A SECOND ORDER NUMERICAL METHODS FOR REISZ-FRACTIONAL ELLIPTIC EQUATION ON GRADED MESH[*]

JIANXING HAN[†] AND MINGHUA CHEN[‡]

Abstract. This is an example SIAM LATEX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

Key words. example, LATEX

MSC codes. ??????????????????

1. Introduction. For $\Omega = (0, 2T)$, $1 < \alpha < 2$,

$$(1.1) \qquad \begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases}$$

where

$$(1.2) \qquad (-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy$$

$$(1.3) \qquad \kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0$$

2. Preliminaries: Numeric scheme and main results.

2.1. Numeric Format.

$$(2.1) \qquad x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases}$$

where $r \ge 1$. And let

$$(2.2) \qquad h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$(2.3) \qquad \phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

And then, define the piecewise linear interpolant of the true solution u to be

$$(2.4) \qquad \Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x)$$

For convience, we denote

(2.5) $$I^{2-\alpha}u(x) := \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x-y|^{1-\alpha} u(y) dy$$

and

(2.6) $$D_h^2 u(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} u(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) u(x_i) + \frac{1}{h_{i+1}} u(x_{i+1}) \right)$$

Now, we discretise (1.1) by replacing $u(x)$ by a continuous piecewise linear function

(2.7) $$u_h(x) := \sum_{j=1}^{2N-1} u_j \phi_j(x)$$

whose nodal values u_j are to be determined by collocation at each mesh point x_i for $i = 1, 2, ..., 2N - 1$:

(2.8) $$-\kappa_\alpha D_h^\alpha u_h(x_i) := -\kappa_\alpha D_h^2 I^{2-\alpha} u_h(x_i) = f(x_i) =: f_i$$

Here,

(2.9) $$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^2 I^{2-\alpha} \phi_j(x_i)\, u_j = \sum_{j=1}^{2N-1} a_{ij}\, u_j$$

where

(2.10) $$a_{ij} = -\kappa_\alpha D_h^2 I^{2-\alpha} \phi_j(x_i) \quad \text{for} \quad i, j = 1, 2, ..., 2N - 1$$

We have replaced $(-\Delta)^{\alpha/2} u(x_i) = f(x_i)$ in (1.1) by $-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i)$ in (2.8), with truncation error

(2.11) $$\tau_i := -\kappa_\alpha \left(D_h^\alpha \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i) \right) \quad \text{for} \quad i = 1, 2, ..., 2N - 1$$

where $-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^\alpha \phi_j(x_i) u(x_j) = \sum_{j=1}^{2N-1} a_{ij} u(x_j)$.

The discrete equation (2.8) can be written in matrix form

(2.12) $$AU = F$$

where $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$, $U = (u_1, \cdots, u_{2N-1})^T$ is unknown and $F = (f_1, \cdots, f_{2N-1})^T$.

We can deduce a_{ij},

(2.13) $$\begin{aligned} a_{ij} &= -\kappa_\alpha D_h^2 I^{2-\alpha} \phi_j(x_i) \\ &= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \end{aligned}$$

where
(2.14)
$$\begin{aligned} \tilde{a}_{ij} &= I^{2-\alpha} \phi_j(x_i) \\ &= \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \end{aligned}$$

2.2. Regularity of the true solution. For any $\beta > 0$, we use the standard notation $C^\beta(\bar{\Omega}), C^\beta(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^\beta(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous[1] with exponent β' in Ω, where $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on open set Ω.

DEFINITION 2.1 (delta dependent norm [2]). ...

THEOREM 2.2. *Let $f \in C^\beta(\Omega), \beta > 2$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

(2.15)
$$|f^{(l)}(x)| \leq \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \leq T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \leq x < 2T \end{cases}$$

THEOREM 2.3 (Regularity up to the boundary [2]). *Let Ω be a bounded domain, and $\beta > 0$ be such that neither β nor $\beta + \alpha$ is an integer. Let $f \in C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R}^n)$ be a solution of (1.1). Then, $u \in C^{\beta+\alpha}(\Omega)$ and*

(2.16)
$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right)$$

COROLLARY 2.4. *Let u be a solution of (1.1) where $f \in L^\infty(\Omega)$ and $\|f\|_\beta^{(\alpha/2)} < \infty$. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

(2.17)
$$|u^{(l)}(x)| \leq \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases}$$

And in this paper bellow, without special instructions, we allways assume that

(2.18)
$$f \in L^\infty(\Omega) \cap C^\beta(\Omega) \quad \text{and} \quad \|f\|_\beta^{(\alpha/2)} < \infty, \text{ with } \alpha + \beta > 4$$

2.3. Main results. Here we state our main results; the proof is deferred to section 3 and section 4.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 2.5 (Local Truncation Error). *If $u(x)$ is a solution of the equation (1.1) where f satisfy the regular condition (2.18), then there exists $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$ and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$, such that the truncation error (2.11) satisfies*

(2.19)
$$|\tau_i| := |-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i)|$$
$$\leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} \begin{cases} x_i^{-\alpha}, & 1 \leq i \leq N \\ (2T-x_i)^{-\alpha}, & N < i \leq 2N-1 \end{cases}$$
$$+ C_2(r-1)h^2 \begin{cases} |T-x_{i-1}|^{1-\alpha}, & 1 \leq i \leq N \\ |T-x_{i+1}|^{1-\alpha}, & N < i \leq 2N-1 \end{cases}$$

THEOREM 2.6 (Global Error). *The discrete equation* (2.8) *has sulotion and there exists a positive constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ *such that the error between the numerial solution* U *with the exact solution* $u(x_i)$ *satisfies*

(2.20)
$$\max_{1 \le i \le 2N-1} |u_i - u(x_i)| \le Ch^{\min\{\frac{r\alpha}{2}, 2\}}$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

3. Local Truncation Error.

3.1. Proof of Theorem 2.5. The truncation error of the discrete format can be written as

(3.1)
$$-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i) = -\kappa_\alpha (D_h^2 I^{2-\alpha} \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i))$$
$$= -\kappa_\alpha D_h^2 I^{2-\alpha}(\Pi_h u - u)(x_i) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i)$$

THEOREM 3.1. *There exits a constant* $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$ *such that*

(3.2)
$$\left| -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \right| \le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases}$$

Proof. Since $f \in C^2(\Omega)$ and

(3.3)
$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha}(x)) = f(x), \quad x \in \Omega,$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.3) of Lemma A.1, for $1 \le i \le 2N-1$, we have

(3.4)
$$-\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i)$$
$$+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right)$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

(3.5)
$$\left| \frac{h_{i+1} - h_i}{3} f'(x_i) \right| \le \frac{C(r-1)\|f\|_{\beta}^{(\alpha/2)}}{3} h^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{-\alpha/2-2/r}, & N < i \le 2N-1 \end{cases}$$

2. See Proof 25, there is a constant $C = C(T, \alpha, r, \|f\|_{\beta}^{\alpha/2})$ such that

(3.6)
$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right)$$
$$\le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases}$$

Summarizes, we get the result. $\square$

And define

(3.7)
$$R_i := D_h^2 I^{2-\alpha}(u - \Pi_h u)(x_i)$$

We have some results about the estimate of R_i

THEOREM 3.2. *For $1 \le i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

(3.8)
$$R_i \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

THEOREM 3.3. *For $N/2 \le i \le N$, there exists constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

(3.9)
$$R_i \le C(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

And for $N < i \le 2N - 1$, it is symmetric to the previous case.

Combine Theorem 3.1, Theorem 3.2 and Theorem 3.3, the proof of Theorem 2.5 completed.

We prove Theorem 3.2 and Theorem 3.3 in next subsections below.

3.2. Proof of Theorem 3.2.

(3.10)
$$D_h^2 I^{2-\alpha}(u - \Pi_h u)(x_i) = D_h^2 \left(\int_0^{2T} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \right)$$

For convience, let's denote

(3.11)
$$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy, \quad i = 0, \cdots, 2N, \ j = 1, \cdots, 2N$$

Also for simplicity, we denote

DEFINITION 3.4.

(3.12)
$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right)$$

then

(3.13)
$$R_i = \sum_{j=1}^{2N} S_{ij}$$

LEMMA 3.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \le i < N/2$,*

(3.14)
$$\sum_{j=\max\{2i+1,i+3\}}^{N} S_{ij} \le Ch^2 x_i^{-\alpha/2-2/r}$$

Proof. Let

$$K_y(x) = \frac{|y - x|^{1-\alpha}}{\Gamma(2 - \alpha)}$$

For $\max\{2i + 1, i + 3\} \leq j \leq N$, by Lemma C.1 and Lemma C.2

(3.15)
$$\begin{aligned}
S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \\
&\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2 - 2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy \\
&= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2 - 2/r - 1} dy
\end{aligned}$$

Therefore,

(3.16)
$$\begin{aligned}
\sum_{j=\max\{2i+1,i+3\}}^{N} S_{ij} &\leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2 - 2/r - 1} dy \\
&= \frac{C}{\alpha/2 + 2/r} h^2 (x_{2i}^{-\alpha/2 - 2/r} - T^{-\alpha/2 - 2/r}) \qquad \square \\
&\leq \frac{C}{\alpha/2 + 2/r} 2^{r(-\alpha/2 - 2/r)} h^2 x_i^{-\alpha/2 - 2/r}
\end{aligned}$$

LEMMA 3.6. *Thert exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $1 \leq i < N/2$,

(3.17)
$$\sum_{j=N+1}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

Proof. For $1 \leq i < N/2, N + 1 \leq j \leq 2N - 1$, by equation (C.2) and Lemma C.2

$$\begin{aligned}
S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \\
&\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2 - 2/r} y^{-1-\alpha} dy \\
&\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2 - 2/r} dy
\end{aligned}$$

140

$$\sum_{j=N+1}^{2N-1} S_{ij} \leq CT^{-1-\alpha}h^2 \int_{x_N}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r}dy$$

141 (3.18)

$$\leq CT^{-1-\alpha}h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1}T^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2-2/r+1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|}h_{2N}^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 < 0 \end{cases}$$

$$= \begin{cases} \frac{C}{\alpha/2-2/r+1}T^{-\alpha/2-2/r}h^2, & \alpha/2-2/r+1 > 0 \\ CrT^{-1-\alpha}h^2\ln(N), & \alpha/2-2/r+1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|}T^{-\alpha/2-2/r}h^{r\alpha/2+r}, & \alpha/2-2/r+1 < 0 \end{cases}$$

142 And by Lemma A.3

143
$$S_{i,2N} \leq CT^{-1-\alpha}h_{2N}^{\alpha/2+1} = CT^{-\alpha/2}h^{r\alpha/2+r}$$

144 And when $\alpha/2 - 2/r + 1 \geq 0$,

145
$$h^{r\alpha/2+r} \leq h^2$$

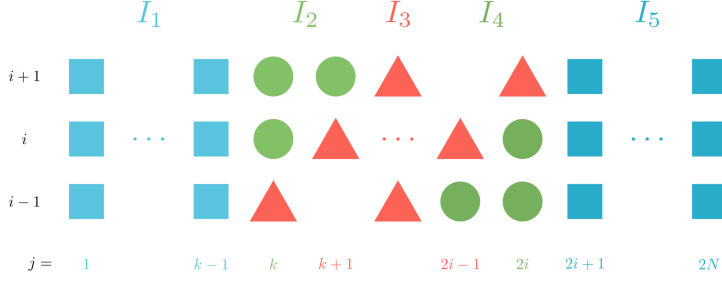146 Summarizes, we get the result. □

147 For $i = 1, 2$.

148 LEMMA 3.7. *By Lemma* C.5 *, Lemma* 3.5 *and Lemma* 3.6 *we get*

$$R_1 = \sum_{j=1}^{3} S_{1j} + \sum_{j=4}^{2N} S_{1j}$$

149 (3.19)

$$\leq Ch^2 x_1^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1 > 0 \\ Ch^2\ln(N), & \alpha/2-2/r+1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1 < 0 \end{cases}$$

150

$$R_2 = \sum_{j=1}^{4} S_{2j} + \sum_{j=5}^{2N} S_{2j}$$

151 (3.20)

$$\leq Ch^2 x_2^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1 > 0 \\ Ch^2\ln(N), & \alpha/2-2/r+1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1 < 0 \end{cases}$$

152 For $3 \leq i < N/2$, we have a new separation of R_i, Let's denote $k = \lceil \frac{i}{2} \rceil$.

153

$$R_i = \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right)$$

$$= \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right)$$

$$+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right)$$

154   (3.21)
$$+ \sum_{j=k+1}^{2i-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right)$$

$$+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right)$$

$$+ \sum_{j=2i+1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right)$$

$$= I_1 + I_2 + I_3 + I_4 + I_5$$

155

156   LEMMA 3.8. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \le$*
157   *$i \le N, k = \lceil \frac{i}{2} \rceil$*

158   (3.22)
$$|I_1| = |\sum_{j=1}^{k-1} S_{ij}| \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

159   *Proof.* by Lemma A.3 , Lemma C.3

160   (3.23) $$S_{i1} \le C x_1^{\alpha/2} x_1 x_i^{-1-\alpha} = C x_1^{\alpha/2+1} x_i^{-1-\alpha} = C T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha}$$

161 For $2 \leq j \leq k-1$, by Lemma C.1 and Lemma C.3

$$S_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy$$

162 (3.24)
$$\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} dy$$

$$= Ch^2 x_i^{-1-\alpha} \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} dy$$

163 Therefore,

$$I_1 = \sum_{j=1}^{k-1} S_{ij} = S_{i1} + \sum_{j=2}^{k-1} S_{ij}$$

164 (3.25)
$$\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil -1}} y^{\alpha/2-2/r} dy$$

$$\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy$$

165 But

166 (3.26) $$\int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \leq \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

167 So we have

168 (3.27) $$I_1 \leq \begin{cases} \frac{C}{\alpha/2-2/r+1} h^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} h^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \qquad \square$$

169 DEFINITION 3.9. *For convience, let's denote*

170 (3.28) $$V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right)$$

171

172 THEOREM 3.10. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for*
173 $3 \leq i < N/2, k = \lceil \frac{i}{2} \rceil,$

174 (3.29) $$I_3 = \sum_{j=k+1}^{2i-1} V_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r}$$

175 To estimete V_{ij}, we need some preparations.

176 LEMMA 3.11. *For* $y \in [x_{j-1}, x_j],$ *we can rewrite* $y = x_{j-1} + \theta h_j = (1 - \theta)x_{j-1} +$

177 $\theta x_j =: y_j^\theta$, $\theta \in [0,1]$, by Lemma A.2

178 (3.30)
$$
\begin{aligned}
T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
&= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\
&= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
&\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta
\end{aligned}
$$

179 where $\eta_{j1}^\theta \in [x_{j-1}, y_j^\theta], \eta_{j2}^\theta \in [y_j^\theta, x_j]$.

