A SECOND ORDER NUMERICAL METHODS FOR REISZ-FRACTIONAL ELLIPTIC EQUATION ON GRADED MESH[*]

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Key words. example, LaTeX

MSC codes. ??????????????????

1. Introduction.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$,

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

2. Preliminaries: Numeric scheme and main results.

2.1. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases} \tag{2.1}$$

where $r \ge 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N \tag{2.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

And then, define the piecewise linear interpolant of the true solution u to be

$$\Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{2.4}$$

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For convience, we denote

$$I^{2-\alpha}u(x) := \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x-y|^{1-\alpha} u(y)dy \tag{2.5}$$

and

$$D_h^2 u(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} u(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) u(x_i) + \frac{1}{h_{i+1}} u(x_{i+1}) \right) \tag{2.6}$$

Now, we discretise (1.1) by replacing $u(x)$ by a continuous piecewise linear function

$$u_h(x) := \sum_{j=1}^{2N-1} u_j \phi_j(x) \tag{2.7}$$

whose nodal values u_j are to be determined by collocation at each mesh point x_i for $i = 1, 2, ..., 2N-1$:

$$-\kappa_\alpha D_h^\alpha u_h(x_i) := -\kappa_\alpha D_h^2 I^{2-\alpha} u_h(x_i) = f(x_i) =: f_i \tag{2.8}$$

Here,

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^2 I^{2-\alpha} \phi_j(x_i)\ u_j = \sum_{j=1}^{2N-1} a_{ij}\ u_j \tag{2.9}$$

where

$$a_{ij} = -\kappa_\alpha D_h^2 I^{2-\alpha} \phi_j(x_i) \quad \text{for} \quad i, j = 1, 2, ..., 2N-1 \tag{2.10}$$

We have replaced $(-\Delta)^{\alpha/2} u(x_i) = f(x_i)$ in (1.1) by $-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i)$ in (2.8), with truncation error

$$\tau_i := -\kappa_\alpha \left(D_h^\alpha \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i) \right) \quad \text{for} \quad i = 1, 2, ..., 2N-1 \tag{2.11}$$

where $-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^\alpha \phi_j(x_i) u(x_j) = \sum_{j=1}^{2N-1} a_{ij} u(x_j)$.

The discrete equation (2.8) can be written in matrix form

$$AU = F \tag{2.12}$$

where $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$, $U = (u_1, \cdots, u_{2N-1})^T$ is unknown and $F = (f_1, \cdots, f_{2N-1})^T$.

We can deduce a_{ij},

$$\begin{aligned} a_{ij} &= -\kappa_\alpha D_h^2 I^{2-\alpha} \phi_j(x_i) \\ &= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \tilde{a}_{ij} &= I^{2-\alpha} \phi_j(x_i) \\ &= \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \end{aligned} \tag{2.14}$$

We shall finally introduce some notations.

For convenience, we use the notation $\simeq$. That $x_1 \simeq y_1$, means that $c_1 x_1 \leq y_1 \leq C_1 x_1$ for some constants c_1 and C_1 that are independent of N.

Meanwhile, let's define kernel functions

$$K_y(x) := \frac{|y-x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{2.15}$$

We define the difference quotients

$$D_h g(x_i) := \frac{g(x_{i+1}) - g(x_i)}{h_{i+1}}, \quad D_{\bar{h}} g(x_i) := \frac{g(x_i) - g(x_{i-1})}{h_i} \tag{2.16}$$

Thus

$$D_h g(x_i) = D_{\bar{h}} g(x_{i+1})$$

$$D_h^2 g(x_i) = \frac{2}{h_i + h_{i+1}} \left(D_h g(x_i) - D_{\bar{h}} g(x_i)\right) = \frac{2}{h_i + h_{i+1}} \left(D_h g(x_i) - D_h g(x_{i-1})\right)$$

And for $j = 1, 2, ..., 2N$, we define

$$y_j^\theta = (1-\theta) x_{j-1} + \theta x_j, \quad \theta \in (0,1) \tag{2.17}$$

2.2. Regularity of the true solution. For any $\beta > 0$, we use the standard notation $C^\beta(\bar{\Omega}), C^\beta(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^\beta(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous[1] with exponent β' in Ω, where $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on open set Ω.

Definition 2.1 (delta dependent norm [2]). ...

Lemma 2.2. *Let $f \in C^\beta(\Omega), \beta > 2$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

$$|f^{(l)}(x)| \leq \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \leq T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \leq x < 2T \end{cases} \tag{2.18}$$

Theorem 2.3 (Regularity up to the boundary [2]). *Let Ω be a bounded domain, and $\beta > 0$ be such that neither β nor $\beta + \alpha$ is an integer. Let $f \in C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R}^n)$ be a solution of (1.1). Then, $u \in C^{\beta+\alpha}(\Omega)$ and*

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)}\right) \tag{2.19}$$

where C is a constant depending only on Ω, α, and β.

Corollary 2.4. *Let u be a solution of (1.1) where $f \in L^\infty(\Omega)$ and $\|f\|_\beta^{(\alpha/2)} < \infty$. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \leq \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases} \tag{2.20}$$

And in this paper bellow, without special instructions, we allways assume that

$$f \in L^\infty(\Omega) \cap C^\beta(\Omega) \quad \text{and} \quad \|f\|_\beta^{(\alpha/2)} < \infty, \text{ with } \alpha + \beta > 4 \tag{2.21}$$

2.3. Main results. Here we state our main results; the proof is deferred to section 3 and section 4.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 2.5 (Local Truncation Error). *If $u(x)$ is a solution of the equation* (1.1) *where f satisfy the regular condition* (2.21)*, then there exists $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$, such that the truncation error* (2.11) *satisfies*

$$
\begin{aligned}
|\tau_i| :=& |-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i)| \\
\le& C_1 h^{\min\{\frac{r\alpha}{2},2\}} \begin{cases} x_i^{-\alpha}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha}, & N < i \le 2N-1 \end{cases} \\
&+ C_2(r-1)h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \le i \le N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \le 2N-1 \end{cases}
\end{aligned}
\tag{2.22}
$$

THEOREM 2.6 (Global Error). *The discrete equation* (2.8) *has sulotion and there exists a positive constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ such that the error between the numerial solution U with the exact solution $u(x_i)$ satisfies*

$$
\max_{1 \le i \le 2N-1} |u_i - u(x_i)| \le C h^{\min\{\frac{r\alpha}{2},2\}}
\tag{2.23}
$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$*.*

Remark 2.7. ...

3. Local Truncation Error.

3.1. Proof of Theorem 2.5. The truncation error of the discrete format can be written as

$$
\begin{aligned}
-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I^{2-\alpha} \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i)) \\
&= -\kappa_\alpha D_h^2 I^{2-\alpha} (\Pi_h u - u)(x_i) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i)
\end{aligned}
\tag{3.1}
$$

THEOREM 3.1. *There exits a constant $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$ such that*

$$
\left| -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) \right| \le C h^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases}
\tag{3.2}
$$

Proof. Since $f \in C^2(\Omega)$ and

$$
\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha} u(x)) = f(x), \quad x \in \Omega,
\tag{3.3}
$$

we have $I^{2-\alpha} u \in C^4(\Omega)$. Therefore, using equation (A.2) of Lemma A.1, for $1 \le i \le$

$2N-1$, we have

(3.4)
$$-\kappa_\alpha(D_h^2 - \frac{d^2}{dx^2})I^{2-\alpha}u(x_i) = \frac{h_{i+1}-h_i}{3}f'(x_i)$$
$$+\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy + \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}} f''(y)\frac{(y-x_{i+1})^3}{3!}dy\right)$$

By Lemma B.2, Lemma 2.2 and Lemma B.3, we get the result. ◻

And now define

$$R_i := D_h^2 I^{2-\alpha}(u-\Pi_h u)(x_i), \quad 1 \le i \le 2N-1 \tag{3.5}$$

We have some results about the estimate of R_i

THEOREM 3.2. *For $1 \le i < N/2$, there exists $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$|R_i| \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1>0 \\ Ch^2(x_i^{-1-\alpha}\ln(i)+\ln(N)), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r}x_i^{-1-\alpha}, & \alpha/2-2/r+1<0 \end{cases} \tag{3.6}$$

THEOREM 3.3. *For $N/2 \le i \le N$, there exists constant $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$|R_i| \le C(r-1)h^2|T-x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2-2/r+1>0 \\ Ch^2\ln(N), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0 \end{cases} \tag{3.7}$$

And for $N < i \le 2N-1$, it is symmetric to the previous case.

Combine Theorem 3.1, Theorem 3.2 and Theorem 3.3, and for $1 \le i \le N$, we have

$$h^2 x_i^{-\alpha/2-2/r} \le T^{\alpha/2-2/r}h^{\min\{\frac{r\alpha}{2},2\}}x_i^{-\alpha} \tag{3.8}$$
$$h^{r\alpha/2+r}x_i^{-1-\alpha} \le T^{-1}h^{r\alpha/2}x_i^{-\alpha} \tag{3.9}$$
$$h^r x_i^{-1}\ln(i) = T^{-1}\frac{\ln(i)}{i^r} \le T^{-1}, \quad h^r\ln(N) = \frac{\ln(N)}{N^r} \le 1 \tag{3.10}$$

the proof of Theorem 2.5 completed.

We prove Theorem 3.2 and Theorem 3.3 in next subsections.

3.2. Outlines and Mesh Transport Functions. For convience, let's denote

DEFINITION 3.4.

$$T_{ij} = \int_{x_{j-1}}^{x_j}(u(y)-\Pi_h u(y))\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy, \quad i=0,\cdots,2N,\ j=1,\cdots,2N \tag{3.11}$$

Also, we denote vertical difference quotients of T_{ij}

$$V_{ij} = \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}T_{i-1,j} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)T_{i,j} + \frac{1}{h_{i+1}}T_{i+1,j}\right) \tag{3.12}$$

And skew difference quotients of T_{ij}

$$S_{ij} = \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}T_{i-1,j-1} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)T_{i,j} + \frac{1}{h_{i+1}}T_{i+1,j+1}\right) \tag{3.13}$$

then $R_i = \sum_{j=1}^{2N} V_{ij}$.

Our main idea is to depart R_i by V_{ij} and S_{ij}. For $3 \le i < N/2$, let's denote $k = \lceil \frac{i}{2} \rceil$, and take some suitable integer m, then

$$\begin{aligned} R_i =& \sum_{j=1}^{2N} V_{ij} \\ =& \sum_{j=1}^{k-1} V_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\ &+ \sum_{j=k+1}^{m-1} S_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,m} + T_{i-1,m-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,m} \right) \\ &+ \sum_{j=m+1}^{2N} V_{ij} \\ =& I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned} \tag{3.14}$$

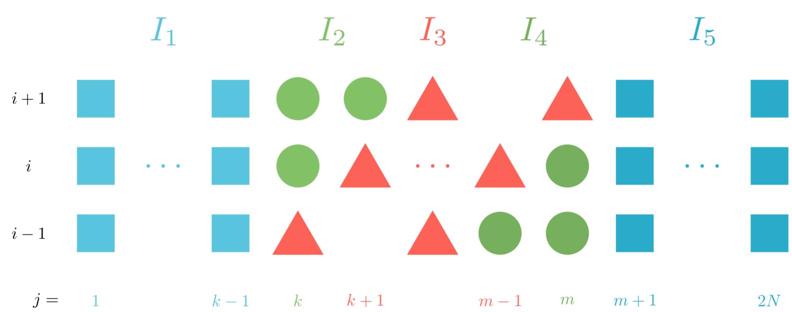

FIG. 1. *The departure of R_i for $i \ge 3$*

and discuss $i = 1, 2$ separately, where

$$R_1 = \sum_{i=1}^{3} V_{1,j} + \sum_{i=4}^{N} V_{i,j}, \quad R_2 = \sum_{i=1}^{4} V_{1,j} + \sum_{i=5}^{N} V_{i,j} \tag{3.15}$$

The difficulty for esitmating S_{ij} is that $T_{i-1,j-1}, T_{i,j}$ and $T_{i+1,j+1}$ have different integral region. We first make them normalized.

LEMMA 3.5. *For $y \in (x_{j-1}, x_j)$, we can rewrite $y = y_j^\theta$, from* (3.11)*, and Lemma* A.2*,*

$$\begin{aligned} T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta \end{aligned} \tag{3.16}$$

where $\eta_{j1}^\theta \in (x_{j-1}, y_j^\theta), \eta_{j2}^\theta \in (y_j^\theta, x_j)$.

Since j changes with i at indices of elements in S_{ij} by (3.13), we create some functions satisfy the property.

