SECOND-ORDER DIFFERENCE-QUADRATURE APPROACH ON GRADED MESHES FOR FRACTIONAL LAPLACIAN

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Abstract. This paper deals with the *integral-differential* version of the fractional Laplacian via the Riesz fractional derivative. The numerical analysis presents significant challenges, partly because the solution to the problem has a weak singularity at the boundary, and the model equation can involve a singular source term. In such cases, many prevalent numerical methods may suffer from a severe order reduction. To fill in this gap, we combine finite difference method and numerical quadrature, called difference-quadrature method, to approximate the *differential* and *integral* operator of the fractional Laplacian on graded meshes, respectively. We design a grid mapping function and a natural-skew ordering to handle local truncation errors, and construct an appropriate right-preconditioner for the resulting matrix algebraic equation. By utilizing the Hölder regularity of the data, we prove that the proposed scheme is second-order convergence on graded meshes even if the source term is hypersingular. Numerical experiments illustrate the theoretical results.

1. Introduction

Fractional Laplacian is a powerful tool in modeling phenomena for anomalous diffusion, which appears naturally in the α-stable Lévy process instead of the standard Brownian motion [1, 3, 17, 35, 24]. It can be found in many applications, such as porous media flow [11], image processing [16], biophysics [2]. In this work, we study a second-order difference-quadrature scheme on graded meshes for the integral-differential version of the fractional Laplacian

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x) & x \in \Omega, \\ u(x) = 0 & x \in \mathbb{R} \setminus \Omega. \end{cases} \tag{1.1}$$

Here $(-\Delta)^{\frac{\alpha}{2}}$ is the integral-differential fractional Laplacian, in terms of the Riesz (left and right Riemann-Liouville) fractional derivative [1, 20, 23, 33], defined by

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{d^2}{dx^2} I^{2-\alpha} u(x) \quad \text{with} \quad 1 < \alpha < 2. \tag{1.2}$$

Note that the Riesz fractional integration can be realized in the form of the Riesz potential defined as the Fourier convolution of the form [32, p. 174], namely,

$$I^{2-\alpha} u(x) = \int_\Omega K(x-y) u(y) dy \quad \text{with} \quad K(x) = \frac{|x|^{1-\alpha}}{2\cos((2-\alpha)\pi/2)\Gamma(2-\alpha)}. \tag{1.3}$$

The fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ can be defined in several equivalent ways [21, 23] on the whole space $\mathbb{R}^n$. For example, it can be defined as a pseudo-differential

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operator via the Fourier transform

$$\mathcal{F}[(-\Delta)^{\frac{\alpha}{2}} u](\xi) = |\xi|^{\alpha} \mathcal{F}[u](\xi),$$

or in terms of the hypersingular integral operator

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{n,\alpha} \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy. \tag{$*$}$$

The major challenges of fractional Laplacian arise partly because typical solutions u have a weak singularity at the boundary; for example in the special case where Ω is a bounded interval $(a,b) \subset \mathbb{R}$ and $f \equiv 1$, the exact solution is [17, 20, 25]

$$u(x) = \frac{2^{-\alpha} \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right) \Gamma\left(\frac{1+\alpha}{2}\right)} \left[(x-a)(b-x)\right]^{\frac{\alpha}{2}}.$$

Moreover, the model equation (1.1) can involve a singular/hypersingualr source term, even if the exact solution u is absolutely continuous [29, 30, 39]. This leads to a severe order reduction for many numerical methods.

Among various techniques for approximating *integral* version of the fractional Laplacian ($*$), numerical quadrature with piecewise linear polynomials (collocation) is the simplest, since it only need a single integration and are much simpler to implement on a computer. In [20], Huang and Oberman first proposed a quadrature-based finite difference method for solving the 1 dimensional (1D) integral fractional Laplacian. The numerical solution obtained from this method is $\mathcal{O}\left(h^{2-\alpha}\right)$ accurate in the discrete $L^{\infty}(\mathbb{R}^n)$ norm if the solution is sufficiently smooth, while this accuracy reduces to $\mathcal{O}\left(h^{\alpha/2}\right)$ in the case $f \equiv 1$, since u has a boundary singularity. Inspired by [20], $\mathcal{O}\left(|\log h| h^{2-\alpha/2}\right)$ convergence for $0 < \alpha < 2$ and $\mathcal{O}\left(h^{\alpha}\right)$ for $\alpha \leq 4/3$, respectively, is proved [19] in the discrete $L^{\infty}(\mathbb{R}^n)$ norm on graded meshes for $n = 1, 2$ by means of a discrete barrier function. Recently, $\mathcal{O}\left(h^{2-\alpha}\right)$ convergence for $0 < \alpha < 1$ is given in [9] by collocation method on graded meshes, where it remains to be proved for $1 < \alpha < 2$. It seems that achieving a second-order accurate scheme using piecewise linear polynomials collocation method for fractional Laplacian ($*$) with $1 < \alpha < 2$ is not an easy task.

Nevertheless, there are already some important progress for numerically solving *integral-differential* version of the fractional Laplacian (1.2) with $1 < \alpha < 2$ via the Riesz (left and right Riemann-Liouville) fractional derivative. Take, for example, the finite difference method [5, 14, 27, 28, 37, 8, 7, 6, 31, 34, 38], finite element method [4, 15, 13], and spectral method [10, 12, 36]. However, these methods may suffer from a severe order reduction when the exact solution has a weak singularity at the boundary and the source term is singular/hypersingualr.

How to design/restore the second-order convergence with a singualr/hypersingular source term for the model (1.1) still has not been addressed in the literature. To fill in this gap, we combine finite difference method and numerical quadrature, called difference-quadrature method, to approximate the *differential* and *integral* operator of the fractional Laplacian on graded meshes. This method was proposed by the authors for solving the fractional partial differential equations on uniform mesh [7, 10] when the solution is smooth with $u \in C^4(\bar{\Omega})$. In this work, we design a grid mapping function and a natural-skew ordering to handle local truncation errors, and construct an appropriate right-preconditioner for the resulting matrix algebraic equation. By utilizing the Hölder regularity of the data, we prove that the proposed scheme is second-order convergence on graded meshes even if the source term is hypersingular. Numerical experiments illustrate the theoretical results.

2. The main results

In this section, we describe the difference-quadrature scheme on graded meshes for fractional Laplacian (1.1) via the Riesz fractional derivative and state our main results about the convergence rate of the numerical solutions.

2.1. **Difference-quadrature scheme.** To keep the expressions simple below we assume we are on the interval $\Omega = (0, 2T)$, but everything can be shifted to an arbitrary interval (a, b). Partition Ω by the graded mesh

$$\pi_h : 0 = x_0 < x_1 < x_2 < \cdots < x_{2N-1} < x_{2N} = 2T,$$

where we set

$$x_j = \begin{cases} T\left(\frac{j}{N}\right)^r & \text{for } j = 0, 1, ..., N, \\ 2T - T\left(\frac{2N-j}{N}\right)^r & \text{for } j = N+1, N+2, ..., 2N, \end{cases} \tag{2.1}$$

with the user-chosen grading exponent $r \geq 1$. When $r > 1$, the mesh points are clustered near $x = 0$ and $x = 2T$.

Set $h_j = x_j - x_{j-1}$ for $j = 1, 2, ..., 2N$ and define $h := \frac{1}{N}$. Let S_h be the space of globally continuous piecewise linear functions on the mesh π_h that vanish at $x = 0, 2T$. In this space, we choose as a basis the standard hat functions

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}) & \text{for } x_{j-1} \leq x \leq x_j, \\ \frac{1}{h_{j+1}}(x_{j+1} - x) & \text{for } x_j \leq x \leq x_{j+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Then, define the piecewise linear interpolant of the true solution u to be

$$\Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x). \tag{2.3}$$

Now, we discretise (1.1) by replacing $u(x)$ by a continuous piecewise linear function

$$u_h(x) := \sum_{j=1}^{2N-1} u_j \phi_j(x), \tag{2.4}$$

whose nodal values u_j are to be determined by collocation at each mesh point x_i for $i = 1, 2, ..., 2N-1$:

$$-D_h^\alpha u_h(x_i) := -D_h^2 I^{2-\alpha} u_h(x_i) = f(x_i) =: f_i. \tag{2.5}$$

Here the approximation of second order derivatives can be found by interpolating by a quadratic function and differentiating twice [22, eq. (1.14)]

$$D_h^2 u(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} u(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) u(x_i) + \frac{1}{h_{i+1}} u(x_{i+1}) \right). \tag{2.6}$$

Moreover, the Riesz fractional derivatives in (1.2) can be approximated by

$$-D_h^\alpha u_h(x_i) = -D_h^2 I^{2-\alpha} \sum_{j=1}^{2N-1} u_j \phi_j(x_i) = \sum_{j=1}^{2N-1} a_{ij}\, u_j. \tag{2.7}$$

We have replaced $-\frac{d^2}{dx^2}I^{2-\alpha}u(x_i) = f(x_i)$ in (1.2) by $-D_h^\alpha u_h(x_i) = f(x_i)$ in (2.5), with truncation error

$$\tau_i := -D_h^\alpha \Pi_h u(x_i) - f(x_i) \quad \text{for} \quad i = 1, 2, ..., 2N-1, \tag{2.8}$$

where

$$-D_h^\alpha \Pi_h u(x_i) = -\sum_{j=1}^{2N-1} D_h^\alpha \phi_j(x_i)\, u(x_j) = \sum_{j=1}^{2N-1} a_{ij} u(x_j). \tag{2.9}$$

The discrete equation (2.5) can be written in matrix form

$$AU = F, \tag{2.10}$$

where the coefficient matrix A and the vectors U and F are defined by $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$, $U = (u_1, \cdots, u_{2N-1})^T$ and $F = (f_1, \cdots, f_{2N-1})^T$. In particular, the coefficient a_{ij} can be explicitly expressed as

$$\begin{aligned} a_{ij} &= -D_h^2 I^{2-\alpha}\phi_j(x_i) \\ &= -\frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_i}\tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right)\tilde{a}_{i,j} + \frac{1}{h_{i+1}}\tilde{a}_{i+1,j}\right) \end{aligned} \tag{2.11}$$

with the quadrature coefficients

$$\begin{aligned} \tilde{a}_{ij} &= I^{2-\alpha}\phi_j(x_i) \\ &= \frac{\kappa_\alpha}{\Gamma(4-\alpha)}\left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \left(\frac{1}{h_j} + \frac{1}{h_{j+1}}\right)|x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}}\right), \end{aligned}$$

and $\kappa_\alpha = \frac{1}{2\cos((2-\alpha)\pi/2)} = -\frac{1}{2\cos(\alpha\pi/2)} > 0$.

2.2. **Regularity of the true solution.** For any $\beta > 0$, we use the standard notation $C^\beta(\bar{\Omega}), C^\beta(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^\beta(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous [18, p. 52] with exponent β' in Ω. Here, $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on open set Ω.

For convenience, we define

$$\delta(x) = \operatorname{dist}(x, \partial\Omega) = \begin{cases} x & 0 < x \le T, \\ 2T - x & T < x < 2T, \end{cases} \tag{2.12}$$

and $\delta(x,y) = \min\{\delta(x), \delta(y)\}$. To bound the derivatives of u, we introduce the following δ-dependent Hölder norms.