180 Now Let's construct a series of functions to represent T_{ij}.

181   DEFINITION 3.12. For $2 \le i, j \le N-1$,

182 (3.31)
$$
y_{j-i}(x) = (x^{1/r} + Z_{j-i})^r, \quad Z_{j-i} = T^{1/r} \frac{j-i}{N}
$$

183

184 (3.32)
$$
y_{j-i}^\theta(x) = (1-\theta) y_{j-1-i}(x) + \theta y_{j-i}(x)
$$

185

186 (3.33)
$$
h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x)
$$

187 Now, we define

188 (3.34)
$$
P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}
$$

189

190 (3.35)
$$
Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}
$$

191 And now we can rewrite T_{ij}

192   LEMMA 3.13. For $2 \le i \le N, 2 \le j \le N$,

193 (3.36)
$$
\begin{aligned}
T_{ij} &= \int_0^1 -\frac{\theta(1-\theta)}{2} P_{j-i}^\theta(x_i) d\theta \\
&\quad + \int_0^1 \frac{\theta(1-\theta)}{3!} Q_{j-i}^\theta(x_i)(\theta^2 u'''(\eta_{j,1}^\theta) - (1-\theta)^2 u'''(\eta_{j,2}^\theta)) d\theta
\end{aligned}
$$

194   Immediately, we can see from (3.28) that

195   LEMMA 3.14. *For $3 \leq i, j \leq N - 1$,*

(3.37)
$$V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right)$$

$$= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^\theta(x_i) d\theta$$

$$+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1,1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta$$

$$- \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i)u'''(\eta_{j,1}^\theta) - Q_{j-i}^\theta(x_{i-1})u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta$$

$$- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1,2}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta$$

$$+ \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i)u'''(\eta_{j,2}^\theta) - Q_{j-i}^\theta(x_{i-1})u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta$$

197   To estimate V_{ij}, we first estimate $D_h^2 P_{j-i}^\theta(x_i)$, but By Lemma A.1,

198   (3.38)
$$D_h^2 P_{j-i}^\theta(x_i) = P_{j-i}^\theta{}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

199   By Leibniz formula, we calculate and estimate the derivations of $h_{j-i}^3(x)$, $u''(y_{j-i}^\theta(x))$

200   and $\frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$ separately.

201   Firstly, we have

202   LEMMA 3.15. *There exists a constant $C = C(T, r)$ such that For $3 \leq i \leq N -$*

203   *$1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i - 1, N - 1\}, \xi \in [x_{i-1}, x_{i+1}]$,*

204   (3.39)
$$h_{j-i}^3(\xi) \leq Ch^2 x_i^{2-2/r} h_j$$

205   (3.40)
$$(h_{j-i}^3(\xi))' \leq C(r-1)h^2 x_i^{1-2/r} h_j$$

206   (3.41)
$$(h_{j-i}^3(\xi))'' \leq C(r-1)h^2 x_i^{-2/r} h_j$$

207   The proof of this theorem see Lemma C.6 and Lemma C.7

208   Second,

209   LEMMA 3.16. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For*

210   *$3 \leq i \leq N - 1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i - 1, N - 1\}, \xi \in [x_{i-1}, x_{i+1}]$,*

211   (3.42)
$$u''(y_{j-i}^\theta(\xi)) \leq Cx_i^{\alpha/2-2}$$

212   (3.43)
$$(u''(y_{j-i}^\theta(\xi)))' \leq Cx_i^{\alpha/2-3}$$

213   (3.44)
$$(u''(y_{j-i}^\theta(\xi)))'' \leq Cx_i^{\alpha/2-4}$$

214   The proof of this theorem see Proof 31

215   And Finally, we have

216   LEMMA 3.17. *There exists a constant $C = C(T, \alpha, r)$ such that For $3 \leq i \leq$*

217   *$N - 1, 1 \leq j \leq \min\{2i - 1, N - 1\}, \xi \in [x_{i-1}, x_{i+1}]$,*

218   (3.45)
$$|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}$$

(3.46)
$$(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' \le C|y_j^\theta - x_i|^{1-\alpha} x_i^{-1}$$

(3.47)
$$(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' \le C|y_j^\theta - x_i|^{1-\alpha} x_i^{-2}$$

where $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

The proof of this theorem see Proof 32

LEMMA 3.18. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$,

(3.48)
$$D_h^2 P_{j-i}^\theta(x_i) \le Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j$$

where $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

Proof. Since

(3.49)
$$D_h^2 P_{j-i}^\theta(x_i) = P_{j-i}^\theta{}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

From (3.34), using Leibniz formula and Lemma 3.15, Lemma 3.16 and Lemma 3.17□

LEMMA 3.19. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i < N, k = \lceil \frac{i}{2} \rceil$.
For $k \le j \le \min\{2i-1, N-1\}$,

(3.50)
$$\frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right)$$
$$\le Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j$$

And for $k+1 \le j \le \min\{2i, N\}$,

(3.51)
$$\frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta) - Q_{j-i}^\theta(x_{i-1})u'''(\eta_{j-1}^\theta)}{h_i} \right)$$
$$\le Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j$$

where $\eta_j^\theta \in [x_{j-1}, x_j]$.

proof see Proof 33

LEMMA 3.20. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i < N, k = \lceil \frac{i}{2} \rceil, k+1 \le j \le \min\{2i-1, N-1\}$,

(3.52)
$$V_{ij} \le Ch^2 \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j d\theta$$
$$= Ch^2 \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy$$

244 *Proof.* Since Lemma 3.14, by Lemma 3.18 and Lemma 3.19, we get the result
245 immediately. □

246 Now we can prove Theorem 3.10 using Lemma 3.20, $k = \lceil \frac{i}{2} \rceil$

247 (3.53)
$$\begin{aligned}
I_3 = \sum_{k+1}^{2i-1} V_{ij} &\leq Ch^2 \int_{x_k}^{x_{2i-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\
&= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{2i-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\
&\leq Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r}
\end{aligned}$$

248

LEMMA 3.21.

249 (3.54)
$$D_h P_{j-i}^\theta(x_i) := \frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} = P_{j-i}^\theta{}'(\xi), \quad \xi \in [x_i, x_{i+1}]$$

250 Then, for $3 \leq i \leq N-1$, $k = \lceil \frac{i}{2} \rceil$,

251 (3.55)
$$D_h P_{k-i}^\theta(x_i) \leq Ch^2 x_i^{-\alpha/2-2/r} h_j$$

252

253 *Proof.* Using Leibniz formula, by Lemma 3.15, Lemma 3.16 and Lemma 3.17, we
254 take $j = k+1, i = i+1$, we get

255 (3.56)
$$\begin{aligned}
D_h P_{k-i}^\theta(x_i) &\leq Ch^2 x_{i+1}^{\alpha/2-2/r-1} |y_{k+1}^\theta - x_{i+1}|^{1-\alpha} h_{j+1} \\
&\leq Ch^2 x_i^{\alpha/2-2/r-1} |y_k^\theta - x_i|^{1-\alpha} h_j \\
&\leq Ch^2 x_i^{-\alpha/2-2/r} h_j
\end{aligned}$$
□

256

257 LEMMA 3.22. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for*
258 $3 \leq i < N, k = \lceil \frac{i}{2} \rceil$,
(3.57)
259
$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

260 *And for $3 \leq i < N/2$,*
(3.58)
261
$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

262 *Proof.* In fact,

263 (3.59)
$$\begin{aligned}
&\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \\
&= \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i}) T_{i,k}
\end{aligned}$$

While, by Lemma A.2

$$\frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) = \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y)) \frac{|x_{i+1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_{i+1}\Gamma(2-\alpha)} dy$$

(3.60)
$$\leq \int_{x_{k-1}}^{x_k} h_k^2 u''(\eta) \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy$$
$$\leq Ch_k h^2 x_k^{2-2/r} x_{k-1}^{\alpha/2-2} |x_i - x_k|^{-\alpha}$$
$$\leq Ch_k h^2 x_i^{-\alpha/2-2/r}$$

Thus,

(3.61)
$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

For
(3.62)
$$\frac{1}{h_{i+1}}(T_{i+1,k+1} - T_{i,k}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} d\theta$$
$$+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{Q_{k-i}^\theta(x_{i+1})u'''(\eta_{k+1,1}^\theta) - Q_{k-i}^\theta(x_i)u'''(\eta_{k,1}^\theta)}{h_{i+1}} d\theta$$
$$- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{Q_{k-i}^\theta(x_{i+1})u'''(\eta_{k+1,2}^\theta) - Q_{k-i}^\theta(x_i)u'''(\eta_{k,2}^\theta)}{h_{i+1}} d\theta$$

And by Lemma 3.21

(3.63)
$$\frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} \leq Ch^2 x_i^{-\alpha/2-2/r} h_k$$

And with Lemma 3.19, we can get

(3.64)
$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}}(T_{i+1,k+1} - T_{i,k}) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

(3.65)
$$\frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,k} \leq h_i^{-3} h^2 x_i^{1-2/r} h_k Ch_k^2 x_{k-1}^{\alpha/2-2} |x_k - x_i|^{1-\alpha}$$
$$\leq Ch^2 x_i^{-\alpha/2-2/r}$$

Summarizes, we have

(3.66)
$$I_2 \leq Ch^2 x_i^{-\alpha/2-2/r}$$

The case for I_4 is similar. □

Now combine Lemma 3.8, Lemma 3.22, Theorem 3.10, Lemma 3.5 and Lemma 3.6 to get the final result.

For $3 \leq i < N/2$

$$R_i = I_1 + I_2 + I_3 + I_4 + I_5$$

(3.67)
$$\leq Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases}$$

283   Combine with $i = 1, 2$, we get for $1 \le i < N/2$

284   (3.68)
$$R_i \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r}x_i^{-1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases}$$

285   **3.3. Proof of Theorem 3.3.** For $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$, we have

(3.69)
$$R_i = \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right)$$

$$= \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right)$$

$$+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}}(T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,k} \right)$$

286
$$+ \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right)$$

$$+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right)$$

$$+ \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right)$$

$$= I_1 + I_2 + I_3^1 + I_3^2 + I_3^3 + I_4 + I_5$$

287   We have estimate I_1 in Lemma 3.8 and I_2 in Lemma 3.22. We can control I_3 in
288   similar with Theorem 3.10 by Lemma 3.20 where $2i - 1 \ge N - 1$

289   LEMMA 3.23. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for*
290    $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$,

291   (3.70)
$$I_3 = \sum_{j=k+1}^{N-1} V_{ij} \le Ch^2 \int_{x_k}^{x_{N-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy$$

$$= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{N-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r}$$

$$\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r}$$

292   Let's study I_5 before I_4.

293   (3.71)
$$I_5 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} V_{ij}$$

294   Similarly, Let's define a new series of functions

295   DEFINITION 3.24. *For* $i < N, j \ge N$, *with no confusion, we also denote in this*
296   *section*

297   (3.72) $$y_{j-i}(x) = 2T - (Z_{2N-j+i} - x^{1/r})^r, \quad Z_{2N-j+i} = T^{1/r}\frac{2N-j+i}{N}$$

298

(3.73)
$$y_{j-i}{}'(x) = (2T - y_{j-i}(x))^{1-1/r} x^{1/r-1}$$

(3.74)
$$y_{j-i}{}''(x) = \frac{1-r}{r}(2T - y_{j-i}(x))^{1-2/r} x^{1/r-2} Z_{2N-j+i}$$

(3.75)

