SECOND-ORDER ERROR ANALYSIS FOR FRACTIONAL LAPLACIAN VIA RIESZ DERIVATIVES ON GRADED MESHES[*]

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Key words. example, LATEX

MSC codes. ?????????????????

1. Introduction.
For $\Omega = (0, 2T)$, $1 < \alpha < 2$,

$$(1.1) \qquad \begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases}$$

where

$$(1.2) \qquad (-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy$$

$$(1.3) \qquad \kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0$$

2. Preliminaries: Numeric scheme and main results.

2.1. Numeric Format.

$$(2.1) \qquad x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases}$$

where $r \ge 1$. And let

$$(2.2) \qquad h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$(2.3) \qquad \phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

And then, define the piecewise linear interpolant of the true solution u to be

$$(2.4) \qquad \Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x)$$

For convience, we denote

(2.5)
$$I^{2-\alpha}u(x) := \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x-y|^{1-\alpha}u(y)dy$$

and

(2.6)
$$D_h^2 u(x_i) := \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_i}u(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) u(x_i) + \frac{1}{h_{i+1}}u(x_{i+1}) \right)$$

Now, we discretise (1.1) by replacing $u(x)$ by a continuous piecewise linear function

(2.7)
$$u_h(x) := \sum_{j=1}^{2N-1} u_j \phi_j(x)$$

whose nodal values u_j are to be determined by collocation at each mesh point x_i for $i = 1, 2, ..., 2N-1$:

(2.8)
$$-\kappa_\alpha D_h^\alpha u_h(x_i) := -\kappa_\alpha D_h^2 I^{2-\alpha}u_h(x_i) = f(x_i) =: f_i$$

Here,

(2.9)
$$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i)\, u_j = \sum_{j=1}^{2N-1} a_{ij}\, u_j$$

where

(2.10)
$$a_{ij} = -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i) \quad \text{for} \quad i,j = 1, 2, ..., 2N-1$$

We have replaced $(-\Delta)^{\alpha/2}u(x_i) = f(x_i)$ in (1.1) by $-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i)$ in (2.8), with truncation error

(2.11)
$$\tau_i := -\kappa_\alpha \left(D_h^\alpha \Pi_h u(x_i) - \frac{d^2}{dx^2}I^{2-\alpha}u(x_i) \right) \quad \text{for} \quad i = 1, 2, ..., 2N-1$$

where $-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^\alpha \phi_j(x_i)u(x_j) = \sum_{j=1}^{2N-1} a_{ij}u(x_j)$.

The discrete equation (2.8) can be written in matrix form

(2.12)
$$AU = F$$

where $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$, $U = (u_1, \cdots, u_{2N-1})^T$ is unknown and $F = (f_1, \cdots, f_{2N-1})^T$.

We can deduce a_{ij},

(2.13)
$$a_{ij} = -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i)$$
$$= -\kappa_\alpha \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_i}\tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}}\tilde{a}_{i+1,j} \right)$$

where

(2.14)
$$I^{2-\alpha}\Pi_h u(x_i) = \sum_{j=1}^{2N-1} I^{2-\alpha}\phi_j(x_i)u(x_j) = \sum_{j=1}^{2N-1} \tilde{a}_{ij}u(x_j)$$

and
(2.15)
$$\tilde{a}_{ij} = I^{2-\alpha}\phi_j(x_i)$$
$$= \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right)$$

2.2. Regularity of the true solution. For any $\beta > 0$, we use the standard notation $C^\beta(\bar{\Omega}), C^\beta(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^\beta(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous[1] with exponent β' in Ω, where $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on open set Ω.

DEFINITION 2.1 (delta dependent norm [2]).

(2.16) $$\delta(x) = dist(x, \partial\Omega) = \begin{cases} x, & 0 < x \le T \\ 2T - x, & T < x < 2T \end{cases}, \quad x \in \Omega$$

(2.17) $$\delta(x, y) = \min\{\delta(x), \delta(y)\}, \quad x, y \in \Omega$$

LEMMA 2.2. *Let* $f \in C^\beta(\Omega), \beta > 2$ *be such that* $\|f\|_\beta^{(\alpha/2)} < \infty$, *then for* $l = 0, 1, 2$

(2.18) $$|f^{(l)}(x)| \le \|f\|_\beta^{(\alpha/2)} \delta(x)^{-l-\alpha/2}$$

THEOREM 2.3 (Regularity up to the boundary [2]). *Let* Ω *be a bounded domain, and* $\beta > 0$ *be such that neither* β *nor* $\beta + \alpha$ *is an integer. Let* $f \in C^\beta(\Omega)$ *be such that* $\|f\|_\beta^{(\alpha/2)} < \infty$, *and* $u \in C^{\alpha/2}(\mathbb{R}^n)$ *be a solution of* (1.1). *Then,* $u \in C^{\beta+\alpha}(\Omega)$ *and*

(2.19) $$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right)$$

where C is a constant depending only on Ω, α, and β.

COROLLARY 2.4. *Let u be a solution of* (1.1) *where* $f \in L^\infty(\Omega)$ *and* $\|f\|_\beta^{(\alpha/2)} < \infty$. *Then, for any* $x \in \Omega$ *and* $l = 0, 1, 2, 3, 4$

(2.20) $$|u^{(l)}(x)| \le \|u\|_{\beta+\alpha}^{(-\alpha/2)} \delta(x)^{\alpha/2-l}$$

And in this paper bellow, without special instructions, we allways assume that

(2.21) $$f \in L^\infty(\Omega) \cap C^\beta(\Omega) \quad \text{and} \quad \|f\|_\beta^{(\alpha/2)} < \infty, \text{ with } \alpha + \beta > 4$$

2.3. Main results. Here we state our main results; the proof is deferred to section 3 and section 4.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 2.5 (Local Truncation Error). *If $u(x)$ is a solution of the equation* (1.1) *where f satisfy the regular condition* (2.21), *then there exists* $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$ *and* $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$, *such that the truncation error* (2.11) *satisfies*

(2.22) $$\begin{aligned} |\tau_i| :=& | -\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i)| \\ \le& C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} \delta(x_i)^{-\alpha} + C_2(r-1)h^2(T - \delta(x_i) + h_N)^{1-\alpha} \end{aligned}$$

THEOREM 2.6 (Global Error). *The discrete equation (2.8) has sulotion and there exists a positive constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ such that the error between the numerial solution U with the exact solution $u(x_i)$ satisfies*

$$(2.23) \qquad \max_{1 \le i \le 2N-1} |u_i - u(x_i)| \le Ch^{\min\{\frac{r\alpha}{2}, 2\}}$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

Remark 2.7. ...

3. Local Truncation Error. We shall first introduce some notations.

For convenience, we use the notation $\simeq$. That $x_1 \simeq y_1$, means that $c_1 x_1 \le y_1 \le C_1 x_1$ for some positive constants c_1 and C_1 that are independent of N.

And for $1 \le j \le 2N$, we define

$$(3.1) \qquad y_j^\theta = (1 - \theta)x_{j-1} + \theta x_j, \quad \theta \in (0, 1)$$

Then we have

LEMMA 3.1. *For $1 \le i \le 2N - 1$*

$$(3.2) \qquad h_i \simeq h_{i+1} \simeq h\delta(x_i)^{1-1/r}, \quad \delta(x_i) \simeq \delta(x_{i+1}) \simeq \delta(y_{i+1}^\theta)$$

Since $i^r - (i-1)^r \simeq i^{r-1}$, for $i \ge 1$, where $\theta \in (0, 1)$.

