

# A SECOND ORDER NUMERICAL METHODS FOR REISZ-FRACTIONAL ELLIPTIC EQUATION ON GRADED MESH\*

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**Abstract.** This is an example SIAM L<sup>A</sup>T<sub>E</sub>X article. This can be used as a template for new articles. Abstracts must be able to stand alone and so cannot contain citations to the paper's references, equations, etc. An abstract must consist of a single paragraph and be concise. Because of online formatting, abstracts must appear as plain as possible. Any equations should be inline.

**Key words.** example, L<sup>A</sup>T<sub>E</sub>X

**MSC codes.** ??????????????????

**1. Introduction.** For  $\Omega = (0, 2T)$ ,  $1 < \alpha < 2$ ,

$$(1.1) \quad \begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R} \setminus \Omega \end{cases}$$

where

$$(1.2) \quad (-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\kappa_\alpha \frac{d^2}{dx^2} \int_\Omega \frac{|x-y|^{1-\alpha}}{\Gamma(2-\alpha)} u(y) dy$$

$$(1.3) \quad \kappa_\alpha = -\frac{1}{2 \cos(\alpha\pi/2)} > 0$$

**2. Preliminaries: Numeric scheme and main results.**

**2.1. Numeric Format.**

$$(2.1) \quad x_i = \begin{cases} T \left( \frac{i}{N} \right)^r, & 0 \leq i \leq N \\ 2T - T \left( \frac{2N-i}{N} \right)^r, & N \leq i \leq 2N \end{cases}$$

where  $r \geq 1$ . And let

$$(2.2) \quad h_j = x_j - x_{j-1}, \quad 1 \leq j \leq 2N$$

Let  $\{\phi_j(x)\}_{j=1}^{2N-1}$  be standard hat functions, which are basis of the piecewise linear function space.

$$(2.3) \quad \phi_j(x) = \begin{cases} \frac{1}{h_j}(x - x_{j-1}), & x_{j-1} \leq x \leq x_j \\ \frac{1}{h_{j+1}}(x_{j+1} - x), & x_j \leq x \leq x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

And then, we can approximate  $u(x)$  with

$$(2.4) \quad \Pi_h u(x) := \sum_{j=1}^{2N-1} u(x_j) \phi_j(x)$$

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For convenience, we denote

$$(2.5) \quad I_h^{2-\alpha} u(x_i) := I^{2-\alpha} \Pi_h u(x_i) = \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} \Pi_h u(y) dy$$

And now, we can approximate the operator (1.2) at  $x_i$  with

$$(2.6) \quad D_h^\alpha u(x_i) := D_h^2 I_h^{2-\alpha} u(x_i) \\ = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} I_h^{2-\alpha} u(x_{i-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) I_h^{2-\alpha} u(x_i) + \frac{1}{h_{i+1}} I_h^{2-\alpha} u(x_{i+1}) \right)$$

Finally, we approximate the equation (1.1) with

$$(2.7) \quad -\kappa_\alpha D_h^\alpha \Pi_h u(x_i) = f(x_i), \quad 1 \leq i \leq 2N-1$$

The discrete equation (2.7) can be written in matrix form

$$(2.8) \quad AU = F$$

where  $U$  is unknown,  $F = (f(x_1), \dots, f(x_{2N-1}))$ . The matrix  $A$  is constructed as follows: Since

$$(2.9) \quad I_h^{2-\alpha} u(x_i) = \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} \Pi_h u(y) dy \\ = \sum_{j=1}^{2N-1} \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x_i - y|^{1-\alpha} u(x_j) \phi_j(y) dy \\ = \sum_{j=1}^{2N-1} u(x_j) \frac{1}{\Gamma(2-\alpha)} \int_{x_{j-1}}^{x_{j+1}} |x_i - y|^{1-\alpha} \phi_j(y) dy \\ = \sum_{j=1}^{2N-1} \frac{u(x_j)}{\Gamma(4-\alpha)} \left( \frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right) \\ =: \sum_{j=1}^{2N-1} \tilde{a}_{ij} u(x_j), \quad 0 \leq i \leq 2N$$

Then, substitute in (2.6), we have

$$(2.10) \quad -\kappa_\alpha D_h^\alpha \Pi_h u(x_i) = \sum_{j=1}^{2N-1} a_{ij} u(x_j)$$

where

$$(2.11) \quad a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \tilde{a}_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right), \quad 1 \leq i \leq 2N-1$$

**2.2. Regularity.** For any  $\beta > 0$ , we use the standard notation  $C^\beta(\bar{\Omega}), C^\beta(\mathbb{R})$ , etc., for Hölder spaces and their norms and seminorms. When no confusion is possible, we use the notation  $C^\beta(\Omega)$  to refer to  $C^{k,\beta'}(\Omega)$ , where  $k$  is the greatest integer such that  $k < \beta$  and where  $\beta' = \beta - k$ . The Hölder spaces  $C^{k,\beta'}(\Omega)$  are defined as the

subspaces of  $C^k(\Omega)$  consisting of functions whose  $k$ -th order partial derivatives are locally Hölder continuous[1] with exponent  $\beta'$  in  $\Omega$ , where  $C^k(\Omega)$  is the set of all  $k$ -times continuously differentiable functions on open set  $\Omega$ .

**DEFINITION 2.1** (delta dependent norm [2]). ...

**THEOREM 2.2.** Let  $f \in C^\beta(\Omega)$ ,  $\beta > 2$  be such that  $\|f\|_\beta^{(\alpha/2)} < \infty$ , then for  $l = 0, 1, 2$

$$(2.12) \quad |f^{(l)}(x)| \leq \|f\|_\beta^{(\alpha/2)} \begin{cases} x^{-l-\alpha/2}, & \text{if } 0 < x \leq T \\ (2T-x)^{-l-\alpha/2}, & \text{if } T \leq x < 2T \end{cases}$$

**THEOREM 2.3** (Regularity up to the boundary [2]). Let  $\Omega$  be a bounded domain, and  $\beta > 0$  be such that neither  $\beta$  nor  $\beta + \alpha$  is an integer. Let  $f \in C^\beta(\Omega)$  be such that  $\|f\|_\beta^{(\alpha/2)} < \infty$ , and  $u \in C^{\alpha/2}(\mathbb{R}^n)$  be a solution of (1.1). Then,  $u \in C^{\beta+\alpha}(\Omega)$  and

$$(2.13) \quad \|u\|_{\beta+\alpha}^{(-\alpha/2)} \leq C \left( \|u\|_{C^{\alpha/2}(\mathbb{R})} + \|f\|_\beta^{(\alpha/2)} \right)$$

**COROLLARY 2.4.** Let  $u$  be a solution of (1.1) on  $\Omega$ . Then, for any  $x \in \Omega$  and  $l = 0, 1, 2, 3, 4$

$$(2.14) \quad |u^{(l)}(x)| \leq \|u\|_{\beta+\alpha}^{(-\alpha/2)} \begin{cases} x^{\alpha/2-l}, & \text{if } 0 < x \leq T \\ (2T-x)^{\alpha/2-l}, & \text{if } T \leq x < 2T \end{cases}$$

The paper is organized as follows. Our main results are in subsection 2.3, experimental results are in section 5. Readers would better see section 4 before section 3 to avoid details.

**2.3. Main results.** Here we state our main results; the proof is deferred to section 3 and section 4.

Let's denote  $h = \frac{1}{N}$ , we have

**THEOREM 2.5** (Truncation Error). If  $f$  satisfy that  $f \in C^\beta(\Omega)$ ,  $\beta > 4 - \alpha$ ,  $\|f\|_\beta^{(\alpha/2)} < \infty$ ,  $\alpha \in (1, 2)$ , and  $u(x)$  is a solution of the equation (1.1), where  $\|u\|_{\beta+\alpha}^{(-\alpha/2)} < \infty$ , then there exists constants  $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$ ,  $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$ , such that the truncation error of the discrete format satisfies

$$(2.15) \quad \begin{aligned} \tau_i &:= | -\kappa_\alpha D_h^\alpha \Pi_h u(x_i) - f(x_i) | \\ &\leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} \begin{cases} x_i^{-\alpha}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha}, & N < i \leq 2N - 1 \end{cases} \\ &\quad + C_2(r-1)h^2 \begin{cases} |T - x_{i-1}|^{1-\alpha}, & 1 \leq i \leq N \\ |T - x_{i+1}|^{1-\alpha}, & N < i \leq 2N - 1 \end{cases} \end{aligned}$$

**THEOREM 2.6** (Convergence). The discrete equation (2.7) has solution  $U$ , and there exists a positive constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_\beta^{(\alpha/2)})$  such that the error between the numerical solution  $U$  with the exact solution  $u(x_i)$  satisfies

$$(2.16) \quad \max_{1 \leq i \leq 2N-1} |U_i - u(x_i)| \leq Ch^{\min\{\frac{r\alpha}{2}, 2\}}$$

77 *That means the numerical method has convergence order  $\min\{\frac{r\alpha}{2}, 2\}$ .*

78 **3. Proof of Theorem 2.5.** For convience, let's denote

$$79 \quad (3.1) \quad I^{2-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)} \int_{\Omega} |x-y|^{1-\alpha} u(y) dy$$

80 Then, the truncation error of the discrete format can be written as  
(3.2)

$$\begin{aligned} 81 \quad -\kappa_{\alpha} D_h^{\alpha} \Pi_h u(x_i) - f(x_i) &= -\kappa_{\alpha} (D_h^2 I_h^{2-\alpha} u(x_i) - \frac{d^2}{dx^2} I^{2-\alpha}(x_i)) \\ &= -\kappa_{\alpha} D_h^2 (I_h^{2-\alpha} u - I^{2-\alpha})(x_i) - \kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \end{aligned}$$

82 **3.1. Estimate of  $-\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i)$ .**

83 **THEOREM 3.1.** *There exists a constant  $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$  such that*

$$84 \quad (3.3) \quad \left| -\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) \right| \leq Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N-1 \end{cases}$$

85 *Proof.* Since  $f \in C^2(\Omega)$  and

$$86 \quad (3.4) \quad \frac{d^2}{dx^2} (-\kappa_{\alpha} I^{2-\alpha}(x)) = f(x), \quad x \in \Omega,$$

87 we have  $I^{2-\alpha} \in C^4(\Omega)$ . Therefore, using equation (A.3) of Lemma A.1, for  $1 \leq i \leq$   
88  $2N-1$ , we have

$$\begin{aligned} 89 \quad (3.5) \quad -\kappa_{\alpha} (D_h^2 - \frac{d^2}{dx^2}) I^{2-\alpha}(x_i) &= \frac{h_{i+1} - h_i}{3} f'(x_i) \\ &\quad + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right) \end{aligned}$$

90 where  $\eta_1 \in [x_{i-1}, x_i]$ ,  $\eta_2 \in [x_i, x_{i+1}]$ . By Lemma B.2 and Theorem 2.2 we have 1.  
(3.6)

$$91 \quad \left| \frac{h_{i+1} - h_i}{3} f'(x_i) \right| \leq \frac{C(r-1)\|f\|_{\beta}^{(\alpha/2)}}{3} h^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{-\alpha/2-2/r}, & N < i \leq 2N-1 \end{cases}$$

92 2. See Proof 25, there is a constant  $C = C(T, \alpha, r, \|f\|_{\beta}^{(\alpha/2)})$  such that

$$\begin{aligned} 93 \quad (3.7) \quad &\frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f''(y) \frac{(y - x_{i-1})^3}{3!} dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f''(y) \frac{(y - x_{i+1})^3}{3!} dy \right) \\ &\leq Ch^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N-1 \end{cases} \end{aligned}$$

94 Summarizes, we get the result.  $\square$

**3.2. Estimate of  $R_i$ .** Now, we study the first part of (3.2)

$$(3.8) \quad D_h^2(I^{2-\alpha} - I_h^{2-\alpha}u)(x_i) = D_h^2\left(\int_0^{2T} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy\right)$$

For convience, let's denote

$$(3.9) \quad T_{ij} = \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy, \quad i = 0, \dots, 2N, j = 1, \dots, 2N$$

And define

$$(3.10) \quad R_i := D_h^2(I^{2-\alpha} - I_h^{2-\alpha}u)(x_i) \\ = \frac{2}{h_i + h_{i+1}} \sum_{j=1}^{2N} \left( \frac{1}{h_i} T_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right), \quad 1 \leq i \leq 2N-1$$

We have some results about the estimate of  $R_i$

**THEOREM 3.2.** *For  $1 \leq i < N/2$ , there exists  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

$$(3.11) \quad R_i \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2(x_i^{-1-\alpha} \ln(i) + \ln(N)), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

**THEOREM 3.3.** *For  $N/2 \leq i \leq N$ , there exists constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

$$(3.12) \quad R_i \leq C(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

And for  $N < i \leq 2N-1$ , it is symmetric to the previous case.

To prove these results, we need some utils. Also for simplicity, we denote

**DEFINITION 3.4.**

$$(3.13) \quad S_{ij} = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} T_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_{i+1}} T_{i+1,j} \right)$$

then

$$(3.14) \quad R_i = \sum_{j=1}^{2N} S_{ij}$$

**3.3. Proof of Theorem 3.2.**

**LEMMA 3.5.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $1 \leq i < N/2$ ,*

$$(3.15) \quad \sum_{j=\max\{2i+1, i+3\}}^N S_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r}$$

*Proof.* Let

$$K_y(x) = \frac{|y-x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

For  $\max\{2i+1, i+3\} \leq j \leq N$ , by Lemma C.1 and Lemma C.2

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \\ &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} dy \\ &= Ch^2 \int_{x_{j-1}}^{x_j} y^{-\alpha/2-2/r-1} dy \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{j=\max\{2i+1, i+3\}}^N S_{ij} &\leq Ch^2 \int_{x_{2i}}^{x_N} y^{-\alpha/2-2/r-1} dy \\ &= \frac{C}{\alpha/2+2/r} h^2 (x_{2i}^{-\alpha/2-2/r} - T^{-\alpha/2-2/r}) \\ &\leq \frac{C}{\alpha/2+2/r} 2^{r(-\alpha/2-2/r)} h^2 x_i^{-\alpha/2-2/r} \end{aligned} \quad \square$$

LEMMA 3.6. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $1 \leq i < N/2$ ,*

$$\sum_{j=N+1}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

*Proof.* For  $1 \leq i < N/2, N+1 \leq j \leq 2N-1$ , by equation (C.2) and Lemma C.2

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \\ &\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T-y)^{\alpha/2-2/r} y^{-1-\alpha} dy \\ &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T-y)^{\alpha/2-2/r} dy \end{aligned}$$

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$$\begin{aligned}
\sum_{j=N+1}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{x_N}^{x_{2N-1}} (2T-y)^{\alpha/2-2/r} dy \\
(3.19) \quad &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 > 0 \\ \ln(T) - \ln(h_{2N}), & \alpha/2-2/r+1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 < 0 \end{cases} \\
&= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r} h^2, & \alpha/2-2/r+1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2-2/r+1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r} h^{r\alpha/2+r}, & \alpha/2-2/r+1 < 0 \end{cases}
\end{aligned}$$

131 And by Lemma A.3

$$132 \quad S_{i,2N} \leq CT^{-1-\alpha} h_{2N}^{\alpha/2+1} = CT^{-\alpha/2} h^{r\alpha/2+r}$$

133 And when  $\alpha/2 - 2/r + 1 \geq 0$ ,

$$134 \quad h^{r\alpha/2+r} \leq h^2$$

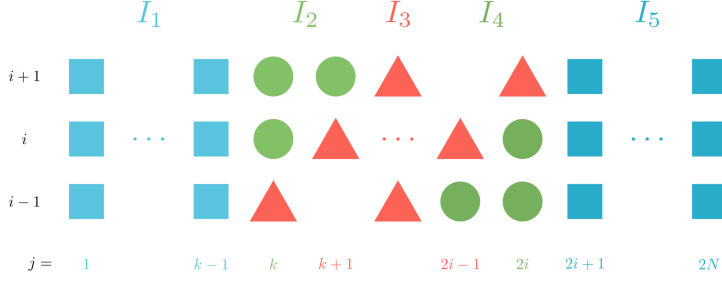
135 Summarizes, we get the result. □136 For  $i = 1, 2$ .137 LEMMA 3.7. *By Lemma C.5 , Lemma 3.5 and Lemma 3.6 we get*

$$\begin{aligned}
R_1 &= \sum_{j=1}^3 S_{1j} + \sum_{j=4}^{2N} S_{1j} \\
(3.20) \quad &\leq Ch^2 x_1^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1 > 0 \\ Ch^2 \ln(N), & \alpha/2-2/r+1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1 < 0 \end{cases}
\end{aligned}$$

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$$\begin{aligned}
R_2 &= \sum_{j=1}^4 S_{2j} + \sum_{j=5}^{2N} S_{2j} \\
(3.21) \quad &\leq Ch^2 x_2^{-\alpha/2-2/r} + \begin{cases} Ch^2, & \alpha/2-2/r+1 > 0 \\ Ch^2 \ln(N), & \alpha/2-2/r+1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2-2/r+1 < 0 \end{cases}
\end{aligned}$$