(3.76)
$$y_{j-i}^{\theta}(x) = (1-\theta)y_{j-i-1}(x) + \theta y_{j-i}(x)$$

(3.77)
$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x)$$

(3.78)
$$P_{j-i}^{\theta}(x) = (h_{j-i}(x))^3 u''(y_{j-i}^{\theta}(x)) \frac{|y_{j-i}^{\theta}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

(3.79)
$$Q_{j-i}^{\theta}(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^{\theta}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

Now we have, for $i < N, j \geq N + 2$,

(3.80)
$$
\begin{aligned}
V_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^{\theta}(x_i) d\theta \\
&\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1,1}^{\theta}) - Q_{j-i}^{\theta}(x_i)u'''(\eta_{j,1}^{\theta})}{h_{i+1}} \right) d\theta \\
&\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_i)u'''(\eta_{j,1}^{\theta}) - Q_{j-i}^{\theta}(x_{i-1})u'''(\eta_{j-1,1}^{\theta})}{h_i} \right) d\theta \\
&\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1,2}^{\theta}) - Q_{j-i}^{\theta}(x_i)u'''(\eta_{j,2}^{\theta})}{h_{i+1}} \right) d\theta \\
&\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^{\theta}(x_i)u'''(\eta_{j,2}^{\theta}) - Q_{j-i}^{\theta}(x_{i-1})u'''(\eta_{j-1,2}^{\theta})}{h_i} \right) d\theta
\end{aligned}
$$

Similarly, we first estimate

(3.81)
$$D_h^2 P_{j-i}^{\theta}(\xi) = P_{j-i}^{\theta}{}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

Combine lemmas Lemma C.8, Lemma C.9 and Lemma C.10 , we have

LEMMA 3.25. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \leq i < N$, $N + 2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in [x_{i-1}, x_{i+1}]$, *we have*

(3.82)
$$
\begin{aligned}
|P_{j-i}^{\theta}{}''(\xi)| \leq C h_j h^2 (&|y_j^{\theta} - x_i|^{1-\alpha} \\
&+ |y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N) \\
&+ |y_j^{\theta} - x_i|^{-1-\alpha}(|2T - x_i - y_j^{\theta}| + h_N)^2 \\
&+ (r-1)|y_j^{\theta} - x_i|^{-\alpha})
\end{aligned}
$$

318   And

319   LEMMA 3.26. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For*
320    $N/2 \le i < N,$, $\xi \in [x_{i-1}, x_{i+1}]$, *we have for* $N + 1 \le j \le 2N - \lceil \frac{N}{2} \rceil$

321   (3.83) $$\frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right)$$
$$\le Ch^2 h_j(|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N))$$

322   *for* $N + 2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$

323   (3.84) $$\frac{2}{h_i + h_{i+1}} \left(\frac{Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta) - Q_{j-i}^\theta(x_{i-1})u'''(\eta_{j-1}^\theta)}{h_{i+1}} \right)$$
$$\le Ch^2 h_j(|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N))$$

324   The proof see Proof 37.
325   Combine (3.80), Lemma 3.25 and Lemma 3.26, we have

326   THEOREM 3.27. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For*
327    $N/2 \le i < N,$ $N + 2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$

328   (3.85) $$V_{ij} \le Ch^2 \int_{x_{j-1}}^{x_j} (|y - x_i|^{1-\alpha}$$
$$+ |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2$$
$$+ (r - 1)|y - x_i|^{-\alpha})dy$$

329   We can esitmate I_5 Now.

330   THEOREM 3.28. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For*
331    $N/2 \le i < N$, *we have*

332   (3.86) $$I_5 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} V_{ij} \le Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha}$$

    Proof.

333   (3.87) $$I_5 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} V_{ij}$$
$$\le Ch^2 \int_{x_{N+1}}^{x_{2N-i}} + \int_{x_{2N-i}}^{x_{2N-\lceil \frac{N}{2} \rceil}} (|y - x_i|^{1-\alpha}$$
$$+ |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2$$
$$+ (r - 1)|y - x_i|^{-\alpha})dy$$
$$= J_1 + J_2$$

334   While $x_{N+1} \le y \le x_{2N-i} = 2T - x_i,$

335   (3.88) $$T - x_{i-1} \le x_{N+1} - x_i \le y - x_i \le x_{2N-i} - x_i \le 2(T - x_{i-1})$$

336   and

337   (3.89) $2T - x_i - y + h_N \leq 2T - x_i - x_{N+1} + h_N = T - x_i \leq T - x_{i-1}$

338   So

$$
\begin{aligned}
J_1 &\leq Ch^2(x_{2N-i} - x_{N+1})(|T - x_{i-1}|^{1-\alpha} + (r-1)|T - x_{i-1}|^{-\alpha}) \\
&\leq Ch^2(|T - x_{i-1}|^{2-\alpha} + (r-1)|T - x_{i-1}|^{1-\alpha}) \\
&\leq Ch^2 T^{2-\alpha} + C(r-1)h^2|T - x_{i-1}|^{1-\alpha}
\end{aligned}
$$

339   (3.90)

340   Otherwise, when $x_{2N-i} \leq y \leq x_{2N-\lceil \frac{N}{2} \rceil}$

341   (3.91) $x_i + y - 2T + h_N \leq y - x_i$

342

$$
\begin{aligned}
J_2 &\leq Ch^2 \int_{x_{2N-i}}^{(2-2^{-r})T} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} \\
&\leq Ch^2(T^{2-\alpha} + (r-1)|x_{2N-i} - x_i|^{1-\alpha}) \\
&= Ch^2 + C(r-1)h^2|T - x_i|^{1-\alpha} \leq Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha}
\end{aligned}
$$

343   (3.92)

344   Summarizes two cases, we get the result. $\square$

For I_4, we have

THEOREM 3.29. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that, for $N/2 \le i \le N - 1$*

(3.93)
$$V_{iN} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,N+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,N} + \frac{1}{h_i} T_{i-1,N-1} \right)$$
$$\le Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha}$$

Proof. We use the similar skill in the last section, but more complicated. for $j = N$, Let

(3.94)
$$_L y_{N-1-i}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r} \frac{N-1-i}{N}$$

(3.95)
$$_0 y_{N-i}(x) = \frac{x^{1/r} - Z_i}{Z_1} h_N + T, \quad Z_i = T^{1/r} \frac{i}{N}, x_N = T$$

and

(3.96)
$$_R y_{N+1-i}(x) = 2T - (Z_{N-1+i} - x^{1/r})^r, \quad Z_{N-1+i} = T^{1/r} \frac{N-1+i}{N}$$

Thus,

$$_L y_{N-1-i}(x_{i-1}) = x_{N-2}, \quad _L y_{N-1-i}(x_i) = x_{N-1}, \quad _L y_{N-1-i}(x_{i+1}) = x_N$$
$$_0 y_{N-i}(x_{i-1}) = x_{N-1}, \quad _0 y_{N-i}(x_i) = x_N, \quad _0 y_{N-i}(x_{i+1}) = x_{N+1}$$
$$_R y_{N+1-i}(x_{i-1}) = x_N, \quad _R y_{N+1-i}(x_i) = x_{N+1}, \quad _R y_{N+1-i}(x_{i+1}) = x_{N+2}$$

Then, define

(3.97)
$$_L y_{N-i}^\theta(x) = \theta _L y_{N-1-i}(x) + (1-\theta) _0 y_{N-i}(x)$$

(3.98)
$$_R y_{N+1-i}^\theta(x) = \theta _0 y_{N-i}(x) + (1-\theta) _R y_{N+1-i}(x)$$

(3.99)
$$_L h_{N-i}(x) = _0 y_{N-i}(x) - _L y_{N-1-i}(x)$$

(3.100)
$$_R h_{N+1-i}(x) = _R y_{N+1-i}(x) - _0 y_{N-i}(x)$$

We have

(3.101)
$$_L y_{N-1-i}'(x) = _L y_{N-1-i}^{1-1/r}(x) x^{1/r-1}$$

(3.102)
$$_L y_{N-1-i}''(x) = \frac{1-r}{r} _L y_{N-1-i}^{1-2/r}(x) x^{1/r-2} Z_{N-1-i}$$

(3.103)
$$_0 y_{N-i}'(x) = \frac{1}{r} \frac{h_N}{Z_1} x^{1/r-1}$$

(3.104)
$$_0 y_{N-i}''(x) = \frac{1-r}{r^2} \frac{h_N}{Z_1} x^{1/r-2}$$

(3.105)
$$_R y_{N+1-i}'(x) = (2T - _R y_{N+1-i}(x))^{1-1/r} x^{1/r-1}$$

(3.106)
$$_R y_{N+1-i}''(x) = \frac{1-r}{r} (2T - _R y_{N+1-i}(x))^{1-2/r} x^{1/r-2} Z_{N-1+i}$$

(3.107)
$$_LP_{N-i}^\theta(x) = (_Lh_{N-i}(x))^3\frac{|_Ly_{N-i}^\theta(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)}u''(_Ly_{N-i}^\theta(x))$$

(3.108)
$$_RP_{N+1-i}^\theta(x) = (_Rh_{N+1-i}(x))^3\frac{|_Ry_{N+1-i}^\theta(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)}u''(_Ry_{N+1-i}^\theta(x))$$

(3.109)
$$_LQ_{N-i}^\theta(x) = (_Lh_{N-i}(x))^4\frac{|_Ly_{N-i}^\theta(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

(3.110)
$$_RQ_{N+1-i}^\theta(x) = (_Rh_{N+1-i}(x))^4\frac{|_Ry_{N+1-i}^\theta(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

Similar with Lemma 3.13, we can get for $l=-1,0,1$,

(3.111)
$$T_{i+l,N+l} = \int_0^1 -\frac{\theta(1-\theta)}{2}{_LP_{N-i}^\theta}(x_{i+l})d\theta$$
$$+ \int_0^1 \frac{\theta(1-\theta)}{3!}{_LQ_{N-i}^\theta}(x_{i+l})(\theta^2u'''(\eta_{N+l,1}^\theta)-(1-\theta)^2u'''(\eta_{N+l,2}^\theta))d\theta$$

(3.112)
$$T_{i+l,N+1+l} = \int_0^1 -\frac{\theta(1-\theta)}{2}{_RP_{N+1-i}^\theta}(x_{i+l})d\theta$$
$$+ \int_0^1 \frac{\theta(1-\theta)}{3!}{_RQ_{N+1-i}^\theta}(x_{i+l})(\theta^2u'''(\eta_{N+1+l,1}^\theta)-(1-\theta)^2u'''(\eta_{N+1+l,2}^\theta))d\theta$$

So we have
(3.113)
$$V_{i,N} = \int_0^1 -\frac{\theta(1-\theta)}{2}D_{hL}^2P_{N-i}^\theta(x_i)d\theta$$
$$+ \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{_LQ_{N-i}^\theta(x_{i+1})u'''(\eta_{N+1,1}^\theta)-{_LQ_{N-i}^\theta}(x_i)u'''(\eta_{N,1}^\theta)}{h_{i+1}}\right)d\theta$$
$$- \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{_LQ_{N-i}^\theta(x_i)u'''(\eta_{N,1}^\theta)-{_LQ_{N-i}^\theta}(x_{i-1})u'''(\eta_{N-1,1}^\theta)}{h_i}\right)d\theta$$
$$- \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{_LQ_{N-i}^\theta(x_{i+1})u'''(\eta_{N+1,2}^\theta)-{_LQ_{N-i}^\theta}(x_i)u'''(\eta_{N,2}^\theta)}{h_{i+1}}\right)d\theta$$
$$+ \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{_LQ_{N-i}^\theta(x_i)u'''(\eta_{N,2}^\theta)-{_LQ_{N-i}^\theta}(x_{i-1})u'''(\eta_{N-1,2}^\theta)}{h_i}\right)d\theta$$

$N+1$ is similar.

We estimate $D_{hL}^2P_{N-i}^\theta(x_i) = {_LP_{N-i}^\theta}{}''(\xi), \xi\in[x_{i-1},x_{i+1}]$,

LEMMA 3.30.

(3.114) $$_Lh_{N-i}^3(\xi) \le Ch_N^3 \le Ch^3$$
(3.115) $$_Rh_{N+1-i}^3(\xi) \le Ch_N^3 \le Ch^3$$

389 (3.116)
$$(_L h_{N-i}^3(\xi))' \le C(r-1) h_N^2 h \le C(r-1) h^3$$

390 (3.117)
$$(_R h_{N+1-i}^3(\xi))' \le C(r-1) h_N^2 h \le C(r-1) h^3$$

391 (3.118)
$$(_L h_{N-i}^3(\xi))'' \le C(r-1) h^2$$

392 (3.119)
$$(_R h_{N+1-i}^3(\xi))'' \le C(r-1) h^2$$

Proof.

