A SECOND ORDER NUMERICAL METHODS FOR REISZ-FRACTIONAL ELLIPTIC EQUATION ON GRADED MESH[*]

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Key words. example, LaTeX

MSC codes. ??????????????????

1. Introduction.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$,

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

2. Preliminaries: Numeric scheme and main results.

2.1. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases} \tag{2.1}$$

where $r \ge 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N \tag{2.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

And then, define the piecewise linear interpolant of the true solution u to be

$$\Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{2.4}$$

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For convience, we denote

$$I^{2-\alpha}u(x) := \frac{1}{\Gamma(2-\alpha)}\int_\Omega |x-y|^{1-\alpha}u(y)dy \tag{2.5}$$

and

$$D_h^2 u(x_i) := \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}u(x_{i-1}) - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)u(x_i) + \frac{1}{h_{i+1}}u(x_{i+1})\right) \tag{2.6}$$

Now, we discretise (1.1) by replacing $u(x)$ by a continuous piecewise linear function

$$u_h(x) := \sum_{j=1}^{2N-1} u_j\phi_j(x) \tag{2.7}$$

whose nodal values u_j are to be determined by collocation at each mesh point x_i for $i = 1, 2, ..., 2N-1$:

$$-\kappa_\alpha D_h^\alpha u_h(x_i) := -\kappa_\alpha D_h^2 I^{2-\alpha}u_h(x_i) = f(x_i) =: f_i \tag{2.8}$$

Here,

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i)\ u_j = \sum_{j=1}^{2N-1} a_{ij}\ u_j \tag{2.9}$$

where

$$a_{ij} = -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i) \quad \text{for} \quad i, j = 1, 2, ..., 2N-1 \tag{2.10}$$

We have replaced $(-\Delta)^{\alpha/2}u(x_i) = f(x_i)$ in (1.1) by $-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i)$ in (2.8), with truncation error

$$\tau_i := -\kappa_\alpha\left(D_h^\alpha \Pi_h u(x_i) - \frac{d^2}{dx^2}I^{2-\alpha}u(x_i)\right) \quad \text{for} \quad i = 1, 2, ..., 2N-1 \tag{2.11}$$

where $-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^\alpha \phi_j(x_i)u(x_j) = \sum_{j=1}^{2N-1} a_{ij}u(x_j)$.

The discrete equation (2.8) can be written in matrix form

$$AU = F \tag{2.12}$$

where $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$, $U = (u_1, \cdots, u_{2N-1})^T$ is unknown and $F = (f_1, \cdots, f_{2N-1})^T$.

We can deduce a_{ij},

$$\begin{aligned} a_{ij} &= -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i) \\ &= -\kappa_\alpha \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}\tilde{a}_{i-1,j} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)\tilde{a}_{i,j} + \frac{1}{h_{i+1}}\tilde{a}_{i+1,j}\right) \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \tilde{a}_{ij} &= I^{2-\alpha}\phi_j(x_i) \\ &= \frac{1}{\Gamma(4-\alpha)}\left(\frac{|x_i-x_{j-1}|^{3-\alpha}}{h_j} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)|x_i-x_j|^{3-\alpha} + \frac{|x_i-x_{j+1}|^{3-\alpha}}{h_{j+1}}\right) \end{aligned} \tag{2.14}$$

We shall finally introduce some notations.

For convenience, we use the notation $\simeq$. That $x_1 \simeq y_1$, means that $c_1 x_1 \le y_1 \le C_1 x_1$ for some constants c_1 and C_1 that are independent of N.

Meanwhile, let's define kernel functions

$$K_y(x) := \frac{|y-x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{2.15}$$

We define the difference quotients

$$D_h g(x_i) := \frac{g(x_{i+1}) - g(x_i)}{h_{i+1}}, \quad D_{\bar{h}} g(x_i) := \frac{g(x_i) - g(x_{i-1})}{h_i} \tag{2.16}$$

Thus

$$D_h g(x_i) = D_{\bar{h}} g(x_{i+1})$$

$$D_h^2 g(x_i) = \frac{2}{h_i + h_{i+1}} \left(D_h g(x_i) - D_{\bar{h}} g(x_i) \right) = \frac{2}{h_i + h_{i+1}} \left(D_h g(x_i) - D_h g(x_{i-1}) \right)$$

And for $j = 1, 2, ..., 2N$, we define

$$y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \quad \theta \in (0,1) \tag{2.17}$$

2.2. Regularity of the true solution. For any $\beta > 0$, we use the standard notation $C^\beta(\bar{\Omega}), C^\beta(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^\beta(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous[1] with exponent β' in Ω, where $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on open set Ω.

Definition 2.1 (delta dependent norm [2]). ...

Lemma 2.2. *Let $f \in C^\beta(\Omega), \beta > 2$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

$$|f^{(l)}(x)| \le \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \le T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \le x < 2T \end{cases} \tag{2.18}$$

Theorem 2.3 (Regularity up to the boundary [2]). *Let Ω be a bounded domain, and $\beta > 0$ be such that neither β nor $\beta + \alpha$ is an integer. Let $f \in C^\beta(\Omega)$ be such that $\|f\|_\beta^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R}^n)$ be a solution of (1.1). Then, $u \in C^{\beta+\alpha}(\Omega)$ and*

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \le C \left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right) \tag{2.19}$$

where C is a constant depending only on Ω, α, and β.

Corollary 2.4. *Let u be a solution of (1.1) where $f \in L^\infty(\Omega)$ and $\|f\|_\beta^{(\alpha/2)} < \infty$. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \le \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \le T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \le x < 2T \end{cases} \tag{2.20}$$

And in this paper bellow, without special instructions, we allways assume that

$$f \in L^\infty(\Omega) \cap C^\beta(\Omega) \quad \text{and} \quad \|f\|_\beta^{(\alpha/2)} < \infty, \text{ with } \alpha + \beta > 4 \tag{2.21}$$

2.3. Main results. Here we state our main results; the proof is deferred to section 3 and section 4.

Let's denote $h = \frac{1}{N}$, we have

THEOREM 2.5 (Local Truncation Error). *If $u(x)$ is a solution of the equation (1.1) where f satisfy the regular condition (2.21), then there exists $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$, such that the truncation error (2.11) satisfies*

$$
\begin{aligned}
|\tau_i| :=& |-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i)| \\
\leq & C_1 h^{\min\{\frac{r\alpha}{2},2\}} \begin{cases} x_i^{-\alpha}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha}, & N < i \leq 2N-1 \end{cases} \\
&+ C_2(r-1)h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \leq i \leq N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \leq 2N-1 \end{cases}
\end{aligned}
\tag{2.22}
$$

THEOREM 2.6 (Global Error). *The discrete equation (2.8) has sulotion and there exists a positive constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ such that the error between the numerial solution U with the exact solution $u(x_i)$ satisfies*

$$
\max_{1 \leq i \leq 2N-1} |u_i - u(x_i)| \leq C h^{\min\{\frac{r\alpha}{2},2\}}
\tag{2.23}
$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

Remark 2.7. ...

3. Local Truncation Error.

3.1. Proof of Theorem 2.5. The truncation error of the discrete format can be written as

$$
\begin{aligned}
-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I^{2-\alpha} \Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i)) \\
&= -\kappa_\alpha D_h^2 I^{2-\alpha} (\Pi_h u - u)(x_i) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i)
\end{aligned}
\tag{3.1}
$$

THEOREM 3.1. *There exits a constant $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$ such that*

$$
\left| -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) \right| \leq C h^2 \begin{cases} x_i^{-\alpha/2 - 2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N-1 \end{cases}
\tag{3.2}
$$

Proof. Since $f \in C^2(\Omega)$ and

$$
\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha} u(x)) = f(x), \quad x \in \Omega,
\tag{3.3}
$$

we have $I^{2-\alpha} u \in C^4(\Omega)$. Therefore, using equation (A.2) of Lemma A.1, for $1 \leq i \leq$

$2N-1$, we have

(3.4)
$$\begin{aligned}&-\kappa_\alpha(D_h^2-\frac{d^2}{dx^2})I^{2-\alpha}u(x_i)=\frac{h_{i+1}-h_i}{3}f'(x_i)\\ &\quad+\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}\int_{x_{i-1}}^{x_i}f''(y)\frac{(y-x_{i-1})^3}{3!}dy+\frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}}f''(y)\frac{(y-x_{i+1})^3}{3!}dy\right)\end{aligned}$$

By Lemma B.2, Lemma 2.2 and Lemma B.3, we get the result. ∎

And now define

$$R_i:=D_h^2I^{2-\alpha}(u-\Pi_hu)(x_i),\quad 1\le i\le 2N-1 \tag{3.5}$$

We have some results about the estimate of R_i

THEOREM 3.2. *For $1\le i<N/2$, there exists $C=C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$|R_i|\le\begin{cases}Ch^2x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1>0\\ Ch^2(x_i^{-1-\alpha}\ln(i)+\ln(N)), & \alpha/2-2/r+1=0\\ Ch^{r\alpha/2+r}x_i^{-1-\alpha}, & \alpha/2-2/r+1<0\end{cases} \tag{3.6}$$

THEOREM 3.3. *For $N/2\le i\le N$, there exists constant $C=C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$|R_i|\le C(r-1)h^2|T-x_{i-1}|^{1-\alpha}+\begin{cases}Ch^2, & \alpha/2-2/r+1>0\\ Ch^2\ln(N), & \alpha/2-2/r+1=0\\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0\end{cases} \tag{3.7}$$

And for $N<i\le 2N-1$, it is symmetric to the previous case.

Combine Theorem 3.1, Theorem 3.2 and Theorem 3.3, and for $1\le i\le N$, we have

$$h^2x_i^{-\alpha/2-2/r}\le T^{\alpha/2-2/r}h^{\min\{\frac{r\alpha}{2},2\}}x_i^{-\alpha} \tag{3.8}$$

$$h^{r\alpha/2+r}x_i^{-1-\alpha}\le T^{-1}h^{r\alpha/2}x_i^{-\alpha} \tag{3.9}$$

$$h^rx_i^{-1}\ln(i)=T^{-1}\frac{\ln(i)}{i^r}\le T^{-1},\quad h^r\ln(N)=\frac{\ln(N)}{N^r}\le 1 \tag{3.10}$$

the proof of Theorem 2.5 completed.

We prove Theorem 3.2 and Theorem 3.3 in next subsections.

3.2. Outlines and Mesh Transport Functions. For convience, let's denote

DEFINITION 3.4.

$$T_{ij}=\int_{x_{j-1}}^{x_j}(u(y)-\Pi_hu(y))\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy,\quad i=0,\cdots,2N,\ j=1,\cdots,2N \tag{3.11}$$

Also, we denote vertical difference quotients of T_{ij}

$$\begin{aligned}V_{ij}&=\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}T_{i-1,j}-\left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)T_{i,j}+\frac{1}{h_{i+1}}T_{i+1,j}\right)\\ &=\int_{x_{i-1}}^{x_i}(u(y)-\Pi_hu(y))D_h^2K_y(x_i)dy\end{aligned} \tag{3.12}$$

And skew difference quotients of T_{ij}

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j-1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j+1} \right) \tag{3.13}$$

then $R_i = \sum_{j=1}^{2N} V_{ij}$.

Our main idea is to depart R_i by V_{ij} and S_{ij}. For $3 \le i < N/2$, let's denote $k = \lceil \frac{i}{2} \rceil$, and take some suitable integer m, then

$$\begin{aligned} R_i =& \sum_{j=1}^{2N} V_{ij} \\ =& \sum_{j=1}^{k-1} V_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\ &+ \sum_{j=k+1}^{m-1} S_{ij} + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,m} + T_{i-1,m-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,m} \right) \\ &+ \sum_{j=m+1}^{2N} V_{ij} \\ =& I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned} \tag{3.14}$$

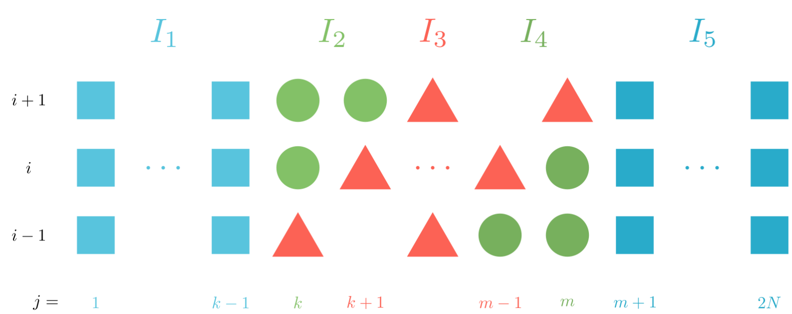

FIG. 1. *The departure of R_i for $i \ge 3$*

and discuss $i = 1, 2$ separately, where

$$R_1 = \sum_{i=1}^{3} V_{1,j} + \sum_{i=4}^{N} V_{i,j}, \quad R_2 = \sum_{i=1}^{4} V_{1,j} + \sum_{i=5}^{N} V_{i,j} \tag{3.15}$$

The difficulty for esitmating S_{ij} is that $T_{i-1,j-1}, T_{i,j}$ and $T_{i+1,j+1}$ have different integral region. We first make them normalized.