DEFINITION 3.6 (Mesh Transport Functions). *For* $1 \le i, j \le 2N-1$.

$$y_{i,j}(x) = \begin{cases} (x^{1/r} + Z_{j-i})^r & i < N, j < N \\ \dfrac{x^{1/r} - Z_i}{Z_1} h_N + x_N & i < N, j = N \\ 2T - (Z_{2N-(j-i)} - x^{1/r})^r & i < N, j > N \\ \left(\dfrac{Z_1}{h_N}(x - x_N) + Z_j\right)^r & i = N, j < N \\ x, & i = N, j = N \\ 2T - \left(\dfrac{Z_1}{h_N}(2T - x - x_N) + Z_{2N-j}\right)^r & i = N, j > N \\ (Z_{2N+j-i} - (2T-x)^{1/r})^r & i > N, j < N \\ \dfrac{Z_{2N-j} - (2T-x)^{1/r}}{Z_1} h_N + x_N & i > N, j = N \\ 2T - ((2T-x)^{1/r} - Z_{j-i})^r & i > N, j > N \end{cases} \tag{3.17}$$

where $Z_j := T^{1/r}\frac{j}{N}$, $x \in [x_{i-1}, x_{i+1}]$. *And*

$$h_{i,j}(x) = y_{i,j}(x) - y_{i,j-1}(x) \tag{3.18}$$

$$y^\theta_{i,j}(x) = (1-\theta) y_{i,j-1}(x) + \theta y_{i,j-1}(x), \quad \theta \in (0,1) \tag{3.19}$$

$$P^\theta_{i,j}(x) = (h_{i,j}(x))^3 \frac{|y^\theta_{i,j}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y^\theta_{i,j}(x)) \tag{3.20}$$

$$Q^\theta_{i,j}(x) = (h_{i,j}(x))^4 \frac{|y^\theta_{i,j}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.21}$$

Obviously,

$$y_{i,j}(x_{i-1}) = x_{j-1}, \quad y_{i,j}(x_i) = x_j, \quad y_{i,j}(x_{i+1}) = x_{j+1} \tag{3.22}$$
$$h_{i,j}(x_{i-1}) = h_{j-1}, \quad h_{i,j}(x_i) = h_j, \quad h_{i,j}(x_{i+1}) = h_{j+1} \tag{3.23}$$
$$y^\theta_{i,j}(x_{i-1}) = y^\theta_{j-1}, \quad y^\theta_{i,j}(x_i) = y^\theta_j, \quad y^\theta_{i,j}(x_{i+1}) = y^\theta_{j+1} \tag{3.24}$$

And now we can rewrite T_{ij}

LEMMA 3.7. *For* $0 \le i \le 2N, 1 \le j \le 2N$,

$$\begin{aligned} T_{ij} = &\int_0^1 -\frac{\theta(1-\theta)}{2} P^\theta_{i,j}(x_i) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} Q^\theta_{i,j}(x_i) \left[\theta^2 u'''(\eta^\theta_{j,1}) - (1-\theta)^2 u'''(\eta^\theta_{j,2})\right] d\theta \end{aligned} \tag{3.25}$$

Immediately, we can see from (3.13) *that For* $1 \le i \le 2N-1, 2 \le j \le 2N-1$,

$$S_{ij} = \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{i,j}^{\theta}(x_i) d\theta + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j}^{\theta}(x_{i+1}) u'''(\eta_{j+1,1}^{\theta}) - Q_{i,j}^{\theta}(x_i) u'''(\eta_{j,1}^{\theta})}{h_{i+1}} \right) d\theta - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j}^{\theta}(x_i) u'''(\eta_{j,1}^{\theta}) - Q_{i,j}^{\theta}(x_{i-1}) u'''(\eta_{j-1,1}^{\theta})}{h_i} \right) d\theta - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j}^{\theta}(x_{i+1}) u'''(\eta_{j+1,2}^{\theta}) - Q_{i,j}^{\theta}(x_i) u'''(\eta_{j,2}^{\theta})}{h_{i+1}} \right) d\theta + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j}^{\theta}(x_i) u'''(\eta_{j,2}^{\theta}) - Q_{i,j}^{\theta}(x_{i-1}) u'''(\eta_{j-1,2}^{\theta})}{h_i} \right) d\theta \tag{3.26}$$

We give some properties of mesh transport functions.

Lemma 3.8. *For* $1 \le i \le 2N-1, 2 \le j \le 2N-1$ *and* $\xi \in (x_{i-1}, x_{i+1})$

$$y_{i,j}(\xi) \simeq x_j, \quad h_{i,j}(\xi) \simeq h_j \tag{3.27}$$

For $1 \le i, j \le 2N-1$,

$$|y_{i,j}(\xi) - \xi| \simeq |x_j - x_i| \tag{3.28}$$

And for $1 \le i \le 2N-1, 2 \le j \le 2N-1$,

$$|y_{i,j}^{\theta}(\xi) - \xi| = (1-\theta)|y_{i,j-1}(\xi) - \xi| + \theta |y_{i,j}(\xi) - \xi| \simeq |y_j^{\theta} - x_i| \tag{3.29}$$

since $y_{i,j-1}(\xi) - \xi$, $y_{i,j}(\xi) - \xi$ *have the same sign* (≥ 0 *or* ≤ 0)

Lemma 3.9.

$$y'_{i,j}(x) = \begin{cases} y_{i,j}^{1-1/r}(x) x^{1/r-1} & i < N, j < N \\ \dfrac{h_N}{rZ_1} x^{1/r-1} & i < N, j = N \\ (2T - y_{i,j}^{1-1/r}(x)) x^{1/r-1} & i < N, j > N \\ y_{i,j}^{1-1/r}(x) \dfrac{rZ_1}{h_N} & i = N, j < N \\ 1 & i = N, j = N \end{cases} \tag{3.30}$$

$$y''_{i,j}(x) = \frac{1-r}{r} \begin{cases} y_{i,j}^{1-2/r}(x) x^{1/r-2} Z_{i-j} & i < N, j < N \\ \dfrac{h_N}{rZ_1} x^{1/r-2} & i < N, j = N \\ (2T - y_{i,j}^{1-2/r}(x)) x^{1/r-2} Z_{2N-j+i} & i < N, j > N \\ -y_{i,j}^{1-2/r}(x) \left(\dfrac{rZ_1}{h_N} \right)^2 & i = N, j < N \\ 0 & i = N, j = N \end{cases} \tag{3.31}$$

LEMMA 3.10. *For $2 \le i \le N, 2 \le j \le 2N-1$*

$$|h'_{i,j}(\xi)| \le C(r-1)hx_i^{1/r-1} \begin{cases} x_j^{1-2/r} & j \le N \\ (2T-x_j)^{1-2/r} & j > N \end{cases} \tag{3.32}$$

And

$$|(y_{i,j}(\xi)-\xi)'| \le C(r-1)|y_{i,j}(\xi)-\xi|\xi^{-1} \simeq (r-1)|x_j - x_i|x_i^{-1} \tag{3.33}$$

3.3. Proof of Theorem 3.2. Then we esrimate each part of (3.14) from easy to hard.

For I_5

LEMMA 3.11. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that for $1 \le i < N/2$,*

$$\sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.34}$$

Proof. For $\max\{2i+1,4\} \le j \le N$, by Lemma A.4 and Lemma B.4 with $y - x_i \simeq y$, we have

$$\begin{aligned} |V_{ij}| &= \left| \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \right| \\ &\le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} y^{-1-\alpha} dy \\ &= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy \end{aligned} \tag{3.35}$$

With $x_i \simeq x_{2i}$,

$$\begin{aligned} \sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| &\le Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\ &= \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\ &\le Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{3.36}$$

∎

LEMMA 3.12. *Thert exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that for $1 \le i < N/2$,*

$$\sum_{j=N+1}^{2N} |V_{ij}| \le \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.37}$$

and for $N/2 \le i \le N$,

$$\sum_{j=N-\lceil \frac{N}{2} \rceil+2}^{2N} |V_{ij}| \le \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.38}$$

Proof. For $1 \le i < N/2, N+1 \le j \le 2N-1$, by Lemma A.4, Lemma B.4 and $y - x_i \simeq T$

$$\begin{aligned}|V_{ij}| &= \left|\int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy\right| \\ &\le C \int_{x_{j-1}}^{x_j} h^2 (2T-y)^{\alpha/2-2/r} |y - x_i|^{-1-\alpha} dy \\ &\le Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T-y)^{\alpha/2-2/r} dy\end{aligned}$$

$$\begin{aligned}\sum_{j=N+1}^{2N-1} |V_{ij}| &\le CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r} dy \\ &\le CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\ &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r}\ h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}\end{aligned} \tag{3.39}$$

And by Lemma A.3

$$|V_{i,2N}| \le CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}$$

Summarizes, we get the result. Similar for the second inequality. ∎

For $i = 1, 2$.

LEMMA 3.13. *From* (3.15)*, by Lemma* B.6*, Lemma* 3.11 *and Lemma* 3.12 *we get for* $i = 1, 2$

$$|R_i| \le Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.40}$$

LEMMA 3.14. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i \le N, k = \lceil \frac{i}{2} \rceil$

$$|I_1| = |\sum_{j=1}^{k-1} V_{ij}| \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.41}$$

Proof. by Lemma A.3 , Lemma B.4

$$|V_{i1}| \le C \int_0^{x_1} x_1^{\alpha/2} |x_i - y|^{-1-\alpha} dy \simeq x_1^{\alpha/2+1} x_i^{-1-\alpha} = T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha} \tag{3.42}$$

For $2 \le j \le k-1$, by Lemma A.4 and Lemma B.4 with $x_i - y \simeq x_i$, we have

$$\begin{aligned} |V_{ij}| &= \left| \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \right| \\ &\le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} x_i^{-1-\alpha} dy \end{aligned} \tag{3.43}$$

Therefore,

$$\sum_{j=2}^{k-1} |V_{ij}| \le Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil -1}} y^{\alpha/2-2/r} dy \tag{3.44}$$

But $x_{\lceil \frac{i}{2} \rceil -1} \le 2^{-r} x_i$, so we have

$$\int_{x_1}^{x_{\lceil \frac{i}{2} \rceil -1}} y^{\alpha/2-2/r} dy \le \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.45}$$

Combine the results above, we get the lemma. ∎

THEOREM 3.15. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \le i < N/2, k = \lceil \frac{i}{2} \rceil$,*

$$|I_3| = \left| \sum_{j=k+1}^{2i-1} S_{ij} \right| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.46}$$

To estimate S_{ij}, we first estimate $D_h^2 P_{i,j}^{\theta}(x_i)$, but By Lemma A.1,

$$D_h^2 P_{i,j}^{\theta}(x_i) = {P_{i,j}^{\theta}}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{3.47}$$

By Leibniz formula, we calculate and estimate the derivations of $h_{j-i}^3(x)$, $u''(y_{j-i}^{\theta}(x))$ and $\frac{|y_{j-i}^{\theta}(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)}$ separately.

Firstly, we have

LEMMA 3.16. *There exists a constant $C = C(T, r)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil \le j \le \min\{2i, N\}$, $\xi \in (x_{i-1}, x_{i+1})$,*

$$h_{j-i}^3(\xi) \le Ch^2 x_i^{2-2/r} h_j \tag{3.48}$$

$$(h_{j-i}^3(\xi))' \le C(r-1)h^2 x_i^{1-2/r} h_j \tag{3.49}$$

$$(h_{j-i}^3(\xi))'' \le C(r-1)h^2 x_i^{-2/r} h_j \tag{3.50}$$

The proof of this theorem see Lemma B.7 and Lemma B.8

Second,

LEMMA 3.17. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil \le j \le \min\{2i, N\}$, $\xi \in (x_{i-1}, x_{i+1})$,*

$$u''(y_{j-i}^{\theta}(\xi)) \le Cx_i^{\alpha/2-2} \tag{3.51}$$

$$(u''(y^\theta_{j-i}(\xi)))' \le C x_i^{\alpha/2-3} \tag{3.52}$$

$$(u''(y^\theta_{j-i}(\xi)))'' \le C x_i^{\alpha/2-4} \tag{3.53}$$

The proof of this theorem see Proof 28

And Finally, we have

LEMMA 3.18. *There exists a constant $C = C(T,\alpha,r)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil \le j \le \min\{2i, N\}$, $\xi \in (x_{i-1}, x_{i+1})$,*

$$|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha} \le C|y^\theta_j - x_i|^{1-\alpha} \tag{3.54}$$

$$\left|(|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha})'\right| \le C|y^\theta_j - x_i|^{1-\alpha} x_i^{-1} \tag{3.55}$$

$$\left|(|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha})''\right| \le C|y^\theta_j - x_i|^{1-\alpha} x_i^{-2} \tag{3.56}$$

where $y^\theta_j = \theta x_{j-1} + (1-\theta)x_j$

The proof of this theorem see Proof 29

LEMMA 3.19. *There exists a constant $C = C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$,*

$$D_h^2 P^\theta_{i,j}(x_i) \le Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \tag{3.57}$$

where $y^\theta_j = \theta x_{j-1} + (1-\theta)x_j$

Proof. Since Lemma A.1

$$D_h^2 P^\theta_{i,j}(x_i) = {P^\theta_{i,j}}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{3.58}$$

From (3.20), using Leibniz formula and Lemma 3.16, Lemma 3.17 and Lemma 3.18∎

LEMMA 3.20. *There exists a constant $C = C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that for $3 \le i \le N-1$.*
For $\lceil \frac{i}{2} \rceil \le j \le \min\{2i-1, N-1\}$,

$$\begin{aligned}\frac{2}{h_i + h_{i+1}} &\left(\frac{Q^\theta_{i,j}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{i,j}(x_i)u'''(\eta^\theta_j)}{h_{i+1}}\right)\\ &\le Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j\end{aligned} \tag{3.59}$$

And for $\lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i, N\}$,

$$\begin{aligned}\frac{2}{h_i + h_{i+1}} &\left(\frac{Q^\theta_{i,j}(x_i)u'''(\eta^\theta_j) - Q^\theta_{i,j}(x_{i-1})u'''(\eta^\theta_{j-1})}{h_i}\right)\\ &\le Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j\end{aligned} \tag{3.60}$$

where $\eta^\theta_j \in (x_{j-1}, x_j)$.

proof see Proof 30

LEMMA 3.21. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that for* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$,

$$\begin{aligned} S_{ij} &\le Ch^2 \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j d\theta \\ &= Ch^2 \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \end{aligned} \tag{3.61}$$

Proof. Since (3.26), by Lemma 3.19 and Lemma 3.20, we get the result immediately. ∎

Now we can prove Theorem 3.15 using Lemma 3.21, $k = \lceil \frac{i}{2} \rceil$

$$\begin{aligned} I_3 = \sum_{k+1}^{2i-1} S_{ij} &\le Ch^2 \int_{x_k}^{x_{2i-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{2i-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\ &\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{3.62}$$

Now we study I_2, I_4.