393 (3.120)
$$_L h_{N-i}(\xi) \le 2h_N, \quad _R h_{N+1-i}(\xi) \le 2h_N$$

394

395 (3.121)
$$(_L h_{N-i}^l(\xi))' = l\,_L h_{N-i}^{l-1}(\xi)(_0 y_{N-i}{}'(\xi) - _L y_{N-1-i}{}'(\xi))$$
$$= l\,_L h_{N-i}^{l-1}(\xi) x_i^{1/r-1}(\frac{1}{r}\frac{h_N}{Z_1} - _L y_{N-1-i}^{1-1/r}(\xi))$$

396 while
(3.122)
$$|\frac{1}{r}\frac{h_N}{Z_1} - _L y_{N-1-i}^{1-1/r}(\xi)| = |\frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}| \quad \eta \in [x_{N-2}, x_N]$$

397
$$= T^{1-1/r}|(\frac{N-t}{N})^{r-1} - (\frac{N-s}{N})^{r-1}| \quad t \in [0,1], s \in [0,2]$$
$$\le T^{1-1/r}|1 - (\frac{N-2}{N})^{r-1}| \le CT^{1-1/r}(r-1)\frac{2}{N}$$

398 Thus,

399 (3.123)
$$(_L h_{N-i}^l(\xi))' \le C(r-1) h_N^{l-1} x_i^{1/r-1} h$$

400 (3.124)
$$(_R h_{N+1-i}^l(\xi))' = l\,_R h_{N+1-i}^{l-1}(\xi)(_R y_{N+1-i}{}'(\xi) - _0 y_{N-i}{}'(\xi))$$
$$= l\,_R h_{N+1-i}^{l-1}(\xi) x_i^{1/r-1}((2T - _R y_{N+1-i}(\xi))^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1})$$

401 Similarly,
(3.125)
$$|(2T - _R y_{N+1-i})^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1}| = |\eta^{1-1/r} - \frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1}| \quad \eta \in [x_{N-2}, x_N]$$

402
$$= T^{1-1/r}|(\frac{N-s}{N})^{r-1} - (\frac{N-t}{N})^{r-1}| \quad t \in [0,1], s \in [0,2]$$
$$\le T^{1-1/r}|(\frac{N-2}{N})^{r-1} - 1| \le CT^{1-1/r}(r-1)\frac{2}{N}$$

403 And
(3.126)
$$(_L h_{N-i}^3(\xi))'' = 3\,_L h_{N-i}^2(\xi)\,_L h_{N-i}{}''(\xi) + 6\,_L h_{N-i}(\xi)(_L h_{N-i}{}'(\xi))^2$$

404
$$\le C h_N^2 \frac{1-r}{r} x_i^{1/r-2}(\frac{1}{r}\frac{h_N}{Z_1} - _L y_{N-1-i}^{1-2/r}(\xi) Z_{N-1-i}) + C h_N(r-1)^2 h^2 x_i^{2/r-2}$$

405
$$|\frac{h_N}{rZ_1} - _L y_{N-1-i}^{1-2/r}(\xi) Z_{N-1-i}| \le T^{1-1/r} + C x_N^{1-2/r} x_N^{1/r} = CT^{1-1/r}$$

So

$$(3.127) \qquad \begin{aligned} (_L h_{N-i}^3(\xi))'' &\leq C h_N^2 \frac{1-r}{r} x_i^{1/r-2} + C(r-1)^2 h_N x_i^{2/r-2} h^2 \\ &\leq C(r-1) h_N^2 \end{aligned}$$

$_R h_{N+1-i}^3(\xi)$ is similar. □

LEMMA 3.31.

$$(3.128) \qquad u''(_L y_{N-i}^\theta(\xi)) \leq C x_{N-2}^{-\alpha/2-2} \leq C$$

$$(3.129) \qquad (u''(_L y_{N-i}^\theta(\xi)))' \leq C$$

$$(3.130) \qquad (u''(_L y_{N-i}^\theta(\xi)))'' \leq C$$

Proof.

$$(3.131) \qquad \begin{aligned} (u''(_L y_{N-i}^\theta(\xi)))' &= u'''(_L y_{N-i}^\theta(\xi))_L y_{N-i}^\theta{}'(\xi) \\ &\leq C(\theta_L y_{N-1-i}'(\xi) + (1-\theta)_0 y_{N-i}'(\xi)) \\ &\leq C x_i^{1/r-1}(\theta_L y_{N-1-i}^{1-1/r}(\xi) + (1-\theta)\frac{h_N}{rZ_1}) \\ &\leq C x_i^{1/r-1} x_N^{1-1/r} \end{aligned}$$

And
$$(3.132)$$
□
$$\begin{aligned} (u''(_L y_{N-i}^\theta(\xi)))'' &= u''''(_L y_{N-i}^\theta(\xi))(_L y_{N-i}^\theta{}'(\xi))^2 + u'''(_L y_{N-i}^\theta(\xi))_L y_{N-i}^\theta{}''(\xi) \\ &\leq C x_i^{2/r-2} x_N^{2-2/r} + C\frac{r-1}{r} x_i^{1/r-2}(\theta x_N^{1-2/r} Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \\ &\leq C x_i^{2/r-2} + C(r-1) x_i^{1/r-2} T^{1-1/r} \end{aligned}$$

LEMMA 3.32.

$$(3.133) \qquad |_L y_{N-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_N^\theta - x_i|^{1-\alpha}$$

$$(3.134) \qquad (|_L y_{N-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C|y_N^\theta - x_i|^{1-\alpha}$$

$$(3.135) \qquad (|_L y_{N-i}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C(r-1)|y_N^\theta - x_i|^{-\alpha} + |y_N^\theta - x_i|^{1-\alpha}$$

Proof.
$$(3.136)$$
$$\begin{aligned} (_L y_{N-i}^\theta(\xi) - \xi)' &= (\theta(_L y_{N-1-i}(\xi) - \xi) + (1-\theta)(_0 y_{N-i}(\xi) - \xi))' \\ &= \theta(_L y_{N-1-i}'(\xi) - 1) + (1-\theta)(_0 y_{N-i}'(\xi) - 1) \\ &= \theta\xi^{1/r-1}(_L y_{N-1-i}^{1-1/r}(\xi) - \xi^{1-1/r}) + (1-\theta)\xi^{1/r-1}(\frac{h_N}{rZ_1} - \xi^{1-1/r}) \end{aligned}$$

$$(3.137) \qquad \begin{aligned} (_L y_{N-i}^\theta(\xi) - \xi)'' &= \theta(_L y_{N-1-i}''(\xi)) + (1-\theta)(_0 y_{N-i}''(\xi)) \\ &= \frac{1-r}{r}\xi^{1/r-2}(\theta_L y_{N-1-i}^{1-2/r}(\xi) Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \leq 0 \end{aligned}$$

And

$$(3.138) \qquad |(_L y_{N-i}^\theta(\xi) - \xi)''| \leq C(r-1)\xi^{1/r-2} T^{1-1/r}$$

We have known

(3.139) $$C|x_{N-1} - x_i| \leq |_L y_{N-1-i}(\xi) - \xi| \leq C|x_{N-1} - x_i|$$

If $\xi \leq x_{N-1}$, then $(_0 y_{N-i}(\xi) - \xi)' \geq 0$, so

(3.140) $$C|x_N - x_i| \leq |x_{N-1} - x_{i-1}| \leq |_L y_{N-i}^\theta(\xi) - \xi| \leq |x_{N+1} - x_{i+1}| \leq C|x_N - x_i|$$

If $i = N - 1$ and $\xi \in [x_{N-1}, x_N]$, then $_0 y_{N-i}(\xi) - \xi$ is concave, bigger than its two neighboring points, which are equal to h_N, so

(3.141) $$h_N = |x_N - x_{N-1}| \leq |_0 y_{N-i}(\xi) - \xi| \leq |x_{N+1} - x_{N-1}| = 2h_N$$

So we have

(3.142) $$|_L y_{N-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_N^\theta - x_i|^{1-\alpha}$$

While

(3.143) $$_L y_{N-1-i}^{1-1/r}(\xi) - \xi^{1-1/r} \leq (_L y_{N-1-i}(\xi) - \xi)\xi^{-1/r}$$

and

(3.144)
$$|\frac{h_N}{rZ_1} - \xi^{1-1/r}| \leq \max\{|\frac{h_N}{rZ_1} - x_{i-1}^{1-1/r}|, |\frac{h_N}{rZ_1} - x_{i+1}^{1-1/r}|\}$$

$$\leq \max \begin{cases} T^{1-1/r} - x_{i-1}^{1-1/r} \leq |x_N - x_{i-1}|T^{-1/r} \leq C|x_N - x_i| \\ |x_{i+1}^{1-1/r} - x_{N-1}^{1-1/r}| \leq |x_{i+1} - x_{N-1}|x_{N-1}^{-1/r} \leq C|x_N - x_i| \end{cases}$$

So we have

(3.145) $$(_L y_{N-i}^\theta(\xi) - \xi)' \leq C|y_N^\theta - x_i|$$

(3.146)
$$(|_L y_{N-i}^\theta(\xi) - \xi|^{1-\alpha})' = |_L y_{N-i}^\theta(\xi) - \xi|^{-\alpha}(_L y_{N-i}^\theta(\xi) - \xi)'$$
$$\leq |y_N^\theta - x_i|^{1-\alpha}$$

Finally,

(3.147)
$$(|_L y_{N-i}^\theta(\xi) - \xi|^{1-\alpha})'' = (1-\alpha)|_L y_{N-i}^\theta(\xi) - \xi|^{-\alpha}(_L y_{N-i}^\theta(\xi) - \xi)''$$
$$+ \alpha(\alpha-1)|_L y_{N-i}^\theta(\xi) - \xi|^{-1-\alpha}((_L y_{N-i}^\theta(\xi) - \xi)')^2 \quad \square$$
$$\leq C(r-1)|y_N^\theta - x_i|^{-\alpha} + C|y_N^\theta - x_i|^{1-\alpha}$$

By the three lemmas above, for $N/2 \leq i \leq N - 1$, we have

LEMMA 3.33.

(3.148)
$$D_{hL}^2 P_{N-i}^\theta(x_i) = {}_L P_{N-i}^\theta{}''(\xi) \quad \xi \in [x_{i-1}, x_{i+1}]$$
$$\leq Ch^3|y_N^\theta - x_i|^{1-\alpha} + C(r-1)(h^3|y_N^\theta - x_i|^{-\alpha} + h^2|y_N^\theta - x_i|^{1-\alpha})$$

And

Lemma 3.34.

(3.149)
$$\frac{2}{h_i + h_{i+1}}\left(\frac{_LQ^\theta_{N-i}(x_{i+1})u'''(\eta^\theta_{N+1}) - _LQ^\theta_{N-i}(x_i)u'''(\eta^\theta_N)}{h_{i+1}}\right)$$
$$\leq Ch^3|y^\theta_N - x_i|^{1-\alpha}$$

And immediately, For $N/2 \leq i \leq N-2$

(3.150)
$$V_{iN} \leq C\int_{x_{N-1}}^{x_N} h^2|y - x_i|^{1-\alpha} + C(r-1)h^2|y - x_i|^{-\alpha} + h|y - x_i|^{1-\alpha}dy$$
$$\leq Ch^2h_N|T - x_i|^{1-\alpha} + C(r-1)h^2|x_{N-1} - x_i|^{1-\alpha} + Chh_N|T - x_i|^{1-\alpha}$$
$$\leq Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha}$$

But expecially, when $i = N-1$,

(3.151)
$$V_{N-1,N} = \int_0^1 -\frac{\theta^{2-\alpha}(1-\theta)}{2}\frac{2}{h_{N-1} + h_N}\left(\frac{1}{h_{N-1}}h^{4-\alpha}_{N-1}u''(y^\theta_{N-1}) - (\frac{1}{h_{N-1}} + \frac{1}{h_N})h^{4-\alpha}_N u''(y^\theta_N) + \frac{1}{h_N}h^{4-\alpha}_{N+1}u''(y^\theta_{N+1})\right)d\theta$$
$$+ \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i + h_{i+1}}\left(\frac{_LQ^\theta_{N-i}(x_{i+1})u'''(\eta^\theta_{N+1,1}) - _LQ^\theta_{N-i}(x_i)u'''(\eta^\theta_{N,1})}{h_{i+1}}\right)d\theta$$
$$- \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i + h_{i+1}}\left(\frac{_LQ^\theta_{N-i}(x_i)u'''(\eta^\theta_{N,1}) - _LQ^\theta_{N-i}(x_{i-1})u'''(\eta^\theta_{N-1,1})}{h_i}\right)d\theta$$
$$- \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i + h_{i+1}}\left(\frac{_LQ^\theta_{N-i}(x_{i+1})u'''(\eta^\theta_{N+1,2}) - _LQ^\theta_{N-i}(x_i)u'''(\eta^\theta_{N,2})}{h_{i+1}}\right)d\theta$$
$$+ \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i + h_{i+1}}\left(\frac{_LQ^\theta_{N-i}(x_i)u'''(\eta^\theta_{N,2}) - _LQ^\theta_{N-i}(x_{i-1})u'''(\eta^\theta_{N-1,2})}{h_i}\right)d\theta$$

while combine Lemma 3.30

(3.152)
$$\frac{2}{h_{N-1} + h_N}\left(\frac{1}{h_{N-1}}h^{4-\alpha}_{N-1}u''(y^\theta_{N-1}) - (\frac{1}{h_{N-1}} + \frac{1}{h_N})h^{4-\alpha}_N u''(y^\theta_N) + \frac{1}{h_N}h^{4-\alpha}_{N+1}u''(y^\theta_{N+1})\right)$$
$$= D^2_h(h^{4-\alpha}_{N-1 \to N}(x_i)u''(y^\theta_{N-1 \to N}(x_i)))$$
$$\leq Ch^{4-\alpha}_N + C(r-1)h^{3-\alpha}_N \leq Ch^{4-\alpha} + C(r-1)h^2|T - x_{N-1-1}|^{1-\alpha}$$