LEMMA 3.5. *For $y \in (x_{j-1}, x_j)$, we can rewrite $y = y_j^\theta$, from* (3.11)*, and Lemma* A.2*,*

$$
\begin{aligned}
T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
&= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\
&= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
&\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta
\end{aligned}
\tag{3.16}
$$

where $\eta_{j1}^\theta \in (x_{j-1}, y_j^\theta), \eta_{j2}^\theta \in (y_j^\theta, x_j)$.

Since j changes with i at indices of elements in S_{ij} by (3.13), we create some functions satisfy the property.

DEFINITION 3.6 (Mesh Transport Functions). *For $1 \le i, j \le 2N - 1$.*

$$
y_{i,j}(x) = \begin{cases}
(x^{1/r} + Z_{j-i})^r & i < N, j < N \\
\dfrac{x^{1/r} - Z_i}{Z_1} h_N + x_N & i < N, j = N \\
2T - (Z_{2N-(j-i)} - x^{1/r})^r & i < N, j > N \\
\left(\dfrac{Z_1}{h_N}(x - x_N) + Z_j\right)^r & i = N, j < N \\
x, & i = N, j = N \\
2T - \left(\dfrac{Z_1}{h_N}(2T - x - x_N) + Z_{2N-j}\right)^r & i = N, j > N \\
(Z_{2N+j-i} - (2T - x)^{1/r})^r & i > N, j < N \\
\dfrac{Z_{2N-j} - (2T - x)^{1/r}}{Z_1} h_N + x_N & i > N, j = N \\
2T - ((2T - x)^{1/r} - Z_{j-i})^r & i > N, j > N
\end{cases}
\tag{3.17}
$$

where $Z_j := T^{1/r} \frac{j}{N}$, $x \in [x_{i-1}, x_{i+1}]$. And

$$
h_{i,j}(x) = y_{i,j}(x) - y_{i,j-1}(x)
\tag{3.18}
$$

$$
y_{i,j}^\theta(x) = (1-\theta) y_{i,j-1}(x) + \theta y_{i,j-1}(x), \quad \theta \in (0,1)
\tag{3.19}
$$

$$
P_{i,j}^\theta(x) = (h_{i,j}(x))^3 \frac{|y_{i,j}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''(y_{i,j}^\theta(x))
\tag{3.20}
$$

$$
Q_{i,j;l}^\theta(x) = (h_{i,j}(x))^l \frac{|y_{i,j}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}
\tag{3.21}
$$

Obviously,

$$
y_{i,j}(x_{i-1}) = x_{j-1}, \quad y_{i,j}(x_i) = x_j, \quad y_{i,j}(x_{i+1}) = x_{j+1}
\tag{3.22}
$$

$$h_{i,j}(x_{i-1}) = h_{j-1}, \quad h_{i,j}(x_i) = h_j, \quad h_{i,j}(x_{i+1}) = h_{j+1} \tag{3.23}$$

$$y^\theta_{i,j}(x_{i-1}) = y^\theta_{j-1}, \quad y^\theta_{i,j}(x_i) = y^\theta_j, \quad y^\theta_{i,j}(x_{i+1}) = y^\theta_{j+1} \tag{3.24}$$

And now we can rewrite T_{ij}

Lemma 3.7. *For $0 \le i \le 2N, 1 \le j \le 2N$,*

$$\begin{aligned} T_{ij} = &\int_0^1 -\frac{\theta(1-\theta)}{2} P^\theta_{i,j}(x_i) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} Q^\theta_{i,j;l}(x_i) \left[\theta^2 u'''(\eta^\theta_{j,1}) - (1-\theta)^2 u'''(\eta^\theta_{j,2})\right] d\theta \end{aligned} \tag{3.25}$$

Immediately, we can see from (3.13) *and Lemma* 3.5 *that For $1 \le i \le 2N-1, 2 \le j \le 2N-1$,*

$$\begin{aligned} S_{ij} = &\int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P^\theta_{i,j}(x_i) d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^\theta_{i,j;4}(x_{i+1}) u'''(\eta^\theta_{j+1,1}) - Q^\theta_{i,j;4}(x_i) u'''(\eta^\theta_{j,1})}{h_{i+1}} \right) d\theta \\ &- \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^\theta_{i,j;4}(x_i) u'''(\eta^\theta_{j,1}) - Q^\theta_{i,j;4}(x_{i-1}) u'''(\eta^\theta_{j-1,1})}{h_i} \right) d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^\theta_{i,j;4}(x_{i+1}) u'''(\eta^\theta_{j+1,2}) - Q^\theta_{i,j;4}(x_i) u'''(\eta^\theta_{j,2})}{h_{i+1}} \right) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^\theta_{i,j;4}(x_i) u'''(\eta^\theta_{j,2}) - Q^\theta_{i,j;4}(x_{i-1}) u'''(\eta^\theta_{j-1,2})}{h_i} \right) d\theta \end{aligned} \tag{3.26}$$

We give some properties of mesh transport functions.

Lemma 3.8. *For $2 \le i \le 2N-2, 2 \le j \le 2N-2$ and $\xi \in (x_{i-1}, x_{i+1})$*

$$\xi \simeq x_i, \quad y_{i,j}(\xi) \simeq x_j, \quad 2T - y_{i,j}(\xi) \simeq 2T - x_j, \quad h_{i,j}(\xi) \simeq h_j \tag{3.27}$$

$$|y_{i,j}(\xi) - \xi| \simeq |x_j - x_i| \tag{3.28}$$

then

$$|y^\theta_{i,j}(\xi) - \xi| = (1-\theta)|y_{i,j-1}(\xi) - \xi| + \theta|y_{i,j}(\xi) - \xi| \simeq |y^\theta_j - x_i| \tag{3.29}$$

since $y_{i,j-1}(\xi) - \xi$, $y_{i,j}(\xi) - \xi$ have the same sign (≥ 0 or ≤ 0)

Lemma 3.9.

$$y'_{i,j}(x) = \begin{cases} y_{i,j}^{1-1/r}(x) x^{1/r-1} & i < N, j < N \\ \dfrac{h_N}{r Z_1} x^{1/r-1} & i < N, j = N \\ (2T - y_{i,j}(x))^{1-1/r} x^{1/r-1} & i < N, j > N \\ y_{i,j}^{1-1/r}(x) \dfrac{r Z_1}{h_N} & i = N, j < N \\ 1 & i = N, j = N \end{cases} \tag{3.30}$$

$$
y''_{i,j}(x) = \frac{1-r}{r}\begin{cases} y_{i,j}^{1-2/r}(x)x^{1/r-2}Z_{j-i} & i<N, j<N \\ \dfrac{h_N}{rZ_1}x^{1/r-2} & i<N, j=N \\ (2T-y_{i,j}(x))^{1-2/r}x^{1/r-2}Z_{2N-j+i} & i<N, j>N \\ -y_{i,j}^{1-2/r}(x)\left(\dfrac{rZ_1}{h_N}\right)^2 & i=N, j<N \\ 0 & i=N, j=N \end{cases} \tag{3.31}
$$

LEMMA 3.10. *For* $2 \le i \le N, 2 \le j \le 2N-2$, $\xi \in (x_{i-1}, x_{i+1})$

$$
|h'_{i,j}(\xi)| \le C(r-1)Z_1 x_i^{1/r-1}\begin{cases} x_j^{1-2/r} & j \le N \\ (2T-x_j)^{1-2/r} & j>N \end{cases} \tag{3.32}
$$

$$
|(y_{i,j}(\xi)-\xi)'| \le Cx_i^{-1}\begin{cases} |x_j - x_i| & j \le N \\ |2T-x_j-x_i| + 2h_N & j>N \end{cases} \tag{3.33}
$$

Proof. From (3.18) and Lemma 3.9, we can see that

$$
\begin{aligned}
h'_{i,j}(x) &= y'_{i,j}(x) - y'_{i,j-1}(x) \\
&= \begin{cases} x^{1/r-1}(y_{i,j}^{1-1/r}(x) - y_{i,j-1}^{1-1/r}(x)) & i<N, j<N \\ x^{1/r-1}(\dfrac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(x)) & i<N, j=N \\ x^{1/r-1}\left((2T-y_{i,N+1}(x))^{1-1/r} - \dfrac{h_N}{rZ_1}\right) & i<N, j=N+1 \\ x^{1/r-1}\left((2T-y_{i,j}(x))^{1-1/r} - (2T-y_{i,j-1}(x))^{1-1/r}\right) & i<N, j>N+1 \\ \dfrac{rZ_1}{h_N}\left(y_{N,j}^{1-1/r}(x) - y_{N,j-1}^{1-1/r}(x)\right) & i=N, j<N \\ \dfrac{rZ_1}{h_N}\left(\dfrac{h_N}{rZ_1} - y_{N,N-1}^{1-1/r}(x)\right) & i=N, j=N \end{cases}
\end{aligned} \tag{3.34}
$$

While for $2 \le i \le N$, if $2 \le j < N$, $\xi \in (x_{i-1}, x_{i+1})$,

$$
\begin{aligned}
y_{i,j}^{1-1/r}(\xi) - y_{i,j-1}^{1-1/r}(\xi) &\le x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r} \\
&= T^{1-1/r}N^{1-r}\left((j+1)^{r-1} - (j-2)^{r-1}\right) \\
&\le CT^{1-1/r}(r-1)N^{1-r}j^{r-2} = C(r-1)Z_1 x_j^{1-2/r}
\end{aligned} \tag{3.35}
$$

if $j=N$, $\xi \in (x_{i-1}, x_{i+1})$, we have $y_{i,N-1}(\xi) \in (x_{N-2}, x_N)$. And

$$
\frac{h_N}{rZ_1} = T^{1-1/r}\frac{1-(1-h)^r}{rh} = \eta^{1-1/r}, \quad \eta \in (x_{N-1}, x_N) \tag{3.36}
$$

Then

$$
|\frac{h_N}{rZ_1} - y_{i,N-1}^{1-1/r}(\xi)| \le x_N^{1-1/r} - x_{N-2}^{1-1/r} \simeq (r-1)Z_1 x_N^{1-2/r} \tag{3.37}
$$

and similar for $j \ge N+1$. Combine with Lemma 3.8, $\eta \simeq x_N$, we get the first result.

For the second estimate, we have

$$(y_{i,j}(x)-x)' = y'_{i,j}(x) - 1 \tag{3.38}$$

Then, for $2 \le i < N$, if $2 \le j < N$, $\xi \in (x_{i-1}, x_{i+1})$, by Lemma A.5

$$\xi^{1/r}|y_{i,j}^{1-1/r}(\xi) - \xi^{1-1/r}| \le |y_{i,j}(\xi) - \xi| \tag{3.39}$$

$j > N$ is symmetric to it, that is

$$\begin{aligned}\xi^{1/r}|(2T-y_{i,j}(\xi))^{1-1/r} - \xi^{1-1/r}| &\le |2T - y_{i,j}(\xi) - \xi| \\ \le |2T - x_j - x_i| + |y_{i,j}(\xi) - x_j| + |\xi - x_i| &\le |2T - x_j - x_i| + 2h_N\end{aligned} \tag{3.40}$$

But if $j = N$, with (3.36) and Lemma A.5,

$$\begin{aligned}\eta^{1/r}|\frac{h_N}{rZ_1} - \xi^{1-1/r}| &\le |\eta - \xi|, \quad \eta \in (x_{N-1}, x_N) \\ &\le |x_N - x_i| + |h_N| + |h_{i+1}| \le C|x_N - x_i|\end{aligned} \tag{3.41}$$

For $i = N$, if $j < N$, similarly with (3.41),

$$\eta^{1/r}|y_{N,j}^{1-1/r}(\xi) - \frac{h_N}{rZ_1}| \le C|x_j - x_N| \tag{3.42}$$

And if $j = N$, it is obviously $\equiv 0$.

