1 问题

对于 $\Omega = (0,1)$, $1 < \alpha < 2$, 假设 $f \in C^2(\Omega)$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1}$$

其中

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = C_R \frac{d^2}{dx^2} \int_\Omega \frac{u(y)}{|x-y|^{\alpha-1}} dy \tag{2}$$

2 数值格式

用线性插值代替原函数，中心差分代替二阶导数，记 $u_h(x)$ 为 $u(x)$ 在网格点上的线性插值。

我们解这样的数值解

$$\begin{aligned} C_R(&\frac{2}{h_{i+1}(h_i+h_{i+1})} \int_\Omega \frac{u_h(x)}{|x_{i+1}-y|^{\alpha-1}} dy \\ &- \frac{2}{h_i h_{i+1}} \int_\Omega \frac{u_h(x)}{|x_i - y|^{\alpha-1}} dy \\ &+ \frac{2}{h_i(h_i+h_{i+1})} \int_\Omega \frac{u_h(x)}{|x_{i-1}-y|^{\alpha-1}} dy) \\ = F_i \end{aligned} \tag{3}$$

矩阵 A 是 M 矩阵，主队角正，其他负，严格对角占优。

3 一致网格

当 $r = 1$, 网格成为一致网格, $x_i = ih, h = \frac{1}{2N}, i = 0, ..., 2N.$

A 等于

$$\begin{aligned} a_{ij} = &\frac{C_R}{(2-\alpha)(3-\alpha)} h^{-\alpha} \\ &\left(|i-j-2|^{3-\alpha} - 4|i-j-1|^{3-\alpha} + 6|i-j|^{3-\alpha} - 4|i-j+1|^{3-\alpha} + |i-j+2|^{3-\alpha}\right) \end{aligned} \tag{4}$$

矩阵行和

$$S_i = \sum_{j=1}^{2N-1} a_{ij} = \frac{C_R}{(2-\alpha)(3-\alpha)} h^{-\alpha}(|i+1|^{3-\alpha} - 3|i|^{3-\alpha} + 3|i-1|^{3-\alpha} - |i-2|^{3-\alpha} + ...2N) \tag{5}$$

我们得到

$$S_i \geq C(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \tag{6}$$

下面估计截断误差 R_i.

$$R_i = \int_0^1 D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \tag{7}$$

目标是

$$R_i \leq Ch^{\alpha/2} S_i \tag{8}$$

这样我们就有

$$\epsilon \leq \max_i \frac{R_i}{S_i} \leq Ch^{\alpha/2} \tag{9}$$

考虑 R_1

$$R_1 = \int_\Omega (u(y) - u_h(y)) \frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} dy \tag{10}$$

我们有

$$R_1 = \int_0^{3h} + \int_{3h}^{1/2} \tag{11}$$

当 $y > 3h$,

$$\frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} \leq C|y|^{-1-\alpha} \tag{12}$$

那么

$$
\begin{aligned}
I_2 &\le C\int_{3h}^{1/2}|y|^{-1-\alpha}u''(y)h^2dy \\
&\le C\int_{3h}^{1}|y|^{-1-\alpha}y^{\alpha/2-2}h^2dy \\
&\le Ch^2\int_{3h}^{1}y^{-3-\alpha/2}dy \\
&\le Ch^2h^{-2-\alpha/2}=Ch^{-\alpha/2} \\
&\le Ch^{\alpha/2}x_1^{-\alpha}\le Ch^{\alpha/2}S_1
\end{aligned}
\tag{13}
$$

在考虑

$$
\begin{aligned}
I_1 &= \int_0^{3h}\frac{u(y)-u_h(y)}{h^2}(|y|^{1-\alpha}-2|y-h|^{1-\alpha}+|y-2h|^{1-\alpha})dy \\
&= \int_0^h+\int_h^{3h}=J_1+J_2
\end{aligned}
\tag{14}
$$

$$
J_2\le Cu''(\eta)h^{2-\alpha}\le Ch^{\alpha/2-2}h^{2-\alpha}\le Ch^{-\alpha/2}
\tag{15}
$$