Similarly with $j = N+1$.   □

454   I_6, I_7 is easy. Similar with Lemma 3.22 and Lemma 3.6, we have

456   THEOREM 3.35. *There is a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For*
457    $N/2 \leq i \leq N$,
(3.153)
458
$$I_6 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right)$$
$$\leq Ch^2$$

459   *Proof.* In fact, let $l = 2N - \lceil \frac{N}{2} \rceil + 1$

460   (3.154)
$$\frac{1}{h_i}(T_{i-1,l} + T_{i-1,l-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,l}$$
$$= \frac{1}{h_i}(T_{i-1,l} - T_{i,l}) + \frac{1}{h_i}(T_{i-1,l-1} - T_{i,l}) + (\frac{1}{h_i} - \frac{1}{h_{i+1}})T_{i,l}$$

461   While, by Lemma A.2

462   (3.155)
$$\frac{1}{h_i}(T_{i-1,l} - T_{i,l}) = \int_{x_{l-1}}^{x_l} (u(y) - \Pi_h u(y))\frac{|x_{i-1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_i \Gamma(2-\alpha)}dy$$
$$\leq C \int_{x_{l-1}}^{x_l} h_l^2 u''(\eta) \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)}dy$$
$$\leq Ch_l^3 x_{l-1}^{\alpha/2-2}T^{-\alpha}$$
$$\leq Ch_l^3$$

463   Thus,

464   (3.156)
$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_i}(T_{i-1,l} - T_{i,l}) \leq Ch_l^2$$

465   For
(3.157)
466
$$\frac{1}{h_i}(T_{i-1,l-1} - T_{i,l}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{h_{l-1}^3|y_{l-1}^\theta - x_{i-1}|^{1-\alpha}u''(\eta_{l-1}^\theta) - h_l^3|y_l^\theta - x_i|^{1-\alpha}u''(\eta_l^\theta)}{h_i}d\theta$$

467   And Similar with Lemma 3.19, we can get

468   (3.158)
$$\frac{h_{l-1}^3|y_{l-1}^\theta - x_{i-1}|^{1-\alpha}u''(\eta_{l-1}^\theta) - h_l^3|y_l^\theta - x_i|^{1-\alpha}u''(\eta_l^\theta)}{(h_i + h_{i+1})h_i} \leq Ch_l^2|y_l^\theta - x_i|^{1-\alpha}$$

469   So

470   (3.159)
$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_i}(T_{i-1,l-1} - T_{i,l}) \leq Ch^2$$

471   For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

472   (3.160)
$$\frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}}T_{i,l} \leq h_i^{-3}h^2 x_i^{1-2/r} h_l Ch_l^2 x_{l-1}^{\alpha/2-2}|x_l - x_i|^{1-\alpha}$$
$$\leq Ch^2$$

473   Summarizes, we have

474   (3.161)
$$I_6 \leq Ch^2 \qquad\qquad\qquad \Box$$

And

LEMMA 3.36. *There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq i \leq N$,*

(3.162)
$$
\begin{aligned}
I_7 &= \sum_{j=2N-\lceil\frac{N}{2}\rceil+2}^{2N} S_{ij} \\
&\leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\end{aligned}
$$

Proof. For $i \leq N, j \geq 2N - \lceil\frac{N}{2}\rceil + 2$, we have

$$
\begin{aligned}
S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) dy \\
&\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2-2/r} |y - x_{i+1}^{-1-\alpha} dy \\
&\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy
\end{aligned}
$$

(3.163)
$$
\begin{aligned}
\sum_{j=2N-\lceil\frac{N}{2}\rceil+2}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{(2-2^{-r})T}^{x_{2N-1}} (2T - y)^{\alpha/2-2/r} dy \\
&\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r}T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\
&= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r} h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r} h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\end{aligned}
$$

Now we can conclude a part of the theorem Theorem 3.3 at the beginning of this section.

By Lemma 3.8 Lemma 3.22 Lemma 3.23 Theorem 3.29 Theorem 3.28 Theorem 3.35 Lemma 3.36 , we have

THEOREM 3.37. *there exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $N/2 \leq i < N$,*

(3.164)
$$
\begin{aligned}
R_i &= \sum_{j=1}^{7} I_j \\
&\leq C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\end{aligned}
$$

490   And what we left is the case $i = N$. Fortunately, we can use the same department
491 of R_i above, and it is symmetric. Most of the item has been esitmated by Lemma 3.8
492 and Theorem 3.35, we just need to consider I_3, I_4.

493

494   THEOREM 3.38. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

495   (3.165)
$$I_3 = \sum_{j=\lceil \frac{N}{2} \rceil+1}^{N-1} V_{Nj} \leq Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha}$$

496   *Proof.*   DEFINITION 3.39. *For $N/2 \leq j < N$, Let's define*

497   (3.166)
$$y_j(x) = (\frac{Z_1}{h_N}(x - x_N) + Z_j)^r, \quad Z_j = T^{1/r}\frac{j}{N}$$

498 *We can see that is the inverse of the function $_0 y_{N-i}(x)$ defined in Theorem 3.29.*

499   (3.167)
$$y_j'(x) = y_j^{1-1/r}(x)\frac{rZ_1}{h_N}$$

500   (3.168)
$$y_j''(x) = y_j^{1-2/r}(x)\frac{r(r-1)Z_1}{h_N}$$

501   With the scheme we used several times, we can get

502   LEMMA 3.40. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For*
503 *$N/2 \leq j < N$, $\xi \in [x_{N-1}, x_{N+1}]$,*

504   (3.169)
$$h_j(\xi)^3 \leq Ch^3$$

505   (3.170)
$$(h_j^3(\xi))' \leq C(r-1)h^3$$

506   (3.171)
$$(h_j^3(\xi))'' \leq C(r-1)h^3$$

507

508   (3.172)
$$u''(y_j^\theta(\xi)) \leq C$$

509   (3.173)
$$(u''(y_j^\theta(\xi)))' \leq C$$

510   (3.174)
$$(u''(y_j^\theta(\xi)))'' \leq C$$

511

512   (3.175)
$$|\xi - y_j^\theta(\xi)|^{1-\alpha} \leq C|x_N - y_j^\theta|^{1-\alpha}$$

513   (3.176)
$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})' \leq C|x_N - y_j^\theta|^{1-\alpha}$$

514   (3.177)
$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})'' \leq C|x_N - y_j^\theta|^{1-\alpha} + C(r-1)|x_N - y_j^\theta|^{-\alpha}$$

515   LEMMA 3.41. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For*
516 *$N/2 \leq j < N$,*

517   (3.178)
$$V_{Nj} \leq Ch^2 \int_{x_{j-1}}^{x_j} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy$$

518   Therefore,

519   (3.179)
$$I_3 \leq Ch^2 \int_{x_{\lceil \frac{N}{2} \rceil}}^{x_{N-1}} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy$$
$$\leq Ch^2(|T - x_{N-1}|^{2-\alpha} + (r-1)|T - x_{N-1}|^{1-\alpha})$$

□

For $j = N$,

LEMMA 3.42.
(3.180)

$$V_{N,N} = \frac{1}{h_N^2} \left(T_{N-1,N-1} - 2T_{N,N} + T_{N+1,N+1}\right) \leq Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha}$$

Proof.
(3.181)   □

$$V_{N,N} = \int_0^1 -\frac{\theta(1-\theta)^{2-\alpha}}{2} \frac{1}{h_N^2} \left(h_{N-1}^{4-\alpha} u''(y_{N-1}^\theta) - 2h_N^{4-\alpha} u''(y_N^\theta) + h_{N+1}^{4-\alpha} u''(y_{N+1}^\theta)\right) d\theta$$

$$+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{1}{h_N} \left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,1}^\theta) - Q_{N\to N}^\theta(x_i)u'''(\eta_{N,1}^\theta)}{h_N}\right) d\theta$$

$$- \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{1}{h_N} \left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,1}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,1}^\theta)}{h_N}\right) d\theta$$

$$- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{1}{h_N} \left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,2}^\theta) - Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta)}{h_N}\right) d\theta$$

$$+ \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{1}{h_N} \left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,2}^\theta)}{h_N}\right) d\theta$$

So combine Lemma 3.8, Theorem 3.35, Theorem 3.38, Lemma 3.42 We have

LEMMA 3.43.

(3.182) $$R_N \leq C(r-1)h^2|T - x_{N-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

and with Theorem 3.37 we prove the Theorem 3.3

526 **4. Convergence analysis.**

527 **4.1. Properties of some Matrices.** Review subsection 2.1, we have got (2.10).

528 DEFINITION 4.1. *We call one matrix an M matrix , which means its entries are*
529 *positive on major diagonal and nonpositive on others, and strictly diagonally dominant*
530 *in rows.*

531 Now we have

532 LEMMA 4.2. *Matrix A defined by (2.12) where (2.13) is an M matrix. And there*
533 *exists a constant $C_A = C(T, \alpha, r)$ such that*

534 (4.1)
$$S_i := \sum_{j=1}^{2N-1} a_{ij} \geq C_A(x_i^{-\alpha} + (2T - x_i)^{-\alpha})$$

535 *Proof.* From (2.14), we have

(4.2)
536
$$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right)$$

537 Let

538 (4.3)
$$g(x) = g_0(x) + g_{2N}(x)$$

539 where

540
$$g_0(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x - x_0|^{3-\alpha} - |x - x_1|^{3-\alpha}}{h_1}$$

541
$$g_{2N}(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N} - x|^{3-\alpha} - |x_{2N-1} - x|^{3-\alpha}}{h_{2N}}$$

542 Thus

543
$$-\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

544 Then

545 (4.4)
$$S_i := \sum_{j=1}^{2N-1} a_{ij}$$
$$= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right)$$
$$= D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i)$$

546 When $i = 1$

547 (4.5)
$$D_h^2 g_0(x_1) = \frac{2}{h_1 + h_2} \left(\frac{1}{h_2} g_0(x_2) - (\frac{1}{h_1} + \frac{1}{h_2}) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right)$$
$$= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1 h_2}$$
$$= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1^{1-\alpha}h_2} h_1^{-\alpha}$$
$$= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha}}{2^r(2^r - 1)} h_1^{-\alpha}$$

but

(4.6)
$$1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha} > 0$$

While for $i \geq 2$

(4.7)
$$
\begin{aligned}
D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \\
&= -\kappa_\alpha \frac{|\xi - x_0|^{1-\alpha} - |\xi - x_1|^{1-\alpha}}{\Gamma(2-\alpha) h_1} \\
&= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} |\xi - \eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\
&\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha}
\end{aligned}
$$

So

(4.8)
$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g_0(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g_0(x_i) + \frac{1}{h_i} g_0(x_{i-1}) \right) \geq C x_i^{-\alpha}$$

symmetricly,

(4.9)
$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g_{2N}(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g_{2N}(x_i) + \frac{1}{h_i} g_{2N}(x_{i-1}) \right) \geq C(\alpha, r)(2T - x_i)^{-\alpha} \qquad \blacksquare$$

Let

(4.10)
$$g(x) = \begin{cases} x, & 0 < x \leq T \\ 2T - x, & T < x < 2T \end{cases}$$

And define

(4.11)
$$G = \mathrm{diag}(g(x_1), ..., g(x_{2N-1}))$$

Then

LEMMA 4.3. *The matrix $B := AG$, the major diagnal is positive, and nonpositive on others. And there is a constant $C_{AG}, C = C(\alpha, r)$ such that*

(4.12) $$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C_{AG}(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases}$$

Proof.

$$b_{ij} = a_{ij} g(x_j) = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \tilde{a}_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) \tilde{a}_{i,j} + \frac{1}{h_i} \tilde{a}_{i-1,j} \right) g(x_j)$$

Since

(4.13)
$$g(x) \equiv \Pi_h g(x)$$

567   by **??**, we have

$$\tilde{M}_i := \sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} \tilde{a}_{ij} g(x_j)$$

568   (4.14)
$$= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} \Pi_h g(y) dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} g(y) dy$$

$$= \frac{-2}{\Gamma(4-\alpha)} |T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)} (x_i^{3-\alpha} + (2T - x_i)^{3-\alpha})$$

$$:= w(x_i) = p(x_i) + q(x_i)$$

569   Thus,

$$M_i := \sum_{j=1}^{2N-1} b_{ij} = \sum_{j=1}^{2N-1} a_{ij} g(x_j)$$

570   (4.15)
$$= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \tilde{M}_{i+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) \tilde{M}_i + \frac{1}{h_i} \tilde{M}_{i-1} \right)$$

$$= D_h^2(-\kappa_\alpha p)(x_i) - \kappa_\alpha D_h^2 q(x_i)$$

571   for $1 \le i \le N - 1$, by Lemma A.1
(4.16)
$$D_h^2(-\kappa_\alpha p)(x_i) := -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} p(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) p(x_i) + \frac{1}{h_i} p(x_{i-1}) \right)$$