Meanwhile, let's define kernel functions

$$(3.3) \qquad K_y(x) := \frac{|y - x|^{1-\alpha}}{\Gamma(2 - \alpha)}$$

3.1. Proof of Theorem 2.5. The truncation error of the discrete format can be written as

$$(3.4)$$
$$-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i) = -\kappa_\alpha \left(D_h^2 I^{2-\alpha} \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i)\right)$$
$$= -\kappa_\alpha D_h^2 I^{2-\alpha}(\Pi_h u - u)(x_i) - \kappa_\alpha \left(D_h^2 - \frac{d^2}{dx^2}\right) I^{2-\alpha} u(x_i)$$

THEOREM 3.2. *There exits a constant $C = C(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$ such that*

$$(3.5) \qquad \left| -\kappa_\alpha \left(D_h^2 - \frac{d^2}{dx^2}\right) I^{2-\alpha} u(x_i) \right| \le Ch^2 \delta(x_i)^{-\alpha/2 - 2/r}$$

Proof. Since $f \in C^2(\Omega)$ and

$$(3.6) \qquad \frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha} u(x)) = f(x), \quad x \in \Omega,$$

we have $I^{2-\alpha} u \in C^4(\Omega)$. Therefore, using equation (A.2) of Lemma A.1, for $1 \le i \le 2N - 1$, we have

$$(3.7)$$
$$-\kappa_\alpha \left(D_h^2 - \frac{d^2}{dx^2}\right) I^{2-\alpha} u(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i)$$
$$+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right)$$

117 By Lemma B.1, Lemma 2.2 and Lemma B.2, we get the result. □

118   And now define

119 (3.8) $$R_i := D_h^2 I^{2-\alpha}(u - \Pi_h u)(x_i), \quad 1 \le i \le 2N-1$$

120   We have some results about the estimate of R_i

121   THEOREM 3.3. *For $1 \le i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

122 (3.9) $$|R_i| \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

123

124   THEOREM 3.4. *For $N/2 \le i \le N$, there exists constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$*
125 *such that*

126 (3.10) $$|R_i| \le C(r-1)h^2(T - x_i + h_N)^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

127   And for $N < i \le 2N - 1$, it is symmetric to the previous case.
128   Combine Theorem 3.2, Theorem 3.3 and Theorem 3.4, and for $1 \le i \le N$, we
129 have

130 (3.11) $$h^2 x_i^{-\alpha/2-2/r} \le T^{\alpha/2-2/r} h^{\min\{\frac{r\alpha}{2}, 2\}} x_i^{-\alpha}$$

131 (3.12) $$h^{r\alpha/2+r} x_i^{-1-\alpha} \le T^{-1} h^{r\alpha/2} x_i^{-\alpha}$$

132 (3.13) $$h^r x_i^{-1} \ln(i) = T^{-1} \frac{\ln(i)}{i^r} \le T^{-1}, \quad h^r \ln(N) = \frac{\ln(N)}{N^r} \le 1$$

133 the proof of Theorem 2.5 completed.
134   We prove Theorem 3.3 and Theorem 3.4 in next subsections.

135   **3.2. Outlines and Mesh Transport Functions.** For convience, let's denote
  DEFINITION 3.5.

136 (3.14) $$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy, \quad i = 0, \cdots, 2N, \ j = 1, \cdots, 2N$$

137 *Also, we denote vertical difference quotients of T_{ij}*

138 (3.15) $$V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right)$$
$$= \int_{x_{i-1}}^{x_i} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy$$

139 *And skew difference quotients of T_{ij}*

140 (3.16) $$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j-1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j+1} \right)$$

then $R_i = \sum_{j=1}^{2N} V_{ij}$.

Our main idea is to depart R_i by V_{ij} and S_{ij}. For $3 \leq i < N/2$, let's denote $k = \lceil \frac{i}{2} \rceil$, and take some suitable integer m, then

$$
\begin{aligned}
R_i &= \sum_{j=1}^{2N} V_{ij} \\
&= \sum_{j=1}^{k-1} V_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}}(T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,k} \right) \\
&\quad + \sum_{j=k+1}^{m-1} S_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,m} + T_{i-1,m-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,m} \right) \\
&\quad + \sum_{j=m+1}^{2N} V_{ij} \\
&= I_1 + I_2 + I_3 + I_4 + I_5
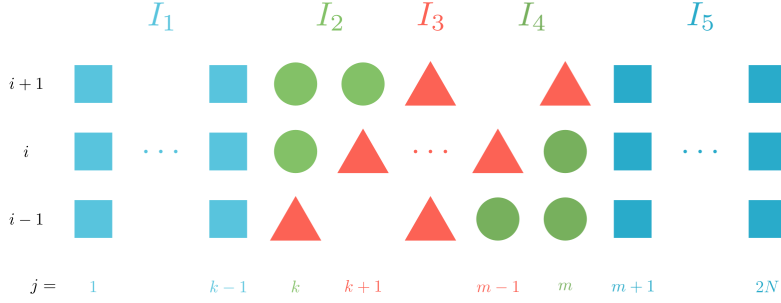\end{aligned}
$$
(3.17)

FIG. 1. *The departure of R_i for $i \geq 3$*

and discuss $i = 1, 2$ separately, where

$$
R_1 = \sum_{i=1}^{3} V_{1,j} + \sum_{i=4}^{N} V_{i,j}, \quad R_2 = \sum_{i=1}^{4} V_{1,j} + \sum_{i=5}^{N} V_{i,j}
$$
(3.18)

The difficulty for esitmating S_{ij} is that $T_{i-1,j-1}, T_{i,j}$ and $T_{i+1,j+1}$ have different integral region. We first make them normalized.

LEMMA 3.6. *For $y \in (x_{j-1}, x_j)$, we can rewrite $y = y_j^\theta$, from (3.14), and Lemma A.2,*

$$
\begin{aligned}
T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
&= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\
&= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
&\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta
\end{aligned}
$$
(3.19)

where $\eta_{j1}^\theta \in (x_{j-1}, y_j^\theta), \eta_{j2}^\theta \in (y_j^\theta, x_j)$.

152 Since j changes with i at indices of elements in S_{ij} by (3.16), we create some
153 functions satisfy the property.

155 DEFINITION 3.7 (Mesh Transport Functions). *For $1 \leq i, j \leq 2N - 1$.*

156 (3.20)
$$y_{i,j}(x) = \begin{cases} (x^{1/r} + Z_{j-i})^r & i < N, j < N \\ \dfrac{x^{1/r} - Z_i}{Z_1} h_N + x_N & i < N, j = N \\ 2T - (Z_{2N-(j-i)} - x^{1/r})^r & i < N, j > N \\ \left(\dfrac{Z_1}{h_N}(x - x_N) + Z_j\right)^r & i = N, j < N \\ x, & i = N, j = N \\ 2T - \left(\dfrac{Z_1}{h_N}(2T - x - x_N) + Z_{2N-j}\right)^r & i = N, j > N \\ (Z_{2N+j-i} - (2T - x)^{1/r})^r & i > N, j < N \\ \dfrac{Z_{2N-j} - (2T - x)^{1/r}}{Z_1} h_N + x_N & i > N, j = N \\ 2T - ((2T - x)^{1/r} - Z_{j-i})^r & i > N, j > N \end{cases}$$

157 *where $Z_j := T^{1/r} \frac{j}{N}$. And*

158 (3.21)
$$h_{i,j}(x) = y_{i,j}(x) - y_{i,j-1}(x)$$

160 (3.22)
$$y_{i,j}^{\theta}(x) = (1 - \theta)y_{i,j-1}(x) + \theta y_{i,j-1}(x), \quad \theta \in (0, 1)$$

162 (3.23)
$$P_{i,j}^{\theta}(x) = (h_{i,j}(x))^3 \frac{|y_{i,j}^{\theta}(x) - x|^{1-\alpha}}{\Gamma(2 - \alpha)} u''(y_{i,j}^{\theta}(x))$$

164 (3.24)
$$Q_{i,j;l}^{\theta}(x) = (h_{i,j}(x))^l \frac{|y_{i,j}^{\theta}(x) - x|^{1-\alpha}}{\Gamma(2 - \alpha)}$$

165 Obviously,

166 (3.25) $\quad y_{i,j}(x_{i-1}) = x_{j-1}, \quad y_{i,j}(x_i) = x_j, \quad y_{i,j}(x_{i+1}) = x_{j+1}$
167 (3.26) $\quad h_{i,j}(x_{i-1}) = h_{j-1}, \quad h_{i,j}(x_i) = h_j, \quad h_{i,j}(x_{i+1}) = h_{j+1}$
168 (3.27) $\quad y_{i,j}^{\theta}(x_{i-1}) = y_{j-1}^{\theta}, \quad y_{i,j}^{\theta}(x_i) = y_j^{\theta}, \quad y_{i,j}^{\theta}(x_{i+1}) = y_{j+1}^{\theta}$

169 And now we can rewrite T_{ij}

LEMMA 3.8.

