

1 问题

对于 $\Omega = (0, 1)$, $1 < \alpha < 2$, 假设 $f \in C^2(\Omega)$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \quad (1)$$

其中

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = C_R \frac{d^2}{dx^2} \int_{\Omega} \frac{u(y)}{|x-y|^{\alpha-1}} dy \quad (2)$$

2 数值格式

用线性插值代替原函数, 中心差分代替二阶导数, 记 $u_h(x)$ 为 $u(x)$ 在网格点上的线性插值。

我们解这样的数值解

$$\begin{aligned} & C_R \left(\frac{2}{h_{i+1}(h_i + h_{i+1})} \int_{\Omega} \frac{u_h(x)}{|x_{i+1} - y|^{\alpha-1}} dy \right. \\ & \quad - \frac{2}{h_i h_{i+1}} \int_{\Omega} \frac{u_h(x)}{|x_i - y|^{\alpha-1}} dy \\ & \quad \left. + \frac{2}{h_i(h_i + h_{i+1})} \int_{\Omega} \frac{u_h(x)}{|x_{i-1} - y|^{\alpha-1}} dy \right) \\ & = F_i \end{aligned} \quad (3)$$

矩阵 A 是 M 矩阵, 主对角正, 其他负, 严格对角占优。

3 一致网格

当 $r = 1$, 网格成为一致网格, $x_i = ih, h = \frac{1}{2N}, i = 0, \dots, 2N$.

A 等于

$$\begin{aligned} a_{ij} &= \frac{C_R}{(2-\alpha)(3-\alpha)} h^{-\alpha} \\ & \quad (|i-j-2|^{3-\alpha} - 4|i-j-1|^{3-\alpha} + 6|i-j|^{3-\alpha} - 4|i-j+1|^{3-\alpha} + |i-j+2|^{3-\alpha}) \end{aligned} \quad (4)$$

矩阵行和

$$S_i = \sum_{j=1}^{2N-1} a_{ij} = \frac{C_R}{(2-\alpha)(3-\alpha)} h^{-\alpha} (|i+1|^{3-\alpha} - 3|i|^{3-\alpha} + 3|i-1|^{3-\alpha} - |i-2|^{3-\alpha} + \dots + 2N) \quad (5)$$

我们得到

$$S_i \geq C(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \quad (6)$$

下面估计截断误差 R_i .

$$R_i = \int_0^1 D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \quad (7)$$

目标是

$$R_i \leq Ch^{\alpha/2} S_i \quad (8)$$

这样我们就有

$$\epsilon \leq \max_i \frac{R_i}{S_i} \leq Ch^{\alpha/2} \quad (9)$$

考虑 R_1

$$R_1 = \int_{\Omega} (u(y) - u_h(y)) \frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} dy \quad (10)$$

我们有

$$R_1 = \int_0^{3h} + \int_{3h}^{1/2} \quad (11)$$

当 $y > 3h$,

$$\frac{|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}}{h^2} \leq C|y|^{-1-\alpha} \quad (12)$$

那么

$$\begin{aligned}
I_2 &\leq C \int_{3h}^{1/2} |y|^{-1-\alpha} u''(y) h^2 dy \\
&\leq C \int_{3h}^1 |y|^{-1-\alpha} y^{\alpha/2-2} h^2 dy \\
&\leq Ch^2 \int_{3h}^1 y^{-3-\alpha/2} dy \\
&\leq Ch^2 h^{-2-\alpha/2} = Ch^{-\alpha/2} \\
&\leq Ch^{\alpha/2} x_1^{-\alpha} \leq Ch^{\alpha/2} S_1
\end{aligned} \tag{13}$$

在考虑

$$\begin{aligned}
I_1 &= \int_0^{3h} \frac{u(y) - u_h(y)}{h^2} (|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}) dy \\
&= \int_0^h + \int_h^{3h} = J_1 + J_2
\end{aligned} \tag{14}$$

$$J_2 \leq Cu''(\eta) h^{2-\alpha} \leq Ch^{\alpha/2-2} h^{2-\alpha} \leq Ch^{-\alpha/2} \tag{15}$$

因为

$$\begin{aligned}
|u(x) - u_h(x)| &\leq \int_0^{x_1} |u'(y)| dy \\
&\leq C \int_0^{x_1} y^{\alpha/2-1} dy \\
&\leq C x_1^{\alpha/2}, \quad x \in (0, h)
\end{aligned} \tag{16}$$

$$\begin{aligned}
J_1 &= \int_0^h \frac{u(y) - u_h(y)}{h^2} (|y|^{1-\alpha} - 2|y-h|^{1-\alpha} + |y-2h|^{1-\alpha}) dy \\
&\leq Ch^{\alpha/2-2} h^{2-\alpha} = Ch^{-\alpha/2}
\end{aligned} \tag{17}$$