141 For  $3 \leq i < N/2$ , we have a new separation of  $R_i$ , Let's denote  $k = \lceil \frac{i}{2} \rceil$ .



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$$\begin{aligned}
R_i &= \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&= \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&\quad + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \\
&\quad + \sum_{j=k+1}^{2i-1} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\
&\quad + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \\
&\quad + \sum_{j=2i+1}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\
&= I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}
\tag{3.22}$$

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145 **LEMMA 3.8.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $3 \leq$*   
146  *$i \leq N, k = \lceil \frac{i}{2} \rceil$*

$$|I_1| = \left| \sum_{j=1}^{k-1} S_{ij} \right| \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2 - 2/r + 1 < 0 \end{cases}
\tag{3.23}$$

148 *Proof.* by Lemma A.3 , Lemma C.3

$$S_{i1} \leq C x_1^{\alpha/2} x_1 x_i^{-1-\alpha} = C x_1^{\alpha/2+1} x_i^{-1-\alpha} = C T^{\alpha/2+1} h^{r\alpha/2+r} x_i^{-1-\alpha}
\tag{3.24}$$



For  $2 \leq j \leq k-1$ , by Lemma C.1 and Lemma C.3

$$\begin{aligned}
 S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 K_y(x_i) dy \\
 &\leq Ch^2 \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} dy \\
 &= Ch^2 x_i^{-1-\alpha} \int_{x_{j-1}}^{x_j} y^{\alpha/2-2/r} dy
 \end{aligned}
 \tag{3.25}$$

Therefore,

$$\begin{aligned}
 I_1 &= \sum_{j=1}^{k-1} S_{ij} = S_{i1} + \sum_{j=2}^{k-1} S_{ij} \\
 &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{x_{\lceil \frac{i}{2} \rceil - 1}} y^{\alpha/2-2/r} dy \\
 &\leq Ch^{r\alpha/2+r} x_i^{-1-\alpha} + Ch^2 x_i^{-1-\alpha} \int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy
 \end{aligned}
 \tag{3.26}$$

But

$$\int_{x_1}^{2^{-r} x_i} y^{\alpha/2-2/r} dy \leq \begin{cases} \frac{1}{\alpha/2-2/r+1} (2^{-r} x_i)^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 > 0 \\ \ln(2^{-r} x_i) - \ln(x_1), & \alpha/2-2/r+1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} x_1^{\alpha/2-2/r+1}, & \alpha/2-2/r+1 < 0 \end{cases}
 \tag{3.27}$$

So we have

$$I_1 \leq \begin{cases} \frac{C}{\alpha/2-2/r+1} h^2 x_i^{-\alpha/2-2/r}, & \alpha/2-2/r+1 > 0 \\ Ch^2 x_i^{-1-\alpha} \ln(i), & \alpha/2-2/r+1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} h^{r\alpha/2+r} x_i^{-1-\alpha}, & \alpha/2-2/r+1 < 0 \end{cases} \quad \square
 \tag{3.28}$$

DEFINITION 3.9. For convience, let's denote

$$V_{ij} = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right)
 \tag{3.29}$$

THEOREM 3.10. There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for

$$3 \leq i < N/2, k = \lceil \frac{i}{2} \rceil,
 \tag{3.30}$$

$$I_3 = \sum_{j=k+1}^{2i-1} V_{ij} \leq Ch^2 x_i^{-\alpha/2-2/r}
 \tag{3.30}$$

To estimate  $V_{ij}$ , we need some preparations.

LEMMA 3.11. For  $y \in [x_{j-1}, x_j]$ , we can rewrite  $y = x_{j-1} + \theta h_j = (1-\theta)x_{j-1} +$

166  $\theta x_j =: y_j^\theta$ ,  $\theta \in [0, 1]$ , by Lemma A.2

$$\begin{aligned}
 T_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\
 &= \int_0^1 (u(y_j^\theta) - \Pi_h u(y_j^\theta)) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j d\theta \\
 &= \int_0^1 -\frac{\theta(1-\theta)}{2} h_j^3 u''(y_j^\theta) \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \\
 &\quad + \frac{\theta(1-\theta)}{3!} h_j^4 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} (\theta^2 u'''(\eta_{j,1}^\theta) - (1-\theta)^2 u'''(\eta_{j,2}^\theta)) d\theta
 \end{aligned}
 \tag{3.31}$$

168 where  $\eta_{j,1}^\theta \in [x_{j-1}, y_j^\theta]$ ,  $\eta_{j,2}^\theta \in [y_j^\theta, x_j]$ .

169 Now Let's construct a series of functions to represent  $T_{ij}$ .

170 DEFINITION 3.12. For  $2 \leq i, j \leq N-1$ ,

$$y_{j-i}(x) = (x^{1/r} + Z_{j-i})^r, \quad Z_{j-i} = T^{1/r} \frac{j-i}{N}
 \tag{3.32}$$

172

$$y_{j-i}^\theta(x) = (1-\theta)y_{j-1-i}(x) + \theta y_{j-i}(x)
 \tag{3.33}$$

174

$$h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x)
 \tag{3.34}$$

176 Now, we define

$$P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}
 \tag{3.35}$$

178

$$Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}
 \tag{3.36}$$

180 And now we can rewrite  $T_{ij}$

181 LEMMA 3.13. For  $2 \leq i \leq N, 2 \leq j \leq N$ ,

$$\begin{aligned}
 T_{ij} &= \int_0^1 -\frac{\theta(1-\theta)}{2} P_{j-i}^\theta(x_i) d\theta \\
 &\quad + \int_0^1 \frac{\theta(1-\theta)}{3!} Q_{j-i}^\theta(x_i) (\theta^2 u'''(\eta_{j,1}^\theta) - (1-\theta)^2 u'''(\eta_{j,2}^\theta)) d\theta
 \end{aligned}
 \tag{3.37}$$

183 Immediately, we can see from (3.29) that

LEMMA 3.14. For  $3 \leq i, j \leq N - 1$ ,  
(3.38)

$$\begin{aligned} V_{ij} &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^\theta(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,2}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta \end{aligned}$$

To estimate  $V_{ij}$ , we first estimate  $D_h^2 P_{j-i}^\theta(x_i)$ , but By Lemma A.1,

$$(3.39) \quad D_h^2 P_{j-i}^\theta(x_i) = P_{j-i}^{\theta''}(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

By Leibniz formula, we calculate and estimate the derivations of  $h_{j-i}^3(x)$ ,  $u''(y_{j-i}^\theta(x))$

and  $\frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$  separately.

Firstly, we have

LEMMA 3.15. There exists a constant  $C = C(T, r)$  such that For  $3 \leq i \leq N - 1$ ,  $\lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i - 1, N - 1\}$ ,  $\xi \in [x_{i-1}, x_{i+1}]$ ,

$$(3.40) \quad h_{j-i}^3(\xi) \leq C h^2 x_i^{2-2/r} h_j$$

$$(3.41) \quad (h_{j-i}^3(\xi))' \leq C(r-1) h^2 x_i^{1-2/r} h_j$$

$$(3.42) \quad (h_{j-i}^3(\xi))'' \leq C(r-1) h^2 x_i^{-2/r} h_j$$

The proof of this theorem see Lemma C.6 and Lemma C.7

Second,

LEMMA 3.16. There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $3 \leq i \leq N - 1$ ,  $\lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i - 1, N - 1\}$ ,  $\xi \in [x_{i-1}, x_{i+1}]$ ,

$$(3.43) \quad u''(y_{j-i}^\theta(\xi)) \leq C x_i^{\alpha/2-2}$$

$$(3.44) \quad (u''(y_{j-i}^\theta(\xi)))' \leq C x_i^{\alpha/2-3}$$

$$(3.45) \quad (u''(y_{j-i}^\theta(\xi)))'' \leq C x_i^{\alpha/2-4}$$

The proof of this theorem see Proof 31

And Finally, we have

LEMMA 3.17. There exists a constant  $C = C(T, \alpha, r)$  such that For  $3 \leq i \leq N - 1$ ,  $1 \leq j \leq \min\{2i - 1, N - 1\}$ ,  $\xi \in [x_{i-1}, x_{i+1}]$ ,

$$(3.46) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C |y_j^\theta - x_i|^{1-\alpha}$$

$$(3.47) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C|y_j^\theta - x_i|^{1-\alpha}x_i^{-1}$$

$$(3.48) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C|y_j^\theta - x_i|^{1-\alpha}x_i^{-2}$$

where  $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

The proof of this theorem see Proof 32

LEMMA 3.18. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $3 \leq i \leq N-1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i-1, N-1\}$ ,*

$$(3.49) \quad D_h^2 P_{j-i}^\theta(x_i) \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j$$

where  $y_j^\theta = \theta x_{j-1} + (1-\theta)x_j$

*Proof.* Since

$$(3.50) \quad D_h^2 P_{j-i}^\theta(x_i) = P_{j-i}^{\theta''}(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

From (3.35), using Leibniz formula and Lemma 3.15, Lemma 3.16 and Lemma 3.17□

LEMMA 3.19. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $3 \leq i < N, k = \lceil \frac{i}{2} \rceil$ . For  $k \leq j \leq \min\{2i-1, N-1\}$ ,*

$$(3.51) \quad \begin{aligned} & \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ & \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned}$$

And for  $k+1 \leq j \leq \min\{2i, N\}$ ,

$$(3.52) \quad \begin{aligned} & \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta) - Q_{j-i}^\theta(x_{i-1})u'''(\eta_{j-1}^\theta)}{h_i} \right) \\ & \leq Ch^2 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned}$$

where  $\eta_j^\theta \in [x_{j-1}, x_j]$ .

proof see Proof 33

LEMMA 3.20. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $3 \leq i < N, k = \lceil \frac{i}{2} \rceil, k+1 \leq j \leq \min\{2i-1, N-1\}$ ,*

$$(3.53) \quad \begin{aligned} V_{ij} & \leq Ch^2 \int_0^1 \frac{|y_j^\theta - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j d\theta \\ & = Ch^2 \int_{x_{j-1}}^{x_j} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \end{aligned}$$

*Proof.* Since Lemma 3.14, by Lemma 3.18 and Lemma 3.19, we get the result immediately.  $\square$

Now we can prove Theorem 3.10 using Lemma 3.20,  $k = \lceil \frac{i}{2} \rceil$

$$\begin{aligned}
 I_3 &= \sum_{k+1}^{2i-1} V_{ij} \leq Ch^2 \int_{x_k}^{x_{2i-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\
 &= Ch^2 \left( \frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{2i-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\
 &\leq Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r}
 \end{aligned}$$

LEMMA 3.21.

$$D_h P_{j-i}^\theta(x_i) := \frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} = P_{j-i}^{\theta'}(\xi), \quad \xi \in [x_i, x_{i+1}]$$

Then, for  $3 \leq i \leq N-1$ ,  $k = \lceil \frac{i}{2} \rceil$ ,

$$D_h P_{k-i}^\theta(x_i) \leq Ch^2 x_i^{-\alpha/2-2/r} h_j$$

*Proof.* Using Leibniz formula, by Lemma 3.15, Lemma 3.16 and Lemma 3.17, we take  $j = k+1$ ,  $i = i+1$ , we get

$$\begin{aligned}
 D_h P_{k-i}^\theta(x_i) &\leq Ch^2 x_{i+1}^{\alpha/2-2/r-1} |y_{k+1}^\theta - x_{i+1}|^{1-\alpha} h_{j+1} \\
 &\leq Ch^2 x_i^{\alpha/2-2/r-1} |y_k^\theta - x_i|^{1-\alpha} h_j \\
 &\leq Ch^2 x_i^{-\alpha/2-2/r} h_j
 \end{aligned}$$

LEMMA 3.22. There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $3 \leq i < N$ ,  $k = \lceil \frac{i}{2} \rceil$ ,

$$I_2 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

And for  $3 \leq i < N/2$ ,

$$I_4 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,2i} + T_{i-1,2i-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2i} \right) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

*Proof.* In fact,

$$\begin{aligned}
 &\frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \\
 &= \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) + \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) + \left( \frac{1}{h_{i+1}} - \frac{1}{h_i} \right) T_{i,k}
 \end{aligned}$$

While, by Lemma A.2

$$\begin{aligned}
 \frac{1}{h_{i+1}}(T_{i+1,k} - T_{i,k}) &= \int_{x_{k-1}}^{x_k} (u(y) - \Pi_h u(y)) \frac{|x_{i+1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_{i+1}\Gamma(2-\alpha)} dy \\
 &\leq \int_{x_{k-1}}^{x_k} h_k^2 u''(\eta) \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy \\
 &\leq Ch_k h^2 x_k^{2-2/r} x_{k-1}^{\alpha/2-2} |x_i - x_k|^{-\alpha} \\
 &\leq Ch_k h^2 x_i^{-\alpha/2-2/r}
 \end{aligned}$$

Thus,

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} (T_{i+1,k} - T_{i,k}) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

For

(3.63)

$$\begin{aligned}
 \frac{1}{h_{i+1}}(T_{i+1,k+1} - T_{i,k}) &= \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} d\theta \\
 &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{Q_{k-i}^\theta(x_{i+1})u'''(\eta_{k+1,1}^\theta) - Q_{k-i}^\theta(x_i)u'''(\eta_{k,1}^\theta)}{h_{i+1}} d\theta \\
 &- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{Q_{k-i}^\theta(x_{i+1})u'''(\eta_{k+1,2}^\theta) - Q_{k-i}^\theta(x_i)u'''(\eta_{k,2}^\theta)}{h_{i+1}} d\theta
 \end{aligned}$$

And by Lemma 3.21

$$\frac{P_{k-i}^\theta(x_{i+1}) - P_{k-i}^\theta(x_i)}{h_{i+1}} \leq Ch^2 x_i^{-\alpha/2-2/r} h_k$$

And with Lemma 3.19, we can get

$$\frac{2}{h_i + h_{i+1}} \frac{1}{h_{i+1}} (T_{i+1,k+1} - T_{i,k}) \leq Ch^2 x_i^{-\alpha/2-2/r}$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$\begin{aligned}
 \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i,k} &\leq h_i^{-3} h^2 x_i^{1-2/r} h_k Ch_k^2 x_{k-1}^{\alpha/2-2} |x_k - x_i|^{1-\alpha} \\
 &\leq Ch^2 x_i^{-\alpha/2-2/r}
 \end{aligned}$$

Summarizes, we have

$$I_2 \leq Ch^2 x_i^{-\alpha/2-2/r}$$

The case for  $I_4$  is similar.  $\square$

Now combine Lemma 3.8, Lemma 3.22, Theorem 3.10, Lemma 3.5 and Lemma 3.6 to get the final result.