170 (3.28)
$$T_{ij} = \int_0^1 -\frac{\theta(1 - \theta)}{2} P_{i,j}^{\theta}(x_i) d\theta$$
$$+ \int_0^1 \frac{\theta(1 - \theta)}{3!} Q_{i,j;l}^{\theta}(x_i) \left[\theta^2 u'''(\eta_{j,1}^{\theta}) - (1 - \theta)^2 u'''(\eta_{j,2}^{\theta})\right] d\theta$$

Immediately, we can see from (3.16) and Lemma 3.6 that For $1 \leq i \leq 2N - 1$, $2 \leq j \leq 2N - 1$,

(3.29)

$$
\begin{aligned}
S_{ij} = & \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{i,j}^\theta(x_i) d\theta \\
& + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j;4}^\theta(x_{i+1})u'''(\eta_{j+1,1}^\theta) - Q_{i,j;4}^\theta(x_i)u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\
& - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j;4}^\theta(x_i)u'''(\eta_{j,1}^\theta) - Q_{i,j;4}^\theta(x_{i-1})u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\
& - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j;4}^\theta(x_{i+1})u'''(\eta_{j+1,2}^\theta) - Q_{i,j;4}^\theta(x_i)u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\
& + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{i,j;4}^\theta(x_i)u'''(\eta_{j,2}^\theta) - Q_{i,j;4}^\theta(x_{i-1})u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta
\end{aligned}
$$

We give some properties of mesh transport functions.

LEMMA 3.9. *For $2 \leq i, j \leq 2N - 2$ and $\xi \in (x_{i-1}, x_{i+1})$*

(3.30)
$$\xi \simeq x_i, \quad \delta(y_{i,j}(\xi)) \simeq \delta(x_j), \quad h_{i,j}(\xi) \simeq h_j$$

(3.31)
$$|y_{i,j}(\xi) - \xi| \simeq |x_j - x_i|, \quad |y_{i,j-1}(\xi) - \xi| \simeq |x_{j-1} - x_i|$$

then

(3.32)
$$|y_{i,j}^\theta(\xi) - \xi| = (1-\theta)|y_{i,j-1}(\xi) - \xi| + \theta|y_{i,j}(\xi) - \xi| \simeq |y_j^\theta - x_i|$$

since $y_{i,j-1}(\xi) - \xi$, $y_{i,j}(\xi) - \xi$ have the same sign (≥ 0 or ≤ 0)

LEMMA 3.10.

(3.33)
$$
y_{i,j}'(x) = \begin{cases}
y_{i,j}^{1-1/r}(x)x^{1/r-1} & i < N, j < N \\
\dfrac{h_N}{rZ_1}x^{1/r-1} & i < N, j = N \\
(2T - y_{i,j}(x))^{1-1/r}x^{1/r-1} & i < N, j > N \\
y_{i,j}^{1-1/r}(x)\dfrac{rZ_1}{h_N} & i = N, j < N \\
1 & i = N, j = N
\end{cases}
$$

(3.34)
$$
y_{i,j}''(x) = \frac{1-r}{r} \begin{cases}
y_{i,j}^{1-2/r}(x)x^{1/r-2}Z_{j-i} & i < N, j < N \\
\dfrac{h_N}{rZ_1}x^{1/r-2} & i < N, j = N \\
(2T - y_{i,j}(x))^{1-2/r}x^{1/r-2}Z_{2N-j+i} & i < N, j > N \\
-y_{i,j}^{1-2/r}(x)\left(\dfrac{rZ_1}{h_N}\right)^2 & i = N, j < N \\
0 & i = N, j = N
\end{cases}
$$

LEMMA 3.11. *For* $2 \leq i \leq N, 2 \leq j \leq 2N - 2$, $\xi \in (x_{i-1}, x_{i+1})$

(3.35) $$|h'_{i,j}(\xi)| \leq C(r-1)Z_1 x_i^{1/r-1}\delta(x_j)^{1-2/r} \leq C(r-1)h_j x_i^{1/r-1}\delta(x_j)^{-1/r}$$

(3.36) $$|(y_{i,j}(\xi) - \xi)'| \leq C x_i^{-1}|x_j - x_i|$$

Proof. From (3.21) and Lemma 3.10, we can see that

(3.37)
$$h'_{i,j}(x) = y'_{i,j}(x) - y'_{i,j-1}(x)$$

$$= \begin{cases} x^{1/r-1}(y_{i,j}^{1-1/r}(x) - y_{i,j-1}^{1-1/r}(x)) & i < N, j < N \\ x^{1/r-1}\left(\dfrac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(x)\right) & i < N, j = N \\ x^{1/r-1}\left((2T - y_{i,N+1}(x))^{1-1/r} - \dfrac{h_N}{rZ_1}\right) & i < N, j = N+1 \\ x^{1/r-1}\left((2T - y_{i,j}(x))^{1-1/r} - (2T - y_{i,j-1}(x))^{1-1/r}\right) & i < N, j > N+1 \\ \dfrac{rZ_1}{h_N}\left(y_{N,j}^{1-1/r}(x) - y_{N,j-1}^{1-1/r}(x)\right) & i = N, j < N \\ \dfrac{rZ_1}{h_N}\left(\dfrac{h_N}{rZ_1} - y_{N,N-1}^{1-1/r}(x)\right) & i = N, j = N \end{cases}$$

While for $2 \leq i \leq N$, if $2 \leq j < N$, $\xi \in (x_{i-1}, x_{i+1})$,

(3.38)
$$y_{i,j}^{1-1/r}(\xi) - y_{i,j-1}^{1-1/r}(\xi) \leq x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r}$$
$$= T^{1-1/r}N^{1-r}\left((j+1)^{r-1} - (j-2)^{r-1}\right)$$
$$\leq CT^{1-1/r}(r-1)N^{1-r}j^{r-2} = C(r-1)Z_1 x_j^{1-2/r}$$

if $j = N$, $\xi \in (x_{i-1}, x_{i+1})$, we have $y_{i,N-1}(\xi) \in (x_{N-2}, x_N)$. And

(3.39) $$\frac{h_N}{rZ_1} = T^{1-1/r}\frac{1-(1-h)^r}{rh} = \eta^{1-1/r} \simeq x_N^{1-1/r}, \quad \eta \in (x_{N-1}, x_N)$$

Then

(3.40) $$|\frac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(\xi)| \leq x_N^{1-1/r} - x_{N-2}^{1-1/r} \simeq (r-1)Z_1 x_N^{1-2/r}$$

and similar for $j \geq N + 1$. Combine with Lemma 3.1, Lemma 3.9, $\eta \simeq x_N$, we get the first result.

