AN EXAMPLE ARTICLE*

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Abstract. This is an example SIAM LaTeX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

Key words. example, LaTeX

MSC codes. 68Q25, 68R10, 68U05

1. Introduction.

The introduction introduces the context and summarizes the manuscript. It is importantly to clearly state the contributions of this piece of work.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$, suppose $f \in C^\beta(\Omega), \beta > 4 - \alpha, \|f\|_\beta^{(\alpha/2)} < \infty$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

and the solution $u \in C^{\alpha/2}(\Omega)$.

2. Regularity.

Remark 2.1. 1. $C^k(U)$ is the set of all k-times continuously differentiable functions on open set U.
2. $C^\beta(U)$ is the collection of function f which for any $V \subset\subset U$ $f|_V \in C^\beta(\bar{V})$.

THEOREM 2.2. *If $f \in C^\beta(\Omega), \beta > 2$ and $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

$$|f^{(l)}(x)| \le \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \le T \\ (2T - x)^{-l-\alpha/2}, & \text{if } T \le x < 2T \end{cases} \tag{2.1}$$

THEOREM 2.3 (Regularity up to the boundary [1]).

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right) \tag{2.2}$$

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*Submitted to the editors DATE.

Funding: This work was funded by the Fog Research Institute under contract no. FRI-454.

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COROLLARY 2.4. *Let u be a solution of (1.1) on Ω. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \leq \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases} \tag{2.3}$$

The paper is organized as follows. Our main results are in section 4, experimental results are in section 7, and the conclusions follow in section 8.

3. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \leq i \leq N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \leq i \leq 2N \end{cases} \tag{3.1}$$

where $r \geq 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \leq j \leq 2N \tag{3.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \leq x \leq x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \leq x \leq x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

And then, we can approximate $u(x)$ with

$$u_h(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{3.4}$$

For convience, we denote

$$I_h^{2-\alpha}(x_i) := \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y) dy \tag{3.5}$$

And now, we can approximate the operator (1.2) at x_i with

$$\begin{aligned} D_h^\alpha u_h(x_i) &:= D_h^2 I_h^{2-\alpha}(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} I_h^{2-\alpha}(x_{i-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) I_h^{2-\alpha}(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha}(x_{i+1}) \right) \end{aligned} \tag{3.6}$$

Finally, we approximate the equation (1.1) with

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i), \quad 1 \leq i \leq 2N-1 \tag{3.7}$$

The discrete equation (3.7) can be written in matrix form

$$AU = F \tag{3.8}$$

where U is unknown, $F = (f(x_1), \cdots, f(x_{2N-1}))$. The matrix A is constructed as follows: Since

$$
\begin{aligned}
I_h^{2-\alpha}(x_i) &= \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u_h(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_\Omega |x_i - y|^{1-\alpha} u(x_j) \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y) dy \\
&= \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\
&=: \sum_{j=1}^{2N-1} \tilde{a}_{ij}\, u(x_j), \quad 0 \le i \le 2N
\end{aligned}
\tag{3.9}
$$

Then, substitute in (3.6), we have

$$
-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} a_{ij}\, u(x_j) \tag{3.10}
$$

where

$$
a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \tilde{a}_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right) \tag{3.11}
$$

4. Main results.

Here we state our main results; the proof is deferred to section 5 and section 6.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 4.1 (Truncation Error). *If $f \in C^2(\Omega)$ and $\alpha \in (1,2)$, and $u(x)$ is a solution of the equation (1.1), then there exists a constant $C_1, C_2 = C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{C^2(\Omega)}), C_2(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$, such that the truncation error of the discrete format satisfies*

$$
\begin{aligned}
|-\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i)| \le & C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} (x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \\
& + C_2 h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \le i \le N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \le 2N-1 \end{cases}
\end{aligned}
\tag{4.1}
$$

where $C_2 = 0$ if $r = 1$.

THEOREM 4.2 (Convergence). *The discrete equation (3.7) has sulotion U, and there exists a positive constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$ such that the error between the numerial solution U with the exact solution $u(x_i)$ satisfies*

$$
\max_{1 \le i \le 2N-1} |U_i - u(x_i)| \le C h^{\min\{\frac{r\alpha}{2}, 2\}} \tag{4.2}
$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

5. Proof of Theorem 4.1.

For convience, let's denote

$$I^{2-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x-y|^{1-\alpha} u(y) dy \tag{5.1}$$

Then, the truncation error of the discrete format can be written as

$$\begin{aligned} -\kappa_\alpha D_h^\alpha u_h(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I_h^{2-\alpha}(x_i) - \frac{d^2}{dx^2} I^{2-\alpha}(x_i)) \\ &= -\kappa_\alpha D_h^2 (I_h^{2-\alpha} - I^{2-\alpha})(x_i) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \end{aligned} \tag{5.2}$$

5.1. Estimate of $-\kappa_\alpha(D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i)$.

Theorem 5.1. *There exits a constant* $C = C(T, \alpha, r, \|f\|_\beta^{(\alpha/2)})$ *such that*

$$\left| -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \right| \le Ch^2 (x_i^{-\alpha/2-2/r} + (2T - x_i)^{-\alpha/2-2/r}) \tag{5.3}$$

Proof. Since $f \in C^2(\Omega)$ and

$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha}(x)) = f(x), \quad x \in \Omega, \tag{5.4}$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.3) of Lemma A.1, for $1 \le i \le 2N-1$, we have

$$-\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) = \frac{h_{i+1} - h_i}{3} f'(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \tag{5.5}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

$$\left| \frac{h_{i+1} - h_i}{3} f'(x_i) \right| \le \frac{\|f\|_\beta^{(\alpha/2)}}{3} Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{-\alpha/2-2/r}, & N < i \le 2N-1 \end{cases} \tag{5.6}$$

2. See Proof 11, there is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

$$\begin{aligned} &\left| \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \right| \\ &\le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases} \end{aligned} \tag{5.7}$$

