A SECOND ORDER NUMERICAL METHODS FOR REISZ-FRACTIONAL ELLIPTIC EQUATION ON GRADED MESH*

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Abstract. This is an example SIAM LaTeX article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

Key words. example, LaTeX

MSC codes. ??????????????????

1. Introduction.

For $\Omega = (0, 2T)$, $1 < \alpha < 2$,

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases} \tag{1.1}$$

where

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy \tag{1.2}$$

$$\kappa_\alpha = -\frac{1}{2\cos(\alpha\pi/2)} > 0 \tag{1.3}$$

2. Preliminaries: Numeric scheme and main results.

2.1. Numeric Format.

$$x_i = \begin{cases} T\left(\frac{i}{N}\right)^r, & 0 \le i \le N \\ 2T - T\left(\frac{2N-i}{N}\right)^r, & N \le i \le 2N \end{cases} \tag{2.1}$$

where $r \ge 1$. And let

$$h_j = x_j - x_{j-1}, \quad 1 \le j \le 2N \tag{2.2}$$

Let $\{\phi_j(x)\}_{j=1}^{2N-1}$ be standard hat functions, which are basis of the piecewise linear function space.

$$\phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \le x \le x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

And then, define the piecewise linear interpolant of the true solution u to be

$$\Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j)\phi_j(x) \tag{2.4}$$

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For convience, we denote

$$I^{2-\alpha}u(x) := \frac{1}{\Gamma(2-\alpha)}\int_{\Omega}|x-y|^{1-\alpha}u(y)dy \tag{2.5}$$

and

$$D_h^2u(x_i) := \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}u(x_{i-1}) - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)u(x_i) + \frac{1}{h_{i+1}}u(x_{i+1})\right) \tag{2.6}$$

Now, we discretise (1.1) by replacing $u(x)$ by a continuous piecewise linear function

$$u_h(x) := \sum_{j=1}^{2N-1} u_j\phi_j(x) \tag{2.7}$$

whose nodal values u_j are to be determined by collocation at each mesh point x_i for $i = 1, 2, ..., 2N-1$:

$$-\kappa_\alpha D_h^\alpha u_h(x_i) := -\kappa_\alpha D_h^2 I^{2-\alpha}u_h(x_i) = f(x_i) =: f_i \tag{2.8}$$

Here,

$$-\kappa_\alpha D_h^\alpha u_h(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i)\; u_j = \sum_{j=1}^{2N-1} a_{ij}\; u_j \tag{2.9}$$

where

$$a_{ij} = -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i) \quad \text{for} \quad i,j = 1,2,...,2N-1 \tag{2.10}$$

We have replaced $(-\Delta)^{\alpha/2}u(x_i) = f(x_i)$ in (1.1) by $-\kappa_\alpha D_h^\alpha u_h(x_i) = f(x_i)$ in (2.8), with truncation error

$$\tau_i := -\kappa_\alpha\left(D_h^\alpha \Pi_h u(x_i) - \frac{d^2}{dx^2}I^{2-\alpha}u(x_i)\right) \quad \text{for} \quad i = 1,2,...,2N-1 \tag{2.11}$$

where $-\kappa_\alpha D_h^\alpha \Pi_h u(x_i) = \sum_{j=1}^{2N-1} -\kappa_\alpha D_h^\alpha \phi_j(x_i)u(x_j) = \sum_{j=1}^{2N-1} a_{ij}u(x_j)$.

The discrete equation (2.8) can be written in matrix form

$$AU = F \tag{2.12}$$

where $A = (a_{ij}) \in \mathbb{R}^{(2N-1)\times(2N-1)}$, $U = (u_1, \cdots, u_{2N-1})^T$ is unknown and $F = (f_1, \cdots, f_{2N-1})^T$.

We can deduce a_{ij},

$$\begin{aligned} a_{ij} &= -\kappa_\alpha D_h^2 I^{2-\alpha}\phi_j(x_i) \\ &= -\kappa_\alpha \frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i}\tilde{a}_{i-1,j} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)\tilde{a}_{i,j} + \frac{1}{h_{i+1}}\tilde{a}_{i+1,j}\right) \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \tilde{a}_{ij} &= I^{2-\alpha}\phi_j(x_i) \\ &= \frac{1}{\Gamma(4-\alpha)}\left(\frac{|x_i-x_{j-1}|^{3-\alpha}}{h_j} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)|x_i-x_j|^{3-\alpha} + \frac{|x_i-x_{j+1}|^{3-\alpha}}{h_{j+1}}\right) \end{aligned} \tag{2.14}$$

2.2. Regularity of the true solution. For any $\beta > 0$, we use the standard notation $C^{\beta}(\bar{\Omega}), C^{\beta}(\mathbb{R})$, etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation $C^{\beta}(\Omega)$ to refer to $C^{k,\beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. The Hölder spaces $C^{k,\beta'}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are locally Hölder continuous[1] with exponent β' in Ω, where $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on open set Ω.

Definition 2.1 (delta dependent norm [2]). ...

Theorem 2.2. *Let $f \in C^{\beta}(\Omega), \beta > 2$ be such that $\|f\|_{\beta}^{(\alpha/2)} < \infty$, then for $l = 0, 1, 2$*

$$|f^{(l)}(x)| \leq \|f\|_{\beta}^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \leq T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \leq x < 2T \end{cases} \tag{2.15}$$

Theorem 2.3 (Regularity up to the boundary [2]). *Let Ω be a bounded domain, and $\beta > 0$ be such that neither β nor $\beta + \alpha$ is an integer. Let $f \in C^{\beta}(\Omega)$ be such that $\|f\|_{\beta}^{(\alpha/2)} < \infty$, and $u \in C^{\alpha/2}(\mathbb{R}^n)$ be a solution of (1.1). Then, $u \in C^{\beta+\alpha}(\Omega)$ and*

$$\|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C\left(\|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_{\beta}^{(\alpha/2)}\right) \tag{2.16}$$

Corollary 2.4. *Let u be a solution of (1.1) where $f \in L^{\infty}(\Omega)$ and $\|f\|_{\beta}^{(\alpha/2)} < \infty$. Then, for any $x \in \Omega$ and $l = 0, 1, 2, 3, 4$*

$$|u^{(l)}(x)| \leq \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases} \tag{2.17}$$

And in this paper bellow, without special instructions, we allways assume that

$$f \in L^{\infty}(\Omega) \cap C^{\beta}(\Omega) \quad \text{and} \quad \|f\|_{\beta}^{(\alpha/2)} < \infty, \text{ with } \alpha + \beta > 4 \tag{2.18}$$

2.3. Main results. Here we state our main results; the proof is deferred to section 3 and section 4.

Let's denote $h = \frac{1}{N}$, we have

Theorem 2.5 (Local Truncation Error). *If $u(x)$ is a solution of the equation (1.1) where f satisfy the regular condition (2.18), then there exists $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ and $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$, such that the truncation error (2.11) satisfies*

$$\begin{aligned} |\tau_i| :=& |-\kappa_{\alpha} D_h^{\alpha} \Pi_h u(x_i) - f(x_i)| \\ \leq& C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} \begin{cases} x_i^{-\alpha}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha}, & N < i \leq 2N-1 \end{cases} \\ &+ C_2(r-1)h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \leq i \leq N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \leq 2N-1 \end{cases} \end{aligned} \tag{2.19}$$

THEOREM 2.6 (Global Error). *The discrete equation* (2.8) *has sulotion and there exists a positive constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ *such that the error between the numerial solution* U *with the exact solution* $u(x_i)$ *satisfies*

$$\max_{1\le i\le 2N-1} |u_i - u(x_i)| \le Ch^{\min\{\frac{r\alpha}{2},2\}} \tag{2.20}$$

That means the numerial method has convergence order $\min\{\frac{r\alpha}{2}, 2\}$.

3. Local Truncation Error.

3.1. Proof of Theorem 2.5. The truncation error of the discrete format can be written as

$$\begin{aligned} -\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i) &= -\kappa_\alpha (D_h^2 I^{2-\alpha}\Pi_h u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha} u(x_i)) \\ &= -\kappa_\alpha D_h^2 I^{2-\alpha}(\Pi_h u - u)(x_i) - \kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha} u(x_i) \end{aligned} \tag{3.1}$$

THEOREM 3.1. *There exits a constant* $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$ *such that*

$$\left| -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \right| \le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases} \tag{3.2}$$

Proof. Since $f \in C^2(\Omega)$ and

$$\frac{d^2}{dx^2}(-\kappa_\alpha I^{2-\alpha}(x)) = f(x), \quad x \in \Omega, \tag{3.3}$$

we have $I^{2-\alpha} \in C^4(\Omega)$. Therefore, using equation (A.3) of Lemma A.1, for $1 \le i \le 2N-1$, we have

$$\begin{aligned} -\kappa_\alpha (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) &= \frac{h_{i+1} - h_i}{3} f'(x_i) \\ &+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right) \end{aligned} \tag{3.4}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$. By Lemma B.2 and Theorem 2.2 we have 1.

$$\left| \frac{h_{i+1} - h_i}{3} f'(x_i) \right| \le \frac{C(r-1)\|f\|_{\beta}^{(\alpha/2)}}{3} h^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{-\alpha/2-2/r}, & N < i \le 2N-1 \end{cases} \tag{3.5}$$

2. See Proof 23, there is a constant $C = C(T, \alpha, r, \|f\|_{\beta}^{\alpha/2})$ such that

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right) \\ &\le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases} \end{aligned} \tag{3.6}$$

Summarizes, we get the result. ∎

And define

$$R_i := D_h^2 I^{2-\alpha}(u - \Pi_h u)(x_i) \tag{3.7}$$

We have some results about the estimate of R_i

THEOREM 3.2. *For $1 \le i < N/2$, there exists $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$R_i \le \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2(x_i^{-1-\alpha}\ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.8}$$

THEOREM 3.3. *For $N/2 \le i \le N$, there exists constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$R_i \le C(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.9}$$

And for $N < i \le 2N - 1$, it is symmetric to the previous case.

Combine Theorem 3.1, Theorem 3.2 and Theorem 3.3, the proof of Theorem 2.5 completed.

We prove Theorem 3.2 and Theorem 3.3 in next subsections below.

3.2. Proof of Theorem 3.2.

$$D_h^2 I^{2-\alpha}(u - \Pi_h u)(x_i) = D_h^2\left(\int_0^{2T} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy\right) \tag{3.10}$$

For convience, let's denote

$$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy, \quad i = 0, \cdots, 2N,\ j = 1, \cdots, 2N \tag{3.11}$$

Also for simplicity, we denote

DEFINITION 3.4.

$$S_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} T_{i-1,j} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right) \tag{3.12}$$

then

$$R_i = \sum_{j=1}^{2N} S_{ij} \tag{3.13}$$

LEMMA 3.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \le i < N/2$,*

$$\sum_{j=\max\{2i+1, i+3\}}^{N} S_{ij} \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.14}$$

Proof. Let

$$K_y(x) = \frac{|y-x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

For $\max\{2i+1, i+3\} \le j \le N$, by Lemma C.1 and Lemma C.2

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \\ &\le Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy \end{aligned} \tag{3.15}$$

Therefore,

$$\begin{aligned} \sum_{j=\max\{2i+1,i+3\}}^{N} S_{ij} &\le Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\ &= \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\ &\le \frac{C}{\alpha/2+2/r} 2^{r(-\alpha/2-2/r)} h^2 x_i^{-\alpha/2-2/r} \end{aligned} \tag{3.16}$$

□

LEMMA 3.6. *Thert exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $1 \le i < N/2$,*

$$\sum_{j=N+1}^{2N} S_{ij} \le \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.17}$$

Proof. For $1 \le i < N/2, N+1 \le j \le 2N-1$, by equation (C.2) and Lemma C.2

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \\ &\le \int_{x_{j-1}}^{x_j} Ch^2 (2T-y)^{\alpha/2-2/r} y^{-1-\alpha} dy \\ &\le Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T-y)^{\alpha/2-2/r} dy \end{aligned}$$

$$
\begin{aligned}
\sum_{j=N+1}^{2N-1} S_{ij} &\le CT^{-1-\alpha}h^2\int_{x_N}^{x_{2N-1}}(2T-y)^{\alpha/2-2/r}dy \\
&\le CT^{-1-\alpha}h^2\begin{cases}\frac{1}{\alpha/2-2/r+1}T^{\alpha/2-2/r+1}, & \alpha/2-2/r+1>0\\ \ln(T)-\ln(h_{2N}), & \alpha/2-2/r+1=0\\ \frac{1}{|\alpha/2-2/r+1|}h_{2N}^{\alpha/2-2/r+1}, & \alpha/2-2/r+1<0\end{cases} \\
&= \begin{cases}\frac{C}{\alpha/2-2/r+1}T^{-\alpha/2-2/r}\ h^2, & \alpha/2-2/r+1>0\\ CrT^{-1-\alpha}h^2\ln(N), & \alpha/2-2/r+1=0\\ \frac{C}{|\alpha/2-2/r+1|}T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2-2/r+1<0\end{cases}
\end{aligned} \tag{3.18}
$$

And by Lemma A.3

$$S_{i,2N}\le CT^{-1-\alpha}h_{2N}^{\alpha/2+1}=CT^{-\alpha/2}h^{r\alpha/2+r}$$

And when $\alpha/2-2/r+1\ge 0$,

$$h^{r\alpha/2+r}\le h^2$$

Summarizes, we get the result. ∎

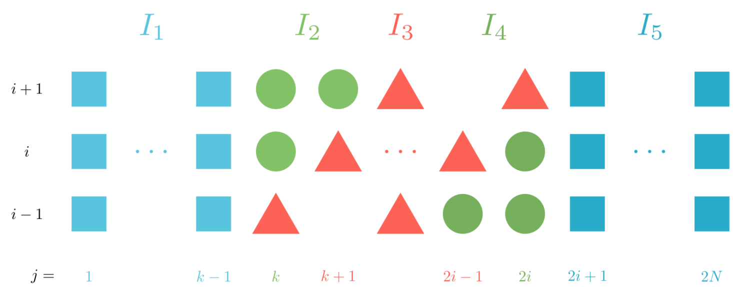
For $i=1,2$.