For the second estimate, we have

(3.41) $$(y_{i,j}(x) - x)' = y'_{i,j}(x) - 1$$

Then, for $2 \leq i < N$, if $2 \leq j < N$, $\xi \in (x_{i-1}, x_{i+1})$, by Lemma A.5

(3.42) $$\xi^{1/r}|y_{i,j}^{1-1/r}(\xi) - \xi^{1-1/r}| \leq |y_{i,j}(\xi) - \xi|$$

$j > N$ is symmetric to it, that is

(3.43)
$$\xi^{1/r}|(2T - y_{i,j}(\xi))^{1-1/r} - \xi^{1-1/r}| \leq |2T - y_{i,j}(\xi) - \xi|$$
$$\leq |2T - x_j - x_i| + |y_{i,j}(\xi) - x_j| + |\xi - x_i| \leq |2T - x_j - x_i| + 2h_N$$
$$\leq |x_j - T| + |T - x_i| + 2h_N \leq 2|x_j - x_i|$$

But if $j = N$, with (3.39) and Lemma A.5,

(3.44)
$$\eta^{1/r}|\frac{h_N}{rZ_1} - \xi^{1-1/r}| \leq |\eta - \xi|, \quad \eta \in (x_{N-1}, x_N)$$
$$\leq |x_N - x_i| + |h_N| + |h_{i+1}| \leq 3|x_N - x_i|$$

For $i = N$, if $j < N$, similarly with (3.44),

(3.45)
$$\eta^{1/r}|y_{N,j}^{1-1/r}(\xi) - \frac{h_N}{rZ_1}| \leq C|x_j - x_N|$$

And if $j = N$, it is obviously $\equiv 0$.

Similarly, by Lemma 3.10 and Lemma 3.9, we get the second result. $\square$

LEMMA 3.12. *For* $2 \leq i \leq N, 2 \leq j \leq 2N - 2$, $\xi \in (x_{i-1}, x_{i+1})$

(3.46)
$$|y_{i,j}''(\xi)| \leq C(r-1) \begin{cases} x_j^{-1/r} x_i^{1/r-2}|x_j - x_i| & i < N, j < N \\ x_N^{1-1/r} x_i^{1/r-2} & i < N, j = N \\ \delta(x_j)^{1-2/r} x_i^{1/r-2} x_N^{1/r} & i < N, j > N \\ \delta(x_j)^{1-2/r} x_N^{2/r-2} & i = N, j \neq N \\ 0 & i = N, j = N \end{cases}$$

And $2 \leq i \leq N, 3 \leq j \leq 2N - 2$, $\xi \in (x_{i-1}, x_{i+1})$
(3.47)

$$|h_{i,j}''(\xi)| \leq C(r-1) \begin{cases} Z_1 x_i^{1/r-2} x_j^{-2/r}(|x_j - x_i| + x_j) & i < N, j < N \\ x_i^{1/r-2} x_N^{1-1/r} & i < N, j = N, N+1 \\ Z_1 x_i^{1/r-2} \delta(x_j)^{1-3/r} x_N^{1/r} & i < N, j > N+1 \\ Z_1 x_N^{2/r-2} \delta(x_j)^{1-3/r} & i = N, j < N \text{ or } j > N+1 \\ x_N^{-1} & i = N, j = N \end{cases}$$

Proof. Since by Lemma A.5, for $2 \leq i, j < N$

(3.48)
$$x_j^{1-1/r}|Z_{j-i}| = x_j^{1-1/r}|x_j^{1/r} - x_i^{1/r}| \leq |x_j - x_i|$$

and by (3.39), $\frac{h_N}{rZ_1} \simeq x_N^{1-1/r}$. And

(3.49)
$$Z_{2N-j+i} \leq Z_{2N} = 2T^{1/r}$$

Then by Lemma 3.10 and Lemma 3.9, we get the first result.

For the second part, by Lemma 3.10

(3.50)
$$h_{i,j}''(x) = y_{i,j}''(x) - y_{i,j-1}''(x)$$

while for $2 \leq i < N$, if $3 \leq j < N$, $\xi \in (x_{i-1}, x_{i+1})$,
(3.51)
$$y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1} = \left(y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi)\right)Z_{j-i} + y_{i,j-1}^{1-2/r}(\xi)Z_1$$

where $y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi) \simeq (r-2)Z_1 x_j^{1-3/r}$ similar with (3.38). Combine with (3.48), we get

(3.52) $\quad |y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1}| \leq CZ_1\left(|r-2|x_j^{-2/r}|x_j - x_i| + x_j^{1-2/r}\right)$

if $j = N$,

(3.53) $\qquad |h_{i,N}''(x)| \leq |y_{i,N}''(x)| + |y_{i,N-1}''(x)| \leq C(r-1)x_i^{1/r-2}x_N^{1-1/r}$

similarly if $j = N + 1$.

However, if $j > N + 1$, similar with (3.51), we get

(3.54)
$(2T - y_{i,j}(\xi))^{1-2/r}Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)}$

$= \left((2T - y_{i,j}(\xi))^{1-2/r} - (2T - y_{i,j-1}(\xi))^{1-2/r}\right)Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r}Z_1$

thus,

(3.55)
$\left|(2T - y_{i,j}(\xi))^{1-2/r}Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)}\right|$

$\leq CZ_1\left(|r-2|(2T-x_j)^{1-3/r}x_N^{1/r} + (2T-x_j)^{1-2/r}\right) \leq CZ_1(2T-x_j)^{1-3/r}x_N^{1/r}$

For $i = N$, it's obvious. Combine with Lemma 3.10 and Lemma 3.9, we get the second result. $\qquad\square$

3.3. Proof of Theorems. Then we esrimate each part of (3.17). And We take $m = 2i$ for $3 \leq i < N/2$, and $m = N - \lceil N/2 \rceil + 1$ for $N/2 \leq i \leq N$.

For I_5

LEMMA 3.13. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

Case 1. For $1 \leq i < N/2$,

(3.56) $$\sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| \leq Ch^2 x_i^{-\alpha/2-2/r}$$

Case 2. For $1 \leq i < N/2$,

(3.57) $$\sum_{j=N+1}^{2N} |V_{ij}| \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

Case 3. For $N/2 \leq i \leq N$,

(3.58) $$\sum_{j=N-\lceil\frac{N}{2}\rceil+2}^{2N} |V_{ij}| \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

Proof. For i, j in each case, by (3.15), Lemma A.3 and Lemma B.3, we have

(3.59) $$|V_{ij}| \leq Ch^2 \int_{x_{j-1}}^{x_j} \delta(y)^{\alpha/2-2/r}|y - x_i|^{-1-\alpha}dy$$

For Case 1, with $x_i \simeq x_{2i}$,

$$\sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| \leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1}dy$$

(3.60)
$$= \frac{C}{\alpha/2 + 2/r}h^2(x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r})$$

$$\leq Ch^2 x_i^{-\alpha/2-2/r}$$

For Case 2 , by (3.15), Lemma A.3, Lemma B.3 and $y - x_i \simeq T$

$$|V_{ij}| \leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2 - 2/r} dy$$

$$\sum_{j=N+1}^{2N-1} |V_{ij}| \leq CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T - y)^{\alpha/2 - 2/r} dy$$

(3.61)
$$\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2 - 2/r + 1} T^{\alpha/2 - 2/r + 1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2 - 2/r + 1|} h_{2N}^{\alpha/2 - 2/r + 1}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

$$= \begin{cases} \frac{C}{\alpha/2 - 2/r + 1} T^{-\alpha/2 - 2/r} h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2 - 2/r + 1|} T^{-\alpha/2 - 2/r} h^{r\alpha/2 + r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

And by Lemma A.4

$$|V_{i,2N}| \leq CT^{-1-\alpha} h_{2N}^{\alpha/2 + 1} = CT^{-\alpha/2} h^{r\alpha/2 + r}$$

Summarizes, we get the result. Similar for Case 3. □

For $i = 1, 2$.