所以有

$$R_1 \leq Ch^{-\alpha/2} \leq Ch^{\alpha/2} h^{-\alpha} \leq Ch^{\alpha/2} S_1, \quad (S_1 \geq C x_1^{-\alpha}) \tag{18}$$

R_1, R_2, R_3 全部类似。

3.1 猜想

$$R_i \leq Ch^{\alpha/2+1}(x_i^{-\alpha-1} + (1-x_i)^{-\alpha-1}) \quad (\text{then } \leq Ch^{\alpha/2}S_i) \quad (19)$$

为了简便, 我们记 $D(y) := u(y) - u_h(y)$.

当 $3 < i \leq N$ 时,

$$\begin{aligned}
R_i &= \int_0^1 D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
&= \int_0^{x_1} D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
&\quad + \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
&\quad + \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} \frac{D(y+h) - D(y)}{h^2} |y-x_i|^{1-\alpha} + D(y) \frac{|y-x_{i+1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2} dy \\
&\quad + \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y-x_i|^{1-\alpha} dy \\
&\quad + \int_{x_i}^{x_{N+\lfloor \frac{i}{2} \rfloor-1}} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y-x_i|^{1-\alpha} dy \\
&\quad + \int_{x_{N+\lfloor \frac{i}{2} \rfloor-1}}^{x_{N+\lfloor \frac{i}{2} \rfloor}} \frac{D(y-h) - D(y)}{h^2} |y-x_i|^{1-\alpha} + D(y) \frac{|y-x_{i-1}|^{1-\alpha} - |y-x_i|^{1-\alpha}}{h^2} dy \\
&\quad + \int_{x_{N+\lfloor \frac{i}{2} \rfloor}}^{x_{2N-1}} + \int_{x_{2N-1}}^1 D(y) \frac{|y-x_{i-1}|^{1-\alpha} - 2|y-x_i|^{1-\alpha} + |y-x_{i+1}|^{1-\alpha}}{h^2} dy \\
&= I_1 + I_2 + I_3 + I_4 + \cdots
\end{aligned} \tag{20}$$

1.

$$\begin{aligned}
I_1 &= \int_0^{x_1} (u(y) - u_h(y)) \frac{|y - x_{i-1}|^{1-\alpha} - 2|y - x_i|^{1-\alpha} + |y - x_{i+1}|^{1-\alpha}}{h^2} dy \\
&\leq Ch^{\alpha/2} \int_0^h |y - x_i|^{-1-\alpha} dy \\
&\leq Ch^{\alpha/2+1} x_i^{-1-\alpha}
\end{aligned} \tag{21}$$

2.

$$\begin{aligned}
I_2 &= \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} D(y) \frac{|y - x_{i-1}|^{1-\alpha} - 2|y - x_i|^{1-\alpha} + |y - x_{i+1}|^{1-\alpha}}{h^2} dy \\
&\leq C \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2} h^2 |x_i - y|^{-1-\alpha} dy \\
&\leq Ch^{\alpha/2-1} h^2 x_i^{-1-\alpha} \leq Ch^{\alpha/2+1} x_i^{-1-\alpha}
\end{aligned} \tag{22}$$

3.

$$\begin{aligned}
I_3 &= \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} \frac{D(y+h) - D(y)}{h^2} |y - x_i|^{1-\alpha} + D(y) \frac{|y - x_{i+1}|^{1-\alpha} - |y - x_i|^{1-\alpha}}{h^2} dy \\
&\leq \int_{x_{\lceil \frac{i}{2} \rceil}}^{x_{\lceil \frac{i}{2} \rceil+1}} u'''(\eta_1) h |x_i - y|^{1-\alpha} + u''(\eta_2) h |x_i - y|^{-\alpha} dy \\
&\leq Ch^2 x_i^{-2-\alpha/2} \leq Ch^{1+\alpha/2} x_i^{-1-\alpha}
\end{aligned} \tag{23}$$

4.