Similarly, by Lemma 3.9 and Lemma 3.8, we get the second result. ∎

LEMMA 3.11. *For* $2 \le i \le N, 2 \le j \le 2N-2$, $\xi \in (x_{i-1}, x_{i+1})$

$$|y''_{i,j}(\xi)| \le C(r-1)\begin{cases} x_j^{-1/r}x_i^{1/r-2}|x_j - x_i| & i<N, j<N \\ x_N^{1-1/r}x_i^{1/r-2} & i<N, j=N \\ (2T-x_j)^{1-2/r}x_i^{1/r-2}x_N^{1/r} & i<N, j>N \\ x_j^{1-2/r}x_N^{2/r-2} & i=N, j<N \\ 0 & i=N, j=N \end{cases} \tag{3.43}$$

And $2 \le i \le N, 3 \le j \le 2N-2$, $\xi \in (x_{i-1}, x_{i+1})$

$$|h''_{i,j}(\xi)| \le C(r-1)\begin{cases} Z_1 x_i^{1/r-2}x_j^{-2/r}(|x_j - x_i| + x_j) & i<N, j<N \\ x_i^{1/r-2}x_N^{1-1/r} & i<N, j=N, N+1 \\ Z_1 x_i^{1/r-2}(2T-x_j)^{1-3/r}x_N^{1/r} & i<N, j>N+1 \\ Z_1 x_N^{2/r-2}x_j^{1-3/r} & i=N, j<N \\ x_N^{-1} & i=N, j=N \end{cases} \tag{3.44}$$

Proof. Since by Lemma A.5, for $2 \le i, j < N$

$$x_j^{1-1/r}|Z_{j-i}| = x_j^{1-1/r}|x_j^{1/r} - x_i^{1/r}| \le |x_j - x_i| \tag{3.45}$$

and by (3.36), $\dfrac{h_N}{rZ_1} \simeq x_N^{1-1/r}$. And

$$Z_{2N-j+i} \le Z_{2N} = 2T^{1/r} \tag{3.46}$$

Then by Lemma 3.9 and Lemma 3.8, we get the first result.

For the second part, by Lemma 3.9

$$h''_{i,j}(x) = y''_{i,j}(x) - y''_{i,j-1}(x) \tag{3.47}$$

while for $2 \le i < N$, if $3 \le j < N$, $\xi \in (x_{i-1}, x_{i+1})$,

$$y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1} = \left(y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi)\right) Z_{j-i} + y_{i,j-1}^{1-2/r}(\xi)Z_1 \tag{3.48}$$

where $y_{i,j}^{1-2/r}(\xi) - y_{i,j-1}^{1-2/r}(\xi) \simeq (r-2)Z_1 x_j^{1-3/r}$ similar with (3.35). Combine with (3.45), we get

$$|y_{i,j}^{1-2/r}(\xi)Z_{j-i} - y_{i,j-1}^{1-2/r}(\xi)Z_{j-i-1}| \le CZ_1 \left(|r-2|x_j^{-2/r}|x_j - x_i| + x_j^{1-2/r}\right) \tag{3.49}$$

if $j = N$,

$$|h''_{i,N}(x)| \le |y''_{i,N}(x)| + |y''_{i,N-1}(x)| \le C(r-1)x_i^{1/r-2}x_N^{1-1/r} \tag{3.50}$$

similarly if $j = N+1$.

However, if $j > N+1$, similar with (3.48), we get

$$\begin{aligned} &(2T - y_{i,j}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)} \\ &= \left((2T - y_{i,j}(\xi))^{1-2/r} - (2T - y_{i,j-1}(\xi))^{1-2/r}\right) Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r} Z_1 \end{aligned} \tag{3.51}$$

thus,

$$\begin{aligned} &\left|(2T - y_{i,j}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{i,j-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)}\right| \\ &\le CZ_1 \left(|r-2|(2T - x_j)^{1-3/r}x_N^{1/r} + (2T - x_j)^{1-2/r}\right) \le CZ_1(2T - x_j)^{1-3/r}x_N^{1/r} \end{aligned} \tag{3.52}$$

For $i = N$, it's obvious. Combine with Lemma 3.9 and Lemma 3.8, we get the second result. ∎

3.3. Proof of Theorem 3.2. Then we esrimate each part of (3.14) from easy to hard. And We take $m = 2i$ for $3 \le i < N/2$, and $m = N - \lceil N/2 \rceil + 1$ for $N/2 \le i \le N$.

For I_5

LEMMA 3.12. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \le i < N/2$,*

$$\sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.53}$$

Proof. For $\max\{2i+1, 4\} \le j \le N$, by (3.12), Lemma A.4 and Lemma B.4 with $y - x_i \simeq y$, we have

$$\begin{aligned} |V_{ij}| &\le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} y^{-1-\alpha} dy \\ &= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy \end{aligned} \tag{3.54}$$

With $x_i \simeq x_{2i}$,

$$\sum_{j=\max\{2i+1,4\}}^{N} |V_{ij}| \le Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy$$

$$= \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \tag{3.55}$$

$$\le Ch^2 x_i^{-\alpha/2-2/r}$$

□

LEMMA 3.13. *Thert exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \le i < N/2$,*

$$\sum_{j=N+1}^{2N} |V_{ij}| \le \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.56}$$

and for $N/2 \le i \le N$,

$$\sum_{j=N-\lceil \frac{N}{2} \rceil+2}^{2N} |V_{ij}| \le \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.57}$$

Proof. For $1 \le i < N/2, N+1 \le j \le 2N-1$, by (3.12), Lemma A.4, Lemma B.4 and $y - x_i \simeq T$

$$|V_{ij}| \le C \int_{x_{j-1}}^{x_j} h^2 (2T-y)^{\alpha/2-2/r} |y-x_i|^{-1-\alpha} dy$$

$$\le Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T-y)^{\alpha/2-2/r} dy$$

$$\sum_{j=N+1}^{2N-1} |V_{ij}| \le CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r} dy$$

$$\le CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.58}$$

$$= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r}\ h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

And by Lemma A.3

$$|V_{i,2N}| \le CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}$$

Summarizes, we get the result. Similar for the second inequality. □

For $i = 1, 2$.

LEMMA 3.14. *From* (3.15)*, by Lemma* B.5*, Lemma* 3.12 *and Lemma* 3.13 *we get for* $i = 1, 2$

$$|R_i| \leq Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.59}$$

LEMMA 3.15. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil$

$$|I_1| = |\sum_{j=1}^{k-1} V_{ij}| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.60}$$

Proof. by (3.12), Lemma A.3 , Lemma B.4

$$|V_{i1}| \leq C \int_0^{x_1} x_1^{\alpha/2} |x_i - y|^{-1-\alpha} dy \simeq x_1^{\alpha/2+1} x_i^{-1-\alpha} = T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha} \tag{3.61}$$

For $2 \leq j \leq k-1$, by Lemma A.4 and Lemma B.4 with $x_i - y \simeq x_i$, we have

$$|V_{ij}| \leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} x_i^{-1-\alpha} dy \tag{3.62}$$

Therefore,

$$\sum_{j=2}^{k-1} |V_{ij}| \leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \tag{3.63}$$

But $x_{\lceil \frac{i}{2} \rceil - 1} \leq 2^{-r} x_i$, so we have

$$\int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \leq \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.64}$$

Combine the results above, we get the lemma. ∎

LEMMA 3.16. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$,

$$|D_h^2 P_{i,j}^\theta (x_i)| \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \tag{3.65}$$

where $y_j^\theta = \theta x_{j-1} + (1-\theta) x_j$

Proof. Since $\mathrm{sign}(y_{i,j}^\theta(\xi) - \xi)$ is independent of ξ, we can derivate it. Then by Lemma A.1

$$D_h^2 P_{i,j}^\theta (x_i) = {P_{i,j}^\theta}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{3.66}$$

From (3.20), using Leibniz formula and chain rule, and Lemma 3.8, Lemma 3.10, Lemma 3.11, Corollary 2.4, Lemma B.1 and $x_j \simeq x_i$ we get the result. ∎

Lemma 3.17. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i \le N$.
For $\lceil \frac{i}{2} \rceil \le j \le \min\{2i-1, N-1\}$,

$$\left| \frac{Q^{\theta}_{i,j;l}(x_{i+1})u^{(l-1)}(\eta^{\theta}_{j+1}) - Q^{\theta}_{i,j;l}(x_i)u^{(l-1)}(\eta^{\theta}_{j})}{h_{i+1}} \right| \le Ch^2 \frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2/r+2-l} h_j^{l-2} \tag{3.67}$$

And for $\lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i, N\}$,

$$\left| \frac{Q^{\theta}_{i,j;l}(x_{i})u^{(l-1)}(\eta^{\theta}_{j}) - Q^{\theta}_{i,j;l}(x_{i-1})u^{(l-1)}(\eta^{\theta}_{j-1})}{h_{i}} \right| \le Ch^2 \frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2/r+2-l} h_j^{l-2} \tag{3.68}$$

where $\eta^{\theta}_j \in (x_{j-1}, x_j)$.

Proof.

$$\begin{aligned} &\frac{Q^{\theta}_{i,j;l}(x_{i+1})u'''(\eta^{\theta}_{j+1}) - Q^{\theta}_{i,j;l}(x_i)u'''(\eta^{\theta}_{j})}{h_{i+1}} \\ &= \frac{Q^{\theta}_{i,j;l}(x_{i+1}) - Q^{\theta}_{i,j;l}(x_i)}{h_{i+1}} u'''(\eta^{\theta}_{j+1}) + Q^{\theta}_{i,j;l}(x_i) \frac{u'''(\eta^{\theta}_{j+1}) - u'''(\eta^{\theta}_{j})}{h_{i+1}} \end{aligned} \tag{3.69}$$

Using mean value theorem

$$D_h Q^{\theta}_{i,j;l}(x_i) = \frac{Q^{\theta}_{i,j;l}(x_{i+1}) - Q^{\theta}_{i,j;l}(x_i)}{h_{i+1}} = {Q^{\theta}_{i,j;l}}'(\xi), \quad \xi \in (x_i, x_{i+1}) \tag{3.70}$$

From (3.21) and Leibniz rule, by Lemma 3.8, Lemma 3.10 and Lemma B.1, we have

$$|{Q^{\theta}_{i,j;l}}'(\xi)| \le Ch^2 x_i^{1-2/r} \frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^{l-2} \tag{3.71}$$

$$Q^{\theta}_{i,j;l}(x_i) = h_j^l \frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \simeq Ch^2 x_i^{2-2/r} \frac{|y^{\theta}_j - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^{l-2} \tag{3.72}$$

With $\eta^{\theta}_j \in (x_{j-1}, x_j)$

$$u^{(l-1)}(\eta^{\theta}_{j+1}) \le C(\eta^{\theta}_{j+1})^{\alpha/2-l+1} \simeq x_j^{\alpha/2-l+1} \simeq x_i^{\alpha/2-l+1}$$

and

$$\begin{aligned} \frac{u^{(l-1)}(\eta^{\theta}_{j+1}) - u^{(l-1)}(\eta^{\theta}_{j})}{h_{i+1}} &= u^{(l)}(\eta) \frac{\eta^{\theta}_{j+1} - \eta^{\theta}_j}{h_{i+1}}, \quad \eta \in (x_{j-1}, x_{j+1}) \\ &\le C\eta^{\alpha/2-l} \frac{x_{j+1} - x_{j-1}}{h_{i+1}} = C\eta^{\alpha/2-l} \frac{h_{j+1} + h_j}{h_{i+1}} \\ &\simeq x_j^{\alpha/2-l} \simeq x_i^{\alpha/2-l} \end{aligned}$$

with $h_j \simeq h_i$, we get the first term. While, the later is similar. □

LEMMA 3.18. *There exists a constant* $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil + 1 \le j \le \min\{2i-1, N-1\}$,

$$\begin{aligned} |S_{ij}| &\le Ch^2 \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j d\theta \\ &= Ch^2 \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \end{aligned} \tag{3.73}$$

Proof. Since (3.26), by Lemma 3.16 and Lemma 3.17, we get the result immediately. ∎

THEOREM 3.19. *There exists a constant* $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i \le N-1, k = \lceil \frac{i}{2} \rceil$,

$$\sum_{j=k+1}^{\min\{2i-1,N-1\}} |S_{ij}| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.74}$$

Proof. By Lemma 3.18, while $x_k \simeq x_i \simeq x_{\min\{2i-1,N-1\}}$, we have

$$\begin{aligned} \sum_{k+1}^{\min\{2i-1,N-1\}} |S_{ij}| &\le Ch^2 \int_{x_k}^{x_{\min\{2i-1,N-1\}}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{3.75}$$

∎

Now we study I_2, I_4.

