# 1 问题

对于 $\Omega = (0,1)$, $1 < \alpha < 2$, 假设 $f \in C^2(\Omega)$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

其中

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{u(y)}{|x-y|^{\alpha-1}} dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)} > 0 \tag{1.3}$$

# 2 数值格式

用线性插值代替原函数，中心差分代替二阶导数，记 $u_h(x)$ 为 $u(x)$ 在网格点上的线性插值。

我们解这样的数值解

$$\begin{aligned} -\kappa_\alpha( & \frac{2}{h_{i+1}(h_i+h_{i+1})} \int_\Omega \frac{u_h(x)}{|x_{i+1}-y|^{\alpha-1}} dy \\ & - \frac{2}{h_i h_{i+1}} \int_\Omega \frac{u_h(x)}{|x_i - y|^{\alpha-1}} dy \\ & + \frac{2}{h_i(h_i+h_{i+1})} \int_\Omega \frac{u_h(x)}{|x_{i-1}-y|^{\alpha-1}} dy) \\ = F_i & \end{aligned} \tag{2.1}$$

矩阵 $A$ 是 $M$ 矩阵，主队角正，其他负，严格对角占优。

# 3 一致网格

当 $r = 1$ , 网格成为一致网格, $x_i = ih, h = \frac{1}{2N}, i = 0, ..., 2N.$

$A$ 等于

$$a_{ij} = \frac{-\kappa_\alpha}{(2-\alpha)(3-\alpha)} h^{-\alpha} \big(|i-j-2|^{3-\alpha} - 4|i-j-1|^{3-\alpha} + 6|i-j|^{3-\alpha} - 4|i-j+1|^{3-\alpha} + |i-j+2|^{3-\alpha}\big) \tag{3.1}$$

矩阵行和

$$S_i = \sum_{j=1}^{2N-1} a_{ij} = \frac{-\kappa_\alpha}{(2-\alpha)(3-\alpha)} h^{-\alpha} (|i+1|^{3-\alpha} - 3|i|^{3-\alpha} + 3|i-1|^{3-\alpha} - |i-2|^{3-\alpha} + ...2N) \tag{3.2}$$

我们得到

$$S_i \geq C(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \tag{3.3}$$

下面估计截断误差 $R_i$.

$$R_i = \int_0^1 D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \tag{3.4}$$

目标是

$$R_i \leq C h^{\alpha/2} S_i \tag{3.5}$$

这样我们就有

$$\epsilon \leq \max_i \frac{R_i}{S_i} \leq C h^{\alpha/2} \tag{3.6}$$

考虑 $R_1$

$$R_1 = \int_\Omega (u(y) - u_h(y)) \frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} dy \tag{3.7}$$

我们有

$$R_1 = \int_0^{3h} + \int_{3h}^{1/2} \tag{3.8}$$

当 $y > 3h$,

$$\frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} \leq C|y|^{-1-\alpha} \tag{3.9}$$

那么

$$\begin{aligned} I_2 &\leq C \int_{3h}^{1/2} |y|^{-1-\alpha} u''(y) h^2 dy \\ &\leq C \int_{3h}^{1} |y|^{-1-\alpha} y^{\alpha/2-2} h^2 dy \\ &\leq Ch^2 \int_{3h}^{1} y^{-3-\alpha/2} dy \\ &\leq Ch^2 h^{-2-\alpha/2} = Ch^{-\alpha/2} \\ &\leq Ch^{\alpha/2} x_1^{-\alpha} \leq Ch^{\alpha/2} S_1 \end{aligned} \tag{3.10}$$

在考虑

$$\begin{aligned} I_1 &= \int_0^{3h} \frac{u(y) - u_h(y)}{h^2} (|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}) dy \\ &= \int_0^{h} + \int_h^{3h} = J_1 + J_2 \end{aligned} \tag{3.11}$$

$$J_2 \leq Cu''(\eta) h^{2-\alpha} \leq Ch^{\alpha/2-2} h^{2-\alpha} \leq Ch^{-\alpha/2} \tag{3.12}$$

因为

$$\begin{aligned} |u(x) - u_h(x)| &\leq \int_0^{x_1} |u'(y)| dy \\ &\leq C \int_0^{x_1} y^{\alpha/2-1} dy \\ &\leq Cx_1^{\alpha/2} \quad , x \in (0, h) \end{aligned} \tag{3.13}$$

$$
\begin{aligned}
J_1 &= \int_0^h \frac{u(y)-u_h(y)}{h^2}(|y|^{1-\alpha}-2|y-h|^{1-\alpha}+|y-2h|^{1-\alpha})dy \\
&\le Ch^{\alpha/2-2}h^{2-\alpha} = Ch^{-\alpha/2}
\end{aligned} \tag{3.14}
$$

所以有

$$
R_1 \le Ch^{-\alpha/2} \le Ch^{\alpha/2}h^{-\alpha} \le Ch^{\alpha/2}S_1, \quad (S_1 \ge Cx_1^{-\alpha}) \tag{3.15}
$$

$R_1, R_2, R_3$ 全部类似。

## 3.1 猜想

$$
R_i \le Ch^{\alpha/2+1}(x_i^{-\alpha-1}+(1-x_i)^{-\alpha-1}) \quad (then \le Ch^{\alpha/2}S_i) \tag{3.16}
$$

为了简便，我们记 $D(y) := u(y) - u_h(y)$.
当 $3 < i \le N$ 时，

$$
\begin{aligned}
R_i =& \int_0^1 D(y)\frac{|y-x_{i-1}|^{1-\alpha}-2|y-x_i|^{1-\alpha}+|y-x_{i+1}|^{1-\alpha}}{h^2}dy\\
=& \int_0^{x_1} D(y)\frac{|y-x_{i-1}|^{1-\alpha}-2|y-x_i|^{1-\alpha}+|y-x_{i+1}|^{1-\alpha}}{h^2}dy\\
&+\int_{x_1}^{x_{\lceil\frac{i}{2}\rceil}} D(y)\frac{|y-x_{i-1}|^{1-\alpha}-2|y-x_i|^{1-\alpha}+|y-x_{i+1}|^{1-\alpha}}{h^2}dy\\
&+\int_{x_{\lceil\frac{i}{2}\rceil}}^{x_{\lceil\frac{i}{2}\rceil+1}} \frac{D(y+h)-D(y)}{h^2}|y-x_i|^{1-\alpha}+D(y)\frac{|y-x_{i+1}|^{1-\alpha}-|y-x_i|^{1-\alpha}}{h^2}dy\\
&+\int_{x_{\lceil\frac{i}{2}\rceil+1}}^{x_i} \frac{D(y-h)-2D(y)+D(y+h)}{h^2}|y-x_i|^{1-\alpha}dy\\
&+\int_{x_i}^{x_{N+\lfloor\frac{i}{2}\rfloor-1}} \frac{D(y-h)-2D(y)+D(y+h)}{h^2}|y-x_i|^{1-\alpha}dy\\
&+\int_{x_{N+\lfloor\frac{i}{2}\rfloor-1}}^{x_{N+\lfloor\frac{i}{2}\rfloor}} \frac{D(y-h)-D(y)}{h^2}|y-x_i|^{1-\alpha}+D(y)\frac{|y-x_{i-1}|^{1-\alpha}-|y-x_i|^{1-\alpha}}{h^2}dy\\
&+\int_{x_{N+\lfloor\frac{i}{2}\rfloor}}^{x_{2N-1}}+\int_{x_{2N-1}}^{1} D(y)\frac{|y-x_{i-1}|^{1-\alpha}-2|y-x_i|^{1-\alpha}+|y-x_{i+1}|^{1-\alpha}}{h^2}dy\\
=& I_1+I_2+I_3+I_4+\cdots
\end{aligned}
\tag{3.17}
$$

1.