LEMMA 3.14. *From (3.18), by Lemma B.4, Lemma 3.13 Case 1 2, we get for $i = 1, 2$*

(3.62)
$$|R_i| \leq Ch^2 x_i^{-\alpha/2 - 2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2 + r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

LEMMA 3.15. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil$*

(3.63)
$$|I_1| = |\sum_{j=1}^{k-1} V_{ij}| \leq \begin{cases} Ch^2 x_i^{-\alpha/2 - 2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2 + r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

Proof. by (3.15), Lemma A.4 , Lemma B.3

(3.64) $$|V_{i1}| \leq C \int_0^{x_1} x_1^{\alpha/2} |x_i - y|^{-1-\alpha} dy \simeq x_1^{\alpha/2 + 1} x_i^{-1-\alpha} = T^{\alpha/2 + 1} h^{r\alpha/2 + r} x_i^{-1-\alpha}$$

For $2 \leq j \leq k - 1$, by Lemma A.3 and Lemma B.3 with $x_i - y \simeq x_i$, we have

(3.65) $$|V_{ij}| \leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2 - 2/r} x_i^{-1-\alpha} dy$$

Therefore,

(3.66) $$\sum_{j=2}^{k-1} |V_{ij}| \leq Ch^{r\alpha/2 + r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2 - 2/r} dy$$

271   But $x_{\lceil \frac{i}{2} \rceil - 1} \leq 2^{-r} x_i$, so we have

272   (3.67) $\int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2 - 2/r} dy \leq \begin{cases} \frac{1}{\alpha/2 - 2/r + 1}(2^{-r} x_i)^{\alpha/2 - 2/r + 1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2 - 2/r + 1|} x_1^{\alpha/2 - 2/r + 1}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$

273   Combine the results above, we get the lemma.   □

274

275   LEMMA 3.16. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta + \alpha}^{(-\alpha/2)})$ such that*

276   *Case 1. For $3 \leq i < N, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i - 1, N - 1\}$,*

277   (3.68) $$|D_h^2 P_{i,j}^\theta(x_i)| \leq C h_j^3 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2 - \alpha)} x_i^{\alpha/2 - 4}$$

278   *Case 2. For $N/2 \leq i \leq N$, $j = N, N + 1$*

279   (3.69) $|D_h^2 P_{i,j}^\theta(\xi)| \leq C h_j^3 |y_j^\theta - x_i|^{1-\alpha} + C(r-1) h_j^2 \left(|y_j^\theta - x_i|^{1-\alpha} + h_j |y_j^\theta - x_i|^{-\alpha} \right)$

280   *Case 3. For $N/2 \leq i \leq N$, $N + 2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil$,*

281   (3.70) $$|D_h^2 P_{i,j}^\theta(\xi)| \leq C h_j^3 \left(|y_j^\theta - x_i|^{1-\alpha} + (r-1)|y_j^\theta - x_i|^{-\alpha} \right)$$

282   *Proof.* Since $\text{sign}(y_{i,j}^\theta(\xi) - \xi)$ is independent of ξ, we can derivate it. Then by
283   Lemma A.1

284   (3.71) $$D_h^2 P_{i,j}^\theta(x_i) = P_{i,j}^{\theta}{}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1})$$

285   From (3.23), using Leibniz formula and chain rules, and Lemma 3.9, Lemma 3.10,
286   Lemma 3.11, Lemma 3.12, Corollary 2.4, Lemma 3.1
287   For every case, we have $x_i \simeq \delta(x_j)$, so we have

288   (3.72) $$h_{i,j}(\xi) \leq C h_j, \quad |h_{i,j}'(\xi)| \leq C(r-1) h_j x_i^{-1}$$
289

290   (3.73) $$|y_{i,j}^\theta(\xi) - \xi| \leq C|y_j^\theta - x_i|, \quad \left| (y_{i,j}^\theta(\xi) - x_i)' \right| \leq C|y_j^\theta - x_i| x_i^{-1}$$
291

      (3.74)
292   $|u''(y_{i,j}^\theta(\xi))| \leq C x_i^{\alpha/2 - 2}, \quad \left| (u''(y_{i,j}^\theta(\xi)))' \right| \leq C x_i^{\alpha/2 - 3}, \quad \left| (u''(y_{i,j}^\theta(\xi)))'' \right| \leq C x_i^{\alpha/2 - 4}$

293   By Lemma 3.12, we have
294   For Case 1,

295   (3.75) $$|h_{i,j}''(\xi)| \leq C(r-1) h_j x_i^{-2}, \quad \left| (y_{i,j}^\theta(\xi) - x_i)'' \right| \leq C(r-1)|y_j^\theta - x_i| x_i^{-2}$$

296   For Case 2, since $x_i \simeq x_j \simeq T$

297   (3.76) $$|h_{i,j}''(\xi)| \leq C(r-1), \quad \left| (y_{i,j}^\theta(\xi) - x_i)'' \right| \leq C(r-1)$$

298   For Case 3, since $x_i \simeq \delta(x_j) \simeq T$, we have

299   (3.77) $$|h_{i,j}''(\xi)| \leq C(r-1) h_j, \quad \left| (y_{i,j}^\theta(\xi) - x_i)'' \right| \leq C(r-1)$$ □

300   Combine them, we get the result.

301

LEMMA 3.17. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $2 \leq i \leq N$, $2 \leq j \leq 2N - 2$,

(3.78)
$$\left| \frac{Q_{i,j;l}^{\theta}(x_{i+1})u^{(l-1)}(\eta_{j+1}^{\theta}) - Q_{i,j;l}^{\theta}(x_i)u^{(l-1)}(\eta_j^{\theta})}{h_{i+1}} \right| \leq Ch_j^l \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{-1} \delta(x_j)^{\alpha/2-l+1-1/r}(x_i^{1/r} + \delta(x_j)^{1/r})$$

And

(3.79)
$$\left| \frac{Q_{i,j;l}^{\theta}(x_i)u^{(l-1)}(\eta_j^{\theta}) - Q_{i,j;l}^{\theta}(x_{i-1})u^{(l-1)}(\eta_{j-1}^{\theta})}{h_i} \right| \leq Ch_j^l \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{-1} \delta(x_j)^{\alpha/2-l+1-1/r}(x_i^{1/r} + \delta(x_j)^{1/r})$$

where $\eta_j^{\theta} \in (x_{j-1}, x_j)$.

Proof.

(3.80)
$$\frac{Q_{i,j;l}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta}) - Q_{i,j;l}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}}$$
$$= \frac{Q_{i,j;l}^{\theta}(x_{i+1}) - Q_{i,j;l}^{\theta}(x_i)}{h_{i+1}}u'''(\eta_{j+1}^{\theta}) + Q_{i,j;l}^{\theta}(x_i)\frac{u'''(\eta_{j+1}^{\theta}) - u'''(\eta_j^{\theta})}{h_{i+1}}$$

Using mean value theorem

(3.81)
$$D_h Q_{i,j;l}^{\theta}(x_i) := \frac{Q_{i,j;l}^{\theta}(x_{i+1}) - Q_{i,j;l}^{\theta}(x_i)}{h_{i+1}} = Q_{i,j;l}^{\theta}{}'(\xi), \quad \xi \in (x_i, x_{i+1})$$

From (3.24) and Leibniz rule, by Lemma 3.9, Lemma 3.11 and Lemma 3.1, we have

(3.82)
$$|Q_{i,j;l}^{\theta}{}'(\xi)| \leq Ch_j^l \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}(x_i^{-1} + x_i^{1/r-1}\delta(x_j)^{-1/r})$$

(3.83)
$$Q_{i,j;l}^{\theta}(x_i) = h_j^l \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}$$

With Lemma 3.1 and Corollary 2.4

$$|u^{(l-1)}(\eta_{j+1}^{\theta})| \leq C(\eta_{j+1}^{\theta})^{\alpha/2-l+1} \simeq \delta(x_j)^{\alpha/2-l+1}$$

and by Lemma 3.1

$$\frac{|u^{(l-1)}(\eta_{j+1}^{\theta}) - u^{(l-1)}(\eta_j^{\theta})|}{h_{i+1}} = |u^{(l)}(\eta)|\frac{\eta_{j+1}^{\theta} - \eta_j^{\theta}}{h_{i+1}}, \quad \eta \in (x_{j-1}, x_{j+1})$$
$$\leq C\delta(\eta)^{\alpha/2-l}\frac{x_{j+1} - x_{j-1}}{h_{i+1}} = C\delta(\eta)^{\alpha/2-l}\frac{h_{j+1} + h_j}{h_{i+1}}$$
$$\simeq x_i^{1/r-1}\delta(x_j)^{\alpha/2-l+1-1/r}$$