Definition 2.1 (δ-dependent Hölder norms [26]). For any $\beta > 0$, write $\beta = k + \beta'$, where k is an integer and $\beta' \in (0, 1]$. Given $\sigma \ge -\beta$, define the seminorm

$$|w|_\beta^{(\sigma)} = \sup_{x,y\in\Omega}\left(\delta(x,y)^{\beta+\sigma}\frac{|w^{(k)}(x) - w^{(k)}(y)|}{|x-y|^{\beta'}}\right).$$

For $\sigma > -1$, we also define the norm $\|\cdot\|_\beta^{(\sigma)}$ as follows: in case that $\sigma \ge 0$,

$$\|w\|_\beta^{(\sigma)} = \sum_{l=0}^{k}\sup_{x\in\Omega}\left(\delta(x)^{l+\sigma}|w^{(l)}(x)|\right) + |w|_\beta^{(\sigma)},$$

while for $-1 < \sigma < 0$,

$$\|w\|_{\beta}^{(\sigma)} = \|w\|_{C^{-\sigma}(\bar{\Omega})} + \sum_{l=1}^{k} \sup_{x\in\Omega} \left(\delta(x)^{l+\sigma} |w^{(l)}(x)|\right) + |w|_{\beta}^{(\sigma)}.$$

Lemma 2.2. [26, pp. 276-277] *Let $f \in L^{\infty}(\Omega)$ and u be a solution of (1.1). Then, $u \in C^{\alpha/2}(\mathbb{R})$ and $u/\delta^{\alpha/2} \in C^{\sigma}(\bar{\Omega})$ for some $\sigma \in (0, 1-\alpha/2)$, $\alpha \in (1,2)$ with*

$$\|u\|_{C^{\alpha/2}(\mathbb{R})} \le C\|f\|_{L^{\infty}(\Omega)} \quad \text{and} \quad \|u/\delta^{\alpha/2}\|_{C^{\sigma}(\bar{\Omega})} \le C\|f\|_{L^{\infty}(\Omega)}$$

for some positive constant $C = C(\Omega, \alpha)$.

In particular, this result says that if $f \in L^{\infty}(\Omega)$, then

$$|u(x)| \le C\delta(x)^{\alpha/2} \quad \text{for all } x \in \bar{\Omega}. \tag{2.13}$$

Lemma 2.3. [26, Proposition 1.4] *Let $\beta > 0$ be such that neither β nor $\beta + \alpha$ is an integer. Let $f \in C^{\beta}(\Omega)$ be such that $\|f\|_{\beta}^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R})$ be a solution of (1.1). Then, $u \in C^{\beta+\alpha}(\Omega)$ and*

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_{\beta}^{(\alpha/2)}\right)$$

for some positive constant $C = C(\Omega, \alpha, \beta)$.

By definition of δ-dependent Hölder norms, we have following results obviously.

Lemma 2.4. *Let $\beta = 4-\alpha+\gamma$ with $0 < \gamma < \alpha - 1$. Assume that $f \in L^{\infty}(\Omega) \cap C^{\beta}(\Omega)$ be such that $\|f\|_{\beta}^{(\alpha/2)} < \infty$, and u be a solution of (1.1). Then*

$$\begin{aligned} |u^{(l)}(x)| &\le C\delta(x)^{\alpha/2-l} \quad \text{for } x \in \Omega \text{ and } l = 0,1,2,3,4, \\ |f^{(l)}(x)| &\le C\delta(x)^{-\alpha/2-l} \ \text{for } x \in \Omega \text{ and } l = 0,1,2, \end{aligned}$$

for some positive constant $C = C(\Omega, \alpha, \beta, f)$.

Proof. Our hypotheses imply that $2 < \beta < 3$, and $4 < \beta + \alpha < 5$. By Lemma 2.3, we have

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_{\beta}^{(\alpha/2)}\right).$$

By Definition 2.1 and Lemma 2.2, it yields

$$\sum_{l=1}^{4} \sup_{x\in\Omega} \left(\delta(x)^{l-\alpha/2} \left|u^{(l)}(x)\right|\right) \le C\left(\|f\|_{L^{\infty}(\Omega)} + \|f\|_{\beta}^{(\alpha/2)}\right),$$

which is desired result $l = 1,2,3,4$. The case $l = 0$ is covered by (2.13).

The second inequality can be obtained by Definition 2.1, namely,

$$\sum_{l=0}^{2} \sup_{x\in\Omega} \left(\delta(x)^{l+\alpha/2} |f^{(l)}(x)|\right) \le \|f\|_{\beta}^{(\alpha/2)}.$$

The proof is completed. $\square$

2.3. **Main results.** The main results of this paper consist of the following theorems, which will be proved in Section 3 and Section 4, respectively.

Theorem 2.5 (Local Truncation Error). *Let $\alpha \in (1,2)$ and $f \in L^\infty(\Omega) \cap C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, where $\beta = 4 - \alpha + \gamma$ with $0 < \gamma < \alpha - 1$. Then,*

$$\begin{aligned}|\tau_i| &= |-D_h^\alpha \Pi_h u(x_i) - f(x_i)|\\ &\le Ch^{\min\{\frac{r\alpha}{2},2\}}\delta(x_i)^{-\alpha} + C(r-1)h^2(T - \delta(x_i) + h_N)^{1-\alpha}\end{aligned}$$

for some positive constant $C = C(\Omega, \alpha, \beta, r, f)$.

Theorem 2.6 (Global Error). *Let $\alpha \in (1,2)$ and $f \in L^\infty(\Omega) \cap C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, where $\beta = 4 - \alpha + \gamma$ with $0 < \gamma < \alpha - 1$. Let u_i be the approximate solution of $u(x_i)$ computed by the discretization scheme* (2.5)*. Then,*

$$\max_{1 \le i \le 2N-1} |u_i - u(x_i)| \le Ch^{\min\{\frac{r\alpha}{2},2\}}$$

for some positive constant $C = C(\Omega, \alpha, \beta, r, f)$.

3. Local Truncation Error

For convenience, we use the notation $\simeq$, where $x \simeq y$ means that $C_1 x \le y \le C_2 x$ for some positive constants C_1 and C_2 independent of h.

For $1 \le j \le 2N$, we define the combination of adjacent grid points as

$$y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \quad \theta \in (0,1). \tag{3.1}$$

Then, using the definition of grid points $\{x_j\}$ in (2.1), it follows that

Lemma 3.1. *Let $h = \frac{1}{N}$ and $\delta(x_j)$ be defined by* (2.12)*. Then we have*

$$\begin{aligned}&h_j \simeq h_{j+1} \simeq h\delta(x_j)^{1-1/r}, \quad 1 \le j \le 2N-1,\\ &\delta(x_j) \simeq \delta(x_{j+1}) \simeq \delta(y_{j+1}^\theta), \ \ 1 \le j \le 2N-2.\end{aligned}$$

We next give a detailed analysis of the local truncation error.

3.1. **Proof of Theorem 2.5.** The local truncation error (2.8) can be expressed by

$$\begin{aligned}\tau_i &= -D_h^2 I^{2-\alpha}\Pi_h u(x_i) + \frac{d^2}{dx^2} I^{2-\alpha} u(x_i)\\ &= D_h^2 I^{2-\alpha}\left(u - \Pi_h u\right)(x_i) - \left(D_h^2 - \frac{d^2}{dx^2}\right) I^{2-\alpha} u(x_i).\end{aligned} \tag{3.2}$$

We estimate each component of this partition.

Theorem 3.2. *There exists a constant C such that*

$$\left|\left(D_h^2 - \frac{d^2}{dx^2}\right) I^{2-\alpha} u(x_i)\right| \le Ch^2\delta(x_i)^{-\alpha/2-2/r}. \tag{3.3}$$

Proof. Since $f \in C^2(\Omega)$ and $-\frac{d^2}{dx^2} I^{2-\alpha} u(x) = f(x)$ for $x \in \Omega$, it implies $I^{2-\alpha} u \in C^4(\Omega)$. From Lemma A.1 in Appendix A, we have for $1 \le i \le 2N-1$,

$$\begin{aligned}&-\left(D_h^2 - \frac{d^2}{dx^2}\right) I^{2-\alpha} u(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i)\\ &\quad + \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i} f''(y)\frac{(y - x_{i-1})^3}{3!}dy + \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}} f''(y)\frac{(y - x_{i+1})^3}{3!}dy\right).\end{aligned}$$

According to Lemmas 2.4, B.1 and B.2, the desired result is obtained. □

Now we consider the first term of the local truncation error in (3.2), which we denote for simplicity

$$R_i := D_h^2 I^{2-\alpha}(u-\Pi_h u)(x_i), \quad 1 \le i \le 2N-1. \tag{3.4}$$

We have derived the following results concerning the estimation of R_i including Theorems 3.3 and 3.4, which will be demonstrated in Subsection 3.3.

Theorem 3.3. *For $1 \le i < N/2$, there exists a constant C such that*

$$|R_i| \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1>0, \\ Ch^2(x_i^{-1-\alpha}\ln(i)+\ln(N)), & \alpha/2-2/r+1=0, \\ Ch^{r\alpha/2+r}x_i^{-1-\alpha}, & \alpha/2-2/r+1<0. \end{cases}$$

Theorem 3.4. *For $N/2 \le i \le N$, there exists a constant C such that*

$$|R_i| \le C(r-1)h^2(T-x_i+h_N)^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2-2/r+1>0, \\ Ch^2\ln(N), & \alpha/2-2/r+1=0, \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0. \end{cases}$$

Remark 3.5. And for $N < i \le 2N-1$, observe first that the mesh (2.1) is symmetric about $x=T$ (i.e., $x=x_i$ is a mesh point if and only if $x=2T-x_i=x_{2N-i}$ is a mesh point), and the a priori derivative bounds of Lemma 2.4 are also symmetric about $x=T$. But the locations of the mesh points and these bounds on derivatives are the only ingredients used in the analysis of the case $1 \le i \le N$. Thus, one can define $\tilde{u}(x)=u(2T-x)$, and now, the truncation error of $u(x)$ at $x=x_i$ for $i=N+1,N+2,...,2N-1$ is exactly the same as the truncation error of $\tilde{u}(x)$ at $x=x_i$ for $i=N-1,N-2,...1$, which can be handled in exactly the same way as the truncation error analysis of $u(x)$ for $i=1,2,...,N-1$. Transforming back via $x \mapsto 2T-x$, we get the desired result for $i=N+1,N+2,...,2N-1$. This technique will be used several times.

Combine Theorems 3.2 to 3.4 and remark 3.5, and for $1 \le i \le N$, we have

$$h^2 x_i^{-\alpha/2-2/r} \le T^{\alpha/2-2/r} h^{\min\{\frac{r\alpha}{2},2\}} x_i^{-\alpha},$$
$$h^{r\alpha/2+r} x_i^{-1-\alpha} \le T^{-1} h^{r\alpha/2} x_i^{-\alpha},$$
$$h^r x_i^{-1}\ln(i) = T^{-1}\frac{\ln(i)}{i^r} \le T^{-1}, \quad h^r\ln(N) = \frac{\ln(N)}{N^r} \le 1,$$

the proof of Theorem 2.5 completed.