因为

$$
\begin{aligned}
|u(x)-u_h(x)| &\le \int_0^{x_1}|u'(y)|dy \\
&\le C\int_0^{x_1}y^{\alpha/2-1}dy \\
&\le Cx_1^{\alpha/2} \quad ,x\in(0,h)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
J_1 &= \int_0^h\frac{u(y)-u_h(y)}{h^2}(|y|^{1-\alpha}-2|y-h|^{1-\alpha}+|y-2h|^{1-\alpha})dy \\
&\le Ch^{\alpha/2-2}h^{2-\alpha}=Ch^{-\alpha/2}
\end{aligned}
\tag{17}
$$

所以有

$$
R_1\le Ch^{-\alpha/2}\le Ch^{\alpha/2}h^{-\alpha}\le Ch^{\alpha/2}S_1,\quad (S_1\ge Cx_1^{-\alpha})
\tag{18}
$$

R_1,R_2,R_3 全部类似。

3.1 猜想

$$R_i \le Ch^{\alpha/2+1}(x_i^{-\alpha-1} + (1-x_i)^{-\alpha-1}) \quad (then \le Ch^{\alpha/2} S_i) \tag{19}$$

为了简便，我们记 $D(y) := u(y) - u_h(y)$.

当 $3 < i \le N$ 时，

$$\begin{aligned}
R_i =& \int_0^1 D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
=& \int_0^{x_1} D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
&+ \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
&+ \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} \frac{D(y+h) - D(y)}{h^2} |y-x_i|^{1-\alpha} + D(y) \frac{|y-x_{i+1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2} dy \\
&+ \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y-x_i|^{1-\alpha} dy \\
&+ \int_{x_i}^{x_{N+\lfloor \frac{i}{2} \rfloor-1}} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y-x_i|^{1-\alpha} dy \\
&+ \int_{x_{N+\lfloor \frac{i}{2} \rfloor-1}}^{x_{N+\lfloor \frac{i}{2} \rfloor}} \frac{D(y-h) - D(y)}{h^2} |y-x_i|^{1-\alpha} + D(y) \frac{|y-x_{i-1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2} dy \\
&+ \int_{x_{N+\lfloor \frac{i}{2} \rfloor}}^{x_{2N-1}} + \int_{x_{2N-1}}^1 D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
=& I_1 + I_2 + I_3 + I_4 + \cdots
\end{aligned} \tag{20}$$

1.

$$
\begin{aligned}
I_1 &= \int_0^{x_1} (u(y) - u_h(y)) \frac{|y - x_{i-1}|^{1-\alpha} - 2|y - x_i|^{1-\alpha} + |y - x_{i+1}|^{1-\alpha}}{h^2} dy \\
&\le Ch^{\alpha/2} \int_0^h |y - x_i|^{-1-\alpha} dy \\
&\le Ch^{\alpha/2+1} x_i^{-1-\alpha}
\end{aligned} \tag{21}
$$

2.

$$
\begin{aligned}
I_2 &= \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} D(y) \frac{|y - x_{i-1}|^{1-\alpha} - 2|y - x_i|^{1-\alpha} + |y - x_{i+1}|^{1-\alpha}}{h^2} dy \\
&\le C \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2} h^2 |x_i - y|^{-1-\alpha} dy \\
&\le Ch^{\alpha/2-1} h^2 x_i^{-1-\alpha} \le Ch^{\alpha/2+1} x_i^{-1-\alpha}
\end{aligned} \tag{22}
$$

3.

$$
\begin{aligned}
I_3 &= \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil + 1}} \frac{D(y+h) - D(y)}{h^2} |y - x_i|^{1-\alpha} + D(y) \frac{|y - x_{i+1}|^{1-\alpha} - |y - x_i|^{1-\alpha}}{h^2} dy \\
&\le \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil + 1}} u'''(\eta_1) h |x_i - y|^{1-\alpha} + u''(\eta_2) h |x_i - y|^{-\alpha} dy \\
&\le Ch^2 x_i^{-2-\alpha/2} \le Ch^{1+\alpha/2} x_i^{-1-\alpha}
\end{aligned} \tag{23}
$$

4.