Combine the results above, we get the first term. While, the later is similar.   □

320

LEMMA 3.18. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that*

Case 1. For $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$,

(3.84)
$$\begin{aligned}|S_{ij}| &\leq Ch_j^2 x_i^{\alpha/2-4} \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\ &= Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy\end{aligned}$$

Thus,

(3.85)
$$\sum_{j=k+1}^{\min\{2i-1,N-1\}} |S_{ij}| \leq Ch^2 x_i^{-\alpha/2-2/r}$$

Case 2. For $N/2 \leq i \leq N$, $j = N, N+1$, since $\theta(1-\theta)h_j \leq |y_j^\theta - x_i|$, we have

(3.86)
$$|S_{ij}| \leq C(h^3 + (r-1)h^2)(T - x_i + h_N)^{1-\alpha}$$

Case 3. For $N/2 \leq i \leq N$, $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil$,

(3.87)
$$|S_{ij}| \leq Ch^2 \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy$$

Thus,

(3.88)
$$\sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} |S_{ij}| \leq Ch^2 + C(r-1)h^2(T - x_i + h_N)^{1-\alpha}$$

Expecially, for $i = N$, the estimate of $\lceil \frac{N}{2} \rceil + 1 \leq j \leq N-1$ is symmetric with $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil$.

Proof. Since (3.29), by $x_i \simeq x_j$, Lemma 3.1, Lemma 3.16, Lemma 3.17
For Case 1, we get the first result immediately. While $x_k \simeq x_i \simeq x_{\min\{2i-1,N-1\}}$, we have

(3.89)
$$\begin{aligned}\sum_{k+1}^{\min\{2i-1,N-1\}} |S_{ij}| &\leq Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_k}^{x_{\min\{2i-1,N-1\}}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &\leq Ch^2 x_i^{\alpha/2-2-2/r} x_i^{2-\alpha} = Ch^2 x_i^{-\alpha/2-2/r}\end{aligned}$$

For Case 2,

(3.90)
$$\begin{aligned}|S_{ij}| &\leq C(h_j^3 + (r-1)h_j^2) \int_0^1 |y_j^\theta - x_i|^{1-\alpha} d\theta \\ &= C(h_j^2 + (r-1)h_j) \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} dy\end{aligned}$$

however,

(3.91)
$$\begin{aligned}\int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} dy &= \frac{1}{2-\alpha} \left((x_j - x_i)^{2-\alpha} - (x_{j-1} - x_i)^{2-\alpha} \right) \\ &\simeq h_N(|x_j - x_i + h_N|)^{1-\alpha}\end{aligned}$$

For Case 3,

$$(3.92) \quad |S_{ij}| \leq Ch_j^2 \int_0^1 \left(|y_j^\theta - x_i|^{1-\alpha} + (r-1)|y_j^\theta - x_i|^{-\alpha} \right) h_j d\theta$$

$$\leq Ch^2 \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy$$

Thus,

$$(3.93) \quad \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} |S_{ij}| = Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil \frac{N}{2} \rceil}} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy$$

$$\leq Ch^2(T^{2-\alpha} + (r-1)(T - x_i + h_N)^{1-\alpha})$$

□

Now we study I_2, I_4.

LEMMA 3.19. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that*

Case 1. For $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil$,

$$(3.94)$$
$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}}(T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,k} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

Case 2. For $3 \leq i < N/2$,

$$(3.95)$$
$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,2i} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

Case 3. For $N/2 \leq i \leq N$, $m = N - \lceil \frac{N}{2} \rceil + 1$,

$$(3.96) \quad I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,m} + T_{i-1,m-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,m} \right) \leq Ch^2$$

Proof. In fact,

$$(3.97) \quad \frac{1}{h_{i+1}}(T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,k}$$

$$= \frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}}(T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i})T_{i,k}$$

While, by Lemma A.3, Lemma B.3, Lemma 3.1 and $x_k \simeq x_i$, we have

$$(3.98) \quad \frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) = \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y))D_h K_y(x_i)dy$$

$$\leq Ch_k^2 x_k^{\alpha/2-2} h_k|x_i - x_k|^{-\alpha} \leq Ch^2 x_i^{-\alpha/2-2/r} h_k$$

Thus,

$$(3.99) \quad \frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}}|T_{i+1,k} - T_{i,k}| \leq Ch^2 x_i^{-\alpha/2-2/r}$$

From (3.14), Lemma A.2 and normalization, we have

$$(3.100)$$
$$\frac{1}{h_{i+1}}(T_{i+1,k+1} - T_{i,k}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{Q_{i,k;3}^\theta(x_{i+1})u''(\eta_{k+1}^\theta) - Q_{i,k;3}^\theta(x_i)u''(\eta_k^\theta)}{h_{i+1}} d\theta$$

362   where $\eta_k^\theta \in (x_{k-1}, x_k)$ and $\eta_{k+1}^\theta \in (x_k, x_{k+1})$. And with Lemma 3.17, we can get

363   (3.101)
$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} |T_{i+1,k+1} - T_{i,k}| \leq Ch^2 x_i^{-\alpha/2 - 2/r}$$

364   For the third term, by Lemma 3.1, Lemma B.1, Lemma A.3 and $x_k \simeq x_i$, we have

365   (3.102)
$$\frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,k} \leq h_i^{-3} h^2 x_i^{1 - 2/r} Ch_k^3 x_k^{\alpha/2 - 2} |x_k - x_i|^{1 - \alpha}$$
$$\leq Ch^2 x_i^{-\alpha/2 - 2/r}$$

366   Summarizes, we have

367   (3.103)
$$I_2 \leq Ch^2 x_i^{-\alpha/2 - 2/r}$$

368   The case for I_4 is similar.   □

369   Now we have study everr part to prove Theorem 3.3 and Theorem 3.4.

370   For $1 \leq i < N/2$, combine Lemma 3.14, Lemma 3.15, Lemma 3.19 Cases 1 2,
371   Lemma 3.18 Case 1, Lemma 3.13 Case 1 2, we get Theorem 3.3.

372   For $N/2 \leq i \leq N$, we take $m = 2N - \lceil \frac{N}{2} \rceil + 1$. And depart I_3 to three parts:

373   (3.104)
$$I_3 = \sum_{j=k+1}^{m} S_{ij} = \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{m-1} S_{ij}$$

374   combine Lemma 3.15, Lemma 3.19 Cases 1 3, Lemma 3.18, Lemma 3.13 Case 1 2, we
375   get Theorem 3.4.