LEMMA 3.22. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that for* $3 \le i \le N-1, k = \lceil \frac{i}{2} \rceil$,

$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.63}$$

And for $3 \le i < N/2$,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.64}$$

Proof. In fact,

$$\begin{aligned} &\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \\ &= \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i}) T_{i,k} \end{aligned} \tag{3.65}$$

While, by Lemma A.2 and Lemma B.1

$$\begin{aligned} \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) &= \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y)) \frac{|x_{i+1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_{i+1} \Gamma(2-\alpha)} dy \\ &\le h_k^2 \max_{\eta \in (x_{k-1}, x_k)} |u''(\eta)| \int_{x_{k-1}}^{x_k} \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy, \quad \xi \in (x_i, x_{i+1}) \\ &\le Ch^2 x_k^{2-2/r} x_{k-1}^{\alpha/2-2}\ h_k |x_i - x_k|^{-\alpha} \\ &\le Ch^2 x_i^{-\alpha/2-2/r} h_k \end{aligned} \tag{3.66}$$

Thus,

$$\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}|T_{i+1,k}-T_{i,k}| \leq Ch^2 x_i^{-\alpha/2-2/r} \tag{3.67}$$

From (3.25)

$$\begin{aligned}\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k}) = &\int_0^1 -\frac{\theta(1-\theta)}{2}\frac{P^\theta_{k-i}(x_{i+1})-P^\theta_{k-i}(x_i)}{h_{i+1}}d\theta \\ &+\int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{Q^\theta_{k-i}(x_{i+1})u'''(\eta^\theta_{k+1,1})-Q^\theta_{k-i}(x_i)u'''(\eta^\theta_{k,1})}{h_{i+1}}d\theta \\ &-\int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{Q^\theta_{k-i}(x_{i+1})u'''(\eta^\theta_{k+1,2})-Q^\theta_{k-i}(x_i)u'''(\eta^\theta_{k,2})}{h_{i+1}}d\theta\end{aligned} \tag{3.68}$$

and

$$D_h P^\theta_{k-i}(x_i) := \frac{P^\theta_{k-i}(x_{i+1})-P^\theta_{k-i}(x_i)}{h_{i+1}} = {P^\theta_{k-i}}'(\xi), \quad \xi \in (x_i, x_{i+1}) \tag{3.69}$$

Similar with Lemma 3.19, from (3.25), using Leibniz formula, by Lemma B.7, Lemma 3.17 and Lemma 3.18 we get

$$|D_h P^\theta_{k-i}(x_i)| \leq Ch^2 x_i^{-\alpha/2-2/r} h_k \tag{3.70}$$

And with Lemma 3.20, we can get

$$\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}|T_{i+1,k+1}-T_{i,k}| \leq Ch^2 x_i^{-\alpha/2-2/r} \tag{3.71}$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}\frac{h_{i+1}-h_i}{h_ih_{i+1}}T_{i,k} &\leq h_i^{-3}h^2x_i^{1-2/r}h_kCh_k^2x_{k-1}^{\alpha/2-2}|x_k-x_i|^{1-\alpha} \\ &\leq Ch^2x_i^{-\alpha/2-2/r}\end{aligned} \tag{3.72}$$

Summarizes, we have

$$I_2 \leq Ch^2 x_i^{-\alpha/2-2/r} \tag{3.73}$$

The case for I_4 is similar. ∎

Now combine Lemma 3.13, Lemma 3.14, Lemma 3.22, Theorem 3.15, Lemma 3.11 and Lemma 3.12, we get Theorem 3.2.

3.4. Proof of Theorem 3.3. For $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$, we have

$$
\begin{aligned}
R_i =& \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
& + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\
& + \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
& + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2N-\lceil \frac{N}{2} \rceil + 1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2N-\lceil \frac{N}{2} \rceil + 1} \right) \\
& + \sum_{j=2N-\lceil \frac{N}{2} \rceil + 2}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& I_1 + I_2 + I_3^1 + I_3^2 + I_3^3 + I_4 + I_5
\end{aligned} \tag{3.74}
$$

We have estimate I_1 in Lemma 3.14 and I_2 in Lemma 3.22. We can control I_3^1 similar with Theorem 3.15 by Lemma 3.21 where $2i - 1 \ge N - 1$

Lemma 3.23. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$,*

$$
\begin{aligned}
I_3^1 =& \sum_{j=k+1}^{N-1} S_{ij} \le Ch^2 \int_{x_k}^{x_{N-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\
=& Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{N-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\
\le& Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{3.75}
$$

Let's study I_3^3 before I_3^2.

$$
I_3^3 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} S_{ij} \tag{3.76}
$$

Similarly, Let's define a new series of functions

Definition 3.24. *For $i \le N-1, j \ge N+1$, with no confusion, we also denote in this section*

$$
y_{j-i}(x) = 2T - (Z_{2N-j+i} - x^{1/r})^r, \quad Z_{2N-j+i} = T^{1/r} \frac{2N - j + i}{N} \tag{3.77}
$$

Particularly

$$
y_{j-i}(x_{i-1}) = x_{j-1}, \quad y_{j-i}(x_i) = x_j, \quad y_{j-i}(x_{i+1}) = x_{j+1}
$$

$y \to z$?

$$y_{j-i}{}'(x) = (2T - y_{j-i}(x))^{1-1/r} x^{1/r-1} \tag{3.78}$$

$$y_{j-i}{}''(x) = \frac{1-r}{r}(2T - y_{j-i}(x))^{1-2/r} x^{1/r-2} Z_{2N-j+i} \tag{3.79}$$

(3.80)

$$y_{j-i}^{\theta}(x) = (1-\theta) y_{j-i-1}(x) + \theta y_{j-i}(x) \tag{3.81}$$

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{3.82}$$

$$P_{i,j}^{\theta}(x) = (h_{j-i}(x))^3 u''(y_{j-i}^{\theta}(x)) \frac{|y_{j-i}^{\theta}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.83}$$

$$Q_{i,j}^{\theta}(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^{\theta}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.84}$$

Now we have the same formula (3.26) for $i \leq N-1, j \geq N+2$,
Similarly, we first estimate

$$D_h^2 P_{i,j}^{\theta}(\xi) = P_{i,j}^{\theta}{}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{3.85}$$

Combine Definition 3.24, Lemma B.9, Lemma B.10 and Lemma B.11, using Leibniz formula, we have

Lemma 3.25. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i \leq N-1$, $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$,, we have*

$$\begin{aligned} |D_h^2 P_{i,j}^{\theta}(\xi)| \leq C h_j h^2 \Big(& |y_j^{\theta} - x_i|^{1-\alpha} \\ & + |y_j^{\theta} - x_i|^{-\alpha} (|2T - x_i - y_j^{\theta}| + h_N) \\ & + |y_j^{\theta} - x_i|^{-1-\alpha} (|2T - x_i - y_j^{\theta}| + h_N)^2 \\ & + (r-1) |y_j^{\theta} - x_i|^{-\alpha} \Big) \end{aligned} \tag{3.86}$$

And

Lemma 3.26. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i \leq N-1$, $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil$, $\xi \in (x_{i-1}, x_{i+1})$, we have*

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} \left| \frac{Q_{i,j}^{\theta}(x_{i+1}) u'''(\eta_{j+1}^{\theta}) - Q_{i,j}^{\theta}(x_i) u'''(\eta_j^{\theta})}{h_{i+1}} \right| \\ & \qquad \leq C h^2 h_j \left(|y_j^{\theta} - x_i|^{1-\alpha} + |y_j^{\theta} - x_i|^{-\alpha} (|2T - x_i - y_j^{\theta}| + h_N) \right) \end{aligned} \tag{3.87}$$

and

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j}^{\theta}(x_i) u'''(\eta_j^{\theta}) - Q_{i,j}^{\theta}(x_{i-1}) u'''(\eta_{j-1}^{\theta})}{h_{i+1}} \right) \\ & \qquad \leq C h^2 h_j (|y_j^{\theta} - x_i|^{1-\alpha} + |y_j^{\theta} - x_i|^{-\alpha} (|2T - x_i - y_j^{\theta}| + h_N)) \end{aligned} \tag{3.88}$$

Proof. From Definition 3.24, by Lemma B.9 and Lemma B.11, for $\xi \in (x_i, x_{i+1})$, by Leibniz formula, we have

$$\left|{Q^{\theta}_{i,j}}'(\xi)\right| \leq Ch^2 h_j^2((r-1)|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N)) \tag{3.89}$$

$$\left|Q^{\theta}_{i,j}(\xi)\right| \leq Ch^2 h_j^2 |y_j^\theta - x_i|^{1-\alpha} \tag{3.90}$$

So use the skill in Proof 30 with Lemma B.10

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}}\left(\frac{Q^{\theta}_{i,j}(x_{i+1})u'''(\eta^{\theta}_{j+1}) - Q^{\theta}_{i,j}(x_i)u'''(\eta^{\theta}_j)}{h_{i+1}}\right) \\ &\leq Ch^2 h_j(|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N)) \end{aligned} \tag{3.91}$$

□

Combine Lemma 3.25, Lemma 3.26 and formula (3.26) for $i \leq N-1, j \geq N+2$, we have

LEMMA 3.27. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \leq i \leq N-1$, $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$*

$$\begin{aligned} S_{ij} \leq & Ch^2 \int_{x_{j-1}}^{x_j} \Big(|y - x_i|^{1-\alpha} \\ &+ |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2 \\ &+ (r-1)|y - x_i|^{-\alpha}\Big)dy \end{aligned} \tag{3.92}$$

We can esitmate I_3^3 Now.

LEMMA 3.28. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \leq i \leq N-1$, we have*

$$I_3^3 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} S_{ij} \leq Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \tag{3.93}$$

Proof.

$$\begin{aligned} I_3^3 = & \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} S_{ij} \\ \leq & Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil \frac{N}{2} \rceil}} \Big(|y - x_i|^{1-\alpha} \\ &+ |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2 \\ &+ (r-1)|y - x_i|^{-\alpha}\Big)dy \end{aligned} \tag{3.94}$$

Since

$$|2T - x_i - y| + h_N \leq y - x_i \tag{3.95}$$

$$\begin{aligned} I_3^3 &\leq Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil \frac{N}{2} \rceil}} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} \\ &\leq Ch^2(T^{2-\alpha} + (r-1)|x_{N+1} - x_i|^{1-\alpha}) \\ &\leq Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned} \tag{3.96}$$

□

For I_3^2, we have

THEOREM 3.29. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that, for $N/2 \le i \le N-1$*

$$V_{iN} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,N+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,N} + \frac{1}{h_i} T_{i-1,N-1} \right) \le Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \tag{3.97}$$

Proof. We use the similar skill in the last section, but more complicated. for $j = N$, Let

$$_L y_{N-1-i}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r} \frac{N-1-i}{N} \tag{3.98}$$

$$_0 y_{N-i}(x) = \frac{x^{1/r} - Z_i}{Z_1} h_N + T, \quad Z_i = T^{1/r} \frac{i}{N}, x_N = T \tag{3.99}$$

and

$$_R y_{N+1-i}(x) = 2T - (Z_{N-1+i} - x^{1/r})^r, \quad Z_{N-1+i} = T^{1/r} \frac{N-1+i}{N} \tag{3.100}$$

Thus,

$$
\begin{gathered}
_L y_{N-1-i}(x_{i-1}) = x_{N-2}, \quad _L y_{N-1-i}(x_i) = x_{N-1}, \quad _L y_{N-1-i}(x_{i+1}) = x_N \\
_0 y_{N-i}(x_{i-1}) = x_{N-1}, \quad _0 y_{N-i}(x_i) = x_N, \quad _0 y_{N-i}(x_{i+1}) = x_{N+1} \\
_R y_{N+1-i}(x_{i-1}) = x_N, \quad _R y_{N+1-i}(x_i) = x_{N+1}, \quad _R y_{N+1-i}(x_{i+1}) = x_{N+2}
\end{gathered}
$$