$$\begin{aligned}
I_4 &= \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} \frac{D(y-h) - 2D(y) + D(y+h)}{h^2} |y - x_i|^{1-\alpha} dy \\
&\leq \int_{x_{\lceil \frac{i}{2} \rceil+1}}^{x_i} u''''(\eta) h^2 |x_i - y|^{1-\alpha} dy \\
&\leq C x_i^{\alpha/2-4} h^2 x_i^{2-\alpha} \\
&\leq Ch^2 x_i^{-2-\alpha/2} \leq Ch^{1+\alpha/2} x_i^{-1-\alpha}
\end{aligned} \tag{24}$$

猜想证毕，一致网格证完。

4 非一致

$r > 1$,

$$\begin{cases} x_i = \frac{1}{2} \left(\frac{i}{N} \right)^r, & 0 \leq i \leq N \\ x_i = 1 - \frac{1}{2} \left(\frac{2N-i}{N} \right)^r, & N \leq i \leq 2N \end{cases} \quad (25)$$

令 $h = \frac{1}{2N}$, 那么

当 $i < N, x_i < \frac{1}{2}$ 时

$$h_i = \frac{1}{2} \left(\left(\frac{i}{N} \right)^r - \left(\frac{i-1}{N} \right)^r \right) \leq C(r) \left(\frac{i}{N} \right)^{r-1} \frac{1}{N} = Ch x_i^{(r-1)/r} \quad (26)$$

当 $i \geq N, x_i \geq \frac{1}{2}$ 时

$$h_i = \frac{1}{2} \left(\left(\frac{2N-i+1}{N} \right)^r - \left(\frac{2N-i}{N} \right)^r \right) \leq C(r) \left(\frac{2N-i+1}{N} \right)^{r-1} \frac{1}{N} = Ch(1-x_{i-1})^{(r-1)/r} \quad (27)$$

我们声明

$$\begin{aligned} S_i &= \sum_{j=1}^{2N-1} a_{ij} = \frac{C_R}{(2-\alpha)(3-\alpha)} \frac{2}{h_i + h_{i+1}} \\ &\quad \left(\frac{1}{h_{i+1}} \frac{|x_{i+1} - x_0|^{3-\alpha} - |x_{i+1} - x_1|^{3-\alpha}}{x_1 - x_0} \right. \\ &\quad \left. - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{x_1 - x_0} \right. \\ &\quad \left. + \frac{1}{h_i} \frac{|x_{i-1} - x_0|^{3-\alpha} - |x_{i-1} - x_1|^{3-\alpha}}{x_1 - x_0} \right) \\ &\geq C(x_i^{-\alpha} + (1-x_i)^{-\alpha}) \end{aligned} \quad (28)$$

$$R_i = \int_0^1 D(y) \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} |x_{i+1} - y|^{1-\alpha} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) |x_i - y|^{1-\alpha} + \frac{1}{h_i} |x_{i-1} - y|^{1-\alpha} \right) dy \quad (29)$$

下面讨论 R_1

$$\begin{aligned}
R_1 &= \int_0^{x_1} + \int_{x_1}^{x_3} + \int_{x_3}^{1/2} + \int_{1/2}^{x_{2N-1}} + \int_{x_{2N-1}}^1 \\
&\quad D(y) \frac{2}{h_1 + h_2} \left(\frac{1}{h_2} |x_2 - y|^{1-\alpha} - \left(\frac{1}{h_1} + \frac{1}{h_2} \right) |x_1 - y|^{1-\alpha} + \frac{1}{h_1} |y|^{1-\alpha} \right) dy \\
&:= I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned} \tag{30}$$

与一致网格时相似,

1.

$$\begin{aligned}
|u(x) - u_h(x)| &\leq \int_0^{x_1} |u'(y)| dy \\
&\leq C \int_0^{x_1} y^{\alpha/2-1} dy \\
&\leq C x_1^{\alpha/2}, \quad x \in (0, x_1)
\end{aligned} \tag{31}$$

因为 $1 - \alpha > -1$

$$\begin{aligned}
I_1 &\leq C \int_0^{x_1} \frac{D(y)}{h_1^2} (|x_2 - y|^{1-\alpha} + 2|x_1 - y|^{1-\alpha} + |y|^{1-\alpha}) dy \\
&\leq C x_1^{\alpha/2-2} x_1^{2-\alpha} = C x_1^{-\alpha/2} = C h^{-r\alpha/2}
\end{aligned} \tag{32}$$

2.

$$I_2 \leq C u''(\eta) x_3^{2-\alpha} \leq C x_1^{\alpha/2-2} x_3^{2-\alpha} \leq C h^{-r\alpha/2} \tag{33}$$

3.