572
$$= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1})$$

$$\ge \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - x_{i-1}|^{1-\alpha}$$

573
(4.17)
$$D_h^2(-\kappa_\alpha p)(x_N) := -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left(\frac{1}{h_{N+1}} p(x_{N+1}) - (\frac{1}{h_N} + \frac{1}{h_{N+1}}) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right)$$

574
$$= \frac{4\kappa_\alpha}{\Gamma(4-\alpha) h_N^2} h_N^{3-\alpha}$$

$$= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - x_{N-1})^{1-\alpha}$$

575   Symmetricly for $i \ge N$, we get

576   (4.18)
$$D_h^2(-\kappa_\alpha p)(x_i) \ge \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \le N \\ |x_{i+1} - T|^{1-\alpha}, & i \ge N \end{cases}$$

577   Similarly, we can get

578   (4.19)
$$D_h^2 q(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} q(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right)$$

$$\le \frac{2^{r(\alpha-1)+1}}{\Gamma(2-\alpha)} (x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}), \quad i = 1, \cdots, 2N - 1$$

□

579   So, we get the result.

580   Notice that

581   (4.20) $$x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha}$$

582   We can get

583   THEOREM 4.4. *There exists a real* $\lambda = \lambda(T, \alpha, r) > 0$ *and* $C = C(T, \alpha, r) > 0$
584   *such that* $B := A(\lambda I + G)$ *is an M matrix. And*

585   (4.21) $$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases}$$

586   *Proof.* By Lemma 4.2 with C_A and Lemma 4.3 with C_{AG} , it's sufficient to take
587    $\lambda = (C + 2TC_{AG})/C_A$, then

588   (4.22) $$M_i \geq C \left((x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \right)$$ □

589   **4.2. Proof of Theorem 2.6.** For equation

590   (4.23) $$AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1} U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1} U = F$$

591   which means

592   (4.24) $$\sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + g(x_j)} = -\tau_i$$

593   where $\epsilon_i = u(x_i) - u_i$.
594   And if

595   (4.25) $$\left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| = \max_{1 \leq i \leq 2N-1} \left| \frac{\epsilon_i}{\lambda + g(x_i)} \right|$$

596   Then, since $B = A(\lambda I + G)$ is an M matrix, it is Strictly diagonally dominant. Thus,

$$
\begin{aligned}
|\tau_{i_0}| &= \left| \sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + g(x_j)} \right| \\
&\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_j}{\lambda + g(x_j)} \right| \\
&\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \\
&= \sum_{j=1}^{2N-1} b_{i_0,j} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \\
&= M_{i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right|
\end{aligned}
$$

597   (4.26)

598   By Theorem 2.5 and Theorem 4.4,
599   We kown that there exists constants $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$,
600   and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that

601   (4.27) $$\left| \frac{\epsilon_i}{\lambda + g(x_i)} \right| \leq \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} + C_2(r - 1)h^2$$

602 as $\lambda + g(x_i) \leq \lambda + T$

603   So, we can get

604 (4.28)
$$|\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{r\alpha}{2}, 2\}}$$

605   The convergency has been proved.

606   Remarks:

5. Experimental results.

5.1. $f \equiv 1.$

5.2. $f = x^\gamma, \gamma < 0.$ **Appendix A. Approximate of difference quotients.**

LEMMA A.1. *If $g(x) \in C^2\Omega$, there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

(A.1)
$$D_h^2 g(x_i) := \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right)$$
$$= g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1})$$

(A.2)
$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right)$$
$$= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right)$$

And if $g(x) \in C^4(\Omega)$, then
(A.3)
$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right)$$
$$= g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2))$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$.

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})$$

Subsitute them in the left side of (A.1), we have

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right)$$
$$= \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2)$$

Now, using intermediate value theorem , there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy$$

626   And the last equation can be obtained by

627
$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

628
$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \; \blacksquare$$

629   Expecially,

630   (A.4)
$$\int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy = \frac{h_i^4}{4!} g''''(\eta_1)$$
$$\int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy = \frac{h_{i+1}^4}{4!} g''''(\eta_2)$$

631   where $\eta_1 \in (x_{i-1}, x_i), \eta_2 \in (x_i, x_{i+1})$. Subsitute them to the left side of (A.3), we can
632   get the result.   □

633   LEMMA A.2. *Denote* $y_j^\theta = \theta x_{j-1} + (1 - \theta) x_j, \theta \in [0, 1]$,

634   (A.5)
$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1 - \theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j]$$

635   (A.6)

636
$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1 - \theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1 - \theta)}{3!} h_j^3 ((1 - \theta)^2 u'''(\eta_1) - \theta^2 u'''(\eta_2))$$

637   *where* $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$.

638   *Proof.* By Taylor expansion, we have

639
$$u(x_{j-1}) = u(y_j^\theta) - (1 - \theta) h_j u'(y_j^\theta) + \frac{(1 - \theta)^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$

640
$$u(x_j) = u(y_j^\theta) + \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

641   Thus

642
$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = u(y_j^\theta) - \theta u(x_{j-1}) - (1 - \theta) u(x_j)$$
$$= -\frac{\theta(1 - \theta)}{2} h_j^2 ((1 - \theta) u''(\xi_1) + \theta u''(\xi_2))$$
$$= -\frac{\theta(1 - \theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2]$$

643   The second equation is similar,

644
$$u(x_{j-1}) = u(y_j^\theta) - (1 - \theta) h_j u'(y_j^\theta) + \frac{(1 - \theta)^2 h_j^2}{2!} u''(y_j^\theta) - \frac{(1 - \theta)^3 h_j^3}{3!} u'''(\eta_1)$$

645
$$u(x_j) = u(y_j^\theta) + \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) + \frac{\theta^3 h_j^3}{3!} u'''(\eta_2)$$

646   where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$. Thus

    □

647
$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = u(y_j^\theta) - \theta u(x_{j-1}) - (1 - \theta) u(x_j)$$
$$= -\frac{\theta(1 - \theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1 - \theta)}{3!} h_j^3 ((1 - \theta)^2 u'''(\eta_1) - \theta^2 u'''(\eta_2))$$

648   LEMMA A.3. *For $x \in [x_{j-1}, x_j]$*

649   (A.7)
$$|u(x) - \Pi_h u(x)| = \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^x u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_x^{x_j} u'(y)dy \right|$$
$$\leq \int_{x_{j-1}}^{x_j} |u'(y)|dy$$

650 *If $x \in [0, x_1]$, with Corollary 2.4, we have*

651   (A.8) $\qquad |u(x) - \Pi_h u(x)| \leq \int_0^{x_1} |u'(y)|dy \leq \int_0^{x_1} Cy^{\alpha/2-1}dy \leq C\frac{2}{\alpha}x_1^{\alpha/2}$

652 *Similarly, if $x \in [x_{2N-1}, 1]$, we have*

653   (A.9) $\qquad\qquad |u(x) - \Pi_h u(x)| \leq C\frac{2}{\alpha}(2T - x_{2N-1})^{\alpha/2} = C\frac{2}{\alpha}x_1^{\alpha/2}$

654   **Appendix B. Inequality.** For convenience, we use the notation and $\simeq$. That
655 $x_1 \simeq y_1$, means that $c_1 x_1 \leq y_1 \leq C_1 x_1$ for some constants c_1 and C_1 that are
656 independent of mesh parameters.
657

    LEMMA B.1.

658   (B.1) $\qquad\qquad h_i \leq rT^{1/r}h \begin{cases} x_i^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_{i-1})^{1-1/r}, & N < i \leq 2N - 1 \end{cases}$

659

660   (B.2) $\qquad\qquad h_i \geq rT^{1/r}h \begin{cases} x_{i-1}^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_i)^{1-1/r}, & N < i \leq 2N - 1 \end{cases}$

661   *Proof.* For $1 \leq i \leq N$,

$$h_i = T\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right)$$
662
$$\leq rT\frac{1}{N}\left(\frac{i}{N}\right)^{r-1} = rT^{1/r}hx_i^{1-1/r}$$

663

664 $$h_i \geq rT\frac{1}{N}\left(\frac{i-1}{N}\right)^{r-1} = rT^{1/r}hx_{i-1}^{1-1/r}$$

665 For $N < i \leq 2N$,

$$h_i = T\left(\left(\frac{2N-i+1}{N}\right)^r - \left(\frac{2N-i}{N}\right)^r\right)$$
666
$$\leq rT\frac{1}{N}\left(\frac{2N-i+1}{N}\right)^{r-1} = rT^{1/r}h(2T - x_{i-1})^{1-1/r}$$

667

668 $$h_i \geq rT\frac{1}{N}\left(\frac{2N-i}{N}\right)^{r-1} = rT^{1/r}h(2T - x_i)^{1-1/r} \qquad\qquad \square$$

669

670   LEMMA B.2. *There is a constant $C = 2^{|r-2|}r(r-1)T^{2/r}$ such that for all $i \in$*
671    $\{1, 2, \cdots, 2N-1\}$

672   (B.3) $|h_{i+1} - h_i| \leq Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases}$

       Proof.

673    $h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \leq i \leq 2N-1 \end{cases}$

674   For $i = 1$,

675        $h_2 - h_1 = T(2^r - 2)\left(\frac{1}{N}\right)^r = (2^r - 2)T^{2/r}h^2 x_1^{1-2/r}$

676   For $2 \leq i \leq N-1$, by Lemma A.1, we have

677        $h_{i+1} - h_i = r(r-1)T\,N^{-2}\eta^{r-2}, \quad \eta \in [\frac{i-1}{N}, \frac{i+1}{N}]$
              $= C(r-1)h^2 x_i^{1-2/r}$

678   Summarizes the inequalities, we can get

679   (B.4) $|h_{i+1} - h_i| \leq 2^{|r-2|}r(r-1)T^{2/r}h^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases}$ □

680   **Appendix C. Proofs of some technical details.**

681   *Additional proof of Theorem 3.1.* For $2 \leq i \leq N-1$,

682    $\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2))$
              $\leq C\frac{2}{h_i + h_{i+1}}(h_i^3 x_{i-1}^{-2-\alpha/2} + h_{i+1}^3 x_i^{-2-\alpha/2})$
              $\leq 2C(h_i^2 x_{i-1}^{-2-\alpha/2} + h_{i+1}^2 x_i^{-2-\alpha/2})$

683   There is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

684    $\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq Ch^2 x_i^{-\alpha/2-2/r}, \quad 2 \leq i \leq N-1$

685   For $i = 1$, by (A.4)

686    $\frac{1}{4!}\frac{2}{h_1 + h_2}(h_1^3 f''(\eta_1) + h_2^3 f''(\eta_2))$
              $= \frac{2}{h_1 + h_2}\left(\frac{1}{h_1}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy + \frac{1}{4!}h_2^3 f''(\eta_2)\right)$

687   We have proved above that

$$\frac{2}{h_1 + h_2} h_2^3 f''(\eta_2) \leq Ch^2 x_1^{-\alpha/2 - 2/r}$$

689   and we can get

$$\int_0^{x_1} f''(y) \frac{y^3}{3!} dy \leq C \frac{1}{3!} \int_0^{x_1} y^{1-\alpha/2} dy$$
$$= C \frac{1}{3!(2 - \alpha/2)} x_1^{2-\alpha/2}$$

691   so

$$\frac{2}{h_1 + h_2} \frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy = \frac{C2^{1-r}}{3!(2 - \alpha/2)} x_1^{-\alpha/2} = \frac{C2^{1-r}}{3!(2 - \alpha/2)} T^{2/r} h^2 x_1^{-\alpha/2 - 2/r}$$

693   And for $i = N$, we have

$$\frac{2}{h_N + h_{N+1}} (h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2))$$
$$= h_N^2 (f''(\eta_1) + f''(\eta_2))$$
$$\leq r^2 T^{2/r} h^2 x_N^{2-2/r} \, 2C x_{N-1}^{-2-\alpha/2}$$
$$\leq 2r^2 T^{2/r} C2^{-r(2-\alpha/2)} \, h^2 x_N^{-\alpha/2 - 2/r}$$

695   Finally, $N + 1 \leq i \leq 2N - 1$ is symmetric to the first half of the proof, so we can
696   conclude that

□

$$\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq Ch^2 \begin{cases} x_i^{-\alpha/2 - 2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2 - 2/r}, & N \leq i \leq 2N - 1 \end{cases}$$

698   LEMMA C.1. *By a standard error estimate for linear interpolation, and Corol-*
699   *lary 2.4, There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ for $2 \leq j \leq N$,*

700   (C.1) $|u(y) - \Pi_h u(y)| \leq Ch^2 y^{\alpha/2 - 2/r}, \quad \text{for } y \in [x_{j-1}, x_j]$

701   *symmetricly, for $N < j \leq 2N - 1$, we have*

702   (C.2) $|u(y) - \Pi_h u(y)| \leq Ch^2 (2T - y)^{\alpha/2 - 2/r}$

703   LEMMA C.2. *There is a constant $C = C(\alpha, r)$ such that for all $1 \leq i < N/2$,*
704    $\max\{2i + 1, i + 3\} \leq j \leq 2N$, *we have*