$$
\begin{aligned}
I_1 =& \int_0^{x_1}(u(y)-u_h(y))\frac{|y-x_{i-1}|^{1-\alpha}-2|y-x_i|^{1-\alpha}+|y-x_{i+1}|^{1-\alpha}}{h^2}dy\\
&\le Ch^{\alpha/2}\int_0^h |y-x_i|^{-1-\alpha}dy\\
&\le Ch^{\alpha/2+1}x_i^{-1-\alpha}
\end{aligned}
\tag{3.18}
$$

2.

$$\begin{aligned}
I_2 &= \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
&\le C \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2} h^2 |x_i - y|^{-1-\alpha} dy \\
&\le C h^{\alpha/2-1} h^2 x_i^{-1-\alpha} \le C h^{\alpha/2+1} x_i^{-1-\alpha}
\end{aligned} \tag{3.19}$$

3.

$$\begin{aligned}
I_3 &= \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} \frac{D(y+h)-D(y)}{h^2} |y-x_i|^{1-\alpha} + D(y) \frac{|y-x_{i+1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2} dy \\
&\le \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} u'''(\eta_1) h |x_i - y|^{1-\alpha} + u''(\eta_2) h |x_i - y|^{-\alpha} dy \\
&\le C h^2 x_i^{-2-\alpha/2} \le C h^{1+\alpha/2} x_i^{-1-\alpha}
\end{aligned} \tag{3.20}$$

4.

$$\begin{aligned}
I_4 &= \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y-x_i|^{1-\alpha} dy \\
&\le \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} u''''(\eta) h^2 |x_i - y|^{1-\alpha} dy \\
&\le C x_i^{\alpha/2-4} h^2 x_i^{2-\alpha} \\
&\le C h^2 x_i^{-2-\alpha/2} \le C h^{1+\alpha/2} x_i^{-1-\alpha}
\end{aligned} \tag{3.21}$$

猜想证毕，一致网格证完。

# 4 非一致

$r > 1$,

$$\begin{cases}
x_i = \frac{1}{2}\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\
x_i = 1 - \frac{1}{2}\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N
\end{cases} \tag{4.1}$$

令 $h=\frac{1}{2N}$ , 那么
当 $i\le N, x_i<\frac{1}{2}$ 时

$$h_i=\frac{1}{2}\left(\left(\frac{i}{N}\right)^r-\left(\frac{i-1}{N}\right)^r\right)\le\frac{r}{2}\frac{1}{N}\left(\frac{i}{N}\right)^{r-1}=Chx_i^{(r-1)/r} \tag{4.2}$$

当 $i>N, x_i\ge\frac{1}{2}$ 时

$$h_i=\frac{1}{2}\left(\left(\frac{2N-i+1}{N}\right)^r-\left(\frac{2N-i}{N}\right)^r\right)\le\frac{r}{2}\left(\frac{2N-i+1}{N}\right)^{r-1}\frac{1}{N}=Ch(1-x_{i-1})^{(r-1)/r} \tag{4.3}$$

$$R_i=\int_0^1 D(y)\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}|x_{i+1}-y|^{1-\alpha}-\left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)|x_i-y|^{1-\alpha}+\frac{1}{h_i}|x_{i-1}-y|^{1-\alpha}\right)dy \tag{4.4}$$

我们声明下面的命题并在这一节中证明

**Theorem 4.1.**

$$R_i\le C\left(h^{r\alpha/2+r}x_i^{-1-\alpha}+h^2x_i^{-\alpha/2-2/r}+h^2\begin{cases}|\frac{1}{2}-x_{i-1}|^{1-\alpha}, & i\le N\\ |\frac{1}{2}-x_{i+1}|^{1-\alpha}, & N<i\le 2N\end{cases}\right) \tag{4.5}$$

为了简单，我们令

$$D(x):=u(x)-u_h(x) \tag{4.6}$$

$$T_{ij}=\int_{x_{j-1}}^{x_j}D(y)|x_i-y|^{1-\alpha}dy \tag{4.7}$$

**Lemma 4.2.** 当 $x\in[0,x_1]$ 时，

$$\begin{aligned}|D(x)|=|u(x)-u_h(x)|&\le\int_0^{x_1}|u'(y)|dy\\&\le C\int_0^{x_1}y^{\alpha/2-1}dy\\&\le Cx_1^{\alpha/2}\end{aligned} \tag{4.8}$$

同理当 $x \in [x_{2N-1}, x_{2N}(=1)]$ 时，

$$\begin{aligned}|D(x)| = |u(x) - u_h(x)| &\le \int_{x_{2N-1}}^{1} |u'(y)| dy \\ &\le C\int_{x_{2N-1}}^{1} (1-y)^{\alpha/2-1} dy \\ &\le C(1-x_{2N-1})^{\alpha/2} = Cx_1^{\alpha/2}\end{aligned} \tag{4.9}$$

**Lemma 4.3.** 对于 $y \in [x_{j-1}, x_j]$，我们记 $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$，则

*1.* 当 $2 \le j \le N$ 时

$$\begin{aligned}D(y_j^\theta) &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\eta), \quad \eta \in [x_{j-1}, x_j] \\ &\le \frac{\theta(1-\theta)}{2} h_j^2 C(\eta(1-\eta))^{\alpha/2-2} \\ &\le Ch_j^2 \eta^{\alpha/2-2} \\ &\le C2^{-r(\alpha/2-2)} h_j^2 (y_j^\theta)^{\alpha/2-2} \le Ch_j^2 (y_j^\theta)^{\alpha/2-2}\end{aligned} \tag{4.10}$$

所以存在 $C = C(\alpha, r)$ 使得

$$\frac{D(y_j^\theta)}{h_j^2} \le C(y_j^\theta)^{\alpha/2-2}, \quad 2 \le j \le N \tag{4.11}$$

同理

$$\frac{D(y_j^\theta)}{h_j^2} \le C(1-y_j^\theta)^{\alpha/2-2}, \quad N \le j \le 2N-1 \tag{4.12}$$

*2.* 当 $2 \le j \le N$ 时

$$\begin{aligned}h_j &\le \frac{r}{2} h x_j^{(r-1)/r} \\ &\le \frac{r}{2} h 2^{r-1} (y_j^\theta)^{(r-1)/r}\end{aligned} \tag{4.13}$$

因此我们有

$$D(y_j^\theta) \le Ch^2 (y_j^\theta)^{\alpha/2-2/r}, \quad 2 \le j \le N \tag{4.14}$$

同理

$$D(y_j^\theta) \le Ch^2 (1-y_j^\theta)^{\alpha/2-2/r}, \quad N \le j \le 2N-1 \tag{4.15}$$

**Lemma 4.4.**

$$D(y_j^\theta) = -\frac{\theta(1-\theta)}{2}h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!}h_j^3(\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \tag{4.16}$$

其中 $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j], \eta_1 < \eta_2$.

## 4.1 i=1

下面讨论 $R_1$，令

$$\begin{aligned} S_{ij} &= \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,j} + \frac{1}{h_i}T_{i-1,j}\right) \\ &= \int_{x_{j-1}}^{x_j} D(y)\frac{2}{h_1+h_2}(\frac{1}{h_2}|x_2 - y|^{1-\alpha} - (\frac{1}{h_1} + \frac{1}{h_2})|x_1 - y|^{1-\alpha} + \frac{1}{h_1}|y|^{1-\alpha})dy \end{aligned} \tag{4.17}$$

$$R_1 = \sum_{j=1}^{2N} S_{1j} \tag{4.18}$$

与一致网格时相似，

1.