3.2. **Grid mapping functions.** In this subsection, we offer an overview of the framework for estimating R_i, where we introduce the *natural-skew ordering* and *grid mapping functions.*

From (1.3) and (3.4), we know that

$$I^{2-\alpha}(u-\Pi_h u)(x_i) = \sum_{j=1}^{2N}\int_{x_{j-1}}^{x_j}(u(y)-\Pi_h u(y))K(x_i-y)dy = \sum_{j=1}^{2N} T_{ij} \tag{3.5}$$

with

$$T_{ij} = \int_{x_{j-1}}^{x_j}(u(y)-\Pi_h u(y))K(x_i-y)dy, \quad i=0,\cdots,2N,\ j=1,\cdots,2N. \tag{3.6}$$

To estimate R_i more precisely, we define the *vertical difference quotients* of T_{ij}

$$V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right), \tag{3.7}$$

and the *skew difference quotients* of T_{ij}

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j-1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j+1} \right). \tag{3.8}$$

From (3.4), (3.5) and (3.6), we have

$$R_1 = \sum_{j=1}^{3} V_{1,j} + \sum_{j=4}^{2N} V_{1,j} \quad \text{and} \quad R_2 = \sum_{j=1}^{4} V_{2,j} + \sum_{j=5}^{2N} V_{2,j}. \tag{3.9}$$

Moreover, using (3.4)-(3.8), we can express R_i based on the natural-skew ordering, as shown in Figure 1:

$$R_i = I_1 + I_2 + I_3 + I_4 + I_5 \quad \text{for} \quad 3 \leq i \leq N. \tag{3.10}$$

Here,

$$I_1 = \sum_{j=1}^{k-1} V_{ij}, \quad I_3 = \sum_{j=k+1}^{m-1} S_{ij}, \quad I_5 = \sum_{j=m+1}^{2N} V_{ij} \quad \text{for} \quad k = \lceil i/2 \rceil,$$

and

$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right),$$
$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,m} + T_{i-1,m-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,m} \right)$$

with

$$m = \begin{cases} 2i, & 3 \leq i < N/2, \\ 2N - \lceil N/2 \rceil + 1, & N/2 \leq i \leq N. \end{cases} \tag{3.11}$$

Noted that I_1 and I_5 along with V_{ij} as defined in (3.7), represent natural (vertical) ordering, while I_3, along with S_{ij} as defined in (3.8), represents skew ordering, which is referred to as natural-skew ordering here.

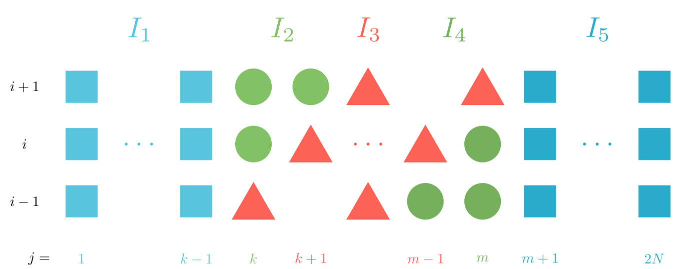


FIGURE 1. Natural-Skew ordering of R_i.

The complexity in estimating S_{ij} in (3.8) lies in the fact that the integral domains for $T_{i-1,j-1}, T_{i,j}$ and $T_{i+1,j+1}$ are distinct. We first normalize T_{ij} to the unit interval.

Lemma 3.6. *For any $y \in (x_{j-1}, x_j)$, there exits*

$$
\begin{aligned}
T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) K(x_i - y) dy \\
&= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) K(x_i - y_j^\theta) h_j d\theta \\
&= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) K(x_i - y_j^\theta) d\theta \\
&\quad + \int_0^1 \frac{\theta(1-\theta)}{3!} h_j^4 K(x_i - y_j^\theta) \left(\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)\right) d\theta
\end{aligned}
$$

with $\eta_{j1}^\theta \in (x_{j-1}, y_j^\theta), \eta_{j2}^\theta \in (y_j^\theta, x_j)$.

Proof. By (3.6) and Lemma A.2, the desired result is obtained. □

To estimate the local truncation error more concisely, we construct the following grid mapping functions.

Definition 3.7. For $1 \le i, j \le 2N-1$, we define the grid mapping functions

$$
y_{i,j}(x) = \begin{cases}
(x^{1/r} + Z_{j-i})^r & i < N, j < N, \\
\frac{x^{1/r} - Z_i}{Z_1} h_N + x_N & i < N, j = N, \\
2T - (Z_{2N-(j-i)} - x^{1/r})^r & i < N, j > N, \\
\left(\frac{Z_1}{h_N}(x - x_N) + Z_j\right)^r & i = N, j < N, \\
x & i = N, j = N, \\
2T - \left(\frac{Z_1}{h_N}(2T - x - x_N) + Z_{2N-j}\right)^r & i = N, j > N, \\
(Z_{2N+j-i} - (2T - x)^{1/r})^r & i > N, j < N, \\
\frac{Z_{2N-j} - (2T-x)^{1/r}}{Z_1} h_N + x_N & i > N, j = N, \\
2T - ((2T - x)^{1/r} - Z_{j-i})^r & i > N, j > N
\end{cases} \tag{3.12}
$$

with $Z_j := T^{1/r} \frac{j}{N}$.

Let us further define

$$h_{i,j}(x) = y_{i,j}(x) - y_{i,j-1}(x), \tag{3.13}$$

$$y_{i,j}^\theta(x) = (1-\theta) y_{i,j-1}(x) + \theta y_{i,j-1}(x), \quad \theta \in (0,1), \tag{3.14}$$

$$P_{i,j}^\theta(x) = (h_{i,j}(x))^3 K(x - y_{i,j}^\theta(x)) u''(y_{i,j}^\theta(x)), \tag{3.15}$$

$$Q_{i,j,l}^\theta(x) = (h_{i,j}(x))^l K(x - y_{i,j}^\theta(x)) u''(y_{i,j}^\theta(x)), \quad l = 3, 4. \tag{3.16}$$

Then, we can check that

$$
\begin{aligned}
&y_{i,j}(x_{i-1}) = x_{j-1}, \quad y_{i,j}(x_i) = x_j, \quad y_{i,j}(x_{i+1}) = x_{j+1}, \\
&h_{i,j}(x_{i-1}) = h_{j-1}, \quad h_{i,j}(x_i) = h_j, \quad h_{i,j}(x_{i+1}) = h_{j+1}, \\
&y_{i,j}^\theta(x_{i-1}) = y_{j-1}^\theta, \quad y_{i,j}^\theta(x_i) = y_j^\theta, \quad y_{i,j}^\theta(x_{i+1}) = y_{j+1}^\theta.
\end{aligned} \tag{3.17}
$$

Now, we can rewrite T_{ij} by (3.15) in (3.6) as

$$\begin{aligned} T_{ij} = & \int_0^1 -\frac{\theta(1-\theta)}{2} P^{\theta}_{i,j}(x_i) d\theta \\ & + \int_0^1 \frac{\theta(1-\theta)}{3!} Q^{\theta}_{i,j,4}(x_i) \left[\theta^2 u'''(\eta^{\theta}_{j,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{j,2})\right] d\theta. \end{aligned} \tag{3.18}$$

From (2.6), (3.8) and (3.18), for $1 \le i \le 2N-1$, $2 \le j \le 2N-1$, we have

$$\begin{aligned} S_{ij} = & \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P^{\theta}_{i,j}(x_i) d\theta \\ & + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{i,j,4}(x_{i+1}) u'''(\eta^{\theta}_{j+1,1}) - Q^{\theta}_{i,j,4}(x_i) u'''(\eta^{\theta}_{j,1})}{h_{i+1}} \right) d\theta \\ & - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{i,j,4}(x_i) u'''(\eta^{\theta}_{j,1}) - Q^{\theta}_{i,j,4}(x_{i-1}) u'''(\eta^{\theta}_{j-1,1})}{h_i} \right) d\theta \\ & - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{i,j,4}(x_{i+1}) u'''(\eta^{\theta}_{j+1,2}) - Q^{\theta}_{i,j,4}(x_i) u'''(\eta^{\theta}_{j,2})}{h_{i+1}} \right) d\theta \\ & + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{i,j,4}(x_i) u'''(\eta^{\theta}_{j,2}) - Q^{\theta}_{i,j,4}(x_{i-1}) u'''(\eta^{\theta}_{j-1,2})}{h_i} \right) d\theta. \end{aligned} \tag{3.19}$$

The derivatives of the grid mapping functions are calculated as follows.

Lemma 3.8. *For $1 \le i, j \le 2N-1$, there exist,*

$$y'_{i,j}(x) = \begin{cases} y_{i,j}^{1-1/r}(x) x^{1/r-1} & i < N, j < N, \\ \dfrac{h_N}{rZ_1} x^{1/r-1} & i < N, j = N, \\ (2T - y_{i,j}(x))^{1-1/r} x^{1/r-1} & i < N, j > N, \\ y_{i,j}^{1-1/r}(x) \dfrac{rZ_1}{h_N} & i = N, j < N, \\ 1 & i = N, j = N, \end{cases}$$

and

$$y''_{i,j}(x) = \frac{1-r}{r} \begin{cases} y_{i,j}^{1-2/r}(x) x^{1/r-2} Z_{j-i} & i < N, j < N, \\ \dfrac{h_N}{rZ_1} x^{1/r-2} & i < N, j = N, \\ (2T - y_{i,j}(x))^{1-2/r} x^{1/r-2} Z_{2N-j+i} & i < N, j > N, \\ -y_{i,j}^{1-2/r}(x) \left(\dfrac{rZ_1}{h_N}\right)^2 & i = N, j < N, \\ 0 & i = N, j = N. \end{cases}$$

Proof. The desired results can be obtained by Definition 3.7 directly. ☐

The following lemmas about the grid mapping functions will be used in next subsection. They are proved in Appendix C.

Lemma 3.9. *For any $\xi \in (x_{i-1}, x_{i+1})$, $2 \le i, j \le 2N-2$, there exist*

$$\xi \simeq x_i, \quad \delta(y_{i,j}(\xi)) \simeq \delta(x_j), \quad h_{i,j}(\xi) \simeq h_j,$$
$$|y_{i,j}(\xi) - \xi| \simeq |x_j - x_i|, \quad |y_{i,j-1}(\xi) - \xi| \simeq |x_{j-1} - x_i|,$$
$$|y^\theta_{i,j}(\xi) - \xi| = (1-\theta)|y_{i,j-1}(\xi) - \xi| + \theta|y_{i,j}(\xi) - \xi| \simeq |y^\theta_j - x_i|.$$

Lemma 3.10. *For any $\xi \in (x_{i-1}, x_{i+1})$, $2 \le i \le N, 2 \le j \le 2N-2$, there exist*

$$|h'_{i,j}(\xi)| \le C(r-1)Z_1 x_i^{1/r-1}\delta(x_j)^{1-2/r} \le C(r-1)h_j x_i^{1/r-1}\delta(x_j)^{-1/r},$$
$$|(y_{i,j}(\xi) - \xi)'| \le C x_i^{-1}|x_j - x_i|.$$

Lemma 3.11. *For any $\xi \in (x_{i-1}, x_{i+1})$, $2 \le i \le N, 2 \le j \le 2N-2$, there exist*

$$|y''_{i,j}(\xi)| \le C(r-1)\begin{cases} x_j^{-1/r}x_i^{1/r-2}|x_j - x_i| & i < N, j < N, \\ x_N^{1-1/r}x_i^{1/r-2} & i < N, j = N, \\ \delta(x_j)^{1-2/r}x_i^{1/r-2}x_N^{1/r} & i < N, j > N, \\ \delta(x_j)^{1-2/r}x_N^{2/r-2} & i = N, j \ne N, \\ 0 & i = N, j = N. \end{cases}$$

For $2 \le i \le N, 3 \le j \le 2N-2$, there exist

$$|h''_{i,j}(\xi)| \le C(r-1)\begin{cases} Z_1 x_i^{1/r-2}x_j^{-2/r}(|x_j - x_i| + x_j) & i < N, j < N, \\ x_i^{1/r-2}x_N^{1-1/r} & i < N, j = N, N+1, \\ Z_1 x_i^{1/r-2}\delta(x_j)^{1-3/r}x_N^{1/r} & i < N, j > N+1, \\ Z_1 x_N^{2/r-2}\delta(x_j)^{1-3/r} & i = N, j \ne N, N+1, \\ x_N^{-1} & i = N, j = N. \end{cases}$$

Lemma 3.12. *Let $P^\theta_{i,j}(x_i)$ be defined by (3.15) and the difference quotient operator D_h^2 be defined by (2.6). Then we have*

Case 1. *For $3 \le i < N, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, there exists*

$$|D_h^2 P^\theta_{i,j}(x_i)| \le C h_j^3 |y^\theta_j - x_i|^{1-\alpha} x_i^{\alpha/2-4}.$$