705   (C.3) $D_h^2 K_y(x_i) \leq C \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}, \quad y \in [x_{j-1}, x_j]$

706   *Proof.* Sinec $y \geq x_{j-1} > x_{i+1}$, by Lemma A.1, if $j - 1 > i + 1$

□

$$D_h^2 K_y(x_i) = K_y''(\xi) = \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}]$$
$$\leq \frac{(y - x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)}$$
$$\leq (1 - (\frac{2}{3})^r)^{-1-\alpha} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}$$

708   LEMMA C.3. *There is a constant $C = C(\alpha, r)$ such that for all $3 \le i \le N, k =$*
709     $\lceil \frac{i}{2} \rceil, 1 \le j \le k-1$ *and* $y \in [x_{j-1}, x_j]$, *we have*

710   (C.4)
$$D_h^2 K_y(x_i) \le C \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)}$$

711   *Proof.* Sinec $y \le x_j < x_{i-1}$, by Lemma A.1,

$$D_h^2 K_y(x_i) = \frac{|\xi - y|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}]$$

712
$$\le \frac{(x_{i-1} - x_j)^{-1-\alpha}}{\Gamma(-\alpha)} \le \frac{(x_{i-1} - x_{k-1})^{-1-\alpha}}{\Gamma(-\alpha)}$$

$$\le \left(\left(\frac{2}{3}\right)^r - \left(\frac{1}{2}\right)^r\right)^{-1-\alpha} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)}$$

713

714   LEMMA C.4. *While $0 \le i < N/2$, By Lemma A.3*

$$|T_{i1}| \le C \int_0^{x_1} x_1^{\alpha/2} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} dy$$

715   (C.5)
$$= C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2} \left| x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha} \right|$$

$$\le C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2+2-\alpha} = C \frac{1}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1$$

716   *For $2 \le j \le N$, by Lemma A.2 and Corollary 2.4*

$$|T_{ij}| \le \frac{C}{4} \int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy$$

717   (C.6)
$$\le \frac{C}{4\Gamma(3-\alpha)} h_j^2 x_{j-1}^{\alpha/2-2} \left| |x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha} \right|$$

718   LEMMA C.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

719   (C.7)
$$\sum_{j=1}^3 S_{1j} \le C h^2 x_1^{-\alpha/2-2/r}$$

720

721   (C.8)
$$\sum_{j=1}^4 S_{2j} \le C h^2 x_2^{-\alpha/2-2/r}$$

722

    Proof.

723
$$S_{1j} = \frac{2}{x_2} \left(\frac{1}{x_1} T_{0j} - \left(\frac{1}{x_1} + \frac{1}{h_2} \right) T_{1j} + \frac{1}{h_2} T_{2j} \right)$$

724 So, by Lemma C.4

725
$$S_{11} \le \frac{2}{x_2 x_1} 4 \frac{C}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \le C x_1^{-\alpha/2}$$

$$S_{12} \leq \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_2^2 x_1^{\alpha/2-2} \left(x_2^{2-\alpha} + 2h_2^{2-\alpha} + h_2^{2-\alpha} \right) \leq C x_1^{-\alpha/2}$$

$$S_{13} \leq \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_3^2 x_2^{\alpha/2-2} \left(x_3^{2-\alpha} + 2x_3^{2-\alpha} + h_3^{2-\alpha} \right) \leq C x_1^{-\alpha/2}$$

But

$$x_1^{-\alpha/2} = T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

$i = 2$ is similar.   □

LEMMA C.6. *There exists a constant* $C = C(T, r, l)$ *such that For* $3 \leq i \leq N - 1, k = \lceil \frac{i}{2} \rceil, k \leq j \leq \min\{2i - 1, N - 1\}$,
when $\xi \in [x_{i-1}, x_{i+1}]$,

(C.9) $(h_{j-i}^3(\xi))' \leq (r-1)Ch^2 x_i^{1-2/r} h_j$

(C.10) $(h_{j-i}^4(\xi))' \leq (r-1)Ch^2 x_i^{1-2/r} h_j^2$

Proof. From (3.31)

(C.11) $y_{j-i}'(x) = y_{j-i}^{1-1/r}(x) x^{1/r-1}$

(C.12) $y_{j-i}''(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i}$

For $\xi \in (x_{i-1}, x_{i+1})$ and $2 \leq k \leq j \leq \min\{2i-1, N-1\}$, using Lemma B.1

$$\xi \simeq x_i \simeq x_j$$

$$h_{j-i}(\xi) \simeq h_j \simeq h x_j^{1-1/r} \simeq h x_i^{1-1/r}$$

(C.13) $\begin{aligned} h_{j-i}'(\xi) &= y_{j-i}'(\xi) - y_{j-i-1}'(\xi) \\ &= \xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \end{aligned}$

Since

(C.14) $\begin{aligned} y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) &\leq x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r} \\ &= T^{1-1/r} N^{1-r}((j+1)^{r-1} - (j-2)^{r-1}) \\ &\leq C(r-1)j^{r-2} N^{1-r} \\ &= C(r-1)h x_j^{1-2/r} \end{aligned}$

Therefore,

(C.15) $h_{j-i}'(\xi) \leq C x_i^{1/r-1}(r-1)h x_j^{1-2/r} \simeq (r-1)h x_i^{-1/r}$

for $l = 3, 4$

(C.16)
$$\begin{aligned}
(h_{j-i}^l(\xi))' &= l h_{j-i}^{l-1}(\xi) h_{j-i}'(\xi) \\
&\leq C h_{j-i}^{l-1}(\xi)(r-1)h x_i^{-1/r} \\
&\simeq C h_j^{l-2} h x_j^{1-1/r}(r-1)h x_i^{-1/r} \\
&\simeq C(r-1)h^2 x_i^{1-2/r} h_j^{l-2}
\end{aligned}$$

Meanwhile, we can get

(C.17)
$$h_{j-i}^3(\xi) \simeq h_j^3 \leq C h^2 x_i^{2-2/r} h_j$$

(C.18)
$$h_{j-i}^4(\xi) \simeq h_j^4 \leq C h^2 x_i^{2-2/r} h_j^2 \qquad \Box$$

LEMMA C.7. *There exists a constant* $C = C(T, r, l)$ *such that For* $3 \leq i \leq N - 1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$, *when* $\xi \in (x_{i-1}, x_{i+1})$,

(C.19)
$$(h_{j-i}^3(\xi))'' \leq C(r-1)h^2 x_i^{-2/r} h_j$$

Proof.

(C.20)
$$(h_{j-i}^3(\xi))'' = 6 h_{j-i}(\xi)(h_{j-i}'(\xi))^2 + 3 h_{j-i}^2(\xi) h_{j-i}''(\xi)$$

By (C.15)

(C.21)
$$h_{j-i}(\xi)(h_{j-i}'(\xi))^2 \leq C h_j(r-1)^2 h^2 x_i^{-2/r}$$

For the second partial

(C.22)
$$\begin{aligned}
h_{j-i}''(\xi) &= y_{j-i}''(\xi) - y_{j-i-1}''(\xi) \\
&= \frac{1-r}{r} \xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\
&= \frac{1-r}{r} \xi^{1/r-2}((y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi))Z_{j-i} + y_{j-i-1}^{1-2/r}(\xi)Z_1)
\end{aligned}$$

but

(C.23)
$$\begin{aligned}
|y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi)| &\leq |x_{j+1}^{1-2/r} - x_{j-2}^{1-2/r}| \\
&= T^{1-2/r}N^{2-r}|(j+1)^{r-2} - (j-2)^{r-2}| \\
&\leq C|r-2|N^{2-r}j^{r-3} \\
&= C|r-2|h x_j^{1-3/r}
\end{aligned}$$

So we can get

(C.24)
$$\begin{aligned}
|h_{j-i}''(\xi)| &\leq C(r-1)x_i^{1/r-2}(|r-2|h x_i^{1-3/r} x_i^{1/r} + x_i^{1-2/r} h) \\
&\leq C(r-1)h x_i^{-1-1/r}
\end{aligned}$$

Summarizes, we have

(C.25)
$$(h_{j-i}^3(\xi))'' \leq C(r-1)h^2 x_i^{-2/r} h_j \qquad \Box$$

proof of Lemma 3.16. From (3.31)

(C.26) $$y'_{j-i}(x) = y^{1-1/r}_{j-i}(x)x^{1/r-1}$$

(C.27) $$y''_{j-i}(x) = \frac{1-r}{r}y^{1-2/r}_{j-i}(x)x^{1/r-2}Z_{j-i}$$

Since

$$y^{\theta}_{j-i}(\xi) \simeq x_j \simeq x_i$$

We have known

(C.28) $$u''(y^{\theta}_{j-i}(\xi)) \le C(y^{\theta}_{j-i}(\xi))^{\alpha/2-2} \simeq x^{\alpha/2-2}_j \simeq x^{\alpha/2-2}_i$$

(C.29) $$\begin{aligned}(u''(y^{\theta}_{j-i}(\xi)))' &= u'''(y^{\theta}_{j-i}(\xi))(y^{\theta}_{j-i}(\xi))' \\ &\le Cx^{\alpha/2-3}_i\xi^{1/r-1}y^{1-1/r}_{j-i}(\xi) \\ &\simeq x^{\alpha/2-3}_i x^{1/r-1}_i x^{1-1/r}_i = Cx^{\alpha/2-3}_i\end{aligned}$$

(C.30) $$\begin{aligned}(u''(y^{\theta}_{j-i}(\xi)))'' &= u''''(y^{\theta}_{j-i}(\xi))(y^{\theta}_{j-i}{}'(\xi))^2 + u'''(y^{\theta}_{j-i}(\xi))y^{\theta}_{j-i}{}''(\xi) \\ &\le Cx^{\alpha/2-4}_i + Cx^{\alpha/2-3}_i\frac{r-1}{r}x^{1-2/r}_i x^{1/r-2}_i Z_{|j-i|+1} \\ &\le Cx^{\alpha/2-4}_i + C\frac{r-1}{r}x^{\alpha/2-3}_i x^{-1-1/r}_i x^{1/r}_i \\ &= Cx^{\alpha/2-4}_i\end{aligned}$$ □

Proof of Lemma 3.17.

(C.31) $$\begin{aligned}|y^{\theta}_{j-i}(\xi) - \xi| &= |\theta(y_{j-i-1}(\xi) - \xi) + (1-\theta)(y_{j-i}(\xi) - \xi)| \\ &= \theta|y_{j-i-1}(\xi) - \xi| + (1-\theta)|y_{j-i}(\xi) - \xi|\end{aligned}$$

where $y_{j-i-1}(\xi) - \xi$ and $y_{j-i}(\xi) - \xi$ have the same sign (≥ 0 or ≤ 0).
Since $|y_{j-i}(\xi) - \xi| \simeq |x_j - x_i|$, we have

$$|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha} \simeq |y^{\theta}_j - x_i|^{1-\alpha}$$

Next,
(C.32)
$$\begin{aligned}(|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})' &= (1-\alpha)|y^{\theta}_{j-i}(\xi) - \xi|^{-\alpha}|\xi^{1/r-1}(\theta y^{1-1/r}_{j-i-1}(\xi) + (1-\theta)y^{1-1/r}_{j-i}(\xi)) - 1| \\ &\le C|y^{\theta}_j - x_i|^{-\alpha}\xi^{1/r-1}|\theta y^{1-1/r}_{j-i-1}(\xi) + (1-\theta)y^{1-1/r}_{j-i}(\xi) - \xi^{1-1/r}|\end{aligned}$$

Similar with (**??**), we have

(C.33) $$\begin{aligned}|y^{1-1/r}_{j-i}(\xi) - \xi^{1-1/r}| &\le C|x^{1-1/r}_j - x^{1-1/r}_i| \\ &\le C|x_j - x_i|x^{-1/r}_i\end{aligned}$$

792   So we can get

$$\begin{aligned}
&|\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| \\
&\leq Cx_i^{-1/r}(\theta|x_{j-1} - x_i| + (1-\theta)|x_j - x_i|) \\
&= Cx_i^{-1/r}|y_j^\theta - x_i|
\end{aligned}$$

793   (C.34)

794   Combine them, we get

795   (C.35)
$$\begin{aligned}
(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' &\leq C|y_j^\theta - x_i|^{-\alpha}x_i^{1/r-1}x_i^{-1/r}|y_j^\theta - x_i| \\
&= C|y_j^\theta - x_i|^{1-\alpha}x_i^{-1}
\end{aligned}$$

796   Finally, we have
(C.36)
$$\begin{aligned}
(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' &= \alpha(\alpha-1)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi)) - 1)^2 \\
&\quad + (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}\frac{1-r}{r}\xi^{1/r-2}|\theta y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1} + (1-\theta)y_{j-i}^{1-2/r}(\xi)Z_{j-i}|
\end{aligned}$$

797

798   Using the inequalities above ,we have

799   (C.37)
$$\begin{aligned}
&|y_{j-i}^\theta(\xi) - \xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi)) - 1)^2 \\
&\leq C|y_j^\theta - x_i|^{-\alpha-1}(x_i^{-1}|y_j^\theta - x_i|)^2 \\
&= C|y_j^\theta - x_i|^{1-\alpha}x_i^{-2}
\end{aligned}$$

800   And by

801   (C.38)
$$|Z_{j-i}| = |x_j^{1/r} - x_i^{1/r}| \leq |x_j - x_i|x_i^{1/r-1}$$

802   we have

803   (C.39)
$$\begin{aligned}
&|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}\xi^{1/r-2}|\theta y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1} + (1-\theta)y_{j-i}^{1-2/r}(\xi)Z_{j-i}| \\
&\leq C|y_j^\theta - x_i|^{-\alpha}x_i^{1/r-2}x_i^{1-2/r}|\theta Z_{j-i-1} + (1-\theta)Z_{j-i}| \\
&\leq C|y_j^\theta - x_i|^{-\alpha}x_i^{-2}|y_j^\theta - x_i| \\
&= C|y_j^\theta - x_i|^{1-\alpha}x_i^{-2}
\end{aligned}$$ □

804   *proof of Lemma 3.19.* For $k \leq j < \min\{2i-1, N-1\}$

805   (C.40)
$$\begin{aligned}
&\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \\
&\frac{Q_{j-i}^\theta(x_{i+1}) - Q_{j-i}^\theta(x_i)}{h_{i+1}}u'''(\eta_{j+1}^\theta) + Q_{j-i}^\theta(x_i)\frac{u'''(\eta_{j+1}^\theta) - u'''(\eta_j^\theta)}{h_{i+1}} \\
&\leq Q_{j-i}^{\theta}{}'(\xi)Cx_j^{\alpha/2-3} + Q_{j-i}^\theta(x_i)Cu''''(\eta)\frac{h_i + h_{i+1}}{h_{i+1}}
\end{aligned}$$

806   where $\xi \in [x_i, x_{i+1}], \eta \in [x_{j-1}, x_{j+1}]$.