因为 $1-\alpha > -1$，应用引理 4.2，

$$\begin{aligned} S_{11} &\le 2^{1-r}\int_0^{x_1} \frac{|D(y)|}{h_1^2}(|x_2 - y|^{1-\alpha} + 2|x_1 - y|^{1-\alpha} + |y|^{1-\alpha})dy \\ &\le Cx_1^{\alpha/2-2}(x_1 h_2^{1-\alpha} + \frac{3}{2-\alpha}x_1^{2-\alpha}) \le Cx_1^{-\alpha/2} = Ch^{-r\alpha/2} \end{aligned} \tag{4.19}$$

2.

$$\begin{aligned} S_{12} &\le 2^{1-r}\int_{x_1}^{x_2} \frac{|D(y)|}{h_1^2}(|x_2 - y|^{1-\alpha} + 2|x_1 - y|^{1-\alpha} + |y|^{1-\alpha})dy \\ &\le Cx_1^{\alpha/2-2}(3h_2^{2-\alpha} + h_2 h_1^{1-\alpha}) \le Ch^{-r\alpha/2} \end{aligned} \tag{4.20}$$

$S_{1,3}$ 同理。

3.

$$
\begin{aligned}
I_3 &= \sum_{j=4}^{N} S_{1j} \\
&= \int_{x_3}^{1/2} D(y)\frac{2}{h_1+h_2}(\frac{1}{h_2}|y-x_2|^{1-\alpha} - (\frac{1}{h_1}+\frac{1}{h_2})|y-x_1|^{1-\alpha} + \frac{1}{h_1}|y|^{1-\alpha})dy \\
&\leq C\int_{x_3}^{1/2} h^2 y^{\alpha/2-2/r} y^{-1-\alpha} dy \quad by\ 4.3\ A.1 \\
&\leq Ch^2 \int_{x_3}^{1/2} y^{\alpha/2-2/r-1-\alpha} dy \\
&\leq Ch^2 (h^r)^{-2/r-\alpha/2} = Ch^{-r\alpha/2}
\end{aligned}
\tag{4.21}
$$

4.

$$
\begin{aligned}
I_4 &= \int_{1/2}^{x_{2N-1}} D(y)\frac{2}{h_1+h_2}(\frac{1}{h_2}|x_2-y|^{1-\alpha} - (\frac{1}{h_1}+\frac{1}{h_2})|x_1-y|^{1-\alpha} + \frac{1}{h_1}|y|^{1-\alpha})dy \\
&\leq C\int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2}(h(1-y)^{(r-1)/r})^2 y^{-1-\alpha} dy \\
&\leq C2^{1+\alpha}h^2 \int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2+2-2/r} dy \\
&\leq Ch^2(C + h_{2N}^{\alpha/2-2/r+1}) \\
&= Ch^2(C + h^{r\alpha/2-2+r}) \leq Ch^{\min\{2, r\alpha/2+r\}}
\end{aligned}
\tag{4.22}
$$

5.

$$
I_5 \leq Ch_{2N}^{\alpha/2+1} \leq Ch^{r\alpha/2+r} \tag{4.23}
$$

综合有

$$
R_1 \leq Ch^{-r\alpha/2} \tag{4.24}
$$

$R_1, R_2, R_3$ 一样。

$R_i, 3 < i < N$ 比较困难。

## 4.2 i<N/2

当 $3 < i < N/2$ , 即 $x_i < (\frac{1}{4})^r$ 时。

$$\begin{aligned}
R_i =& \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& \sum_{j=1}^{i/2} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&+ \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,i/2+1} + T_{i+1,i/2+2}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,i/2+1} \right) \\
&+ \sum_{j=i/2+2}^{2i-1} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&+ \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i-1}} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \\
&+ \sum_{j=2i+1}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
I_1 =& \int_0^{x_1} + \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} \\
& D(y) \frac{2}{h_i + h_{i+1}} (\frac{1}{h_{i+1}} |x_{i+1} - y|^{1-\alpha} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) |x_i - y|^{1-\alpha} + \frac{1}{h_i} |x_{i-1} - y|^{1-\alpha}) dy
\end{aligned} \tag{4.26}$$

1.

$$J_1 \le C x_1^{\alpha/2+1} x_i^{-1-\alpha} \le C h^{r\alpha/2+r} x_i^{-1-\alpha} \tag{4.27}$$

2.

$$
\begin{aligned}
J_2 &\le C\int_{x_1}^{x_{\lceil \frac{i}{2}\rceil}} y^{\alpha/2-2}(hy^{(r-1)/r})^2|x_i-y|^{-1-\alpha}dy \\
&\le Ch^2x_i^{-1-\alpha}\int_{x_1}^{x_{\lceil \frac{i}{2}\rceil}} y^{\alpha/2-2/r}dy \\
&\le Ch^2x_i^{-1-\alpha}(h^{r\alpha/2-2+r}+x_i^{\alpha/2-2/r+1}) \\
&= C(h^{r\alpha/2+r}x_i^{-1-\alpha}+h^2x_i^{-\alpha/2-2/r})
\end{aligned} \tag{4.28}
$$

我们先研究 $I_3$，考虑

$$
\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j+1}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,j}+\frac{1}{h_i}T_{i-1,j-1}\right) \tag{4.29}
$$

$$
\begin{aligned}
T_{ij} &= \int_{x_{j-1}}^{x_j} D(y)|x_i-y|^{1-\alpha}dy \\
&= \int_0^1 \frac{\theta(1-\theta)}{2}h_j^3u''(y_j^\theta)|x_i-y_j^\theta|^{1-\alpha}d\theta \\
&\quad + \int_0^1 \frac{\theta(1-\theta)}{3!}h_j^4|x_i-y_j^\theta|^{1-\alpha}(\theta^2u'''(\eta_{1,j}^\theta)-(1-\theta)^2u'''(\eta_{2,j}^\theta))d\theta
\end{aligned} \tag{4.30}
$$

现在回到原来的问题，我们要研究

$$
\begin{aligned}
&\frac{2}{h_i+h_{i+1}}(\frac{1}{h_{i+1}}h_{j+1}^3u''(y_{j+1}^\theta)|x_{i+1}-y_{j+1}^\theta|^{1-\alpha} \\
&\quad -(\frac{1}{h_i}+\frac{1}{h_{i+1}})h_j^3\ u''(y_j^\theta)|x_i-y_j^\theta|^{1-\alpha} \\
&\quad\quad +\frac{1}{h_i}h_{j-1}^3u''(y_{j-1}^\theta)|x_{i-1}-y_{j-1}^\theta|^{1-\alpha})
\end{aligned} \tag{4.31}
$$

我们希望把他看成一个函数的二阶导，注意到当 $i/2\le j\le 2i$ 时

$$
x_i^{1/r}-x_j^{1/r}=x_{i+1}^{1/r}-x_{j+1}^{1/r}=2^{-1/r}\frac{i-j}{N} \tag{4.32}
$$

那么我们将其他的相都表示成 $x_i$ 的函数。

$$y_L(x) = (x^{1/r} + z_L)^r, \quad y_R(x) = (x^{1/r} + z_R)^r \tag{4.33}$$

其中 $z_L = 2^{-1/r\frac{j-i-1}{N}}, z_R = 2^{-1/r\frac{j-i}{N}}$.

$$y_R(x_i) = x_j, \quad y_R(x_{i+1}) = x_{j+1}, \quad y_R(x_{i-1}) = x_{j-1} \tag{4.34}$$

$$y_L(x_i) = x_{j-1}, \quad y_L(x_{i+1}) = x_j, \quad y_L(x_{i-1}) = x_{j-2} \tag{4.35}$$

$$y_\theta(x) = \theta y_L(x) + (1-\theta) y_R(x) \tag{4.36}$$

$$h_J(x) = y_R(x) - y_L(x) \tag{4.37}$$

那么我么要研究的就是函数

$$K_1(x) = h_J^3(x)|x - y_\theta(x)|^{1-\alpha} u''(y_\theta(x)) \tag{4.38}$$

在网格 $x_{i-1}, x_i, x_{i+1}$ 的数值二阶差商。

$$\frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}K_1(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right)K_1(x_i) + \frac{1}{h_i}K_1(x_{i-1})\right) = K_1''(\xi),\ \xi \in [x_{i-1}, x_{i+1}] \tag{4.39}$$

由 Leibniz 公式

$$(uvw)'' = u''vw + uv''w + uvw'' + 2u'v'w + 2uv'w' + 2u'vw' \tag{4.40}$$

由 $y_R^{1/r} = x^{1/r} + z_R$, 我们得到

$$\frac{dy_R}{dx} = x^{1/r-1} y_R^{1-1/r} \tag{4.41}$$

$$\frac{d^2 y_R}{dx^2} = \frac{1-r}{r} x^{1/r-2} y_R^{1-2/r} z_R \tag{4.42}$$

因此

1.