$$
\begin{aligned}
I_4 &= \int_{x_{\lceil \frac{i}{2} \rceil + 1}}^{x_i} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y - x_i|^{1-\alpha} dy \\
&\le \int_{x_{\lceil \frac{i}{2} \rceil + 1}}^{x_i} u''''(\eta) h^2 |x_i - y|^{1-\alpha} dy \\
&\le C x_i^{\alpha/2-4} h^2 x_i^{2-\alpha} \\
&\le Ch^2 x_i^{-2-\alpha/2} \le Ch^{1+\alpha/2} x_i^{-1-\alpha}
\end{aligned} \tag{24}
$$

猜想证毕，一致网格证完。

4 非一致

$r>1$,

$$\begin{cases} x_i = \frac{1}{2}\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ x_i = 1-\frac{1}{2}\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases} \tag{25}$$

令 $h=\frac{1}{2N}$, 那么
当 $i<N, x_i<\frac{1}{2}$ 时

$$h_i = \frac{1}{2}\left(\left(\frac{i}{N}\right)^r - \left(\frac{i-1}{N}\right)^r\right) \le C(r)\left(\frac{i}{N}\right)^{r-1}\frac{1}{N} = Chx_i^{(r-1)/r} \tag{26}$$

当 $i \ge N, x_i \ge \frac{1}{2}$ 时

$$h_i = \frac{1}{2}\left(\left(\frac{2N-i+1}{N}\right)^r - \left(\frac{2N-i}{N}\right)^r\right) \le C(r)\left(\frac{2N-i+1}{N}\right)^{r-1}\frac{1}{N} = Ch(1-x_{i-1})^{(r-1)/r} \tag{27}$$

我们声明

$$\begin{aligned} S_i = \sum_{j=1}^{2N-1} a_{ij} =& \frac{C_R}{(2-\alpha)(3-\alpha)}\frac{2}{h_i+h_{i+1}} \\ & (\frac{1}{h_{i+1}}\frac{|x_{i+1}-x_0|^{3-\alpha}-|x_{i+1}-x_1|^{3-\alpha}}{x_1-x_0} \\ & -(\frac{1}{h_i}+\frac{1}{h_{i+1}})\frac{|x_i-x_0|^{3-\alpha}-|x_i-x_1|^{3-\alpha}}{x_1-x_0} \\ & +\frac{1}{h_i}\frac{|x_{i-1}-x_0|^{3-\alpha}-|x_{i-1}-x_1|^{3-\alpha}}{x_1-x_0}) \\ \ge& C(x_i^{-\alpha}+(1-x_i)^{-\alpha}) \end{aligned} \tag{28}$$

$$R_i = \int_0^1 D(y)\frac{2}{h_i+h_{i+1}}(\frac{1}{h_{i+1}}|x_{i+1}-y|^{1-\alpha} - (\frac{1}{h_i}+\frac{1}{h_{i+1}})|x_i-y|^{1-\alpha} + \frac{1}{h_i}|x_{i-1}-y|^{1-\alpha})dy \tag{29}$$

下面讨论 R_1

$$
\begin{aligned}
R_1 = &\int_0^{x_1} + \int_{x_1}^{x_3} + \int_{x_3}^{1/2} + \int_{1/2}^{x_{2N-1}} + \int_{x_{2N-1}}^{1} \\
&D(y)\frac{2}{h_1+h_2}(\frac{1}{h_2}|x_2-y|^{1-\alpha} - (\frac{1}{h_1}+\frac{1}{h_2})|x_1-y|^{1-\alpha} + \frac{1}{h_1}|y|^{1-\alpha})dy \\
:= &I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}
\tag{30}
$$

与一致网格时相似，

1.

$$
\begin{aligned}
|u(x) - u_h(x)| &\le \int_0^{x_1} |u'(y)|dy \\
&\le C\int_0^{x_1} y^{\alpha/2-1}dy \\
&\le C x_1^{\alpha/2} \quad , x \in (0, x_1)
\end{aligned}
\tag{31}
$$