$$\begin{aligned}
I_3 &= \int_{x_3}^{1/2} D(y) \frac{2}{h_1 + h_2} \left(\frac{1}{h_2} |x_2 - y|^{1-\alpha} - \left(\frac{1}{h_1} + \frac{1}{h_2} \right) |x_1 - y|^{1-\alpha} + \frac{1}{h_1} |y|^{1-\alpha} \right) dy \\
&\leq C \int_{x_3}^{1/2} y^{\alpha/2-2} (h y^{(r-1)/r})^2 y^{-1-\alpha} dy \\
&\leq C h^2 \int_{x_3}^{1/2} y^{\alpha/2-2/r-1-\alpha} dy \\
&\leq C h^2 (h^r)^{-2/r-\alpha/2} = C h^{-r\alpha/2}
\end{aligned} \tag{34}$$

4.

$$\begin{aligned}
I_4 &= \int_{1/2}^{x_{2N-1}} D(y) \frac{2}{h_1 + h_2} \left(\frac{1}{h_2} |x_2 - y|^{1-\alpha} - \left(\frac{1}{h_1} + \frac{1}{h_2} \right) |x_1 - y|^{1-\alpha} + \frac{1}{h_1} |y|^{1-\alpha} \right) dy \\
&\leq C \int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2} (h(1-y)^{(r-1)/r})^2 y^{-1-\alpha} dy \\
&\leq Ch^2 \int_{1/2}^{x_{2N-1}} (1-y)^{\alpha/2-2+2-2/r} dy \\
&\leq Ch^2 (C + h_{2N}^{\alpha/2-2/r+1}) \\
&= Ch^2 (C + h^{r\alpha/2-2+r}) \leq Ch^{\min\{2, r\alpha/2+r\}}
\end{aligned} \tag{35}$$

5.

$$I_5 \leq Ch_{2N}^{\alpha/2+1} \leq Ch^{r\alpha/2+r} \tag{36}$$

综合有

$$R_1 \leq Ch^{-r\alpha/2} \tag{37}$$

 R_1, R_2, R_3 一样。

4.1 一般的 i

 $R_i, 3 < i < N$ 比较困难。 $i = N$ 再处理。我们记 $D(y) = u(y) - u_h(y)$

$$T_{ij} = \int_{x_{j-1}}^{x_j} D(y) |x_i - y|^{1-\alpha} dy \tag{38}$$

那么

$$\begin{aligned}
R_i &= \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&= \sum_{j=1}^{i/2} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,i/2+1} + T_{i+1,i/2+2}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) (T_{i,i/2+1}) \right) \\
&\quad + \sum_{j=i/2+2}^i \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&\quad + \sum_{j=i+1}^{N-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,N+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,N} + \frac{1}{h_i} T_{i-1,N-1} \right) \\
&\quad + \sum_{j=N+1}^{N+i/2-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&\quad + \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i-1}} (T_{i-1,N+i/2} + T_{i-1,N+i/2-1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) (T_{i,N+i/2}) \right) \\
&\quad + \sum_{j=N+i/2+1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&= I_1 + I_2 + I_3 + \dots
\end{aligned} \tag{39}$$

$$\begin{aligned}
I_1 &= \int_0^{x_1} + \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} \\
&\quad D(y) \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} |x_{i+1} - y|^{1-\alpha} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) |x_i - y|^{1-\alpha} + \frac{1}{h_i} |x_{i-1} - y|^{1-\alpha} \right) dy
\end{aligned} \tag{40}$$

1.

$$J_1 \leq C x_1^{\alpha/2+1} x_i^{-1-\alpha} \leq C h^{r\alpha/2+r} x_i^{-1-\alpha} \tag{41}$$

2.

$$\begin{aligned}
J_2 &\leq C \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2} (hy^{(r-1)/r})^2 |x_i - y|^{-1-\alpha} dy \\
&\leq Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil}} y^{\alpha/2-2/r} dy \\
&\leq Ch^2 x_i^{-1-\alpha} (h^{r\alpha/2-2+r} + x_i^{\alpha/2-2/r+1})
\end{aligned} \tag{42}$$

我们先研究 I_3 , 考虑

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \tag{43}$$

在此之前我们做一些准备工作。

对于 $y \in [x_{j-1}, x_j]$, 我们记 $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

$$D(y_j^\theta) = \frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \tag{44}$$