Case 2. *For $N/2 \le i \le N$, $j = N, N+1$, there exists*

$$|D_h^2 P^\theta_{i,j}(\xi)| \le C h_j^3 |y^\theta_j - x_i|^{1-\alpha} + C(r-1)h_j^2\Big(|y^\theta_j - x_i|^{1-\alpha} + h_j|y^\theta_j - x_i|^{-\alpha}\Big).$$

Case 3. *For $N/2 \le i \le N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil$, there exists*

$$|D_h^2 P^\theta_{i,j}(\xi)| \le C h_j^3\Big(|y^\theta_j - x_i|^{1-\alpha} + (r-1)|y^\theta_j - x_i|^{-\alpha}\Big).$$

Lemma 3.13. *Let $Q^\theta_{i,j,l}(x_i)$ be defined by (3.16). Then we have for $2 \le i \le N$, $2 \le j \le 2N-2$, $l = 3, 4$, there exist*

$$\left|\frac{Q^\theta_{i,j,l}(x_{i+1})u^{(l-1)}(\eta^\theta_{j+1}) - Q^\theta_{i,j,l}(x_i)u^{(l-1)}(\eta^\theta_j)}{h_{i+1}}\right|$$
$$\le C h_j^l |y^\theta_j - x_i|^{1-\alpha} x_i^{-1}\delta(x_j)^{\alpha/2-l+1-1/r}(x_i^{1/r} + \delta(x_j)^{1/r}),$$

and

$$\left|\frac{Q^\theta_{i,j,l}(x_i)u^{(l-1)}(\eta^\theta_j) - Q^\theta_{i,j,l}(x_{i-1})u^{(l-1)}(\eta^\theta_{j-1})}{h_i}\right|$$
$$\le C h_j^l |y^\theta_j - x_i|^{1-\alpha} x_i^{-1}\delta(x_j)^{\alpha/2-l+1-1/r}(x_i^{1/r} + \delta(x_j)^{1/r})$$

with $\eta_j^\theta \in (x_{j-1}, x_j)$.

3.3. **Error analysis of R_i.** In this subsection, we estimate the first term of the local truncation error R_i in (3.4) through (3.9) and (3.10). We denote

$$K_y(x) := K(x-y) = \frac{\kappa_\alpha}{\Gamma(2-\alpha)}|x-y|^{1-\alpha}, \quad 1<\alpha<2, \tag{3.20}$$

where the kernel function $K(x)$ is given in (1.3) and κ_α is given in (2.11).

Lemma 3.14. *Let $I_5 = \sum_{j=m+1}^{2N} V_{ij}$ be defined by* (3.10). *Then we have*
Case 1. *For $1 \le i < N/2$ and $m = \max\{2i, 3\}$, there exists*

$$\sum_{j=m+1}^{2N} |V_{ij}| \le Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1>0,\\ Ch^2\ln(N), & \alpha/2-2/r+1=0,\\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0. \end{cases}$$

Case 2. *For $N/2 \le i \le N$ and $m = 2N - \lceil \frac{N}{2} \rceil + 1$, there exists*

$$\sum_{j=m+1}^{2N} |V_{ij}| \le \begin{cases} Ch^2, & \alpha/2-2/r+1>0,\\ Ch^2\ln(N), & \alpha/2-2/r+1=0,\\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0. \end{cases}$$

Proof. For $1 \le i < N/2$, $m+1 \le j \le 2N$ with $m = \max\{2i, 3\}$, using (3.6), (3.7), (3.20), Lemmas A.3 and B.3, we have

$$\begin{aligned} |V_{ij}| &= \left| \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y))\, D_h^2 K_y(x_i) dy \right| \\ &\le Ch^2 \int_{x_{j-1}}^{x_j} \delta(y)^{\alpha/2-2/r} |x_i - y|^{-1-\alpha} dy. \end{aligned}$$

Since $y \ge x_{j-1} \ge x_{2i}$, $y - x_i \simeq y$, and $x_i \simeq x_{2i}$, it yields

$$\begin{aligned} \sum_{j=m+1}^{N} |V_{ij}| &\le Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\ &= \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\ &\le Ch^2 x_i^{-\alpha/2-2/r}. \end{aligned}$$

On the other hand, since $y - x_i \simeq T$ if $y \ge x_N = T$, there exist

$$\begin{aligned} \sum_{j=N+1}^{2N-1} |V_{ij}| &\le CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r} dy \\ &\le \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r}\ h^2, & \alpha/2-2/r+1>0,\\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2-2/r+1=0,\\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2-2/r+1<0. \end{cases} \end{aligned}$$

Finally, by Lemma A.4, one has

$$|V_{i,2N}| \le CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}.$$

Then, the desired result in Case 1 is obtained. We can similarly prove for Case 2, the details are omitted here. □

Immediately, we can calculate R_1, R_2 from (3.9).

Lemma 3.15. *For $i = 1, 2$, we have*

$$|R_i| \le Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

Proof. According to (3.9), Lemmas 3.14 and B.4, the desired result is obtained. □

For R_i with $3 \le i \le N$, the terms $\{I_1, I_2, I_3, I_4\}$ in (3.10) remain to be estimated.

Lemma 3.16. *Let $I_1 = \sum_{j=1}^{k-1} V_{ij}$ be defined by (3.10). Then we have, for $3 \le i \le N, k = \lceil \frac{i}{2} \rceil$,*

$$\sum_{j=1}^{k-1} |V_{ij}| \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0, \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0, \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

Proof. According to (3.7), Lemmas A.4 and B.3, it yields

$$|V_{i1}| \le C \int_0^{x_1} x_1^{\alpha/2} |x_i - y|^{-1-\alpha} dy \simeq x_1^{\alpha/2+1} x_i^{-1-\alpha} = T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha}.$$

Using Lemma A.3, Lemma B.3 and $y \le x_{k-1} < 2^{-r} x_i$, $x_i - y \simeq x_i$, we have

$$|V_{ij}| \le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} x_i^{-1-\alpha} dy, \quad 2 \le j \le k-1,$$

and

$$\sum_{j=2}^{k-1} |V_{ij}| \le Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy.$$

Moreover we can check that

$$\int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \le \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1} & \alpha/2 - 2/r + 1 > 0, \\ \ln(2^{-r} x_i) - \ln(x_1) & \alpha/2 - 2/r + 1 = 0, \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1} & \alpha/2 - 2/r + 1 < 0. \end{cases}$$

The proof is completed. □

Subsequently, we turn our attention to $I_3 = \sum_{j=k+1}^{m-1} S_{ij}$ with $m = 2i$ for $3 \le i < N/2$ and $m = 2N - \lceil N/2 \rceil + 1$ for $N/2 \le i \le N$ in (3.11).

Lemma 3.17. *Let $I_3 = \sum_{j=k+1}^{m-1} S_{ij}$ be defined by (3.10). Then we have*
Case 1. *For $N/2 \le i \le N$, $m = 2N - \lceil N/2 \rceil + 1$, there exist*

$$|S_{ij}| \le C(h^3 + (r-1)h^2)(T - x_i + h_N)^{1-\alpha}, \quad j = N, N+1,$$

and

$$\sum_{j=N+2}^{m-1} |S_{ij}| \le Ch^2 + C(r-1)h^2 (T - x_i + h_N)^{1-\alpha}.$$

Case 2. *For* $3 \le i \le N-1$, $k = \lceil \frac{i}{2} \rceil$, *there exist*

$$\sum_{j=k+1}^{\min\{m-1,N-1\}} |S_{ij}| \le Ch^2 x_i^{-\alpha/2-2/r},$$

and

$$\sum_{j=\lceil \frac{N}{2} \rceil+1}^{N-1} |S_{Nj}| \le Ch^2 + C(r-1)h^2 h_N^{1-\alpha}.$$

Proof. Case 1: From (3.19), using $\theta(1-\theta)h_j \le |y_j^\theta - x_i|$, Lemmas 3.1, 3.12 and 3.13, it yields

$$|S_{ij}| \le C(h_j^3 + (r-1)h_j^2) \int_0^1 |y_j^\theta - x_i|^{1-\alpha} d\theta, \quad j = N, N+1$$

with

$$\int_0^1 |y_j^\theta - x_i|^{1-\alpha} dy \simeq (|x_j - x_i| + h_N)^{1-\alpha}.$$

On the other hand, for $j \ge N+2$, $x_i \simeq x_j \simeq T$, we have

$$\begin{aligned} |S_{ij}| &\le Ch_j^2 \int_0^1 \left(|y_j^\theta - x_i|^{1-\alpha} + (r-1)|y_j^\theta - x_i|^{-\alpha} \right) h_j d\theta \\ &\le Ch^2 \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy. \end{aligned}$$

It implies that

$$\begin{aligned} \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} |S_{ij}| &= Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil \frac{N}{2} \rceil}} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} dy \\ &\le Ch^2 \left(T^{2-\alpha} + (r-1)(T - x_i + h_N)^{1-\alpha} \right). \end{aligned}$$

Case 2: for $3 \le i \le N-1$, $k+1 \le j \le \min\{m-1, N-1\}$, using Lemmas 3.1, 3.12 and 3.13, $x_i \simeq x_j$ and $h_i \simeq h_j$, we have

$$\begin{aligned} |S_{ij}| &\le Ch_j^2 x_i^{\alpha/2-4} \int_0^1 |y_j^\theta - x_i|^{1-\alpha} h_j d\theta \\ &= Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_{j-1}}^{x_j} |y - x_i|^{1-\alpha} dy, \end{aligned}$$

and

$$\begin{aligned} \sum_{k+1}^{\min\{2i-1,N-1\}} |S_{ij}| &\le Ch^2 x_i^{\alpha/2-2-2/r} \int_{x_k}^{x_{\min\{2i-1,N-1\}}} |y - x_i|^{1-\alpha} dy \\ &\le Ch^2 x_i^{\alpha/2-2-2/r} x_i^{2-\alpha} = Ch^2 x_i^{-\alpha/2-2/r}. \end{aligned}$$

We can similarly prove the last inequality by Case 1. The proof is completed. □

Finally, we focus our error analysis on the terms I_2 and I_4.

Lemma 3.18. *Let* I_2, I_4 *be defined by* (3.10)*. Then we have*
Case 1. *For* $3 \le i \le N$, $k = \lceil \frac{i}{2} \rceil$, *there exists*

$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \le Ch^2 x_i^{-\alpha/2-2/r}.$$

Case 2. *For $3 \le i < N/2$, $m = 2i$, there exists*

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,2i} + T_{i-1,2i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \le Ch^2 x_i^{-\alpha/2-2/r}.$$

Case 3. *For $N/2 \le i \le N$, $m = N - \lceil \frac{N}{2} \rceil + 1$, there exists*

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,m} + T_{i-1,m-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,m} \right) \le Ch^2.$$

Proof. Since

$$\begin{aligned}
&\frac{1}{h_{i+1}} \left(T_{i+1,k} + T_{i+1,k+1} \right) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \\
&= \frac{1}{h_{i+1}} \left(T_{i+1,k} - T_{i,k} \right) + \frac{1}{h_{i+1}} \left(T_{i+1,k+1} - T_{i,k} \right) + \left(\frac{1}{h_{i+1}} - \frac{1}{h_i} \right) T_{i,k}.
\end{aligned} \tag{3.21}$$

According to $x_i - x_k \simeq x_i \simeq x_k$, Lemmas 3.1, A.3 and B.3, we have

$$\begin{aligned}
\frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) &= \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y)) D_h K_y(x_i) dy \\
&\le Ch_k^2 x_k^{\alpha/2-2}\ h_k |x_i - x_k|^{-\alpha} \le Ch^2 x_i^{-\alpha/2-2/r} h_k.
\end{aligned}$$