Then, define

$$_L y^\theta_{N-i}(x) = \theta\, _L y_{N-1-i}(x) + (1-\theta)\, _0 y_{N-i}(x) \tag{3.101}$$

$$_R y^\theta_{N+1-i}(x) = \theta\, _0 y_{N-i}(x) + (1-\theta)\, _R y_{N+1-i}(x) \tag{3.102}$$

$$_L h_{N-i}(x) = {_0 y_{N-i}}(x) - {_L y_{N-1-i}}(x) \tag{3.103}$$

$$_R h_{N+1-i}(x) = {_R y_{N+1-i}}(x) - {_0 y_{N-i}}(x) \tag{3.104}$$

We have

$$_L y_{N-1-i}{}'(x) = {_L y^{1-1/r}_{N-1-i}}(x) x^{1/r-1} \tag{3.105}$$

$$_L y_{N-1-i}{}''(x) = \frac{1-r}{r} {_L y^{1-2/r}_{N-1-i}}(x) x^{1/r-2} Z_{N-1-i} \tag{3.106}$$

$$_0 y_{N-i}{}'(x) = \frac{1}{r} \frac{h_N}{Z_1} x^{1/r-1} \tag{3.107}$$

$$_0 y_{N-i}{}''(x) = \frac{1-r}{r^2} \frac{h_N}{Z_1} x^{1/r-2} \tag{3.108}$$

$$_R y_{N+1-i}{}'(x) = (2T - {_R y_{N+1-i}}(x))^{1-1/r} x^{1/r-1} \tag{3.109}$$

$$_R y_{N+1-i}{}''(x) = \frac{1-r}{r} (2T - {_R y_{N+1-i}}(x))^{1-2/r} x^{1/r-2} Z_{N-1+i} \tag{3.110}$$

$$ {}_L P^{\theta}_{N-i}(x) = ({}_L h_{N-i}(x))^3 \frac{|{}_L y^{\theta}_{N-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_L y^{\theta}_{N-i}(x)) \tag{3.111} $$

$$ {}_R P^{\theta}_{N+1-i}(x) = ({}_R h_{N+1-i}(x))^3 \frac{|{}_R y^{\theta}_{N+1-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_R y^{\theta}_{N+1-i}(x)) \tag{3.112} $$

$$ {}_L Q^{\theta}_{N-i}(x) = ({}_L h_{N-i}(x))^4 \frac{|{}_L y^{\theta}_{N-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.113} $$

$$ {}_R Q^{\theta}_{N+1-i}(x) = ({}_R h_{N+1-i}(x))^4 \frac{|{}_R y^{\theta}_{N+1-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.114} $$

Similar with (3.25), we can get for $l = -1, 0, 1$,

$$ \begin{aligned} T_{i+l,N+l} = & \int_0^1 -\frac{\theta(1-\theta)}{2} {}_L P^{\theta}_{N-i}(x_{i+l}) d\theta \\ & + \int_0^1 \frac{\theta(1-\theta)}{3!} {}_L Q^{\theta}_{N-i}(x_{i+l}) (\theta^2 u'''(\eta^{\theta}_{N+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+l,2})) d\theta \end{aligned} \tag{3.115} $$

$$ \begin{aligned} T_{i+l,N+1+l} = & \int_0^1 -\frac{\theta(1-\theta)}{2} {}_R P^{\theta}_{N+1-i}(x_{i+l}) d\theta \\ & + \int_0^1 \frac{\theta(1-\theta)}{3!} {}_R Q^{\theta}_{N+1-i}(x_{i+l}) (\theta^2 u'''(\eta^{\theta}_{N+1+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+1+l,2})) d\theta \end{aligned} \tag{3.116} $$

So we have

$$ \begin{aligned} V_{i,N} = & \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 {}_L P^{\theta}_{N-i}(x_i) d\theta \\ & + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_L Q^{\theta}_{N-i}(x_{i+1}) u'''(\eta^{\theta}_{N+1,1}) - {}_L Q^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,1})}{h_{i+1}} \right) d\theta \\ & - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_L Q^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,1}) - {}_L Q^{\theta}_{N-i}(x_{i-1}) u'''(\eta^{\theta}_{N-1,1})}{h_i} \right) d\theta \\ & - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_L Q^{\theta}_{N-i}(x_{i+1}) u'''(\eta^{\theta}_{N+1,2}) - {}_L Q^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,2})}{h_{i+1}} \right) d\theta \\ & + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_L Q^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,2}) - {}_L Q^{\theta}_{N-i}(x_{i-1}) u'''(\eta^{\theta}_{N-1,2})}{h_i} \right) d\theta \end{aligned} \tag{3.117} $$

$N+1$ is similar.

We estimate $D_h^2 {}_L P^{\theta}_{N-i}(x_i) = {}_L P^{\theta\,\prime\prime}_{N-i}(\xi), \xi \in (x_{i-1}, x_{i+1})$,

Lemma 3.30.

$$ {}_L h^3_{N-i}(\xi) \le C h_N^3 \le C h^3 \tag{3.118} $$

$$ {}_R h^3_{N+1-i}(\xi) \le C h_N^3 \le C h^3 \tag{3.119} $$

$$({}_Lh^3_{N-i}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{3.120}$$

$$({}_Rh^3_{N+1-i}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{3.121}$$

$$({}_Lh^3_{N-i}(\xi))'' \le C(r-1)h^2 \tag{3.122}$$

$$({}_Rh^3_{N+1-i}(\xi))'' \le C(r-1)h^2 \tag{3.123}$$

Proof.

$$ {}_Lh_{N-i}(\xi) \le 2(C?)h_N, \quad {}_Rh_{N+1-i}(\xi) \le 2h_N \tag{3.124}$$

$$\begin{aligned}({}_Lh^l_{N-i}(\xi))' &= l\,{}_Lh^{l-1}_{N-i}(\xi)({}_0y_{N-i}{}'(\xi) - {}_Ly_{N-1-i}{}'(\xi))\\ &= l\,{}_Lh^{l-1}_{N-i}(\xi)\xi^{1/r-1}(\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-1/r}_{N-1-i}(\xi))\end{aligned} \tag{3.125}$$

while

(3.126)

$$\begin{aligned}|\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-1/r}_{N-1-i}(\xi)| &= |\frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}| \quad \eta \in [x_{N-2}, x_N]\\ &= T^{1-1/r}|(\frac{N-t}{N})^{r-1} - (\frac{N-s}{N})^{r-1}| \quad t \in [0,1], s \in [0,2]\\ &\le T^{1-1/r}|1 - (\frac{N-2}{N})^{r-1}| \le CT^{1-1/r}(r-1)\frac{2}{N}\end{aligned}$$

Thus,

$$({}_Lh^l_{N-i}(\xi))' \le C(r-1)h_N^{l-1}x_i^{1/r-1}h \tag{3.127}$$

And

(3.128)

$$\begin{aligned}({}_Lh^3_{N-i}(\xi))'' &= 3{}_Lh^2_{N-i}(\xi){}_Lh_{N-i}{}''(\xi) + 6{}_Lh_{N-i}(\xi)({}_Lh_{N-i}{}'(\xi))^2\\ &\le Ch_N^2\frac{1-r}{r}x_i^{1/r-2}(\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i}) + Ch_N(r-1)^2h^2x_i^{2/r-2}\end{aligned}$$

$$|\frac{h_N}{rZ_1} - {}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i}| \le T^{1-1/r} + Cx_N^{1-2/r}x_N^{1/r} = CT^{1-1/r}$$

So

$$\begin{aligned}({}_Lh^3_{N-i}(\xi))'' &\le Ch_N^2\frac{1-r}{r}x_i^{1/r-2} + C(r-1)^2h_Nx_i^{2/r-2}h^2\\ &\le C(r-1)h_N^2\end{aligned} \tag{3.129}$$

${}_Rh^3_{N+1-i}(\xi)$ is similar. ∎

Lemma 3.31.

$$u''({}_Ly^\theta_{N-i}(\xi)) \le Cx_{N-2}^{-\alpha/2-2} \le C \tag{3.130}$$

$$(u''({}_Ly^\theta_{N-i}(\xi)))' \le C \tag{3.131}$$

$$(u''({}_Ly^\theta_{N-i}(\xi)))'' \le C \tag{3.132}$$

Proof.

$$
\begin{aligned}
(u''({}_Ly^{\theta}_{N-i}(\xi)))' &= u'''({}_Ly^{\theta}_{N-i}(\xi))\,{}_Ly^{\theta}_{N-i}{}'(\xi)\\
&\le C(\theta\, {}_Ly_{N-1-i}{}'(\xi) + (1-\theta)\,{}_0y_{N-i}{}'(\xi))\\
&\le Cx_i^{1/r-1}(\theta\, {}_Ly^{1-1/r}_{N-1-i}(\xi) + (1-\theta)\frac{h_N}{rZ_1})\\
&\le Cx_i^{1/r-1}x_N^{1-1/r}
\end{aligned} \tag{3.133}
$$

And

$$
\begin{aligned}
(u''({}_Ly^{\theta}_{N-i}(\xi)))'' &= u''''({}_Ly^{\theta}_{N-i}(\xi))({}_Ly^{\theta}_{N-i}{}'(\xi))^2 + u'''({}_Ly^{\theta}_{N-i}(\xi))\,{}_Ly^{\theta}_{N-i}{}''(\xi)\\
&\le Cx_i^{2/r-2}x_N^{2-2/r} + C\frac{r-1}{r}x_i^{1/r-2}(\theta x_N^{1-2/r}Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1})\\
&\le Cx_i^{2/r-2} + C(r-1)x_i^{1/r-2}T^{1-1/r}
\end{aligned} \tag{3.134}
$$

∎

LEMMA 3.32.

$$
|{}_Ly^{\theta}_{N-i}(\xi) - \xi|^{1-\alpha} \le C|y^{\theta}_N - x_i|^{1-\alpha} \tag{3.135}
$$

$$
(|{}_Ly^{\theta}_{N-i}(\xi) - \xi|^{1-\alpha})' \le C|y^{\theta}_N - x_i|^{1-\alpha} \tag{3.136}
$$

$$
(|{}_Ly^{\theta}_{N-i}(\xi) - \xi|^{1-\alpha})'' \le C(r-1)|y^{\theta}_N - x_i|^{-\alpha} + |y^{\theta}_N - x_i|^{1-\alpha} \tag{3.137}
$$

Proof.

$$
\begin{aligned}
({}_Ly^{\theta}_{N-i}(\xi) - \xi)' &= (\theta({}_Ly_{N-1-i}(\xi) - \xi) + (1-\theta)({}_0y_{N-i}(\xi) - \xi))'\\
&= \theta({}_Ly_{N-1-i}{}'(\xi) - 1) + (1-\theta)({}_0y_{N-i}{}'(\xi) - 1)\\
&= \theta\xi^{1/r-1}({}_Ly^{1-1/r}_{N-1-i}(\xi) - \xi^{1-1/r}) + (1-\theta)\xi^{1/r-1}(\frac{h_N}{rZ_1} - \xi^{1-1/r})
\end{aligned} \tag{3.138}
$$

$$
\begin{aligned}
({}_Ly^{\theta}_{N-i}(\xi) - \xi)'' &= \theta({}_Ly_{N-1-i}{}''(\xi)) + (1-\theta)({}_0y_{N-i}{}''(\xi))\\
&= \frac{1-r}{r}\xi^{1/r-2}(\theta\, {}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \le 0
\end{aligned} \tag{3.139}
$$

And

$$
|({}_Ly^{\theta}_{N-i}(\xi) - \xi)''| \le C(r-1)\xi^{1/r-2}T^{1-1/r} \tag{3.140}
$$

We have known

$$
C|x_{N-1} - x_i| \le |{}_Ly_{N-1-i}(\xi) - \xi| \le C|x_{N-1} - x_i| \tag{3.141}
$$

If $\xi \le x_{N-1}$, then $({}_0y_{N-i}(\xi) - \xi)' \ge 0$, so

$$
C|x_N - x_i| \le |x_{N-1} - x_{i-1}| \le |{}_Ly^{\theta}_{N-i}(\xi) - \xi| \le |x_{N+1} - x_{i+1}| \le C|x_N - x_i| \tag{3.142}
$$

If $i = N-1$ and $\xi \in [x_{N-1}, x_N]$, then ${}_0y_{N-i}(\xi) - \xi$ is concave, bigger than its two neighboring points, which are equal to h_N, so

$$
h_N = |x_N - x_{N-1}| \le |{}_0y_{N-i}(\xi) - \xi| \le |x_{N+1} - x_{N-1}| = 2h_N \tag{3.143}
$$

So we have

$$|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha}\le C|y^{\theta}_N-x_i|^{1-\alpha} \tag{3.144}$$

While

$$ {}_Ly^{1-1/r}_{N-1-i}(\xi)-\xi^{1-1/r}\le({}_Ly_{N-1-i}(\xi)-\xi)\xi^{-1/r} \tag{3.145}$$

and

$$\begin{aligned}|\frac{h_N}{rZ_1}-\xi^{1-1/r}|&\le\max\{|\frac{h_N}{rZ_1}-x^{1-1/r}_{i-1}|,|\frac{h_N}{rZ_1}-x^{1-1/r}_{i+1}|\}\\&\le\max\begin{cases}T^{1-1/r}-x^{1-1/r}_{i-1}\le|x_N-x_{i-1}|T^{-1/r}\le C|x_N-x_i|\\|x^{1-1/r}_{i+1}-x^{1-1/r}_{N-1}|\le|x_{i+1}-x_{N-1}|x^{-1/r}_{N-1}\le C|x_N-x_i|\end{cases}\end{aligned} \tag{3.146}$$