For  $3 \leq i < N/2$

$$R_i = I_1 + I_2 + I_3 + I_4 + I_5$$

$$\leq Ch^2 x_i^{-\alpha/2-2/r} + \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2 (x_i^{1-\alpha} \ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases}$$

Combine with  $i = 1, 2$ , we get for  $1 \leq i < N/2$

$$(3.69) \quad R_i \leq \begin{cases} Ch^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ Ch^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ Ch^{r\alpha/2+r} x_i^{-1-\alpha}, & r\alpha/2 + r - 2 < 0 \end{cases}$$

**3.4. Proof of Theorem 3.3.** For  $N/2 \leq i < N, k = \lceil \frac{i}{2} \rceil$ , we have

$$(3.70) \quad \begin{aligned} R_i &= \sum_{j=1}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &= \sum_{j=1}^{k-1} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &\quad + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} (T_{i+1,k} + T_{i+1,k+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,k} \right) \\ &\quad + \sum_{j=k+1}^{N-1} + \sum_{j=N}^{N+1} + \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &\quad + \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1,2N-\lceil \frac{N}{2} \rceil+1} + T_{i-1,2N-\lceil \frac{N}{2} \rceil}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,2N-\lceil \frac{N}{2} \rceil+1} \right) \\ &\quad + \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j} \right) \\ &= I_1 + I_2 + I_3^1 + I_3^2 + I_3^3 + I_4 + I_5 \end{aligned}$$

We have estimate  $I_1$  in Lemma 3.8 and  $I_2$  in Lemma 3.22. We can control  $I_3$  in similar with Theorem 3.10 by Lemma 3.20 where  $2i - 1 \geq N - 1$

**LEMMA 3.23.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $N/2 \leq i < N, k = \lceil \frac{i}{2} \rceil$ ,*

$$(3.71) \quad \begin{aligned} I_3 &= \sum_{j=k+1}^{N-1} V_{ij} \leq Ch^2 \int_{x_k}^{x_{N-1}} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} dy \\ &= Ch^2 \left( \frac{|x_k - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{|x_{N-1} - x_i|^{2-\alpha}}{\Gamma(3-\alpha)} \right) x_i^{\alpha/2-2-2/r} \\ &\leq Ch^2 x_i^{2-\alpha} x_i^{\alpha/2-2-2/r} = Ch^2 x_i^{-\alpha/2-2/r} \end{aligned}$$

Let's study  $I_5$  before  $I_4$ .

$$(3.72) \quad I_5 = \sum_{j=N+2}^{2N-\lceil \frac{N}{2} \rceil} V_{ij}$$

Similarly, Let's define a new series of functions

**DEFINITION 3.24.** *For  $i < N, j \geq N$ , with no confusion, we also denote in this section*

$$(3.73) \quad y_{j-i}(x) = 2T - (Z_{2N-j+i} - x^{1/r})^r, \quad Z_{2N-j+i} = T^{1/r} \frac{2N-j+i}{N}$$

287

$$(3.74) \quad y_{j-i}'(x) = (2T - y_{j-i}(x))^{1-1/r} x^{1/r-1}$$

$$(3.75) \quad y_{j-i}''(x) = \frac{1-r}{r} (2T - y_{j-i}(x))^{1-2/r} x^{1/r-2} Z_{2N-j+i}$$

$$(3.76)$$

291

$$(3.77) \quad y_{j-i}^\theta(x) = (1-\theta)y_{j-i-1}(x) + \theta y_{j-i}(x)$$

293

$$(3.78) \quad h_{j-i}(x) = y_{j-i}(x) - y_{j-i-1}(x)$$

295

$$(3.79) \quad P_{j-i}^\theta(x) = (h_{j-i}(x))^3 u''(y_{j-i}^\theta(x)) \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

297

$$(3.80) \quad Q_{j-i}^\theta(x) = (h_{j-i}(x))^4 \frac{|y_{j-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

299

Now we have, for  $i < N, j \geq N+2$ ,

(3.81)

$$\begin{aligned} V_{ij} &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1,j+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i,j} + \frac{1}{h_i} T_{i-1,j-1} \right) \\ &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_h^2 P_{j-i}^\theta(x_i) d\theta \\ &\quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta)}{h_{i+1}} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,1}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,1}^\theta)}{h_i} \right) d\theta \\ &\quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1,2}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta)}{h_{i+1}} \right) d\theta \\ &\quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i) u'''(\eta_{j,2}^\theta) - Q_{j-i}^\theta(x_{i-1}) u'''(\eta_{j-1,2}^\theta)}{h_i} \right) d\theta \end{aligned}$$

300

301

Similarly, we first estimate

$$(3.82) \quad D_h^2 P_{j-i}^\theta(\xi) = P_{j-i}^{\theta''}(\xi), \quad \xi \in [x_{i-1}, x_{i+1}]$$

303

Combine lemmas Lemma C.8, Lemma C.9 and Lemma C.10, we have

304

LEMMA 3.25. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For*

305

 *$N/2 \leq i < N, N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1, \xi \in [x_{i-1}, x_{i+1}]$ , we have*

$$(3.83)$$

$$\begin{aligned} |P_{j-i}^{\theta''}(\xi)| &\leq Ch_j h^2 (|y_j^\theta - x_i|^{1-\alpha} \\ &\quad + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N) \\ &\quad + |y_j^\theta - x_i|^{-1-\alpha} (|2T - x_i - y_j^\theta| + h_N)^2 \\ &\quad + (r-1) |y_j^\theta - x_i|^{-\alpha}) \end{aligned}$$

306



And

LEMMA 3.26. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i < N$ ,  $\xi \in [x_{i-1}, x_{i+1}]$ , we have for  $N+1 \leq j \leq 2N - \lceil \frac{N}{2} \rceil$*

$$(3.84) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ \leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N))$$

for  $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$

$$(3.85) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta) - Q_{j-i}^\theta(x_{i-1})u'''(\eta_{j-1}^\theta)}{h_{i+1}} \right) \\ \leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N))$$

The proof see Proof 37.

Combine (3.81), Lemma 3.25 and Lemma 3.26, we have

THEOREM 3.27. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i < N$ ,  $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$*

$$(3.86) \quad V_{ij} \leq Ch^2 \int_{x_{j-1}}^{x_j} (|y - x_i|^{1-\alpha} \\ + |y - x_i|^{-\alpha} (|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha} (|2T - x_i - y| + h_N)^2 \\ + (r-1)|y - x_i|^{-\alpha}) dy$$

We can esitmate  $I_5$  Now.

THEOREM 3.28. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i < N$ , we have*

$$(3.87) \quad I_5 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \leq Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha}$$

*Proof.*

$$(3.88) \quad I_5 = \sum_{j=N+2}^{2N - \lceil \frac{N}{2} \rceil} V_{ij} \\ \leq Ch^2 \int_{x_{N+1}}^{x_{2N-i}} + \int_{x_{2N-i}}^{x_{2N - \lceil \frac{N}{2} \rceil}} (|y - x_i|^{1-\alpha} \\ + |y - x_i|^{-\alpha} (|2T - x_i - y| + h_N) + |y - x_i|^{-1-\alpha} (|2T - x_i - y| + h_N)^2 \\ + (r-1)|y - x_i|^{-\alpha}) dy \\ = J_1 + J_2$$

While  $x_{N+1} \leq y \leq x_{2N-i} = 2T - x_i$ ,

$$(3.89) \quad T - x_{i-1} \leq x_{N+1} - x_i \leq y - x_i \leq x_{2N-i} - x_i \leq 2(T - x_{i-1})$$

325 and

$$326 \quad (3.90) \quad 2T - x_i - y + h_N \leq 2T - x_i - x_{N+1} + h_N = T - x_i \leq T - x_{i-1}$$

327 So

$$\begin{aligned} 328 \quad (3.91) \quad J_1 &\leq Ch^2(x_{2N-i} - x_{N+1})(|T - x_{i-1}|^{1-\alpha} + (r-1)|T - x_{i-1}|^{-\alpha}) \\ &\leq Ch^2(|T - x_{i-1}|^{2-\alpha} + (r-1)|T - x_{i-1}|^{1-\alpha}) \\ &\leq Ch^2T^{2-\alpha} + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned}$$

329 Otherwise, when  $x_{2N-i} \leq y \leq x_{2N-\lceil \frac{N}{2} \rceil}$

$$330 \quad (3.92) \quad x_i + y - 2T + h_N \leq y - x_i$$

331

$$\begin{aligned} 332 \quad (3.93) \quad J_2 &\leq Ch^2 \int_{x_{2N-i}}^{(2-2^{-r})T} |y - x_i|^{1-\alpha} + (r-1)|y - x_i|^{-\alpha} \\ &\leq Ch^2(T^{2-\alpha} + (r-1)|x_{2N-i} - x_i|^{1-\alpha}) \\ &= Ch^2 + C(r-1)h^2|T - x_i|^{1-\alpha} \leq Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned}$$

333 Summarizes two cases, we get the result. □

For  $I_4$ , we have

**THEOREM 3.29.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that, for  $N/2 \leq i \leq N-1$*

$$(3.94) \quad \begin{aligned} V_{iN} &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} T_{i+1, N+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, N} + \frac{1}{h_i} T_{i-1, N-1} \right) \\ &\leq Ch^2 + C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} \end{aligned}$$

*Proof.* We use the similar skill in the last section, but more complicated. for  $j = N$ , Let

$$(3.95) \quad {}_L y_{N-1-i}(x) = (x^{1/r} + Z_{N-1-i})^r, \quad Z_{N-1-i} = T^{1/r} \frac{N-1-i}{N}$$

$$(3.96) \quad {}_0 y_{N-i}(x) = \frac{x^{1/r} - Z_i}{Z_1} h_N + T, \quad Z_i = T^{1/r} \frac{i}{N}, x_N = T$$

and

$$(3.97) \quad {}_R y_{N+1-i}(x) = 2T - (Z_{N-1+i} - x^{1/r})^r, \quad Z_{N-1+i} = T^{1/r} \frac{N-1+i}{N}$$

Thus,

$$\begin{aligned} {}_L y_{N-1-i}(x_{i-1}) &= x_{N-2}, \quad {}_L y_{N-1-i}(x_i) = x_{N-1}, \quad {}_L y_{N-1-i}(x_{i+1}) = x_N \\ {}_0 y_{N-i}(x_{i-1}) &= x_{N-1}, \quad {}_0 y_{N-i}(x_i) = x_N, \quad {}_0 y_{N-i}(x_{i+1}) = x_{N+1} \\ {}_R y_{N+1-i}(x_{i-1}) &= x_N, \quad {}_R y_{N+1-i}(x_i) = x_{N+1}, \quad {}_R y_{N+1-i}(x_{i+1}) = x_{N+2} \end{aligned}$$

Then, define

$$(3.98) \quad {}_L y_{N-i}^\theta(x) = \theta {}_L y_{N-1-i}(x) + (1-\theta) {}_0 y_{N-i}(x)$$

$$(3.99) \quad {}_R y_{N+1-i}^\theta(x) = \theta {}_0 y_{N-i}(x) + (1-\theta) {}_R y_{N+1-i}(x)$$

$$(3.100) \quad {}_L h_{N-i}(x) = {}_0 y_{N-i}(x) - {}_L y_{N-1-i}(x)$$

$$(3.101) \quad {}_R h_{N+1-i}(x) = {}_R y_{N+1-i}(x) - {}_0 y_{N-i}(x)$$

We have

$$(3.102) \quad {}_L y_{N-1-i}'(x) = {}_L y_{N-1-i}^{1-1/r}(x) x^{1/r-1}$$

$$(3.103) \quad {}_L y_{N-1-i}''(x) = \frac{1-r}{r} {}_L y_{N-1-i}^{1-2/r}(x) x^{1/r-2} Z_{N-1-i}$$

$$(3.104) \quad {}_0 y_{N-i}'(x) = \frac{1}{r} \frac{h_N}{Z_1} x^{1/r-1}$$

$$(3.105) \quad {}_0 y_{N-i}''(x) = \frac{1-r}{r^2} \frac{h_N}{Z_1} x^{1/r-2}$$

$$(3.106) \quad {}_R y_{N+1-i}'(x) = (2T - {}_R y_{N+1-i}(x))^{1-1/r} x^{1/r-1}$$

$$(3.107) \quad {}_R y_{N+1-i}''(x) = \frac{1-r}{r} (2T - {}_R y_{N+1-i}(x))^{1-2/r} x^{1/r-2} Z_{N-1+i}$$

362

$$363 \quad (3.108) \quad {}_L P_{N-i}^\theta(x) = ({}_L h_{N-i}(x))^3 \frac{|{}_L y_{N-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_L y_{N-i}^\theta(x))$$

$$364 \quad (3.109) \quad {}_R P_{N+1-i}^\theta(x) = ({}_R h_{N+1-i}(x))^3 \frac{|{}_R y_{N+1-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)} u''({}_R y_{N+1-i}^\theta(x))$$

$$365 \quad (3.110) \quad {}_L Q_{N-i}^\theta(x) = ({}_L h_{N-i}(x))^4 \frac{|{}_L y_{N-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$366 \quad (3.111) \quad {}_R Q_{N+1-i}^\theta(x) = ({}_R h_{N+1-i}(x))^4 \frac{|{}_R y_{N+1-i}^\theta(x) - x|^{1-\alpha}}{\Gamma(2-\alpha)}$$

367 Similar with Lemma 3.13, we can get for  $l = -1, 0, 1$ ,

$$368 \quad (3.112) \quad \begin{aligned} T_{i+l, N+l} &= \int_0^1 -\frac{\theta(1-\theta)}{2} {}_L P_{N-i}^\theta(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} {}_L Q_{N-i}^\theta(x_{i+l}) (\theta^2 u'''(\eta_{N+l,1}^\theta) - (1-\theta)^2 u'''(\eta_{N+l,2}^\theta)) d\theta \end{aligned}$$

369

$$(3.113) \quad \begin{aligned} T_{i+l, N+1+l} &= \int_0^1 -\frac{\theta(1-\theta)}{2} {}_R P_{N+1-i}^\theta(x_{i+l}) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)}{3!} {}_R Q_{N+1-i}^\theta(x_{i+l}) (\theta^2 u'''(\eta_{N+1+l,1}^\theta) - (1-\theta)^2 u'''(\eta_{N+1+l,2}^\theta)) d\theta \end{aligned}$$

371 So we have

$$(3.114) \quad \begin{aligned} V_{i,N} &= \int_0^1 -\frac{\theta(1-\theta)}{2} D_{hL}^2 {}_L P_{N-i}^\theta(x_i) d\theta \\ &+ \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_{i+1}) u'''(\eta_{N+1,1}^\theta) - {}_L Q_{N-i}^\theta(x_i) u'''(\eta_{N,1}^\theta)}{h_{i+1}} \right) d\theta \\ &- \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_i) u'''(\eta_{N,1}^\theta) - {}_L Q_{N-i}^\theta(x_{i-1}) u'''(\eta_{N-1,1}^\theta)}{h_i} \right) d\theta \\ &- \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_{i+1}) u'''(\eta_{N+1,2}^\theta) - {}_L Q_{N-i}^\theta(x_i) u'''(\eta_{N,2}^\theta)}{h_{i+1}} \right) d\theta \\ &+ \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_i) u'''(\eta_{N,2}^\theta) - {}_L Q_{N-i}^\theta(x_{i-1}) u'''(\eta_{N-1,2}^\theta)}{h_i} \right) d\theta \end{aligned}$$

373  $N+1$  is similar.

374 We estimate  $D_{hL}^2 {}_L P_{N-i}^\theta(x_i) = {}_L P_{N-i}^{\theta''}(\xi)$ ,  $\xi \in [x_{i-1}, x_{i+1}]$ ,

375

LEMMA 3.30.

$$376 \quad (3.115) \quad {}_L h_{N-i}^3(\xi) \leq C h_N^3 \leq C h^3$$

$$377 \quad (3.116) \quad {}_R h_{N+1-i}^3(\xi) \leq C h_N^3 \leq C h^3$$

$$(3.117) \quad ({}_L h_{N-i}^3(\xi))' \leq C(r-1)h_N^2 h \leq C(r-1)h^3$$

$$(3.118) \quad ({}_R h_{N+1-i}^3(\xi))' \leq C(r-1)h_N^2 h \leq C(r-1)h^3$$

$$(3.119) \quad ({}_L h_{N-i}^3(\xi))'' \leq C(r-1)h^2$$

$$(3.120) \quad ({}_R h_{N+1-i}^3(\xi))'' \leq C(r-1)h^2$$

*Proof.*

$$(3.121) \quad {}_L h_{N-i}(\xi) \leq 2h_N, \quad {}_R h_{N+1-i}(\xi) \leq 2h_N$$

383

$$(3.122) \quad \begin{aligned} ({}_L h_{N-i}^l(\xi))' &= {}_L h_{N-i}^{l-1}(\xi)({}_0 y_{N-i}'(\xi) - {}_L y_{N-1-i}'(\xi)) \\ &= {}_L h_{N-i}^{l-1}(\xi)x_i^{1/r-1}(\frac{1}{r} \frac{h_N}{Z_1} - {}_L y_{N-1-i}^{1-1/r}(\xi)) \end{aligned}$$

385 while

(3.123)

$$\begin{aligned} |\frac{1}{r} \frac{h_N}{Z_1} - {}_L y_{N-1-i}^{1-1/r}(\xi)| &= |\frac{1}{r} \frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1} - \eta^{1-1/r}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r} |(\frac{N-t}{N})^{r-1} - (\frac{N-s}{N})^{r-1}| \quad t \in [0, 1], s \in [0, 2] \\ &\leq T^{1-1/r} |1 - (\frac{N-2}{N})^{r-1}| \leq CT^{1-1/r}(r-1)\frac{2}{N} \end{aligned}$$

386

387 Thus,

$$(3.124) \quad ({}_L h_{N-i}^l(\xi))' \leq C(r-1)h_N^{l-1}x_i^{1/r-1}h$$

$$(3.125) \quad \begin{aligned} ({}_R h_{N+1-i}^l(\xi))' &= {}_R h_{N+1-i}^{l-1}(\xi)({}_R y_{N+1-i}'(\xi) - {}_0 y_{N-i}'(\xi)) \\ &= {}_R h_{N+1-i}^{l-1}(\xi)x_i^{1/r-1}((2T - {}_R y_{N+1-i}(\xi))^{1-1/r} - \frac{1}{r} \frac{h_N}{Z_1}) \end{aligned}$$

390 Similarly,

(3.126)

$$\begin{aligned} |(2T - {}_R y_{N+1-i})^{1-1/r} - \frac{1}{r} \frac{h_N}{Z_1}| &= |\eta^{1-1/r} - \frac{1}{r} \frac{x_N - (x_N^{1/r} - Z_1)^r}{Z_1}| \quad \eta \in [x_{N-2}, x_N] \\ &= T^{1-1/r} |(\frac{N-s}{N})^{r-1} - (\frac{N-t}{N})^{r-1}| \quad t \in [0, 1], s \in [0, 2] \\ &\leq T^{1-1/r} |(\frac{N-2}{N})^{r-1} - 1| \leq CT^{1-1/r}(r-1)\frac{2}{N} \end{aligned}$$

