i)^{-\alpha/2-2/r}, & N\le i\le 2N-1 \end{cases}$$

LEMMA C.1. *There is a constant $C = C(T,\alpha,r,f)$ for $2\le j\le N$, if $y\in [x_{j-1},x_j]$,*

$$|u(y)-u_h(y)| \le Ch^2 y^{\alpha/2-2/r} \tag{C.1}$$

Proof. For $2 \le j \le N$, we have

$$x_j \le 2^r y, \quad x_{j-1} \ge 2^{-r} y$$

And by Lemma A.2, Lemma B.1 and Corollary 2.4, we have

$$\begin{aligned} u(y)-u_h(y) &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi\in[x_{j-1},x_j] \\ &\le \frac{C}{4} h^2 x_j^{2-2/r} x_{j-1}^{\alpha/2-2} \\ &\le \frac{C}{4} h^2\ 2^{2r-2} y^{2-2/r}\ 2^{-r(\alpha/2-2)} y^{\alpha/2-2} \\ &= C 2^{-r\alpha/2+4r-2}\ h^2 y^{\alpha/2-2/r} \end{aligned}$$

symmetricly, for $N < j \le 2N-1$, we have

$$|u(y)-u_h(y)| \le Ch^2(2T-y)^{\alpha/2-2/r} \tag{C.2}$$

□

LEMMA C.2. *There is a constant $C = C(\alpha, r)$ such that for all $1 \le i < N/2$, $\max\{2i+1, i+3\} \le j \le 2N$ and $y\in[x_{j-1},x_j]$, we have*

$$D_h^2\left(\frac{|y-\cdot|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) \le C\frac{y^{-1-\alpha}}{\Gamma(-\alpha)} \tag{C.3}$$

Proof. Sinec $y \ge x_{j-1} > x_{i+1}$, by Lemma A.1, if $j-1 > i+1$

□

$$\begin{aligned} D_h^2\left(\frac{|y-\cdot|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) &= \frac{|y-\xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi\in[x_{i-1},x_{i+1}] \\ &\le \frac{(y-x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\le (1-(\frac{2}{3})^r)^{-1-\alpha}\frac{y^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

LEMMA C.3. *There is a constant $C = C(\alpha, r)$ such that for all $3 \le i < N/2, k = \lceil \frac{i}{2} \rceil$, $1 \le j \le k-1$ and $y\in[x_{j-1},x_j]$, we have*

$$D_h^2\left(\frac{|\cdot-y|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) \le C\frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \tag{C.4}$$

Proof. Sinec $y \le x_j < x_{i-1}$, by Lemma A.1,

$$\begin{aligned} D_h^2\left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) &= \frac{|\xi - y|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\le \frac{(x_{i-1} - x_j)^{-1-\alpha}}{\Gamma(-\alpha)} \le \frac{(x_{i-1} - x_{k-1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\le \left(\left(\frac{2}{3}\right)^r - \left(\frac{1}{2}\right)^r\right)^{-1-\alpha} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

□

LEMMA C.4. *While $0 \le i < N/2$, By Lemma* A.3

$$\begin{aligned} |T_{i1}| &\le C \int_0^{x_1} x_1^{\alpha/2} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2} \left| x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha} \right| \\ &\le C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2+2-\alpha} = C \frac{1}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1 \end{aligned} \tag{C.5}$$

For $2 \le j \le N$, by Lemma A.2 *and Corollary* 2.4

$$\begin{aligned} |T_{ij}| &\le \frac{C}{4} \int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &\le \frac{C}{4\Gamma(3-\alpha)} h_j^2 x_{j-1}^{\alpha/2-2} \left| |x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha} \right| \end{aligned} \tag{C.6}$$

LEMMA C.5. *There exists a constant $C = C(T, \alpha, r, f)$ such that*

$$\sum_{j=1}^{3} S_{1j} \le C h^2 x_1^{-\alpha/2-2/r} \tag{C.7}$$

$$\sum_{j=1}^{4} S_{2j} \le C h^2 x_2^{-\alpha/2-2/r} \tag{C.8}$$

Proof.

$$S_{1j} = \frac{2}{x_2}\left(\frac{1}{x_1} T_{0j} - \left(\frac{1}{x_1} + \frac{1}{h_2}\right) T_{1j} + \frac{1}{h_2} T_{2j}\right)$$

So, by Lemma C.4

$$S_{11} \le \frac{2}{x_2 x_1} 4 \frac{C}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \le C x_1^{-\alpha/2}$$

$$S_{12} \le \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_2^2 x_1^{\alpha/2-2} \left(x_2^{2-\alpha} + 2h_2^{2-\alpha} + h_2^{2-\alpha}\right) \le C x_1^{-\alpha/2}$$

$$S_{13} \le \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_3^2 x_2^{\alpha/2-2} \left(x_3^{2-\alpha} + 2x_3^{2-\alpha} + h_3^{2-\alpha}\right) \le C x_1^{-\alpha/2}$$

But

$$x_1^{-\alpha/2} = T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

For $i = 2$, Sorry $\square$

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