So we have

$$({}_Ly^{\theta}_{N-i}(\xi)-\xi)'\le C|y^{\theta}_N-x_i| \tag{3.147}$$

$$\begin{aligned}(|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha})'&=|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{-\alpha}({}_Ly^{\theta}_{N-i}(\xi)-\xi)'\\&\le|y^{\theta}_N-x_i|^{1-\alpha}\end{aligned} \tag{3.148}$$

Finally,

$$\begin{aligned}(|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha})''&=(1-\alpha)|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{-\alpha}({}_Ly^{\theta}_{N-i}(\xi)-\xi)''\\&\quad+\alpha(\alpha-1)|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{-1-\alpha}(({}_Ly^{\theta}_{N-i}(\xi)-\xi)')^2\\&\le C(r-1)|y^{\theta}_N-x_i|^{-\alpha}+C|y^{\theta}_N-x_i|^{1-\alpha}\end{aligned} \tag{3.149}$$ □

By the three lemmas above, for $N/2\le i\le N-1$, we have

Lemma 3.33.

$$\begin{aligned}D^2_h{}_LP^{\theta}_{N-i}(x_i)&={}_LP^{\theta}_{N-i}{}''(\xi)\quad \xi\in(x_{i-1},x_{i+1})\\&\le Ch^3|y^{\theta}_N-x_i|^{1-\alpha}+C(r-1)(h^3|y^{\theta}_N-x_i|^{-\alpha}+h^2|y^{\theta}_N-x_i|^{1-\alpha})\end{aligned} \tag{3.150}$$

while $\theta h_N=y^{\theta}_N-x_{N-1}\le y^{\theta}_N-x_i$, *we have*

$$\theta D^2_h{}_LP^{\theta}_{N-i}(x_i)\le Ch^3|y^{\theta}_N-x_i|^{1-\alpha}+C(r-1)(h^2|y^{\theta}_N-x_i|^{1-\alpha}) \tag{3.151}$$

And

Lemma 3.34.

$$\begin{aligned}&\frac{2}{h_i+h_{i+1}}\left(\frac{{}_LQ^{\theta}_{N-i}(x_{i+1})u'''(\eta^{\theta}_{N+1})-{}_LQ^{\theta}_{N-i}(x_i)u'''(\eta^{\theta}_N)}{h_{i+1}}\right)\\&\le Ch^3|y^{\theta}_N-x_i|^{1-\alpha}\end{aligned} \tag{3.152}$$

And immediately with (3.26), For $N/2\le i\le N-1$

$$\begin{aligned}V_{iN}&\le C\int_{x_{N-1}}^{x_N}h^2|y-x_i|^{1-\alpha}+C(r-1)h|y-x_i|^{1-\alpha}dy\\&\le Ch^2h_N|T-x_i|^{1-\alpha}+C(r-1)h^2|x_N-x_i|^{1-\alpha}\\&\le Ch^2+C(r-1)h^2|T-x_{i-1}|^{1-\alpha}\end{aligned} \tag{3.153}$$

Similarly with $j=N+1$. □

I_4, I_5 is easy. Similar with Lemma 3.22 and Lemma 3.12, we have

THEOREM 3.35. *There is a constant $C = C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \le i \le N$,*

(3.154)
$$I_4 = \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}(T_{i-1,2N-\lceil\frac{N}{2}\rceil+1}+T_{i-1,2N-\lceil\frac{N}{2}\rceil}) - (\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,2N-\lceil\frac{N}{2}\rceil+1}\right) \le Ch^2$$

Proof. Similar with Lemma 3.22. In fact, let $m = 2N - \lceil\frac{N}{2}\rceil + 1$

$$\begin{aligned} &\frac{1}{h_i}(T_{i-1,l}+T_{i-1,l-1}) - (\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,l} \\ &= \frac{1}{h_i}(T_{i-1,l}-T_{i,l}) + \frac{1}{h_i}(T_{i-1,l-1}-T_{i,l}) + (\frac{1}{h_i}-\frac{1}{h_{i+1}})T_{i,l} \end{aligned} \tag{3.155}$$

While, by Lemma A.2

$$\begin{aligned} \frac{1}{h_i}(T_{i-1,l}-T_{i,l}) &= \int_{x_{l-1}}^{x_l}(u(y)-\Pi_h u(y))\frac{|x_{i-1}-y|^{1-\alpha}-|x_i-y|^{1-\alpha}}{h_i\Gamma(2-\alpha)}dy \\ &\le C\int_{x_{l-1}}^{x_l} h_l^2 u''(\eta)\,\frac{|\xi-y|^{-\alpha}}{\Gamma(1-\alpha)}dy, \quad \xi\in(x_{i-1},x_i) \\ &\le Ch_l^3(2T-x_{l-1})^{\alpha/2-2}T^{-\alpha} \\ &\le Ch_l^3 \end{aligned} \tag{3.156}$$

Thus,

$$\frac{2}{h_i+h_{i+1}}\frac{1}{h_i}(T_{i-1,l}-T_{i,l}) \le Ch_l^2 \tag{3.157}$$

For

(3.158)
$$\frac{1}{h_i}(T_{i-1,l-1}-T_{i,l}) = \int_0^1 -\frac{\theta(1-\theta)}{2}\frac{h_{l-1}^3|y_{l-1}^\theta-x_{i-1}|^{1-\alpha}u''(\eta_{l-1}^\theta) - h_l^3|y_l^\theta-x_i|^{1-\alpha}u''(\eta_l^\theta)}{h_i}d\theta$$

And Similar with Lemma 3.20, we can get

$$\frac{h_{l-1}^3|y_{l-1}^\theta-x_{i-1}|^{1-\alpha}u''(\eta_{l-1}^\theta) - h_l^3|y_l^\theta-x_i|^{1-\alpha}u''(\eta_l^\theta)}{(h_i+h_{i+1})h_i} \le Ch_l^2|y_l^\theta-x_i|^{1-\alpha} \tag{3.159}$$

So

$$\frac{2}{h_i+h_{i+1}}\frac{1}{h_i}(T_{i-1,l-1}-T_{i,l}) \le Ch^2 \tag{3.160}$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned} \frac{2}{h_i+h_{i+1}}\frac{h_{i+1}-h_i}{h_ih_{i+1}}T_{i,l} &\le h_i^{-3}h^2x_i^{1-2/r}h_lCh_l^2x_{l-1}^{\alpha/2-2}|x_l-x_i|^{1-\alpha} \\ &\le Ch^2 \end{aligned} \tag{3.161}$$

Summarizes, we have

$$I_4 \le Ch^2 \tag{3.162}$$

□

Now we can conclude a part of the theorem Theorem 3.3 at the beginning of this section.

By Lemma 3.14, Lemma 3.22, Lemma 3.23, Theorem 3.29, Lemma 3.28, Theorem 3.35, Lemma 3.12, we have

THEOREM 3.36. *there exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $N/2 \leq i \leq N-1$,

$$\begin{aligned} R_i &= I_1 + I_2 + I_3^1 + I_3^2 + I_3^3 + I_4 + I_5 \\ &\leq C(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned} \tag{3.163}$$

And what we left is the case $i = N$. Fortunately, we can use the same department of R_i above, and it is symmetric. Most of the item has been esitmated by Lemma 3.14 and Theorem 3.35, we just need to consider I_3, I_4.

THEOREM 3.37. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that*

$$I_3 = \sum_{j=\lceil \frac{N}{2} \rceil+1}^{N-1} V_{Nj} \le Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha} \tag{3.164}$$

Proof. DEFINITION 3.38. *For* $N/2 \le j < N$ *, Let's define*

$$y_j(x) = (\frac{Z_1}{h_N}(x - x_N) + Z_j)^r, \quad Z_j = T^{1/r}\frac{j}{N} \tag{3.165}$$

We can see that is the inverse of the function ${}_0y_{N-i}(x)$ *defined in Theorem* 3.29.

$$y_j'(x) = y_j^{1-1/r}(x)\frac{rZ_1}{h_N} \tag{3.166}$$

$$y_j''(x) = y_j^{1-2/r}(x)\frac{r(r-1)Z_1}{h_N} \tag{3.167}$$

With the scheme we used several times, we can get

LEMMA 3.39. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le j < N$, $\xi \in [x_{N-1}, x_{N+1}]$,

$$h_j(\xi)^3 \le Ch^3 \tag{3.168}$$

$$(h_j^3(\xi))' \le C(r-1)h^3 \tag{3.169}$$

$$(h_j^3(\xi))'' \le C(r-1)h^3 \tag{3.170}$$

$$u''(y_j^\theta(\xi)) \le C \tag{3.171}$$

$$(u''(y_j^\theta(\xi)))' \le C \tag{3.172}$$

$$(u''(y_j^\theta(\xi)))'' \le C \tag{3.173}$$

$$|\xi - y_j^\theta(\xi)|^{1-\alpha} \le C|x_N - y_j^\theta|^{1-\alpha} \tag{3.174}$$

$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})' \le C|x_N - y_j^\theta|^{1-\alpha} \tag{3.175}$$

$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})'' \le C|x_N - y_j^\theta|^{1-\alpha} + C(r-1)|x_N - y_j^\theta|^{-\alpha} \tag{3.176}$$

LEMMA 3.40. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le j < N$,

$$V_{Nj} \le Ch^2 \int_{x_{j-1}}^{x_j} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \tag{3.177}$$

Therefore,

$$\begin{aligned} I_3 &\le Ch^2 \int_{x_{\lceil \frac{N}{2} \rceil}}^{x_{N-1}} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \\ &\le Ch^2(|T - x_{N-1}|^{2-\alpha} + (r-1)|T - x_{N-1}|^{1-\alpha}) \end{aligned} \tag{3.178}$$

□

For $j = N$,

LEMMA 3.41.

$$V_{N,N} = \frac{1}{h_N^2}\left(T_{N-1,N-1} - 2T_{N,N} + T_{N+1,N+1}\right) \le Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha} \tag{3.179}$$

Proof.

$$\begin{aligned} V_{N,N} = &\int_0^1 -\frac{\theta(1-\theta)^{2-\alpha}}{2}\frac{1}{h_N^2}\left(h_{N-1}^{4-\alpha}u''(y_{N-1}^\theta) - 2h_N^{4-\alpha}u''(y_N^\theta) + h_{N+1}^{4-\alpha}u''(y_{N+1}^\theta)\right)d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,1}^\theta) - Q_{N\to N}^\theta(x_i)u'''(\eta_{N,1}^\theta)}{h_N}\right)d\theta \\ &- \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,1}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,1}^\theta)}{h_N}\right)d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,2}^\theta) - Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta)}{h_N}\right)d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,2}^\theta)}{h_N}\right)d\theta \end{aligned} \tag{3.180}$$

□

So combine Lemma 3.14, Theorem 3.35, Theorem 3.37, Lemma 3.41 We have

LEMMA 3.42.

$$R_N \le C(r-1)h^2|T - x_{N-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.181}$$

and with Theorem 3.36 we prove the Theorem 3.3

4. Convergence analysis.

4.1. Properties of some Matrices.

Review subsection 2.1, we have got (2.10).

DEFINITION 4.1. *We call one matrix an M matrix , which means its entries are positive on major diagonal and nonpositive on others, and strictly diagonally dominant in rows.*

Now we have

LEMMA 4.2. *Matrix A defined by (2.12) where (2.13) is an M matrix. And there exists a constant $C_A = C(T, \alpha, r)$ such that*

$$S_i := \sum_{j=1}^{2N-1} a_{ij} \geq C_A(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \tag{4.1}$$

Proof. From (2.14), we have

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right) \tag{4.2}$$

Let

$$g(x) = g_0(x) + g_{2N}(x) \tag{4.3}$$

where

$$g_0(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x - x_0|^{3-\alpha} - |x - x_1|^{3-\alpha}}{h_1}$$

$$g_{2N}(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N} - x|^{3-\alpha} - |x_{2N-1} - x|^{3-\alpha}}{h_{2N}}$$

Thus

$$-\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

Then

$$\begin{aligned} S_i := & \sum_{j=1}^{2N-1} a_{ij} \\ = & \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = & D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i) \end{aligned} \tag{4.4}$$

When $i = 1$

$$\begin{aligned} D_h^2 g_0(x_1) = & \frac{2}{h_1 + h_2} \left(\frac{1}{h_2} g_0(x_2) - (\frac{1}{h_1} + \frac{1}{h_2}) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right) \\ = & \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2) h_1 h_2} \\ = & \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2) h_1^{1-\alpha} h_2} h_1^{-\alpha} \\ = & \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha}}{2^r (2^r - 1)} h_1^{-\alpha} \end{aligned} \tag{4.5}$$

but

$$1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha} > 0 \tag{4.6}$$

While for $i \geq 2$

$$\begin{aligned} D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \\ &= -\kappa_\alpha \frac{|\xi - x_0|^{1-\alpha} - |\xi - x_1|^{1-\alpha}}{\Gamma(2-\alpha) h_1} \\ &= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} |\xi - \eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\ &\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha} \end{aligned} \tag{4.7}$$

So

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g_0(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g_0(x_i) + \frac{1}{h_i} g_0(x_{i-1}) \right) \geq C x_i^{-\alpha} \tag{4.8}$$

symmetricly,

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g_{2N}(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g_{2N}(x_i) + \frac{1}{h_i} g_{2N}(x_{i-1}) \right) \geq C(\alpha, r)(2T - x_i)^{-\alpha} \tag{4.9}$$