391

392 And

(3.127)

$$\begin{aligned} ({}_L h_{N-i}^3(\xi))'' &= 3{}_L h_{N-i}^2(\xi){}_L h_{N-i}''(\xi) + 6{}_L h_{N-i}(\xi)({}_L h_{N-i}'(\xi))^2 \\ &\leq Ch_N^2 \frac{1-r}{r} x_i^{1/r-2} (\frac{1}{r} \frac{h_N}{Z_1} - {}_L y_{N-1-i}^{1-2/r}(\xi)Z_{N-1-i}) + Ch_N(r-1)^2 h^2 x_i^{2/r-2} \end{aligned}$$

393

$$(3.128) \quad |\frac{h_N}{rZ_1} - {}_L y_{N-1-i}^{1-2/r}(\xi)Z_{N-1-i}| \leq T^{1-1/r} + Cx_N^{1-2/r}x_N^{1/r} = CT^{1-1/r}$$

394

395 So

$$\begin{aligned}
 (3.128) \quad (Lh_{N-i}^3(\xi))'' &\leq Ch_N^2 \frac{1-r}{r} x_i^{1/r-2} + C(r-1)^2 h_N x_i^{2/r-2} h^2 \\
 &\leq C(r-1) h_N^2
 \end{aligned}$$

397  $Rh_{N+1-i}^3(\xi)$  is similar. □

LEMMA 3.31.

$$398 \quad (3.129) \quad u''(Ly_{N-i}^\theta(\xi)) \leq Cx_{N-2}^{-\alpha/2-2} \leq C$$

$$399 \quad (3.130) \quad (u''(Ly_{N-i}^\theta(\xi)))' \leq C$$

$$400 \quad (3.131) \quad (u''(Ly_{N-i}^\theta(\xi)))'' \leq C$$

*Proof.*

$$\begin{aligned}
 (u''(Ly_{N-i}^\theta(\xi)))' &= u'''(Ly_{N-i}^\theta(\xi))Ly_{N-i}^{\theta'}(\xi) \\
 &\leq C(\theta Ly_{N-1-i}'(\xi) + (1-\theta)_0 y_{N-i}'(\xi)) \\
 401 \quad (3.132) \quad &\leq Cx_i^{1/r-1}(\theta Ly_{N-1-i}^{1-1/r}(\xi) + (1-\theta)\frac{h_N}{rZ_1}) \\
 &\leq Cx_i^{1/r-1}x_N^{1-1/r}
 \end{aligned}$$

402 And

(3.133)

$$\begin{aligned}
 (u''(Ly_{N-i}^\theta(\xi)))'' &= u''''(Ly_{N-i}^\theta(\xi))(Ly_{N-i}^{\theta'}(\xi))^2 + u'''(Ly_{N-i}^\theta(\xi))Ly_{N-i}^{\theta''}(\xi) \\
 403 \quad &\leq Cx_i^{2/r-2}x_N^{2-2/r} + C\frac{r-1}{r}x_i^{1/r-2}(\theta x_N^{1-2/r}Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \\
 &\leq Cx_i^{2/r-2} + C(r-1)x_i^{1/r-2}T^{1-1/r}
 \end{aligned}$$

LEMMA 3.32.

$$404 \quad (3.134) \quad |Ly_{N-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_N^\theta - x_i|^{1-\alpha}$$

$$405 \quad (3.135) \quad (|Ly_{N-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C|y_N^\theta - x_i|^{1-\alpha}$$

$$406 \quad (3.136) \quad (|Ly_{N-i}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C(r-1)|y_N^\theta - x_i|^{-\alpha} + |y_N^\theta - x_i|^{1-\alpha}$$

*Proof.*

(3.137)

$$\begin{aligned}
 (Ly_{N-i}^\theta(\xi) - \xi)' &= (\theta(Ly_{N-1-i}(\xi) - \xi) + (1-\theta)(_0y_{N-i}(\xi) - \xi))' \\
 407 \quad &= \theta(Ly_{N-1-i}'(\xi) - 1) + (1-\theta)(_0y_{N-i}'(\xi) - 1) \\
 &= \theta\xi^{1/r-1}(Ly_{N-1-i}^{1-1/r}(\xi) - \xi^{1-1/r}) + (1-\theta)\xi^{1/r-1}(\frac{h_N}{rZ_1} - \xi^{1-1/r})
 \end{aligned}$$

408

$$\begin{aligned}
 (Ly_{N-i}^\theta(\xi) - \xi)'' &= \theta(Ly_{N-1-i}''(\xi)) + (1-\theta)(_0y_{N-i}''(\xi)) \\
 409 \quad (3.138) \quad &= \frac{1-r}{r}\xi^{1/r-2}(\theta Ly_{N-1-i}^{1-2/r}(\xi)Z_{N-1-i} + (1-\theta)\frac{h_N}{rZ_1}) \leq 0
 \end{aligned}$$

410 And

$$411 \quad (3.139) \quad |(Ly_{N-i}^\theta(\xi) - \xi)''| \leq C(r-1)\xi^{1/r-2}T^{1-1/r}$$

We have known

$$(3.140) \quad C|x_{N-1} - x_i| \leq |{}_L y_{N-1-i}(\xi) - \xi| \leq C|x_{N-1} - x_i|$$

If  $\xi \leq x_{N-1}$ , then  $({}_0 y_{N-i}(\xi) - \xi)' \geq 0$ , so

$$(3.141) \quad C|x_N - x_i| \leq |x_{N-1} - x_{i-1}| \leq |{}_L y_{N-i}^\theta(\xi) - \xi| \leq |x_{N+1} - x_{i+1}| \leq C|x_N - x_i|$$

If  $i = N - 1$  and  $\xi \in [x_{N-1}, x_N]$ , then  ${}_0 y_{N-i}(\xi) - \xi$  is concave, bigger than its two neighboring points, which are equal to  $h_N$ , so

$$(3.142) \quad h_N = |x_N - x_{N-1}| \leq |{}_0 y_{N-i}(\xi) - \xi| \leq |x_{N+1} - x_{N-1}| = 2h_N$$

So we have

$$(3.143) \quad |{}_L y_{N-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_N^\theta - x_i|^{1-\alpha}$$

While

$$(3.144) \quad {}_L y_{N-1-i}^{1-1/r}(\xi) - \xi^{1-1/r} \leq ({}_L y_{N-1-i}(\xi) - \xi)\xi^{-1/r}$$

and

$$(3.145) \quad \begin{aligned} \left| \frac{h_N}{rZ_1} - \xi^{1-1/r} \right| &\leq \max\left\{ \left| \frac{h_N}{rZ_1} - x_{i-1}^{1-1/r} \right|, \left| \frac{h_N}{rZ_1} - x_{i+1}^{1-1/r} \right| \right\} \\ &\leq \max \begin{cases} T^{1-1/r} - x_{i-1}^{1-1/r} \leq |x_N - x_{i-1}|T^{-1/r} \leq C|x_N - x_i| \\ |x_{i+1}^{1-1/r} - x_{N-1}^{1-1/r}| \leq |x_{i+1} - x_{N-1}|x_{N-1}^{-1/r} \leq C|x_N - x_i| \end{cases} \end{aligned}$$

So we have

$$(3.146) \quad ({}_L y_{N-i}^\theta(\xi) - \xi)' \leq C|y_N^\theta - x_i|$$

$$(3.147) \quad \begin{aligned} (|{}_L y_{N-i}^\theta(\xi) - \xi|^{1-\alpha})' &= |{}_L y_{N-i}^\theta(\xi) - \xi|^{-\alpha} ({}_L y_{N-i}^\theta(\xi) - \xi)' \\ &\leq |y_N^\theta - x_i|^{1-\alpha} \end{aligned}$$

Finally,

$$(3.148) \quad \begin{aligned} (|{}_L y_{N-i}^\theta(\xi) - \xi|^{1-\alpha})'' &= (1-\alpha)|{}_L y_{N-i}^\theta(\xi) - \xi|^{-\alpha} ({}_L y_{N-i}^\theta(\xi) - \xi)'' \\ &\quad + \alpha(\alpha-1)|{}_L y_{N-i}^\theta(\xi) - \xi|^{-1-\alpha} ({}_L y_{N-i}^\theta(\xi) - \xi')^2 \quad \square \\ &\leq C(r-1)|y_N^\theta - x_i|^{-\alpha} + C|y_N^\theta - x_i|^{1-\alpha} \end{aligned}$$

By the three lemmas above, for  $N/2 \leq i \leq N - 1$ , we have

LEMMA 3.33.

$$(3.149) \quad \begin{aligned} D_h^2 {}_L P_{N-i}^\theta(x_i) &= {}_L P_{N-i}^{\theta''}(\xi) \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq Ch^3|y_N^\theta - x_i|^{1-\alpha} + C(r-1)(h^3|y_N^\theta - x_i|^{-\alpha} + h^2|y_N^\theta - x_i|^{1-\alpha}) \end{aligned}$$

And

LEMMA 3.34.

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_{i+1})u'''(\eta_{N+1}^\theta) - {}_L Q_{N-i}^\theta(x_i)u'''(\eta_N^\theta)}{h_{i+1}} \right) \\ & \leq Ch^3|y_N^\theta - x_i|^{1-\alpha} \end{aligned} \quad (3.150)$$

And immediately, For  $N/2 \leq i \leq N-2$

$$\begin{aligned} V_{iN} & \leq C \int_{x_{N-1}}^{x_N} h^2|y - x_i|^{1-\alpha} + C(r-1)h^2|y - x_i|^{-\alpha} + h|y - x_i|^{1-\alpha} dy \\ & \leq Ch^2h_N|T - x_i|^{1-\alpha} + C(r-1)h^2|x_{N-1} - x_i|^{1-\alpha} + Chh_N|T - x_i|^{1-\alpha} \\ & \leq Ch^2 + C(r-1)h^2|T - x_{i-1}|^{1-\alpha} \end{aligned} \quad (3.151)$$

But expecially, when  $i = N-1$ ,

$$\begin{aligned} V_{N-1,N} & = \int_0^1 -\frac{\theta^{2-\alpha}(1-\theta)}{2} \frac{2}{h_{N-1} + h_N} \left( \frac{1}{h_{N-1}} h_{N-1}^{4-\alpha} u''(y_{N-1}^\theta) - \left( \frac{1}{h_{N-1}} + \frac{1}{h_N} \right) h_N^{4-\alpha} u''(y_N^\theta) + \frac{1}{h_N} h_{N+1}^{4-\alpha} u''(y_{N+1}^\theta) \right) d\theta \\ & \quad + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_{i+1})u'''(\eta_{N+1,1}^\theta) - {}_L Q_{N-i}^\theta(x_i)u'''(\eta_{N,1}^\theta)}{h_{i+1}} \right) d\theta \\ & \quad - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_i)u'''(\eta_{N,1}^\theta) - {}_L Q_{N-i}^\theta(x_{i-1})u'''(\eta_{N-1,1}^\theta)}{h_i} \right) d\theta \\ & \quad - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_{i+1})u'''(\eta_{N+1,2}^\theta) - {}_L Q_{N-i}^\theta(x_i)u'''(\eta_{N,2}^\theta)}{h_{i+1}} \right) d\theta \\ & \quad + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{2}{h_i + h_{i+1}} \left( \frac{{}_L Q_{N-i}^\theta(x_i)u'''(\eta_{N,2}^\theta) - {}_L Q_{N-i}^\theta(x_{i-1})u'''(\eta_{N-1,2}^\theta)}{h_i} \right) d\theta \end{aligned} \quad (3.152)$$

while combine Lemma 3.30

$$\begin{aligned} & \frac{2}{h_{N-1} + h_N} \left( \frac{1}{h_{N-1}} h_{N-1}^{4-\alpha} u''(y_{N-1}^\theta) - \left( \frac{1}{h_{N-1}} + \frac{1}{h_N} \right) h_N^{4-\alpha} u''(y_N^\theta) + \frac{1}{h_N} h_{N+1}^{4-\alpha} u''(y_{N+1}^\theta) \right) \\ & = D_h^2(h_{N-1 \rightarrow N}^{4-\alpha}(x_i)u''(y_{N-1 \rightarrow N}^\theta(x_i))) \\ & \leq Ch_N^{4-\alpha} + C(r-1)h_N^{3-\alpha} \leq Ch^{4-\alpha} + C(r-1)h^2|T - x_{N-1-1}|^{1-\alpha} \end{aligned} \quad (3.153)$$

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442 Similarly with  $j = N+1$ . □



$I_6, I_7$  is easy. Similar with Lemma 3.22 and Lemma 3.6, we have

**THEOREM 3.35.** *There is a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For*  
 $N/2 \leq i \leq N,$   
(3.154)

$$I_6 = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} (T_{i-1, 2N - \lceil \frac{N}{2} \rceil + 1} + T_{i-1, 2N - \lceil \frac{N}{2} \rceil}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, 2N - \lceil \frac{N}{2} \rceil + 1} \right) \\ \leq Ch^2$$

*Proof.* In fact, let  $l = 2N - \lceil \frac{N}{2} \rceil + 1$

$$(3.155) \quad \begin{aligned} & \frac{1}{h_i} (T_{i-1, l} + T_{i-1, l-1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) T_{i, l} \\ &= \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) + \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) + \left( \frac{1}{h_i} - \frac{1}{h_{i+1}} \right) T_{i, l} \end{aligned}$$

While, by Lemma A.2

$$(3.156) \quad \begin{aligned} \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) &= \int_{x_{l-1}}^{x_l} (u(y) - \Pi_h u(y)) \frac{|x_{i-1} - y|^{1-\alpha} - |x_i - y|^{1-\alpha}}{h_i \Gamma(2-\alpha)} dy \\ &\leq C \int_{x_{l-1}}^{x_i} h_l^2 u''(\eta) \frac{|\xi - y|^{-\alpha}}{\Gamma(1-\alpha)} dy \\ &\leq Ch_l^3 x_{l-1}^{\alpha/2-2} T^{-\alpha} \\ &\leq Ch_l^3 \end{aligned}$$

Thus,

$$(3.157) \quad \frac{2}{h_i + h_{i+1}} \frac{1}{h_i} (T_{i-1, l} - T_{i, l}) \leq Ch_l^2$$

For

$$(3.158) \quad \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) = \int_0^1 -\frac{\theta(1-\theta)}{2} \frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{h_i} d\theta$$

And Similar with Lemma 3.19, we can get

$$(3.159) \quad \frac{h_{l-1}^3 |y_{l-1}^\theta - x_{i-1}|^{1-\alpha} u''(\eta_{l-1}^\theta) - h_l^3 |y_l^\theta - x_i|^{1-\alpha} u''(\eta_l^\theta)}{(h_i + h_{i+1}) h_i} \leq Ch_l^2 |y_l^\theta - x_i|^{1-\alpha}$$

So

$$(3.160) \quad \frac{2}{h_i + h_{i+1}} \frac{1}{h_i} (T_{i-1, l-1} - T_{i, l}) \leq Ch^2$$

For the third term, by Lemma B.1, Lemma B.2 and Lemma A.2

$$(3.161) \quad \begin{aligned} \frac{2}{h_i + h_{i+1}} \frac{h_{i+1} - h_i}{h_i h_{i+1}} T_{i, l} &\leq h_i^{-3} h^2 x_i^{1-2/r} h_l C h_l^2 x_{l-1}^{\alpha/2-2} |x_l - x_i|^{1-\alpha} \\ &\leq Ch^2 \end{aligned}$$

Summarizes, we have

$$(3.162) \quad I_6 \leq Ch^2$$

□

And

LEMMA 3.36. *There is a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i \leq N$ ,*

$$I_7 = \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N} S_{ij} \leq \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

*Proof.* For  $i \leq N, j \geq 2N - \lceil \frac{N}{2} \rceil + 2$ , we have

$$\begin{aligned} S_{ij} &= \int_{x_{j-1}}^{x_j} (u(y) - \Pi_h u(y)) D_h^2 \left( \frac{|y - \cdot|^{1-\alpha}}{\Gamma(2-\alpha)} \right) (x_i) dy \\ &\leq \int_{x_{j-1}}^{x_j} Ch^2 (2T - y)^{\alpha/2-2/r} |y - x_{i+1}|^{-1-\alpha} dy \\ &\leq Ch^2 T^{-1-\alpha} \int_{x_{j-1}}^{x_j} (2T - y)^{\alpha/2-2/r} dy \end{aligned}$$

$$\begin{aligned} \sum_{j=2N-\lceil \frac{N}{2} \rceil+2}^{2N-1} S_{ij} &\leq CT^{-1-\alpha} h^2 \int_{(2-2^{-r})T}^{x_{2N-1}} (2T - y)^{\alpha/2-2/r} dy \\ &\leq CT^{-1-\alpha} h^2 \begin{cases} \frac{1}{\alpha/2-2/r+1} T^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 > 0 \\ \ln(2^{-r}T) - \ln(h_{2N}), & \alpha/2 - 2/r + 1 = 0 \\ \frac{1}{|\alpha/2-2/r+1|} h_{2N}^{\alpha/2-2/r+1}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \\ &= \begin{cases} \frac{C}{\alpha/2-2/r+1} T^{-\alpha/2-2/r} h^2, & \alpha/2 - 2/r + 1 > 0 \\ CrT^{-1-\alpha} h^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ \frac{C}{|\alpha/2-2/r+1|} T^{-\alpha/2-2/r} h^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned}$$

Now we can conclude a part of the theorem Theorem 3.3 at the beginning of this section.