LEMMA 3.7. *By Lemma* C.5 , *Lemma* 3.5 *and Lemma* 3.6 *we get*

$$
\begin{aligned}
R_1 &= \sum_{j=1}^{3}S_{1j}+\sum_{j=4}^{2N}S_{1j} \\
&\le Ch^2x_1^{-\alpha/2-2/r}+\begin{cases}Ch^2, & \alpha/2-2/r+1>0\\ Ch^2\ln(N), & \alpha/2-2/r+1=0\\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0\end{cases}
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
R_2 &= \sum_{j=1}^{4}S_{2j}+\sum_{j=5}^{2N}S_{2j} \\
&\le Ch^2x_2^{-\alpha/2-2/r}+\begin{cases}Ch^2, & \alpha/2-2/r+1>0\\ Ch^2\ln(N), & \alpha/2-2/r+1=0\\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1<0\end{cases}
\end{aligned} \tag{3.20}
$$

For $3\le i<N/2$, we have a new separation of R_i, Let's denote $k=\lceil\frac{i}{2}\rceil$.



$$
\begin{aligned}
R_i =&\sum_{j=1}^{2N} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=&\sum_{j=1}^{k-1} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&+ \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\
&+ \sum_{j=k+1}^{2i-1} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&+ \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \\
&+ \sum_{j=2i+1}^{2N} \frac{2}{h_i+h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=&I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned} \tag{3.21}
$$

LEMMA 3.8. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that for $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil$*

$$
|I_1| = |\sum_{j=1}^{k-1} S_{ij}| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.22}
$$

Proof. by Lemma A.3 , Lemma C.3

$$
S_{i1} \leq C x_1^{\alpha/2} x_1 x_i^{-1-\alpha} = C x_1^{\alpha/2+1} x_i^{-1-\alpha} = C T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha} \tag{3.23}
$$

For $2 \leq j \leq k-1$, by Lemma C.1 and Lemma C.3

$$
\begin{aligned}
S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \\
&\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} dy \\
&= Ch^2 x_i^{-1-\alpha} \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} dy
\end{aligned} \tag{3.24}
$$

Therefore,

$$
\begin{aligned}
I_1 &= \sum_{j=1}^{k-1} S_{ij} = S_{i1} + \sum_{j=2}^{k-1} S_{ij} \\
&\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil -1}} y^{\alpha/2-2/r} dy \\
&\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy
\end{aligned} \tag{3.25}
$$

But

$$
\int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \leq \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.26}
$$

So we have

$$
I_1 \leq \begin{cases} \frac{C}{\alpha/2-2/r+1} h^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} h^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.27}
$$

□

DEFINITION 3.9. *For convience, let's denote*

$$
V_{ij} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \tag{3.28}
$$

THEOREM 3.10. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \leq i < N/2, k = \lceil \frac{i}{2} \rceil$,*

$$
I_3 = \sum_{j=k+1}^{2i-1} V_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r} \tag{3.29}
$$

To estimete V_{ij}, we need some preparations.

LEMMA 3.11. *For $y \in (x_{j-1}, x_j)$, we can rewrite*

$$
y = x_{j-1} + \theta h_j = (1-\theta) x_{j-1} + \theta x_j =: y_j^\theta, \; \theta \in (0,1) \tag{3.30}
$$

by Lemma A.2,

$$T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \tag{3.31}$$
$$= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta$$
$$= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)}$$
$$+ \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j1}^\theta) - (1-\theta)^2 u'''(\eta_{j2}^\theta)) d\theta$$

where $\eta_{j1}^\theta \in (x_{j-1}, y_j^\theta), \eta_{j2}^\theta \in (y_j^\theta, x_j)$.

Now Let's construct a series of functions to represent T_{ij}.

Definition 3.12.

$$y_{j-i}(x) = (x^{1/r} + Z_{j-i})^r, \quad Z_{j-i} = T^{1/r} \frac{j-i}{N} \tag{3.32}$$

$$y_{j-i}^\theta(x) = (1-\theta) y_{j-1-i}(x) + \theta y_{j-i}(x) \tag{3.33}$$

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{3.34}$$

Now, we define

$$P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.35}$$

$$Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.36}$$

And now we can rewrite T_{ij}

Lemma 3.13. *For* $2 \le i \le N, 2 \le j \le N$,

$$T_{ij} = \int_0^1 -\frac{\theta(1-\theta)}{2} P_{j-i}^\theta(x_i) d\theta \tag{3.37}$$
$$+ \int_0^1 \frac{\theta(1-\theta)}{3!} Q_{j-i}^\theta(x_i) \left[\theta^2 u'''(\eta_{j,1}^\theta) - (1-\theta)^2 u'''(\eta_{j,2}^\theta)\right] d\theta$$

Immediately, we can see from (3.28) that

LEMMA 3.14. *For* $3 \le i, j \le N-1$,

$$V_{ij} = \frac{2}{h_i + h_{i+1}}\left(\frac{1}{h_{i+1}}T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right)T_{i,j} + \frac{1}{h_i}T_{i-1,j-1}\right) \tag{3.38}$$
$$= \int_0^1 -\frac{\theta(1-\theta)}{2}D_h^2 P_{j-i}^{\theta}(x_i)d\theta$$
$$+ \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1,1}^{\theta}) - Q_{j-i}^{\theta}(x_i)u'''(\eta_{j,1}^{\theta})}{h_{i+1}}\right)d\theta$$
$$- \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{j-i}^{\theta}(x_i)u'''(\eta_{j,1}^{\theta}) - Q_{j-i}^{\theta}(x_{i-1})u'''(\eta_{j-1,1}^{\theta})}{h_i}\right)d\theta$$
$$- \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1,2}^{\theta}) - Q_{j-i}^{\theta}(x_i)u'''(\eta_{j,2}^{\theta})}{h_{i+1}}\right)d\theta$$
$$+ \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{Q_{j-i}^{\theta}(x_i)u'''(\eta_{j,2}^{\theta}) - Q_{j-i}^{\theta}(x_{i-1})u'''(\eta_{j-1,2}^{\theta})}{h_i}\right)d\theta$$

To estimate V_{ij}, we first estimate $D_h^2 P_{j-i}^{\theta}(x_i)$, but By Lemma A.1,

$$D_h^2 P_{j-i}^{\theta}(x_i) = {P_{j-i}^{\theta}}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{3.39}$$

By Leibniz formula, we calculate and estimate the derivations of $h_{j-i}^3(x)$, $u''(y_{j-i}^{\theta}(x))$ and $\frac{|y_{j-i}^{\theta}(x)-x|^{1-\alpha}}{\Gamma(2-\alpha)}$ separately.

Firstly, we have

LEMMA 3.15. *There exists a constant* $C = C(T, r)$ *such that For* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil \le j \le \min\{2i, N\}$, $\xi \in (x_{i-1}, x_{i+1})$,

$$h_{j-i}^3(\xi) \le Ch^2 x_i^{2-2/r} h_j \tag{3.40}$$
$$(h_{j-i}^3(\xi))' \le C(r-1)h^2 x_i^{1-2/r} h_j \tag{3.41}$$
$$(h_{j-i}^3(\xi))'' \le C(r-1)h^2 x_i^{-2/r} h_j \tag{3.42}$$

The proof of this theorem see Lemma C.6 and Lemma C.7

Second,

LEMMA 3.16. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil \le j \le \min\{2i, N\}$, $\xi \in (x_{i-1}, x_{i+1})$,

$$u''(y_{j-i}^{\theta}(\xi)) \le Cx_i^{\alpha/2-2} \tag{3.43}$$
$$(u''(y_{j-i}^{\theta}(\xi)))' \le Cx_i^{\alpha/2-3} \tag{3.44}$$
$$(u''(y_{j-i}^{\theta}(\xi)))'' \le Cx_i^{\alpha/2-4} \tag{3.45}$$

The proof of this theorem see Proof 29

And Finally, we have

LEMMA 3.17. *There exists a constant* $C = C(T, \alpha, r)$ *such that For* $3 \le i \le N-1, \lceil \frac{i}{2} \rceil \le j \le \min\{2i, N\}$, $\xi \in (x_{i-1}, x_{i+1})$,

$$|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha} \le C|y_j^{\theta} - x_i|^{1-\alpha} \tag{3.46}$$

$$(|y_{j-i}^{\theta}(\xi)-\xi|^{1-\alpha})' \le C|y_j^{\theta}-x_i|^{1-\alpha}x_i^{-1} \tag{3.47}$$

$$(|y_{j-i}^{\theta}(\xi)-\xi|^{1-\alpha})'' \le C|y_j^{\theta}-x_i|^{1-\alpha}x_i^{-2} \tag{3.48}$$

where $y_j^{\theta}=\theta x_{j-1}+(1-\theta)x_j$

The proof of this theorem see Proof 30

Lemma 3.18. *There exists a constant* $C=C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $3\le i\le N-1, \lceil \frac{i}{2}\rceil+1\le j\le \min\{2i-1,N-1\}$,

$$D_h^2 P_{j-i}^{\theta}(x_i) \le Ch^2\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j \tag{3.49}$$

where $y_j^{\theta}=\theta x_{j-1}+(1-\theta)x_j$

Proof. Since Lemma A.1

$$D_h^2 P_{j-i}^{\theta}(x_i) = {P_{j-i}^{\theta}}''(\xi), \quad \xi\in(x_{i-1},x_{i+1}) \tag{3.50}$$

From (3.35), using Leibniz formula and Lemma 3.15, Lemma 3.16 and Lemma 3.17□

Lemma 3.19. *There exists a constant* $C=C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3\le i\le N-1$.
For $\lceil \frac{i}{2}\rceil\le j\le \min\{2i-1,N-1\}$,

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}&\left(\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta})-Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}}\right)\\ &\le Ch^2\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j\end{aligned} \tag{3.51}$$

And for $\lceil \frac{i}{2}\rceil+1\le j\le \min\{2i,N\}$,

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}&\left(\frac{Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})-Q_{j-i}^{\theta}(x_{i-1})u'''(\eta_{j-1}^{\theta})}{h_i}\right)\\ &\le Ch^2\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j\end{aligned} \tag{3.52}$$

where $\eta_j^{\theta}\in(x_{j-1},x_j)$.

proof see Proof 31

Lemma 3.20. *There exists a constant* $C=C(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that for* $3\le i\le N-1, \lceil \frac{i}{2}\rceil+1\le j\le \min\{2i-1,N-1\}$,

$$\begin{aligned}V_{ij}&\le Ch^2\int_0^1\frac{|y_j^{\theta}-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}h_j d\theta\\ &= Ch^2\int_{x_{j-1}}^{x_j}\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}x_i^{\alpha/2-2-2/r}dy\end{aligned} \tag{3.53}$$

Proof. Since Lemma 3.14, by Lemma 3.18 and Lemma 3.19, we get the result immediately. ∎

Now we can prove Theorem 3.10 using Lemma 3.20, $k = \lceil \frac{i}{2} \rceil$

$$
\begin{aligned}
I_3 = \sum_{k+1}^{2i-1} V_{ij} &\le Ch^2 \int_{x_k}^{x_{2i-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\
&= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{2i-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\
&\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{3.54}
$$

Now we study I_2, I_4.

Lemma 3.21. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $3 \le i \le N-1, k = \lceil \frac{i}{2} \rceil$,*

$$
I_2 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.55}
$$

And for $3 \le i < N/2$,

$$
I_4 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2i} \right) \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.56}
$$

Proof. In fact,

$$
\begin{aligned}
&\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \\
&= \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + (\frac{1}{h_{i+1}} - \frac{1}{h_i}) T_{i,k}
\end{aligned} \tag{3.57}
$$

While, by Lemma A.2 and Lemma B.1

$$
\begin{aligned}
\frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) &= \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y)) \frac{|x_{i+1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_{i+1} \Gamma(2-\alpha)} dy \\
&\le h_k^2 \max_{\eta \in (x_{k-1}, x_k)} |u''(\eta)| \int_{x_{k-1}}^{x_k} \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy, \quad \xi \in (x_i, x_{i+1}) \\
&\le Ch^2 x_k^{2-2/r} x_{k-1}^{\alpha/2-2} \, h_k |x_i - x_k|^{-\alpha} \\
&\le Ch^2 x_i^{-\alpha/2-2/r} h_k
\end{aligned} \tag{3.58}
$$

Thus,

$$
\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} |T_{i+1,k} - T_{i,k}| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.59}
$$

From Lemma 3.13

$$\begin{aligned}\frac{1}{h_{i+1}}(T_{i+1,k+1}-T_{i,k}) = &\int_0^1 -\frac{\theta(1-\theta)}{2}\frac{P_{k-i}^{\theta}(x_{i+1})-P_{k-i}^{\theta}(x_i)}{h_{i+1}}d\theta \\ &+\int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{Q_{k-i}^{\theta}(x_{i+1})u'''(\eta_{k+1,1}^{\theta})-Q_{k-i}^{\theta}(x_i)u'''(\eta_{k,1}^{\theta})}{h_{i+1}}d\theta \\ &-\int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{Q_{k-i}^{\theta}(x_{i+1})u'''(\eta_{k+1,2}^{\theta})-Q_{k-i}^{\theta}(x_i)u'''(\eta_{k,2}^{\theta})}{h_{i+1}}d\theta\end{aligned} \tag{3.60}$$

and

$$D_h P_{k-i}^{\theta}(x_i) := \frac{P_{k-i}^{\theta}(x_{i+1})-P_{k-i}^{\theta}(x_i)}{h_{i+1}} = {P_{k-i}^{\theta}}'(\xi), \quad \xi\in(x_i,x_{i+1}) \tag{3.61}$$

Similar with Lemma 3.18, from Lemma 3.13, using Leibniz formula, by Lemma C.6, Lemma 3.16 and Lemma 3.17 we get

$$|D_h P_{k-i}^{\theta}(x_i)| \le Ch^2 x_i^{-\alpha/2-2/r}h_k \tag{3.62}$$

And with Lemma 3.19, we can get

$$\frac{2}{h_i+h_{i+1}}\frac{1}{h_{i+1}}|T_{i+1,k+1}-T_{i,k}| \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.63}$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned}\frac{2}{h_i+h_{i+1}}\frac{h_{i+1}-h_i}{h_ih_{i+1}}T_{i,k} &\le h_i^{-3}h^2x_i^{1-2/r}h_kCh_k^2x_{k-1}^{\alpha/2-2}|x_k-x_i|^{1-\alpha} \\ &\le Ch^2x_i^{-\alpha/2-2/r}\end{aligned} \tag{3.64}$$

Summarizes, we have

$$I_2 \le Ch^2 x_i^{-\alpha/2-2/r} \tag{3.65}$$

The case for I_4 is similar. ∎

Now combine Lemma 3.7, Lemma 3.8, Lemma 3.21, Theorem 3.10, Lemma 3.5 and Lemma 3.6, we get Theorem 3.2.