Let

$$g(x) = \begin{cases} x, & 0 < x \leq T \\ 2T - x, & T < x < 2T \end{cases} \tag{4.10}$$

And define

$$G = \mathrm{diag}(g(x_1), ..., g(x_{2N-1})) \tag{4.11}$$

Then

LEMMA 4.3. *The matrix $B := AG$, the major diagnal is positive, and nonpositive on others. And there is a constant $C_{AG}, C = C(\alpha, r)$ such that*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C_{AG}(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \tag{4.12}$$

Proof.

$$b_{ij} = a_{ij} g(x_j) = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \tilde{a}_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) \tilde{a}_{i,j} + \frac{1}{h_i} \tilde{a}_{i-1,j} \right) g(x_j)$$

Since

$$g(x) \equiv \Pi_h g(x) \tag{4.13}$$

by **??**, we have

$$
\begin{aligned}
\tilde{M}_i &:= \sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} \tilde{a}_{ij} g(x_j) \\
&= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} \Pi_h g(y) dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} g(y) dy \\
&= \frac{-2}{\Gamma(4-\alpha)} |T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)} (x_i^{3-\alpha} + (2T - x_i)^{3-\alpha}) \\
&:= w(x_i) = p(x_i) + q(x_i)
\end{aligned} \tag{4.14}
$$

Thus,

$$
\begin{aligned}
M_i &:= \sum_{j=1}^{2N-1} b_{ij} = \sum_{j=1}^{2N-1} a_{ij} g(x_j) \\
&= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \tilde{M}_{i+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) \tilde{M}_i + \frac{1}{h_i} \tilde{M}_{i-1} \right) \\
&= D_h^2(-\kappa_\alpha p)(x_i) - \kappa_\alpha D_h^2 q(x_i)
\end{aligned} \tag{4.15}
$$

for $1 \le i \le N-1$,by Lemma A.1

$$
\begin{aligned}
D_h^2(-\kappa_\alpha p)(x_i) &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} p(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) p(x_i) + \frac{1}{h_i} p(x_{i-1}) \right) \\
&= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1}) \\
&\ge \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - x_{i-1}|^{1-\alpha}
\end{aligned} \tag{4.16}
$$

$$
\begin{aligned}
D_h^2(-\kappa_\alpha p)(x_N) &:= -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left(\frac{1}{h_{N+1}} p(x_{N+1}) - (\frac{1}{h_N} + \frac{1}{h_{N+1}}) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right) \\
&= \frac{4\kappa_\alpha}{\Gamma(4-\alpha) h_N^2} h_N^{3-\alpha} \\
&= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - x_{N-1})^{1-\alpha}
\end{aligned} \tag{4.17}
$$

Symmetricly for $i \ge N$, we get

$$
D_h^2(-\kappa_\alpha p)(x_i) \ge \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \le N \\ |x_{i+1} - T|^{1-\alpha}, & i \ge N \end{cases} \tag{4.18}
$$

Similarly, we can get

$$
\begin{aligned}
D_h^2 q(x_i) &:= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} q(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right) \\
&\le \frac{2^{r(\alpha-1)+1}}{\Gamma(2-\alpha)} (x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}), \quad i = 1, \cdots, 2N-1
\end{aligned} \tag{4.19}
$$

□

So, we get the result.

Notice that

$$x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha} \tag{4.20}$$

We can get

THEOREM 4.4. *There exists a real $\lambda = \lambda(T, \alpha, r) > 0$ and $C = C(T, \alpha, r) > 0$ such that $B := A(\lambda I + G)$ is an M matrix. And*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \tag{4.21}$$

Proof. By Lemma 4.2 with C_A and Lemma 4.3 with C_{AG} , it's sufficient to take $\lambda = (C + 2TC_{AG})/C_A$, then

$$M_i \geq C \left((x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \right) \tag{4.22}$$

□

4.2. Proof of Theorem 2.6. For equation

$$AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1}U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1}U = F \tag{4.23}$$

which means

$$\sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + g(x_j)} = -\tau_i \tag{4.24}$$

where $\epsilon_i = u(x_i) - u_i$.

And if

$$\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| = \max_{1 \leq i \leq 2N-1} \left|\frac{\epsilon_i}{\lambda + g(x_i)}\right| \tag{4.25}$$

Then, since $B = A(\lambda I + G)$ is an M matrix, it is Strictly diagonally dominant. Thus,

$$\begin{aligned} |\tau_{i_0}| &= \left|\sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + g(x_j)}\right| \\ &\geq b_{i_0,i_0}\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| - \sum_{j \neq i_0} |b_{i_0,j}|\left|\frac{\epsilon_j}{\lambda + g(x_j)}\right| \\ &\geq b_{i_0,i_0}\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| - \sum_{j \neq i_0} |b_{i_0,j}|\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| \\ &= \sum_{j=1}^{2N-1} b_{i_0,j}\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| \\ &= M_{i_0}\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| \end{aligned} \tag{4.26}$$

By Theorem 2.5 and Theorem 4.4,

We kown that there exists constants $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$, and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that

$$\left|\frac{\epsilon_i}{\lambda + g(x_i)}\right| \leq \left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| \leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} + C_2(r-1)h^2 \tag{4.27}$$

as $\lambda + g(x_i) \leq \lambda + T$

So, we can get

$$|\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{r\alpha}{2},2\}} \tag{4.28}$$

The convergency has been proved.

Remarks:

5. Experimental results.

5.1. $f \equiv 1$.

5.2. $f = x^\gamma, \gamma < 0$. Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x) \in C^2(\Omega)$, there exists $\xi \in (x_{i-1}, x_{i+1})$ such that*

$$D_h^2 g(x_i) = g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{A.1}$$

And if $g(x) \in C^4(\Omega)$, then

$$\begin{aligned} D_h^2 g(x_i) &= g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) \\ &+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right) \end{aligned} \tag{A.2}$$

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})$$

Subsitute them in the left side of (A.1), we have

$$\begin{aligned} D_h^2 g(x_i) &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (g(x_{i+1}) - g(x_i) + \frac{1}{h_i} (g(x_{i-1}) - g(x_i)) \right) \\ &= \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) \end{aligned}$$

Now, using intermediate value theorem, there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy$$

Expecially,

$$\begin{aligned} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy &= \frac{h_i^4}{4!} g''''(\eta_1) \\ \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy &= \frac{h_{i+1}^4}{4!} g''''(\eta_2) \end{aligned} \tag{A.3}$$

where $\eta_1 \in (x_{i-1}, x_i), \eta_2 \in (x_i, x_{i+1})$. ∎

LEMMA A.2. *Denote* $y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \theta \in (0,1)$,

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in (x_{j-1}, x_j) \tag{A.4}$$

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3(\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \tag{A.5}$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in (x_{j-1}, y_j^\theta)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in (y_j^\theta, x_j)$$

Thus

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta) u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$. Thus

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3(\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

∎

LEMMA A.3. *For* $x \in [x_{j-1}, x_j]$

$$\begin{aligned} |u(x) - \Pi_h u(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_x^{x_j} u'(y)dy \right| \\ &\le \int_{x_{j-1}}^{x_j} |u'(y)|dy \end{aligned} \tag{A.6}$$

If $x \in [0, x_1]$, *with Corollary* 2.4, *we have*

$$|u(x) - \Pi_h u(x)| \le \int_0^{x_1} |u'(y)|dy \le \int_0^{x_1} C y^{\alpha/2-1} dy \le C\frac{2}{\alpha} x_1^{\alpha/2} = C\frac{2}{\alpha} h_1^{\alpha/2} \tag{A.7}$$

Similarly, if $x \in [x_{2N-1}, 1]$, *we have*

$$|u(x) - \Pi_h u(x)| \le C\frac{2}{\alpha}(2T - x_{2N-1})^{\alpha/2} = C\frac{2}{\alpha} h_{2N}^{\alpha/2} \tag{A.8}$$

LEMMA A.4. *By Lemma* A.2*, Corollary* 2.4 *and Lemma* B.1 *, There is a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *for* $2 \le j \le N$*,*

$$|u(y) - \Pi_h u(y)| \le h_j^2 \max_{\xi \in [x_{j-1}, x_j]} |u''(\xi)| \le C h^2 y^{\alpha/2-2/r}, \quad \textit{for } y \in (x_{j-1}, x_j) \tag{A.9}$$

symmetricly, for $N < j \le 2N-1$*, we have*

$$|u(y) - \Pi_h u(y)| \le C h^2 (2T - y)^{\alpha/2-2/r}, \quad \textit{for } y \in (x_{j-1}, x_j) \tag{A.10}$$

LEMMA A.5.

$$b^{1-\theta}|a^\theta - b^\theta| \le |a-b| \ (\textit{ also } a^{1-\theta}|a^\theta - b^\theta| \le |a-b|), \quad a, b \ge 0, \ \theta \in [0,1] \tag{A.11}$$

Appendix B. Proofs of some technical details. Review that $h = \frac{1}{N}$ and the defination of $\simeq$ in subsection 2.1

LEMMA B.1.

$$h_i \simeq \begin{cases} h x_i^{1-1/r}, & 1 \le i \le N \\ h(2T - x_{i-1})^{1-1/r}, & N < i \le 2N \end{cases} \tag{B.1}$$

Since $i^r - (i-1)^r \simeq i^{r-1}$*, for* $i \ge 1$*.*

And

$$h_i \simeq h_{i+1}, \quad x_i \simeq x_{i+1} \simeq y_i^\theta, \quad \textit{for } 1 \le i \le 2N-1, \ \theta \in (0,1) \tag{B.2}$$

LEMMA B.2. *There is a constant* C *such that for* $i = 1, 2, \cdots, 2N-1$

$$|h_{i+1} - h_i| \le C h^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N-1 \end{cases} \tag{B.3}$$

Proof. By (2.2),

(B.4)

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \le i \le N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \le i \le 2N-1 \end{cases}$$

Since

$$(i+1)^r - 2i^r + (i-1)^r \simeq r(r-1)i^{r-2}, \quad \text{for } i \ge 1 \tag{B.5}$$

We get the result. ∎

LEMMA B.3. *there is a constant* $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ *such that*

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right| \\ &\le C h^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases} \end{aligned} \tag{B.6}$$

Proof. By Lemma 2.2, we have for $1 \le i \le N$

$$\left|\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy\right| \le \frac{\|f\|_\beta^{(\alpha/2)}}{3!}\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y-x_{i-1})^3dy \tag{B.7}$$

For $i=1$,

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y-x_{i-1})^3dy = \int_0^{x_1} y^{1-\alpha/2}dy = \frac{1}{2-\alpha/2}x_1^{2-\alpha/2} = \frac{1}{2-\alpha/2}x_1^{-\alpha/2-2}h_1^4$$

And for $2 \le i \le N$, since $x_i \simeq x_{i-1} \le y \le x_i$, we have

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y-x_{i-1})^3dy \simeq \int_{x_{i-1}}^{x_i} x_i^{-\alpha/2-2}(y-x_{i-1})^3dy = \frac{1}{4!}x_i^{-\alpha/2-2}h_i^4$$

So for $1 \le i \le N$, we have

$$\left|\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy\right| \le Cx_i^{-\alpha/2-2}h_i^4 \tag{B.8}$$

and similarly,

$$\left|\int_{x_i}^{x_{i+1}} f''(y)\frac{(x_{i+1}-y)^3}{3!}dy\right| \le Cx_i^{-\alpha/2-2}h_{i+1}^4 \tag{B.9}$$

Thus for $1 \le i \le N$, with Lemma B.1 we have

$$\begin{aligned}&\frac{2}{h_i+h_{i+1}}\left|\frac{1}{h_i}\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy + \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}} f''(y)\frac{(y-x_{i+1})^3}{3!}dy\right| \\ &\le Cx_i^{-\alpha/2-2}\frac{2}{h_i+h_{i+1}}(h_i^3+h_{i+1}^3) \simeq x_i^{-\alpha/2-2}h_i^2 \simeq x_i^{-\alpha/2-2}h^2x_i^{2-2/r} \\ &= Ch^2x_i^{-\alpha/2-2/r}\end{aligned} \tag{B.10}$$

It's symmetric for $N < i \le 2N-1$. ∎

LEMMA B.4. *There is a constant $C = C(\alpha, r)$ such that for all $1 \le i \le 2N-1$, $1 \le j \le 2N$ s.t. $\min\{|j-i|, |j-1-i|\} \ge 2$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_h^2K_y(x_i) \simeq |y-x_i|^{-1-\alpha} \tag{B.11}$$

Proof. Sinec $y-x_{i-1}, y-x_i, y-x_{i+1}$ have the same sign, by Lemma A.1,

$$D_h^2K_y(x_i) = \frac{|y-\xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in (x_{i-1}, x_{i+1})$$

however, $|y-\xi| \simeq |y-x_i|$, we get the result. ∎

LEMMA B.5. *While $0 \le i < N/2$, By Lemma* A.3

$$\begin{aligned}|T_{i1}| &\le C\int_0^{x_1} x_1^{\alpha/2}\frac{|x_i-y|^{1-\alpha}}{\Gamma(2-\alpha)}dy \\ &= C\frac{1}{\Gamma(3-\alpha)}x_1^{\alpha/2}\left|x_i^{2-\alpha} - |x_i-x_1|^{2-\alpha}\right| \\ &\le C\frac{1}{\Gamma(3-\alpha)}x_1^{\alpha/2+2-\alpha} = C\frac{1}{\Gamma(3-\alpha)}x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1\end{aligned} \tag{B.12}$$