$$h_J^3 \sim h^3 y_R^{3-3/r} \sim h^3 x^{3-3/r} \tag{4.43}$$

$$\begin{aligned}(h_J^3)' &= 3h_J^2(y_R' - y_L') \\ &= 3h_J^2 x^{1/r-1}(y_R^{1-1/r} - y_L^{1-1/r}) \\ &\sim h^3 y_R^{2-2/r} x^{1/r-1} y_R^{1-2/r} \\ &\sim h^3 x^{2-3/r}\end{aligned} \tag{4.44}$$

$$\begin{aligned}(h_J^3)'' &= 6h_J x^{2/r-2}(y_R^{1-1/r} - y_L^{1-1/r})^2 + 3h_J^2 \frac{1-r}{r} x^{1/r-2}(y_R^{1-2/r} z_R - y_L^{1-2/r} z_L) \\ &\sim h y_R^{1-1/r} x^{2/r-2}(h y_R^{1-2/r})^2 + \frac{1-r}{r} h^2 y_R^{2-2/r} x^{1/r-2}(z_R h y_R^{1-3/r} + h y_L^{1-2/r}) \\ &\sim h^3 y_R^{3-5/r} x^{2/r-2} + h^3(y_R^{3-5/r} x^{1/r-2} z_R + y_R^{3-4/r} x^{1/r-2}) \\ &\sim h^3(y_R^{3-5/r} x^{2/r-2} + y_R^{3-4/r} x^{1/r-2} + y_R^{3-5/r} x^{1/r-2} z_R) \\ &\sim h^3 x^{1-3/r}\end{aligned} \tag{4.45}$$

2.

由于

$$x - y_L = (x^{1/r})^r - (x^{1/r} + z_L)^r = -z_L \xi^{r-1} \sim z_L x^{(r-1)/r} \tag{4.46}$$

$$\begin{aligned}|x - y_\theta|^{1-\alpha} &= |x - \theta y_L - (1-\theta) y_R|^{1-\alpha} \sim |(\theta z_L + (1-\theta) z_R) \xi^{1-1/r}|^{1-\alpha} \\ &\sim z_\theta^{1-\alpha} \xi^{1-\alpha+(\alpha-1)/r}, \quad \xi \in [y_L, x]\end{aligned} \tag{4.47}$$

$$\begin{aligned}(|x - y_\theta|^{1-\alpha})' &= \mathrm{sign}(x - y_\theta)(1-\alpha)|x - y_\theta|^{-\alpha}(1 - x^{1/r-1}(\theta y_L^{1-1/r} + (1-\theta) y_R^{1-1/r})) \\ &= (1-\alpha)|x - y_\theta|^{-\alpha} x^{1/r-1}(x^{1-1/r} - (\theta y_L^{1-1/r} + (1-\theta) y_R^{1-1/r})) \\ &\sim |x - y_\theta|^{-\alpha} x^{1/r-1}(\theta z_L + (1-\theta) z_R) \xi_2^{1-2/r} \\ &\sim |(\theta z_L + (1-\theta) z_R) \xi_1^{1-1/r}|^{-\alpha} x^{1/r-1}(\theta z_L + (1-\theta) z_R) \xi_2^{1-2/r} \\ &\sim z_\theta^{1-\alpha} x^{-\alpha+(\alpha-1)/r}\end{aligned} \tag{4.48}$$

$$
\begin{aligned}
(|x-y_\theta|^{1-\alpha})'' =&\alpha(\alpha-1)|x-y_\theta|^{-1-\alpha}(1-x^{1/r-1}(\theta y_L^{1-1/r}+(1-\theta)y_R^{1-1/r}))^2\\
&-\mathrm{sign}(x-y_\theta)(1-\alpha)|x-y_\theta|^{-\alpha}(\frac{1-r}{r}x^{1/r-2}(\theta y_L^{1-2/r}z_L+(1-\theta)y_R^{1-2/r}z_R))\\
\sim&|(\theta z_L+(1-\theta)z_R)\xi_1^{1-1/r}|^{-1-\alpha}x^{2/r-2}\xi_2^{2-4/r}(\theta z_L+(1-\theta)z_R)^2\\
&+|(\theta z_L+(1-\theta)z_R)\xi_1^{1-1/r}|^{-\alpha}x^{1/r-2}(\theta z_L+(1-\theta)z_R)y_R^{1-2/r}\\
\sim&z_\theta^{1-\alpha}x^{-1-\alpha+(\alpha-1)/r}
\end{aligned}
\tag{4.49}
$$

3.

$$
u''(y_\theta)\le Cy_\theta^{\alpha/2-2}\sim x^{\alpha/2-2} \tag{4.50}
$$

$$
\begin{aligned}
(u''(y_\theta))' &= u'''(y_\theta)x^{1/r-1}(\theta y_L^{1-1/r}+(1-\theta)y_R^{1-1/r})\\
&\le Cy_\theta^{\alpha/2-3}x^{1/r-1}y_R^{1-1/r}\sim x^{\alpha/2-3}
\end{aligned}
\tag{4.51}
$$

$$
\begin{aligned}
(u''(y_\theta))'' =&u''''(y_\theta)(x^{1/r-1}(\theta y_L^{1-1/r}+(1-\theta)y_R^{1-1/r}))^2\\
&+u'''(y_\theta)\frac{1-r}{r}x^{1/r-2}(\theta y_L^{1-2/r}z_L+(1-\theta)y_R^{1-2/r}z_R)\\
\sim&y_\theta^{\alpha/2-4}(x^{1/r-1}y_R^{1-1/r})^2+z_\theta y_R^{\alpha/2-3+1-2/r}x^{1/r-2}\\
<&x^{\alpha/2-4}
\end{aligned}
\tag{4.52}
$$

$$u''vw \sim h^3 x^{1-3/r}\ z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r}\ x^{\alpha/2-2} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \tag{4.53}$$

$$uv''w \sim h^3 x^{3-3/r}\ z_\theta^{1-\alpha} x^{-1-\alpha+(\alpha-1)/r}\ x^{\alpha/2-2} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \tag{4.54}$$

$$uvw'' \sim h^3 x^{3-3/r}\ z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r}\ x^{\alpha/2-4} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \tag{4.55}$$

$$u'v'w \sim h^3 x^{2-3/r}\ z_\theta^{1-\alpha} x^{-\alpha+(\alpha-1)/r}\ x^{\alpha/2-2} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \tag{4.56}$$

$$uv'w' \sim h^3 x^{3-3/r}\ z_\theta^{1-\alpha} x^{-\alpha+(\alpha-1)/r}\ x^{\alpha/2-3} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \tag{4.57}$$

$$u'vw' \sim h^3 x^{2-3/r}\ z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r}\ x^{\alpha/2-3} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \tag{4.58}$$

因此

$$K_1''(\xi) \sim h^3 z_\theta^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r}, \xi \in [x_{i-1}, x_{i+1}] \tag{4.59}$$

现在我们处理第二部分

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}(&\frac{1}{h_{i+1}} h_{j+1}^4 u'''(\eta_{1,j+1}^\theta)|x_{i+1}-y_{j+1}^\theta|^{1-\alpha} \\ &-(\frac{1}{h_i}+\frac{1}{h_{i+1}})h_j^4\ u'''(\eta_{1,j}^\theta)|x_i-y_j^\theta|^{1-\alpha} \\ &\qquad + \frac{1}{h_i} h_{j-1}^4 u'''(\eta_{1,j-1}^\theta)|x_{i-1}-y_{j-1}^\theta|^{1-\alpha})\end{aligned} \tag{4.60}$$

这次我们只用一阶差分

$$\frac{1}{h_i}(h_j^4 u'''(\eta_{1,j}^\theta)|x_i-y_j^\theta|^{1-\alpha} - h_{j-1}^4\ u'''(\eta_{1,j-1}^\theta)|x_{i-1}-y_{j-1}^\theta|^{1-\alpha}) \tag{4.61}$$