4. Convergence analysis.

4.1. Properties of some Matrices. Review subsection 2.1, we have got (2.10).

378   DEFINITION 4.1. *We call one matrix an M matrix , which means its entries are*
379   *positive on major diagonal and nonpositive on others, and strictly diagonally dominant*
380   *in rows.*

381   Now we have

382   LEMMA 4.2. *Matrix A defined by (2.12) where (2.13) is an M matrix. And there*
383   *exists a constant $C_A = C(T, \alpha, r)$ such that*

384   (4.1)
$$S_i := \sum_{j=1}^{2N-1} a_{ij} \geq C_A (x_i^{-\alpha} + (2T - x_i)^{-\alpha})$$

385   *Proof.* From (2.15), we have
       (4.2)
386    $$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4 - \alpha)} \left(\frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right)$$

387   Let

388   (4.3)
$$g(x) = g_0(x) + g_{2N}(x)$$

389   where

$$g_0(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x-x_0|^{3-\alpha} - |x-x_1|^{3-\alpha}}{h_1}$$

$$g_{2N}(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N}-x|^{3-\alpha} - |x_{2N-1}-x|^{3-\alpha}}{h_{2N}}$$

392   Thus

$$-\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

394   Then

395   (4.4)
$$\begin{aligned}
S_i &:= \sum_{j=1}^{2N-1} a_{ij} \\
&= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\
&= D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i)
\end{aligned}$$

396   When $i = 1$

397   (4.5)
$$\begin{aligned}
D_h^2 g_0(x_1) &= \frac{2}{h_1 + h_2} \left(\frac{1}{h_2} g_0(x_2) - (\frac{1}{h_1} + \frac{1}{h_2}) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right) \\
&= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2) h_1 h_2} \\
&= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2) h_1^{1-\alpha} h_2} h_1^{-\alpha} \\
&= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha}}{2^r (2^r - 1)} h_1^{-\alpha}
\end{aligned}$$

398   but

399   (4.6) $$1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha} > 0$$

400   While for $i \geq 2$

401   (4.7)
$$\begin{aligned}
D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \\
&= -\kappa_\alpha \frac{|\xi - x_0|^{1-\alpha} - |\xi - x_1|^{1-\alpha}}{\Gamma(2-\alpha) h_1} \\
&= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} |\xi - \eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\
&\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha}
\end{aligned}$$

402   So

403   (4.8) $$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g_0(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g_0(x_i) + \frac{1}{h_i} g_0(x_{i-1}) \right) \geq C x_i^{-\alpha}$$

404 symmetricly,

(4.9)

$$\frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}g_{2N}(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})g_{2N}(x_i) + \frac{1}{h_i}g_{2N}(x_{i-1})\right) \geq C(\alpha,r)(2T-x_i)^{-\alpha}$$

406 Let

407 (4.10) $$G = \mathrm{diag}(\delta(x_1), ..., \delta(x_{2N-1}))$$

408 Then

409 LEMMA 4.3. *The matrix $B := AG$, the major diagnal is positive, and nonpositive*
410 *on others. And there is a constant $C_{AG}, C = C(\alpha,r)$ such that*

411 (4.11) $$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C_{AG}(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}) + C(T - \delta(x_i) + h_N)^{1-\alpha}$$

  Proof.

412 $$b_{ij} = a_{ij}\delta(x_j) = -\kappa_\alpha \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}\tilde{a}_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})\tilde{a}_{i,j} + \frac{1}{h_i}\tilde{a}_{i-1,j}\right)\delta(x_j)$$

413 Since

414 (4.12) $$\delta(x) \equiv \Pi_h \delta(x)$$

415 by (2.14) and (2.5), we have

$$\tilde{M}_i := \sum_{j=1}^{2N-1} \tilde{b}_{ij} := \sum_{j=1}^{2N-1} \tilde{a}_{ij}\delta(x_j)$$

416 (4.13) $$= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)}\Pi_h\delta(y)dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)}\delta(y)dy$$

$$= \frac{-2}{\Gamma(4-\alpha)}|T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)}(x_i^{3-\alpha} + (2T - x_i)^{3-\alpha})$$

$$:= w(x_i) = p(x_i) + q(x_i)$$

417 Thus,

$$M_i := \sum_{j=1}^{2N-1} b_{ij} = \sum_{j=1}^{2N-1} a_{ij}\delta(x_j)$$

418 (4.14) $$= -\kappa_\alpha \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}\tilde{M}_{i+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})\tilde{M}_i + \frac{1}{h_i}\tilde{M}_{i-1}\right)$$

$$= D_h^2(-\kappa_\alpha p)(x_i) - \kappa_\alpha D_h^2 q(x_i)$$

419 for $1 \leq i \leq N - 1$, by Lemma A.1

(4.15)

$$D_h^2(-\kappa_\alpha p)(x_i) := -\kappa_\alpha \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}p(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})p(x_i) + \frac{1}{h_i}p(x_{i-1})\right)$$

420 $$= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}|T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1})$$

$$\geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}(T - \delta(x_i) + h_N)^{1-\alpha}$$

(4.16)

$$D_h^2(-\kappa_\alpha p)(x_N) := -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left(\frac{1}{h_{N+1}} p(x_{N+1}) - (\frac{1}{h_N} + \frac{1}{h_{N+1}})p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right)$$

$$= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)h_N^2} h_N^{3-\alpha} = \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - \delta(x_N) + h_N)^{1-\alpha}$$

Symmetricly for $i \geq N$, we get

(4.17) $$D_h^2(-\kappa_\alpha p)(x_i) \geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} (T - \delta(x_i) + h_N)^{1-\alpha}$$

Similarly, we can get

(4.18)
$$D_h^2 q(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} q(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right)$$

$$\leq \frac{2^{r(\alpha-1)+1}}{\Gamma(2-\alpha)} (x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}), \quad i = 1, \cdots, 2N-1$$ □

So, we get the result.

Notice that

(4.19) $$x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha}$$

We can get

THEOREM 4.4. *There exists a real $\lambda = \lambda(T, \alpha, r) > 0$ and $C = C(T, \alpha, r) > 0$ such that $B := A(\lambda I + G)$ is an M matrix. And*

(4.20) $$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) + C(T - \delta(x_i) + h_N)^{1-\alpha}$$

Proof. By Lemma 4.2 with C_A and Lemma 4.3 with C_{AG}, it's sufficient to take $\lambda = (C + 2TC_{AG})/C_A$, then

(4.21) $$M_i \geq C \left((x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + (T - \delta(x_i) + h_N)^{1-\alpha} \right)$$ □

4.2. Proof of Theorem 2.6. For equation

(4.22) $$AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1}U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1}U = F$$

which means

(4.23) $$\sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + \delta(x_j)} = -\tau_i$$

where $\epsilon_i = u(x_i) - u_i$.

And if

(4.24) $$\left| \frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})} \right| = \max_{1 \leq i \leq 2N-1} \left| \frac{\epsilon_i}{\lambda + \delta(x_i)} \right|$$

444   Then, since $B = A(\lambda I + G)$ is an M matrix, it is Strictly diagonally dominant. Thus,

$$
\begin{aligned}
|\tau_{i_0}| &= |\sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + \delta(x_j)}| \\
&\geq b_{i_0,i_0}|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}| - \sum_{j \neq i_0} |b_{i_0,j}||\frac{\epsilon_j}{\lambda + \delta(x_j)}| \\
&\geq b_{i_0,i_0}|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}| - \sum_{j \neq i_0} |b_{i_0,j}||\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}| \\
&= \sum_{j=1}^{2N-1} b_{i_0,j}|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}| \\
&= M_{i_0}|\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}|
\end{aligned}
$$

445   (4.25)

446   By Theorem 2.5 and Theorem 4.4,

447   We kown that there exists constants $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$,

448   and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that

449   (4.26)
$$
|\frac{\epsilon_i}{\lambda + \delta(x_i)}| \leq |\frac{\epsilon_{i_0}}{\lambda + \delta(x_{i_0})}| \leq C_1 h^{\min\{\frac{r\alpha}{2},2\}} + C_2(r-1)h^2
$$

450   as $\lambda + \delta(x_i) \leq \lambda + T$

451   So, we can get

452   (4.27)
$$
|\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{r\alpha}{2},2\}}
$$

453   The convergency has been proved.

454   Remarks:

5. Experimental results.

5.1. $f \equiv 1$.