For $2 \le j \le N$, by Lemma A.2 and Corollary 2.4

$$|T_{ij}| \le \frac{C}{4}\int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2}\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy \le \frac{C}{4\Gamma(3-\alpha)}h_j^2 x_{j-1}^{\alpha/2-2}\left||x_j-x_i|^{2-\alpha}-|x_{j-1}-x_i|^{2-\alpha}\right| \tag{B.13}$$

Lemma B.6. *There exists a constant $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$\sum_{j=1}^{3} V_{1j} \le Ch^2 x_1^{-\alpha/2-2/r} \tag{B.14}$$

$$\sum_{j=1}^{4} V_{2j} \le Ch^2 x_2^{-\alpha/2-2/r} \tag{B.15}$$

Proof. For $0 \le i \le 3, 1 \le j \le 4$, by Lemma A.3, Lemma A.4 and (3.11)

$$T_{ij} \le Cx_1^{2-\alpha/2} \simeq h_1^2\, h^2 x_1^{-\alpha/2-2/r} \simeq h_1^2\, h^2 x_2^{-\alpha/2-2/r} \tag{B.16}$$

Therefore, by (3.12), we get the result. ∎

Lemma B.7. *There exists a constant $C = C(T,r,l)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2}\rceil \le j \le \min\{2i, N\}$,*
when $\xi \in (x_{i-1}, x_{i+1})$,

$$(h_{j-i}^3(\xi))' \le (r-1)Ch^2 x_i^{1-2/r} h_j \tag{B.17}$$

$$(h_{j-i}^4(\xi))' \le (r-1)Ch^2 x_i^{1-2/r} h_j^2 \tag{B.18}$$

Proof. From (**??**)

$$y_{j-i}'(x) = y_{j-i}^{1-1/r}(x)x^{1/r-1} \tag{B.19}$$

$$y_{j-i}''(x) = \frac{1-r}{r}y_{j-i}^{1-2/r}(x)x^{1/r-2}Z_{j-i} \tag{B.20}$$

For $\xi \in (x_{i-1}, x_{i+1})$ and $2 \le k \le j \le \min\{2i-1, N-1\}$, using Lemma B.1

$$\xi \simeq x_i \simeq x_j$$

$$h_{j-i}(\xi) \simeq h_j \simeq hx_j^{1-1/r} \simeq hx_i^{1-1/r}$$

$$h_{j-i}'(\xi) = y_{j-i}'(\xi) - y_{j-i-1}'(\xi) = \xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \tag{B.21}$$

Since

$$\begin{aligned} y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) &\le x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r} \\ &= T^{1-1/r} N^{1-r}((j+1)^{r-1} - (j-2)^{r-1}) \\ &\le C(r-1) j^{r-2} N^{1-r} \\ &= C(r-1) h x_j^{1-2/r} \end{aligned} \tag{B.22}$$

Therefore,

$$h'_{j-i}(\xi) \le C x_i^{1/r-1}(r-1) h x_j^{1-2/r} \simeq (r-1) h x_i^{-1/r} \tag{B.23}$$

for $l = 3, 4$

$$\begin{aligned} (h_{j-i}^l(\xi))' &= l h_{j-i}^{l-1}(\xi) h'_{j-i}(\xi) \\ &\le C h_{j-i}^{l-1}(\xi)(r-1) h x_i^{-1/r} \\ &\simeq C h_j^{l-2} h x_j^{1-1/r}(r-1) h x_i^{-1/r} \\ &\simeq C(r-1) h^2 x_i^{1-2/r} h_j^{l-2} \end{aligned} \tag{B.24}$$

Meanwhile, we can get

$$h_{j-i}^3(\xi) \simeq h_j^3 \le C h^2 x_i^{2-2/r} h_j \tag{B.25}$$

$$h_{j-i}^4(\xi) \simeq h_j^4 \le C h^2 x_i^{2-2/r} h_j^2 \tag{B.26}$$

□

Lemma B.8. *There exists a constant $C = C(T, r, l)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil \le j \le \min\{2i, N\}$, when $\xi \in (x_{i-1}, x_{i+1})$,*

$$(h_{j-i}^3(\xi))'' \le C(r-1) h^2 x_i^{-2/r} h_j \tag{B.27}$$

Proof.

$$(h_{j-i}^3(\xi))'' = 6 h_{j-i}(\xi)(h'_{j-i}(\xi))^2 + 3 h_{j-i}^2(\xi) h''_{j-i}(\xi) \tag{B.28}$$

By (B.23)

$$h_{j-i}(\xi)(h'_{j-i}(\xi))^2 \le C h_j (r-1)^2 h^2 x_i^{-2/r} \tag{B.29}$$

For the second partial

$$\begin{aligned} h''_{j-i}(\xi) &= y''_{j-i}(\xi) - y''_{j-i-1}(\xi) \\ &= \frac{1-r}{r} \xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi) Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi) Z_{j-i-1}) \\ &= \frac{1-r}{r} \xi^{1/r-2}((y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi)) Z_{j-i} + y_{j-i-1}^{1-2/r}(\xi) Z_1) \end{aligned} \tag{B.30}$$

but

$$\begin{aligned} |y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi)| &\le |x_{j+1}^{1-2/r} - x_{j-2}^{1-2/r}| \\ &= T^{1-2/r} N^{2-r} |(j+1)^{r-2} - (j-2)^{r-2}| \\ &\le C|r-2| N^{2-r} j^{r-3} \\ &= C|r-2| h x_j^{1-3/r} \end{aligned} \tag{B.31}$$

So we can get

$$\begin{aligned} |h''_{j-i}(\xi)| &\le C(r-1)x_i^{1/r-2}(|r-2|hx_i^{1-3/r}x_i^{1/r}+x_i^{1-2/r}h) \\ &\le C(r-1)hx_i^{-1-1/r} \end{aligned} \tag{B.32}$$

Summarizes, we have

$$(h^3_{j-i}(\xi))'' \le C(r-1)h^2x_i^{-2/r}h_j \tag{B.33}$$ □

proof of Lemma 3.17. From (**??**)

$$y'_{j-i}(x) = y_{j-i}^{1-1/r}(x)x^{1/r-1} \tag{B.34}$$

$$y''_{j-i}(x) = \frac{1-r}{r}y_{j-i}^{1-2/r}(x)x^{1/r-2}Z_{j-i} \tag{B.35}$$

Since

$$y^\theta_{j-i}(\xi) \simeq x_j \simeq x_i$$

We have known

$$u''(y^\theta_{j-i}(\xi)) \le C(y^\theta_{j-i}(\xi))^{\alpha/2-2} \simeq x_j^{\alpha/2-2} \simeq x_i^{\alpha/2-2} \tag{B.36}$$

$$\begin{aligned} (u''(y^\theta_{j-i}(\xi)))' &= u'''(y^\theta_{j-i}(\xi))(y^\theta_{j-i}(\xi))' \\ &\le Cx_i^{\alpha/2-3}\xi^{1/r-1}y_{j-i}^{1-1/r}(\xi) \\ &\simeq x_i^{\alpha/2-3}x_i^{1/r-1}x_i^{1-1/r} = Cx_i^{\alpha/2-3} \end{aligned} \tag{B.37}$$

$$\begin{aligned} (u''(y^\theta_{j-i}(\xi)))'' &= u''''(y^\theta_{j-i}(\xi))({y^\theta_{j-i}}'(\xi))^2 + u'''(y^\theta_{j-i}(\xi)){y^\theta_{j-i}}''(\xi) \\ &\le Cx_i^{\alpha/2-4} + Cx_i^{\alpha/2-3}\frac{r-1}{r}x_i^{1-2/r}x_i^{1/r-2}Z_{|j-i|+1} \\ &\le Cx_i^{\alpha/2-4} + C\frac{r-1}{r}x_i^{\alpha/2-3}x_i^{-1-1/r}x_i^{1/r} \\ &= Cx_i^{\alpha/2-4} \end{aligned} \tag{B.38}$$ □

Proof of Lemma 3.18.

$$\begin{aligned} |y^\theta_{j-i}(\xi)-\xi| &= |\theta(y_{j-i-1}(\xi)-\xi)+(1-\theta)(y_{j-i}(\xi)-\xi)| \\ &= \theta|y_{j-i-1}(\xi)-\xi| + (1-\theta)|y_{j-i}(\xi)-\xi| \end{aligned} \tag{B.39}$$

where $y_{j-i-1}(\xi)-\xi$ and $y_{j-i}(\xi)-\xi$ have the same sign (≥ 0 or ≤ 0), independent with ξ.

Since $|y_{j-i}(\xi)-\xi| = \mathrm{sign}(j-i)(y_{j-i}(\xi)-\xi)$ is increasing with ξ,

$$(\frac{i-1}{i})^r|x_j-x_i| \le |x_{j-1}-x_{i-1}| \le |y_{j-i}(\xi)-\xi| \le |x_{j+1}-x_{i+1}| \le (\frac{i+1}{i})^r|x_j-x_i| \tag{B.40}$$

we have

$$|y_{j-i}(\xi)-\xi| \simeq |x_j-x_i| \tag{B.41}$$

Similarly, $|y_{j-1-i}(\xi)-\xi| \simeq |x_{j-1}-x_i|$. Thus, with (B.39) , (B.41) and (2.17) we get

$$|y^{\theta}_{j-i}(\xi)-\xi| \simeq |y^{\theta}_j - x_i| \tag{B.42}$$

Next, since $|y^{\theta}_{j-i}(\xi)-\xi| = \mathrm{sign}(j-i-1+\theta)(y^{\theta}_{j-i}(\xi)-\xi)$, so we can derivate it.

$$|(|y^{\theta}_{j-i}(\xi)-\xi|^{1-\alpha})'| = (\alpha-1)|y^{\theta}_{j-i}(\xi)-\xi|^{-\alpha}|(y^{\theta}_{j-i}(\xi))'-1| \tag{B.43}$$

While, similar with (B.39), we have

$$|(y^{\theta}_{j-i}(\xi))'-1| = (1-\theta)|y'_{j-i-1}(\xi)-1| + \theta|y'_{j-i}(\xi)-1| \tag{B.44}$$

By Lemma A.5 and (B.41), we have

$$\begin{aligned} |y'_{j-i}(\xi)-1| &= \xi^{1/r-1}|y^{1-1/r}_{j-i}(\xi)-\xi^{1-1/r}| \\ &\le \xi^{-1}|y_{j-i}(\xi)-\xi| \\ &\simeq x_i^{-1}|x_j-x_i| \end{aligned} \tag{B.45}$$

So similar with (B.42), we can get

$$|(y^{\theta}_{j-i}(\xi))'-1| \le Cx_i^{-1}|y^{\theta}_j - x_i| \tag{B.46}$$

Combine with (B.42), we get

$$|(|y^{\theta}_{j-i}(\xi)-\xi|^{1-\alpha})'| \le C|y^{\theta}_j-x_i|^{-\alpha}x_i^{-1}|y^{\theta}_j-x_i| = C|y^{\theta}_j-x_i|^{1-\alpha}x_i^{-1} \tag{B.47}$$

Finally, we have

$$\begin{aligned} (|y^{\theta}_{j-i}(\xi)-\xi|^{1-\alpha})'' = \alpha(\alpha-1)|y^{\theta}_{j-i}(\xi)-\xi|^{-\alpha-1}((y^{\theta}_{j-i}(\xi))'-1)^2 \\ + \mathrm{sign}(j-i-1+\theta)(1-\alpha)|y^{\theta}_{j-i}(\xi)-\xi|^{-\alpha}(y^{\theta}_{j-i}(\xi))'' \end{aligned} \tag{B.48}$$

For

$$(y^{\theta}_{j-i}(\xi))'' = (1-\theta)y''_{j-i-1}(\xi) + \theta y''_{j-i}(\xi) \tag{B.49}$$

and

$$\begin{aligned} y''_{j-i}(\xi) &= \frac{1-r}{r}y^{1-2/r}_{j-i}(x)x^{1/r-2}Z_{j-i} \\ &\simeq \frac{1-r}{r}x_j^{1-2/r}x_i^{1/r-2}Z_{j-i} \end{aligned} \tag{B.50}$$

while by Lemma A.5

$$|Z_{j-i}| = |x_j^{1/r}-x_i^{1/r}| \le |x_j-x_i|x_i^{1/r-1} \tag{B.51}$$

we have

$$|y''_{j-i}(\xi)| \le C(r-1)x_i^{-2}|x_j-x_i| \tag{B.52}$$

Therefore

$$|(y^{\theta}_{j-i}(\xi))''| \le C(r-1)x_i^{-2}|y^{\theta}_j-x_i| \tag{B.53}$$

Then, combine with (B.46),

$$|(|y^{\theta}_{j-i}(\xi)-\xi|^{1-\alpha})''| \le C|y^{\theta}_j-x_i|^{1-\alpha}x_i^{-2} \tag{B.54}$$