因为 $1-\alpha > -1$

$$
\begin{aligned}
I_1 &\le C\int_0^{x_1} \frac{D(y)}{h_1^2}(|x_2-y|^{1-\alpha} + 2|x_1-y|^{1-\alpha} + |y|^{1-\alpha})dy \\
&\le C x_1^{\alpha/2-2} x_1^{2-\alpha} = C x_1^{-\alpha/2} = C h^{-r\alpha/2}
\end{aligned}
\tag{32}
$$

2.

$$
I_2 \le C u''(\eta) x_3^{2-\alpha} \le C x_1^{\alpha/2-2} x_3^{2-\alpha} \le C h^{-r\alpha/2} \tag{33}
$$

3.

$$
\begin{aligned}
I_3 = &\int_{x_3}^{1/2} D(y)\frac{2}{h_1+h_2}(\frac{1}{h_2}|x_2-y|^{1-\alpha} - (\frac{1}{h_1}+\frac{1}{h_2})|x_1-y|^{1-\alpha} + \frac{1}{h_1}|y|^{1-\alpha})dy \\
\le &C\int_{x_3}^{1/2} y^{\alpha/2-2}(hy^{(r-1)/r})^2 y^{-1-\alpha}dy \\
\le &Ch^2\int_{x_3}^{1/2} y^{\alpha/2-2/r-1-\alpha}dy \\
\le &Ch^2(h^r)^{-2/r-\alpha/2} = Ch^{-r\alpha/2}
\end{aligned}
\tag{34}
$$

4.

$$\begin{aligned}
I_4 &= \int_{1/2}^{x_{2N-1}} D(y)\frac{2}{h_1+h_2}(\frac{1}{h_2}|x_2-y|^{1-\alpha}-(\frac{1}{h_1}+\frac{1}{h_2})|x_1-y|^{1-\alpha}+\frac{1}{h_1}|y|^{1-\alpha})dy \\
&\leq C\int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2}(h(1-y)^{(r-1)/r})^2 y^{-1-\alpha}dy \\
&\leq Ch^2\int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2+2-2/r} \\
&\leq Ch^2(C+h_{2N}^{\alpha/2-2/r+1}) \\
&= Ch^2(C+h^{r\alpha/2-2+r}) \leq Ch^{\min\{2,r\alpha/2+r\}}
\end{aligned} \tag{35}$$

5.

$$I_5 \leq Ch_{2N}^{\alpha/2+1} \leq Ch^{r\alpha/2+r} \tag{36}$$

综合有

$$R_1 \leq Ch^{-r\alpha/2} \tag{37}$$

R_1, R_2, R_3 一样。

4.1 一般的 i

$R_i, 3 < i \leq N$ 比较困难。

我们记

$$T_{ij} = \int_{x_{j-1}}^{x_j} D(y)|x_i-y|^{1-\alpha}dy \tag{38}$$

那么

$$
\begin{aligned}
R_i =&\sum_{j=1}^{2N} T_{ij}\\
=&\sum_{j=1}^{2N} \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,j}+\frac{1}{h_i}T_{i-1,j}\right)\\
=&\sum_{j=1}^{i/2} \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,j}+\frac{1}{h_i}T_{i-1,j}\right)\\
&+\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}(T_{i+1,i/2+1}+T_{i+1,i/2+2})-(\frac{1}{h_i}+\frac{1}{h_{i+1}})(T_{i,i/2+1})\right)\\
&+\sum_{j=i/2+2}^{i} \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j+1}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,j}+\frac{1}{h_i}T_{i-1,j-1}\right)\\
&+...\\
=&I_1+I_2+I_3+...
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
I_1 =&\int_0^{x_1}+\int_{x_1}^{x_{\lceil \frac{i}{2}\rceil}}\\
&D(y)\frac{2}{h_i+h_{i+1}}(\frac{1}{h_{i+1}}|x_{i+1}-y|^{1-\alpha}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})|x_i-y|^{1-\alpha}+\frac{1}{h_i}|x_{i-1}-y|^{1-\alpha})dy
\end{aligned}
\tag{40}
$$

1.

$$
J_1\le Cx_1^{\alpha/2+1}x_i^{-1-\alpha}\le Ch^{r\alpha/2+r}x_i^{-1-\alpha} \tag{41}
$$