From Lemmas 3.6 and A.2 and (3.16), we can obtain

$$\frac{1}{h_{i+1}} \left(T_{i+1,k+1} - T_{i,k} \right) = \int_0^1 \frac{\theta(\theta-1)}{2} \frac{Q^\theta_{i,k;3}(x_{i+1}) u''(\eta^\theta_{k+1}) - Q^\theta_{i,k;3}(x_i) u''(\eta^\theta_k)}{h_{i+1}} d\theta$$

with $\eta^\theta_k \in (x_{k-1}, x_k)$ and $\eta^\theta_{k+1} \in (x_k, x_{k+1})$. Using Lemmas 3.1 and 3.13, we have

$$\frac{1}{h_{i+1}} |T_{i+1,k+1} - T_{i,k}| \le Ch^2 x_i^{-\alpha/2-2/r} h_k.$$

For the third term in (3.21), using $h_i \simeq h_k$, Lemmas 3.1, A.3 and B.1, it yields

$$\begin{aligned}
\frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,k} &\le C(r-1) h_i^{-2} h^2 x_i^{1-2/r}\ h_k^3 x_k^{\alpha/2-2} |x_k - x_i|^{1-\alpha} \\
&\le C(r-1) h^2 x_i^{-\alpha/2-2/r} h_k.
\end{aligned}$$

Then, the desired result of Case 1 is obtained. The Case 2 and Case 3 for I_4 can be similarly proven as the way in Case 1; the details are omitted here. □

Proof of Theorem 3.3. For $1 \le i < N/2$ with $m = 2i$ in (3.10), combining Lemma 3.15, Lemma 3.16, Cases 1 and 2 of Lemma 3.18, Case 2 of Lemma 3.17 and Case 1 of Lemma 3.14, the proof is completed. □

Proof of Theorem 3.4. For $N/2 \le i \le N$ with $m = 2N - \lceil N/2 \rceil + 1$ in (3.10), we split I_3 as

$$I_3 = \sum_{j=k+1}^{m-1} S_{ij} = \sum_{j=k+1}^{N-1} S_{ij} + (S_{iN} + S_{i,N+1}) + \sum_{j=N+2}^{m-1} S_{ij}. \tag{3.22}$$

According to Lemma 3.16, Cases 1 and 3 of Lemma 3.18, Lemma 3.17 and Case 2 of Lemma 3.14, the desired result is obtained. □

4. Convergence analysis

We can now prove our main convergence result for Theorem 2.6.

4.1. **Some properties of the stiffness matrix.** In this subsection, we show some properties of the stiffness matrix A defined by (2.10) and construct an appropriate right-preconditioner for the resulting matrix algebraic equation.

Lemma 4.1. *The stiffness matrix A defined by* (2.10) *is strictly diagonally dominant, with positive entries on the main diagonal and negative off-diagonal entries. In particular, there exists a constant C_A such that*

$$\sum_{j=1}^{2N-1} a_{ij} \geq C_A(x_i^{-\alpha} + (2T - x_i)^{-\alpha})$$

with $C_A = \frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)} 2^{-r\alpha}$.

Proof. Let $C_j := \left(\frac{1}{h_j}, -\frac{1}{h_j} - \frac{1}{h_{j+1}}, \frac{1}{h_{j+1}}\right)$ and

$$D_{ij} := \begin{pmatrix} |x_{i-1} - x_{j-1}|^{3-\alpha} & |x_{i-1} - x_j|^{3-\alpha} & |x_{i-1} - x_{j+1}|^{3-\alpha} \\ |x_i - x_{j-1}|^{3-\alpha} & |x_i - x_j|^{3-\alpha} & |x_i - x_{j+1}|^{3-\alpha} \\ |x_{i+1} - x_{j-1}|^{3-\alpha} & |x_{i+1} - x_j|^{3-\alpha} & |x_{i+1} - x_{j+1}|^{3-\alpha} \end{pmatrix}.$$

From (2.11), we have

$$a_{ij} = \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{2}{h_i + h_{i+1}} C_i D_{ij} C_j^T$$

with $\mathrm{sign}(a_{ij}) = \mathrm{sign}(a_{ji})$. For $i = j$, there exists

$$a_{ii} = \frac{\kappa_\alpha}{\Gamma(4-\alpha)} \frac{4}{h_i h_{i+1}} \left(h_i^{2-\alpha} + h_{i+1}^{2-\alpha} - (h_i + h_{i+1})^{2-\alpha}\right) > 0,$$

where we use $1 + t^\theta > (1+t)^\theta$ with $t = \frac{h_{i+1}}{h_i}$ for $\theta \in (0,1)$.

For $j = i - 1$, we can check that

$$\begin{aligned} C_i D_{i,i-1} C_{i-1}^T = & \frac{1}{h_{i-1} h_i h_{i+1}} \Big(h_{i+1} h_{i-1}^{3-\alpha} - (h_i + h_{i+1})(h_{i-1} + h_i)^{3-\alpha} \\ & + h_i (h_{i-1} + h_i + h_{i-1})^{3-\alpha} + (h_{i-1} + h_i)(h_i + h_{i+1}) h_i^{2-\alpha} \\ & - (h_{i-1} + h_i)(h_i + h_{i+1})^{3-\alpha} + h_{i-1} h_{i+1} h_i^{2-\alpha} + h_{i-1} h_{i+1}^{3-\alpha}\Big). \end{aligned}$$

Let $s = \frac{h_{i-1}}{h_i}$ and $t = \frac{h_{i+1}}{h_i}$. Then by Lemma D.2, we have

$$\begin{aligned} C_i D_{i,i-1} C_{i-1}^T = & \frac{h_i^{3-\alpha}}{h_{i-1} h_{i+1}} \Big(st(1 + s^{2-\alpha} + t^{2-\alpha}) + (1+s+t)^{3-\alpha} \\ & - (1+s)(1+t)\left((1+s)^{2-\alpha} + (1+t)^{2-\alpha} - 1\right)\Big) > 0, \end{aligned}$$

which implies that

$$a_{i,i-1} = \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{2}{h_i + h_{i+1}} C_i D_{i,i-1} C_{i-1}^T < 0.$$

For $|i - j| \geq 2$, $x_{i+1} - y$, $x_i - y$ and $x_{i-1} - y$ have the same sign (> 0 or < 0) for $y \in (x_{j-1}, x_{j+1})$, it yields

$$\frac{h_i}{h_i + h_{i+1}} |x_{i+1} - y| + \frac{h_{i+1}}{h_i + h_{i+1}} |x_{i-1} - y| = |x_i - y|.$$

Since $x^{1-\alpha}$ is a convex function for $\alpha \in (1,2)$, by Jensen's inequality, we have

$$\frac{h_i}{h_i+h_{i+1}}|x_{i+1}-y|^{1-\alpha}+\frac{h_{i+1}}{h_i+h_{i+1}}|x_{i-1}-y|^{1-\alpha} > |x_i-y|^{1-\alpha},$$

which implies that $D_h^2 K_y(x_i) > 0$ by (2.6) and (3.20). Thus, from (2.11), we get

$$a_{ij} = -D_h^2 I^{2-\alpha}\phi_j(x_i) = -\int_{x_{j-1}}^{x_{j+1}} \phi_j(y) D_h^2 K_y(x_i)dy < 0.$$

We next prove that the stiffness matrix A defined by (2.10) is strictly diagonally dominant. For the quadrature coefficients $\tilde{a}_{ij}$ in (2.11), we calculate that

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = g_0(x_i) + g_{2N}(x_i)$$

with

$$\begin{aligned} g_0(x) &= \frac{-\kappa_\alpha}{\Gamma(4-\alpha)}\frac{|x-x_0|^{3-\alpha}-|x-x_1|^{3-\alpha}}{h_1}, \\ g_{2N}(x) &= \frac{-\kappa_\alpha}{\Gamma(4-\alpha)}\frac{|x_{2N}-x|^{3-\alpha}-|x_{2N-1}-x|^{3-\alpha}}{h_{2N}}. \end{aligned}$$

It implies that

$$\sum_{j=1}^{2N-1} a_{ij} = D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i).$$

For $i = 1$, there exists

$$\begin{aligned} D_h^2 g_0(x_1) &= \frac{2}{h_1+h_2}\left(\frac{1}{h_2}g_0(x_2) - (\frac{1}{h_1}+\frac{1}{h_2})g_0(x_1) + \frac{1}{h_1}g_0(x_0)\right) \\ &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)}\frac{1+(2^r-1)^{3-\alpha}+2(2^r-1)-(2^r)^{3-\alpha}}{2^r(2^r-1)}x_1^{-\alpha} \\ &\geq \frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)}2^{-r\alpha}x_1^{-\alpha}. \end{aligned}$$

For $i \geq 2$, using Lemma A.1, it leads to

$$\begin{aligned} D_h^2 g_0(x_i) = g_0''(\xi), &\quad \xi \in (x_{i-1}, x_{i+1}) \\ &= -\kappa_\alpha \frac{|\xi-x_0|^{1-\alpha}-|\xi-x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1} \\ &= \frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)}|\xi-\eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\ &\geq \frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)}x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha(\alpha-1)}{\Gamma(2-\alpha)}2^{-r\alpha}x_i^{-\alpha}. \end{aligned}$$

Then we have $D_h^2 g_0(x_i) \geq C_A x_i^{-\alpha}$ for $i \geq 1$. We can similarly prove $D_h^2 g_{2N}(x_i) \geq C_A(2T-x_i)^{-\alpha}$. The proof is completed. □

Let us first introduce the quasi-preconditioner

$$G = \mathrm{diag}(\delta(x_1), ..., \delta(x_{2N-1})), \tag{4.1}$$

where $\delta(x)$ is defined by (2.12). Then we have

Lemma 4.2. *Let $\tilde{B} := AG$ and A be defined by (2.10). Then the matrix $\tilde{B} = (\tilde{b}_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$ has positive entries on the main diagonal and negative off-diagonal entries. In particular, there exist constants $C_{\tilde{B}}, C_B$ such that*

$$\sum_{j=1}^{2N-1} \tilde{b}_{ij} \geq C_B(T-\delta(x_i)+h_N)^{1-\alpha} - C_{\tilde{B}}(x_i^{1-\alpha} + (2T-x_i)^{1-\alpha}).$$

with $C_B = \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}$, $C_{\tilde{B}} = \frac{\kappa_\alpha}{\Gamma(2-\alpha)} 2^{r(\alpha-1)}$.

Proof. From (2.11) and (4.1), it yields

$$\tilde{b}_{ij} = a_{ij}\delta(x_j) = -\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}\tilde{a}_{i+1,j} - (\frac{1}{h_i}+\frac{1}{h_{i+1}})\tilde{a}_{i,j} + \frac{1}{h_i}\tilde{a}_{i-1,j}\right)\delta(x_j).$$

Since $\delta(x) \equiv \Pi_h\delta(x) = \sum_{j=1}^{2N-1}\phi_j(x)\delta(x_j)$ by (2.3) and (2.12), from the definition of the quadrature coefficients $\tilde{a}_{ij}$ in (2.11), we have

$$\sum_{j=1}^{2N-1}\tilde{a}_{ij}\delta(x_j) = \sum_{j=1}^{2N-1} I^{2-\alpha}\phi_j(x_i)\delta(x_j) = I^{2-\alpha}\delta(x_i) = p(x_i) + q(x_i)$$

with

$$p(x) = \frac{-2\kappa_\alpha}{\Gamma(4-\alpha)}|T-x|^{3-\alpha} \quad \text{and} \quad q(x) = \frac{\kappa_\alpha}{\Gamma(4-\alpha)}\left(x^{3-\alpha} + (2T-x)^{3-\alpha}\right).$$

Thus, we have

$$\sum_{j=1}^{2N-1}\tilde{b}_{ij} = \sum_{j=1}^{2N-1} a_{ij}\delta(x_j) = -D_h^2 p(x_i) - D_h^2 q(x_i).$$

For $i \neq N$, by Lemma A.1, it leads to

$$\begin{aligned} -D_h^2 p(x_i) &= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}|T-\xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1}) \\ &\geq C_B(T-\delta(x_i)+h_N)^{1-\alpha} \text{ with } C_B = \frac{2\kappa_\alpha}{\Gamma(2-\alpha)}, \end{aligned}$$

and for $i = N$, it yields

$$-D_h^2 p(x_N) = \frac{4\kappa_\alpha}{\Gamma(4-\alpha)h_N^2} h_N^{3-\alpha} = \frac{4\kappa_\alpha}{\Gamma(4-\alpha)}(T-\delta(x_N)+h_N)^{1-\alpha}.$$

We can similarly prove the following inequality.

$$D_h^2 q(x_i) \leq C_{\tilde{B}}(x_i^{1-\alpha} + (2T-x_i)^{1-\alpha}), \quad i = 1, \cdots, 2N-1$$

The proof is completed. $\square$

Noted that $\tilde{B} = AG$ in Lemma 4.2 is not diagonally dominant, e.g., $\sum_{j=1}^{2N-1}\tilde{b}_{ij} < 0$ if x_i is near the boundary. We introduce the preconditioner $\lambda I + \mu G$ as following.