3.3. Proof of Theorem 3.3. For $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$, we have

(3.66)
$$
\begin{aligned}
R_i =& \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,k} \right) \\
&+ \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&+ \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} (T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right) \\
&+ \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
=& I_1 + I_2 + I_3^1 + I_3^2 + I_3^3 + I_4 + I_5
\end{aligned}
$$

We have estimate I_1 in Lemma 3.8 and I_2 in Lemma 3.21. We can control I_3 in similar with Theorem 3.10 by Lemma 3.20 where $2i - 1 \ge N - 1$

LEMMA 3.22. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that for $N/2 \le i < N, k = \lceil \frac{i}{2} \rceil$,*

$$
\begin{aligned}
I_3 = \sum_{j=k+1}^{N-1} V_{ij} &\le Ch^2 \int_{x_k}^{x_{N-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\
&= Ch^2 \left(\frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{N-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\
&\le Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r}
\end{aligned} \tag{3.67}
$$

Let's study I_5 before I_4.

$$
I_5 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} V_{ij} \tag{3.68}
$$

Similarly, Let's define a new series of functions

DEFINITION 3.23. *For $i < N, j \ge N$, with no confusion, we also denote in this section*

$$
y_{j-i}(x) = 2T - (Z_{2N-j+i} - x^{1/r})^r, \quad Z_{2N-j+i} = T^{1/r} \frac{2N - j + i}{N} \tag{3.69}
$$

$$
{y_{j-i}}'(x) = (2T - y_{j-i}(x))^{1-1/r} x^{1/r-1} \tag{3.70}
$$

$$ {y_{j-i}}''(x) = \frac{1-r}{r}(2T - y_{j-i}(x))^{1-2/r} x^{1/r-2} Z_{2N-j+i} \tag{3.71} $$

(3.72)

$$ y^{\theta}_{j-i}(x) = (1-\theta) y_{j-i-1}(x) + \theta y_{j-i}(x) \tag{3.73} $$

$$ h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x) \tag{3.74} $$

$$ P^{\theta}_{j-i}(x) = (h_{j-i}(x))^3 u''(y^{\theta}_{j-i}(x)) \frac{|y^{\theta}_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.75} $$

$$ Q^{\theta}_{j-i}(x) = (h_{j-i}(x))^4 \frac{|y^{\theta}_{j-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.76} $$

Now we have, for $i < N, j \geq N+2$,

$$ \begin{aligned} V_{ij} &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,j+1} - \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P^{\theta}_{j-i}(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_{i+1}) u'''(\eta^{\theta}_{j+1,1}) - Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,1})}{h_{i+1}} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,1}) - Q^{\theta}_{j-i}(x_{i-1}) u'''(\eta^{\theta}_{j-1,1})}{h_i} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_{i+1}) u'''(\eta^{\theta}_{j+1,2}) - Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,2})}{h_{i+1}} \right) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{Q^{\theta}_{j-i}(x_i) u'''(\eta^{\theta}_{j,2}) - Q^{\theta}_{j-i}(x_{i-1}) u'''(\eta^{\theta}_{j-1,2})}{h_i} \right) d\theta \end{aligned} \tag{3.77} $$

Similarly, we first estimate

$$ D_h^2 P^{\theta}_{j-i}(\xi) = {P^{\theta}_{j-i}}''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \tag{3.78} $$

Combine lemmas Lemma C.8, Lemma C.9 and Lemma C.10 , we have

LEMMA 3.24. *There exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \leq i < N$, $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in (x_{i-1}, x_{i+1})$, we have*

$$ \begin{aligned} |{P^{\theta}_{j-i}}''(\xi)| \leq C h_j h^2 (&|y^{\theta}_j - x_i|^{1-\alpha} \\ &+ |y^{\theta}_j - x_i|^{-\alpha} (|2T - x_i - y^{\theta}_j| + h_N) \\ &+ |y^{\theta}_j - x_i|^{-1-\alpha} (|2T - x_i - y^{\theta}_j| + h_N)^2 \\ &+ (r-1) |y^{\theta}_j - x_i|^{-\alpha}) \end{aligned} \tag{3.79} $$

And

LEMMA 3.25. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le i < N$, , $\xi \in (x_{i-1}, x_{i+1})$, *we have for* $N+1 \le j \le 2N - \lceil \frac{N}{2} \rceil$

$$\begin{aligned}\frac{2}{h_i + h_{i+1}} &\left(\frac{Q_{j-i}^{\theta}(x_{i+1}) u'''(\eta_{j+1}^{\theta}) - Q_{j-i}^{\theta}(x_i) u'''(\eta_j^{\theta})}{h_{i+1}} \right) \\ &\le Ch^2 h_j (|y_j^{\theta} - x_i|^{1-\alpha} + |y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N))\end{aligned} \tag{3.80}$$

for $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$

$$\begin{aligned}\frac{2}{h_i + h_{i+1}} &\left(\frac{Q_{j-i}^{\theta}(x_i) u'''(\eta_j^{\theta}) - Q_{j-i}^{\theta}(x_{i-1}) u'''(\eta_{j-1}^{\theta})}{h_{i+1}} \right) \\ &\le Ch^2 h_j (|y_j^{\theta} - x_i|^{1-\alpha} + |y_j^{\theta} - x_i|^{-\alpha}(|2T - x_i - y_j^{\theta}| + h_N))\end{aligned} \tag{3.81}$$

The proof see Proof 35.

Combine (3.77), Lemma 3.24 and Lemma 3.25, we have

THEOREM 3.26. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le i < N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$

$$\begin{aligned}V_{ij} \le& Ch^2 \int_{x_{j-1}}^{x_j} (|y - x_i|^{1-\alpha} \\ &+ |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2 \\ &+ (r-1)|y - x_i|^{-\alpha}) dy\end{aligned} \tag{3.82}$$

We can esitmate I_5 Now.

THEOREM 3.27. *There exists a constant* $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ *such that For* $N/2 \le i < N$, *we have*

$$I_5 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \le Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \tag{3.83}$$

Proof.

$$\begin{aligned}I_5 =& \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \\ \le& Ch^2 \int_{x_{N+1}}^{x_{2N-i}} + \int_{x_{2N-i}}^{x_{2N - \lceil \frac{N}{2} \rceil}} (|y - x_i|^{1-\alpha} \\ &+ |y - x_i|^{-\alpha}(|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha}(|2T - x_i - y| + h_N)^2 \\ &+ (r-1)|y - x_i|^{-\alpha}) dy \\ =& J_1 + J_2\end{aligned} \tag{3.84}$$

While $x_{N+1} \le y \le x_{2N-i} = 2T - x_i$,

$$T - x_{i-1} \le x_{N+1} - x_i \le y - x_i \le x_{2N-i} - x_i \le 2(T - x_{i-1}) \tag{3.85}$$

and

$$2T - x_i - y + h_N \le 2T - x_i - x_{N+1} + h_N = T - x_i \le T - x_{i-1} \tag{3.86}$$

So

$$
\begin{aligned}
J_1 &\le Ch^2(x_{2N-i}-x_{N+1})(|T-x_{i-1}|^{1-\alpha}+(r-1)|T-x_{i-1}|^{-\alpha}) \\
&\le Ch^2(|T-x_{i-1}|^{2-\alpha}+(r-1)|T-x_{i-1}|^{1-\alpha}) \\
&\le Ch^2T^{2-\alpha}+C(r-1)h^2|T-x_{i-1}|^{1-\alpha}
\end{aligned} \tag{3.87}
$$

Otherwise, when $x_{2N-i} \le y \le x_{2N-\lceil \frac{N}{2} \rceil}$

$$
x_i+y-2T+h_N \le y-x_i \tag{3.88}
$$

$$
\begin{aligned}
J_2 &\le Ch^2\int_{x_{2N-i}}^{(2-2^{-r})T} |y-x_i|^{1-\alpha}+(r-1)|y-x_i|^{-\alpha} \\
&\le Ch^2(T^{2-\alpha}+(r-1)|x_{2N-i}-x_i|^{1-\alpha}) \\
&= Ch^2+C(r-1)h^2|T-x_i|^{1-\alpha} \le Ch^2+C(r-1)h^2|T-x_{i-1}|^{1-\alpha}
\end{aligned} \tag{3.89}
$$

Summarizes two cases, we get the result. ∎

For I_4, we have

THEOREM 3.28. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that, for $N/2 \le i \le N-1$*

$$V_{iN} = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} T_{i+1,N+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) T_{i,N} + \frac{1}{h_i} T_{i-1,N-1} \right) \le Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \tag{3.90}$$

Proof. We use the similar skill in the last section, but more complicated. for $j = N$, Let

$$_L y_{N-1-i}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r} \frac{N-1-i}{N} \tag{3.91}$$

$$_0 y_{N-i}(x) = \frac{x^{1/r} - Z_i}{Z_1} h_N + T, \quad Z_i = T^{1/r} \frac{i}{N}, x_N = T \tag{3.92}$$

and

$$_R y_{N+1-i}(x) = 2T - (Z_{N-1+i} - x^{1/r})^r, \quad Z_{N-1+i} = T^{1/r} \frac{N-1+i}{N} \tag{3.93}$$

Thus,

$$\begin{aligned}
_L y_{N-1-i}(x_{i-1}) &= x_{N-2}, \quad {}_L y_{N-1-i}(x_i) = x_{N-1}, \quad {}_L y_{N-1-i}(x_{i+1}) = x_N \\
_0 y_{N-i}(x_{i-1}) &= x_{N-1}, \quad {}_0 y_{N-i}(x_i) = x_N, \quad {}_0 y_{N-i}(x_{i+1}) = x_{N+1} \\
_R y_{N+1-i}(x_{i-1}) &= x_N, \quad {}_R y_{N+1-i}(x_i) = x_{N+1}, \quad {}_R y_{N+1-i}(x_{i+1}) = x_{N+2}
\end{aligned}$$

Then, define

$$_L y^\theta_{N-i}(x) = \theta {}_L y_{N-1-i}(x) + (1-\theta) {}_0 y_{N-i}(x) \tag{3.94}$$

$$_R y^\theta_{N+1-i}(x) = \theta {}_0 y_{N-i}(x) + (1-\theta) {}_R y_{N+1-i}(x) \tag{3.95}$$

$$_L h_{N-i}(x) = {}_0 y_{N-i}(x) - {}_L y_{N-1-i}(x) \tag{3.96}$$

$$_R h_{N+1-i}(x) = {}_R y_{N+1-i}(x) - {}_0 y_{N-i}(x) \tag{3.97}$$

We have

$$_L y_{N-1-i}{}'(x) = {}_L y^{1-1/r}_{N-1-i}(x) x^{1/r-1} \tag{3.98}$$

$$_L y_{N-1-i}{}''(x) = \frac{1-r}{r} {}_L y^{1-2/r}_{N-1-i}(x) x^{1/r-2} Z_{N-1-i} \tag{3.99}$$

$$_0 y_{N-i}{}'(x) = \frac{1}{r} \frac{h_N}{Z_1} x^{1/r-1} \tag{3.100}$$

$$_0 y_{N-i}{}''(x) = \frac{1-r}{r^2} \frac{h_N}{Z_1} x^{1/r-2} \tag{3.101}$$

$$_R y_{N+1-i}{}'(x) = (2T - {}_R y_{N+1-i}(x))^{1-1/r} x^{1/r-1} \tag{3.102}$$

$$_R y_{N+1-i}{}''(x) = \frac{1-r}{r} (2T - {}_R y_{N+1-i}(x))^{1-2/r} x^{1/r-2} Z_{N-1+i} \tag{3.103}$$

$$ {}_LP^{\theta}_{N-i}(x) = ({}_Lh_{N-i}(x))^3 \frac{|{}_Ly^{\theta}_{N-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_Ly^{\theta}_{N-i}(x)) \tag{3.104} $$

$$ {}_RP^{\theta}_{N+1-i}(x) = ({}_Rh_{N+1-i}(x))^3 \frac{|{}_Ry^{\theta}_{N+1-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_Ry^{\theta}_{N+1-i}(x)) \tag{3.105} $$

$$ {}_LQ^{\theta}_{N-i}(x) = ({}_Lh_{N-i}(x))^4 \frac{|{}_Ly^{\theta}_{N-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.106} $$

$$ {}_RQ^{\theta}_{N+1-i}(x) = ({}_Rh_{N+1-i}(x))^4 \frac{|{}_Ry^{\theta}_{N+1-i}(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} \tag{3.107} $$

Similar with Lemma 3.13, we can get for $l = -1, 0, 1$,

$$ \begin{aligned} T_{i+l,N+l} = &\int_0^1 -\frac{\theta(1-\theta)}{2} {}_LP^{\theta}_{N-i}(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} {}_LQ^{\theta}_{N-i}(x_{i+l}) (\theta^2 u'''(\eta^{\theta}_{N+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+l,2})) d\theta \end{aligned} \tag{3.108} $$

(3.109)

$$ \begin{aligned} T_{i+l,N+1+l} = &\int_0^1 -\frac{\theta(1-\theta)}{2} {}_RP^{\theta}_{N+1-i}(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} {}_RQ^{\theta}_{N+1-i}(x_{i+l}) (\theta^2 u'''(\eta^{\theta}_{N+1+l,1}) - (1-\theta)^2 u'''(\eta^{\theta}_{N+1+l,2})) d\theta \end{aligned} $$

So we have

(3.110)

$$ \begin{aligned} V_{i,N} = &\int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 {}_LP^{\theta}_{N-i}(x_i) d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_LQ^{\theta}_{N-i}(x_{i+1}) u'''(\eta^{\theta}_{N+1,1}) - {}_LQ^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,1})}{h_{i+1}} \right) d\theta \\ &- \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_LQ^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,1}) - {}_LQ^{\theta}_{N-i}(x_{i-1}) u'''(\eta^{\theta}_{N-1,1})}{h_i} \right) d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_LQ^{\theta}_{N-i}(x_{i+1}) u'''(\eta^{\theta}_{N+1,2}) - {}_LQ^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,2})}{h_{i+1}} \right) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left(\frac{{}_LQ^{\theta}_{N-i}(x_i) u'''(\eta^{\theta}_{N,2}) - {}_LQ^{\theta}_{N-i}(x_{i-1}) u'''(\eta^{\theta}_{N-1,2})}{h_i} \right) d\theta \end{aligned} $$

$N+1$ is similar.