□

proof of Lemma 3.20. For $\lceil \frac{i}{2} \rceil \leq j \leq \min\{2i-1, N-1\}$

$$\begin{aligned}
&\frac{Q^\theta_{i,j}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{i,j}(x_i)u'''(\eta^\theta_j)}{h_{i+1}} \\
&= \frac{Q^\theta_{i,j}(x_{i+1}) - Q^\theta_{i,j}(x_i)}{h_{i+1}} u'''(\eta^\theta_{j+1}) + Q^\theta_{i,j}(x_i) \frac{u'''(\eta^\theta_{j+1}) - u'''(\eta^\theta_j)}{h_{i+1}}
\end{aligned} \tag{B.55}$$

Since mean value theorem

$$\frac{Q^\theta_{i,j}(x_{i+1}) - Q^\theta_{i,j}(x_i)}{h_{i+1}} = {Q^\theta_{i,j}}'(\xi), \quad \xi \in (x_i, x_{i+1}) \tag{B.56}$$

From (3.21) and Leibniz rule, by Lemma B.7 and Lemma 3.18, we have

$$|{Q^\theta_{i,j}}'(\xi)| \leq Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 \tag{B.57}$$

And by **??** and Lemma B.1

$$Q^\theta_{i,j}(x_i) = h_j^4 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \simeq Ch^2 x_i^{2-2/r} \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 \tag{B.58}$$

With $\eta^\theta_j \in (x_{j-1}, x_j)$

$$u'''(\eta^\theta_{j+1}) \leq C(\eta^\theta_{j+1})^{\alpha/2-3} \simeq x_j^{\alpha/2-3} \simeq x_i^{\alpha/2-3}$$

and

$$\begin{aligned}
\frac{u'''(\eta^\theta_{j+1}) - u'''(\eta^\theta_j)}{h_{i+1}} &= u''''(\eta) \frac{\eta^\theta_{j+1} - \eta^\theta_j}{h_{i+1}} \\
&\leq C\eta^{\alpha/2-4} \frac{x_{j+1} - x_{j-1}}{h_{i+1}} = C\eta^{\alpha/2-4} \frac{h_{j+1} + h_j}{h_{i+1}} \\
&\simeq x_j^{\alpha/2-4} \simeq x_i^{\alpha/2-4}
\end{aligned}$$

So we have

$$\begin{aligned}
&\frac{Q^\theta_{i,j}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{i,j}(x_i)u'''(\eta^\theta_j)}{h_{i+1}} \\
&\leq Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 \, x_i^{\alpha/2-3} + Ch^2 x_i^{2-2/r} \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 x_{j-1}^{\alpha/2-4} \\
&= Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j^2
\end{aligned} \tag{B.59}$$

while $h_j \simeq h_i$, subsitute into the inequality above, we get the goal

$$\begin{aligned}
&\frac{2}{h_i + h_{i+1}} \left(\frac{Q^\theta_{i,j}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{i,j}(x_i)u'''(\eta^\theta_j)}{h_{i+1}} \right) \\
&\leq \frac{1}{h_i} Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j h_i \\
&= Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j
\end{aligned} \tag{B.60}$$

While, the later is similar. ∎

LEMMA B.9. *There exists a constant $C = C(T, r)$ such that For $N/2 \le i \le N-1$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $l = 3, 4$, $\xi \in (x_{i-1}, x_{i+1})$, we have*

$$h^l_{j-i}(\xi) \le Ch^l_j \le Ch^2 h^{l-2}_j \tag{B.61}$$

$$(h^l_{j-i-1}(\xi))' \le C(r-1)h^2 h^{l-2}_j \tag{B.62}$$

$$(h^3_{j-i}(\xi))'' \le C(r-1)h^2 h_j \tag{B.63}$$

Proof.

$$\begin{aligned}(h_{j-i}(\xi))' &= {y_{j-i}}'(\xi) - {y_{j-i-1}}'(\xi) \\ &= \xi^{1/r-1}((2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}) \le 0\end{aligned} \tag{B.64}$$

Thus,

$$Ch_j \le h_{j+1} \le h_{j-i}(\xi) \le h_{j-i}(x_{i-1}) = h_{j-1} \le Ch_j \tag{B.65}$$

So as $4^{-r}T \le 2T - x_j \le T, 2^{-r}T \le x_i \le T$, we have

$$h^l_{j-i}(\xi) \le Ch^l_j \le Ch^2(2T - x_j)^{2-2/r} h^{l-2}_j \le Ch^2 h^{l-2}_j \tag{B.66}$$

Since

$$\begin{aligned}&|(2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}| \\ &\quad = |(Z_{2N-(j-i)} - \xi^{1/r})^{r-1} - (Z_{2N-(j-1-i)} - \xi^{1/r})^{r-1}| \\ &\quad = (r-1)Z_1(Z_{2N-(j-i-\gamma)} - \xi^{1/r})^{r-2} \quad \gamma \in [0, 1] \\ &\quad \le C(r-1)h(2T - x_j)^{1-2/r}\end{aligned} \tag{B.67}$$

we have

$$|(h_{j-i}(\xi))'| \le C(r-1)h(2T - x_j)^{1-2/r} x_i^{1/r-1} \tag{B.68}$$

And

$$\begin{aligned}(h^l_{j-i}(\xi))' &= lh^{l-1}_{j-i}(\xi){h_{j-i}}'(\xi) \\ &\le C(r-1)h^{l-1}_j\ h(2T - x_j)^{1-2/r} x_i^{1/r-1} \\ &\le C(r-1)h^2 h^{l-2}_j (2T - x_j)^{2-3/r} x_i^{1-1/r} \\ &\le C(r-1)h^2 h^{l-2}_j\end{aligned} \tag{B.69}$$

∎

(B.70)

$$\begin{aligned}(h^3_{j-i}(\xi))'' &= 6h_{j-i}(\xi)({y_{j-i}}'(\xi) - {y_{j-i-1}}'(\xi))^2 + 3h^2_{j-i}(\xi)({y_{j-i}}''(\xi) - {y_{j-i-1}}''(\xi)) \\ &\le C(r-1)h_j h^2 + Ch^2_j \frac{1-r}{r} \xi^{1/r-2}((2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-1-i)}) \\ &\le C(r-1)h_j h^2 + C(r-1)h^2_j (C(r-2)h(2T - x_j)^{1-3/r} Z_{2N-(j-i)} + Z_1(2T - x_{j-1})^{1-2/r}) \\ &\le C(r-1)h_j h^2 + C(r-1)h^2_j h = Ch^2 h_j\end{aligned}$$

Lemma B.10. *There exists a constant* $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le i \le N-1$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in (x_{i-1}, x_{i+1})$, *we have*

$$u''(y^{\theta}_{j-i}(\xi)) \le C \tag{B.71}$$

$$(u''(y^{\theta}_{j-i}(\xi)))' \le C \tag{B.72}$$

$$(u''(y^{\theta}_{j-i}(\xi)))'' \le C \tag{B.73}$$

Proof.

$$x_{j-2} \le y^{\theta}_{j-i}(\xi) \le x_{j+1} \Rightarrow 4^{-r}T \le 2T - y^{\theta}_{j-i}(\xi) \le T \tag{B.74}$$

Thus, for $l = 2,3,4$,

$$u^{(l)}(y^{\theta}_{j-i}(\xi)) \le C(2T - y^{\theta}_{j-i}(\xi))^{\alpha/2-l} \le C \tag{B.75}$$

and

$$\begin{aligned}(y^{\theta}_{j-i}(\xi))' &= \theta {y_{j-1-i}}'(\xi) + (1-\theta){y_{j-i-1}}'(\xi)\\ &= \xi^{1/r-1}(\theta(2T - y_{j-1-i}(\xi))^{1-1/r} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-1/r})\\ &\le C(2T - x_{j-2})^{1-1/r} \le C\end{aligned} \tag{B.76}$$

With

$$Z_{2N-j-i} \le 2T^{1/r} \tag{B.77}$$

$$\begin{aligned}(y^{\theta}_{j-i}(\xi))'' &= \theta {y_{j-1-i}}''(\xi) + (1-\theta){y_{j-i-1}}''(\xi)\\ &= \frac{1-r}{r}\xi^{1/r-2}(\theta(2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)} + (1-\theta)(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)})\\ &\le C(r-1)\end{aligned} \tag{B.78}$$

Therefore,

$$\begin{aligned}(u''(y^{\theta}_{j-i}(\xi)))' &= u'''(y^{\theta}_{j-i}(\xi))(y^{\theta}_{j-i}(\xi))'\\ &\le C\end{aligned} \tag{B.79}$$

$$\begin{aligned}(u''(y^{\theta}_{j-i}(\xi)))'' &= u'''(y^{\theta}_{j-i}(\xi))({y^{\theta}_{j-i}}'(\xi))^2 + u''''(y^{\theta}_{j-i}(\xi)){y^{\theta}_{j-i}}''(\xi)\\ &\le C + C(r-1) = C\end{aligned} \tag{B.80}$$

∎

Lemma B.11. *There exists a constant* $C = C(T,\alpha,r)$ *such that For* $N/2 \le i \le N-1$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in (x_{i-1}, x_{i+1})$

$$|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha} \le C|y^{\theta}_{j} - x_i|^{1-\alpha} \tag{B.81}$$

$$\left|(|y^{\theta}_{j-i}(\xi) - \xi)^{1-\alpha}|'\right| \le C|y^{\theta}_{j} - x_i|^{-\alpha}(|2T - x_i - y^{\theta}_{j}| + h_N) \tag{B.82}$$

$$\left|(|y^{\theta}_{j-i}(\xi) - \xi)^{1-\alpha}|''\right| \le C(r-1)|y^{\theta}_{j} - x_i|^{-\alpha} + C|y^{\theta}_{j} - x_i|^{-1-\alpha}(|2T - x_i - y^{\theta}_{j}| + h_N)^2 \tag{B.83}$$

Proof. Since $y_{j-i-1}(\xi) > x_{j-2} \geq x_N > \xi$

$$y^{\theta}_{j-i}(\xi) - \xi = (1-\theta)(y_{j-1-i}(\xi) - \xi) + \theta(y_{j-i}(\xi) - \xi) > 0 \tag{B.84}$$

$$\begin{aligned}(y_{j-i}(\xi) - \xi)'' &= {y_{j-i}}''(\xi) \\ &= \frac{1-r}{r}\xi^{1/r-2}(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} \leq 0\end{aligned} \tag{B.85}$$

It's concave, so

$$y_{j-i}(\xi) - \xi \geq \min_{\xi \in \{x_{i-1}, x_{i+1}\}} y_{j-i}(\xi) - \xi = \min\{x_{j+1} - x_{i+1}, x_{j-1} - x_{i-1}\} \geq C(x_j - x_i) \tag{B.86}$$

With (B.84), we have

$$|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha} \leq C|y^{\theta}_j - x_i|^{1-\alpha} \tag{B.87}$$

By Lemma A.5

$$\begin{aligned}|{y_{j-i}}'(\xi) - 1| &= \xi^{1/r-1}|(2T - y_{j-i}(\xi))^{1-1/r} - \xi^{1-1/r}| \\ &\leq \xi^{-1}|2T - y_{j-i}(\xi) - \xi|\end{aligned} \tag{B.88}$$

$$\begin{aligned}|2T - \xi - y_{j-i}(\xi)| &\leq |2T - x_i - x_j| + |x_i - \xi| + |x_j - y_{j-i}(\xi)| \\ &\leq |2T - x_i - x_j| + h_{i+1} + h_j \\ &\leq C\left(|2T - x_i - x_j| + h_N\right)\end{aligned} \tag{B.89}$$

With $\xi \simeq x_i \simeq 1$,

$$|{y_{j-i}}'(\xi) - 1| \leq C\left(|2T - x_i - x_j| + h_N\right) \tag{B.90}$$

Thus,

$$\begin{aligned}|(y^{\theta}_{j-i}(\xi))' - 1| &\leq (1-\theta)|{y_{j-i-1}}'(\xi) - 1| + \theta|{y_{j-i}}'(\xi) - 1| \\ &\leq C\left((1-\theta)|2T - x_i - x_{j-1}| + \theta|2T - x_i - x_j| + h_N\right) \\ &= C\left(|2T - x_i - y^{\theta}_j| + h_N\right)\end{aligned} \tag{B.91}$$

So

$$\begin{aligned}\left|(|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})'\right| &= |1-\alpha||y^{\theta}_{j-i}(\xi) - \xi|^{-\alpha}|(y^{\theta}_{j-i}(\xi))' - 1| \\ &\leq C|y^{\theta}_j - x_i|^{-\alpha}(|2T - x_i - y^{\theta}_j| + h_N)\end{aligned} \tag{B.92}$$

$$\begin{aligned}\left|(|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})''\right| &\leq |1-\alpha||y^{\theta}_{j-i}(\xi) - \xi|^{-\alpha}|(y^{\theta}_{j-i}(\xi) - \xi)''| + \alpha(\alpha-1)|y^{\theta}_{j-i}(\xi) - \xi|^{-1-\alpha}({y^{\theta}_{j-i}}'(\xi) - 1)^2 \\ &\leq C(r-1)|y^{\theta}_j - x_i|^{-\alpha} + C|y^{\theta}_j - x_i|^{-1-\alpha}(|2T - x_i - y^{\theta}_j| + h_N)^2\end{aligned} \tag{B.93}$$

∎

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