5.2. $f = x^\gamma, \gamma < 0$. **Appendix A. Approximate of difference quotients.**

LEMMA A.1. *If $g(x) \in C^2(\Omega)$, there exists $\xi \in (x_{i-1}, x_{i+1})$ such that*

(A.1) $$D_h^2 g(x_i) = g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1})$$

And if $g(x) \in C^4(\Omega)$, then

(A.2)
$$D_h^2 g(x_i) = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i)$$
$$+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right)$$

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})$$

Subsitute them in the left side of (A.1), we have

$$D_h^2 g(x_i) = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (g(x_{i+1}) - g(x_i) + \frac{1}{h_i}(g(x_{i-1}) - g(x_i)) \right)$$
$$= \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2)$$

Now, using intermediate value theorem, there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy$$

Expecially,

(A.3)
$$\int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy = \frac{h_i^4}{4!} g''''(\eta_1)$$
$$\int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy = \frac{h_{i+1}^4}{4!} g''''(\eta_2)$$

where $\eta_1 \in (x_{i-1}, x_i), \eta_2 \in (x_i, x_{i+1})$. □

474   LEMMA A.2. *Denote* $y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \theta \in (0,1)$,

475   (A.4) $$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2}h_j^2 u''(\xi), \quad \xi \in (x_{j-1}, x_j)$$

476

(A.5)

477    $$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2}h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!}h_j^3(\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

478   *where* $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$.

479   *Proof.* By Taylor expansion, we have

480    $$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!}u''(\xi_1), \quad \xi_1 \in (x_{j-1}, y_j^\theta)$$

481    $$u(x_j) = u(y_j^\theta) + (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!}u''(\xi_2), \quad \xi_2 \in (y_j^\theta, x_j)$$

482   Thus

483
$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j)$$
$$= -\frac{\theta(1-\theta)}{2}h_j^2(\theta u''(\xi_1) + (1-\theta)u''(\xi_2))$$
$$= -\frac{\theta(1-\theta)}{2}h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2]$$

484   The second equation is similar,

485    $$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!}u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!}u'''(\eta_1)$$

486    $$u(x_j) = u(y_j^\theta) + (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!}u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!}u'''(\eta_2)$$

487   where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$. Thus

488
$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j)$$
$$= -\frac{\theta(1-\theta)}{2}h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!}h_j^3(\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

$\square$

489   LEMMA A.3. *By Lemma A.2, Corollary 2.4 and Lemma 3.1 , There is a constant*
490    $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *for* $2 \leq j \leq 2N - 1$,

491   (A.6) $|u(y) - \Pi_h u(y)| \leq h_j^2 \max\limits_{\xi \in [x_{j-1}, x_j]} |u''(\xi)| \leq Ch^2 \delta(y)^{\alpha/2-2/r}, \quad$ *for* $y \in (x_{j-1}, x_j)$

492   LEMMA A.4. *For* $x \in [x_{j-1}, x_j]$

493   (A.7)
$$|u(x) - \Pi_h u(x)| = \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^x u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_x^{x_j} u'(y)dy \right|$$
$$\leq \int_{x_{j-1}}^{x_j} |u'(y)|dy$$

If $x \in [0, x_1]$, with Corollary 2.4, we have

(A.8) $\quad |u(x) - \Pi_h u(x)| \leq \int_0^{x_1} |u'(y)| dy \leq \int_0^{x_1} Cy^{\alpha/2-1} dy \leq C\frac{2}{\alpha} x_1^{\alpha/2} = C\frac{2}{\alpha} h_1^{\alpha/2}$

Similarly, if $x \in [x_{2N-1}, 1]$, we have

(A.9) $\qquad\qquad |u(x) - \Pi_h u(x)| \leq C\frac{2}{\alpha}(2T - x_{2N-1})^{\alpha/2} = C\frac{2}{\alpha} h_{2N}^{\alpha/2}$

LEMMA A.5.

(A.10) $\quad b^{1-\theta}|a^\theta - b^\theta| \leq |a - b| \ (\text{ also } a^{1-\theta}|a^\theta - b^\theta| \leq |a - b|), \quad a, b \geq 0, \ \theta \in [0, 1]$

Appendix B. Proofs of some technical details. Review that $h = \frac{1}{N}$ and the defination of $\simeq$ in subsection 2.1

LEMMA B.1. *There is a constant C such that for $i = 1, 2, \cdots, 2N - 1$*

(B.1) $\qquad\qquad\qquad |h_{i+1} - h_i| \leq Ch^2 \delta(x_i)^{1-2/r}$

Proof. By (2.2),

(B.2)
$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \leq i \leq 2N-1 \end{cases}$$

Since

(B.3) $\qquad\qquad (i+1)^r - 2i^r + (i-1)^r \simeq r(r-1)i^{r-2}, \quad \text{for } i \geq 1$

We get the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

LEMMA B.2. *there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that*

(B.4) $\quad \frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right|$
$\qquad\qquad \leq Ch^2 \delta(x_i)^{-\alpha/2-2/r}$

Proof. By Lemma 2.2, we have for $1 \leq i \leq N$

(B.5) $\qquad \left| \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy \right| \leq \frac{\|f\|_\beta^{(\alpha/2)}}{3!} \int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y - x_{i-1})^3 dy$

For $i = 1$,

$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y - x_{i-1})^3 dy = \int_0^{x_1} y^{1-\alpha/2} dy = \frac{1}{2 - \alpha/2} x_1^{2-\alpha/2} = \frac{1}{2 - \alpha/2} x_1^{-\alpha/2-2} h_1^4$

And for $2 \leq i \leq N$, since $x_i \simeq x_{i-1} \leq y \leq x_i$, we have

$\qquad \int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y - x_{i-1})^3 dy \simeq \int_{x_{i-1}}^{x_i} x_i^{-\alpha/2-2}(y - x_{i-1})^3 dy = \frac{1}{4!} x_i^{-\alpha/2-2} h_i^4$

So for $1 \leq i \leq N$, we have

(B.6)
$$\left| \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy \right| \leq C x_i^{-\alpha/2-2} h_i^4$$

and similarly,

(B.7)
$$\left| \int_{x_i}^{x_{i+1}} f''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right| \leq C x_i^{-\alpha/2-2} h_{i+1}^4$$

Thus for $1 \leq i \leq N$, with Lemma 3.1 we have

(B.8)
$$\frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right|$$
$$\leq C x_i^{-\alpha/2-2} \frac{2}{h_i + h_{i+1}} (h_i^3 + h_{i+1}^3) \simeq x_i^{-\alpha/2-2} h_i^2 \simeq x_i^{-\alpha/2-2} h^2 x_i^{2-2/r}$$
$$= C h^2 x_i^{-\alpha/2-2/r}$$

It's symmetric for $N < i \leq 2N - 1$. □

LEMMA B.3. *There is a constant $C = C(\alpha, r)$ such that for all $1 \leq i \leq 2N - 1$, $1 \leq j \leq 2N$ s.t. $\min\{|j - i|, |j - 1 - i|\} \geq 2$ and $y \in [x_{j-1}, x_j]$, we have*

(B.9)
$$D_h K_y(x_i) \simeq |y - x_i|^{-\alpha}, \quad D_h^2 K_y(x_i) \simeq |y - x_i|^{-1-\alpha}$$

Proof. Sinec $y - x_{i-1}, y - x_i, y - x_{i+1}$ have the same sign, by mean value theorem and Lemma A.1,

$$D_h K_y(x_i) = \frac{|y - \xi|^{-\alpha}}{\Gamma(1 - \alpha)}, \quad \xi \in (x_i, x_{i+1})$$
$$D_h^2 K_y(x_i) = \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in (x_{i-1}, x_{i+1})$$

however, $|y - \xi| \simeq |y - x_i|$, we get the result. □

LEMMA B.4. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

(B.10)
$$\sum_{j=1}^{3} V_{1j} \leq C h^2 x_1^{-\alpha/2-2/r}$$

(B.11)
$$\sum_{j=1}^{4} V_{2j} \leq C h^2 x_2^{-\alpha/2-2/r}$$

Proof. For $0 \leq i \leq 3, 1 \leq j \leq 4$, by Lemma A.4, Lemma A.3 and (3.14)

(B.12)
$$T_{ij} \leq C x_1^{2-\alpha/2} \simeq h_1^2 h^2 x_1^{-\alpha/2-2/r} \simeq h_1^2 h^2 x_2^{-\alpha/2-2/r}$$

Therefore, by (3.15), we get the result. □

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