那么

$$\begin{aligned}
T_{ij} &= \int_{x_{j-1}}^{x_j} D(y) |x_i - y|^{1-\alpha} dy \\
&= \int_0^1 \frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) |x_i - y_j^\theta|^{1-\alpha} d\theta \\
&\quad + \int_0^1 \frac{\theta(1-\theta)}{2} h_j^4 |x_i - y_j^\theta|^{1-\alpha} (\theta^2 u'''(\eta_{1,j}^\theta) - (1-\theta)^2 u'''(\eta_{2,j}^\theta)) d\theta
\end{aligned} \tag{45}$$

现在回到原来的问题, 我们要研究

$$\begin{aligned}
&\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} h_{j+1}^3 u''(y_{j+1}^\theta) |x_{i+1} - y_{j+1}^\theta|^{1-\alpha} \right. \\
&\quad \left. - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) h_j^3 u''(y_j^\theta) |x_i - y_j^\theta|^{1-\alpha} \right. \\
&\quad \left. + \frac{1}{h_i} h_{j-1}^3 u''(y_{j-1}^\theta) |x_{i-1} - y_{j-1}^\theta|^{1-\alpha} \right)
\end{aligned} \tag{46}$$

我们希望把他看成一个函数的二阶导，注意到当 $j \leq i \leq N$ 时

$$x_i^{1/r} - x_j^{1/r} = x_{i+1}^{1/r} - x_{j+1}^{1/r} = 2^{-1/r} \frac{i-j}{N} \quad (47)$$

那么我们将其他的相都表示成 x_i 的函数。

$$y_L(x) = (x^{1/r} - z_L)^r, \quad y_R(x) = (x^{1/r} - z_R)^r \quad (48)$$

其中 $z_R = 2^{-1/r} \frac{i-j}{N}$, $z_L = 2^{-1/r} \frac{i-j+1}{N}$.

$$y_R(x_i) = x_j, \quad y_R(x_{i+1}) = x_{j+1}, \quad y_R(x_{i-1}) = x_{j-1} \quad (49)$$

$$y_L(x_i) = x_{j-1}, \quad y_L(x_{i+1}) = x_j, \quad y_L(x_{i-1}) = x_{j-2} \quad (50)$$

$$y_\theta(x) = \theta y_L(x) + (1 - \theta) y_R(x) \quad (51)$$

$$h_J(x) = y_R(x) - y_L(x) \quad (52)$$

那么我要研究的函数

$$K_1(x) = h_J^3(x) |x - y_\theta(x)|^{1-\alpha} u''(y_\theta(x)) \quad (53)$$

在网格 x_{i-1}, x_i, x_{i+1} 的数值二阶差商。

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} K_1(x_{i+1}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) K_1(x_i) + \frac{1}{h_i} K_1(x_{i-1}) \right) = K_1''(\xi), \quad \xi \in [x_{i-1}, x_{i+1}] \quad (54)$$

由 Leibniz 公式

$$(uvw)'' = u''vw + uv''w + uvw'' + 2u'v'w + 2uv'w' + 2u'vw' \quad (55)$$

由 $y_R^{1/r} = x^{1/r} - z_R$, 我们得到

$$\frac{dy_R}{dx} = x^{1/r-1} y_R^{1-1/r} \quad (56)$$

$$\frac{d^2 y_R}{dx^2} = \frac{r-1}{r} x^{1/r-2} y_R^{1-2/r} z_R \quad (57)$$

因此

1.

$$h_J^3 \sim h^3 y_R^{3-3/r} \sim h^3 x^{3-3/r} \quad (58)$$

$$\begin{aligned} (h_J^3)' &= 3h_J^2(y_R' - y_L') \\ &= 3h_J^2 x^{1/r-1} (y_R^{1-1/r} - y_L^{1-1/r}) \\ &\sim h^3 y_R^{2-2/r} x^{1/r-1} y_R^{1-2/r} \\ &\sim h^3 x^{2-3/r} \end{aligned} \quad (59)$$

$$\begin{aligned} (h_J^3)'' &= 6h_J x^{2/r-2} (y_R^{1-1/r} - y_L^{1-1/r})^2 + 3h_J^2 \frac{r-1}{r} x^{1/r-2} (y_R^{1-2/r} z_R - y_L^{1-2/r} z_L) \\ &\sim h y_R^{1-1/r} x^{2/r-2} (h y_R^{1-2/r})^2 + \frac{r-1}{r} h^2 y_R^{2-2/r} x^{1/r-2} (z_R h y_R^{1-3/r} - h y_L^{1-2/r}) \\ &\sim h^3 x^{1-3/r} + \frac{r-1}{r} h^3 (x^{1-4/r} z_R - x^{1-3/r}) \\ &\sim h^3 (x^{1-3/r} + x^{1-4/r} z_R) \\ &< h^3 x^{1-3/r} \end{aligned} \quad (60)$$

2.