Summarizes, we get the result. □

5.2. Estimate of R_i.

Now, we study the first part of (5.2)

$$D_h^2 (I^{2-\alpha} - I_h^{2-\alpha})(x_i) = D_h^2 (\int_0^{2T} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy) \tag{5.8}$$

For convience, let's denote

$$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \tag{5.9}$$

And define

$$\begin{aligned} R_i &:= D_h^2(I^{2-\alpha} - I_h^{2-\alpha})(x_i) \\ &= \frac{2}{h_i + h_{i+1}} \sum_{j=1}^{2N} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \end{aligned} \tag{5.10}$$

We have some results about the estimate of R_i

THEOREM 5.2. *For $1 \leq i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$R_i \leq C(h^{r\alpha/2+r} x_i^{-1-\alpha} + h^2 x_i^{-\alpha/2-2/r}) \tag{5.11}$$

THEOREM 5.3. *For $N/2 \leq i \leq N$, there exists constant C, C_2 such that*

$$R_i \leq Ch^2 x_i^{-\alpha/2-2/r} + C_2 h^2 |T - x_{i-1}|^{1-\alpha} \tag{5.12}$$

where $C_2 = 0$ if $r = 1$.

And for $N < i \leq 2N - 1$, it is symmetric to the previous case.

To prove these results, we need some utils. Also for simplicity, we denote

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \tag{5.13}$$

then

$$R_i = \sum_{j=1}^{2N} S_{ij} \tag{5.14}$$

5.3. Proof of Theorem 5.2.

LEMMA 5.4. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \leq i < N/2$,*

$$\sum_{j=\max\{2i+1, i+3\}}^{N} S_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r} \tag{5.15}$$

Proof. For $\max\{2i+1, i+3\} \leq j \leq N$, by Lemma C.1 and Lemma C.2

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) dy \\ &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy \end{aligned} \tag{5.16}$$

Therefore,

$$\begin{aligned} \sum_{j=\max\{2i+1, i+3\}}^{N} S_{ij} &\leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\ &= \frac{C}{\alpha/2 + 2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\ &\leq \frac{C}{\alpha/2 + 2/r} 2^{r(-\alpha/2-2/r)} h^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{5.17}$$

□

Lemma 5.5. *Thert exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $1 \leq i < N/2$,

$$\sum_{j=N+1}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{5.18}$$

Proof. For $1 \leq i < N/2, N+1 \leq j \leq 2N-1$, by equation (C.2) and Lemma C.2

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right)(x_i) dy \\ &\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2 - 2/r} y^{-1-\alpha} dy \\ &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy \end{aligned}$$

$$\begin{aligned} \sum_{j=N+1}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r} dy \\ &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\ &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r}\ h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned} \tag{5.19}$$

And by Lemma A.3

$$S_{i,2N} \leq CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}$$

And when $\alpha/2 - 2/r + 1 \geq 0$,

$$h^{r\alpha/2+r} \leq h^2$$

Summarizes, we get the result. ∎

For $i = 1, 2$.

Lemma 5.6. *By Lemma* C.5 , *Lemma* 5.4 *and Lemma* 5.5 *we get*

$$\begin{aligned} R_1 &= \sum_{j=1}^{3} S_{1j} + \sum_{j=4}^{2N} S_{1j} \\ &\leq Ch^2 x_1^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned} \tag{5.20}$$

$$\begin{aligned} R_2 &= \sum_{j=1}^{4} S_{2j} + \sum_{j=5}^{2N} S_{2j} \\ &\leq Ch^2 x_2^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1>0 \\ Ch^2\ln(N), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0 \end{cases} \end{aligned} \tag{5.21}$$

For $3 \leq i < N/2$, we have a new separation of R_i, Let's denote $k = \lceil \frac{i}{2} \rceil$.

$$\begin{aligned} R_i =& \sum_{j=1}^{2N} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& \sum_{j=1}^{k-1} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &+ \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\ &+ \sum_{j=k+1}^{2i-1} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &+ \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i-1}} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \\ &+ \sum_{j=2i+1}^{2N} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ =& I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned} \tag{5.22}$$

LEMMA 5.7. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \leq i < N/2, k = \lceil \frac{i}{2} \rceil$*

$$|I_1| = |\sum_{j=1}^{k-1} S_{ij}| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1>0 \\ Ch^2 x_i^{-1-\alpha}\ln(i), & \alpha/2-2/r+1=0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2-2/r+1<0 \end{cases} \tag{5.23}$$

Proof. For $2 \leq j \leq k-1$, by Lemma C.1 and Lemma C.3

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) D_h^2 \left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\ &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 x_i^{-1-\alpha} \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} dy \end{aligned} \tag{5.24}$$

And by Lemma A.3 , Lemma C.3

$$S_{i1} \leq Cx_1^{\alpha/2} x_1 x_i^{-1-\alpha} = Cx_1^{\alpha/2+1} x_i^{-1-\alpha} = CT^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha} \tag{5.25}$$

Therefore,

$$\begin{aligned} I_1 &= \sum_{j=1}^{k-1} S_{ij} = S_{i1} + \sum_{j=2}^{k-1} S_{ij} \\ &\le Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil -1}} y^{\alpha/2-2/r} dy \\ &\le Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{2^{-r}x_i} y^{\alpha/2-2/r} dy \end{aligned} \tag{5.26}$$

But

$$\int_{x_1}^{2^{-r}x_i} y^{\alpha/2-2/r} dy \le \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r}x_i)^{\alpha/2-2/r+1}, & \alpha/2-2/r+1>0 \\ \ln(2^{-r}x_i) - \ln(x_1), & \alpha/2-2/r+1=0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2-2/r+1<0 \end{cases} \tag{5.27}$$

So we have

$$I_1 \le \begin{cases} \frac{C}{\alpha/2-2/r+1} h^2 x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1>0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2-2/r+1=0 \\ \frac{C}{|\alpha/2-2/r+1|} h^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2-2/r+1<0 \end{cases} \qquad \square \tag{5.28}$$

For convience, let's denote

$$V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \tag{5.29}$$

THEOREM 5.8. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i < N/2, k = \lceil \frac{i}{2} \rceil$,

$$I_3 = \sum_{j=k+1}^{2i-1} V_{ij} \le Ch^2 x_i^{-\alpha/2-2/r} := Sorry \tag{5.30}$$

To estimete V_{ij}, we need some preparations.