LEMMA 3.20. *There exists a constant* $C = C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3 \le i \le N, k = \lceil \frac{i}{2} \rceil$,

$$I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.76}$$

And for $3 \le i < N/2$,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.77}$$

Proof. In fact,

$$\begin{aligned} &\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \\ &= \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i}) T_{i,k} \end{aligned} \tag{3.78}$$

While, by Lemma A.4, Lemma B.4, Lemma B.1 and $x_k \simeq x_i$, we have

$$\begin{aligned} \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) &= \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y)) D_h K_y(x_i) dy \\ &\le Ch_k^2 x_k^{\alpha/2-2} h_k |x_i - x_k|^{-\alpha} \le Ch^2 x_i^{-\alpha/2-2/r} h_k \end{aligned} \tag{3.79}$$

Thus,

$$\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}|T_{i+1,k}-T_{i,k}| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.80}$$

From (3.11), Lemma A.2 and normalizzation, we have

$$\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k}) = \int_0^1 -\frac{\theta(1-\theta)}{2}\frac{Q^\theta_{i,k;3}(x_{i+1})u''(\eta^\theta_{k+1})-Q^\theta_{i,k;3}(x_i)u''(\eta^\theta_k)}{h_{i+1}}d\theta \tag{3.81}$$

where $\eta^\theta_k \in (x_{k-1},x_k)$ and $\eta^\theta_{k+1} \in (x_k,x_{k+1})$. And with Lemma 3.17, we can get

$$\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}|T_{i+1,k+1}-T_{i,k}| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.82}$$

For the third term, by Lemma B.1, Lemma B.2, Lemma A.4 and $x_k \simeq x_i$, we have

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}\frac{h_{i+1}-h_i}{h_ih_{i+1}}T_{i,k} &\le h_i^{-3}h^2x_i^{1-2/r}Ch_k^3x_k^{\alpha/2-2}|x_k-x_i|^{1-\alpha}\\ &\le Ch^2x_i^{-\alpha/2-2/r}\end{aligned} \tag{3.83}$$

Summarizes, we have

$$I_2 \le Ch^2x_i^{-\alpha/2-2/r} \tag{3.84}$$

The case for I_4 is similar. ∎

Now combine Lemma 3.14, Lemma 3.15, Lemma 3.20, Theorem 3.19, Lemma 3.12 and Lemma 3.13, we get Theorem 3.2.

For $N/2 \le i < N$, we take $m = 2N - \lceil \frac{N}{2} \rceil + 1$. And depart I_3 to three parts:

$$\begin{aligned} I_3 = \sum_{j=k+1}^{m} S_{ij} &= \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{m-1} S_{ij}\\ &= I_3^1 + I_3^2 + I_3^3\end{aligned} \tag{3.85}$$

We have estimated I_3^1 in Theorem 3.19.

Combine Lemma 3.8, Lemma 3.10, Lemma 3.11, Lemma B.1, using Leibniz formula, we have

Lemma 3.21. *There exists a constant $C = C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \le i \le N-1$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$,, we have*

$$\begin{aligned}|D_h^2P^\theta_{i,j}(\xi)| \le Ch_jh^2\Big(&|y^\theta_j - x_i|^{1-\alpha}\\ &+ |y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_N)\\ &+ |y^\theta_j - x_i|^{-1-\alpha}(|2T - x_i - y^\theta_j| + h_N)^2\\ &+ (r-1)|y^\theta_j - x_i|^{-\alpha}\Big)\end{aligned} \tag{3.86}$$

And

LEMMA 3.22. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le i \le N-1,\ N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil,\ \xi \in (x_{i-1}, x_{i+1})$ *, we have*

$$\begin{aligned}\frac{2}{h_i + h_{i+1}} &\left| \frac{Q^\theta_{i,j;l}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{i,j;l}(x_i)u'''(\eta^\theta_j)}{h_{i+1}} \right| \\ &\le Ch^2 h_j \left(|y^\theta_j - x_i|^{1-\alpha} + |y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_N) \right)\end{aligned} \tag{3.87}$$

and

$$\begin{aligned}\frac{2}{h_i + h_{i+1}} &\left(\frac{Q^\theta_{i,j;l}(x_i)u'''(\eta^\theta_j) - Q^\theta_{i,j;l}(x_{i-1})u'''(\eta^\theta_{j-1})}{h_{i+1}} \right) \\ &\le Ch^2 h_j (|y^\theta_j - x_i|^{1-\alpha} + |y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_N))\end{aligned} \tag{3.88}$$

Proof. From **??**, by Lemma B.6 and Lemma B.8, for $\xi \in (x_i, x_{i+1})$, by Leibniz formula, we have

$$\left| {Q^\theta_{i,j;l}}'(\xi) \right| \le Ch^2 h_j^2 ((r-1)|y^\theta_j - x_i|^{1-\alpha} + |y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_N)) \tag{3.89}$$

$$\left| Q^\theta_{i,j;l}(\xi) \right| \le Ch^2 h_j^2 |y^\theta_j - x_i|^{1-\alpha} \tag{3.90}$$

So use the skill in Proof 8 with Lemma B.7

$$\begin{aligned}\frac{2}{h_i + h_{i+1}} &\left(\frac{Q^\theta_{i,j;l}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{i,j;l}(x_i)u'''(\eta^\theta_j)}{h_{i+1}} \right) \\ &\le Ch^2 h_j (|y^\theta_j - x_i|^{1-\alpha} + |y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_N))\end{aligned} \tag{3.91}$$

□

Combine Lemma 3.21, Lemma 3.22 and formula (3.26) for $i \le N-1, j \ge N+2$, we have

LEMMA 3.23. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le i \le N-1,\ N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$

$$\begin{aligned}S_{ij} \le & Ch^2 \int_{x_{j-1}}^{x_j} \Big(|y - x_i|^{1-\alpha} \\ & + |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2 \\ & + (r-1)|y - x_i|^{-\alpha} \Big) dy\end{aligned} \tag{3.92}$$

We can esitmate I_3^3 Now.

LEMMA 3.24. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le i \le N-1$*, we have*

$$I_3^3 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} S_{ij} \le Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \tag{3.93}$$

Proof.

$$
\begin{aligned}
I_3^3 = & \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} S_{ij} \\
\le & Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil \frac{N}{2} \rceil}} \Big(|y-x_i|^{1-\alpha} \\
& + |y-x_i|^{-\alpha}(|2T-x_i-y|+h_N) + |y-x_i|^{-1-\alpha}(|2T-x_i-y|+h_N)^2 \\
& + (r-1)|y-x_i|^{-\alpha} \Big) dy
\end{aligned} \tag{3.94}
$$

Since

$$
|2T-x_i-y| + h_N \le y - x_i \tag{3.95}
$$

$$
\begin{aligned}
I_3^3 & \le Ch^2 \int_{x_{N+1}}^{x_{2N-\lceil \frac{N}{2} \rceil}} |y-x_i|^{1-\alpha} + (r-1)|y-x_i|^{-\alpha} \\
& \le Ch^2 (T^{2-\alpha} + (r-1)|x_{N+1}-x_i|^{1-\alpha}) \\
& \le Ch^2 + C(r-1)h^2 |T-x_{i-1}|^{1-\alpha}
\end{aligned} \tag{3.96}
$$

∎

For I_3^2, we have

THEOREM 3.25. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that, for $N/2 \le i \le N-1$*

$$V_{iN} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,N+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,N} + \frac{1}{h_i} T_{i-1,N-1} \right) \le Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \tag{3.97}$$

Proof. We use the similar skill in the last section, but more complicated. for $j = N$, Let

$$_L y_{N-1-i}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r} \frac{N-1-i}{N} \tag{3.98}$$

$$_0 y_{N-i}(x) = \frac{x^{1/r} - Z_i}{Z_1} h_N + T, \quad Z_i = T^{1/r} \frac{i}{N}, x_N = T \tag{3.99}$$

and

$$_R y_{N+1-i}(x) = 2T - (Z_{N-1+i} - x^{1/r})^r, \quad Z_{N-1+i} = T^{1/r} \frac{N-1+i}{N} \tag{3.100}$$

Thus,

$$\begin{gathered} _L y_{N-1-i}(x_{i-1}) = x_{N-2}, \quad _L y_{N-1-i}(x_i) = x_{N-1}, \quad _L y_{N-1-i}(x_{i+1}) = x_N \\ _0 y_{N-i}(x_{i-1}) = x_{N-1}, \quad _0 y_{N-i}(x_i) = x_N, \quad _0 y_{N-i}(x_{i+1}) = x_{N+1} \\ _R y_{N+1-i}(x_{i-1}) = x_N, \quad _R y_{N+1-i}(x_i) = x_{N+1}, \quad _R y_{N+1-i}(x_{i+1}) = x_{N+2} \end{gathered}$$

Then, define

$$_L y_{N-i}^{\theta}(x) = \theta {}_L y_{N-1-i}(x) + (1-\theta) {}_0 y_{N-i}(x) \tag{3.101}$$

$$_R y_{N+1-i}^{\theta}(x) = \theta {}_0 y_{N-i}(x) + (1-\theta) {}_R y_{N+1-i}(x) \tag{3.102}$$

$$_L h_{N-i}(x) = {}_0 y_{N-i}(x) - {}_L y_{N-1-i}(x) \tag{3.103}$$

$$_R h_{N+1-i}(x) = {}_R y_{N+1-i}(x) - {}_0 y_{N-i}(x) \tag{3.104}$$

We have

$$_L y_{N-1-i}{}'(x) = {}_L y_{N-1-i}^{1-1/r}(x) x^{1/r-1} \tag{3.105}$$

$$_L y_{N-1-i}{}''(x) = \frac{1-r}{r} {}_L y_{N-1-i}^{1-2/r}(x) x^{1/r-2} Z_{N-1-i} \tag{3.106}$$

$$_0 y_{N-i}{}'(x) = \frac{1}{r} \frac{h_N}{Z_1} x^{1/r-1} \tag{3.107}$$

$$_0 y_{N-i}{}''(x) = \frac{1-r}{r^2} \frac{h_N}{Z_1} x^{1/r-2} \tag{3.108}$$

$$_R y_{N+1-i}{}'(x) = (2T - {}_R y_{N+1-i}(x))^{1-1/r} x^{1/r-1} \tag{3.109}$$

$$_R y_{N+1-i}{}''(x) = \frac{1-r}{r} (2T - {}_R y_{N+1-i}(x))^{1-2/r} x^{1/r-2} Z_{N-1+i} \tag{3.110}$$

$$
{}_LP^{\theta}_{N-i}(x) = ({}_Lh_{N-i}(x))^3 \frac{|{}_Ly^{\theta}_{N-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_Ly^{\theta}_{N-i}(x)) \tag{3.111}
$$

$$
{}_RP^{\theta}_{N+1-i}(x) = ({}_Rh_{N+1-i}(x))^3 \frac{|{}_Ry^{\theta}_{N+1-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_Ry^{\theta}_{N+1-i}(x)) \tag{3.112}
$$

$$
{}_LQ^{\theta}_{N-i}(x) = ({}_Lh_{N-i}(x))^4 \frac{|{}_Ly^{\theta}_{N-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.113}
$$

$$
{}_RQ^{\theta}_{N+1-i}(x) = ({}_Rh_{N+1-i}(x))^4 \frac{|{}_Ry^{\theta}_{N+1-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.114}
$$

Similar with (3.25), we can get for $l = -1, 0, 1$,

$$
\begin{aligned}
T_{i+l,N+l} = &\int_0^1 -\frac{\theta(1-\theta)}{2} {}_LP^{\theta}_{N-i}(x_{i+l}) d\theta \\
&+ \int_0^1 \frac{\theta(1-\theta)}{3!} {}_LQ^{\theta}_{N-i}(x_{i+l}) (\theta^2 u'''(\eta^{\theta}_{N+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+l,2})) d\theta
\end{aligned} \tag{3.115}
$$

(3.116)

$$
\begin{aligned}
T_{i+l,N+1+l} = &\int_0^1 -\frac{\theta(1-\theta)}{2} {}_RP^{\theta}_{N+1-i}(x_{i+l}) d\theta \\
&+ \int_0^1 \frac{\theta(1-\theta)}{3!} {}_RQ^{\theta}_{N+1-i}(x_{i+l}) (\theta^2 u'''(\eta^{\theta}_{N+1+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+1+l,2})) d\theta
\end{aligned}
$$

So we have

(3.117)

$$
\begin{aligned}
V_{i,N} = &\int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 {}_LP^{\theta}_{N-i}(x_i) d\theta \\
&+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_LQ^{\theta}_{N-i}(x_{i+1}) u'''(\eta^{\theta}_{N+1,1}) - {}_LQ^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,1})}{h_{i+1}} \right) d\theta \\
&- \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_LQ^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,1}) - {}_LQ^{\theta}_{N-i}(x_{i-1}) u'''(\eta^{\theta}_{N-1,1})}{h_i} \right) d\theta \\
&- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_LQ^{\theta}_{N-i}(x_{i+1}) u'''(\eta^{\theta}_{N+1,2}) - {}_LQ^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,2})}{h_{i+1}} \right) d\theta \\
&+ \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_LQ^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,2}) - {}_LQ^{\theta}_{N-i}(x_{i-1}) u'''(\eta^{\theta}_{N-1,2})}{h_i} \right) d\theta
\end{aligned}
$$

$N+1$ is similar.