By Lemma 3.8 Lemma 3.22 Lemma 3.23 Theorem 3.29 Theorem 3.28 Theorem 3.35 Lemma 3.36, we have

THEOREM 3.37. *there exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that for  $N/2 \leq i < N$ ,*

$$\begin{aligned} R_i &= \sum_{j=1}^7 I_j \\ &\leq C(r-1)h^2 |T - x_{i-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases} \end{aligned}$$

And what we left is the case  $i = N$ . Fortunately, we can use the same department of  $R_i$  above, and it is symmetric. Most of the item has been esitimated by Lemma 3.8 and Theorem 3.35, we just need to consider  $I_3, I_4$ .

**THEOREM 3.38.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

$$(3.166) \quad I_3 = \sum_{j=\lceil \frac{N}{2} \rceil + 1}^{N-1} V_{Nj} \leq Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha}$$

*Proof.* **DEFINITION 3.39.** *For  $N/2 \leq j < N$ , Let's define*

$$(3.167) \quad y_j(x) = \left( \frac{Z_1}{h_N}(x - x_N) + Z_j \right)^r, \quad Z_j = T^{1/r} \frac{j}{N}$$

*We can see that is the inverse of the function  ${}_0y_{N-i}(x)$  defined in Theorem 3.29.*

$$(3.168) \quad y'_j(x) = y_j^{1-1/r}(x) \frac{rZ_1}{h_N}$$

$$(3.169) \quad y''_j(x) = y_j^{1-2/r}(x) \frac{r(r-1)Z_1}{h_N}$$

With the scheme we used several times, we can get

**LEMMA 3.40.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq j < N$ ,  $\xi \in [x_{N-1}, x_{N+1}]$ ,*

$$(3.170) \quad h_j(\xi)^3 \leq Ch^3$$

$$(3.171) \quad (h_j^3(\xi))' \leq C(r-1)h^3$$

$$(3.172) \quad (h_j^3(\xi))'' \leq C(r-1)h^3$$

$$(3.173) \quad u''(y_j^\theta(\xi)) \leq C$$

$$(3.174) \quad (u''(y_j^\theta(\xi)))' \leq C$$

$$(3.175) \quad (u''(y_j^\theta(\xi)))'' \leq C$$

$$(3.176) \quad |\xi - y_j^\theta(\xi)|^{1-\alpha} \leq C|x_N - y_j^\theta|^{1-\alpha}$$

$$(3.177) \quad (|\xi - y_j^\theta(\xi)|^{1-\alpha})' \leq C|x_N - y_j^\theta|^{1-\alpha}$$

$$(3.178) \quad (|\xi - y_j^\theta(\xi)|^{1-\alpha})'' \leq C|x_N - y_j^\theta|^{1-\alpha} + C(r-1)|x_N - y_j^\theta|^{-\alpha}$$

**LEMMA 3.41.** *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq j < N$ ,*

$$(3.179) \quad V_{Nj} \leq Ch^2 \int_{x_{j-1}}^{x_j} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy$$

Therefore,

$$(3.180) \quad \begin{aligned} I_3 &\leq Ch^2 \int_{\lceil \frac{N}{2} \rceil}^{x_{N-1}} |x_N - y|^{1-\alpha} + (r-1)|x_N - y|^{-\alpha} dy \\ &\leq Ch^2(|T - x_{N-1}|^{2-\alpha} + (r-1)|T - x_{N-1}|^{1-\alpha}) \end{aligned}$$

□

For  $j = N$ ,

LEMMA 3.42.

(3.181)

$$V_{N,N} = \frac{1}{h_N^2} (T_{N-1,N-1} - 2T_{N,N} + T_{N+1,N+1}) \leq Ch^2 + C(r-1)h^2|T - x_{N-1}|^{1-\alpha}$$

*Proof.*

(3.182)

□

$$\begin{aligned} V_{N,N} = & \int_0^1 -\frac{\theta(1-\theta)^{2-\alpha}}{2} \frac{1}{h_N^2} (h_{N-1}^{4-\alpha} u''(y_{N-1}^\theta) - 2h_N^{4-\alpha} u''(y_N^\theta) + h_{N+1}^{4-\alpha} u''(y_{N+1}^\theta)) d\theta \\ & + \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{1}{h_N} \left( \frac{Q_{N \rightarrow N}^\theta(x_{N+1}) u'''(\eta_{N+1,1}^\theta) - Q_{N \rightarrow N}^\theta(x_i) u'''(\eta_{N,1}^\theta)}{h_N} \right) d\theta \\ & - \int_0^1 \frac{\theta^3(1-\theta)}{3!} \frac{1}{h_N} \left( \frac{Q_{N \rightarrow N}^\theta(x_N) u'''(\eta_{N,1}^\theta) - Q_{N \rightarrow N}^\theta(x_{N-1}) u'''(\eta_{N-1,1}^\theta)}{h_N} \right) d\theta \\ & - \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{1}{h_N} \left( \frac{Q_{N \rightarrow N}^\theta(x_{N+1}) u'''(\eta_{N+1,2}^\theta) - Q_{N \rightarrow N}^\theta(x_N) u'''(\eta_{N,2}^\theta)}{h_N} \right) d\theta \\ & + \int_0^1 \frac{\theta(1-\theta)^3}{3!} \frac{1}{h_N} \left( \frac{Q_{N \rightarrow N}^\theta(x_N) u'''(\eta_{N,2}^\theta) - Q_{N \rightarrow N}^\theta(x_{N-1}) u'''(\eta_{N-1,2}^\theta)}{h_N} \right) d\theta \end{aligned}$$

So combine Lemma 3.8, Theorem 3.35, Theorem 3.38, Lemma 3.42 We have

LEMMA 3.43.

$$R_N \leq C(r-1)h^2|T - x_{N-1}|^{1-\alpha} + \begin{cases} Ch^2, & \alpha/2 - 2/r + 1 > 0 \\ Ch^2 \ln(N), & \alpha/2 - 2/r + 1 = 0 \\ Ch^{r\alpha/2+r}, & \alpha/2 - 2/r + 1 < 0 \end{cases}$$

and with Theorem 3.37 we prove the Theorem 3.3

**3.5. Truncation error.** combine Theorem 3.1, Theorem 3.2 and Theorem 3.3

we get For  $1 \leq i \leq N$

(3.184)

$$R_i \leq C_2(r-1)h^2|T - x_{i-1}|^{1-\alpha} + \begin{cases} C_1 h^2 x_i^{-\alpha/2-2/r}, & r\alpha/2 + r - 2 > 0 \\ C_1 h^2 (x_i^{-1-\alpha} \ln(i) + \ln(N)), & r\alpha/2 + r - 2 = 0 \\ C_1 h^{r\alpha/2+r} x_i^{-1-\alpha/2}, & r\alpha/2 + r - 2 < 0 \end{cases}$$

But,

$$h^2 x_i^{-\alpha/2-2/r} \leq T^{\alpha/2-2/r} \begin{cases} h^2 x_i^{-\alpha}, & \text{if } r\alpha/2 - 2 \geq 0 \\ h^{r\alpha/2} x_i^{-\alpha}, & \text{if } r\alpha/2 - 2 \leq 0 \end{cases}$$

$$h^{r\alpha/2+r} x_i^{-1-\alpha} \leq T^{-1} h^{r\alpha/2} x_i^{-\alpha}, \quad \text{if } r\alpha/2 - 2 \leq 0$$

(3.187)

And when  $r\alpha/2 - 2 = -r < 0$ ,

$$\begin{aligned} h^2 x_i^{-1-\alpha} \ln(i) h^{-r\alpha/2} x_i^\alpha &= h^r x_i^{-1} \ln(i) \\ &= T^{-1} \frac{\ln(i)}{i^r} \leq C(T, r) \end{aligned}$$

(3.188)

524 and

$$525 \quad (3.189) \quad h^2 \ln(N) h^{-r\alpha/2} x_i^\alpha = h^r \ln(N) x_i^\alpha \leq T^\alpha \frac{\ln(N)}{N^r} \leq C(T, \alpha, r)$$

526 So for  $1 \leq i \leq N$ ,

$$527 \quad (3.190) \quad R_i \leq C_2(r-1)h^2|T-x_{i-1}|^{1-\alpha} + C_1h^{\min\{\frac{r\alpha}{2}, 2\}}x_i^{-\alpha}$$

528 And for  $i \geq N$ , it is symmetric for  $i$  and  $2N-i$ .

529 The proof of Theorem 2.5 completed.

**4. Proof of Theorem 2.6.** Review subsection 2.1, we have (2.9) and (2.11),

$$(4.1) \quad \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left( \frac{|x_i - x_{j-1}|^{3-\alpha}}{h_j} - \frac{h_j + h_{j+1}}{h_j h_{j+1}} |x_i - x_j|^{3-\alpha} + \frac{|x_i - x_{j+1}|^{3-\alpha}}{h_{j+1}} \right)$$

$$(4.2) \quad a_{ij} = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \tilde{a}_{i-1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} \right)$$

Thus

LEMMA 4.1.

$$(4.3) \quad \sum_{j=1}^{2N-1} \tilde{a}_{ij} = \frac{1}{\Gamma(4-\alpha)} \left( \frac{|x_i - x_0|^{3-\alpha} - |x_i - x_1|^{3-\alpha}}{h_1} + \frac{|x_{2N} - x_i|^{3-\alpha} - |x_{2N-1} - x_i|^{3-\alpha}}{h_{2N}} \right)$$

DEFINITION 4.2. We call one matrix a *M matrix*, which means its entries are positive on major diagonal and nonpositive on others, and Strictly diagonally dominant in rows.

Now we have

LEMMA 4.3. The matrix *A* defined by (2.11) is a *M matrix*. and

$$(4.4) \quad \begin{aligned} S_i &:= \sum_{j=1}^{2N-1} a_{ij} \\ &= -\kappa_\alpha \sum_{j=1}^{2N-1} \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_i} \tilde{a}_{i-1,j} \right) \\ &\geq C_A (x_i^{-\alpha} + (2T - x_i)^{-\alpha}) \end{aligned}$$

*Proof.* Let

$$(4.5) \quad g(x) = g_0(x) + g_{2N}(x)$$

where

$$\begin{aligned} g_0(x) &:= \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x - x_0|^{3-\alpha} - |x - x_1|^{3-\alpha}}{h_1} \\ g_{2N}(x) &:= \frac{-\kappa_\alpha}{\Gamma(4-\alpha)} \frac{|x_{2N} - x|^{3-\alpha} - |x_{2N-1} - x|^{3-\alpha}}{h_{2N}} \end{aligned}$$

Thus

$$-\kappa_\alpha \sum_{j=1}^{2N-1} \tilde{a}_{ij} = g(x_i)$$

Then

$$(4.6) \quad \begin{aligned} S_i &:= \sum_{j=1}^{2N-1} a_{ij} \\ &= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ &= D_h^2 g_0(x_i) + D_h^2 g_{2N}(x_i) \end{aligned}$$

When  $i = 1$

$$\begin{aligned}
 D_h^2 g_0(x_1) &= \frac{2}{h_1 + h_2} \left( \frac{1}{h_2} g_0(x_2) - \left( \frac{1}{h_1} + \frac{1}{h_2} \right) g_0(x_1) + \frac{1}{h_1} g_0(x_0) \right) \\
 &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1h_2} \\
 &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{h_1^{3-\alpha} + h_2^{3-\alpha} + 2h_1^{2-\alpha}h_2 - (h_1 + h_2)^{3-\alpha}}{(h_1 + h_2)h_1^{1-\alpha}h_2} h_1^{-\alpha} \\
 &= \frac{2\kappa_\alpha}{\Gamma(4-\alpha)} \frac{1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha}}{2^r(2^r - 1)} h_1^{-\alpha}
 \end{aligned}
 \tag{4.7}$$

but

$$1 + (2^r - 1)^{3-\alpha} + 2(2^r - 1) - (2^r)^{3-\alpha} > 0$$

While for  $i \geq 2$

$$\begin{aligned}
 D_h^2 g_0(x_i) &= g_0''(\xi), \quad \xi \in (x_{i-1}, x_{i+1}) \\
 &= -\kappa_\alpha \frac{|\xi - x_0|^{1-\alpha} - |\xi - x_1|^{1-\alpha}}{\Gamma(2-\alpha)h_1} \\
 &= \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} |\xi - \eta|^{-\alpha}, \quad \eta \in [x_0, x_1] \\
 &\geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} x_{i+1}^{-\alpha} \geq \frac{\kappa_\alpha}{-\Gamma(1-\alpha)} 2^{-r\alpha} x_i^{-\alpha}
 \end{aligned}
 \tag{4.9}$$

So

$$\frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g_0(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g_0(x_i) + \frac{1}{h_i} g_0(x_{i-1}) \right) \geq C x_i^{-\alpha}$$

symmetricly,

$$\frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g_{2N}(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g_{2N}(x_i) + \frac{1}{h_i} g_{2N}(x_{i-1}) \right) \geq C(\alpha, r)(2T - x_i)^{-\alpha} \quad \square$$

Let

$$g(x) = \begin{cases} x, & 0 < x \leq T \\ 2T - x, & T < x < 2T \end{cases}$$

And define

$$G = \text{diag}(g(x_1), \dots, g(x_{2N-1}))$$

Then

LEMMA 4.4. *The matrix  $B := AG$ , the major diagonal is positive, and nonpositive on others. And there is a constant  $C_{AG}, C = C(\alpha, r)$  such that*

$$M_i := \sum_{j=1}^{2N-1} b_{ij} \geq -C_{AG}(x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases}$$

*Proof.*

$$b_{ij} = a_{ij}g(x_j) = -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} \tilde{a}_{i+1,j} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{a}_{i,j} + \frac{1}{h_i} \tilde{a}_{i-1,j} \right) g(x_j)$$

Since

$$(4.15) \quad g(x) \equiv \Pi_h g(x)$$

by (2.9), we have

$$\begin{aligned} \tilde{M}_i &:= \sum_{j=1}^{2N-1} \tilde{b}_{ij} = \sum_{j=1}^{2N-1} \tilde{a}_{ij} g(x_j) \\ &= \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} \Pi_h g(y) dy = \int_0^{2T} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} g(y) dy \\ &= \frac{-2}{\Gamma(4-\alpha)} |T - x_i|^{3-\alpha} + \frac{1}{\Gamma(4-\alpha)} (x_i^{3-\alpha} + (2T - x_i)^{3-\alpha}) \\ &:= w(x_i) = p(x_i) + q(x_i) \end{aligned}$$

Thus,

$$\begin{aligned} M_i &:= \sum_{j=1}^{2N-1} b_{ij} = \sum_{j=1}^{2N-1} a_{ij} g(x_j) \\ &= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} \tilde{M}_{i+1} - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) \tilde{M}_i + \frac{1}{h_i} \tilde{M}_{i-1} \right) \\ &= D_h^2(-\kappa_\alpha p)(x_i) - \kappa_\alpha D_h^2 q(x_i) \end{aligned}$$

for  $1 \leq i \leq N-1$ , by Lemma A.1

$$\begin{aligned} D_h^2(-\kappa_\alpha p)(x_i) &:= -\kappa_\alpha \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} p(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) p(x_i) + \frac{1}{h_i} p(x_{i-1}) \right) \\ &= \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - \xi|^{1-\alpha} \quad \xi \in (x_{i-1}, x_{i+1}) \\ &\geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} |T - x_{i-1}|^{1-\alpha} \end{aligned}$$

(4.19)

$$\begin{aligned} D_h^2(-\kappa_\alpha p)(x_N) &:= -\kappa_\alpha \frac{2}{h_N + h_{N+1}} \left( \frac{1}{h_{N+1}} p(x_{N+1}) - \left( \frac{1}{h_N} + \frac{1}{h_{N+1}} \right) p(x_N) + \frac{1}{h_N} p(x_{N-1}) \right) \\ &= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} h_N^{2-\alpha} \\ &= \frac{4\kappa_\alpha}{\Gamma(4-\alpha)} (T - x_{N-1})^{1-\alpha} \end{aligned}$$

Symmetrically for  $i \geq N$ , we get

$$(4.20) \quad D_h^2(-\kappa_\alpha p)(x_i) \geq \frac{2\kappa_\alpha}{\Gamma(2-\alpha)} \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases}$$



Similarly, we can get

$$(4.21) \quad \begin{aligned} D_h^2 q(x_i) &:= \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} q(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) q(x_i) + \frac{1}{h_i} q(x_{i-1}) \right) \\ &\leq \frac{2^{r(\alpha-1)+1}}{\Gamma(2-\alpha)} (x_i^{1-\alpha} + (2T - x_i)^{1-\alpha}), \quad i = 1, \dots, 2N-1 \end{aligned} \quad \square$$

So, we get the result.