We estimate $D_h^2 {}_LP^{\theta}_{N-i}(x_i) = {}_LP^{\theta\,\prime\prime}_{N-i}(\xi), \xi \in (x_{i-1}, x_{i+1})$,

LEMMA 3.29.

$$ {}_Lh^3_{N-i}(\xi) \le Ch_N^3 \le Ch^3 \tag{3.111} $$

$$ {}_Rh^3_{N+1-i}(\xi) \le Ch_N^3 \le Ch^3 \tag{3.112} $$

$$({}_Lh^3_{N-i}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{3.113}$$

$$({}_Rh^3_{N+1-i}(\xi))' \le C(r-1)h_N^2 h \le C(r-1)h^3 \tag{3.114}$$

$$({}_Lh^3_{N-i}(\xi))'' \le C(r-1)h^2 \tag{3.115}$$

$$({}_Rh^3_{N+1-i}(\xi))'' \le C(r-1)h^2 \tag{3.116}$$

Proof.

$$_Lh_{N-i}(\xi) \le 2h_N, \quad _Rh_{N+1-i}(\xi) \le 2h_N \tag{3.117}$$

$$\begin{aligned}({}_Lh^l_{N-i}(\xi))' &= l\,{}_Lh^{l-1}_{N-i}(\xi)({}_0y_{N-i}{}'(\xi) - {}_Ly_{N-1-i}{}'(\xi)) \\ &= l\,{}_Lh^{l-1}_{N-i}(\xi)x_i^{1/r-1}(\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-1/r}_{N-1-i}(\xi))\end{aligned} \tag{3.118}$$

while

$$\begin{aligned}|\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-1/r}_{N-1-i}(\xi)| &= |\frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r}|(\frac{N-t}{N})^{r-1} - (\frac{N-s}{N})^{r-1}| \quad t \in [0,1], s \in [0,2] \\ &\le T^{1-1/r}|1 - (\frac{N-2}{N})^{r-1}| \le CT^{1-1/r}(r-1)\frac{2}{N}\end{aligned} \tag{3.119}$$

Thus,

$$({}_Lh^l_{N-i}(\xi))' \le C(r-1)h_N^{l-1}x_i^{1/r-1}h \tag{3.120}$$

$$\begin{aligned}({}_Rh^l_{N+1-i}(\xi))' &= l\,{}_Rh^{l-1}_{N+1-i}(\xi)({}_Ry_{N+1-i}{}'(\xi) - {}_0y_{N-i}{}'(\xi)) \\ &= l\,{}_Rh^{l-1}_{N+1-i}(\xi)x_i^{1/r-1}((2T - {}_Ry_{N+1-i}(\xi))^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1})\end{aligned} \tag{3.121}$$

Similarly,

$$\begin{aligned}|(2T - {}_Ry_{N+1-i})^{1-1/r} - \frac{1}{r}\frac{h_N}{Z_1}| &= |\eta^{1-1/r} - \frac{1}{r}\frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r}|(\frac{N-s}{N})^{r-1} - (\frac{N-t}{N})^{r-1}| \quad t \in [0,1], s \in [0,2] \\ &\le T^{1-1/r}|(\frac{N-2}{N})^{r-1} - 1| \le CT^{1-1/r}(r-1)\frac{2}{N}\end{aligned} \tag{3.122}$$

And

$$\begin{aligned}({}_Lh^3_{N-i}(\xi))'' &= 3\,{}_Lh^2_{N-i}(\xi)\,{}_Lh_{N-i}{}''(\xi) + 6\,{}_Lh_{N-i}(\xi)({}_Lh_{N-i}{}'(\xi))^2 \\ &\le Ch_N^2\frac{1-r}{r}x_i^{1/r-2}(\frac{1}{r}\frac{h_N}{Z_1} - {}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i}) + Ch_N(r-1)^2h^2x_i^{2/r-2}\end{aligned} \tag{3.123}$$

$$|\frac{h_N}{rZ_1} - {}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i}| \le T^{1-1/r} + Cx_N^{1-2/r}x_N^{1/r} = CT^{1-1/r}$$

So

$$(3.124)\quad \begin{aligned}({}_Lh^3_{N-i}(\xi))'' &\le Ch_N^2\frac{1-r}{r}x_i^{1/r-2}+C(r-1)^2h_Nx_i^{2/r-2}h^2\\ &\le C(r-1)h_N^2\end{aligned}$$

${}_Rh^3_{N+1-i}(\xi)$ is similar. ∎

Lemma 3.30.

$$(3.125)\quad u''({}_Ly^\theta_{N-i}(\xi))\le Cx_{N-2}^{-\alpha/2-2}\le C$$

$$(3.126)\quad (u''({}_Ly^\theta_{N-i}(\xi)))'\le C$$

$$(3.127)\quad (u''({}_Ly^\theta_{N-i}(\xi)))''\le C$$

Proof.

$$(3.128)\quad \begin{aligned}(u''({}_Ly^\theta_{N-i}(\xi)))' &= u'''({}_Ly^\theta_{N-i}(\xi)){}_L{y^\theta_{N-i}}'(\xi)\\ &\le C(\theta{}_L{y_{N-1-i}}'(\xi)+(1-\theta){}_0{y_{N-i}}'(\xi))\\ &\le Cx_i^{1/r-1}(\theta{}_Ly^{1-1/r}_{N-1-i}(\xi)+(1-\theta)\frac{h_N}{rZ_1})\\ &\le Cx_i^{1/r-1}x_N^{1-1/r}\end{aligned}$$

And

$$(3.129)\quad \begin{aligned}(u''({}_Ly^\theta_{N-i}(\xi)))'' &= u''''({}_Ly^\theta_{N-i}(\xi))({}_L{y^\theta_{N-i}}'(\xi))^2+u'''({}_Ly^\theta_{N-i}(\xi)){}_L{y^\theta_{N-i}}''(\xi)\\ &\le Cx_i^{2/r-2}x_N^{2-2/r}+C\frac{r-1}{r}x_i^{1/r-2}(\theta x_N^{1-2/r}Z_{N-1-i}+(1-\theta)\frac{h_N}{rZ_1})\\ &\le Cx_i^{2/r-2}+C(r-1)x_i^{1/r-2}T^{1-1/r}\end{aligned}$$

∎

Lemma 3.31.

$$(3.130)\quad |{}_Ly^\theta_{N-i}(\xi)-\xi|^{1-\alpha}\le C|y_N^\theta-x_i|^{1-\alpha}$$

$$(3.131)\quad (|{}_Ly^\theta_{N-i}(\xi)-\xi|^{1-\alpha})'\le C|y_N^\theta-x_i|^{1-\alpha}$$

$$(3.132)\quad (|{}_Ly^\theta_{N-i}(\xi)-\xi|^{1-\alpha})''\le C(r-1)|y_N^\theta-x_i|^{-\alpha}+|y_N^\theta-x_i|^{1-\alpha}$$

Proof.

$$(3.133)\quad \begin{aligned}({}_Ly^\theta_{N-i}(\xi)-\xi)' &= (\theta({}_Ly_{N-1-i}(\xi)-\xi)+(1-\theta)({}_0y_{N-i}(\xi)-\xi))'\\ &= \theta({}_L{y_{N-1-i}}'(\xi)-1)+(1-\theta)({}_0{y_{N-i}}'(\xi)-1)\\ &= \theta\xi^{1/r-1}({}_Ly^{1-1/r}_{N-1-i}(\xi)-\xi^{1-1/r})+(1-\theta)\xi^{1/r-1}(\frac{h_N}{rZ_1}-\xi^{1-1/r})\end{aligned}$$

$$(3.134)\quad \begin{aligned}({}_Ly^\theta_{N-i}(\xi)-\xi)'' &= \theta({}_L{y_{N-1-i}}''(\xi))+(1-\theta)({}_0{y_{N-i}}''(\xi))\\ &= \frac{1-r}{r}\xi^{1/r-2}(\theta{}_Ly^{1-2/r}_{N-1-i}(\xi)Z_{N-1-i}+(1-\theta)\frac{h_N}{rZ_1})\le 0\end{aligned}$$

And

$$(3.135)\quad |({}_Ly^\theta_{N-i}(\xi)-\xi)''|\le C(r-1)\xi^{1/r-2}T^{1-1/r}$$

We have known

$$C|x_{N-1}-x_i| \le |{}_Ly_{N-1-i}(\xi)-\xi| \le C|x_{N-1}-x_i| \tag{3.136}$$

If $\xi \le x_{N-1}$, then $({}_0y_{N-i}(\xi)-\xi)' \ge 0$, so

$$C|x_N-x_i| \le |x_{N-1}-x_{i-1}| \le |{}_Ly^{\theta}_{N-i}(\xi)-\xi| \le |x_{N+1}-x_{i+1}| \le C|x_N-x_i| \tag{3.137}$$

If $i=N-1$ and $\xi \in [x_{N-1},x_N]$, then ${}_0y_{N-i}(\xi)-\xi$ is concave, bigger than its two neighboring points, which are equal to h_N, so

$$h_N = |x_N-x_{N-1}| \le |{}_0y_{N-i}(\xi)-\xi| \le |x_{N+1}-x_{N-1}| = 2h_N \tag{3.138}$$

So we have

$$|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha} \le C|y^{\theta}_N-x_i|^{1-\alpha} \tag{3.139}$$

While

$${}_Ly^{1-1/r}_{N-1-i}(\xi)-\xi^{1-1/r} \le ({}_Ly_{N-1-i}(\xi)-\xi)\xi^{-1/r} \tag{3.140}$$

and

$$\begin{aligned}
&|\frac{h_N}{rZ_1}-\xi^{1-1/r}| \le \max\{|\frac{h_N}{rZ_1}-x^{1-1/r}_{i-1}|,|\frac{h_N}{rZ_1}-x^{1-1/r}_{i+1}|\}\\
&\qquad \le \max\begin{cases} T^{1-1/r}-x^{1-1/r}_{i-1} \le |x_N-x_{i-1}|T^{-1/r} \le C|x_N-x_i| \\ |x^{1-1/r}_{i+1}-x^{1-1/r}_{N-1}| \le |x_{i+1}-x_{N-1}|x^{-1/r}_{N-1} \le C|x_N-x_i| \end{cases}
\end{aligned} \tag{3.141}$$

So we have

$$({}_Ly^{\theta}_{N-i}(\xi)-\xi)' \le C|y^{\theta}_N-x_i| \tag{3.142}$$

$$\begin{aligned}
(|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha})' &= |{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{-\alpha}({}_Ly^{\theta}_{N-i}(\xi)-\xi)'\\
&\le |y^{\theta}_N-x_i|^{1-\alpha}
\end{aligned} \tag{3.143}$$

Finally,

$$\begin{aligned}
(|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{1-\alpha})'' &= (1-\alpha)|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{-\alpha}({}_Ly^{\theta}_{N-i}(\xi)-\xi)''\\
&\quad +\alpha(\alpha-1)|{}_Ly^{\theta}_{N-i}(\xi)-\xi|^{-1-\alpha}(({}_Ly^{\theta}_{N-i}(\xi)-\xi)')^2\\
&\le C(r-1)|y^{\theta}_N-x_i|^{-\alpha}+C|y^{\theta}_N-x_i|^{1-\alpha}
\end{aligned} \tag{3.144}$$ ∎

By the three lemmas above, for $N/2 \le i \le N-1$, we have

Lemma 3.32.

$$\begin{aligned}
&D^2_h {}_LP^{\theta}_{N-i}(x_i) = {}_LP^{\theta\,\prime\prime}_{N-i}(\xi) \quad \xi \in (x_{i-1},x_{i+1})\\
&\qquad \le Ch^3|y^{\theta}_N-x_i|^{1-\alpha}+C(r-1)(h^3|y^{\theta}_N-x_i|^{-\alpha}+h^2|y^{\theta}_N-x_i|^{1-\alpha})
\end{aligned} \tag{3.145}$$

And

LEMMA 3.33.

$$\frac{2}{h_i+h_{i+1}}\left(\frac{{}_LQ^\theta_{N-i}(x_{i+1})u'''(\eta^\theta_{N+1})-{}_LQ^\theta_{N-i}(x_i)u'''(\eta^\theta_N)}{h_{i+1}}\right)\\ \leq Ch^3|y^\theta_N-x_i|^{1-\alpha} \tag{3.146}$$

And immediately, For $N/2\leq i\leq N-2$

$$\begin{aligned} V_{iN} &\leq C\int_{x_{N-1}}^{x_N} h^2|y-x_i|^{1-\alpha}+C(r-1)h^2|y-x_i|^{-\alpha}+h|y-x_i|^{1-\alpha}dy\\ &\leq Ch^2h_N|T-x_i|^{1-\alpha}+C(r-1)h^2|x_{N-1}-x_i|^{1-\alpha}+Chh_N|T-x_i|^{1-\alpha}\\ &\leq Ch^2+C(r-1)h^2|T-x_{i-1}|^{1-\alpha} \end{aligned} \tag{3.147}$$

But expecially, when $i=N-1$,

$$\begin{aligned} V_{N-1,N} = &\int_0^1 -\frac{\theta^{2-\alpha}(1-\theta)}{2}\frac{2}{h_{N-1}+h_N}\left(\frac{1}{h_{N-1}}h^{4-\alpha}_{N-1}u''(y^\theta_{N-1})-(\frac{1}{h_{N-1}}+\frac{1}{h_N})h^{4-\alpha}_N u''(y^\theta_N)+\frac{1}{h_N}h^{4-\alpha}_{N+1}u''(y^\theta_{N+1})\right)d\theta\\ &+\int_0^1\frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{{}_LQ^\theta_{N-i}(x_{i+1})u'''(\eta^\theta_{N+1,1})-{}_LQ^\theta_{N-i}(x_i)u'''(\eta^\theta_{N,1})}{h_{i+1}}\right)d\theta\\ &-\int_0^1\frac{\theta^3(1-\theta)}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{{}_LQ^\theta_{N-i}(x_i)u'''(\eta^\theta_{N,1})-{}_LQ^\theta_{N-i}(x_{i-1})u'''(\eta^\theta_{N-1,1})}{h_i}\right)d\theta\\ &-\int_0^1\frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{{}_LQ^\theta_{N-i}(x_{i+1})u'''(\eta^\theta_{N+1,2})-{}_LQ^\theta_{N-i}(x_i)u'''(\eta^\theta_{N,2})}{h_{i+1}}\right)d\theta\\ &+\int_0^1\frac{\theta(1-\theta)^3}{3!}\frac{2}{h_i+h_{i+1}}\left(\frac{{}_LQ^\theta_{N-i}(x_i)u'''(\eta^\theta_{N,2})-{}_LQ^\theta_{N-i}(x_{i-1})u'''(\eta^\theta_{N-1,2})}{h_i}\right)d\theta \end{aligned} \tag{3.148}$$

while combine Lemma 3.29

$$\begin{aligned} &\frac{2}{h_{N-1}+h_N}\left(\frac{1}{h_{N-1}}h^{4-\alpha}_{N-1}u''(y^\theta_{N-1})-(\frac{1}{h_{N-1}}+\frac{1}{h_N})h^{4-\alpha}_N u''(y^\theta_N)+\frac{1}{h_N}h^{4-\alpha}_{N+1}u''(y^\theta_{N+1})\right)\\ &=D^2_h(h^{4-\alpha}_{N-1\to N}(x_i)u''(y^\theta_{N-1\to N}(x_i)))\\ &\leq Ch^{4-\alpha}_N+C(r-1)h^{3-\alpha}_N\leq Ch^{4-\alpha}+C(r-1)h^2|T-x_{N-1-1}|^{1-\alpha} \end{aligned} \tag{3.149}$$