We estimate $D_h^2 {}_LP^{\theta}_{N-i}(x_i) = {}_LP^{\theta}_{N-i}{}''(\xi), \xi \in (x_{i-1}, x_{i+1})$,

Lemma 3.26.

$$
{}_Lh^3_{N-i}(\xi) \le Ch^3_N \le Ch^3 \tag{3.118}
$$

$$
{}_Rh^3_{N+1-i}(\xi) \le Ch^3_N \le Ch^3 \tag{3.119}
$$

$$({}_Lh^3_{N-i}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{3.120}$$

$$({}_Rh^3_{N+1-i}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{3.121}$$

$$({}_Lh^3_{N-i}(\xi))'' \le C(r-1)h^2 \tag{3.122}$$

$$({}_Rh^3_{N+1-i}(\xi))'' \le C(r-1)h^2 \tag{3.123}$$

Proof.

$$ {}_Lh_{N-i}(\xi) \le 2(C?)h_N, \quad {}_Rh_{N+1-i}(\xi) \le 2h_N \tag{3.124}$$

$$\begin{aligned}({}_Lh^l_{N-i}(\xi))' &= l\,{}_Lh^{l-1}_{N-i}(\xi)({}_0y_{N-i}'(\xi) - {}_Ly_{N-1-i}'(\xi)) \\ &= l\,{}_Lh^{l-1}_{N-i}(\xi)\xi^{1/r-1}(\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-1/r}_{N-1-i}(\xi))\end{aligned} \tag{3.125}$$

while

$$\begin{aligned}|\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-1/r}_{N-1-i}(\xi)| &= |\frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r}|(\frac{N-t}{N})^{r-1} - (\frac{N-s}{N})^{r-1}| \quad t \in [0,1], s \in [0,2] \\ &\le T^{1-1/r}|1 - (\frac{N-2}{N})^{r-1}| \le CT^{1-1/r}(r-1)\frac{2}{N}\end{aligned} \tag{3.126}$$

Thus,

$$({}_Lh^l_{N-i}(\xi))' \le C(r-1)h_N^{l-1}x_i^{1/r-1}h \tag{3.127}$$

And

$$\begin{aligned}({}_Lh^3_{N-i}(\xi))'' &= 3\,{}_Lh^2_{N-i}(\xi)\,{}_Lh_{N-i}''(\xi) + 6\,{}_Lh_{N-i}(\xi)({}_Lh_{N-i}'(\xi))^2 \\ &\le Ch_N^2\frac{1-r}{r}x_i^{1/r-2}(\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i}) + Ch_N(r-1)^2h^2x_i^{2/r-2}\end{aligned} \tag{3.128}$$

$$|\frac{h_N}{rZ_1} - {}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i}| \le T^{1-1/r} + Cx_N^{1-2/r}x_N^{1/r} = CT^{1-1/r}$$

So

$$\begin{aligned}({}_Lh^3_{N-i}(\xi))'' &\le Ch_N^2\frac{1-r}{r}x_i^{1/r-2} + C(r-1)^2h_Nx_i^{2/r-2}h^2 \\ &\le C(r-1)h_N^2\end{aligned} \tag{3.129}$$

${}_Rh^3_{N+1-i}(\xi)$ is similar. ∎

LEMMA 3.27.

$$u''({}_Ly^\theta_{N-i}(\xi)) \le Cx_{N-2}^{-\alpha/2-2} \le C \tag{3.130}$$

$$(u''({}_Ly^\theta_{N-i}(\xi)))' \le C \tag{3.131}$$

$$(u''({}_Ly^\theta_{N-i}(\xi)))'' \le C \tag{3.132}$$

Proof.

$$(u''({}_Ly^{\theta}_{N-i}(\xi)))' = u'''({}_Ly^{\theta}_{N-i}(\xi)){}_L{y^{\theta}_{N-i}}'(\xi) \le C(\theta{}_L{y_{N-1-i}}'(\xi) + (1-\theta){}_0{y_{N-i}}'(\xi)) \le Cx_i^{1/r-1}(\theta{}_Ly^{1-1/r}_{N-1-i}(\xi) + (1-\theta)\frac{h_N}{rZ_1}) \le Cx_i^{1/r-1}x_N^{1-1/r} \tag{3.133}$$

And

$$(u''({}_Ly^{\theta}_{N-i}(\xi)))'' = u''''({}_Ly^{\theta}_{N-i}(\xi))({}_L{y^{\theta}_{N-i}}'(\xi))^2 + u'''({}_Ly^{\theta}_{N-i}(\xi)){}_L{y^{\theta}_{N-i}}''(\xi) \le Cx_i^{2/r-2}x_N^{2-2/r} + C\frac{r-1}{r}x_i^{1/r-2}(\theta x_N^{1-2/r}Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \le Cx_i^{2/r-2} + C(r-1)x_i^{1/r-2}T^{1-1/r} \tag{3.134}$$

∎

Lemma 3.28.

$$|{}_Ly^{\theta}_{N-i}(\xi) - \xi|^{1-\alpha} \le C|y^{\theta}_N - x_i|^{1-\alpha} \tag{3.135}$$

$$(|{}_Ly^{\theta}_{N-i}(\xi) - \xi|^{1-\alpha})' \le C|y^{\theta}_N - x_i|^{1-\alpha} \tag{3.136}$$

$$(|{}_Ly^{\theta}_{N-i}(\xi) - \xi|^{1-\alpha})'' \le C(r-1)|y^{\theta}_N - x_i|^{-\alpha} + |y^{\theta}_N - x_i|^{1-\alpha} \tag{3.137}$$

Proof.

$$({}_Ly^{\theta}_{N-i}(\xi) - \xi)' = (\theta({}_Ly_{N-1-i}(\xi) - \xi) + (1-\theta)({}_0y_{N-i}(\xi) - \xi))' = \theta({}_L{y_{N-1-i}}'(\xi) - 1) + (1-\theta)({}_0{y_{N-i}}'(\xi) - 1) = \theta\xi^{1/r-1}({}_Ly^{1-1/r}_{N-1-i}(\xi) - \xi^{1-1/r}) + (1-\theta)\xi^{1/r-1}(\frac{h_N}{rZ_1} - \xi^{1-1/r}) \tag{3.138}$$

$$({}_Ly^{\theta}_{N-i}(\xi) - \xi)'' = \theta({}_L{y_{N-1-i}}''(\xi)) + (1-\theta)({}_0{y_{N-i}}''(\xi)) = \frac{1-r}{r}\xi^{1/r-2}(\theta{}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \le 0 \tag{3.139}$$

And

$$|({}_Ly^{\theta}_{N-i}(\xi) - \xi)''| \le C(r-1)\xi^{1/r-2}T^{1-1/r} \tag{3.140}$$

We have known

$$C|x_{N-1} - x_i| \le |{}_Ly_{N-1-i}(\xi) - \xi| \le C|x_{N-1} - x_i| \tag{3.141}$$

If $\xi \le x_{N-1}$, then $({}_0y_{N-i}(\xi) - \xi)' \ge 0$, so

$$C|x_N - x_i| \le |x_{N-1} - x_{i-1}| \le |{}_Ly^{\theta}_{N-i}(\xi) - \xi| \le |x_{N+1} - x_{i+1}| \le C|x_N - x_i| \tag{3.142}$$

If $i = N-1$ and $\xi \in [x_{N-1}, x_N]$, then ${}_0y_{N-i}(\xi) - \xi$ is concave, bigger than its two neighboring points, which are equal to h_N, so

$$h_N = |x_N - x_{N-1}| \le |{}_0y_{N-i}(\xi) - \xi| \le |x_{N+1} - x_{N-1}| = 2h_N \tag{3.143}$$

So we have

$$|_L y_{N-i}^{\theta}(\xi)-\xi|^{1-\alpha}\leq C|y_N^{\theta}-x_i|^{1-\alpha} \tag{3.144}$$

While

$$_L y_{N-1-i}^{1-1/r}(\xi)-\xi^{1-1/r}\leq ({_L y_{N-1-i}}(\xi)-\xi)\xi^{-1/r} \tag{3.145}$$

and

$$\begin{aligned}|\frac{h_N}{rZ_1}-\xi^{1-1/r}|&\leq \max\{|\frac{h_N}{rZ_1}-x_{i-1}^{1-1/r}|,|\frac{h_N}{rZ_1}-x_{i+1}^{1-1/r}|\}\\&\leq \max\begin{cases}T^{1-1/r}-x_{i-1}^{1-1/r}\leq |x_N-x_{i-1}|T^{-1/r}\leq C|x_N-x_i|\\|x_{i+1}^{1-1/r}-x_{N-1}^{1-1/r}|\leq |x_{i+1}-x_{N-1}|x_{N-1}^{-1/r}\leq C|x_N-x_i|\end{cases}\end{aligned} \tag{3.146}$$

So we have

$$({_L y_{N-i}^{\theta}}(\xi)-\xi)'\leq C|y_N^{\theta}-x_i| \tag{3.147}$$

$$\begin{aligned}(|_L y_{N-i}^{\theta}(\xi)-\xi|^{1-\alpha})'&=|_L y_{N-i}^{\theta}(\xi)-\xi|^{-\alpha}({_L y_{N-i}^{\theta}}(\xi)-\xi)'\\&\leq |y_N^{\theta}-x_i|^{1-\alpha}\end{aligned} \tag{3.148}$$

Finally,

$$\begin{aligned}(|_L y_{N-i}^{\theta}(\xi)-\xi|^{1-\alpha})''&=(1-\alpha)|_L y_{N-i}^{\theta}(\xi)-\xi|^{-\alpha}({_L y_{N-i}^{\theta}}(\xi)-\xi)''\\&\quad+\alpha(\alpha-1)|_L y_{N-i}^{\theta}(\xi)-\xi|^{-1-\alpha}(({_L y_{N-i}^{\theta}}(\xi)-\xi)')^2\\&\leq C(r-1)|y_N^{\theta}-x_i|^{-\alpha}+C|y_N^{\theta}-x_i|^{1-\alpha}\end{aligned} \tag{3.149}$$ ◻

By the three lemmas above, for $N/2\leq i\leq N-1$, we have

LEMMA 3.29.

$$\begin{aligned}D_h^2{_L P_{N-i}^{\theta}}(x_i)&={_L P_{N-i}^{\theta}}''(\xi)\quad \xi\in(x_{i-1},x_{i+1})\\&\leq Ch^3|y_N^{\theta}-x_i|^{1-\alpha}+C(r-1)(h^3|y_N^{\theta}-x_i|^{-\alpha}+h^2|y_N^{\theta}-x_i|^{1-\alpha})\end{aligned} \tag{3.150}$$

while $\theta h_N=y_N^{\theta}-x_{N-1}\leq y_N^{\theta}-x_i$, *we have*

$$\theta D_h^2{_L P_{N-i}^{\theta}}(x_i)\leq Ch^3|y_N^{\theta}-x_i|^{1-\alpha}+C(r-1)(h^2|y_N^{\theta}-x_i|^{1-\alpha}) \tag{3.151}$$

And

LEMMA 3.30.

$$\begin{aligned}&\frac{2}{h_i+h_{i+1}}\left(\frac{{_L Q_{N-i}^{\theta}}(x_{i+1})u'''(\eta_{N+1}^{\theta})-{_L Q_{N-i}^{\theta}}(x_i)u'''(\eta_N^{\theta})}{h_{i+1}}\right)\\&\leq Ch^3|y_N^{\theta}-x_i|^{1-\alpha}\end{aligned} \tag{3.152}$$

And immediately with (3.26), For $N/2\leq i\leq N-1$

$$\begin{aligned}V_{iN}&\leq C\int_{x_{N-1}}^{x_N}h^2|y-x_i|^{1-\alpha}+C(r-1)h|y-x_i|^{1-\alpha}dy\\&\leq Ch^2h_N|T-x_i|^{1-\alpha}+C(r-1)h^2|x_N-x_i|^{1-\alpha}\\&\leq Ch^2+C(r-1)h^2|T-x_{i-1}|^{1-\alpha}\end{aligned} \tag{3.153}$$

Similarly with $j=N+1$. ◻

I_4, I_5 is easy. Similar with Lemma 3.20 and Lemma 3.13, we have

THEOREM 3.31. *There is a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le i \le N$,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right) \le Ch^2 \tag{3.154}$$

Proof. Similar with Lemma 3.20. In fact, let $m = 2N - \lceil \frac{N}{2} \rceil + 1$

$$\begin{aligned} &\frac{1}{h_i}(T_{i-1,l} + T_{i-1,l-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,l} \\ &= \frac{1}{h_i}(T_{i-1,l} - T_{i,l}) + \frac{1}{h_i}(T_{i-1,l-1} - T_{i,l}) + (\frac{1}{h_i} - \frac{1}{h_{i+1}}) T_{i,l} \end{aligned} \tag{3.155}$$

While, by Lemma A.2

$$\begin{aligned} \frac{1}{h_i}(T_{i-1,l} - T_{i,l}) &= \int_{x_{l-1}}^{x_l} (u(y) - \Pi_h u(y)) \frac{|x_{i-1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_i \Gamma(2-\alpha)} dy \\ &\le C \int_{x_{l-1}}^{x_l} h_l^2 u''(\eta) \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy, \quad \xi \in (x_{i-1}, x_i) \\ &\le C h_l^3 (2T - x_{l-1})^{\alpha/2-2} T^{-\alpha} \\ &\le C h_l^3 \end{aligned} \tag{3.156}$$

Thus,

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_i} (T_{i-1,l} - T_{i,l}) \le C h_l^2 \tag{3.157}$$

For

$$\frac{1}{h_i}(T_{i-1,l-1} - T_{i,l}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{h_i} d\theta \tag{3.158}$$

And Similar with Lemma 3.17, we can get

$$\frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{(h_i + h_{i+1}) h_i} \le C h_l^2 |y_l^\theta - x_i|^{1-\alpha} \tag{3.159}$$

So

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_i} (T_{i-1,l-1} - T_{i,l}) \le Ch^2 \tag{3.160}$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned} \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,l} &\le h_i^{-3} h^2 x_i^{1-2/r} h_l C h_l^2 x_{l-1}^{\alpha/2-2} |x_l - x_i|^{1-\alpha} \\ &\le Ch^2 \end{aligned} \tag{3.161}$$

Summarizes, we have

$$I_4 \le Ch^2 \tag{3.162}$$

□

Now we can conclude a part of the theorem Theorem 3.3 at the beginning of this section.