由于

$$x - y_L = (x^{1/r})^r - (x^{1/r} - z_L)^r = z_L \xi^{r-1} \sim z_L x^{(r-1)/r} \quad (61)$$

$$\begin{aligned} |x - y_\theta|^{1-\alpha} &= |x - \theta y_L - (1-\theta) y_R|^{1-\alpha} \sim |(\theta z_L + (1-\theta) z_R) x^{1-1/r}|^{1-\alpha} \\ &\sim z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} \end{aligned} \quad (62)$$

$$\begin{aligned}
(|x - y_\theta|^{1-\alpha})' &= (1 - \alpha)|x - y_\theta|^{-\alpha}(1 - x^{1/r-1}(\theta y_L^{1-1/r} + (1 - \theta)y_R^{1-1/r})) \\
&= (1 - \alpha)|x - y_\theta|^{-\alpha}x^{1/r-1}(x^{1-1/r} - (\theta y_L^{1-1/r} + (1 - \theta)y_R^{1-1/r})) \\
&\sim |x - y_\theta|^{-\alpha}x^{1/r-1}(\theta z_L + (1 - \theta)z_R)x^{1-2/r} \\
&\sim |(\theta z_L + (1 - \theta)z_R)x^{1-1/r}|^{-\alpha}x^{-1/r}(\theta z_L + (1 - \theta)z_R) \\
&\sim z_\theta^{1-\alpha}x^{-\alpha+(\alpha-1)/r}
\end{aligned} \tag{63}$$

$$\begin{aligned}
(|x - y_\theta|^{1-\alpha})'' &= \alpha(\alpha - 1)|x - y_\theta|^{-1-\alpha}(1 - x^{1/r-1}(\theta y_L^{1-1/r} + (1 - \theta)y_R^{1-1/r}))^2 \\
&\quad + (1 - \alpha)|x - y_\theta|^{-\alpha}\left(\frac{1-r}{r}x^{1/r-2}(\theta y_L^{1-2/r}z_L + (1 - \theta)y_R^{1-2/r}z_R)\right) \\
&\sim |(\theta z_L + (1 - \theta)z_R)x^{1-1/r}|^{-1-\alpha}x^{-2/r}(\theta z_L + (1 - \theta)z_R)^2 \\
&\quad + |(\theta z_L + (1 - \theta)z_R)x^{1-1/r}|^{-\alpha}x^{-1-1/r}(\theta z_L + (1 - \theta)z_R) \\
&\sim z_\theta^{1-\alpha}x^{-1-\alpha+(\alpha-1)/r}
\end{aligned} \tag{64}$$

3.

$$u''(y_\theta) \leq C y_\theta^{\alpha/2-2} \sim x^{\alpha/2-2} \tag{65}$$

$$\begin{aligned}
(u''(y_\theta))' &= u'''(y_\theta)x^{1/r-1}(\theta y_L^{1-1/r} + (1 - \theta)y_R^{1-1/r}) \\
&\leq C y_\theta^{\alpha/2-3} \sim x^{\alpha/2-3}
\end{aligned} \tag{66}$$

$$\begin{aligned}
(u''(y_\theta))'' &= u''''(y_\theta)(x^{1/r-1}(\theta y_L^{1-1/r} + (1 - \theta)y_R^{1-1/r}))^2 \\
&\quad + u'''(y_\theta)\frac{r-1}{r}x^{1/r-2}(\theta y_L^{1-2/r}z_L + (1 - \theta)y_R^{1-2/r}z_R) \\
&\sim x^{\alpha/2-4} + (\theta z_L + (1 - \theta)z_R)x^{\alpha/2-3-1-1/r} \\
&< x^{\alpha/2-4}
\end{aligned} \tag{67}$$

$$u''vw \sim h^3 x^{1-3/r} z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} x^{\alpha/2-2} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \quad (68)$$

$$uv''w \sim h^3 x^{3-3/r} z_\theta^{1-\alpha} x^{-1-\alpha+(\alpha-1)/r} x^{\alpha/2-2} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \quad (69)$$

$$uvw'' \sim h^3 x^{3-3/r} z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} x^{\alpha/2-4} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \quad (70)$$

$$u'v'w \sim h^3 x^{2-3/r} z_\theta^{1-\alpha} x^{-\alpha+(\alpha-1)/r} x^{\alpha/2-2} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \quad (71)$$

$$uv'w' \sim h^3 x^{3-3/r} z_\theta^{1-\alpha} x^{-\alpha+(\alpha-1)/r} x^{\alpha/2-3} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \quad (72)$$

$$u'vw' \sim h^3 x^{2-3/r} z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} x^{\alpha/2-3} \sim h^3 z_\theta^{1-\alpha} x^{-\alpha/2-2/r+(\alpha-2)/r} \quad (73)$$