Similarly with $j=N+1$. ∎

I_6, I_7 is easy. Similar with Lemma 3.21 and Lemma 3.6, we have

THEOREM 3.34. *There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \le i \le N$,*

(3.150)
$$I_6 = \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i}(T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right) \le Ch^2$$

Proof. In fact, let $l = 2N - \lceil \frac{N}{2} \rceil + 1$

$$\begin{aligned} &\frac{1}{h_i}(T_{i-1,l} + T_{i-1,l-1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}})T_{i,l} \\ &= \frac{1}{h_i}(T_{i-1,l} - T_{i,l}) + \frac{1}{h_i}(T_{i-1,l-1} - T_{i,l}) + (\frac{1}{h_i} - \frac{1}{h_{i+1}})T_{i,l} \end{aligned} \tag{3.151}$$

While, by Lemma A.2

$$\begin{aligned} \frac{1}{h_i}(T_{i-1,l} - T_{i,l}) &= \int_{x_{l-1}}^{x_l} (u(y) - \Pi_h u(y)) \frac{|x_{i-1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_i \Gamma(2-\alpha)} dy \\ &\le C \int_{x_{l-1}}^{x_l} h_l^2 u''(\eta) \, \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy \\ &\le C h_l^3 x_{l-1}^{\alpha/2-2} T^{-\alpha} \\ &\le C h_l^3 \end{aligned} \tag{3.152}$$

Thus,

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_i}(T_{i-1,l} - T_{i,l}) \le C h_l^2 \tag{3.153}$$

For

(3.154)
$$\frac{1}{h_i}(T_{i-1,l-1} - T_{i,l}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{h_i} d\theta$$

And Similar with Lemma 3.19, we can get

$$\frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{(h_i + h_{i+1})h_i} \le C h_l^2 |y_l^\theta - x_i|^{1-\alpha} \tag{3.155}$$

So

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_i}(T_{i-1,l-1} - T_{i,l}) \le Ch^2 \tag{3.156}$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned} \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,l} &\le h_i^{-3} h^2 x_i^{1-2/r} h_l C h_l^2 x_{l-1}^{\alpha/2-2} |x_l - x_i|^{1-\alpha} \\ &\le Ch^2 \end{aligned} \tag{3.157}$$

Summarizes, we have

$$I_6 \le Ch^2 \tag{3.158}$$

□

And

LEMMA 3.35. *There is a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \leq i \leq N$,*

$$I_7 = \sum_{j=2N-\lceil\frac{N}{2}\rceil+2}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.159}$$

Proof. For $i \leq N, j \geq 2N - \lceil\frac{N}{2}\rceil + 2$, we have

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 \left(\frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)}\right)(x_i) dy \\ &\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2 - 2/r} |y - x_{i+1}^{-1-\alpha} dy \\ &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy \end{aligned}$$

$$\begin{aligned} \sum_{j=2N-\lceil\frac{N}{2}\rceil+2}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{(2-2^{-r})T}^{x_{2N-1}} (2T - y)^{\alpha/2-2/r} dy \\ &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r}T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\ &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r}\ h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r}\ h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned} \tag{3.160}$$

∎

Now we can conclude a part of the theorem Theorem 3.3 at the beginning of this section.

By Lemma 3.8 Lemma 3.21 Lemma 3.22 Theorem 3.28 Theorem 3.27 Theorem 3.34 Lemma 3.35 , we have

THEOREM 3.36. *there exists a constant $C = C(T, \alpha, r, \|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that for $N/2 \leq i < N$,*

$$\begin{aligned} R_i &= \sum_{j=1}^{7} I_j \\ &\leq C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned} \tag{3.161}$$

And what we left is the case $i = N$. Fortunately, we can use the same department of R_i above, and it is symmetric. Most of the item has been esitmated by Lemma 3.8 and Theorem 3.34, we just need to consider I_3, I_4.

THEOREM 3.37. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$I_3 = \sum_{j=\lceil \frac{N}{2} \rceil+1}^{N-1} V_{Nj} \leq Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha} \tag{3.162}$$

Proof. DEFINITION 3.38. *For $N/2 \leq j < N$, Let's define*

$$y_j(x) = (\frac{Z_1}{h_N}(x - x_N) + Z_j)^r, \quad Z_j = T^{1/r}\frac{j}{N} \tag{3.163}$$

We can see that is the inverse of the function ${}_0y_{N-i}(x)$ defined in Theorem 3.28.

$$y_j'(x) = y_j^{1-1/r}(x)\frac{rZ_1}{h_N} \tag{3.164}$$

$$y_j''(x) = y_j^{1-2/r}(x)\frac{r(r-1)Z_1}{h_N} \tag{3.165}$$

With the scheme we used several times, we can get

LEMMA 3.39. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq j < N$, $\xi \in [x_{N-1}, x_{N+1}]$,*

$$h_j(\xi)^3 \leq Ch^3 \tag{3.166}$$

$$(h_j^3(\xi))' \leq C(r-1)h^3 \tag{3.167}$$

$$(h_j^3(\xi))'' \leq C(r-1)h^3 \tag{3.168}$$

$$u''(y_j^\theta(\xi)) \leq C \tag{3.169}$$

$$(u''(y_j^\theta(\xi)))' \leq C \tag{3.170}$$

$$(u''(y_j^\theta(\xi)))'' \leq C \tag{3.171}$$

$$|\xi - y_j^\theta(\xi)|^{1-\alpha} \leq C|x_N - y_j^\theta|^{1-\alpha} \tag{3.172}$$

$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})' \leq C|x_N - y_j^\theta|^{1-\alpha} \tag{3.173}$$

$$(|\xi - y_j^\theta(\xi)|^{1-\alpha})'' \leq C|x_N - y_j^\theta|^{1-\alpha} + C(r-1)|x_N - y_j^\theta|^{-\alpha} \tag{3.174}$$

LEMMA 3.40. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that For $N/2 \leq j < N$,*

$$V_{Nj} \leq Ch^2 \int_{x_{j-1}}^{x_j} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \tag{3.175}$$

Therefore,

$$\begin{aligned} I_3 &\leq Ch^2 \int_{x_{\lceil \frac{N}{2} \rceil}}^{x_{N-1}} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \\ &\leq Ch^2(|T - x_{N-1}|^{2-\alpha} + (r-1)|T - x_{N-1}|^{1-\alpha}) \end{aligned} \tag{3.176}$$

□

For $j = N$,

LEMMA 3.41.

$$V_{N,N} = \frac{1}{h_N^2}\left(T_{N-1,N-1} - 2T_{N,N} + T_{N+1,N+1}\right) \leq Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha} \tag{3.177}$$

Proof. $\square$

$$\begin{aligned} V_{N,N} = &\int_0^1 -\frac{\theta(1-\theta)^{2-\alpha}}{2}\frac{1}{h_N^2}\left(h_{N-1}^{4-\alpha}u''(y_{N-1}^\theta) - 2h_N^{4-\alpha}u''(y_N^\theta) + h_{N+1}^{4-\alpha}u''(y_{N+1}^\theta)\right)d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,1}^\theta) - Q_{N\to N}^\theta(x_i)u'''(\eta_{N,1}^\theta)}{h_N}\right)d\theta \\ &- \int_0^1 \frac{\theta^3(1-\theta)}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,1}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,1}^\theta)}{h_N}\right)d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_{N+1})u'''(\eta_{N+1,2}^\theta) - Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta)}{h_N}\right)d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)^3}{3!}\frac{1}{h_N}\left(\frac{Q_{N\to N}^\theta(x_N)u'''(\eta_{N,2}^\theta) - Q_{N\to N}^\theta(x_{N-1})u'''(\eta_{N-1,2}^\theta)}{h_N}\right)d\theta \end{aligned} \tag{3.178}$$

So combine Lemma 3.8, Theorem 3.34, Theorem 3.37, Lemma 3.41 We have

LEMMA 3.42.

$$R_N \leq C(r-1)h^2|T - x_{N-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2\ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \tag{3.179}$$

and with Theorem 3.36 we prove the Theorem 3.3

4. Convergence analysis.

4.1. Properties of some Matrices. Review subsection 2.1, we have got (2.10).

DEFINITION 4.1. *We call one matrix an M matrix , which means its entries are positive on major diagonal and nonpositive on others, and strictly diagonally dominant in rows.*

Now we have

LEMMA 4.2. *Matrix A defined by* (2.12) *where* (2.13) *is an M matrix. And there exists a constant $C_A = C(T, \alpha, r)$ such that*

$$S_i := \sum_{j=1}^{2N-1} a_{ij} \geq C_A(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \tag{4.1}$$

Proof. From (2.14), we have

$$\sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left(\frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right) \tag{4.2}$$

Let

$$g(x) = g_0(x) + g_{2N}(x) \tag{4.3}$$

where

$$g_0(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x - x_0|^{3-\alpha} - |x - x_1|^{3-\alpha}}{h_1}$$

$$g_{2N}(x) := \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N} - x|^{3-\alpha} - |x_{2N-1} - x|^{3-\alpha}}{h_{2N}}$$

Thus

$$-\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

Then

$$\begin{aligned} S_i :=& \sum_{j=1}^{2N-1} a_{ij} \\ =& \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ =& D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i) \end{aligned} \tag{4.4}$$

When $i = 1$

$$\begin{aligned} D_h^2 g_0(x_1) = & \frac{2}{h_1 + h_2} \left(\frac{1}{h_2} g_0(x_2) - (\frac{1}{h_1} + \frac{1}{h_2}) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right) \\ =& \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2) h_1 h_2} \\ =& \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha} h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2) h_1^{1-\alpha} h_2} h_1^{-\alpha} \\ =& \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha}}{2^r (2^r - 1)} h_1^{-\alpha} \end{aligned} \tag{4.5}$$

but

$$1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha} > 0 \tag{4.6}$$

While for $i \geq 2$

$$\begin{aligned} D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \\ &= -\kappa_\alpha \frac{|\xi - x_0|^{1-\alpha} - |\xi - x_1|^{1-\alpha}}{\Gamma(2-\alpha) h_1} \\ &= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} |\xi - \eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\ &\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha} \end{aligned} \tag{4.7}$$

So

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g_0(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g_0(x_i) + \frac{1}{h_i} g_0(x_{i-1}) \right) \geq C x_i^{-\alpha} \tag{4.8}$$

symmetricly,

$\square$

(4.9)

$$\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g_{2N}(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g_{2N}(x_i) + \frac{1}{h_i} g_{2N}(x_{i-1}) \right) \geq C(\alpha, r)(2T - x_i)^{-\alpha}$$

Let

$$g(x) = \begin{cases} x, & 0 < x \leq T \\ 2T - x, & T < x < 2T \end{cases} \tag{4.10}$$

And define

$$G = \mathrm{diag}(g(x_1), ..., g(x_{2N-1})) \tag{4.11}$$

Then

LEMMA 4.3. *The matrix $B := AG$, the major diagnal is positive, and nonpositive on others. And there is a constant $C_{AG}, C = C(\alpha, r)$ such that*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C_{AG}(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \tag{4.12}$$

Proof.

$$b_{ij} = a_{ij} g(x_j) = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \tilde{a}_{i+1,j} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) \tilde{a}_{i,j} + \frac{1}{h_i} \tilde{a}_{i-1,j} \right) g(x_j)$$

Since

$$g(x) \equiv \Pi_h g(x) \tag{4.13}$$

by **??**, we have

$$
\begin{aligned}
\tilde{M}_i &:= \sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} \tilde{a}_{ij} g(x_j) \\
&= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} \Pi_h g(y) dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} g(y) dy \\
&= \frac{-2}{\Gamma(4-\alpha)} |T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)} (x_i^{3-\alpha} + (2T - x_i)^{3-\alpha}) \\
&:= w(x_i) = p(x_i) + q(x_i)
\end{aligned}
\tag{4.14}
$$

Thus,

$$
\begin{aligned}
M_i &:= \sum_{j=1}^{2N-1} b_{ij} = \sum_{j=1}^{2N-1} a_{ij} g(x_j) \\
&= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} \tilde{M}_{i+1} - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) \tilde{M}_i + \frac{1}{h_i} \tilde{M}_{i-1} \right) \\
&= D_h^2(-\kappa_\alpha p)(x_i) - \kappa_\alpha D_h^2 q(x_i)
\end{aligned}
\tag{4.15}
$$

for $1 \le i \le N-1$,by Lemma A.1

$$
\begin{aligned}
D_h^2(-\kappa_\alpha p)(x_i) &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} p(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) p(x_i) + \frac{1}{h_i} p(x_{i-1}) \right) \\
&= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1}) \\
&\ge \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - x_{i-1}|^{1-\alpha}
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
D_h^2(-\kappa_\alpha p)(x_N) &:= -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left(\frac{1}{h_{N+1}} p(x_{N+1}) - (\frac{1}{h_N} + \frac{1}{h_{N+1}}) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right) \\
&= \frac{4\kappa_\alpha}{\Gamma(4-\alpha) h_N^2} h_N^{3-\alpha} \\
&= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - x_{N-1})^{1-\alpha}
\end{aligned}
\tag{4.17}
$$

Symmetricly for $i \ge N$, we get

$$
D_h^2(-\kappa_\alpha p)(x_i) \ge \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \le N \\ |x_{i+1} - T|^{1-\alpha}, & i \ge N \end{cases}
\tag{4.18}
$$

Similarly, we can get

$$
\begin{aligned}
D_h^2 q(x_i) &:= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} q(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right) \\
&\le \frac{2^{r(\alpha-1)+1}}{\Gamma(2-\alpha)} (x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}), \quad i = 1, \cdots, 2N-1
\end{aligned}
\tag{4.19}
$$

$\square$

So, we get the result.