By Lemma 3.15, Lemma 3.20, **??**, Theorem 3.25, Lemma 3.24, Theorem 3.31, Lemma 3.13, we have

THEOREM 3.32. *there exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $N/2 \le i \le N-1$,

$$\begin{aligned} R_i &= I_1 + I_2 + I_3^1 + I_3^2 + I_3^3 + I_4 + I_5 \\ &\le C(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned} \tag{3.163}$$

And what we left is the case $i = N$. Fortunately, we can use the same department of R_i above, and it is symmetric. Most of the item has been esitmated by Lemma 3.15 and Theorem 3.31, we just need to consider I_3, I_4.

THEOREM 3.33. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$I_3 = \sum_{j=\lceil \frac{N}{2} \rceil+1}^{N-1} V_{Nj} \le Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha} \tag{3.164}$$

Proof. DEFINITION 3.34. *For $N/2 \le j < N$, Let's define*

$$y_j(x) = (\frac{Z_1}{h_N}(x - x_N) + Z_j)^r, \quad Z_j = T^{1/r}\frac{j}{N} \tag{3.165}$$

We can see that is the inverse of the function ${}_0y_{N-i}(x)$ defined in Theorem 3.25.

$$y_j'(x) = y_j^{1-1/r}(x)\frac{rZ_1}{h_N} \tag{3.166}$$

$$y_j''(x) = y_j^{1-2/r}(x)\frac{r(r-1)Z_1}{h_N} \tag{3.167}$$

With the scheme we used several times, we can get

LEMMA 3.35. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le j < N$, $\xi \in [x_{N-1}, x_{N+1}]$,*

$$h_j(\xi)^3 \le Ch^3 \tag{3.168}$$

$$(h_j^3(\xi))' \le C(r-1)h^3 \tag{3.169}$$

$$(h_j^3(\xi))'' \le C(r-1)h^3 \tag{3.170}$$

$$u''(y_j^\theta(\xi)) \le C \tag{3.171}$$

$$(u''(y_j^\theta(\xi)))' \le C \tag{3.172}$$

$$(u''(y_j^\theta(\xi)))'' \le C \tag{3.173}$$

$$|\xi - y_j^\theta(\xi)|^{1-\alpha} \le C|x_N - y_j^\theta|^{1-\alpha} \tag{3.174}$$

$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})' \le C|x_N - y_j^\theta|^{1-\alpha} \tag{3.175}$$

$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})'' \le C|x_N - y_j^\theta|^{1-\alpha} + C(r-1)|x_N - y_j^\theta|^{-\alpha} \tag{3.176}$$

LEMMA 3.36. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le j < N$,*

$$V_{Nj} \le Ch^2 \int_{x_{j-1}}^{x_j} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \tag{3.177}$$

Therefore,

$$\begin{aligned} I_3 &\le Ch^2 \int_{x_{\lceil \frac{N}{2} \rceil}}^{x_{N-1}} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \\ &\le Ch^2(|T - x_{N-1}|^{2-\alpha} + (r-1)|T - x_{N-1}|^{1-\alpha}) \end{aligned} \tag{3.178}$$

$\square$

For $j = N$,

LEMMA 3.37.

$$V_{N,N} = \frac{1}{h_N^2}\left(T_{N-1,N-1} - 2T_{N,N} + T_{N+1,N+1}\right) \leq Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha} \tag{3.179}$$

Proof. ∎

$$\begin{aligned}
V_{N,N} = &\int_0^1 -\frac{\theta(1-\theta)^{2-\alpha}}{2}\frac{1}{h_N^2}\left(h_{N-1}^{4-\alpha}u''(y_{N-1}^\theta) - 2h_N^{4-\alpha}u''(y_N^\theta) + h_{N+1}^{4-\alpha}u''(y_{N+1}^\theta)\right)d\theta \\
&+ \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,1}^\theta) - Q_{N\to N}^\theta(x_i)u'''(\eta_{N,1}^\theta)}{h_N}\right)d\theta \\
&- \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,1}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,1}^\theta)}{h_N}\right)d\theta \\
&- \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,2}^\theta) - Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta)}{h_N}\right)d\theta \\
&+ \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,2}^\theta)}{h_N}\right)d\theta
\end{aligned} \tag{3.180}$$

So combine Lemma 3.15, Theorem 3.31, Theorem 3.33, Lemma 3.37 We have

LEMMA 3.38.

$$R_N \leq C(r-1)h^2|T - x_{N-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.181}$$

and with Theorem 3.32 we prove the Theorem 3.3

4. Convergence analysis.

4.1. Properties of some Matrices.

Review subsection 2.1, we have got (2.10).

DEFINITION 4.1. *We call one matrix an M matrix , which means its entries are positive on major diagonal and nonpositive on others, and strictly diagonally dominant in rows.*

Now we have

LEMMA 4.2. *Matrix A defined by* (2.12) *where* (2.13) *is an M matrix. And there exists a constant $C_A = C(T, \alpha, r)$ such that*

$$S_i := \sum_{j=1}^{2N-1} a_{ij} \geq C_A(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \tag{4.1}$$

Proof. From (2.14), we have

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right) \tag{4.2}$$

Let

$$g(x) = g_0(x) + g_{2N}(x) \tag{4.3}$$

where

$$g_0(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x - x_0|^{3-\alpha} - |x - x_1|^{3-\alpha}}{h_1}$$

$$g_{2N}(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N} - x|^{3-\alpha} - |x_{2N-1} - x|^{3-\alpha}}{h_{2N}}$$

Thus

$$-\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

Then

$$\begin{aligned} S_i :=& \sum_{j=1}^{2N-1} a_{ij} \\ =& \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ =& D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i) \end{aligned} \tag{4.4}$$

When $i = 1$

$$\begin{aligned} D_h^2 g_0(x_1) = & \frac{2}{h_1 + h_2} \left(\frac{1}{h_2} g_0(x_2) - (\frac{1}{h_1} + \frac{1}{h_2}) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right) \\ =& \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2) h_1 h_2} \\ =& \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2) h_1^{1-\alpha} h_2} h_1^{-\alpha} \\ =& \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha}}{2^r (2^r - 1)} h_1^{-\alpha} \end{aligned} \tag{4.5}$$

but

$$1+(2^r-1)^{3-\alpha}+2(2^r-1)-(2^r)^{3-\alpha}>0 \tag{4.6}$$

While for $i\geq 2$

$$\begin{aligned} D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi\in(x_{i-1},x_{i+1}) \\ &= -\kappa_\alpha \frac{|\xi-x_0|^{1-\alpha}-|\xi-x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1} \\ &= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)}|\xi-\eta|^{-\alpha}, \quad \eta\in[x_0,x_1] \\ &\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)}x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)}2^{-r\alpha}x_i^{-\alpha} \end{aligned} \tag{4.7}$$

So

$$\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}g_0(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})g_0(x_i)+\frac{1}{h_i}g_0(x_{i-1})\right)\geq Cx_i^{-\alpha} \tag{4.8}$$

symmetricly,

$$\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}g_{2N}(x_{i+1})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})g_{2N}(x_i)+\frac{1}{h_i}g_{2N}(x_{i-1})\right)\geq C(\alpha,r)(2T-x_i)^{-\alpha} \tag{4.9}$$

$\square$

Let

$$g(x)=\begin{cases} x, & 0<x\leq T \\ 2T-x, & T<x<2T \end{cases} \tag{4.10}$$

And define

$$G=\mathrm{diag}(g(x_1),...,g(x_{2N-1})) \tag{4.11}$$

Then

LEMMA 4.3. *The matrix $B:=AG$, the major diagnal is positive, and nonpositive on others. And there is a constant $C_{AG}, C=C(\alpha,r)$ such that*

$$M_i:=\sum_{j=1}^{2N-1}b_{ij}\geq -C_{AG}(x_i^{1-\alpha}+(2T-x_i)^{1-\alpha})+C\begin{cases}|T-x_{i-1}|^{1-\alpha}, & i\leq N \\ |x_{i+1}-T|^{1-\alpha}, & i\geq N\end{cases} \tag{4.12}$$

Proof.

$$b_{ij}=a_{ij}g(x_j)=-\kappa_\alpha\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}\tilde{a}_{i+1,j}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})\tilde{a}_{i,j}+\frac{1}{h_i}\tilde{a}_{i-1,j}\right)g(x_j)$$

Since

$$g(x)\equiv\Pi_h g(x) \tag{4.13}$$

by **??**, we have

$$
\begin{aligned}
\tilde{M}_i &:= \sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} \tilde{a}_{ij} g(x_j) \\
&= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} \Pi_h g(y) dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} g(y) dy \\
&= \frac{-2}{\Gamma(4-\alpha)} |T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)} (x_i^{3-\alpha} + (2T - x_i)^{3-\alpha}) \\
&:= w(x_i) = p(x_i) + q(x_i)
\end{aligned} \tag{4.14}
$$

Thus,

$$
\begin{aligned}
M_i &:= \sum_{j=1}^{2N-1} b_{ij} = \sum_{j=1}^{2N-1} a_{ij} g(x_j) \\
&= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \tilde{M}_{i+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) \tilde{M}_i + \frac{1}{h_i} \tilde{M}_{i-1} \right) \\
&= D_h^2(-\kappa_\alpha p)(x_i) - \kappa_\alpha D_h^2 q(x_i)
\end{aligned} \tag{4.15}
$$

for $1 \leq i \leq N-1$,by Lemma A.1

$$
\begin{aligned}
D_h^2(-\kappa_\alpha p)(x_i) &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} p(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) p(x_i) + \frac{1}{h_i} p(x_{i-1}) \right) \\
&= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1}) \\
&\geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - x_{i-1}|^{1-\alpha}
\end{aligned} \tag{4.16}
$$

$$
\begin{aligned}
D_h^2(-\kappa_\alpha p)(x_N) &:= -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left(\frac{1}{h_{N+1}} p(x_{N+1}) - (\frac{1}{h_N} + \frac{1}{h_{N+1}}) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right) \\
&= \frac{4\kappa_\alpha}{\Gamma(4-\alpha) h_N^2} h_N^{3-\alpha} \\
&= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - x_{N-1})^{1-\alpha}
\end{aligned} \tag{4.17}
$$

Symmetricly for $i \geq N$, we get

$$
D_h^2(-\kappa_\alpha p)(x_i) \geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \tag{4.18}
$$

Similarly, we can get

$$
\begin{aligned}
D_h^2 q(x_i) &:= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} q(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right) \\
&\leq \frac{2^{r(\alpha-1)+1}}{\Gamma(2-\alpha)} (x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}), \quad i = 1, \cdots, 2N-1
\end{aligned} \tag{4.19}
$$

□

So, we get the result.