为了方便计算，我们还是用辅助函数来对上面一项进行估计。

$$K_2(x) = h_J^4(x)|x-y_\theta(x)|^{1-\alpha} \tag{4.62}$$

$$
\begin{aligned}
K_2'(x) &= (h_J^4)'|x-y_\theta(x)|^{1-\alpha} + h_J^4(|x-y_\theta(x)|^{1-\alpha})' \\
&\sim h^3x^{3-3/r}x^{1/r-1}hy_R^{1-2/r}\ z_\theta^{1-\alpha}x^{1-\alpha+(\alpha-1)/r} \\
&\quad + h^4y_R^{4-4/r}z_\theta^{1-\alpha}x^{-\alpha+(\alpha-1)/r} \\
&\sim h^4z_\theta^{1-\alpha}x^{4-5/r-\alpha+\alpha/r}
\end{aligned} \tag{4.63}
$$

那么，上面就等于

$$
\begin{aligned}
&\frac{1}{h_i}(K_2(x_i)u'''(\eta_{1,j}^\theta) - K_2(x_{i-1})u'''(\eta_{1,j-1}^\theta)) \\
=&\frac{1}{h_i}K_2(x_i)(u'''(\eta_{1,j}^\theta) - u'''(\eta_{1,j-1}^\theta)) + \frac{1}{h_i}(K_2(x_i) - K_2(x_{i-1})u'''(\eta_{1,j-1}^\theta)) \\
\le&h_i^{-1}K_2(x_i)u''''(\eta_j^\theta)(x_j-x_{j-2}) + K_2'(\xi)u'''(\eta_{1,j-1}^\theta) \quad (\eta_j^\theta \in [x_{j-2},x_j], \xi \in [x_{j-1},x_j]) \\
\sim&h_i^{-1}h_j^4|x_i-y_j^\theta|^{1-\alpha}\ C(\eta_j^\theta)^{\alpha/2-4}2h_j \\
&+h^4z_\theta^{1-\alpha}\xi^{4-5/r-\alpha+\alpha/r}(\eta_j^\theta)^{\alpha/2-3} \\
\sim&h^4x_i^{4-4/r}z_\theta^{1-\alpha}x_i^{1-\alpha+(\alpha-1)/r}x_i^{\alpha/2-4} + h^4z_\theta^{1-\alpha}x_i^{4-5/r-\alpha+\alpha/r}x_i^{\alpha/2-3} \\
\sim&hx_i^{1-1/r}\ h^3z_\theta^{1-\alpha}x_i^{-\alpha/2-2/r+(\alpha-2)/r}
\end{aligned} \tag{4.64}
$$

因此，

$$
\begin{aligned}
\frac{2}{h_i+h_{i+1}}\frac{1}{h_i}(K_2(x_i)u'''(\eta_{1,j}^\theta) - K_2(x_{i-1})u'''(\eta_{1,j-1}^\theta)) \\
\sim h^3z_\theta^{1-\alpha}x_i^{-\alpha/2-2/r+(\alpha-2)/r}
\end{aligned} \tag{4.65}
$$

最终我们得到，当 $i/2+2 \le j \le 2i-1 < N$ 时，有

$$
\begin{aligned}
\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j+1} - (\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,j} + \frac{1}{h_i}T_{i-1,j-1}\right) \\
\le Ch^3\left(\frac{|i-j|+1}{N}\right)^{1-\alpha}x_i^{-\alpha/2-2/r+(\alpha-2)/r}
\end{aligned} \tag{4.66}
$$

那么我们得到

$$
\begin{aligned}
I_3 &\le C \sum_{j=i/2+2}^{2i-1} \left(\frac{1}{N}\right)^3 \left(\frac{|i-j|+1}{N}\right)^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r} \\
&\le C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r+(\alpha-2)/r} \left(\frac{2i}{N}\right)^{2-\alpha} \\
&\le C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r+(\alpha-2)/r} x_i^{(2-\alpha)/r} \\
&= C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r}
\end{aligned}
\tag{4.67}
$$

现在我们处理 $I_2$, 记 $k = i/2+1$

$$
\begin{aligned}
I_2 &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,i/2+1} + T_{i+1,i/2+2}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})(T_{i,i/2+1}) \right) \\
&= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i}) T_{i,k} \right) \\
&= J_1 + J_2 + J_3
\end{aligned}
\tag{4.68}
$$

$$
\begin{aligned}
J_1 &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) \right) \\
&= \frac{2}{h_i + h_{i+1}} \int_{x_{k-1}}^{x_k} D(y) \frac{|x_{i+1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_{i+1}} dy \\
&\le C x_i^{\alpha/2-2} h_k^2 x_i^{-\alpha} \\
&\le C h^2 x_i^{-\alpha/2-2/r}
\end{aligned}
\tag{4.69}
$$

$$
\begin{aligned}
J_2 &= \frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) \\
&= \frac{2}{h_i + h_{i+1}} \int_0^1 \frac{h_{k+1} D(y_{k+1}^\theta) |x_{i+1} - y_{k+1}^\theta|^{1-\alpha} - h_k D(y_k^\theta) |x_i - y_k^\theta|^{1-\alpha}}{h_{i+1}} d\theta
\end{aligned}
\tag{4.70}
$$

我们看他的两个积分项

$$
\begin{aligned}
\frac{K_1(x_{i+1})-K_1(x_i)}{h_{i+1}} &= K_1'(\xi) \\
&\sim h^3x^{2-3/r}\ z_\theta^{1-\alpha}x^{1-\alpha+(\alpha-1)/r}\ x^{\alpha/2-2} \\
&\quad + h^3x^{3-3/r}\ z_\theta^{1-\alpha}x^{-\alpha+(\alpha-1)/r}\ x^{\alpha/2-2} \\
&\quad + h^3x^{3-3/r}\ z_\theta^{1-\alpha}x^{1-\alpha+(\alpha-1)/r}\ x^{\alpha/2-3} \\
&\sim hx^{1-1/r}h^2z_\theta^{1-\alpha}x^{-\alpha/2+\alpha/r-3/r} \\
&\sim hx^{1-1/r}h^2x^{(1-\alpha)/r}x^{-\alpha/2+\alpha/r-3/r} \\
&\sim hx^{1-1/r}h^2x^{-\alpha/2-2/r}
\end{aligned}
\tag{4.71}
$$

第二部分研究过了

$$
\begin{aligned}
&\frac{1}{h_i}(K_2(x_{i+1})u'''(\eta_{1,k+1}^\theta)-K_2(x_i)u'''(\eta_{1,k}^\theta)) \\
&\sim hx_i^{1-1/r}\ h^3z_\theta^{1-\alpha}x_i^{-\alpha/2-2/r+(\alpha-2)/r} \\
&\sim hx_i^{1-1/r}\ h^3x_i^{(1-\alpha)/r}x_i^{-\alpha/2-2/r+(\alpha-2)/r} \\
&\sim hx_i^{1-1/r}\ h^3x_i^{-\alpha/2-2/r-1/r}
\end{aligned}
\tag{4.72}
$$

因此

$$J_2 \le Ch^2x^{-\alpha/2-2/r} \tag{4.73}$$

现在考虑 $J_3$

$$
\begin{aligned}
J_3 &= \frac{2}{h_i+h_{i+1}}(\frac{1}{h_{i+1}}-\frac{1}{h_i})T_{i,k} \\
&= -\frac{2}{h_i+h_{i+1}}\frac{h_{i+1}-h_i}{h_ih_{i+1}}\int_{x_{k-1}}^{x_k}D(y_k^\theta)|x_i-y_k^\theta|^{1-\alpha}dy \\
&\sim h_i^{-1}x_i^{-1}h_k^3x_i^{\alpha/2-2}x_i^{1-\alpha} \\
&\sim h^2x_i^{-\alpha/2-2/r}
\end{aligned}
\tag{4.74}
$$