LEMMA 5.9. *Denote* $y_j^\theta = \theta x_{j-1} + (1-\theta) x_j, \theta \in [0,1]$, *by Lemma* A.2

$$\begin{aligned} T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - u_h(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= \int_{x_{j-1}}^{x_j} -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!} h_j^3 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) dy_j^\theta \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta \end{aligned} \tag{5.31}$$

where $\eta_{j1}^\theta \in [x_{j-1}, y_j^\theta], \eta_{j2}^\theta \in [y_j^\theta, x_j]$.

Now Let's construct a series of functions to represent T_{ij}.

$$y_{j-i}(x) = (x^{1/r} + Z_{j-i})^r, \quad Z_{j-i} = T^{1/r}\frac{j-i}{N} \tag{5.32}$$

$$y^{\theta}_{j-i}(x) = \theta y_{j-1-i}(x) + (1-\theta) y_{j-i}(x) \tag{5.33}$$

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{5.34}$$

Now, we define

$$P^{\theta}_{j-i}(x) = (h_{j-i}(x))^3 u''(y^{\theta}_{j-i}(x)) \frac{|y^{\theta}_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.35}$$

$$Q^{\theta}_{j-i}(x) = (h_{j-i}(x))^4 \frac{|y^{\theta}_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{5.36}$$

And now we can rewrite T_{ij}

LEMMA 5.10. *For* $2 \le i \le N, 2 \le j \le N$,

$$\begin{aligned} T_{ij} = &\int_0^1 -\frac{\theta(1-\theta)}{2} P^{\theta}_{j-i}(x_i) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} (\theta^2 Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j1}) - (1-\theta)^2 Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j2})) d\theta \end{aligned} \tag{5.37}$$

Immediately, we can see from (5.29) that

LEMMA 5.11. *For* $3 \le i \le N-1$, $3 \le j \le N-1$,

$$\begin{aligned} V_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P^{\theta}_{j-i}(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_{i+1}) u'''(\eta^{\theta}_{j+1,1}) - Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,1})}{h_{i+1}} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,1}) - Q^{\theta}_{j-i}(x_{i-1}) u'''(\eta^{\theta}_{j-1,1})}{h_i} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_{i+1}) u'''(\eta^{\theta}_{j+1,2}) - Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,2})}{h_{i+1}} \right) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,2}) - Q^{\theta}_{j-i}(x_{i-1}) u'''(\eta^{\theta}_{j-1,2})}{h_i} \right) d\theta \end{aligned} \tag{5.38}$$

To estimate V_{ij}, we first estimate $D_h^2 P^{\theta}_{j-i}(x_i)$, but By Lemma A.1,

$$D_h^2 P^{\theta}_{j-i}(x_i) = {P^{\theta}_{j-i}}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \tag{5.39}$$

By Leibniz formula, we calculate and estimate the derivations of h^3_{j-i}, $u''(y^{\theta}_{j-i}(x))$ and $\frac{|y^{\theta}_{j-i}(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)}$ separately.

Firstly, we have

Lemma 5.12. *There exists a constant* $C = C(T, r)$ *such that For* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$h^3_{j-i}(\xi) \le Ch^2 x_i^{2-2/r} h_j \tag{5.40}$$

$$(h^3_{j-i}(\xi))' \le C(r-1)h^2 x_i^{1-2/r} h_j \tag{5.41}$$

$$(h^3_{j-i}(\xi))'' \le C(r-1)h^2 x_i^{-2/r} h_j \tag{5.42}$$

The proof of this theorem see Lemma C.6 and Lemma C.7

Second,

Lemma 5.13. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that For* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$u''(y^\theta_{j-i}(\xi)) \le Cx_i^{\alpha/2-2} \tag{5.43}$$

$$(u''(y^\theta_{j-i}(\xi)))' \le Cx_i^{\alpha/2-3} \tag{5.44}$$

$$(u''(y^\theta_{j-i}(\xi)))'' \le Cx_i^{\alpha/2-4} \tag{5.45}$$

The proof of this theorem see Proof 18

And Finally, we have

Lemma 5.14. *There exists a constant* $C = C(T, \alpha, r)$ *such that For* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, $\xi \in [x_{i-1}, x_{i+1}]$,

$$|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha} \le C|y^\theta_j - x_i|^{1-\alpha} \tag{5.46}$$

$$(|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha})' \le C|y^\theta_j - x_i|^{1-\alpha} x_i^{-1} \tag{5.47}$$

$$(|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha})'' \le C|y^\theta_j - x_i|^{1-\alpha} x_i^{-2} \tag{5.48}$$

where $y^\theta_j = \theta x_{j-1} + (1-\theta)x_j$

The proof of this theorem see Proof 19

Lemma 5.15. *There exists a constant* $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ *such that For* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$,

$$D^2_h P^\theta_{j-i}(x_i) \le Ch^2 \frac{|y^\theta_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \tag{5.49}$$

where $y^\theta_j = \theta x_{j-1} + (1-\theta)x_j$

Proof. Since

$$D^2_h P^\theta_{j-i}(x_i) = {P^\theta_{j-i}}''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \tag{5.50}$$

From (5.35), using Leibniz formula and Lemma 5.12, Lemma 5.13 and Lemma 5.14 ∎

Lemma 5.16. *There exists a constant* $C = C(T, \alpha, r, f)$ *such that for* $3 \le i < N, k = \lceil \frac{i}{2} \rceil, k+1 \le j \le \min\{2i-1, N-1\}$,

$$V_{ij} \le \mathit{Sorry} \tag{5.51}$$

Proof.

6. Proof of Theorem 4.2.

7. Experimental results.

8. Conclusions.

Some conclusions here.

Appendix A. Approximate of difference quotients.