因此

$$K_1''(\xi) \sim h^3 z_\theta^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r}, \xi \in [x_{i-1}, x_{i+1}] \quad (74)$$

现在我们处理第二部分

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} h_{j+1}^4 u'''(\eta_{1,j+1}^\theta) |x_{i+1} - y_{j+1}^\theta|^{1-\alpha} \right. \\ & \quad - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) h_j^4 u'''(\eta_{1,j}^\theta) |x_i - y_j^\theta|^{1-\alpha} \\ & \quad \left. + \frac{1}{h_i} h_{j-1}^4 u'''(\eta_{1,j-1}^\theta) |x_{i-1} - y_{j-1}^\theta|^{1-\alpha} \right) \end{aligned} \quad (75)$$

这次我们只用一阶差分

$$\frac{1}{h_i} (h_j^4 u'''(\eta_{1,j}^\theta) |x_i - y_j^\theta|^{1-\alpha} - h_{j-1}^4 u'''(\eta_{1,j-1}^\theta) |x_{i-1} - y_{j-1}^\theta|^{1-\alpha}) \quad (76)$$

为了方便计算，我们还是用辅助函数来对上面一项进行估计。

$$K_2(x) = h_j^4(x) |x - y_\theta(x)|^{1-\alpha} \quad (77)$$

$$\begin{aligned}
K_2'(x) &= (h_j^4)'|x - y_\theta(x)|^{1-\alpha} + h_j^4(|x - y_\theta(x)|^{1-\alpha})' \\
&\sim h^3 x^{3-3/r} x^{1/r-1} h y_R^{1-2/r} z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} \\
&\quad + h^4 y_R^{4-4/r} z_\theta^{1-\alpha} x^{-\alpha+(\alpha-1)/r} \\
&\sim h^4 z_\theta^{1-\alpha} x^{4-5/r-\alpha+\alpha/r}
\end{aligned} \tag{78}$$

那么, 上面就等于

$$\begin{aligned}
&\frac{1}{h_i}(K_2(x_i)u'''(\eta_{1,j}^\theta) - K_2(x_{i-1})u'''(\eta_{1,j-1}^\theta)) \\
&= \frac{1}{h_i}K_2(x_i)(u'''(\eta_{1,j}^\theta) - u'''(\eta_{1,j-1}^\theta)) + \frac{1}{h_i}(K_2(x_i) - K_2(x_{i-1}))u'''(\eta_{1,j-1}^\theta) \\
&= K_2(x_i)u''''(\eta_j^\theta)(x_j - x_{j-2}) + K_2'(\xi)u'''(\eta_{1,j-1}^\theta) \quad (\eta_j^\theta \in [x_{j-2}, x_j], \xi \in [x_{j-1}, x_j]) \\
&\sim h_i^{-1}h_j^4|x_i - y_j^\theta|^{1-\alpha} C(\eta_j^\theta)^{\alpha/2-4}2h_j \\
&\quad + h^4 z_\theta^{1-\alpha} \xi^{4-5/r-\alpha+\alpha/r} (\eta_j^\theta)^{\alpha/2-3} \\
&\sim h^4 x_i^{4-4/r} z_\theta^{1-\alpha} x_i^{1-\alpha+(\alpha-1)/r} x_i^{\alpha/2-4} + h^4 z_\theta^{1-\alpha} x_i^{4-5/r-\alpha+\alpha/r} x_i^{\alpha/2-3} \\
&\sim h x_i^{1-1/r} h^3 z_\theta^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r}
\end{aligned} \tag{79}$$

因此,

$$\begin{aligned}
&\frac{2}{h_i + h_{i+1}} \frac{1}{h_i} (K_2(x_i)u'''(\eta_{1,j}^\theta) - K_2(x_{i-1})u'''(\eta_{1,j-1}^\theta)) \\
&\sim h^3 z_\theta^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r}
\end{aligned} \tag{80}$$