Lemma 4.3. *Let $B := A(\lambda I + \mu G)$ with $\lambda = 1 + 2TC_{\tilde{B}}/C_B$ and $\mu = C_A/C_B$. Then the matrix $B = (b_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$ is strictly diagonally dominant, with positive entries on the main diagonal and negative off-diagonal entries. In particular, there exists*

$$\sum_{j=1}^{2N-1} b_{ij} \geq C_A\left((x_i^{-\alpha} + (2T-x_i)^{-\alpha}) + (T-\delta(x_i)+h_N)^{1-\alpha}\right).$$

Proof. From Lemmas 4.1 and 4.2, we have

$$\begin{aligned}\sum_{j=1}^{2N-1} b_{ij} &= \sum_{j=1}^{2N-1} \left(\lambda a_{ij} + \mu \tilde{b}_{ij}\right) \\ &\geq \lambda C_A \left(x_i^{-\alpha} + (2T - x_i)^{-\alpha}\right) - \mu C_{\tilde{B}} \left(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}\right) \\ &\quad + \mu C_B \left(T - \delta(x_i) + h_N\right)^{1-\alpha}.\end{aligned}$$

Since $2T > x_i$, the proof is completed. □

4.2. **Proof of Theorem 2.6.** Let $\epsilon_i = u(x_i) - u_i$ with $\epsilon_0 = \epsilon_{2N} = 0$. Subtracting (2.7) from (2.9), we get

$$A\epsilon = \tau, \tag{4.2}$$

where $\epsilon = [\epsilon_1, \epsilon_2, ..., \epsilon_{2N-1}]^T$ and $\tau = [\tau_1, \tau_2, ..., \tau_{2N-1}]^T$ with τ_i in (2.8).

Let $\lambda I + \mu G$ be the right-preconditioner and $B = A(\lambda I + \mu G)$ defined in Lemma 4.3. Then we can rewrite (4.2) as

$$B(\lambda I + \mu G)^{-1}\epsilon = \tau, \quad \text{i.e.} \quad \sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + \mu\delta(x_j)} = \tau_i.$$

Assume that

$$\left|\frac{\epsilon_{i_0}}{\lambda + \mu\delta(x_{i_0})}\right| = \max_{1\leq j\leq 2N-1} \left|\frac{\epsilon_j}{\lambda + \mu\delta(x_j)}\right|.$$

From Lemma 4.3 with $b_{ii} > 0$ and $b_{ij} < 0, i \neq j$, it yields

$$\begin{aligned}|\tau_{i_0}| &= \left|\sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + \mu\delta(x_j)}\right| \\ &\geq b_{i_0,i_0} \left|\frac{\epsilon_{i_0}}{\lambda + \mu\delta(x_{i_0})}\right| - \sum_{j\neq i_0} |b_{i_0,j}| \left|\frac{\epsilon_j}{\lambda + \mu\delta(x_j)}\right| \\ &\geq b_{i_0,i_0} \left|\frac{\epsilon_{i_0}}{\lambda + \mu\delta(x_{i_0})}\right| - \sum_{j\neq i_0} |b_{i_0,j}| \left|\frac{\epsilon_{i_0}}{\lambda + \mu\delta(x_{i_0})}\right| \\ &= \sum_{j=1}^{2N-1} b_{i_0,j} \left|\frac{\epsilon_{i_0}}{\lambda + \mu\delta(x_{i_0})}\right|.\end{aligned}$$

According to the above inequality, Theorem 2.5 and Lemma 4.3, we have

$$\left|\frac{\epsilon_i}{\lambda + \mu\delta(x_i)}\right| \leq \left|\frac{\epsilon_{i_0}}{\lambda + \mu\delta(x_{i_0})}\right| \leq \frac{|\tau_{i_0}|}{\sum_{j=1}^{2N-1} b_{i_0,j}} \leq Ch^{\min\{\frac{r\alpha}{2},2\}} + C(r-1)h^2.$$

Since $\lambda + \mu\delta(x_i) \leq \lambda + \mu T$, we can derive

$$|\epsilon_i| \leq C(\lambda + \mu T)h^{\min\{\frac{r\alpha}{2},2\}}.$$

The proof of convergency is completed.

Remark 4.4 (Weaker regularity on the derivatives of u). Suppose that the bound of Lemma 2.4 is replaced by the more general weaker regularity bound

$$|u^{(l)}(x)| \leq C\delta(x)^{\sigma-l}, \quad l = 0, 1, 2, 3, 4,$$

where $\sigma \in (0, \frac{\alpha}{2}]$ is fixed. Then

$$I^{2-\alpha}u(x) = \int_0^{x/2} + \int_{x/2}^{T+x/2} + \int_{T+x/2}^{2T} u(y)K_y(x)dx.$$

For $l = 1, 2, 3, 4$, we have

$$\begin{aligned}\frac{d^l}{dx^l} I^{2-\alpha}u(x) = & \int_0^{x/2} + \int_{T+x/2}^{2T} u(y)K_y^{(l)}(x)dy \\ & + \sum_{k=0}^{l-1} \Big(u^{(k)}(\frac{x}{2})K_{x/2}^{(l-1-k)}(x) - u^{(k)}(T + \frac{x}{2})K_{T+x/2}^{(l-1-k)}(x) \Big) \\ & + \int_{x/2}^{T+x/2} u^{(l)}(y)K_y(x)dy,\end{aligned}$$

Thus, we can get

$$|f^l(x)| \le C\delta(x)^{\sigma-\alpha-l}, \quad l = 0, 1, 2.$$

Examine the proof above, by replacing the regularity condition with the weaker one, we can get the similar results:

$$\begin{aligned}|\tau_i| &= | - \kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i)| \\ &\le Ch^{\min\{r\sigma,2\}}\delta(x_i)^{-\alpha} + C(r-1)h^2(T - \delta(x_i) + h_N)^{1-\alpha}.\end{aligned}$$

The convergence result of Theorem 2.6 is changed to

$$\max_{1\le i\le 2N-1} |u_i - u(x_i)| \le Ch^{\min\{r\sigma,2\}}.$$

5. Experimental results

We use the scheme (2.10) to solve the fractional Laplacian boundary value problem (1.1) on the interval $\Omega = (0, 1)$.

5.1. $f \equiv 1$.

If $f \equiv 1$, The exact (Getoor) solution of this problem is

$$u(x) = \frac{2^{-\alpha}\Gamma(\frac{1}{2})}{\Gamma(1+\frac{\alpha}{2})\Gamma(\frac{1+\alpha}{2})} [x(1-x)]^{\frac{\alpha}{2}}, \quad x \in \Omega.$$

In the numerical experiments of this example, we measure the numerical errors by using the maximum nodal error (i.e., the discrete L^∞ norm):

$$E^N := \max_{0\le i\le 2N} |u(x_i) - u_i|.$$

The rate of convergence of E^N is computed in the usual way, viz.,

$$Rate^N = \log_2\left(\frac{E^N}{E^{2N}}\right)$$

In Table 1 and Table 2, we choose different α, and take the mesh grading parameter $r = 1, \frac{4}{\alpha}$ by Theorem 2.6 , then display the values of E^N and $Rate^N$ for various N. Our chosen values of α are

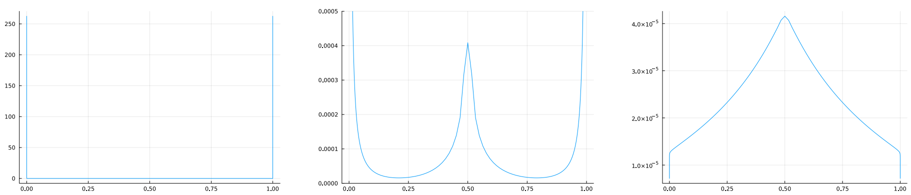
And Figure 2(a) and Figure 2(b) show the $|\tau_i|$, whose difference is just ylim, and Figure 2(c) shows the global error $|u_i - u(x_i)|$. And that is the Figure 2(c) suggests the technique we used in Subsection 4.2

TABLE 1. $r = 1$:

α \ $2N$	200	400	800	1600
1.2	1.127e-3	7.428e-4	4.899e-4	3.231e-4
		0.6013	0.6006	0.6003
1.5	2.500e-4	1.488e-4	8.849e-5	5.263e-5
		0.7487	0.7494	0.7497
1.8	2.732e-5	1.483e-5	7.997e-6	4.299e-6
		0.8815	0.8909	0.8955

TABLE 2. $r = \frac{4}{\alpha}$:

α \ $2N$	200	400	800	1600
1.2	4.158e-5	1.063e-5	2.692e-6	6.782e-7
		1.9682	1.9811	1.9888
1.5	2.068e-5	5.379e-5	1.382e-6	3.524e-7
		1.9429	1.9601	1.9720
1.8	7.642e-6	2.065e-6	5.501e-7	1.450e-7
		1.8880	1.9083	1.9240



(a) LTE.1 (b) LTE.2 (c) GE

FIGURE 2. truncation error and global error for $f = 1$, where $\alpha = 1.2$, $r = 4/\alpha$, $2N = 200$.

5.2. f **is singular.** While by Remark 4.4, we take $f = x^{\sigma-\alpha}$, where $\sigma \in (0, \frac{\alpha}{2}]$. In these cases, we do not know the exact solution, so we calculate the rate of convergence by

$$Rate^N = \log_2\left(\frac{RE^N}{RE^{2N}}\right)$$

where

$$RE^N = \max_{1\le i\le 2N-1} |u_i^N - u_{2i}^{2N}|$$

Let $\sigma = 0.4$, and take $r = 1, \frac{2}{\sigma}$ and various α, then display the values of RE^N and $Rate^N$ for various N in Table 3 and Table 4.