Lemma A.1. *If $g(x)$ is twice differentable continous function on open set Ω, there exists $\xi \in [x_{i-1}, x_{i+1}]$ such that*

$$\begin{aligned} D_h^2 g(x_i) &:= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &= g''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \end{aligned} \tag{A.1}$$

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\quad = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right) \end{aligned} \tag{A.2}$$

And if $g(x) \in C^4(\Omega)$, then

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\qquad = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2)) \end{aligned} \tag{A.3}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$.

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in [x_{i-1}, x_i]$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in [x_i, x_{i+1}]$$

Subsitute them in the left side of (A.1), we have

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\qquad = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) \end{aligned}$$

Now, using intermediate value theorem , there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y)dy$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \frac{h_i^4}{4!} g''''(\eta_1)$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \frac{h_{i+1}^4}{4!} g''''(\eta_2)$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. Expecially,

$$\begin{aligned} \frac{h_i^4}{4!} g''''(\eta_1) &= \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy \\ \frac{h_{i+1}^4}{4!} g''''(\eta_2) &= \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \end{aligned} \tag{A.4}$$

Subsitute them to the left side of (A.3), we can get the result. ∎

LEMMA A.2. *If $y \in [x_{j-1}, x_j]$, denote $y = \theta x_{j-1} + (1-\theta)x_j, \theta \in [0,1]$,*

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \tag{A.5}$$

(A.6)

$$u(y_j^\theta) - u_h(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

Thus

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta) u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$. Thus ∎

$$\begin{aligned} u(y_j^\theta) - u_h(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

LEMMA A.3. *For $x \in [x_{j-1}, x_j]$*

$$\begin{aligned} |u(x) - u_h(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_{x}^{x_j} u'(y)dy \right| \\ &\leq \int_{x_{j-1}}^{x_j} |u'(y)|dy \end{aligned} \tag{A.7}$$

If $x \in [0, x_1]$, with Corollary 2.4, we have

$$|u(x) - u_h(x)| \leq \int_0^{x_1} |u'(y)|dy \leq \int_0^{x_1} Cy^{\alpha/2-1}dy \leq C\frac{2}{\alpha}x_1^{\alpha/2} \tag{A.8}$$

Similarly, if $x \in [x_{2N-1}, 1]$, we have

$$|u(x) - u_h(x)| \leq C\frac{2}{\alpha}(2T - x_{2N-1})^{\alpha/2} = C\frac{2}{\alpha}x_1^{\alpha/2} \tag{A.9}$$

Appendix B. Inequality.

LEMMA B.1.

$$h_i \leq rT^{1/r}h \begin{cases} x_i^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_{i-1})^{1-1/r}, & N < i \leq 2N-1 \end{cases} \tag{B.1}$$

$$h_i \geq rT^{1/r}h \begin{cases} x_{i-1}^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_i)^{1-1/r}, & N < i \leq 2N-1 \end{cases} \tag{B.2}$$

Proof. For $1 \leq i \leq N$,

$$\begin{aligned} h_i &= T\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right) \\ &\leq rT\frac{1}{N}\left(\frac{i}{N}\right)^{r-1} = rT^{1/r}hx_i^{1-1/r} \end{aligned}$$

$$h_i \geq rT\frac{1}{N}\left(\frac{i-1}{N}\right)^{r-1} = rT^{1/r}hx_{i-1}^{1-1/r}$$

For $N < i \leq 2N$,

$$\begin{aligned} h_i &= T\left(\left(\frac{2N-i+1}{N}\right)^r - \left(\frac{2N-i}{N}\right)^r\right) \\ &\leq rT\frac{1}{N}\left(\frac{2N-i+1}{N}\right)^{r-1} = rT^{1/r}h(2T - x_{i-1})^{1-1/r} \end{aligned}$$

$$h_i \geq rT\frac{1}{N}\left(\frac{2N-i}{N}\right)^{r-1} = rT^{1/r}h(2T - x_i)^{1-1/r}$$

□

Lemma B.2. *There is a constant* $C = 2^{|r-2|}r(r-1)T^{2/r}$ *such that for all* $i \in \{1, 2, \cdots, 2N-1\}$

$$|h_{i+1} - h_i| \le Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N-1 \end{cases} \tag{B.3}$$

Proof.

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \le i \le N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \le i \le 2N-1 \end{cases}$$

For $i = 1$,

$$h_2 - h_1 = T(2^r - 2)\left(\frac{1}{N}\right)^r = (2^r - 2)T^{2/r}h^2x_1^{1-2/r}$$

For $2 \le i \le N-1$,

$$h_{i+1} - h_i = r(r-1)T\ N^{-2}\eta^{r-2}, \quad \eta \in [\frac{i-1}{N}, \frac{i+1}{N}]$$

If $r \in [1, 2]$,

$$\begin{aligned} h_{i+1} - h_i = r(r-1)T\ N^{-2}\eta^{r-2} &\le r(r-1)T\ h^2\left(\frac{i-1}{N}\right)^{r-2} \\ &\le r(r-1)T\ h^2 2^{2-r}\left(\frac{i}{N}\right)^{r-2} \\ &= 2^{2-r}r(r-1)T^{2/r}h^2x_i^{1-2/r} \end{aligned}$$

else if $r > 2$,

$$\begin{aligned} h_{i+1} - h_i = r(r-1)T\ N^{-2}\eta^{r-2} &\le r(r-1)T\ h^2\left(\frac{i+1}{N}\right)^{r-2} \\ &\le r(r-1)T\ h^2 2^{r-2}\left(\frac{i}{N}\right)^{r-2} \\ &= 2^{r-2}r(r-1)T^{2/r}h^2x_i^{1-2/r} \end{aligned}$$

Since

$$2^r - 2 \le 2^{|r-2|}r(r-1), \quad r \ge 1$$

we have

$$h_{i+1} - h_i \le 2^{|r-2|}r(r-1)T^{2/r}h^2x_i^{1-2/r}, \quad 1 \le i \le N-1$$

For $i = N$, $h_{N+1} - h_N = 0$. For $N < i \le 2N-1$, it's central symmetric to the first half of the proof, which is

$$h_i - h_{i+1} \le 2^{|r-2|}r(r-1)T^{2/r}h^2(2T - x_i)^{1-2/r}$$

Summarizes the inequalities, we can get

$$\text{(B.4)} \qquad |h_{i+1}-h_i| \le 2^{|r-2|}r(r-1)T^{2/r}h^2 \begin{cases} x_i^{1-2/r}, & 1\le i\le N-1 \\ 0, & i=N \\ (2T-x_i)^{1-2/r}, & N<i\le 2N-1 \end{cases} \qquad \square$$

Appendix C. Proofs of some technical details.