因此我们有

$$I_2 \le Ch^2x_i^{-\alpha/2-2/r} \tag{4.75}$$

$I_4$ 类似。

现在考虑 $I_5$

$$
\begin{aligned}
I_5 =& \sum_{j=2i+1}^{N} + \sum_{j=N+1}^{2N-1} + \sum_{2N} \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j} - (\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,j} + \frac{1}{h_i}T_{i-1,j}\right)\\
=& \int_{x_{2i}}^{x_N} + \int_{x_N}^{x_{2N-1}} + \int_{x_{2N-1}}^{2N}\\
& D(y)\frac{2}{h_i+h_{i+1}}(\frac{1}{h_{i+1}}|x_{i+1}-y|^{1-\alpha} - (\frac{1}{h_i}+\frac{1}{h_{i+1}})|x_i-y|^{1-\alpha} + \frac{1}{h_i}|x_{i-1}-y|^{1-\alpha})dy\\
=& J_1+J_2+J_3
\end{aligned}
\tag{4.76}
$$

$$
\begin{aligned}
J_1 =& \int_{x_{2i}}^{1/2} D(y)\frac{2}{h_i+h_{i+1}}(\frac{1}{h_{i+1}}|x_{i+1}-y|^{1-\alpha} - (\frac{1}{h_i}+\frac{1}{h_{i+1}})|x_i-y|^{1-\alpha} + \frac{1}{h_i}|x_{i-1}-y|^{1-\alpha})dy\\
\le& C\int_{x_{2i}}^{1/2} y^{\alpha/2-2}(hy^{1-1/r})^2|y-x_i|^{-1-\alpha}dy\\
\le& C\int_{x_{2i}}^{1/2} h^2 y^{\alpha/2-2+2-2/r-1-\alpha}dy\\
\le& Ch^2x_i^{-\alpha/2-2/r}
\end{aligned}
\tag{4.77}
$$

$$
\begin{aligned}
J_2 =& \int_{1/2}^{x_{2N-1}} D(y)\frac{2}{h_i+h_{i+1}}(\frac{1}{h_{i+1}}|x_{i+1}-y|^{1-\alpha} - (\frac{1}{h_i}+\frac{1}{h_{i+1}})|x_i-y|^{1-\alpha} + \frac{1}{h_i}|x_{i-1}-y|^{1-\alpha})dy\\
\le& C\int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2}(h(1-y)^{1-1/r})^2|y-x_i|^{-1-\alpha}dy\\
\le& C\int_{1/2}^{x_{2N-1}} h^2(1-y)^{\alpha/2-2+2-2/r}dy\\
\le& Ch^2(C+h^{r(\alpha/2-2/r+1)})\\
\le& Ch^2+Ch^{r\alpha/2+r}
\end{aligned}
\tag{4.78}
$$

$$
J_3 \le Ch_{2N}^{\alpha/2+1} \le Ch^{r\alpha/2+r} \tag{4.79}
$$

全部加起来，我们得到

$$
R_i \le Ch^{r\alpha/2+r}x_i^{-1-\alpha} + Ch^2x_i^{-\alpha/2-2/r} \tag{4.80}
$$

## 4.3 N/2<i<N

当 $N/2 < i < N$ , 即 $(\frac{1}{4})^r < x_i < \frac{1}{2}$ 时。

$$
\begin{aligned}
R_i =&\sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=&\sum_{j=1}^{i/2} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&+ \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,i/2+1} + T_{i+1,i/2+2}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,i/2+1} \right) \\
&+ \sum_{j=i/2+2}^{3N/2-1} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&+ \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i-1}} (T_{i-1,3N/2} + T_{i-1,3N/2-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,3N/2} \right) \\
&+ \sum_{j=3N/2+1}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=&I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned} \tag{4.81}
$$

$I_1, I_2, I_4, I_5$ 略。

$$
\begin{aligned}
I_3 =& \sum_{j=i/2+2}^{N-1} + \sum_{j=N}^{N+1} + \sum_{N+2}^{2N-i} + \sum_{2N-i+1}^{3N/2-1} \\
& \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
=& J_1 + J_2 + J_3 + J_4
\end{aligned} \tag{4.82}
$$

$$
\begin{aligned}
J_1 &\le C\sum_{j=i/2+2}^{N-1}\left(\frac{1}{N}\right)^3\left(\frac{|i-j|+1}{N}\right)^{1-\alpha}x_i^{-\alpha/2-2/r+(\alpha-2)/r}\\
&\le C\left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r+(\alpha-2)/r}(\left(\frac{i}{2N}\right)^{2-\alpha}+\left(\frac{N-i}{N}\right)^{2-\alpha})\\
&\le C\left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r+(\alpha-2)/r}(x_i^{(2-\alpha)/r}+|\frac{1}{2}-x_i|^{2-\alpha}x_i^{(2-\alpha)(1/r-1)})\\
&= Ch^2|\frac{1}{2}-x_i|^{2-\alpha}+Ch^2\le Ch^2
\end{aligned}
\tag{4.83}
$$

对于 $J_3, J_4$，由于 $i < N < j-1$

$$
x_i^{1/r}+(1-x_j)^{1/r}=2^{-1/r}(\frac{i}{N}+\frac{2N-j}{N})=2^{-1/r}(2-\frac{j-i}{N})=x_{i-1}^{1/r}+(1-x_{j-1})^{1/r}
\tag{4.84}
$$

那么令

$$
y_L=1-(z_L-x^{1/r})^r,\quad y_R=1-(z_R-x^{1/r})^r \tag{4.85}
$$

其中

$$
z_L=2^{-1/r}(2-\frac{j-i-1}{N}),\quad z_R=2^{-1/r}(2-\frac{j-i}{N}) \tag{4.86}
$$

那么类似的

$$
\frac{dy_R}{dx}=x^{1/r-1}(1-y_R)^{1-1/r} \tag{4.87}
$$

$$
\frac{d^2y_R}{dx^2}=\frac{1-r}{r}x^{1/r-2}(1-y_R)^{1-2/r}z_R \tag{4.88}
$$

那么我们考虑

$$
K_1(x)=h_J^3|y_\theta-x|^{1-\alpha}u''(y_\theta) \tag{4.89}
$$

的二阶导数

其中 $\frac{1}{4}<x<\frac{1}{2}, \frac{1}{2}<y_\theta<\frac{3}{4}$.

1.

$$h_J^3 \sim h^3(1-y_R)^{3-3/r} \sim h^3 \tag{4.90}$$

$$\begin{aligned}(h_J^3)' &= 3h_J^2(y_R' - y_L') \\ &= 3h_J^2 x^{1/r-1}((1-y_R)^{1-1/r} - (1-y_L)^{1-1/r}) \\ &\sim h^3(1-y_R)^{2-2/r}x^{1/r-1}(1-y_R)^{1-2/r} \\ &\sim h^3\end{aligned} \tag{4.91}$$

$$\begin{aligned}(h_J^3)'' &= 6h_J x^{2/r-2}((1-y_R)^{1-1/r} - (1-y_L)^{1-1/r})^2 \\ &\quad + 3h_J^2\frac{1-r}{r}x^{1/r-2}((1-y_R)^{1-2/r}z_R - (1-y_L)^{1-2/r}z_L) \\ &\sim h(1-y_R)^{1-1/r}x^{2/r-2}(h(1-y_R)^{1-2/r})^2 \\ &\quad - h^2(1-y_R)^{2-2/r}x^{1/r-2}(z_R h(1-y_R)^{1-3/r} + h(1-y_L)^{1-2/r}) \\ &\sim h^3\end{aligned} \tag{4.92}$$

2.