TABLE 3. $r = 1$:

α \ $2N$	200	400	800	1600
1.2		0.2262	0.01744	0.01339
			0.3755	0.3804
1.5		0.03107	0.02372	0.01806
			0.3895	0.3934
1.8		0.04347	0.03311	0.02516
			0.3926	0.3962

TABLE 4. $r = \frac{2}{\sigma}$:

α \ $2N$	200	400	800	1600
1.2		6.963e-4	1.742e-4	4.356e-5
			1.9992	1.9996
1.5		8.015e-4	2.022e-4	5.095e-5
			1.9867	1.9889
1.8		1.319e-3	3.416e-4	8.769e-5
			1.9492	1.9617

APPENDIX A. APPROXIMATE OF DIFFERENCE QUOTIENTS

Lemma A.1. *If $g(x) \in C^2(\Omega)$, there exists $\xi \in (x_{i-1}, x_{i+1})$ such that*

$$D_h^2 g(x_i) = g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}).$$

If $g(x) \in C^4(\Omega)$, then

$$D_h^2 g(x_i) = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i)$$
$$+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right).$$

Proof. By Taylor expansion

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i),$$
$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1}).$$

Substitute them into the operator D_h^2, we have

$$\begin{aligned} D_h^2 g(x_i) &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (g(x_{i+1}) - g(x_i) + \frac{1}{h_i} (g(x_{i-1}) - g(x_i)) \right) \\ &= \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2). \end{aligned}$$

Now, using intermediate value theorem, there exists some $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi).$$

The second equation can be also obtained by Taylor expansion similarly. Especially,

$$\begin{aligned}\int_{x_{i-1}}^{x_i} g''''(y)\frac{(y-x_{i-1})^3}{3!}dy &= \frac{h_i^4}{4!}g''''(\eta_1)\\ \int_{x_i}^{x_{i+1}} g''''(y)\frac{(x_{i+1}-y)^3}{3!}dy &= \frac{h_{i+1}^4}{4!}g''''(\eta_2)\end{aligned}\tag{A.1}$$

where $\eta_1 \in (x_{i-1}, x_i), \eta_2 \in (x_i, x_{i+1})$. □

Lemma A.2. *Denote* $y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \theta \in (0,1)$, *we have*

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2}h_j^2 u''(\xi), \quad \xi \in (x_{j-1}, x_j),$$

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2}h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!}h_j^3\left(\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)\right),$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$.

Proof. By Taylor expansion, we have

$$\begin{aligned}u(x_{j-1}) &= u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!}u''(\xi_1), \quad \xi_1 \in (x_{j-1}, y_j^\theta),\\ u(x_j) &= u(y_j^\theta) + (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!}u''(\xi_2), \quad \xi_2 \in (y_j^\theta, x_j).\end{aligned}$$

Thus,

$$\begin{aligned}u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j)\\ &= -\frac{\theta(1-\theta)}{2}h_j^2(\theta u''(\xi_1) + (1-\theta)u''(\xi_2))\\ &= -\frac{\theta(1-\theta)}{2}h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2].\end{aligned}$$

The second equation can be got by

$$\begin{aligned}u(x_{j-1}) &= u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!}u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!}u'''(\eta_1),\\ u(x_j) &= u(y_j^\theta) + (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!}u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!}u'''(\eta_2),\end{aligned}$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$. □

Lemma A.3. *For any* $y \in (x_{j-1}, x_j)$, $2 \le j \le 2N-1$, *there exists*

$$|u(y) - \Pi_h u(y)| \le h_j^2 \max_{\xi \in [x_{j-1}, x_j]} |u''(\xi)| \le Ch^2 \delta(y)^{\alpha/2 - 2/r}.$$

Proof. By Lemmas 2.4, 3.1 and A.2, the desired result is obtained. □

Lemma A.4. *For any* $x \in [x_{j-1}, x_j]$, *there exist*

$$\begin{aligned}|u(x) - \Pi_h u(x)| &= \left|\frac{x_j - x}{h_j}\int_{x_{j-1}}^x u'(y)dy - \frac{x - x_{j-1}}{h_j}\int_x^{x_j} u'(y)dy\right|\\ &\le \int_{x_{j-1}}^{x_j} |u'(y)|dy.\end{aligned}$$

If $x \in [0, x_1]$, there exist

$$|u(x) - \Pi_h u(x)| \leq \int_0^{x_1} |u'(y)|dy \leq \int_0^{x_1} C y^{\alpha/2-1} dy \leq C\frac{2}{\alpha} x_1^{\alpha/2} = C\frac{2}{\alpha} h_1^{\alpha/2}.$$

Similarly, if $x \in [x_{2N-1}, 2T]$, there exist

$$|u(x) - \Pi_h u(x)| \leq C\frac{2}{\alpha}(2T - x_{2N-1})^{\alpha/2} = C\frac{2}{\alpha} h_{2N}^{\alpha/2}.$$

Appendix B. Proofs of some technical details

Review that $h = \frac{1}{N}$ and the defination of $\simeq$ in Subsection 2.1

Lemma B.1. *There is a constant $C = C(T, r)$ such that for $i = 1, 2, \cdots, 2N-1$*

$$|h_{i+1} - h_i| \leq C(r-1)h^2\delta(x_i)^{1-2/r}.$$

Proof. By definition of h_i, we have

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \leq i \leq N-1, \\ 0, & i = N, \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \leq i \leq 2N-1. \end{cases}$$

Since $(i+1)^r - 2i^r + (i-1)^r \simeq r(r-1)i^{r-2}$ for $i \geq 1$, the desired result is obtained. $\square$

Lemma B.2. *There is a constant $C = C(T, \alpha, \beta, r, f)$ such that*

$$\frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y)\frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y)\frac{(y - x_{i+1})^3}{3!} dy \right|$$
$$\leq Ch^2\delta(x_i)^{-\alpha/2-2/r}.$$

Proof. By Lemma 2.4, for $1 \leq i \leq 2N-1$, we have

$$\left| \int_{x_{i-1}}^{x_i} f''(y)\frac{(y - x_{i-1})^3}{3!} dy \right| \leq \frac{\|f\|_\beta^{(\alpha/2)}}{3!} \int_{x_{i-1}}^{x_i} \delta(y)^{-\alpha/2-2}(y - x_{i-1})^3 dy.$$

For $i = 1$,

$$\int_{x_{i-1}}^{x_i} \delta(y)^{-\alpha/2-2}(y - x_{i-1})^3 dy = \int_0^{x_1} y^{1-\alpha/2} dy = \frac{1}{2 - \alpha/2} x_1^{2-\alpha/2} \simeq x_1^{-\alpha/2-2} h_1^4.$$

And for $2 \leq i \leq 2N-1$, by Lemma 3.1, we have

$$\int_{x_{i-1}}^{x_i} \delta(y)^{-\alpha/2-2}(y - x_{i-1})^3 dy \simeq \int_{x_{i-1}}^{x_i} \delta(x_i)^{-\alpha/2-2}(y - x_{i-1})^3 dy \simeq \delta(x_i)^{-\alpha/2-2} h_i^4$$

So for $1 \leq i \leq 2N-1$, we have

$$\left| \int_{x_{i-1}}^{x_i} f''(y)\frac{(y - x_{i-1})^3}{3!} dy \right| \leq C\delta(x_i)^{-\alpha/2-2} h_i^4.$$

Similarly,

$$\left| \int_{x_i}^{x_{i+1}} f''(y)\frac{(x_{i+1} - y)^3}{3!} dy \right| \leq C\delta(x_i)^{-\alpha/2-2} h_{i+1}^4.$$

Thus for $1 \le i \le 2N-1$, with Lemma 3.1 we have

$$\frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y-x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y-x_{i+1})^3}{3!} dy \right|$$
$$\le C\delta(x_i)^{-\alpha/2-2} \frac{h_i^3 + h_{i+1}^3}{h_i + h_{i+1}} \simeq \delta(x_i)^{-\alpha/2-2} h_i^2 \simeq h^2 \delta(x_i)^{-\alpha/2-2/r}.$$

□

Lemma B.3. *For all $1 \le i \le 2N-1$, $1 \le j \le 2N$, there exist*

$$-D_h K_y(x_i) \simeq |x_i - y|^{-\alpha}, \quad [x_{j-1}, x_j] \cap [x_i, x_{i+1}] = \varnothing,$$
$$D_h^2 K_y(x_i) \simeq |x_i - y|^{-1-\alpha}, \quad [x_{j-1}, x_j] \cap [x_{i-1}, x_{i+1}] = \varnothing.$$

Proof. Since $x_{i-1} - y$, $x_i - y$ and $x_{i+1} - y$ have the same sign, by the mean value theorem and Lemma A.1,

$$D_h K_y(x_i) = \frac{\kappa_\alpha}{\Gamma(1-\alpha)} |\xi - y|^{-\alpha}, \quad \xi \in (x_i, x_{i+1}),$$
$$D_h^2 K_y(x_i) = \frac{\kappa_\alpha}{\Gamma(-\alpha)} |\xi - y|^{-1-\alpha}, \quad \xi \in (x_{i-1}, x_{i+1}).$$

however, $|\xi - y| \simeq |x_i - y|$, we get the result. □

Lemma B.4. *There exist*

$$\sum_{j=1}^{3} V_{1j} \le Ch^2 x_1^{-\alpha/2-2/r} \quad \textit{and} \quad \sum_{j=1}^{4} V_{2j} \le Ch^2 x_2^{-\alpha/2-2/r}.$$

Proof. For $0 \le i \le 3, 1 \le j \le 4$, by Lemma A.4, Lemma A.3 and (3.6)

$$T_{ij} \le C x_1^{2-\alpha/2} \simeq h_1^2 \, h^2 x_1^{-\alpha/2-2/r} \simeq h_1^2 \, h^2 x_2^{-\alpha/2-2/r}.$$

Therefore, by (3.7), the desired results are obtained. □

Appendix C. Proofs of grid mapping functions

Proof of Lemma 3.9. The initial two approximations can be readily derived, provided that no element approaching the boundary of Ω.

Then we consider $|y_{i,j}(\xi) - \xi| = \mathrm{sign}(j-i)(y_{i,j}(\xi) - \xi)$. It is obvious that $y_{i,i}(\xi) - \xi = 0$. Otherwise, without loss of generality, set $i < j$, then $y_{i,j}(\xi) - \xi \le x_{j+1} - x_{i-1} \simeq x_j - x_i$. On the other hand, $|y_{i,j}(\xi) - \xi|$ is concave by Lemma 3.8. So $|y_{i,j}(\xi) - \xi| \ge \min\{x_{j-1} - x_{i-1}, x_{j+1} - x_{i+1}\} \simeq |x_j - x_i|$.

Thus, $h_{i,j}(\xi) = y_{i,j}(\xi) - y_{i,j-1}(\xi) = y_{j-1,j}(y_{i,j-1}(\xi)) - y_{i,j-1}(\xi) \simeq x_j - x_{j-1}$.

The final estimate can be obtained since $y_{i,j-1}(\xi) - \xi$, $y_{i,j}(\xi) - \xi$ have the same sign (≥ 0 or ≤ 0). □

Proof of Lemma 3.10. From (3.13) and Lemma 3.8, we can see that

$$
\begin{aligned}
&h'_{i,j}(x) = y'_{i,j}(x) - y'_{i,j-1}(x)\\
&= \begin{cases}
x^{1/r-1}(y_{i,j}^{1-1/r}(x) - y_{i,j-1}^{1-1/r}(x)) & i<N, j<N,\\
x^{1/r-1}(\frac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(x)) & i<N, j=N,\\
x^{1/r-1}\left((2T-y_{i,N+1}(x))^{1-1/r} - \frac{h_N}{rZ_1}\right) & i<N, j=N+1,\\
x^{1/r-1}\left((2T-y_{i,j}(x))^{1-1/r} - (2T-y_{i,j-1}(x))^{1-1/r}\right) & i<N, j>N+1,\\
\frac{rZ_1}{h_N}\left(y_{N,j}^{1-1/r}(x) - y_{N,j-1}^{1-1/r}(x)\right) & i=N, j<N,\\
\frac{rZ_1}{h_N}\left(\frac{h_N}{rZ_1} - y_{N,N-1}^{1-1/r}(x)\right) & i=N, j=N.
\end{cases}
\end{aligned}
$$

While for $2 \le i \le N$, if $2 \le j < N$, $\xi \in (x_{i-1}, x_{i+1})$,

$$
\begin{aligned}
y_{i,j}^{1-1/r}(\xi) - y_{i,j-1}^{1-1/r}(\xi) &\le x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r}\\
&= T^{1-1/r}N^{1-r}\left((j+1)^{r-1} - (j-2)^{r-1}\right) \qquad \text{(C.1)}\\
&\le CT^{1-1/r}(r-1)N^{1-r}j^{r-2} = C(r-1)Z_1 x_j^{1-2/r}.
\end{aligned}
$$

If $j = N$, $\xi \in (x_{i-1}, x_{i+1})$, we have $y_{i,N-1}(\xi) \in (x_{N-2}, x_N)$. And

$$
\frac{h_N}{rZ_1} = T^{1-1/r}\frac{1-(1-h)^r}{rh} = \eta^{1-1/r} \simeq x_N^{1-1/r}, \quad \eta \in (x_{N-1}, x_N). \tag{C.2}
$$

Then

$$
|\frac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(\xi)| \le x_N^{1-1/r} - x_{N-2}^{1-1/r} \simeq (r-1)Z_1 x_N^{1-2/r}.
$$

And similar if $j \ge N+1$. Combine with Lemma 3.1, Lemma 3.9, $\eta \simeq x_N$, we get the first result.