Notice that

$$x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha} \tag{4.20}$$

We can get

THEOREM 4.4. *There exists a real $\lambda = \lambda(T, \alpha, r) > 0$ and $C = C(T, \alpha, r) > 0$ such that $B := A(\lambda I + G)$ is an M matrix. And*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \tag{4.21}$$

Proof. By Lemma 4.2 with C_A and Lemma 4.3 with C_{AG} , it's sufficient to take $\lambda = (C + 2TC_{AG})/C_A$, then

$$M_i \geq C \left((x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \right) \tag{4.22}$$

□

4.2. Proof of Theorem 2.6. For equation

$$AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1} U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1} U = F \tag{4.23}$$

which means

$$\sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + g(x_j)} = -\tau_i \tag{4.24}$$

where $\epsilon_i = u(x_i) - u_i$.

And if

$$|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| = \max_{1 \leq i \leq 2N-1} |\frac{\epsilon_i}{\lambda + g(x_i)}| \tag{4.25}$$

Then, since $B = A(\lambda I + G)$ is an M matrix, it is Strictly diagonally dominant. Thus,

$$\begin{aligned} |\tau_{i_0}| &= |\sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + g(x_j)}| \\ &\geq b_{i_0,i_0} |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| - \sum_{j \neq i_0} |b_{i_0,j}| |\frac{\epsilon_j}{\lambda + g(x_j)}| \\ &\geq b_{i_0,i_0} |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| - \sum_{j \neq i_0} |b_{i_0,j}| |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| \\ &= \sum_{j=1}^{2N-1} b_{i_0,j} |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| \\ &= M_{i_0} |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| \end{aligned} \tag{4.26}$$

By Theorem 2.5 and Theorem 4.4,

We kown that there exists constants $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$, and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that

$$|\frac{\epsilon_i}{\lambda + g(x_i)}| \leq |\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}| \leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} + C_2(r-1)h^2 \tag{4.27}$$

as $\lambda + g(x_i) \leq \lambda + T$

So, we can get

$$|\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{r\alpha}{2},2\}} \tag{4.28}$$

The convergency has been proved.

Remarks:

5. Experimental results.

5.1. $f \equiv 1$.

5.2. $f = x^\gamma, \gamma < 0$. Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x) \in C^2(\Omega)$, there exists $\xi \in (x_{i-1}, x_{i+1})$ such that*

$$D_h^2 g(x_i) = g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{A.1}$$

And if $g(x) \in C^4(\Omega)$, then

$$\begin{aligned} D_h^2 g(x_i) &= g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) \\ &\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy \right) \end{aligned} \tag{A.2}$$

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})$$

Subsitute them in the left side of (A.1), we have

$$\begin{aligned} D_h^2 g(x_i) &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (g(x_{i+1}) - g(x_i) + \frac{1}{h_i} (g(x_{i-1}) - g(x_i)) \right) \\ &= \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) \end{aligned}$$

Now, using intermediate value theorem, there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy$$

Expecially,

$$\begin{aligned} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy &= \frac{h_i^4}{4!} g''''(\eta_1) \\ \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy &= \frac{h_{i+1}^4}{4!} g''''(\eta_2) \end{aligned} \tag{A.3}$$

where $\eta_1 \in (x_{i-1}, x_i), \eta_2 \in (x_i, x_{i+1})$. ∎

LEMMA A.2. *Denote* $y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \theta \in (0,1)$,

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in (x_{j-1}, x_j) \tag{A.4}$$

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \tag{A.5}$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in (x_{j-1}, y_j^\theta)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in (y_j^\theta, x_j)$$

Thus

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta) u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$. Thus

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta)u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

∎

LEMMA A.3. *For* $x \in [x_{j-1}, x_j]$

$$\begin{aligned} |u(x) - \Pi_h u(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y) dy - \frac{x - x_{j-1}}{h_j} \int_x^{x_j} u'(y) dy \right| \\ &\le \int_{x_{j-1}}^{x_j} |u'(y)| dy \end{aligned} \tag{A.6}$$

If $x \in [0, x_1]$, *with Corollary* 2.4, *we have*

$$|u(x) - \Pi_h u(x)| \le \int_0^{x_1} |u'(y)| dy \le \int_0^{x_1} C y^{\alpha/2-1} dy \le C \frac{2}{\alpha} x_1^{\alpha/2} = C \frac{2}{\alpha} h_1^{\alpha/2} \tag{A.7}$$

Similarly, if $x \in [x_{2N-1}, 1]$, *we have*

$$|u(x) - \Pi_h u(x)| \le C \frac{2}{\alpha} (2T - x_{2N-1})^{\alpha/2} = C \frac{2}{\alpha} h_{2N}^{\alpha/2} \tag{A.8}$$

LEMMA A.4. *By Lemma* A.2, *Corollary* 2.4 *and Lemma* B.1 , *There is a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *for* $2 \le j \le N$,

$$|u(y) - \Pi_h u(y)| \le h_j^2 \max_{\xi \in [x_{j-1}, x_j]} |u''(\xi)| \le Ch^2 y^{\alpha/2-2/r}, \quad \text{for } y \in (x_{j-1}, x_j) \tag{A.9}$$

symmetricly, for $N < j \le 2N - 1$, *we have*

$$|u(y) - \Pi_h u(y)| \le Ch^2(2T - y)^{\alpha/2-2/r}, \quad \text{for } y \in (x_{j-1}, x_j) \tag{A.10}$$

LEMMA A.5.

$$b^{1-\theta}|a^\theta - b^\theta| \le |a - b| \ (\text{ also } a^{1-\theta}|a^\theta - b^\theta| \le |a - b|), \quad a, b \ge 0, \ \theta \in [0, 1] \tag{A.11}$$

Appendix B. Proofs of some technical details. Review that $h = \frac{1}{N}$ and the defination of $\simeq$ in subsection 2.1

LEMMA B.1.

$$h_i \simeq \begin{cases} h x_i^{1-1/r}, & 1 \le i \le N \\ h(2T - x_{i-1})^{1-1/r}, & N < i \le 2N \end{cases} \tag{B.1}$$

Since $i^r - (i-1)^r \simeq i^{r-1}$, *for* $i \ge 1$.

And

$$h_i \simeq h_{i+1}, \quad x_i \simeq x_{i+1} \simeq y_i^\theta, \quad \text{for } 1 \le i \le 2N - 1, \ \theta \in (0, 1) \tag{B.2}$$

LEMMA B.2. *There is a constant* C *such that for* $i = 1, 2, \cdots, 2N - 1$

$$|h_{i+1} - h_i| \le Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N - 1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N - 1 \end{cases} \tag{B.3}$$

Proof. By (2.2),

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \le i \le N - 1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N + 1 \le i \le 2N - 1 \end{cases} \tag{B.4}$$

Since

$$(i+1)^r - 2i^r + (i-1)^r \simeq r(r-1)i^{r-2}, \quad \text{for } i \ge 1 \tag{B.5}$$

We get the result. ∎

LEMMA B.3. *there is a constant* $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ *such that*

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left| \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right| \\ &\le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N - 1 \end{cases} \end{aligned} \tag{B.6}$$

Proof. By Lemma 2.2, we have for $1 \le i \le N$

$$\left|\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy\right| \le \frac{\|f\|_\beta^{(\alpha/2)}}{3!}\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y-x_{i-1})^3 dy \tag{B.7}$$

For $i=1$,

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y-x_{i-1})^3 dy = \int_0^{x_1} y^{1-\alpha/2}dy = \frac{1}{2-\alpha/2}x_1^{2-\alpha/2} = \frac{1}{2-\alpha/2}x_1^{-\alpha/2-2}h_1^4$$

And for $2 \le i \le N$, since $x_i \simeq x_{i-1} \le y \le x_i$, we have

$$\int_{x_{i-1}}^{x_i} y^{-\alpha/2-2}(y-x_{i-1})^3 dy \simeq \int_{x_{i-1}}^{x_i} x_i^{-\alpha/2-2}(y-x_{i-1})^3 dy = \frac{1}{4!}x_i^{-\alpha/2-2}h_i^4$$

So for $1 \le i \le N$, we have

$$\left|\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy\right| \le Cx_i^{-\alpha/2-2}h_i^4 \tag{B.8}$$

and similarly,

$$\left|\int_{x_i}^{x_{i+1}} f''(y)\frac{(x_{i+1}-y)^3}{3!}dy\right| \le Cx_i^{-\alpha/2-2}h_{i+1}^4 \tag{B.9}$$

Thus for $1 \le i \le N$, with Lemma B.1 we have

$$\begin{aligned}
&\frac{2}{h_i+h_{i+1}}\left|\frac{1}{h_i}\int_{x_{i-1}}^{x_i} f''(y)\frac{(y-x_{i-1})^3}{3!}dy + \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}} f''(y)\frac{(y-x_{i+1})^3}{3!}dy\right| \\
&\le Cx_i^{-\alpha/2-2}\frac{2}{h_i+h_{i+1}}(h_i^3+h_{i+1}^3) \simeq x_i^{-\alpha/2-2}h_i^2 \simeq x_i^{-\alpha/2-2}h^2x_i^{2-2/r} \\
&= Ch^2x_i^{-\alpha/2-2/r}
\end{aligned} \tag{B.10}$$

It's symmetric for $N < i \le 2N-1$. ∎

Lemma B.4. *There is a constant $C = C(\alpha, r)$ such that for all $1 \le i \le 2N-1$, $1 \le j \le 2N$ s.t. $\min\{|j-i|, |j-1-i|\} \ge 2$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_hK_y(x_i) \simeq |y-x_i|^{-\alpha}, \quad D_h^2K_y(x_i) \simeq |y-x_i|^{-1-\alpha} \tag{B.11}$$

Proof. Sinec $y-x_{i-1}, y-x_i, y-x_{i+1}$ have the same sign, by mean value theorem and Lemma A.1,

$$\begin{aligned}
D_hK_y(x_i) &= \frac{|y-\xi|^{-\alpha}}{\Gamma(1-\alpha)}, \quad \xi \in (x_i, x_{i+1}) \\
D_h^2K_y(x_i) &= \frac{|y-\xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in (x_{i-1}, x_{i+1})
\end{aligned}$$

however, $|y-\xi| \simeq |y-x_i|$, we get the result. ∎

Lemma B.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$\sum_{j=1}^{3} V_{1j} \le Ch^2x_1^{-\alpha/2-2/r} \tag{B.12}$$

$$\sum_{j=1}^{4} V_{2j} \le Ch^2 x_2^{-\alpha/2-2/r} \tag{B.13}$$

Proof. For $0 \le i \le 3, 1 \le j \le 4$, by Lemma A.3, Lemma A.4 and (3.11)

$$T_{ij} \le Cx_1^{2-\alpha/2} \simeq h_1^2\, h^2 x_1^{-\alpha/2-2/r} \simeq h_1^2\, h^2 x_2^{-\alpha/2-2/r} \tag{B.14}$$

Therefore, by (3.12), we get the result. ∎

Proof of **??**.

$$\begin{aligned} |y^{\theta}_{j-i}(\xi) - \xi| &= |\theta(y_{j-i-1}(\xi) - \xi) + (1-\theta)(y_{j-i}(\xi) - \xi)| \\ &= \theta|y_{j-i-1}(\xi) - \xi| + (1-\theta)|y_{j-i}(\xi) - \xi| \end{aligned} \tag{B.15}$$

where $y_{j-i-1}(\xi) - \xi$ and $y_{j-i}(\xi) - \xi$ have the same sign (≥ 0 or ≤ 0), independent with ξ.