由于

$$y - x \sim \begin{cases} \frac{1}{2} - x, & 1 < y < 1-x \\ \frac{1}{2} - y, & y > 1-x \end{cases} \tag{4.93}$$

$$\begin{aligned}(|y_\theta - x|^{1-\alpha})' &= (1-\alpha)|y_\theta - x|^{-\alpha}(x^{1/r-1}(\theta(1-y_L)^{1-1/r} + (1-\theta)(1-y_R)^{1-1/r}) - 1) \\ &= (1-\alpha)|y_\theta - x|^{-\alpha}x^{1/r-1}(\theta(1-y_L)^{1-1/r} + (1-\theta)(1-y_R)^{1-1/r} - (1-(1-x))^{1-1/r}) \\ &\sim |y_\theta - x|^{-\alpha}(y_\theta - (1-x))\end{aligned} \tag{4.94}$$

$$\begin{aligned}(|y_\theta - x|^{1-\alpha})'' =& \alpha(\alpha-1)|y_\theta - x|^{-1-\alpha}(x^{1/r-1}(\theta(1-y_L)^{1-1/r} + (1-\theta)(1-y_R)^{1-1/r}) - 1)^2 \\ & - (1-\alpha)|y_\theta - x|^{-\alpha}(\frac{1-r}{r}x^{1/r-2}(\theta(1-y_L)^{1-2/r}z_L + (1-\theta)(1-y_R)^{1-2/r}z_R)) \\ \sim& |y_\theta - x|^{-1-\alpha}(y_\theta - (1-x))^2 + |y_\theta - x|^{-\alpha}\end{aligned} \tag{4.95}$$

3.

$$u''(y_\theta) \le C(1-y_\theta)^{\alpha/2-2} \sim 1 \tag{4.96}$$

$$\begin{aligned}(u''(y_\theta))' &= u'''(y_\theta)x^{1/r-1}(\theta(1-y_L)^{1-1/r}+(1-\theta)(1-y_R)^{1-1/r})\\ &\sim 1\end{aligned} \tag{4.97}$$

$$\begin{aligned}(u''(y_\theta))'' =& u''''(y_\theta)(x^{1/r-1}(\theta(1-y_L)^{1-1/r}+(1-\theta)(1-y_R)^{1-1/r}))^2\\ &+u'''(y_\theta)\frac{1-r}{r}x^{1/r-2}(\theta(1-y_L)^{1-2/r}z_L+(1-\theta)(1-y_R)^{1-2/r}z_R)\\ \sim& 1\end{aligned} \tag{4.98}$$

那么 大概齐，要补上细节

$$\begin{aligned}J_3 &\le C\sum_{j=N+2}^{2N-i} h^3(|\frac{1}{2}-x_i|^{1-\alpha}+|\frac{1}{2}-x_i|^{-\alpha})\\ &\le Ch^2|\frac{1}{2}-x_i|^{-\alpha}\int_{1/2}^{1-x_i} 1\, dy\\ &\le Ch^2|\frac{1}{2}-x_i|^{1-\alpha}\end{aligned} \tag{4.99}$$

$$\begin{aligned}J_4 &\le C\int_{1-x_i}^{3/4} h^2|y-x_i|^{-\alpha}\\ &\le Ch^2+Ch^2|\frac{1}{2}-x_i|^{1-\alpha}\end{aligned} \tag{4.100}$$

综上，

$$I_3 \le Ch^2|\frac{1}{2}-x_i|^{1-\alpha} \tag{4.101}$$

### 4.4 i=N

$h_N = h_{N+1}, \quad x_N = \frac{1}{2}.$

$$R_N = \frac{1}{h_N^2}\int_0^1 D(y)(|x_{N+1}-y|^{1-\alpha}-2|x_N-y|^{1-\alpha}+|x_{N-1}-y|^{1-\alpha})dy \quad (4.102)$$

# 5 收敛性分析

**Lemma 5.1.**

$$\sum_{j=1}^{2N-1}\tilde{a}_{ij} = \frac{1}{(2-\alpha)(3-\alpha)}\left(\frac{|x_i-x_0|^{3-\alpha}-|x_i-x_1|^{3-\alpha}}{h_1}+\frac{|x_{2N}-x_i|^{3-\alpha}-|x_{2N-1}-x_i|^{3-\alpha}}{h_{2N}}\right) \quad (5.1)$$

令

$$g(x) = g_0(x) + g_{2N}(x) \quad (5.2)$$

其中

$$\begin{aligned} g_0(x) &:= \frac{-\kappa_\alpha}{(2-\alpha)(3-\alpha)}\frac{|x_i-x_0|^{3-\alpha}-|x_i-x_1|^{3-\alpha}}{h_1} \\ g_{2N}(x) &:= \frac{-\kappa_\alpha}{(2-\alpha)(3-\alpha)}\frac{|x_{2N}-x_i|^{3-\alpha}-|x_{2N-1}-x_i|^{3-\alpha}}{h_{2N}} \end{aligned}$$

**Lemma 5.2.** $A$ 是 $M$ 矩阵。且

$$\begin{aligned} S_i :=& \sum_{j=1}^{2N-1} a_{ij} \\ =& \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}g(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})g(x_i)+\frac{1}{h_i}g(x_{i-1})\right) \\ \geq& C(x_i^{-\alpha}+(1-x_i)^{-\alpha}) \end{aligned} \quad (5.3)$$

证明. 事实上，当 $i \geq 2$ 时，

$$\begin{aligned} &\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}g_0(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})g_0(x_i)+\frac{1}{h_i}g_0(x_{i-1})\right) \\ &=g_0''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \end{aligned} \quad (5.4)$$

又有

$$\begin{aligned} g_0''(\xi) &= -\kappa_\alpha \frac{|\xi - x_0|^{1-\alpha} - |\xi - x_1|^{1-\alpha}}{h_1} \\ &= \kappa_\alpha(\alpha - 1)|\xi - \eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\ &\geq \kappa_\alpha(\alpha - 1)x_{i+1}^{-\alpha} \geq \kappa_\alpha(\alpha - 1)2^{-r\alpha}x_i^{-\alpha} \end{aligned} \tag{5.5}$$

当 $i = 1$ 时

$$\begin{aligned} &\frac{2}{h_1 + h_2}\left(\frac{1}{h_2}g_0(x_2) - (\frac{1}{h_1} + \frac{1}{h_2})g_0(x_1) + \frac{1}{h_1}g_0(x_0)\right) \\ =&\frac{2\kappa_\alpha}{(2-\alpha)(3-\alpha)}\frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1h_2} \\ =&\frac{2\kappa_\alpha}{(2-\alpha)(3-\alpha)}\frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1^{1-\alpha}h_2}h_1^{-\alpha} \\ =&\frac{2\kappa_\alpha}{(2-\alpha)(3-\alpha)}\frac{1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha}}{2^r(2^r - 1)}h_1^{-\alpha} \end{aligned} \tag{5.6}$$

因为 $3 - \alpha > 1$，我们有 $1 + (2^r - 1)^{3-\alpha} \geq (2^r)^{3-\alpha}$，因此

$$RHS \geq \frac{2\kappa_\alpha}{(2-\alpha)(3-\alpha)}2^{1-r}x_1^{-\alpha} \tag{5.7}$$

错！ 关于 $g_{2N}(x)$，完全对称，所以存在 $C = C(\alpha, r)$ 使得

$$S_i \geq \kappa_\alpha(\alpha - 1)2^{-r\alpha}(x_i^{-\alpha} + (1 - x_i)^{-\alpha}) \tag{5.8}$$

□

令

$$G = \text{diag}(x_1, ..., x_N(= 1 - x_N), 1 - x_{N+1}, ..., 1 - x_{2N-1}) \tag{5.9}$$

**Lemma 5.3.** 矩阵 $B := AG$ 主对角元为正，其他元为负。且存在 $C = C(\alpha, r) > 0$ 使得

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C(x_i^{1-\alpha} + (1 - x_i)^{1-\alpha}) + C\begin{cases} |\frac{1}{2} - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - \frac{1}{2}|^{1-\alpha}, & i \geq N \end{cases} \tag{5.10}$$

证明. 令

$$g(x) = \begin{cases} x, & 0 < x \le 1/2 \\ 1 - x, & 1/2 < x < 1 \end{cases} \tag{5.11}$$