Additional proof of Theorem 5.1. For $2\le i\le N-1$,

$$\begin{aligned} &\frac{2}{h_i+h_{i+1}}(h_i^3f''(\eta_1)+h_{i+1}^3f''(\eta_2)) \\ &\qquad \le C\frac{2}{h_i+h_{i+1}}(h_i^3x_{i-1}^{-2-\alpha/2}+h_{i+1}^3x_i^{-2-\alpha/2}) \\ &\qquad \le 2C(h_i^2x_{i-1}^{-2-\alpha/2}+h_{i+1}^2x_i^{-2-\alpha/2}) \end{aligned}$$

Since Lemma B.1, we have

$$\begin{aligned} h_i &\le rT^{1/r}hx_i^{1-1/r}, \quad 1\le i\le N \\ h_{i+1} &\le rT^{1/r}hx_{i+1}^{1-1/r}, \quad 1\le i\le N-1 \end{aligned}$$

and

$$\begin{aligned} x_{i-1}^{-2-\alpha/2} &\le 2^{-r(-2-\alpha/2)}x_i^{-2-\alpha/2} \quad 2\le i\le N-1 \\ x_{i+1}^{1-1/r} &\le 2^{r-1}x_i^{1-1/r} \quad 1\le i\le N-1 \end{aligned}$$

So there is a constant $C=C(T,\alpha,r,\|f\|_\beta^{\alpha/2})$ such that

$$\frac{2}{h_i+h_{i+1}}(h_i^3f''(\eta_1)+h_{i+1}^3f''(\eta_2)) \le Ch^2x_i^{-\alpha/2-2/r}, \quad 2\le i\le N-1$$

For $i=1$, by (A.4)

$$\begin{aligned} &\frac{1}{4!}\frac{2}{h_1+h_2}(h_1^3f''(\eta_1)+h_2^3f''(\eta_2)) \\ &\qquad = \frac{2}{h_1+h_2}\left(\frac{1}{h_1}\int_0^{x_1}f''(y)\frac{y^3}{3!}dy+\frac{1}{4!}h_2^3f''(\eta_2)\right) \end{aligned}$$

We have proved above that

$$\frac{2}{h_1+h_2}h_2^3f''(\eta_2) \le Ch^2x_1^{-\alpha/2-2/r}$$

and we can get

$$\begin{aligned} \int_0^{x_1}f''(y)\frac{y^3}{3!}dy &\le C\frac{1}{3!}\int_0^{x_1}y^{1-\alpha/2}dy \\ &= C\frac{1}{3!(2-\alpha/2)}x_1^{2-\alpha/2} \end{aligned}$$

so

$$\frac{2}{h_1+h_2}\frac{1}{h_1}\int_0^{x_1}f''(y)\frac{y^3}{3!}dy = \frac{C2^{1-r}}{3!(2-\alpha/2)}x_1^{-\alpha/2} = \frac{C2^{1-r}}{3!(2-\alpha/2)}T^{2/r}h^2x_1^{-\alpha/2-2/r}$$

And for $i = N$, we have

$$\begin{aligned}
&\frac{2}{h_N + h_{N+1}}(h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2)) \\
&\qquad = h_N^2(f''(\eta_1) + f''(\eta_2)) \\
&\qquad \le r^2 T^{2/r} h^2 x_N^{2-2/r}\ 2C x_{N-1}^{-2-\alpha/2} \\
&\qquad \le 2r^2 T^{2/r} C 2^{-r(-2-\alpha/2)}\ h^2 x_N^{-\alpha/2-2/r}
\end{aligned}$$

Finally, $N + 1 \le i \le 2N - 1$ is symmetric to the first half of the proof, so we can conclude that

□

$$\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N - 1 \end{cases}$$

Lemma C.1. *There is a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *for* $2 \le j \le N$, *if* $y \in [x_{j-1}, x_j]$,

$$|u(y) - u_h(y)| \le Ch^2 y^{\alpha/2-2/r} \tag{C.1}$$

Proof. For $2 \le j \le N$, we have

$$x_j \le 2^r y, \quad x_{j-1} \ge 2^{-r} y$$

And by Lemma A.2, Lemma B.1 and Corollary 2.4, we have

$$\begin{aligned}
u(y) - u_h(y) &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j] \\
&\le \frac{\|u\|_{\beta+\alpha}^{(-\alpha/2)}}{4} r^2 T^{2/r}\ h^2 x_j^{2-2/r} x_{j-1}^{\alpha/2-2} \\
&\le Ch^2\ 2^{2r-2} y^{2-2/r}\ 2^{-r(\alpha/2-2)} y^{\alpha/2-2} \\
&= C 2^{-r\alpha/2+4r-2}\ h^2 y^{\alpha/2-2/r}
\end{aligned}$$

symmetricly, for $N < j \le 2N - 1$, we have

$$|u(y) - u_h(y)| \le Ch^2 (2T - y)^{\alpha/2-2/r} \tag{C.2}$$

□

Lemma C.2. *There is a constant* $C = C(\alpha, r)$ *such that for all* $1 \le i < N/2$, $\max\{2i + 1, i + 3\} \le j \le 2N$ *and* $y \in [x_{j-1}, x_j]$, *we have*

$$D_h^2\left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) \le C \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} \tag{C.3}$$

Proof. Sinec $y \ge x_{j-1} > x_{i+1}$, by Lemma A.1, if $j - 1 > i + 1$

□

$$\begin{aligned}
D_h^2\left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) &= \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\
&\le \frac{(y - x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)} \\
&\le (1 - (\frac{2}{3})^r)^{-1-\alpha} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}
\end{aligned}$$