For the second estimate, we have

$$
(y_{i,j}(x) - x)' = y'_{i,j}(x) - 1.
$$

Then, for $2 \le i < N$, if $2 \le j < N$, $\xi \in (x_{i-1}, x_{i+1})$, by Lemma D.1 and Lemma 3.9

$$
\xi^{1/r}|y_{i,j}^{1-1/r}(\xi) - \xi^{1-1/r}| \le |y_{i,j}(\xi) - \xi| \simeq |x_j - x_i|.
$$

If $j > N$

$$
\begin{aligned}
&\xi^{1/r}|(2T - y_{i,j}(\xi))^{1-1/r} - \xi^{1-1/r}| \le |2T - y_{i,j}(\xi) - \xi|\\
&\quad \le |2T - x_j - x_i| + |y_{i,j}(\xi) - x_j| + |\xi - x_i| \le |2T - x_j - x_i| + 2h_N\\
&\quad \le |x_j - T| + |T - x_i| + 2h_N \le 2|x_j - x_i|.
\end{aligned}
$$

But if $j = N$, with (C.2) and Lemma D.1,

$$
\begin{aligned}
\eta^{1/r}|\frac{h_N}{rZ_1} - \xi^{1-1/r}| &\le |\eta - \xi|, \quad \eta \in (x_{N-1}, x_N) \qquad \text{(C.3)}\\
&\le |x_N - x_i| + |h_N| + |h_{i+1}| \le 3|x_N - x_i|.
\end{aligned}
$$

For $i = N$, if $j < N$, similarly with (C.3),

$$
\eta^{1/r}|y_{N,j}^{1-1/r}(\xi) - \frac{h_N}{rZ_1}| \le C|x_j - x_N|.
$$

And if $j = N$, it is obviously $\equiv 0$, and similar if $j > N$. So, by Lemma 3.8 and Lemma 3.9, we get the second result. $\square$

Proof of Lemma 3.11. By Lemma 3.8, for $2 \le i, j < N$, using Lemma D.1,

$$x_j^{1-1/r}|Z_{j-i}| = x_j^{1-1/r}|x_j^{1/r} - x_i^{1/r}| \le |x_j - x_i|, \tag{C.4}$$

and by (C.2) $\dfrac{h_N}{rZ_1} \simeq x_N^{1-1/r}$, with $Z_{2N-j+i} \le Z_{2N} = 2T^{1/r}$, and Lemma 3.9, we get the first result.

For the second part, by Lemma 3.8

$$h''_{i,j}(x) = y''_{i,j}(x) - y''_{i,j-1}(x),$$

while for $2 \le i < N$, if $3 \le j < N$, $\xi \in (x_{i-1}, x_{i+1})$,

$$y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1} = \left(y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi)\right) Z_{j-i} + y_{i,j-1}^{1-2/r}(\xi)Z_1, \tag{C.5}$$

where $y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi) \simeq (r-2)Z_1 x_j^{1-3/r}$ similar with (C.1). Combine with (C.4), we get

$$|y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1}| \le CZ_1 \left(|r-2|x_j^{-2/r}|x_j - x_i| + x_j^{1-2/r}\right).$$

If $j = N$,

$$|h''_{i,N}(x)| \le |y''_{i,N}(x)| + |y''_{i,N-1}(x)| \le C(r-1)x_i^{1/r-2}x_N^{1-1/r}.$$

Similarly if $j = N+1$.

However, if $j > N+1$, similar with (C.5), by Lemma 3.1 we get

$$\begin{aligned}
&\left|\delta(y_{i,j}(\xi))^{1-2/r}Z_{2N-(j-i)} - \delta(y_{i,j-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)}\right| \\
&= \left|\left(\delta(y_{i,j}(\xi))^{1-2/r} - \delta(y_{i,j-1}(\xi))^{1-2/r}\right) Z_{2N-(j-i)} - \delta(y_{i,j-1}(\xi))^{1-2/r}Z_1\right| \\
&\le CZ_1\left(|r-2|\delta(x_j)^{1-3/r}x_N^{1/r} + \delta(x_j)^{1-2/r}\right) \le CZ_1\delta(x_j)^{1-3/r}x_N^{1/r}.
\end{aligned}$$

For $i = N$, it's obvious. Combined with Lemma 3.8 and Lemma 3.9, we get the second result. □

Proof of Lemma 3.12. Since the sign of $y^\theta_{i,j}(\xi) - \xi$ is independent of ξ, it can be differentiated. Then by Lemma A.1

$$D_h^2 P^\theta_{i,j}(x_i) = {P^\theta_{i,j}}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}).$$

From (3.15), using Leibniz formula and chain rules, Lemmas 2.4, 3.1 and 3.8 to 3.11 with $x_i \simeq \delta(x_j)$ in each cases, we have

$$h_{i,j}(\xi) \le Ch_j, \quad |h'_{i,j}(\xi)| \le C(r-1)h_j x_i^{-1},$$

$$|y^\theta_{i,j}(\xi) - \xi| \le C|y^\theta_j - x_i|, \quad \left|(y^\theta_{i.j}(\xi) - x_i)'\right| \le C|y^\theta_j - x_i|x_i^{-1},$$

$$\left|u''(y^\theta_{i,j}(\xi))\right| \le Cx_i^{\alpha/2-2}, \ \left|\left(u''(y^\theta_{i,j}(\xi))\right)'\right| \le Cx_i^{\alpha/2-3}, \ \left|\left(u''(y^\theta_{i,j}(\xi))\right)''\right| \le Cx_i^{\alpha/2-4}.$$

By Lemma 3.11, we have

For Case 1,

$$|h''_{i,j}(\xi)| \le C(r-1)h_j x_i^{-2}, \quad \left|(y^\theta_{i,j}(\xi) - x_i)''\right| \le C(r-1)|y^\theta_j - x_i|x_i^{-2}.$$

For Case 2, since $x_i \simeq x_j \simeq T$

$$|h''_{i,j}(\xi)| \le C(r-1), \quad \left|(y^\theta_{i,j}(\xi) - x_i)''\right| \le C(r-1).$$

For Case 3, since $x_i \simeq \delta(x_j) \simeq T$, we have

$$|h''_{i,j}(\xi)| \le C(r-1)h_j, \quad \left|(y^\theta_{i,j}(\xi) - x_i)''\right| \le C(r-1).$$

Combine them, the desired results are obtained. □

Proof of Lemma 3.13. We have

$$\begin{aligned}(C.6)\qquad &\frac{Q^\theta_{i,j,l}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{i,j,l}(x_i)u'''(\eta^\theta_j)}{h_{i+1}} \\ &= \frac{Q^\theta_{i,j,l}(x_{i+1}) - Q^\theta_{i,j,l}(x_i)}{h_{i+1}}u'''(\eta^\theta_{j+1}) + Q^\theta_{i,j,l}(x_i)\frac{u'''(\eta^\theta_{j+1}) - u'''(\eta^\theta_j)}{h_{i+1}}.\end{aligned}$$

Using mean value theorem

$$\frac{Q^\theta_{i,j,l}(x_{i+1}) - Q^\theta_{i,j,l}(x_i)}{h_{i+1}} = {Q^\theta_{i,j,l}}'(\xi), \quad \xi \in (x_i, x_{i+1}).$$

From (3.16) and Leibniz rule, by Lemmas 3.1, 3.9 and 3.10, we have

$$|{Q^\theta_{i,j,l}}'(\xi)| \le Ch^l_j|y^\theta_j - x_i|^{1-\alpha}(x_i^{-1} + x_i^{1/r-1}\delta(x_j)^{-1/r}),$$

$$Q^\theta_{i,j,l}(x_i) = Ch^l_j|y^\theta_j - x_i|^{1-\alpha}.$$

By Lemmas 2.4 and 3.1, we have

$$|u^{(l-1)}(\eta^\theta_{j+1})| \le C(\eta^\theta_{j+1})^{\alpha/2-l+1} \simeq \delta(x_j)^{\alpha/2-l+1}.$$

By Lemma 3.1

$$\begin{aligned}\frac{|u^{(l-1)}(\eta^\theta_{j+1}) - u^{(l-1)}(\eta^\theta_j)|}{h_{i+1}} &= |u^{(l)}(\eta)|\frac{\eta^\theta_{j+1} - \eta^\theta_j}{h_{i+1}}, \quad \eta \in (x_{j-1}, x_{j+1}) \\ &\le C\delta(\eta)^{\alpha/2-l}\frac{x_{j+1} - x_{j-1}}{h_{i+1}} = C\delta(\eta)^{\alpha/2-l}\frac{h_{j+1} + h_j}{h_{i+1}} \\ &\simeq x_i^{1/r-1}\delta(x_j)^{\alpha/2-l+1-1/r}.\end{aligned}$$

Combine the cases above, we get the first term. While, the later is similar. □

Appendix D. Additional inequalities

Lemma D.1.

$$b^{1-\theta}|a^\theta - b^\theta| \le |a-b| \ (\ \textit{also}\ a^{1-\theta}|a^\theta - b^\theta| \le |a-b|), \quad a, b \ge 0,\ \theta \in [0,1].$$

Lemma D.2. *Let*

$$\begin{aligned}f(x,y) =& xy(1 + x^{2-\alpha} + y^{2-\alpha}) + (1+x+y)^{3-\alpha} \\ &- (1+x)(1+y)\left((1+x)^{2-\alpha} + (1+y)^{2-\alpha} - 1\right),\end{aligned}$$

with $\alpha \in (1,2)$. *Then* $f(x,y) > 0$ *for* $x > 0, y \ge 1$.

Proof. It is obvious that $f(x,y) = f(y,x)$ and $f(0,y) = f(x,0) = 0$. The second derivatives of f is

$$\begin{aligned}\partial_x f(x,y) =& (3-\alpha)\left(x^{2-\alpha}y + (1+x+y)^{2-\alpha} - (1+x)^{2-\alpha}(1+y)\right) \\ &+ 1 + 2y + y^{3-\alpha} - (1+y)^{3-\alpha}, \\ \partial^2_x f(x,y) =& (3-\alpha)(2-\alpha)\left(yx^{1-\alpha} + (1+x+y)^{1-\alpha} - (1+y)(1+x)^{1-\alpha}\right).\end{aligned}$$

Since $x^{1-\alpha}$ is convex for $x > 0$, using Jensen's inequality we have

$$\frac{y}{1+y}x^{1-\alpha} + \frac{1}{1+y}(1+x+y)^{1-\alpha} > (1+x)^{1-\alpha},$$

which implies $\partial_x^2 f(x,y) > 0$ and $\partial_y^2 f(x,y) > 0$.

Since $f(x,0) = 0$, it is sufficient to prove $f(x,1) > 0$. While $f(0,1) = 0$ and $\partial_x^2 f(x,1) > 0$, we only need to prove that $\partial_x f(0,1) > 0$, where

$$\partial_x f(0,1) = 4 - 2(3-\alpha) - (\alpha-1)2^{2-\alpha} > 0.$$

The proof is completed. □

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