807   From (3.35), by Lemma C.6 and Lemma 3.17, we have

808   (C.41)
$$\begin{aligned}
Q_{j-i}^{\theta}{}'(\xi) &\leq Ch^2\frac{|y_{j+1}^\theta - x_{i+1}|^{1-\alpha}}{\Gamma(2-\alpha)}x_{i+1}^{1-2/r}h_{j+1}^2 \\
&\leq Ch^2\frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{1-2/r}h_j^2
\end{aligned}$$

809   And by defination

810   (C.42) $Q_{j-i}^{\theta}(x_i) = h_j^4 \dfrac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \leq Ch^2 x_i^{2-2/r} \dfrac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2$

811   With , we have

812   (C.43) $4^{-r} x_i \leq x_{k-1} \leq x_{j-1} < x_j \leq x_{2i-1} \leq 2^r x_i$

813   So we have

$$\dfrac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta}) - Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}}$$

814   (C.44) $\leq Ch^2 \dfrac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 \, x_i^{\alpha/2-3} + Ch^2 x_i^{2-2/r} \dfrac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 x_{j-1}^{\alpha/2-4}$

$= Ch^2 \dfrac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j^2$

815   while

816                        $h_j \leq h_{2i-1} \leq 2^r h_i$

817   Subsitute into the inequality above, we get the goal

$$\dfrac{2}{h_i + h_{i+1}} \left(\dfrac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta}) - Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}} \right)$$

818   (C.45) $\leq \dfrac{1}{h_i} Ch^2 \dfrac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j 2^r h_i$

$= Ch^2 \dfrac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j$

819   While, the later is similar.   □

820

821   LEMMA C.8. *There exists a constant $C = C(T,r)$ such that For $N/2 \leq i < N$,*
822   *$N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$, $l = 3,4$, $\xi \in [x_{i-1}, x_{i+1}]$, we have*

823   (C.46) $h_{j-i}^l(\xi) \leq Ch_j^l \leq Ch^2 h_j^{l-2}$
824   (C.47) $(h_{j-i-1}^l(\xi))' \leq C(r-1)h^2 h_j^{l-2}$
825   (C.48) $(h_{j-i}^3(\xi))'' \leq C(r-1)h^2 h_j$

    Proof.

826   (C.49) $(h_{j-i}(\xi))' = y_{j-i}{}'(\xi) - y_{j-i-1}{}'(\xi)$
$= \xi^{1/r-1}((2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}) \leq 0$

827   Thus,

828   (C.50) $Ch_j \leq h_{j+1} \leq h_{j-i}(\xi) \leq h_{j-i}(x_{i-1}) = h_{j-1} \leq Ch_j$

829   So as $4^{-r}T \leq 2T - x_j \leq T, 2^{-r}T \leq x_i \leq T$, we have

830   (C.51) $$h^l_{j-i}(\xi) \leq Ch^l_j \leq Ch^2(2T - x_j)^{2-2/r}h^{l-2}_j \leq Ch^2h^{l-2}_j$$

831   Since

832   (C.52) $$\begin{aligned} |(2T - y_{j-i}(\xi))^{1-1/r} &- (2T - y_{j-i-1}(\xi))^{1-1/r}| \\ &= |(Z_{2N-(j-i)} - \xi^{1/r})^{r-1} - (Z_{2N-(j-1-i)} - \xi^{1/r})^{r-1}| \\ &= (r-1)Z_1(Z_{2N-(j-i-\gamma)} - \xi^{1/r})^{r-2} \quad \gamma \in [0, 1] \\ &\leq C(r-1)h(2T - x_j)^{1-2/r} \end{aligned}$$

833   we have

834   (C.53) $$|(h_{j-i}(\xi))'| \leq C(r-1)h(2T - x_j)^{1-2/r}x^{1/r-1}_i$$

835   And

836   (C.54) $$\begin{aligned} (h^l_{j-i}(\xi))' &= lh^{l-1}_{j-i}(\xi)h_{j-i}'(\xi) \\ &\leq C(r-1)h^{l-1}_j\, h(2T - x_j)^{1-2/r}x^{1/r-1}_i \\ &\leq C(r-1)h^2h^{l-2}_j(2T - x_j)^{2-3/r}x^{1-1/r}_i \\ &\leq C(r-1)h^2h^{l-2}_j \end{aligned}$$

(C.55) □

837    $$\begin{aligned} (h^3_{j-i}(\xi))'' &= 6h_{j-i}(\xi)(y_{j-i}'(\xi) - y_{j-i-1}'(\xi))^2 + 3h^2_{j-i}(\xi)(y_{j-i}''(\xi) - y_{j-i-1}''(\xi)) \\ &\leq C(r-1)h_jh^2 + Ch^2_j\frac{1-r}{r}\xi^{1/r-2}((2T - y_{j-i}(\xi))^{1-2/r}Z_{2N-(j-i)} - (2T - y_{j-i-1}(\xi))^{1-2/r}Z_{2N-(j-1-i)}) \\ &\leq C(r-1)h_jh^2 + C(r-1)h^2_j(C(r-2)h(2T - x_j)^{1-3/r}Z_{2N-(j-i)} + Z_1(2T - x_{j-1})^{1-2/r}) \\ &\leq C(r-1)h_jh^2 + C(r-1)h^2_jh = Ch^2h_j \end{aligned}$$

838

839   **LEMMA C.9.** *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that For*
840    $N/2 \leq i < N$, $N + 2 \leq j \leq 2N - \lceil\frac{N}{2}\rceil + 1$, $\xi \in [x_{i-1}, x_{i+1}]$, *we have*

841   (C.56) $$u''(y^\theta_{j-i}(\xi)) \leq C$$
842   (C.57) $$(u''(y^\theta_{j-i}(\xi)))' \leq C$$
843   (C.58) $$(u''(y^\theta_{j-i}(\xi)))'' \leq C$$

Proof.

844   (C.59) $$x_{j-2} \leq y^\theta_{j-i}(\xi) \leq x_{j+1} \Rightarrow 4^{-r}T \leq 2T - y^\theta_{j-i}(\xi) \leq T$$

845   Thus, for $l = 2, 3, 4$,

846   (C.60) $$u^{(l)}(y^\theta_{j-i}(\xi)) \leq C(2T - y^\theta_{j-i}(\xi))^{\alpha/2-l} \leq C$$

847   and

848   (C.61) $$\begin{aligned} (y^\theta_{j-i}(\xi))' &= \theta y_{j-1-i}'(\xi) + (1-\theta)y_{j-i-1}'(\xi) \\ &= \xi^{1/r-1}(\theta(2T - y_{j-1-i}(\xi))^{1-1/r} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-1/r}) \\ &\leq C(2T - x_{j-2})^{1-1/r} \leq C \end{aligned}$$

With

$$Z_{2N-j-i} \leq 2T^{1/r} \tag{C.62}$$

(C.63)
$$
\begin{aligned}
(y_{j-i}^{\theta}(\xi))'' &= \theta y_{j-1-i}''(\xi) + (1-\theta) y_{j-i-1}''(\xi) \\
&= \frac{1-r}{r} \xi^{1/r-2} (\theta(2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)} + (1-\theta)(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)}) \\
&\leq C(r-1)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(u''(y_{j-i}^{\theta}(\xi)))' &= u'''(y_{j-i}^{\theta}(\xi))(y_{j-i}^{\theta}(\xi))' \\
&\leq C
\end{aligned} \tag{C.64}
$$

$$
\begin{aligned}
(u''(y_{j-i}^{\theta}(\xi)))'' &= u'''(y_{j-i}^{\theta}(\xi))(y_{j-i}^{\theta}{}'(\xi))^2 + u''''(y_{j-i}^{\theta}(\xi)) y_{j-i}^{\theta}{}''(\xi) \\
&\leq C + C(r-1) = C
\end{aligned} \tag{C.65}
$$

□

LEMMA C.10. *There exists a constant $C = C(T, \alpha, r)$ such that*

$$|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha} \leq C|y_j^{\theta} - x_i|^{1-\alpha} \tag{C.66}$$

$$(|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha})' \leq C|y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N) \tag{C.67}$$

(C.68)
$$(|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha})'' \leq C(r-1)|y_j^{\theta} - x_i|^{-\alpha} + C|y_j^{\theta} - x_i|^{-1-\alpha}(|2T - x_i - y_j^{\theta}| + h_N)^2$$

Proof.

$$(y_{j-i}^{\theta}(\xi) - \xi)' = \theta y_{j-1-i}'(\xi) + (1-\theta) y_{j-i}'(\xi) - 1 \tag{C.69}$$

$$
\begin{aligned}
|y_{j-i}'(\xi) - 1| &= \xi^{1/r-1}|(2T - y_{j-i}(\xi))^{1-1/r} - \xi^{1-1/r}| \\
&\leq \xi^{1/r-1}|2T - \xi - y_{j-i}(\xi)|\xi^{-1/r}
\end{aligned} \tag{C.70}
$$

$$
\begin{aligned}
|2T - \xi - y_{j-i}(\xi)| &\leq \max \begin{cases} |2T - x_{i-1} - x_{j-1}| \\ |2T - x_{i+1} - x_{j+1}| \end{cases} \\
&\leq |2T - x_i - x_j| + h_{i+1} + h_j
\end{aligned} \tag{C.71}
$$

(C.72)
$$
\begin{aligned}
(y_{j-i}^{\theta}(\xi) - \xi)'' &= \theta y_{j-1-i}''(\xi) + (1-\theta) y_{j-i}''(\xi) \\
&= \frac{1-r}{r} \xi^{1/r-2} (\theta(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)}) \leq 0
\end{aligned}
$$

It's concave, so

$$y_{j-i}(\xi) - \xi \geq \min\{x_{j+1} - x_{i+1}, x_{j-1} - x_{i-1}\} \geq C(x_j - x_i) \tag{C.73}$$

We have

$$(C.74) \qquad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \le C|y_j^\theta - x_i|^{1-\alpha}$$

$$\begin{aligned}(C.75) \qquad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' &= (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}(y_{j-i}^\theta(\xi) - \xi)' \\ &\le C|y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_{i+1} + h_{j-1})\end{aligned}$$

$$(C.76)$$
$$\begin{aligned}(|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' &= (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}(y_{j-i}^\theta(\xi) - \xi)'' + \alpha(\alpha-1)|y_{j-i}^\theta(\xi) - \xi|^{-1-\alpha}(y_{j-i}^{\theta'}(\xi) - 1)^2 \\ &\le C(r-1)|y_j^\theta - x_i|^{-\alpha} + C|y_j^\theta - x_i|^{-1-\alpha}(|2T - x_i - y_j^\theta| + h_{i+1} + h_{j-1})^2\end{aligned}$$

Proof. From (3.24), by Lemma C.8 and Lemma C.10, we have $\xi \in [x_i, x_{i+1}]$

$$(C.77) \quad Q_{j-i}^{\theta'}(\xi) \le Ch^2 h_j^2((r-1)|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N))$$

$$(C.78) \qquad Q_{j-i}^\theta(\xi) \le Ch^2 h_j^2 |y_j^\theta - x_i|^{1-\alpha}$$

So use the skill in Proof 33 with Lemma C.9

$$\begin{aligned}(C.79) \qquad \frac{2}{h_i + h_{i+1}} &\left(\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ &\le Ch^2 h_j(|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N))\end{aligned}$$

$$(C.80) \qquad a^{1-\theta}|a^\theta - b^\theta| \le |a - b|, \theta \in [0,1]$$

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