Lemma C.3. *There is a constant $C = C(\alpha, r)$ such that for all $3 \le i < N/2, k = \lceil \frac{i}{2} \rceil$, $1 \le j \le k-1$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_h^2\left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) \le C\frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \tag{C.4}$$

Proof. Sinec $y \le x_j < x_{i-1}$, by Lemma A.1, □

$$\begin{aligned} D_h^2\left(\frac{|\cdot - y|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) &= \frac{|\xi - y|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\le \frac{(x_{i-1} - x_j)^{-1-\alpha}}{\Gamma(-\alpha)} \le \frac{(x_{i-1} - x_{k-1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\le \left(\left(\frac{2}{3}\right)^r - \left(\frac{1}{2}\right)^r\right)^{-1-\alpha} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

Lemma C.4. *While $0 \le i < N/2$, By Lemma A.3*

$$\begin{aligned} |T_{i1}| &\le C \int_0^{x_1} x_1^{\alpha/2} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= C\frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2} \left| x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha} \right| \\ &\le C\frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2+2-\alpha} = C\frac{1}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1 \end{aligned} \tag{C.5}$$

For $2 \le j \le N$, by Lemma A.2 and Corollary 2.4

$$\begin{aligned} |T_{ij}| &\le \frac{C}{4} \int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &\le \frac{C}{4\Gamma(3-\alpha)} h_j^2 x_{j-1}^{\alpha/2-2} \left| |x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha} \right| \end{aligned} \tag{C.6}$$

Lemma C.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$\sum_{j=1}^{3} S_{1j} \le Ch^2 x_1^{-\alpha/2-2/r} \tag{C.7}$$

$$\sum_{j=1}^{4} S_{2j} \le Ch^2 x_2^{-\alpha/2-2/r} \tag{C.8}$$

Proof.

$$S_{1j} = \frac{2}{x_2}\left(\frac{1}{x_1}T_{0j} - \left(\frac{1}{x_1} + \frac{1}{h_2}\right)T_{1j} + \frac{1}{h_2}T_{2j}\right)$$

So, by Lemma C.4

$$S_{11} \le \frac{2}{x_2 x_1} 4 \frac{C}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \le C x_1^{-\alpha/2}$$

$$S_{12} \le \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_2^2 x_1^{\alpha/2-2} \left(x_2^{2-\alpha} + 2h_2^{2-\alpha} + h_2^{2-\alpha} \right) \le C x_1^{-\alpha/2}$$

$$S_{13} \le \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_3^2 x_2^{\alpha/2-2} \left(x_3^{2-\alpha} + 2x_3^{2-\alpha} + h_3^{2-\alpha} \right) \le C x_1^{-\alpha/2}$$

But

$$x_1^{-\alpha/2} = T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

For $i = 2$, Sorry $\square$

Lemma C.6. *There exists a constant $C = C(T, r, l)$ such that For $3 \le i \le N - 1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, $l = 3, 4$, when $\xi \in [x_{i-1}, x_{i+1}]$,*

$$(h_{j-i}^3(\xi))' \le (r-1) C h^2 x_i^{1-2/r} h_j \tag{C.9}$$

$$(h_{j-i}^4(\xi))' \le (r-1) C h^3 x_i^{2-3/r} h_j \tag{C.10}$$

Proof. From (5.32)

$$y'_{j-i}(x) = y_{j-i}^{1-1/r}(x) x^{1/r-1} \tag{C.11}$$

$$y''_{j-i}(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i} \tag{C.12}$$

for $l = 3, 4$, by (5.34)

$$\begin{aligned} (h_{j-i}^l(\xi))' &= l\, h_{j-i}^{l-1}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi)) \\ &= l\, h_{j-i}^{l-1}(\xi) \xi^{1/r-1} (y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \ge 0 \end{aligned} \tag{C.13}$$

For $\xi \in [x_{i-1}, x_{i+1}]$ and $3 \le \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$, using Lemma B.1

$$\begin{aligned} h_{j-i}(\xi) \le h_{j-i}(x_{i+1}) &= h_{j+1} \\ &\le r T^{1/r}\, h x_{j+1}^{1-1/r} \le r T^{1/r} 2^{r-1}\, h x_i^{1-1/r} \end{aligned}$$

And

$$2^{-r} x_i \le x_{i-1} \le \xi \le x_{i+1} \le 2^r x_i \tag{C.14}$$

We have

$$\xi^{1/r-m} \le 2^{|mr-1|} x_i^{1/r-m}, \quad m = 1, 2 \tag{C.15}$$

but

$$\begin{aligned} y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) &= (\xi^{1/r} + Z_{j-i})^{r-1} - (\xi^{1/r} + Z_{j-i-1})^{r-1} \\ &= (r-1) Z_1 (\xi^{1/r} + Z_{j-i-\gamma})^{r-2}, \quad \gamma \in [0, 1] \\ &= (r-1) T^{1/r} h y_{j-i-\gamma}^{1-2/r}(\xi) \end{aligned} \tag{C.16}$$

And

$$4^{-r}x_i \le x_{\lceil \frac{i}{2}\rceil -1} \le x_{j-2} = y_{j-i-1}(x_{i-1}) \le y_{j-i-\gamma}(\xi) \le y_{j-i}(x_{i+1}) = x_{j+1} \le x_{2i} \le 2^r x_i \tag{C.17}$$

Therefore,

$$y_{j-i-\gamma}^{1-2/r}(\xi) \le 2^{2|r-2|} x_i^{1-2/r} \tag{C.18}$$

So we can get

$$y'_{j-i}(\xi) - y'_{j-i-1}(\xi) \le (r-1)C(T,r)hx_i^{-1/r} \tag{C.19}$$

We get

$$(h_{j-i}^l(\xi))' \le l(r-1)C\ h_{j+1}^{l-1} h x_i^{-1/r} \tag{C.20}$$

And by Lemma B.1,

$$h_{j+1} \le rTh\left(\frac{j+1}{N}\right)^{r-1} \le rTh2^{r-1}\left(\frac{j-1}{N}\right) = 2^{r-1}h_j \tag{C.21}$$

$$h_{j+1} \le rT^{1/r}hx_{j+1}^{1-1/r} \le rT^{1/r}hx_{2i}^{1-1/r} \le rT^{1/r}2^{r-1}hx_i^{1-1/r} \tag{C.22}$$