Notice that

$$x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha} \tag{4.20}$$

We can get

THEOREM 4.4. *There exists a real $\lambda = \lambda(T,\alpha,r) > 0$ and $C = C(T,\alpha,r) > 0$ such that $B := A(\lambda I + G)$ is an M matrix. And*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \tag{4.21}$$

Proof. By Lemma 4.2 with C_A and Lemma 4.3 with C_{AG} , it's sufficient to take $\lambda = (C + 2TC_{AG})/C_A$, then

$$M_i \geq C\left((x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \right) \tag{4.22}$$

∎

4.2. Proof of Theorem 2.6. For equation

$$AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1}U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1}U = F \tag{4.23}$$

which means

$$\sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + g(x_j)} = -\tau_i \tag{4.24}$$

where $\epsilon_i = u(x_i) - u_i$.

And if

$$\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| = \max_{1 \leq i \leq 2N-1} \left|\frac{\epsilon_i}{\lambda + g(x_i)}\right| \tag{4.25}$$

Then, since $B = A(\lambda I + G)$ is an M matrix, it is Strictly diagonally dominant. Thus,

$$\begin{aligned} |\tau_{i_0}| &= \left|\sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + g(x_j)}\right| \\ &\geq b_{i_0,i_0}\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| - \sum_{j \neq i_0} |b_{i_0,j}|\left|\frac{\epsilon_j}{\lambda + g(x_j)}\right| \\ &\geq b_{i_0,i_0}\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| - \sum_{j \neq i_0} |b_{i_0,j}|\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| \\ &= \sum_{j=1}^{2N-1} b_{i_0,j}\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| \\ &= M_{i_0}\left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| \end{aligned} \tag{4.26}$$

By Theorem 2.5 and Theorem 4.4,

We kown that there exists constants $C_1(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$, and $C_2(T,\alpha,r,\|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that

$$\left|\frac{\epsilon_i}{\lambda + g(x_i)}\right| \leq \left|\frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})}\right| \leq C_1 h^{\min\{\frac{r\alpha}{2},2\}} + C_2(r-1)h^2 \tag{4.27}$$

as $\lambda + g(x_i) \leq \lambda + T$

So, we can get

$$|\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{r\alpha}{2},2\}} \tag{4.28}$$

The convergency has been proved.

Remarks:

5. Experimental results.

5.1. $f \equiv 1$.

5.2. $f = x^\gamma, \gamma < 0$. Appendix A. Approximate of difference quotients.

LEMMA A.1. *If $g(x) \in C^2(\Omega)$, there exists $\xi \in (x_{i-1}, x_{i+1})$ such that*

$$\begin{aligned} D_h^2 g(x_i) &:= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &= g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \end{aligned} \tag{A.1}$$

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &= \frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right) \end{aligned} \tag{A.2}$$

And if $g(x) \in C^4(\Omega)$, then

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\qquad = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2)) \end{aligned} \tag{A.3}$$

where $\eta_1 \in [x_{i-1}, x_i], \eta_2 \in [x_i, x_{i+1}]$.

Proof.

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})$$

Subsitute them in the left side of (A.1), we have

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}} \left(\frac{1}{h_{i+1}} g(x_{i+1}) - (\frac{1}{h_i} + \frac{1}{h_{i+1}}) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &\qquad = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) \end{aligned}$$

Now, using intermediate value theorem , there exists $\xi \in [\xi_1, \xi_2]$ such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1}) g'(x_i) + \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i) g'(x_i) + \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y-x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1}-y)^3}{3!} dy$$

Expecially,

$$\begin{aligned} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y-x_{i-1})^3}{3!} dy &= \frac{h_i^4}{4!} g''''(\eta_1) \\ \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1}-y)^3}{3!} dy &= \frac{h_{i+1}^4}{4!} g''''(\eta_2) \end{aligned} \tag{A.4}$$

where $\eta_1 \in (x_{i-1}, x_i), \eta_2 \in (x_i, x_{i+1})$. Subsitute them to the left side of (A.3), we can get the result. □

Lemma A.2. *Denote* $y_j^\theta = (1-\theta)x_{j-1} + \theta x_j, \theta \in (0,1)$,

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in (x_{j-1}, x_j) \tag{A.5}$$

(A.6)

$$u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2))$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$.

Proof. By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in (x_{j-1}, y_j^\theta)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in (y_j^\theta, x_j)$$

Thus

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta) u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 (\theta u''(\xi_1) + (1-\theta) u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) - \frac{\theta^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + (1-\theta) h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) + \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_2)$$

where $\eta_1 \in (x_{j-1}, y_j^\theta), \eta_2 \in (y_j^\theta, x_j)$. Thus □

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - (1-\theta) u(x_{j-1}) - \theta u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 (\theta^2 u'''(\eta_1) - (1-\theta)^2 u'''(\eta_2)) \end{aligned}$$

LEMMA A.3. *For $x \in [x_{j-1}, x_j]$*

$$|u(x) - \Pi_h u(x)| = \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^{x} u'(y)dy - \frac{x - x_{j-1}}{h_j} \int_{x}^{x_j} u'(y)dy \right| \leq \int_{x_{j-1}}^{x_j} |u'(y)|dy \tag{A.7}$$

If $x \in [0, x_1]$, with Corollary 2.4, we have

$$|u(x) - \Pi_h u(x)| \leq \int_0^{x_1} |u'(y)|dy \leq \int_0^{x_1} C y^{\alpha/2-1} dy \leq C \frac{2}{\alpha} x_1^{\alpha/2} \tag{A.8}$$

Similarly, if $x \in [x_{2N-1}, 1]$, we have

$$|u(x) - \Pi_h u(x)| \leq C \frac{2}{\alpha} (2T - x_{2N-1})^{\alpha/2} = C \frac{2}{\alpha} x_1^{\alpha/2} \tag{A.9}$$

LEMMA A.4.

$$b^{1-\theta} |a^\theta - b^\theta| \leq |a - b| \text{ (also } a^{1-\theta} |a^\theta - b^\theta| \leq |a - b|), \quad a, b \geq 0, \ \theta \in [0, 1] \tag{A.10}$$

Appendix B. Inequality. For convenience, we use the notation and $\simeq$. That $x_1 \simeq y_1$, means that $c_1 x_1 \leq y_1 \leq C_1 x_1$ for some constants c_1 and C_1 that are independent of mesh parameters.

LEMMA B.1.

$$h_i \simeq \begin{cases} h x_i^{1-1/r}, & 1 \leq i \leq N \\ h(2T - x_i)^{1-1/r}, & N < i \leq 2N-1 \end{cases} \tag{B.1}$$

Since, $i^r - (i-1)^r \simeq i^{r-1}$, for $i \geq 1$

LEMMA B.2. *There is a constant $C = 2^{|r-2|} r(r-1) T^{2/r}$ such that for all $i \in \{1, 2, \cdots, 2N-1\}$*

$$|h_{i+1} - h_i| \leq C h^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases} \tag{B.2}$$

Proof.

$$h_{i+1} - h_i = \begin{cases} T\left(\left(\frac{i+1}{N}\right)^r - 2\left(\frac{i}{N}\right)^r + \left(\frac{i-1}{N}\right)^r\right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T\left(\left(\frac{2N-i-1}{N}\right)^r - 2\left(\frac{2N-i}{N}\right)^r + \left(\frac{2N-i+1}{N}\right)^r\right), & N+1 \leq i \leq 2N-1 \end{cases}$$

For $i = 1$,

$$h_2 - h_1 = T(2^r - 2)\left(\frac{1}{N}\right)^r = (2^r - 2) T^{2/r} h^2 x_1^{1-2/r}$$

For $2 \le i \le N-1$, by Lemma A.1, we have

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T\,N^{-2}\eta^{r-2}, \quad \eta \in [\frac{i-1}{N}, \frac{i+1}{N}] \\ &= C(r-1)h^2 x_i^{1-2/r} \end{aligned}$$

Summarizes the inequalities, we can get

$$\text{(B.3)} \qquad |h_{i+1} - h_i| \le 2^{|r-2|} r(r-1) T^{2/r} h^2 \begin{cases} x_i^{1-2/r}, & 1 \le i \le N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \le 2N-1 \end{cases} \qquad \square$$

Appendix C. Proofs of some technical details.

Additional proof of Theorem 3.1. For $2 \le i \le N-1$,

$$\begin{aligned} &\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\ &\le C\frac{2}{h_i + h_{i+1}}(h_i^3 x_{i-1}^{-2-\alpha/2} + h_{i+1}^3 x_i^{-2-\alpha/2}) \\ &\le 2C(h_i^2 x_{i-1}^{-2-\alpha/2} + h_{i+1}^2 x_i^{-2-\alpha/2}) \end{aligned}$$

There is a constant $C = C(T, \alpha, r, \|f\|_\beta^{\alpha/2})$ such that

$$\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \le C h^2 x_i^{-\alpha/2-2/r}, \quad 2 \le i \le N-1$$

For $i = 1$, by (A.4)

$$\begin{aligned} &\frac{1}{4!}\frac{2}{h_1 + h_2}(h_1^3 f''(\eta_1) + h_2^3 f''(\eta_2)) \\ &= \frac{2}{h_1 + h_2}\left(\frac{1}{h_1}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy + \frac{1}{4!}h_2^3 f''(\eta_2)\right) \end{aligned}$$

We have proved above that

$$\frac{2}{h_1 + h_2} h_2^3 f''(\eta_2) \le C h^2 x_1^{-\alpha/2-2/r}$$

and we can get

$$\begin{aligned} \int_0^{x_1} f''(y)\frac{y^3}{3!}dy &\le C\frac{1}{3!}\int_0^{x_1} y^{1-\alpha/2}dy \\ &= C\frac{1}{3!(2-\alpha/2)}x_1^{2-\alpha/2} \end{aligned}$$

so

$$\frac{2}{h_1 + h_2}\frac{1}{h_1}\int_0^{x_1} f''(y)\frac{y^3}{3!}dy = \frac{C2^{1-r}}{3!(2-\alpha/2)}x_1^{-\alpha/2} = \frac{C2^{1-r}}{3!(2-\alpha/2)}T^{2/r}h^2 x_1^{-\alpha/2-2/r}$$

And for $i = N$, we have

$$\begin{aligned}
&\frac{2}{h_N + h_{N+1}}(h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2)) \\
&\quad = h_N^2(f''(\eta_1) + f''(\eta_2)) \\
&\quad \le r^2 T^{2/r} h^2 x_N^{2-2/r}\ 2C x_{N-1}^{-2-\alpha/2} \\
&\quad \le 2r^2 T^{2/r} C 2^{-r(-2-\alpha/2)}\ h^2 x_N^{-\alpha/2-2/r}
\end{aligned}$$

Finally, $N+1 \le i \le 2N-1$ is symmetric to the first half of the proof, so we can conclude that

∎

$$\frac{2}{h_i + h_{i+1}}(h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \le Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \le i \le N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \le i \le 2N-1 \end{cases}$$

LEMMA C.1. *By a standard error estimate for linear interpolation, and Corollary 2.4, There is a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ for $2 \le j \le N$,*

$$|u(y) - \Pi_h u(y)| \le Ch^2 y^{\alpha/2-2/r}, \quad \textit{for } y \in [x_{j-1}, x_j] \tag{C.1}$$

symmetricly, for $N < j \le 2N-1$, we have

$$|u(y) - \Pi_h u(y)| \le Ch^2 (2T - y)^{\alpha/2-2/r} \tag{C.2}$$

LEMMA C.2. *There is a constant $C = C(\alpha, r)$ such that for all $1 \le i < N/2$, $\max\{2i+1, i+3\} \le j \le 2N$, we have*

$$D_h^2 K_y(x_i) \le C \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}, \quad y \in [x_{j-1}, x_j] \tag{C.3}$$

Proof. Sinec $y \ge x_{j-1} > x_{i+1}$, by Lemma A.1, if $j - 1 > i + 1$

∎

$$\begin{aligned}
D_h^2 K_y(x_i) = K_y''(\xi) &= \frac{|y-\xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in (x_{i-1}, x_{i+1}) \\
&\le \frac{(y - x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)} \\
&\le (1 - (\frac{2}{3})^r)^{-1-\alpha} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}
\end{aligned}$$

LEMMA C.3. *There is a constant $C = C(\alpha, r)$ such that for all $3 \le i \le N, k = \lceil \frac{i}{2} \rceil$, $1 \le j \le k-1$ and $y \in [x_{j-1}, x_j]$, we have*

$$D_h^2 K_y(x_i) \le C \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \tag{C.4}$$

Proof. Sinec $y \le x_j < x_{i-1}$, by Lemma A.1,

∎

$$\begin{aligned}
D_h^2 K_y(x_i) &= \frac{|\xi - y|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in (x_{i-1}, x_{i+1}) \\
&\le \frac{(x_{i-1} - x_j)^{-1-\alpha}}{\Gamma(-\alpha)} \le \frac{(x_{i-1} - x_{k-1})^{-1-\alpha}}{\Gamma(-\alpha)} \\
&\le ((\frac{2}{3})^r - (\frac{1}{2})^r)^{-1-\alpha} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)}
\end{aligned}$$

LEMMA C.4. *While $0 \le i < N/2$, By Lemma* A.3

$$
\begin{aligned}
|T_{i1}| &\le C\int_0^{x_1} x_1^{\alpha/2}\frac{|x_i-y|^{1-\alpha}}{\Gamma(2-\alpha)}dy \\
&= C\frac{1}{\Gamma(3-\alpha)}x_1^{\alpha/2}\left|x_i^{2-\alpha}-|x_i-x_1|^{2-\alpha}\right| \\
&\le C\frac{1}{\Gamma(3-\alpha)}x_1^{\alpha/2+2-\alpha} = C\frac{1}{\Gamma(3-\alpha)}x_1^{2-\alpha/2} \quad 0<2-\alpha<1
\end{aligned}
\tag{C.5}
$$