最终我们得到, 当 $i/2 + 2 \leq j \leq i < N$ 时, 有

$$\begin{aligned}
&\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&\leq C h^3 \left(\frac{i-j+1}{N} \right)^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r}
\end{aligned} \tag{81}$$

那么我们得到

$$\begin{aligned}
I_3 &\leq C \sum_{j=i/2+2}^i \left(\frac{1}{N}\right)^3 \left(\frac{i-j+1}{N}\right)^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r} \\
&\leq C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r+(\alpha-2)/r} \left(\frac{i}{2N}\right)^{2-\alpha} \\
&\leq C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r+(\alpha-2)/r} x_i^{(2-\alpha)/r} \\
&= C \left(\frac{1}{N}\right)^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{82}$$

最后我们处理 I_2 , 记 $k = i/2 + 1$

$$\begin{aligned}
I_2 &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,i/2+1} + T_{i+1,i/2+2}) - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right) (T_{i,i/2+1}) \right) \\
&= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + \left(\frac{1}{h_{i+1}} - \frac{1}{h_i}\right) T_{i,k} \right) \\
&= J_1 + J_2 + J_3
\end{aligned} \tag{83}$$

$$\begin{aligned}
J_1 &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) \right) \\
&= \frac{2}{h_i + h_{i+1}} \int_{x_{k-1}}^{x_k} D(y) \frac{|x_{i+1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_{i+1}} dy \\
&\leq C x_i^{\alpha/2-2} h_k^2 x_i^{-\alpha} \\
&\leq C h^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{84}$$

$$\begin{aligned}
J_2 &= \frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) \\
&= \frac{2}{h_i + h_{i+1}} \int_0^1 \frac{h_{k+1} D(y_{k+1}^\theta) |x_{i+1} - y_{k+1}^\theta|^{1-\alpha} - h_k D(y_k^\theta) |x_i - y_k^\theta|^{1-\alpha}}{h_{i+1}} d\theta
\end{aligned} \tag{85}$$

我们看他的两个积分项

$$\begin{aligned}
\frac{K_1(x_{i+1}) - K_1(x_i)}{h_{i+1}} &= K_1'(\xi) \\
&\sim h^3 x^{2-3/r} z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} x^{\alpha/2-2} \\
&\quad + h^3 x^{3-3/r} z_\theta^{1-\alpha} x^{-\alpha+(\alpha-1)/r} x^{\alpha/2-2} \\
&\quad + h^3 x^{3-3/r} z_\theta^{1-\alpha} x^{1-\alpha+(\alpha-1)/r} x^{\alpha/2-3} \\
&\sim h x^{1-1/r} h^2 z_\theta^{1-\alpha} x^{-\alpha/2+\alpha/r-3/r} \\
&\sim h x^{1-1/r} h^2 x^{(1-\alpha)/r} x^{-\alpha/2+\alpha/r-3/r} \\
&\sim h x^{1-1/r} h^2 x^{-\alpha/2-2/r}
\end{aligned} \tag{86}$$

第二部分研究过了

$$\begin{aligned}
&\frac{1}{h_i} (K_2(x_{i+1}) u'''(\eta_{1,k+1}^\theta) - K_2(x_i) u'''(\eta_{1,k}^\theta)) \\
&\sim h x_i^{1-1/r} h^3 z_\theta^{1-\alpha} x_i^{-\alpha/2-2/r+(\alpha-2)/r} \\
&\sim h x_i^{1-1/r} h^3 x_i^{(1-\alpha)/r} x_i^{-\alpha/2-2/r+(\alpha-2)/r} \\
&\sim h x_i^{1-1/r} h^3 x_i^{-\alpha/2-2/r-1/r}
\end{aligned} \tag{87}$$

因此

$$J_2 \leq C h^2 x^{-\alpha/2-2/r} \tag{88}$$

现在考虑 J_3

$$\begin{aligned}
J_3 &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} - \frac{1}{h_i} \right) T_{i,k} \\
&= -\frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} \int_{x_{k-1}}^{x_k} D(y_k^\theta) |x_i - y_k^\theta|^{1-\alpha} dy \\
&\sim h_i^{-1} x_i^{-1} h_k^3 x_i^{\alpha/2-2} x_i^{1-\alpha} \\
&\sim h^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{89}$$

因此我们有

$$I_2 \leq C h^2 x_i^{-\alpha/2-2/r} \tag{90}$$

全部加起来，我们得到

错了！不对称，要补上