2.

$$
\begin{aligned}
J_2 &\le C\int_{x_1}^{x_{\lceil \frac{i}{2}\rceil}} y^{\alpha/2-2}(hy^{(r-1)/r})^2|x_i-y|^{-1-\alpha}dy\\
&\le Ch^2x_i^{-1-\alpha}\int_{x_1}^{x_{\lceil \frac{i}{2}\rceil}} y^{\alpha/2-2/r}dy\\
&\le Ch^2x_i^{-1-\alpha}(h^{r\alpha/2-2+r}+x_i^{\alpha/2-2/r+1})
\end{aligned}
\tag{42}
$$

我们先研究 I_3，考虑

$$\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j+1}-(\frac{1}{h_i}+\frac{1}{h_{i+1}})T_{i,j}+\frac{1}{h_i}T_{i-1,j-1}\right) \tag{43}$$

在此之前我们做一些准备工作。

对于 $y\in[x_{j-1},x_j]$，我们记 $y_j^\theta=\theta x_{j-1}+(1-\theta)x_j$

$$\begin{aligned}D(y_j^\theta)=&\frac{\theta(1-\theta)}{2}h_j^2u''(y_j^\theta)+\frac{\theta(1-\theta)(1-2\theta)}{3!}h_j^3u'''(y_j^\theta)\\&+\frac{\theta(1-\theta)}{4!}h_j^4(\theta^3u'''(\eta_1)+(1-\theta)^3u'''(\eta_2))\end{aligned} \tag{44}$$

那么

$$\begin{aligned}T_{ij}&=\int_{x_{j-1}}^{x_j}D(y)|x_i-y|^{1-\alpha}dy\\&=\int_0^1\frac{\theta(1-\theta)}{2}h_j^3u''(y_j^\theta)|x_i-y_j^\theta|^{1-\alpha}d\theta+...\end{aligned} \tag{45}$$

现在回到原来的问题，我们要研究

$$\begin{aligned}&\frac{2}{h_i+h_{i+1}}(\frac{1}{h_{i+1}}h_{j+1}^3u''(y_{j+1}^\theta)|x_{i+1}-y_{j+1}^\theta|^{1-\alpha}\\&\quad-(\frac{1}{h_i}+\frac{1}{h_{i+1}})h_j^3\ u''(y_j^\theta)|x_i-y_j^\theta|^{1-\alpha}\\&\qquad+\frac{1}{h_i}h_{j-1}^3u''(y_{j-1}^\theta)|x_{i-1}-y_{j-1}^\theta|^{1-\alpha})\end{aligned} \tag{46}$$

我们希望把他看成一个函数的二阶导，注意到当 $j\le i\le N$ 时

$$x_i^{1/r}-x_j^{1/r}=x_{i+1}^{1/r}-x_{j+1}^{1/r}=2^{-1/r}\frac{i-j}{N} \tag{47}$$

那么我们将其他的相都表示成 x_i 的函数。

$$y_L=(x^{1/r}-z_1)^r,\quad y_R=(x^{1/r}-z)^r \tag{48}$$

其中 $z=2^{-1/r}\frac{i-j}{N},z_1=2^{-1/r}\frac{i-j+1}{N}$

$$y_\theta = \theta y_L + (1-\theta) y_R \tag{49}$$
$$h_J = y_R - y_L \tag{50}$$

那么我么要研究的就是函数

$$h_J^3 |x - y_\theta|^{1-\alpha} u''(y^\theta) \tag{51}$$

在网格 x_{i-1}, x_i, x_{i+1} 的数值二阶差商。

由 Leibniz 公式

$$(uvw)'' = u''vw + uv''w + uvw'' + 2u'v'w + 2uv'w' + 2u'vw' \tag{52}$$

由 $y_R^{1/r} = x^{1/r} - z$, 我们得到

$$\frac{dy_R}{dx} = x^{1/r-1} y_R^{1-1/r} \tag{53}$$
$$\frac{d^2 y_R}{dx^2} = \frac{r-1}{r} x^{1/r-2} y_R^{1-2/r} z \tag{54}$$
$$\tag{55}$$