For $2 \le j \le N$, by Lemma A.2 *and Corollary* 2.4

$$
\begin{aligned}
|T_{ij}| &\le \frac{C}{4}\int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2}\frac{|y-x_i|^{1-\alpha}}{\Gamma(2-\alpha)}dy \\
&\le \frac{C}{4\Gamma(3-\alpha)}h_j^2 x_{j-1}^{\alpha/2-2}\left||x_j-x_i|^{2-\alpha}-|x_{j-1}-x_i|^{2-\alpha}\right|
\end{aligned}
\tag{C.6}
$$

LEMMA C.5. *There exists a constant $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ such that*

$$
\sum_{j=1}^{3} S_{1j} \le Ch^2 x_1^{-\alpha/2-2/r}
\tag{C.7}
$$

$$
\sum_{j=1}^{4} S_{2j} \le Ch^2 x_2^{-\alpha/2-2/r}
\tag{C.8}
$$

Proof.

$$
S_{1j} = \frac{2}{x_2}\left(\frac{1}{x_1}T_{0j} - \left(\frac{1}{x_1}+\frac{1}{h_2}\right)T_{1j} + \frac{1}{h_2}T_{2j}\right)
$$

So, by Lemma C.4

$$
S_{11} \le \frac{2}{x_2 x_1}4\frac{C}{\Gamma(3-\alpha)}x_1^{2-\alpha/2} \le Cx_1^{-\alpha/2}
$$

$$
S_{12} \le \frac{2}{x_2 x_1}\frac{C}{4\Gamma(3-\alpha)}h_2^2 x_1^{\alpha/2-2}\left(x_2^{2-\alpha}+2h_2^{2-\alpha}+h_2^{2-\alpha}\right) \le Cx_1^{-\alpha/2}
$$

$$
S_{13} \le \frac{2}{x_2 x_1}\frac{C}{4\Gamma(3-\alpha)}h_3^2 x_2^{\alpha/2-2}\left(x_3^{2-\alpha}+2x_3^{2-\alpha}+h_3^{2-\alpha}\right) \le Cx_1^{-\alpha/2}
$$

But

$$
x_1^{-\alpha/2} = T^{2/r}h^2 x_1^{-\alpha/2-2/r}
$$

$i = 2$ is similar. ∎

LEMMA C.6. *There exists a constant $C = C(T, r, l)$ such that For $3 \le i \le N-1, \lceil \frac{i}{2} \rceil \le j \le \min\{2i, N\}$, when $\xi \in (x_{i-1}, x_{i+1})$,*

$$(h^3_{j-i}(\xi))' \le (r-1)Ch^2 x_i^{1-2/r} h_j \tag{C.9}$$

$$(h^4_{j-i}(\xi))' \le (r-1)Ch^2 x_i^{1-2/r} h_j^2 \tag{C.10}$$

Proof. From (3.32)

$$y'_{j-i}(x) = y_{j-i}^{1-1/r}(x) x^{1/r-1} \tag{C.11}$$

$$y''_{j-i}(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i} \tag{C.12}$$

For $\xi \in (x_{i-1}, x_{i+1})$ and $2 \le k \le j \le \min\{2i-1, N-1\}$, using Lemma B.1

$$\xi \simeq x_i \simeq x_j$$

$$h_{j-i}(\xi) \simeq h_j \simeq h x_j^{1-1/r} \simeq h x_i^{1-1/r}$$

$$\begin{aligned} h'_{j-i}(\xi) &= y'_{j-i}(\xi) - y'_{j-i-1}(\xi) \\ &= \xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \end{aligned} \tag{C.13}$$

Since

$$\begin{aligned} y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) &\le x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r} \\ &= T^{1-1/r} N^{1-r}((j+1)^{r-1} - (j-2)^{r-1}) \\ &\le C(r-1) j^{r-2} N^{1-r} \\ &= C(r-1) h x_j^{1-2/r} \end{aligned} \tag{C.14}$$

Therefore,

$$h'_{j-i}(\xi) \le C x_i^{1/r-1}(r-1) h x_j^{1-2/r} \simeq (r-1) h x_i^{-1/r} \tag{C.15}$$

for $l = 3, 4$

$$\begin{aligned} (h^l_{j-i}(\xi))' &= l h^{l-1}_{j-i}(\xi) h'_{j-i}(\xi) \\ &\le C h^{l-1}_{j-i}(\xi)(r-1) h x_i^{-1/r} \\ &\simeq C h_j^{l-2} h x_j^{1-1/r}(r-1) h x_i^{-1/r} \\ &\simeq C(r-1) h^2 x_i^{1-2/r} h_j^{l-2} \end{aligned} \tag{C.16}$$

Meanwhile, we can get

$$h^3_{j-i}(\xi) \simeq h_j^3 \le Ch^2 x_i^{2-2/r} h_j \tag{C.17}$$

$$h^4_{j-i}(\xi) \simeq h_j^4 \le Ch^2 x_i^{2-2/r} h_j^2 \tag{C.18}$$

$\square$

Lemma C.7. *There exists a constant $C = C(T, r, l)$ such that For $3 \le i \le N - 1, \lceil \frac{i}{2} \rceil \le j \le \min\{2i, N\}$, when $\xi \in (x_{i-1}, x_{i+1})$,*

$$(h_{j-i}^3(\xi))'' \le C(r-1)h^2 x_i^{-2/r} h_j \tag{C.19}$$

Proof.

$$(h_{j-i}^3(\xi))'' = 6h_{j-i}(\xi)(h'_{j-i}(\xi))^2 + 3h_{j-i}^2(\xi)h''_{j-i}(\xi) \tag{C.20}$$

By (C.15)

$$h_{j-i}(\xi)(h'_{j-i}(\xi))^2 \le Ch_j(r-1)^2 h^2 x_i^{-2/r} \tag{C.21}$$

For the second partial

$$\begin{aligned} h''_{j-i}(\xi) &= y''_{j-i}(\xi) - y''_{j-i-1}(\xi) \\ &= \frac{1-r}{r}\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\ &= \frac{1-r}{r}\xi^{1/r-2}((y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi))Z_{j-i} + y_{j-i-1}^{1-2/r}(\xi)Z_1) \end{aligned} \tag{C.22}$$

but

$$\begin{aligned} |y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi)| \le |x_{j+1}^{1-2/r} - x_{j-2}^{1-2/r}| \\ = T^{1-2/r}N^{2-r}|(j+1)^{r-2} - (j-2)^{r-2}| \\ \le C|r-2|N^{2-r}j^{r-3} \\ = C|r-2|hx_j^{1-3/r} \end{aligned} \tag{C.23}$$

So we can get

$$\begin{aligned} |h''_{j-i}(\xi)| &\le C(r-1)x_i^{1/r-2}(|r-2|hx_i^{1-3/r}x_i^{1/r} + x_i^{1-2/r}h) \\ &\le C(r-1)hx_i^{-1-1/r} \end{aligned} \tag{C.24}$$

Summarizes, we have

$$(h_{j-i}^3(\xi))'' \le C(r-1)h^2 x_i^{-2/r} h_j \tag{C.25}$$

∎

proof of Lemma 3.16. From (3.32)

$$y'_{j-i}(x) = y_{j-i}^{1-1/r}(x)x^{1/r-1} \tag{C.26}$$

$$y''_{j-i}(x) = \frac{1-r}{r}y_{j-i}^{1-2/r}(x)x^{1/r-2}Z_{j-i} \tag{C.27}$$

Since

$$y_{j-i}^\theta(\xi) \simeq x_j \simeq x_i$$

We have known

$$u''(y_{j-i}^\theta(\xi)) \le C(y_{j-i}^\theta(\xi))^{\alpha/2-2} \simeq x_j^{\alpha/2-2} \simeq x_i^{\alpha/2-2} \tag{C.28}$$

$$\begin{aligned}(u''(y_{j-i}^{\theta}(\xi)))' &= u'''(y_{j-i}^{\theta}(\xi))(y_{j-i}^{\theta}(\xi))' \\ &\le Cx_i^{\alpha/2-3}\xi^{1/r-1}y_{j-i}^{1-1/r}(\xi) \\ &\simeq x_i^{\alpha/2-3}x_i^{1/r-1}x_i^{1-1/r} = Cx_i^{\alpha/2-3}\end{aligned} \tag{C.29}$$

$$\begin{aligned}(u''(y_{j-i}^{\theta}(\xi)))'' &= u''''(y_{j-i}^{\theta}(\xi))({y_{j-i}^{\theta}}'(\xi))^2 + u'''(y_{j-i}^{\theta}(\xi)){y_{j-i}^{\theta}}''(\xi) \\ &\le Cx_i^{\alpha/2-4} + Cx_i^{\alpha/2-3}\frac{r-1}{r}x_i^{1-2/r}x_i^{1/r-2}Z_{|j-i|+1} \\ &\le Cx_i^{\alpha/2-4} + C\frac{r-1}{r}x_i^{\alpha/2-3}x_i^{-1-1/r}x_i^{1/r} \\ &= Cx_i^{\alpha/2-4}\end{aligned} \tag{C.30}$$

□

Proof of Lemma 3.17.

$$\begin{aligned}|y_{j-i}^{\theta}(\xi) - \xi| &= |\theta(y_{j-i-1}(\xi) - \xi) + (1-\theta)(y_{j-i}(\xi) - \xi)| \\ &= \theta|y_{j-i-1}(\xi) - \xi| + (1-\theta)|y_{j-i}(\xi) - \xi|\end{aligned} \tag{C.31}$$

where $y_{j-i-1}(\xi) - \xi$ and $y_{j-i}(\xi) - \xi$ have the same sign (≥ 0 or ≤ 0), independent with ξ.

Since $|y_{j-i}(\xi) - \xi| = \text{sign}(j-i)(y_{j-i}(\xi) - \xi)$ is increasing with ξ,

$$(\frac{i-1}{i})^r|x_j - x_i| \le |x_{j-1} - x_{i-1}| \le |y_{j-i}(\xi) - \xi| \le |x_{j+1} - x_{i+1}| \le (\frac{i+1}{i})^r|x_j - x_i| \tag{C.32}$$

we have

$$|y_{j-i}(\xi) - \xi| \simeq |x_j - x_i| \tag{C.33}$$

Similarly, $|y_{j-1-i}(\xi) - \xi| \simeq |x_{j-1} - x_i|$. Thus, with (C.31) , (C.33) and (3.30) we get

$$|y_{j-i}^{\theta}(\xi) - \xi| \simeq |y_j^{\theta} - x_i| \tag{C.34}$$

Next, since $|y_{j-i}^{\theta}(\xi) - \xi| = \text{sign}(j-i-1+\theta)(y_{j-i}^{\theta}(\xi) - \xi)$, so we can derivate it.

$$|(|y_{j-i}^{\theta}(\xi) - \xi|^{1-\alpha})'| = (\alpha-1)|y_{j-i}^{\theta}(\xi) - \xi|^{-\alpha}|(y_{j-i}^{\theta}(\xi))' - 1| \tag{C.35}$$

While, similar with (C.31), we have

$$|(y_{j-i}^{\theta}(\xi))' - 1| = (1-\theta)|y_{j-i-1}'(\xi) - 1| + \theta|y_{j-i}'(\xi) - 1| \tag{C.36}$$

By Lemma A.4 and (C.33), we have

$$\begin{aligned}|y_{j-i}'(\xi) - 1| &= \xi^{1/r-1}|y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| \\ &\le \xi^{-1}|y_{j-i}(\xi) - \xi| \\ &\simeq x_i^{-1}|x_j - x_i|\end{aligned} \tag{C.37}$$

So similar with (C.34), we can get

$$|(y_{j-i}^{\theta}(\xi))' - 1| \le Cx_i^{-1}|y_j^{\theta} - x_i| \tag{C.38}$$

Combine with (C.34), we get

$$|(|y_{j-i}^\theta(\xi)-\xi|^{1-\alpha})'| \le C|y_j^\theta - x_i|^{-\alpha} x_i^{-1}|y_j^\theta - x_i| = C|y_j^\theta - x_i|^{1-\alpha}x_i^{-1} \tag{C.39}$$

Finally, we have

$$\begin{aligned}(|y_{j-i}^\theta(\xi)-\xi|^{1-\alpha})'' = \alpha(\alpha-1)&|y_{j-i}^\theta(\xi)-\xi|^{-\alpha-1}((y_{j-i}^\theta(\xi))'-1)^2 \\ &+\operatorname{sign}(j-i-1+\theta)(1-\alpha)|y_{j-i}^\theta(\xi)-\xi|^{-\alpha}(y_{j-i}^\theta(\xi))''\end{aligned} \tag{C.40}$$

For

$$(y_{j-i}^\theta(\xi))'' = (1-\theta)y_{j-i-1}''(\xi) + \theta y_{j-i}''(\xi) \tag{C.41}$$

and

$$\begin{aligned} y_{j-i}''(\xi) &= \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i} \\ &\simeq \frac{1-r}{r} x_j^{1-2/r} x_i^{1/r-2} Z_{j-i}\end{aligned} \tag{C.42}$$

while by Lemma A.4

$$|Z_{j-i}| = |x_j^{1/r} - x_i^{1/r}| \le |x_j - x_i| x_i^{1/r-1} \tag{C.43}$$

we have

$$|y_{j-i}''(\xi)| \le C(r-1)x_i^{-2}|x_j - x_i| \tag{C.44}$$

Therefore

$$|(y_{j-i}^\theta(\xi))''| \le C(r-1)x_i^{-2}|y_j^\theta - x_i| \tag{C.45}$$

Then, combine with (C.38),

$$|(|y_{j-i}^\theta(\xi)-\xi|^{1-\alpha})''| \le C|y_j^\theta - x_i|^{1-\alpha}x_i^{-2} \tag{C.46}$$

□

proof of Lemma 3.19. For $\lceil \frac{i}{2} \rceil \le j \le \min\{2i-1, N-1\}$

$$\begin{aligned}&\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \\ &\quad = \frac{Q_{j-i}^\theta(x_{i+1}) - Q_{j-i}^\theta(x_i)}{h_{i+1}} u'''(\eta_{j+1}^\theta) + Q_{j-i}^\theta(x_i)\frac{u'''(\eta_{j+1}^\theta) - u'''(\eta_j^\theta)}{h_{i+1}}\end{aligned} \tag{C.47}$$