Notice that

$$(4.22) \quad x_i^{-\alpha} \geq (2T)^{-1} x_i^{1-\alpha}$$

We can get

THEOREM 4.5. *There exists a real  $\lambda = \lambda(T, \alpha, r) > 0$  and  $C = C(T, \alpha, r) > 0$  such that  $B := A(\lambda I + G)$  is an  $M$  matrix. And*

$$(4.23) \quad M_i := \sum_{j=1}^{2N-1} b_{ij} \geq C(x_i^{-\alpha} + (2T - x_i)^{-\alpha}) + C \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases}$$

*Proof.* By Lemma 4.3 with  $C_A$  and Lemma 4.4 with  $C_{AG}$ , it's sufficient to take  $\lambda = (C + 2TC_{AG})/C_A$ , then

$$(4.24) \quad M_i \geq C \left( (x_i^{-\alpha} + (1 - x_i)^{-\alpha}) + \begin{cases} |T - x_{i-1}|^{1-\alpha}, & i \leq N \\ |x_{i+1} - T|^{1-\alpha}, & i \geq N \end{cases} \right) \quad \square$$

Now, we can prove the convergency Theorem 2.6.

For equation

$$(4.25) \quad AU = F \Leftrightarrow A(\lambda I + G)(\lambda I + G)^{-1}U = F \quad \text{i.e.} \quad B(\lambda I + G)^{-1}U = F$$

which means

$$(4.26) \quad \sum_{j=1}^{2N-1} b_{ij} \frac{\epsilon_j}{\lambda + g(x_j)} = -\tau_i$$

where  $\epsilon_i = u(x_i) - u_i$ .

And if

$$(4.27) \quad \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| = \max_{1 \leq i \leq 2N-1} \left| \frac{\epsilon_i}{\lambda + g(x_i)} \right|$$

Then, since  $B = A(\lambda I + G)$  is an  $M$  matrix, it is Strictly diagonally dominant. Thus,

$$(4.28) \quad \begin{aligned} |\tau_{i_0}| &= \left| \sum_{j=1}^{2N-1} b_{i_0,j} \frac{\epsilon_j}{\lambda + g(x_j)} \right| \\ &\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_j}{\lambda + g(x_j)} \right| \\ &\geq b_{i_0,i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| - \sum_{j \neq i_0} |b_{i_0,j}| \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \\ &= \sum_{j=1}^{2N-1} b_{i_0,j} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \\ &= M_{i_0} \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \end{aligned}$$

By Theorem 2.5 and Theorem 4.5,

We know that there exists constants  $C_1(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)}, \|f\|_{\beta}^{(\alpha/2)})$ ,

and  $C_2(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that

$$(4.29) \quad \left| \frac{\epsilon_i}{\lambda + g(x_i)} \right| \leq \left| \frac{\epsilon_{i_0}}{\lambda + g(x_{i_0})} \right| \leq C_1 h^{\min\{\frac{r\alpha}{2}, 2\}} + C_2(r-1)h^2$$

as  $\lambda + g(x_i) \leq \lambda + T$

So, we can get

$$(4.30) \quad |\epsilon_i| \leq C(\lambda + T)h^{\min\{\frac{r\alpha}{2}, 2\}}$$

The convergency has been proved.

## 5. Experimental results.

### 6. Remarks. some remarks.

In Theorem 2.3 If  $f \in L^\infty(\Omega)$  then  $u \in C_{\alpha/2}(\Omega)$ , which is Proposition 1.1 in [2].

When  $\|f\|_\beta^{(\gamma)} < \infty$ , where  $\beta > 2 - \alpha$  and  $\gamma \in [-\alpha, -\alpha/2]$ , we observed convergent order  $\min\{r(\alpha+\gamma), 2\}$  in numerical experiments. And we can prove that kind theorems with the techneque we used in this paper.

## Appendix A. Approximate of difference quotients.

LEMMA A.1. If  $g(x) \in C^2\Omega$ , there exists  $\xi \in [x_{i-1}, x_{i+1}]$  such that

$$(A.1) \quad D_h^2 g(x_i) := \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = g''(\xi), \quad \xi \in (x_{i-1}, x_{i+1})$$

$$(A.2) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_i} \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy + \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y) dy \right)$$

And if  $g(x) \in C^4(\Omega)$ , then

$$(A.3) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = g''(x_i) + \frac{h_{i+1} - h_i}{3} g'''(x_i) + \frac{1}{4!} \frac{2}{h_i + h_{i+1}} (h_i^3 g''''(\eta_1) + h_{i+1}^3 g''''(\eta_2))$$

where  $\eta_1 \in [x_{i-1}, x_i]$ ,  $\eta_2 \in [x_i, x_{i+1}]$ .

*Proof.*

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \frac{(x_i - x_{i-1})^2}{2} g''(\xi_1), \quad \xi_1 \in (x_{i-1}, x_i)$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \frac{(x_{i+1} - x_i)^2}{2} g''(\xi_2), \quad \xi_2 \in (x_i, x_{i+1})$$

Substitute them in the left side of (A.1), we have

$$\frac{2}{h_i + h_{i+1}} \left( \frac{1}{h_{i+1}} g(x_{i+1}) - \left( \frac{1}{h_i} + \frac{1}{h_{i+1}} \right) g(x_i) + \frac{1}{h_i} g(x_{i-1}) \right) \\ = \frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2)$$

Now, using **intermediate value theorem**, there exists  $\xi \in [\xi_1, \xi_2]$  such that

$$\frac{h_i}{h_i + h_{i+1}} g''(\xi_1) + \frac{h_{i+1}}{h_i + h_{i+1}} g''(\xi_2) = g''(\xi)$$

For the second equation, similarly

$$g(x_{i-1}) = g(x_i) - (x_i - x_{i-1})g'(x_i) + \int_{x_{i-1}}^{x_i} g''(y)(y - x_{i-1}) dy$$

$$g(x_{i+1}) = g(x_i) + (x_{i+1} - x_i)g'(x_i) + \int_{x_i}^{x_{i+1}} g''(y)(x_{i+1} - y)dy$$

And the last equation can be obtained by

$$g(x_{i-1}) = g(x_i) - h_i g'(x_i) + \frac{h_i^2}{2} g''(x_i) - \frac{h_i^3}{3!} g'''(x_i) + \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy$$

$$g(x_{i+1}) = g(x_i) + h_{i+1} g'(x_i) + \frac{h_{i+1}^2}{2} g''(x_i) + \frac{h_{i+1}^3}{3!} g'''(x_i) + \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy$$

Especially,

$$(A.4) \quad \begin{aligned} \int_{x_{i-1}}^{x_i} g''''(y) \frac{(y - x_{i-1})^3}{3!} dy &= \frac{h_i^4}{4!} g''''(\eta_1) \\ \int_{x_i}^{x_{i+1}} g''''(y) \frac{(x_{i+1} - y)^3}{3!} dy &= \frac{h_{i+1}^4}{4!} g''''(\eta_2) \end{aligned}$$

where  $\eta_1 \in (x_{i-1}, x_i)$ ,  $\eta_2 \in (x_i, x_{i+1})$ . Subsitute them to the left side of (A.3), we can get the result.  $\square$

LEMMA A.2. Denote  $y_j^\theta = \theta x_{j-1} + (1 - \theta)x_j$ ,  $\theta \in [0, 1]$ ,

$$(A.5) \quad u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [x_{j-1}, x_j]$$

$$(A.6) \quad u(y_j^\theta) - \Pi_h u(y_j^\theta) = -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 ((1-\theta)^2 u'''(\eta_1) - \theta^2 u'''(\eta_2))$$

where  $\eta_1 \in [x_{j-1}, y_j^\theta]$ ,  $\eta_2 \in [y_j^\theta, x_j]$ .

*Proof.* By Taylor expansion, we have

$$u(x_{j-1}) = u(y_j^\theta) - (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(\xi_1), \quad \xi_1 \in [x_{j-1}, y_j^\theta]$$

$$u(x_j) = u(y_j^\theta) + \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(\xi_2), \quad \xi_2 \in [y_j^\theta, x_j]$$

Thus

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - \theta u(x_{j-1}) - (1-\theta)u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 ((1-\theta)u''(\xi_1) + \theta u''(\xi_2)) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(\xi), \quad \xi \in [\xi_1, \xi_2] \end{aligned}$$

The second equation is similar,

$$u(x_{j-1}) = u(y_j^\theta) - (1-\theta)h_j u'(y_j^\theta) + \frac{(1-\theta)^2 h_j^2}{2!} u''(y_j^\theta) - \frac{(1-\theta)^3 h_j^3}{3!} u'''(\eta_1)$$

$$u(x_j) = u(y_j^\theta) + \theta h_j u'(y_j^\theta) + \frac{\theta^2 h_j^2}{2!} u''(y_j^\theta) + \frac{\theta^3 h_j^3}{3!} u'''(\eta_2)$$

where  $\eta_1 \in [x_{j-1}, y_j^\theta], \eta_2 \in [y_j^\theta, x_j]$ . Thus □

$$\begin{aligned} u(y_j^\theta) - \Pi_h u(y_j^\theta) &= u(y_j^\theta) - \theta u(x_{j-1}) - (1-\theta)u(x_j) \\ &= -\frac{\theta(1-\theta)}{2} h_j^2 u''(y_j^\theta) + \frac{\theta(1-\theta)}{3!} h_j^3 ((1-\theta)^2 u'''(\eta_1) - \theta^2 u'''(\eta_2)) \end{aligned}$$

LEMMA A.3. For  $x \in [x_{j-1}, x_j]$

$$\begin{aligned} |u(x) - \Pi_h u(x)| &= \left| \frac{x_j - x}{h_j} \int_{x_{j-1}}^x u'(y) dy - \frac{x - x_{j-1}}{h_j} \int_x^{x_j} u'(y) dy \right| \\ &\leq \int_{x_{j-1}}^{x_j} |u'(y)| dy \end{aligned} \quad (\text{A.7})$$

If  $x \in [0, x_1]$ , with Corollary 2.4, we have

$$|u(x) - \Pi_h u(x)| \leq \int_0^{x_1} |u'(y)| dy \leq \int_0^{x_1} C y^{\alpha/2-1} dy \leq C \frac{2}{\alpha} x_1^{\alpha/2} \quad (\text{A.8})$$

Similarly, if  $x \in [x_{2N-1}, 1]$ , we have

$$|u(x) - \Pi_h u(x)| \leq C \frac{2}{\alpha} (2T - x_{2N-1})^{\alpha/2} = C \frac{2}{\alpha} x_1^{\alpha/2} \quad (\text{A.9})$$

**Appendix B. Inequality.** For convenience, we use the notation and  $\simeq$ . That  $x_1 \simeq y_1$ , means that  $c_1 x_1 \leq y_1 \leq C_1 x_1$  for some constants  $c_1$  and  $C_1$  that are independent of mesh parameters.

LEMMA B.1.

$$h_i \leq r T^{1/r} h \begin{cases} x_i^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_{i-1})^{1-1/r}, & N < i \leq 2N-1 \end{cases} \quad (\text{B.1})$$

$$h_i \geq r T^{1/r} h \begin{cases} x_{i-1}^{1-1/r}, & 1 \leq i \leq N \\ (2T - x_i)^{1-1/r}, & N < i \leq 2N-1 \end{cases} \quad (\text{B.2})$$

*Proof.* For  $1 \leq i \leq N$ ,

$$\begin{aligned} h_i &= T \left( \left( \frac{i}{N} \right)^r - \left( \frac{i-1}{N} \right)^r \right) \\ &\leq r T \frac{1}{N} \left( \frac{i}{N} \right)^{r-1} = r T^{1/r} h x_i^{1-1/r} \end{aligned} \quad (\text{B.3})$$

$$h_i \geq r T \frac{1}{N} \left( \frac{i-1}{N} \right)^{r-1} = r T^{1/r} h x_{i-1}^{1-1/r} \quad (\text{B.4})$$

For  $N < i \leq 2N$ ,

$$\begin{aligned} h_i &= T \left( \left( \frac{2N-i+1}{N} \right)^r - \left( \frac{2N-i}{N} \right)^r \right) \\ &\leq r T \frac{1}{N} \left( \frac{2N-i+1}{N} \right)^{r-1} = r T^{1/r} h (2T - x_{i-1})^{1-1/r} \end{aligned} \quad (\text{B.5})$$

$$h_i \geq rT \frac{1}{N} \left( \frac{2N-i}{N} \right)^{r-1} = rT^{1/r} h(2T - x_i)^{1-1/r}$$

□

LEMMA B.2. *There is a constant  $C = 2^{|r-2|}r(r-1)T^{2/r}$  such that for all  $i \in \{1, 2, \dots, 2N-1\}$*

$$(B.3) \quad |h_{i+1} - h_i| \leq Ch^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases}$$

*Proof.*

$$h_{i+1} - h_i = \begin{cases} T \left( \left( \frac{i+1}{N} \right)^r - 2 \left( \frac{i}{N} \right)^r + \left( \frac{i-1}{N} \right)^r \right), & 1 \leq i \leq N-1 \\ 0, & i = N \\ -T \left( \left( \frac{2N-i-1}{N} \right)^r - 2 \left( \frac{2N-i}{N} \right)^r + \left( \frac{2N-i+1}{N} \right)^r \right), & N+1 \leq i \leq 2N-1 \end{cases}$$

For  $i = 1$ ,

$$h_2 - h_1 = T(2^r - 2) \left( \frac{1}{N} \right)^r = (2^r - 2)T^{2/r} h^2 x_1^{1-2/r}$$

For  $2 \leq i \leq N-1$ , by Lemma A.1, we have

$$\begin{aligned} h_{i+1} - h_i &= r(r-1)T N^{-2} \eta^{r-2}, \quad \eta \in \left[ \frac{i-1}{N}, \frac{i+1}{N} \right] \\ &= C(r-1)h^2 x_i^{1-2/r} \end{aligned}$$

Summarizes the inequalities, we can get

$$(B.4) \quad |h_{i+1} - h_i| \leq 2^{|r-2|}r(r-1)T^{2/r} h^2 \begin{cases} x_i^{1-2/r}, & 1 \leq i \leq N-1 \\ 0, & i = N \\ (2T - x_i)^{1-2/r}, & N < i \leq 2N-1 \end{cases}$$

□

### Appendix C. Proofs of some technical details.

*Additional proof of Theorem 3.1.* For  $2 \leq i \leq N-1$ ,

$$\begin{aligned} & \frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \\ & \leq C \frac{2}{h_i + h_{i+1}} (h_i^3 x_{i-1}^{-2-\alpha/2} + h_{i+1}^3 x_i^{-2-\alpha/2}) \\ & \leq 2C (h_i^2 x_{i-1}^{-2-\alpha/2} + h_{i+1}^2 x_i^{-2-\alpha/2}) \end{aligned}$$

There is a constant  $C = C(T, \alpha, r, \|f\|_{\beta}^{\alpha/2})$  such that

$$\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq Ch^2 x_i^{-\alpha/2-2/r}, \quad 2 \leq i \leq N-1$$

For  $i = 1$ , by (A.4)

$$\begin{aligned} & \frac{1}{4!} \frac{2}{h_1 + h_2} (h_1^3 f''(\eta_1) + h_2^3 f''(\eta_2)) \\ &= \frac{2}{h_1 + h_2} \left( \frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy + \frac{1}{4!} h_2^3 f''(\eta_2) \right) \end{aligned}$$

We have proved above that

$$\frac{2}{h_1 + h_2} h_2^3 f''(\eta_2) \leq C h^2 x_1^{-\alpha/2-2/r}$$

and we can get

$$\begin{aligned} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy &\leq C \frac{1}{3!} \int_0^{x_1} y^{1-\alpha/2} dy \\ &= C \frac{1}{3!(2-\alpha/2)} x_1^{2-\alpha/2} \end{aligned}$$

so

$$\frac{2}{h_1 + h_2} \frac{1}{h_1} \int_0^{x_1} f''(y) \frac{y^3}{3!} dy = \frac{C 2^{1-r}}{3!(2-\alpha/2)} x_1^{-\alpha/2} = \frac{C 2^{1-r}}{3!(2-\alpha/2)} T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