Since $|y_{j-i}(\xi) - \xi| = \mathrm{sign}(j-i)(y_{j-i}(\xi) - \xi)$ is increasing with ξ,

$$(\frac{i-1}{i})^r|x_j - x_i| \le |x_{j-1} - x_{i-1}| \le |y_{j-i}(\xi) - \xi| \le |x_{j+1} - x_{i+1}| \le (\frac{i+1}{i})^r|x_j - x_i| \tag{B.16}$$

we have

$$|y_{j-i}(\xi) - \xi| \simeq |x_j - x_i| \tag{B.17}$$

Similarly, $|y_{j-1-i}(\xi) - \xi| \simeq |x_{j-1} - x_i|$. Thus, with (B.15) , (B.17) and (2.17) we get

$$|y^{\theta}_{j-i}(\xi) - \xi| \simeq |y^{\theta}_j - x_i| \tag{B.18}$$

Next, since $|y^{\theta}_{j-i}(\xi) - \xi| = \mathrm{sign}(j-i-1+\theta)(y^{\theta}_{j-i}(\xi) - \xi)$, so we can derivate it.

$$|(|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})'| = (\alpha-1)|y^{\theta}_{j-i}(\xi) - \xi|^{-\alpha}|(y^{\theta}_{j-i}(\xi))' - 1| \tag{B.19}$$

While, similar with (B.15), we have

$$|(y^{\theta}_{j-i}(\xi))' - 1| = (1-\theta)|y'_{j-i-1}(\xi) - 1| + \theta|y'_{j-i}(\xi) - 1| \tag{B.20}$$

By Lemma A.5 and (B.17), we have

$$\begin{aligned} |y'_{j-i}(\xi) - 1| &= \xi^{1/r-1}|y^{1-1/r}_{j-i}(\xi) - \xi^{1-1/r}| \\ &\le \xi^{-1}|y_{j-i}(\xi) - \xi| \\ &\simeq x_i^{-1}|x_j - x_i| \end{aligned} \tag{B.21}$$

So similar with (B.18), we can get

$$|(y^{\theta}_{j-i}(\xi))' - 1| \le Cx_i^{-1}|y^{\theta}_j - x_i| \tag{B.22}$$

Combine with (B.18), we get

$$|(|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})'| \le C|y^{\theta}_j - x_i|^{-\alpha}x_i^{-1}|y^{\theta}_j - x_i| = C|y^{\theta}_j - x_i|^{1-\alpha}x_i^{-1} \tag{B.23}$$

Finally, we have

$$\begin{aligned} (|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})'' &= \alpha(\alpha-1)|y^{\theta}_{j-i}(\xi) - \xi|^{-\alpha-1}((y^{\theta}_{j-i}(\xi))' - 1)^2 \\ &\quad + \mathrm{sign}(j-i-1+\theta)(1-\alpha)|y^{\theta}_{j-i}(\xi) - \xi|^{-\alpha}(y^{\theta}_{j-i}(\xi))'' \end{aligned} \tag{B.24}$$

For

$$(y^{\theta}_{j-i}(\xi))'' = (1-\theta)y''_{j-i-1}(\xi) + \theta y''_{j-i}(\xi) \tag{B.25}$$

and

$$\begin{aligned} y''_{j-i}(\xi) &= \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i} \\ &\simeq \frac{1-r}{r} x_j^{1-2/r} x_i^{1/r-2} Z_{j-i} \end{aligned} \tag{B.26}$$

while by Lemma A.5

$$|Z_{j-i}| = |x_j^{1/r} - x_i^{1/r}| \le |x_j - x_i| x_i^{1/r-1} \tag{B.27}$$

we have

$$|y''_{j-i}(\xi)| \le C(r-1)x_i^{-2}|x_j - x_i| \tag{B.28}$$

Therefore

$$|(y^{\theta}_{j-i}(\xi))''| \le C(r-1)x_i^{-2}|y^{\theta}_j - x_i| \tag{B.29}$$

Then, combine with (B.22),

$$|(|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})''| \le C|y^{\theta}_j - x_i|^{1-\alpha} x_i^{-2} \tag{B.30}$$

□

Lemma B.6. *There exists a constant $C = C(T,r)$ such that For $N/2 \le i \le N-1$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $l = 3,4$, $\xi \in (x_{i-1}, x_{i+1})$, we have*

$$h^{l}_{j-i}(\xi) \le Ch^{l}_j \le Ch^2 h^{l-2}_j \tag{B.31}$$

$$(h^{l}_{j-i-1}(\xi))' \le C(r-1)h^2 h^{l-2}_j \tag{B.32}$$

$$(h^{3}_{j-i}(\xi))'' \le C(r-1)h^2 h_j \tag{B.33}$$

Proof.

$$\begin{aligned} (h_{j-i}(\xi))' &= {y_{j-i}}'(\xi) - {y_{j-i-1}}'(\xi) \\ &= \xi^{1/r-1}((2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}) \le 0 \end{aligned} \tag{B.34}$$

Thus,

$$Ch_j \le h_{j+1} \le h_{j-i}(\xi) \le h_{j-i}(x_{i-1}) = h_{j-1} \le Ch_j \tag{B.35}$$

So as $4^{-r}T \le 2T - x_j \le T, 2^{-r}T \le x_i \le T$, we have

$$h^{l}_{j-i}(\xi) \le Ch^{l}_j \le Ch^2(2T - x_j)^{2-2/r} h^{l-2}_j \le Ch^2 h^{l-2}_j \tag{B.36}$$

Since

$$\begin{aligned} &|(2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}| \\ &\quad = |(Z_{2N-(j-i)} - \xi^{1/r})^{r-1} - (Z_{2N-(j-1-i)} - \xi^{1/r})^{r-1}| \\ &\quad = (r-1)Z_1(Z_{2N-(j-i-\gamma)} - \xi^{1/r})^{r-2} \quad \gamma \in [0,1] \\ &\quad \le C(r-1)h(2T - x_j)^{1-2/r} \end{aligned} \tag{B.37}$$

we have

$$|(h_{j-i}(\xi))'| \le C(r-1)h(2T-x_j)^{1-2/r}x_i^{1/r-1} \tag{B.38}$$

And

$$\begin{aligned}(h^l_{j-i}(\xi))' &= l h^{l-1}_{j-i}(\xi) h_{j-i}{}'(\xi)\\ &\le C(r-1)h_j^{l-1}\, h(2T-x_j)^{1-2/r}x_i^{1/r-1}\\ &\le C(r-1)h^2 h_j^{l-2}(2T-x_j)^{2-3/r}x_i^{1-1/r}\\ &\le C(r-1)h^2h_j^{l-2}\end{aligned} \tag{B.39}$$

(B.40)
$$\begin{aligned}(h^3_{j-i}(\xi))'' &= 6h_{j-i}(\xi)(y_{j-i}{}'(\xi)-y_{j-i-1}{}'(\xi))^2 + 3h^2_{j-i}(\xi)(y_{j-i}{}''(\xi)-y_{j-i-1}{}''(\xi))\\ &\le C(r-1)h_jh^2 + Ch_j^2\frac{1-r}{r}\xi^{1/r-2}((2T-y_{j-i}(\xi))^{1-2/r}Z_{2N-(j-i)} - (2T-y_{j-i-1}(\xi))^{1-2/r}Z_{2N-(j-1-i)})\\ &\le C(r-1)h_jh^2 + C(r-1)h_j^2(C(r-2)h(2T-x_j)^{1-3/r}Z_{2N-(j-i)} + Z_1(2T-x_{j-1})^{1-2/r})\\ &\le C(r-1)h_jh^2 + C(r-1)h_j^2h = Ch^2h_j\end{aligned}$$

∎

LEMMA B.7. *There exists a constant $C = C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \le i \le N-1$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in (x_{i-1}, x_{i+1})$, we have*

$$u''(y^\theta_{j-i}(\xi)) \le C \tag{B.41}$$

$$(u''(y^\theta_{j-i}(\xi)))' \le C \tag{B.42}$$

$$(u''(y^\theta_{j-i}(\xi)))'' \le C \tag{B.43}$$

Proof.

$$x_{j-2} \le y^\theta_{j-i}(\xi) \le x_{j+1} \Rightarrow 4^{-r}T \le 2T - y^\theta_{j-i}(\xi) \le T \tag{B.44}$$

Thus, for $l = 2,3,4$,

$$u^{(l)}(y^\theta_{j-i}(\xi)) \le C(2T - y^\theta_{j-i}(\xi))^{\alpha/2-l} \le C \tag{B.45}$$

and

$$\begin{aligned}(y^\theta_{j-i}(\xi))' &= \theta y_{j-1-i}{}'(\xi) + (1-\theta)y_{j-i-1}{}'(\xi)\\ &= \xi^{1/r-1}(\theta(2T-y_{j-1-i}(\xi))^{1-1/r} + (1-\theta)(2T-y_{j-i-1}(\xi))^{1-1/r})\\ &\le C(2T-x_{j-2})^{1-1/r} \le C\end{aligned} \tag{B.46}$$

With

$$Z_{2N-j-i} \le 2T^{1/r} \tag{B.47}$$

(B.48)
$$\begin{aligned}(y^\theta_{j-i}(\xi))'' &= \theta y_{j-1-i}{}''(\xi) + (1-\theta)y_{j-i-1}{}''(\xi)\\ &= \frac{1-r}{r}\xi^{1/r-2}(\theta(2T-y_{j-i-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)} + (1-\theta)(2T-y_{j-i}(\xi))^{1-2/r}Z_{2N-(j-i)})\\ &\le C(r-1)\end{aligned}$$

Therefore,

$$(u''(y^\theta_{j-i}(\xi)))' = u'''(y^\theta_{j-i}(\xi))(y^\theta_{j-i}(\xi))' \\ \leq C \tag{B.49}$$

$$(u''(y^\theta_{j-i}(\xi)))'' = u'''(y^\theta_{j-i}(\xi))({y^\theta_{j-i}}'(\xi))^2 + u''''(y^\theta_{j-i}(\xi)){y^\theta_{j-i}}''(\xi) \\ \leq C + C(r-1) = C \tag{B.50}$$

∎

LEMMA B.8. *There exists a constant $C = C(T, \alpha, r)$ such that For $N/2 \leq i \leq N-1$, $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in (x_{i-1}, x_{i+1})$*

$$|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha} \leq C|y^\theta_j - x_i|^{1-\alpha} \tag{B.51}$$

$$\left|(|y^\theta_{j-i}(\xi) - \xi)^{1-\alpha}|'\right| \leq C|y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_N) \tag{B.52}$$

$$\left|(|y^\theta_{j-i}(\xi) - \xi)^{1-\alpha}|''\right| \leq C(r-1)|y^\theta_j - x_i|^{-\alpha} + C|y^\theta_j - x_i|^{-1-\alpha}(|2T - x_i - y^\theta_j| + h_N)^2 \tag{B.53}$$

Proof. Since $y_{j-i-1}(\xi) > x_{j-2} \geq x_N > \xi$

$$y^\theta_{j-i}(\xi) - \xi = (1-\theta)(y_{j-1-i}(\xi) - \xi) + \theta(y_{j-i}(\xi) - \xi) > 0 \tag{B.54}$$

$$(y_{j-i}(\xi) - \xi)'' = {y_{j-i}}''(\xi) \\ = \frac{1-r}{r}\xi^{1/r-2}(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} \leq 0 \tag{B.55}$$

It's concave, so

$$y_{j-i}(\xi) - \xi \geq \min_{\xi \in \{x_{i-1}, x_{i+1}\}} y_{j-i}(\xi) - \xi = \min\{x_{j+1} - x_{i+1}, x_{j-1} - x_{i-1}\} \geq C(x_j - x_i) \tag{B.56}$$

With (B.54), we have

$$|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha} \leq C|y^\theta_j - x_i|^{1-\alpha} \tag{B.57}$$

By Lemma A.5

$$|{y_{j-i}}'(\xi) - 1| = \xi^{1/r-1}|(2T - y_{j-i}(\xi))^{1-1/r} - \xi^{1-1/r}| \\ \leq \xi^{-1}|2T - y_{j-i}(\xi) - \xi| \tag{B.58}$$

$$|2T - \xi - y_{j-i}(\xi)| \leq |2T - x_i - x_j| + |x_i - \xi| + |x_j - y_{j-i}(\xi)| \\ \leq |2T - x_i - x_j| + h_{i+1} + h_j \\ \leq C\left(|2T - x_i - x_j| + h_N\right) \tag{B.59}$$

With $\xi \simeq x_i \simeq 1$,

$$|{y_{j-i}}'(\xi) - 1| \leq C\left(|2T - x_i - x_j| + h_N\right) \tag{B.60}$$

Thus,

$$
\begin{aligned}
|(y^\theta_{j-i}(\xi))' - 1| &\le (1-\theta)|{y_{j-i-1}}'(\xi) - 1| + \theta|{y_{j-i}}'(\xi) - 1| \\
&\le C\left((1-\theta)|2T - x_i - x_{j-1}| + \theta|2T - x_i - x_j| + h_N\right) \\
&= C\left(|2T - x_i - y^\theta_j| + h_N\right)
\end{aligned}
\tag{B.61}
$$

So

$$
\begin{aligned}
\left|(|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha})'\right| &= |1-\alpha||y^\theta_{j-i}(\xi) - \xi|^{-\alpha}|(y^\theta_{j-i}(\xi))' - 1| \\
&\le C|y^\theta_j - x_i|^{-\alpha}(|2T - x_i - y^\theta_j| + h_N)
\end{aligned}
\tag{B.62}
$$

$$
\begin{aligned}
\left|(|y^\theta_{j-i}(\xi) - \xi|^{1-\alpha})''\right| &\le |1-\alpha||y^\theta_{j-i}(\xi) - \xi|^{-\alpha}|(y^\theta_{j-i}(\xi) - \xi)''| + \alpha(\alpha-1)|y^\theta_{j-i}(\xi) - \xi|^{-1-\alpha}({y^\theta_{j-i}}'(\xi) - 1)^2 \\
&\le C(r-1)|y^\theta_j - x_i|^{-\alpha} + C|y^\theta_j - x_i|^{-1-\alpha}(|2T - x_i - y^\theta_j| + h_N)^2
\end{aligned}
\tag{B.63}
$$

$\square$

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