We can get

$$\begin{aligned}(h_{j-i}^l(\xi))' &\le l(r-1)C\ h_j h_{j+1}^{l-2} h x_i^{-1/r}\\ &\le l(r-1)Chh_j(hx_i^{1-1/r})^{l-2}x_i^{-1/r}\\ &= (r-1)C\ h^{l-1}x_i^{l-2-(l-1)/r}h_j\end{aligned} \tag{C.23}$$

Meanwhile, we can get

$$h_{j-i}^3(\xi) \le h_{j+1}^3 \le Ch^2 x_i^{2-2/r} h_j \tag{C.24}$$

□

Lemma C.7. *There exists a constant $C = C(T,r,l)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2}\rceil + 1 \le j \le \min\{2i-1, N-1\}$, when $\xi \in [x_{i-1}, x_{i+1}]$,*

$$(h_{j-i}^3(\xi))'' \le C(r-1)h^2 x_i^{-2/r} h_j \tag{C.25}$$

Proof. From (C.11)

$$\begin{aligned}(h_{j-i}^3(\xi))'' &= 6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 + 3h_{j-i}^2(\xi)(y''_{j-i}(\xi) - y''_{j-i-1}(\xi))\\ &= 6h_{j-i}(\xi)\xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi))\\ &\quad + 3\frac{1-r}{r}h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1})\end{aligned} \tag{C.26}$$

Using the inequalities of the proof of Lemma C.6

$$\begin{aligned}&6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2\\ &\le 6h_{j+1}((r-1)Chx_i^{-1/r})^2\\ &\le C(r-1)^2\ h^2x_i^{-2/r}h_j\end{aligned} \tag{C.27}$$

For the second partial

$$\begin{aligned} &h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\ &\qquad \le Ch_{j+1}^2 x_i^{1/r-2}((y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi))Z_{j-i} + y_{j-i-1}^{1-2/r}(\xi)Z_1) \end{aligned} \tag{C.28}$$

but

$$\begin{aligned} y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi) &= (\xi^{1/r} + Z_{j-i})^{r-2} - (\xi^{1/r} + Z_{j-i-1})^{r-2} \\ &= (r-2)Z_1(\xi^{1/r} + Z_{j-i-\gamma})^{r-3} \\ &= (r-2)T^{-r}hy_{j-i-\gamma}^{1-3/r}(\xi) \\ &\le C(r-2)hx_i^{1-3/r} \end{aligned} \tag{C.29}$$

So we can get

$$\begin{aligned} &h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\ &\qquad \le Ch_jhx_i^{1-1/r}x_i^{1/r-2}(C(r-2)hx_i^{1-3/r}Z_{j-i} + Cx_i^{1-2/r}T^{1/r}h) \\ &\qquad \le Ch^2((r-2)x_i^{-3/r}x_{|j-i|}^{1/r} + x_i^{-2/r})h_j \\ &\qquad \le Ch^2x_i^{-2/r}h_j \end{aligned} \tag{C.30}$$

Summarizes, we have

$$(h_{j-i}^3(\xi))'' \le C(r-1)h^2x_i^{-2/r}h_j \tag{C.31}$$

□

proof of Lemma 5.13. From (5.32)

$$y_{j-i}'(x) = y_{j-i}^{1-1/r}(x)x^{1/r-1} \tag{C.32}$$

$$y_{j-i}''(x) = \frac{1-r}{r}y_{j-i}^{1-2/r}(x)x^{1/r-2}Z_{j-i} \tag{C.33}$$

Since

$$x_{j-2} \le y_{j-i-1}(x_{i-1}) \le y_{j-i}^{\theta}(\xi) \le y_{j-i-1}^{\theta}(x_{i+1}) \le x_{j+1}$$

We have known (C.17)

$$u''(y_{j-i}^{\theta}(\xi)) \le C(y_{j-i}^{\theta}(\xi))^{\alpha/2-2} \le Cx_{j-2}^{\alpha/2-2} \le Cx_{\lceil \frac{i}{2}\rceil -1}^{\alpha/2-2} \le C4^{r(2-\alpha/2)}x_i^{\alpha/2-2} \tag{C.34}$$

$$\begin{aligned} (u''(y_{j-i}^{\theta}(\xi)))' &= u'''(y_{j-i}^{\theta}(\xi)){y_{j-i}^{\theta}}'(\xi) \\ &\le Cx_i^{\alpha/2-3}\xi^{1/r-1}y_{j-i}^{1-1/r}(\xi) \\ &\le Cx_i^{\alpha/2-3}x_i^{1/r-1}x_i^{1-1/r} = Cx_i^{\alpha/2-3} \end{aligned} \tag{C.35}$$

$$\begin{aligned} (u''(y_{j-i}^{\theta}(\xi)))'' &= u''''(y_{j-i}^{\theta}(\xi))({y_{j-i}^{\theta}}''(\xi))^2 + u'''(y_{j-i}^{\theta}(\xi)){y_{j-i}^{\theta}}''(\xi) \\ &\le Cx_i^{\alpha/2-4} + Cx_i^{\alpha/2-3}\frac{r-1}{r}x_i^{1-2/r}x_i^{1/r-2}Z_{|j-i|+1} \\ &\le Cx_i^{\alpha/2-4} + C\frac{r-1}{r}x_i^{\alpha/2-3}x_i^{-1/r}x_i^{1/r} \\ &= Cx_i^{\alpha/2-4} \end{aligned} \tag{C.36}$$

□

Proof of Lemma 5.14.