因为 $g(x)$ 的线性插值就是他自身 $g(x) \equiv g_h(x)$，所以有

$$\begin{aligned}
\tilde{M}_i &:= \sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} \tilde{a}_{ij} g(x_j) \\
&= \int_0^1 |x_i - y|^{1-\alpha} g(y) dy \\
&= \frac{-2}{(2-\alpha)(3-\alpha)} |\frac{1}{2} - x_i|^{3-\alpha} + \frac{1}{(2-\alpha)(3-\alpha)} (x_i^{3-\alpha} + (1 - x_i)^{3-\alpha}) \\
&:= w(x_i) = p(x_i) + q(x_i)
\end{aligned} \tag{5.12}$$

所以

$$\begin{aligned}
M_i &:= \sum_{j=1}^{2N-1} a_{ij} g(x_j) \\
&= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} w(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) w(x_i) + \frac{1}{h_i} w(x_{i-1}) \right)
\end{aligned} \tag{5.13}$$

特别的,

$$\begin{aligned}
P_N &:= -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left( \frac{1}{h_{N+1}} p(x_{N+1}) - (\frac{1}{h_N} + \frac{1}{h_{N+1}}) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right) \\
&= \frac{4\kappa_\alpha}{(2-\alpha)(3-\alpha) h_N^2} h_N^{3-\alpha} \\
&= \frac{4\kappa_\alpha}{(2-\alpha)(3-\alpha)} (\frac{1}{2} - x_{N-1})^{1-\alpha}
\end{aligned} \tag{5.14}$$

$$
\begin{aligned}
P_{N-1} &:= \frac{-2\kappa_\alpha}{h_{N-1}+h_N}\left(\frac{1}{h_N}p(x_N)-(\frac{1}{h_{N-1}}+\frac{1}{h_N})p(x_{N-1})+\frac{1}{h_{N-1}}p(x_{N-2})\right)\\
&= \frac{2\kappa_\alpha}{(2-\alpha)(3-\alpha)}\frac{2}{h_{N-1}+h_N}\left(-(\frac{1}{h_{N-1}}+\frac{1}{h_N})h_N^{3-\alpha}+\frac{1}{h_{N-1}}(h_{N-1}+h_N)^{3-\alpha}\right)\\
&= \frac{4\kappa_\alpha}{(2-\alpha)(3-\alpha)h_{N-1}}\left(-h_N^{2-\alpha}+(h_{N-1}+h_N)^{2-\alpha}\right)\\
&= \frac{4\kappa_\alpha}{(3-\alpha)}\xi^{1-\alpha} \quad \xi\in[h_N,h_{N-1}+h_N]\\
&\geq \frac{4\kappa_\alpha}{(3-\alpha)}(h_{N-1}+h_N)^{1-\alpha}=\frac{4\kappa_\alpha}{(3-\alpha)}(\frac{1}{2}-x_{N-2})^{1-\alpha}
\end{aligned} \tag{5.15}
$$

对于 $1\leq i<N-1$，应用引理 A.1 得

$$
\begin{aligned}
P_i &= -\kappa_\alpha\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}p(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})p(x_i)+\frac{1}{h_i}p(x_{i-1})\right)\\
&= 2\kappa_\alpha|\frac{1}{2}-\xi|^{1-\alpha} \quad \xi\in[x_{i-1},x_{i+1}]\\
&\geq 2\kappa_\alpha|\frac{1}{2}-x_{i-1}|^{1-\alpha}
\end{aligned} \tag{5.16}
$$

而 $2N-i$ 完全对称，综合起来有

$$
P_i \geq 2\kappa_\alpha\begin{cases}|\frac{1}{2}-x_{i-1}|^{1-\alpha}, & i\leq N\\ |x_{i+1}-\frac{1}{2}|^{1-\alpha}, & i\geq N\end{cases} \tag{5.17}
$$

并且对于 $1<i\leq N$, 我们有不等式

$$
\begin{aligned}
Q_i &:= -\kappa_\alpha\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}q(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})q(x_i)+\frac{1}{h_i}q(x_{i-1})\right)\\
&= -\kappa_\alpha q''(\xi) \quad \xi\in[x_{i-1},x_{i+1}]\\
&= -\kappa_\alpha(\xi^{1-\alpha}+(1-\xi)^{1-\alpha})\\
&\geq -\kappa_\alpha(x_{i-1}^{1-\alpha}+(1-x_{i+1})^{1-\alpha})\\
&\geq -\kappa_\alpha 2^{-r(1-\alpha)}(x_i^{1-\alpha}+(1-x_i)^{1-\alpha})
\end{aligned} \tag{5.18}
$$

$$
\begin{aligned}
{}_LQ_1 &= \frac{-\kappa_\alpha}{(2-\alpha)(3-\alpha)} \frac{2}{h_1+h_2} \left( -(\frac{1}{h_1}+\frac{1}{h_2}) h_1^{3-\alpha} + \frac{1}{h_2}(h_1+h_2)^{3-\alpha} \right) \\
&= \frac{-2\kappa_\alpha}{(2-\alpha)(3-\alpha)h_2} \left( -h_1^{2-\alpha} + (h_1+h_2)^{2-\alpha} \right) \\
&= \frac{-2\kappa_\alpha}{(3-\alpha)} \xi^{1-\alpha} \quad \xi \in [h_1, h_1+h_2] \\
&\geq \frac{-2\kappa_\alpha}{(3-\alpha)} h_1^{1-\alpha}
\end{aligned} \tag{5.19}
$$

能得到

$$
Q_i \geq -2^{r(\alpha-1)+1}\kappa_\alpha (x_{i-1}^{1-\alpha} + (1-x_{i+1})^{1-\alpha}) \tag{5.20}
$$

□

**Theorem 5.4.** *存在 $\lambda > 0$ ，以及 $C = C(\alpha, r) > 0$ ，使得 $B := A(\lambda I + G)$ 也是 $M$ 矩阵，且*

$$
M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (1-x_i)^{-\alpha}) + C \begin{cases} |\frac{1}{2} - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - \frac{1}{2}|^{1-\alpha}, & i \geq N \end{cases} \tag{5.21}
$$

证明. 由引理 5.3 以及定理 5.2 只需令 $\lambda = (2^{r\alpha+1} + 2^{2r\alpha-r+1})/(\alpha-1)$, 则

$$
\begin{aligned}
M_i &\geq 2\kappa_\alpha \begin{cases} |\frac{1}{2} - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - \frac{1}{2}|^{1-\alpha}, & i \geq N \end{cases} - 2^{r(\alpha-1)+1}\kappa_\alpha (x_i^{1-\alpha} + (1-x_i)^{1-\alpha}) \\
&\quad + \lambda\kappa_\alpha(\alpha-1)2^{-r\alpha}(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \\
&\geq 2\kappa_\alpha \left( \begin{cases} |\frac{1}{2} - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - \frac{1}{2}|^{1-\alpha}, & i \geq N \end{cases} + \alpha^{-r}(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \right)
\end{aligned} \tag{5.22}
$$

□

**Theorem 5.5.** *那么由定理 4.1 以及定理 5.4 我们可以得到*

$$
\max_i |\frac{\epsilon_i}{\lambda + g(x_i)}| = |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| \leq \frac{|R_{i_0}|}{M_{i_0}} \leq Ch^{\min\{2, r\alpha/2\}} \tag{5.23}
$$

从而

$$|\epsilon_i| \le C(\lambda + \frac{1}{2})h^{\min\{2,r\alpha/2\}} \tag{5.24}$$

# A 引理

**Lemma A.1.** 若 $g(x)$ 三阶连续可微，那么

*1.*

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \frac{(x_i - x_{i-1})^2}{2}g''(\xi_1), \quad \xi_1 \in [x_{i-1}, x_i]$$
$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2}g''(\xi_2), \quad \xi_2 \in [x_i, x_{i+1}]$$

*2.*

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})g(x_i) + \frac{1}{h_i}g(x_{i-1})\right) \\ &= \frac{h_i}{h_i + h_{i+1}}g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}}g''(\xi_2) \\ &= g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \end{aligned} \tag{A.1}$$

# B 不等式