And for  $i = N$ , we have

$$\begin{aligned} & \frac{2}{h_N + h_{N+1}} (h_N^3 f''(\eta_1) + h_{N+1}^3 f''(\eta_2)) \\ &= h_N^2 (f''(\eta_1) + f''(\eta_2)) \\ &\leq r^2 T^{2/r} h^2 x_N^{2-2/r} 2C x_{N-1}^{-2-\alpha/2} \\ &\leq 2r^2 T^{2/r} C 2^{-r(-2-\alpha/2)} h^2 x_N^{-\alpha/2-2/r} \end{aligned}$$

Finally,  $N + 1 \leq i \leq 2N - 1$  is symmetric to the first half of the proof, so we can conclude that  $\square$

$$\frac{2}{h_i + h_{i+1}} (h_i^3 f''(\eta_1) + h_{i+1}^3 f''(\eta_2)) \leq C h^2 \begin{cases} x_i^{-\alpha/2-2/r}, & 1 \leq i \leq N \\ (2T - x_i)^{-\alpha/2-2/r}, & N \leq i \leq 2N - 1 \end{cases}$$

LEMMA C.1. By a standard error estimate for linear interpolation, and Corollary 2.4, There is a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  for  $2 \leq j \leq N$ ,

$$(C.1) \quad |u(y) - \Pi_h u(y)| \leq C h^2 y^{\alpha/2-2/r}, \quad \text{for } y \in [x_{j-1}, x_j]$$

symmetricly, for  $N < j \leq 2N - 1$ , we have

$$(C.2) \quad |u(y) - \Pi_h u(y)| \leq C h^2 (2T - y)^{\alpha/2-2/r}$$

LEMMA C.2. There is a constant  $C = C(\alpha, r)$  such that for all  $1 \leq i < N/2$ ,  $\max\{2i + 1, i + 3\} \leq j \leq 2N$ , we have

$$(C.3) \quad D_h^2 K_y(x_i) \leq C \frac{y^{-1-\alpha}}{\Gamma(-\alpha)}, \quad y \in [x_{j-1}, x_j]$$

*Proof.* Since  $y \geq x_{j-1} > x_{i+1}$ , by Lemma A.1, if  $j - 1 > i + 1$

□

$$\begin{aligned} D_h^2 K_y(x_i) &= K_y''(\xi) = \frac{|y - \xi|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq \frac{(y - x_{i+1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\leq (1 - (\frac{2}{3})^r)^{-1-\alpha} \frac{y^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

LEMMA C.3. *There is a constant  $C = C(\alpha, r)$  such that for all  $3 \leq i \leq N, k = \lceil \frac{i}{2} \rceil, 1 \leq j \leq k - 1$  and  $y \in [x_{j-1}, x_j]$ , we have*

$$(C.4) \quad D_h^2 K_y(x_i) \leq C \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)}$$

*Proof.* Since  $y \leq x_j < x_{i-1}$ , by Lemma A.1,

□

$$\begin{aligned} D_h^2 K_y(x_i) &= \frac{|\xi - y|^{-1-\alpha}}{\Gamma(-\alpha)}, \quad \xi \in [x_{i-1}, x_{i+1}] \\ &\leq \frac{(x_{i-1} - x_j)^{-1-\alpha}}{\Gamma(-\alpha)} \leq \frac{(x_{i-1} - x_{k-1})^{-1-\alpha}}{\Gamma(-\alpha)} \\ &\leq ((\frac{2}{3})^r - (\frac{1}{2})^r)^{-1-\alpha} \frac{x_i^{-1-\alpha}}{\Gamma(-\alpha)} \end{aligned}$$

LEMMA C.4. *While  $0 \leq i < N/2$ , By Lemma A.3*

$$\begin{aligned} |T_{i1}| &\leq C \int_0^{x_1} x_1^{\alpha/2} \frac{|x_i - y|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &= C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2} |x_i^{2-\alpha} - |x_i - x_1|^{2-\alpha}| \\ &\leq C \frac{1}{\Gamma(3-\alpha)} x_1^{\alpha/2+2-\alpha} = C \frac{1}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \quad 0 < 2-\alpha < 1 \end{aligned}$$

For  $2 \leq j \leq N$ , by Lemma A.2 and Corollary 2.4

$$\begin{aligned} |T_{ij}| &\leq \frac{C}{4} \int_{x_{j-1}}^{x_j} h_j^2 x_{j-1}^{\alpha/2-2} \frac{|y - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} dy \\ &\leq \frac{C}{4\Gamma(3-\alpha)} h_j^2 x_{j-1}^{\alpha/2-2} ||x_j - x_i|^{2-\alpha} - |x_{j-1} - x_i|^{2-\alpha}| \end{aligned}$$

LEMMA C.5. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that*

$$(C.7) \quad \sum_{j=1}^3 S_{1j} \leq C h^2 x_1^{-\alpha/2-2/r}$$

$$(C.8) \quad \sum_{j=1}^4 S_{2j} \leq C h^2 x_2^{-\alpha/2-2/r}$$



*Proof.*

$$S_{1j} = \frac{2}{x_2} \left( \frac{1}{x_1} T_{0j} - \left( \frac{1}{x_1} + \frac{1}{h_2} \right) T_{1j} + \frac{1}{h_2} T_{2j} \right)$$

So, by Lemma C.4

$$S_{11} \leq \frac{2}{x_2 x_1} 4 \frac{C}{\Gamma(3-\alpha)} x_1^{2-\alpha/2} \leq C x_1^{-\alpha/2}$$

$$S_{12} \leq \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_2^2 x_1^{\alpha/2-2} (x_2^{2-\alpha} + 2h_2^{2-\alpha} + h_2^{2-\alpha}) \leq C x_1^{-\alpha/2}$$

$$S_{13} \leq \frac{2}{x_2 x_1} \frac{C}{4\Gamma(3-\alpha)} h_3^2 x_2^{\alpha/2-2} (x_3^{2-\alpha} + 2x_3^{2-\alpha} + h_3^{2-\alpha}) \leq C x_1^{-\alpha/2}$$

But

$$x_1^{-\alpha/2} = T^{2/r} h^2 x_1^{-\alpha/2-2/r}$$

$i = 2$  is similar. □

LEMMA C.6. *There exists a constant  $C = C(T, r, l)$  such that For  $3 \leq i \leq N - 1$ ,  $k = \lceil \frac{i}{2} \rceil$ ,  $k \leq j \leq \min\{2i - 1, N - 1\}$ , when  $\xi \in [x_{i-1}, x_{i+1}]$ ,*

$$(C.9) \quad (h_{j-i}^3(\xi))' \leq (r-1) C h^2 x_i^{1-2/r} h_j$$

$$(C.10) \quad (h_{j-i}^4(\xi))' \leq (r-1) C h^2 x_i^{1-2/r} h_j^2$$

*Proof.* From (3.32)

$$(C.11) \quad y'_{j-i}(x) = y_{j-i}^{1-1/r}(x) x^{1/r-1}$$

$$(C.12) \quad y''_{j-i}(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x) x^{1/r-2} Z_{j-i}$$

for  $l = 3, 4$ , by (3.34)

$$(C.13) \quad \begin{aligned} (h_{j-i}^l(\xi))' &= l h_{j-i}^{l-1}(\xi) (y'_{j-i}(\xi) - y'_{j-i-1}(\xi)) \\ &= l h_{j-i}^{l-1}(\xi) \xi^{1/r-1} (y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)) \end{aligned}$$

For  $\xi \in (x_{i-1}, x_{i+1})$  and  $2 \leq k \leq j \leq \min\{2i - 1, N - 1\}$ , using Lemma B.1

$$\xi \simeq x_i \simeq x_j$$

$$\begin{aligned} h_{j-i}(\xi) &\leq h_{j-i}(x_{i+1}) = h_{j+1} \simeq h_j \\ &\simeq h x_j^{1-1/r} \simeq h x_i^{1-1/r} \end{aligned}$$

758

$$759 \quad h_{j-i}(\xi) \simeq h x_i^{1-1/r}$$

760 And

$$761 \quad (C.14) \quad 2^{-r} x_i \leq x_{i-1} \leq \xi \leq x_{i+1} \leq 2^r x_i$$

762 We have

$$763 \quad (C.15) \quad \xi^{1/r-m} \leq 2^{|mr-1|} x_i^{1/r-m}, \quad m = 1, 2$$

764 but

$$765 \quad (C.16) \quad x_j \simeq x_i, \quad \text{for } k-1 \leq j \leq \min\{2i-1, N-1\}$$

766 Then

$$\begin{aligned} 767 \quad (C.17) \quad y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) &\leq x_{j+1}^{1-1/r} - x_{j-2}^{1-1/r} \\ &= T^{1-1/r} N^{1-r} ((j+1)^{r-1} - (j-2)^{r-1}) \\ &\leq C(r-1) j^{r-2} N^{1-r} \\ &= C(r-1) h x_j^{1-2/r} \end{aligned}$$

768 Therefore,

$$769 \quad (C.18) \quad y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi) \leq C(r-1) h x_j^{1-2/r} \leq C(r-1) h x_i^{1-2/r}$$

770

771 But expecially for  $i = 3, j = 2$ ,

$$772 \quad y_{-1}^{1-1/r}(\xi) - y_{-2}^{1-1/r}(\xi) \leq \max \begin{cases} x_3^{1-1/r} - x_2^{1-1/r} \\ x_1^{1-1/r} - 0 \end{cases} \leq C(r-1) x_1^{1-1/r} \leq C(r-1) h x_3^{1-2/r}$$

773 So we can get

$$774 \quad (C.19) \quad y'_{j-i}(\xi) - y'_{j-i-1}(\xi) \leq C(r-1) h x_i^{-1/r}$$

775 We get

$$776 \quad (C.20) \quad (h_{j-i}^l(\xi))' \leq l(r-1) C h_j^{l-1} h x_i^{-1/r}$$

777 And by Lemma B.1,

$$778 \quad (C.21) \quad h_{j+1} \leq r T h \left( \frac{j+1}{N} \right)^{r-1} \leq r T h 2^{r-1} \left( \frac{j-1}{N} \right) = 2^{r-1} h_j$$

779

$$780 \quad (C.22) \quad h_{j+1} \leq r T^{1/r} h x_{j+1}^{1-1/r} \leq r T^{1/r} h x_{2i}^{1-1/r} \leq r T^{1/r} 2^{r-1} h x_i^{1-1/r}$$

781 We can get

$$\begin{aligned} 782 \quad (C.23) \quad (h_{j-i}^l(\xi))' &\leq l(r-1) C h_j^{l-2} h x_i^{-1/r} \\ &\leq l(r-1) C h h_j^{l-2} (h x_i^{1-1/r}) x_i^{-1/r} \\ &= (r-1) C h^2 x_i^{1-2/r} h_j^{l-2} \end{aligned}$$

783 Meanwhile, we can get

$$784 \quad (C.24) \quad h_{j-i}^3(\xi) \simeq h_j^3 \leq Ch^2 x_i^{2-2/r} h_j$$

$$785 \quad (C.25) \quad h_{j-i}^4(\xi) \simeq h_j^4 \leq Ch^2 x_i^{2-2/r} h_j^2 \quad \square$$

786

787 **LEMMA C.7.** *There exists a constant  $C = C(T, r, l)$  such that For  $3 \leq i \leq N -$*   
 788  *$1, \lceil \frac{i}{2} \rceil + 1 \leq j \leq \min\{2i - 1, N - 1\},$*   
 789 *when  $\xi \in (x_{i-1}, x_{i+1})$ ,*

$$790 \quad (C.26) \quad (h_{j-i}^3(\xi))'' \leq C(r-1)h^2 x_i^{-2/r} h_j$$

791 *Proof.* From (C.11)

$$\begin{aligned} (h_{j-i}^3(\xi))'' &= 6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 + 3h_{j-i}^2(\xi)(y''_{j-i}(\xi) - y''_{j-i-1}(\xi)) \\ &= 6h_{j-i}(\xi)(\xi^{1/r-1}(y_{j-i}^{1-1/r}(\xi) - y_{j-i-1}^{1-1/r}(\xi)))^2 \\ &\quad + 3\frac{1-r}{r}h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \end{aligned}$$

793 Using the inequalities of the proof of Lemma C.6

$$\begin{aligned} 6h_{j-i}(\xi)(y'_{j-i}(\xi) - y'_{j-i-1}(\xi))^2 \\ \leq Ch_j((r-1)Chx_i^{-1/r})^2 \\ \leq C(r-1)^2 h^2 x_i^{-2/r} h_j \end{aligned}$$

795 For the second partial

$$\begin{aligned} h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\ \leq Ch_{j+1}^2 x_i^{1/r-2}((y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi))Z_{j-i} + y_{j-i-1}^{1-2/r}(\xi)Z_1) \end{aligned}$$

797 but

$$\begin{aligned} y_{j-i}^{1-2/r}(\xi) - y_{j-i-1}^{1-2/r}(\xi) &= (\xi^{1/r} + Z_{j-i})^{r-2} - (\xi^{1/r} + Z_{j-i-1})^{r-2} \\ &= (r-2)Z_1(\xi^{1/r} + Z_{j-i-\gamma})^{r-3} \\ &= (r-2)T^{1/r}hy_{j-i-\gamma}^{1-3/r}(\xi) \\ &\leq C(r-2)hx_i^{1-3/r} \end{aligned}$$

799 So we can get

$$\begin{aligned} h_{j-i}^2(\xi)\xi^{1/r-2}(y_{j-i}^{1-2/r}(\xi)Z_{j-i} - y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1}) \\ \leq Ch_j hx_i^{1-1/r} x_i^{1/r-2}(C(r-2)hx_i^{1-3/r}Z_{j-i} + Cx_i^{1-2/r}T^{1/r}h) \\ \leq Ch^2((r-2)x_i^{-3/r}x_{|j-i|}^{1/r} + x_i^{-2/r})h_j \\ \leq Ch^2 x_i^{-2/r} h_j \end{aligned}$$

801 Summarizes, we have

$$802 \quad (C.32) \quad (h_{j-i}^3(\xi))'' \leq C(r-1)h^2 x_i^{-2/r} h_j \quad \square$$

proof of Lemma 3.16. From (3.32)

$$(C.33) \quad y'_{j-i}(x) = y_{j-i}^{1-1/r}(x)x^{1/r-1}$$

$$(C.34) \quad y''_{j-i}(x) = \frac{1-r}{r} y_{j-i}^{1-2/r}(x)x^{1/r-2} Z_{j-i}$$

Since

$$y_{j-i}^\theta(\xi) \simeq x_j \simeq x_i$$

We have

$$(C.35) \quad u''(y_{j-i}^\theta(\xi)) \leq C(y_{j-i}^\theta(\xi))^{\alpha/2-2} \leq Cx_j^{\alpha/2-2} \leq Cx_i^{\alpha/2-2}$$

$$(C.36) \quad \begin{aligned} (u''(y_{j-i}^\theta(\xi)))' &= u'''(y_{j-i}^\theta(\xi))y_{j-i}^{\theta-1}(\xi) \\ &\leq Cx_i^{\alpha/2-3}\xi^{1/r-1}y_{j-i}^{1-1/r}(\xi) \\ &\leq Cx_i^{\alpha/2-3}x_i^{1/r-1}x_i^{1-1/r} = Cx_i^{\alpha/2-3} \end{aligned}$$

$$(C.37) \quad \begin{aligned} (u''(y_{j-i}^\theta(\xi)))'' &= u''''(y_{j-i}^\theta(\xi))(y_{j-i}^{\theta-1}(\xi))^2 + u'''(y_{j-i}^\theta(\xi))y_{j-i}^{\theta-2}(\xi) \\ &\leq Cx_i^{\alpha/2-4} + Cx_i^{\alpha/2-3}\frac{r-1}{r}x_i^{1-2/r}x_i^{1/r-2}Z_{|j-i|+1} \\ &\leq Cx_i^{\alpha/2-4} + C\frac{r-1}{r}x_i^{\alpha/2-3}x_i^{-1/r}x_i^{1/r} \\ &= Cx_i^{\alpha/2-4} \end{aligned} \quad \square$$

*Proof of Lemma 3.17.*

$$(C.38) \quad \begin{aligned} |y_{j-i}^\theta(\xi) - \xi| &= |\theta(y_{j-i-1}(\xi) - \xi) + (1-\theta)(y_{j-i}(\xi) - \xi)| \\ &= \theta|y_{j-i-1}(\xi) - \xi| + (1-\theta)|y_{j-i}(\xi) - \xi| \end{aligned}$$

Since  $|y_{j-i}(\xi) - \xi|$  is increasing about  $\xi$ , we have

$$(C.39) \quad \left(\frac{i-1}{i}\right)^r |x_j - x_i| \leq |x_{j-1} - x_{i-1}| \leq |y_{j-i}(\xi) - \xi| \leq |x_{j+1} - x_{i+1}| \leq \left(\frac{i+1}{i}\right)^r |x_j - x_i|$$