Since mean value theorem

$$\frac{Q_{j-i}^\theta(x_{i+1}) - Q_{j-i}^\theta(x_i)}{h_{i+1}} = {Q_{j-i}^\theta}'(\xi), \quad \xi \in (x_i, x_{i+1}) \tag{C.48}$$

From (3.36) and Leibniz rule, by Lemma C.6 and Lemma 3.17, we have

$$|{Q_{j-i}^\theta}'(\xi)| \le Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 \tag{C.49}$$

And by Definition 3.12 and Lemma B.1

$$Q^\theta_{j-i}(x_i) = h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \simeq Ch^2 x_i^{2-2/r} \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 \tag{C.50}$$

With $\eta_j^\theta \in (x_{j-1}, x_j)$

$$u'''(\eta^\theta_{j+1}) \le C(\eta^\theta_{j+1})^{\alpha/2-3} \simeq x_j^{\alpha/2-3} \simeq x_i^{\alpha/2-3}$$

and

$$\begin{aligned}\frac{u'''(\eta^\theta_{j+1}) - u'''(\eta^\theta_j)}{h_{i+1}} &= u''''(\eta)\frac{\eta^\theta_{j+1} - \eta^\theta_j}{h_{i+1}} \\ &\le C\eta^{\alpha/2-4}\frac{x_{j+1} - x_{j-1}}{h_{i+1}} = C\eta^{\alpha/2-4}\frac{h_{j+1} + h_j}{h_{i+1}} \\ &\simeq x_j^{\alpha/2-4} \simeq x_i^{\alpha/2-4}\end{aligned}$$

So we have

$$\begin{aligned}&\frac{Q^\theta_{j-i}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{j-i}(x_i)u'''(\eta^\theta_j)}{h_{i+1}} \\ &\le Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 \, x_i^{\alpha/2-3} + Ch^2 x_i^{2-2/r}\frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 x_{j-1}^{\alpha/2-4} \\ &= Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j^2\end{aligned} \tag{C.51}$$

while $h_j \simeq h_i$, subsitute into the inequality above, we get the goal

$$\begin{aligned}&\frac{2}{h_i + h_{i+1}}\left(\frac{Q^\theta_{j-i}(x_{i+1})u'''(\eta^\theta_{j+1}) - Q^\theta_{j-i}(x_i)u'''(\eta^\theta_j)}{h_{i+1}}\right) \\ &\le \frac{1}{h_i} Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j h_i \\ &= Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j\end{aligned} \tag{C.52}$$

While, the later is similar. ∎

Lemma C.8. *There exists a constant $C = C(T, r)$ such that For $N/2 \le i < N$, $N + 2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $l = 3, 4$, $\xi \in (x_{i-1}, x_{i+1})$, we have*

$$h^l_{j-i}(\xi) \le Ch^l_j \le Ch^2 h^{l-2}_j \tag{C.53}$$

$$(h^l_{j-i-1}(\xi))' \le C(r-1)h^2 h^{l-2}_j \tag{C.54}$$

$$(h^3_{j-i}(\xi))'' \le C(r-1)h^2 h_j \tag{C.55}$$

Proof.

$$\begin{aligned}(h_{j-i}(\xi))' &= {y_{j-i}}'(\xi) - {y_{j-i-1}}'(\xi) \\ &= \xi^{1/r-1}((2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}) \le 0\end{aligned} \tag{C.56}$$

Thus,

$$Ch_j \le h_{j+1} \le h_{j-i}(\xi) \le h_{j-i}(x_{i-1}) = h_{j-1} \le Ch_j \tag{C.57}$$

So as $4^{-r}T \le 2T - x_j \le T, 2^{-r}T \le x_i \le T$, we have

$$h^l_{j-i}(\xi) \le Ch^l_j \le Ch^2(2T-x_j)^{2-2/r}h_j^{l-2} \le Ch^2h_j^{l-2} \tag{C.58}$$

Since

$$\begin{aligned} &|(2T-y_{j-i}(\xi))^{1-1/r} - (2T-y_{j-i-1}(\xi))^{1-1/r}| \\ &\quad = |(Z_{2N-(j-i)} - \xi^{1/r})^{r-1} - (Z_{2N-(j-1-i)} - \xi^{1/r})^{r-1}| \\ &\quad = (r-1)Z_1(Z_{2N-(j-i-\gamma)} - \xi^{1/r})^{r-2} \quad \gamma \in [0,1] \\ &\quad \le C(r-1)h(2T-x_j)^{1-2/r} \end{aligned} \tag{C.59}$$

we have

$$|(h_{j-i}(\xi))'| \le C(r-1)h(2T-x_j)^{1-2/r}x_i^{1/r-1} \tag{C.60}$$

And

$$\begin{aligned} (h^l_{j-i}(\xi))' = lh^{l-1}_{j-i}(\xi){h_{j-i}}'(\xi) & \\ \le C(r-1)h_j^{l-1}\ h(2T-x_j)^{1-2/r}x_i^{1/r-1} & \\ \le C(r-1)h^2h_j^{l-2}(2T-x_j)^{2-3/r}x_i^{1-1/r} & \\ \le C(r-1)h^2h_j^{l-2} & \end{aligned} \tag{C.61}$$

□

(C.62)

$$\begin{aligned} (h^3_{j-i}(\xi))'' &= 6h_{j-i}(\xi)({y_{j-i}}'(\xi) - {y_{j-i-1}}'(\xi))^2 + 3h^2_{j-i}(\xi)({y_{j-i}}''(\xi) - {y_{j-i-1}}''(\xi)) \\ &\le C(r-1)h_jh^2 + Ch_j^2\frac{1-r}{r}\xi^{1/r-2}((2T-y_{j-i}(\xi))^{1-2/r}Z_{2N-(j-i)} - (2T-y_{j-i-1}(\xi))^{1-2/r}Z_{2N-(j-1-i)}) \\ &\le C(r-1)h_jh^2 + C(r-1)h_j^2(C(r-2)h(2T-x_j)^{1-3/r}Z_{2N-(j-i)} + Z_1(2T-x_{j-1})^{1-2/r}) \\ &\le C(r-1)h_jh^2 + C(r-1)h_j^2h = Ch^2h_j \end{aligned}$$

Lemma C.9. *There exists a constant $C = C(T,\alpha,r,\|u\|^{(-\alpha/2)}_{\beta+\alpha})$ such that For $N/2 \le i < N$, $N+2 \le j \le 2N - \lceil \frac{N}{2} \rceil + 1$, $\xi \in (x_{i-1}, x_{i+1})$, we have*

$$u''(y^\theta_{j-i}(\xi)) \le C \tag{C.63}$$

$$(u''(y^\theta_{j-i}(\xi)))' \le C \tag{C.64}$$

$$(u''(y^\theta_{j-i}(\xi)))'' \le C \tag{C.65}$$

Proof.

$$x_{j-2} \le y^\theta_{j-i}(\xi) \le x_{j+1} \Rightarrow 4^{-r}T \le 2T - y^\theta_{j-i}(\xi) \le T \tag{C.66}$$

Thus, for $l = 2,3,4$,

$$u^{(l)}(y^\theta_{j-i}(\xi)) \le C(2T - y^\theta_{j-i}(\xi))^{\alpha/2-l} \le C \tag{C.67}$$

and

$$(\mathrm{C}.68)\quad \begin{aligned}(y^{\theta}_{j-i}(\xi))' &= \theta y_{j-1-i}{}'(\xi) + (1-\theta)y_{j-i-1}{}'(\xi)\\ &= \xi^{1/r-1}(\theta(2T - y_{j-1-i}(\xi))^{1-1/r} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-1/r})\\ &\le C(2T - x_{j-2})^{1-1/r} \le C\end{aligned}$$

With

$$(\mathrm{C}.69)\quad Z_{2N-j-i} \le 2T^{1/r}$$

$$(\mathrm{C}.70)\quad \begin{aligned}(y^{\theta}_{j-i}(\xi))'' &= \theta y_{j-1-i}{}''(\xi) + (1-\theta)y_{j-i-1}{}''(\xi)\\ &= \frac{1-r}{r}\xi^{1/r-2}(\theta(2T - y_{j-i-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)} + (1-\theta)(2T - y_{j-i}(\xi))^{1-2/r}Z_{2N-(j-i)})\\ &\le C(r-1)\end{aligned}$$

Therefore,

$$(\mathrm{C}.71)\quad \begin{aligned}(u''(y^{\theta}_{j-i}(\xi)))' &= u'''(y^{\theta}_{j-i}(\xi))(y^{\theta}_{j-i}(\xi))'\\ &\le C\end{aligned}$$

$$(\mathrm{C}.72)\quad \begin{aligned}(u''(y^{\theta}_{j-i}(\xi)))'' &= u'''(y^{\theta}_{j-i}(\xi))(y^{\theta}_{j-i}{}'(\xi))^2 + u''''(y^{\theta}_{j-i}(\xi))y^{\theta}_{j-i}{}''(\xi)\\ &\le C + C(r-1) = C\end{aligned} \qquad \square$$

Lemma C.10. *There exists a constant $C = C(T,\alpha,r)$ such that*

$$(\mathrm{C}.73)\quad |y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha} \le C|y^{\theta}_j - x_i|^{1-\alpha}$$

$$(\mathrm{C}.74)\quad (|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})' \le C|y^{\theta}_j - x_i|^{-\alpha}(|2T - x_i - y^{\theta}_j| + h_N)$$

$$(\mathrm{C}.75)\quad (|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})'' \le C(r-1)|y^{\theta}_j - x_i|^{-\alpha} + C|y^{\theta}_j - x_i|^{-1-\alpha}(|2T - x_i - y^{\theta}_j| + h_N)^2$$

Proof.

$$(\mathrm{C}.76)\quad (y^{\theta}_{j-i}(\xi) - \xi)' = \theta y_{j-1-i}{}'(\xi) + (1-\theta)y_{j-i}{}'(\xi) - 1$$

$$(\mathrm{C}.77)\quad \begin{aligned}|y_{j-i}{}'(\xi) - 1| &= \xi^{1/r-1}|(2T - y_{j-i}(\xi))^{1-1/r} - \xi^{1-1/r}|\\ &\le \xi^{1/r-1}|2T - \xi - y_{j-i}(\xi)|\xi^{-1/r}\end{aligned}$$

$$(\mathrm{C}.78)\quad \begin{aligned}|2T - \xi - y_{j-i}(\xi)| &\le \max\begin{cases}|2T - x_{i-1} - x_{j-1}|\\ |2T - x_{i+1} - x_{j+1}|\end{cases}\\ &\le |2T - x_i - x_j| + h_{i+1} + h_j\end{aligned}$$

$$(\mathrm{C}.79)\quad \begin{aligned}(y^{\theta}_{j-i}(\xi) - \xi)'' &= \theta y_{j-1-i}{}''(\xi) + (1-\theta)y_{j-i}{}''(\xi)\\ &= \frac{1-r}{r}\xi^{1/r-2}(\theta(2T - y_{j-i}(\xi))^{1-2/r}Z_{2N-(j-i)} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-2/r}Z_{2N-(j-i-1)}) \le 0\end{aligned}$$

It's concave, so

$$y_{j-i}(\xi) - \xi \geq \min\{x_{j+1} - x_{i+1}, x_{j-1} - x_{i-1}\} \geq C(x_j - x_i) \tag{C.80}$$

We have

$$|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha} \leq C|y^{\theta}_j - x_i|^{1-\alpha} \tag{C.81}$$

$$\begin{aligned}(|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})' &= (1-\alpha)|y^{\theta}_{j-i}(\xi) - \xi|^{-\alpha}(y^{\theta}_{j-i}(\xi) - \xi)' \\ &\leq C|y^{\theta}_j - x_i|^{-\alpha}(|2T - x_i - y^{\theta}_j| + h_{i+1} + h_{j-1})\end{aligned} \tag{C.82}$$

(C.83) $\square$

$$\begin{aligned}(|y^{\theta}_{j-i}(\xi) - \xi|^{1-\alpha})'' &= (1-\alpha)|y^{\theta}_{j-i}(\xi) - \xi|^{-\alpha}(y^{\theta}_{j-i}(\xi) - \xi)'' + \alpha(\alpha-1)|y^{\theta}_{j-i}(\xi) - \xi|^{-1-\alpha}({y^{\theta}_{j-i}}'(\xi) - 1)^2 \\ &\leq C(r-1)|y^{\theta}_j - x_i|^{-\alpha} + C|y^{\theta}_j - x_i|^{-1-\alpha}(|2T - x_i - y^{\theta}_j| + h_{i+1} + h_{j-1})^2\end{aligned}$$

Proof. From (3.23), by Lemma C.8 and Lemma C.10, we have $\xi \in [x_i, x_{i+1}]$

$${Q^{\theta}_{j-i}}'(\xi) \leq Ch^2 h_j^2((r-1)|y^{\theta}_j - x_i|^{1-\alpha} + |y^{\theta}_j - x_i|^{-\alpha}(|2T - x_i - y^{\theta}_j| + h_N)) \tag{C.84}$$

$$Q^{\theta}_{j-i}(\xi) \leq Ch^2 h_j^2 |y^{\theta}_j - x_i|^{1-\alpha} \tag{C.85}$$

So use the skill in Proof 31 with Lemma C.9

$$\begin{aligned}&\frac{2}{h_i + h_{i+1}}\left(\frac{Q^{\theta}_{j-i}(x_{i+1})u'''(\eta^{\theta}_{j+1}) - Q^{\theta}_{j-i}(x_i)u'''(\eta^{\theta}_j)}{h_{i+1}}\right) \\ &\leq Ch^2 h_j(|y^{\theta}_j - x_i|^{1-\alpha} + |y^{\theta}_j - x_i|^{-\alpha}(|2T - x_i - y^{\theta}_j| + h_N))\end{aligned} \tag{C.86}$$

$\square$

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REFERENCES

[1] D. Gilbarg and N. S. Trudinger, *Poisson's Equation and the Newtonian Potential*, Springer Berlin Heidelberg, Berlin, Heidelberg, 1977, pp. 50–67, https://doi.org/10.1007/978-3-642-96379-7_4, https://doi.org/10.1007/978-3-642-96379-7_4.

[2] X. Ros-Oton and J. Serra, *The dirichlet problem for the fractional laplacian: Regularity up to the boundary*, Journal de Mathématiques Pures et Appliquées, 101 (2014), pp. 275–302, https://doi.org/https://doi.org/10.1016/j.matpur.2013.06.003, https://www.sciencedirect.com/science/article/pii/S0021782413000895.