$$
\begin{aligned}
|y^\theta_{j-i}(\xi)-\xi| &= |\theta(y_{j-i-1}(\xi)-\xi)+(1-\theta)(y_{j-i}(\xi)-\xi)| \\
&= \theta|y_{j-i-1}(\xi)-\xi|+(1-\theta)|y_{j-i}(\xi)-\xi|
\end{aligned}
\tag{C.37}
$$

Since $|y_{j-i}(\xi)-\xi|$ is increasing about ξ, we have

$$
(\frac{i-1}{i})^r|x_j-x_i| \le |x_{j-1}-x_{i-1}| \le |y_{j-i}(\xi)-\xi| \le |x_{j+1}-x_{i+1}| \le (\frac{i+1}{i})^r|x_j-x_i|
\tag{C.38}
$$

Thus,

$$
(\frac{2}{3})^r|y^\theta_j-x_i| \le |y^\theta_{j-i}(\xi)-\xi| \le (\frac{3}{4})^r(\theta|x_j-x_i|+(1-\theta)|x_{j-1}-x_i|) = (\frac{3}{4})^r|y^\theta_j-x_i|
\tag{C.39}
$$

$$
|y^\theta_{j-i}(\xi)-\xi|^{1-\alpha} \le C|y^\theta_j-x_i|^{1-\alpha}
\tag{C.40}
$$

Next,

$$
\begin{aligned}
(|y^\theta_{j-i}(\xi)-\xi|^{1-\alpha})' &= (1-\alpha)|y^\theta_{j-i}(\xi)-\xi|^{-\alpha}|\xi^{1/r-1}(\theta y^{1-1/r}_{j-i-1}(\xi)+(1-\theta)y^{1-1/r}_{j-i}(\xi))-1| \\
&\le C|y^\theta_j-x_i|^{-\alpha}\xi^{1/r-1}|\theta y^{1-1/r}_{j-i-1}(\xi)+(1-\theta)y^{1-1/r}_{j-i}(\xi)-\xi^{1-1/r}|
\end{aligned}
\tag{C.41}
$$

Similar with (C.39), we have

$$
\begin{aligned}
|y^{1-1/r}_{j-i}(\xi)-\xi^{1-1/r}| &\le C|x^{1-1/r}_j-x^{1-1/r}_i| \\
&\le C|x_j-x_i|x_i^{-1/r}
\end{aligned}
\tag{C.42}
$$

So we can get

$$
\begin{aligned}
&|\theta y^{1-1/r}_{j-i-1}(\xi)+(1-\theta)y^{1-1/r}_{j-i}(\xi)-\xi^{1-1/r}| \\
&\quad \le Cx_i^{-1/r}(\theta|x_{j-1}-x_i|+(1-\theta)|x_j-x_i|) \\
&\quad = Cx_i^{-1/r}|y^\theta_j-x_i|
\end{aligned}
\tag{C.43}
$$

Combine them, we get

$$
\begin{aligned}
(|y^\theta_{j-i}(\xi)-\xi|^{1-\alpha})' &\le C|y^\theta_j-x_i|^{-\alpha}x_i^{1/r-1}x_i^{-1/r}|y^\theta_j-x_i| \\
&= C|y^\theta_j-x_i|^{1-\alpha}x_i^{-1}
\end{aligned}
\tag{C.44}
$$

Finally, we have

$$
\begin{aligned}
(|y^\theta_{j-i}(\xi)-\xi|^{1-\alpha})'' &= \alpha(\alpha-1)|y^\theta_{j-i}(\xi)-\xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y^{1-1/r}_{j-i-1}(\xi)+(1-\theta)y^{1-1/r}_{j-i}(\xi))-1)^2 \\
&\quad +(1-\alpha)|y^\theta_{j-i}(\xi)-\xi|^{-\alpha}\frac{1-r}{r}\xi^{1/r-2}|\theta y^{1-2/r}_{j-i-1}(\xi)Z_{j-i-1}+(1-\theta)y^{1-2/r}_{j-i}(\xi)Z_{j-i}|
\end{aligned}
\tag{C.45}
$$

Using the inequalities above ,we have

$$
\begin{aligned}
&|y^\theta_{j-i}(\xi)-\xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y^{1-1/r}_{j-i-1}(\xi)+(1-\theta)y^{1-1/r}_{j-i}(\xi))-1)^2 \\
&\quad \le C|y^\theta_j-x_i|^{-\alpha-1}(x_i^{-1}|y^\theta_j-x_i|)^2 \\
&\quad = C|y^\theta_j-x_i|^{1-\alpha}x_i^{-2}
\end{aligned}
\tag{C.46}
$$

And by

$$|Z_{j-i}| = |x_j^{1/r} - x_i^{1/r}| \leq |x_j - x_i| x_i^{1/r-1} \tag{C.47}$$

we have

$$\begin{aligned}
&|y^{\theta}_{j-i}(\xi) - \xi|^{-\alpha} \xi^{1/r-2} |\theta y^{1-2/r}_{j-i-1}(\xi) Z_{j-i-1} + (1-\theta) y^{1-2/r}_{j-i}(\xi) Z_{j-i}| \\
&\leq C |y^{\theta}_j - x_i|^{-\alpha} x_i^{1/r-2} x_i^{1-2/r} |\theta Z_{j-i-1} + (1-\theta) Z_{j-i}| \\
&\leq C |y^{\theta}_j - x_i|^{-\alpha} x_i^{-2} |y^{\theta}_j - x_i| \\
&= C |y^{\theta}_j - x_i|^{1-\alpha} x_i^{-2}
\end{aligned} \tag{C.48}$$

□

Acknowledgments. We would like to acknowledge the assistance of volunteers in putting together this example manuscript and supplement.

REFERENCES

[1] X. Ros-Oton and J. Serra, *The dirichlet problem for the fractional laplacian: Regularity up to the boundary*, Journal de Mathématiques Pures et Appliquées, 101 (2014), pp. 275–302, https://doi.org/https://doi.org/10.1016/j.matpur.2013.06.003, https://www.sciencedirect.com/science/article/pii/S0021782413000895.