Thus,

$$(C.40) \quad \left(\frac{2}{3}\right)^r |y_j^\theta - x_i| \leq |y_{j-i}^\theta(\xi) - \xi| \leq \left(\frac{3}{4}\right)^r (\theta|x_j - x_i| + (1-\theta)|x_{j-1} - x_i|) = \left(\frac{3}{4}\right)^r |y_j^\theta - x_i|$$

$$(C.41) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}$$

Next,

$$(C.42) \quad \begin{aligned} (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' &= (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}|\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi)) - 1| \\ &\leq C|y_j^\theta - x_i|^{-\alpha}\xi^{1/r-1}|\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| \end{aligned}$$

823 Similar with (C.39), we have

$$824 \quad (C.43) \quad \begin{aligned} |y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| &\leq C|x_j^{1-1/r} - x_i^{1-1/r}| \\ &\leq C|x_j - x_i|x_i^{-1/r} \end{aligned}$$

825 So we can get

$$826 \quad (C.44) \quad \begin{aligned} &|\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi) - \xi^{1-1/r}| \\ &\leq Cx_i^{-1/r}(\theta|x_{j-1} - x_i| + (1-\theta)|x_j - x_i|) \\ &= Cx_i^{-1/r}|y_j^\theta - x_i| \end{aligned}$$

827 Combine them, we get

$$828 \quad (C.45) \quad \begin{aligned} (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' &\leq C|y_j^\theta - x_i|^{-\alpha}x_i^{1/r-1}x_i^{-1/r}|y_j^\theta - x_i| \\ &= C|y_j^\theta - x_i|^{1-\alpha}x_i^{-1} \end{aligned}$$

829 Finally, we have

$$830 \quad (C.46) \quad \begin{aligned} (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' &= \alpha(\alpha-1)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi)) - 1)^2 \\ &\quad + (1-\alpha)|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}\frac{1-r}{r}\xi^{1/r-2}|\theta y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1} + (1-\theta)y_{j-i}^{1-2/r}(\xi)Z_{j-i}| \end{aligned}$$

831 Using the inequalities above, we have

$$832 \quad (C.47) \quad \begin{aligned} &|y_{j-i}^\theta(\xi) - \xi|^{-\alpha-1}(\xi^{1/r-1}(\theta y_{j-i-1}^{1-1/r}(\xi) + (1-\theta)y_{j-i}^{1-1/r}(\xi)) - 1)^2 \\ &\leq C|y_j^\theta - x_i|^{-\alpha-1}(x_i^{-1}|y_j^\theta - x_i|)^2 \\ &= C|y_j^\theta - x_i|^{1-\alpha}x_i^{-2} \end{aligned}$$

833 And by

$$834 \quad (C.48) \quad |Z_{j-i}| = |x_j^{1/r} - x_i^{1/r}| \leq |x_j - x_i|x_i^{1/r-1}$$

835 we have

$$836 \quad (C.49) \quad \begin{aligned} &|y_{j-i}^\theta(\xi) - \xi|^{-\alpha}\xi^{1/r-2}|\theta y_{j-i-1}^{1-2/r}(\xi)Z_{j-i-1} + (1-\theta)y_{j-i}^{1-2/r}(\xi)Z_{j-i}| \\ &\leq C|y_j^\theta - x_i|^{-\alpha}x_i^{1/r-2}x_i^{1-2/r}|\theta Z_{j-i-1} + (1-\theta)Z_{j-i}| \\ &\leq C|y_j^\theta - x_i|^{-\alpha}x_i^{-2}|y_j^\theta - x_i| \\ &= C|y_j^\theta - x_i|^{1-\alpha}x_i^{-2} \end{aligned} \quad \square$$

837 *proof of Lemma 3.19.* For  $k \leq j < \min\{2i-1, N-1\}$

$$838 \quad (C.50) \quad \begin{aligned} &\frac{Q_{j-i}^\theta(x_{i+1})u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i)u'''(\eta_j^\theta)}{h_{i+1}} \\ &\frac{Q_{j-i}^\theta(x_{i+1}) - Q_{j-i}^\theta(x_i)}{h_{i+1}}u'''(\eta_{j+1}^\theta) + Q_{j-i}^\theta(x_i)\frac{u'''(\eta_{j+1}^\theta) - u'''(\eta_j^\theta)}{h_{i+1}} \\ &\leq Q_{j-i}^{\theta'}(\xi)Cx_j^{\alpha/2-3} + Q_{j-i}^\theta(x_i)Cu''''(\eta)\frac{h_i + h_{i+1}}{h_{i+1}} \end{aligned}$$

839 where  $\xi \in [x_i, x_{i+1}]$ ,  $\eta \in [x_{j-1}, x_{j+1}]$ .

840 From (3.36), by Lemma C.6 and Lemma 3.17, we have

$$\begin{aligned} 841 \quad (C.51) \quad Q_{j-i}^{\theta}{}'(\xi) &\leq Ch^2 \frac{|y_{j+1}^{\theta} - x_{i+1}|^{1-\alpha}}{\Gamma(2-\alpha)} x_{i+1}^{1-2/r} h_{j+1}^2 \\ &\leq Ch^2 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 \end{aligned}$$

842 And by defination

$$843 \quad (C.52) \quad Q_{j-i}^{\theta}(x_i) = h_j^4 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} \leq Ch^2 x_i^{2-2/r} \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2$$

844 With , we have

$$845 \quad (C.53) \quad 4^{-r} x_i \leq x_{k-1} \leq x_{j-1} < x_j \leq x_{2i-1} \leq 2^r x_i$$

846 So we have

$$\begin{aligned} 847 \quad (C.54) \quad &\frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta}) - Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}} \\ &\leq Ch^2 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{1-2/r} h_j^2 x_i^{\alpha/2-3} + Ch^2 x_i^{2-2/r} \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} h_j^2 x_{j-1}^{\alpha/2-4} \\ &= Ch^2 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j^2 \end{aligned}$$

848 while

$$849 \quad h_j \leq h_{2i-1} \leq 2^r h_i$$

850 Subsitute into the inequality above, we get the goal

$$\begin{aligned} 851 \quad (C.55) \quad &\frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^{\theta}(x_{i+1})u'''(\eta_{j+1}^{\theta}) - Q_{j-i}^{\theta}(x_i)u'''(\eta_j^{\theta})}{h_{i+1}} \right) \\ &\leq \frac{1}{h_i} Ch^2 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j 2^r h_i \\ &= Ch^2 \frac{|y_j^{\theta} - x_i|^{1-\alpha}}{\Gamma(2-\alpha)} x_i^{\alpha/2-2-2/r} h_j \end{aligned}$$

852 While, the later is similar. □

853

854 **LEMMA C.8.** *There exists a constant  $C = C(T, r)$  such that For  $N/2 \leq i < N$ ,*  
 855  *$N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$ ,  $l = 3, 4$ ,  $\xi \in [x_{i-1}, x_{i+1}]$ , we have*

$$856 \quad (C.56) \quad h_{j-i}^l(\xi) \leq Ch_j^l \leq Ch^2 h_j^{l-2}$$

$$857 \quad (C.57) \quad (h_{j-i-1}^l(\xi))' \leq C(r-1)h^2 h_j^{l-2}$$

$$858 \quad (C.58) \quad (h_{j-i}^3(\xi))'' \leq C(r-1)h^2 h_j$$

*Proof.*

$$(h_{j-i}(\xi))' = y_{j-i}'(\xi) - y_{j-i-1}'(\xi) \\ = \xi^{1/r-1}((2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}) \leq 0$$

Thus,

$$Ch_j \leq h_{j+1} \leq h_{j-i}(\xi) \leq h_{j-i}(x_{i-1}) = h_{j-1} \leq Ch_j$$

So as  $4^{-r}T \leq 2T - x_j \leq T$ ,  $2^{-r}T \leq x_i \leq T$ , we have

$$h_{j-i}^l(\xi) \leq Ch_j^l \leq Ch^2(2T - x_j)^{2-2/r} h_j^{l-2} \leq Ch^2 h_j^{l-2}$$

Since

$$|(2T - y_{j-i}(\xi))^{1-1/r} - (2T - y_{j-i-1}(\xi))^{1-1/r}| \\ = |(Z_{2N-(j-i)} - \xi^{1/r})^{r-1} - (Z_{2N-(j-1-i)} - \xi^{1/r})^{r-1}| \\ = (r-1)Z_1(Z_{2N-(j-i-\gamma)} - \xi^{1/r})^{r-2} \quad \gamma \in [0, 1] \\ \leq C(r-1)h(2T - x_j)^{1-2/r}$$

we have

$$|(h_{j-i}(\xi))'| \leq C(r-1)h(2T - x_j)^{1-2/r} x_i^{1/r-1}$$

And

$$(h_{j-i}^l(\xi))' = lh_{j-i}^{l-1}(\xi)h_{j-i}'(\xi) \\ \leq C(r-1)h_j^{l-1} h(2T - x_j)^{1-2/r} x_i^{1/r-1} \\ \leq C(r-1)h^2 h_j^{l-2} (2T - x_j)^{2-3/r} x_i^{1-1/r} \\ \leq C(r-1)h^2 h_j^{l-2}$$

(C.65)

$$(h_{j-i}^3(\xi))'' = 6h_{j-i}(\xi)(y_{j-i}'(\xi) - y_{j-i-1}'(\xi))^2 + 3h_{j-i}^2(\xi)(y_{j-i}''(\xi) - y_{j-i-1}''(\xi)) \\ \leq C(r-1)h_j h^2 + Ch_j^2 \frac{1-r}{r} \xi^{1/r-2} ((2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} - (2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-1-i)}) \\ \leq C(r-1)h_j h^2 + C(r-1)h_j^2 (C(r-2)h(2T - x_j)^{1-3/r} Z_{2N-(j-i)} + Z_1(2T - x_{j-1})^{1-2/r}) \\ \leq C(r-1)h_j h^2 + C(r-1)h_j^2 h = Ch^2 h_j$$

871

LEMMA C.9. *There exists a constant  $C = C(T, \alpha, r, \|u\|_{\beta+\alpha}^{(-\alpha/2)})$  such that For  $N/2 \leq i < N$ ,  $N+2 \leq j \leq 2N - \lceil \frac{N}{2} \rceil + 1$ ,  $\xi \in [x_{i-1}, x_{i+1}]$ , we have*

$$(C.66) \quad u''(y_{j-i}^\theta(\xi)) \leq C$$

$$(C.67) \quad (u''(y_{j-i}^\theta(\xi)))' \leq C$$

$$(C.68) \quad (u''(y_{j-i}^\theta(\xi)))'' \leq C$$

*Proof.*

$$(C.69) \quad x_{j-2} \leq y_{j-i}^\theta(\xi) \leq x_{j+1} \Rightarrow 4^{-r}T \leq 2T - y_{j-i}^\theta(\xi) \leq T$$

Thus, for  $l = 2, 3, 4$ ,

$$(C.70) \quad u^{(l)}(y_{j-i}^\theta(\xi)) \leq C(2T - y_{j-i}^\theta(\xi))^{\alpha/2-l} \leq C$$

and

$$(C.71) \quad \begin{aligned} (y_{j-i}^\theta(\xi))' &= \theta y_{j-1-i}'(\xi) + (1-\theta)y_{j-i-1}'(\xi) \\ &= \xi^{1/r-1}(\theta(2T - y_{j-1-i}(\xi))^{1-1/r} + (1-\theta)(2T - y_{j-i-1}(\xi))^{1-1/r}) \\ &\leq C(2T - x_{j-2})^{1-1/r} \leq C \end{aligned}$$

With

$$(C.72) \quad Z_{2N-j-i} \leq 2T^{1/r}$$

$$(C.73) \quad \begin{aligned} (y_{j-i}^\theta(\xi))'' &= \theta y_{j-1-i}''(\xi) + (1-\theta)y_{j-i-1}''(\xi) \\ &= \frac{1-r}{r} \xi^{1/r-2}(\theta(2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)} + (1-\theta)(2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)}) \\ &\leq C(r-1) \end{aligned}$$

Therefore,

$$(C.74) \quad \begin{aligned} (u''(y_{j-i}^\theta(\xi)))' &= u'''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta(\xi))' \\ &\leq C \end{aligned}$$

$$(C.75) \quad \begin{aligned} (u''(y_{j-i}^\theta(\xi)))'' &= u'''(y_{j-i}^\theta(\xi))(y_{j-i}^\theta(\xi))'^2 + u''''(y_{j-i}^\theta(\xi))y_{j-i}^{\theta''}(\xi) \\ &\leq C + C(r-1) = C \end{aligned} \quad \square$$

LEMMA C.10. *There exists a constant  $C = C(T, \alpha, r)$  such that*

$$(C.76) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C|y_j^\theta - x_i|^{1-\alpha}$$

$$(C.77) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' \leq C|y_j^\theta - x_i|^{-\alpha}(|2T - x_i - y_j^\theta| + h_N)$$

$$(C.78) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' \leq C(r-1)|y_j^\theta - x_i|^{-\alpha} + C|y_j^\theta - x_i|^{-1-\alpha}(|2T - x_i - y_j^\theta| + h_N)^2$$

*Proof.*

$$(C.79) \quad (y_{j-i}^\theta(\xi) - \xi)' = \theta y_{j-1-i}'(\xi) + (1-\theta)y_{j-i-1}'(\xi) - 1$$

$$(C.80) \quad \begin{aligned} |y_{j-i}'(\xi) - 1| &= \xi^{1/r-1}|(2T - y_{j-i}(\xi))^{1-1/r} - \xi^{1-1/r}| \\ &\leq \xi^{1/r-1}|2T - \xi - y_{j-i}(\xi)|\xi^{-1/r} \end{aligned}$$

$$(C.81) \quad \begin{aligned} |2T - \xi - y_{j-i}(\xi)| &\leq \max \begin{cases} |2T - x_{i-1} - x_{j-1}| \\ |2T - x_{i+1} - x_{j+1}| \end{cases} \\ &\leq |2T - x_i - x_j| + h_{i+1} + h_j \end{aligned}$$



(C.82)

$$(y_{j-i}^\theta(\xi) - \xi)'' = \theta y_{j-1-i}''(\xi) + (1 - \theta) y_{j-i}''(\xi) \\ = \frac{1-r}{r} \xi^{1/r-2} (\theta (2T - y_{j-i}(\xi))^{1-2/r} Z_{2N-(j-i)} + (1 - \theta) (2T - y_{j-i-1}(\xi))^{1-2/r} Z_{2N-(j-i-1)}) \leq 0$$

It's concave, so

$$(C.83) \quad y_{j-i}(\xi) - \xi \geq \min\{x_{j+1} - x_{i+1}, x_{j-1} - x_{i-1}\} \geq C(x_j - x_i)$$

We have

$$(C.84) \quad |y_{j-i}^\theta(\xi) - \xi|^{1-\alpha} \leq C |y_j^\theta - x_i|^{1-\alpha}$$

$$(C.85) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})' = (1 - \alpha) |y_{j-i}^\theta(\xi) - \xi|^{-\alpha} (y_{j-i}^\theta(\xi) - \xi)' \\ \leq C |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_{i+1} + h_{j-1})$$

$$(C.86) \quad (|y_{j-i}^\theta(\xi) - \xi|^{1-\alpha})'' = (1 - \alpha) |y_{j-i}^\theta(\xi) - \xi|^{-\alpha} (y_{j-i}^\theta(\xi) - \xi)'' + \alpha(\alpha - 1) |y_{j-i}^\theta(\xi) - \xi|^{-1-\alpha} (y_{j-i}^\theta(\xi) - \xi)'(\xi) - 1)^2 \\ \leq C(r - 1) |y_j^\theta - x_i|^{-\alpha} + C |y_j^\theta - x_i|^{-1-\alpha} (|2T - x_i - y_j^\theta| + h_{i+1} + h_{j-1})^2$$

*Proof.* From (3.24), by Lemma C.8 and Lemma C.10, we have  $\xi \in [x_i, x_{i+1}]$ 

$$(C.87) \quad Q_{j-i}^{\theta'}(\xi) \leq Ch^2 h_j^2 ((r - 1) |y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N))$$

$$(C.88) \quad Q_{j-i}^\theta(\xi) \leq Ch^2 h_j^2 |y_j^\theta - x_i|^{1-\alpha}$$

So use the skill in Proof 33 with Lemma C.9

$$(C.89) \quad \frac{2}{h_i + h_{i+1}} \left( \frac{Q_{j-i}^\theta(x_{i+1}) u'''(\eta_{j+1}^\theta) - Q_{j-i}^\theta(x_i) u'''(\eta_j^\theta)}{h_{i+1}} \right) \\ \leq Ch^2 h_j (|y_j^\theta - x_i|^{1-\alpha} + |y_j^\theta - x_i|^{-\alpha} (|2T - x_i - y_j^\theta| + h_N))$$

$$(C.90) \quad a^{1-\theta} |a^\theta - b^\theta| \leq |a - b|, \theta \in [0, 1]$$

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